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Chapter 45

Perfect measures and disintegrations

One of the most remarkable features of countably additive measures is that they provide us with a framework for probability theory, as described in Chapter 27. The extraordinary achievements of probability theory since Kolmogorov are to a large extent possible because of the rich variety of probability measures which can be constructed. We have already seen image measures (234C[1]) and product measures (§254). The former are elementary, but a glance at the index will confirm that they have many surprises to offer; the latter are obviously fundamental to any idea of what probability theory means. In this chapter I will look at some further constructions. The most important are those associated with 'disintegrations' or 'regular conditional probabilities' (§§452-453) and methods for confirming the existence of measures on product spaces with given images on subproducts (§454, 455A). We find that these constructions have to be based on measure spaces of special types; the measures involved in the principal results are the Radon measures of Chapter 41 (of course), the compact and perfect measures of Chapter 34, and an intermediate class, the 'countably compact' measures of Marczewski 53 (451B). So the first section of this chapter is a systematic discussion of compact, countably compact and perfect measures.

A 'disintegration', when present, is likely to provide us with a particularly effective instrument for studying a measure, analogous to Fubini's theorem for product measures (see 452F). §§452-453 therefore concentrate on theorems guaranteeing the existence of disintegrations compatible with some pre-existing structure, typically an inverse-measure-preserving function (452I, 452O, 453K) or a product structure (452M). Both depend on the existence of suitable liftings, and for the topological version in §453 we need a 'strong' lifting, so much of that section is devoted to the study of such liftings.

One of the central concerns of probability theory is to understand 'stochastic processes', that is, models of systems evolving randomly over time. If we think of our state space as consisting of functions, so that a whole possible history is described by a random function of time, it is natural to think of our functions as members of some set $\prod_{n\in\mathbb{N}} Z_n$ (if we think of observations as being taken at discrete time intervals) or $\prod_{t\in[0,\infty[} Z_t$ (if we regard our system as evolving continuously), where Z_t represents the set of possible states of the system at time t. We are therefore led to consider measures on such product spaces, and the new idea is that we may have some definite intuition concerning the joint distribution of *finite* strings $(f(t_0),\dots,f(t_n))$ of values of our random function, that is to say, we may think we know something about the image measures on finite products $\prod_{i\le n} Z_{t_i}$. So we come immediately to a fundamental question: given a (probability) measure μ_J on $\prod_{i\in J} Z_i$ for each finite $J\subseteq T$, when will there be a measure on $\prod_{i\in T} Z_i$ compatible with every μ_J? In §454 I give the most important generally applicable existence theorems for such measures, and in 455A-455E I show how they can be applied to a general construction for models of Markov processes. These models enable us to discuss the Markov property either in terms of disintegrations or in terms of conditional expectations (455C, 455O), and for Lévy processes, in terms of inverse-measure-preserving functions (455U).

The abstract theory of §454 yields measures on product spaces which, from the point of view of a probabilist, are unnaturally large, often much larger than intuition suggests. Some of the most powerful results in the theory of Markov processes, such as the strong Markov property (455O), depend on moving to much smaller spaces; most notably the space of càdlàg functions (455G), but the larger space of càllàl functions is also of interest. The most important example, Brownian motion, will have to wait for Chapter 47, but I give the basic general theory of Lévy processes in complete metric groups.

One of the defining characteristics of Brownian motion is the fact that all its finite-dimensional marginals are Gaussian distributions. Stochastic processes with this property form a particularly interesting class,

[1]Formerly 112E.

which I examine in §456. From the point of view of this volume, one of their most striking properties is Talagrand's theorem that, regarded as measures on powers $\mathbb{R}^I$, they are τ-additive (456O).

The next two sections look again at some of the ideas of the previous sections when interpreted as answers to questions of the form 'can all the measures in such-and-such a family be simultaneously extended to a single measure?' If we seek only a *finitely* additive common extension, there is a reasonably convincing general result (457A); but countably additive measures remain puzzling even in apparently simple circumstances (457Z). In §458 I introduce 'relatively independent' families of σ-algebras, with the associated concept of 'relative product' of measures, and the corresponding concepts for probability algebras. Finally, in §459, I give some basic results on symmetric measures and exchangeable random variables, with De Finetti's theorem (459C) and corresponding theorems on representing permutation-invariant measures on products as mixtures of product measures (459E, 459H).

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451 Perfect, compact and countably compact measures

In §§342-343 I introduced 'compact' and 'perfect' measures as part of a study of the representation of homomorphisms of measure algebras by functions between measure spaces. An intermediate class of 'countably compact' measures (the 'compact' measures of MARCZEWSKI 53) has appeared in the exercises. It is now time to collect these ideas together in a more systematic way. In this section I run through the standard properties of compact, countably compact and perfect measures (451A-451J), with a couple of simple examples of their interaction with topologies (451M-451P). An example of a perfect measure space which is not countably compact is in 451U. Some new ideas, involving non-trivial set theory, show that measurable functions from compact totally finite measure spaces to metrizable spaces have 'essentially separable ranges' (451R); consequently, any measurable function from a Radon measure space to a metrizable space is almost continuous (451T).

451A Let me begin by recapitulating the principal facts already covered.

(a) A family $\mathcal{K}$ of sets is a **compact class** if $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property. If $\mathcal{K} \subseteq \mathcal{P}X$, then $\mathcal{K}$ is a compact class iff there is a compact topology on X for which every member of $\mathcal{K}$ is closed (342D). A subfamily of a compact class is compact (342Ab).

(b) A measure on a set X is **compact** if it is inner regular with respect to some compact class of sets; equivalently, if it is inner regular with respect to the closed sets for some compact topology on X (342F). All Radon measures are compact measures (416Wa). If (X, Σ, μ) is a semi-finite compact measure space with measure algebra $\mathfrak{A}$, (Y, T, ν) is a complete strictly localizable measure space with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a function $g : Y \to X$ such that $g^{-1}[E] \in \mathrm{T}$ and $g^{-1}[E]^\bullet = \pi(E^\bullet)$ for every $E \in \Sigma$ (343B).

(c) A family $\mathcal{K}$ of sets is a **countably compact class** if $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ whenever $\langle K_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{i\le n} K_i \neq \emptyset$ for every $n \in \mathbb{N}$. Any subfamily of a countably compact class is countably compact. If $\mathcal{K}$ is a countably compact class, then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ which is closed under finite unions and countable intersections (413T).

(d) A measure space (X, Σ, μ) is **perfect** if whenever $f : X \to \mathbb{R}$ is measurable, $E \in \Sigma$ and $\mu E > 0$, there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$. A countably separated semi-finite measure space is compact iff it is perfect (343K). A measure space (X, Σ, μ) is isomorphic to the unit interval with Lebesgue measure iff it is an atomless complete countably separated perfect probability space (344Ka).

451B Now for the new class of measures.

Definition Let (X, Σ, μ) be a measure space. Then (X, Σ, μ), or μ, is **countably compact** if μ is inner regular with respect to some countably compact class of sets.

Evidently compact measures are also countably compact. A simple example of a countably compact measure which is not compact is the countable-cocountable measure on an uncountable set (342M). For an example of a perfect measure which is not countably compact, see 451U.

Note that if μ is inner regular with respect to a countably compact class $\mathcal{K}$, then it is also inner regular with respect to $\mathcal{K}\cap\Sigma$ (411B), and $\mathcal{K}\cap\Sigma$ is still countably compact.

451C Proposition (Ryll-Nardzewski 53) Any semi-finite countably compact measure is perfect.

proof The central idea is the same as in 342L, but we need to refine the second half of the argument.

(a) Let (X,Σ,μ) be a countably compact measure space, $f:X\to\mathbb{R}$ a measurable function, and $E\in\Sigma$ a set of positive measure. Let $\mathcal{K}$ be a countably compact class such that μ is inner regular with respect to $\mathcal{K}$; by 451Ac, we may suppose that $\mathcal{K}$ is closed under finite unions and countable intersections.

Because μ is semi-finite, there is a measurable set $F\subseteq E$ such that $0<\mu F<\infty$; replacing F by a set of the form $F\cap f^{-1}[[-n,n]]$ if necessary, we may suppose that $f[F]$ is bounded; finally, we may suppose that $F\in\mathcal{K}$. Let $\langle\epsilon_q\rangle_{q\in\mathbb{Q}}$ be a family of strictly positive real numbers such that $\sum_{q\in\mathbb{Q}}\epsilon_q<\frac{1}{2}\mu F$. For each $q\in\mathbb{Q}$, set $E_q=\{x:x\in F,\ f(x)\le q\}$, $E'_q=\{x:x\in F,\ f(x)>q\}$, and choose K_q, $K'_q\in\mathcal{K}\cap\Sigma$ such that $K_q\subseteq E_q$, $K'_q\subseteq E'_q$ and $\mu(E_q\setminus K_q)\le\epsilon_q$, $\mu(E'_q\setminus K'_q)\le\epsilon_q$. Then $K=\bigcap_{q\in\mathbb{Q}}(K_q\cup K'_q)\in\mathcal{K}\cap\Sigma$, $K\subseteq F$ and

$$\mu(F\setminus K)\le\sum\nolimits_{q\in\mathbb{Q}}\mu(E_q\setminus K_q)+\mu(E'_q\setminus K'_q)<\mu F,$$

so $\mu K>0$.

(b) Take any $t\in\overline{f[K]}$. Enumerate $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$ and define $\langle L_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ by the rule

$$\begin{aligned}L_n&=K_{q_n}\text{ if }t<q_n,\\&=K'_{q_n}\text{ if }t>q_n,\\&=F\text{ if }t=q_n.\end{aligned}$$

Now $\bigcap_{i\le n}L_i\ne\emptyset$ for every $n\in\mathbb{N}$. **P** Because $t\in\overline{f[K]}$, there must be some $s\in f[K]$ such that $s<q_i$ whenever $i\le n$ and $t<q_i$, while $s>q_i$ whenever $i\le n$ and $t>q_i$. Let $x\in K$ be such that $f(x)=s$. Then, for any $i\le n$,

either $t<q_i$, $f(x)<q_i$ so $x\notin K'_{q_i}$ and $x\in K_{q_i}=L_i$

or $t>q_i$, $f(x)>q_i$ so $x\notin K_{q_i}$ and $x\in K'_{q_i}=L_i$,

or $t=q_i$ and $x\in F=L_i$.

So $x\in\bigcap_{i\le n}L_i$. **Q**

As $\mathcal{K}$ is a countably compact class, there must be some $x\in\bigcap_{n\in\mathbb{N}}L_n$. But this means that, for any $n\in\mathbb{N}$,

if $t>q_n$ then $x\in K'_{q_n}$ and $f(x)>q_n$,

if $t<q_n$ then $x\in K_{q_n}$ and $f(x)<q_n$.

So in fact $f(x)=t$. Accordingly $t\in f[K]$.

(c) What this shows is that $\overline{f[K]}\subseteq f[K]$ and $f[K]$ is closed. Because (by the choice of F) it is also bounded, it is compact (2A2F). Of course we now have $f[K]\subseteq f[E]$, while $\mu f^{-1}[f[K]]\ge\mu K>0$. As f and E are arbitrary, μ is perfect.

451D Proposition Let (X,Σ,μ) be a measure space, and $E\in\Sigma$; let μ_E be the subspace measure on E.

(a) If μ is compact, so is μ_E.

(b) If μ is countably compact, so is μ_E.

(c) If μ is perfect, so is μ_E.

proof (a)-(b) Let $\mathcal{K}$ be a (countably) compact class such that μ is inner regular with respect to $\mathcal{K}$. Then μ_E is inner regular with respect to $\mathcal{K}$ (412Oa), so is (countably) compact.

(c) Suppose that $f:E\to\mathbb{R}$ is Σ_E-measurable, where $\Sigma_E=\Sigma\cap\mathcal{P}E$ is the subspace σ-algebra, and $F\subseteq E$ is such that $\mu F>0$. Set

$$g(x) = \arctan f(x) \text{ if } x \in E,$$
$$= 2 \text{ if } x \in X \setminus E.$$

Then g is Σ-measurable, so there is a compact set $K \subseteq g[F]$ such that $\mu g^{-1}[K] > 0$. Set $L = \{\tan t : t \in K\}$; then $L \subseteq f[F]$ is compact and $f^{-1}[L] = g^{-1}[K]$ has non-zero measure. As f and F are arbitrary, μ_E is perfect.

451E Proposition Let (X, Σ, μ) be a perfect measure space.

(a) If (Y, T, ν) is another measure space and $f : X \to Y$ is an inverse-measure-preserving function, then ν is perfect.

(b) In particular, $\mu{\restriction}\mathrm{T}$ is perfect for any σ-subalgebra T of Σ.

proof (a) Suppose that $g : Y \to \mathbb{R}$ is T-measurable and $F \in \mathrm{T}$ is such that $\nu F > 0$. Then $gf : X \to \mathbb{R}$ is Σ-measurable and $\mu f^{-1}[F] > 0$. So there is a compact set $K \subseteq (gf)[f^{-1}[F]]$ such that $\mu(gf)^{-1}[K] > 0$. But now $K \subseteq g[F]$ and $\nu g^{-1}[K] > 0$. As g and F are arbitrary, ν is perfect.

(b) Apply (a) to $Y = X$, $\nu = \mu{\restriction}\mathrm{T}$ and f the identity function.

Remark We shall see in 452R that there is a similar result for countably compact measures; but for compact measures, there is not (342Xf, 451Xh).

451F Lemma (Sazonov 66) Let (X, Σ, μ) be a semi-finite measure space. Then the following are equiveridical:

(i) μ is perfect;

(ii) $\mu{\restriction}\mathrm{T}$ is compact for every countably generated σ-subalgebra T of Σ;

(iii) $\mu{\restriction}\mathrm{T}$ is perfect for every countably generated σ-subalgebra T of Σ;

(iv) for every countable set $\mathcal{E} \subseteq \Sigma$ there is a σ-algebra $\mathrm{T} \supseteq \mathcal{E}$ such that $\mu{\restriction}\mathrm{T}$ is perfect.

proof (a)(i)$\Rightarrow$(ii) Suppose that μ is perfect, and that T is a countably generated σ-subalgebra of Σ. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a sequence in T which σ-generates it, and define $f : X \to \mathbb{R}$ by setting $f(x) = \sum_{n=0}^{\infty} 3^{-n}\chi E_n(x)$ for every $x \in X$. Then f is measurable. Set $\mathcal{K} = \{f^{-1}[L] : L \subseteq f[X] \text{ is compact}\}$. Then $\mathcal{K}$ is a compact class. **P** If $\mathcal{K}' \subseteq \mathcal{K}$ is non-empty and has the finite intersection property, then $\mathcal{L}' = \{L : L \subseteq f[X] \text{ is compact}, f^{-1}[L] \in \mathcal{K}'\}$ is also a non-empty family with the finite intersection property. So there is an $\alpha \in \bigcap \mathcal{L}'$; since $\alpha \in f[X]$, there is an x such that $f(x) = \alpha$, and now $x \in \bigcap \mathcal{K}'$. As $\mathcal{K}'$ is arbitrary, $\mathcal{K}$ is a compact class. **Q**

Observe next that, for any $n \in \mathbb{N}$,

$$E_n = \{x : \exists\, I \subseteq n, \textstyle\sum_{i\in I} 3^{-i} + 3^{-n} \le f(x) < \sum_{i\in I} 3^{-i} + 3^{-n+1}\}.$$

So $\mathrm{T}' = \{f^{-1}[F] : F \subseteq \mathbb{R}\}$ contains every E_n; as it is a σ-algebra of subsets of X, it includes T.

Now $\mu{\restriction}\mathrm{T}$ is inner regular with respect to $\mathcal{K}$. **P** If $E \in \mathrm{T}$ and $\mu E > 0$, there is a set $F \subseteq \mathbb{R}$ such that $E = f^{-1}[F]$. Because f is Σ-measurable and μ is perfect, there is a compact set $L \subseteq f[E]$ such that $\mu f^{-1}[L] > 0$. But now $f^{-1}[L] \in \mathcal{K} \cap \mathrm{T}$, and $f^{-1}[L] \subseteq E$ because $L \subseteq F$. Because $\mathcal{K}$ is closed under finite unions, this is enough to show that $\mu{\restriction}\mathrm{T}$ is inner regular with respect to $\mathcal{K}$. **Q**

Thus $\mathcal{K}$ witnesses that $\mu{\restriction}\mathrm{T}$ is a compact measure.

(b)(ii)$\Rightarrow$(i) Now suppose that $\mu{\restriction}\mathrm{T}$ is compact for every countably generated σ-algebra $\mathrm{T} \subseteq \Sigma$, that $f : X \to \mathbb{R}$ is a measurable function, and that $\mu E > 0$. Let $F \subseteq E$ be a measurable set of non-zero finite measure, and T the σ-algebra generated by $\{F\} \cup \{f^{-1}[\,]{-\infty}, q[\,] : q \in \mathbb{Q}\}$, so that T is countably generated and f is T-measurable. Because $\mu{\restriction}\mathrm{T}$ is compact, so is the subspace measure $(\mu{\restriction}\mathrm{T})_F$ (451Da); but this is now perfect (342L or 451C), while $F \in \mathrm{T}$ and $\mu F > 0$, so there is a compact set $L \subseteq f[F] \subseteq f[E]$ such that $\mu f^{-1}[L] > 0$. As f and E are arbitrary, μ is perfect.

(c)(i)$\Rightarrow$(iv) is trivial.

(d)(iv)$\Rightarrow$(iii) If (iv) is true, and T is a countably generated σ-subalgebra of Σ, let $\mathcal{E}$ be a countable set generating it. Then there is a σ-algebra $\mathrm{T}_1 \supseteq \mathcal{E}$ such that $\mu{\restriction}\mathrm{T}_1$ is perfect. By 451Eb, $\mu{\restriction}\mathrm{T} = (\mu{\restriction}\mathrm{T}_1){\restriction}\mathrm{T}$ is compact, therefore perfect.

(e)(iii)⇒(ii) If (iii) is true, and T is a countably generated σ-subalgebra of Σ, then $\mu\restriction\mathrm{T}$ is perfect; but as (i)⇒(ii), and T is a countably generated σ-subalgebra of itself, $\mu\restriction\mathrm{T}$ is compact.

451G Proposition Let (X,Σ,μ) be a measure space. Let $(X,\hat{\Sigma},\hat{\mu})$ be its completion and $(X,\tilde{\Sigma},\tilde{\mu})$ its c.l.d. version. Then

(a)(i) if μ is compact, so are $\hat{\mu}$ and $\tilde{\mu}$;

(ii) if μ is semi-finite and either $\hat{\mu}$ or $\tilde{\mu}$ is compact, then μ is compact.

(b)(i) If μ is countably compact, so are $\hat{\mu}$ and $\tilde{\mu}$;

(ii) if μ is semi-finite and either $\hat{\mu}$ or $\tilde{\mu}$ is countably compact, then μ is countably compact.

(c)(i) If μ is perfect, so are $\hat{\mu}$ and $\tilde{\mu}$;

(ii) if $\hat{\mu}$ is perfect, then μ is perfect;

(iii) if μ is semi-finite and $\tilde{\mu}$ is perfect, then μ is perfect.

proof (a)-(b) The arguments for $\hat{\mu}$ and $\tilde{\mu}$ run very closely together. Write $\check{\mu}$ for either of them, and $\check{\Sigma}$ for its domain.

(i) If μ is inner regular with respect to $\mathcal{K}$, so is $\check{\mu}$ (412Ha). So if μ is (countably) compact, so is $\check{\mu}$.

(ii) Now suppose that μ is semi-finite. The point is that if $\mathcal{K}$ is closed under countable intersections and $\check{\mu}$ is inner regular with respect to $\mathcal{K}$, so is μ. **P** Suppose that $E\in\Sigma$ and that $\mu E>\gamma$. Choose sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ and $\mathcal{K}_n$ in $\mathcal{K}$ inductively, as follows. E_0 is to be such that $E_0\subseteq E$ and $\gamma<\mu E_0<\infty$. Given that $\gamma<\mu E_n<\infty$, let $K_n\in\mathcal{K}\cap\check{\Sigma}$ be such that $K_n\subseteq E_n$ and $\check{\mu}K_n>\gamma$; now take $E_{n+1}\in\Sigma$ such that $E_{n+1}\subseteq K_n$ and $\mu E_{n+1}=\check{\mu}K_n$ (212C or 213Fc), and continue. At the end of the induction, $\bigcap_{n\in\mathbb{N}}K_n=\bigcap_{n\in\mathbb{N}}E_n$ is a member of $\Sigma\cap\mathcal{K}$ included in E and of measure at least γ. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$. **Q**

It follows that if $\check{\mu}$ is compact or countably compact, so is μ. **P** Let $\mathcal{K}$ be a (countably) compact class such that $\check{\mu}$ is inner regular with respect to $\mathcal{K}$; by 451Aa or 451Ac, there is a (countably) compact class $\mathcal{K}^*$, including $\mathcal{K}$, which is closed under countable intersections, so that μ is inner regular with respect to $\mathcal{K}^*$, and is itself (countably) compact. **Q**

(c)(i)($\boldsymbol{\alpha}$) Let $f:X\to\mathbb{R}$ be $\hat{\Sigma}$-measurable, and $E\in\hat{\Sigma}$ such that $\hat{\mu}E>0$. Then there are a μ-conegligible set $F_0\in\Sigma$ such that $f\restriction F_0$ is Σ-measurable (212Fa), and an $F_1\in\Sigma$ such that $F_1\subseteq E$ and $\hat{\mu}(E\setminus F_1)=0$. Set $F=F_0\cap F_1$. By 451Dc, the subspace measure μ_F is perfect, while $f\restriction F$ is Σ_F-measurable; so there is a compact set $K\subseteq f[F]$ such that $\mu(F\cap f^{-1}[K])>0$. But now $K\subseteq f[E]$ and $\hat{\mu}f^{-1}[K]>0$. As f and E are arbitrary, $\hat{\mu}$ is perfect.

($\boldsymbol{\beta}$) Let $f:X\to\mathbb{R}$ be $\tilde{\Sigma}$-measurable, and $E\in\tilde{\Sigma}$ such that $\tilde{\mu}E>0$. Then there is a set $F\in\Sigma$ such that $\mu F<\infty$ and $\hat{\mu}(F\cap E)$ is defined and greater than 0 (213D). In this case, $\hat{\mu}$ and $\tilde{\mu}$ induce the same subspace measure $\hat{\mu}_F$ on F. Accordingly $f\restriction F$ is $\hat{\Sigma}$-measurable. Because $\hat{\mu}$ is perfect (by (α) just above), so is $\hat{\mu}_F$ (451Dc), and there is a compact set $K\subseteq f[F\cap E]$ such that $\hat{\mu}_F(f\restriction F)^{-1}[K]>0$. But now, of course, $K\subseteq f[E]$ and $\tilde{\mu}f^{-1}[K]>0$. As f and E are arbitrary, $\tilde{\mu}$ is perfect.

(ii) Suppose that $\hat{\mu}$ is perfect. Since $\mu=\hat{\mu}\restriction\Sigma$, μ is perfect, by 451Eb.

(iii) Similarly, if $\tilde{\mu}$ is perfect and μ is semi-finite, then $\mu=\tilde{\mu}\restriction\Sigma$, by 213Hc, so μ is perfect.

451H Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X. Suppose that $\mathcal{K}_i\subseteq\mathcal{P}X_i$ for each $i\in I$, and set $\mathcal{K}=\{\pi_i^{-1}[K]:i\in I,\,K\in\mathcal{K}_i\}$, where $\pi_i:X\to X_i$ is the coordinate map for each $i\in I$. Then

(a) if every $\mathcal{K}_i$ is a compact class, so is $\mathcal{K}$;

(b) if every $\mathcal{K}_i$ is a countably compact class, so is $\mathcal{K}$.

proof (a) For each $i\in I$, let $\mathfrak{T}_i$ be a compact topology on X_i such that every member of $\mathcal{K}_i$ is closed. Then the product topology $\mathfrak{T}$ on X is compact (3A3J), and every member of $\mathcal{K}$ is $\mathfrak{T}$-closed, so $\mathcal{K}$ is a compact class.

(b) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{k\le n}K_k\ne\emptyset$ for every $n\in\mathbb{N}$, then we must be able to express each K_n as $\pi_{j_n}^{-1}[L_n]$, where $j_n\in I$ and $L_n\in\mathcal{K}_{j_n}$ for every n. Now, for $i\in I$, $\mathcal{L}_i=\{K_{j_n}:n\in\mathbb{N},\,j_n=i\}$ is a countable subset of $\mathcal{K}_i$, and any finite subfamily of $\mathcal{L}_i$ has non-empty intersection. Since $K_0 \neq \emptyset$, $X_i \neq \emptyset$; so, whether $\mathcal{L}_i$ is empty or not, $X_i \cap \bigcap \mathcal{L}_i$ is non-empty. Accordingly

$$\bigcap_{k\in\mathbb{N}} K_k = \prod_{i\in I}(X_i \cap \bigcap \mathcal{L}_i)$$

is not empty. As $\langle K_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}$ is countably compact.

451I Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with c.l.d. product $(X\times Y,\Lambda,\lambda)$.

(a) If μ and ν are compact, so is λ.

(b) If μ and ν are countably compact, so is λ.

(c) If μ and ν are perfect, so is λ.

proof (a)-(b) Let $\mathcal{K} \subseteq \mathcal{P}X$, $\mathcal{L} \subseteq \mathcal{P}Y$ be (countably) compact classes such that μ is inner regular with respect to $\mathcal{K}$ and ν is inner regular with respect to $\mathcal{L}$. Set $\mathcal{M}_0 = \{K\times Y : K\in\mathcal{K}\}\cup\{X\times L : L\in\mathcal{L}\}$. Then $\mathcal{M}_0$ is (countably) compact, by 451H. By 451Aa/451Ac, there is a (countably) compact class $\mathcal{M}\supseteq\mathcal{M}_0$ which is closed under finite unions and countable intersections. By 412R, λ is inner regular with respect to $\mathcal{M}$, so is (countably) compact.

(c)(i) Let $f : X\times Y\to\mathbb{R}$ be Λ-measurable, and $V\in\Lambda$ a set of positive measure. Then there are $G\in\Sigma$, $H\in\mathrm{T}$ such that μG, νH are both finite and $\lambda(V\cap(G\times H)) > 0$. Recall that the subspace measure $\lambda_{G\times H}$ on $G\times H$ is just the product of the subspace measures μ_G and μ_H (251P(ii-α)), and is the completion of its restriction θ to the σ-algebra $\Sigma_G\widehat{\otimes}\mathrm{T}_H$ generated by $\{E\times F : E\in\Sigma_G,\ F\in\mathrm{T}_H\}$, where Σ_G and T_H are the subspace σ-algebras on G, H respectively, the domains of μ_G and μ_H (251K). Next, for any $W\in\Sigma_G\widehat{\otimes}\mathrm{T}_H$, there are countable families $\mathcal{E}\subseteq\Sigma_G$, $\mathcal{F}\subseteq\mathrm{T}_H$ such that W belongs to the σ-algebra of subsets of $G\times H$ generated by $\{E\times F : E\in\mathcal{E},\ F\in\mathcal{F}\}$ (331Gd).

(ii) The point is that θ is perfect. **P** Let Λ' be any countably generated σ-subalgebra of $\Sigma_G\widehat{\otimes}\mathrm{T}_H$; let $\langle W_n\rangle_{n\in\mathbb{N}}$ be a sequence in Λ' generating it. Then there are countable families $\mathcal{E}\subseteq\Sigma_G$, $\mathcal{F}\subseteq\mathrm{T}_H$ such that every W_n belongs to the σ-algebra generated by $\{E\times F : E\in\mathcal{E},\ F\in\mathcal{F}\}$. Let Σ', T' be the σ-algebras of subsets of G and H generated by $\mathcal{E}$ and $\mathcal{F}$ respectively; then every W_n belongs to $\Sigma'\widehat{\otimes}\mathrm{T}'$, so $\Lambda'\subseteq\Sigma'\widehat{\otimes}\mathrm{T}'$. Let λ' be the product of the measures $\mu\restriction\Sigma' = \mu_G\restriction\Sigma'$ and $\nu\restriction\mathrm{T}'$. Then λ' is the completion of its restriction to $\Sigma'\widehat{\otimes}\mathrm{T}'$.

Now trace through the results above. μ_G and ν_H are perfect (451Dc), so $\mu_G\restriction\Sigma'$ and $\nu_H\restriction\mathrm{T}'$ are compact (451F), so λ' is compact ((a) of this theorem), so λ' is perfect (342L or 451C again). But θ must agree with λ' on Λ', by Fubini's theorem (252D), or otherwise, so $\theta\restriction\Lambda'$ is a restriction of λ', and is perfect (451Eb).

Thus $\theta\restriction\Lambda'$ is perfect for every countably generated σ-subalgebra Λ' of $\operatorname{dom}\theta$. By 451F, θ is perfect. **Q**

(iii) By 451G(c-i), $\lambda_{G\times H}$ is perfect. Now $f\restriction G\times H$ is measurable, and $\lambda_{G\times H}(V\cap(G\times H)) > 0$, so there is a compact set $K\subseteq f[V\cap(G\times H)]$ such that $\lambda_{G\times H}((G\times H)\cap f^{-1}[K]) > 0$; in which case $K\subseteq f[V]$ and $\lambda f^{-1}[K] > 0$.

As f and V are arbitary, λ is perfect.

451J Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X,Σ,μ).

(a) If every μ_i is compact, so is μ.

(b) (Marczewski 53) If every μ_i is countably compact, so is μ.

(c) If every μ_i is perfect, so is μ.

proof The same strategy as in 451I is again effective.

(a)-(b) For each $i\in I$, let $\mathcal{K}_i\subseteq\mathcal{P}X_i$ be a (countably) compact class such that μ_i is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{M}_0 = \{\pi_i^{-1}[K] : i\in I,\ K\in\mathcal{K}_i\}$, so that $\mathcal{M}_0$ is (countably) compact. Let $\mathcal{M}\supseteq\mathcal{M}_0$ be a (countably) compact class which is closed under finite unions and countable intersections. By 412T, μ is inner regular with respect to $\mathcal{M}$, so is (countably) compact.

(c) Let Λ' be a countably generated σ-subalgebra of $\widehat{\bigotimes}_{i\in I}\Sigma_i$, the σ-algebra of subsets of X generated by the sets $\{x : x(i)\in E\}$ for $i\in I$ and $E\in\Sigma_i$. Then $\lambda\restriction\Lambda'$ is perfect. **P** For every $W\in\widehat{\bigotimes}_{i\in I}\Sigma_i$, we must be able to find countable subsets $\mathcal{E}_i$ of Σ_i such that W is in the σ-algebra generated by $\{\pi_i^{-1}[E] : i\in I,\ E\in\Sigma_i\}$; so there are in fact countable sets $\mathcal{E}_i\subseteq\Sigma_i$ such that the σ-algebra generated by $\{\pi_i^{-1}[E] : i\in I,\ E\in\Sigma_i\}$

includes Λ'. Let T_i be the σ-subalgebra of Σ_i generated by $\mathcal{E}_i$, so that $\mu_i\restriction\mathrm{T}_i$ is compact. Let λ' be the product of $\langle\mu_i\restriction\mathrm{T}_i\rangle_{i\in I}$; then λ' is compact, by (a) above, therefore perfect. Now λ is an extension of λ', by 254G or otherwise, so λ' is an extension of $\lambda\restriction\Lambda'$, and $\lambda\restriction\Lambda'$ is perfect. **Q** As Λ' is arbitrary, $\lambda\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$ is perfect, and its completion λ (254Ff) also is perfect.

Remark This theorem is generalized in 454Ab.

451K The following result is interesting because it can be reached from an unexpectedly weak hypothesis; it will be useful in §455.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X, and Σ_i a σ-algebra of subsets of X_i for each i. Let λ be a perfect totally finite measure with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$. Set $\pi_J(x)=x\restriction J$ for $x\in X$ and $J\subseteq I$.

(a) Let $\mathcal{K}$ be the set $\{V:V\subseteq X,\,\pi_J[V]\in\widehat{\bigotimes}_{i\in J}\Sigma_i$ for every $J\subseteq I\}$. Then λ is inner regular with respect to $\mathcal{K}$.

(b) Let $\hat\lambda$ be the completion of λ.

(i) For any $J\subseteq I$, the completion of the image measure $\lambda\pi_J^{-1}$ on $\prod_{i\in J}X_i$ is the image measure $\hat\lambda\pi_J^{-1}$.

(ii) If W is measured by $\hat\lambda$ and W is determined by coordinates in $J\subseteq I$, then there is a $V\in\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $V\subseteq W$, V is determined by coordinates in J and $W\setminus V$ is λ-negligible.

proof (a)(i) Take $W\in\widehat{\bigotimes}_{i\in I}\Sigma_i$. Then we can find a family $\langle\mathrm{T}_i\rangle_{i\in I}$ such that T_i is a countably generated σ-subalgebra of Σ_i for each i and $W\in\widehat{\bigotimes}_{i\in I}\mathrm{T}_i$. For each $i\in I$ and $E\in\mathrm{T}_i$ set $\lambda_iE=\lambda\{x:x\in X,$ $x(i)\in E\}$; then λ_i is perfect (451Ea). Because T_i is countably generated, λ_i is compact (451F); let $\mathcal{K}_i$ be a compact class such that λ_i is inner regular with respect to $\mathcal{K}_i$. By 342D, we may suppose that $\mathcal{K}_i$ is the family of closed sets for a compact topology $\mathfrak{T}_i$ on X_i.

(ii) Let $\mathcal{V}$ be the family of all sets $V\subseteq X$ expressible in the form

$$V=\bigcap_{n\in\mathbb{N}}\bigcup_{i\in J_n}\{x:x\in X,\,x(i)\in K_{ni}\}$$

where $\langle J_n\rangle_{n\in\mathbb{N}}$ is a sequence of finite subsets of I and $K_{ni}\in\mathcal{K}_i\cap\mathrm{T}_i$ whenever $n\in\mathbb{N}$ and $i\in J_n$. Given V expressed in this form, set $V_n=\bigcap_{m\le n}\bigcup_{i\in J_m}\{x:x(i)\in K_{mi}\}$ for each n. Then $\pi_J[V]=\bigcap_{n\in\mathbb{N}}\pi_J[V_n]$ for every $J\subseteq I$. **P** The product topology $\mathfrak{T}$ on X is compact, and all the V_n are $\mathfrak{T}$-closed. If $z\in\bigcap_{n\in\mathbb{N}}\pi_J[V_n]$, then for each $n\in\mathbb{N}$ there is an $x_n\in V_n$ such that $\pi_J(x)=z$. Let x be a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$. The topologies are not Hausdorff, so we do not know at once that $\pi_J(x)=z$; but if we define x' by saying that

$$\begin{aligned}x'(i)&=z(i)\text{ if }i\in J,\\&=x(i)\text{ if }i\in I\setminus J,\end{aligned}$$

then any neighbourhood U of x' must include a neighbourhood of the form $\{y:y(i)\in U_i$ for $i\in K\}$ where $K\subseteq I$ is finite and U_i is a neighbourhood of $x'(i)$ for each $i\in K$. In this case, $\{y:y\in U_i$ for $i\in K\setminus J\}$ is a neighbourhood of x, so

$$\{n:x_n\in U\}\supseteq\{n:x_n(i)\in U_i\text{ for }i\in K\setminus J\}$$

is infinite. Thus x' also is a cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$, while $\pi_J(x')=z$. Since $x'\in\overline{\{x_m:m\ge n\}}\subseteq V_n$ for every n, $x\in V$, and $z\in\pi_J[V]$. Thus $\bigcap_{n\in\mathbb{N}}\pi_J[V_n]\subseteq\pi_J[V]$. Since surely $\pi_J[V]\subseteq\bigcap_{n\in\mathbb{N}}\pi_J[V_n]$, we have equality. **Q**

It follows that $V\in\mathcal{K}$. **P** If $J\subseteq I$ and $n\in\mathbb{N}$, then V_n belongs to the algebra of subsets of X generated by sets of the form $\{x:x(i)\in H\}$ where $i\in I$ and $H\in\Sigma_i$, which we can identify with the free product $\bigotimes_{i\in I}\Sigma_i$ (315Ma[2]). This means that V_n can be expressed as a finite union of cylinder sets of the form $C=\prod_{i\in I}H_i$ where $H_i\in\Sigma_i$ for every i and $\{i:H_i\ne X_i\}$ is finite (315Kb[3]). But in this case $\pi_J[C]$ is either empty or $\prod_{i\in J}H_i$, and in either case belongs to $\widehat{\bigotimes}_{i\in J}\Sigma_i$. So $\pi_J[V_n]$, being a finite union of such sets, also belongs to $\widehat{\bigotimes}_{i\in J}\Sigma_i$. As this is true for every $n\in\mathbb{N}$, $\pi_J[V]=\bigcap_{n\in\mathbb{N}}\pi_J[V_n]$ belongs to $\widehat{\bigotimes}_{i\in J}\Sigma_i$. As J is arbitrary, $V\in\mathcal{K}$. **Q**

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[2]Formerly 315L.

[3]Formerly 315J.

(iii) Observe next that $\mathcal{V}$ is closed under finite unions. **P** If V', $V''\in\mathcal{V}$, express them as

$$V'=\bigcap_{n\in\mathbb{N}}\bigcup_{i\in J'_n}\{x:x(i)\in K'_{ni}\}$$

$$V''=\bigcap_{n\in\mathbb{N}}\bigcup_{i\in J''_n}\{x:x(i)\in K''_{ni}\}$$

where, for each n, J'_n, $J''_n\subseteq I$ are finite, $K'_{ni}\in\mathcal{K}_{ni}\cap\Sigma_i$ for $i\in J'_n$ and $K''_{ni}\in\mathcal{K}_{ni}\cap\Sigma_i$ for $i\in J''_n$. For m, $n\in\mathbb{N}$, set $J_{mn}=J'_m\cup J''_n$ and

$$\begin{aligned}K_{mni}&=K'_{mi}\cup K''_{ni}\text{ if }i\in J'_m\cap J''_n,\\&=K'_{mi}\text{ if }i\in J'_m\setminus J''_n,\\&=K''_{ni}\text{ if }i\in J''_n\setminus J''_m.\end{aligned}$$

Then

$$V'\cap V''=\bigcap_{m,n\in\mathbb{N}}\bigcup_{i\in J_{mn}}\{x:x(i)\in K_{mni}\}\in\mathcal{V}.\ \mathbf{Q}$$

We see also, immediately from its definition, that $\mathcal{V}$ is closed under countable intersections.

(iv) Now consider the family $\mathcal{A}$ of sets of the form $\{x:x(i)\in E\}$ where $i\in I$ and $E\in\mathrm{T}_i$. If $A\in\mathcal{A}$ is expressed in this form, then

$$\sup\{\lambda V:V\in\mathcal{V},\,V\subseteq A\}\ge\sup\{\lambda_iK:K\in\mathcal{K}_i\cap\mathrm{T}_i,\,K\subseteq E\}=\lambda_iE=\lambda A.$$

By 412C, $\lambda\restriction\widehat{\bigotimes}_{i\in I}\mathrm{T}_i$ is inner regular with respect to $\mathcal{V}$. In particular, returning to our original set W,

$$\mu W=\sup\{\lambda V:V\in\mathcal{V},\,V\subseteq W\}=\sup\{\lambda K:K\in\mathcal{K},\,K\subseteq W\}.$$

As W is arbitrary, λ is inner regular with respect to $\mathcal{K}$.

(b)(i) Write $\lambda_J=\lambda\pi_J^{-1}$ and $\hat\lambda_J$ for its completion. Since $\pi_J:X\to\prod_{i\in J}X_J$ is inverse-measure-preserving for λ and λ_J, it is inverse-measure-preserving for $\hat\lambda$ and $\hat\lambda_J$ (234Ba[4]), that is, $\hat\lambda\pi_J^{-1}$ extends $\hat\lambda_J$. Now suppose that V is measured by $\hat\lambda\pi_J^{-1}$. Since λ is inner regular with respect to $\mathcal{K}$, so is $\hat\lambda$ (412Ha again), so

$$\begin{aligned}\hat\lambda\pi_J^{-1}[V]&=\sup\{\lambda K:K\in\mathcal{K},\,K\subseteq\pi_J^{-1}[V]\}\\&\le\sup\{\lambda\pi_J^{-1}[\pi_J[K]]:K\in\mathcal{K},\,K\subseteq\pi_J^{-1}[V]\}\\&\le\sup\{\lambda\pi_J^{-1}[F]:F\in\widehat{\bigotimes}_{i\in J}\Sigma_i,\,F\subseteq V\}.\end{aligned}$$

As V is arbitrary, $\hat\lambda\pi_J^{-1}$ is inner regular with respect to $\widehat{\bigotimes}_{i\in J}\Sigma_i$. By 412Mb (or otherwise), $\hat\lambda\pi_J^{-1}=\hat\lambda_J$.

(ii) Because $\pi_J^{-1}[\pi_J[W]]=W$, $\pi_J[W]$ is measured by $\hat\lambda\pi_J^{-1}=\hat\lambda_J$. So there is a $V'\subseteq\pi_J[W]$, measured by λ_J, such that

$$0=\hat\lambda_J(\pi_J[W]\setminus V')=\hat\lambda(W\setminus\pi_J^{-1}[V']),$$

and we can take $V=\pi_J^{-1}[V']$.

***451L** The next result is sometimes useful, as a fractionally weaker sufficient condition for compactness or countable compactness of a measure.

Proposition (Borodulin-Nadzieja & Plebanek 05) Let (X,Σ,μ) be a strictly localizable measure space. Let us say that a family $\mathcal{E}\subseteq\Sigma$ is ***μ*-centered** if $\mu(\bigcap\mathcal{E}_0)>0$ for every non-empty finite $\mathcal{E}_0\subseteq\mathcal{E}$.

(i) Suppose that μ is inner regular with respect to some $\mathcal{K}\subseteq\Sigma$ such that every μ-centered subset of $\mathcal{K}$ has non-empty intersection. Then μ is compact.

(ii) Suppose that μ is inner regular with respect to some $\mathcal{K}\subseteq\Sigma$ such that every countable μ-centered subset of $\mathcal{K}$ has non-empty intersection. Then μ is countably compact.

[4]Formerly 235Hc.

proof I take the two arguments together, as follows. The case $\mu X = 0$ is trivial; suppose henceforth that $\mu X > 0$. Let $\hat{\mu}$ be the completion of μ. Then $\hat{\mu}$ is still strictly localizable (212Gb) so has a lifting $\phi : \Sigma \to \Sigma$ (341K). Let $\mathcal{K}_1$ be the set of all those $K \in \Sigma$ for which there is some sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that

$$K = \bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} \phi K_n.$$

Then μ is inner regular with respect to $\mathcal{K}_1$. **P** Suppose that $E \in \Sigma$ and $0 \le \gamma < \mu E$. Because μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. Choose $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ inductively, as follows. K_0 is to be such that $K_0 \subseteq F$ and $\mu K_0 > \gamma$. Given that $\mu K_n > \gamma$, then $\hat{\mu}(K_n \cap \phi K_n) = \hat{\mu} K_n > \gamma$; also $\hat{\mu}$ is inner regular with respect to $\mathcal{K}$ (412Ha once more), so there is a $K_{n+1} \in \mathcal{K}$ such that $K_{n+1} \subseteq K_n \cap \phi K_n$ and $\mu K_{n+1} = \hat{\mu} K_{n+1} > \gamma$. Continue. At the end of the induction,

$$K = \bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} \phi K_n$$

belongs to $\mathcal{K}_1$, is included in E and has measure at least γ. **Q**

Now $\mathcal{K}_1$ is (countably) compact. **P** Let $\mathcal{K}' \subseteq \mathcal{K}_1$ be a [countable] set with the finite intersection property. For each $K \in \mathcal{K}'$, let $\mathcal{E}_K \subseteq \mathcal{K}$ be a countable set such that $K = \bigcap \mathcal{E}_K \subseteq \bigcap\{\phi E : E \in \mathcal{E}_K\}$; set $\mathcal{E} = \bigcup_{K\in\mathcal{K}'} \mathcal{E}_K$. If $\mathcal{E}_0 \subseteq \mathcal{E}$ is finite and not empty, then $\phi(\bigcap \mathcal{E}_0) = \bigcap_{E\in\mathcal{E}_0} \phi E$ includes the intersection of a finite subfamily of $\mathcal{K}'$, so is not empty, and $\mu(\bigcap \mathcal{E}_0) = \hat{\mu}(\bigcap \mathcal{E}_0)$ is non-zero. Thus $\mathcal{E} \subseteq \mathcal{K}$ is a [countable] μ-centered set and must have non-empty intersection. But now $\bigcap \mathcal{K}' = \bigcap \mathcal{E}$ is non-empty. As $\mathcal{K}'$ is arbitrary, $\mathcal{K}_1$ is (countably) compact. **Q**

So $\mathcal{K}_1$ witnesses that μ is (countably) compact, as claimed.

451M The following is one of the basic ways in which we can find ourselves with a compact measure.

Proposition Let (X, Σ) be a standard Borel space. Then any semi-finite measure μ with domain Σ is compact, therefore perfect.

proof If $\mathfrak{T}$ is a Polish topology on X with respect to which Σ is the Borel σ-algebra, then μ is inner regular with respect to the family $\mathcal{K}$ of $\mathfrak{T}$-compact sets (433Ca), which is a compact class.

451N Proposition Let (X, Σ, μ) be a perfect measure space and $\mathfrak{T}$ a T_0 topology on X with a countable network consisting of measurable sets. (For instance, μ might be a topological measure on a regular space with a countable network (4A2Ng), or a second-countable space. In particular, X might be a separable metrizable space.) Then μ is inner regular with respect to the compact sets.

proof This is a refinement of 343K. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a sequence in Σ running over a network for $\mathfrak{T}$. Define $g : X \to \mathbb{R}$ by setting $g = \sum_{n=0}^{\infty} 3^{-n} \chi H_n$ (cf. 343E). Then g is measurable, because every χE_n is. Writing $\alpha_I = \sum_{i\in I} 3^{-i}$ for $I \subseteq \mathbb{N}$, and

$$H_n = \bigcup_{I\subseteq n} \left]\alpha_I + \tfrac{1}{2} 3^{-n}, \alpha_I + 3^{-n+1}\right[,$$

we see that $E_n = g^{-1}[H_n]$ for each $n \in \mathbb{N}$. This shows that g is injective, because if x, y are distinct points in X there is an open set containing one but not the other, and now there is an $n \in \mathbb{N}$ such that E_n contains that one and not the other, so that just one of $g(x)$, $g(y)$ belongs to H_n. Also $g^{-1} : g[X] \to X$ is continuous, since $(g^{-1})^{-1}[E_n] = g[E_n] = H_n \cap g[X]$ is relatively open in $g[X]$ for every $n \in \mathbb{N}$ (4A2B(a-ii)).

Now suppose that $E \in \Sigma$ and $\mu E > 0$. Then there is a compact set $K \subseteq g[E]$ such that $\mu g^{-1}[K] > 0$. But as g is injective, $g^{-1}[K] \subseteq E$, and as g^{-1} is continuous, $g^{-1}[K]$ is compact. By 412B, this is enough to show that μ is inner regular with respect to the compact sets.

451O Corollary Let (X, Σ, μ) be a complete perfect measure space, Y a Hausdorff space with a countable network consisting of Borel sets and $f : X \to Y$ a measurable function. If the image measure μf^{-1} is locally finite, it is a Radon measure.

proof Because f is measurable, μf^{-1} is a topological measure; by 451Ea, it is perfect; by 451N, it is tight; and it is complete because μ is. Because Y has a countable network, it is Lindelöf (4A2Nb), and μf^{-1} is σ-finite (411Ge), therefore locally determined. So it is a Radon measure.

451P Corollary Let (X, Σ, μ) be a perfect measure space, Y a separable metrizable space, and $f : X \to Y$ a measurable function.

(a) If $E \in \Sigma$ and $\gamma < \mu E$, there is a compact set $K \subseteq f[E]$ such that $\mu(E \cap f^{-1}[K]) \geq \gamma$.
(b) If $\nu = \mu f^{-1}$ is the image measure, then $\mu_* f^{-1}[B] = \nu_* B$ for every $B \subseteq Y$.
(c) If moreover μ is σ-finite, then $\mu^* f^{-1}[B] = \nu^* B$ for every $B \subseteq Y$.

proof (a) Consider the subspace measure μ_E, the measurable function $f\restriction E$ from E to the separable metrizable space $f[E]$, and the image measure $\nu' = \mu_E(f\restriction E)^{-1}$ on $f[E]$. By 451Dc, 451Ea and 451N, this is tight, while $\nu' f[E] = \mu E$; so there is a compact set $K \subseteq f[E]$ such that $\nu' K \geq \gamma$, and this serves.

(b)(i) If $F \in \operatorname{dom}\nu$ and $F \subseteq B$, then

$$\nu F = \mu f^{-1}[F] \leq \mu_* f^{-1}[B];$$

as F is arbitrary, $\mu_* f^{-1}[B] \geq \nu_* B$. (ii) If $E \in \Sigma$ and $E \subseteq f^{-1}[B]$ and $\gamma < \mu E$, then (a) tells us that there is a compact set $K \subseteq f[E]$ such that $\mu(E \cap f^{-1}[K]) \geq \gamma$, in which case

$$\nu_* B \geq \nu K \geq \gamma.$$

As E and γ are arbitrary, $\nu_* B \geq \mu_* f^{-1}[B]$.

(c)(i) If $F \in \operatorname{dom}\nu$ and $F \supseteq B$, then

$$\nu F = \mu f^{-1}[F] \geq \mu^* f^{-1}[B];$$

as F is arbitrary, $\mu^* f^{-1}[B] \leq \nu^* B$. (ii) If $\mu^* f^{-1}[B] = \infty$, then of course $\mu^* f^{-1}[B] = \nu^* B$. Otherwise, because μ is σ-finite, we can find a disjoint sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of subsets of X of finite measure, covering X, such that $E_0 \supseteq f^{-1}[B]$ and $\mu E_0 = \mu^* f^{-1}[B]$. Let $\epsilon > 0$. For each $n \geq 1$, (a) tells us that there is a compact set $K_n \subseteq f[E_n]$ such that $\mu f^{-1}[E_n \setminus K_n] \leq 2^{-n}\epsilon$. Set $H = Y \setminus \bigcup_{n\geq 1} K_n$; then $\nu H \leq \mu E + \epsilon$, and $B \subseteq H$. So

$$\nu^* B \leq \nu H \leq \mu E + \epsilon = \mu^* f^{-1}[B] + \epsilon.$$

As ϵ is arbitrary, $\nu^* B \leq \mu^* f^{-1}[B]$.

451Q I turn now to a remarkable extension of the idea above to general metric spaces Y.

Lemma Let (X, Σ, μ) be a semi-finite compact measure space, and $\langle E_i \rangle_{i\in I}$ a disjoint family of subsets of X such that $\bigcup_{i\in J} E_i \in \Sigma$ for every $J \subseteq I$. Then $\mu(\bigcup_{i\in I} E_i) = \sum_{i\in I} \mu E_i$.

proof (a) To begin with (down to the end of part (d) of the proof) assume that μ is complete and totally finite and that every E_i is negligible. Set $X_0 = \bigcup_{i\in I} E_i$, and let μ_0 be the subspace measure on X_0. Define $f : X_0 \to I$ by setting $f(x) = i$ if $i \in I$, $x \in E_i$, and let ν be the image measure $\mu_0 f^{-1}$, so that $\nu J = \mu(\bigcup_{i\in J} E_i)$ for $J \subseteq I$; then $(I, \mathcal{P}I, \nu)$ is a totally finite measure space.

(b) ν is purely atomic. **P?** Suppose, if possible, otherwise; that there is a $K \subseteq I$ such that $\nu K > 0$ and the subspace measure $\nu\restriction\mathcal{P}K$ is atomless. In this case there is an inverse-measure-preserving function $g : K \to [0, \gamma]$, where $\gamma = \nu K$ and $[0, \gamma]$ is given Lebesgue measure (343Cc); write λ for Lebesgue measure on $[0, \gamma]$. Set $X_1 = f^{-1}[K] = \bigcup_{i\in K} E_i$ and let μ_1 be the subspace measure on X_1. Now $gf : X_1 \to [0, \gamma]$ is inverse-measure-preserving for μ_1 and λ. Because μ is compact, so is μ_1 (451Da), so μ_1 is perfect (342L or 451C once more). By 451O, the image measure $\lambda_1 = \mu_1(gf)^{-1}$ is a Radon measure. But λ_1 must be an extension of Lebesgue measure λ, because gf is inverse-measure-preserving for μ_1 and λ, and λ_1 and λ must agree on all compact sets. By 416E(b-ii), λ_1 and λ are identical, and, in particular, have the same domains. Now for any set $A \subseteq [0, \gamma]$, $(gf)^{-1}[A] = \bigcup_{i\in J} E_i \in \Sigma$, where $J = g^{-1}[A] \subseteq I$; so $A \in \operatorname{dom}\lambda_1 = \operatorname{dom}\lambda$. But we know from 134D or 419I that not every subset of $[0, \gamma]$ can be Lebesgue measurable. **XQ**

(c) But ν is also atomless. **P?** Suppose, if possible, that $M \subseteq I$ is an atom for ν. Set $\gamma = \nu M = \mu(\bigcup_{i\in M} E_i)$,

$$\mathcal{F} = \{F : F \subseteq M,\ \nu(M \setminus F) = 0\}.$$

Because νF is defined for every $F \subseteq M$, and M is an atom, $\mathcal{F}$ is an ultrafilter on M; and because ν is countably additive, the intersection of any sequence in $\mathcal{F}$ belongs to $\mathcal{F}$, that is, $\mathcal{F}$ is ω_1-complete (definition: 4A1Ib). Also $\mathcal{F}$ must be non-principal, because we are supposing that $\nu\{i\} = 0$ for every $i \in M$. By 4A1K, there are a regular uncountable cardinal κ and a function $h : M \to \kappa$ such that the image filter $\mathcal{H} = h[[\mathcal{F}]]$ is normal.

For each $\xi < \kappa$, $\kappa \setminus \xi \in \mathcal{H}$, so

$$G_\xi = (hf)^{-1}[\kappa \setminus \xi] = \bigcup\{E_i : h(i) \ge \xi\} \in \Sigma, \quad \mu G_\xi = \nu h^{-1}[\kappa \setminus \xi] = \gamma > 0.$$

At this point I apply the full strength of the hypothesis that μ is a compact measure. Let $\mathcal{K} \subseteq \Sigma$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$, and for each $\xi < \kappa$ choose $K_\xi \in \mathcal{K}$ such that $K_\xi \subseteq G_\xi$ and $\mu K_\xi \ge \frac{1}{2}\gamma$. Let $S \subseteq [\kappa]^{<\omega}$ be the family of those finite sets $L \subseteq \kappa$ such that $\bigcap_{\xi \in L} K_\xi = \emptyset$. Because $\mathcal{H}$ is a normal ultrafilter, there is an $H \in \mathcal{H}$ such that, for every $n \in \mathbb{N}$, $[H]^n$ is either a subset of S or disjoint from S (4A1L).

If we look at $\{G_\xi : \xi \in H\}$, we see that it has empty intersection, because $h(f(x)) \ge \xi$ for every $x \in G_\xi$, and $\sup H = \kappa$. So $\bigcap_{\xi \in H} K_\xi = \emptyset$. Because all the K_ξ belong to the compact class $\mathcal{K}$, there must be a finite set $L_0 \subseteq H$ such that $\bigcap_{\xi \in L_0} K_\xi = \emptyset$, that is, $L_0 \in S$. But this means that $[H]^n \cap S \ne \emptyset$, where $n = \#(L_0)$, so that $[H]^n \subseteq S$, by the choice of H. However, H is surely infinite, so we can find distinct $\xi_0, \dots, \xi_{2n}$ in H. If we now look at $K_{\xi_0}, \dots, K_{\xi_{2n}}$, we see that $\#(\{i : i \le 2n,\, x \in K_{\xi_i}\}) < n$ for every $x \in X$, so

$$\sum_{i=0}^{2n} \chi K_{\xi_i} \le (n-1)\chi G_0, \quad \sum_{i=0}^{2n} \int \chi K_{\xi_i} \ge \frac{1}{2}\gamma(2n+1),$$

which is impossible, because $\mu G_0 = \gamma$. **XQ**

(d) Thus ν is simultaneously atomless and purely atomic, which means that $\nu I = 0$, that is, that $\mu(\bigcup_{i \in I} E_i) = 0 = \sum_{i \in I} \mu E_i$.

(e) Now let us return to the general case. Of course

$$\sum_{i \in I} \mu E_i = \sup_{J \subseteq I \text{ is finite}} \sum_{i \in J} \mu E_i \le \mu(\bigcup_{i \in I} E_i).$$

? Suppose, if possible, that $\sum_{i \in I} \mu E_i < \mu(\bigcup_{i \in I} E_i)$. Because μ is semi-finite, there is a set $F \subseteq \bigcup_{i \in I} E_i$ such that $\sum_{i \in I} \mu E_i < \mu F < \infty$. Set $L = \{i : i \in I,\, \mu E_i > 0\}$; then L must be countable, so $\mu(\bigcup_{i \in J} E_i) = \sum_{i \in J} \mu E_i < \mu F$, and $\mu G > 0$, where $G = F \setminus \bigcup_{i \in L} E_i$. Set $E'_i = G \cap E_i$ for every $i \in I$, and let $\hat\mu_G$ be the completion of the subspace measure μ_G on G. Then $\hat\mu_G$ is compact (451Da, 451G(a-i)) and totally finite, $\hat\mu_G E'_i = 0$ for every $i \in I$, $\bigcup_{i \in J} E'_i = G \cap \bigcup_{i \in J} E_i$ is measured by $\hat\mu_G$ for every $J \subseteq I$, every E'_i is $\hat\mu_G$-negligible, but $\hat\mu_G(\bigcup_{i \in I} E'_i) = \mu G$ is not zero; which contradicts the result of (a)-(d) above. **X**

So $\sum_{i \in I} \mu E_i = \mu(\bigcup_{i \in I} E_i)$, as required.

451R Lemma Let (X, Σ, μ) be a totally finite compact measure space, Y a metrizable space, and $f : X \to Y$ a measurable function. Then there is a closed separable subspace Y_0 of Y such that $f^{-1}[Y \setminus Y_0]$ is negligible.

proof (a) (Cf. 438D.) By 4A2L(g-ii), there is a σ-disjoint base $\mathcal{U}$ for the topology of Y. Express $\mathcal{U}$ as $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ where $\mathcal{U}_n$ is disjoint for each n. Then $\langle f^{-1}[U] \rangle_{U \in \mathcal{U}_n}$ is disjoint, so $\sum_{U \in \mathcal{U}_n} \mu f^{-1}[U] \le \mu X$ is finite, and $\mathcal{V}_n = \{V : V \in \mathcal{U}_n,\, \mu f^{-1}[V] > 0\}$ is countable for each n.

If $\mathcal{W} \subseteq \mathcal{U}_n \setminus \mathcal{V}_n$, then

$$\mu(\bigcup_{U \in \mathcal{W}} f^{-1}[U]) = f^{-1}[\bigcup \mathcal{W}]$$

is measurable. By 451Q,

$$\mu f^{-1}[\bigcup(\mathcal{U}_n \setminus \mathcal{V}_n)] = \mu(\bigcup_{U \in \mathcal{U}_n \setminus \mathcal{V}_n} f^{-1}[U]) = \sum_{U \in \mathcal{U}_n \setminus \mathcal{V}_n} \mu f^{-1}[U] = 0.$$

Set

$$\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n, \quad Y_0 = Y \setminus \bigcup(\mathcal{U} \setminus \mathcal{V}).$$

Then Y_0 is closed, and

$$f^{-1}[Y \setminus Y_0] \subseteq \bigcup_{n \in \mathbb{N}} f^{-1}[\bigcup(\mathcal{U}_n \setminus \mathcal{V}_n)]$$

is negligible, so $f^{-1}[Y_0]$ is conegligible. On the other hand, Y_0 is separable. **P** Because $\mathcal{U}$ is a base for the topology of X, $\{Y \cap U : U \in \mathcal{U}\}$ is a base for the topology of Y (4A2B(a-vi)). But this is included in the countable family $\{Y \cap V : V \in \mathcal{V}\} \cup \{\emptyset\}$, so Y is second-countable, therefore separable (4A2Oc). **Q**

So we have found an appropriate Y_0.

451S Proposition Let (X,Σ,μ) be a semi-finite compact measure space, Y a metrizable space and $f:X\to Y$ a measurable function.

(a) The image measure $\nu=\mu f^{-1}$ is tight.

(b) If ν is locally finite and μ is complete and locally determined, ν is a Radon measure.

proof (a) Take $F\subseteq Y$ such that $\nu F>0$. Then $\mu f^{-1}[F]>0$. Because μ is semi-finite, there is an $E\in\Sigma$ such that $E\subseteq f^{-1}[F]$ and $0<\mu E<\infty$.

Consider the subspace measure μ_E and the restriction $f\restriction E$. μ_E is a totally finite compact measure and $f\restriction E$ is measurable, so 451R tells us that there is a closed separable subspace $Y_0\subseteq Y$ such that $\mu(E\setminus f^{-1}[Y_0])=0$. Set $E_1=E\cap f^{-1}[Y_0]$, so that $\mu E_1>0$. Again, the subspace measure μ_{E_1} is a totally finite compact measure, therefore perfect, while $f[E_1]\subseteq Y_0$. So the image measure $\mu_{E_1}(f\restriction E_1)^{-1}$ on Y_0 is perfect (451Ea), therefore tight (451N), and there is a compact set $K\subseteq Y_0\cap F$ such that $\nu K=\mu f^{-1}[K]>0$. By 412B, this is enough to show that ν is tight.

(b) ν is complete because μ is. Now suppose that $H\subseteq Y$ is such that $H\cap F$ belongs to the domain T of ν whenever $\mu F<\infty$. In this case μ is inner regular with respect to $\mathcal{E}=\{E:E\in\Sigma,\,E\cap f^{-1}[H]\in\Sigma\}$. **P** Suppose that $E\in\Sigma$ and that $\mu E>0$. Applying (a) to μ_E and $f\restriction E$, there is a compact set $K\subseteq f[E]$ such that $\mu f^{-1}[K]>0$. Now $\nu K<\infty$, because ν is locally finite, so $K\cap H\in\mathrm{T}$ and $f^{-1}[K]\cap f^{-1}[H]\in\Sigma$. Thus $f^{-1}[K]$ is a non-negligible member of $\mathcal{E}$ included in E. Since $\mathcal{E}$ is closed under finite unions, this is enough to show that μ is inner regular with respect to $\mathcal{E}$. **Q**

Accordingly $f^{-1}[H]\in\Sigma$, by 412Ja. As H is arbitrary, ν is locally determined, therefore a Radon measure.

451T Theorem (Fremlin 81, Koumoullis & Prikry 83) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and Y a metrizable space. Then a function $f:X\to Y$ is measurable iff it is almost continuous.

proof If f is almost continuous it is surely measurable, by 418E. Now suppose that f is measurable and that $E\in\Sigma$ and $\gamma<\mu E$. Let $E_0\subseteq E$ be such that $E_0\in\Sigma$ and $\gamma<\mu E_0<\infty$. Applying 451R to the subspace measure μ_{E_0} and the restricted function $f\restriction E_0$, we see that there is a closed separable subspace Y_0 of Y such that $\mu(E_0\setminus f^{-1}[Y_0])=0$. Set $E_1=E_0\cap f^{-1}[Y_0]$; then $\mu E_1>\gamma$. Applying 418J to μ_{E_1} and $f\restriction E_1:E_1\to Y_0$, we can find a measurable set $F\subseteq E_1$ such that $f\restriction F$ is continuous and $\mu F\ge\gamma$. As E and γ are arbitrary, f is almost continuous.

451U Example (Vinokurov & Makhkamov 73, Musiał 76) There is a perfect completion regular quasi-Radon probability space which is not countably compact.

proof (a) Let Ω be the set of non-zero countable limit ordinals. For each $\xi\in\Omega$, let $\langle\theta_\xi(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in ξ with supremum ξ, and set

$$Q_\xi=\{x:x\in\{0,1\}^{\omega_1},\,x(\theta_\xi(n))=0\text{ for every }n\in\mathbb{N}\}.$$

Write

$$X=\{0,1\}^{\omega_1}\setminus\bigcup_{\xi\in\Omega}Q_\xi.$$

Let ν_{ω_1} be the usual measure on $\{0,1\}^{\omega_1}$, and T_{ω_1} its domain; let μ be the subspace measure on X, and $\Sigma=\operatorname{dom}\mu$.

(b) It is convenient to note immediately the following fact: for every countable set $J\subseteq\omega_1$, the set $\pi_J[X]$ is conegligible in $\{0,1\}^J$, where $\pi_J(x)=x\restriction J$ for $x\in\{0,1\}^{\omega_1}$. **P** Set

$$A=\{\xi:\xi\in\Omega,\,\theta_\xi(n)\in J\text{ for every }n\in\mathbb{N}\}.$$

Then A is countable, because $\xi\le\sup J$ for every $\xi\in A$. So

$$D=\bigcup_{\xi\in A}\{y:y\in\{0,1\}^J,\,y(\theta_\xi(n))=0\text{ for every }n\in\mathbb{N}\}$$

is negligible in $\{0,1\}^J$, being a countable union of negligible sets. If $y\in\{0,1\}^J\setminus D$, define $x\in\{0,1\}^{\omega_1}$ by setting $x(\eta)=y(\eta)$ for $\eta\in J$, $x(\eta)=1$ for $\eta\in\omega_1\setminus J$. Then $x\notin Q_\xi$ for any $\xi\in A$, because $x\restriction J=y\restriction J$, while $x\notin Q_\xi$ for any $\xi\in\Omega\setminus A$ by the definition of A. So $x\in X$. As y is arbitrary, $\pi_J[X]\supseteq\{0,1\}^J\setminus D$ is conegligible. **Q**

(c) μ is a completion regular quasi-Radon measure because ν_{ω_1} is (415E, 415B, 412Pd). Also $\mu X = 1$. **P** Let $F \in \mathrm{T}_{\omega_1}$ be a measurable envelope for X. Then there is a countable $J \subseteq \omega_1$ such that $\nu_J\pi_J[F]$ is defined and equal to $\nu_{\omega_1}F$ (254Od), where ν_J is the usual measure on $\{0,1\}^J$. But we know that $\nu_J\pi_J[X] = 1$, so

$$\mu X = \nu^*_{\omega_1}X = \nu_{\omega_1}F = \nu_J\pi_J F = 1. \text{ **Q**}$$

(d) μ is perfect. **P** Take $E \in \Sigma$ such that $\mu E > 0$, and a measurable function $f : E \to \mathbb{R}$. Set $f_1(x) = \frac{f(x)}{1+|f(x)|}$ for $x \in E$, 1 for $x \in X \setminus E$; then $f_1 : X \to \mathbb{R}$ is measurable. Let $g : \{0,1\}^{\omega_1} \to \mathbb{R}$ be a measurable function extending f_1. By 254Pb, there are a countable set $J \subseteq \omega_1$, a conegligible set $W \subseteq \{0,1\}^J$, and a measurable $h : W \to \mathbb{R}$ such that g extends $h\pi_J$. By (b), $W' = W \cap \pi_J[X]$ is conegligible, while $W'' = \{z : z \in W',\ h_1(z) < 1\}$ is measurable and not negligible. Because W'' is a non-negligible measurable subset of the perfect measure space $\{0,1\}^J$, there is a compact set $K_1 \subseteq h[W'']$ such that $\nu_J h^{-1}[K_1] > 0$. Set $K = \{\frac{t}{1-|t|} : t \in K_1\}$; then K is compact, and we have

$$K_1 \subseteq h[W''] = h[W \cap \pi_J[X]] \cap \,]-\infty,1[\, \subseteq g[X] \cap \,]-\infty,1[\, = f_1[X] \cap \,]-\infty,1[\, = f_1[E],$$

$$K \subseteq f[E],$$

while f_1, g and $h\pi_J$ all agree on the μ-conegligible set $X \cap \pi_J^{-1}[W]$, so

$$\begin{aligned}\mu f^{-1}[K] &= \mu f_1^{-1}[K_1] = \mu(X \cap (h\pi_J)^{-1}[K_1])\\ &= \nu^*_{\omega_1}(X \cap (h\pi_J)^{-1}[K_1]) = \nu_{\omega_1}(h\pi_J)^{-1}[K_1]\end{aligned}$$

(because $\nu^*_{\omega_1}X = 1$ and $(h\pi_J)^{-1}[K_1]$ is measurable)

$$= \nu_J h^{-1}[K_1] > 0.$$

As f is arbitrary, μ is perfect. **Q**

(e) ? Suppose, if possible, that μ is countably compact. Let $\mathcal{K}$ be a countably compact class of sets such that μ is inner regular with respect to $\mathcal{K}$; we may suppose that $\mathcal{K} \subseteq \Sigma$.

(i) For $I \subseteq \omega_1$ set

$$U(I) = \{x : x \in X,\ x(\eta) = 0 \text{ for every } \eta \in I\}.$$

It will be helpful to know that if $E \in \Sigma$ and $\mu E > 0$, there is a $\gamma < \omega_1$ such that $\mu(E \cap U(I)) > 0$ for every finite $I \subseteq \omega_1 \setminus \gamma$. **P** Express E as $X \cap F$ where $F \in \mathrm{T}_{\omega_1}$. Let $J \subseteq \omega_1$ be a countable set such that $\nu_{\omega_1}(F' \setminus F) = 0$, where $F' = \pi_J^{-1}[\pi_J[F]]$ (254Od again), and $\gamma < \omega_1$ such that $J \subseteq \gamma$. If $I \subseteq \omega_1 \setminus \gamma$ is finite, then $I \cap J = \emptyset$, while $U(I)$ is determined by coordinates in I and F' is determined by coordinates in J; so

$$\begin{aligned}\mu(E \cap U(I)) &= \nu^*_{\omega_1}(X \cap F \cap U(I)) = \nu_{\omega_1}(F \cap U(I))\\ &= \nu_{\omega_1}(F' \cap U(I)) = \nu_{\omega_1}F' \cdot \nu_{\omega_1}U(I) = \mu E \cdot \nu_{\omega_1}U(I) > 0.\end{aligned}$$

Thus this γ serves. **Q**

(ii) Let $\mathcal{M}$ be the family of countable subsets M of $\omega_1 \cup \mathcal{K}$ such that

(α) if $I \subseteq M \cap \omega_1$ is finite there is a $K \in M \cap \mathcal{K}$ such that $K \subseteq U(I)$ and $\mu K > 0$;

(β) if $K \in M \cap \mathcal{K}$, $I \subseteq M \cap \omega_1$ is finite and $\mu(K \cap U(I)) > 0$, then there is a $K' \in M \cap \mathcal{K}$ such that $K' \subseteq K \cap U(I)$ and $\mu K' > 0$;

(γ) if $\gamma \in M \cap \omega_1$ then $\gamma \subseteq M$;

(δ) if $K \in M \cap \mathcal{K}$ and $\mu K > 0$ then there is a $\gamma \in M \cap \omega_1$ such that $\mu(K \cap U(I)) > 0$ whenever $I \subseteq \omega_1 \setminus \gamma$ is finite.

Then every countable $M \subseteq \omega_1 \cup \mathcal{K}$ is included in some member M' of $\mathcal{M}$.

P Choose $\langle N_n \rangle_{n \in \mathbb{N}}$ as follows. $N_0 = M$. Given that N_n is a countable subset of $\omega_1 \cup \mathcal{K}$ then let $N_{n+1} \subseteq \omega_1 \cup \mathcal{K}$ be a countable set such that

(α) if $I \subseteq N_n \cap \omega_1$ is finite there is a $K \in N_{n+1} \cap \mathcal{K}$ such that $K \subseteq U(I)$ and $\mu K > 0$;

(β) if $K \in N_n \cap \mathcal{K}$, $I \subseteq N_n \cap \omega_1$ is finite and $\mu(K \cap U(I)) > 0$, then there is a $K' \in N_{n+1} \cap \mathcal{K}$ such that $K' \subseteq K \cap U(I)$ and $\mu K' > 0$;

(γ) if $\gamma \in N_n \cap \omega_1$ then $\gamma \subseteq N_{n+1}$;

(δ) if $K \in N_n \cap \mathcal{K}$ and $\mu K > 0$ then there is a $\gamma \in N_{n+1} \cap \omega_1$ such that $\mu(K \cap U(I)) > 0$ whenever $I \subseteq \omega_1 \setminus \gamma$ is finite;

(ϵ) $N_n \subseteq N_{n+1}$.

On completing the induction, set $M' = \bigcup_{n\in\mathbb{N}} N_n$; this serves (because every finite subset of M' is a subset of some N_n). **Q**

(iii) Choose a sequence $\langle M_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}$ such that, for each n, $M_n \cup \{\sup(M_n \cap \omega_1) + 1\} \subseteq M_{n+1}$. Set $\gamma_n = \sup(M_n \cap \omega_1)$ for each n. Note that $\gamma_n \subseteq M_n$, because if $\eta < \gamma_n$ then there is some $\xi \in M_n$ such that $\eta < \xi$; now $\xi \subseteq M_n$ because $M_n \in \mathcal{M}$, so $\eta \in M_n$. Also $\gamma_n + 1 \in M_{n+1}$ for each n, so $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ is strictly increasing, and $\xi = \sup_{n\in\mathbb{N}} \gamma_n$ belongs to Ω.

Set $J = \{\theta_\xi(n) : n \in \mathbb{N}\}$. Then $J \cap \eta$ is finite for every $\eta < \xi$, and in particular $J \cap \gamma_n$ is finite for every n. Set $I_0 = J \cap \gamma_0$ and $I_n = J \cap \gamma_n \setminus \gamma_{n-1}$ for $n \geq 1$. Then $\bigcap_{n\in\mathbb{N}} U(I_n) = Q_\xi$ is disjoint from X.

Choose a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ as follows. Because I_0 is a finite subset of $M_0 \cap \omega_1$, there is a $K_0 \in M_0 \cap \mathcal{K}$ such that $K_0 \subseteq U(I_0)$ and $\mu K_0 > 0$. Given that $K_n \in M_n \cap \mathcal{K}$ and $\mu K_n > 0$, then there is a $\beta \in M_n \cap \omega_1$ such that $\mu(K_n \cap U(I)) > 0$ for every finite $I \subseteq \omega_1 \setminus \beta$; now $\beta \leq \gamma_n$ and $I_{n+1} \cap \gamma_n = \emptyset$, so $\mu(K_n \cap U(I_{n+1})) > 0$. But $K_n \in M_{n+1} \cap \mathcal{K}$ and I_{n+1} is a finite subset of $M_{n+1} \cap \omega_1$, so there is a $K_{n+1} \in M_{n+1} \cap \mathcal{K}$ such that $K_{n+1} \subseteq K_n \cap U(I_{n+1})$ and $\mu K_{n+1} > 0$. Continue.

In this way we find a non-increasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $K_n \subseteq U(I_n)$ for every n and no K_n is empty. But in this case $\bigcap_{i\leq n} K_i = K_n$ is non-empty for every n, while $\bigcap_{n\in\mathbb{N}} K_n \subseteq X \cap \bigcap_{n\in\mathbb{N}} U(I_n)$ is empty. So $\mathcal{K}$ is not a countably compact class. **X**

(f) Thus μ is not countably compact, and has all the properties claimed.

***451V Weakly α-favourable spaces** There is an interesting variation on the concept of 'countably compact' measure space, as follows. For any measure space (X, Σ, μ) we can imagine an infinite game for two players, whom I will call 'Empty' and 'Nonemepty'. Empty chooses a non-negligible measurable set E_0; Nonempty chooses a non-negligible measurable set $F_0 \subseteq E_0$; Empty chooses a non-negligible measurable set $E_1 \subseteq F_0$; Nonempty chooses a non-negligible measurable set $F_1 \subseteq E_1$, and so on. At the end of the game, Empty wins if $\bigcap_{n\in\mathbb{N}} E_n = \bigcap_{n\in\mathbb{N}} F_n$ is empty; otherwise Nonempty wins. (If $\mu X = 0$, so that Empty has no legal initial move, I declare Nonempty the winner by default.) If you have seen 'Banach-Mazur' games, you will recognise this as a similar construction, in which open sets are replaced by non-negligible measurable sets.

A **strategy** for Nonempty is a rule to determine his moves in terms of the preceding moves for Empty; that is, a function $\sigma : \bigcup_{n\in\mathbb{N}} (\Sigma \setminus \mathcal{N})^{n+1} \to \Sigma \setminus \mathcal{N}$, where $\mathcal{N}$ is the ideal of negligible sets, such that $\sigma(E_0, E_1, \dots, E_n) \subseteq E_n$, at least whenever $E_0, \dots, E_n \in \Sigma \setminus \mathcal{N}$ are such that $E_{k+1} \subseteq \sigma(E_0, \dots, E_k)$ for every $k < n$; since it never matters what Nonempty does if Empty has already broken the rules, we usually just demand that $\sigma(E_0, \dots, E_n) \subseteq E_n$ for all $E_0, \dots, E_n \in \Sigma \setminus \mathcal{N}$. σ is a **winning strategy** if $\bigcap_{n\in\mathbb{N}} E_n \neq \emptyset$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\Sigma \setminus \mathcal{N}$ such that $E_{n+1} \subseteq \sigma(E_0, \dots, E_n)$ for every $n \in \mathbb{N}$. In terms of the game, we interpret this as saying that Nonempty will win if he plays $F_n = \sigma(E_0, \dots, E_n)$ whenever faced with the position $(E_0, F_0, E_1, F_1, \dots, F_{n-1}, E_n)$. (Since it is supposed that Nonempty will use the same strategy throughout the game, the moves $F_0, \dots, F_{n-1}$ are determined by $E_0, \dots, E_{n-1}$ and there is no advantage in taking them separately into account when choosing F_n.)

Now we say that the measure space (X, Σ, μ) is **weakly α-favourable** if there is such a winning strategy for Nonempty.

It turns out that the class of weakly α-favourable spaces behaves in much the same way as the class of countably compact spaces. For the moment, however, I leave the details to the exercises (451Yh-451Yr). See Fremlin 00.

451X Basic exercises (a) (i) Show that any purely atomic measure space is perfect. (ii) Show that any strictly localizable purely atomic measure space is countably compact. (iii) Show that the space of 342N is not countably compact.

>(b) Show that a compact measure space in which singleton sets are negligible is atomless.

>(c) Let (X,Σ,μ) be a measure space, and ν an indefinite-integral measure over μ (234J[5]). Show that ν is compact, or countably compact, or perfect if μ is.

(d) In 413Xn, show that μ is a countably compact measure. (*Hint*: show that the algebra Σ there is a countably compact class.)

(e) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X,Σ,μ). Show that μ is compact, or countably compact, or perfect iff every μ_i is.

(f) Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu_*K>0$. (i) Show that if $\mathcal{K}$ is a compact class then μ is a compact measure. (ii) Show that if $\mathcal{K}$ is a countably compact class then μ is a countably compact measure.

(g) Let (X,Σ,μ) be a measure space. For $A\subseteq X$, write μ_A for the subspace measure on A. Suppose that whenever $E\in\Sigma$ and $\mu E>0$ there is a set $A\subseteq X$ such that μ_A is perfect and $\mu^*(A\cap E)>0$. Show that μ is perfect.

(h)(i) Give an example of a compact probability space (X,Σ,μ) and a σ-subalgebra T of Σ such that $\mu\restriction\mathrm{T}$ is not compact. (ii) Give an example of a compact probability space (X,Σ,μ), a set Y and a function $f:X\to Y$ such that the image measure μf^{-1} is not compact. (*Hint*: 342M, 342Xf, 439Xa.)

(i) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, with product X. Suppose that $\mathcal{K}_i\subseteq\mathcal{P}X_i$ for each i, and set $\mathcal{K}=\{\prod_{i\in I}K_i:K_i\in\mathcal{K}_i$ for each $i\}$. (i) Show that if $\mathcal{K}_i$ is a compact class for each i, so is $\mathcal{K}$. (ii) Show that if $\mathcal{K}_i$ is a countably compact class for each i, so is $\mathcal{K}$.

(j) Let $A\subseteq[0,1]$ be a set with outer Lebesgue measure 1 and inner measure 0. Show that there is a Borel measure λ on $A\times[0,1]$ such that λ is not inner regular with respect to sets which have Borel measurable projections on the factor spaces.

(k) Let X be a Polish space and E a subset of X. Show that the following are equiveridical: (i) E is universally measurable; (ii) every Borel probability measure on E is perfect; (iii) every σ-finite Borel measure on E is compact.

(l) In 451N, show that μ is a compact measure.

(m) Find a Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$, a continuous function $f:X\to[0,1]$ and a set $B\subseteq[0,1]$ such that $\mu^*(f^{-1}[B])<(\mu f^{-1})^*B$.

(n) Let (X,Σ,μ) be a σ-finite measure space. Show that it is perfect iff whenever $f:X\to\mathbb{R}$ is measurable there is a K_σ set $H\subseteq f[X]$ such that $f^{-1}[H]$ is conegligible.

(o) Let X be a metrizable space, and μ a semi-finite topological measure on X which (regarded as a measure) is compact. Show that μ is τ-additive.

(p) Let (X,Σ,μ) be a compact strictly localizable measure space (e.g., any Radon measure space), (Y,T,ν) a σ-finite measure space, and $f:X\to L^0(\nu)$ a function. Show that the following are equiveridical: (i) f is measurable, when $L^0(\nu)$ is given its topology of convergence in measure; (ii) there is a function $h\in\mathcal{L}^0(\lambda)$, where λ is the c.l.d. product measure on $X\times Y$, such that $f(x)=h_x^\bullet$ for almost every $x\in X$, where $h_x(y)=h(x,y)$. (*Hint*: 418R.)

>(q) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space. Show that $\Sigma=\mathcal{P}X$ iff μ is purely atomic. (*Hint*: if $\Sigma=\mathcal{P}X$, apply 451T with $Y=X$, the discrete topology on Y and the identity function from X to Y.)

[5]Formerly 234B.

(r) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and U a normed space. Show that if f, $g:X\to U$ are measurable functions, then $f+g$ is measurable. (Cf. 418Xk.)

(s) Show that in all three of the constructions of 439A, the measure ν is countably compact. (*Hint*: for the 'third construction', consider $\{f^{-1}[F]:F\subseteq\{0,1\}^{\mathfrak{c}}$ is a zero set$\}$.)

451Y Further exercises (a) Show that for any probability space (X,Σ,μ), there is a compact probability space (Y,T,ν) with a subspace isomorphic to (X,Σ,μ).

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i a σ-algebra of subsets of X_i for each i. Suppose that for each finite $J\subseteq I$ we are given a finitely additive functional ν_J on $X_J=\prod_{i\in J}X_i$, with domain the algebra $\mathrm{T}_J=\bigotimes_{i\in J}\Sigma_i$ generated by sets of the form $\{x:x\in X_J,\,x(i)\in E\}$ for $i\in J$, $E\in\Sigma_i$, and that (α) $\nu_K\{x:x\in X_K,\,x{\restriction}J\in W\}=\nu_JW$ whenever $J\subseteq K\in[I]^{<\omega}$ and $W\in\mathrm{T}_J$ (β) $\mu_i=\nu_{\{i\}}$ is a countably compact probability measure for every $i\in I$. Show that there is a countably compact measure μ on $X=X_I$ such that $\mu\{x:x\in X,\,x{\restriction}J\in W\}=\nu_JW$ whenever $J\in[I]^{<\omega}$ and $W\in\mathrm{T}_J$. (*Hint*: 454D.) (Compare 418M.)

(c) Describe μ in the case of 451Yb in which $I=[0,1]$, $X_i=[0,1]\setminus\{i\}$, Σ_i is the algebra of Lebesgue measurable subsets of X_i, and $\nu_JE=\mu_L\{t:z_{Jt}\in E\}$ for every $E\in\bigotimes_{i\in J}\Sigma_i$, where $z_{Jt}(i)=t$ for $i\in J$, $t\in[0,1]$. Contrast this with the difficulty encountered in 418Xx.

(d) Let (X,Σ,μ) be a semi-finite compact measure space, and $\langle E_i\rangle_{i\in I}$ a point-finite family of measurable subsets of X such that $\bigcup_{i\in J}E_i\in\Sigma$ for every $J\subseteq I$. Show that $\mu(\bigcup_{i\in I}E_i)=\sup_{J\subseteq I\text{ is finite}}\mu(\bigcup_{i\in J}E_i)$. (*Hint*: 438Ya.)

(e) Let X be a hereditarily metacompact space, and μ a semi-finite topological measure on X which (regarded as a measure) is compact. Show that μ is τ-additive.

(f) Let (X,Σ,μ) be a compact measure space, V a Banach space and $f:X\to V$ a measurable function such that $\|f\|:X\to[0,\infty[$ is integrable. Show that f is Bochner integrable (253Yf).

(g) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space. Suppose that Y is a separable metrizable space and Z is a metrizable space, and that $f:X\times Y\to Z$ is a function such that $x\mapsto f(x,y)$ is measurable for every $y\in Y$ and $y\mapsto f(x,y)$ is continuous for every $x\in X$. Show that μ is inner regular with respect to $\{F:F\subseteq X,$ $f{\restriction}F\times Y$ is continuous$\}$. (*Hint*: 418Yk.)

(h) Show that any purely atomic measure space is weakly α-favourable, so that the space of 342N is weakly α-favourable but not countably compact.

(i) Show that the direct sum of a family of weakly α-favourable measure spaces is weakly α-favourable.

(j) Show that an indefinite-integral measure over a weakly α-favourable measure is weakly α-favourable.

(k)(i) Show that a countably compact measure space is weakly α-favourable. (ii) Show that a semi-finite weakly α-favourable measure space is perfect.

(l) Show that any measurable subspace of a weakly α-favourable measure space is weakly α-favourable.

(m) Let (X,Σ,μ) be a weakly α-favourable measure space, (Y,T,ν) a semi-finite measure space, and $f:X\to Y$ a (Σ,T)-measurable function such that $f^{-1}[F]$ is negligible whenever $F\subseteq Y$ is negligible. Show that (Y,T,ν) is weakly α-favourable.

(n)(i) Show that a measure space is weakly α-favourable iff its completion is weakly α-favourable. (ii) Show that a semi-finite measure space is weakly α-favourable iff its c.l.d. version is weakly α-favourable.

(o) Show that the c.l.d. product of two weakly α-favourable measure spaces is weakly α-favourable.

(p) Show that the product of any family of weakly α-favourable probability measures is weakly α-favourable.

(q) Show that the space of 451U is not weakly α-favourable.

(r) Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}$ a lower density for μ such that $\underline{\phi}X = X$; let $\mathfrak{T}$ be the corresponding density topology (414P). Show that (X,Σ,μ) is weakly α-favourable iff $(X,\mathfrak{T})$ is weakly α-favourable (definition: 4A2A).

(s) Let X be a set, and $\langle\mu_i\rangle_{i\in I}$ a family of weakly α-favourable measures on X with sum μ (234G[6]). Show that if μ is semi-finite, it is weakly α-favourable.

(t) Let X and Y be locally compact Hausdorff groups and $\phi : X \to Y$ a group homomorphism which is Haar measurable in the sense of 411L, that is, $\phi^{-1}[H]$ is Haar measurable for every open $H \subseteq Y$. Show that ϕ is continuous.

(u) Let μ be a quasi-Radon measure on the Sorgenfrey line (415Xc). Show that μ is weakly α-favourable.

451 Notes and comments For a useful survey of results on countably compact and perfect measures, with historical notes, see Ramachandran 02.

The concepts of 'compact', 'countably compact' and 'perfect' measure space can all be regarded as attempts to understand and classify the special properties of Lebesgue measure on $[0,1]$, regarded as a measure space. Because a countably separated perfect probability space is very nearly isomorphic to Lebesgue measure (451Ad), we can think of a perfect measure space as one in which the countably-generated σ-subalgebras look like Lebesgue measure (451F). The arguments of 451Ic and 451Jc already hint at the kind of results we can hope for. When we form a product measure, each measurable set in the product will depend, in effect, on sequences of measurable sets in the factors, and therefore can be studied in terms of countably generated subalgebras; so that many results about products of perfect measures will be derivable, if we wish to take that route, from results about products of copies of Lebesgue measure. Of course my normal approach in this treatise is to go straight for the general result; but like anyone else I often start from a picture based on the familiar special case. In the next section we shall have some theorems for which countable compactness, rather than perfectness, seems to be the relevant property.

The first half of the section (down to 451P) is essentially a matter of tidying up the theory of compact and perfect measures, and showing that the same ideas will cover the new class of countably compact measures. (You may like to go back to 342G, in which I worked through the basic properties of compact measures, and contrast the arguments used there with the slightly more sophisticated ones above.) In 451Q-451T I enter new territory, showing that for compact measures (and therefore for Radon measures) the theory of measurable functions into metric spaces is particularly simple, without making any assumptions about measure-free cardinals.

Version of 6.11.08

452 Integration and disintegration of measures

A standard method of defining measures is through a formula

$$\mu E = \int \mu_y E\, \nu(dy)$$

where (Y,T,ν) is a measure space and $\langle\mu_y\rangle_{y\in Y}$ is a family of measures on another set X. In practice these constructions commonly involve technical problems concerning the domain of μ (as in 452Xi), which is why I have hardly used them so far in this treatise. There are not-quite-trivial examples in 417Yb, 434R and 436F, and the indefinite-integral measures of §234 can also be expressed in this way (452Xf); for a case in which this approach is worked out fully, see 453N. But when a formula of this kind is valid, as in Fubini's theorem, it is likely to be so useful that it dominates further investigation of the topic. In this section I give one of the two most important theorems guaranteeing the existence of appropriate families $\langle\mu_y\rangle_{y\in Y}$

[6]Formerly 112Ya.

when μ and ν are given (452I); the other will follow in the next section (453K). They both suppose that we are provided with a suitable function $f : X \to Y$, and rely heavily on the Lifting Theorem (§341) and on considerations of inner regularity from Chapter 41.

The formal definition of a 'disintegration' (which is nearly the same thing as a 'regular conditional probability') is in 452E. The main theorem depends, for its full generality, on the concept of 'countably compact measure' (451B). It can be strengthened when μ is actually a Radon measure (452O).

The greater part of the section is concerned with general disintegrations, in which the measures μ_y are supposed to be measures on X and are not necessarily related to any particular structure on X. However a natural, and obviously important, class of applications has $X = Y \times Z$ and each μ_y based on the section $\{y\} \times Z$, so that it can be regarded as a measure on Z. Mostly there is very little more to be said in this case (see 452B-452D); but in 452M we find that there is an interesting variation in the way that countable compactness can be used.

452A Lemma Let (Y, T, ν) be a measure space, X a set, and $\langle\mu_y\rangle_{y\in Y}$ a family of measures on X. Let $\mathcal{A}$ be the family of subsets A of X such that $\theta E = \int \mu_y E\,\nu(dy)$ is defined in $\mathbb{R}$. Suppose that $X \in \mathcal{A}$.

(a) $\mathcal{A}$ is a Dynkin class.

(b) If Σ is any σ-subalgebra of $\mathcal{A}$ then $\mu = \theta\restriction\Sigma$ is a measure on X.

(c) Suppose now that every μ_y is complete. If, in (b), $\hat\mu$ is the completion of μ and $\hat\Sigma$ its domain, then $\hat\Sigma \subseteq \mathcal{A}$ and $\hat\mu = \theta\restriction\hat\Sigma$.

proof For (a) and (b), we have only to look at the definitions of 'Dynkin class' and 'measure' and apply the elementary properties of the integral. For (c), if $E \in \hat\Sigma$, then there are E', $E'' \in \Sigma$ such that $E' \subseteq E \subseteq E''$ and $\theta E' = \theta E''$. So $\mu_y E' = \mu_y E''$ for ν-almost every y; since all the μ_y are supposed to be complete, $\mu_y E$ is defined and equal to $\mu_y E'$ for almost every y, and θE is defined and equal to $\theta E' = \mu E' = \hat\mu E$.

452B Theorem (a) Let X be a set, (Y, T, ν) a measure space, and $\langle\mu_y\rangle_{y\in Y}$ a family of measures on X such that $\int \mu_y X\,\nu(dy)$ is defined and finite. Let $\mathcal{E}$ be a family of subsets of X, closed under finite intersections, such that $\int \mu_y E\,\nu(dy)$ is defined in $\mathbb{R}$ for every $E \in \mathcal{E}$.

(i) If Σ is the σ-algebra of subsets of X generated by $\mathcal{E}$, we have a totally finite measure μ on X, with domain Σ, given by the formula $\mu E = \int \mu_y E\,\nu(dy)$ for every $E \in \Sigma$.

(ii) If $\hat\mu$ is the completion of μ and $\hat\Sigma$ its domain, then $\int \hat\mu_y E\,\nu(dy)$ is defined and equal to $\hat\mu E$ for every $E \in \hat\Sigma$, where $\hat\mu_y$ is the completion of μ_y for each $y \in Y$.

(b) Let Z be a set, (Y, T, ν) a measure space, and $\langle\mu_y\rangle_{y\in Y}$ a family of measures on Z such that $\int \mu_y Z\,\nu(dy)$ is defined and finite. Let $\mathcal{H}$ be a family of subsets of Z, closed under finite intersections, such that $\int \mu_y H\,\nu(dy)$ is defined in $\mathbb{R}$ for every $H \in \mathcal{H}$.

(i) If Υ is the σ-algebra of subsets of Z generated by $\mathcal{H}$, we have a totally finite measure μ on $Y \times Z$, with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, defined by setting $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$.

(ii) If $\hat\mu$ is the completion of μ and $\hat\Sigma$ its domain, then $\int \hat\mu_y E[\{y\}]\nu(dy)$ is defined and equal to $\hat\mu E$ for every $E \in \hat\Sigma$, where $\hat\mu_y$ is the completion of μ_y for each $y \in Y$.

proof (a) Define $\mathcal{A} \subseteq \mathcal{P}X$ as in 452A. Then $\mathcal{E} \subseteq \mathcal{A}$, so by the Monotone Class Theorem (136B) $\Sigma \subseteq \mathcal{A}$ and we have (i). Applying 452Ac to $\langle\hat\mu_y\rangle_{y\in Y}$ we have (ii).

(b) Set $X = Y \times Z$. For $y \in Y$, let μ'_y be the measure on X defined by setting $\mu'_y E = \mu_y E[\{y\}]$ whenever this is defined; that is, μ'_y is the image of μ_y under the function $z \mapsto (y, z) : Z \to X$. Set $\mathcal{E} = \{F \times H : F \in \mathrm{T}, H \in \mathcal{H}\}$. Then $\mathcal{E}$ is a family of subsets of X closed under finite intersections, and

$$\int \mu'_y(F \times H)\nu(dy) = \int \chi F(y)\mu_y H\,\nu(dy)$$

is defined whenever $F \in \mathrm{T}$ and $H \in \mathcal{H}$. By (a), we have a measure μ on X, with domain the σ-algebra Σ generated by $\mathcal{E}$, defined by writing

$$\mu E = \int \mu'_y E\,\nu(dy) = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \Sigma$. Of course Σ includes $\mathrm{T}\widehat{\otimes}\Upsilon$ (the set $\{H : Y \times H \in \Sigma\}$ is a σ-algebra of subsets of Z including $\mathcal{H}$, so includes Υ) and is therefore equal to $\mathrm{T}\widehat{\otimes}\Upsilon$.

This proves (i). If now $E \in \hat{\Sigma}$, (a-ii) tells us that

$$\hat{\mu}E = \int \hat{\mu}'_y E\,\nu(dy) = \int \hat{\mu}_y E[\{y\}]\nu(dy).$$

452C Theorem (a) Let Y be a topological space, ν a τ-additive topological measure on Y, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a family of τ-additive topological measures on X such that $\int \mu_y X\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$.

(i) We can define a τ-additive Borel measure μ on X by writing $\mu E = \int \mu_y E\,\nu(dy)$ for every Borel set $E \subseteq X$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E\,\nu(dy)$ is defined and equal to $\hat{\mu}E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

(b) Let Y be a topological space, ν a τ-additive topological measure on Y, $(Z, \mathfrak{U})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a family of τ-additive topological measures on Z such that $\int \mu_y Z\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y \mapsto \mu_y V$ is lower semi-continuous for every $V \in \mathcal{V}$.

(i) We can define a τ-additive Borel measure μ on $Y \times Z$ by writing $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every Borel set $E \subseteq Y \times Z$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E[\{y\}]\nu(dy)$ is defined and equal to $\hat{\mu}E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

proof (a) For $A \subseteq X$, set $f_A(y) = \mu_y A$ when this is defined. We may suppose that $\emptyset \in \mathcal{U}$. If $\mathcal{W} \subseteq \mathcal{U}$ is a non-empty upwards-directed set with union G, $\langle f_W \rangle_{W \in \mathcal{W}}$ is an upwards-directed family of lower semi-continuous functions with supremum f_G, because every μ_y is τ-additive. So f_G is lower semi-continuous, and also $\int f_G d\nu = \sup_{W \in \mathcal{W}} \int f_W d\nu$, by 414Ba. Taking $\mathcal{E}$ to be the family of open subsets of X in 452Ba, we see that we have a τ-additive Borel measure μ on X such that $\mu E = \int \mu_y E\,\nu(dy)$ for every Borel set $E \subseteq X$. Moreover, if $\mathcal{G}$ is a non-empty upwards-directed family of open subsets of X with union G^*, then $\mathcal{W} = \{W : W \in \mathcal{U},\, W \subseteq G \text{ for some } G \in \mathcal{G}\}$ is an upwards-directed family with union G^*, so

$$\mu G^* = \int f_{G^*} d\nu = \sup_{W \in \mathcal{W}} \int f_W d\nu \le \sup_{G \in \mathcal{G}} \mu G \le \mu G^*.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. This proves (i); (ii) follows immediately, as in 452Ba.

(b) Let $\mathcal{U}$ be the family of sets expressible as $\bigcup_{i \le n} H_i \times V_i$ where $H_i \subseteq Y$ is open and $V_i \in \mathcal{V}$ for every $i \le n$. Because $\mathcal{V}$ is a base for $\mathfrak{U}$, $\mathcal{U}$ is a base for the topology of $X = Y \times Z$. For $y \in Y$ let μ'_y be the measure on X defined by saying that $\mu'_y E = \mu_y E[\{y\}]$ whenever this is defined. Then μ'_y is a τ-additive topological probability measure on X, by 418Ha or otherwise. If $U \in \mathcal{U}$, $y \mapsto \mu'_y U$ is lower semi-continuous. **P** Express U as $\bigcup_{i \le n} H_i \times V_i$ where $H_i \subseteq Y$ is open and $V_i \in \mathcal{V}$ for each i. Suppose that $y \in Y$ and $\gamma < \mu_y U$. Set $I = \{i : i \le n,\, y \in H_i\}$, $H = Y \cap \bigcap_{i \in I} H_i$ and $V = \bigcup_{i \in I} V_i$. Then $U[\{y\}] = V \subseteq U[\{y'\}]$ for every $y' \in H$. Also $H' = \{y' : \mu_{y'} V > \gamma\}$ is a neighbourhood of y. So $H \cap H'$ is a neighbourhood of y, and $\mu'_{y'} U > \gamma$ for every $y' \in H \cap H'$. As y and γ are arbitrary, we have the result. **Q**

Now applying (a) to $\langle \mu'_y \rangle_{y \in Y}$ we see that (b) is true.

452D Theorem (a) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon measure space, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a uniformly tight (definition: 437O) family of Radon measures on X such that $\int \mu_y X\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$. Then we have a totally finite Radon measure $\tilde{\mu}$ on X defined by saying that that $\tilde{\mu}E = \int \mu_y E\,\nu(dy)$ whenever $\tilde{\mu}$ measures E.

(b) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon measure space, $(Z, \mathfrak{U})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a uniformly tight family of Radon measures on Z such that $\int \mu_y Z\,\nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y \mapsto \mu_y V$ is lower semi-continuous for every $V \in \mathcal{V}$. Then we have a totally finite Radon measure $\tilde{\mu}$ on $Y \times Z$ such that $\tilde{\mu}E = \int \mu_y E[\{y\}]\nu(dy)$ whenever $\tilde{\mu}$ measures E.

proof I take the two parts together. In (b), write X for $Y \times Z$. By 452C we have a τ-additive Borel measure μ satisfying the appropriate formula. Now for any $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu K \ge \mu X - 2\epsilon$. **P** In (a), take $\eta > 0$ such that $\int \min(\eta, \mu_y X)\nu(dy) \le 2\epsilon$, and K such that $\mu_y(X \setminus K) \le \eta$

for every $y \in Y$. In (b), take $\eta > 0$ such that $\int \min(\eta, \mu_y Z)\nu(dy) \le \epsilon$. Now let $K_1 \subseteq Y$ and $K_2 \subseteq Z$ be compact sets such that

$$\int_{K_1} \mu_y Z\,\nu(dy) \ge \int_Y \mu_y Z\,\nu(dy) - \epsilon, \quad \mu_y(Z \setminus K_2) \le \eta \text{ for every } y \in Y.$$

Then $K = K_1 \times K_2$ is compact and

$$\begin{aligned}\mu((Y \times Z) \setminus K) &\le \int_{Y \setminus K_1} \mu_y Z\,\nu(dy) + \int_{K_1} \mu_y(Z \setminus K_2)\nu(dy) \\ &\le \epsilon + \int_{K_1} \min(\eta, \mu_y Z)\nu(dy) \le 2\epsilon. \ \mathbf{Q}\end{aligned}$$

Since μ is totally finite it is surely locally finite and effectively locally finite, so the conditions of 416F(iv) are satisfied and the c.l.d. version $\tilde\mu$ of μ is a Radon measure on X. But of course $\tilde\mu$ is just the completion of μ, so 452C(a-ii) or 452C(b-ii) tells us that the declared formula also applies to $\tilde\mu$.

452E All the constructions above can be thought of as special cases of the following.

Definition Let (X, Σ, μ) and (Y, T, ν) be measure spaces. A **disintegration** of μ over ν is a family $\langle \mu_y \rangle_{y \in Y}$ of measures on X such that $\int \mu_y E\,\nu(dy)$ is defined in $[0, \infty]$ and equal to μE for every $E \in \Sigma$. If $f : X \to Y$ is an inverse-measure-preserving function, a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν is **consistent** with f if, for each $F \in \mathrm{T}$, $\mu_y f^{-1}[F] = 1$ for ν-almost every $y \in F$. $\langle \mu_y \rangle_{y \in Y}$ is **strongly consistent** with f if, for almost every $y \in Y$, μ_y is a probability measure for which $f^{-1}[\{y\}]$ is conegligible.

A trivial example of a disintegration is when ν is a probability measure and $\mu_y = \mu$ for every y. Of course this is of little interest. The archetypal disintegration is 452Bb when all the μ_y are the same, in which case Fubini's theorem tells us that we are looking at a product measure on $X = Y \times Z$. If μ is a probability measure then this disintegration is strongly consistent.

The phrase **regular conditional probability** is used for special types of disintegration; typically, when μ and ν and every μ_y are probabilities, and sometimes supposing that every μ_y has the same domain as μ. I have seen the word **decomposition** used for what I call a disintegration.

452F Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces and $\langle \mu_y \rangle_{y \in Y}$ a disintegration of μ over ν. Then $\iint f(x)\mu_y(dx)\nu(dy)$ is defined and equal to $\int f d\mu$ for every $[-\infty, \infty]$-valued function f such that $\int f d\mu$ is defined in $[-\infty, \infty]$.

proof (a) Suppose first that f is non-negative. Let $H \in \Sigma$ be a conegligible set such that $f{\restriction}H$ is Σ-measurable. For $n \in \mathbb{N}$ set

$$E_{nk} = \{x : x \in H,\ 2^{-n}k \le f(x)\} \text{ for } k \ge 1, \quad f_n = 2^{-n}\textstyle\sum_{k=1}^{4^n} \chi E_{nk}.$$

Then $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of functions with $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in H$. Now $\int \mu_y(X \setminus H)\nu(dy) = 0$, so $X \setminus H$ is μ_y-negligible for almost every y. Set

$$V = \{y : \mu_y(X \setminus H) = 0,\ E_{nk} \in \operatorname{dom} \mu_y \text{ for every } n \in \mathbb{N},\ k \ge 1\};$$

then V is ν-conegligible. For $y \in V$,

$$\int f d\mu_y = \lim_{n\to\infty} \int f_n d\mu_y = \lim_{n\to\infty} 2^{-n}\textstyle\sum_{k=1}^{4^n} \mu_y E_{nk},$$

while each function $y \mapsto \mu_y E_{nk}$ is ν-virtually measurable, so $y \mapsto \int f d\mu_y$ is ν-virtually measurable and

$$\begin{aligned}\iint f d\mu_y \nu(dy) &= \lim_{n\to\infty} \iint f_n d\mu_y \nu(dy) = \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{4^n} \int \mu_y E_{nk}\,\nu(dy) \\ &= \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{4^n} \mu E_{nk} = \lim_{n\to\infty} \int f_n d\mu = \int f d\mu.\end{aligned}$$

(b) For general f we now have

$$\iint f(x)\mu_y(dx)\nu(dy) = \iint f^+(x)\mu_y(dx)\nu(dy) - \iint f^-(x)\mu_y(dx)\nu(dy)$$
$$= \int f^+ d\mu - \int f^- d\mu = \int f d\mu,$$

where f^+, f^- are the positive and negative parts of f.

Remark When $X = Y \times Z$ and our disintegration is a family $\langle \mu'_y \rangle_{y \in Y}$ of measures on X defined from a family $\langle \mu_y \rangle_{y \in Y}$ of probability measures on Z, as in 452Bb, we can more naturally write $\int f(y,z)\mu_y(dz)$ in place of $\int f(x)\mu'_y(dx)$, and we get

$$\iint f(y,z)\mu_y(dz)\nu(dy) = \int f d\mu \text{ whenever the latter is defined in } [-\infty, \infty]$$

as in 252B.

452G The most useful theorems about disintegrations of course involve some restrictions on their form, most commonly involving consistency with some kind of projection. I clear the path with statements of some elementary facts.

Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, $f : X \to Y$ an inverse-measure-preserving function, and $\langle \mu_y \rangle_{y \in Y}$ a disintegration of μ over ν.

(a) If $\langle \mu_y \rangle_{y \in Y}$ is consistent with f, and $F \in \mathrm{T}$, then $\mu_y f^{-1}[F] = \chi F(y)$ for ν-almost every $y \in Y$; in particular, almost every μ_y is a probability measure.

(b) If $\langle \mu_y \rangle_{y \in Y}$ is strongly consistent with f it is consistent with f.

(c) If ν is countably separated (definition: 343D) and $\langle \mu_y \rangle_{y \in Y}$ is consistent with f, then it is strongly consistent with f.

proof (a) We have $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$. Since also

$$\mu_y(X \setminus f^{-1}[F]) = \mu_y f^{-1}[Y \setminus F] = 1, \quad \mu_y X = \mu_y f^{-1}[Y] = 1$$

for almost every $y \in Y \setminus F$, $\mu_y f^{-1}[F] = 0$ for almost every $y \in X \setminus F$.

(b) If $F \in \mathrm{T}$, then $f^{-1}[F] \supseteq f^{-1}[\{y\}]$ is μ_y-conegligible for almost every $y \in F$; since we are also told that $\mu_y X = 1$ for almost every y, $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$.

(c) There is a countable $\mathcal{F} \subseteq \mathrm{T}$ separating the points of Y; we may suppose that $Y \in \mathcal{F}$ and that $Y \setminus F \in \mathcal{F}$ for every $F \in \mathcal{F}$. Now

$$H_F = F \setminus \{y : \mu_y f^{-1}[F] \text{ is defined and equal to } 1\}$$

is negligible for every $F \in \mathcal{F}$, so that

$$Z = Y \setminus \bigcup_{F \in \mathcal{F}} H_F$$

is conegligible. For $y \in Z$, set $\mathcal{F}_y = \{F : y \in F \in \mathcal{F}\}$; then

$$\{y\} = \bigcap \mathcal{F}_y, \quad f^{-1}[\{y\}] = \bigcap \{f^{-1}[F] : F \in \mathcal{F}_y\},$$

while $\mu_y f^{-1}[F] = 1$ for every $F \in \mathcal{F}_y$. Because $\mathcal{F}_y$ is countable, $\mu_y f^{-1}[\{y\}] = 1$. This is true for almost every y, so $\langle \mu_y \rangle_{y \in Y}$ is strongly consistent with f.

452H Lemma Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $T : L^\infty(\mu) \to L^\infty(\nu)$ a positive linear operator such that $T(\chi X^\bullet) = \chi Y^\bullet$ and $\int Tu = \int u$ whenever $u \in L^\infty(\mu)^+$. Let $\mathcal{K}$ be a countably compact class of subsets of X, closed under finite unions and countable intersections, such that μ is inner regular with respect to $\mathcal{K}$. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν such that

(i) μ_y is a complete probability measure on X, inner regular with respect to $\mathcal{K}$ and measuring every member of $\mathcal{K}$, for every $y \in Y$;

(ii) setting $h_g(y) = \int g\, d\mu_y$ whenever $g \in \mathcal{L}^\infty(\mu)$ and $y \in Y$ are such that the integral is defined, $h_g \in \mathcal{L}^\infty(\nu)$ and $T(g^\bullet) = h_g^\bullet$ for every $g \in \mathcal{L}^\infty(\mu)$.

proof (a) Completing ν does not change $\mathcal{L}^\infty(\nu)$ or $L^\infty(\nu)$, nor does it change the families which are disintegrations over ν; so we may assume throughout that ν is complete. It therefore has a lifting $\theta : \mathfrak{B} \to \mathrm{T}$, where $\mathfrak{B}$ is the measure algebra of ν, which gives rise to a Riesz homomorphism S from $L^\infty(\nu) \cong L^\infty(\mathfrak{B})$ to the space $L^\infty(\mathrm{T})$ of bounded T-measurable real-valued functions on Y such that $(Sv)^\bullet = v$ for every $v \in L^\infty(\nu)$ (363I, 363F, 363H).

(b) For $y \in Y$ and $E \in \Sigma$, set $\psi_y E = (ST(\chi E^\bullet))(y)$. Because $0 \le T(\chi E^\bullet) \le \chi Y^\bullet$ in $L^\infty(\nu)$, $0 \le \psi_y E \le 1$. The maps

$$E \mapsto \chi E \mapsto \chi E^\bullet \mapsto T(\chi E^\bullet) \mapsto ST(\chi E^\bullet)$$

are all additive, so $\psi_y : \Sigma \to [0,1]$ is additive for each $y \in Y$. For fixed $E \in \Sigma$,

$$\mu E = \int \chi E \, d\mu = \int (\chi E^\bullet) = \int T(\chi E^\bullet) = \int ST(\chi E^\bullet) = \int \psi_y E \, \nu(dy).$$

(c) Recall that μ is supposed to be inner regular with respect to the countably compact class $\mathcal{K}$. By 413Ua, there is for every $y \in Y$ a complete measure μ'_y on X such that $\mu'_y X \le \psi_y X \le 1$, $\mathcal{K} \subseteq \operatorname{dom} \mu'_y$, and $\mu'_y K \ge \psi_y K$ for every $K \in \mathcal{K} \cap \Sigma$.

(d) Now, for any fixed $E \in \Sigma$, $\mu'_y E$ is defined and equal to $\psi_y E$ for almost every $y \in Y$. **P** Let $\langle K_n \rangle_{n\in\mathbb{N}}$, $\langle K'_n \rangle_{n\in\mathbb{N}}$ be sequences in $\mathcal{K} \cap \Sigma$ such that $K_n \subseteq E$ and $K'_n \subseteq X \setminus E$ for every n, while $\mu E = \sup_{n\in\mathbb{N}} \mu K_n$ and $\mu(X \setminus E) = \sup_{n\in\mathbb{N}} \mu K'_n$. Set $L = \bigcup_{n\in\mathbb{N}} K_n$, $L' = \bigcap_{n\in\mathbb{N}} (X \setminus K'_n)$. Then both L and L' belong to the domain of every μ'_y, and

$$\begin{aligned} \sup_{n\in\mathbb{N}} \psi_y K_n &\le \sup_{n\in\mathbb{N}} \mu'_y K_n \le \mu'_y L \le \mu'_y L' \\ &\le \inf_{n\in\mathbb{N}} \mu'_y (X \setminus K'_n) = \mu'_y X - \sup_{n\in\mathbb{N}} \mu'_y K'_n \le 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n \end{aligned}$$

for every y. On the other hand,

$$\begin{aligned} \int (1 - \sup_{n\in\mathbb{N}} \psi_y K'_n) \nu(dy) &\le \nu Y - \sup_{n\in\mathbb{N}} \int \psi_y (K'_n) \nu(dy) = \mu X - \sup_{n\in\mathbb{N}} \mu K'_n = \mu E \\ &= \sup_{n\in\mathbb{N}} \mu K_n = \sup_{n\in\mathbb{N}} \int \psi_y K_n \nu(dy) \le \int \sup_{n\in\mathbb{N}} \psi_y K_n \nu(dy). \end{aligned}$$

So

$$\sup_{n\in\mathbb{N}} \psi_y K_n = \mu'_y L = \mu'_y L' = 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n$$

for almost every y. Because $L \subseteq E \subseteq L'$ and μ'_y is complete, $E \in \operatorname{dom} \mu'_y$ and

$$\mu'_y E = 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n \ge 1 - \psi_y (X \setminus E) \ge \psi_y E$$

for almost every $y \in Y$. Similarly, $\mu'_y (X \setminus E) \ge \psi_y (X \setminus E)$ for almost every y. But as

$$\mu'_y E + \mu'_y (X \setminus E) = \mu'_y X \le \psi_y X \le 1$$

whenever the left-hand side is defined, we must have $\mu'_y E = \psi_y E$ for almost every y, as claimed. **Q**

It follows at once that

$$\int \mu'_y E \, \nu(dy) = \int \psi_y E \, \nu(dy) = \mu E$$

for every $E \in \Sigma$, and $\langle \mu'_y \rangle_{y\in Y}$ is a disintegration of μ over ν.

(e) At this point observe that

$$\int \mu'_y X \, \nu(dy) = \mu X = \int \chi X^\bullet = \int T(\chi X^\bullet) = \nu Y,$$

so $F_0 = \{y : \mu'_y X < 1\}$ is negligible. Taking any $y_0 \in Y \setminus F_0$ and setting

$$\begin{aligned} \mu_y &= \mu'_{y_0} \text{ for } y \in F_0 \\ &= \mu'_y \text{ for } y \in Y \setminus F_0, \end{aligned}$$

we find ourselves with a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν with the same properties as $\langle \mu'_y \rangle_{y\in Y}$, but now consisting entirely of probability measures.

(f) For $g \in \mathcal{L}^\infty(\mu)$, set $h_g(y) = \int g\,d\mu_y$ whenever $y \in Y$ is such that the integral is defined. Consider the set V of those $g \in \mathcal{L}^\infty(\mu)$ such that $h_g \in \mathcal{L}^\infty(\nu)$ and $Tg^\bullet = h_g^\bullet$ in $L^\infty(\nu)$. If $E \in \Sigma$, then $h_{\chi E}(y) = \psi_y E$ for almost every y, so

$$h_{\chi E}^\bullet = (ST(\chi E^\bullet))^\bullet = T(\chi E^\bullet);$$

accordingly $\chi E \in V$. It is easy to check that V is closed under addition and scalar multiplication, so it contains all simple functions. Next, if $\langle g_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions with limit $g \in \mathcal{L}^\infty(\nu)$, then $h_g = \sup_{n\in\mathbb{N}} h_{g_n}$ wherever the right-hand side is defined. Also T is order-continuous, because it preserves integrals, so

$$Tg^\bullet = \sup_{n\in\mathbb{N}} Tg_n^\bullet = \sup_{n\in\mathbb{N}} h_{g_n}^\bullet = h_g^\bullet$$

and $g \in V$. Finally, if $g \in \mathcal{L}^\infty(\mu)$ is zero almost everywhere, there is a negligible $E \in \Sigma$ such that $g(x) = 0$ for every $x \in X \setminus E$; $\mu_y E = 0$ for almost every y, so $h_g(y) = \int g\,d\mu_y = 0$ for almost every y and again $g \in V$. Putting these together, we see that $V = \mathcal{L}^\infty(\nu)$, as required by (ii) as stated above.

452I Theorem (Pachl 78) Let (X,Σ,μ) be a non-empty countably compact measure space, (Y,T,ν) a σ-finite measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν, consistent with f, such that μ_y is a complete probability measure on X for every $y \in Y$. Moreover,

(i) if $\mathcal{K}$ is a countably compact class of subsets of X such that μ is inner regular with respect to $\mathcal{K}$, then we can arrange that $\mathcal{K} \subseteq \operatorname{dom}\mu_y$ for every $y \in Y$;

(ii) if, in (i), $\mathcal{K}$ is closed under finite unions and countable intersections, then we can arrange that $\mathcal{K} \subseteq \operatorname{dom}\mu_y$ and μ_y is inner regular with respect to $\mathcal{K}$ for every $y \in Y$.

proof (a) Consider first the case in which ν and μ are probability measures and we are provided with a class $\mathcal{K}$ as in (ii). In this case, for each $u \in L^\infty(\mu)$, $F \mapsto \int_{f^{-1}[F]} u$ is countably additive. So we have an operator $T : L^\infty(\mu) \to L^\infty(\nu)$ defined by saying that $\int_F Tu = \int_{f^{-1}[F]} u$ whenever $u \in L^\infty(\mu)$ and $F \in \mathrm{T}$. Of course T is linear and positive and $\int Tu = \int u$ whenever $u \in L^\infty(\mu)$.

By 452H, there is a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν such that

(α) for every $y \in Y$, $\mu_y X = 1$, $\mathcal{K} \subseteq \operatorname{dom}\mu_y$ and μ_y is inner regular with respect to $\mathcal{K}$;

(β) $T(g^\bullet) = h_g^\bullet$ whenever $g \in \mathcal{L}^\infty(\mu)$ and $h_g(y) = \int g\,d\mu_y$ when the integral is defined.

If now $F \in \mathrm{T}$, set $g = \chi f^{-1}[F]$ in (β); then $Tg^\bullet$ is defined by saying that

$$\int_H Tg^\bullet = \int_{f^{-1}[H]} g = \mu f^{-1}[F \cap H] = \nu(F \cap H)$$

for every $H \in \mathrm{T}$, so that $Tg^\bullet = \chi F^\bullet$ and we must have $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$. Thus $\langle \mu_y \rangle_{y\in Y}$ is a consistent distribution.

(b) The theorem is formulated in a way to make it quotable in parts without committing oneself to a particular class $\mathcal{K}$. But if we are given a class satisfying (i), we can extend it to one satisfying (ii), by 413T; and if we are told only that μ is countably compact, we know from the definition that we shall be able to choose a countably compact class satisfying (i).

(c) This proves the theorem on the assumption that μ and ν are probability measures. If $\mu X = \nu Y = 0$ then the result is trivial, as we can take every μ_y to be the zero measure. Otherwise, because ν is σ-finite, there is a partition $\langle Y_n \rangle_{n\in\mathbb{N}}$ of Y into measurable sets of finite measure. Let $\langle \gamma_n \rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^\infty \gamma_n \nu Y_n = 1$, and write

$$\nu' F = \sum_{n=0}^\infty \gamma_n \nu(F \cap Y_n) \text{ for } F \in \mathrm{T},$$

$$\mu' E = \sum_{n=0}^\infty \gamma_n \mu(E \cap X_n) \text{ for } E \in \Sigma.$$

It is easy to check (α) that ν' and μ' are probability measures (β) that f is inverse-measure-preserving for μ' and ν' (γ) that if μ is inner regular with respect to $\mathcal{K}$ so is μ'. Note that ν' and ν have the same negligible sets. By (a)-(b), μ' has a disintegration $\langle \mu_y \rangle_{y\in Y}$ over ν' which is consistent with f, and (if appropriate) has the properties demanded in (i) or (ii). Now, if $E \in \Sigma$,

$$\begin{aligned}\mu E &= \sum_{n=0}^{\infty} \gamma_n^{-1} \mu'(E \cap X_n) = \sum_{n=0}^{\infty} \gamma_n^{-1} \int \mu_y(E \cap X_n)\nu'(dy) \\ &= \sum_{n=0}^{\infty} \gamma_n^{-1} \int_{Y_n} \mu_y E\, \nu'(dy)\end{aligned}$$

(because $\mu_y X = 1$, $\mu_y X_n = (\chi Y_n)(y)$ for ν'-almost every y, every n)

$$= \sum_{n=0}^{\infty} \int_{Y_n} \mu_y E\, \nu(dy) = \sum_{n=0}^{\infty} \int \mu_y(E \cap X_n)\nu(dy) = \int \mu_y E\, \nu(dy).$$

So $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν. If $F \in \mathrm{T}$, then $\mu_y f^{-1}[F] = 1$ for ν'-almost every y, that is, for ν-almost every y, so $\langle \mu_y \rangle_{y \in Y}$ is still consistent with f with respect to the measure ν.

452J Remarks (a) In the theorem above, I have carefully avoided making any promises about the domains of the μ_y beyond that in (i). If Σ_0 is the σ-algebra generated by $\mathcal{K} \cap \Sigma$, then whenever $E \in \Sigma$ there are E', $E'' \in \Sigma_0$ such that $E' \subseteq E \subseteq E''$ and $\mu(E'' \setminus E') = 0$. (For μ, like ν, must be σ-finite, so we can choose E' to be a countable union of members of $\mathcal{K} \cap \Sigma$, and E'' to be the complement of such a union.) Thus we shall have a σ-algebra on which every μ_y is defined and which will be adequate to describe nearly everything about μ. The example of Lebesgue measure on the square (452E) shows that we cannot ordinarily expect the μ_y to be defined on the whole of Σ itself. In many important cases, of course, we can say more (452Xl).

(b) Necessarily (as remarked in the course of the proof) $\mu_y X = 1$ for almost every y. In some applications it seems right to change μ_y for a negligible set of y's so that every μ_y is a probability measure. Of course this cannot be done if $X = \emptyset \ne Y$, but this case is trivial (we should have to have $\nu Y = 0$). In other cases, we can make sure that any new μ_y is equal to some old one, so that a property required by (i) or (ii) remains true of the new disintegration. If we want to have '$\mu_y f^{-1}[\{y\}] = \mu_y X = 1$ for every $y \in Y$', strengthening 'strongly consistent', we shall of course have to begin by checking that f is surjective.

(c) The question of whether 'σ-finite' can be weakened to 'strictly localizable' in the hypotheses of 452I is related to the Banach-Ulam problem (452Yb). See also 452O.

452K Example The hypothesis 'countably compact' in 452I is in fact essential (452Ye). To see at least that it cannot be omitted, we have the following elementary example. Set $Y = [0,1]$, and let ν be Lebesgue measure on Y, with domain T. Let $X \subseteq [0,1]$ have outer measure 1 and inner measure 0 (134D, 419I); let μ be the subspace measure on X. Set $f(x) = x$ for $x \in X$. Then there is no disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν which is consistent with f.

P? Suppose, if possible, that $\langle \mu_y \rangle_{y \in [0,1]}$ is such a disintegration. Then, in particular, the sets

$$H_q = [0,q] \setminus \{y : X \cap [0,q] \in \operatorname{dom}\mu_y,\ \mu_y(X \cap [0,q]) = 1\},$$

$$H'_q = [q,1] \setminus \{y : X \cap [q,1] \in \operatorname{dom}\mu_y,\ \mu_y(X \cap [q,1]) = 1\}$$

are negligible for every $q \in [0,1]$. Set $G = [0,1] \setminus \bigcup_{q \in \mathbb{Q} \cap [0,1]} (H_q \cup H'_q)$, so that G is ν-conegligible. Then there must be some $y \in G \setminus X$. Now $\mu_y(X \cap [0,q']) = \mu_y(X \cap [q,1]) = 1$ whenever q, $q' \in \mathbb{Q}$ and $0 \le q < y < q' \le 1$, so that $\mu_y(X \cap \{y\}) = 1$. But $X \cap \{y\} = \emptyset$. **XQ**

452L The same ideas as in 452I can be used to prove a result on the disintegration of measures on product spaces. It will help to have a definition.

Definition Let $\langle X_i \rangle_{i \in I}$ be a family of sets, and λ a measure on $X = \prod_{i \in I} X_i$. For each $i \in I$ set $\pi_i(x) = x(i)$ for $x \in X$. Then the image measure $\lambda \pi_i^{-1}$ is the **marginal measure** of λ on X_i.

452M I return to the context of 452B-452D.

Theorem Let Y and Z be sets and $\mathrm{T} \subseteq \mathcal{P}Y$, $\Upsilon \subseteq \mathcal{P}Z$ σ-algebras. Let μ be a non-zero totally finite measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, and ν the marginal measure of μ on Y. Suppose that the marginal measure λ of μ on Z is inner regular with respect to a countably compact class $\mathcal{K} \subseteq \mathcal{P}Z$ which is closed under finite unions and countable intersections. Then there is a family $\langle\mu_y\rangle_{y\in Y}$ of complete probability measures on Z, all measuring every member of $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that

$$\mu E = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$, and

$$\int f d\mu = \iint f(y,z)\mu_y(dz)\nu(dy)$$

whenever f is a $[-\infty,\infty]$-valued function such that $\int f d\mu$ is defined in $[-\infty,\infty]$.

proof (a) To begin with, assume that μ is a probability measure and that ν is complete. Let $\mathfrak{B}$ be the measure algebra of ν and $\theta : \mathfrak{B} \to \mathrm{T}$ a lifting. For $H \in \Upsilon$ and $F \in \mathrm{T}$ set $\nu_H F = \mu(F \times H)$; then $\nu_H : \mathrm{T} \to [0,1]$ is countably additive and $\nu_H F \le \nu F$ for every $F \in \mathrm{T}$, so there is a $v_H \in L^1(\nu)$ such that $\int_F v_H = \nu_H F$ for every $F \in \mathrm{T}$ and $0 \le v_H \le \chi 1$. We can therefore think of v_H as a member of $L^\infty(\nu) \cong L^\infty(\mathfrak{B})$. Let $T : L^\infty(\mathfrak{B}) \to L^\infty(\mathrm{T})$ be the Riesz homomorphism associated with θ, and set $\psi_y H = (Tv_H)(y)$ for every $y \in Y$.

Each $\psi_y : \Upsilon \to [0,\infty[$ is finitely additive. So we have a complete measure μ_y on Z such that $\mu_y Z \le \psi_y Z = 1$, $\mathcal{K} \subseteq \operatorname{dom}\mu_y$, μ_y is inner regular with respect to $\mathcal{K}$ and $\mu_y K \ge \psi_y K$ for every $K \in \mathcal{K}$ (413Ua, as before).

For $H \in \Upsilon$, $F \in \mathrm{T}$ we have

$$\int_F \psi_y H\,\nu(dy) = \int_F Tv_H = \int_F v_H = \nu_H F = \mu(F \times H).$$

So

$$\underline{\int}\mu_y K \cdot \chi F(y)\nu(dy) \ge \int_F \psi_y K\,\nu(dy) = \mu(F \times K)$$

for every $K \in \mathcal{K}$. Now note that, for any $H \in \Upsilon$ and $F \in \mathrm{T}$,

$$\begin{aligned}\mu(F \times H) - \sup_{K\in\mathcal{K},K\subseteq H}\mu(F \times K) &= \inf_{K\in\mathcal{K},K\subseteq H}\mu(F \times (H \setminus K))\\ &\le \inf_{K\in\mathcal{K},K\subseteq H}\lambda(H \setminus K) = 0\end{aligned}$$

because λ is inner regular with respect to $\mathcal{K}$ (and, like μ, is a probability measure). So

$$\begin{aligned}\underline{\int}(\mu_y)_* H \cdot \chi F(y)\nu(dy) &\ge \sup_{K\in\mathcal{K},K\subseteq H}\underline{\int}\mu_y K \cdot \chi F(y)\nu(dy)\\ &\ge \sup_{K\in\mathcal{K},K\subseteq H}\mu(F \times K) = \mu(F \times H).\end{aligned}$$

In particular,

$$\underline{\int}(\mu_y)_* H\,\nu(dy) \ge \mu(Y \times H) = \lambda H,$$

and similarly $\underline{\int}(\mu_y)_*(Z \setminus H)\nu(dy) \ge \lambda(Z \setminus H)$.

Taking ν-integrable functions g_1, g_2 such that $g_1(y) \le (\mu_y)_* H$ and $g_2(y) \le (\mu_y)_*(Z \setminus H)$ for almost every y, $\int g_1 d\nu = \underline{\int}(\mu_y)_* H\,\nu(dy)$ and $\int g_2 d\nu = \underline{\int}(\mu_y)_*(Z \setminus H)\nu(dy)$ (133Ja), we must have

$$g_1(y) + g_2(y) \le (\mu_y)_* H + (\mu_y)_*(Z \setminus H) \le \mu_y Z \le 1$$

for almost every y, while $\int g_1 + g_2\,d\nu \ge 1$; so that, for almost all y,

$$g_1(y) + g_2(y) = (\mu_y)_* H + (\mu_y)_*(Z \setminus H) = \mu_y Z = 1,$$

and (because μ_y is complete) $\mu_y H$ is defined and equal to $g_1(y)$ (413Ec, 413Ef). It now follows that

$$\int_F \mu_y H\,\nu(dy) = \int_F g_1(y)\nu(dy) = \underline{\int}(\mu_y)_* H \cdot \chi F(y)\nu(dy) \ge \mu(F \times H)$$

for every $F \in \mathrm{T}$. But since also

$$\int_F \mu_y(Z \setminus H)\nu(dy) \geq \mu(F \times (Z \setminus H)),$$

$$\int_F \mu_y H + \mu_y(Z \setminus H)\,\nu(dy) \leq \nu F = \mu(F \times H) + \mu(F \times (Z \setminus H)),$$

we must actually have $\int_F \mu_y H\,\nu(dy) = \mu(F \times H)$.

All this is true whenever $F \in \mathrm{T}$ and $H \in \Upsilon$. But now, setting

$$\mathcal{E} = \{E : E \in \mathrm{T}\widehat{\otimes}\Upsilon,\ \mu E = \int \mu_y E[\{y\}]\nu(dy)\},$$

we see that $\mathcal{E}$ is a Dynkin class and includes $\mathcal{I} = \{F \times H : F \in \mathrm{T},\ H \in \Upsilon\}$, which is closed under finite intersections; so that the Monotone Class Theorem tells us that $\mathcal{E}$ includes the σ-algebra generated by $\mathcal{I}$, and is the whole of $\mathrm{T}\widehat{\otimes}\Upsilon$.

(b) The rest is just tidying up. (i) The construction in (a) allows $\mu_y Z$ to be less than 1 for a ν-negligible set of y; but of course all we have to do, if that happens, is to amend μ_y arbitrarily on that set to any of the 'ordinary' values of μ_y. (ii) If the original measure ν is not complete, let $\hat\mu$ and $\hat\nu$ be the completions of μ and ν, and $\hat{\mathrm{T}}$ the domain of $\hat\nu$. The projection onto Y is inverse-measure-preserving for μ and ν, so is inverse-measure-preserving for $\hat\mu$ and $\hat\nu$ (234Ba[7]), and $\hat\mu$ measures every member of $\hat{\mathrm{T}}\widehat{\otimes}\Upsilon$; set $\mu' = \hat\mu\restriction\hat{\mathrm{T}}\widehat{\otimes}\Upsilon$. Next, the marginal measure of μ' on Z is still λ (since both must have domain Υ). So we can apply (a) to μ' to get the result. (iii) If the original measure μ is not a probability measure, apply the arguments so far to suitable scalar multiples of μ and ν.

(c) Thus we have the formula

$$\mu E = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$. The second formula announced follows as in the remark following 452F.

452N Corollary Let Y and Z be sets and $\mathrm{T} \subseteq \mathcal{P}Y$, $\Upsilon \subseteq \mathcal{P}Z$ σ-algebras. Let μ be a probability measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, and ν the marginal measure of μ on Y. Suppose that

either Υ is the Baire σ-algebra with respect to a compact Hausdorff topology on Z
or Υ is the Borel σ-algebra with respect to an analytic Hausdorff topology on Z
or (Z, Υ) is a standard Borel space.

Then there is a family $\langle\mu_y\rangle_{y\in Y}$ of probability measures on Z, all with domain Υ, such that

$$\mu E = \int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$, and

$$\int f d\mu = \iint f(y,z)\mu_y(dz)\nu(dy)$$

whenever f is a $[-\infty,\infty]$-valued function such that $\int f d\mu$ is defined in $[-\infty,\infty]$.

proof In each case, the marginal measure of μ on Z is tight (that is, inner regular with respect to the closed compact sets) for a Hausdorff topology on Z. (Use 412D when Υ is the Baire σ-algebra on a compact Hausdorff space and 433Ca when it is the Borel σ-algebra on an analytic Hausdorff space; when (Z, Υ) is a standard Borel space, take any appropriate Polish topology on Z and use 423Ba.) So 452M tells us that we can achieve the formulae sought with Radon probability measures μ_y. Since (in all three cases) $\operatorname{dom}\mu_y$ will include Υ for every y, we can get the result as stated by replacing each μ_y by $\mu_y\restriction\Upsilon$.

452O Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν, consistent with f, such that every μ_y is a Radon measure on X.

proof (a) Let $\langle Y_i\rangle_{i\in I}$ be a decomposition of Y. For each $i \in I$, let ν_i be the subspace measure on Y_i and λ_i the subspace measure on $X_i = f^{-1}[Y_i]$. Then $f_i = f\restriction X_i$ is inverse-measure-preserving for λ_i and ν_i. Let $\mathcal{K}_i$ be the family of compact subsets of X_i; of course $\mathcal{K}_i$ is a (countably) compact class and λ_i is inner regular

[7]Formerly 235Hc.

with respect to $\mathcal{K}_i$ (412Oa). By 452I, we can choose, for each $i \in I$, a disintegration $\langle \tilde{\mu}_y \rangle_{y \in Y_i}$ of λ_i over ν_i, consistent with $f{\restriction}X_i$, such that $\tilde{\mu}_y$ measures every compact subset of X_i and is inner regular with respect to $\mathcal{K}_i$ for every $y \in Y_i$. Adjusting any which are not probability measures, and completing them if necessary, we can suppose that every $\tilde{\mu}_y$ is a complete probability measure. By 412Ja, $\tilde{\mu}_y$ measures every relatively closed subset of X_i for every $y \in Y_i$.

For $i \in I$ and $y \in Y_i$, set

$$\mu_y E = \tilde{\mu}_y(E \cap X_i)$$

whenever $E \subseteq X$ and $E \cap X_i$ is measured by $\tilde{\mu}_y$. Then μ_y is a complete totally finite measure on X; it is inner regular with respect to $\mathcal{K}_i$ and measures every closed subset of X. It follows at once that it is tight and measures every Borel set, that is, is a Radon measure on X.

(b) Now $\mu E = \int \mu_y E\,\nu(dy)$ for every $E \in \Sigma$. **P** $\bigcup_{i \in J} E \cap X_i = E \cap f^{-1}[\bigcup_{i \in J} Y_i]$ belongs to Σ for every $J \subseteq I$. By 451Q, $\mu E = \sum_{i \in I} \mu(E \cap X_i)$. For $i \in I$, we have $\int_{Y_i} \tilde{\mu}_y(E \cap X_i)\nu_i(dy) = \mu(E \cap X_i)$. So

$$\begin{aligned}\mu E &= \sum_{i \in I} \mu(E \cap X_i) = \sum_{i \in I} \int_{Y_i} \tilde{\mu}_y(E \cap X_i)\nu_i(dy) \\ &= \sum_{i \in I} \int_{Y_i} \mu_y E\,\nu(dy) = \int \mu_y E\,\nu(dy)\end{aligned}$$

by 214N. **Q**

Thus $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν.

(c) Finally, if $F \in \mathrm{T}$ and $i \in I$, then

$$\begin{aligned}Y_i \cap F \setminus \{y : \mu_y f^{-1}[F] \text{ is defined and equal to } 1\} \\ = (F \cap Y_i) \setminus \{y : y \in Y_i,\ \tilde{\mu}_y f^{-1}[F \cap Y_i] = 1\}\end{aligned}$$

is negligible for every i, so $\mu_y f^{-1}[F] = 1$ for almost every y. Thus $\langle \mu_y \rangle_{y \in Y}$ is consistent with f.

452P Corollary (cf. Blackwell 56) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ an analytic Radon measure space and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, strongly consistent with f, such that every μ_y is a Radon measure on X.

proof By 433B, ν is countably separated; now put 452O and 452Gc together.

452Q Disintegrations and conditional expectations Fubini's theorem provides a relatively concrete description of the conditional expectation of a function on a product of probability spaces with respect to the σ-algebra defined by one of the factors, by means of the formula $g(x, y) = \int f(x, z)dz$ (253H). This generalizes straightforwardly to measures with disintegrations, as follows.

Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces and $f : X \to Y$ an inverse-measure-preserving function. Suppose that $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν which is consistent with f, and that g is a μ-integrable real-valued function.

(a) Setting $h_0(y) = \int g\,d\mu_y$ whenever $y \in Y$ and the integral is defined in $\mathbb{R}$, h_0 is a Radon-Nikodým derivative of the functional $F \mapsto \int_{f^{-1}[F]} g\,d\mu : \mathrm{T} \to \mathbb{R}$.

(b) Now suppose that ν is complete. Setting $h_1(x) = \int g\,d\mu_{f(x)}$ whenever $x \in X$ and the integral is defined in $\mathbb{R}$, then h_1 is a conditional expectation of g on the σ-algebra $\Sigma_0 = \{f^{-1}[F] : F \in \mathrm{T}\}$.

proof (a) If $F \in \mathrm{T}$, then $f^{-1}[F]$ is μ_y-conegligible for almost every $y \in F$, and μ_y-negligible for almost every $y \in Y \setminus F$, so $\int g \times \chi f^{-1}[F] d\mu_y = h_0(y) \times \chi F(y)$ for almost every y, and

$$\int_F h_0 d\nu = \iint g \times \chi f^{-1}[F] d\mu_y \nu(dy) = \int_{f^{-1}[F]} g\,d\mu$$

(452F). As F is arbitrary, we have the result.

(b) Of course Σ_0 is a σ-algebra (111Xd), and it is included in Σ because f is inverse-measure-preserving. By 452F, $Y_0 = \{y : g \text{ is } \mu_y\text{-integrable}\}$ is conegligible, so $\operatorname{dom} h_1 = f^{-1}[Y_0]$ is conegligible. If $\alpha \in \mathbb{R}$, then

$$F = \{y : y \in Y_0,\ \int g\,d\mu_y \ge \alpha\}$$

belongs to T because $y \mapsto \int g\,d\mu_y$ is ν-virtually measurable and ν is complete. So

$$\{x : x \in \operatorname{dom} h_1,\ h_1(x) \ge \alpha\} = f^{-1}[F]$$

belongs to Σ_0, and h_1 is Σ_0-measurable. If $F \in \mathrm{T}$, then

$$\int_{f^{-1}[F]} h_1\,d\mu = \int_{f^{-1}[F]} \int g\,d\mu_{f(x)}\mu(dx) = \int_F \int g\,d\mu_y \nu(dy)$$

(235G[8])

$$= \int_F h_0 d\nu = \int_{f^{-1}[F]} g\,d\mu$$

as in (a). As F is arbitrary, h_1 is a conditional expectation of g on Σ_0, as claimed.

***452R** I take the opportunity to interpolate an interesting result about countably compact measures. It demonstrates the power of 452I to work in unexpected ways.

Theorem (Pachl 79) Let (X, Σ, μ) be a countably compact measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then ν is countably compact.

proof (a) For most of the proof (down to the end of (b) below) I suppose that μ and ν are totally finite.

Let Z be the Stone space of the Boolean algebra T. (I am *not* using the measure algebra here!) For $F \in \mathrm{T}$, let F^* be the corresponding open-and-closed subset of Z. For each $y \in Y$, the map $F \mapsto \chi F(y)$ is a Boolean homomorphism from T to $\{0, 1\}$, so belongs to Z; define $g : Y \to Z$ by saying that $g(y)(F) = \chi F(y)$ for $y \in Y$, $F \in \mathrm{T}$, that is, $g^{-1}[F^*] = F$ for every $F \in \mathrm{T}$. Let $\mathcal{Z}$ be the family of zero sets in Z, and Λ the Baire σ-algebra of Z.

The set

$$\{W : W \subseteq Z,\ g^{-1}[W] \in \mathrm{T}\}$$

is a σ-algebra of subsets of Z containing all the open-and-closed sets, so contains every zero set (4A3Od) and includes Λ. Set $\lambda W = \nu g^{-1}[W]$ for $W \in \Lambda$. Then λ is a Baire measure on Z, so is inner regular with respect to $\mathcal{Z}$ (412D).

Set $h = gf : X \to Z$. Then h is a composition of inverse-measure-preserving functions, so is inverse-measure-preserving. By 452I, there is a disintegration $\langle \mu_z \rangle_{z \in Z}$ of μ over λ which is consistent with h.

(b) Let $\mathcal{K} \subseteq \mathcal{P}Y$ be the family of sets

$$\{g^{-1}[V] : V \in \mathcal{Z},\ \mu_z h^{-1}[V] = \mu_z X = 1 \text{ for every } z \in V\}.$$

(i) $\mathcal{K}$ is a countably compact class of sets. **P** Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}$ such that $\bigcap_{i \le n} K_i \ne \emptyset$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $V_n \in \mathcal{Z}$ be such that $K_n = g^{-1}[V_n]$ and $\mu_z h^{-1} V_n = \mu_z X = 1$ for every $z \in V_n$. Then

$$g^{-1}[\bigcap_{i \le n} V_i] = \bigcap_{i \le n} K_i \ne \emptyset$$

for every $n \in \mathbb{N}$, so $\{V_n : n \in \mathbb{N}\}$ has the finite intersection property and (because Z is compact) there is a $z \in \bigcap_{n \in \mathbb{N}} V_n$. Now

$$\mu_z h^{-1}[V_n] = \mu_z X = 1$$

for every $n \in \mathbb{N}$, so

$$\emptyset \ne \bigcap_{n \in \mathbb{N}} h^{-1}[V_n] = f^{-1}[\bigcap_{n \in \mathbb{N}} K_n].$$

Thus $\bigcap_{n \in \mathbb{N}} K_n$ is non-empty. As $\langle K_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mathcal{K}$ is a countably compact class. **Q**

[8]Formerly 235I.

(ii) ν is inner regular with respect to $\mathcal{K}$. **P** Suppose that $F \in \mathrm{T}$ and $\gamma < \nu F$. Choose a sequence $\langle V_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{Z}$ as follows. Start with $V_0 = F^*$, so that

$$\lambda V_0 = \nu g^{-1}[V_0] = \nu F > \gamma.$$

Given that $V_n \in \mathcal{Z}$ and $\lambda V_n > \gamma$, then we know that $\mu_z h^{-1}[V_n] = \mu_z X = 1$ for λ-almost every $z \in V_n$; because λ is inner regular with respect to $\mathcal{Z}$, there is a $V_{n+1} \in \mathcal{Z}$ such that $V_{n+1} \subseteq V_n$, $\lambda V_{n+1} > \gamma$ and $\mu_z h^{-1}[V_n] = \mu_z X = 1$ for every $z \in V_{n+1}$. Continue.

At the end of the induction, set $V = \bigcap_{n\in\mathbb{N}} V_n$. Then $V \in \mathcal{Z}$. If $z \in V$, then

$$\mu_z h^{-1}[V] = \lim_{n\to\infty} \mu_z h^{-1}[V_n] = 1 = \mu_z X,$$

so $g^{-1}[V] \in \mathcal{K}$. Because $V \subseteq V_0 = F^*$, $g^{-1}[V] \subseteq F$, and

$$\nu g^{-1}[V] = \lambda V = \lim_{n\to\infty} \lambda V_n \geq \gamma.$$

As F and γ are arbitrary, ν is inner regular with respect to $\mathcal{K}$. **Q**

Thus $\mathcal{K}$ witnesses that ν is countably compact.

(c) For the general case, let $\langle Y_i \rangle_{i\in I}$ be a decomposition of Y. For each $i \in I$, set $X_i = f^{-1}[Y_i]$; let μ_i be the subspace measure on X_i and ν_i the subspace measure on Y_i. Then μ_i is countably compact (451Db) and $f\restriction X_i : X_i \to Y_i$ is inverse-measure-preserving for μ_i and ν_i, so ν_i is countably compact, by (a)-(b) above. Let $\mathcal{K}_i \subseteq \mathcal{P}Y_i$ be a countably compact class such that ν_i is inner regular with respect to $\mathcal{K}_i$. Then $\mathcal{K} = \bigcup_{i\in I} \mathcal{K}_i$ is a countably compact class (because any sequence in $\mathcal{K}$ with the finite intersection property must lie within a single $\mathcal{K}_i$). By 413T, there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ which is closed under finite unions; by 412Aa, ν is inner regular with respect to $\mathcal{K}^*$, so is countably compact. This completes the proof.

***452S Corollary** (Pachl 78) If (X, Σ, μ) is a countably compact totally finite measure space, and T is any σ-subalgebra of Σ, then $\mu\restriction\mathrm{T}$ is countably compact.

452T In 452E, I remarked in passing that Fubini's theorem on a product space $X = Y \times Z$ can be thought of as giving us a disintegration of the product measure on X over the factor measure on Y. There are other contexts in which we find that a canonical disintegration is provided for a structure (X, μ, Y, ν) without calling on the Lifting Theorem. Here I will describe an important case arising naturally in the theory of group actions.

Theorem Let X be a locally compact Hausdorff space, G a compact Hausdorff topological group and $\bullet$ a continuous action of G on X. Suppose that μ is a G-invariant Radon probability measure on X. For $x \in X$, write $f(x)$ for the corresponding orbit $\{a\bullet x : a \in G\}$ of the action. Let $Y = f[X]$ be the set of orbits, with the topology $\{W : W \subseteq Y,\ f^{-1}[W] \text{ is open in } X\}$. Write ν for the image measure μf^{-1} on Y.

(a) Y is locally compact and Hausdorff, and ν is a Radon probability measure.

(b) For each $\boldsymbol{y} \in Y$, there is a unique G-invariant Radon probability $\mu_{\boldsymbol{y}}$ on X such that $\mu_{\boldsymbol{y}}(\boldsymbol{y}) = 1$.

(c) $\langle \mu_{\boldsymbol{y}} \rangle_{\boldsymbol{y}\in Y}$ is a disintegration of μ over ν, strongly consistent with f.

proof (a) By 4A5Ja, Y is locally compact and Hausdorff, and f is an open map. By 418I, ν is a Radon measure.

(b) Let λ be the unique Haar probability measure on G (442Id). By 443Ub-443Ud, applied to the action $\bullet\restriction G \times \boldsymbol{y}$ of G on $\boldsymbol{y}$, we have a unique G-invariant Radon probability measure $\mu'_{\boldsymbol{y}}$ on $\boldsymbol{y}$ defined by saying that $\mu'_{\boldsymbol{y}} E = \lambda\{g : g\bullet x \in E\}$ for every $x \in \boldsymbol{y}$ and Borel set $E \subseteq \boldsymbol{y}$. Now $\mu_{\boldsymbol{y}}$ must be the unique extension of $\mu'_{\boldsymbol{y}}$ to X. Of course we still have $\mu_{\boldsymbol{y}} E = \lambda\{g : g\bullet x \in E\}$ for every $x \in \boldsymbol{y}$ and Borel set $E \subseteq X$.

(c)(i) Let $V \subseteq X$ be an open set, and set $h_V(\boldsymbol{y}) = \mu_{\boldsymbol{y}} V$ for $\boldsymbol{y} \in Y$. Then h_V is lower semi-continuous. **P** Suppose that $\boldsymbol{y} \in Y$ and $\alpha \in \mathbb{R}$ are such that $h_V(\boldsymbol{y}) > \alpha$. Then there is a compact set $K \subseteq V$ such that $\mu_{\boldsymbol{y}} K > \alpha$. Fix $x \in \boldsymbol{y}$, and set $L = \{g : g\bullet x \in K\}$, so that L is a compact subset of G and $\lambda L > \alpha$. The set $\{(g, x') : g \in L,\ x' \in X,\ g\bullet x' \notin V\}$ is closed in $L \times X$, so its projection $\{x' : \exists\, g \in L,\ g\bullet x' \notin V\}$ is closed (4A2Gm) and $U = \{x' : g\bullet x' \in V \text{ for every } g \in L\}$ is open in X. Now $f[U]$ is open in Y, because f is an open map. Of course $x \in U$ and $\boldsymbol{y} \in f[U]$. But if $\boldsymbol{y}' \in f[U]$, there is an $x' \in U$ such that $f(x') = \boldsymbol{y}'$, and now

$$h_V(\boldsymbol{y}') = \mu_{\boldsymbol{y}'}V = \lambda\{g : g\bullet x' \in V\} \ge \lambda L > \alpha.$$

As $\boldsymbol{y}$ and α are arbitrary, h_V is lower semi-continuous. **Q**

(ii) In particular, h_V is Borel measurable; because f is inverse-measure-preserving for μ and ν,

$$\int h_V\,d\nu = \int h_V(f(x))\mu(dx)$$

(235G again)

$$= \int \lambda\{g : g\bullet x \in V\}\mu(dx) = \int \mu\{x : g\bullet x \in V\}\lambda(dg)$$

(by 417G, because μ and λ are totally finite Radon measures and $\{(g,x) : g\bullet x \in V\}$ is an open set in $G \times X$)

$$= \int \mu(g^{-1}\bullet V)\lambda(dg) = \int \mu V\,\lambda(dg)$$

(because μ is G-invariant)

$$= \mu V.$$

By the Monotone Class Theorem, as usual, it follows that $\int \mu_{\boldsymbol{y}}E\,\nu(d\boldsymbol{y}) = \mu E$ for every Borel set $E \subseteq X$ (apply 136C to μ and $E \mapsto \int \mu_{\boldsymbol{y}}E\,\nu(d\boldsymbol{y})$), and therefore (because every $\mu_{\boldsymbol{y}}$ is complete and μ is the completion of a Borel measure) for every $E \in \operatorname{dom}\mu$. So $\langle\mu_{\boldsymbol{y}}\rangle_{\boldsymbol{y}\in Y}$ is a disintegration of μ over ν. Since

$$\mu_{\boldsymbol{y}}f^{-1}[\{\boldsymbol{y}\}] = \mu_{\boldsymbol{y}}(\boldsymbol{y}) = 1$$

for every $\boldsymbol{y} \in Y$, the disintegration is strongly consistent with f.

452X Basic exercises (a) Let Y be a first-countable topological space, ν a topological probability measure on Y, Z a topological space, and $\langle\mu_y\rangle_{y\in Y}$ a family of topological probability measures on Z such that $y \mapsto \mu_y V$ is lower semi-continuous for every open set $V \subseteq Z$. Show that there is a Borel probability measure μ on $Y \times Z$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every Borel set $E \subseteq Y \times Z$. (*Hint*: 434R.)

(b) Let (Y, T, ν) be a probability space, Z a topological space and P the set of topological probability measures on Z with its narrow topology (437Jd). Let $y \mapsto \mu_y : Y \to P$ be a function which is measurable in the sense of 411L. Show that, writing $\mathcal{B}(Z)$ for the Borel σ-algebra of Z, we have a probability measure μ defined on $\mathrm{T}\widehat{\otimes}\mathcal{B}(Z)$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every $E \in \mathrm{T}\widehat{\otimes}\mathcal{B}(Z)$.

(c) Let (Y, T, ν) be a probability space, Z a topological space and $P_{\mathcal{B}\mathfrak{a}}$ the set of Baire probability measures on Z with its vague topology (437Jc). Let $y \mapsto \mu_y : Y \to P_{\mathcal{B}\mathfrak{a}}$ be a measurable function. Show that, writing $\mathcal{B}\mathfrak{a}(Z)$ for the Baire σ-algebra of Z, we have a probability measure μ defined on $\mathrm{T}\widehat{\otimes}\mathcal{B}\mathfrak{a}(Z)$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every $E \in \mathrm{T}\widehat{\otimes}\mathcal{B}\mathfrak{a}(Z)$.

(d) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon probability space, $(X, \mathfrak{T})$ a topological space, and $\langle\mu_y\rangle_{y\in Y}$ a family of Radon probability measures on X. Suppose that (i) there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$ (ii) ν is inner regular with respect to the family $\{K : K \subseteq Y,\ \{\mu_y : y \in K\}$ is uniformly tight$\}$. Show that we have a Radon probability measure $\tilde\mu$ on X such that $\tilde\mu E = \int \mu_y E\nu(dy)$ whenever $\tilde\mu$ measures E.

(e) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon probability space, $(Z, \mathfrak{U})$ a Prokhorov Hausdorff space (437U), and P the space of Radon probability measures on Z with its narrow topology. Suppose that $y \mapsto \mu_y : Y \to P$ is almost continuous. Show that we have a Radon probability measure $\tilde\mu$ on $Y \times Z$ such that $\tilde\mu E = \int \mu_y E[\{y\}]\nu(dy)$ whenever $\tilde\mu$ measures E.

(f) Let (X, T, ν) be a measure space, and μ an indefinite-integral measure over ν (234J[9]). Show that there is a disintegration $\langle\mu_x\rangle_{x\in X}$ of μ over ν such that $\mu_x\{x\} = \mu_x X$ for every $x \in X$.

[9]Formerly 234B.

>(g) Let (X,Σ,μ) and (Y,T,ν) be measure spaces and $\langle\mu_y\rangle_{y\in Y}$ a disintegration of μ over ν. Show that $\langle\hat\mu_y\rangle_{y\in Y}$ is a disintegration of $\hat\mu$ over ν, where $\hat\mu_y$ and $\hat\mu$ are the completions of μ_y and μ respectively.

>(h) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and ν' an indefinite-integral measure over ν, defined from a ν-virtually measurable function $g:Y\to[0,\infty[$. Suppose that $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν'. Show that $\langle g(y)\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν.

(i) Let (Y,T,ν) be a probability space, X a set and $\langle\mu_y\rangle_{y\in Y}$ a family of probability measures on X. Set $\theta A=\overline{\int}\mu_y^*(A)\,\nu(dy)$ for every $A\subseteq X$. (i) Show that θ is an outer measure on X. (ii) Let μ be the measure on X defined from θ by Carathéodory's construction. Show that $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν. (iii) Suppose that $X=[0,1]^2$, ν is Lebesgue measure on $[0,1]=Y$ and $\mu_yE=\nu\{x:(x,y)\in E\}$ whenever this is defined. Show that, for any E measured by μ, $\mu_yE\in\{0,1\}$ for ν-almost every y.

(j) Explore connexions between 452F and the formula $\int f d\mu=\iint f d\nu_z\lambda(dz)$ of 443Qe.

(k) Let (X,Σ,μ) be a countably compact σ-finite measure space, (Y,T,ν) a σ-finite measure space, and $f:X\to Y$ a (Σ,T)-measurable function such that $f^{-1}[F]$ is μ-negligible whenever $F\subseteq Y$ is ν-negligible. Show that there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν such that, for each $F\in\mathrm{T}$, $\mu_y(X\setminus f^{-1}[F])=0$ for almost every $y\in F$. (*Hint*: Reduce to the case in which μ is totally finite, and disintegrate μ over $\nu'=(\mu f^{-1})\restriction\mathrm{T}$.)

>(l) Let (X,Σ,μ) be a non-empty countably compact measure space such that Σ is countably generated (as σ-algebra), (Y,T,ν) a σ-finite measure space, and $f:X\to Y$ an inverse-measure-preserving function. (i) Show that there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν, consistent with f, such that every μ_y is a probability measure with domain Σ. (ii) Show that if $\langle\mu'_y\rangle_{y\in Y}$ is any other disintegration of μ over ν which is consistent with f, then $\mu_y=\mu'_y\restriction\Sigma$ for almost every y.

(m) Let (X,Σ) be a non-empty standard Borel space, μ a measure with domain Σ, (Y,T,ν) a σ-finite measure space, and $f:X\to Y$ an inverse-measure-preserving function. (i) Show that there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν, consistent with f, such that every μ_y is a probability measure with domain Σ. (ii) Show that if $\langle\mu'_y\rangle_{y\in Y}$ is any other disintegration of μ over ν which is consistent with f, then $\mu_y=\mu'_y\restriction\Sigma$ for almost every y.

(n) Let (X,Σ,μ) be a totally finite countably compact measure space and $\mathrm{T}\subseteq\Sigma$ a countably-generated σ-algebra; set $\nu=\mu\restriction\mathrm{T}$. Show that there is a disintegration $\langle\mu_x\rangle_{x\in X}$ of μ over ν such that $\mu_xH_x=\mu_xX=1$ for every $x\in X$, where $H_x=\bigcap\{F:x\in F\in\mathrm{T}\}$ for every x. (*Hint*: apply 452I with $Y=\{H_x:x\in X\}$.)

(o) Show that 452I can be deduced from 452M. (*Hint*: start with the case $\nu Y=1$; set $\lambda W=\mu\{x:(x,f(x))\in W\}$ for $W\in\Sigma\widehat{\otimes}\mathrm{T}$.)

(p) Show that, in 452M, we shall have $\hat\mu E=\int\mu_yE[\{y\}]\nu(dy)$ whenever the completion $\hat\mu$ of μ measures E.

>(q) Let T be the Borel σ-algebra of $[0,1]$, ν the restriction of Lebesgue measure to T, $Z\subseteq[0,1]$ a set with inner measure 0 and outer measure 1, and Υ the Borel σ-algebra of Z. Show that there is a probability measure μ on $[0,1]\times Z$ defined by setting $\mu E=\nu^*\{y:(y,y)\in E\}$ for $E\in\mathrm{T}\widehat{\otimes}\Upsilon$. Show that there is no disintegration of μ over ν which is consistent with the projection $(y,z)\mapsto y$.

>(r) Let (X,Σ,μ) be a complete totally finite countably compact measure space and T a σ-subalgebra of Σ containing all negligible sets. Show that there is a family $\langle\mu_x\rangle_{x\in X}$ of probability measures on X such that (i) $x\mapsto\mu_xE$ is T-measurable and $\int\mu_xE\,\mu(dx)=\mu E$ for every $E\in\Sigma$ (ii) if $F\in\mathrm{T}$, then $\mu_xF=1$ for almost every $x\in F$. Show that if g is any μ-integrable real-valued function, then g is μ_x-integrable for almost every x, and $x\mapsto\int g\,d\mu_x$ is a conditional expectation of g on T.

(s) Let (X_0,Σ_0,μ_0) and (X_1,Σ_1,μ_1) be σ-finite measure spaces. For each i, let (Y_i,T_i,ν_i) be a measure space and $\langle\mu_y^{(i)}\rangle_{y\in Y_i}$ a disintegration of μ_i over ν_i. Show that $\langle\mu_{y_0}^{(0)}\times\mu_{y_1}^{(1)}\rangle_{(y_0,y_1)\in Y_0\times Y_1}$ is a disintegration of $\mu_0\times\mu_1$ over $\nu_0\times\nu_1$, where each product here is a c.l.d. product measure.

(t) In 452M, suppose that Z is a metrizable space and $\mathcal{K}$ is the family of compact subsets of Z, and let $(Y, \hat{\mathrm{T}}, \hat{\nu})$ be the completion of (Y, T, ν). Show that $y \mapsto \mu_y$ is a $\hat{\mathrm{T}}$-measurable function from Y to the set of Radon probability measures on Z with its narrow topology. (*Hint*: 437Rh.)

(u) $SU(r)$, for $r \geq 2$, is the set of $r \times r$ matrices T with complex coefficients such that $\det T = 1$ and $TT^* = I$, where T^* is the complex conjugate of the transpose of T. (i) Show that under the natural action $(T, u) \mapsto Tu : SU(r) \times \mathbb{C}^r \to \mathbb{C}^r$ the orbits are the spheres $\{u : u \,.\, \bar{u} = \gamma\}$, for $\gamma > 0$, together with $\{0\}$. (ii) Show that if a Borel set $C \subseteq \mathbb{C}^r$ is such that $\gamma C \subseteq C$ for every $\gamma > 0$, and μ_0, μ_1 are two $SU(r)$-invariant Radon probability measures on $\mathbb{C}^r$ such that $\mu_0\{0\} = \mu_1\{0\}$, than $\mu_0 C = \mu_1 C$.[10]

(v) Let $\langle X_i \rangle_{i \in I}$ be a family of compact Hausdorff spaces with product X, and μ a completion regular topological measure on X. Show that all the marginal measures of μ are completion regular. (*Hint*: 434U.)

452Y Further exercises (a) Let Z be a set, (Y, T, ν) a measure space, and $\langle \mu_y \rangle_{y \in Y}$ a family of measures on Z. Let Υ be a σ-algebra of subsets of Z such that, for every $H \in \Upsilon$, $y \mapsto \mu_y H : Y \to [0, \infty]$ is defined ν-a.e. and is ν-virtually measurable. For $F \in \mathrm{T}$, set $\mathcal{H}_F = \{H : H \in \Upsilon,\ \mu_y H$ is defined for every $y \in F$ and $\sup_{y \in F} \mu_y H < \infty\}$. Show that there is a measure μ on $Y \times Z$, with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, defined by setting

$$\mu E = \sup\{\sum_{i=0}^{n} \int_{F_i} \mu_y(E[\{y\}] \cap H_i)\nu(dy) : F_0, \dots, F_n \in \mathrm{T} \text{ are disjoint},$$
$$\nu F_i < \infty \text{ and } H_i \in \mathcal{H}_{F_i} \text{ for every } i \leq n\}$$

for $E \in \mathrm{T}\widehat{\otimes}\Upsilon$.

(b) Let (X, Σ, μ) be a semi-finite countably compact measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Suppose that the magnitude of ν (definition: 332Ga) is finite or a measure-free cardinal (definition: 438A). Show that there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν which is consistent with f.

(c) Give an example to show that the phrase 'strictly localizable' in the statements of 452O and 452Yb cannot be dispensed with.

(d) Give an example to show that, in 452M, we cannot always arrange that $\Upsilon \subseteq \operatorname{dom} \mu_y$ for ν-almost every $y \in Y$.

(e) Let (X, Σ, μ) be a probability space such that whenever (Y, T, ν) is a probability space and $f : X \to Y$ is an inverse-measure-preserving function, there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν which is consistent with f. Show that μ is countably compact. (*Hint*: 452R, or Pachl 78.)

(f) Let X be a K-analytic Hausdorff space and μ a totally finite measure on X which is inner regular with respect to the closed sets. Show that μ is countably compact. (*Hint*: 432D.)

(g) Let X be a set, and $\langle \mu_i \rangle_{i \in I}$ a family of countably compact measures on X with sum μ (234G[11]). Show that if μ is semi-finite, it is countably compact.

(h) Let X be a locally compact Hausdorff space, G a compact Hausdorff group, and $\bullet$ a continuous action of G on X. Let H be another group and $\circ$ a continuous action of H on X which commutes with $\bullet$ in the sense that $g\bullet(h\circ x) = h\circ(g\bullet x)$ for all $g \in G$, $h \in H$ and $x \in X$. (i) Show that $((g, h), x) \to g\bullet(h\circ x) : (G \times H) \times X \to X$ is a continuous action of the product group $G \times H$ on X. (ii) Suppose that the action in (i) is transitive. Show that if μ, μ' are G-invariant Radon probability measures on X and $E \subseteq X$ is a Borel set such that $h\circ E = E$ for every $h \in H$, then $\mu E = \mu' E$.

[10] I am grateful to G.Vitillaro for bringing this to my attention.
[11] Formerly 112Ya.

452 Notes and comments 452B and 452C correspond respectively to the ordinary and τ-additive product measures of §§251 and 417. I have not attempted to find a suitable general formulation for the constructions when the measures involved are not totally finite. In 452Ya I set out a possible version which at least agrees with the c.l.d. product measure when all the μ_y are the same. Any product measure which has an associated Fubini theorem can be expected to be generalizable in the same way; for instance, 434R becomes 452Xa.

The hypotheses in 452B are closely matched with the conclusion, and clearly cannot be relaxed substantially if the theorem is to remain true. 452C and 452D are a rather different matter. While the condition '$y \mapsto \mu_y V$ is lower semi-continuous' is a natural one, and plainly necessary for the argument given, the integrated measure μ can be τ-additive or Radon for other reasons. In particular, the most interesting specific example in this book of a Radon measure constructed through these formulae (453N below) does not satisfy the lower semi-continuity condition for the section measures.

Early theorems on disintegrations concentrated on cases in which all the measure spaces involved were 'standard' in that the measures were defined on standard Borel algebras, or were the completions of such measures. Theorem 452I here is the end (so far) of a long search for ways to escape from topological considerations. As usual, of course, the most important applications (in probability theory) are still rooted in the standard case. Being countably separated, such spaces automatically yield disintegrations which are concentrated on fibers, in the sense that $\mu_y f^{-1}[\{y\}] = \mu_y X = 1$ for almost every y (452P). The general question of when we can expect to find disintegrations of this type is an important one to which I will return in the next section.

452I and 452O, as stated, assume that the functions $f : X \to Y$ controlling the disintegrations are inverse-measure-preserving. In fact it is easy to weaken this assumption (452Xk). Note the constructions for conditional expectations in 452Q and 452Xr.

Obviously 452I and 452M are nearly the same theorem; but I write out formally independent proofs because the constructions needed to move between them are not quite trivial. In fact I think it is easier to deduce 452I from 452M than the other way about (452Xo). The point of 452N is that the spaces (X,Σ) there have the 'countably compact measure property', that is, any totally finite measure with domain Σ is countably compact. I will return to this in the exercises to §454 (454Xf *et seq.*).

The method of 452R, due to J.Pachl, may have inspired the proof of (vi)$\Rightarrow$(i) in 343B. In the general introduction to this work I wrote 'I have very little confidence in anything I have ever read concerning the history of ideas'. We have here a case indicating the difficulties a historian faces. I proved 343B in the winter of 1996-97, while a guest of the University of Wisconsin at Madison. Around that time I was renewing my acquaintance with Pachl 78. I know I ran my eye over the proof of 452R, without, I may say, understanding it, as became plain when I came to write the first draft of the present section in the summer of 1997; whether I had understood it twenty years earlier I do not know. It is entirely possible that a subterranean percolation of Pachl's idea was what dislodged an obstacle to my attempts to prove 343B, but I was not at the time conscious of any connexion.

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453 Strong liftings

The next step involves the concept of 'strong' lifting on a topological measure space (453A); I devote a few pages to describing the principal cases in which strong liftings are known to exist (453B-453J). When we have *Radon* measures μ and ν, with an *almost continuous* inverse-measure-preserving function between them, and a *strong* lifting for ν, we can hope for a disintegration $\langle\mu_y\rangle_{y\in Y}$ such that (almost) every μ_y lives on the appropriate fiber. This is the content of 453K. I end the section with a note on the relation between strong liftings and Stone spaces (453M) and with V.Losert's example of a space with no strong lifting (453N).

Much of the work here is based on ideas in Ionescu Tulcea & Ionescu Tulcea 69.

453A The proof of the first disintegration theorem I presented, 452H, depended on two essential steps: the use of a lifting for (Y,T,ν) to define the finitely additive functionals ψ_y, and the use of a countably compact class to convert these into countably additive functionals. In 452O I observed that if our countably compact class is the family of compact sets in a Hausdorff space, we can get Radon measures in our disintegration. Similarly, if we have a lifting of a special type, we can hope for special properties of the disintegration. A particularly important kind of lifting, in this context, is the following.

Definition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space. A lifting $\phi:\Sigma\to\Sigma$ is **strong** or **of local type** if $\phi G\supseteq G$ for every open set $G\subseteq X$, that is, if $\phi F\subseteq F$ for every closed set $F\subseteq X$. I will say that ϕ is **almost strong** if $\bigcup_{G\in\mathfrak{T}} G\setminus\phi G$ is negligible.

Similarly, if $\mathfrak{A}$ is the measure algebra of μ, a lifting $\theta:\mathfrak{A}\to\Sigma$ is **strong** if $\theta G^\bullet\supseteq G$ for every open set $G\subseteq X$, and **almost strong** if $\bigcup_{G\in\mathfrak{T}} G\setminus\theta G^\bullet$ is negligible.

Obviously a strong lifting is almost strong.

453B We already have the machinery to describe a particularly striking class of strong liftings.

Theorem Let X be a topological group with a Haar measure μ, and Σ its algebra of Haar measurable sets.

(a) If $\phi:\Sigma\to\Sigma$ is a left-translation-invariant lifting, in the sense of 447A, then ϕ is strong.

(b) μ has a strong lifting.

proof (a) Apply 447B with $Y=\{e\}$ and $\underline{\phi}=\phi$.

(b) For there is a left-translation-invariant lifting (447J).

Remark In particular, translation-invariant liftings on $\mathbb{R}^r$ or $\{0,1\}^I$ (§345) are strong.

453C Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space and $\phi:\Sigma\to\Sigma$ a lifting. Write $\mathcal{L}^\infty$ for the space of bounded Σ-measurable real-valued functions on X, so that $\mathcal{L}^\infty$ can be identified with $L^\infty(\Sigma)$ (363H) and the Boolean homomorphism $\phi:\Sigma\to\Sigma$ gives rise to a Riesz homomorphism $T:\mathcal{L}^\infty\to\mathcal{L}^\infty$ (363F).

(a) If ϕ is a strong lifting, then $Tf=f$ for every bounded continuous function $f:X\to\mathbb{R}$.

(b) If $(X,\mathfrak{T})$ is completely regular and $Tf=f$ for every $f\in C_b(X)$, then ϕ is strong.

proof (a) Suppose first that $f\ge 0$. For $\alpha\in\mathbb{R}$, set $G_\alpha=\{x:x\in X,\ f(x)>\alpha\}$; then G_α is open, so $\phi G_\alpha\supseteq G_\alpha$. We have $f\ge\alpha\chi G_\alpha$, so

$$Tf\ge\alpha T(\chi G_\alpha)=\alpha\chi(\phi G_\alpha)\ge\alpha\chi G_\alpha,$$

that is, $(Tf)(x)\ge\alpha$ whenever $f(x)>\alpha$. As α is arbitrary, $Tf\ge f$. At the same time, setting $\gamma=\|f\|_\infty$, we have

$$T(\gamma\chi X-f)\ge\gamma\chi X-f,\quad T(\gamma\chi X)=\gamma\chi(\phi X)=\gamma\chi X,$$

so $Tf\le f$ and $Tf=f$.

For general $f\in C_b(X)$,

$$Tf=T(f^+-f^-)=Tf^+-Tf^-=f^+-f^-=f,$$

where f^+ and f^- are the positive and negative parts of f.

(b) Let $G\subseteq X$ be open and x any point of G. Then there is an $f\in C_b(X)$ such that $f\le\chi G$ and $f(x)=1$. In this case

$$f=Tf\le T(\chi G)=\chi(\phi G),$$

so $x\in\phi G$. As x is arbitrary, $G\subseteq\phi G$; as G is arbitrary, ϕ is strong.

453D Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space.

(a) If μ has a strong lifting it is strictly positive (definition: 411Nf).

(b) If μ is strictly positive and complete, and has an almost strong lifting, it has a strong lifting.

(c) If μ has an almost strong lifting it is τ-additive, so has a support.

(d) If μ is complete and $\mu X>0$ and the subspace measure μ_E has an almost strong lifting for some conegligible set $E\subseteq X$, then μ has an almost strong lifting.

proof (a) If $\phi:\Sigma\to\Sigma$ is a strong lifting, then $G\subseteq\phi G=\emptyset$ whenever G is a negligible open set, so μ is strictly positive.

(b) If μ is strictly positive and complete and $\phi : \Sigma \to \Sigma$ is an almost strong lifting, set $A = \bigcup_{G \in \mathfrak{T}} G \setminus \phi G$. For each $x \in A$, let $\mathcal{I}_x$ be the ideal of subsets of X generated by

$$\{F : F \subseteq X \text{ is closed, } x \notin F\} \cup \{B : B \subseteq X \text{ is negligible}\}.$$

Then $X \notin \mathcal{I}_x$, because μ is strictly positive, so a closed set not containing x cannot be conegligible. There is therefore a Boolean homomorphism $\psi_x : \mathcal{P}X \to \{0, 1\}$ such that $\psi_x F = 0$ for every $F \in \mathcal{I}_x$ (311D). Set

$$\tilde{\phi} E = (\phi E \setminus A) \cup \{x : x \in A, \, \psi_x E = 1\}$$

for $E \in \Sigma$. It is easy to check that $\tilde{\phi} : \Sigma \to \mathcal{P}X$ is a Boolean homomorphism. (Compare the proof of 341J.) If $E \in \Sigma$, then

$$E \triangle \tilde{\phi} E \subseteq (E \triangle \phi E) \cup A$$

is negligible, so (because μ is complete) $\tilde{\phi} E \in \Sigma$. If E is negligible, then $E \in \mathcal{I}_x$ and $\psi_x E = 0$ for every $x \in A$, so $\tilde{\phi} E = \phi E = \emptyset$. Thus $\tilde{\phi}$ is a lifting. Now suppose that $x \in G \in \mathfrak{T}$. If $x \in A$, then $X \setminus G \in \mathcal{I}_x$, so $\psi_x(X \setminus G) = 0$, $\psi_x G = 1$ and $x \in \tilde{\phi} G$. If $x \notin A$, then $x \in \phi G$ and again $x \in \tilde{\phi} G$. As x and G are arbitrary, $\tilde{\phi}$ is a strong lifting.

(c) Suppose that $\phi : \Sigma \to \Sigma$ is an almost strong lifting. Let $\mathcal{G}$ be a non-empty upwards-directed family of open sets with union H. If $\sup_{G \in \mathcal{G}} \mu G = \infty$, this is surely equal to μH. Otherwise, there is a non-decreasing sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ such that $G \setminus G^*$ is negligible for every $G \in \mathcal{G}$, where $G^* = \bigcup_{n \in \mathbb{N}} G_n$ (215Ab). Then $\phi G \subseteq \phi G^*$ for every $G \in \mathcal{G}$. This means that

$$H \setminus \phi G^* \subseteq \bigcup_{G \in \mathcal{G}} G \setminus \phi G$$

is negligible, because ϕ is almost strong, and

$$\mu H \le \mu(\phi G^*) = \mu G^* = \lim_{n \to \infty} \mu G_n = \sup_{G \in \mathcal{G}} \mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. By 411Nd, it has a support.

(d) Now suppose that μ is complete, that $\mu X > 0$ and that there is a conegligible $E \subseteq X$ such that μ_E has an almost strong lifting ϕ. Let $\psi : \mathcal{P}X \to \{\emptyset, X\}$ be any Boolean homomorphism such that $\psi A = \emptyset$ whenever A is negligible. (This is where I use the hypothesis that X is not negligible.) Define $\tilde{\phi} : \Sigma \to \mathcal{P}X$ by setting

$$\tilde{\phi} F = \phi(E \cap F) \cup (\psi F \setminus E).$$

Then $\tilde{\phi}$ is a Boolean homomorphism because ϕ and ψ are;

$$F \triangle \tilde{\phi} F \subseteq ((E \cap F) \triangle \phi(E \cap F)) \cup (X \setminus E)$$

is negligible, so $\tilde{\phi} F \in \Sigma$, for every $F \in \Sigma$, because μ is complete; and if F is negligible, then $\phi(E \cap F) = \psi F = \emptyset$ so $\tilde{\phi} F = \emptyset$. Thus $\tilde{\phi}$ is a lifting. Finally,

$$\bigcup_{G \in \mathfrak{T}} G \setminus \tilde{\phi} G \subseteq (X \setminus E) \cup \bigcup_{G \in \mathfrak{T}} ((G \cap E) \setminus \phi(G \cap E))$$

is negligible because ϕ is almost strong and E is conegligible.

453E Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete strictly localizable topological measure space with an almost strong lifting, and $A \subseteq X$ a non-negligible set. Then the subspace measure μ_A has an almost strong lifting.

proof Let $\phi : \Sigma \to \Sigma$ be an almost strong lifting. Because μ is strictly localizable, A has a measurable envelope W say (put 213J and 213L together). Write Σ_A for the subspace σ-algebra on A. Let $\psi : \Sigma_A \to \{\emptyset, A\}$ be any Boolean homomorphism such that $\psi H = \emptyset$ for every negligible set $H \subseteq A$.

If E, $F \in \Sigma$ and $E \cap A = F \cap A$, then $\phi E \cap \phi W = \phi F \cap \phi W$. **P**

$$\mu((E \triangle F) \cap W) = \mu^*((E \triangle F) \cap A) = 0,$$

so

$$(\phi E \cap \phi W) \triangle (\phi F \cap \phi W) = \phi((E \triangle F) \cap W) = \emptyset. \text{ } \mathbf{Q}$$

We can therefore define a function $\tilde\phi : \Sigma_A \to \mathcal{P}A$ by setting

$$\tilde\phi H = (\phi E \cap \phi W \cap A) \cup (\psi H \setminus \phi W)$$

whenever $E \in \Sigma$ and $H = E \cap A$. It is easy to check that $\tilde\phi$ is a Boolean homomorphism. If $E \in \Sigma$ then

$$(E \cap A)\triangle\tilde\phi(E \cap A) \subseteq (E\triangle\phi E) \cup (A \setminus \phi W) \subseteq (E\triangle\phi E) \cup (W \setminus \phi W)$$

is negligible, so $\tilde\phi(E \cap A) \in \Sigma_A$ (because μ and μ_A are complete). If $H \in \Sigma_A$ is negligible, then

$$\tilde\phi H \subseteq \phi H \cup \psi H = \emptyset,$$

so $\tilde\phi$ is a lifting for μ_A.

Now set $B = (A \setminus \phi W) \cup \bigcup_{G\in\mathfrak{T}} G \setminus \phi G$. Because ϕ is almost strong, B is negligible. If $H \subseteq A$ is relatively open, then $H \setminus \tilde\phi H \subseteq B$. **P** Take $x \in H \setminus \tilde\phi H$. Express H as $G \cap A$ where $G \subseteq X$ is open. If $x \in \phi W$, then $x \notin \phi G$ so $x \in B$; if $x \notin \phi W$, then of course $x \in B$. **Q** Thus

$$\bigcup\{H \setminus \tilde\phi H : H \subseteq A \text{ is relatively open}\} \subseteq B$$

is negligible and $\tilde\phi$ is almost strong.

453F Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete strictly localizable topological measure space.

(a) If $\mathfrak{T}$ has a countable network, any lifting for μ is almost strong.

(b) Suppose that $\mu X > 0$ and μ is inner regular with respect to

$$\mathcal{K} = \{K : K \in \Sigma,\ \mu_K \text{ has an almost strong lifting}\},$$

where μ_K is the subspace measure on K. Then μ has an almost strong lifting.

proof (a) Let $\mathcal{E}$ be a countable network for $\mathfrak{T}$, and $\phi : \Sigma \to \Sigma$ a lifting. For each $E \in \mathcal{E}$, let $\hat E$ be a measurable envelope of E (213J/213L again). Then

$$\bigcup_{G\in\mathfrak{T}} G \setminus \phi G = \bigcup_{G\in\mathfrak{T}, E\in\mathcal{E}, E\subseteq G} E \setminus \phi G \subseteq \bigcup_{G\in\mathfrak{T}, E\in\mathcal{E}, E\subseteq G} \hat E \setminus \phi \hat E$$

(because if $E \subseteq G \in \mathfrak{T}$, then $G \in \Sigma$, so $\mu(\hat E \setminus G) = 0$ and $\phi\hat E \subseteq \phi G$)

$$\subseteq \bigcup_{E\in\mathcal{E}} \hat E \setminus \phi\hat E$$

is negligible, so ϕ is almost strong.

(b) Let $\mathcal{L} \subseteq \mathcal{K}$ be a disjoint family such that $\mu^* A = \sum_{L\in\mathcal{L}} \mu^*(A \cap L)$ for every $A \subseteq X$ (412Ib). For each $L \in \mathcal{L}$, let Σ_L be the corresponding subspace σ-algebra and $\phi_L : \Sigma_L \to \Sigma_L$ an almost strong lifting. Set $E = \bigcup\mathcal{L}$; then

$$\mu^*(X \setminus E) = \textstyle\sum_{L\in\mathcal{L}} \mu(L \setminus E) = 0,$$

so E is conegligible. For $F \in \Sigma_E$ set $\phi F = \bigcup_{L\in\mathcal{L}} \phi_L(F \cap L)$; then ϕ is a Boolean homomorphism from Σ_E to $\mathcal{P}E$. If $F \in \Sigma_E$, then

$$\mu^*(F\triangle\phi F) = \textstyle\sum_{L\in\mathcal{L}} \mu^*(L \cap (F\triangle\phi F)) = \sum_{L\in\mathcal{L}} \mu^*((F \cap L)\triangle\phi_L(F \cap L)) = 0,$$

while if $\mu F = 0$ then $\phi_L(F \cap L) = \emptyset$ for every L, so $\phi F = \emptyset$. Thus ϕ is a lifting. Now set

$$A = \bigcup\{H \setminus \phi H : H \subseteq E \text{ is relatively open}\}.$$

If $L \in \mathcal{L}$, then

$$A \cap L = \bigcup\{(H \cap L) \setminus \phi_L(H \cap L) : H \subseteq E \text{ is relatively open}\}$$

is negligible, because ϕ_L is almost strong; thus ϕ is an almost strong lifting for μ_E. By 453Dd, μ also has an almost strong lifting.

453G Corollary (a) A non-zero quasi-Radon measure on a separable metrizable space has an almost strong lifting.

(b) A non-zero Radon measure μ on an analytic Hausdorff space X has an almost strong lifting.

proof (a) A quasi-Radon measure is complete and strictly localizable (415A), so, if non-zero, has a lifting (341K). A separable metrizable space has a countable network (4A2P(a-iii)), so this lifting must be almost strong.

(b) If $K \subseteq X$ is compact and non-negligible, it is metrizable (423Dc), so that the subspace measure μ_K has an almost strong lifting, by (a); as μ is tight (that is, inner regular with respect to the closed compact sets), it has an almost strong lifting, by 453Fb.

Remark In particular, Lebesgue measure on $\mathbb{R}^r$ has an almost strong lifting and therefore, by 453Db, a strong lifting, as already noted in 453B.

453H Lemma Let (X, Σ, μ) be a complete locally determined measure space and $\mathfrak{T}$ a topology on X generated by a family $\mathcal{U} \subseteq \Sigma$. Suppose that $\phi : \Sigma \to \Sigma$ is a lifting such that $\phi U \supseteq U$ for every $U \in \mathcal{U}$. Then μ is a τ-additive topological measure, and ϕ is a strong lifting.

proof Of course ϕ is a lower density, and $\phi X = X$, so by 414P we have a density topology

$$\mathfrak{T}_d = \{E : E \in \Sigma,\, E \subseteq \phi E\}$$

with respect to which μ is a τ-additive topological measure. But our hypothesis is that $\mathcal{U} \subseteq \mathfrak{T}_d$, so $\mathfrak{T} \subseteq \mathfrak{T}_d$ and μ is a τ-additive topological measure with respect to $\mathfrak{T}$. Also, of course, $\phi G \supseteq G$ for every $G \in \mathfrak{T}$, so ϕ is a strong lifting.

453I Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of topological probability spaces such that every $\mathfrak{T}_i$ has a countable network and every μ_i is strictly positive. Let λ be the (ordinary) complete product measure on $X = \prod_{i\in I} X_i$. Then λ is a τ-additive topological measure and has a strong lifting.

proof (a) The strategy of the proof is as follows. We may suppose that $I = \kappa$ is a cardinal. Write Λ for the domain of λ, and for each $\xi \le \kappa$ let Λ_ξ be the σ-algebra of members of Λ determined by coordinates less than ξ; write $\pi_\xi : X \to X_\xi$ for the canonical map. I seek to define a lifting $\phi : \Lambda \to \Lambda$ such that $\phi W \supseteq W$ for every open set $W \in \Lambda$. This will be the last in a family $\langle \phi_\xi\rangle_{\xi\le\kappa}$ of partial liftings, constructed inductively as in the proof of 341H, with $\operatorname{dom}\phi_\xi = \Lambda_\xi$ for each ξ. The inductive hypothesis will be that ϕ_ξ extends ϕ_η whenever $\eta \le \xi$, and $\phi_\xi \pi_\eta^{-1}[G] \supseteq \pi_\eta^{-1}[G]$ for every $\eta < \xi$ and every open $G \subseteq X_\eta$.

The induction starts with $\Lambda_0 = \{\emptyset, X\}$, $\phi_0\emptyset = \emptyset$, $\phi_0 X = X$. For $\xi \le \kappa$, set $\mathfrak{B}_\xi = \{W^\bullet : W \in \Lambda_\xi\}$.

(b) *Inductive step to a successor ordinal* $\xi + 1$ Suppose that ϕ_ξ has been defined, where $\xi < \kappa$.

(i) By 341Nb, there is a lifting $\phi'_\xi : \Lambda \to \Lambda$ extending ϕ_ξ. Let $\mathcal{E}_\xi$ be a countable network for $\mathfrak{T}_\xi$. For each $E \in \mathcal{E}_\xi$ let $\hat{E}$ be a measurable envelope of E. Set

$$Q = \bigcup\{\pi_\xi^{-1}[\hat{E}] \setminus \phi'_\xi(\pi_\xi^{-1}[\hat{E}]) : E \in \mathcal{E}_\xi\};$$

then Q is negligible.

(ii) For $x \in Q$, let $\mathcal{I}_x \subseteq \Lambda$ be the ideal generated by

$$\{W : W \in \Lambda_\xi,\, x \notin \phi_\xi W\} \cup \{\pi_\xi^{-1}[F] : F \subseteq X_\xi \text{ is closed, } \pi_\xi(x) \notin F\} \cup \{W : \lambda W = 0\}.$$

Then $X \notin \mathcal{I}_x$. **P?** Otherwise, there are a $W \in \Lambda_\xi$, a closed $F \subseteq X_\xi$ and a negligible $W' \in \Lambda$ such that $W \cup W' \cup \pi_\xi^{-1}[F] = X$ while $x \notin \phi_\xi W \cup \pi_\xi^{-1}[F]$. But in this case

$$\begin{aligned} 0 = \lambda W' &\ge \lambda((X \setminus W) \cap (X \setminus \pi_\xi^{-1}[F])) \\ &= \lambda(X \setminus W) \cdot \lambda(X \setminus \pi_\xi^{-1}[F]) = \lambda(X \setminus W) \cdot \mu_\xi(X_\xi \setminus F) > 0 \end{aligned}$$

because μ_ξ is strictly positive and $\phi_\xi W \ne X$. **XQ**

There is therefore a Boolean homomorphism $\psi_x : \Lambda \to \{0,1\}$ which is zero on $\mathcal{I}_x$.

(iii) Set

$$\phi_{\xi+1} W = (\phi'_\xi W \setminus Q) \cup \{x : x \in Q,\, \psi_x W = 1\}$$

for every $W \in \Lambda_{\xi+1}$. Then $\phi_{\xi+1}$ is a Boolean homomorphism from $\Lambda_{\xi+1}$ to $\mathcal{P}X$. Because $\phi_{\xi+1}W\triangle\phi'_\xi W \subseteq Q$ is negligible, $\phi_{\xi+1}W \in \Lambda$ and $W\triangle\phi_{\xi+1}W$ is negligible for every $W \in \Lambda_{\xi+1}$. If $\lambda W = 0$ then $\phi'_\xi W = \emptyset$ and $\psi_x W = 0$ for every $x \in Q$, so $\phi_{\xi+1}W = \emptyset$; thus $\phi_{\xi+1} : \Lambda_{\xi+1} \to \Lambda$ is a partial lifting. If $W \in \Lambda_\xi$, then, for $x \in Q$,

$$\begin{aligned} x \in \phi_\xi W &\implies x \notin \phi_\xi(X \setminus W) \implies X \setminus W \in \mathcal{I}_x \\ &\implies \psi_x(X \setminus W) = 0 \implies \psi_x W = 1 \iff x \in \phi_{\xi+1}W \\ &\implies W \notin \mathcal{I}_x \implies x \in \phi_\xi W, \end{aligned}$$

so $\phi_{\xi+1}W = \phi_\xi W$. Thus $\phi_{\xi+1}$ extends ϕ_ξ.

(iv) Suppose that $\eta \le \xi$ and $G \subseteq X_\eta$ is open. If $\eta < \xi$ then

$$\phi_{\xi+1}(\pi_\eta^{-1}[G]) = \phi_\xi(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G]$$

by the inductive hypothesis. If $\eta = \xi$, take any $x \in \pi_\xi^{-1}[G]$. If $x \in Q$, then $X\setminus\pi_\xi^{-1}[G] \in \mathcal{I}_x$, so $\psi_x(\pi_\xi^{-1}[G]) = 1$ and $x \in \phi_{\xi+1}(\pi_\xi^{-1}[G])$. If $x \notin Q$, there is an $E \in \mathcal{E}_\xi$ such that $x(\xi) \in E \subseteq G$. In this case, $x(\xi) \in \hat{E}$, so

$$x \in \pi_\xi^{-1}[\hat{E}] \setminus Q \subseteq \phi'_\xi(\pi_\xi^{-1}[\hat{E}]) \setminus Q \subseteq \phi_{\xi+1}(\pi_\xi^{-1}[\hat{E}]) \subseteq \phi_{\xi+1}(\pi_\xi^{-1}[G])$$

because $\hat{E} \setminus G$ and $\pi_\xi^{-1}[\hat{E}] \setminus \pi_\xi^{-1}[G]$ are negligible. As x is arbitary, $\pi_\eta^{-1}[G] \subseteq \phi_{\xi+1}(\pi_\eta^{-1}[G])$ in this case also.
Thus the induction continues.

(c) *Inductive step to a non-zero limit ordinal ξ of countable cofinality* Suppose that $0 < \xi \le \kappa$, that $\operatorname{cf}\xi = \omega$ and that ϕ_η has been defined for every $\eta < \xi$. Let $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in ξ with limit ξ. Then $\mathfrak{B}_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_{\zeta_n}$ (using 254N and 254Fe, or otherwise). By 341G, there is a partial lower density $\underline{\phi} : \Lambda_\xi \to \Lambda$ extending every ϕ_{ζ_n}, and therefore extending ϕ_η for every $\eta < \xi$. By 341Jb (applied to $\lambda\restriction\widehat{\Lambda}_\xi$, where $\widehat{\Lambda}_\xi$ is the σ-subalgebra of Λ generated by $\Lambda_\xi \cup \{W : \lambda W = 0\}$), there is a partial lifting $\phi_\xi : \Lambda_\xi \to \Lambda$ such that $\underline{\phi}W \subseteq \phi_\xi W$ for every $W \in \Lambda_\xi$.
If $\eta < \xi$ and $W \in \Lambda_\eta$, then

$$\phi_\eta W = \underline{\phi}W \subseteq \phi_\xi W, \quad X \setminus \phi_\eta W = \phi_\eta(X \setminus W) \subseteq \phi_\xi(X \setminus W) = X \setminus \phi_\xi W,$$

so ϕ_ξ extends ϕ_η. If $\eta < \xi$ and $G \subseteq X_\eta$ is open,

$$\phi_\xi(\pi_\eta^{-1}[G]) = \phi_{\eta+1}(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G].$$

So again the induction continues.

(d) *Inductive step to a limit ordinal ξ of uncountable cofinality* In this case, $\mathfrak{B}_\xi = \bigcup_{\eta<\xi}\mathfrak{B}_\eta$, as in the proof of 341H; so there will be a unique partial lifting $\phi_\xi : \Lambda_\xi \to \Lambda$ extending ϕ_η for every $\eta < \xi$ (set $\phi_\xi W = \phi_\eta W'$ whenever $W \in \Lambda_\xi$, $\eta < \xi$, $W' \in \Lambda_\eta$ and $W\triangle W'$ is negligible). As in (c), we again have

$$\phi_\xi(\pi_\eta^{-1}[G]) = \phi_{\eta+1}(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G]$$

whenever $\eta < \xi$ and $G \subseteq X_\eta$ is open.

(e) At the end of the induction, we have a lifting $\phi = \phi_\kappa$ of Λ such that $\phi U \supseteq U$ for every $U \in \mathcal{U}$, where $\mathcal{U} = \{\pi_\xi^{-1}[G] : \xi < \kappa, G \in \mathfrak{T}_\xi\}$. By 453H, λ is a τ-additive topological measure and ϕ is a strong lifting.

453J Corollary Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces such that every $\mathfrak{T}_i$ has a countable network consisting of measurable sets and every μ_i is strictly positive. Then the ordinary product measure λ on $X = \prod_{i\in I} X_i$ is quasi-Radon and has a strong lifting. If every X_i is compact and Hausdorff, then λ is a Radon measure.

proof We have just seen that λ is a τ-additive topological measure with a strong lifting; but also it is inner regular with respect to the closed sets, by 412Ua, so it is a quasi-Radon measure. If all the X_i are compact and Hausdorff, so is X, so λ is a Radon measure (416G).

453K We come now to the construction of disintegrations from strong liftings.

Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces and $f:X\to Y$ an almost continuous inverse-measure-preserving function. Suppose that ν has an almost strong lifting. Then there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν such that every μ_y is a Radon measure and $\mu_yX=\mu_yf^{-1}[\{y\}]=1$ for almost every $y\in Y$.

proof (a)(i) Suppose first that X is compact, μ is a probability measure and that f is continuous.

Turn back to the proofs of 452H-452I. In part (a) of the proof of 452H, suppose that the lifting $\theta:\mathfrak{B}\to\mathrm{T}$ corresponds to an almost strong lifting $\phi:\mathrm{T}\to\mathrm{T}$ (see 341Ba). Set $B=\bigcup_{H\in\mathfrak{S}}H\setminus\phi H$, so that B is negligible. In part (c) of the proof of 452H, take $\mathcal{K}$ to be the family of compact subsets of X. Then all the μ_y, as constructed in 452H, will be Radon probability measures. For every y, $f^{-1}[\{y\}]$ is a closed set, so is necessarily measured by μ_y. But also it is μ_y-conegligible for every $y\in Y\setminus B$. **P** Let $K\subseteq X\setminus f^{-1}[\{y\}]$ be a compact set. Then $f[K]$ is a compact set not containing y. Because Y is Hausdorff, there is an open set H containing y such that $\overline{H}\cap f[K]=\emptyset$ (4A2F(h-i)). Now

$$y\in H\setminus B\subseteq\phi H\subseteq\phi\overline{H}.$$

Let E be the compact set $f^{-1}[\overline{H}]$. Taking $T:L^\infty(\mu)\to L^\infty(\nu)$ as in part (a) of the proof of 452I, $T(\chi E^\bullet)=\chi\overline{H}^\bullet$, so

$$\psi_yE=(ST(\chi E^\bullet))(y)=(S(\chi\overline{H}^\bullet))(y)=(\chi(\phi\overline{H}))(y)=1.$$

Because $E\in\mathcal{K}$, $\mu_yE\ge\psi_yE$; since we always have $\mu_yX=1$, E is μ_y-conegligible. But $K\cap E=\emptyset$, so $\mu_yK=0$. As K is arbitrary, $\mu_y(X\setminus f^{-1}[\{y\}])=0$. **Q**

Thus $\mu_yf^{-1}[\{y\}]=1$ for almost every $y\in Y$, while $\mu_yX\le1$ for every y.

(ii) The result for totally finite μ and ν and continuous f follows at once.

(b) Now suppose that μ and ν are totally finite, and that f is almost continuous.

(i) Let $\mathcal{K}$ be the family of subsets $K\subseteq X$ such that

K is compact and $f{\restriction}K$ is continuous,

whenever $F\in\mathrm{T}$ and $\nu(F\cap f[K])>0$ then $\mu(K\cap f^{-1}[F])>0$,

either $K=\emptyset$ or $\mu K>0$.

Take any $E\in\Sigma$ such that $\mu E>0$. Then there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu K>0$. **P** Let $K_0\subseteq E$ be a compact set such that $f{\restriction}K_0$ is continuous and $\mu K_0>0$. Let $\delta>0$ be such that $\mu K_0-\delta\nu Y>0$. For compact sets $K\subseteq K_0$ set $q(K)=\mu K-\delta\nu f[K]$. Choose $\langle\alpha_n\rangle_{n\in\mathbb{N}}$, $\langle K_n\rangle_{n\ge1}$ as follows. Given that K_n is a compact subset of K_0, where $n\in\mathbb{N}$, set

$$\alpha_n=\sup\{q(K):K\subseteq K_n\text{ is compact}\},$$

and choose a compact subset K_{n+1} of K_n such that $q(K_{n+1})\ge\max(q(K_n),\alpha_n-2^{-n})$. Continue.

Set $K=\bigcap_{n\in\mathbb{N}}K_n$. We have

$$\begin{aligned}q(K)&=\mu K-\delta\nu f[K]\\&\ge\lim_{n\to\infty}\mu K_n-\delta\inf_{n\in\mathbb{N}}\nu f[K_n]=\lim_{n\to\infty}q(K_n)=\sup_{n\in\mathbb{N}}q(K_n)\end{aligned}$$

because $\langle q(K_n)\rangle_{n\in\mathbb{N}}$ is non-decreasing. Of course $K\subseteq E$,

$$\mu K\ge q(K)\ge q(K_0)>0,$$

and $f{\restriction}K$ is continuous because $K\subseteq K_0$.

? If there is an $F\in\mathrm{T}$ such that $\nu(F\cap f[K])>0$ but $\mu(K\cap f^{-1}[F])=0$, take a compact set $K'\subseteq K\setminus f^{-1}[F]$ such that $\mu K'>\mu K-\delta\nu(F\cap f[K])$. Then $f[K']\subseteq f[K]\setminus F$, so

$$q(K')=\mu K'-\delta\nu f[K']\ge\mu K'-\delta(\nu f[K]-\nu(F\cap f[K]))>\mu K-\delta\nu f[K]=q(K).$$

Let $n\in\mathbb{N}$ be such that $q(K')>q(K)+2^{-n}$; then K' is a compact subset of K_n, so

$$\alpha_n\ge q(K')>q(K)+2^{-n}\ge q(K_{n+1})+2^{-n}\ge\alpha_n,$$

which is impossible. **X** Thus K belongs to $\mathcal{K}$ and will serve. **Q**

(ii) By 342B, there is a countable disjoint set $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $\mu(X \setminus \bigcup \mathcal{K}_0) = 0$. Enumerate $\mathcal{K}_0$ as $\langle K_n \rangle_{n<\#(\mathcal{K}_0)}$; for convenience of notation, if $\mathcal{K}_0$ is finite, set $K_n = \emptyset$ for $n \geq \#(\mathcal{K}_0)$, so that every K_n belongs to $\mathcal{K}$ and $\mu E = \sum_{n=0}^{\infty} \mu(E \cap K_n)$ for every $E \in \Sigma$.

(iii) For each $n \in \mathbb{N}$, define $\lambda_n : \mathrm{T} \to \mathbb{R}$ by setting $\lambda_n F = \mu(K_n \cap f^{-1}[F])$ for every $F \in \mathrm{T}$. Then λ_n is a measure dominated by ν, so there is a T-measurable $g_n : Y \to [0,1]$ such that $\lambda_n F = \int_F g_n$ for every $F \in \mathrm{T}$, by the Radon-Nikodým theorem. Because $\lambda_n(Y \setminus f[K_n]) = 0$, we may suppose that $g_n(y) = 0$ for $y \notin f[K_n]$. We have

$$\int_F \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} \int_F g_n = \sum_{n=0}^{\infty} \mu(K_n \cap f^{-1}[F]) = \mu f^{-1}[F] = \nu F$$

for every $F \in \mathrm{T}$, so $\sum_{n=0}^{\infty} g_n(y) = 1$ for ν-almost every y. Reducing the g_n further on a set of measure zero, if need be, we may suppose that $\sum_{n=0}^{\infty} g_n(y) \leq 1$ for every y.

(iv) For each $n \in \mathbb{N}$, let $\tilde{\lambda}_n$ be the subspace measure on $f[K_n]$ induced by λ_n, and $\tilde{\mu}_n$ the subspace measure on K_n induced by μ. Then $f\restriction K_n$ is inverse-measure-preserving for $\tilde{\mu}_n$ and $\tilde{\lambda}_n$. Also, $\tilde{\lambda}_n$ has an almost strong lifting. **P** If $K_n = \emptyset$, this is trivial. Otherwise, $\nu f[K_n] \geq \mu K_n > 0$, so the subspace measure $\tilde{\nu}_n$ induced by ν on $f[K_n]$ has an almost strong lifting, by 453E. But $\tilde{\nu}_n$ and $\tilde{\lambda}_n$ have the same domain $\mathrm{T} \cap \mathcal{P}(f[K_n])$ and the same null ideal, because $K_n \in \mathcal{K}$; so an almost strong lifting for $\tilde{\nu}_n$ is an almost strong lifting for $\tilde{\lambda}_n$. **Q**

By (a) above, we can find a disintegration $\langle \mu_{ny} \rangle_{y \in f[K_n]}$ of $\tilde{\mu}_n$ over $\tilde{\lambda}_n$ such that every μ_{ny} is a Radon measure on K_n, $\mu_{ny} K_n \leq 1$ for every y and

$$\mu_{ny}\{x : x \in K_n,\ f(x) = y\} = 1$$

for $\tilde{\lambda}_n$-almost every $y \in f[K_n]$, that is, for ν-almost every $y \in f[K_n]$. For $y \in Y \setminus f[K_n]$, let μ_{ny} be the zero measure on K_n.

(v) Now, for $y \in Y$, set

$$\mu_y E = \sum_{n=0}^{\infty} g_n(y) \mu_{ny}(E \cap K_n)$$

for all those $E \subseteq X$ such that the sum is defined. Then μ_y is a Radon measure and $\mu_y X \leq 1$. **P** Because every μ_{ny} is a complete measure, so is μ_y. We have

$$\mu_y X = \sum_{n=0}^{\infty} g_n(y) \mu_{ny} K_n \leq \sum_{n=0}^{\infty} g_n(y) \leq 1$$

by the choice of the g_n. If $G \subseteq X$ is open then μ_{ny} measures $G \cap K_n$ for every n, so μ_y measures G; accordingly μ_y measures every compact set. If $\mu_y E > 0$, there is some $n \in \mathbb{N}$ such that $g_n(y) > 0$ and $\mu_{ny}(E \cap K_n) > 0$; now there is a compact set $K \subseteq E \cap K_n$ such that $\mu_{ny} K > 0$, in which case $\mu_y K > 0$. By 412B, μ_y is tight, and is a Radon measure. **Q**

(vi) $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν. **P** If $E \in \Sigma$ then

$$\mu E = \sum_{n=0}^{\infty} \mu(E \cap K_n)$$

(by the choice of the K_n in (ii) above)

$$= \sum_{n=0}^{\infty} \tilde{\mu}_n(E \cap K_n) = \sum_{n=0}^{\infty} \int_{f[K_n]} \mu_{ny}(E \cap K_n) \tilde{\lambda}_n(dy)$$

(because $\langle \mu_{ny} \rangle_{y \in f[K_n]}$ is a disintegration of $\tilde{\mu}_n$ over $\tilde{\lambda}_n$)

$$= \sum_{n=0}^{\infty} \int_{f[K_n]} \mu_{ny}(E \cap K_n) \lambda_n(dy) = \sum_{n=0}^{\infty} \int \mu_{ny}(E \cap K_n) \lambda_n(dy)$$

(because $\lambda_n(Y \setminus f[K_n]) = 0$)

$$= \sum_{n=0}^{\infty} \int g_n(y) \mu_{ny}(E \cap K_n) \nu(dy)$$

(235A)

$$= \int \sum_{n=0}^{\infty} g_n(y)\mu_{ny}(E \cap K_n)\nu(dy) = \int \mu_y E\,\nu(dy).\ \mathbf{Q}$$

(vii) It follows that $\mu_y f^{-1}[\{y\}] = 1$ for almost every y. **P**

$$\begin{aligned}\{y : \mu_y f^{-1}[\{y\}] \neq 1\} &\subseteq \{y : \mu_y X \neq 1\} \cup \{y : \mu_y^*(X \setminus f^{-1}[\{y\}]) > 0\}\\ &\subseteq \{y : \mu_y X \neq 1\} \cup \bigcup_{n\in\mathbb{N}} \{y : y \in f[K_n],\, \mu_{ny}(K_n \setminus f^{-1}[\{y\}]) > 0\}\end{aligned}$$

is negligible. **Q**

(c) Now let us turn to the general case. This proceeds just as in 452O. Let $\langle Y_i\rangle_{i\in I}$ be a decomposition of Y. For each $i \in I$, take X_i, λ_i and ν_i as in the proof of 452O. Note that λ_i and ν_i are Radon measures, so that we can apply (b) above to find a disintegration $\langle\tilde\mu_y\rangle_{y\in Y_i}$ of λ_i over ν_i such that every $\tilde\mu_y$ is a Radon measure and $\tilde\mu_y X_i = \tilde\mu_y f^{-1}[\{y\}] = 1$ for ν_i-almost every $y \in Y_i$. Just as in 452O, we can set

$$\mu_y E = \tilde\mu_y(E \cap X_i)$$

whenever $y \in Y_i$ and μ_y measures $E \cap X_i$, to obtain a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν in which every μ_y is a Radon measure and $\mu_y X = 1$ for almost every y; and this time

$$\{y : y \in Y,\, \mu_y f^{-1}[\{y\}] \neq 1\} = \bigcup_{i\in I}\{y : y \in Y_i,\, \tilde\mu_y f^{-1}[\{y\}] \neq 1\}$$

is negligible. So we have a disintegration of the required type.

453L Remark If f is surjective, we can arrange that every μ_y is a Radon probability measure for which $X_y = f^{-1}[\{y\}]$ is μ_y-conegligible, just by changing some of the μ_y to Dirac measures. If f is not surjective, then we can still (if X itself is not empty) arrange that every μ_y is a Radon probability measure; but it might be more appropriate to make some of the μ_y the zero measure, so that X_y is always μ_y-conegligible.

I have continued to express this theorem in terms of measures μ_y on the whole space X. Of course, if we take it that X_y is to be μ_y-conegligible for every y, it will sometimes be easier to think of μ_y as a measure on X_y; this is very much what we do in the case of Fubini's theorem, where all the X_y are, in effect, the same.

453M Strong liftings and Stone spaces Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ. For $E \in \Sigma$ let $E^* \subseteq Z$ be the open-and-closed set corresponding to the equivalence class $E^\bullet \in \mathfrak{A}$. Let R be the relation

$$\bigcap_{F\subseteq X \text{ is closed}}\{(z, x) : z \in Z \setminus F^* \text{ or } x \in F\} \subseteq Z \times X$$

(415Q). For every lifting $\phi : \Sigma \to \Sigma$ we have a unique function $g_\phi : X \to Z$ such that $\phi E = g_\phi^{-1}[E^*]$ for every $E \in \Sigma$ (see 341P). Now we have the following easy facts.

(a) ϕ is strong iff $(g_\phi(x), x) \in R$ for every $x \in X$. **P**

$$\begin{aligned}&(g_\phi(x), x) \in R \text{ for every } x \in X\\ &\qquad\iff x \in F \text{ whenever } F \text{ is closed and } g_\phi(x) \in F^*\\ &\qquad\iff g_\phi^{-1}[F^*] \subseteq F \text{ for every closed set } F \subseteq X\\ &\qquad\iff \phi F \subseteq F \text{ for every closed set } F \subseteq X\\ &\qquad\iff \phi \text{ is strong. } \mathbf{Q}\end{aligned}$$

(b) If $\mathfrak{T}$ is Hausdorff, so that R is the graph of a function f (415Ra), then ϕ is strong iff $fg_\phi(x) = x$ for every $x \in X$. (For $(g_\phi(x), x) \in R$ iff $fg_\phi(x) = x$.)

453N Losert's example (Losert 79) There is a compact Hausdorff space with a strictly positive completion regular Radon probability measure which has no strong lifting.

proof (a) Let ν be the usual measure on $\{0,1\}^{\mathbb{N}} = Y$. Let $M \subseteq Y$ be a closed nowhere dense set such that $\nu M > 0$ (cf. 419B), and ν_1 a Radon probability measure on Y such that $\nu_1 M = 1$ (e.g., a Dirac measure concentrated at some point of M).

Let I be any set with cardinal at least ω_2 such that $I \cap (I \times I) = \emptyset$. Let λ be the product measure on Y^I, giving each factor the measure ν; of course λ can be identified with the usual measure on $\{0,1\}^{\mathbb{N}\times I}$ (254N). Note that λ and ν are both strictly positive. For $i \in I$ write $M_i = \{z : z \in Y^I,\, z(i) \in M\}$; then M_i is closed in Y^I.

Set $A = \{(i,j) : i,\, j \in I,\, i \neq j\}$. For $z \in Y^I$ and $(i,j) \in A$ let $\nu_{ij}^{(z)}$ be the Radon probability measure on Y given by setting

$$\begin{aligned} \nu_{ij}^{(z)} &= \nu_1 \text{ if } z \in M_i \cap M_j, \\ &= \nu \text{ otherwise.} \end{aligned}$$

Now, for $z \in Y^I$, let λ_z be the Radon product measure of $\langle \nu_{ij}^{(z)} \rangle_{(i,j)\in A}$ on Y^A.

(b) Let $\mathcal{U}$ be the family of sets $U \subseteq Y^A$ of the form $\{u : u(i,j) \in U_{ij}$ for $(i,j) \in B\}$, where $B \subseteq A$ is finite and $U_{ij} \subseteq Y$ is open-and-closed for every $(i,j) \in B$. Then the function $z \mapsto \lambda_z U : Y^I \to [0,1]$ is Borel measurable for every $U \in \mathcal{U}$. **P** Express U in the given form. For $C \subseteq B$ set

$$E_C = \{z : z \in Y^I,\, C = \{(i,j) : (i,j) \in B,\, z \in M_i \cap M_j\}\},$$

so that $\langle E_C \rangle_{C \subseteq B}$ is a partition of Y^I into Borel sets. For any $C \subseteq B$,

$$\lambda_z U = \prod_{(i,j)\in B} \nu_{ij}^{(z)}(U_{ij}) = \prod_{(i,j)\in C} \nu_1 U_{ij} \cdot \prod_{(i,j)\in B\setminus C} \nu U_{ij}$$

is constant for $z \in E_C$. **Q**

(c) There is a Radon measure μ on $X = Y^I \times Y^A$ specified by the formula

$$\mu E = \int \lambda_z E[\{z\}] \lambda(dz)$$

for every Baire set $E \subseteq X$. **P** Let $\mathcal{E}$ be the class of those sets $E \subseteq X$ such that $\int \lambda_z E[\{z\}]\lambda(dz)$ is defined. Then $\mathcal{E}$ is closed under monotone limits of sequences, and $E \setminus E' \in \mathcal{E}$ whenever $E,\, E' \in \mathcal{E}$ and $E' \subseteq E$; also $\mathcal{E}$ contains all the basic open-and-closed sets in X of the form $V \times U$, where $V \subseteq Y^I$ is open-and-closed and $U \in \mathcal{U}$. By the Monotone Class Theorem (136B), $\mathcal{E}$ includes the σ-algebra generated by such sets, which is the Baire σ-algebra $\mathcal{B}\mathfrak{a}$ of X (4A3Of). Of course $E \mapsto \int \lambda_z E[\{z\}]\lambda(dz)$ is countably additive on $\mathcal{B}\mathfrak{a}$, so is a Baire measure on X, and has a unique extension to a Radon measure, by 432F. **Q**

μ is strictly positive. **P** Let $W \subseteq X$ be any non-empty open set. Then it includes an open set of the form $V \times U$ where $V = \{z : z \in Y^I,\, z(i) \in V_i$ for every $i \in J\}$, $U = \{u : u \in Y^A,\, u(j,k) \in U_{jk}$ for every $(j,k) \in B\}$, $J \subseteq I$ and $B \subseteq A$ are finite sets, and $V_i,\, U_{jk} \subseteq Y$ are non-empty open sets for every $i \in J$ and $(j,k) \in B$. Now ν is strictly positive, so $\lambda V' > 0$, where

$$V' = \{z : z \in V,\, z \notin M_j \text{ whenever } (j,k) \in B\}.$$

(This is where we need to know that the M_j are nowhere dense.) But if $z \in V'$ then $\nu_{jk}^{(z)} = \nu$ for every $(j,k) \in B$, so

$$\lambda_z U = \prod\nolimits_{(j,k)\in B} \nu U_{jk} > 0.$$

Accordingly

$$\mu W \geq \int_{V'} \lambda_z U \lambda(dz) > 0.$$

As W is arbitrary, μ is strictly positive. **Q**

Write Σ for the domain of μ.

(d) Fix on a self-supporting compact set $K \subseteq X$. I seek to show that, regarded as a subset of $Y^{I\cup A}$, K is determined by coordinates in some countable set.

(i) There is a zero set $L \supseteq K$ such that $\mu L = \mu K$. **P** Let $\langle K_n \rangle_{n\in\mathbb{N}}$ be a sequence of compact subsets of $X \setminus K$ such that $\lim_{n\to\infty} \mu K_n = \mu(X \setminus K)$. For each $n \in \mathbb{N}$ there is a continuous function $f_n : X \to [0,1]$

which is zero on K and 1 on K_n; now $L=\{x:f_n(x)=0$ for every $n\in\mathbb{N}\}$ is a zero set including K and of the same measure as K. **Q**

(ii) By 4A3Nc, L is determined by coordinates in a countable subset of $I\cup A$, that is, there are countable sets $J_0\subseteq I$, $B_0\subseteq A$ such that whenever $(z,u)\in L$, $(z',u')\in X$, $z\restriction J_0=z'\restriction J_0$ and $u\restriction B_0=u'\restriction B_0$ we shall have $(z',u')\in L$. Set

$$J=J_0\cup\{i:(i,j)\in B_0\}\cup\{j:(i,j)\in B_0\},\quad B=A\cap(J\times J);$$

then $J\supseteq J_0$ and $B\supseteq B_0$ are still countable, and L is determined by coordinates in $J\cup B$.

(iii) Take any $(z_0,u_0)\in X\setminus K$. Because K is closed, we can find finite sets $J_1\subseteq I$ and $B_1\subseteq A$, open-and-closed sets $V_i\subseteq Y$ for $i\in J_1$, and open-and-closed sets $U_{ij}\subseteq Y$ for $(i,j)\in B_1$, such that

$$W=\{(z,u):z(i)\in V_i\text{ for every }i\in J_1,\ u(i,j)\in U_{ij}\text{ for every }(i,j)\in B_1\}$$

contains (z_0,u_0) and is disjoint from K. Set

$$\begin{aligned}W_1=\{(z,u):(z,u)\in X,z(i)\in V_i\text{ for every }i\in J_1\cap J,\\ u(i,j)\in U_{ij}\text{ for every }(i,j)\in B_1\cap B\},\end{aligned}$$

$$Q=\{z:z\in Y^I,\ \lambda_z((L\cap W_1)[\{z\}])>0\},$$

so that W_1 is an open-and-closed set in X and Q is a Borel set in Y^I ((b) above). Now Q is determined by coordinates in J. **P** Suppose that $z\in Q$, $z'\in Y^I$ and $z\restriction J=z'\restriction J$. Because both L and W_1 are determined by coordinates in $J\cup B$, $(L\cap W_1)[\{z\}]=(L\cap W_1)[\{z'\}]=H$ say, and H is determined by coordinates in B. At the same time, for any $(i,j)\in B$, $M_i\cap M_j$ is determined by coordinates in J, so contains z iff it contains z', and $\nu_{ij}^{(z)}=\nu_{ij}^{(z')}$. This means that, writing λ'_z and $\lambda'_{z'}$ for the products of $\langle\nu_{ij}^{(z)}\rangle_{(i,j)\in B}$ and $\langle\nu_{ij}^{(z')}\rangle_{(i,j)\in B}$ on Y^B, $\lambda'_z=\lambda'_{z'}$. So

$$\lambda_{z'}((L\cap W_1)[\{z'\}])=\lambda_{z'}H=\lambda'_{z'}H'=\lambda'_zH'=\lambda_zH=\lambda_z((L\cap W_1)[\{z\}])>0,$$

where $H'=\{u\restriction B:u\in H\}$ (254Ob), and $z'\in Q$. **Q**

(iv) Set

$$J_2=(\{i:(i,j)\in B_1\setminus B\}\cup\{j:(i,j)\in B_1\setminus B\})\setminus J.$$

Then J_2 is a finite subset of $I\setminus J$, and $B_1\subseteq(J\cup J_2)\times(J\cup J_2)$. Set

$$G=\{z:z\in Y^I,\ z(i)\notin M\text{ for every }i\in J_2\},$$

so that G is a dense open subset of Y^I. Set

$$G_1=\{z:z\in Y^I,\ z(i)\in V_i\text{ for every }i\in J_1\setminus J\}.$$

Then G_1 is a non-empty open set, so $G\cap G_1\neq\emptyset$ and $\lambda(G\cap G_1)>0$.

(v) Set

$$U=\{u:u\in Y^A,\ u(i,j)\in U_{ij}\text{ for every }(i,j)\in B_1\setminus B\}.$$

If $z\in G$, then $z\notin M_i\cap M_j$ whenever $(i,j)\in B_1\setminus B$, so $\nu_{ij}^{(z)}=\nu$ for every $(i,j)\in B_1\setminus B$, and

$$\lambda_zU=\prod_{(i,j)\in B_1\setminus B}\nu U_{ij}>0.$$

(vi) ? Suppose, if possible, that $\lambda Q>0$. Because Q is determined by coordinates in J and $G\cap G_1$ is determined by coordinates in $J_2\cup(J_1\setminus J)$,

$$\lambda(Q\cap G\cap G_1)=\lambda Q\cdot\lambda(G\cap G_1)>0.$$

If $z\in Q\cap G\cap G_1$,

$$\lambda_z((L\cap W)[\{z\}])=\lambda_z(U\cap(L\cap W_1)[\{z\}])$$

(because $W=W_1\cap(Y^I\times U)\cap(G_1\times Y^A)$, and $z\in G_1$)

$$= \lambda_z U \cdot \lambda_z((L \cap W_1)[\{z\}])$$

(because $(L \cap W_1)[\{z\}]$ is determined by coordinates in B, while U is determined by coordinates in $B_1 \setminus B$, and λ_z is a product measure)

$$> 0$$

because $z \in G \cap Q$. But this means that

$$0 < \int \lambda_z((L \cap W)[\{z\}])\lambda(dz) = \mu(L \cap W) = \mu(K \cap W) = \mu\emptyset,$$

which is absurd. **X**

Thus λQ must be zero.

(vii) Consequently

$$\mu(K \cap W_1) = \mu(L \cap W_1) = \int \lambda_z((L \cap W_1)[\{z\}])\lambda(dz) = 0;$$

because K is self-supporting, $K \cap W_1 = \emptyset$. And W_1 contains (z_0, u_0) and is determined by coordinates in $J \cup B$.

(viii) What this means is that there can be no $(z, u) \in K$ such that $z \restriction J = z_0 \restriction J$ and $u \restriction B = u_0 \restriction B$. At this point, recall that (z_0, u_0) was an arbitrary point of $X \setminus K$. So what must be happening is that K is determined by coordinates in the countable set $J \cup B$. By 4A3Nc again, in the other direction, K is a zero set.

(e) Part (d) shows that every self-supporting compact subset of X is a zero set. Since μ is certainly inner regular with respect to the self-supporting compact sets, it is inner regular with respect to the zero sets, that is, is completion regular.

It follows that whenever $E \in \Sigma$ there is an $E' \subseteq E$, determined by coordinates in a countable subset of $I \cup A$, such that $E \setminus E'$ is negligible. (Take E' to be a countable union of self-supporting compact sets.)

(f) ? Now suppose, if possible, that we could find a strong lifting ϕ for μ. For each $i \in I$, take a set $E_i \subseteq \phi(M_i \times Y^A)$ such that $\mu E_i = \mu\phi(M_i \times Y^A)$ and E_i is determined by coordinates in $J_i \cup B_i$, where $J_i \subseteq I$ and $B_i \subseteq A$ are countable. Set

$$J_i^* = \{j : (j, k) \in B_i\} \cup \{k : (j, k) \in B_i\},$$

so that J_i^* also is countable. Because $\#(I) \geq \omega_2$, there are distinct i, $j \in I$ such that $i \notin J_j^*$ and $j \notin J_i^*$ (4A1Ea). So $(i, j) \notin B_i \cup B_j$.

Set

$$F = \{u : u \in Y^A,\ u(i, j) \in M\}.$$

Then $\mu((M_i \cap M_j) \times (Y^A \setminus F)) = 0$. **P** If $z \in M_i \cap M_j$, then

$$\lambda_z(Y^A \setminus F) = \nu_{ij}^{(z)}(Y \setminus M) = 0.$$

But $(M_i \cap M_j) \times (Y^A \setminus F)$ is a Baire set, so

$$\begin{aligned}\mu((M_i \cap M_j) \times (Y^A \setminus F)) &= \int \lambda_z((M_i \cap M_j) \times (Y^A \setminus F))[\{z\}]\lambda(dz) \\ &= \int_{M_i \cap M_j} \lambda_z(Y^A \setminus F)\lambda(dz) = 0. \text{ } \mathbf{Q}\end{aligned}$$

Accordingly

$$E_i \cap E_j \subseteq \phi(M_i \times Y^A) \cap \phi(M_j \times Y^A) = \phi((M_i \cap M_j) \times Y^A) \subseteq \phi(Y^I \times F)$$

(because $((M_i \cap M_j) \times Y^A) \setminus (Y^I \times F)$ is negligible)

$$\subseteq Y^I \times F$$

because $Y^I \times F$ is closed and ϕ is supposed to be strong. However, $E_i \cap E_j$ is determined by coordinates in $J_i \cup J_j \cup B_i \cup B_j$, while $Y^I \times F$ is determined by coordinates in $\{(i,j)\}$, which does not meet $B_i \cup B_j$. So either $E_i \cap E_j$ is empty or $F = Y^A$. But $F \neq Y^A$ because $M \neq Y$, while

$$\begin{aligned}\mu(E_i \cap E_j) &= \mu(\phi(M_i \times Y^A) \cap \phi(M_j \times Y^A)) = \mu((M_i \cap M_j) \times Y^A)\\ &= \lambda(M_i \cap M_j) = (\nu M)^2 > 0,\end{aligned}$$

so $E_i \cap E_j \neq \emptyset$. **X**

(g) Thus μ has no strong lifting, as claimed.

453X Basic exercises >(a) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $(Z, \mathfrak{T}, \Sigma, \mu)$ its Stone space. Show that the canonical lifting for μ (341O) is strong.

(b) Show that there is a strong lifting for Lebesgue measure on the Sorgenfrey line (415Xc). (*Hint*: set $\underline{\phi}E = \{x : \lim_{\delta\downarrow 0}\frac{1}{\delta}\mu(E \cap [x, x+\delta]) = 1\}$, and use 341Jb.)

>(c) Let μ be the usual measure on the split interval (343J, 419L). Show that μ has a strong lifting.

(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space such that μ is inner regular with respect to the closed sets, and $\phi : \Sigma \to \Sigma$ a strong lifting. Show that μ is a quasi-Radon measure with respect to the lifting topology $\mathfrak{T}_l$ (414Q). Show that if $\mathfrak{T}$ is regular then $\mathfrak{T}_l \supseteq \mathfrak{T}$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space which has an almost strong lifting. Show that any non-zero indefinite-integral measure over μ (234J[12]) has an almost strong lifting.

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space such that (X, Σ, μ) is countably separated and $\mu X > 0$; for example, $(X, \mathfrak{T})$ could be an analytic space (433B). Show that μ is inner regular with respect to the compact metrizable subsets of X, so has an almost strong lifting. (*Hint*: there is an injective measurable $f : X \to \mathbb{R}$, which must be almost continuous.)

(g) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space such that μ is effectively locally finite and inner regular with respect to the closed sets, and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}G \supseteq G$ for every open $G \subseteq X$. Show that μ is a quasi-Radon measure with respect to both $\mathfrak{T}$ and the density topology associated with $\underline{\phi}$.

(h) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}G \supseteq G$ for every open $G \subseteq X$. Let $\langle G_x\rangle_{x\in X}$ be a family of open sets in X such that $x \notin \underline{\phi}(X \setminus G_x)$ for every $x \in X$. (i) Show that $A \setminus \bigcup_{x\in A}(G_x \cap U_x)$ is negligible whenever $A \subseteq X$ and U_x is a neighbourhood of x for every $x \in A$. (ii) Let $\mathfrak{S}$ be the topology on X generated by $\mathfrak{T} \cup \{\{x\} \cup G_x : x \in X\}$. Show that μ is quasi-Radon with respect to $\mathfrak{S}$.

(i) Let X and Y be Hausdorff spaces, and μ a Radon probability measure on $X \times Y$; set $\pi(x, y) = y$ for $x \in X$, $y \in Y$, and let ν be the image measure $\mu\pi^{-1}$. Suppose that ν has an almost strong lifting. Show that there is a family $\langle\mu_y\rangle_{y\in Y}$ of Radon probability measures on X such that $\mu E = \int \mu_y(E^{-1}[\{y\}])\nu(dy)$ for every $E \in \operatorname{dom}\mu$.

(j) Use 453Xe to simplify part (b) of the proof of 453K.

(k) In 453N, show that $\int \lambda_z E[\{z\}]\lambda(dz)$ is defined and equal to μE whenever μ measures E.

453Y Further exercises (a) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon measure space such that $\nu Y > 0$ and whenever $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space and $f : X \to Y$ is an almost continuous inverse-measure-preserving function, then there is a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν such that $\mu_y f^{-1}[\{y\}] = 1$ for almost every y. Show that ν has an almost strong lifting. (*Hint*: Start with the case in which Y is compact. Take $f : X \to Y$ to be the function described in 415R, 416V and 453Mb. Set $\underline{\phi}E = \{y : \mu_y E^* = \mu_y X = 1\}$.)

[12] Formerly 234B.

453Z Problems (a) If $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ are compact Radon measure spaces with strong liftings, does their product necessarily have a strong lifting? What if they are both Stone spaces?

(b) If $(X,\mathfrak{T},\Sigma,\mu)$ is a Radon probability space with countable Maharam type, must it have an almost strong lifting?

453 Notes and comments As I noted in §452, early theorems on disintegrations concentrated on cases in which all the measure spaces involved were 'standard' in that the measures were defined on standard Borel spaces (§424), or were the completions of such measures. Under these conditions the distinction between 452I and 453K becomes blurred; measures (when completed) have to be Radon measures (433Cb), liftings have to be almost strong (453F) and disintegrations have to be concentrated on fibers (452Gc). Theorem 453K provides disintegrations concentrated on fibers without any limitation on the size of the spaces involved, though making strong topological assumptions.

The strength of 453K derives from the remarkable variety of the (Radon) measure spaces which have strong liftings, as in 453F, 453G, 453I and 453J. For some ten years there were hopes that every strictly positive Radon measure had a strong lifting, which were finally dashed by Losert 79; I give a version of the example in 453N. This is a special construction, and it remains unclear whether some much more direct approach might yield another example (453Za). I should perhaps remark straight away that if the continuum hypothesis is true, then any strictly positive Radon measure with Maharam type at most ω_1 has a strong lifting (see 535I in Volume 5). In particular, subject to the continuum hypothesis, $Z\times Z$ has a strong lifting, where Z is the Stone space of the Lebesgue measure algebra, and we have a positive answer to 453Zb.

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454 Measures on product spaces

A central concern of probability theory is the study of 'processes', that is, families $\langle X_t\rangle_{t\in T}$ of random variables thought of as representing the evolution of a system in time. Kolmogorov's successful representation of such processes as measurable functions on an abstract probability space was one of the foundations on which the modern concept of 'random variable' was built. In this section I give a version of Kolmogorov's theorem on the extension of consistent families of measures on subproducts to a measure on the whole product (454D). It turns out that some restriction on the marginal measures is necessary, and 'perfectness' seems to be an appropriate hypothesis, necessarily satisfied if the factor spaces are standard Borel spaces or the marginal measures are Radon measures. If we have marginal measures with stronger properties then we shall be able to infer corresponding properties of the measure on the product space (454A, generalizing 451J).

The apparatus here makes it easy to describe joint distributions of arbitrary families of real-valued random variables (454J-454P), extending the ideas of §271. For the sake of the theorem that almost all Brownian paths are continuous (477B) I briefly investigate measures on $C(T)$, where T is a Polish space (454Q-454S).

454A Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a non-empty family of totally finite measure spaces. Set $X=\prod_{i\in I}X_i$ and let μ be a measure on X which is inner regular with respect to the σ-algebra $\widehat{\bigotimes}_{i\in I}\Sigma_i$ generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$, where $\pi_i:X\to X_i$ is the coordinate map for each $i\in I$. Suppose that every π_i is inverse-measure-preserving.

(a) If $\mathcal{K}\subseteq\mathcal{P}X$ is a family of sets which is closed under finite unions and countable intersections, and μ_i is inner regular with respect to $\mathcal{K}_i=\{K:K\subseteq X_i,\,\pi_i^{-1}[K]\in\mathcal{K}\}$ for every $i\in I$, then μ is inner regular with respect to $\mathcal{K}$.

(b)(i) If every μ_i is a compact measure, so is μ;

(ii) if every μ_i is a countably compact measure, so is μ;

(iii) if every μ_i is a perfect measure, so is μ.

proof If X is empty this is all trivial, so we may suppose that $X\neq\emptyset$.

(a) Set $\mathcal{A} = \{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$. If $A \in \mathcal{A}$, $V \in \Sigma$ and $\mu(A \cap V) > 0$, there is a $K \in \mathcal{K} \cap \mathcal{A}$ such that $K \subseteq A$ and $\mu(K \cap V) > 0$. **P** Express A as $\pi_i^{-1}[E]$, where $E \in \Sigma_i$; take $L \in \mathcal{K}_i$ such that $L \subseteq E$ and $\mu_i L > \mu_i E - \mu(A \cap V)$, and set $K = \pi_i^{-1}[L]$. **Q**

By 412C, $\mu \restriction \widehat{\bigotimes}_{i \in I} \Sigma_i$ is inner regular with respect to $\mathcal{K}$; by 412Ab, so is μ.

(b)(i)-(ii) Suppose that every μ_i is (countably) compact. Then for each $i \in I$ we can find a (countably) compact class $\mathcal{K}_i \subseteq \mathcal{P}X_i$ such that μ_i is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{L} = \{\pi_i^{-1}[K] : i \in I, K \in \mathcal{K}_i\}$. Then $\mathcal{L}$ is (countably) compact (451H). So there is a (countably) compact $\mathcal{K} \supseteq \mathcal{L}$ which is closed under finite unions and countable intersections (342Da, 413T). Now μ is inner regular with respect to $\mathcal{K}$, by (a), and therefore (countably) compact.

(iii) Let T_0 be a countably generated σ-subalgebra of $\widehat{\bigotimes}_{i \in I} \Sigma_i$. Then there must be some countable subfamily $\mathcal{E}$ of $\{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$ such that T_0 is included in the σ-algebra generated by $\mathcal{E}$ (use 331Gd). Set $\mathcal{E}_i = \{E : E \in \Sigma_i, \pi_i^{-1}[E] \in \mathcal{E}\}$ for each i, so that $\mathcal{E}_i$ is countable, and let Σ_i' be the σ-algebra generated by $\mathcal{E}_i$. Then $\mu_i \restriction \Sigma_i'$ is compact (451F(ii)). Applying (i), we see that $\mu \restriction \widehat{\bigotimes}_{i \in I} \Sigma_i'$ is compact, therefore perfect; while $\mathrm{T}_0 \subseteq \widehat{\bigotimes}_{i \in I} \Sigma_i'$. As T_0 is arbitrary, $\mu \restriction \widehat{\bigotimes}_{i \in I} \Sigma_i$ is perfect (451F(i)). But as the completion of μ is exactly the completion of $\mu \restriction \widehat{\bigotimes}_{i \in I} \Sigma_i$, μ also is perfect, by 451Gc.

454B Corollary Let $\langle X_i \rangle_{i \in I}$ be a family of Polish spaces with product X. Then any totally finite Baire measure on X is a compact measure.

proof If μ is a Baire measure on X, then its domain $\mathcal{B}\mathfrak{a}(X)$ is $\widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$, where $\mathcal{B}(X_i)$ is the Borel σ-algebra of X_i for each $i \in I$ (4A3Na). So each image measure μ_i on X_i is a Borel measure, therefore tight (that is, inner regular with respect to the closed compact sets, 433Ca), and by 454A(b-i) μ is compact.

454C Theorem (Marczewski & Ryll-Nardzewski 53) Let (X, Σ, μ) be a perfect totally finite measure space and (Y, T, ν) any totally finite measure space. Let $\Sigma \otimes \mathrm{T}$ be the algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma, F \in \mathrm{T}\}$. If $\lambda_0 : \Sigma \otimes \mathrm{T} \to [0, \infty[$ is a non-negative finitely additive functional such that $\lambda_0(E \times Y) = \mu E$ and $\lambda_0(X \times F) = \nu F$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$, then λ_0 has a unique extension to a measure defined on the σ-algebra $\Sigma \widehat{\otimes} \mathrm{T}$ generated by $\Sigma \otimes \mathrm{T}$.

proof (a) By 413Lb, it will be enough to show that $\lim_{n \to \infty} \lambda_0 W_n = 0$ for every non-increasing sequence $\langle W_n \rangle_{n \in \mathbb{N}}$ in $\Sigma \otimes \mathrm{T}$ with empty intersection. Take such a sequence. Each W_n must belong to the algebra generated by some finite subset of $\{E \times F : E \in \Sigma, F \in \mathrm{T}\}$, so there must be a countable set $\mathcal{E} \subseteq \Sigma$ such that every W_n belongs to the algebra generated by $\{E \times F : E \in \mathcal{E}, F \in \mathrm{T}\}$; let Σ_0 be the σ-subalgebra of Σ generated by $\mathcal{E}$, so that every W_n belongs to $\Sigma_0 \widehat{\otimes} \mathrm{T}$.

(b) By 451F, $\mu \restriction \Sigma_0$ is a compact measure; let $\mathcal{K} \subseteq \mathcal{P}X$ be a compact class such that $\mu \restriction \Sigma_0$ is inner regular with respect to $\mathcal{K}$. We may suppose that $\mathcal{K}$ is the family of closed sets for a compact topology on X (342Da). Let $\mathcal{W}$ be the family of those elements W of $\Sigma_0 \otimes \mathrm{T}$ such that every horizontal section $W^{-1}[\{y\}]$ belongs to $\mathcal{K}$. Then $\mathcal{W}$ is closed under finite unions and intersections.

(c) If $W \in \Sigma_0 \otimes \mathrm{T}$ and $\epsilon > 0$, then there is a $W' \in \mathcal{W}$ such that $W' \subseteq W$ and $\lambda_0(W \setminus W') \le \epsilon$. **P** Express W as $\bigcup_{i \le n} E_i \times F_i$, where $E_i \in \Sigma_0$ and $F_i \in \mathrm{T}$ for each $i \le n$. (Cf. 315Kb.) For each $i \le n$, take $K_i \in \mathcal{K} \cap \Sigma_0$ such that $\mu(E_i \setminus K_i) \le \frac{1}{n+1}\epsilon$, and set $W' = \bigcup_{i \le n} K_i \times F_i$. Then $W' \in \mathcal{W}$, $W' \subseteq W$ and

$$\begin{aligned}\lambda_0(W \setminus W') &\le \sum_{i=0}^{n} \lambda_0((E_i \times F_i) \setminus (K_i \times F_i)) \le \sum_{i=0}^{n} \lambda_0((E_i \setminus K_i) \times Y) \\ &= \sum_{i=0}^{n} \mu_0(E_i \setminus K_i) \le \epsilon. \text{ } \mathbf{Q}\end{aligned}$$

(d) Take any $\epsilon > 0$. Then for each $n \in \mathbb{N}$ we can find $W_n' \in \mathcal{W}$ such that $W_n' \subseteq W_n$ and $\lambda_0(W_n \setminus W_n') \le 2^{-n}\epsilon$. Set $V_n = \bigcap_{i \le n} W_i'$, so that $V_n \in \mathcal{W}$ and

$$\lambda_0(W_n \setminus V_n) \le \sum_{i=0}^{n} \lambda_0(W_i \setminus W_i') \le 2\epsilon$$

for each n, and $\langle V_n\rangle_{n\in\mathbb{N}}$ is non-increasing, with empty intersection.

Because $V_n \in \Sigma_0 \otimes \mathrm{T}$, its projection $H_n = V_n[X]$ belongs to T, for each n. Of course $\langle H_n\rangle_{n\in\mathbb{N}}$ is non-increasing; also $\bigcap_{n\in\mathbb{N}} H_n = \emptyset$. **P** If $y \in Y$, then $\langle V_n^{-1}[\{y\}]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, because $\bigcap_{n\in\mathbb{N}} V_n \subseteq \bigcap_{n\in\mathbb{N}} W_n$ is empty. But $\mathcal{K}$ is a compact class, so there must be some n such that $V_n^{-1}[\{y\}]$ is empty, that is, $y \notin H_n$. **Q** Accordingly $\lim_{n\to\infty} \nu H_n = 0$. But as $V_n \subseteq X \times H_n$, $\lim_{n\to\infty} \lambda_0 V_n = 0$.

This means that $\lim_{n\to\infty} \lambda_0 W_n \le 2\epsilon$. But as ϵ is arbitrary, $\lim_{n\to\infty} \lambda_0 W_n = 0$, as required.

454D Theorem (Kolmogorov 1933, §III.4) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of totally finite perfect measure spaces. Set $X = \prod_{i\in I} X_i$, and write $\bigotimes_{i\in I} \Sigma_i$ for the algebra of subsets of X generated by $\{\pi_i^{-1}[E] : i \in I,\, E \in \Sigma_i\}$, where $\pi_i : X \to X_i$ is the coordinate map for each $i \in I$. Suppose that $\lambda_0 : \bigotimes_{i\in I} \Sigma_i \to [0,\infty[$ is a non-negative finitely additive functional such that $\lambda_0\pi_i^{-1}[E] = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$. Then λ_0 has a unique extension to a measure λ with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$, and λ is perfect.

proof (a) The argument follows the same pattern as that of 454C. This time, take a non-increasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\bigotimes_{i\in I} \Sigma_i$ with empty intersection. Each W_n belongs to the algebra generated by some finite subset of $\{\pi_i^{-1}[E] : i \in I,\, E \in \Sigma_i\}$, so we can find countable sets $\mathcal{E}_i \subseteq \Sigma_i$ such that every W_n belongs to the subalgebra generated by $\{\pi_i^{-1}[E] : i \in I,\, E \in \mathcal{E}_i\}$. Let T_i be the σ-subalgebra of Σ_i generated by $\mathcal{E}_i$, so that every W_n belongs to $\bigotimes_{i\in I} \mathrm{T}_i$.

(b) For each $i \in I$, $\mu_i{\restriction}\mathrm{T}_i$ is compact (451F); let $\mathfrak{T}_i$ be a compact topology on X_i such that $\mu_i{\restriction}\mathrm{T}_i$ is inner regular with respect to the closed sets (342F). Let $\mathfrak{T}$ be the product topology on X, so that $\mathfrak{T}$ is compact (3A3J). Let $\mathcal{W}$ be the family of closed sets in X belonging to $\bigotimes_{i\in I} \mathrm{T}_i$.

(c) If $W \in \bigotimes_{i\in I} \mathrm{T}_i$ and $\epsilon > 0$, there is a $W' \in \mathcal{W}$ such that $W' \subseteq W$ and $\lambda_0(W \setminus W') \le \epsilon$. **P** We can express W as $\bigcup_{k\le n} \bigcap_{i\in J_k} \pi_i^{-1}[E_{ki}]$ where each J_k is a finite subset of I and $E_{ki} \in \Sigma_i$ for $k \le n$, $i \in J_k$ (again as in 315Kb). Let $\langle\epsilon_{ki}\rangle_{k\le n, i\in J_k}$ be a family of strictly positive numbers with sum at most ϵ. For each $k \le n$, $i \in J_k$ take a closed set $K_{ki} \in \mathrm{T}_i$ such that $K_{ki} \subseteq E_{ki}$ and $\mu_i(E_{ki} \setminus K_{ki}) \le \epsilon_{ki}$, and set $W' = \bigcup_{k\le n} \bigcap_{i\in J_k} \pi_i^{-1}[K_{ki}]$. **Q**

(d) Take any $\epsilon > 0$. Then for each $n \in \mathbb{N}$ we can find $W_n' \in \mathcal{W}$ such that $W_n' \subseteq W_n$ and $\lambda_0(W_n \setminus W_n') \le 2^{-n}\epsilon$. Set $V_n = \bigcap_{i\le n} W_i'$. Then $\langle V_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets in the compact space X, and has empty intersection, so there is some n such that V_n is empty, and

$$\lambda_0 W_n \le \sum_{i=0}^{n} \lambda_0(W_i \setminus W_i') \le 2\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty} \lambda_0 W_n = 0$.

(e) As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, λ_0 has a unique countably additive extension to $\widehat{\bigotimes}_{i\in I}\Sigma_i$, by 413Lb, as before. Of course the extension is perfect, by 454A(b-iii).

454E Corollary Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of perfect measure spaces. Let $\mathcal{C}$ be the family of subsets of $X = \prod_{i\in I} X_i$ expressible in the form $X \cap \bigcap_{i\in J} \pi_i^{-1}[E_i]$ where $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for every $i \in I$, writing $\pi_i(x) = x(i)$ for $x \in X$, $i \in I$. Suppose that $\lambda_0 : \mathcal{C} \to \mathbb{R}$ is a functional such that (i) $\lambda_0\pi_i^{-1}[E] = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$ (ii) $\lambda_0 C = \lambda_0(C \cap \pi_i^{-1}[E]) + \lambda_0(C \setminus \pi_i^{-1}[E])$ whenever $C \in \mathcal{C}$, $i \in I$ and $E \in \Sigma_i$. Then λ_0 has a unique extension to a measure on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, which is necessarily perfect.

proof By 326E, λ_0 has an extension to an additive functional on $\bigotimes_{i\in I} \Sigma_i$, so we can apply 454D.

454F Corollary Let $\langle(X_i, \Sigma_i)\rangle_{i\in I}$ be a family of standard Borel spaces. Set $X = \prod_{i\in I} X_i$, and let $\bigotimes_{i\in I} \Sigma_i$ be the algebra of subsets of X generated by $\{\pi_i^{-1}[E] : i \in I,\, E \in \Sigma_i\}$, where $\pi_i : X \to X_i$ is the coordinate map for each i. Let $\lambda_0 : \bigotimes_{i\in I} \Sigma_i \to [0,\infty[$ be a non-negative finitely additive functional such that all the marginal functionals $E \mapsto \lambda_0\pi_i^{-1}[E] : \Sigma_i \to [0,\infty[$ are countably additive. Then λ_0 has a unique extension to a measure defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, which is a compact measure.

proof This follows immediately from 454D and 454A if we note that all the measures $\lambda_0\pi_i^{-1}$ are necessarily compact, therefore perfect (451M).

454G Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i a σ-algebra of subsets of X_i for each $i\in I$. Suppose that for each finite set $J\subseteq I$ we are given a totally finite measure μ_J on $Z_J=\prod_{i\in J}X_i$ with domain $\widehat{\bigotimes}_{i\in J}\Sigma_i$ such that (i) whenever J, K are finite subsets of I and $J\subseteq K$, then the canonical projection from Z_K to Z_J is inverse-measure-preserving (ii) every marginal measure $\mu_{\{i\}}$ on $Z_{\{i\}}\cong X_i$ is perfect. Then there is a unique measure μ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that the canonical projection $\tilde\pi_J:\prod_{i\in I}X_i\to Z_J$ is inverse-measure-preserving for every finite $J\subseteq I$.

proof All we need to observe is that

$$\bigotimes_{i\in I}\Sigma_i=\{\tilde\pi_J^{-1}[V]:J\in[I]^{<\omega},\,V\in\bigotimes_{i\in J}\Sigma_i\}.$$

Because all the canonical projections from X_K onto X_J are inverse-measure-preserving, we have $\mu_JV=\mu_KV'$ whenever J, K are finite subsets of I, $V\in\bigotimes_{i\in J}\Sigma_i$, $V'\in\bigotimes_{i\in K}\Sigma_i$ and $\tilde\pi_J^{-1}[V]=\tilde\pi_K^{-1}[V']$. So we have a functional $\lambda_0:\bigotimes_{i\in I}\Sigma_i\to[0,\infty[$ such that $\lambda_0\tilde\pi_J^{-1}[V]=\mu_JV$ whenever $J\subseteq I$ is finite and $V\in\bigotimes_{i\in J}\Sigma_i$. It is easy to check that λ_0 is finitely additive and satisfies the conditions of 454D. So λ_0 can be extended to a measure μ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$.

If $J\subseteq I$ is finite, then μ_J and $\mu\tilde\pi_J^{-1}$ agree on $\bigotimes_{i\in J}\Sigma_J$ and therefore (by the Monotone Class Theorem, 136C) on $\widehat{\bigotimes}_{i\in J}\Sigma_i$; that is, $\tilde\pi_J$ is inverse-measure-preserving. To see that μ itself is unique, observe that the conditions define its values on $\bigotimes_{i\in I}\Sigma_i$ and therefore on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, by the Monotone Class Theorem once more.

454H Corollary Let $\langle(X_n,\Sigma_n)\rangle_{n\in\mathbb{N}}$ be a sequence of standard Borel spaces. For each $n\in\mathbb{N}$ set $Z_n=\prod_{i<n}X_i$ and $\mathrm{T}_n=\widehat{\bigotimes}_{i<n}\Sigma_i$. (For $n=0$, we have $Z_0=\{\emptyset\}$, $\mathrm{T}_0=\{\emptyset,Z_0\}$.) For $n\in\mathbb{N}$, $W\in\mathrm{T}_{n+1}$ and $z\in Z_n$ write $W[\{z\}]=\{\xi:\xi\in X_n,\,(z,\xi)\in W\}$; set $X=\prod_{n\in\mathbb{N}}X_n$ and write $\tilde\pi_n$ for the canonical projection of X onto Z_n. Suppose that for each $n\in\mathbb{N}$ and $z\in Z_n$ we are given a probability measure ν_z on X_n with domain Σ_n such that $z\mapsto\nu_z(E)$ is T_n-measurable for every $E\in\Sigma_n$.

(a) We have a sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ of probability measures such that, for each $n\in\mathbb{N}$, μ_n has domain T_n and

$$\mu_{n+1}(W)=\int\nu_zW[\{z\}]\mu_n(dz)$$

for every $W\in\mathrm{T}_{n+1}$, and

$$\int fd\mu_{n+1}=\iint\ldots\iint f(\xi_0,\ldots,\xi_n)\nu_{(\xi_0,\ldots,\xi_{n-1})}(d\xi_n)\,\nu_{(\xi_0,\ldots,\xi_{n-2})}(d\xi_{n-1})\ldots\nu_{\xi_0}(d\xi_1)\nu_\emptyset(d\xi_0)$$

for every $n\in\mathbb{N}$ and μ_{n+1}-integrable real-valued function f.

(b) There is a unique probability measure μ on $X=\prod_{n\in\mathbb{N}}X_n$, with domain $\widehat{\bigotimes}_{n\in\mathbb{N}}\Sigma_n$, such that μ_n is the image measure $\mu\tilde\pi_n^{-1}$ on Z_n for every $n\in\mathbb{N}$

proof (a) Of course this is an induction on n.

(i) μ_0 must be the unique probability measure on the singleton set Z_0. Given that $\operatorname{dom}\mu_n=\mathrm{T}_n$, the class $\mathcal{W}$ of sets $W\subseteq Z_{n+1}$ for which $\int\nu_zW[\{z\}]\mu_n(dz)$ is defined will contain all sets of the form $\prod_{i\le n}E_i$ where $E_i\in\Sigma_i$ for every $i\le n$, just because the function $z\mapsto\nu_zE_n$ is T_n-measurable. Since $\mathcal{W}$ is closed under increasing sequential unions and differences of comparable sets, the Monotone Class Theorem (136B) tells us that it includes the σ-algebra generated by the cylinder sets, which is T_{n+1}.

(ii) As for the integrals, we start with the elementary case $Z_1=X_0$, $\mu_1=\nu_\emptyset$ and

$$\int fd\mu_1=\int f(\xi_0)\nu_\emptyset(d\xi_0)$$

for $f:X_0\to\mathbb{R}$. For the inductive step to $n\ge 1$, $\langle\nu'_z\rangle_{z\in Z_n}$ is a disintegration of μ_{n+1} over μ_n, where ν'_z is the measure with domain T_{n+1} defined by writing $\nu'_z(W)=\nu_zW[\{z\}]$ for $W\in\mathrm{T}_{n+1}$, $z\in Z_n$. By 452F,

$$\begin{aligned}\int_{Z_{n+1}} f d\mu_{n+1} &= \int_{Z_n} \int_{Z_{n+1}} f(w)\nu'_z(dw)\mu_n(dz)\\ &= \int_{Z_n} \int_{X_n} f(z,\xi_n)\nu_z(d\xi_n)\mu_n(dz)\end{aligned}$$

for every μ_{n+1}-integrable function f. (The second equality can be regarded as an application of the change-of-variable formula 235Gb applied to the (ν_z,ν'_z)-inverse-measure-preserving function $\xi \mapsto (z,\xi) : X_n \to Z_{n+1}$.)

(b)(i) The canonical maps from Z_{n+1} to Z_n are all inverse-measure-preserving, just because every ν_z is a probability measure. We therefore have a well-defined functional $\lambda_0 : \bigotimes_{n\in\mathbb{N}} \Sigma_n \to [0,1]$ defined by setting $\lambda_0\tilde{\pi}_n^{-1}[W] = \mu_n W$ whenever $n \in \mathbb{N}$ and $W \in \bigotimes_{i<n} \Sigma_i$, and this λ_0 is finitely additive; moreover, each marginal measure $\lambda_0\pi_n^{-1}$, where $\pi_n : X \to X_n$ is the coordinate map, is countably additive, because it is expressible as an image measure of μ_{n+1} on X_n.

(ii) Everything so far has been valid for any sequence $\langle(X_n,\Sigma_n)\rangle_{n\in\mathbb{N}}$ of sets with attached σ-algebras. But at this point we note that every marginal measure $\lambda_0\pi_n^{-1}$ must be perfect, because (X_n,Σ_n) is a standard Borel space. So Theorem 454D gives the result.

454I Remarks In 454F and 454H the hypotheses call for 'standard Borel spaces' (X_i,Σ_i). As the proofs make clear, what is needed in each case is that 'every totally finite measure with domain Σ_i must be perfect'. We have already seen other ways in which this can be true: for instance, if X is any Radon Hausdorff space (434C), and Σ its Borel σ-algebra. Further examples are in 454Xd, 454Xh-454Xi and 454Yb-454Yc. Indeed, even weaker hypotheses can be fully adequate. In 454H, for instance, it will be quite enough if all the marginal measures on the factors X_n are perfect; in view of 454A and 451E, this will be so iff all the measures $\tilde{\mu}_n$ on the partial products Z_n are perfect. It may be difficult to be sure of this unless either we have some argument from the nature of the factor spaces (X_n,Σ_n), as suggested above, or a clear understanding of the marginal measures. In applications such as 455A below, however, there may be other approaches available.

454J Distributions of random processes For the next few paragraphs I shift to probabilists' notation.

Proposition Let (Ω,Σ,μ) be a probability space and $\langle X_i\rangle_{i\in I}$ a family of real-valued random variables on Ω (see §271).

(i) There is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that

$$\nu\{x : x \in \mathbb{R}^I,\, x(i_r) \le \alpha_r \text{ for every } r \le n\} = \Pr(X_{i_r} \le \alpha_r \text{ for every } r \le n)$$

whenever $i_0,\dots,i_n \in I$ and $\alpha_0,\dots,\alpha_n \in \mathbb{R}$.

(ii) If $i_0,\dots,i_n \in I$ and $\tilde{\pi}(x) = (x(i_0),\dots,x(i_n))$ for $x \in \mathbb{R}^I$, then the image measure $\nu\tilde{\pi}^{-1}$ on $\mathbb{R}^{n+1}$ is the joint distribution of $X_{i_0},\dots,X_{i_n}$ as defined in 271C.

(iii) ν is a compact measure. If I is countable then ν is a Radon measure.

(iv) If every X_i is defined everywhere on Ω, then the function $\omega \mapsto \langle X_i(\omega)\rangle_{i\in I} : \Omega \to \mathbb{R}^I$ is inverse-measure-preserving for $\hat{\mu}$ and ν, where $\hat{\mu}$ is the completion of μ.

proof (a) Completing μ, and adjusting the X_i on negligible sets, does not change any of the joint distributions of families $X_{i_0},\dots,X_{i_n}$ (271Ad), so we may suppose henceforth that μ is complete and that every X_i is defined on the whole of Ω. Set $\phi(\omega) = \langle X_i(\omega)\rangle_{i\in I}$ for $\omega \in \Omega$. Then $\{F : F \subseteq \mathbb{R}^I,\ \phi^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of $\mathbb{R}^I$ containing $\{x : x(i) \le \alpha\}$ whenever $i \in I$ and $\alpha \in \mathbb{R}$, so includes the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ of $\mathbb{R}^I$ (4A3Na again). If we define $\nu_0 F = \mu\phi^{-1}[F]$ for $F \in \mathcal{B}\mathfrak{a}(\mathbb{R}^I)$, ν_0 is a Baire measure on $\mathbb{R}^I$ for which ϕ is inverse-measure-preserving. We are supposing that μ is complete, so ϕ is still inverse-measure-preserving for μ and the completion ν of ν_0 (234Ba). Since ν_0 is inner regular with respect to the zero sets (412D), so is ν (412Ha), and of course ν measures every Baire set. By 454B, ν_0 is compact, so ν also is (451Ga).

(b) If $i_0,\dots,i_n \in I$ and $\alpha_0,\dots,\alpha_n \in \mathbb{R}$, then

$$\begin{aligned}\Pr(X_{i_r} \le \alpha_r \text{ for every } r \le n) &= \mu\{\omega : X_{i_r}(\omega) \le \alpha_r \text{ for } r \le n\}\\ &= \mu\{\omega : \phi(\omega)(i_r) \le \alpha_r \text{ for } r \le n\}\\ &= \nu\{x : x(i_r) \le \alpha_r \text{ for } r \le n\}.\end{aligned}$$

(c) If $i_0, \dots, i_n \in I$ and we set $\tilde\pi(x) = (x(i_0), \dots, x(i_n))$ for $x \in \mathbb{R}^I$, then $\nu\tilde\pi^{-1}$ is a Radon measure (451O). Since it agrees with the distribution of $X_{i_0}, \dots, X_{i_n}$ on all sets of the form $\{z : z(r) \le \alpha_r \text{ for } r \le n\}$, it must be exactly the distribution of $X_{i_0}, \dots, X_{i_n}$ (271Ba).

(d) If I is countable, then $\mathbb{R}^I$ is Polish, so ν is a Radon measure (433Cb).

(e) The only point I have not covered is the uniqueness of ν. But suppose that ν' is another measure on $\mathbb{R}^I$ with the properties described in (i). If $i_0, \dots, i_n \in I$ and $\tilde\pi(x) = (x(i_0), \dots, x(i_n))$ for $x \in \mathbb{R}^I$, then the image measures $\nu\tilde\pi^{-1}$ and $\nu'\tilde\pi^{-1}$ on $\mathbb{R}^{n+1}$ are both the distribution of $X_{i_0}, \dots, X_{i_n}$, by the argument of (c) above. This means that ν and ν' agree on the algebra of subsets of $\mathbb{R}^I$ generated by sets of the form $\{x : x(i) \in E\}$ where $i \in I$ and $E \subseteq \mathbb{R}$ is Borel. By 4A3Na and 454D, they agree on all zero sets, and must be equal (412M).

454K Definition In the context of 454J, I will call ν the (**joint**) **distribution** of the process $\langle X_i\rangle_{i\in I}$.

Note that if $I = n \in \mathbb{N} \setminus \{0\}$, then ν is a Radon measure on $\mathbb{R}^n$, so is the distribution of $\langle X_i\rangle_{i<n}$ in the sense of 271C.

454L Independence With this extension of the notion of 'distribution' we have a straightforward reformulation of the characterization of independence in 272G.

Theorem Let (Ω, Σ, μ) be a probability space and $\langle X_i\rangle_{i\in I}$ a family of real-valued random variables on Ω, with distribution ν on $\mathbb{R}^I$. Then $\langle X_i\rangle_{i\in I}$ is independent iff ν is the c.l.d. product of the marginal measures on $\mathbb{R}$.

proof (a) For $i \in I$, write ν_i for the marginal measure $\mu\pi_i^{-1}$ on $\mathbb{R}$, taking $\pi_i(x) = x(i)$ as usual. If $J \subseteq I$ is finite, and $\tilde\pi_J(x) = x{\restriction}J$, then $\nu\tilde\pi_J^{-1}$ is the distribution (in the sense of Chapter 27) of $\langle X_i\rangle_{i\in J}$, by 454J(iii). In particular, ν_i is the distribution of X_i for each i.

(b) If ν is the product measure $\prod_{i\in I}\nu_i$, and $J \subseteq I$ is finite, then $\nu\tilde\pi_J^{-1}$ is the product measure $\prod_{i\in J}\nu_i$ (254Oa), so $\langle X_i\rangle_{i\in J}$ is independent (272G). As J is arbitrary, $\langle X_i\rangle_{i\in I}$ is independent (272Bb).

(c) Conversely, if $\langle X_i\rangle_{i\in I}$ is independent, then ν agrees with $\lambda = \prod_{i\in I}\nu_i$ on all sets of the form $\{x : x(i) \le \alpha_i \text{ for } i \in J\}$ where $J \subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in J} \in \mathbb{R}^J$. By the uniqueness assertion in 454J(i), $\nu = \lambda$.

454M The fundamental existence theorem 454G takes a more direct form in this context.

Proposition Let I be a set, and suppose that for each finite $J \subseteq I$ we are given a Radon probability measure ν_J on $\mathbb{R}^J$ such that whenever K is a finite subset of I and $J \subseteq K$, then the canonical projection from $\mathbb{R}^K$ to $\mathbb{R}^J$ is inverse-measure-preserving. Then there is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that the canonical projection from $\mathbb{R}^I$ to $\mathbb{R}^J$ is inverse-measure-preserving for every finite $J \subseteq I$.

proof For finite $J \subseteq I$, let μ_J be the restriction of ν_J to the Borel σ-algebra $\mathcal{B}(\mathbb{R}^J)$. Then the canonical projection from $\mathbb{R}^K$ to $\mathbb{R}^J$ is inverse-measure-preserving for μ_K and μ_J whenever $J \subseteq K$ are finite subsets of I. Moreover, $\mu_{\{i\}}$ is a Borel measure on $\mathbb{R}$, therefore perfect, for every $i \in I$. By 454G, we have a unique Baire probability measure μ on $\mathbb{R}^I$ such that the projections $\mathbb{R}^I \to \mathbb{R}^J$ are (μ, μ_J)-inverse-measure-preserving for all finite $J \subseteq I$. Let ν be the completion of μ; then the projections are (ν, ν_J)-inverse-measure-preserving because ν_J is always the completion of μ_J. Finally, ν is unique because $\nu{\restriction}\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ must have the defining property for μ.

454N We know that Radon measures are often determined by the integrals they give to continuous functions (415I). If we look at distributions we get a stronger result for probability measures.

Proposition Let Ω be a Hausdorff space, μ and ν two Radon probability measures on Ω, and $\langle X_i\rangle_{i\in I}$ a family of continuous functions separating the points of Ω. If μ and ν give $\langle X_i\rangle_{i\in I}$ the same distribution, they are equal.

proof (a) If K and L are disjoint compact subsets of Ω, there is an open set G such that $K\subseteq G\subseteq X\setminus L$ and $\mu G=\nu G$. **P** $W_i=\{(\omega,\omega'):X_i(\omega)\ne X_i(\omega')\}$ is an open subset of $\Omega\times\Omega$, and $\bigcup_{i\in I}W_i$ includes the compact set $K\times L$. So there is a finite set $J\subseteq I$ such that $K\times L\subseteq\bigcup_{i\in J}W_i$. Define $f:\Omega\to\mathbb{R}^J$ by setting $f(\omega)(i)=X_i(\omega)$ for $\omega\in\Omega$ and $i\in J$; then f is continuous, and $f[K]\cap f[L]=\emptyset$. Also the image measures μf^{-1} and νf^{-1} must be the same, because they are both the common distribution of $\langle X_i\rangle_{i\in J}$. Set $G=\Omega\setminus f^{-1}[L]$; this works. **Q**

(b) Now if $E\subseteq\Omega$ is a Borel set, and $\epsilon>0$, there are compact sets $K\subseteq E$, $L\subseteq\Omega\setminus E$ such that $\mu K\ge\mu E-\epsilon$ and $\nu L\ge\nu(\Omega\setminus E)-\epsilon$. Let G be an open set such that $\mu G=\nu G$ and $K\subseteq G\subseteq\Omega\setminus L$. Then

$$\begin{aligned}\mu E&\le\epsilon+\mu K\le\epsilon+\mu G=\epsilon+\nu G\le\epsilon+\nu(\Omega\setminus L)\\&=\epsilon+1-\nu L\le 2\epsilon+1-\nu(\Omega\setminus E)=2\epsilon+\nu E.\end{aligned}$$

As ϵ is arbitrary, $\mu E\le\nu E$; similarly, $\nu E\le\mu E$. As E is arbitrary, μ and ν agree on the Borel sets and must coincide.

454O What distributions determine Suppose that we have two families $\langle X_i\rangle_{i\in I}$, $\langle X_i'\rangle_{i\in I}$ on possibly different probability spaces, and we are told that they have the same distribution. Then $f(\langle X_i\rangle_{i\in I})$ and $f(\langle X_i'\rangle_{i\in I})$ will have the same distribution for any Baire measurable function $f:\mathbb{R}^I\to\mathbb{R}$. More generally, we have the following.

Proposition Let (Ω,Σ,μ), (Ω',Σ',μ') probability spaces, $\langle X_i\rangle_{i\in I}$ a family of random variables on Ω and $\langle X_i'\rangle_{i\in I}$ a family of random variables on Ω', both with the same distribution ν on R^I. Suppose that $\langle I_j\rangle_{j\in J}$ is a family of countable subsets of I, and that for each $j\in I$ we have a Borel measurable function $f_j:\mathbb{R}^{I_j}$ to $\mathbb{R}$. For $j\in J$ define Y_j, Y_j' by saying that

$$Y_j(\omega)=f_j(\langle X_i(\omega)\rangle_{i\in I_j})\text{ for }\omega\in\Omega\cap\bigcap_{i\in I_j}\operatorname{dom}X_i,$$

$$Y_j'(\omega')=f_j(\langle X_i'(\omega')\rangle_{i\in I_j})\text{ for }\omega'\in\Omega'\cap\bigcap_{i\in I_j}\operatorname{dom}X_i'.$$

Then $\langle Y_j\rangle_{j\in J}$ and $\langle Y_j'\rangle_{j\in J}$ have the same distribution.

proof For each $j\in J$ the domain $\Omega\cap\bigcap_{i\in I_j}\operatorname{dom}X_i$ is a conegligible subset of Ω and Y_j is Σ-measurable (418Bd), so Y_j is a real-valued random variable on Ω; similarly, every Y_j' is a real-valued random variable on Ω', and we can speak of the distributions of $\langle Y_j\rangle_{j\in J}$ and $\langle Y_j'\rangle_{j\in J}$. Let ν be the common distribution of $\langle X_i\rangle_{i\in I}$ and $\langle X_i'\rangle_{i\in I}$. For each $i\in I$ let $\hat X_i$ be any extension of X_i to a function from Ω to $\mathbb{R}$, and write $\hat\mu$ for the completion of μ. If $j_0,\dots,j_n\in J$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$,

$$\begin{aligned}\Pr(Y_{j_r}\le\alpha_r\text{ for every }r\le n)&=\mu\{\omega:\omega\in\Omega\cap\bigcap_{r\le n}\bigcap_{i\in I_{j_r}}\operatorname{dom}X_i,\\&\qquad f_{j_r}(\langle X_i(\omega)\rangle_{i\in I_j})\le\alpha_r\text{ for every }r\le n\}\\&=\hat\mu\{\omega:\omega\in\Omega,f_{j_r}(\langle\hat X_i(\omega)\rangle_{i\in I_j})\le\alpha_r\text{ for every }r\le n\}\\&=\nu\{x:x\in\mathbb{R}^I,\ f_{j_r}(x\restriction I_{j_r})\le\alpha_r\text{ for every }r\le n\}\end{aligned}$$

(454J(iv))

$$=\Pr(Y_{j_r}'\le\alpha_r\text{ for every }r\le n).$$

By 454J(i), the distributions of $\langle Y_j\rangle_{j\in J}$ and $\langle Y_j'\rangle_{j\in J}$ coincide.

454P Theorem Let I be a set.

(a) Let ν and ν' be Baire probability measures on $\mathbb{R}^I$ such that $\int e^{if(x)}\nu(dx)=\int e^{if(x)}\nu'(dx)$ for every continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$. Then $\nu=\nu'$.

(b) Let $\langle X_j\rangle_{j\in I}$ and $\langle Y_j\rangle_{j\in I}$ be two families of random variables such that

$$\mathbb{E}(\exp(i\sum_{r=0}^n \alpha_r X_{j_r})) = \mathbb{E}(\exp(i\sum_{r=0}^n \alpha_r Y_{j_r}))$$

whenever $j_0,\dots,j_n \in I$ and $\alpha_0,\dots,\alpha_n \in \mathbb{R}$. Then $\langle X_j\rangle_{j\in I}$ and $\langle Y_j\rangle_{j\in I}$ have the same distribution.

proof (a) For each finite set $J \subseteq I$, write $\tilde\pi_J(x) = x\restriction J$ for $x \in X$. Then we have Radon probability measures μ_J and μ'_J on $\mathbb{R}^J$ defined by saying that $\mu_J F = \nu\tilde\pi_J^{-1}[F]$, $\mu'_J F = \nu'\tilde\pi_J^{-1}[F]$ for Borel sets $F \subseteq \mathbb{R}^J$. If $\langle\alpha_j\rangle_{j\in J} \in \mathbb{R}^J$, then

$$\begin{aligned}\int \exp(i\sum_{j\in J}\alpha_j z(j))\mu_J(dz) &= \int \exp(i\sum_{j\in J}\alpha_j x(j))\nu(dx)\\ &= \int \exp(i\sum_{j\in J}\alpha_j x(j))\nu'(dx) = \int \exp(i\sum_{j\in J}\alpha_j z(j))\mu'_J(dz),\end{aligned}$$

so μ_J and μ'_J have the same characteristic function, therefore are equal (285M). This is true for every J, so ν and ν' are equal, by 454D.

(b) Taking ν and ν' to be the two distributions, (a) (with 454O) tells us that their restrictions to the Baire σ-algebra of $\mathbb{R}^I$ are the same, so they must be identical.

454Q Continuous processes The original, and still by far the most important, context for 454D is when every (X_i,Σ_i) is $\mathbb{R}$ with its Borel σ-algebra, so that $X = \prod_{i\in I} X_i$ can be identified with $\mathbb{R}^I$. In the discussion so far, the set I has been an abstract set, except in the very special case of 454H. But some of the most important applications (to which I shall come in §455) involve index sets carrying a topological structure; for instance, I could be the unit interval $[0,1]$ or the half-line $[0,\infty[$. In such a case, we have a wide variety of subspaces of $\mathbb{R}^I$ (for instance, the space of continuous functions) marked out as special, and it is important to know when, and in what sense, our measures on the product space $\mathbb{R}^I$ can be regarded as, or replaced by, measures on the subspace of interest. In the next few paragraphs I look briefly at spaces of continuous functions on Polish spaces.

Lemma Let T be a separable metrizable space and (X,Σ,μ) a semi-finite measure space. Let $\mathfrak{T}$ be a topology on X such that μ is inner regular with respect to the closed sets.

(a) Let $\phi : X\times T \to \mathbb{R}$ be a function such that (i) for each $x \in X$, $t \mapsto \phi(x,t)$ is continuous (ii) for each $t \in T$, $x \mapsto \phi(x,t)$ is Σ-measurable. Then μ is inner regular with respect to $\mathcal{K} = \{K : K \subseteq X,\ \phi\restriction K\times T$ is continuous$\}$.

(b) Let $\theta : X \to C(T)$ be a function such that $x \mapsto \theta(x)(t)$ is Σ-measurable for every $t \in T$. Give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T. Then θ is almost continuous.

proof The result is trivial if T is empty, so we may suppose that $T \ne \emptyset$.

(a) Take $E \in \Sigma$ and $\gamma < \mu E$; take $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. Let $\mathcal{U}$ be a countable base for the topology of T consisting of non-empty sets, D a countable dense subset of T and $\mathcal{V}$ a countable base for the topology of $\mathbb{R}$. For $U \in \mathcal{U}$, $V \in \mathcal{V}$ set

$$E_{UV} = \{x : \phi(x,t) \in V \text{ for every } t \in U\cap D\};$$

then $E_{UV} \in \Sigma$. Let $\langle\epsilon_{UV}\rangle_{U\in\mathcal{U},V\in\mathcal{V}}$ be a family of strictly positive numbers with sum at most $\mu F - \gamma$. For each $U \in \mathcal{U}$, $V \in \mathcal{V}$ take a closed set $F_{UV} \subseteq F\setminus E_{UV}$ such that $\mu F_{UV} \ge \mu(F\setminus E_{UV}) - \epsilon_{UV}$. Consider

$$K = \bigcap_{U\in\mathcal{U},V\in\mathcal{V}} F_{UV}\cup(F\cap E_{UV}).$$

Then $K \subseteq E$ and $\mu K \ge \gamma$.

If $x \in K$, $t \in T$ and $\phi(x,t) \in V_0 \in \mathcal{V}$, let $V \in \mathcal{V}$ be such that $\phi(x,t) \in V$ and $\overline{V} \subseteq V_0$. Then $\{t' : \phi(x,t') \in V\}$ is an open set containing t, so there is some $U \in \mathcal{U}$ such that $t \in U$ and $\phi(x,t') \in V$ for every $t' \in U$. This means that $x \in E_{UV}$, so that $(K\setminus F_{UV})\times U$ contains (x,t), and is a relatively open set in $K\times T$. If $(x',t') \in (K\setminus F_{UV})\times U$, then $x' \in E_{UV}$, so $\phi(x',t'') \in V$ whenever $t'' \in U\cap D$; as D is dense, $\phi(x',t'') \in \overline{V}$ whenever $t'' \in U$; in particular, $\phi(x',t') \in \overline{V} \subseteq V_0$. This shows that $(K\times T)\cap\phi^{-1}[V_0]$ is relatively open in $K\times T$; as V_0 is arbitrary, $\phi\restriction K\times T$ is continuous.

So $K \in \mathcal{K}$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

(b) Set $\phi(x,t) = \theta(x)(t)$ for $x \in X$, $t \in T$. Because $\theta(x) \in C(T)$ for every x, ϕ is continuous in the second variable; and the hypothesis on θ is just that ϕ is measurable in the first variable. So μ is inner regular with respect to $\mathcal{K}$ as described in (a). But $\theta\restriction K$ is continuous for every $K \in \mathcal{K}$, by 4A2G(g-ii). So θ is almost continuous.

454R Proposition Let T be an analytic metrizable space (e.g., a Polish space, or any Souslin-F subset of a Polish space), and μ a probability measure on $C(T)$ with domain the σ-algebra Σ generated by the evaluation functionals $f \mapsto f(t) : C(T) \to \mathbb{R}$ for $t \in T$. Give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T. Then the completion of μ is a $\mathfrak{T}_c$-Radon measure.

proof If T is empty this is trivial, so let us suppose henceforth that $T \neq \emptyset$.

(a) Let D be a countable dense subset of T. Let $\pi : C(T) \to \mathbb{R}^D$ be the restriction map. Set $X = \pi[C(T)] \subseteq \mathbb{R}^D$; then X, with the topology it inherits from $\mathbb{R}^D$, is a separable metrizable space. Note that, because D is dense, π is injective.

We need to know that π is an isomorphism between $(C(T), \Sigma)$ and $(X, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of X. **P** Since the Borel σ-algebra of $\mathbb{R}^D$ is just the σ-algebra generated by the functionals $g \mapsto g(t) : \mathbb{R}^D \to \mathbb{R}$ as t runs over D (4A3Dc/4A3E), $\mathcal{B}$ is the σ-algebra of subsets of X generated by the functionals $g \mapsto g(t) : X \to \mathbb{R}$ for $t \in D$. So π is surely $(\Sigma, \mathcal{B})$-measurable. On the other hand, if $t \in X$, there is a sequence $\langle t_n \rangle_{n\in\mathbb{N}}$ in D converging to t, so that $\pi^{-1}(g)(t) = \lim_{n\to\infty} g(t_n)$ for every $g \in X$, and $g \mapsto \pi^{-1}(g)(t) : X \to \mathbb{R}$ is $\mathcal{B}$-measurable. Accordingly π^{-1} is $(\mathcal{B}, \Sigma)$-measurable. **Q**

(b) Let $\langle U_n \rangle_{n\in\mathbb{N}}$ be a sequence running over a base for the topology of T, with no U_n empty. For each $n \in \mathbb{N}$, $g \in \mathbb{R}^D$ set

$$\omega_n(g) = \sup_{t,u\in U_n\cap D} \min(1, g(t) - g(u)),$$

so that $\omega_n : \mathbb{R}^D \to [0,1]$ is T-measurable, where T is the Borel (or Baire) algebra of $\mathbb{R}^D$. For $g \in \mathbb{R}^D$, $g \in X$ iff g has an extension to a continuous function on T, that is,

for every $t \in T$, $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $t \in U_n$ and $\omega_n(g) \le 2^{-k}$.

Turning this round, $\mathbb{R}^D \setminus X$ is the projection onto the first coordinate of the set

$$Q = \bigcup_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \{(g,t)\text{: either } t \notin U_n \text{ or } \omega_n(g) > 2^{-k}\} \subseteq \mathbb{R}^D \times T.$$

But (because every U_n is an open set and every ω_n is Borel measurable) Q is a Borel set in the analytic space $\mathbb{R}^D \times T$. So Q and $\mathbb{R}^D \setminus X$ are analytic (423Eb, 423Bb). Since $\mathbb{R}^D$, being Polish (4A2Qc), is a Radon space (434Kb), X is a Radon space (434Fd).

(c) The image measure $\nu = \mu\pi^{-1}$ on X is a Borel probability measure. Because X is a Radon space, ν is tight, and its completion $\hat\nu$ is a Radon measure.

By 454Qb, $\pi^{-1} : X \to C(T)$ is almost continuous if we give $C(T)$ the topology $\mathfrak{T}_c$. So the image measure $\lambda = \hat\nu(\pi^{-1})^{-1}$ is a Radon measure for $\mathfrak{T}_c$ (418I). But of course λ is the completion of μ, just because π is a bijection and $\hat\nu$ is the completion of ν.

454S Corollary Let T be an analytic metrizable space.

(a) $C(T)$, with either the topology $\mathfrak{T}_p$ of uniform convergence on finite subsets of T or the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T, is a measure-compact Radon space.

(b) Let μ be a Baire probability measure on $\mathbb{R}^T$ such that $\mu^* C(T) = 1$. Then the subspace measure $\hat\mu_C$ on $C(T)$ induced by the completion of μ is a Radon measure on $C(T)$ if $C(T)$ is given either $\mathfrak{T}_p$ or $\mathfrak{T}_c$. μ itself is τ-additive and has a unique extension $\tilde\mu$ which is a Radon measure on $\mathbb{R}^T$; $\hat\mu_C$ is the subspace measure on $C(T)$ induced by $\tilde\mu$.

proof (a) Let μ be a probability measure on $C(T)$ which is either a Baire measure or a Borel measure with respect to either $\mathfrak{T}_p$ or $\mathfrak{T}_c$. Let $\tilde\mu$ be the completion of $\mu\restriction\Sigma$, where Σ is the σ-algebra generated by the functionals $f \mapsto f(t)$; because these are $\mathfrak{T}_p$-continuous, Σ is certainly included in the Baire σ-algebra for $\mathfrak{T}_p$, so that $\Sigma \subseteq \operatorname{dom}\mu$. 454R tells us that $\tilde\mu$ is a Radon measure for $\mathfrak{T}_c$. Because $\mathfrak{T}_p$ is a coarser Hausdorff topology, $\tilde\mu$ is also a Radon measure for $\mathfrak{T}_p$. Also $\tilde\mu$ must extend μ, because its domain includes that of μ and the completion of μ must extend $\tilde\mu$ (in fact, of course, this means that $\tilde\mu$ is actually the completion of

μ). Now $\tilde{\mu}$ is τ-additive (for either topology), so μ also is; as μ is arbitrary, $C(T)$ is measure-compact (for either topology). On the other hand, if μ is a Borel measure (for either topology), it must be tight for that topology; so that $C(T)$ is a Radon space.

(b) Write μ_C for the subspace measure on $C(T)$. Recall that the domain Σ of μ is just the σ-algebra generated by the functionals $f \mapsto f(t) : \mathbb{R}^T \to \mathbb{R}$, as t runs over T (4A3Na once more), so that the domain Σ_C of μ_C is the σ-algebra of subsets of $C(T)$ generated by the functionals $f \mapsto f(t) : C(T) \to \mathbb{R}$. By 454R again, the completion of μ_C is a Radon measure on $C(T)$ if we give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T, and therefore also for the coarser Hausdorff topology $\mathfrak{T}_p$. Because the μ_C-negligible sets for μ_C are just the intersections of $C(T)$ with μ-negligible sets (214Cb), the completion of μ_C is the subspace measure $\hat{\mu}_C$ induced by the completion of μ (214Ib).

The embedding $C(T) \subsetneqq \mathbb{R}^T$ is of course continuous for $\mathfrak{T}_c$ and the product topology on $\mathbb{R}^T$, so we have a Radon image measure $\tilde{\mu}$ on $\mathbb{R}^T$ defined by saying that $\tilde{\mu}E = \hat{\mu}_C(E \cap C(T))$ whenever $E \cap C(T)$ is measured by $\hat{\mu}_C$. If $E \in \Sigma$, then

$$\tilde{\mu}E = \hat{\mu}_C(E \cap C(T)) = \mu_C(E \cap C(T)) = \mu^*(E \cap C(T)) = \mu E$$

because $\mu^* C(T) = 1$, so $\tilde{\mu}$ extends μ. Of course $\tilde{\mu}C(T) = 1$ and the subspace measure on $C(T)$ induced by $\tilde{\mu}$ is just $\hat{\mu}_C$.

Finally, because μ has an extension to a Radon measure, it must itself be τ-additive. Because Σ includes a base for the topology of $\mathbb{R}^T$, μ can have only one extension to a Radon measure on $\mathbb{R}^T$ (415H(iv)).

***454T Convergence of distributions (a)** Let I be a set. Write M for the set of distributions on $\mathbb{R}^I$, that is, the set of completions of probability measures with domain $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$. For any $\nu \in M$, the integral $\int f\,d\nu$ is defined for every bounded continuous function $f : \mathbb{R}^I \to \mathbb{R}$, just because such functions are Baire measurable. I will say that the **vague topology** on M is the topology generated by the functionals $\nu \mapsto \int f d\nu$ as f runs over the space $C_b(\mathbb{R}^I)$ of bounded continuous real-valued functions on $\mathbb{R}^I$. (Compare 437Jc.)

(b) The vague topology on M is Hausdorff. **P** If ν, $\nu' \in M$ are different, then $\nu\restriction\mathcal{B}\mathfrak{a}(X) \neq \nu'\restriction\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$. $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ is the σ-algebra generated by the family $\mathcal{Z}$ of zero subsets of $\mathbb{R}^I$ (4A3Kb); by the Monotone Class Theorem (136C), there is an $F \in \mathcal{Z}$ such that $\nu F \neq \nu' F$. Suppose that $\nu F < \nu' F$. Let $f : \mathbb{R}^I \to \mathbb{R}$ be a continuous function such that $F = \{x : x \in \mathbb{R}^I,\ f(x) = 0\}$. Then there is a $\delta > 0$ such that $\nu\{x : x \in \mathbb{R}^I,\ |f(x)| < \delta\} < \nu' F$. Set $g(x) = \min(\delta, |f(x)|)$ for $x \in \mathbb{R}^I$, so that g is a bounded continuous function. Now

$$\int g\,d\nu' \le \delta\nu'\{x : |f(x)| > 0\} < \delta\nu\{x : |f(x)| \ge \delta\} \le \int g\,d\nu$$

and ν, ν' are separated by the vague topology. **Q**

***454U Theorem** Let (Ω, Σ, μ) be a probability space, and I a set. Let M be the set of distributions on $\mathbb{R}^I$; for a family $\boldsymbol{X} = \langle X_i \rangle_{i \in I}$ of real-valued random variables on Ω, let $\nu_{\boldsymbol{X}}$ be its distribution (454K). Then the function $\boldsymbol{X} \mapsto \nu_{\boldsymbol{X}} : \mathcal{L}^0(\mu)^I \to M$ is continuous for the product topology on $\mathcal{L}^0(\mu)^I$ corresponding to the topology of convergence in measure on $\mathcal{L}^0(\mu)$ (245A) and the vague topology on M (454Ta).

proof By the definition of 'vague topology' we need to prove that $\boldsymbol{X} \mapsto \int f d\nu_{\boldsymbol{X}} : \mathcal{L}^0(\mu)^I \to \mathbb{R}$ is continuous for every bounded continuous function $f : \mathbb{R}^I \to \mathbb{R}$.

(a) Consider first the case in which I is countable and we are given $\boldsymbol{Y} = \langle Y_i \rangle_{i \in I}$ and a sequence $\langle \boldsymbol{X}_n \rangle_{n \in \mathbb{N}} = \langle \langle X_{ni} \rangle_{i \in I} \rangle_{n \in \mathbb{N}}$ in $\mathcal{L}^0(\mu)^I$ such that $\langle X_{ni} \rangle_{n \in \mathbb{N}}$ converges a.e. to Y_i for every $i \in I$. Then $\nu_{\boldsymbol{Y}} = \lim_{n\to\infty} \nu_{\boldsymbol{X}_n}$. **P** As I is countable, the set

$$F = \{\omega : \omega \in \bigcap_{n \in \mathbb{N}, i \in I} \operatorname{dom} X_{ni} \cap \operatorname{dom} Y_i,\ Y_i(\omega) = \lim_{n\to\infty} X_{ni}(\omega)\}$$

is μ-conegligible.

Now if $J \subseteq I$ is finite, $E_i \subseteq \mathbb{R}$ is a Borel set for each $i \in J$ and $W = \{z : z \in \mathbb{R}^I,\ z(i) \in E_i \text{ for every } i \in J\}$,

$$\nu_{\boldsymbol{Y}} W = \hat{\mu}(\Omega \cap \bigcap_{i \in J} \{\omega : Y_i(\omega) \in E_i\}) = \hat{\mu} Y^{-1}[W]$$

where $Y(\omega) = \langle Y_i(\omega) \rangle_{i \in I}$ for $\omega \in F$ and $\hat{\mu}$ is the completion of μ. By the Monotone Class Theorem, $\nu_{\boldsymbol{Y}} W = \hat{\mu} Y^{-1}[W]$ for every $W \in \widehat{\bigotimes}_{i \in I} \mathcal{B}(\mathbb{R}) = \mathcal{B}\mathfrak{a}(\mathbb{R}^I)$. So if $f : \mathbb{R}^I \to \mathbb{R}$ is a bounded continuous function, $\int f d\nu_{\boldsymbol{Y}} = \int fY\,d\mu$.

Similarly, if we set $X_n(\omega) = \langle X_{ni}(\omega)\rangle_{i\in I}$ for $\omega \in \Omega$, $\int f d\nu_{\boldsymbol{X}_n} = \int fX_n d\mu$ for every n. But now observe that $Y(\omega) = \lim_{n\to\infty} X_n(\omega)$ in $\mathbb{R}^I$ for every $\omega \in F$, so $fY(\omega) = \lim_{n\to\infty} fX_n(\omega)$ for every $\omega \in F$, and

$$\int f d\nu_{\boldsymbol{Y}} = \int fY d\mu = \lim_{n\to\infty} fX_n d\mu = \lim_{n\to\infty} \int f d\nu_{\boldsymbol{X}_n}$$

by Lebesgue's Dominated Convergence Theorem. As f is arbitrary, $\nu_{\boldsymbol{Y}} = \lim_{n\to\infty} \nu_{\boldsymbol{X}_n}$. **Q**

(b) Next suppose that I is countable and we are given $\boldsymbol{Y} = \langle Y_i\rangle_{i\in I}$ and a sequence $\langle \boldsymbol{X}_n\rangle_{n\in\mathbb{N}} = \langle\langle X_{ni}\rangle_{i\in I}\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0(\mu)^I$ such that $\langle X_{ni}\rangle_{n\in\mathbb{N}}$ converges in measure to Y_i for every $i \in I$. Then $\nu_{\boldsymbol{Y}} = \lim_{n\to\infty} \nu_{\boldsymbol{X}_n}$. **P?** Otherwise, I is surely not empty and there are a continuous bounded $f : \mathbb{R}^I \to \mathbb{R}$ and an $\epsilon > 0$ such that $J = \{n : n \in \mathbb{N}, |\int f d\nu_{\boldsymbol{X}_n} - \int f d\nu_{\boldsymbol{Y}}| \ge \epsilon\}$ is infinite. Let $\langle i_k\rangle_{k\in\mathbb{N}}$ be a sequence running over I, and $\langle m(n)\rangle_{n\in\mathbb{N}}$ a strictly increasing sequence in J such that

$$\hat\mu\{\omega : |X_{m(n),i_k}(\omega) - Y_{i_k}(\omega)| \ge 2^{-n}\} \le 2^{-n}$$

whenever $k \le n \in \mathbb{N}$. Now $Y_i =_{\text{a.e.}} \lim_{n\to\infty} X_{m(n),i}$ for every $i \in I$, so $\nu_{\boldsymbol{Y}} = \lim_{n\to\infty} \nu_{\boldsymbol{X}_{m(n)}}$, by (a); but $|\int f d\nu_{\boldsymbol{X}_{m(n)}} - \int f d\nu_{\boldsymbol{Y}}| \ge \epsilon$ for every n. **XQ**

(c) This shows that if I is countable, $\boldsymbol{X} \mapsto \nu_{\boldsymbol{X}}$ is sequentially continuous. But as the topology of convergence in measure on $\mathcal{L}^0(\mu)$ is pseudometrizable (see 245Eb), the product topology on $\mathcal{L}^0(\mu)^I$ is also pseudometrizable (4A2Lh), and sequentially continuous functions on $\mathcal{L}^0(\mu)^I$ are continuous (4A2Ld). So in this case $\boldsymbol{X} \mapsto \nu_{\boldsymbol{X}}$ is continuous.

(d) This deals with the case of countable I. For the general case, given I and a continuous bounded $f : \mathbb{R}^I \to \mathbb{R}$, there are a countable set $J \subseteq I$ and a continuous $g : \mathbb{R}^J \to \mathbb{R}$ such that $f = g\pi_J$ where $\pi_J(x) = x{\restriction}J$ for every $x \in \mathbb{R}^I$ (put 4A2E(a-iii) and 4A2F(b-ii) together). For any $\boldsymbol{X} = \langle X_i\rangle_{i\in I} \in \mathcal{L}^0(\Sigma)^I$ with distribution $\nu_{\boldsymbol{X}}$, write $\boldsymbol{X}'$ for $\langle X_i\rangle_{i\in J}$ and $\nu_{\boldsymbol{X}'}$ for its distribution. Then $\pi_J : \mathbb{R}^I \to \mathbb{R}^J$ is inverse-measure-preserving for $\nu_{\boldsymbol{X}}$ and $\nu_{\boldsymbol{X}'}$. **P** If $K \subseteq J$ is finite and $E_i \in \mathcal{B}(\mathbb{R})$ for $i \in K$,

$$\begin{aligned}\nu_{\boldsymbol{X}}\pi_J^{-1}\{y : y \in \mathbb{R}^J,\, y(i) \in E_i \text{ for } i \in K\} &= \nu_{\boldsymbol{X}}\{x : x \in \mathbb{R}^I,\, x(i) \in E_i \text{ for } i \in K\}\\ &= \mu\{\omega : \omega \in \Omega,\, X_i(\omega) \in E_i \text{ for } i \in K\}\\ &= \nu_{\boldsymbol{X}'}\{y : y \in \mathbb{R}^J,\, y(i) \in E_i \text{ for } i \in K\}.\end{aligned}$$

By the Monotone Class Theorem (136C), $\nu_{\boldsymbol{X}}\pi_J^{-1}[H] = \nu_{\boldsymbol{X}'}H$ for every $H \in \mathcal{B}\mathfrak{a}(\mathbb{R}^J)$; as $\nu_{\boldsymbol{X}}$ is complete and $\nu_{\boldsymbol{X}'}$ is the completion of its restriction to $\mathcal{B}\mathfrak{a}(\mathbb{R}^J)$, π_J is inverse-measure-preserving. **Q**

It follows that

$$\int f d\nu_{\boldsymbol{X}} = \int g\pi_J d\nu_{\boldsymbol{X}'} = \int g\, d\nu_{\boldsymbol{X}'}$$

(235G), and this is true for every $\boldsymbol{X} \in \mathcal{L}^0(\mu)^I$.

Now observe that $\boldsymbol{X} \mapsto \boldsymbol{X}' = \boldsymbol{X}{\restriction}J : L^0(\mu)^I \to L^0(\mu)^J$ is continuous, while $\boldsymbol{X}' \to \int g\, d\nu_{\boldsymbol{X}'}$ is continuous by (a)-(c) above; so $\boldsymbol{X} \mapsto \int f d\nu_{\boldsymbol{X}}$ is continuous. As f is arbitrary. $\boldsymbol{X} \mapsto \nu_{\boldsymbol{X}}$ is continuous, and the result is true in this case also.

***454V** In this volume I am deliberately leaving some of the central concerns of Volume 3 to one side. But the concept of 'joint distribution' has a natural, and in some contexts important, alternative expression in the language of §364, as follows.

(a)(i) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, I is a set, and $u \in L^0(\mathfrak{A})^I$, we have a sequentially order-continuous Boolean homomorphism $E \mapsto [\![u \in E]\!] : \mathcal{B}\mathfrak{a}(\mathbb{R}^I) \to \mathfrak{A}$ defined by saying that

$$[\![u \in \{x : x \in \mathbb{R}^I,\, x(i) \le \alpha\}]\!] = [\![u(i) \le \alpha]\!]$$

whenever $i \in I$ and $\alpha \in \mathbb{R}$. **P** It is enough to consider the case in which $\mathfrak{A} = \Sigma/\mathcal{I}$, where Σ is a σ-algebra of subsets of a set X and $\mathcal{I}$ is a σ-ideal of Σ. In this case each $u(i)$ can be identified with the equivalence class of a Σ-measurable function $f_i : X \to \mathbb{R}$ (364C); setting $f(x) = \langle f_i(x)\rangle_{i\in I}$ for $x \in X$, f is $(\Sigma, \mathcal{B}\mathfrak{a}(\mathbb{R}^I))$-measurable (4A3Ne) and we have a corresponding function $E \mapsto f^{-1}[E]^\bullet : \mathcal{B}\mathfrak{a}(\mathbb{R}^I) \to \mathfrak{A}$ which has the required properties. Since $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ is the σ-algebra generated by $\mathcal{E} = \{\{x : x(i) \le \alpha\} : i \in I, \alpha \in \mathbb{R}\}$, there is only one sequentially order-continuous homomorphism with the right values on $\mathcal{E}$. **Q**

(ii) If $h:\mathbb{R}^I\to\mathbb{R}$ is a Baire measurable function, there is a function $\bar h:L^0(\mathfrak{A})^I\to L^0(\mathfrak{A})$ defined by saying that $[\![\bar h(u)\in E]\!]=[\![u\in h^{-1}[E]]\!]$ for every Borel set $E\subseteq\mathbb{R}$. **P** $E\mapsto[\![u\in h^{-1}[E]]\!]$ is a sequentially order-continuous Boolean homomorphism so we can use 364F. **Q**

(b) Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra, I is a set and $u\in L^0(\mathfrak{A})^I$. Then there is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that

$$\nu\{x:x\in\mathbb{R}^I,\,x(i)\in E_i\text{ for every }i\in J\}=\bar\mu(\inf_{i\in J}[\![u(i)\in E_i]\!])$$

whenever $J\subseteq I$ is finite and $E_i\subseteq\mathbb{R}$ is a Borel set for every $i\in J$. **P** Express $(\mathfrak{A},\bar\mu)$ as the measure algebra of a probability space (Ω,Σ,μ) (321J), and for each $i\in I$ choose a measurable function $X_i:\Omega\to\mathbb{R}$ such that $u(i)$ can be identified with $X_i^\bullet\in L^0(\mu)$ (364Ic); now the distribution ν of $\langle X_i\rangle_{i\in I}$, as defined in 454J-454K, has the required properties. The argument of part (e) of the proof of 454J still applies, so ν is unique. **Q**

(c) In this context, I will call ν the **(joint) distribution** of u. (Compare 364Gb[13].)

(d) Translating 454O into this language, we get the following. Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A}',\bar\mu')$ be probability algebras, and $u\in L^0(\mathfrak{A})^I$, $u'\in L^0(\mathfrak{A}')^I$ families with the same distribution. Suppose that $\langle h_j\rangle_{j\in J}$ is a family of Baire measurable functions from $\mathbb{R}^I$ to $\mathbb{R}$. Then $\langle\bar h_j(u)\rangle_{j\in J}$ and $\langle\bar h_j(u')\rangle_{j\in J}$ have the same distribution. **P** If $J\subseteq I$ is non-empty and finite and E_j is a Borel subset of $\mathbb{R}$ for $j\in J$,

$$\bar\mu(\inf_{j\in J}[\![h_j(u)\in E_j]\!])=\bar\mu([\![u\in\textstyle\bigcap_{j\in J}h_j^{-1}[E_j]]\!])=\nu(\displaystyle\bigcap_{j\in J}h_j^{-1}[E_j])$$

(where ν is the common distribution of u and u')

$$=\bar\mu'(\inf_{j\in J}[\![h_j(u')\in E_j]\!]).\ \mathbf{Q}$$

(e) Similarly, if $(\mathfrak{A},\bar\mu)$ is a probability algebra, I a set, and we write ν_u for the distribution of $u\in L^0(\mathfrak{A})^I$, $u\mapsto\nu_u$ is continuous for the product topology on $L^0(\mathfrak{A})^I$ corresponding to the topology of convergence in measure on $L^0(\mathfrak{A})$ (367L) and the vague topology on the space M of distributions on $\mathbb{R}^I$. **P** It is enough to consider the case in which $(\mathfrak{A},\bar\mu)$ is the measure algebra of a probability space (Ω,Σ,μ), so that $L^0(\mathfrak{A})$ can be identified with $L^0(\mu)$. Let $\phi:L^0(\mathfrak{A})\to\mathcal{L}^0(\mu)$ be any function such that $\phi(w)^\bullet=w$ for every $w\in L^0(\mathfrak{A})$. Then ϕ is continuous for the topologies of convergence in measure on $L^0(\mathfrak{A})\cong L^0(\mu)$ and $\mathcal{L}^0(\mu)$ (see 245B) and the corresponding map $u\mapsto\phi u=\langle\phi(u(i))\rangle_{i\in I}:L^0(\mathfrak{A})^I\to\mathcal{L}^0(\mu)^I$ is continuous. As $[\![w\in E]\!]=\{\omega:\phi(w)(\omega)\in E\}^\bullet$ for any $w\in L^0(\mathfrak{A})$ and $E\in\mathcal{B}(\mathbb{R})$, the distributions ν_u, $\nu_{\phi u}$ of u and ϕu are the same for every $u\in L^0(\mathfrak{A})^I$. Since $\boldsymbol{X}\mapsto\nu_{\boldsymbol{X}}:\mathcal{L}^0(\mu)^I\to M$ is continuous (454U), so is $u\mapsto\nu_u:L^0(\mathfrak{A})^I\to M$. **Q**

454X Basic exercises >(a) Let μ be Lebesgue measure on $[0,1]$, and Σ its domain. Let X_0, $X_1\subseteq[0,1]$ be disjoint sets of full outer measure. For each i, let Σ_i be the relative σ-algebra on X_i. Show that we have a finitely additive functional λ defined on $\Sigma_0\otimes\Sigma_1$ by the formula

$$\lambda((E\cap X_0)\times(F\cap X_1))=\mu(E\cap F)\text{ for all }E,\,F\in\Sigma,$$

and that λ has no extension to a measure on $X_0\times X_1$.

(b) Adapt the example of 419K to provide a counter-example for 454G if we omit the hypothesis that the marginal measures $\mu_{\{i\}}$ must be perfect.

(c) Adapt the example of 419K/454Xb to provide a counter-example for 454H if we omit the hypothesis that the (X_n,Σ_n) must be standard Borel spaces. (*Hint*: if $z\in\prod_{i\le n}X_i$, try $\nu_z(E)=1$ if $z(n)\in E$, 0 otherwise.)

[13]Formerly 364Xd.

>(d) Let X be a set and Σ a σ-algebra of subsets of X. Let us say that (X,Σ) has the **perfect measure property** if every totally finite measure with domain Σ is perfect. Show that (i) if (X,Σ) has the perfect measure property, so does (E,Σ_E) for any $E\in\Sigma$, where Σ_E is the subspace σ-algebra on E (ii) if $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ is a family of spaces with the perfect measure property, then $(\prod_{i\in I}X_i,\widehat{\bigotimes}_{i\in I}\Sigma_i)$ has the perfect measure property.

(e) Let (X,Σ) be a space with the perfect measure property, and T the smallest σ-algebra including Σ and closed under Souslin's operation. Show that (X,T) has the perfect measure property.

(f) Let X be a set and Σ a σ-algebra of subsets of X. Let us say that (X,Σ) has the **countably compact measure property** if every totally finite measure with domain Σ is countably compact. Show that (i) if (X,Σ) has the countably compact measure property it has the perfect measure property (ii) if (X,Σ) has the countably compact measure property so does (E,Σ_E) for every $E\in\Sigma$, where Σ_E is the subspace σ-algebra on E (iii) if $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ is a family of spaces with the countably compact measure property, then $(\prod_{i\in I}X_i,\widehat{\bigotimes}_{i\in I}\Sigma_i)$ has the countably compact measure property.

(g) Suppose that (X,Σ) has the countably compact measure property. (i) Let μ be a totally finite measure with domain Σ, (Y,T,ν) a measure space, and $f:X\to Y$ an inverse-measure-preserving function. Show that μ has a disintegration $\langle\mu_y\rangle_{y\in Y}$ over ν which is consistent with f. (ii) Let Y be any set, T a σ-algebra of subsets of Y, and λ a probability measure with domain $\Sigma\widehat{\otimes}\mathrm{T}$. Show that there is a family $\langle\mu_y\rangle_{y\in Y}$ of probability measures on X such that $\lambda W=\int\mu_yW^{-1}[\{y\}]\nu(dy)$ for every $W\in\Sigma\widehat{\otimes}\mathrm{T}$, where ν is the marginal measure of λ on Y. (*Hint*: 452M.)

(h)(i) Let X be any set, and Σ the countable-cocountable algebra on X. Show that (X,Σ) has the countably compact measure property. (ii) Show that any standard Borel space has the countably compact measure property.

(i) Let X be a Radon Hausdorff space, and Σ_{um} the algebra of universally measurable sets in X (434D). Show that (X,Σ_{um}) has the countably compact measure property.

>(j) Let $\langle X_i\rangle_{i\in I}$ be an independent family of normal random variables. Show that its distribution is a quasi-Radon measure on $\mathbb{R}^I$. (*Hint*: 415E.)

(k) Give an example of a metrizable space Ω with a continuous injective function $X:\Omega\to[0,1]$ and two different quasi-Radon probability measures μ, ν on Ω giving the same distribution to the random variable X.

(l) Let I be a set and ν, ν' two quasi-Radon measures on $\mathbb{R}^I$ such that $\int e^{if(x)}\nu(dx)=\int e^{if(x)}\nu'(dx)$ for every continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$. Show that $\nu=\nu'$.

>(m) Let Σ be the σ-algebra of subsets of $C([0,\infty[)$ generated by the functionals $f\mapsto f(t)$ for $t\ge0$. Give $C([0,\infty[)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact sets. (i) Show that $\mathfrak{T}_c$ is Polish, and that $\Sigma\cap\mathfrak{T}_c$ is a base for $\mathfrak{T}_c$ which generates Σ as σ-algebra. (ii) Use this to give a quick proof of 454R in this case.

(n) Let T be a Polish space, and $\mathfrak{T}_c$ the topology on $C(T)$ of uniform convergence on compact sets. Show that if $\mathfrak{T}$ is any Hausdorff topology on $C(T)$, coarser than $\mathfrak{T}_c$, such that all the functionals $f\mapsto f(t)$, for $t\in T$, are Baire measurable for $\mathfrak{T}$, then $(C(T),\mathfrak{T})$ is a measure-compact Radon space.

454Y Further exercises (a) In 454Ab, show that μ is weakly α-favourable (definition: 451V) if every μ_i is.

(b) Let Σ be the algebra of Lebesgue measurable subsets of $\mathbb{R}$. Show that $(\mathbb{R},\Sigma)$ has the perfect measure property (454Xd) iff $\mathfrak{c}$ is measure-free.

(c) Let $\mathcal{B}$ be the Borel σ-algebra of ω_1 with its order topology. Show that $(\omega_1, \mathcal{B})$ has the perfect measure property. (*Hint*: 439Xn.)

(d) Let (X, Σ, μ) be a semi-finite measure space with a topology such that μ is inner regular with respect to the closed sets, T a second-countable space and Y a separable metrizable space. Suppose that $\phi : X \times T \to Y$ is continuous in the second variable and measurable in the first, as in 454Q. Show that μ is inner regular with respect to $\mathcal{K} = \{K : K \subseteq X,\ \phi\restriction K \times T \text{ is continuous}\}$.

454 Notes and comments 454A generalizes Theorem 451J, which gave the same result (with essentially the same proof) for product measures. One of the themes of this section is the idea that we can deduce properties of measures on product spaces from properties of their marginal measures, that is, the image measures on the factors. The essence of 'compactness', 'countable compactness' and 'perfectness' is that we can find enough points in the measure space to do what we want. (See, for instance, the characterization of local compactness in 343B, or Pachl's characterization of countable compactness in 452Ye.) Since the canonical feature of a product space is that we put in every point the Axiom of Choice provides us with, it's perhaps not surprising that such properties can be inherited by measures on product spaces.

Theorems 454C and 454D can be regarded as further variations on the same theme. A finitely additive non-negative functional on an algebra of sets will have an extension to a measure if, and only if, it is sequentially smooth in the sense that the measures of a decreasing sequence of sets with empty intersection converge to zero (413L). If we have a decreasing sequence of sets, with measures bounded away from zero, but with empty intersection, one interpretation of the phenomenon is that some points which ought to have been present got left out of the sets. What 454D tells us is that perfectness (and countable additivity) of the marginal measures is enough to ensure that there are enough points in the product to stop this happening. In effect, 454C tells us that it will be enough if every marginal but one is perfect.

These results are of course associated with the projective limit constructions in 418M-418Q. In the theorems there we had Radon measures, so that they were actually compact rather than perfect; in return for the stronger hypothesis on the measures, we could handle projective limits corresponding to rather small subsets of the product spaces (see the formulae in 418O-418Q). Just as in §418, the patterns change when we have countable rather than uncountable families to deal with (418P-418Q, 454H).

In 454J-454P, I insist rather arbitrarily that 'the' joint distribution of a family $\langle X_i\rangle_{i\in I}$ of real-valued random variables is the completion of a Baire measure on $\mathbb{R}^I$. Of course all the ideas can also be expressed in terms of the Baire measure itself, but I have sought a formulation which is consistent with the rules set out in §271. When I is countable, we get a Radon measure (454J(iii)), as in the finite-dimensional case. There are other cases in which the distribution is a quasi-Radon measure (454Xj). As always, we can ask whether the distribution is τ-additive; in this case it will have a canonical extension to a quasi-Radon measure (415N). Important examples of this phenomenon are described in 454Xj, 455H and 456O. Because $\mathbb{R}^I$ has a linear topological space structure, we have a notion of 'characteristic function' for any probability measure on $\mathbb{R}^I$ measuring the zero sets, and the characteristic function of a Baire measure determines that measure (454P, 454Xl).

In 454R, $C(T)$, with $\mathfrak{T}_c$, has a countable network (4A2Oe), so the subspace measure μ_C induced by μ on $C(T)$ must be a τ-additive topological measure with respect to $\mathfrak{T}_c$ (414O) and has a unique extension to a quasi-Radon measure on $C(T)$ (415M). The hard bit is the next step, showing that $C(T)$, under $\mathfrak{T}_c$, is a Radon space; this is the real point of 454Q-454R. For the most important case, in which $T = [0, \infty[$, we have a useful simplification, because $\mathfrak{T}_c$ is actually Polish (454Xm). Even in this case, however, we need to observe that the measure we are seeking is a little more complicated than a simple completion of a measure on $\mathbb{R}^T$. We must complete the *subspace* measure on $C(T)$, and $C(T)$ is far from being a measurable set. The measure $\tilde{\mu}$ of 454S will not as a rule be completion regular, for instance. Spaces of continuous functions are so important that it is worth noticing that the results here will be valid for various topologies on $C(T)$ (454Xn).

I suppose that pretty well every result on distributions in Chapter 27 corresponds to some significant development expressible in the language of this section. 454T-454U take up the idea of Exercise 274Yf. Looking at the facts here from the point of view of Volume 3 we get the alternative versions in 454V.

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455 Markov and Lévy processes

For a 'Markov' process, in which the evolution of the system after a time t depends only on the state at time t, the general theory of §454 leads to a straightforward existence theorem (at least for random variables taking values in standard Borel spaces) dependent only on a natural consistency condition on the transitional probabilities (455A, 455E). The formulation leads naturally to descriptions of the 'Markov property' (for stopping times taking only countably many values) in terms of disintegrations and conditional expectations (455C, 455Ec). With appropriate continuity conditions, we find that the process can be represented either by a Radon measure (455H) or by a measure on the set of càdlàg paths (455Gc) for which we have a formulation of the strong Markov property (for general stopping times) in terms of disintegrations (455O). These conditions are satisfied by Lévy processes (455P-455R). For these, we have an alternative expression of the strong Markov property in terms of inverse-measure-preserving functions (455U). By far the most important example of a continuous-time Markov process is Brownian motion, but I defer discussion of this to §477.

455A Theorem Let T be a totally ordered set with least element t^*, and for each $t \in T$ let Ω_t be a non-empty set and T_t a σ-algebra of subsets of Ω_t containing all singleton subsets of Ω_t. Set $\Omega = \prod_{t \in T} \Omega_t$ and for $t \in T$, $\omega \in \Omega$ set $X_t(\omega) = \omega(t)$. Fix $x^* \in \Omega_{t^*}$. Suppose that we are given, for each pair $s < t$ in T, a family $\langle \nu_x^{(s,t)} \rangle_{x \in \Omega_s}$ of perfect probability measures on Ω_t, all with domain T_t, and suppose that

($\dagger$) whenever $s < t < u$ in T and $x \in \Omega_s$, then $\langle \nu_y^{(t,u)} \rangle_{y \in \Omega_t}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$.

For $J \subseteq T$ write π_J for the canonical map from Ω onto $Z_J = \prod_{t \in J} \Omega_t$. Then there is a unique probability measure μ on Ω, with domain $\widehat{\bigotimes}_{t \in T} \mathrm{T}_t$, such that, writing λ_J for the image measure $\mu \pi_J^{-1}$,

$$\begin{aligned} \int f d\lambda_J &= \int f(\omega(t^*), \omega(t_1), \dots, \omega(t_n)) \mu(d\omega) \\ &= \int \dots \iint f(x^*, x_1, \dots, x_n) \nu_{x_{n-1}}^{(t_{n-1}, t_n)}(dx_n) \\ &\qquad\qquad \nu_{x_{n-2}}^{(t_{n-2}, t_{n-1})}(dx_{n-1}) \dots \nu_{x^*}^{(t^*, t_1)}(dx_1) \end{aligned}$$

whenever $t^* < t_1 < \dots < t_n$, $J = \{t^*, t_1, \dots, t_n\}$ and f is λ_J-integrable. μ is perfect, and the marginal measure $\mu_t = \mu X_t^{-1}$ is equal to $\nu_{x^*}^{(t^*,t)}$, if $t > t^*$, while $\mu_{t^*}\{x^*\} = 1$.

proof (a) For $I \subseteq T$, write $\mathrm{T}_I = \widehat{\bigotimes}_{t \in I} \mathrm{T}_t$. If $I = \{t_0, t_1, \dots, t_n\}$ is a finite subset of T with $t^* = t_0 < t_1 < \dots < t_n$, then we have a probability measure λ_I on Z_I with domain T_I such that

$$\begin{aligned} \int f d\lambda_I = \int \dots \iint f(x^*, x_1, \dots, x_n) &\nu_{x_{n-1}}^{(t_{n-1}, t_n)}(dx_n) \\ &\nu_{x_{n-2}}^{(t_{n-2}, t_{n-1})}(dx_{n-1}) \dots \nu_{x^*}^{(t^*, t_1)}(dx_1) \end{aligned}$$

for every λ_I-integrable function f. **P** Use 454Ha on the finite sequence $(\Omega_{t_0}, \dots, \Omega_{t_n})$. The measures ν_z required by 454H must be constructed by the rule

$$\nu_z = \nu_{z(t_m)}^{(t_m, t_{m+1})}$$

for $m < n$, $z \in \prod_{i \le m} \Omega_{t_i}$, while of course $\nu_\emptyset\{x^*\} = 1$. Having a finite sequence rather than an infinite one clearly makes things easier.) **Q**

When $I = \{t^*\}$, so that Z_I can be identified with Ω_{t^*}, I mean to interpret these formulae in such a way that $\lambda_I\{x^*\} = 1$. When $J = \{t^*, t\}$, with $t^* < t$, and $E \in \mathrm{T}_t$, then we can apply the formula above to the function $z \mapsto \chi E(z(t))$ to get $\lambda_J\{z : z(t) \in E\} = \nu_{x^*}^{(t^*,t)}(E)$.

(b) Of course the point of this is that these measures λ_I form a consistent family; if $t^* \in I \subseteq J \in [T]^{<\omega}$, then the canonical projection $\pi_{IJ} : Z_J \to Z_I$ is inverse-measure-preserving. **P** It is enough to consider the case in which J has just one more point than I, since then we can induce on $\#(J \setminus I)$. In this case, express J as $\{t_0, \ldots, t_n\}$ where $t^* = t_0 < \ldots < t_n$, and suppose that $I = J \setminus \{t_m\}$. If $W \in \mathrm{T}_I$, then

$$\begin{aligned}\lambda_J \pi_{IJ}^{-1}[W] &= \int \ldots \iint \ldots \int \chi W(x^*, x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n) \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \\ &\qquad \ldots \nu_{x_m}^{(t_m,t_{m+1})}(dx_{m+1}) \nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m) \ldots \nu_{x^*}^{(t^*,t_1)}(dx_1) \\ &= \int \ldots \iint g_{(x_1,\ldots,x_{m-1})}(x_{m+1}) \nu_{x_m}^{(t_m,t_{m+1})}(dx_{m+1}) \qquad (*) \\ &\qquad \nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m) \ldots \nu_{x^*}^{(t^*,t_1)}(dx_1)\end{aligned}$$

where

$$\begin{aligned}g_{(x_1,\ldots,x_{m-1})}(x_{m+1}) = \int \ldots \int \chi W(x^*, x_1 \ldots, x_{m-1}, x_{m+1}, \ldots, x_n) \\ \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \ldots \nu_{x_{m+1}}^{(t_{m+1},t_{m+2})}(dx_{m+2}).\end{aligned}$$

Here, of course, we use the hypothesis (†); since $\langle \nu_y^{(t_m,t_{m+1})} \rangle_{y \in \Omega_{t_m}}$ is a disintegration of $\nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}$ over $\nu_{x_{m-1}}^{(t_{m-1},t_m)}$, and $g_{(x_1,\ldots,x_{m-1})}$ is bounded and $\nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}$-integrable (by 454H),

$$\begin{aligned}&\int g_{(x_1,\ldots,x_{m-1})}(x_{m+1}) \nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}(dx_{m+1}) \\ &\qquad = \iint g_{(x_1,\ldots,x_{m-1})}(x_{m+1}) \nu_{x_m}^{(t_m,t_{m+1})}(dx_{m+1}) \nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m)\end{aligned}$$

(452F). Substituting this into (*) above,

$$\begin{aligned}\lambda_J \pi_{IJ}^{-1}[W] &= \int \ldots \iint g_{(x_1,\ldots,x_{m-1})}(x_{m+1}) \\ &\qquad \nu_{x_m}^{(t_m,t_{m+1})}(dx_{m+1}) \nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m) \ldots \nu_{x^*}^{(t^*,t_1)}(dx_1) \\ &= \int \ldots \int g_{(x_1,\ldots,x_{m-1})}(x_{m+1}) \nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}(dx_{m+1}) \ldots \nu_{x^*}^{(t^*,t_1)}(dx_1) \\ &= \int \ldots \int \ldots \int \chi W(x^*, x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n) \\ &\qquad \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \ldots \nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}(dx_{m+1}) \ldots \nu_{x^*}^{(t^*,t_1)}(dx_1) \\ &= \lambda_I W,\end{aligned}$$

applying the formula in (a) again. **Q**

(Some of the formulae here are inappropriate if $m = n > 1$. In this case, of course,

$$\begin{aligned}\lambda_J \pi_{IJ}^{-1}[W] &= \int \ldots \int \chi W(x^*, x_1, \ldots, x_{n-1}) \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \ldots \nu_{x^*}^{(t^*,t_1)}(dx_1) \\ &= \int \ldots \int \chi W(x^*, x_1, \ldots, x_{n-1}) \nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1}) \ldots \nu_{x^*}^{(t^*,t_1)}(dx_1) = \lambda_I W.\end{aligned}$$

If $m = 1 < n$, there is a collapse of a different kind; we must look at

$$\begin{aligned}\lambda_J \pi_{IJ}^{-1}[W] &= \int \ldots \iint \chi W(x^*, x_2, \ldots, x_n) \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \ldots \nu_{x_1}^{(t_1,t_2)}(dx_2) \nu_{x^*}^{(t^*,t_1)}(dx_1) \\ &= \int \ldots \iint \chi W(x^*, x_2, \ldots, x_n) \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \ldots \nu_{x^*}^{(t^*,t_2)}(dx_2) = \lambda_I W.\end{aligned}$$

If $m = n = 1$ then

$$\lambda_J \pi_{IJ}^{-1}[W] = \int \chi W(x^*) \nu_{x^*}^{(t^*,t_1)}(dx_1) = \chi W(x^*) = \lambda_I W.)$$

(c) Part (b) tells us that we have a consistent family of measures on the finite products Z_J, and therefore have a functional λ on $\bigotimes_{t\in T} \mathrm{T}_t$ defined by setting $\lambda\pi_J^{-1}[W] = \lambda_J W$ for every finite $J \subseteq T$ containing t^* and $W \in \bigotimes_{t\in J} \mathrm{T}_t$. λ is finitely additive, and its images $\mu_t = \lambda X_t^{-1}$ are all countably additive and perfect because $\mu_t = \nu_{x^*}^{(t^*,t)}$ for $t > t^*$, while μ_{t^*} is concentrated at $\{x^*\}$.

By 454D, we have a perfect measure μ extending λ. We have to check that each λ_J is the image measure $\mu\pi_J^{-1}$; but this is true because they agree on $\bigotimes_{t\in J} \mathrm{T}_t$ (using the Monotone Class Theorem in the form 136C, as always). So the integral formula sought for λ_J is just that obtained in part (a). By the last remark in (a), we have the declared formulae for the marginal measures μ_t.

455B Lemma Suppose that T, t^*, $\langle(\Omega_t, \mathrm{T}_t)\rangle_{t\in T}$, Ω, x^* and $\langle\nu_x^{(s,t)}\rangle_{s<t, x\in\Omega_s}$ are as in 455A.

(a) Suppose that μ is constructed from x^* and $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ as in 455A. If $F \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ is determined by coordinates in $[t^*, t_0]$ and $H^* = \{\omega : \omega(t_i) \in E_i \text{ for } 1 \le i \le n\}$ where $t_0 < t_1 \ldots < t_n$ and $E_i \in \mathrm{T}_{t_i}$ for $1 \le i \le n$, then

$$\mu(H^* \cap F) = \int_F \int \ldots \int \chi H(y_1, \ldots, y_n) \nu_{y_{n-1}}^{(t_{n-1},t_n)}(dy_n) \ldots \nu_{\omega(t_0)}^{(t_0,t_1)}(dy_1) \mu(d\omega) \qquad (*)$$

where $H = \prod_{1\le i\le n} E_i$.

(b) Suppose that $\omega \in \Omega$ and $a \in T \cup \{\infty\}$, where ∞ is taken to be greater than every element of T. For $s < t$ in T and $x \in \Omega_s$ set

$$\begin{aligned}\nu_{\omega ax}^{(s,t)} &= \nu_x^{(s,t)} \text{ if } a < s,\\ &= \nu_{\omega(a)}^{(a,t)} \text{ if } s \le a < t,\\ &= \delta_{\omega(t)}^{(t)} \text{ if } t \le a,\end{aligned}$$

here writing $\delta_x^{(t)}$ for the probability measure with domain T_t such that $\delta_x^{(t)}(\{x\}) = 1$.

(i) $\nu_{\omega ax}^{(s,t)}$ is always a perfect probability measure with domain T_t, and $\langle\nu_{\omega ay}^{(t,u)}\rangle_{y\in\Omega_t}$ is a disintegration of $\nu_{\omega ax}^{(s,u)}$ over $\nu_{\omega ax}^{(s,t)}$ whenever $s < t < u$ in T and $x \in \Omega_s$.

(ii) Taking $\mu_{\omega a}$ to be the measure on Ω defined from $\omega(t^*)$ and $\langle\nu_{\omega ax}^{(s,t)}\rangle_{s<t,x\in\Omega_t}$ by the method of 455A, then $\{\omega' : \omega' \in \Omega,\ \omega'{\restriction}D = \omega{\restriction}D\}$ is $\mu_{\omega a}$-conegligible for every countable $D \subseteq T \cap [t^*, a]$.

(iii) If ω, $\omega' \in \Omega$ and $\omega{\restriction}[t^*, a] = \omega'{\restriction}[t^*, a]$ then $\mu_{\omega a} = \mu_{\omega' a}$.

proof (a)(i) Suppose first that F is of the form $\{\omega : \omega(s_i) \in F_i \text{ for } i \le m\}$ where $t^* = s_0 < \ldots < s_m = t_0$. For $x \in \Omega_{t_0}$ set

$$f(x) = \int \ldots \int \chi H(y_1, \ldots, y_n) \nu_{y_{n-1}}^{(t_{n-1},t_n)}(dy_n) \ldots \nu_x^{(t_0,t_1)}(dy_1).$$

Writing $G = \prod_{i\le m} F_i$, we have

$$\begin{aligned}\mu(H^* \cap F) &= \int \ldots \iint \ldots \int \chi G(x^*, x_1, \ldots, x_m) \chi H(y_1, \ldots, y_n) \nu_{y_{n-1}}^{(t_{n-1},t_n)}(dy_n)\\ &\qquad \ldots \nu_{x_m}^{(s_m,t_1)}(dy_1) \nu_{x_{m-1}}^{(s_{m-1},s_m)}(dx_m) \ldots \nu_{x^*}^{(t^*,s_1)}(dx_1)\\ &= \int \ldots \int \chi G(x^*, x_1, \ldots, x_m) f(x_m) \nu_{x_{m-1}}^{(s_{m-1},s_m)}(dx_m) \ldots \nu_{x^*}^{(t^*,s_1)}(dx_1)\\ &= \int g\, d\lambda_J\end{aligned}$$

(where $J = \{t^*, s_1, \ldots, s_m\}$, $g(z) = \chi G(z(t^*), \ldots, z(s_m)) f(z(s_m))$ for $z \in \prod_{s\in J} \Omega_s$, and λ_J is defined as in 455A)

$$= \int g\pi_J d\mu = \int_F f(\omega(t_0))\mu(d\omega)$$

(because $g\pi_J(\omega) = f(\omega(s_m)) = f(\omega(t_0))$ if $\omega \in F$, 0 otherwise)

$$= \int_F \int \ldots \int \chi H(y_1, \ldots , y_n)\nu^{(t_{n-1},t_n)}_{y_{n-1}}(dy_n) \ldots \nu^{(t_0,t_1)}_{\omega(t_0)}(dy_1)\mu(d\omega).$$

(ii) Let $\mathcal{I}$ be the family of sets F of the type dealt with in (a). Since the intersection of two members of $\mathcal{I}$ belongs to $\mathcal{I}$, the Monotone Class Theorem tells us that (*) is true for all sets in the σ-algebra T generated by $\mathcal{I}$. But any member of $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ determined by coordinates in $[t^*, t_0]$ belongs to T. **P** Fix $v \in \prod_{s\in T\setminus[t^*,t_0]} \Omega_s$. For $\omega \in \Omega$ define $f(\omega) \in \Omega$ by setting

$$f(\omega)(s) = \omega(s) \text{ if } s \le t_0,$$
$$= v(s) \text{ if } s > t_0.$$

Then $\mathrm{T}' = \{F : F \subseteq \Omega,\ f^{-1}[F] \in \mathrm{T}\}$ is a σ-algebra of subsets of Ω containing $\{\omega : \omega(t) \in E\}$ whenever $t \in T$ and $E \in \mathrm{T}_t$, so includes $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$. If $F \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ and F is determined by coordinates in $[t^*, t_0]$, then $F = f^{-1}[F] \in \mathrm{T}$. **Q**

So (*) is true of every $F \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$, as claimed.

(b)(i) Of course every $\nu^{(s,t)}_{\omega ax}$ is a perfect probability measure with domain T_t. If $s < t < u$ and $E \in \mathrm{T}_u$, then

$$\int_{\Omega_t} \nu^{(t,u)}_{\omega ay}(E)\nu^{(s,t)}_{\omega ax}(dy) = \int_{\Omega_t} \nu^{(t,u)}_{y}(E)\nu^{(s,t)}_{x}(dy) = \nu^{(s,u)}_{x}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } a < s,$$
$$= \int_{\Omega_t} \nu^{(t,u)}_{y}(E)\nu^{(a,t)}_{\omega(a)}(dy) = \nu^{(a,u)}_{\omega(a)}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } s \le a < t,$$
$$= \int_{\Omega_t} \nu^{(t,u)}_{\omega(t)}(E)\delta^{(t)}_{\omega(t)}(dy) = \nu^{(t,u)}_{\omega(t)}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } a = t,$$
$$= \int_{\Omega_t} \nu^{(a,u)}_{\omega(a)}(E)\delta^{(t)}_{\omega(t)}(dy) = \nu^{(a,u)}_{\omega(a)}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } t < a < u,$$
$$= \int_{\Omega_t} \delta^{(u)}_{\omega(u)}(E)\delta^{(t)}_{\omega(t)}(dy) = \delta^{(u)}_{\omega(u)}(E) = \nu^{(s,u)}_{\omega ax}(E)$$
$$\text{if } u \le a.$$

(ii) Consider first the case $D = \{t\}$, where $t^* < t \le a$. Then

$$\mu_{\omega a}\{\omega' : \omega'(t) = \omega(t)\} = \nu^{(t^*,t)}_{\omega,a,\omega(t^*)}\{\omega(t)\} = \delta^{(t)}_{\omega(t)}\{\omega(t)\} = 1.$$

As for $D = \{t^*\}$, $\mu_{\omega a}$ starts at $\omega(t^*)$, so (as noted in the last clause of the statement of 455A) $\mu_{\omega a}\{\omega' : \omega'(t^*) = \omega(t^*)\} = 1$.

For general D, we have an intersection of countably many sets of these types, which will be $\mu_{\omega a}$-conegligible.

(iii) Looking at the definition, we see that $\nu^{(s,t)}_{\omega' ax} = \nu^{(s,t)}_{\omega ax}$ for all s, t and x, and of course $\omega'(t^*) = \omega(t^*)$, so $\mu_{\omega' a} = \mu_{\omega a}$.

455C Theorem Suppose that T, t^*, $\langle(\Omega_t, \mathrm{T}_t)\rangle_{t\in T}$, Ω, x^*, $\langle\nu^{(s,t)}_x\rangle_{s<t,x\in\Omega_s}$ and μ are as in 455A. Adjoin a point ∞ to T above any point of T, and let $\tau : \Omega \to T \cup \{\infty\}$ be a function taking countably many values and such that $\{\omega : \tau(\omega) \le s\}$ belongs to $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ and is determined by coordinates in $[t^*, s]$ for every $s \in T$.

(a) For $\omega \in \Omega$ define $\nu^{(s,t)}_{\omega,\tau(\omega),x}$, for $s < t$ and $x \in \Omega_s$, as in 455Bb, and let $\mu_{\omega,\tau(\omega)}$ be the corresponding measure on Ω. Then $\langle \mu_{\omega,\tau(\omega)} \rangle_{\omega \in \Omega}$ is a disintegration of μ over itself.

(b) Let Σ_τ be the set of those $E \in \widehat{\bigotimes}_{t \in T} \mathrm{T}_t$ such that $E \cap \{\omega : \tau(\omega) \le t\}$ is determined by coordinates in $[t^*, t]$ for every $t \in T$. Then Σ_τ is a σ-subalgebra of $\widehat{\bigotimes}_{t \in T} \mathrm{T}_t$. If f is any μ-integrable real-valued function, and we set $g_f(\omega) = \int f d\mu_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$, then g_f is a conditional expectation of f on Σ_τ.

proof (a)(i) Set $F_t = \{\omega : \omega \in \Omega,\, \tau(\omega) = t\}$ for $t \in T \cup \{\infty\}$; note that $F_t \in \widehat{\bigotimes}_{t \in T} \mathrm{T}_t$ for every $t \in T \cup \{\infty\}$, and that F_t is determined by coordinates in $[t^*, t]$ for $t \in T$.

(ii) Consider first the case in which τ takes only finitely many values. Suppose that $J \subseteq T$ is a finite set including $\{t^*\} \cup (T \cap \tau[\Omega])$. Enumerate J as $\langle t_i \rangle_{i \le n}$. Suppose that $E_i \in \mathrm{T}_{t_i}$ for $i \le n$ and set $H^* = \{\omega : \omega(t_i) \in E_i \text{ for every } i \le n\}$. We need to calculate $\int_\Omega \mu_{\omega,\tau(\omega)}(H^*) \mu(d\omega)$.

Set $H = \prod_{i \le n} E_i$,

$$H_j = \prod_{j < i \le n} E_j, \quad H_j^* = \{\omega : \omega \in \Omega,\, \omega(t_i) \in E_i \text{ for } j < i \le n\},$$

$$G_j^* = \{\omega : \omega(t_i) \in E_i \text{ for } i \le j\}$$

for $j \le n$. If $i < n$, $j \le n$, $\omega \in F_{t_j}$ and $x \in \Omega_{t_i}$, then

$$\begin{aligned}
\nu^{(t_i,t_{i+1})}_{\omega,\tau(\omega),x} &= \nu^{(t_i,t_{i+1})}_x \text{ if } i > j, \\
&= \nu^{(t_j,t_{j+1})}_{\omega(t_j)} \text{ if } i = j, \\
&= \delta^{(t_{i+1})}_{\omega(t_{j+1})} \text{ if } i < j.
\end{aligned}$$

So if $j \le n$ and $\omega \in F_{t_j}$,

$$\begin{aligned}
\mu_{\omega,\tau(\omega)}(H^*) &= \int \dots \int \chi H(\omega(t^*), x_1, \dots, x_n) \nu^{(t_{n-1},t_n)}_{\omega,\tau(\omega),x_{n-1}}(dx_n) \dots \nu^{(t^*,t_1)}_{\omega,\tau(\omega),\omega(t^*)}(dx_1) \\
&= \iint \dots \int \chi H(x_0, \dots, x_n) \nu^{(t_{n-1},t_n)}_{\omega,\tau(\omega),x_{n-1}}(dx_n) \dots \nu^{(t^*,t_1)}_{\omega,\tau(\omega),x_0}(dx_1) \delta^{(t^*)}_{\omega(t^*)}(dx_0) \\
&= \iint \dots \iiint \dots \int \chi H(x_0, \dots, x_n) \nu^{(t_{n-1},t_n)}_{\omega,\tau(\omega),x_{n-1}}(dx_n) \\
&\qquad \dots \nu^{(t_{j+1},t_{j+2})}_{\omega,\tau(\omega),x_{j+1}}(dx_{j+2}) \nu^{(t_j,t_{j+1})}_{\omega,\tau(\omega),x_j}(dx_{j+1}) \nu^{(t_{j-1},t_j)}_{\omega,\tau(\omega),x_{j-1}}(dx_j) \\
&\qquad \dots \delta^{(t_1)}_{\omega(t_1)}(dx_1) \delta^{(t^*)}_{\omega(t^*)}(dx_0) \\
&= \int \dots \iiint \dots \int \chi H(x_0, \dots, x_n) \nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n) \\
&\qquad \dots \nu^{(t_j,t_{j+1})}_{x_j}(dx_{j+1}) \nu^{(t_j,t_{j+1})}_{\omega(t_j)}(dx_{j+1}) \delta^{(t_j)}_{\omega(t_j)}(dx_j) \dots \delta^{(t^*)}_{\omega(t^*)}(dx_0) \\
&= \int \dots \int \chi H(\omega(t^*), \dots, \omega(t_j), x_{j+1}, \dots, x_n) \nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n) \\
&\qquad \dots \nu^{(t_j,t_{j+1})}_{\omega(t_j)}(dx_{j+1}) \\
&= \int \dots \int \chi H_j(x_{j+1}, \dots, x_n) \nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n) \dots \nu^{(t_j,t_{j+1})}_{\omega(t_j)}(dx_{j+1}) \\
&\qquad \qquad \text{if } \omega \in G_j^*, \\
&= 0 \text{ otherwise.}
\end{aligned}$$

As noted in 455B(b-ii), $\mu_{\omega,\tau(\omega)}(H^*) = \chi H^*(\omega)$ if $\tau(\omega) = \infty$.

Now

$$\begin{aligned}\int_\Omega \mu_{\omega,\tau(\omega)}(H^*)\mu(d\omega) &= \sum_{j=0}^n \int_{F_{t_j}} \mu_{\omega,\tau(\omega)}(H^*)\mu(d\omega) + \int_{F_\infty} \mu_{\omega,\tau(\omega)}(H^*)\mu(d\omega)\\ &= \sum_{j=0}^n \int_{F_{t_j}\cap G_j^*} \int \ldots \int \chi H_j(x_{j+1},\ldots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\ &\qquad \ldots \nu_{\omega(t_j)}^{(t_j,t_{j+1})}(dx_{j+1})\mu(d\omega) + \mu(F_\infty\cap H^*)\\ &= \sum_{j=0}^n \mu(F_{t_j}\cap G_j^*\cap H_j^*) + \mu(H^*\cap F_\infty)\end{aligned}$$

(by 455Ba)

$$= \sum_{j=0}^n \mu(F_{t_j}\cap H^*) + \mu(F_\infty\cap H^*) = \mu H^*.$$

Thus we have the formula we need when E is of the special form $\{\omega : \omega(t) \in E_t$ for every $t \in J\}$, $J \subseteq T$ being a finite set and E_t being a member of T_t for every $t \in J$. By the Monotone Class Theorem (136B), we shall have $\int \mu_{\omega,\tau(\omega)}(E)\mu(d\omega) = \mu E$ for every $E \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$, so that $\langle\mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of μ over itself.

(iii) If τ takes infinitely many values, enumerate them as $\langle t_n\rangle_{n\in\mathbb{N}}$, and for $n \in \mathbb{N}$ define $\tau_n : T \to T\cup\{\infty\}$ by setting

$$\begin{aligned}\tau_n(\omega) &= t_i \text{ if } i \le n \text{ and } \tau(\omega) = t_i,\\ &= \infty \text{ if } \tau(\omega) \notin \{t_i : i \le n\}.\end{aligned}$$

Then τ_n takes only finitely many values, and $\{\omega : \tau_n(\omega) \le t\} \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ is determined by coordinates in $[0,t]$ for every $t \in T$. So we shall have

$$\int \mu_{\omega,\tau_n(\omega)}(E)\mu(d\omega) = \mu E$$

for every $E \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$. Now observe that $\mu_{\omega,\tau_n(\omega)} = \mu_{\omega,\tau(\omega)}$ whenever $\tau(\omega) = \tau_n(\omega)$. So, for each ω, $\mu_{\omega,\tau_n(\omega)} = \mu_{\omega,\tau(\omega)}$ for all but finitely many n. This means that, for every $E \in \widehat{\bigotimes}_{t\in T}\mathrm{T}_t$,

$$\mu_{\omega,\tau(\omega)}(E) = \lim_{n\to\infty} \mu_{\omega,\tau_n(\omega)}(E)$$

for every $\omega \in \Omega$, and

$$\int \mu_{\omega,\tau(\omega)}(E)\mu(d\omega) = \lim_{n\to\infty} \int \mu_{\omega,\tau_n(\omega)}(E)\mu(d\omega) = \mu E,$$

as required.

(b)(i) Since $\{\omega : \tau(\omega) \le t\}$ is determined by coordinates in $[t^*,t]$ for every $t \in T$, $\Omega \in \Sigma_\tau$, and it is now elementary to confirm that Σ_τ is a σ-algebra.

(ii) I had better note that g_f is defined almost everywhere; this is because, by (a) above and 452F,

$$\int g_f d\mu = \iint f d\mu_{\omega,\tau(\omega)}\mu(d\omega) = \int f d\mu.$$

(iii) If ω, $\omega' \in \Omega$ and $\omega'\restriction[t^*,\tau(\omega)] = \omega\restriction[t^*,\tau(\omega)]$, then $g_f(\omega) = g_f(\omega')$ if either is defined. **P** Since $F_{\tau(\omega)}$ is determined by coordinates in $[t^*,\tau(\omega)]$, $\tau(\omega') = \tau(\omega)$. By 455B(b-ii), $\mu_{\omega',\tau(\omega')} = \mu_{\omega,\tau(\omega)}$, so $g_f(\omega) = g_f(\omega')$ if either is defined. **Q**

(iv) If $F \in \Sigma_\tau$ and $\omega \in \Omega$, then $\mu_{\omega,\tau(\omega)}F = 1$ if $\omega \in F$, 0 otherwise. **P** Setting $b = \tau(\omega)$, $F \cap F_b$ and $F_b \setminus F$ are determined by coordinates in a countable subset of $T \cap [t^*,b]$, so by 455B(b-ii) we have $\mu_{\omega b}F = 1$ if $\omega \in F_b \cap F$ and $\mu_{\omega b}(F_b \setminus F) = 1$ if $\omega \in F_b \setminus F$. **Q**

It follows that if f is μ-integrable and $F \in \Sigma_\tau$, then $g_{f\times\chi F} = g_f \times \chi F$. **P** If $\omega \in F$, then $\mu_{\omega,\tau(\omega)}F = 1$ and

$$g_{f\times\chi F}(\omega)=\int_F f d\mu_{\omega,\tau(\omega)}=\int f d\mu_{\omega,\tau(\omega)};$$

if $\omega\notin F$ then $\mu_{\omega,\tau(\omega)}F=0$ and

$$g_{f\times\chi F}(\omega)=\int_F f d\mu_{\omega,\tau(\omega)}=0.\ \mathbf{Q}$$

(v) Now let f be any μ-integrable real-valued function. Then there is a Σ_τ-measurable function $g'_f:\Omega\to\,]-\infty,\infty]$ such that $g'_f=_{\text{a.e.}}g_f$, $g'_f(\omega)\le g_f(\omega)$ for every $\omega\in\operatorname{dom}g_f$, and $g'_f(\omega)=-\infty$ for every $\omega\in\Omega\setminus\operatorname{dom}g_f$. **P** For $q\in\mathbb{Q}$, set $W_q=\{\omega:\omega\in\operatorname{dom}g_f,\ g_f(\omega)\ge q\}$. For $q\in\mathbb{Q}$ and $b\in\tau[\Omega]$, consider $W_{bq}=W_q\cap F_b$. W_{bq} is measured by the completion $\hat\mu$ of μ, and is determined by coordinates in $T\cap[t^*,b]$, by (iii). By 451K(b-ii) there is a $W'_{bq}\in\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ such that $W'_{bq}\subseteq W_{bq}$, $W_{bq}\setminus W'_{bq}$ is negligible and W'_{bq} is determined by coordinates in $T\cap[t^*,b]$.

Having defined the family $\langle W'_{bq}\rangle_{b\in\tau[\Omega],q\in\mathbb{Q}}$, set $W'_q=\bigcup_{b\in\tau[\Omega]}W'_{bq}$ for $q\in\mathbb{Q}$. Then $W'_q\in\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$ and $W'_q\cap F_b=W'_{bq}$ is determined by coordinates in $T\cap[t^*,b]$ for every $b\in\tau[\Omega]$, so $W'_q\in\Sigma_\tau$. Also $W'_q\subseteq W_q$ and $W_q\setminus W'_q$ is negligible.

Set

$$g'_f(\omega)=\sup\{q:q\in\mathbb{Q},\ \omega\in W'_q\}$$

for $\omega\in\Omega$, counting $\sup\emptyset$ as $-\infty$. Then g'_f is Σ_τ-measurable, $g'_f(\omega)=-\infty$ for $\omega\notin\operatorname{dom}g_f$, $g'_f(\omega)\le g_f(\omega)$ for $\omega\in\operatorname{dom}g_f$, and $g'_f=g_f$ on $\operatorname{dom}g_f\setminus\bigcup_{q\in\mathbb{Q}}W_q\setminus W'_q$, so $g'_f=_{\text{a.e.}}g_f$. **Q**

(vi) Continuing from (v), we find that g'_f is a conditional expectation of f on Σ_τ.[14] **P** I have already shown that g'_f is Σ_τ-measurable. If $F\in\Sigma_\tau$ then

$$\int_F g'_f d\mu=\int g'_f\times\chi F\,d\mu=\int g_f\times\chi F\,d\mu$$

(because $g_f=_{\text{a.e.}}g'_f$)

$$=\int g_{f\times\chi F}d\mu$$

(by (iv))

$$=\iint f\times\chi F\,d\mu_{\omega,\tau(\omega)}\mu(d\omega)=\int f\times\chi F\,d\mu$$

(452F once again)

$$=\int_F f d\mu.\ \mathbf{Q}$$

(vii) Similarly, or applying the arguments of (v)-(vi) to $-f$, we see that for any μ-integrable function f there is a conditional expectation g''_f of f on Σ_τ such that $g''_f(\omega)\ge g_f(\omega)$ when $\omega\in\operatorname{dom}g_f$ and $g''_f(\omega)=\infty$ when $g_f(\omega)$ is undefined. Now $g'_f=_{\text{a.e.}}g''_f$ and both are Σ_τ-measurable. It follows that g_f is defined, and equal to both g'_f and g''_f, $(\mu\restriction\Sigma_\tau)$-a.e.; so that g_f itself is also a conditional expectation of f on Σ_τ.

455D Remarks (a) The idea of the construction in 455A is that $\langle X_t\rangle_{t\in T}$ is a family of random variables, and that we start from the assurance that 'history is irrelevant'; if, at time b, we wish to make guesses about the behaviour of X_t, the state of the system at a future time t, then we expect that it will be useful to look at the current state X_b, but once we know the value of X_b then any further information about X_s for $s<b$ will tell us nothing more about X_t. We are given the **transitional probabilities** $\nu_x^{(s,t)}$, which can be thought of as the conditional distributions of X_t given that $X_s=x$. The condition (†) of 455A is plainly necessary if the system is going to make sense at all; the content of the theorem is that it is also sufficient, at least when all the conditional expectations are perfect measures, to ensure that the system as a whole can indeed be represented as a family of random variables, in Kolmogorov's sense, on a suitable probability space.

[14] The definition of 'conditional expectation' in 233D was directed towards real-valued functions, and g'_f is permitted to take the values $\pm\infty$. So what I really mean here is that the restriction of g'_f to the set on which it is finite is a conditional expectation of f.

(b) The statement '$\langle\mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of μ over itself' in 455Ca is not obviously a target worth working very hard for. But the point of this particular family is that not only does $\mu_{\omega,\tau(\omega)}$ follow ω up to and including time $\tau(\omega)$ (455B(b-ii)), but also $\mu_{\omega,\tau(\omega)} = \mu_{\omega',\tau(\omega')}$ whenever $\omega'\restriction[t^*,\tau(\omega)] = \omega\restriction[t^*,\tau(\omega)]$, as noted in (b-iii) of the proof of 455B.

If we take τ in 455C to be constant, with value $b \in T$, then we get a precise description of what it means for 'history to be irrelevant'. In this case, we can take the measures $\mu_{\omega b}$, and project them onto $\prod_{t\ge b}\Omega_t$; let $\lambda^{(\omega)}_{[b,\infty[}$ be the image measure. Then it is easy to check that $\lambda^{(\omega)}_{[b,\infty[}$ is the measure defined from the point $\omega(b)$ and the family $\langle\nu^{(s,t)}_x\rangle_{b\le s<t,x\in\Omega_s}$ by the method of 455A; so that $\lambda^{(\omega)}_{[b,\infty[} = \lambda^{(\omega')}_{[b,\infty[}$ whenever $\omega(b) = \omega'(b)$.

(c) I have called 455C a 'theorem', and there are certainly enough ideas in it to warrant the title. But the restriction to stopping times taking only countably many values means that we are a large step away from a result which is really useful in continuous time. The calculations with sets $\{\omega : \tau(\omega) = b\}$ in the proofs of 455C and 455E are a clear sign that we are not yet ready for continuous stopping times, in which $\{\omega : \tau(\omega) = b\}$ will usually be negligible for every b, except perhaps $b = \infty$. Of course we can use 455C with $T = \mathbb{N}$; but it must be obvious that there are better and cleaner expressions of the result in this case. In the work below, 455C is going to function as a lemma, the first stage in much stronger results (starting with 455O) which depend on special properties of the measures $\nu^{(s,t)}_x$.

(d) In the context of 455A, it seemed to involve fewer explanations to take a fixed σ-algebra T_t for each t and to define μ on $\widehat{\bigotimes}_{t\in T}\mathrm{T}_t$. As you know, I ordinarily have a strong prejudice in favour of completing measures. In the situations most important to us, this is perfectly straightforward, if a touch laborious; I present a version in the next theorem.

455E Theorem Let T be a totally ordered set with least element t^*. Let $\langle\Omega_t\rangle_{t\in T}$ be a family of Hausdorff spaces; suppose that we are given an $x^* \in \Omega_{t^*}$ and, for each pair $s < t$ in T, a family $\langle\nu^{(s,t)}_x\rangle_{x\in\Omega_s}$ of Radon probability measures on Ω_t such that

$\langle\nu^{(t,u)}_y\rangle_{y\in\Omega_s}$ is a disintegration of $\nu^{(s,u)}_x$ over $\nu^{(s,t)}_x$ whenever $s < t < u$ in T and $x \in \Omega_s$.

Write $\Omega = \prod_{t\in T}\Omega_t$; for $t \in T$ let $\mathcal{B}(\Omega_t)$ be the Borel σ-algebra of Ω_t, and $X_t : \Omega \to \Omega_t$ the canonical map; for $J \subseteq T$ write π_J for the canonical map from Ω onto $\prod_{t\in J}\Omega_t$. For $t \in T$ and $x \in \Omega_t$ let $\delta^{(t)}_x$ be the Dirac measure on Ω_t concentrated at x.

(a) There is a unique complete probability measure $\hat\mu$ on Ω, inner regular with respect to $\widehat{\bigotimes}_{t\in T}\mathcal{B}(\Omega_t)$, such that, writing $\hat\lambda_J$ for the image measure $\hat\mu\pi_J^{-1}$,

$$\begin{aligned}\int f d\hat\lambda_J &= \int f(\omega(t^*),\omega(t_1),\dots,\omega(t_n))\hat\mu(d\omega)\\ &= \int\dots\iint f(x^*,x_1,\dots,x_n)\nu^{(t_{n-1},t_n)}_{x_{n-1}}(dx_n)\\ &\qquad\qquad\nu^{(t_{n-2},t_{n-1})}_{x_{n-2}}(dx_{n-1})\dots\nu^{(t^*,t_1)}_{x^*}(dx_1)\end{aligned}$$

whenever $t^* < t_1 < \dots < t_n$ in T, $J = \{t^*,t_1,\dots,t_n\}$ and f is $\hat\lambda_J$-integrable. In particular, the image measure $\hat\mu X_t^{-1}$ is equal to $\nu^{(t^*,t)}_{x^*}$ if $t > t^*$, and to $\delta^{(t^*)}_{x^*}$ if $t = t^*$.

(b)(i) For $\omega \in \Omega$ and $a \in T \cup \{\infty\}$ define $\langle\nu^{(s,t)}_{\omega a x}\rangle_{s<t,x\in\Omega_s}$ by setting

$$\begin{aligned}\nu^{(s,t)}_{\omega a x} &= \nu^{(s,t)}_x \text{ if } a < s,\\ &= \nu^{(a,t)}_{\omega(a)} \text{ if } s \le a < t,\\ &= \delta^{(t)}_{\omega(t)} \text{ if } t \le a.\end{aligned}$$

The family $\langle\nu^{(s,t)}_{\omega a x}\rangle_{s<t,x\in\Omega_s}$, together with the point $\omega(t^*) \in \Omega_{t^*}$, satisfy the conditions of (a), so can be used to define a complete measure $\hat\mu_{\omega a}$ on Ω.

(ii) If $\omega \in \Omega$ and $D \subseteq T \cap [t^*,a]$ is countable, then $\hat\mu_{\omega a}\{\omega' : \omega'\restriction D = \omega\restriction D\} = 1$.

(iii) If ω, $\omega' \in \Omega$ and $\omega'\restriction[t^*,a] = \omega\restriction[t^*,a]$, then $\hat\mu_{\omega' a} = \hat\mu_{\omega a}$.

(c) Let Σ be the domain of $\hat{\mu}$. Suppose that $\tau : \Omega \to T \cup \{\infty\}$ is a function taking countably many values and such that $\{\omega : \tau(\omega) \le t\}$ belongs to Σ and is determined by coordinates in $[t^*, t]$ for every $t \in T$.

(i) $\langle \hat{\mu}_{\omega,\tau(\omega)} \rangle_{\omega \in \Omega}$ is a disintegration of $\hat{\mu}$ over itself.

(ii) Let Σ_τ be the set

$$\{E : E \in \Sigma,\ E \cap \{\omega : \tau(\omega) \le t\} \text{ is determined by coordinates in } [t^*, t] \text{ for every } t \in T\}.$$

Then Σ_τ is a σ-subalgebra of Σ. If f is any $\hat{\mu}$-integrable real-valued function, and we set $g_f(\omega) = \int f d\hat{\mu}_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$, then g_f is a conditional expectation of f on Σ_τ.

proof My aim is to apply 455A-455C to the Borel measures $\grave{\nu}_x^{(s,t)} = \nu_x^{(s,t)} \restriction \mathcal{B}(\Omega_t)$, and take $\hat{\mu}$ to be the completion of the Baire measure μ produced by the method of 455A. The essential discipline is to check carefully that almost every measure ζ is the completion of an appropriate measure $\grave{\zeta}$.

(a) At the start, every Radon probability measure is the completion of the corresponding Borel measure, so that the $\nu_x^{(s,t)}$ are indeed the completions of the $\grave{\nu}_x^{(s,t)}$ defined from them. Since completing a measure does not affect the associated integration (212Fb), the condition

whenever $s < t < u$ in T, $x \in \Omega_s$ and $E \subseteq \Omega_u$ is a Borel set, then $\grave{\nu}_s^{(s,u)}(E) = \int \grave{\nu}_y^{(t,u)}(E) \grave{\nu}_x^{(s,t)}(dy)$

follows at once from

whenever $s < t < u$ in T and $x \in \Omega_s$, then $\langle \nu_y^{(t,u)} \rangle_{y \in \Omega_s}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$.

Also the $\grave{\nu}_x^{(s,t)}$, being tight Borel measures, are all perfect (342L/451C). So we can indeed form a measure μ on Ω with domain $\widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$ by the process in 455A, and complete it.

The next step has a little more content in it: I need to show that for any $J \subseteq T$, the image measure $\hat{\mu}\pi_J^{-1}$ on $\prod_{t \in J} \Omega_t$ is the completion of the image measure $\mu\pi_J^{-1}$. But here we just have to recall that μ is perfect (454D), so that we can use 451Kb. For finite $J \subseteq T$ we can therefore write $\hat{\lambda}_J$ indifferently for the completion of $\lambda_J = \mu\pi_J^{-1}$ and for $\hat{\mu}\pi_J^{-1}$, and the formula for $\int f d\hat{\lambda}_J$ can be read off from 455A, since it deals only with integrals, which are unaffected by completions.

(b) This follows 455Bb. This time we must start by noting that every $\nu_{\omega a x}^{(s,t)}$ is a Radon probability measure.

(i) The formulae of part (i) of the proof of 455Bb can still be applied to show that

$$\int_{\Omega_t} \nu_{\omega a y}^{(t,u)}(E) \nu_{\omega a x}^{(s,t)}(dy) = \nu_{\omega a x}^{(s,u)}(E)$$

whenever $s < t < u$, $x \in \Omega_s$ and $E \in \mathcal{B}(\Omega_u)$. Since any set measured by $\nu_{\omega a x}^{(s,u)}$ can be approximated internally and externally by Borel sets, we see that $\langle \nu_{\omega a y}^{(t,u)} \rangle_{y \in \Omega_t}$ is a disintegration of $\nu_{\omega a x}^{(s,u)}$ over $\nu_{\omega a x}^{(s,t)}$. (Cf. 452Xg.)

(ii) Similarly, the argument of part (ii) of the proof of 455Bb can still be used to show that whenever $\omega \in \Omega$ and $D \subseteq T \cap [t^*, a]$ is countable, then $\omega' \restriction D = \omega \restriction D$ for $\hat{\mu}_{\omega a}$-almost every $\omega' \in \Omega$.

(iii) Once again, we can use the argument from 455B; if $\omega' \restriction [t^*, a] = \omega \restriction [t^*, a]$, then $\nu_{\omega' a x}^{(s,t)} = \nu_{\omega a x}^{(s,t)}$ for all x, s and t, and $\hat{\mu}_{\omega' a} = \hat{\mu}_{\omega a}$.

(c)(i)(α) The key step here is to observe that there is a function $\grave{\tau} : \Omega \to T \cup \{\infty\}$ which satisfies the properties required in 455C and is equal $\hat{\mu}$-almost everywhere to τ. **P** For each $a \in T \cap \tau[\Omega]$, $F_a = \tau^{-1}[\{a\}]$ belongs to Σ and is determined by coordinates in $[t^*, a]$. By 451K(b-ii) again, there is an $F'_a \in \widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$ such that $F'_a \subseteq F_a$, F'_a is determined by coordinates in $[t^*, a]$ and $\hat{\mu}(F_a \setminus F'_a) = 0$. Define $\grave{\tau}$ by setting

$$\grave{\tau}(\omega) = a \text{ if } a \in T \cap \tau[\Omega] \text{ and } \omega \in F'_a,$$
$$= \infty \text{ if } \omega \in \Omega \setminus \bigcup_{a \in T \cap \tau[\Omega]} F'_a.$$

It is easy to check that this $\grave{\tau}$ will serve. **Q**

(β) For $\omega \in \Omega$ and $a \in T$, define $\langle \grave{\nu}^{(s,t)}_{\omega a x} \rangle_{s<t, x \in \Omega_t}$ and $\langle \mu_{\omega a} \rangle_{\omega \in \Omega}$ from $\langle \grave{\nu}^{(s,t)}_x \rangle_{s<t, x \in \Omega_t}$ and $\grave{\tau}$ as in 455Bb. If $\tau(\omega) = \grave{\tau}(\omega)$ then $\grave{\nu}^{(s,t)}_{\omega, \grave{\tau}(\omega), x} = \nu^{(s,t)}_{\omega, \tau(\omega), x} \restriction \mathcal{B}(\Omega_t)$ for all s, t and x, so that $\hat{\mu}_{\omega, \tau(\omega)}$ is the completion of $\mu_{\omega, \grave{\tau}(\omega)}$. This is true for almost all ω. Now we know from 455Ca that $\langle \mu_{\omega, \grave{\tau}(\omega)} \rangle_{\omega \in \Omega}$ is a disintegration of μ over itself, and therefore also over $\hat{\mu}$. It follows that $\langle \hat{\mu}_{\omega, \grave{\tau}(\omega)} \rangle_{\omega \in \Omega}$ is a disintegration of $\hat{\mu}$ over $\hat{\mu}$, by 452B(ii). But $\hat{\mu}_{\omega, \grave{\tau}(\omega)} = \hat{\mu}_{\omega, \tau(\omega)}$ for $\hat{\mu}$-almost every ω, so $\langle \hat{\mu}_{\omega, \tau(\omega)} \rangle_{\omega \in \Omega}$ also is a disintegration of $\hat{\mu}$ over itself.

(ii)(α) Just as in part (b-i) of the proof of 455C, Σ_τ is a σ-algebra because it contains Ω.

(β) Recall the F_a, F'_a in (i-α) above. Set $F_\infty = \tau^{-1}[\{\infty\}]$, and take $F'_\infty \in \widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$ such that $F'_\infty \subseteq F_\infty$ and $F_\infty \setminus F'_\infty$ is negligible. Then $F^* = \bigcup_{a \in \tau[\Omega]} F'_a$ is conegligible in Ω. Write $\grave{\Sigma}_{\grave{\tau}}$ for the set of those $F \in \widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$ such that $F \cap F'_a$ is determined by coordinates in $[t^*, a]$ for every $a \in T \cap \grave{\tau}[\Omega]$. Then $F^* \in \grave{\Sigma}_{\grave{\tau}} \cap \Sigma_\tau$ because $F^* \cap F'_a = F^* \cap F_a = F'_a$ for every $a \in \tau[\Omega]$. In fact we have more. First, $\tau \restriction F^* = \grave{\tau} \restriction F^*$. Next, if $F \subseteq F^*$ and $F \in \grave{\Sigma}_{\grave{\tau}}$, then $F \in \Sigma_\tau$. **P** For any $a \in T \cap \tau[\Omega]$, $F \cap F_a = F \cap F'_a$ belongs to $\widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t) \subseteq \Sigma$ and is determined by coordinates in $[t^*, a]$. **Q** And thirdly, if $F \in \Sigma_\tau$, there is a $G \in \grave{\Sigma}_{\grave{\tau}}$ such that $G \subseteq F$ and $F \setminus G$ is negligible. **P** As in (iv-α), we can find for each $a \in \tau[\Omega]$ a set $G_a \in \widehat{\bigotimes}_{t \in T} \mathcal{B}(\Omega_t)$, determined by coordinates in $T \cap [t^*, a]$, such that $G_a \subseteq F \cap F_a$ and $(F \cap F_a) \setminus G_a$ is negligible. Set $G = \bigcup_{a \in \tau[\Omega]} G_a$. **Q**

(γ) Now take a $\hat{\mu}$-integrable function f. Then it is μ-integrable. By 455Cb, $\grave{g}_f$ is a conditional expectation of f on $\grave{\Sigma}_{\grave{\tau}}$, where

$$\grave{g}_f(\omega) = \int f d\mu_{\omega, \grave{\tau}(\omega)} = \int f d\hat{\mu}_{\omega, \grave{\tau}(\omega)}$$

whenever the integral is defined in $\mathbb{R}$. We know that there is a $\grave{\Sigma}_{\grave{\tau}}$-measurable function $g' : \Omega \to \mathbb{R}$ equal to $\grave{g}_f$ except perhaps on a negligible set H belonging to $\grave{\Sigma}_{\grave{\tau}}$. Replacing g' by $g' \times \chi F^*$ and H by $H \cup (\Omega \setminus F^*)$ if necessary, we can suppose that g' is zero outside F^* and that $\Omega \setminus H \subseteq F^*$. In this case, g' is Σ_τ-measurable. **P** For any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \{\omega : g'(\omega) \ge \alpha\} &= \{\omega : \omega \in F^*, \, g'(\omega) \ge \alpha\} \cup (\Omega \setminus F^*) \text{ if } \alpha \le 0, \\ &= \{\omega : \omega \in F^*, \, g'(\omega) \ge \alpha\} \text{ if } \alpha > 0, \end{aligned}$$

and in either case belongs to Σ_τ, by (β). **Q** At the same time, we note that $H \in \Sigma_\tau$.

If $\omega \in \Omega \setminus H$, then $\omega \in F^*$, $\tau(\omega) = \grave{\tau}(\omega)$, $\hat{\mu}_{\omega, \tau(\omega)} = \hat{\mu}_{\omega, \grave{\tau}(\omega)}$ and

$$g_f(\omega) = \int f d\hat{\mu}_{\omega, \tau(\omega)} = \int f d\hat{\mu}_{\omega, \grave{\tau}(\omega)} = \int f d\mu_{\omega, \grave{\tau}(\omega)} = \grave{g}_f(\omega) = g'(\omega).$$

So g_f is defined and equal to g' and $\grave{g}_f$ except perhaps on the negligible set H belonging to Σ_τ; consequently g_f is defined $(\hat{\mu} \restriction \Sigma_\tau)$-a.e. and is $(\hat{\mu} \restriction \Sigma_\tau)$-virtually measurable.

If $F \in \Sigma_\tau$, there is a $G \in \grave{\Sigma}_{\grave{\tau}}$ such that $G \subseteq F$ and $F \setminus G$ is negligible, by the last remark in (β). So

$$\int_F f d\hat{\mu} = \int_G f d\mu = \int_G \grave{g}_f d\mu$$

(because $\grave{g}_f$ is a conditional expectation of f on $\grave{\Sigma}_{\grave{\tau}}$)

$$= \int_F \grave{g}_f d\hat{\mu} = \int_F g_f d\hat{\mu} = \int_F g_f d(\hat{\mu} \restriction \Sigma_\tau).$$

As F is arbitrary, g_f is a conditional expectation of f on Σ_τ, and the proof is complete.

455F Of course the leading example for the work above is the case in which $T = [0, \infty[$ and $\Omega_t = \mathbb{R}$ for every $t \ge 0$. Moving towards this, a natural intermediate stage is when $T = [0, \infty[$ and all the Ω_t are the same, so that we can regard an element of $\prod_{t \in T} \Omega_t$ as the path of a moving point. In this case we can begin to think about paths which are more or less continuous. The next theorem gives a widely applicable condition for existence of many paths which are one-sidedly continuous. It depends on a fairly strong continuity property for the transitional probabilities.

Definitions (a) Let U be a Hausdorff space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U. I will say that $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is **narrowly continuous** if it is continuous, as a function from $\{(s,t):0\le s<t\}\times U$ to the set of Radon probability measures on U, when the latter is given its narrow topology (437Jd).

Remark I speak of the 'narrow' topology here partly because, in the present treatise, this has become the standard topology on spaces of Radon measures, and partly because the phrase 'vaguely continuous' seems inappropriate. But, as will appear, all the results below will rely on the fact that the vague topology (437Jc) is coarser than the narrow topology. In the present context, in which we have Radon measures on a completely regular Hausdorff space, the two topologies actually coincide (437L). So $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is narrowly continuous iff $(s,t,x)\mapsto\int fd\nu_x^{(s,t)}$ is continuous for every bounded continuous $f:\Omega\to\mathbb{R}$.

(b) Let (U,ρ) be a metric space, and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U. I will say that $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is **uniformly time-continuous on the right** if for every $\epsilon>0$ there is a $\delta>0$ such that $\nu_x^{(s,t)}B(x,\epsilon)\ge 1-\epsilon$ whenever $x\in U$ and $0\le s<t\le s+\delta$.

455G Theorem Let (U,ρ) be a complete metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0\le s<t<u$ and $x\in U$. Take a point $\tilde\omega$ in $\Omega=U^{[0,\infty[}$, and $a\in[0,\infty]$. Let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on Ω defined from $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$, $\tilde\omega$ and a as in 455Eb.

(a) For $\hat\mu_{\tilde\omega a}$-almost every $\omega\in\Omega$, $\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q)$ and $\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ are defined in U for every $t>a$.

(b)(i) If $a\le t<\infty$, then $\omega(t)=\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q)$ for $\hat\mu_{\tilde\omega a}$-almost every $\omega\in\Omega$.

(ii) If $a<t<\infty$, then $\omega(t)=\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ for $\hat\mu_{\tilde\omega a}$-almost every $\omega\in\Omega$.

(c)(i) Let $C^{\Downarrow}$ be the set of càllàl functions from $[0,\infty[$ to U (438S). If $\tilde\omega\in C^{\Downarrow}$, $C^{\Downarrow}$ has full outer measure for $\hat\mu_{\tilde\omega a}$.

(ii) Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U. If $\tilde\omega\in C_{\text{dlg}}$, C_{dlg} has full outer measure for $\hat\mu_{\tilde\omega a}$.

Remark In this result and the ones to follow, I have not spelt out separately what it means if $a=0$; but of course this is the case in which we are starting the process at time $t^*=0$ and value $x^*=\tilde\omega(0)$, just as in the original construction 455A.

proof (a) Of course we can assume in this part of the proof that a is finite.

(i) Suppose that $\eta\in\,]0,1[$ and ϵ, $\delta>0$ are such that $\nu_x^{(s,t)}B(x,\epsilon)\ge 1-\eta$ whenever $x\in U$ and $0\le s<t\le s+\delta$. Then

$$\hat\mu_{\tilde\omega a}\{\omega:\omega\in\Omega,\,\operatorname{diam}\omega[D]\le 4\epsilon\}\ge\tfrac{1-2\eta}{1-\eta}$$

whenever $D\subseteq[a,\infty[$ is a countable set of diameter at most δ.

P **(α)** For finite D, I seek to induce on $\#(D)$. If $\#(D)\le 1$ then of course $\operatorname{diam}\omega[D]\le 4\epsilon$ for every ω and we can stop. So suppose that $D=\{t_0,\dots,t_n\}$ where $n\ge 1$ and $a\le t_0<\ldots<t_n$. To begin with, I go through the formulae when $t_0>0$.

For $k\le n$ set

$$E_k=\{\omega:\rho(\omega(t_k),\omega(t_0))>2\epsilon,\,\rho(\omega(t_i),\omega(t_0))\le 2\epsilon\text{ for }i<k\},$$

$$F_k=\{\omega:\omega\in E_k,\,\rho(\omega(t_n),\omega(t_k))\le\epsilon\},$$

$$G_k=\{(x_0,\dots,x_k):\rho(x_k,x_0)>2\epsilon,\,\rho(x_i,x_0)\le 2\epsilon\text{ for }i<k\}\subseteq U^{k+1}.$$

If $1\le k<n$ then

$$\begin{aligned}\hat\mu_{\tilde\omega a}F_k=\lambda_{\{0,t_0,\dots,t_k,t_n\}}\{(x,x_0,\dots,x_k,x_n):\rho(x_i,x_0)\le 2\epsilon\text{ for }i<k,\\ \rho(x_0,x_k)>2\epsilon,\,\rho(x_k,x_n)\le\epsilon\}\end{aligned}$$

(defining λ_J as the image measure of $\hat\mu_{\tilde\omega a}$ on U^J, as in 455E)

$$= \int \ldots \int \chi G_k(x_0, \ldots , x_k)\chi B(x_k, \epsilon)(x_n)\nu^{(t_k,t_n)}_{\tilde\omega a x_k}(dx_n) \ldots \nu^{(0,t_0)}_{\tilde\omega(0)}(dx_0)$$

(were $\nu^{(s,t)}_{\tilde\omega ax}$ is defined as in 455Eb)

$$= \int \ldots \int \chi G_k(x_0, \ldots , x_k)\chi B(x_k, \epsilon)(x_n)\nu^{(t_k,t_n)}_{x_k}(dx_n) \ldots \nu^{(0,t_0)}_{\tilde\omega(0)}(dx_0)$$

(because $a \le t_0 < \ldots < t_n$)

$$= \int \ldots \int \chi G_k(x_0, \ldots , x_k)\nu^{(t_k,t_n)}_{x_k}(B(x_k, \epsilon))\nu^{(t_{k-1},t_k)}_{x_{k-1}}(dx_k) \ldots \nu^{(0,t_0)}_{\tilde\omega(0)}(dx_0)$$

$$\ge \int \ldots \int (1-\eta)\chi G_k(x_0, \ldots , x_k)\nu^{(t_{k-1},t_k)}_{x_{k-1}}(dx_k) \ldots \nu^{(0,t_0)}_{\tilde\omega(0)}(dx_0)$$

(because $t_k < t_n \le t_k + \delta$, so $\nu^{(t_k,t_n)}_x B(x, \epsilon) \ge 1 - \eta$ for every x)

$$= (1-\eta)\lambda_{\{0,t_0,\ldots,t_k\}}\{(x, x_0, \ldots , x_k, x_n) : \rho(x_i, x_0) \le 2\epsilon \text{ for } i < k, \\ \rho(x_0, x_k) > 2\epsilon\}$$

$$= (1-\eta)\hat\mu_{\tilde\omega a}E_k.$$

If $k = n$, then of course $F_k = E_k$, so again $\hat\mu_{\tilde\omega a}F_k \ge (1-\eta)\hat\mu_{\tilde\omega a}E_k$. Accordingly

$$(1-\eta)\sum_{k=1}^n \hat\mu_{\tilde\omega a}E_k \le \sum_{k=1}^n \hat\mu_{\tilde\omega a}F_k \le \hat\mu_{\tilde\omega a}\{\omega : \rho(\omega(t_n), \omega(t_0)) > \epsilon\}$$

$$= \lambda_{\{0,t_0,t_n\}}\{(x, x_0, x_n) : \rho(x_0, x_n) > \epsilon\}$$

$$= \int \nu^{(t_0,t_n)}_{x_0}(U \setminus B(x_0, \epsilon))\nu^{(0,t_0)}_{\tilde\omega(0)}(dx_0) \le \eta$$

because $t_n - t_0 \le \delta$ so $\nu^{(t_0,t_n)}_x(U \setminus B(x, \epsilon)) \le \eta$ for every x.

But now we have

$$\hat\mu_{\tilde\omega a}\{\omega : \omega \in \Omega, \operatorname{diam}\omega[D] \le 4\epsilon\} \ge \hat\mu_{\tilde\omega a}\{\omega : \rho(\omega(t_k), \omega(t_0)) \le 2\epsilon \text{ for } 1 \le k \le n\}$$

$$= \hat\mu_{\tilde\omega a}(\Omega \setminus \bigcup_{1\le k\le n} E_k) \ge 1 - \tfrac{\eta}{1-\eta} = \tfrac{1-2\eta}{1-\eta}$$

as required.

(β) If $t_0 = a = 0$ the formulae simplify slightly, but the ideas are the same. We have $\omega(0) = \tilde\omega(0)$ for $\hat\mu_{\tilde\omega 0}$-almost every ω, so

$$\hat\mu_{\tilde\omega 0}F_k = \lambda_{0,t_1,\ldots,t_k,t_n}\{(\tilde\omega(0), x_1, \ldots , x_k, x_n) : \rho(x_i, \tilde\omega(0)) \le 2\epsilon \text{ for } i < k, \\ \rho(\tilde\omega(0), x_k) > 2\epsilon,\ \rho(x_k, x_n) \le \epsilon\}$$

$$= \int \ldots \int \chi G_k(\tilde\omega(0), x_1, \ldots , x_k)\chi B(x_k, \epsilon)(x_n)\nu^{(t_k,t_n)}_{\tilde\omega a x_k}(dx_n) \ldots \nu^{(0,t_1)}_{\tilde\omega(0)}(dx_1)$$

$$\ge (1-\eta)\int \ldots \int \chi G_k(\tilde\omega(0), x_1, \ldots , x_k)\nu^{(t_{k-1},t_k)}_{\tilde\omega a x_{k-1}}(dx_k) \ldots \nu^{(0,t_1)}_{\tilde\omega(0)}(dx_1)$$

$$= (1-\eta)\hat\mu_{\tilde\omega 0}E_k$$

for $1 \le k < n$,

$$(1-\eta)\sum_{k=1}^n \hat\mu_{\tilde\omega 0}E_k \le \lambda_{\{0,t_n\}}\{(\tilde\omega(0), x_n) : \rho(\tilde\omega(0), x_n) > \epsilon\}$$

$$= \int \nu^{(t_0,t_n)}_{\tilde\omega(0)}(U \setminus B(\tilde\omega(0), \epsilon)) \le \eta,$$

and the final calculation is unchanged.

(γ) For countably infinite D, let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of finite sets with union D; then $\langle\{\omega:\omega\in\Omega,\ \operatorname{diam}\omega[I_n]\le 4\epsilon\}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with intersection $\{\omega:\omega\in\Omega,\ \operatorname{diam}\omega[D]\le 4\epsilon\}$, so the measure of the limit is the limit of the measures, and is at most $\frac{1-2\eta}{1-\eta}$. **Q**

(ii) For $m\in\mathbb{N}$, $\epsilon>0$ and $A\subseteq[0,\infty[$ let $G(A,\epsilon,m)$ be

$$\{\omega:\omega\in\Omega,\ \text{there are } s_0<s_0'\le s_1<s_1'\le\ldots\le s_m\le s_m' \text{ in } A \text{ such that } \rho(\omega(s_i'),\omega(s_i))>4\epsilon \text{ for every } i\le m\}.$$

Let $\delta>0$ be such that $\nu_x^{(s,t)}B(x,\epsilon)\ge\frac{4}{5}$ whenever $x\in U$ and $s<t\le s+\delta$. Then $\hat\mu_{\tilde\omega a}G(D,\epsilon,m)\le 2^{-m}$ whenever $m\in\mathbb{N}$ and $D\subseteq[a,\infty[$ is a countable set of diameter at most δ.

P **(α)** As in (i), first consider finite D. For these, we can induce on m. If $m=0$ then $G(D,\epsilon,0)=\{\omega:\operatorname{diam}\omega[D]>4\epsilon\}$ so (i), with $\eta=\frac{1}{5}$, tells us that $\hat\mu_{\tilde\omega a}G(D,\epsilon,0)\le\frac{2\eta}{1-\eta}=\frac{1}{2}$. For the inductive step to $m+1$, define $\tau:\Omega\to[0,\infty]$ by setting

$$\begin{aligned}\tau(\omega)&=\min\{t:t\in D,\ \omega\in G(D\cap[a,t],\epsilon,m)\}\text{ if }\omega\in G(D,\epsilon,m),\\&=\infty\text{ otherwise.}\end{aligned}$$

Then τ takes only finitely many values, all strictly greater than a, and $\{\omega:\tau(\omega)=t\}$ belongs to $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)=\widehat{\bigotimes}_{t\in[0,\infty[}\mathcal{B}(U)$ and is determined by coordinates in $[0,t]$ for every $t\ge0$. We can therefore apply 455E(b)-(c).

For each $\omega\in\Omega$, define $\langle\nu^{(s,t)}_{\omega,\tau(\omega),x}\rangle_{s<t,x\in U}$ from $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in U}$ as in 455Eb; let $\langle\tilde\nu^{(s,t)}_{\omega,\tau(\omega),x}\rangle_{s<t,x\in U}$ be the family defined in the same way from $\langle\nu^{(s,t)}_{\tilde\omega ax}\rangle_{s<t,x\in U}$. Let $\hat\mu_{\omega,\tau(\omega)}$ be defined from $\omega(0)$ and $\langle\nu^{(s,t)}_{\omega,\tau(\omega),x}\rangle_{s<t,x\in U}$, and $\hat\mu'_{\omega,\tau(\omega)}$ from $\omega(0)$ and $\langle\tilde\nu^{(s,t)}_{\omega,\tau(\omega),x}\rangle_{s<t,x\in U}$, again as in 455Eb. Then 455E(c-i) tells us that $\langle\hat\mu'_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of $\hat\mu_{\tilde\omega a}$ over itself. But now observe that, for any $\omega\in\Omega$ and $x\in U$,

$$\nu^{(s,t)}_{\omega,\tau(\omega),x}=\nu_x^{(s,t)}=\nu^{(s,t)}_{\tilde\omega ax}=\tilde\nu^{(s,t)}_{\omega,\tau(\omega),x}\text{ if }\tau(\omega)<s<t,$$

(because $a<\tau(\omega)$)

$$\begin{aligned}&=\nu^{(\tau(\omega),t)}_{\omega(\tau(\omega))}=\nu^{(\tau(\omega),t)}_{\tilde\omega,a,\omega(\tau(\omega))}=\tilde\nu^{(s,t)}_{\omega,\tau(\omega),x}\text{ if }s\le\tau(\omega)<t,\\&=\delta^{(t)}_{\omega(t)}=\tilde\nu^{(s,t)}_{\omega,\tau(\omega),x}\text{ if }s<t\le\tau(\omega),\end{aligned}$$

so $\hat\mu_{\omega,\tau(\omega)}=\hat\mu'_{\omega,\tau(\omega)}$. Accordingly $\langle\hat\mu_{\omega,\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of $\hat\mu_{\tilde\omega a}$ over itself.

Now, for $\omega\in\Omega$, consider

$$H_\omega=\{\omega':\omega'\in G(D,\epsilon,m+1),\ \omega'\restriction D\cap[0,\tau(\omega)]=\omega\restriction D\cap[0,\tau(\omega)]\}.$$

If $\omega\notin G(D,\epsilon,m)$ then $\tau(\omega)=\infty$ and $H_\omega=\emptyset$, because $G(D,\epsilon,m+1)\subseteq G(D,\epsilon,m)$ are determined by coordinates in D. If $\omega\in G(D,\epsilon,m)$ and $\tau(\omega)=b$, then

$$H_\omega=\{\omega':\omega'\restriction D\cap[0,b]=\omega\restriction D\cap[0,b]\text{ and }\operatorname{diam}(\omega'[D\cap[b,\infty[])>4\epsilon\},$$

so that $\hat\mu_{\omega,\tau(\omega)}H_\omega\le\frac{1}{2}$ by (i), again with $\eta=\frac{1}{5}$.

So

$$\hat\mu_{\tilde\omega a}G(D,\epsilon,m+1)=\int\hat\mu_{\omega,\tau(\omega)}G(D,\epsilon,m+1)\hat\mu_{\tilde\omega a}(d\omega)=\int\hat\mu_{\omega,\tau(\omega)}H_\omega\hat\mu_{\tilde\omega a}(d\omega)$$

(using 455E(b-ii))

$$=\int_{G(D,\epsilon,m)}\hat\mu_{\omega,\tau(\omega)}H_\omega\hat\mu_{\tilde\omega a}(d\omega)\le\frac{1}{2}\hat\mu_{\tilde\omega a}G(D,\epsilon,m)\le2^{-m-1}$$

by the inductive hypothesis. Thus the induction proceeds.

(β) Now, for countably infinite D, again express D as the union of a non-decreasing sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of finite sets, and observe that $\langle G(I_n,\epsilon,m)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union $G(D,\epsilon,m)$; so

$$\hat\mu_{\tilde\omega a}G(D,\epsilon,m)=\lim_{n\to\infty}\hat\mu_{\tilde\omega a}G(I_n,\epsilon,m)\le 2^{-m}$$

for every $m\in\mathbb{N}$. **Q**

(iii) For $n\in\mathbb{N}$, let $\delta_n>0$ be such that $\nu_x^{(s,t)}B(x,2^{-n})\ge\frac{4}{5}$ whenever $x\in U$ and $s<t\le s+\delta_n$. Consider the set

$$E=\bigcup_{n,k\in\mathbb{N}}\bigcap_{m\in\mathbb{N}}G(\mathbb{Q}\cap[a+k\delta_n,a+(k+1)\delta_n],2^{-n+2},m).$$

Then $\hat\mu_{\tilde\omega a}E=0$. Suppose that $\omega\in\Omega\setminus E$ and $t>a$. **?** If $\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ is undefined, then (because U is complete under ρ) there must be an $n\in\mathbb{N}$ and a strictly increasing sequence $\langle q_i\rangle_{i\in\mathbb{N}}$ in $\mathbb{Q}$, with supremum t, such that $\rho(\omega(q_{i+1}),\omega(q_i))\ge 2^{-n+2}$ for every $i\in\mathbb{N}$. Let $k\in\mathbb{N}$ be such that $t\in\,]a+k\delta_n,a+(k+1)\delta_n]$; let $l\in\mathbb{N}$ be such that $q_l\ge a+k\delta_n$. Then, for every $m\in\mathbb{N}$, $(q_l,q_{l+1},q_{l+1},q_{l+2},\dots,q_{m-1},q_m)$ witnesses that $\omega\in G(\mathbb{Q}\cap[a+k\delta_n,a+(k+1)\delta_n],2^{-n+2},m)$; which is impossible. **X** So $\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ is defined; similarly, $\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q)$ is defined.

As E is $\hat\mu_{\tilde\omega a}$-negligible, this proves (a).

(b)(i) This is actually easier. Consider part (a-i) of the proof above. Given $n\in\mathbb{N}$, we see that there is a $\delta_n>0$ such that $\hat\mu_{\tilde\omega a}\{\omega:\operatorname{diam}\omega[D]\le 2^{-n}\}\ge 1-2^{-n}$ whenever $D\subseteq[a,\infty[$ is a countable set of diameter at most δ_n. Set

$$D_n=\{t\}\cup(\mathbb{Q}\cap[t,t+\delta_n]),\quad E_n=\{\omega:\operatorname{diam}\omega[D_n]\le 2^{-n}\}$$

for each $n\in\mathbb{N}$, and $E=\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}E_m$. Then $\hat\mu_{\tilde\omega a}E=1$, and for $\omega\in E$ we have an $n\in\mathbb{N}$ such that $\rho(\omega(t),\omega(q))\le 2^{-m}$ whenever $m\ge n$ and $q\in\mathbb{Q}\cap[t,t+\delta_m]$, so that $\omega(t)=\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q)$.

(ii) If $t>a$, the same argument applies on the other side of t, taking $D_n=\{t\}\cup(\mathbb{Q}\cap[\max(a,t-\delta_n),t])$, to see that $\omega(t)=\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)$ for $\hat\mu_{\tilde\omega a}$-almost every ω.

(c)(α) Suppose that $E\subseteq\Omega$ and $\hat\mu_{\tilde\omega a}E>0$. Then there is an $\omega^*\in E$ such that

$$\omega^*(t)=\tilde\omega(t)\text{ for every }t\le a,$$

$$\omega^*(t)=\lim_{s\downarrow t}\omega^*(s)\text{ for every }t\ge a,$$

$$\lim_{s\uparrow t}\omega^*(t)\text{ is defined for every }t>a.$$

P Let $E'\in\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ be such that $E'\subseteq E$ and $\hat\mu_{\tilde\omega a}E'>0$. Let $D\subseteq[0,\infty[$ be a countable set such that E' is determined by coordinates in D; we can suppose that $a\in D$ if a is finite. Let F be the set of those $\omega\in\Omega$ such that

$$\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q)\text{ and }\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)\text{ are defined in }U\text{ for every }t>a,$$

$$\omega(t)=\lim_{q\in\mathbb{Q},q\downarrow t}\omega(t)\text{ for every }t\in D\cap[a,\infty[,$$

$$\omega(t)=\tilde\omega(t)\text{ for every }t\in D\cap[0,a].$$

Then (a) and (b), with 455E(b-ii), tell us that F is $\hat\mu_{\tilde\omega a}$-conegligible. So there is an $\omega\in E\cap F$. Define $\omega^*\in\Omega$ by setting

$$\begin{aligned}\omega^*(t)&=\tilde\omega(t)\text{ if }t\le a,\\&=\lim_{q\in\mathbb{Q},q\downarrow t}\omega(t)\text{ if }t\ge a;\end{aligned}$$

note that the definitions of $\omega^*(a)$ are consistent if a is finite, and that $\omega^*\restriction D=\omega\restriction D$, so that $\omega^*\in E'\subseteq E$.

If $t\le a$, then of course $\omega^*(t)=\tilde\omega(t)$. If $t\ge a$ and $\epsilon>0$, there is a $\delta>0$ such that $\rho(\omega(q),\omega^*(t))\le\epsilon$ whenever $q\in\mathbb{Q}\cap]t,t+\delta]$; in which case $\rho(\omega^*(s),\omega^*(t))\le\epsilon$ whenever $s\in[t,t+\delta[$; as ϵ is arbitrary, $\omega^*(t)=\lim_{s\downarrow t}\omega^*(s)$. If $t>a$ and $\epsilon>0$, there is a $\delta>0$ such that $\rho(\omega(q),\omega(q'))\le\epsilon$ whenever $q\in\mathbb{Q}\cap[t-\delta,t[$; in which case $\rho(\omega^*(s),\omega^*(s'))\le\epsilon$ whenever $s\in[t-\delta,t[$; as ϵ is arbitrary and U is complete, $\lim_{s\uparrow t}\omega^*(s)$ is defined in U. So we have an appropriate ω^*. **Q**

(β) Suppose, in (α), that $\tilde\omega \in C^{⫫}$. Then $\omega^* \in C^{⫫}$. **P**

$$\lim_{s\uparrow t}\omega^*(s) = \lim_{s\uparrow t}\tilde\omega(s) \text{ is defined whenever } 0 < t \le a,$$

$$\lim_{s\downarrow t}\omega^*(s) = \lim_{s\downarrow t}\tilde\omega(s) \text{ is defined whenever } 0 \le t < a,$$

$$\text{if } a > 0,\ \lim_{s\downarrow 0}\omega^*(s) = \lim_{s\downarrow 0}\tilde\omega(s) = \tilde\omega(0) = \omega^*(0),$$

if $0 < t < a$, then $\omega^*(t) = \tilde\omega(t)$ is equal to at least one of $\lim_{s\uparrow t}\omega^*(s) = \lim_{s\uparrow t}\tilde\omega(s)$, $\lim_{s\downarrow t}\omega^*(s) = \lim_{s\downarrow t}\tilde\omega(s)$.

Since we already know that

$$\omega^*(t) = \lim_{s\downarrow t}\omega^*(s) \text{ for every } t \ge a,$$

$$\lim_{s\uparrow t}\omega^*(t) \text{ is defined for every } t > a,$$

ω^* is càllàl. **Q**

As E is arbitrary, it follows that if $\tilde\omega \in C^{⫫}$ then $C^{⫫}$ meets every non-negligible $\hat\mu_{\tilde\omega a}$-measurable set, so that $\hat\mu^*_{\tilde\omega a}C^{⫫} = 1$, as required by (i).

(γ) Similarly, if $\tilde\omega \in C_{\text{dlg}}$, then any ω^* with the properties described in (α) also belongs to C_{dlg}. **P** This time, we have

$$\text{if } 0 \le t < a, \text{ then } \omega^*(t) = \tilde\omega(t) = \lim_{s\downarrow t}\omega^*(s) = \lim_{s\downarrow t}\tilde\omega(s),$$

which with the other properties listed is enough to ensure that $\omega^* \in C_{\text{dlg}}$. **Q** Since E is arbitrary, $\hat\mu^*_{\tilde\omega a}C_{\text{dlg}} = 1$.

This completes the proof of part (c).

455H Corollary Let (U,ρ) be a complete metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0 \le s < t < u$ and $x \in U$. Let $C^{⫫}(U)$ be the set of càllàl functions from $[0,\infty[$ to U. Suppose that $\tilde\omega \in C^{⫫}(U)$, and $a \in [0,\infty]$; let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on $\Omega = U^{[0,\infty[}$ defined from $\tilde\omega$, a and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ as in 455Eb. Then $\hat\mu_{\tilde\omega a}$ has a unique extension to a Radon measure $\tilde\mu_{\tilde\omega a}$ on Ω, and $\tilde\mu_{\tilde\omega a}C^{⫫}(U) = 1$.

proof (a) In the language of 455E(b-i), $\nu_{\tilde\omega ax}^{(0,t)}$ is a Radon measure whenever $t > 0$ and $x \in U$, so the image measure defined from $\hat\mu_{\tilde\omega a}$ and the map $\omega \mapsto \omega(t)$ is always a Radon measure on U, and there there is a σ-compact set $H_t \subseteq U$ such that $\omega(t) \in H_t$ for $\hat\mu_{\tilde\omega a}$-almost every ω. Set $U_0 = \overline{\bigcup_{q\in\mathbb{Q}\cap[0,\infty[}H_q}$; then U_0 is separable and $\hat\mu_{\tilde\omega a}E = 1$, where $E = \{\omega : \omega(q) \in U_0 \text{ for every } q \in \mathbb{Q}\cap[0,\infty[\}$. By 455G(c-i), $E \cap C^{⫫}(U)$ has full outer measure; and if $\omega \in E \cap C^{⫫}(U)$, then $\omega(t) \in U_0$ for every $t \ge 0$.

(b) Thus $E \cap C^{⫫}(U)$ is included in $C^{⫫}(U_0)$, the set of càllàl functions from $[0,\infty[$ to the Polish space U_0. So $\hat\mu^*_{\tilde\omega a}C^{⫫}(U_0) = 1$. Let $\hat\mu_C$ be the subspace probability measure on $C^{⫫}(U_0)$.

Since $\hat\mu_{\tilde\omega a}$ is inner regular with respect to $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$, $\hat\mu_C$ is inner regular with respect to the σ-algebra $\Sigma = \{E \cap C^{⫫}(U_0) : E \in \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)\}$ (412Ob). But Σ is just the σ-algebra generated by the maps $\omega \mapsto \omega(t) : C^{⫫}(U_0) \to U_0$ for $t \ge 0$, which is the Baire σ-algebra of $C^{⫫}(U_0)$ (4A3Na, 4A3Nd). Accordingly $\hat\mu_C{\restriction}\Sigma$ is a Baire measure and is inner regular with respect to the closed sets (412D); it follows that its completion $\hat\mu_C$ is inner regular with respect to the closed sets (412Ha).

At this point, recall that $C^{⫫}(U_0)$ is K-analytic (438Sc). So $\hat\mu_C$ has an extension to a Radon measure $\tilde\mu_C$ on $C^{⫫}(U_0)$ (432D). Now $\tilde\mu_C$ has an extension to a Radon probability measure $\tilde\mu_{\tilde\omega a}$ on Ω such that $\tilde\mu_{\tilde\omega a}C^{⫫}(U) = \tilde\mu_{\tilde\omega a}C^{⫫}(U_0) = 1$. And if $\hat\mu_{\tilde\omega a}$ measures E, then

$$\tilde\mu_{\tilde\omega a}E = \tilde\mu_C(E \cap C^{⫫}(U_0)) = \hat\mu_C(E \cap C^{⫫}(U_0)) = \hat\mu^*_{\tilde\omega a}(E \cap C^{⫫}(U_0)) = \hat\mu_{\tilde\omega a}E,$$

so $\tilde\mu_{\tilde\omega a}$ extends $\hat\mu_{\tilde\omega a}$.

(c) As for uniqueness, observe that $\operatorname{dom}\hat{\mu}_{\tilde{\omega}a}$ includes a base for the topology of Ω, so by 415H there can be at most one Radon measure extending $\hat{\mu}_{\tilde{\omega}a}$.

455I In fact we can go farther; the Radon measure $\tilde{\mu}$ is much more closely related to the completed Baire measure it extends than one might expect.

Lemma Let (U,ρ) be a complete separable metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0\le s<t<u$ and $x\in U$. Suppose that $\tilde{\omega}\in\Omega$, and $a\in[0,\infty]$; let $\hat{\mu}_{\tilde{\omega}a}$ be the completed probability measure on $\Omega=U^{[0,\infty[}$ defined from $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$, $\tilde{\omega}$ and a as in 455Eb.

(a) Suppose that $0\le q_0<q_1$ and $\epsilon>0$. For $\omega\in\Omega$, I will say that $]q_0,q_1[$ is an **ϵ-shift interval** of ω with (q_0,q_1,ϵ)**-shift point** t if $\rho(\omega(q_0),\omega(q_1))>2\epsilon$ and

$$\begin{aligned}t&=\sup\{q:q\in\mathbb{Q}\cap\,]q_0,q_1[\,,\ \rho(\omega(q),\omega(q_0))\le\epsilon\}\\&=\inf\{q:q\in\mathbb{Q}\cap\,]q_0,q_1[\,,\ \rho(\omega(q),\omega(q_1))\le\epsilon\}.\end{aligned}$$

Let E be the set of such ω.

(i) $E\in\mathcal{B}\mathfrak{a}(\Omega)=\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$.

(ii) The function $f:E\to\,]q_0,q_1[$ which takes each $\omega\in E$ to its (q_0,q_1,ϵ)-shift point is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable.

(iii) If $q_0\ge a$, the set $\{\omega:\omega\in E,\ f(\omega)=t\}$ is $\hat{\mu}_{\tilde{\omega}a}$-negligible for every $t\in\,]q_0,q_1[$.

(iv) If q_0, $q_1\in\mathbb{Q}$, $\omega\in E$, $\omega'\in\Omega$ and $\omega'{\restriction}\mathbb{Q}=\omega{\restriction}\mathbb{Q}$, then $\omega'\in E$ and $f(\omega')=f(\omega)$.

(b) Suppose that $\langle q_i\rangle_{i\le n}$, $\langle q_i'\rangle_{i\le n}$, $\langle\le_i\rangle_{i\le n}$, $\epsilon>0$, $E\in\mathcal{B}\mathfrak{a}(\Omega)$ and $\langle f_i\rangle_{i\le n}$ are such that, for every $i\le n$,

$$q_i,\ q_i'\in\mathbb{Q},\quad q_i<q_i',\quad \le_i\text{ is either }\le\text{ or }\ge,$$

$$]q_i,q_i'[\text{ is an }\epsilon\text{-shift interval of }\omega\text{ with }(q_i,q_i',\epsilon)\text{-shift point }f_i(\omega),\text{ for every }\omega\in E,$$

and also

$$a\le q_0,\quad q_i'\le q_{i+1}\text{ for every }i<n,$$

$$\text{whenever }\omega,\ \omega'\in E\text{ there is an }i\le n\text{ such that }f_i(\omega')\le_i f_i(\omega).$$

Then E is $\hat{\mu}_{\tilde{\omega}a}$-negligible.

(c) Suppose that $\langle q_i\rangle_{i\le n}$, $\langle q_i'\rangle_{i\le n}$, $\langle\le_i\rangle_{i\le n}$, $\epsilon>0$, $E\in\mathcal{B}\mathfrak{a}(\Omega)$ and $\langle f_i\rangle_{i\le n}$ are such that, for every $i\le n$,

$$q_i,\ q_i'\in\mathbb{Q},\quad q_i<q_i',\quad \le_i\text{ is either }\le\text{ or }\ge,$$

$$]q_i,q_i'[\text{ is an }\epsilon\text{-shift interval of }\omega\text{ with }(q_i,q_i',\epsilon)\text{-shift point }f_i(\omega),\text{ for every }\omega\in E,$$

and also

$$a\le q_0,\quad q_i'\le q_{i+1}\text{ for every }i<n.$$

Then for $\hat{\mu}_{\tilde{\omega}a}$-almost every $\omega\in E$ there is an $\omega'\in E$ such that $f_i(\omega')<_i f_i(\omega)$ for every $i\le n$.

proof (a)(i) Note that by 4A3Na we can identify $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ with the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\Omega)$ of Ω. If s, $t\ge0$, then $\omega\mapsto(\omega(s),\omega(t)):\Omega\to U^2$ is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable, by 418Bb; so $\omega\mapsto\rho(\omega(s),\omega(t))$ is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable. For $\omega\in\Omega$, $\omega\in E$ iff (α) $\rho(\omega(q_0),\omega(q_1))>2\epsilon$ (β) whenever q, $q'\in\mathbb{Q}\cap\,]q_0,q_1[$, $\rho(\omega(q),\omega(q_0))\le\epsilon$ and $\rho(\omega(q'),\omega(q_1))\le\epsilon$ then $q\le q'$ (γ) for every $n\in\mathbb{N}$ there are q, $q'\in\mathbb{Q}\cap\,]q_0,q_1[$ such that $\rho(\omega(q),\omega(q_0))\le\epsilon$, $\rho(\omega(q'),\omega(q_0'))\le\epsilon$ and $q'\le q+2^{-n}$. So $E\in\mathcal{B}\mathfrak{a}(\Omega)$.

(ii) Now, for any t,

$$\{\omega:\omega\in E,\ f(\omega)>t\}=\bigcup_{q\in\mathbb{Q}\cap]t,q_1[}\{\omega:\omega\in E,\ \rho(\omega(q),\omega(q_0))\le\epsilon\}$$

belongs to $\mathcal{B}\mathfrak{a}(\Omega)$, so f is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable.

(iii) Consider the set E' of those $\omega\in\Omega$ such that

$$\lim_{q\in\mathbb{Q},q\uparrow t}\omega(q)=\omega(t)=\lim_{q\in\mathbb{Q},q\downarrow t}\omega(q).$$

If $\omega \in E \cap E'$, at least one of $\rho(\omega(t),\omega(q_0))$, $\rho(\omega(t),\omega(q_1))$ must be greater than ϵ; in the first case, t cannot be $\sup\{q : q \in \mathbb{Q} \cap \,]q_0, q_1[, \ \rho(\omega(q),\omega(q_0)) \le \epsilon\}$; in the second case, t cannot be $\inf\{q : q \in \mathbb{Q} \cap \,]q_0, q_1[, \ \rho(\omega(q),\omega(q_1)) \le \epsilon\}$; so in either case $f(\omega)$ cannot be equal to t. Now $\hat\mu_{\tilde\omega a} E' = 1$, by 455Gb, so $\{\omega : \omega \in E, f(\omega) = t\} \subseteq \Omega \setminus E'$ is $\hat\mu_{\tilde\omega a}$-negligible.

(iv) Immediate from the definitions.

(b) Induce on n. Of course we need consider only the case $E \ne \emptyset$.

(i) If $n = 0$, f_0 must be constant on E, so E must be negligible, by (a-iii).

(ii) For the inductive step to $n \ge 1$, set $E_t = \{\omega : \omega \in E, \ f_0(\omega) = t\}$ for $t \in \,]q_0, q_0'[$; by (a-ii), $E_t \in \mathcal{B}\mathfrak{a}(\Omega)$.

(α) There is a countable set $J \subseteq \,]q_0, q_0'[$ such that whenever $t \in [q_0, q_0'] \setminus J$ and ω, $\omega' \in E_t$ then there is an i such that $1 \le i \le n$ and $f_i(\omega) \le_i f_i(\omega')$. **P** Let $\mathcal{W}$ be a countable base for the topology of $\prod_{1\le i\le n}]q_i, q_i'[$. For $\omega \in E$ set $g(\omega) = \langle f_i(\omega)\rangle_{1\le i\le n}$; note that $g : E \to \prod_{1\le i\le n}]q_i, q_i'[$ is $\mathcal{B}\mathfrak{a}(\Omega)$-measurable. For $W \in \mathcal{W}$, set

$$A_W = \{t : t \in \,]q_0, q_0'[\text{ and there is an } \omega \in E_t \text{ such that } g(\omega) \in W\}.$$

Set

$$J = \{t : t \in \,]q_0, q_0'[, \ t \text{ is either } \inf A_W \text{ or } \sup A_W \text{ for some } W \in \mathcal{W}\}.$$

Then J is a countable subset of $]q_0, q_0'[$. **?** Suppose that $t \in \,]q_0, q_0'[\setminus J$ and ω, $\omega' \in E_t$ are such that $f_i(\omega') <_i f_i(\omega)$ for $1 \le i \le n$. Let $W \in \mathcal{W}$ be such that $g(\omega') \in W$ and $z(i) <_i f_i(\omega)$ whenever $1 \le i \le n$ and $z \in W$. Then ω' witnesses that $t \in A_W$; since t is neither the greatest nor the least element of A_W, there is a $t' \in A_W$ such that $t' <_0 t$; take $\omega'' \in E_{t'}$ such that $g(\omega'') \in W$. Then

$$f_0(\omega'') = t' <_0 t = f_0(\omega),$$

$$f_i(\omega'') = g(\omega'')(i) <_i f_i(\omega) \text{ for } 1 \le i \le n,$$

which is impossible. **X** Thus J has the required property. **Q**

(β) Now consider the family $\langle \hat\mu_{\omega q_1}\rangle_{\omega\in\Omega}$. Because $q_1 > a$, this is a disintegration of $\hat\mu_{\tilde\omega a}$ over itself. **P** As in part (a-ii-α) of the proof of 455G, we can think of each $\hat\mu_{\omega q_1}$ as defined either from $\langle \nu_x^{(s,t)}\rangle_{0\le s<t, x\in U}$ or from $\langle \nu_{\tilde\omega a x}^{(s,t)}\rangle_{a\le s<t, x\in U}$; and in the latter form we can apply 455E(c-i). **Q**

Consider $\mu_{\omega q_1}(E)$ for $\omega \in \Omega$. This time, note that $\{\omega' : \omega'\restriction[0, q_1] \cap \mathbb{Q} = \omega\restriction[0, q_1] \cap \mathbb{Q}\}$ is $\mu_{\omega q_1}$-conegligible. In particular, $\mu_{\omega q_1}(E) = 0$ unless $]q_0, q_0'[$ is an ϵ-shift interval of ω. Next,

$$\{\omega :]q_0, q_0'[\text{ is an } \epsilon\text{-shift interval of } \omega \text{ with } (q_0, q_0', \epsilon)\text{-shift point in } J\}$$

is $\hat\mu_{\tilde\omega a}$-negligible, by (a-iii) again. Finally, suppose that $\omega \in \Omega$ is such that $]q_0, q_0'[$ is an ϵ-shift interval of ω with (q_0, q_0', ϵ)-shift point $t \in \,]q_0, q_0'[\setminus J$. Then

$$\begin{aligned} \mu_{\omega q_1} E &= \mu_{\omega q_1}\{\omega' : \omega' \in E, \ \omega'\restriction\mathbb{Q} \cap [0, q_1] = \omega\restriction\mathbb{Q} \cap [0, q_1]\} \\ &= \mu_{\omega q_1}\{\omega' : \omega' \in E_t\}. \end{aligned}$$

But the choice of J in (α) ensured that E_t would be a set of the same type as E, one level down, determined by intervals starting from q_1, so that $\hat\mu_{\omega q_1} E_t = 0$, by the inductive hypothesis applied to ω and q_1 in place of $\tilde\omega$ and a.

(γ) So we see that $\hat\mu_{\omega q_1} E = 0$ for $\hat\mu_{\tilde\omega a}$-almost every ω, and $\hat\mu_{\tilde\omega a} E = 0$. Thus the induction proceeds.

(c) Let F be the set of those $\omega \in E$ for which there is no $\omega' \in E$ such that $f_i(\omega') <_i f_i(\omega)$ for every $i \le n$. Then $F \in \mathcal{B}\mathfrak{a}(\Omega)$. **P** For each $\omega \in E$ set $f(\omega) = \langle f_i(\omega)\rangle_{i\le n}$ and

$$W_\omega = \{z : z \in \textstyle\prod_{i\le n}]q_i, q_i'[, \ f_i(\omega) <_i z(i) \text{ for every } i \le n\},$$

so that W_ω is open in $\prod_{i\le n}]q_i, q_i'[$. Set $W = \bigcup_{\omega\in E} W_\omega$. Then W is open and $F = \{\omega : \omega \in E, \ f(\omega) \notin W\}$ belongs to $\mathcal{B}\mathfrak{a}(\Omega)$. **Q**

If ω, $\omega' \in F$ then there is surely some $i \le n$ such that $f_i(\omega') \le_i f_i(\omega)$. By (b), $\hat\mu_{\tilde\omega a} F = 0$.

455J Theorem Let (U,ρ) be a complete separable metric space and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ a family of Radon probability measures on U, uniformly time-continuous on the right, such that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $0\le s<t<u$ and $x\in U$. Write $C^{\Downarrow}$ for the set of càllàl functions from $[0,\infty[$ to U. Suppose that $\tilde\omega\in C^{\Downarrow}$, and $a\in[0,\infty]$; let $\hat\mu_{\tilde\omega a}$ be the completed probability measure on $\Omega=U^{[0,\infty[}$ defined from $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$, $\tilde\omega$ and a as in 455Eb, and $\tilde\mu_{\tilde\omega a}$ its extension to a Radon measure on Ω, as in 455H. Then $\tilde\mu_{\tilde\omega a}$ is inner regular with respect to sets of the form $F\cap C^{\Downarrow}$ where $F\subseteq\Omega$ is a zero set.

proof (a) As in 455I, $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ is the Baire σ-algebra of Ω. Let D be $((\{a\}\cup\mathbb{Q})\cap[0,\infty[)\cup\{t:t\ge0,\ \tilde\omega$ is not continuous at $t\}$; then D is countable (438S(a-i)). Let E^* be $\{\omega:\omega\in\Omega,\ \omega\restriction D\cap[0,a]=\tilde\omega\restriction D\cap[0,a]\}$; then E^* is $\hat\mu_{\tilde\omega a}$-conegligible, by 455E(b-ii) once again.

(b) Let $\mathcal{G}$ be a countable base for the topology of U. Let $\mathcal{W}$ be the family of open subsets of Ω of the form $\{\omega:\omega(q)\in G_q$ for every $q\in J\}$ where $J\subseteq D$ is finite and $G_q\in\mathcal{G}$ for every $q\in J$. Let Θ be the set of all strings

$$\theta=(q_0,q_0',\dots,q_n,q_n',\le_0,\dots,\le_n,k,W)$$

such that

$$q_0,\dots,q_n'\in\mathbb{Q},\quad a\le q_0<q_0'\le q_1<q_1'\le\dots\le q_n<q_n',$$

$$\text{for each } i\le n,\ \le_i \text{ is either } \le \text{ or } \ge,$$

$$k\in\mathbb{N},\quad W\in\mathcal{W};$$

then Θ is countable.

(c) Let $K\subseteq E^*\cap C^{\Downarrow}$ be compact. Set $L=\pi_D^{-1}[\pi_D[K]]$, where $\pi_D(\omega)=\omega\restriction D$ for $\omega\in\Omega$; then L is a Baire subset of Ω, because $\pi_D[K]$ is a compact subset of the metrizable space U^D.

(i) For

$$\theta=(q_0,q_0',\dots,q_n,q_n',\le_0,\dots,\le_n,k,W)\in\Theta$$

let E_θ be the set of those $\omega\in L\cap W$ such that, for each $i\le n$, $]q_i,q_i'[$ is a 2^{-k}-shift interval of ω (definition: 455Ia). For $\omega\in E_\theta$ and $i\le n$ let $f_i(\theta,\omega)$ be the $(q_i,q_i',2^{-k})$-shift point of ω. By 455Ia, E_θ is a Baire subset of Ω and $\omega\mapsto f_i(\theta,\omega)$ is Baire measurable. Let F_θ be the set of those $\omega\in E_\theta$ such that there is no $\omega'\in E_\theta$ with $f_i(\theta,\omega')<_i f_i(\theta,\omega)$ for every $i\le n$; by 455Ic, F_θ is $\hat\mu_{\tilde\omega a}$-negligible. So $F^*=\bigcup_{\theta\in\Theta}F_\theta$ is $\hat\mu_{\tilde\omega a}$-negligible.

(ii) Suppose that $\omega\in K\setminus F^*$. Let A be the set of points in $]a,\infty[$ at which ω is discontinuous. If $J\subseteq A$ is finite and $\epsilon_t\in\{-1,1\}$ for each $t\in J$, there is an $\omega'\in K$ such that $\omega'\restriction D=\omega\restriction D$ and ω' is continuous on the right at every point t of J such that $\epsilon_t=1$, while ω' is continuous on the left at every point t of J such that $\epsilon_t=-1$. **P** This is trivial if J is empty. Otherwise, enumerate J in ascending order as $t_0<t_1<\dots<t_n$. Set $x_i=\lim_{t\uparrow t_i}\omega(t)$, $y_i=\lim_{t\downarrow t_i}\omega(t)$; because $\omega\in C^{\Downarrow}$ these are defined, and because ω is not continuous at t they are different.

Let $k\in\mathbb{N}$ be such that $\rho(x_i,y_i)>2^{-k+1}$ for each $i\le n$. For $i\le n$, let let q_i, $q_i'\in\mathbb{Q}$ be such that $q_i<t_i<q_i'$, $\rho(\omega(t),x_i)\le2^{-k-1}$ for $t\in[q_i,t_i[$, and $\rho(\omega(t),y_i)\le2^{-k-1}$ for $t\in]t_i,q_i']$. Of course we can suppose that $a\le q_0$ and that $q_i'\le q_{i+1}$ for $i<n$. Observe that this will ensure that every $]q_i,q_i'[$ is a 2^{-k}-shift interval of ω with $(q_i,q_i',2^{-k})$-shift point t_i.

Let $\le_i$ be $\le$ if $\epsilon_{t_i}=1$, $\ge$ if $\epsilon_{t_i}=-1$. For each $W\in\mathcal{W}$ containing ω, let $\theta_W\in\Theta$ be $(q_0,\dots,q_n',\le_0,\dots,\le_n,k,W)$. Then $\omega\in E_{\theta_W}$, and $f_i(\theta_W,\omega)=t_i$ for each $i\le n$. Because $\omega\notin F_{\theta_W}$, there is an $\omega_W\in E_{\theta_W}$ such that $f_i(\theta_W,\omega_W)<_i f_i(\theta_W,\omega)=t_i$ for every $i\le n$. Let $\omega_W'\in K$ be such that $\omega_W'\restriction D=\omega_W\restriction D$; then $\omega_W'\in E_{\theta_W}$ and

$$f_i(\theta_W,\omega_W')=f_i(\theta_W,\omega_W)<_i t_i$$

for every $i\le n$ (455I(a-iv)).

If $i\le n$ and $\epsilon_{t_i}=1$,

$$\rho(\omega_W'(q),\omega_W'(q_i'))\le2^{-k}$$

for every rational $q \in]f_i(\theta_W, \omega'_W), q'_i]$; because $\omega'_W \in C^{\mathbb{l}}$ and $t_i \in]f_i(\theta_W, \omega'_W), q'_i]$ $\rho(\omega'_W(t_i), \omega'_W(q'_i)) \le 2^{-k}$. Similarly, if $\epsilon_{t_i} = -1$, $\rho(\omega'_W(t_i), \omega'_W(q_i)) \le 2^{-k}$.

Let $\mathcal{F}$ be an ultrafilter on $\mathcal{W}$ containing all sets of the form $\{W : \omega \in W \subseteq W_0\}$ where $\omega \in W_0 \in \mathcal{W}$, and set $\omega' = \lim_{W\to\mathcal{F}} \omega'_W \in K$. Then $\omega'\restriction D = \omega\restriction D$, because ω_W and ω'_W belong to W whenever $\omega \in W \in \mathcal{W}$. If $i \le n$ and $\epsilon_{t_i} = 1$, then

$$\rho(\omega'(t_i), \omega'(q'_i)) = \lim_{W\to\mathcal{F}} \rho(\omega'_W(t_i), \omega'_W(q'_i)) \le 2^{-k},$$

$$\rho(\omega'(q_i), \omega'(q'_i)) = \rho(\omega(q_i), \omega(q'_i)) > 2^{-k+1},$$

so $\rho(\omega'(q_i), \omega'(t_i)) > 2^{-k}$. On the other hand,

$$\rho(\omega'(q_i), \omega'(q)) = \rho(\omega(q_i), \omega(q)) \le 2^{-k}$$

for every rational $q \in [q_i, t_i[$. So ω' cannot be continuous on the left at t_i; because $\omega' \in C^{\mathbb{l}}$, it must be continuous on the right at t_i. Similarly, if $i \le n$ and $\epsilon_{t_i} = -1$, ω' cannot be continuous on the right at t_i and must be continuous on the left at t_i. But this is what we need to know. **Q**

(iii) Suppose that $\omega \in K \setminus F^*$, $\omega' \in C^{\mathbb{l}}$ and $\omega\restriction D = \omega'\restriction D$. Then $\omega' \in K$. **P** Let A be the set of points in $]a, \infty[$ where ω is not continuous, and for $t \in A$ let ϵ_t be 1 if ω' is continuous on the right at t, -1 if ω' is continuous on the left at t. For each finite $J \subseteq A$, (ii) tells us that there is an $\omega_J \in K$ such that $\omega_J\restriction D = \omega\restriction D = \omega'\restriction D$ and, for $t \in J$, ω_J is continuous on the right at t if $\epsilon_t = 1$, and continuous on the left at t if $\epsilon_t = -1$. As both ω_J and ω' are càllàl, this means that $\omega_J(t) = \omega'(t)$ for $t \in J$. Taking a cluster point $\omega^* \in K$ of ω_J as J increases through the finite subsets of A, we see that $\omega^*\restriction(A \cup D) = \omega'\restriction(A \cup D)$.

Now recall that $\omega \in E^*$, so that

$$\omega'\restriction D \cap [0, a] = \omega\restriction D \cap [0, a] = \tilde{\omega}\restriction D \cap [0, a].$$

Since both ω' and $\tilde{\omega}$ are càllàl, $\tilde{\omega}$ is discontinuous at any point of $[0, a[$ at which ω' is discontinuous. Since I arranged that a (if finite) would be in D, $D \cup A$ contains every point at which ω' is discontinuous. But this means that $\omega^* = \omega'$ (438S(a-ii)). So $\omega' \in K$. **Q**

(iv) Suppose that $\tilde{\mu}_{\tilde{\omega}a} K > \gamma \ge 0$. Then there is a zero set $F \subseteq \Omega$ such that $F \cap C^{\mathbb{l}} \subseteq K$ and $\tilde{\mu}_{\tilde{\omega}a}(F \cap C^{\mathbb{l}}) \ge \gamma$. **P** Because $\tilde{\mu}_{\tilde{\omega}a} F^* = \hat{\mu}_{\tilde{\omega}a} F^* = 0$, there is a compact $K' \subseteq K \setminus F^*$ such that $\tilde{\mu}_{\tilde{\omega}a} K' \ge \gamma$. Set $F = \pi_D^{-1}[\pi_D[K']]$; F is a zero set in Ω because $\pi_D[K']$ is a zero set in U^D. By (iii), $F \cap C^{\mathbb{l}} \subseteq K$; and

$$\tilde{\mu}_{\tilde{\omega}a}(F \cap C^{\mathbb{l}}) \ge \tilde{\mu}_{\tilde{\omega}a} K' \ge \gamma. \textbf{ Q}$$

(c) Since E^* and $C^{\mathbb{l}}$ are $\tilde{\mu}_{\tilde{\omega}a}$-conegligible, the Radon measure $\tilde{\mu}_{\tilde{\omega}a}$ is certainly inner regular with respect to the compact subsets of $E^* \cap C^{\mathbb{l}}$; by (b-iv), $\tilde{\mu}_{\tilde{\omega}a}$ is inner regular with respect to the intersections of $C^{\mathbb{l}}$ with zero sets.

455K Corollary Suppose, in 455J, that $\tilde{\omega} \in C_{\mathrm{dlg}}$, the space of càdlàg functions from $[0, \infty[$ to U. Then the subspace measure $\ddot{\mu}_{\tilde{\omega}a}$ on C_{dlg} induced by $\hat{\mu}_{\tilde{\omega}a}$ is a completion regular quasi-Radon measure.

proof The point is that the outer measures $\tilde{\mu}^*_{\tilde{\omega}a}$ and $\hat{\mu}^*_{\tilde{\omega}a}$ agree on subsets of C_{dlg}. **P** Since $\tilde{\mu}_{\tilde{\omega}a}$ extends $\hat{\mu}_{\tilde{\omega}a}$, $\tilde{\mu}^*_{\tilde{\omega}a} A \le \hat{\mu}^*_{\tilde{\omega}a} A$ for every $A \subseteq \Omega$. On the other hand, if $A \subseteq C_{\mathrm{dlg}}$ and $\tilde{\mu}^*_{\tilde{\omega}a} A < \gamma$, there is an $E \supseteq A$ such that $\tilde{\mu}_{\tilde{\omega}a} E < \gamma$. By 455J, there is a zero set $F \subseteq \Omega$ such that $E \cap F \cap C^{\mathbb{l}} = \emptyset$ and $\tilde{\mu}_{\tilde{\omega}a}(F \cap C^{\mathbb{l}}) \ge 1 - \gamma$. Now

$$\hat{\mu}^*_{\tilde{\omega}a} A \le \hat{\mu}^*_{\tilde{\omega}a}(C_{\mathrm{dlg}} \setminus F) = \hat{\mu}_{\tilde{\omega}a}(\Omega \setminus F)$$

(because $\hat{\mu}^*_{\tilde{\omega}a} C_{\mathrm{dlg}} = 1$, by 455G(c-ii))

$$= \tilde{\mu}_{\tilde{\omega}a}(\Omega \setminus F) = \tilde{\mu}_{\tilde{\omega}a}(C^{\mathbb{l}} \setminus F) \le \gamma.$$

As γ is arbitrary, $\hat{\mu}^*_{\tilde{\omega}a} A \le \tilde{\mu}^*_{\tilde{\omega}a} A$. **Q**

Write $\ddot{\tilde{\mu}}_{\tilde{\omega}a}$ for the subspace measure on C_{dlg} induced by $\tilde{\mu}_{\tilde{\omega}a}$. By 214Cd, the outer measures $\ddot{\tilde{\mu}}^*_{\tilde{\omega}a} = \tilde{\mu}^*_{\tilde{\omega}a}\restriction\mathcal{P}C_{\mathrm{dlg}}$ and $\ddot{\mu}^*_{\tilde{\omega}a}$ are the same. Because $\ddot{\tilde{\mu}}_{\tilde{\omega}a}$ and $\ddot{\mu}_{\tilde{\omega}a}$ are both complete probability measures, they must be identical (213C). Because $\tilde{\mu}_{\tilde{\omega}a}$ is a Radon measure, $\ddot{\tilde{\mu}}_{\tilde{\omega}a} = \ddot{\mu}_{\tilde{\omega}a}$ is quasi-Radon (415B). Because $\hat{\mu}_{\tilde{\omega}a}$ is the completion of a Baire measure, therefore inner regular with respect to the zero sets in Ω (412D, 412Ha), $\ddot{\mu}_{\tilde{\omega}a}$ is inner regular with respect to the zero sets in C_{dlg}, by 412Pd, and is completion regular.

455L Stopping times We need the continuous-time version of the concept of 'stopping time' introduced in §275. Let Ω be a set, Σ a σ-algebra of subsets of Ω and $\langle\Sigma_t\rangle_{t\ge0}$ a non-decreasing family of σ-subalgebras of Σ. (Such a family is called a **filtration**.) For $t \ge 0$, set $\Sigma_t^+ = \bigcap_{s>t}\Sigma_s$, so that $\langle\Sigma_t^+\rangle_{t\ge0}$ also is a non-decreasing family of σ-algebras. Of course $\Sigma_t^+ = \bigcap_{s>t}\Sigma_s^+$ for every $t \ge 0$.

(a) A function $\tau : \Omega \to [0,\infty]$ is a **stopping time adapted to** $\langle\Sigma_t\rangle_{t\ge0}$ if $\{\omega : \omega \in \Omega,\, \tau(\omega) \le t\}$ belongs to Σ_t for every $t \ge 0$.

Note that in this case τ will be Σ-measurable.

(b) A function $\tau : \Omega \to [0,\infty]$ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$ iff $\{\omega : \tau(\omega) < t\} \in \Sigma_t$ for every $t \ge 0$. **P** (i) If τ is adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$ and $t \ge 0$, then $\{\omega : \tau(\omega) \le q\} \in \Sigma_q^+ \subseteq \Sigma_t$ whenever $0 \le q < t$, so

$$\{\omega : \tau(\omega) < t\} = \bigcup_{q\in\mathbb{Q}\cap[0,t[}\{\omega : \tau(\omega) \le q\} \in \Sigma_t.$$

Thus τ satisfies the condition. (ii) If τ satisfies the condition and $t \ge 0$, set $t_n = t + 2^{-n}$ for each n. Then

$$\{\omega : \tau(\omega) < t_n\} \in \Sigma_{t_n} \subseteq \Sigma_{t_m}$$

whenever $m \le n$, so

$$\{\omega : \tau(\omega) \le t\} = \bigcap_{n\ge m}\{\omega : \tau(\omega) < t_n\} \in \Sigma_{t_m}$$

for every m, and

$$\{\omega : \tau(\omega) \le t\} \in \bigcap_{m\in\mathbb{N}}\Sigma_{t_m} = \Sigma_t^+.$$

As t is arbitrary, τ is adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$. **Q**

(c)(i) Constant functions on Ω are of course stopping times.

(ii) If τ and τ' are stopping times adapted to $\langle\Sigma_t\rangle_{t\ge0}$, so is $\tau + \tau'$. **P**

$$\{\omega : \tau(\omega) + \tau'(\omega) \le t\} = \bigcap_{q\in\mathbb{Q}\cap[0,t]}\{\omega : \tau(\omega) \le q\} \cup \{\omega : \tau'(\omega) \le t - q\} \in \Sigma_t$$

for every $t \ge 0$. **Q**

(iii) (Compare 455Cb and 455E(c-ii)). If τ is a stopping time adapted to $\langle\Sigma_t\rangle_{t\ge0}$, then

$$\Sigma_\tau = \{E : E \in \Sigma,\, E \cap \{\omega : \tau(\omega) \le t\} \in \Sigma_t \text{ for every } t \ge 0\}$$

is a σ-subalgebra of Σ. (The check is elementary.)

(iv) If $\langle\tau_i\rangle_{i\in I}$ is a countable family of stopping times adapted to $\langle\Sigma_t\rangle_{t\ge0}$, then $\tau = \sup_{i\in I}\tau_i$ is adapted to $\langle\Sigma_t\rangle_{t\ge0}$. **P** For any $t \ge 0$,

$$\{\omega : \tau(\omega) \le t\} = \bigcap_{i\in I}\{\omega : \tau_i(\omega) \le t\} \in \Sigma_t. \text{ } \mathbf{Q}$$

(v) If $\langle\tau_i\rangle_{i\in I}$ is a countable family of stopping times adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$, then $\tau = \inf_{i\in I}\tau_i$ is adapted to $\langle\Sigma_t^+\rangle_{t\ge0}$, because

$$\{\omega : \tau(\omega) < t\} = \bigcup_{i\in I}\{\omega : \tau_i(\omega) < t\} \in \Sigma_t$$

for every $\tau \ge 0$.

(d) Now suppose that Y is a topological space and we have a family $\langle X_t\rangle_{t\ge0}$ of functions from Ω to Y, and that $\tau : \Omega \to [0,\infty]$ is any Σ-measurable function. Set $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ when $\tau(\omega) < \infty$. If $(t,\omega) \mapsto X_t(\omega) : [0,\infty[\,\times\,\Omega \to Y$ is $\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma$-measurable, where $\mathcal{B}([0,\infty[)$ is the Borel σ-algebra of $[0,\infty[$, then $X_\tau : \{\omega : \tau(\omega) < \infty\} \to Y$ is Σ-measurable. **P** Setting $\Omega_0 = \{\omega : \tau(\omega) < \infty\}$, the map $\omega \mapsto (\tau(\omega),\omega) : \Omega_0 \to [0,\infty[\,\times\,\Omega$ is $(\Sigma, \mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma)$-measurable (4A3Bc), so X_τ is the composition of a $(\Sigma, \mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma)$-measurable function with a $\mathcal{B}([0,\infty[)\widehat{\otimes}\Sigma$-measurable function and is Σ-measurable, by 4A3Bb. **Q**

***(e)** Again take a topological space Y, a family $\langle X_t\rangle_{t\ge 0}$ of functions from Ω to Y, and a stopping time $\tau:\Omega\to[0,\infty]$ adapted to $\langle\Sigma_t\rangle_{t\ge 0}$. This time, suppose that $\langle X_t\rangle_{t\ge 0}$ is **progressively measurable**, that is, that $(s,\omega)\mapsto X_s(\omega):[0,t]\times\Omega\to Y$ is $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$-measurable for every $t\ge 0$, and moreover that Σ_t is closed under Souslin's operation (421B) for every t. Then X_τ, as defined in (d), will be Σ_τ-measurable. **P** Suppose that $H\subseteq Y$ is open, and set $E=\{\omega:\omega\in\operatorname{dom}X_\tau,\,X_\tau(\omega)\in H\}$. Of course $\langle X_t\rangle_{t\ge 0}$ satisfies the condition of (d), so $E\in\Sigma$. Take any $t\ge 0$. Then

$$\begin{aligned}\{(s,\omega):0\le s\le t,\,\tau(\omega)=s\}=\bigcap_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}}&\{s:2^{-n}(i-1)<s\le\min(t,2^{-n}i)\}\\&\times\{\omega:2^{-n}(i-1)<\tau(\omega)\le\min(t,2^{-n}i)\}\\ \in\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t,&\end{aligned}$$

$$\{(s,\omega):s\le t,\,X_s(\omega)\in H\}\in\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t,$$

so

$$W=\{(s,\omega):s\le t,\,\tau(\omega)=s,\,X_s(\omega)\in H\}$$

also belongs to $\mathcal{B}([0,t])\widehat{\otimes}\Sigma_t$. Consequently the projection of W onto Ω belongs to $\mathcal{S}(\Sigma_t)=\Sigma_t$ (423O). But this is just

$$\{\omega:\tau(\omega)\le t,\,X_{\tau(\omega)}\in H\}=E\cap\{\omega:\tau(\omega)\le t\}.$$

As t is arbitrary, $E\in\Sigma_\tau$; as H is arbitrary, X_τ is Σ_τ-measurable. **Q**

***(f)** There are some technical points concerning stopping times which are perhaps worth noting here.

(i) Suppose that μ is a probability measure with domain Σ and null ideal $\mathcal{N}(\mu)$. Then we can form the completion $\hat\mu$ with domain $\hat\Sigma$. If we now set $\hat\Sigma_t=\{E\triangle A:E\in\Sigma_t,\,A\in\mathcal{N}(\mu)\}$, $\langle\hat\Sigma_t\rangle_{t\ge 0}$ and $\langle\hat\Sigma_t^+\rangle_{t\ge 0}$ are filtrations, where $\hat\Sigma_t^+=\bigcap_{s>t}\hat\Sigma_s$ for $t\ge 0$.

(ii) We find that $\hat\Sigma_t^+=\{E\triangle A:E\in\Sigma_t^+,\,A\in\mathcal{N}(\mu)\}$ for every $t\ge 0$. **P** Of course

$$\{E\triangle A:E\in\Sigma_t^+,\,A\in\mathcal{N}(\mu)\}\subseteq\textstyle\bigcap_{s>t}\{E\triangle A:E\in\Sigma_s,\,A\in\mathcal{N}(\mu)\}=\hat\Sigma_t^+.$$

If $F\in\hat\Sigma_t^+$, then for every $q\in\mathbb{Q}$ such that $q>t$ there is an $E_q\in\Sigma_q$ such that $F\triangle E_q$ is negligible. Set

$$E=\textstyle\bigcup_{q\in\mathbb{Q},q>t}\bigcap_{q'\in\mathbb{Q},t<q'\le q}E_{q'},\qquad A=F\triangle E;$$

then $E\in\Sigma_t^+$, $A\in\mathcal{N}(\mu)$ and $F=E\triangle A$. **Q**

(iii) Of course every stopping time adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$ is adapted to $\langle\hat\Sigma_t^+\rangle_{t\ge 0}$. Conversely, if $\tau:\Omega\to[0,\infty]$ is a stopping time adapted to $\langle\hat\Sigma_t^+\rangle_{t\ge 0}$, there is a stopping time τ', adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$, such that $\tau=_{\text{a.e.}}\tau'$. **P** For each $q\in\mathbb{Q}\cap[0,\infty[$, set $F_q=\{\omega:\tau(\omega)<q\}$; by (b), $F_q\in\hat\Sigma_q$ and there is an $E_q\in\Sigma_q$ such that $F_q\triangle E_q$ is negligible. For $\omega\in\Omega$, set $\tau'(\omega)=\inf\{q:q\in\mathbb{Q}\cap[0,\infty[,\,\omega\in E_q\}$, counting $\inf\emptyset$ as ∞. Then $\{\omega:\tau'(\omega)<t\}=\bigcup_{q\in\mathbb{Q}\cap[0,t[}E_q$ belongs to Σ_t for every t, so τ' is adapted to $\langle\Sigma_t^+\rangle_{t\ge 0}$. And $\{\omega:\tau'(\omega)\ne\tau(\omega)\}\subseteq\bigcup_{q\in\mathbb{Q}\cap[0,\infty[}E_q\triangle F_q$ is negligible. **Q**

(iv) Continuing from (iii) just above, we find that, defining $\hat\Sigma_\tau^+$ from $\langle\hat\Sigma_t^+\rangle_{t\ge 0}$ and τ and $\Sigma_{\tau'}^+$ from $\langle\Sigma_t^+\rangle_{t\ge 0}$ and τ' by the formula in (c-iii), then $\hat\Sigma_\tau^+=\{F\triangle A:F\in\Sigma_{\tau'}^+,\,A\in\mathcal{N}(\mu)\}$. **P** Let A_0 be the negligible set $\{\omega:\tau(\omega)\ne\tau'(\omega)\}$. ($\alpha$) If $E\in\Sigma_{\tau'}^+$, then for every $t\ge 0$ we have

$$E\cap\{\omega:\tau'(\omega)\le t\}\in\Sigma_t^+,$$

$$(E\cap\{\omega:\tau(\omega)\le t\})\triangle(E\cap\{\omega:\tau'(\omega)\le t\})\subseteq A_0\in\mathcal{N}(\mu),$$

so (using (ii)) $E\cap\{\omega:\tau(\omega)\le t\}\in\hat\Sigma_t^+$; as t is arbitrary, $E\in\hat\Sigma_\tau^+$. (β) If $F\in\hat\Sigma_\tau^+$, then for every $q\in\mathbb{Q}\cap[0,\infty[$ the sets $F\cap\{\omega:\tau(\omega)\le q\}$ and $F\cap\{\omega:\tau'(\omega)\le q\}$ belong to $\hat\Sigma_q^+$, so there is an $E_q\in\Sigma_q^+$ such that $E_q\triangle(F\cap\{\omega:\tau'(\omega)\le q\})$ is negligible. Set $E_q'=\bigcup_{r\in\mathbb{Q}\cap[0,q]}E_r$ for $q\in\mathbb{Q}\cap[0,\infty[$; then $E_q'\in\Sigma_q^+$ and $E_q'\triangle(F\cap\{\omega:\tau'(\omega)\le q\})$ is negligible for each q, while $E_q'\subseteq E_r'$ if $q\le r$ in $\mathbb{Q}\cap[0,\infty[$. It follows that

$$\bigcap_{q\in\mathbb{Q}\cap[t,\infty[} E'_q = \bigcap_{q\in\mathbb{Q}\cap[t,s]} E'_q \in \Sigma^+_s$$

whenever $t < s$ in $[0,\infty[$, so that $\bigcap_{q\in\mathbb{Q}\cap[t,\infty[} E'_q \in \Sigma^+_t$ for every t. Set

$$E = \bigcap_{q\in\mathbb{Q}\cap[0,\infty[} E'_q \cup \{\omega : \tau'(\omega) > q\}.$$

Then

$$\begin{aligned} E \cap \{\omega : \tau'(\omega) \le t\} &= \{\omega : \tau'(\omega) \le t\} \cap \bigcap_{q\in\mathbb{Q}\cap[0,\infty[} (E'_q \cup \{\omega : \tau'(\omega) > q\}) \\ &= \{\omega : \tau'(\omega) \le t\} \cap \bigcap_{q\in\mathbb{Q}\cap[0,t[} (E'_q \cup \{\omega : \tau'(\omega) > q\}) \cap \bigcap_{q\in\mathbb{Q}\cap[t,\infty[} E'_q \\ &\in \Sigma^+_t \end{aligned}$$

for any $t \ge 0$, so $E \in \Sigma^+_{\tau'}$. If we look at $(E\triangle F) \cap \{\omega : \tau'(\omega) < \infty\}$, we see that this is included in the negligible set

$$\bigcup_{q\in\mathbb{Q}\cap[0,\infty[} E'_q \triangle (F \cap \{\omega : \tau'(\omega) \le q\})$$

because $\{\omega : \tau'(\omega) < \infty\} \cap F$ is just

$$\{\omega : \tau'(\omega) < \infty\} \cap \bigcap_{q\in\mathbb{Q}\cap[0,\infty[} (F \cap \{\omega : \tau'(\omega) \le q\}) \cup \{\omega : \tau'(\omega) > q\}.$$

As for the set $H = \{\omega : \tau'(\omega) = \infty\}$, this belongs to Σ, and every subset of H belonging to Σ also belongs to $\Sigma^+_{\tau'}$. Let $H' \in \Sigma$ be such that $H' \subseteq H$ and $H'\triangle(F \cap H)$ is negligible; then $E' = H' \cup (E \setminus H)$ belongs to $\Sigma^+_{\tau'}$ and differs from F by a negligible set. **Q**

455M Hitting times I mention a class of stopping times which is particularly important in applications, and also very helpful in giving an idea of the concept.

Proposition Let U be a Polish space and C_{dlg} the set of càdlàg functions from $[0,\infty[$ to U. Let $A \subseteq U$ be an analytic set, and define $\tau : C_{\mathrm{dlg}} \to [0,\infty]$ by setting

$$\tau(\omega) = \inf\{t : \omega(t) \in A\}$$

for $\omega \in C_{\mathrm{dlg}}$, counting $\inf\emptyset$ as ∞.

(a) Let Σ be a σ-algebra of subsets of C_{dlg} closed under Souslin's operation and including the algebra generated by the functionals $\omega \mapsto \omega(t)$ for $t \ge 0$. Then τ is Σ-measurable.

(b) For $t \ge 0$ let Σ_t be

$$\{F : F \in \Sigma,\ \omega' \in F \text{ whenever } \omega,\ \omega' \in C_{\mathrm{dlg}},\ \omega \in F \text{ and } \omega\restriction[0,t] = \omega'\restriction[0,t]\},$$

and $\Sigma^+_t = \bigcap_{s>t} \Sigma_t$. Then τ is a stopping time adapted to $\langle\Sigma^+_t\rangle_{t\ge0}$.

(c) If A is closed, then τ is adapted to $\langle\Sigma_t\rangle_{t\ge0}$.

proof (a)(i) It will help to recall from 4A3Q that there is a Polish topology $\mathfrak{S}$ on C_{dlg} such that the corresponding Borel σ-algebra $\mathcal{B}(C_{\mathrm{dlg}})$ is just the σ-algebra generated by the coordinate functions $\omega \mapsto \omega(t)$, so is included in Σ. In this case, every $\mathfrak{S}$-analytic set, being $\mathfrak{S}$-Souslin-F (423Eb), belongs to Σ.

(ii) The set

$$W = \{(\omega, t, x) : \omega \in C_{\mathrm{dlg}},\ t \ge 0,\ x \in U,\ \omega(t) = x\}$$

is a Borel subset of $C_{\mathrm{dlg}} \times [0,\infty[\times U$. **P** If ρ is a metric on U inducing its topology and D is a countable dense subset of U,

$$W = \bigcap_{n\in\mathbb{N}} \bigcup_{q\in\mathbb{Q},q\ge0} \bigcup_{y\in D} \{(\omega, t, x) : t \in [q - 2^{-n}, q],$$

$$\rho(\omega(q), y) \le 2^{-n},\ \rho(x, y) \le 2^{-n}\}. \textbf{ Q}$$

(iii) Since C_{dlg}, $[0,\infty[$ and U are all Polish, W is an analytic set. Now, for any $t \ge 0$,

$$W' = \{(\omega, s, x) : s \in [0, t[,\ x \in A,\ (\omega, s, x) \in W\}$$

is analytic and its projection

$$\{\omega : \tau(\omega) < t\} = \{\omega : \text{there are } s,\ x \text{ such that } (\omega, s, x) \in W'\}$$

is analytic and belongs to Σ. As t is arbitrary, τ is Σ-measurable.

(b) Now, given $t \geq 0$, $F = \{\omega : \tau(\omega) < t\}$ belongs to Σ, and if $\omega \in F$, $\omega' \in C_{\mathrm{dlg}}$ are such that $\omega'\restriction[0, t] = \omega\restriction[0, t]$, there is an $s < t$ such that $\omega'(s) = \omega(s) \in A$, so $\tau(\omega') < t$ and $\omega' \in F$. Thus $F \in \Sigma_t$. As t is arbitrary, τ is a stopping time adapted to $\langle\Sigma_t^+\rangle_{t\geq 0}$, by 455Lb.

(c) As A is closed and every member of C_{dlg} is continuous on the right, $\omega(\tau(\omega)) \in A$ whenever $\tau(\omega) < \infty$. So if ω, $\omega' \in C_{\mathrm{dlg}}$, $\tau(\omega) \leq t$ and $\omega'\restriction[0, t] = \omega\restriction[0, t]$, then $\omega'(\tau(\omega)) \in A$ and $\tau(\omega') \leq t$. Thus $\{\omega : \tau(\omega) \leq t\} \in \Sigma_t$ for every t, and τ is adapted to $\langle\Sigma_t\rangle_{t\geq 0}$.

455N We need an elementary fact about narrow (more properly, vague) convergence.

Lemma Let (U, ρ) be a metric space, $n \in \mathbb{N}$ and $f : U^{n+1} \to \mathbb{R}$ a bounded uniformly continuous function. Let $\langle\nu_x^{(k)}\rangle_{k<n,x\in U}$ be a family of topological probability measures on U such that $x \mapsto \nu_x^{(k)}$ is continuous for the narrow topology for every $k < n$. Then

$$y \mapsto \iint \ldots \int f(y, x_1, \ldots , x_n)\nu_{x_{n-1}}^{(n-1)}(dx_n) \ldots \nu_{x_1}^{(1)}(dx_2)\nu_y^{(0)}(dx_1)$$

is defined everywhere on U and continuous.

proof Induce on n. If $n = 0$ the formula is just $y \mapsto f(y)$, so the result is trivial. For the inductive step to $n \geq 1$, set

$$g(y, x_1) = \int \ldots \int f(y, x_1, \ldots , x_n)\nu_{x_{n-1}}^{(n-1)}(dx_n) \ldots \nu_{x_1}^{(1)}(dx_2)$$

for y, $x_1 \in U$; by the inductive hypothesis this is well-defined and $x_1 \mapsto g(y, x_1)$ is continuous. Note that g is bounded because f is. It follows that $h(y) = \int g(y, x_1)\nu_y^{(0)}(dx_1)$ is defined for every y. I need to show that h is continuous. Take any $y \in U$ and $\epsilon > 0$. Then there is a $\delta_0 > 0$ such that $|f(y', x_1, \ldots , x_n) - f(y, x_1, \ldots , x_n)| \leq \epsilon$ whenever $\rho(y', y) \leq \delta_0$ and $x_1, \ldots , x_n \in U$; so that $|g(y', x_1) - g(y, x_1)| \leq \epsilon$ whenever $\rho(y', y) \leq \delta_0$ and $x_1 \in U$. Next, because $x \mapsto \nu_x^{(0)}$ is narrowly continuous, $x \mapsto \int g(y, x_1)\nu_x^{(0)}(dx_1)$ is continuous (437Jf/437Kb), and there is a $\delta \in\ ]0, \delta_0]$ such that $|\int g(y, x_1)\nu_{y'}^{(0)}(dx_1) - \int g(y, x_1)\nu_y^{(0)}(dx_1)| \leq \epsilon$ whenever $\rho(y', y) \leq \delta$. So if $\rho(y', y) \leq \delta$,

$$\begin{aligned} |h(y') - h(y)| &\leq |\int g(y', x_1)\nu_{y'}^{(0)}(dx_1) - \int g(y, x_1)\nu_{y'}^{(0)}(dx_1)| \\ &\qquad + |\int g(y, x_1)\nu_{y'}^{(0)}(dx_1) - \int g(y, x_1)\nu_y^{(0)}(dx_1)| \\ &\leq \int |g(y', x_1) - g(y, x_1)|\nu_{y'}^{(0)}(dx_1) + \epsilon \leq 2\epsilon. \end{aligned}$$

As y and ϵ are arbitrary, h is continuous and the induction proceeds.

455O If both the continuity conditions in 455F are satisfied, we have a version of 455C/455Eb which is much more to the point.

Theorem Suppose that (U, ρ) is a complete metric space, x^* is a point of U, $\langle\nu_x^{(s,t)}\rangle_{0\leq s<t,x\in U}$ is a family of Radon probability measures on U which is both narrowly continuous and uniformly time-continuous on the right, and that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $x \in U$ and $s < t < u$. Let $\hat{\mu}$ be the corresponding completed measure on $\Omega = U^{[0,\infty[}$, as in 455E. Let C_{dlg} be the set of càdlàg functions from $[0, \infty[$ to U, $\ddot{\mu}$ the subspace measure on C_{dlg}, and $\ddot{\Sigma}$ its domain. For $t \geq 0$, let $\ddot{\Sigma}_t$ be

$$\{F : F \in \ddot{\Sigma},\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in C_{\mathrm{dlg}} \text{ and } \omega'\restriction[0, t] = \omega\restriction[0, t]\},$$

and $\ddot{\Sigma}_t^+ = \bigcap_{s>t} \ddot{\Sigma}_t$.

For $\omega \in \Omega$ and $a \ge 0$ let $\hat{\mu}_{\omega a}$ be the completed measure on Ω built from $\omega(0)$ and $\langle \nu_{\omega a x}^{(s,t)} \rangle_{0 \le s < t, x \in U}$ as in 455Eb; let $\ddot{\mu}_{\omega a}$ be the subspace measure on C_{dlg}. Let $\tau : C_{\mathrm{dlg}} \to [0,\infty]$ be a stopping time adapted to $\langle \ddot{\Sigma}_t^+ \rangle_{t \ge 0}$.

(a) $\langle \ddot{\mu}_{\omega,\tau(\omega)} \rangle_{\omega \in C_{\mathrm{dlg}}}$ is a disintegration of $\ddot{\mu}$ over itself.

(b) Set

$$\ddot{\Sigma}_\tau^+ = \{F : F \in \ddot{\Sigma},\, F \cap \{\omega : \tau(\omega) \le t\} \in \ddot{\Sigma}_t^+ \text{ for every } t \ge 0\}.$$

Then $\ddot{\Sigma}_\tau^+$ is a σ-algebra of subsets of C_{dlg}. For a $\ddot{\mu}$-integrable function f on C_{dlg}, write $\ddot{g}_f(\omega) = \int_{C_{\mathrm{dlg}}} f d\ddot{\mu}_{\omega,\tau(\omega)}$ when this is defined in $\mathbb{R}$. Then $\ddot{g}_f$ is a conditional expectation of f on $\ddot{\Sigma}_\tau^+$.

(c) If τ is adapted to $\langle \ddot{\Sigma}_t \rangle_{t \ge 0}$, set

$$\ddot{\Sigma}_\tau = \{F : F \in \ddot{\Sigma},\, F \cap \{\omega : \tau(\omega) \le t\} \in \ddot{\Sigma}_t \text{ for every } t \ge 0\}.$$

Then $\ddot{\Sigma}_\tau$ is a σ-algebra of subsets of C_{dlg}, and $\ddot{g}_f$ is a conditional expectation of f on $\ddot{\Sigma}_\tau$, for every $f \in \mathcal{L}^1(\ddot{\mu})$.

proof (a)(i) I had better begin by checking that the ground is clear. By 455G, $\hat{\mu}^* C_{\mathrm{dlg}} = \hat{\mu}_{\omega a}^* C_{\mathrm{dlg}} = 1$ for every $\omega \in C_{\mathrm{dlg}}$ and $a \ge 0$, so that $\ddot{\mu}$ and $\ddot{\mu}_{\omega a}$ (for $\omega \in C_{\mathrm{dlg}}$) are all probability measures.

Of course $\langle \ddot{\Sigma}_t \rangle_{t \ge 0}$ is a non-decreasing family of σ-subalgebras of $\ddot{\Sigma}$, so that $\langle \ddot{\Sigma}_t^+ \rangle_{t \ge 0}$ is another such family, and we are in the territory explored in 455L.

(ii) Write Σ for the domain of $\hat{\mu}$, and for $t \ge 0$ set

$$\Sigma_t = \{E : E \in \Sigma,\, E \text{ is determined by coordinates in } [0,t]\}.$$

Then $\ddot{\Sigma}_t = \{E \cap C_{\mathrm{dlg}} : E \in \Sigma_t\}$. **P** If $E \in \Sigma_t$, then $E \cap C_{\mathrm{dlg}} \in \ddot{\Sigma}$ and clearly $E \cap C_{\mathrm{dlg}} \in \ddot{\Sigma}_t$. If $F \in \ddot{\Sigma}_t$, let $E \in \Sigma$ be such that $E \cap C_{\mathrm{dlg}} = F$. Applying 455Ec to the stopping time with constant value t, we have

$$\hat{\mu}E = \int_\Omega \hat{\mu}_{\omega t}(E) \hat{\mu}(d\omega).$$

Set

$$E^* = \{\omega : \omega \in \Omega,\, \hat{\mu}_{\omega t}(E) \text{ is defined}\},$$

$$E_0 = \{\omega : \omega \in E^*,\, \hat{\mu}_{\omega t}(E) = 0\}, \quad E_1 = \{\omega : \omega \in E^*,\, \hat{\mu}_{\omega t}(E) = 1\}.$$

Then E^*, E_0 and E_1 are measured by $\hat{\mu}$ and are determined by coordinates in $[0,t]$ (by 455E(b-iii)), and $\hat{\mu}E^* = 1$.

If $\omega \in E^* \cap C_{\mathrm{dlg}}$, then $\hat{\mu}_{\omega t}^* C_{\mathrm{dlg}} = 1$, so

$$\ddot{\mu}_{\omega t}(F) = \hat{\mu}_{\omega t}^*(E \cap C_{\mathrm{dlg}}) = \hat{\mu}_{\omega t}(E).$$

If $\omega \in C_{\mathrm{dlg}}$, let D be a countable dense subset of $[0,t]$ containing t; then

$$\begin{aligned} 1 &= \hat{\mu}_{\omega t}\{\omega' : \omega' \in \Omega,\, \omega' \restriction D = \omega \restriction D\} = \ddot{\mu}_{\omega t}\{\omega' : \omega' \in C_{\mathrm{dlg}},\, \omega' \restriction D = \omega \restriction D\} \\ &= \ddot{\mu}_{\omega t}\{\omega' : \omega' \in C_{\mathrm{dlg}},\, \omega' \restriction [0,t] = \omega \restriction [0,t]\}. \end{aligned}$$

So if $\omega \in E^* \cap C_{\mathrm{dlg}}$,

$$\hat{\mu}_{\omega t}(E) = \ddot{\mu}_{\omega t}(F) = \ddot{\mu}_{\omega t}\{\omega' : \omega' \in F,\, \omega' \restriction [0,t] = \omega \restriction [0,t]\} = \chi F(\omega) \in \{0,1\}$$

because F is determined (relative to C_{dlg}) by coordinates in $[0,t]$. This means that $E_1 \cap C_{\mathrm{dlg}} \subseteq F$ and $E_0 \cap F = \emptyset$, while $E_0 \cup E_1$ is $\hat{\mu}$-conegligible. So if we take

$$E' = E_1 \cup \{\omega : \omega\Omega \setminus E_1 \text{ and there is an } \omega' \in F \text{ such that } \omega' \restriction [0,t] = \omega \restriction [0,t]\},$$

$E' \cap C_{\mathrm{dlg}} = F$, E' is determined by coordinates in $[0,t]$, $E_1 \subseteq E' \subseteq \Omega \setminus E_0$, $\hat{\mu}$ measures E' and $E' \in \Sigma_t$. Thus $\ddot{\Sigma}_t = \{E \cap C_{\mathrm{dlg}} : E \in \Sigma_t\}$, as claimed. **Q**

(iii) Take $n \in \mathbb{N}$, and set $D_n = \{2^{-n}i : i \in \mathbb{N}\}$. Suppose that $\tau : C_{\mathrm{dlg}} \to D_n \cup \{\infty\}$ is a stopping time adapted to $\langle \ddot{\Sigma}_t \rangle_{t \ge 0}$. Then $\langle \hat{\mu}_{\omega,\tau(\omega)} \rangle_{\omega \in C_{\mathrm{dlg}}}$ is a disintegration of $\hat{\mu}$ over $\ddot{\mu}$. **P** For each $i \in \mathbb{N}$, $F_i = \tau^{-1}[\{2^{-n}i\}]$ belongs to $\ddot{\Sigma}_{2^{-n}i}$, so there is an $E_i \in \Sigma$, determined by coordinates in $[0, 2^{-n}i]$, such that $F_i = E_i \cap C_{\mathrm{dlg}}$. For $\omega \in \Omega$, set

$$\check{\tau}(\omega) = \inf\{2^{-n}i : i \in \mathbb{N},\, \omega \in E_i\},$$

counting $\inf\emptyset$ as ∞. Then $\check\tau\restriction C_{\mathrm{dlg}} = \tau$. Also $\check\tau[\Omega] \subseteq D_n \cup \{\infty\}$ is countable, and $\check\tau^{-1}[\{b\}] \in \Sigma$ is determined by coordinates in $[0,b]$ for every $b \in D_n$. By 455Ec, $\langle\hat\mu_{\omega,\check\tau(\omega)}\rangle_{\omega\in\Omega}$ is a disintegration of $\hat\mu$ over itself.

Now take any $E \in \Sigma$. Then

$$\hat\mu E = \int_\Omega \hat\mu_{\omega,\check\tau(\omega)}(E)\hat\mu(d\omega) = \int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\check\tau(\omega)}(E)\ddot\mu(d\omega)$$

(214F)

$$= \int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\tau(\omega)}(E)\ddot\mu(d\omega). \ \mathbf{Q}$$

It follows that $\langle\ddot\mu_{\omega,\tau(\omega)}\rangle_{\omega\in C_{\mathrm{dlg}}}$ is a disintegration of $\ddot\mu$ over itself. $\mathbf{P}$ If $F \in \ddot\Sigma$, there is an $E \in \Sigma$ such that $F = E \cap C_{\mathrm{dlg}}$. Now

$$\ddot\mu F = \hat\mu E = \int_{C_{\mathrm{dlg}}} \hat\mu_{\omega,\tau(\omega)}(E)\ddot\mu(d\omega)$$
$$= \int_{C_{\mathrm{dlg}}} \ddot\mu_{\omega,\tau(\omega)}(E \cap C_{\mathrm{dlg}})\ddot\mu(d\omega) = \int_{C_{\mathrm{dlg}}} \ddot\mu_{\omega,\tau(\omega)}(F)\ddot\mu(d\omega). \ \mathbf{Q}$$

(iv) Now let $\tau : C_{\mathrm{dlg}} \to [0,\infty]$ be any stopping time adapted to $\langle\ddot\Sigma_t^+\rangle_{t\ge0}$. For each $n \in \mathbb{N}$, define $\tau_n : C_{\mathrm{dlg}} \to D_n \cup \{\infty\}$ by setting

$$\tau_n(\omega) = 2^{-n}(i+1) \text{ if } i \in \mathbb{N} \text{ and } 2^{-n}i \le \tau(\omega) < 2^{-n}(i+1),$$
$$= \infty \text{ if } \tau(\omega) = \infty.$$

By 455Lb, $\{\omega : \tau_n(\omega) = t\} \in \ddot\Sigma_t$ for every $t \in D_n$. So (iii) tells us that $\langle\ddot\mu_{\omega,\tau_n(\omega)}\rangle_{\omega\in C_{\mathrm{dlg}}}$ is a disintegration of $\ddot\mu$ over itself.

(v) Suppose that $k \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_k$, $h : U^{k+1} \to \mathbb{R}$ is bounded and uniformly continuous, and $\omega \in C_{\mathrm{dlg}}$. Then

$$\textstyle\int_\Omega h(\omega'(t_0),\ldots,\omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega') = \lim_{n\to\infty}\int_\Omega h(\omega'(t_0),\ldots,\omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega').$$

$\mathbf{P}$ Recall from 455E that

$$\int_\Omega h(\omega'(t_0),\ldots,\omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')$$
$$= \int_U \ldots \int_U h(\omega(0),x_1,\ldots,x_k)\nu^{(t_{k-1},t_k)}_{\omega,\tau(\omega),x_{k-1}}(dx_k)\ldots\nu^{(0,t_1)}_{\omega(0)}(dx_1),$$

and similarly for each τ_n. If $\tau(\omega) \ge t_k$, then

$$\nu^{(t_{i-1},t_i)}_{\omega,\tau_n(\omega),x} = \delta_{\omega(t_i)} = \nu^{(t_{i-1},t_i)}_{\omega,\tau(\omega),x}$$

for $1 \le i \le k$, $n \in \mathbb{N}$ and $x \in U$, so the result is trivial. If $j \le k$ is such that $t_{j-1} \le \tau(\omega) < t_j$, then

$$\nu^{(t_{i-1},t_i)}_{\omega,\tau(\omega),x} = \delta_{\omega(t_i)} \text{ if } i < j,$$
$$= \nu^{(\tau(\omega),t_j)}_{\omega(\tau(\omega))} \text{ if } i = j,$$
$$= \nu^{(t_{i-1},t_i)}_x \text{ if } j < i < k.$$

So

$$\int_\Omega h(\omega'(t_0),\ldots,\omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')$$
$$= \int_U\int_U \ldots \int_U h(\omega(0),\ldots,\omega(j-1),x_j,\ldots,x_k)$$
$$\nu^{(t_{k-1},t_k)}_{x_{k-1}}(dx_k)\ldots\nu^{(t_j,t_{j+1})}_{x_j}(dx_{j+1})\nu^{(\tau(\omega),t_j)}_{\omega(\tau(\omega))}(dx_j).$$

Moreover, there is some n_0 such that $\tau_n(\omega) < t_j$ for every $n \ge n_0$, so that we can use this formula for all such n. Setting

$$g(x) = \int_U \ldots \int_U h(\omega(0), \ldots , \omega(j-1), x, x_{j+1}, \ldots , x_k)\nu_{x_{k-1}}^{(t_{k-1},t_k)}(dx_k) \ldots \nu_x^{(t_j,t_{j+1})}(dx_{j+1})$$

for $x \in U$, we see from 455N that g is continuous, while of course it is also bounded, because h is bounded. At this point, recall that ω is supposed to be continuous on the right, while the system of transitional probabilities is jointly continuous, so that

$$\nu_{\omega(\tau(\omega))}^{(\tau(\omega),t_j)} = \lim_{n\to\infty} \nu_{\omega(\tau_n(\omega))}^{(\tau_n(\omega),t_j)}$$

for the narrow topology, and

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega & h(\omega'(t_0), \ldots , \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega') \\
&= \lim_{n\to\infty} \int_U \int_U \ldots \int_U h(\omega(0), \ldots , \omega(j-1), x_j, \ldots , x_k) \\
&\qquad \nu_{x_{k-1}}^{(t_{k-1},t_k)}(dx_k) \ldots \nu_{x_j}^{(t_j,t_{j+1})}(dx_{j+1})\nu_{\omega(\tau_n(\omega))}^{(\tau_n(\omega),t_j)}(dx_j) \\
&= \lim_{n\to\infty} \int_U g(x_j)\nu_{\omega(\tau_n(\omega))}^{(\tau_n(\omega),t_j)}(dx_j) \\
&= \int_U g(x_j)\nu_{\omega(\tau(\omega))}^{(\tau(\omega),t_j)}(dx_j) \\
&= \int_\Omega h(\omega'(t_0), \ldots , \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega'),
\end{aligned}$$

as claimed. **Q**

(vi) Again suppose that $0 = t_0 < t_1 < \ldots < t_k$. If $h : U^{k+1} \to \mathbb{R}$ is bounded and uniformly continuous, then

$$\int_{C_{\mathrm{dlg}}} \int_\Omega h(\omega'(t_0), \ldots , \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')\ddot\mu(d\omega) = \int_\Omega h(\omega(t_0), \ldots , \omega(t_k))\hat\mu(d\omega).$$

P The point here is that

$$\int_{C_{\mathrm{dlg}}} \int_\Omega h(\omega'(t_0), \ldots , \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega')\ddot\mu(d\omega) = \int_\Omega h(\omega(t_0), \ldots , \omega(t_k))\hat\mu(d\omega)$$

is defined for every $n \in \mathbb{N}$, by (iii) and 452F, as usual. Now the integrands

$$\omega \mapsto \int_\Omega h(\omega'(t_0), \ldots , \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega')$$

converge at every point of C_{dlg}, by (v), and are uniformly bounded, because h is, so that

$$\begin{aligned}
\int_{C_{\mathrm{dlg}}} \int_\Omega & h(\omega'(t_0), \ldots , \omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')\ddot\mu(d\omega) \\
&= \lim_{n\to\infty} \int_{C_{\mathrm{dlg}}} \int_\Omega h(\omega'(t_0), \ldots , \omega'(t_k))\hat\mu_{\omega,\tau_n(\omega)}(d\omega')\ddot\mu(d\omega) \\
&= \int_\Omega h(\omega(t_0), \ldots , \omega(t_k))(d\omega). \text{ } \mathbf{Q}
\end{aligned}$$

If $G \subseteq U^{k+1}$ is open, there is a non-decreasing sequence $\langle h_m\rangle_{m\in\mathbb{N}}$ of uniformly continuous functions from U^{k+1} to $[0,1]$ such that $\chi G = \sup_{m\in\mathbb{N}} h_m$, in which case

$$\begin{aligned}\int_{C_{\rm dlg}} \hat\mu_{\omega,\tau(\omega)}\{\omega' : (\omega'(t_0),\dots,\omega'(t_k)) \in G\}\ddot\mu(d\omega)\\ &= \lim_{m\to\infty}\int_{C_{\rm dlg}}\int_\Omega h_m(\omega'(t_0),\dots,\omega'(t_k))\hat\mu_{\omega,\tau(\omega)}(d\omega')\ddot\mu(d\omega)\\ &= \lim_{m\to\infty}\int_\Omega h_m(\omega(t_0),\dots,\omega(t_k))\hat\mu(d\omega)\\ &= \hat\mu\{\omega : (\omega(t_0),\dots,\omega(t_k)) \in G\}.\end{aligned}$$

By the Monotone Class Theorem, we get

$$\begin{aligned}\int_{C_{\rm dlg}} \hat\mu_{\omega,\tau(\omega)}\{\omega' : (\omega'(t_0),\dots,\omega'(t_k)) \in E\}\ddot\mu(d\omega)\\ &= \hat\mu\{\omega : (\omega(t_0),\dots,\omega(t_k)) \in E\}\end{aligned}$$

for every Borel set $E \subseteq U^{k+1}$. Now recall that $t_0,\dots,t_k$ were any strictly increasing sequence starting at 0, so we can use the Monotone Class Theorem yet again to see that

$$\int_{C_{\rm dlg}} \hat\mu_{\omega,\tau(\omega)}(E)\ddot\mu(d\omega) = \hat\mu(E)$$

for every $E \in \widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ and therefore for every $E \in \Sigma$.

(vii) Finally, if $F \in \ddot\Sigma$, there is an $E \in \Sigma$ such that $F = E \cap C_{\rm dlg}$, so that

$$\ddot\mu(F) = \hat\mu(E) = \int_{C_{\rm dlg}} \hat\mu_{\omega,\tau(\omega)}(E)\ddot\mu(d\omega) = \int_{C_{\rm dlg}} \ddot\mu_{\omega,\tau(\omega)}(F)\ddot\mu(d\omega);$$

which is what we set out to prove.

(b)(i) By 455L(c-iii), $\ddot\Sigma_\tau^+$ is a σ-algebra. If f is a $\ddot\mu$-integrable real-valued function, then $\int_{C_{\rm dlg}} \ddot g_f\ddot\mu = \int_{C_{\rm dlg}} f\ddot\mu$, by (a) and 452F. For $\alpha \in \mathbb{R}$ set

$$E(f,\alpha) = \{\omega : \omega \in C_{\rm dlg},\ \ddot g_f(\omega) \text{ is defined in } \mathbb{R} \text{ and } \ddot g_f(\omega) \le \alpha\},$$

so that $E(f,\alpha) \in \ddot\Sigma$. For $t \ge 0$, set

$$H_t = \{\omega : \omega \in C_{\rm dlg},\ \tau(\omega) \le t\}, \quad H'_t = \{\omega : \omega \in C_{\rm dlg},\ \tau(\omega) < t\},$$

so that $H_t \in \ddot\Sigma_t^+$ and $H'_t \in \ddot\Sigma_t$ (455Lb).

(ii) If ω, $\omega' \in C_{\rm dlg}$ and $s > \tau(\omega)$ are such that $\omega'\restriction[0,s] = \omega\restriction[0,s]$, then $\tau(\omega') = \tau(\omega)$. **P** $H_{\tau(\omega)}$, $H'_{\tau(\omega)}$ and their difference belong to $\ddot\Sigma_s$, so are determined (relative to $C_{\rm dlg}$) by coordinates in $[0,s]$; since $H_{\tau(\omega)} \setminus H'_{\tau(\omega)}$ contains ω, it also contains ω', and $\tau(\omega') = \tau(\omega)$. **Q**

(iii) If f is $\ddot\mu$-integrable, $\alpha \in \mathbb{R}$ and $s > 0$, then $E(f,\alpha) \cap H'_s \in \ddot\Sigma_s$. **P** Certainly $E(f,\alpha) \cap H'_s \in \ddot\Sigma$. If ω, $\omega' \in C_{\rm dlg}$ and $\omega\restriction[0,s] = \omega'\restriction[0,s]$, then

$$\begin{aligned}\omega \in E(f,\alpha) \cap H'_s &\Longrightarrow \tau(\omega) < s \text{ and } \int_{C_{\rm dlg}} fd\ddot\mu_{\omega,\tau(\omega)} \le \alpha\\ &\Longrightarrow \tau(\omega') = \tau(\omega) < s \text{ and } \int_{C_{\rm dlg}} fd\ddot\mu_{\omega,\tau(\omega')} \le \alpha\end{aligned}$$

(by (ii))

$$\Longrightarrow \omega' \in E(f,\alpha) \cap H'_s.$$

So $E(f,\alpha) \cap H'_s$ is determined (relative to $C_{\rm dlg}$) by coordinates in $[0,s]$ and belongs to $\ddot\Sigma_s$. **Q**

Consequently $E(f,\alpha) \cap H_t \in \ddot\Sigma_t^+$ for every $t \ge 0$. **P** $H_t = \bigcap_{n\in\mathbb{N}} H'_{t_n}$ where $t_n = t + 2^{-n}$ for each n, so

$$E(f,\alpha) \cap H_t = \bigcap_{n\ge m} E(f,\alpha) \cap H'_{t_n}$$

belongs to $\ddot{\Sigma}_{t_m}$ for every $m \in \mathbb{N}$, and $E(f,\alpha) \cap H_t \in \ddot{\Sigma}_t^+$. **Q**

Thus $E(f,\alpha) \in \ddot{\Sigma}_\tau^+$ for every α. As α is arbitrary, $\operatorname{dom} \ddot{g}_f \in \ddot{\Sigma}_\tau^+$ and $\ddot{g}_f$ is $\ddot{\Sigma}_\tau^+$-measurable.

(iv) Define $\langle \tau_n \rangle_{n\in\mathbb{N}}$ as in (a-iv) above, so that each τ_n is a stopping time adapted to $\langle \ddot{\Sigma}_t \rangle_{t\ge 0}$ and $\langle \tau_n(\omega) \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with limit $\tau(\omega)$ for every ω. For a $\ddot{\mu}$-integrable real-valued function f on C_{dlg}, $\omega \in C_{\mathrm{dlg}}$ and $n \in \mathbb{N}$, set

$$\ddot{g}_f^{(n)}(\omega) = \int_{C_{\mathrm{dlg}}} f d\ddot{\mu}_{\omega,\tau_n(\omega)}$$

whenever the right-hand side is defined in $\mathbb{R}$. By (a), $\int_{C_{\mathrm{dlg}}} \ddot{g}_f^{(n)} d\ddot{\mu} = \int_{C_{\mathrm{dlg}}} f d\ddot{\mu}$. We have seen also, in (a-iii), that each τ_n has an extension $\dot{\tau}_n$ which is a stopping time on Ω of the type considered in 455Ec. So if we take a $\hat{\mu}$-integrable function $\tilde{f}$ extending f, and set

$$g_{\tilde{f}}^{(n)}(\omega) = \int_\Omega \tilde{f} d\hat{\mu}_{\omega,\dot{\tau}_n(\omega)}$$

whenever $\omega \in \Omega$ is such that the integral is defined in $\mathbb{R}$, $g_{\tilde{f}}^{(n)}$ will be a conditional expectation of $\tilde{f}$ on $\Sigma_{\dot{\tau}_n}$, the algebra of sets $E \in \Sigma$ such that $E \cap \{\omega : \dot{\tau}_n(\omega) \le t\}$ is determined by coordinates in $[0,t]$ for every $t \ge 0$.

If $\omega \in C_{\mathrm{dlg}}$, then C_{dlg} has full outer measure for $\hat{\mu}_{\omega,\tau_n(\omega)} = \hat{\mu}_{\omega,\dot{\tau}_n(\omega)}$, so

$$g_{\tilde{f}}^{(n)}(\omega) = \int_\Omega \tilde{f} d\hat{\mu}_{\omega,\dot{\tau}_n(\omega)} = \int_{C_{\mathrm{dlg}}} f d\ddot{\mu}_{\omega,\tau_n(\omega)} = \ddot{g}_f^{(n)}(\omega)$$

whenever either is defined.

(v) Set

$$\ddot{\Sigma}_{\tau_n} = \{F : F \in \ddot{\Sigma},\ F \cap \{\omega : \tau_n(\omega) \le t\} \in \ddot{\Sigma}_t \text{ for every } t \ge 0\}.$$

Then every $F \in \ddot{\Sigma}_{\tau_n}$ is of the form $\tilde{F} \cap C_{\mathrm{dlg}}$ where $\tilde{F} \in \Sigma_{\dot{\tau}_n}$. **P** Recall that τ_n and $\dot{\tau}_n$ take values in $D_n \cup \{\infty\}$, where $D_n = \{2^{-n}i : i \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, set $F_i = \{\omega : \omega \in F,\ \tau_n(\omega) = 2^{-n}i\}$; then $F_i \in \ddot{\Sigma}_{2^{-n}i}$, so there is an $E_i \in \Sigma_{2^{-n}i}$ such that $F_i = E_i \cap C_{\mathrm{dlg}}$ (a-ii). Let $E_\infty \in \Sigma$ be such that $E_\infty \cap C_{\mathrm{dlg}} = \{\omega : \tau_n(\omega) = \infty\}$, and try

$$\tilde{F} = \bigcup_{i\in\mathbb{N}} (E_i \cap \dot{\tau}_n^{-1}[\{2^{-n}i\}]) \cup (E_\infty \cap \dot{\tau}_n^{-1}[\{\infty\}]).$$

Then $\tilde{F} \cap C_{\mathrm{dlg}} = F$ (because $\dot{\tau}_n$ extends τ_n) and $\tilde{F} \in \Sigma_{\dot{\tau}_n}$ (because

$$\tilde{F} \cap \dot{\tau}_n^{-1}[\{2^{-n}i\}] = E_i \cap \dot{\tau}_n^{-1}[\{2^{-n}i\}] \in \Sigma_{2^{-n}i}$$

for every i). **Q**

(vi) If f is $\ddot{\mu}$-integrable, then $\ddot{g}_f^{(n)}$ is a conditional expectation of f on $\ddot{\Sigma}_{\tau_n}$ for every n. **P** Take $F \in \ddot{\Sigma}_{\tau_n}$. Then there are an $\tilde{F} \in \Sigma_{\dot{\tau}_n}$ such that $F = \tilde{F} \cap C_{\mathrm{dlg}}$, and a $\hat{\mu}$-integrable $\tilde{f}$ such that $f = \tilde{f} \restriction C_{\mathrm{dlg}}$. So

$$\int_F f d\ddot{\mu} = \int_{\tilde{F}} \tilde{f} d\hat{\mu} = \int_{\tilde{F}} g_{\tilde{f}}^{(n)} d\hat{\mu}$$

(455E(c-ii))

$$= \int_{\tilde{F}} \int_\Omega \tilde{f} d\hat{\mu}_{\omega,\dot{\tau}_n(\omega)} \hat{\mu}(d\omega) = \int_F \int_\Omega \tilde{f} d\hat{\mu}_{\omega,\tau_n(\omega)} \ddot{\mu}(d\omega)$$

(because $\hat{\mu}^* C_{\mathrm{dlg}} = 1$, $F = \tilde{F} \cap C_{\mathrm{dlg}}$ and $\tau_n = \dot{\tau}_n \restriction C_{\mathrm{dlg}}$)

$$= \int_F \int_{C_{\mathrm{dlg}}} f d\ddot{\mu}_{\omega,\tau_n(\omega)} \ddot{\mu}(d\omega) = \int_F \ddot{g}_f^{(n)} d\ddot{\mu}. \text{ **Q**}$$

(vii) Let Φ be the set of those $\ddot{\mu}$-integrable real-valued functions f such that $\lim_{n\to\infty} \int_{C_{\mathrm{dlg}}} |\ddot{g}_f - \ddot{g}_f^{(n)}| d\ddot{\mu} = 0$. For $J \subseteq [0,\infty[$ let $\pi_J : \Omega \to U^J$ be the restriction map. By (a-v), $f = h\pi_J \restriction C_{\mathrm{dlg}}$ belongs to Φ whenever $J \subseteq [0,\infty[$ is finite and $h : U^J \to \mathbb{R}$ is bounded and uniformly continuous, since in this case

$$\begin{aligned}\ddot g_f(\omega) &= \int_{C_{\mathrm{dlg}}} h\pi_J d\ddot\mu_{\omega,\tau(\omega)} = \int_\Omega h\pi_J d\hat\mu_{\omega,\tau(\omega)} = \lim_{n\to\infty}\int_\Omega h\pi_J d\hat\mu_{\omega,\tau_n(\omega)}\\ &= \lim_{n\to\infty}\int_{C_{\mathrm{dlg}}} h\pi_J d\ddot\mu_{\omega,\tau_n(\omega)} = \lim_{n\to\infty}\ddot g_f^{(n)}(\omega)\end{aligned}$$

for every $\omega\in C_{\mathrm{dlg}}$. Next, $\ddot g_{\alpha f} =_{\text{a.e.}} \alpha\ddot g_f$, $\ddot g_{f+f'} =_{\text{a.e.}} \ddot g_f + \ddot g_{f'}$ and

$$\int_{C_{\mathrm{dlg}}}|\ddot g_f-\ddot g_{f'}|d\ddot\mu = \int_{C_{\mathrm{dlg}}}|\ddot g_{f-f'}|d\ddot\mu \le \int_{C_{\mathrm{dlg}}}\ddot g_{|f-f'|}d\ddot\mu = \int_{C_{\mathrm{dlg}}}|f-f'|d\ddot\mu$$

for all f, $f'\in\mathcal{L}^1(\ddot\mu)$ and $\alpha\in\mathbb{R}$; and we have similar expressions for every $\ddot g_f^{(n)}$. So $f+f'\in\Phi$ and $\alpha f\in\Phi$ whenever f, $f'\in\Phi$, and moreover $f\in\Phi$ whenever $f\in\mathcal{L}^1(\ddot\mu)$ and there is a sequence $\langle f_k\rangle_{f\in\mathbb{N}}$ in Φ such that $\lim_{k\to\infty}\int_{C_{\mathrm{dlg}}}|f-f_k|d\ddot\mu=0$.

If $J\subseteq[0,\infty[$ is finite and $G\subseteq U^J$, then $(\chi G)\pi_J\restriction C_{\mathrm{dlg}}\in\Phi$. **P** There is a non-decreasing sequence $\langle h_k\rangle_{k\in\mathbb{N}}$ of bounded uniformly continuous functions on U^J with limit χG; now $h_k\pi_J\restriction C_{\mathrm{dlg}}\in\Phi$ and $(\chi G)\pi_J\restriction C_{\mathrm{dlg}} = \lim_{k\to\infty}h_k\pi_J\restriction C_{\mathrm{dlg}}$. **Q** By the Monotone Class Theorem, $(\chi E)\pi_J\restriction C_{\mathrm{dlg}}\in\Phi$ whenever $J\subseteq[0,\infty[$ is finite and $E\in\mathcal{B}(U^J)$. By the Monotone Class Theorem again, $\chi(E\cap C_{\mathrm{dlg}})\in\Phi$ whenever $E\in\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$. Since we surely have $f'\in\Phi$ whenever $f\in\Phi$ and $f'=f$ $\ddot\mu$-a.e., $\chi(E\cap C_{\mathrm{dlg}})\in\Phi$ whenever $E\in\Sigma$, that is, $\chi E\in\Phi$ for every $E\in\ddot\Sigma$. It follows at once that $\Phi=\mathcal{L}^1(\ddot\mu)$.

(viii) We are nearly home. Suppose that $f\in\mathcal{L}^1(\ddot\mu)$ and $F\in\ddot\Sigma_\tau^+$. If $n\in\mathbb{N}$, then $F\in\ddot\Sigma_{\tau_n}$. **P** For any $t>0$,

$$F\cap\{\omega:\tau(\omega)<t\} = \bigcup_{q\in\mathbb{Q},q<t}F\cap\{\omega:\tau(\omega)\le q\}\in\ddot\Sigma_t.$$

So, for any $i\in\mathbb{N}$,

$$F\cap\{\omega:\tau_n(\omega)\le 2^{-n}i\} = F\cap\{\omega:\tau(\omega)<2^{-n}i\}\in\ddot\Sigma_{2^{-n}i}.\ \mathbf{Q}$$

So $\int_F\ddot g_f^{(n)}d\ddot\mu=\int_F fd\ddot\mu$. But $f\in\Phi$, so

$$\int_F\ddot g_f d\ddot\mu = \lim_{n\to\infty}\int_F\ddot g_f^{(n)}d\ddot\mu = \int_F fd\ddot\mu.$$

Since we already know, from (iii) above, that $\operatorname{dom}\ddot g_f\in\ddot\Sigma_\tau^+$ and $\ddot g_f$ is $\ddot\Sigma_\tau^+$-measurable, $\ddot g_f$ is a conditional expectation of f on $\ddot\Sigma_\tau^+$, as claimed.

(c)(i) By 455L(c-iii) again, $\ddot\Sigma_\tau$ is a σ-algebra.

(ii) If ω, $\omega'\in C_{\mathrm{dlg}}$ and $\omega'\restriction[0,\tau(\omega)]=\omega\restriction[0,\tau(\omega)]$ then $\ddot\mu_{\omega',\tau(\omega')}=\ddot\mu_{\omega,\tau(\omega)}$. **P** Set $t=\tau(\omega)$. This time, H_t and H'_t, defined as in (b-i), belong to $\ddot\Sigma_t$, so their difference belongs to $\ddot\Sigma_t$ and is determined (relative to C_{dlg}) by coordinates in $[0,t]$; so $\omega'\in H_t\setminus H'_t$ and $\tau(\omega')=t$. Now, reading off the definition in 455Eb, $\nu^{(s,u)}_{\omega'tx}=\nu^{(s,u)}_{\omega tx}$ for all s, u and x, so $\ddot\mu_{\omega't}=\ddot\mu_{\omega t}$. **Q**

(iii) It follows that if $f\in\mathcal{L}^1(\ddot\mu)$ and $\alpha\in\mathbb{R}$ then $F=\{\omega:\omega\in C_{\mathrm{dlg}},\ \ddot g_f(\omega)$ is defined and at most $\alpha\}$ belongs to $\ddot\Sigma_\tau$. **P** We know from (b-iii) that $F\in\ddot\Sigma$. If $t\ge0$, $\omega\in F$, $\omega'\in C_{\mathrm{dlg}}$, $\tau(\omega)\le t$ and $\omega'\restriction[0,t]=\omega\restriction[0,t]$ then $\ddot\mu_{\omega',\tau(\omega')}=\ddot\mu_{\omega,\tau(\omega)}$, so $\ddot g_f(\omega')=\ddot g_f(\omega)$ and $\omega'\in F$. Thus $F\cap\{\omega:\tau(\omega)\le t\}$ is determined (relative to C_{dlg}) by coordinates in $[0,t]$ and belongs to $\ddot\Sigma_t$. **Q**

(iv) Thus $\operatorname{dom}\ddot g_f\in\ddot\Sigma_\tau$ and $\ddot g_f$ is $\ddot\Sigma_\tau$-measurable. As we already know that it is a conditional expectation of f on $\ddot\Sigma_\tau^+\supseteq\ddot\Sigma_\tau$, it is a conditional expectation of f on $\ddot\Sigma_\tau$.

455P The eventual objective of this section is to provide a foundation for study of the original, and still by far the most important, example of a continuous-time Markov process, Brownian motion. In the language developed above, we shall have $U=\mathbb{R}$ (or, when we come to the applications in §§477-479, $U=\mathbb{R}^r$), and all the transitional probabilities $\nu_x^{(s,t)}$ will be Gaussian. But the techniques so far developed can tell us a great deal about much more general processes with some of the same features.

Theorem Let U be a metrizable topological group which is complete under a right-translation-invariant metric ρ inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that the convolution $\lambda_s * \lambda_t$ (444A) is equal to λ_{s+t} for all s, $t > 0$. Suppose that $\lim_{t\downarrow 0}\lambda_t G = 1$ for every open neighbourhood G of the identity in U. For $x \in U$ and $0 \le s < t$, let $\nu_x^{(s,t)}$ be the Radon probability measure on U defined by saying that $\nu_x^{(s,t)}(E) = \lambda_{t-s}(Ex^{-1})$ whenever λ_{t-s} measures Ex^{-1}.

(a) $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $x \in U$ and $0 \le s < t < u$.

(b) $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is narrowly continuous and uniformly time-continuous on the right.

(c)(i) We can define a complete measure $\hat\mu$ on $U^{[0,\infty[}$ by the method of 455E applied to $x^* = e$ and $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$.

(ii) If C_{dlg} is the space of càdlàg functions from $[0,\infty[$ to U, then $\hat\mu^* C_{\text{dlg}} = 1$, and the subspace measure $\ddot\mu$ on C_{dlg} will have the properties described in 455O, with $\omega(0) = e$ for $\ddot\mu$-almost every $\omega \in C_{\text{dlg}}$.

(iii) $\hat\mu$ has a unique extension to a Radon measure $\tilde\mu$ on $U^{[0,\infty[}$.

proof (a) Note first that $y \mapsto yx$ is inverse-measure-preserving for λ_{t-s} and $\nu_x^{(s,t)}$, so that $\int f(y)\nu_x^{(s,t)}(dy) = \int f(yx)\lambda_{t-s}(dy)$ for any real-valued function on U for which either is defined (235Gb). If $E \subseteq U$ is measured by $\nu_x^{(s,u)}$, then

$$\nu_x^{(s,u)}(E) = \lambda_{u-s}(Ex^{-1}) = (\lambda_{u-t} * \lambda_{t-s})(Ex^{-1}) = \int \lambda_{u-t}(Ex^{-1}y^{-1})\lambda_{t-s}(dy)$$

(444A)

$$= \int \nu_{yx}^{(t,u)}(E)\lambda_{t-s}(dy) = \int \nu_y^{(t,u)}(E)\nu_x^{(s,t)}(dy);$$

as E is arbitrary, $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$.

(b)(i)(α) Suppose that $x \in U$ and $0 \le s < t$; set $u = t - s$. Let $f : U \to \mathbb{R}$ be a bounded continuous function and set $M = \|f\|_\infty$. Take $\epsilon \in \left]0, \frac{1}{2}\right[$. Let $K \subseteq U$ be a compact set such that $\lambda_u K \ge 1-\epsilon$. Then there is a symmetric open neighbourhood V of the identity e of U such that $|f(wyx^{-1}) - f(wx^{-1})| \le 2\epsilon$ whenever $w \in K$ and $y \in V^2$. **P** For each $w \in K$ there is a neighbourhood W_w of e such that $|f(wyx^{-1}) - f(wx^{-1})| \le \epsilon$ whenever $y \in W_w^2$. Because K is compact, there are $w_0, \dots, w_n \in K$ such that $K \subseteq \bigcup_{i\le n} w_i W_{w_i}$; set $W = \bigcap_{i\le n} W_{w_i}$. If $w \in K$ and $y \in W$, there is an $i \le n$ such that $w \in w_i W_{w_i}$, in which case

$$\begin{aligned}|f(wyx^{-1}) - f(wx^{-1})| &\le |f(w_i(w_i^{-1}wy)x^{-1}) - f(w_i x^{-1})| \\ &\qquad + |f(w_i x^{-1}) - f(w_i(w_i^{-1}w)x^{-1})| \\ &\le 2\epsilon\end{aligned}$$

because both $w_i^{-1}wy$ and $w_i^{-1}w$ belong to $W_{w_i}^2$. So if we take a symmetric open neighbourhood V of e such that $V^2 \subseteq W$, this will serve. **Q**

(β) Let $\delta > 0$ be such that $\lambda_v V \ge 1-\epsilon$ whenever $0 < v \le 2\delta$. It will be worth noting that $\lambda_v(KV) \ge 1 - 2\epsilon$ whenever $0 < v < u$ and $u - v \le 2\delta$. **P** In this case, $\lambda_u = \lambda_v * \lambda_{u-v}$. Now $U \setminus K \supseteq (U \setminus KV)V^{-1}$. So

$$\epsilon \ge \lambda_u(U \setminus K) \ge \lambda_v(U \setminus KV)\lambda_{u-v}(V^{-1}) \ge (1-\epsilon)\lambda_v(U \setminus KV)$$

and

$$\lambda_v(KV) \ge 1 - \tfrac{\epsilon}{1-\epsilon} \ge 1 - 2\epsilon. \text{ **Q**}$$

(γ) Suppose that $0 \le s' < t'$ and $y \in U$ are such that $y^{-1}x \in W$, $|s' - s| \le \delta$ and $|t' - t| \le \delta$. Then $|\int f d\nu_y^{(s',t')} - \int f d\nu_x^{(s,t)}| \le (6M + 4)\epsilon$. **P** Set $u' = t' - s'$, so that $|u - u'| \le 2\delta$. We have

$$|\int f d\nu_y^{(s',t')} - \int f d\nu_x^{(s,t)}| = |\int f(wy^{-1})\lambda_{u'}(dw) - \int f(wx^{-1})\lambda_u(dw)|.$$

case 1 Suppose that $u' < u$. Then $\lambda_u = \lambda_{u'} * \lambda_{u-u'}$, so

$$\begin{aligned}|\int fd\nu_y^{(s',t')}-\int fd\nu_x^{(s,t)}|&=|\int f(wy^{-1})\lambda_{u'}(dw)-\int f(wx^{-1})(\lambda_{u'}*\lambda_{u-u'})(dw)|\\&=|\int f(wy^{-1})\lambda_{u'}(dw)-\iint f(wzx^{-1})\lambda_{u-u'}(dz)\lambda_{u'}(dw)|\end{aligned}$$

(444C)

$$\begin{aligned}&\le\int|f(wy^{-1})-\int f(wzx^{-1})\lambda_{u-u'}(dz)|\lambda_{u'}(dw)\\&\le 4M\epsilon+\sup_{w\in KV}|f(wy^{-1})-\int f(wzx^{-1})\lambda_{u-u'}(dz)|\end{aligned}$$

(because $\lambda_{u'}(U\setminus KV)\le 2\epsilon$, by (β), and $|f(wy^{-1})-\int f(wzx^{-1})\lambda_{u-u'}(dz)|\le 2M$ for every w)

$$\begin{aligned}&\le 4M\epsilon+\sup_{w\in KV}\int|f(wy^{-1})-\int f(wzx^{-1})|\lambda_{u-u'}(dz)\\&\le 6M\epsilon+\sup_{w\in KV,z\in V}|f(wy^{-1})-\int f(wzx^{-1})|\end{aligned}$$

(because $\lambda_{u-u'}V\ge 1-\epsilon$)

$$\begin{aligned}&\le 6M\epsilon+\sup_{w\in K,v\in V,z\in V}|f(wvy^{-1}xx^{-1})-f(wvzx^{-1})|\\&\le 6M\epsilon+\sup_{w\in K,v\in V,z\in V}\big(|f(wvy^{-1}xx^{-1})-f(wx^{-1})|\\&\qquad\qquad+|f(wvzx^{-1})-f(wx^{-1})|\big)\\&\le 6M\epsilon+4\epsilon\end{aligned}$$

by the choice of V, because $y^{-1}x\in V$.

case 2 Suppose that $u'=u$. Then

$$\begin{aligned}|\int fd\nu_y^{(s',t')}-\int fd\nu_x^{(s,t)}|&\le\int|f(wy^{-1})-f(wx^{-1})|\lambda_u(dw)\\&\le 2M\epsilon+\sup_{w\in K}|f(wy^{-1}xx^{-1})-f(wx^{-1})|\\&\le 2M\epsilon+\epsilon.\end{aligned}$$

case 3 Suppose that $u'>u$. Then $\lambda_{u'}=\lambda_u*\lambda_{u'-u}$, so

$$\begin{aligned}|\int fd\nu_y^{(s',t')}-\int fd\nu_x^{(s,t)}|&=|\iint f(wzy^{-1})\lambda_{u'-u}(dz)\lambda_u(dw)-\int f(wx^{-1})\lambda_u(dw)|\\&\le\int|\int f(wzy^{-1})\lambda_{u'-u}(dz)-f(wx^{-1})|\lambda_u(dw)\\&\le 2M\epsilon+\sup_{w\in K}|\int f(wzy^{-1})\lambda_{u'-u}(dz)-f(wx^{-1})|\\&\le 2M\epsilon+2M\epsilon+\sup_{w\in K,z\in V}|f(wzy^{-1})-f(wx^{-1})|\\&\le 4M\epsilon+2\epsilon.\end{aligned}$$

So we have the result in all cases. **Q**

(δ) As s, t, x, ϵ and f are arbitrary, $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ is narrowly (= vaguely) continuous.

(ii) Given $\epsilon>0$, there is a $\delta>0$ such that $\lambda_t\{x:\rho(x,e)<\epsilon\}\ge 1-\epsilon$ whenever $0<t\le\delta$. Now suppose that $x\in U$ and $0\le s<t\le s+\delta$. Then

$$\nu_x^{(s,t)}B(x,\epsilon) = \lambda_{t-s}(B(x,\epsilon)x^{-1}) = \lambda_{t-s}B(e,\epsilon)$$

(because ρ is right-translation-invariant)

$$\geq 1-\epsilon.$$

As ϵ is arbitrary, $\langle \nu_x^{(s,t)}\rangle_{0\leq s<t,x\in U}$ is uniformly time-continuous on the right.

(c) This is now just a matter of putting 455O and 455H together, and recalling from 455E that $\omega(0)=e$ for $\hat{\mu}$-almost every $\omega\in\Omega$.

455Q Lévy processes If we approach as probabilists, without prejudices in favour of any particular realization, the processes in 455P manifest themselves as follows. Let U be a separable metrizable topological group with identity e, and consider the following list of properties of a family $\langle X_t\rangle_{t\geq 0}$ of U-valued random variables:

$X_0 = e$ almost everywhere,

$\Pr(X_tX_s^{-1}\in F) = \Pr(X_{t-s}\in F)$ whenever $0\leq s<t$ and $F\subseteq U$ is Borel

(the process is **stationary**),

whenever $0\leq t_0<t_1<\ldots<t_n$, then $X_{t_1}X_{t_0}^{-1}, X_{t_2}X_{t_1}^{-1},\ldots, X_{t_n}X_{t_{n-1}}^{-1}$ are independent

in the sense of 418U (the process has **independent increments**),

$X_t\to e$ in measure as $t\downarrow 0$

(that is, $\lim_{t\downarrow 0}\Pr(X_t\in G)=1$ for every neighbourhood G of the identity). I say here that U should be separable and metrizable in order to ensure that all the functions $X_tX_s^{-1}$ should be measurable (of course it will be enough if U is metrizable and of measure-free weight, as in 438E). Such a family I will call a **Lévy process**.

455R Theorem Let U be a Polish group with identity e which is complete under a right-translation-invariant metric inducing its topology. A family $\langle X_t\rangle_{t\geq 0}$ of U-valued random variables is a Lévy process iff there is a family $\langle\lambda_t\rangle_{t>0}$ of Radon probability measures on U, satisfying the conditions of 455P, such that if we start from $x^*=e$ and build the measure $\hat{\mu}$ on $U^{[0,\infty[}$ as in 455Pc, then

$$\Pr(X_{t_i}\in F_i \text{ for every } i\leq n) = \hat{\mu}\{\omega:\omega(t_i)\in F_i \text{ for every } i\leq n\}$$

whenever $t_0,\ldots,t_n\in[0,\infty[$ and $F_i\subseteq U$ is a Borel set for every $i\leq n$.

proof (a) Suppose we have a family $\langle\lambda_t\rangle_{t>0}$ of Radon probability measures on U such that $\lambda_s*\lambda_t=\lambda_{s+t}$ for all s, $t>0$ and $\lim_{t\downarrow 0}\lambda_tG=1$ for every open neighbourhood G of e in U. Define $\langle\nu_x^{(s,t)}\rangle_{0\leq s<t,x\in U}$ as in 455P, and let $\hat{\mu}$ be the corresponding completed measure on $\Omega=U^{[0,\infty[}$ as in 455Pc. Set $X_t(\omega)=\omega(t)$ for $t\geq 0$ and $\omega\in\Omega$. Then $X_0=e$ a.e. (455Ea) and

$$\Pr(X_t\in F) = \hat{\mu}X_t^{-1}[F] = \nu_0^{(0,t)}F = \lambda_tF$$

for $t>0$ and $F\in\mathcal{B}(U)$ (455Ea again). In particular,

$$\lim_{t\downarrow 0}\Pr(X_t\in G) = \lim_{t\downarrow 0}\lambda_tG = 1$$

for every neighbourhood G of the identity. If $0<s<t$ and $F\in\mathcal{B}(U)$, set $H=\{(e,x,y):yx^{-1}\in F\}\subseteq U^3$. Then

$$\begin{aligned}\Pr(X_tX_s^{-1}\in F) &= \hat{\mu}\{\omega:(\omega(0),\omega(s),\omega(t))\in H\}\\ &= \iint\chi H(e,x,y)\nu_x^{(s,t)}(dy)\nu_e^{(0,s)}(dx)\end{aligned}$$

(455E)

$$\begin{aligned}&= \iint\chi H(e,x,yx)\lambda_{t-s}(dy)\lambda_s(dx)\\ &= \iint\chi F(y)\lambda_{t-s}(dy)\lambda_s(dx) = \lambda_{t-s}(F) = \Pr(X_{t-s}\in F).\end{aligned}$$

If $0=s<t$ then $X_tX_s^{-1} =_{\text{a.e.}} X_t = X_{t-s}$, of course. If $0=t_0<t_1<\ldots<t_n$ and $F_0,\ldots,F_{n-1}\in\mathcal{B}(U)$, set

$$E_k = \{\omega:\omega\in\Omega,\ \omega(t_{i+1})\omega(t_i)^{-1}\in F_i \text{ for every } i<k\},$$

$$H_k = \{(x_0,\ldots,x_k): x_{i+1}x_i^{-1}\in F_i \text{ for every } i<k\}\subseteq U^{k+1}$$

for $k\le n$. Then

$$\hat\mu E_1 = \hat\mu\{\omega:\omega(t_1)\omega(0)^{-1}\in F_0\} = \hat\mu\{\omega:\omega(t_1)\in F_0\} = \nu_e^{(0,t_1)}F_0 = \lambda_{t_1}F_0,$$

and for $k\ge 2$

$$\begin{aligned}\Pr(X_{t_{i+1}}X_{t_i}^{-1}&\in F_i \text{ for every } i<k)\\ &= \hat\mu E_k = \int\ldots\int \chi H_k(e,x_1,\ldots,x_k)\nu_{x_{k-1}}^{(t_{k-1},t_k)}(dx_k)\ldots\nu_e^{(0,t_1)}(dx_1)\end{aligned}$$

(455E)

$$\begin{aligned}&= \int\ldots\iint \chi H_k(e,x_1,\ldots,x_{k-1},x_kx_{k-1})\lambda_{t_k-t_{k-1}}(dx_k)\\ &\qquad \nu_{x_{k-2}}^{(t_{k-2},t_{k-1})}(dx_{k-1})\ldots\nu_e^{(0,t_1)}(dx_1)\\ &= \int\ldots\iint \chi H_{k-1}(e,x_1,\ldots,x_{k-1})\chi F_{k-1}(x_k)\lambda_{t_k-t_{k-1}}(dx_k)\\ &\qquad \nu_{x_{k-2}}^{(t_{k-2},t_{k-1})}(dx_{k-1})\ldots\nu_e^{(0,t_1)}(dx_1)\\ &= \lambda_{t_k-t_{k-1}}(F_{k-1})\int\ldots\int \chi H_{k-1}(e,x_1,\ldots,x_{k-1})\\ &\qquad \nu_{x_{k-2}}^{(t_{k-2},t_{k-1})}(dx_{k-1})\ldots\nu_e^{(0,t_1)}(dx_1)\\ &= \lambda_{t_k-t_{k-1}}(F_{k-1})\cdot\hat\mu E_{k-1}.\end{aligned}$$

So

$$\begin{aligned}\Pr(X_{t_{i+1}}X_{t_i}^{-1}\in F_i \text{ for every } i<n) &= \hat\mu E_n = \prod_{i=0}^{n-1}\lambda_{t_i-t_{i-1}}F_i\\ &= \prod_{i=0}^{n-1}\Pr(X_{t_i-t_{i-1}}\in F_i) = \prod_{i=0}^{n-1}\Pr(X_{t_i}X_{t_{i-1}}^{-1}\in F_i).\end{aligned}$$

As $F_0,\ldots,F_{n-1}$ are arbitrary, $X_{t_1}X_{t_0}^{-1},X_{t_2}X_{t_1}^{-1},\ldots,X_{t_n}X_{t_{n-1}}^{-1}$ are independent. Thus all the conditions of 455Q are satisfied.

(b)(i) In the other direction, given a family $\langle X_t\rangle_{t\ge0}$ with the properties listed in 455Q, then for each $t>0$ there is a Radon measure λ_t on U such that $\lambda_tF=\Pr(X_t\in F)$ for every $F\in\mathcal{B}(U)$, for each $t>0$. **P** U is Polish, therefore analytic, and we can apply 433Cb to the Borel measure $F\mapsto\Pr(X_t\in F)$. **Q** If s, $t>0$, then the distribution of $X_{s+t}X_s^{-1}$ is the same as the distribution of X_t, so is λ_t.

If s, $t>0$ then $\lambda_{s+t}=\lambda_s*\lambda_t$. **P** If F_1, $F_2\in\mathcal{B}(U)$ then

$$\begin{aligned}\Pr((X_t,X_{s+t}X_t^{-1})\in F_1\times F_2) &= \Pr(X_t\in F_1, X_{s+t}X_t^{-1}\in F_2)\\ &= \Pr(X_t\in F_1)\Pr(X_{s+t}X_t^{-1}\in F_2)\end{aligned}$$

(because X_t and $X_{s+t}X_t^{-1}$ are independent)

$$= \lambda_tF_1\cdot\lambda_sF_2 = (\lambda_t\times\lambda_s)(F_1\times F_2).$$

By the Monotone Class Theorem, or otherwise,

$$\Pr((X_t,X_{s+t}X_t^{-1})\in H) = (\lambda_t\times\lambda_s)H$$

for every $H \in \mathcal{B}(U)\widehat{\otimes}\mathcal{B}(U) = \mathcal{B}(U^2)$. So if $F \in \mathcal{B}(U)$ we shall have

$$\begin{aligned}(\lambda_s * \lambda_t)(F) &= (\lambda_s \times \lambda_t)\{(x,y) : xy \in F\} = (\lambda_t \times \lambda_s)\{(y,x) : xy \in F\}\\ &= \Pr(X_{s+t}X_t^{-1}X_t \in F) = \lambda_{s+t}F,\end{aligned}$$

and $\lambda_s * \lambda_t = \lambda_{s+t}$. (Cf. 272T[15].) **Q**

Next, for any neighbourhood G of e,

$$\lim_{t\downarrow 0} \lambda_t G = \lim_{t\downarrow 0} \Pr(X_t \in G) = 1.$$

So $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455P. Let $\hat\mu$ be the corresponding completed measure on $U^{[0,\infty[}$ as in 455Pc.

(ii) $\Pr(X_{t_i} \in F_i$ for every $i \le k) = \hat\mu\{\omega : \omega(t_i) \in F_i$ for every $i \le k\}$ whenever $t_0,\dots,t_n \in [0,\infty[$ and $F_i \in \mathcal{B}(U)$ for every $i \le k$. **P** It is enough to consider the case $0 = t_0 < t_1 < \dots < t_n$. In this case, whenever $E_0,\dots,E_n \in \mathcal{B}(U)$,

$$\begin{aligned}\Pr((X_{t_0},&X_{t_1}X_{t_0}^{-1},\dots,X_{t_n}X_{t_{n-1}}^{-1}) \in E_0 \times \dots \times E_n)\\ &= \Pr(X_{t_0} \in E_0)\Pr(X_{t_1}X_{t_0}^{-1} \in E_1)\dots\Pr(X_{t_n}X_{t_{n-1}}^{-1}) \in E_n)\\ &= \delta_e(E_0)\lambda_{t_1-t_0}(E_1)\dots\lambda_{t_n-t_{n-1}}(E_n)\end{aligned}$$

(where δ_e is the Dirac measure concentrated at e)

$$\begin{aligned}&= \hat\mu\{\omega : \omega(t_0) \in E_0\}\hat\mu\{\omega : \omega(t_1)\omega(t_0)^{-1} \in E_1\}\dots\\ &\qquad\qquad \hat\mu\{\omega : \omega(t_n)\omega(t_{n-1})^{-1} \in E_n\}\\ &= \hat\mu\{\omega : \omega(t_0) \in E_0,\, \omega(t_1)\omega(t_0)^{-1} \in E_1,\, \dots,\omega(t_n)\omega(t_{n-1})^{-1} \in E_n\}\end{aligned}$$

(by (a) above)

$$= \hat\mu\{\omega : (\omega(t_0),\omega(t_1)\omega(t_0)^{-1},\dots,\omega(t_n)\omega(t_{n-1})^{-1}) \in E_0 \times \dots \times E_n\}.$$

So in fact

$$\begin{aligned}\Pr((X_{t_0},X_{t_1}X_{t_0}^{-1},&\dots,X_{t_n}X_{t_{n-1}}^{-1}) \in H)\\ &= \hat\mu\{\omega : (\omega(t_0),\omega(t_1)\omega(t_0)^{-1},\dots,\omega(t_n)\omega(t_{n-1})^{-1}) \in H\}\end{aligned}$$

for every Borel set $H \subseteq U^{n+1}$.

Set

$$\phi(x_0,\dots,x_n) = (x_0, x_1x_0, x_2x_1x_0,\dots,x_nx_{n-1}\dots x_1x_0)$$

for $x_0,\dots,x_n \in U$, so that $\phi : U^{n+1} \to U^{n+1}$ is continuous. If $H \in \mathcal{B}(U^{n+1})$, then

$$\begin{aligned}\Pr(X_{t_0},\dots,X_{t_n} \in H) &= \Pr(\phi(X_{t_0},X_{t_1}X_{t_0}^{-1},\dots,X_{t_n}X_{t_{n-1}}^{-1}) \in H)\\ &= \Pr((X_{t_0},X_{t_1}X_{t_0}^{-1},\dots,X_{t_n}X_{t_{n-1}}^{-1}) \in \phi^{-1}[H])\\ &= \hat\mu\{\omega : ((\omega(t_0),\omega(t_1)\omega(t_0)^{-1},\dots,\omega(t_n)\omega(t_{n-1})^{-1}) \in \phi^{-1}[H]\}\\ &= \hat\mu\{\omega : (\omega(t_0),\dots,\omega(t_n)) \in H\}.\end{aligned}$$

Taking $H = F_0 \times \dots \times F_n$ we have the result. **Q**

455S Lemma Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s * \lambda_t = \lambda_{s+t}$ for all s, $t > 0$ and $\lim_{t\downarrow 0}\lambda_t G = 1$ for every open neighbourhood G of the identity e in U. For $x \in U$ and $0 \le s < t$, let $\nu_x^{(s,t)}$ be the Radon probability measure on U defined by saying that $\nu_x^{(s,t)}(E) = \lambda_{t-s}(Ex^{-1})$ whenever λ_{t-s} measures Ex^{-1}.

[15] Formerly 272S.

(a) If $0 \le t_0 < t_1 < \ldots < t_n$, $z \in U$ and $f : \mathbb{R}^J \to \mathbb{R}$ is a bounded Borel measurable function, where $J = \{t_0, \ldots, t_n\}$, then

$$\iint \ldots \int f(z, x_1, \ldots, x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \ldots \nu_{x_1}^{(t_1,t_2)}(dx_2)\nu_z^{(t_0,t_1)}(dx_1)$$
$$= \iint \ldots \int f(z, x_1 z, \ldots, x_n z)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)$$
$$\ldots \nu_{x_1}^{(t_1,t_2)}(dx_2)\nu_e^{(t_0,t_1)}(dx_1).$$

(b) Take $\omega \in U^{[0,\infty[}$ and $a \ge 0$. Let $\hat\mu$ and $\hat\mu_{\omega a}$ be the measures on $U^{[0,\infty[}$ defined from $\langle \nu_x^{(s,t)} \rangle_{s<t, x\in U}$ by the method of 455E, starting from $x^* = e$. Define $\phi_{\omega a} : U^{[0,\infty[} \to U^{[0,\infty[}$ by setting

$$\phi_{\omega a}(\omega')(t) = \omega(t) \text{ if } t < a,$$
$$= \omega'(t-a)\omega(a) \text{ if } t \ge a.$$

Then $\hat\mu_{\omega a}$ is the image measure $\hat\mu\phi_{\omega a}^{-1}$.

(c) In (b), suppose that ω belongs to the set $C_{\rm dlg}$ of càdlàg functions from $[0,\infty[$ to U. Then $\phi_{\omega a}(\omega') \in C_{\rm dlg}$ for every $\omega' \in C_{\rm dlg}$, and $\phi_{\omega a} : C_{\rm dlg} \to C_{\rm dlg}$ is inverse-measure-preserving for the subspace measures $\ddot\mu$ and $\ddot\mu_{\omega a}$ on $C_{\rm dlg}$.

proof (a)(i) If $x \in U$ and $0 \le s < t$, then $\nu_x^{(s,t)}(E) = \nu_e^{(s,t)}(Ex^{-1})$ for any $E \subseteq U$ such that either is defined; so $\int f(y)\nu_x^{(s,t)}(dy) = \int f(yx)\nu_e^{(s,t)}(dy)$ for any function $f : U \to \mathbb{R}$ for which either is defined. More generally,

$$\int f(y)\nu_{xz}^{(s,t)}(dy) = \int f(yxz)\nu_e^{(s,t)}(dy) = \int f(yz)\nu_x^{(s,t)}(dy)$$

whenever f is such that any of the three integrals is defined.

(ii) Now induce on n. For the case $n = 0$, the natural interpretation of both sides of the formula presented is $f(z)$. For the inductive step to $n+1$, we have

$$\int \ldots \int f(z, x_1, x_2, \ldots, x_{n+1})\nu_{x_n}^{(t_n,t_{n+1})}(dx_{n+1}) \ldots \nu_z^{(t_0,t_1)}(dx_1)$$
$$= \int \ldots \int f(z, x_1 z, \ldots, x_n z, x_{n+1})\nu_{x_n z}^{(t_n,t_{n+1})}(dx_{n+1}) \ldots \nu_e^{(t_0,t_1)}(dx_1)$$

(by the inductive hypothesis applied to $(x_0, x_1, \ldots, x_n) \mapsto \int f(x_0, \ldots, x_n, x_{n+1})\nu_{x_n}^{(t_n,t_{n+1})}(dx_{n+1})$)

$$= \int \ldots \int f(z, x_1 z, \ldots, x_n z, x_{n+1} z)\nu_{x_n}^{(t_n,t_{n+1})}(dx_{n+1}) \ldots \nu_e^{(t_0,t_1)}(dx_1)$$

by (i) applied to the functions $y \mapsto f(z, x_1 z, \ldots, x_n z, y)$ for each $x_1, \ldots, x_n$.

(b)(i) Suppose that $J \subseteq [0,\infty[$ is a finite set containing both 0 and a, enumerated in increasing order as $(t_0, \ldots, t_n)$ with $a = t_j$. Set $z = \omega(a)$. Let $f : \mathbb{R}^J \to \mathbb{R}$ be a function. Then $f\pi_J\phi_{\omega a} = g\pi_K$ where $K = \{0, t_{j+1} - a, \ldots, t_n - a\}$ and $g(x_j, \ldots, x_n) = f(\omega(0), \omega(t_1), \ldots, \omega(t_{j-1}), x_j z, \ldots, x_n z)$ for $x_j, \ldots, x_n \in U$. **P** For $\omega' \in U^{[0,\infty[}$,

$$f\pi_J\phi_{\omega a}(\omega') = (f(\phi_{\omega a}(\omega')(t_0)), \ldots, f(\phi_{\omega a}(\omega')(t_n))$$
$$= (f(\omega(0)), \ldots, f(\omega(t_{j-1})), f(\omega'(0)z), \ldots f(\omega'(t_n - a)z))$$
$$= g(\omega'(0), \ldots, \omega'(t_n - a)) = g\pi_K(\omega'). \mathbf{Q}$$

(ii) Again suppose that $J \subseteq [0,\infty[$ is a finite set containing both 0 and a, enumerated in increasing order as $(t_0, \ldots, t_n)$ with $a = t_j$, and set $z = \omega(a)$. This time, let $f : \mathbb{R}^J \to \mathbb{R}$ be a bounded Borel measurable function. Then

$$\int f\pi_J d\hat\mu_{\omega a} = \int\ldots\int f(e,x_1,\ldots,x_n)\nu_{\omega a x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\ldots\nu_{\omega a0}^{(0,t_1)}(dx_1)$$

$$= \int\ldots\iint\ldots\int f(e,x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_n)$$
$$\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\ldots\nu_z^{(a,t_{j+1})}(dx_{j+1})\delta_z(dx_j)\ldots\delta_{\omega(t_1)}(dx_1)$$

(reading from the formulae in 455E; here each δ_x is a Dirac measure on U)

$$= \int\ldots\int f(e,\omega(t_1),\ldots,z,x_{j+1},\ldots,x_n)$$
$$\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\ldots\nu_z^{(a,t_{j+1})}(dx_{j+1})$$

$$= \int\ldots\int f(e,\omega(t_1),\ldots,z,x_{j+1}z,\ldots,x_n z)$$
$$\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\ldots\nu_e^{(a,t_{j+1})}(dx_{j+1})$$

(applying (a) to the function $(y_0,\ldots,y_{n-j})\mapsto f(e,\ldots,\omega(t_{j-1}),y_0,\ldots,y_{n-j}))$

$$= \int\ldots\int g(e,x_{j+1},\ldots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\ldots\nu_e^{(a,t_{j+1})}(dx_{j+1})$$

(where $g(x_j,\ldots,x_n)=f(\omega(0),\omega(t_1),\ldots,\omega(t_{j-1}),x_jz,\ldots,x_nz)$ for $x_j,\ldots,x_n\in U$)

$$= \int\ldots\int g(e,x_{j+1},\ldots,x_n)\nu_{x_{n-1}}^{(t_{n-1}-a,t_n-a)}(dx_n)\ldots\nu_e^{(0,t_{j+1}-a)}(dx_{j+1})$$

(because $\nu_x^{(s-a,t-a)}E=\lambda_{t-s}(Ex^{-1})=\nu_x^{(s,t)}E$ whenever $E\subseteq\mathbb{R}^K$ is Borel, $x\in U$ and $a\le s<t$)

$$= \int g\pi_K d\hat\mu$$

(where $K=\{0,t_{j+1}-a,\ldots,t_n-a\}$)

$$= \int f\pi_J\phi_{\omega a}d\hat\mu = \int f\pi_J d(\hat\mu\phi_{\omega a}^{-1}).$$

As f and J are arbitrary, $\hat\mu_{\omega a}$ and $\hat\mu\phi_{\omega a}^{-1}$ agree on the algebra $\bigotimes_{[0,\infty[}\mathcal{B}(U)$ generated by sets of the form $\{\omega:\omega(t)\in E\}$ for $t\ge0$ and Borel sets $E\subseteq U$. By the Monotone Class Theorem, the measures agree on the σ-algebra $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ generated by $\bigotimes_{[0,\infty[}\mathcal{B}(U)$; because they are both defined as complete measures inner regular with respect to this σ-algebra, they are identical.

(c) The defining formula for $\phi_{\omega a}$ makes it plain that $\phi_{\omega a}(\omega')$ is càdlàg whenever ω, ω' are càdlàg. If W is measured by $\ddot\mu_{\omega a}$, there is a $W'\in\operatorname{dom}\hat\mu_{\omega a}$ such that $W=W'\cap C_{\mathrm{dlg}}$. In this case, $\phi_{\omega a}^{-1}[W]\cap C_{\mathrm{dlg}}=\phi_{\omega a}^{-1}[W']\cap C_{\mathrm{dlg}}$ while $\phi_{\omega a}^{-1}[W']\in\operatorname{dom}\hat\mu$; so $\ddot\mu\phi_{\omega a}^{-1}[W]$ is defined and equal to

$$\hat\mu\phi_{\omega a}^{-1}[W']=\hat\mu_{\omega a}W'=\ddot\mu_{\omega a}W.$$

455T Corollary Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle\lambda_t\rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s*\lambda_t=\lambda_{s+t}$ for all s, $t>0$ and $\lim_{t\downarrow0}\lambda_tG=1$ for every open neighbourhood G of the identity e in U; let $\hat\mu$ be the measure on $U^{[0,\infty[}$ defined from $\langle\lambda_t\rangle_{t>0}$ by the method of 455Pc. Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U, $\ddot\mu$ the subspace measure on C_{dlg} and $\ddot\Sigma$ its domain. For $t\ge0$, let $\ddot\Sigma_t$ be

$$\{F:F\in\ddot\Sigma,\ \omega'\in F\text{ whenever }\omega\in F,\ \omega'\in C_{\mathrm{dlg}}\text{ and }\omega'\restriction[0,t]=\omega\restriction[0,t]\}$$

and $\hat{\ddot\Sigma}_t=\{F\triangle A:F\in\ddot\Sigma_t,\ \ddot\mu A=0\}$. Then $\hat{\ddot\Sigma}_t=\bigcap_{s>t}\hat{\ddot\Sigma}_s$ includes $\ddot\Sigma_t^+=\bigcap_{s>t}\ddot\Sigma_s$.

proof (a) I show first that $\ddot\Sigma_t^+\subseteq\hat{\ddot\Sigma}_t$. **P** Take $E\in\ddot\Sigma_t^+$. Let $\tau:C_{\mathrm{dlg}}\to[0,\infty]$ be the constant stopping time with value t, and f the characteristic function χE. Set $g(\omega)=\int_{C_{\mathrm{dlg}}}fd\ddot\mu_{\omega t}$ when this is defined in $\mathbb{R}$, where $\ddot\mu_{\omega t}$ is defined as in 455O; then g is a conditional expectation of f on $\ddot\Sigma_\tau$ (455Ob). Since

$$\begin{aligned}\ddot{\Sigma}_\tau^+ &= \{H : H \in \ddot{\Sigma},\ H \cap \{\omega : \tau(\omega) \le s\} \in \ddot{\Sigma}_s^+ \text{ for every } s \ge 0\}\\ &= \{H : H \in \ddot{\Sigma},\ H \in \ddot{\Sigma}_s^+ \text{ for every } s \ge t\} = \ddot{\Sigma}_t^+\end{aligned}$$

contains E, $g =_{\text{a.e.}} \chi E$. Setting $F = \{\omega : \omega \in \operatorname{dom} g,\ g(\omega) = 1\}$, $F \in \ddot{\Sigma}$ (remember that $\ddot{\mu}$ is complete), and $E \triangle F$ is negligible.

Now 455Sc, with 235Gb, tells us that

$$g(\omega) = \int_{C_{\text{dlg}}} f d\ddot{\mu}_{\omega t} = \int_{C_{\text{dlg}}} f \phi_{\omega t} d\ddot{\mu}$$

whenever either integral is defined in $\mathbb{R}$, where

$$\begin{aligned}\phi_{\omega t}(\omega')(s) &= \omega(s) \text{ if } s < t,\\ &= \omega'(s-t)\omega(t) \text{ if } s \ge t.\end{aligned}$$

If ω_0, $\omega_1 \in C_{\text{dlg}}$ and $\omega_0 \restriction [0,t] = \omega_1 \restriction [0,t]$, then $\phi_{\omega_0 t} = \phi_{\omega_1 t}$ so $g(\omega_0) = g(\omega_1)$ if either is defined. It follows that $\omega_0 \in F$ iff $\omega_1 \in F$. As ω_0 and ω_1 are arbitrary, $F \in \ddot{\Sigma}_t$ and $E \in \hat{\ddot{\Sigma}}_t$. **Q**

(b) Of course $\hat{\ddot{\Sigma}}_t \subseteq \hat{\ddot{\Sigma}}_s$ whenever $s > t$. Putting (a) and 455L(f-ii) together,

$$\bigcap_{s>t} \hat{\ddot{\Sigma}}_s = \{E \triangle A : E \in \ddot{\Sigma}_t^+,\ \ddot{\mu}A = 0\} \subseteq \{E \triangle A : E \in \hat{\ddot{\Sigma}}_t,\ \ddot{\mu}A = 0\} = \hat{\ddot{\Sigma}}_t$$

and we have equality.

455U Theorem Let U be a metrizable topological group which is complete under a right-translation-invariant metric inducing its topology. Let $\langle \lambda_t \rangle_{t>0}$ be a family of Radon probability measures on U such that $\lambda_s * \lambda_t = \lambda_{s+t}$ for all s, $t > 0$ and $\lim_{t \downarrow 0} \lambda_t G = 1$ for every open neighbourhood G of the identity e in U; let $\hat{\mu}$ be the measure on $U^{[0,\infty[}$ defined from $\langle \lambda_t \rangle_{t>0}$ by the method of 455Pc. Let C_{dlg} be the set of càdlàg functions from $[0,\infty[$ to U, $\ddot{\mu}$ the subspace measure on C_{dlg} and $\ddot{\Sigma}$ its domain. For $t \ge 0$, let $\ddot{\Sigma}_t$ be

$$\{F : F \in \ddot{\Sigma},\ \omega' \in F \text{ whenever } \omega \in F,\ \omega' \in C_{\text{dlg}} \text{ and } \omega' \restriction [0,t] = \omega \restriction [0,t]\},$$

and $\hat{\ddot{\Sigma}}_t = \{F \triangle A : F \in \ddot{\Sigma}_t,\ \ddot{\mu}A = 0\}$; let $\tau : C_{\text{dlg}} \to [0,\infty]$ be a stopping time adapted to $\langle \hat{\ddot{\Sigma}}_t \rangle_{t \ge 0}$. Define $\phi_\tau : C_{\text{dlg}} \times C_{\text{dlg}} \to C_{\text{dlg}}$ by setting

$$\begin{aligned}\phi_\tau(\omega, \omega')(t) &= \omega'(t - \tau(\omega))\omega(\tau(\omega)) \text{ if } t \ge \tau(\omega),\\ &= \omega(t) \text{ otherwise.}\end{aligned}$$

Then ϕ_τ is inverse-measure-preserving for the product measure $\ddot{\mu} \times \ddot{\mu}$ on $C_{\text{dlg}} \times C_{\text{dlg}}$ and $\ddot{\mu}$ on C_{dlg}.

proof (a) To begin with (down to the end of (c) below), suppose that τ is adapted to $\langle \ddot{\Sigma}_t^+ \rangle_{t \ge 0}$, where $\ddot{\Sigma}_t^+ = \bigcap_{s>t} \ddot{\Sigma}_s$ for $t \ge 0$. In this case we know from 455P that the conditions of 455O are satisfied. I aim to apply 455Oa, using 455S to give a description of the measures $\ddot{\mu}_{\omega,\tau(\omega)}$. Now if f is $\ddot{\mu}$-integrable, we have, in the notation of 455O and 455S,

$$\int_{C_{\text{dlg}}} f d\ddot{\mu} = \int_{C_{\text{dlg}}} \int_{C_{\text{dlg}}} f d\ddot{\mu}_{\omega,\tau(\omega)} \ddot{\mu}(d\omega)$$

(455Oa)

$$= \int_{C_{\text{dlg}}} \int_{C_{\text{dlg}}} f \phi_{\omega,\tau(\omega)}(\omega') \ddot{\mu}(d\omega') \ddot{\mu}(d\omega)$$

(455Sc)

$$= \int_{C_{\text{dlg}}} \int_{C_{\text{dlg}}} f \phi_\tau(\omega, \omega') \ddot{\mu}(d\omega') \ddot{\mu}(d\omega)$$

.

(b) To convert the repeated integral into the product measure, we have still to check for measurability. The point is that, writing Λ for the domain of the product measure $\ddot{\mu}\times\ddot{\mu}$, ϕ_τ is $(\Lambda,\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U))$-measurable.

P(i) Consider first the case in which U is separable. Take $t\ge 0$. Then $E_t=\{\omega:\omega\in C_{\mathrm{dlg}},\,\tau(\omega)\le t\}$ belongs to $\ddot{\Sigma}$. The function

$$\omega\mapsto t-\tau(\omega):E_t\to[0,\infty[$$

is $(\ddot{\Sigma},\mathcal{B}([0,\infty[))$-measurable; the function

$$(\omega',s)\mapsto\omega'(s):C_{\mathrm{dlg}}\times[0,\infty[\to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\mathcal{B}([0,\infty[),\mathcal{B}(U))$-measurable, by 4A3Qc, because U is Polish; so the function

$$(\omega,\omega')\mapsto\omega'(t-\tau(\omega)):E_t\times C_{\mathrm{dlg}}\to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma},\mathcal{B}(U))$-measurable. Next, similarly,

$$\omega\mapsto\omega(\tau(\omega)):E_t\to U$$

is $(\ddot{\Sigma},\mathcal{B}(U))$-measurable, while

$$(y,z)\mapsto yz:U\times U\to U$$

is $(\mathcal{B}(U)\widehat{\otimes}\mathcal{B}(U),\mathcal{B}(U))$-measurable, because U is a second-countable topological group. But this means that

$$(\omega,\omega')\mapsto\omega'(t-\tau(\omega))\omega(\tau(\omega)):E_t\times C_{\mathrm{dlg}}\to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma},\mathcal{B}(U))$-measurable. On the other hand, of course,

$$(\omega,\omega')\mapsto\omega(t):(C_{\mathrm{dlg}}\setminus E_t)\times C_{\mathrm{dlg}}\to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma},\mathcal{B}(U))$-measurable. Putting these together,

$$(\omega,\omega')\mapsto\phi_\tau(\omega,\omega')(t):C_{\mathrm{dlg}}\times C_{\mathrm{dlg}}\to U$$

is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma},\mathcal{B}(U))$-measurable. This is true for every $t\ge 0$, so $\phi_\tau\restriction C_{\mathrm{dlg}}\times C_{\mathrm{dlg}}$ is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma},\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U))$-measurable.

(ii) For the general case, we can use the trick in 455H. There is a separable subgroup U' of U such that $\nu_e^{(0,q)}U'=1$ for every rational $q\ge 0$. We can suppose that U' is a closed subgroup of U. Because U' is closed,

$$\begin{aligned}C'_{\mathrm{dlg}}&=\{\omega:\omega\in C_{\mathrm{dlg}},\,\omega(t)\in U'\text{ for every }t\ge 0\}\\&=\{\omega:\omega\in C_{\mathrm{dlg}},\,\omega(q)\in U'\text{ for every }q\in\mathbb{Q}\cap[0,\infty[\}\end{aligned}$$

is $\ddot{\mu}$-conegligible in C_{dlg}, and because U' is a subgroup, $\phi_\tau(\omega,\omega')\in C'_{\mathrm{dlg}}$ for all ω, $\omega'\in C'_{\mathrm{dlg}}$. Now the argument of (i) shows that $\phi_\tau\restriction C'_{\mathrm{dlg}}\times C'_{\mathrm{dlg}}$ is $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma},\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U'))$-measurable, therefore $(\ddot{\Sigma}\widehat{\otimes}\ddot{\Sigma},\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U))$-measurable. Since $C'_{\mathrm{dlg}}\times C'_{\mathrm{dlg}}$ is $(\ddot{\mu}\times\ddot{\mu})$-conegligible, ϕ_τ is $(\Lambda,\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U))$-measurable. **Q**

(c) It follows that if $E\in\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$ then, setting $f=\chi(E\cap C_{\mathrm{dlg}})$ in (a),

$$(\ddot{\mu}\times\ddot{\mu})\phi_\tau^{-1}[E\cap C_{\mathrm{dlg}}]=\int f(\phi_\tau(\omega,\omega'))\ddot{\mu}(d\omega')\ddot{\mu}(d\omega)=\int f\,d\ddot{\mu}=\ddot{\mu}(E\cap C_{\mathrm{dlg}})$$

by Fubini's theorem. But $\ddot{\mu}$ is the subspace measure generated by the completion of a measure with domain $\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$, so is inner regular with respect to sets of the form $E\cap C_{\mathrm{dlg}}$ with $E\in\widehat{\bigotimes}_{[0,\infty[}\mathcal{B}(U)$; by 412K, ϕ_τ is inverse-measure-preserving.

(d) Now suppose only that τ is adapted to $\langle\hat{\ddot{\Sigma}}_t\rangle_{t\ge 0}$. By 455L(f-iii), there is a stopping time $\tau':C_{\mathrm{dlg}}\to[0,\infty]$, adapted to $\langle\ddot{\Sigma}_t^+\rangle_{t\ge 0}$, such that $\tau'=_{\mathrm{a.e.}}\tau$. Now we see from (a)-(c) that $\phi_{\tau'}$ is inverse-measure-preserving, while

$$\{(\omega,\omega'):\phi_{\tau'}(\omega,\omega')\ne\phi_\tau(\omega,\omega')\}\subseteq\{(\omega,\omega'):\tau'(\omega)\ne\tau(\omega)\}$$

is $(\ddot{\mu}\times\ddot{\mu})$-negligible, so ϕ_τ also is inverse-measure-preserving.

455X Basic exercises (a) Let $\langle A_n\rangle_{n\ge 1}$ be a non-increasing sequence of subsets of $[0,1]$, all with Lebesgue outer measure 1, and with empty intersection. Set $T=\{0\}\cup\{\frac{1}{n}:n\ge 1\}$, $\Omega_0=\{0\}$, $\Omega_{1/n}=A_n$ for $n\ge 1$; for $t\in T$ let T_t be the Borel σ-algebra of Ω_t. For $s<t$ in T and $x\in\Omega_s$ define a Borel measure $\nu_x^{(s,t)}$ on Ω_t by saying that

if $n\ge 1$, then $\nu_0^{(0,1/n)}(E\cap A_n)$ is the Lebesgue measure of E for every Borel set $E\subseteq[0,1]$,

if $0<s<t$, then $\nu_x^{(s,t)}\{x\}=1$.

Show that $\langle\nu_y^{(t,u)}\rangle_{y\in\Omega_t}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $s<t<u$ in T and $x\in\Omega_s$. Taking $x^*=0$, show that there is no measure μ on $\prod_{t\in T}\Omega_t$ with the properties listed in 455A.

(b) Let T, t^*, $\langle(\Omega_t,\mathrm{T}_t)\rangle_{t\in T}$, x^*, $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ and μ be as in 455A. Suppose that we are given a family $\langle(\Omega'_t,\mathrm{T}'_t,\pi_t)\rangle_{t\in T}$ such that (α) Ω'_t is a set, T'_t is a σ-algebra of subsets of Ω'_t and $\pi_t:\Omega_t\to\Omega'_t$ is a surjective $(\mathrm{T}_t,\mathrm{T}'_t)$-measurable function for every $t\in T$ (β) whenever $s<t$ in T and x, $x'\in\Omega_s$ are such that $\pi_s(x)=\pi_s(x')$, then $\nu_x^{(s,t)}$ and $\nu_{x'}^{(s,t)}$ agree on $\{\pi_t^{-1}[F]:F\in\mathrm{T}'_t\}$. (i) Show that if we set $\acute\nu_w^{(s,t)}(F)=\nu_x^{(s,t)}\pi_t^{-1}[F]$ whenever $s<t$ in T, $x\in\Omega_s$, $w=\pi_s(x)$ and $F\in\mathrm{T}'_t$, then every $\acute\nu_w^{(s,t)}$ is a perfect probability measure, and $\langle\acute\nu_z^{(t,u)}\rangle_{z\in\Omega'_t}$ is a disintegration of $\acute\nu_w^{(s,u)}$ over $\acute\nu_w^{(s,t)}$ whenever $s<t<u$ in T and $w\in\Omega'_s$. (ii) Let μ' be the measure on $\Omega'=\prod_{t\in T}\Omega'_t$ defined by the method of 455A from $\pi_{t^*}(x^*)$ and $\langle\acute\nu_w^{(s,t)}\rangle_{s<t,w\in\Omega'_s}$. Show that $\pi:\Omega\to\Omega$ is inverse-measure-preserving for μ and μ', where $\pi(\omega)(t)=\pi_t(\omega(t))$ for $\omega\in\Omega$ and $t\in T$.

(c) In 455E, set $T=\{-1\}\cup[0,\infty[$, let each Ω_t be $\mathbb{R}$, and for $x\in\mathbb{R}$, $0\le s<t$ let $\nu_x^{(s,t)}$ be the Dirac measure on $\mathbb{R}$ concentrated at $\psi(x,t-s)$ on $\mathbb{R}$, where

$$\begin{aligned}\psi(x,t)&=\tfrac{x}{1-xt}\text{ if }xt\ne 1\text{ and }x\ne 0,\\&=0\text{ if }xt=1,\\&=-\frac{1}{t}\text{ if }x=0.\end{aligned}$$

Let ν be any atomless Radon probability measure on $\mathbb{R}$, and complete the definition by setting $\nu_x^{(-1,t)}(E)=\nu\{y:\psi(y,t)\in E\}$ whenever $t\ge 0$ and this is defined; set $x^*=0$. Show that the conditions of 455E are satisfied, that the measure $\hat\mu$ constructed in 455E is a distribution on $\mathbb{R}^T$, and that $\hat\mu$ is not τ-additive. (*Hint*: setting $\phi(y)(-1)=0$, $\phi(y)(t)=\psi(y,t)$ for $t\ge 0$ and $y\in\mathbb{R}$, show that $\phi:\mathbb{R}\to\mathbb{R}^T$ is inverse-measure-preserving for ν_0 and $\hat\mu$, and that every point of $\mathbb{R}^T$ has a neighbourhood of zero measure.)

(d) Let T, t^*, $\langle\Omega_t\rangle_{t\in T}$, x^*, $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ and $\hat\mu$ be as in 455E. Suppose that we are given a family $\langle(\Omega'_t,\pi_t)\rangle_{t\in T}$ such that (α) Ω'_t is a Hausdorff space and $\pi_t:\Omega_t\to\Omega'_t$ is a continuous surjective function for every $t\in T$ (β) whenever $s<t$ in T and x, $x'\in\Omega_s$ are such that $\pi_s(x)=\pi_s(x')$, then the image measures $\nu_x^{(s,t)}\pi_t^{-1}$ and $\nu_{x'}^{(s,t)}\pi_t^{-1}$ on Ω'_t are the same. (i) Show that if we set $\acute\nu_w^{(s,t)}=\nu_x^{(s,t)}\pi_t^{-1}$ whenever $s<t$ in T, $x\in\Omega_s$ and $w=\pi_s(x)$, then every $\acute\nu_w^{(s,t)}$ is a Radon probability measure, and $\langle\acute\nu_z^{(t,u)}\rangle_{z\in\Omega'_t}$ is a disintegration of $\acute\nu_w^{(s,u)}$ over $\acute\nu_w^{(s,t)}$ whenever $s<t<u$ in T and $w\in\Omega'_s$. (ii) Let $\hat\mu'$ be the measure on $\Omega'=\prod_{t\in T}\Omega'_t$ defined by the method of 455E from $\pi_{t^*}(x^*)$ and $\langle\acute\nu_w^{(s,t)}\rangle_{s<t,w\in\Omega'_s}$. Show that $\pi:\Omega\to\Omega$ is inverse-measure-preserving for $\hat\mu$ and $\hat\mu'$, where $\pi(\omega)(t)=\pi_t(\omega(t))$ for $\omega\in\Omega$ and $t\in T$.

(e) Let U be a locally compact metrizable group and ν any Radon probability measure on U. For $t>0$ let λ_t be the Radon probability measure

$$e^{-t}(\delta_e+t\nu+\frac{t^2}{2!}\nu*\nu+\frac{t^3}{3!}\nu*\nu*\nu+\dots),$$

where δ_e is the Dirac measure on U concentrated at the identity e of U, and the sum is defined as in 234G[16]. Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455P (with respect to an appropriate metric on U). (*Hint*: 4A5Mb, 4A5Q(iv).)

[16]Formerly 112Ya.

(f) Let U, $\langle\lambda_t\rangle_{t>0}$ and $\hat\mu$ be as in 455Pc. Let V be a Hausdorff space, z^* a point of V and $\bullet$ a continuous action of U on V; set $\pi(x)=x\bullet z^*$ for $x\in U$. Define $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ from $\langle\lambda_t\rangle_{t>0}$ as in 455P. (i) Show that if x, $x'\in U$ and $\pi(x)=\pi(x')$, then the image measures $\nu_x^{(s,t)}\pi^{-1}$, $\nu_{x'}^{(s,t)}\pi^{-1}$ on V are equal whenever $s<t$. (ii) Let $\hat\mu'$ be the measure on $V^{[0,\infty[}$ defined as in 455Xd. Show that if we define $\tilde\pi:U^{[0,\infty[}\to V^{[0,\infty[}$ by setting $\tilde\pi(\omega)(t)=\omega(t)\bullet z^*$ for every $\omega\in U^{[0,\infty[}$ and $t\ge0$, $\tilde\pi$ is inverse-measure-preserving for $\hat\mu$ and $\hat\mu'$ and also for the Radon measures extending them; moreover, that the restriction of $\tilde\pi$ to $C_{\text{dlg}}(U)$, the space of càdlàg functions from $[0,\infty[$ to U, is inverse-measure-preserving for the subspace measures on $C_{\text{dlg}}(U)$ and $C_{\text{dlg}}(V)$.

>(g) For $t>0$, let λ_t be the normal distribution on $\mathbb{R}$ with expectation 0 and variance t. Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455Q.

>(h) For $t>0$, let λ_t be the Poisson distribution with expectation t, that is, $\lambda_t(E)=e^{-t}\sum_{m\in E\cap\mathbb{N}}t^m/m!$ for $E\subseteq\mathbb{R}$ (cf. 285Q, 285Xr). (i) Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455Q. (ii) Show that if $\tilde\mu$ is the Radon measure defined from $\langle\lambda_t\rangle_{t>0}$ as in 455Pc, then ω is non-decreasing and $\omega[\,[0,\infty[\,]=\mathbb{N}$ for $\tilde\mu$-almost every $\omega\in\mathbb{R}^{[0,\infty[}$.

>(i) For $t>0$, let λ_t be the Cauchy distribution with centre 0 and scale parameter t, that is, the distribution with probability density function $x\mapsto\frac{t}{\pi(x^2+t^2)}$ (285Xp). (i) Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455Q. (ii) Show that if $\hat\mu$ is the corresponding distribution on $\mathbb{R}^{[0,\infty[}$, then $C([0,\infty[)$ is $\hat\mu$-negligible. (*Hint*: estimate $\Pr(|X_{(i+1)/n}-X_{i/n}|\le\epsilon$ for every $i<n$).) (iii) Suppose that $\alpha>0$. Define $T_\alpha:\mathbb{R}^{[0,\infty[}\to\mathbb{R}^{[0,\infty[}$ by setting $(T_\alpha\omega)(t)=\dfrac{1}{\alpha}\omega(\alpha t)$ for $t\ge0$ and $\omega\in\mathbb{R}^{[0,\infty[}$. Show that T_α is inverse-measure-preserving for $\hat\mu$.

>(j)(i) The **standard gamma distribution** with parameter t is the probability distribution λ_t on $\mathbb{R}$ with probability density function $x\mapsto\dfrac{1}{\Gamma(t)}x^{t-1}e^{-x}$ for $x>0$. Show that its expectation is t. (*Hint*: 225Xh(iv).) Show that its variance is t. (ii) Show that $\langle\lambda_t\rangle_{t>0}$ satisfies the conditions of 455Q. (*Hint*: 272U[17], 252Yf.) (iii) Show that $\lim_{t\downarrow0}t\Gamma(t)=1$, so that $\lim_{t\downarrow0}\frac{1}{t}\lambda_t\,[1,\infty[\,=\int_1^\infty\frac{1}{x}te^{-x}dx>0$. (iv) Show that if $\tilde\mu$ is the Radon measure on $\mathbb{R}^{[0,\infty[}$ defined from $\langle\lambda_t\rangle_{t>0}$ as in 455Pc, then $\{\omega:\omega$ is strictly increasing and not continuous$\}$ is $\tilde\mu$-conegligible.

(k) Let U be an abelian Hausdorff topological group. Let $\langle\lambda'_t\rangle_{t>0}$, $\langle\lambda''_t\rangle_{t>0}$ be two families of Radon probability measures on U and set $\lambda_t=\lambda'_t*\lambda''_t$ for $t>0$. (i) Show that if $\lambda'_{s+t}=\lambda'_s*\lambda'_t$ and $\lambda''_{s+t}=\lambda''_s*\lambda''_t$ for all s, $t>0$, then $\lambda_{s+t}=\lambda_s*\lambda_t$ for all s, $t>0$. (ii) Show that if $\lim_{t\downarrow0}\lambda'_tG=\lim_{t\downarrow0}\lambda''_tG=1$ for every open set containing the identity e of U, then $\lim_{t\downarrow0}\lambda_tG=1$ for every open set G containing e. (iii) Now suppose that U is metrizable and complete under a right-translation-invariant metric inducing its topology. Let $\hat\mu'$, $\hat\mu''$ and $\hat\mu$ be the measures on $U^{[0,\infty[}$ defined from $\langle\lambda'_t\rangle_{t>0}$, $\langle\lambda''_t\rangle_{t>0}$ and $\langle\lambda_t\rangle_{t>0}$ as in 455Pc. Set $\theta(\omega,\omega')(t)=\omega(t)\omega'(t)$ for ω, $\omega'\in U^{[0,\infty[}$ and $t\ge0$. Show that $\theta:U^{[0,\infty[}\times U^{[0,\infty[}\to U^{[0,\infty[}$ is inverse-measure-preserving for $\hat\mu'\times\hat\mu''$ and $\hat\mu$. (iv) Repeat (iii) for the subspace measures on the space of càdlàg functions from $[0,\infty[$ to U.

455Y Further exercises (a) Let $\langle X_n\rangle_{n\in\mathbb{Z}}$ be a double-ended sequence of real-valued random variables such that (i) for each $n\in\mathbb{Z}$, $Y_n=X_{n+1}-X_n$ is independent of $\{X_i:i\le n\}$ (ii) $\langle Y_n\rangle_{n\in\mathbb{Z}}$ is identically distributed. Show that the Y_n are essentially constant. (*Hint*: 285Yc.)

(b) For $0\le s<t$ and $x\in\mathbb{R}$ define a Radon probability measure $\nu_x^{(s,t)}$ on $\mathbb{R}$ by saying that

[17] Formerly 272T.

$$\begin{aligned}\nu_x^{(s,t)} &= \tfrac{1-t}{1-s}\delta_0 + \tfrac{t-s}{1-s}\lambda_{[s,t]} \text{ if } x = 0 \text{ and } t \le 1,\\ &= \tfrac{t-1}{1-s}\delta_0 + \tfrac{2-t-s}{1-s}\lambda_{[s,2-t]} \text{ if } x = 0 \text{ and } 1 \le t < 2-s,\\ &= \delta_0 \text{ if } 0 < x < 1 \text{ and } 2-x \le t,\\ &= \delta_x \text{ otherwise,}\end{aligned}$$

writing δ_x for the Dirac measure on $\mathbb{R}$ concentrated at x, and $\lambda_{[s,t]}$ for the uniform distribution based on the interval $[s,t]$. (i) Show that $\langle \nu_x^{(s,t)}\rangle_{s<t,x\in\mathbb{R}}$ satisfies the conditions of 455E. (ii) Starting from $x^* = 0$, let $\hat\mu$ be the corresponding measure on $\mathbb{R}^{[0,\infty[}$. Show that $\hat\mu^* C_{\text{dlg}} = 1$, where C_{dlg} is the space of càdlàg functions from $[0,\infty[$ to $\mathbb{R}$. (iii) Show that $\hat\mu$ has a unique extension to a Radon measure $\tilde\mu$ on $\mathbb{R}^{[0,\infty[}$. (iv) Show that $\tilde\mu C_{\text{dlg}} = 0$. (v) Show that the subspace measure on C_{dlg} induced by $\hat\mu$ is not τ-additive.

(c) A probability distribution λ on $\mathbb{R}$ is **infinitely divisible** if for every $n \ge 1$ it is expressible as a convolution $\nu * \ldots * \nu$ of n copies of a probability distribution. Let ϕ be the characteristic function (§285) of an infinitely divisible distribution λ. (i) Show that for each $n \ge 1$ there is a characteristic function ϕ_n such that $\phi_n^n = \phi$. (ii) Show that if $\delta > 0$ is such that $\phi(y) \ne 0$ for $|y| \le \delta$, then $\lim_{n\to\infty}\phi_n(y) = 1$ for $|y| \le \delta$. (iii) Show that $\lim_{n\to\infty}\phi_n(y) = 1$ for every $y \in \mathbb{R}$. (*Hint*: for any characteristic function ψ, $4\,\mathcal{R}\mathrm{e}\,\psi(y) \le 3 + \mathcal{R}\mathrm{e}\,\psi(2y)$ for every y.) (iv) Show that ϕ is never zero, and that there is a unique family $\langle\lambda_t\rangle_{t>0}$ of distributions satisfying the conditions in 455P and such that $\lambda_1 = \lambda$. (v) Show that if λ has finite expectation, then $\mathbb{E}(\lambda_t)$ is defined and equal to $t\mathbb{E}(\lambda)$ for every $t > 0$. (vi) Show that if λ has finite variance, then $\mathrm{Var}(\lambda_t)$ is defined and equal to $t\,\mathrm{Var}(\lambda)$ for every $t > 0$.

(d) Let U be a Hausdorff topological group and $\langle\lambda_t\rangle_{t>0}$ a family of Radon probability measures on U such that $\lambda_{s+t} = \lambda_s * \lambda_t$ whenever s, $t > 0$. (i) Show that we can define a family $\langle\nu_x^{(s,t)}\rangle_{0\le s<t,x\in U}$ as in 455P, and that $\langle\nu_y^{(t,u)}\rangle_{y\in U}$ is a disintegration of $\nu_x^{(s,u)}$ over $\nu_x^{(s,t)}$ whenever $x \in U$ and $s < t < u$, so that, starting from $x^* = e$ the identity of U, we can apply 455E to obtain a measure $\hat\mu$ on $U^{[0,\infty[}$. (ii) Now suppose that $U = \mathbb{R}^r$ where $r \ge 1$. For $t > 0$, $E \subseteq U$ set $\overset{\leftrightarrow}{\lambda}_t E = \lambda_t(-E)$ whenever λ_t measures $-E$; now set $\lambda_t^{\#} = \lambda_t * \overset{\leftrightarrow}{\lambda}_t$. Show that $\lambda_{s+t}^{\#} = \lambda_s^{\#} * \lambda_t^{\#}$ for all s, $t > 0$, and that $\lim_{t\downarrow 0}\lambda_t^{\#}G = 1$ for every open neighbourhood G of 0. Show that $\hat\mu$ has an extension to a Radon measure on $(\mathbb{R}^r)^{[0,\infty[}$.

(e) Let Y be a metrizable space and C_{dlg} the set of càdlàg functions from $[0,\infty[$ to Y. For $\omega \in C_{\text{dlg}}$ and $t \ge 0$ set $X_t(\omega) = \omega(t)$. Let Σ be a σ-algebra of subsets of C_{dlg} such that $X_t : C_{\text{dlg}} \to Y$ is measurable for every $t \ge 0$. For $t \ge 0$ let Σ_t be

$$\{F : F \in \Sigma,\ \omega' \in F \text{ whenever } \omega,\ \omega' \in C_{\text{dlg}},\ \omega \in F \text{ and } \omega\restriction[0,t] = \omega'\restriction[0,t]\}.$$

Show that $\langle X_t\rangle_{t\ge0}$ is progressively measurable with respect to $\langle\Sigma_t\rangle_{t\ge0}$.

455 Notes and comments This section has grown into the longest in this treatise. There are some big theorems here. I am trying to do two rather different things: sketch the fundamental properties of Markov processes, and work through the details of particular realizations of them. I remarked in the introduction to Chapter 27 that probability theory is not really about measure spaces and measurable functions. It is much more about distributions, and by 'distribution' here I do not really mean a Radon probability measure on $\mathbb{R}^r$, let alone a completed Baire measure on $\mathbb{R}^I$, as in 454K. I mean rather the family of probabilities of the type $\Pr(X_i \le \alpha_i\ \forall\ i \le n)$; everything else is formal structure, offering proofs and (I hope) some kinds of deeper understanding, but essentially secondary. The appalling formulae above ($\nu_{\omega,\tau(\omega),x_j}^{(t_j,t_{j+1})}(dx_{j+1})$, $\ddot\Sigma\widehat\otimes\ddot\Sigma$ and so on) arise from my attempts to distinguish clearly among the host of probability spaces which present themselves to us as relevant.

However one of the messages of this section is that for many stochastic processes it is possible to identify semi-canonical realizations. We already have a crude one in 454J; starting from any family $\langle X_i\rangle_{i\in I}$ of real-valued random variables on any probability space, we can move to a measure on $\mathbb{R}^I$ which is in some sense unique and carries the probabilistic content of the original family. I noted in §454 that when this measure is

τ-additive we have a canonical extension to a quasi-Radon measure, just as good regarded as a realization of the abstract process, and possibly with useful further properties. In 455H we find that many of the most important processes can be represented by Radon measures; I do not think these Radon measures have been much studied, except, of course, in the case of Brownian motion. But 455O and 455U show that for some purposes we are better off with quasi-Radon measures on the set of càdlàg functions. The most important stopping times are the hitting times of 455M, which are adapted to families of the form $\langle\Sigma_t^+\rangle_{t\ge0}$; and for such a stopping time to be approximated by discrete stopping times, as in parts (a-vi) and (b-vii) of the proof of 455O, we need to know that our paths are continuous on the right.

It is of course true that when the complete metric space U, in 455O or later, is separable, then we have a standard Borel structure on the space C_{dlg} of càdlàg functions (4A3Qb), so that the measures $\ddot\mu$ are Radon measures for appropriate Polish topologies on C_{dlg}.

Returning to the detailed exposition, 455A is an attempt at a continuous-time version of 454H. I use the letters t, T to suggest the probabilistic intuitions behind these results; we think of the spaces Ω_t in 455A as being the sets of possible states of a system at 'time' t, so that the measures $\nu_x^{(s,t)}$ are descriptions of how we believe the system is likely to evolve between times s and t, having observed that it is in state x at time s. In the case of 'discrete time', when we observe the system only at clearly separated moments, it is easy to handle non-Markov processes, in which evolution between times $n-1$ and n can depend on the whole history up to time $n-1$; thus in 454H the measures $\nu_z=\nu_z^{(n-1,n)}$ are defined for every $z\in\prod_{i<n}X_i$, but we make no attempt to describe measures $\nu_z^{(n-1,m)}$ for any $m>n$. In 'continuous time' we need to say something about arbitrary time steps, and it is hard to formulate a consistency condition to fill the place of (†) in 455A without limiting the kind of process being examined. At the cost of an appalling increase in complexity, of course, the formulae of 455A can sometimes be adapted to general processes, if we replace the 'current' state space Ω_t by the 'historical' state space $\prod_{t^*\le s\le t}\Omega_s$. (For we can hope that $(\prod_{t^*\le s\le t}\Omega_s,\widehat{\bigotimes}_{t^*\le s\le t}\mathrm{T}_s)$ will have the 'perfect measure property' of 454Xd.) We should finish up with a measure on $\prod_{t\in T}(\prod_{t^*\le s\le t}\Omega_s)$. But the important applications, even when not Markov, are open to more economical and more enlightening approaches. We really do need a least element t^* of T; see 455Ya.

I have not yet come to the reason why this section is such hard work. This is in its attempt to analyze the 'Markov property' of the distributions being examined here. The point about the families $\langle\nu_x^{(s,t)}\rangle_{s<t,x\in\Omega_s}$ of transitional probabilities is that they not only give us stochastic processes, as in 455A, but also recipes for conditional expectations, derived from the truncated families $\langle\nu_x^{(s,t)}\rangle_{a\le s<t,x\in\Omega_s}$. These lead to measures μ'_{ax} on $\prod_{t\ge a}\Omega_t$ which can be thought of as distributions of future paths given that we have reached the point x at time a. It is no surprise that these should provide straightforward descriptions of conditional expectations on algebras of the form

$$\{F: F\text{ is determined by coordinates in }[t^*,t]\}.$$

Without much more trouble, we can extend this to suitable algebras defined from simple 'stopping times', as in 455C. The arguments there have some technical features which you may find annoying (and I invite you to find your own way past the complications), but are essentially elementary, as they have to be in such a general context. It is interesting that we can move to stopping times taking countably many values without further difficulty.

However, we are still only seven pages into the section, and not everything to come is as straightforward as the completion processes described in 455E. An essential aspect of continuous-time Markov processes is the possibility of stopping times which take a continuum of values, as is typically the case in the examples provided by 455M. These are much harder to deal with, and we have to restrict sharply the class of processes we examine. The particular restriction I have chosen is described by the definitions in 455F. I should of course say that these, particularly 455Fb ('uniformly time-continuous on the right') are more limiting than is strictly necessary; in 'Feller processes' (Rogers & Williams 94, III.6) we have a slightly different approach to the same intuitive target. The aim is to find sufficient conditions for the 'strong Markov property', in which we can find disintegrations and conditional expectations associated with general stopping times, as in 455O. To do this, we have to abandon the set $\Omega=U^{[0,\infty[}$ and move to the correct set of full outer measure, the set C_{dlg} of 'càdlàg' functions, which dominates the central part of this section. The first thing the definition 455Fb must do is to ensure that C_{dlg} has full outer measure not only for the distribution on Ω but also for the conditional distributions we shall be using (455G). If U is a Polish space, C_{dlg} has a standard

Borel structure (4A3Qb), which is comforting.

I hope that you are becoming resigned to the view that the notational complexities of this section are not solely due to an inconsiderate disregard for the reader's eyesight. The original probability measures $\nu_x^{(s,t)}$ of 455A really do form a three-parameter family, the conversion of these into finite-dimensional distributions λ_J really is a multiple repeated integral, the derived probabilities $\nu_{\omega ax}^{(s,t)}$ in 455B are a five-parameter system. Without wishing to insist on my use of grave accents in the proof of 455E, it is surely safer to have a way of distinguishing between completed and uncompleted measures, and while the result may be 'obvious', I think there are some twists on the way which not everyone would foresee. Again, if you wish to dispense with the double-dotted symbols from 455O on, you will have to find some other way of reminding yourself that we are looking at a new representation of the process on a new probability space.

This treatise as a whole is theory-heavy and example-light. I assure you that all the theory here is in fact example-driven. You should start with the four examples of Lévy processes in 455Xg-455Xj. Of these, 455Xg is **Brownian motion**, the starting point of the whole theory; I will return to this in §477. A problem with the formalization in 455A is that we have to start with an exact description of the transitional probabilities $\nu_x^{(s,t)}$. It does not help at all in establishing the existence of such families matching some probabilistic intuition. Only in rather special cases do we have elegant formulae for these systems. In 455Xb, 455Xd and 455Xf I try to show how the general theory gives us methods of using one system to build others.

I suppose that 455O is the summit; from here on the going is easier. In 455P I introduce 'Lévy processes', a particularly interesting class intermediate in generality between the continuous processes of 455O and Brownian motion. These have of course mostly been considered in the case $U=\mathbb{R}$, but the extension to Banach spaces U is an obvious one, and we can even manage non-abelian groups if we are careful. (For an elementary example of a process which can really exploit a non-abelian group, see 455Xe.) The 'Poisson process' in 455Xh is by some way the most important example after Brownian motion itself. Lévy processes on $\mathbb{R}$ are well understood; the family $\langle\lambda_t\rangle_{t>0}$ is determined by λ_1, any 'infinitely divisible' distribution can be taken for λ_1 (455Yc), and a complete description of infinitely divisible distributions is provided by the Lévy-Khintchine representation theorem (Fristedt & Gray 97, 16.3). As a final result in the general theory, I give an alternative version of the strong Markov property in 455U. For Lévy processes, we can re-start, following any of the usual stopping times, with an exact copy of the process, and this corresponds to a true inverse-measure-preserving function from $C_{\rm dlg}^2$ to $C_{\rm dlg}$.

A comment on 455T. The idea behind the σ-algebras Σ_t, $\ddot\Sigma_t$ of 455M, 455O and later is that they consist of events 'observable at time t', that is, determined by the path taken up to and including time t. We quickly find ourselves forced to consider augmented algebras $\Sigma_t^+=\bigcap_{s>t}\Sigma_s$, where somehow we are allowed infinitesimal intuitions into the immediate future. (A typical situation is that of 455Mb when the set A is open, so that if $\omega(t)\in\overline{A}$ we can expect that there will be paths which continue immediately into A, and others which do not, and it may not be obvious which, if either, should be regarded as typical.) The question is, whether Σ_t is really different from Σ_t^+. The claim of 455T is that $\ddot\Sigma_t^+$ is included in a kind of completion $\hat{\ddot\Sigma}_t$ of $\ddot\Sigma_t$. Of course the completion is in terms of the measure $\ddot\mu$ on the whole space $C_{\rm dlg}$ of càdlàg paths; we need advance knowledge of which subsets of $C_{\rm dlg}$ are negligible. But if we are interested in the measure algebra $\mathfrak{A}$ of $\ddot\mu$ and its closed subalgebras $\mathfrak{A}_t=\{E^\bullet:E\in\ddot\Sigma_t\}$, 455T tells us that (in the context of Lévy processes) we can expect to have $\mathfrak{A}_t=\bigcap_{s>t}\mathfrak{A}_s$. Turning to the definition of $\ddot\mu$ in 455O as a subspace measure, we see that $\mathfrak{A}$ can be regarded as the measure algebra of the measure $\hat\mu$ on $U^{[0,\infty[}$ defined by the formulae of 455E; and even that $\mathfrak{A}_t$ can be identified with

$$\{E^\bullet:E\in\operatorname{dom}\hat\mu,\ E\text{ is determined by coordinates in }[0,t]\}$$

(see part (a-ii) of the proof of 455O). But I think that this last step will not usually be helpful, because (as noted above) $\ddot\mu$ will commonly be a Radon measure for an appropriate topology, while $\hat\mu$ is likely at best to be the completion of a Baire measure.

I have cast the second half of the section in terms of measures on $C_{\rm dlg}$, because it is reasonably well adapted to Lévy processes in general. When we come to look at particular processes, we often find that there is a smaller class of functions (e.g., continuous functions in the case of Brownian motion, or non-decreasing $\mathbb{N}$-valued functions in the case of the Poisson process) which is fully adequate and easier to focus on. For the detailed study of such processes, as in §477 below, I think it will usually be helpful to make the shift. But there may be rival conegligible subsets of $C_{\rm dlg}$ with different virtues, as in 477Ef.

Version of 19.5.10

456 Gaussian distributions

Uncountable powers of $\mathbb{R}$ are not as a rule measure-compact (439P, 455Xc; see also 533J in Volume 5). Accordingly distributions, in the sense of 454K, need not be τ-additive. But some, at least, of the distributions most important to us are indeed τ-additive, and therefore have interesting canonical extensions. This section is devoted to a remarkable result, taken from Talagrand 81, concerning a class of distributions which are of great importance in probability theory. It demands a combination of techniques from classical probability theory and from the topological measure theory of this volume. I begin with the definition and fundamental properties of what I call 'centered Gaussian distributions' (456A-456I). These are fairly straightforward adaptations of the classical finite-dimensional theory, and will be useful in §477 when we come to study Brownian motion. Another relatively easy idea is that of 'universal' Gaussian distribution (456J-456L). In 456M we come to a much deeper result, a step towards classifying the ways in which a Gaussian family of n-dimensional random variables can accumulate at 0. The ideas are combined in 456N-456O to complete the proof of Talagrand's theorem that Gaussian distributions on powers of $\mathbb{R}$ are τ-additive.

456A Definitions (a) Write μ_G for the Radon probability measure on $\mathbb{R}$ which is the distribution of a standard normal random variable, that is, the probability distribution with density function $x \mapsto \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ (274A). For any set I, write $\mu_G^{(I)}$ for the measure on $\mathbb{R}^I$ which is the product of copies of μ_G; this is always quasi-Radon (415E/453J); if I is countable, it is Radon (417Q); if $I = n \in \mathbb{N} \setminus \{0\}$, it is the probability distribution with density function $x \mapsto (2\pi)^{-n/2}e^{-x\cdot x/2}$ (272I); if $I = \emptyset$, it is the unique probability measure on the singleton set $\mathbb{R}^\emptyset$.

(b) I will use the phrase **centered Gaussian distribution** to mean a measure μ on a power $\mathbb{R}^I$ of $\mathbb{R}$ such that μ is the completion of a Baire measure (that is, is a distribution in the sense of 454K) and every continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$ is either zero almost everywhere or is a normal random variable with zero expectation. (Note that I call the distribution concentrated at the point 0 in $\mathbb{R}^I$ a 'Gaussian distribution'.)

(c) If I is a set and μ is a centered Gaussian distribution on $\mathbb{R}^I$, its **covariance matrix** is the family $\langle\sigma_{ij}\rangle_{i,j\in I}$ where $\sigma_{ij} = \int x(i)x(j)\mu(dx)$ for i, $j \in I$. (The integral is always defined and finite because each function $x \mapsto x(i)$ is either essentially constant or normally distributed, and in either case is square-integrable.)

456B I start with some fundamental facts about Gaussian distributions.

Proposition (a) Suppose that I and J are sets, μ is a centered Gaussian distribution on $\mathbb{R}^I$, and $T : \mathbb{R}^I \to \mathbb{R}^J$ is a continuous linear operator. Then there is a unique centered Gaussian distribution on $\mathbb{R}^J$ for which T is inverse-measure-preserving; if J is countable, this is the image measure μT^{-1}.

(b) Let I be a set, and μ, ν two centered Gaussian distributions on $\mathbb{R}^I$. If they have the same covariance matrices they are equal.

(c) For any set I, $\mu_G^{(I)}$ is the centered Gaussian distribution on $\mathbb{R}^I$ with the identity matrix for its covariance matrix.

(d) Suppose that I is a countable set. Then a measure μ on $\mathbb{R}^I$ is a centered Gaussian distribution iff it is of the form $\mu_G^{(\mathbb{N})}T^{-1}$ where $T : \mathbb{R}^\mathbb{N} \to \mathbb{R}^I$ is a continuous linear operator.

(e) Suppose $\langle I_j\rangle_{j\in J}$ is a disjoint family of sets with union I, and that for each $j \in J$ we have a centered Gaussian distribution ν_j on $\mathbb{R}^{I_j}$. Then the product ν of the measures ν_j, regarded as a measure on $\mathbb{R}^I$, is a centered Gaussian distribution.

(f) Let I be any set, μ a centered Gaussian distribution on $\mathbb{R}^I$ and $E \subseteq \mathbb{R}^I$ a set such that μ measures E. Writing $-E = \{-x : x \in E\}$, $\mu(-E) = \mu E$.

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proof (a)(i) For Baire sets $F \subseteq \mathbb{R}^J$, set $\nu F = \mu T^{-1}[F]$; this is always defined because T is continuous (4A3Kc). This makes ν a Baire probability measure on $\mathbb{R}^J$ for which T is inverse-measure-preserving. Because μ is complete, T is still inverse-measure-preserving for the completion $\hat{\nu}$ of ν (234Ba[18]). If $g : \mathbb{R}^J \to \mathbb{R}$ is a continuous linear functional, so is $gT : \mathbb{R}^I \to \mathbb{R}$; now $\nu\{y : g(y) \le \alpha\} = \mu\{x : gT(x) \le \alpha\}$ for every α, so g and gT have the same distribution, and are both either zero a.e. or normal random variables. As g is arbitrary, $\hat{\nu}$ is a centered Gaussian distribution as defined in 456Ab. Of course it is the only such distribution on $\mathbb{R}^J$ for which T is inverse-measure-preserving.

(ii) Now suppose that J is countable. Then $\mathbb{R}^J$ is Polish (4A2Qc), so ν is a Borel measure and $\hat{\nu}$ is a Radon measure (433Cb). $\mathbb{R}^J$ has a countable network consisting of Borel sets, μ is perfect (454A(b-iii)) and totally finite, and T is measurable (418Bd), so μT^{-1} is a Radon measure (451O). Thus $\hat{\nu}$ and μT^{-1} are Radon measures agreeing on the Borel sets and must be equal.

(b) The point is that $\mu f^{-1} = \nu f^{-1}$ for every continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$. **P** By (a), μf^{-1} and νf^{-1} are Radon measures on $\mathbb{R}$, and by the definition of 'Gaussian distribution' each is either a normal distribution with expectation zero, or is concentrated at 0. By 4A4Be, we can express f in the form $f(x) = \sum_{i\in I} \beta_i x(i)$ for every $x \in \mathbb{R}^I$, where $\{i : \beta_i \ne 0\}$ is finite. In this case

$$\int t^2(\mu f^{-1})(dt) = \int f(x)^2 \mu(dx)$$

(235G[19])

$$= \sum_{i,j\in I} \beta_i\beta_j \int x(i)x(j)\mu(dx)$$

$$= \sum_{i,j\in I} \beta_i\beta_j \int x(i)x(j)\nu(dx) = \int t^2(\nu f^{-1})(dt)$$

because μ and ν have the same covariance matrices. But this means that μf^{-1} and νf^{-1} have the same variance; if this is zero, they both give measure 1 to $\{0\}$; otherwise, they are normal distributions with the same expectation and the same variance, so again are equal. **Q**

By 454P, $\mu = \nu$.

(c) Being a completion regular quasi-Radon probability measure (415E), $\mu_G^{(I)}$ is the completion of a Baire probability measure on $\mathbb{R}^I$. If $f : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional, then it is expressible in the form $f(z) = \sum_{i\in I} \beta_i z(i)$, where $J = \{i : \beta_i \ne 0\}$ is finite. I need to show that f is either zero a.e. or a normal random variable with expectation 0. If $J = \emptyset$ then $f = 0$ everywhere and we can stop. Otherwise, $f = \sum_{i\in J} \beta_i\pi_i$, where $\pi_i(x) = x(i)$ for $i \in I$ and $x \in \mathbb{R}^I$. Now, with respect to the measure $\mu_G^{(I)}$, $\langle\pi_i\rangle_{i\in J}$ is an independent family of normal random variables with zero expectation (272G). So $\langle\beta_i\pi_i\rangle_{i\in J}$ is independent (272E), and $\beta_i\pi_i$ is normal for $i \in J$ (274Ae). By 274B, $f = \sum_{i\in J} \beta_i\pi_i$ is normal, and of course it has zero expectation. As f is arbitrary, $\mu_G^{(I)}$ is a centered Gaussian distribution.

We have

$$\int x(i)x(i)\mu_G^{(I)}(dx) = \int t^2\mu_G(dt) = 1$$

for $i \in I$, and

$$\int x(i)x(j)\mu_G^{(I)}(dx) = \int t\mu_G(dt) \cdot \int t\mu_G(dt) = 0$$

if i, $j \in I$ are distinct. So the covariance matrix of $\mu_G^{(J)}$ is the identity matrix. By (b), it is the only centered Gaussian distribution with this covariance matrix.

(d)(i) It follows from (c) and (a) that if $\mu = \mu_G^{(\mathbb{N})}T^{-1}$, where $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^I$ is a continuous linear operator, then μ is a centered Gaussian distribution.

[18] Formerly 235Hc.
[19] Formerly 235I.

(ii) Now suppose that μ is a centered Gaussian distribution on $\mathbb{R}^I$. Set $\pi_i(x) = x(i)$ for $i \in I$ and $x \in \mathbb{R}^I$; for $i \in I$, set $u_i = \pi_i^\bullet$ in $L^2 = L^2(\mu)$. By 4A4Jh, there is a countable orthonormal family $\langle v_j\rangle_{j\in J}$ in L^2 such that every v_j is a linear combination of the u_i, and every u_i is a linear combination of the v_j. We may suppose that $J \subseteq \mathbb{N}$. For $i \in I$, express u_i as $\sum_{j\in J}\alpha_{ij}v_j$, where $\{j : \alpha_{ij} \ne 0\}$ is finite. Define $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^I$ by setting $(Tz)(i) = \sum_{j\in J}\alpha_{ij}z(j)$ for every $z \in \mathbb{R}^{\mathbb{N}}$ and $i \in I$. Then T is a continuous linear functional. Set $\nu = \mu_G^{(\mathbb{N})}T^{-1}$, so that ν is a centered Gaussian distribution on $\mathbb{R}^I$, by (a). Because $\mathbb{R}^I$ is Polish, both μ and ν must be Radon measures.

(iii) Now μ and ν have the same covariance matrices. **P** If i, $i' \in I$ then

$$\begin{aligned}\int x(i)x(i')\mu(dx) &= (u_i|u_{i'}) = \sum_{j,j'\in J}\alpha_{ij}\alpha_{i'j'}(v_j|v_{j'})\\ &= \sum_{j\in J}\alpha_{ij}\alpha_{i'j} = \sum_{j,j'\in J}\alpha_{ij}\alpha_{i'j'}\int z(j)z(j')\mu_G^{(\mathbb{N})}(dz)\\ &= \int (Tz)(i)(Tz)(i')\mu_G^{(\mathbb{N})}(dz) = \int x(i)x(i')\nu(dx). \ \mathbf{Q}\end{aligned}$$

By (b), $\mu = \nu$ is of the required form.

(e) We must first confirm that ν is the completion of a Baire measure. **P** If we write $\mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$ for the Baire σ-algebra of $\mathbb{R}^{I_j}$, then each ν_j is the completion of its restriction $\nu_j\restriction\mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$, so ν is also the product of the measures $\nu_j\restriction\mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$ (254I), and is therefore the completion of its restriction to $\widehat{\bigotimes}_{j\in J}\mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$ (254Ff). But as $\mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j}) = \widehat{\bigotimes}_{i\in I_j}\mathcal{B}\mathfrak{a}(\mathbb{R})$ for every j (4A3Na), $\widehat{\bigotimes}_{j\in J}\mathcal{B}\mathfrak{a}(\mathbb{R}^{I_j})$ can be identified with $\widehat{\bigotimes}_{i\in I}\mathcal{B}\mathfrak{a}(\mathbb{R}) = \mathcal{B}\mathfrak{a}(\mathbb{R}^I)$, so that ν is indeed the completion of $\nu\restriction\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$. **Q**

Now suppose that $f : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional. Then we can express f in the form $f(x) = \sum_{i\in K}\alpha_i x(i)$ for every $x \in \mathbb{R}^I$, where $K \subseteq I$ is finite. Set $L = \{j : K \cap I_j \ne \emptyset\}$ and $K_j = K \cap I_j$ for $j \in L$, so that L and every K_j are finite; for $j \in L$ and $x \in \mathbb{R}^I$ set $f_j(x) = \sum_{i\in K_j}\alpha_i x(i)$. Now $f = \sum_{j\in L} f_j$.

If we set $g_j(y) = \sum_{i\in K_j}\alpha_i y(i)$ for $y \in \mathbb{R}^{I_j}$, then g_j is either zero a.e. or a normal random variable with respect to the probability measure ν_j. Since

$$\nu\{x : f_j(x) \le \alpha\} = \nu\{x : g_j(x\restriction I_j) \le \alpha\} = \nu_j\{y : g_j(y) \le \alpha\}$$

for every $\alpha \in \mathbb{R}$, f_j (regarded as a random variable on $(\mathbb{R}^I, \nu)$) has the same distribution as g_j (regarded as a random variable on $(\mathbb{R}^{I_j}, \nu_j)$). This is true for every $j \in L$. Moreover, the different f_j, as j runs over L, are independent. So $f = \sum_{j\in L} f_j$ is the sum of independent random variables which are all either normal or essentially constant. By 274B again, f also is either normal or essentially constant. And of course its expectation is zero. As f is arbitrary, this shows that ν is a centered Gaussian distribution.

(f) Set $Tx = -x$ for $x \in \mathbb{R}^I$, so that T is a continuous linear operator and we have a unique centered Gaussian distribution ν on $\mathbb{R}^I$ such that T is inverse-measure-preserving for μ and ν, by (a). For any i, $j \in I$,

$$\int x(i)x(j)\nu(dx) = \int (Tx)(i)(Tx)(j)\mu(dx) = \int x(i)x(j)\mu(dx),$$

so μ and ν have the same covariance matrices and are equal, by (b). Accordingly

$$\mu(-E) = \mu T^{-1}[E] = \nu E = \mu E$$

whenever μ measures E.

456C Since a Gaussian distribution is determined by its covariance matrix (456Bb), we naturally seek descriptions of which matrices can arise.

Theorem Let I be a set and $\langle\sigma_{ij}\rangle_{i,j\in I}$ a family of real numbers. Then the following are equiveridical:

(i) $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^I$;

(ii) there are a (real) Hilbert space U and a family $\langle u_i\rangle_{i\in I}$ in U such that $(u_i|u_j) = \sigma_{ij}$ for all i, $j \in I$;

(iii) for every finite $J \subseteq I$, $\langle\sigma_{ij}\rangle_{i,j\in J}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^J$;

(iv) $\langle\sigma_{ij}\rangle_{i,j\in I}$ is symmetric and positive semi-definite in the sense that $\sigma_{ij}=\sigma_{ji}$ for all i, $j\in I$ and $\sum_{i,j\in J}\alpha_i\alpha_j\sigma_{ij}\ge 0$ whenever $J\subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in J}\in\mathbb{R}^J$.

proof (i)$\Rightarrow$(ii) If μ is a centered Gaussian distribution on $\mathbb{R}^I$ with covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in I}$, then $L^2(\mu)$ is a Hilbert space. Setting $X_i(x)=x(i)$ for $x\in\mathbb{R}^I$, $u_i=X_i^\bullet$ belongs to the Hilbert space $L^2(\mu)$ for every $i\in I$, and

$$(u_i|u_j)=\int X_i\times X_j d\mu=\int x(i)x(j)\mu(dx)=\sigma_{ij}$$

for all i, $j\in I$.

(ii)$\Rightarrow$(iv) In this context,

$$\sigma_{ij}=(u_i|u_j)=(u_j|u_i)=\sigma_{ji},$$

$$\textstyle\sum_{i,j\in J}\alpha_i\alpha_j\sigma_{ij}=\sum_{i,j\in J}\alpha_i\alpha_j(u_i|u_j)=\|\sum_{i\in J}\alpha_i u_i\|^2\ge 0.$$

(iv)$\Rightarrow$(iii) Here we have to know something about symmetric matrices. Given a family $\langle\sigma_{ij}\rangle_{i,j\in I}$ satisfying the conditions of (iv), and a finite set $J\subseteq I$, we have a linear operator $T:\mathbb{R}^J\to\mathbb{R}^J$ defined by saying that $(Tz)(i)=\sum_{j\in J}\sigma_{ij}z(j)$ for $z\in\mathbb{R}^J$ and $i\in J$. Give $\mathbb{R}^J=\ell^2(J)$ its usual inner product, so that $w\centerdot z=\sum_{j\in J}w(j)z(j)$ for w, $z\in\mathbb{R}^J$; then $\mathbb{R}^J$ is a Hilbert space and

$$\textstyle Tw\centerdot z=\sum_{i\in J}\sum_{j\in J}\sigma_{ij}w(j)z(i)=\sum_{j\in J}\sum_{i\in J}\sigma_{ji}z(i)w(j)=w\centerdot Tz$$

for all w, $z\in\mathbb{R}^J$, so that T is self-adjoint. Moreover, if $z\in\mathbb{R}^J$,

$$\textstyle Tz\centerdot z=\sum_{i,j\in J}\sigma_{ij}z(i)z(j)\ge 0$$

by the other condition on $\langle\sigma_{ij}\rangle_{i,j\in I}$.

By 4A4M[20], $\mathbb{R}^J$ has an orthonormal basis consisting of eigenvectors for T; if $\#(J)=n$, we have a basis $\langle u_k\rangle_{k<n}$ and a family $\langle\gamma_k\rangle_{k<n}$ of real numbers such that $Tu_k=\gamma_k u_k$ for each $k<n$. We need to know that $\sum_{k<n}u_k(i)u_k(j)=1$ if $i=j$, 0 otherwise. **P** Let $\langle v_i\rangle_{i\in J}$ be the standard basis of $\mathbb{R}^J$, so that $v_i(j)=1$ if $i=j$, 0 if $i\ne j$. Then

$$\textstyle v_i=\sum_{k<n}(v_i\centerdot u_k)u_k=\sum_{k<n}u_k(i)u_k$$

for $i\in J$, so

$$\sum_{k<n}u_k(i)u_k(j)=\sum_{k,l<n}u_k(i)u_l(j)u_k\centerdot u_l=v_i\centerdot v_j=1 \text{ if } i=j,$$
$$=0 \text{ otherwise. } \mathbf{Q}$$

Now $\gamma_k=Tu_k\centerdot u_k\ge 0$, so $\sqrt{\gamma_k}$ is defined for each k, and we have a linear operator $S:\mathbb{R}^n\to\mathbb{R}^J$ defined by setting $Se_k=\sqrt{\gamma_k}u_k$ for each k, where $\langle e_k\rangle_{k<n}$ is the standard basis of $\mathbb{R}^n$, defined by saying that $e_k(l)=1$ if $k=l$, 0 otherwise.

Taking $\mu_G^{(n)}$ to be the standard Gaussian distribution on $\mathbb{R}^n$, $\mu=\mu_G^{(n)}S^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^J$, by 456Ba. For i, $j\in J$,

[20] Or, rather, its finite-dimensional special case, which is easier; you may know it under the slogan ‘symmetric matrices are diagonalisable’.

$$\begin{aligned}\int w(i)w(j)\mu(dw) &= \int (Sz)(i)(Sz)(j)\mu_G^{(n)}(dz)\\ &= \int \sum_{k<n}\sqrt{\gamma_k}z(k)u_k(i)\cdot\sum_{l<n}\sqrt{\gamma_l}z(l)u_l(j)\mu_G^{(n)}(dz)\\ &= \sum_{k<n}\sum_{l<n}\sqrt{\gamma_k\gamma_l}u_k(i)u_l(j)\int z(k)z(l)\mu_G^{(n)}(dz)\\ &= \sum_{k<n}\gamma_k u_k(i)u_k(j) = \sum_{k<n}Tu_k(i)u_k(j)\\ &= \sum_{k<n,l\in J}\sigma_{il}u_k(l)u_k(j) = \sum_{l\in J}\sigma_{il}\sum_{k<n}u_k(l)u_k(j) = \sigma_{ij}.\end{aligned}$$

So μ is the distribution we are looking for.

(iii)⇒(i) I seek to apply 454M. For each finite $J \subseteq I$, let μ_J be a centered Gaussian distribution on $\mathbb{R}^J$ with covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in J}$; by 456Bb, it is unique. If $K \subseteq I$ is finite and $J \subseteq K$, set $T_{KJ}z = z\restriction J$ for $z \in \mathbb{R}^K$; then $\mu_K T_{KJ}^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^J$, by 456Ba, and its covariance matrix is that of μ_J, so $\mu_J = \mu_K T_{KJ}^{-1}$. By 454M, we have a distribution μ on $\mathbb{R}^I$, the completion of a Baire probability measure, such that $\mu_J = \mu T_J^{-1}$ for every finite $J \subseteq I$, setting $T_J x = x\restriction J$ for $x \in \mathbb{R}^I$.

Applying this with $J = \{i, j\}$, we see that $\int x(i)x(j)\mu(dx) = \sigma_{ij}$ for all i, $j \in I$. To see that μ is a centered Gaussian distribution in the sense of 456Ab, take a continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$. Then there is a finite family $\langle\beta_i\rangle_{i\in J}$ in $\mathbb{R}$ such that $f(x) = \sum_{i\in J}\beta_i x(i)$ for each $x \in \mathbb{R}^I$. Setting $g(z) = \sum_{i\in J}\beta_i z(i)$ for $z \in \mathbb{R}^J$, we have $f = gT_J$, so that the image distribution μf^{-1} on $\mathbb{R}$ is just $\mu_J g^{-1}$, and (because μ_J is a centered Gaussian distribution) is either normal or concentrated at 0. As f is arbitrary, μ itself is a centered Gaussian distribution.

456D Gaussian processes I take a page to spell out the connexion between centered Gaussian distributions, and the processes considered in 454J-454K.

Definition A family $\langle X_i\rangle_{i\in I}$ of real-valued random variables on a probability space is a **centered Gaussian process** if its distribution (454J) is a centered Gaussian distribution.

456E Independence and correlation We have an important characterization of independence of families forming a Gaussian process. The essential idea is in (a) below. I give the more elaborate version (b) for the sake of an application in §477.

Proposition (a) Let $\langle X_i\rangle_{i\in I}$ be a centered Gaussian process. Then $\langle X_i\rangle_{i\in I}$ is independent iff $\mathbb{E}(X_i\times X_j) = 0$ for all distinct i, $j \in I$.

(b) Let $\langle X_i\rangle_{i\in I}$ be a centered Gaussian process on a complete probability space (Ω, Σ, μ), and $\mathcal{J}$ a disjoint family of subsets of I; for $J \in \mathcal{J}$ let Σ_J be the σ-algebra of subsets of Ω generated by $\{X_i^{-1}[F] : i \in J$, $F \subseteq \mathbb{R}$ is Borel$\}$. Suppose that $\mathbb{E}(X_i \times X_j) = 0$ whenever J, J' are distinct members of $\mathcal{J}$, $i \in J$ and $j \in J'$. Then $\langle\Sigma_J\rangle_{J\in\mathcal{J}}$ is independent.

proof (a)(i) If $\langle X_i\rangle_{i\in I}$ is independent, and i, $j \in I$ are distinct, then $\mathbb{E}(X_i \times X_j) = \mathbb{E}(X_i)\mathbb{E}(X_j) = 0$, by 272R[21].

(ii) If $\mathbb{E}(X_i \times X_j) = 0$ for all distinct i, $j \in I$, let μ be the distribution of $\langle X_i\rangle_{i\in I}$ and $\langle\sigma_{ij}\rangle_{i,j\in I}$ its covariance matrix. Then $\sigma_{ij} = 0$ whenever $i \neq j$. So if we take ν_i to be the normal distribution on $\mathbb{R}$ with expectation 0 and variance σ_{ii} (or the distribution concentrated at 0 if $\sigma_{ii} = 0$), the product $\nu = \prod_{i\in I}\nu_i$ will be a centered Gaussian distribution on $\mathbb{R}^I$ (456Be) also with covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in I}$, and is equal to μ, by 456Bb. Thus μ is a product measure and $\langle X_i\rangle_{i\in I}$ is independent (454L).

(b) Set $K = \bigcup\mathcal{J}$. For each $J \in \mathcal{J}$, let ν_J be the distribution of $\langle X_i\rangle_{i\in J}$, and let $\nu = \prod_{J\in\mathcal{J}}\nu_J$ be the product measure on $\prod_{J\in\mathcal{J}}\mathbb{R}^J$, which we can identify with $\mathbb{R}^K$. Then ν is a centered Gaussian distribution, and its covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in K}$ is such that

[21] Formerly 272Q.

$$\sigma_{ij} = \mathbb{E}(X_i \times X_j) \text{ if } i,\ j \text{ belong to the same member of } \mathcal{J},$$
$$= \mathbb{E}(X_i)\mathbb{E}(X_j) = 0 \text{ otherwise;}$$

that is, it is the covariance matrix of the process $\langle X_i\rangle_{i\in K}$. Let $f : \Omega \to \mathbb{R}^K$ be a function such that $f(\omega)(i) = X_i(\omega)$ whenever $i \in K$ and $\omega \in \operatorname{dom} X_i$; because μ is complete, f is $(\Sigma, \mathcal{B}\mathfrak{a}(\mathbb{R}^K))$-measurable. For $J \in \mathcal{J}$ and $\omega \in \Omega$ set $f_J(\omega) = f(\omega)\restriction J$.

Now suppose that $\mathcal{J}_0 \subseteq \mathcal{J}$ is non-empty and finite and that $E_J \in \Sigma_J$ for each $J \in \mathcal{J}_0$. Then for each $J \in \mathcal{J}_0$ there is a Baire set $F_J \subseteq \mathbb{R}^J$ such that $E_J \triangle f_J^{-1}[F_J]$ is μ-negligible, and $\mu E_J = \nu_J F_J$. Next, the distribution of $\langle X_i\rangle_{i\in K}$ is a centered Gaussian distribution on $\mathbb{R}^K$, and has covariance matrix $\langle \mathbb{E}(X_i \times X_j)\rangle_{i,j\in K} = \langle \sigma_{ij}\rangle_{i,j\in K}$, so it must be ν. But this means that, setting $F = \{x : x \in \mathbb{R}^K,\ x\restriction J \in F_J$ for every $J \in \mathcal{J}_0\} \in \mathcal{B}\mathfrak{a}(\mathbb{R}^K)$,

$$\mu(\bigcap_{J\in\mathcal{J}} E_J) = \mu(\bigcap_{J\in\mathcal{J}} f_J^{-1}[F_J]) = \mu f^{-1}[F]$$
$$= \nu F = \prod_{J\in\mathcal{J}} \nu_J F_J = \prod_{J\in\mathcal{J}} \mu E_J.$$

As $\langle E_J\rangle_{J\in\mathcal{J}_0}$ is arbitrary, $\langle \Sigma_J\rangle_{J\in\mathcal{J}}$ is independent.

456F Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of random variables on a probability space (Ω, Σ, μ). Then the following are equiveridical:

(i) the distribution of $\langle X_i\rangle_{i\in I}$, in the sense of 454K, is a centered Gaussian distribution;

(ii) whenever $i_0, \dots, i_n \in I$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ then $\sum_{r=0}^n \alpha_r X_{i_r}$ is either zero a.e. or a normal random variable with zero expectation;

(iii) whenever $i_0, \dots, i_n \in I$ then the joint distribution of $X_{i_0}, \dots, X_{i_n}$, in the sense of 271C, is a centered Gaussian distribution;

(iv) whenever $J \subseteq I$ is finite then there is an independent family $\langle Y_k\rangle_{k\in K}$ of standard normal random variables on Ω such that each X_i, for $i \in J$, is almost everywhere equal to a linear combination of the Y_k.

proof (ii)$\Leftrightarrow$(i) is immediate from the definition in 456Ab and Proposition 454O.

(i)$\Leftrightarrow$(iii) is also direct from 456Ab and the identification of the two concepts of 'distribution' (454K).

(iv)$\Rightarrow$(ii) is direct from 274A-274B.

(i)$\Rightarrow$(iv) For $i \in J$ set $u_i = X_i^\bullet$ in $L^2(\mu)$. By 4A4Jh again there is an orthonormal family $\langle v_k\rangle_{k\in K}$ in $L^2(\mu)$ such that each v_k is a linear combination of the u_i and each u_i is a linear combination of the v_k. Take Y_k such that $Y_k^\bullet = v_k$ for each k; then each X_i is equal almost everywhere to a linear combination of the Y_k, while each Y_k is equal almost everywhere to a linear combination of the X_i. As $\#(K)$ must be the dimension of the linear span of $\{u_i : i \in J\}$, K is finite. Any linear combination of the Y_k is equal almost everywhere to a linear combination of the X_k, so is either zero a.e. or a normal random variable with zero expectation. Because (ii)$\Rightarrow$(iii), $\langle Y_k\rangle_{k\in K}$ has a centered Gaussian distribution ν say. Each Y_k has variance $(v_k|v_k) = 1$, so is a standard normal random variable.

The covariance matrix of ν is given by

$$\int y(j)y(k)\nu(dy) = \mathbb{E}(Y_j \times Y_k) = (v_j|v_k) = 1 \text{ if } j = k,$$
$$= 0 \text{ otherwise.}$$

By 456E, $\langle Y_k\rangle_{k\in K}$ is independent, so we have found a suitable family.

456G Now I start work on material for the main theorem of this section.

Lemma Let I be a finite set and μ a centered Gaussian distribution on $\mathbb{R}^I$. Suppose that $\gamma \geq 0$ and $\alpha = \mu\{x : \sup_{i\in I} |x(i)| \geq \gamma\}$. Then $\mu\{x : \sup_{i\in I} |x(i)| \geq \frac{1}{2}\gamma\} \geq 2\alpha(1-\alpha)^3$.

proof (a) The case $\gamma = 0$, $\alpha = 1$ is trivial; suppose that $\gamma > 0$. We may suppose that I has a total order $\le$. Give $\mathbb{R}^{I\times 4} \cong (\mathbb{R}^I)^4$ the product λ of four copies of μ; then λ is a centered Gaussian distribution (456Be). Define $T : \mathbb{R}^{I\times 4} \to \mathbb{R}^I$ by setting $(Ty)(i) = \frac{1}{2}\sum_{r=0}^{3} y(i,r)$ for $i \in I$, $y \in \mathbb{R}^{I\times 4}$; then λT^{-1} is a centered Gaussian distribution on $\mathbb{R}^I$ (456Ba). Now λT^{-1} has the same covariance matrix as μ. **P** If $i, j \in I$ then

$$\int x(i)x(j)(\lambda T^{-1})(dx) = \int (Ty)(i)(Ty)(j)\lambda(dy) = \frac{1}{4}\sum_{r=0}^{3}\sum_{s=0}^{3}\int y(i,r)y(j,s)\lambda(dy)$$

$$= \frac{1}{4}\sum_{r=0}^{3}\int y(i,r)y(j,r)\lambda(dy) = \frac{1}{4}\sum_{r=0}^{3}\int x(i)x(j)\mu(dx)$$

(because the map $y \mapsto \langle y(i,r)\rangle_{i\in I} : \mathbb{R}^{I\times 4} \to \mathbb{R}^I$ is inverse-measure-preserving for each r)

$$= \int x(i)x(j)\mu(dx). \textbf{ Q}$$

So $\lambda T^{-1} = \mu$, by 456Bb.

(b) Define

$$E_{ir} = \{y : y \in \mathbb{R}^{I\times 4}, |y(i,r)| \ge \gamma\}$$

for $r < 4$ and $i \in I$, and

$$E_r = \bigcup_{i\in I} E_{ir}$$

for $r < 4$, so that

$$\lambda E_r = \lambda\{y : \sup_{i\in I}|y(i,r)| \ge \gamma\} = \mu\{x : \sup_{i\in I}|x(i)| \ge \gamma\} = \alpha.$$

Set $E'_r = E_r \setminus \bigcup_{s\ne r} E_s$, so that

$$\lambda E'_r = \lambda\{y : \sup_{i\in I}|y(i,r)| \ge \gamma, \sup_{i\in I}|y(i,s)| < \gamma \text{ for } s \ne r\}$$

$$= \lambda\{y : \sup_{i\in I}|y(i,r)| \ge \gamma\} \cdot \prod_{s\ne r}\lambda\{y : \sup_{i\in I}|y(i,s)| < \gamma\}$$

(because these are independent events)

$$= \alpha(1-\alpha)^3.$$

(c) Next, for $i \in I$ and $r < 4$, set $E'_{ir} = E_{ir} \setminus (\bigcup_{j<i} E_{jr} \cup \bigcup_{s\ne r} E_s)$, so that $E'_r = \bigcup_{i\in I} E'_{ir}$. Observe that $\langle E'_{ir}\rangle_{i\in I,r<4}$ is disjoint. Set

$$F_{ir} = \{y : y \in E'_{ir}, y(i,r)\textstyle\sum_{s\ne r} y(i,s) \ge 0\}.$$

Then $\nu F_{ir} \ge \frac{1}{2}\nu E'_{ir}$. **P** We can think of $\mathbb{R}^{I\times 4}$ as a product $\mathbb{R}^J \times \mathbb{R}^K$, where $J = I \times \{r\}$ and $K = I \times (4 \setminus \{r\})$. In this case, λ becomes identified with a product $\lambda_J \times \lambda_K$, where λ_J and λ_K are centered Gaussian distributions on $\mathbb{R}^J$ and $\mathbb{R}^K$ respectively, and E'_{ir} is of the form $V \times W$, where

$$V = \{v : v \in \mathbb{R}^J, |v(i,r)| \ge \gamma, |v(j,r)| < \gamma \text{ for } j < i\},$$

$$W = \{w : w \in \mathbb{R}^K, |w(j,s)| < \gamma \text{ for every } j \in I, s \ne r\}.$$

In the same representation, F_{ir} becomes $(V^+ \times W^+) \cup (V^- \times W^-)$, where

$$V^+ = \{v : v \in V, v(i,r) \ge \gamma\}, \quad V^- = \{v : v \in V, v(i,r) \le -\gamma\},$$

$$W^+ = \{w : w \in W, \textstyle\sum_{r\ne s} w(i,s) \ge 0\}, \quad W^- = \{w : w \in W, \textstyle\sum_{r\ne s} w(i,s) \le 0\}.$$

By 456Bf, $\lambda_J V^+ = \lambda_J V^-$ and $\lambda_K W^+ = \lambda_K W^-$; since $V^- = V \setminus V^+$, while $W^+ \cup W^- = W$, we have

$$\lambda_J V^+ = \lambda_J V^- = \frac{1}{2}\lambda_J V, \quad \lambda_K W^+ = \lambda_K W^- \ge \frac{1}{2}\lambda_K W.$$

But this means that

$$\begin{aligned}\lambda F_{ir} &= \lambda(V^+ \times W^+) + \lambda(V^- \times W^-) \\ &= \lambda_J V^+ \cdot \lambda_K W^+ + \lambda_J V^- \cdot \lambda_K W^- \ge \frac{1}{2}\lambda_J V \cdot \lambda_K W = \frac{1}{2}\lambda E'_{ir},\end{aligned}$$

as claimed. **Q**

(d) At this point, observe that if $y \in F_{ir}$ then $|\sum_{s=0}^{3} y(i,s)| \ge |y(i,r)| \ge \gamma$. So

$$\begin{aligned}\mu\{x : \sup_{i\in I}|x(i)| \ge \frac{1}{2}\gamma\}) &= \lambda T^{-1}[\{x : \sup_{i\in I}|x(i)| \ge \frac{1}{2}\gamma\})] \\ &= \lambda\{y : \sup_{i\in I}|\frac{1}{2}\sum_{r=0}^{3} y(i,r)| \ge \frac{1}{2}\gamma\}) \\ &= \lambda\{y : \sup_{i\in I}|\sum_{r=0}^{3} y(i,r)| \ge \gamma) \\ &\ge \lambda(\bigcup_{i\in I, r<4} F_{ir}) = \sum_{i\in I, r<4} \lambda F_{ir} \\ &\ge \frac{1}{2}\sum_{r<4}\sum_{i\in I}\lambda E'_{ir} \ge \frac{1}{2}\sum_{r<4}\lambda E'_r = 2\alpha(1-\alpha)^3,\end{aligned}$$

which is what we set out to prove.

456H The support of a Gaussian distribution: Proposition Let I be a set and μ a centered Gaussian distribution on $\mathbb{R}^I$. Write Z for the set of those $x \in \mathbb{R}^I$ such that $f(x) = 0$ whenever $f : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional and $f = 0$ a.e. Then Z is a self-supporting closed linear subspace of $\mathbb{R}^I$ with full outer measure. If I is countable Z is the support of μ.

proof (a) Being the intersection of a family of closed linear subspaces, of course Z is a closed linear subspace.

(b) Z has full outer measure. **P** Let $F \subseteq \mathbb{R}^I$ be a non-negligible zero set. Let $J \subseteq I$ be a countable set such that F is determined by coordinates in J. For $i \in I$ and $x \in \mathbb{R}^I$ set $\pi_i(x) = x(i)$; then each π_i is either normally distributed or zero almost everywhere, so is square-integrable; set $u_i = \pi_i^\bullet$ in $L^2 = L^2(\mu)$. Let $\langle v_k\rangle_{k\in K}$ be a countable orthonormal family in L^2 such that every v_k is a linear combination of the u_i, for $i \in J$, and every u_i, for $i \in J$, is a linear combination of the v_k (4A4Jh once more). Extend $\langle v_k\rangle_{k\in K}$ to a Hamel basis $\langle v_l\rangle_{l\in L}$ of L^2. For every $i \in I$, we can express u_i as $\sum_{l\in L}\alpha_{il}v_l$, where $\{l : \alpha_{il} \ne 0\}$ is finite; and the construction ensures that $\alpha_{il} = 0$ if $i \in J$ and $l \in L \setminus K$.

Consider the linear operator $T_0 : \mathbb{R}^K \to \mathbb{R}^J$ defined by setting $(T_0 z)(i) = \sum_{k\in K}\alpha_{ik}z(k)$ for $z \in \mathbb{R}^K$ and $i \in J$. If we give $\mathbb{R}^K$ the product measure $\mu_G^{(K)}$, then the image measure $\mu_G^{(K)}T_0^{-1}$ is a Gaussian distribution (456Ba), with covariance matrix

$$\begin{aligned}\sigma_{ii'} &= \int x(i)x(i')(\mu_G^{(K)}T_0^{-1})dx = \int (T_0 z)(i)(T_0 z)(i')\mu_G^{(K)}(dz) \\ &= \sum_{k,k'\in K}\alpha_{ik}\alpha_{i'k'}z(k)z(k')\mu_G^{(K)}(dz) = \sum_{k\in K}\alpha_{ik}\alpha_{i'k} \\ &= \sum_{k,k'\in K}\alpha_{ik}\alpha_{i'k}(v_k|v_{k'}) = (u_i|u_{i'}) = \int x(i)x(i')\mu(dx).\end{aligned}$$

But this means that $\mu_G^{(K)}T_0^{-1}$ has the same covariance matrix as $\mu\tilde{\pi}_J^{-1}$, where $\tilde{\pi}_J x = x{\restriction}J$ for $x \in \mathbb{R}^I$. Since this also is a centered Gaussian distribution, the two measures must be equal (456Bb). We know that $\tilde{\pi}_J[F]$ has non-zero measure, so there is a $z_0 \in \mathbb{R}^K$ such that $T_0 z_0 \in \tilde{\pi}_J[F]$. Extend z_0 arbitrarily to $z_1 \in \mathbb{R}^L$.

Set $x_1(i)=\sum_{l\in L}\alpha_{il}z_1(l)$ for $i\in I$. Then $\tilde\pi_Jx_1=T_0z_0\in\tilde\pi_J[F]$, so $x_1\in F$, because F is determined by coordinates in J. If a continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$ is zero a.e., it can be expressed in the form $f(x)=\sum_{i\in I}\beta_ix(i)$ where $\{i:\beta_i\ne 0\}$ is finite (4A4Be again). In this case,

$$0=f^\bullet=\sum_{i\in I}\beta_iu_i=\sum_{l\in L}\sum_{i\in I}\beta_i\alpha_{il}v_l$$

in L^2. Since $\langle v_l\rangle_{l\in L}$ is linearly independent, $\sum_{i\in I}\beta_i\alpha_{il}=0$ for every $l\in L$. But this means that

$$f(x_1)=\sum_{i\in I,l\in L}\beta_i\alpha_{il}z_1(l)=0.$$

As f is arbitrary, $x_1\in Z$ and $Z\cap F\ne\emptyset$. As F is arbitrary, and μ is inner regular with respect to the zero sets, Z has full outer measure. **Q**

(c) Z is self-supporting. **P** If $W\subseteq\mathbb{R}^I$ is an open set meeting Z, there is an open set V, depending on coordinates in a finite set $J\subseteq I$, such that $V\subseteq W$ and $V\cap Z\ne\emptyset$. Write $\tilde\pi_J(x)=x\restriction J$ for $x\in\mathbb{R}^I$, and ν_J for the image measure $\mu\tilde\pi_J^{-1}$ on $\mathbb{R}^J$; by 456Ba, this is a centered Gaussian distribution. By 456Bd, there is a continuous linear operator $T:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^J$ such that $\nu_J=\mu_G^{(\mathbb{N})}T^{-1}$. Since the support of μ_G is $\mathbb{R}$, the support of $\mu_G^{(\mathbb{N})}$ is $\mathbb{R}^{\mathbb{N}}$ (417E(b-iii), or otherwise), and the support of ν_J is $Z_1=\overline{T[\mathbb{R}^{\mathbb{N}}]}$ (411Ne).

Write Q for the set of linear functionals $g:\mathbb{R}^J\to\mathbb{R}$ (necessarily continuous, because J is finite) which are zero on Z_1. If $g\in Q$, then $gT=0$, so $g=0$ ν_J-a.e. and $g\tilde\pi_J=0$ μ-a.e. This means that $g\tilde\pi_J(x)=0$ for every $x\in Z$, that is, $g(y)=0$ for every $y\in\tilde\pi_J[Z]$. Because Z_1 is a linear subspace of $\mathbb{R}^J$, this is enough to show that $\tilde\pi_J[Z]\subseteq Z_1$.

Now recall that $V\cap Z\ne\emptyset$ so $Z_1\cap\tilde\pi_J[V]\ne\emptyset$, while $V=\tilde\pi_J^{-1}[\tilde\pi_J[V]]$. Since $\tilde\pi_J$ is an open map (4A2B(f-i)), $\tilde\pi_J[V]$ is open and

$$\mu^*(W\cap Z)=\mu W\ge\mu V=\nu_J\tilde\pi_J[V]>0,$$

because Z_1 is the support of ν_J. **Q**

(d) If I is countable, μ is a topological measure so measures Z, and Z is the support of μ.

456I Remarks (a) In the context of 456H, I will call Z the **support** of the centered Gaussian distribution μ, even though μ need not be a topological measure, so the definition 411Nb is not immediately applicable. In 456P we shall see that Z really is the support of a canonical extension of μ.

(b) It is worth making one elementary point at once. If I and J are sets, μ and ν are centered Gaussian distributions on $\mathbb{R}^I$ and $\mathbb{R}^J$ respectively with supports Z and Z', and $T:\mathbb{R}^I\to\mathbb{R}^J$ is an inverse-measure-preserving continuous linear operator, then $Tz\in Z'$ for every $z\in Z$. **P** If $g:\mathbb{R}^J\to\mathbb{R}$ is a continuous linear functional which is zero ν-a.e., then $gT:\mathbb{R}^I\to\mathbb{R}$ is a continuous linear functional which is zero μ-a.e., so $g(Tz)=(gT)(z)=0$. **Q**

456J Universal Gaussian distributions: Definition A centered Gaussian distribution on $\mathbb{R}^I$ is **universal** if its covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the inner product for a Hilbert space structure on I. (See 456Xe.)

456K Proposition Let I be any set, and μ a centered Gaussian distribution on I. Then there are a set J, a universal centered Gaussian distribution ν on $\mathbb{R}^J$, and a continuous inverse-measure-preserving linear operator $T:\mathbb{R}^J\to\mathbb{R}^I$.

proof (a) Set $J=L^2(\mu)$. Then for any finite $K\subseteq J$ there is a centered Gaussian distribution μ_K on $\mathbb{R}^K$ such that $\int x(u)x(v)\mu(dx)=(u|v)$ for all u, $v\in K$. **P** If $K=\emptyset$ or $K=\{0\}$ this is trivial, as we take μ to be the trivial distribution concentrated at 0. Otherwise, let $\langle w_i\rangle_{i<n}$ be an orthonormal basis for the linear subspace of J generated by K. For each $u\in K$, express it as $\sum_{i=0}^{n-1}\alpha_{ui}w_i$. Define $T:\mathbb{R}^n\to\mathbb{R}^K$ by setting $(Tz)(u)=\sum_{i=0}^{n-1}\alpha_{ui}z(i)$ for $z\in\mathbb{R}^n$ and $u\in K$. Set $\mu_K=\mu_G^{(n)}T^{-1}$. Then μ_K is a centered Gaussian distribution, by 456Ba, and its covariance matrix is given by

$$\begin{aligned}\int x(u)x(v)\mu_K(dx) = \int (Ty)(u)(Ty)(v)\mu_G^{(n)}(dy) &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\alpha_{ui}\alpha_{vj}\int y(i)y(j)\mu_G^{(n)}(dy)\\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\alpha_{ui}\alpha_{vj}(w_i|w_j) = (u|v)\end{aligned}$$

for all u, $v \in K$. So μ_K is the distribution we seek. **Q**

(b) If $K \subseteq J$ is finite and $L \subseteq K$, then $\mu_L = \mu_K\pi_{KL}^{-1}$, where $\pi_{KL}(x) = x\restriction L$ for $x \in \mathbb{R}^K$. **P** Since $\mu_K\pi_{KL}^{-1}$ is a centered Gaussian distribution, all we have to do is to check its covariance matrix. But if u, $v \in L$ then

$$\begin{aligned}\int y(u)y(v)(\mu_K\pi_{KL}^{-1})(dy) &= \int (\pi_{KL}x)(u)(\pi_{KL}x)(v)\mu_K(dx)\\ &= \int x(u)x(v)\mu_K(dx) = (u|v) = \int y(u)y(v)\mu_L(dy).\end{aligned}$$

By 456Bb, $\mu_L = \mu_K\pi_{KL}^{-1}$. **Q**

(c) By 454G, there is a Baire measure ν' on $\mathbb{R}^J$ such that $\nu'\pi_{JK}^{-1}[E] = \mu_K E$ for every finite $K \subseteq J$ and every Borel set $E \subseteq \mathbb{R}^K$. Take ν to be the completion of ν'. Then π_{JK} is inverse-measure-preserving for ν and μ_K, for every finite $K \subseteq J$. If $f : \mathbb{R}^J \to \mathbb{R}$ is a continuous linear functional, there are a finite $K \subseteq J$ and a linear functional $g : \mathbb{R}^K \to \mathbb{R}$ such that $f = \pi_{JK}g$, so that

$$\nu\{x : f(x) \le \alpha\} = \mu_K\{x : g(x) \le \alpha\}$$

for every α, and f and g have the same distribution; as g is either normal with zero expectation or zero a.e., so is f. As f is arbitrary, ν is a centered Gaussian distribution.

(d) ν is universal. **P** If u, $v \in J$ set $K = \{u, v\}$. Then

$$\begin{aligned}\int x(u)x(v)\nu(dx) &= \int (\pi_{JK}x)(u)(\pi_{JK}x)(v)\nu(dx)\\ &= \int y(u)y(v)\mu_K(dy) = (u|v).\end{aligned}$$

Thus the covariance matrix of ν is just the inner product of the standard Hilbert space structure of J. **Q**

(e) For $i \in I$, let $u_i \in J$ be the equivalence class of the square-integrable function $x \mapsto x(i) : \mathbb{R}^I \to \mathbb{R}$. Define $T : \mathbb{R}^J \to \mathbb{R}^I$ by setting $(Ty)(i) = y(u_i)$ for every $i \in I$ and $y \in \mathbb{R}^J$. Then there is a centered Gaussian distribution μ' on $\mathbb{R}^I$ such that T is inverse-measure-preserving for ν and μ'. Now the covariance matrix of μ' is defined by

$$\begin{aligned}\int x(i)x(j)\mu'(dx) = \int (Ty)(i)(Ty)(j)\nu(dy) &= \int y(u_i)y(u_j)\nu(dy)\\ &= (u_i|u_j) = \int x(i)x(j)\mu(dx)\end{aligned}$$

for all i, $j \in I$. So μ and μ' are equal and T is inverse-measure-preserving for ν and μ.

456L Lemma Let μ be a universal centered Gaussian distribution on $\mathbb{R}^I$; give I a corresponding Hilbert space structure such that $\int x(i)x(j)\mu(dx) = (i|j)$ for all i, $j \in I$. Let $F \in \operatorname{dom}\mu$ be a set determined by coordinates in J, where $J \subseteq I$ is a closed linear subspace for the Hilbert space structure of I. Let W be the union of all the open subsets of $\mathbb{R}^I$ which meet F in a negligible set, and W' the union of the open subsets of $\mathbb{R}^I$ which meet F in a negligible set and are determined by coordinates in J. If $F \subseteq W$ then $F \subseteq W'$.

proof (a) Let Z be the support of μ in the sense of 456H. We need to know that Z is just the set of all linear functionals from I to $\mathbb{R}$. **P** If $K \subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in K} \in \mathbb{R}^K$ and $f(x) = \sum_{i\in I}\alpha_i x(i)$ for $x \in \mathbb{R}^I$, then

$$\|f\|_2^2 = \sum_{i,j\in K} \alpha_i\alpha_j(i|j) = \|\sum_{i\in K} \alpha_i i\|^2.$$

So, for $x \in \mathbb{R}^I$,

$$\begin{aligned} x \in Z &\iff f(x) = 0 \text{ whenever } f \in (\mathbb{R}^I)^* \text{ and } \|f\|_2 = 0 \\ &\iff \sum_{i\in K} \alpha_i x(i) = 0 \text{ whenever } K \subseteq I \text{ is finite and } \sum_{i\in K} \alpha_i i = 0 \text{ in } I \\ &\iff x : I \to \mathbb{R} \text{ is linear. } \mathbf{Q} \end{aligned}$$

(b) Let $J^\perp$ be the orthogonal complement of J in I, so that $I = J \oplus J^\perp$ (4A4Jf). Give $\mathbb{R}^J$ and $\mathbb{R}^{J^\perp}$ the centered Gaussian distributions μ_J, $\mu_{J^\perp}$ induced by μ and the projections $x \mapsto x{\restriction}J$, $x \mapsto x{\restriction}J^\perp$. Then the product measure λ on $\mathbb{R}^J \times \mathbb{R}^{J^\perp}$ is also a centered Gaussian distribution (456Be). Define $T : \mathbb{R}^J \times \mathbb{R}^{J^\perp} \to \mathbb{R}^I$ by setting $T(u,v)(j+k) = u(j) + v(k)$ whenever $j \in J$, $k \in J^\perp$, $u \in \mathbb{R}^J$ and $v \in \mathbb{R}^{J^\perp}$. Then T is inverse-measure-preserving for λ and μ. **P** T is a continuous linear operator so we have a centered Gaussian distribution μ' on $\mathbb{R}^I$ such that T is inverse-measure-preserving for λ and μ'. If j, $j' \in J$ and k, $k' \in J^\perp$,

$$\begin{aligned} \int x(j+k)x(j'+k')\mu'(dx) &= \int T(u,v)(j+k)T(u,v)(j'+k')\lambda(d(u,v)) \\ &= \int (u(j)+v(k))(u(j')+v(k'))\lambda(d(u,v)) \\ &= \int u(j)u(j')\mu_J(du) + \int v(k)v(k')\mu_{J^\perp}(dv) \\ &= \int x(j)x(j')\mu(dx) + \int x(k)x(k')\mu(dx) \\ &= (j|j') + (k|k') = (j+k|j'+k') \\ &= \int x(j+k)x(j'+k')\mu(dx). \end{aligned}$$

Thus μ and μ' have the same covariance matrix and are equal, and T is inverse-measure-preserving for λ and μ. **Q**

(c) Take any $z \in W \cap Z$. Then there is an open set V, determined by coordinates in a finite set $K_0 \subseteq I$, such that $z \in V$ and $\mu(V \cap F) = 0$. Let $\epsilon > 0$ be such that $y \in V$ whenever $x \in \mathbb{R}^I$ and $|x(i) - z(i)| < 2\epsilon$ for every $i \in K_0$. Express each $k \in K_0$ as $k' + k''$ where $k' \in J$ and $k'' \in J^\perp$. Set

$$V' = \{x : x \in \mathbb{R}^I,\ |x(k') - z(k')| < \epsilon \text{ for every } k \in K_0\}.$$

Then V' is an open set, determined by coordinates in J, and contains z. Also $\mu(V' \cap F) = 0$. **P** Set $V'' = \{x : x \in \mathbb{R}^I,\ |x(k'') - z(k'')| < \epsilon$ for every $k \in K_0\}$. Then $V \supseteq V' \cap V'' \cap Z$. Since Z has full outer measure (456H),

$$\mu(V' \cap V'' \cap F) = \mu^*(V' \cap V'' \cap F \cap Z) \le \mu(V \cap F) = 0.$$

Now

$$\begin{aligned} 0 = \mu(V' \cap V'' \cap F) &= \lambda T^{-1}[V' \cap F \cap V''] \\ &= \lambda\{(x{\restriction}J, x{\restriction}J^\perp) : x \in V' \cap F \cap V''\} \end{aligned}$$

(because $V' \cap F \cap V''$ is determined by coordinates in $J \cup J^\perp$)

$$= \mu_J\{x{\restriction}J : x \in V' \cap F\} \cdot \mu_{J^\perp}\{x{\restriction}J^\perp : x \in V''\}$$

because $V' \cap F$ is determined by coordinates in J, while V'' is determined by coordinates in $J^\perp$. However, $z \in V''$, and $z{\restriction}J^\perp$ belongs to the support Z' of $\mu_{J^\perp}$, by 456Ib; since Z' is self-supporting, and $\{x{\restriction}J^\perp : x \in V''\}$ is open, $\mu_{J^\perp}\{x{\restriction}J^\perp : x \in V''\} > 0$. We conclude that

$$0 = \mu_J\{x{\restriction}J : x \in V' \cap F\} = \mu(V' \cap F). \ \mathbf{Q}$$

(d) This shows that $z \in W'$. As z is arbitrary, $W \cap Z \subseteq W'$.

? Suppose, if possible, that there is a point $z_0 \in F \setminus W'$. If i, $j \in J$ and $\alpha \in \mathbb{R}$, then

$$\{x : x(i+j) \neq x(i) + x(j)\}, \quad \{x : x(\alpha i) \neq \alpha x(i)\}$$

are negligible open sets determined by coordinates in J, so are included in W' and do not contain z_0. Thus $z_0 \restriction J : J \to \mathbb{R}$ is linear. Let $z : I \to \mathbb{R}$ be a linear functional extending $z_0 \restriction J$. Then $z \in Z$ and $z \restriction J = z_0 \restriction J$; as both F and W' are determined by coordinates in J, $z \in F \setminus W'$. But this means that $z \in Z \cap W \setminus W'$, which is impossible. **X**

So $F \subseteq W'$, as claimed.

456M Cluster sets: Lemma Let I be a countable set, $n \geq 1$ an integer and μ a centered Gaussian distribution on $\mathbb{R}^{I \times n}$. For $\epsilon > 0$ set

$$I_\epsilon = \{i : i \in I, \int |x(i,r)|^2 \mu(dx) \leq \epsilon^2 \text{ for every } r < n\};$$

suppose that no I_ϵ is empty.

(a) There is a closed set $F \subseteq \mathbb{R}^n$ such that

$$F = \bigcap_{\epsilon>0} \overline{\{\langle x(i,r) \rangle_{r<n} : i \in I_\epsilon\}}$$

for almost every $x \in \mathbb{R}^{I \times n}$.

(b) If $z \in F$ and $-1 \leq \alpha \leq 1$, then $\alpha z \in F$.

(c) If F is bounded, then there is some $\epsilon > 0$ such that $\sup_{i \in I_\epsilon, r<n} |x(i,r)| < \infty$ for almost every $x \in \mathbb{R}^{I \times n}$.

proof (a)(i) For $x \in \mathbb{R}^{I \times n}$ and $i \in I$ set $S_i(x) = \langle x(i,r) \rangle_{r<n} \in \mathbb{R}^n$. For $x \in \mathbb{R}^{I \times n}$ set

$$F_x = \bigcap_{\epsilon>0} \overline{\{S_i x : i \in I_\epsilon\}},$$

so that F_x is a closed subset of $\mathbb{R}^n$. For $A \subseteq \mathbb{R}^n$ set $E_A = \{x : x \in \mathbb{R}^{I \times n}, A \cap F_x \neq \emptyset\}$.

(ii) By 456Bd, there is a continuous linear operator $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{I \times n}$ such that $\mu = \mu_G^{(\mathbb{N})} T^{-1}$. Set $T_i = S_i T$ for $i \in I$; then $T_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ is a continuous linear operator. For $y \in \mathbb{R}^{\mathbb{N}}$, set

$$\tilde{F}_y = F_{T(y)} = \bigcap_{\epsilon>0} \overline{\{T_i(y) : i \in I_\epsilon\}}.$$

For $A \subseteq \mathbb{R}^n$ set

$$\tilde{E}_A = T^{-1}[E_A] = \{y : y \in \mathbb{R}^{\mathbb{N}}, A \cap \tilde{F}_y \neq \emptyset\}.$$

(iii) If $K \subseteq \mathbb{R}^n$ is compact, then $\tilde{E}_K$ is a Borel subset of $\mathbb{R}^{\mathbb{N}}$. **P** Let $\mathcal{V}$ be a countable base for the topology of $\mathbb{R}^n$, and for $k \geq 1$ let $\mathcal{V}_k$ be the set of members of $\mathcal{V}$ with diameter at most $1/k$ which meet K. Then $T_i^{-1}[V]$ is a Borel set for every $V \in \mathcal{V}$ and $i \in I$, so

$$E' = \bigcap_{k \geq 1} \bigcup_{V \in \mathcal{V}_k} \bigcup_{i \in I_{1/k}} T_i^{-1}[V]$$

is a Borel set.

If $y \in \tilde{E}_K$, take $k \geq 1$. There is a $z \in \tilde{F}_y \cap K$. Let $V \in \mathcal{V}$ be such that $z \in V$ and $\operatorname{diam} V \leq 1/k$; in this case $V \in \mathcal{V}_k$. Because $z \in \overline{\{T_i(y) : i \in I_{1/k}\}}$, there is an $i \in I_{1/k}$ such that $T_i(y) \in V$. As k is arbitrary, this shows that $y \in E'$; thus $\tilde{E}_K \subseteq E'$.

If $y \notin \tilde{E}_K$, then K is a compact set disjoint from the closed set $\tilde{F}_y$. There is therefore some $\epsilon > 0$ such that $K \cap \overline{\{T_i(y) : i \in I_\epsilon\}} = \emptyset$ (since these form a downwards-directed family of compact sets with empty intersection). Next, there is a $\delta > 0$ such that $B(z,\delta) \cap \overline{\{T_i(y) : i \in I_\epsilon\}} = \emptyset$ for every $z \in K$ (2A2Ed). Let $k \geq 1$ be such that $1/k \leq \min(\epsilon, \delta)$. If $V \in \mathcal{V}_k$ and $i \in I_{1/k}$, there is some $z \in K \cap V$ so $V \subseteq B(z,\delta)$ and $T_i(y) \notin V$. This shows that $y \notin E'$. As y is arbitrary, $E' \subseteq \tilde{E}_K$.

So $\tilde{E}_K = E'$ is a Borel set. **Q**

It follows at once that $\tilde{E}_H$ is a Borel set for every K_σ set H, in particular, for any open or closed set H.

(iv) We need a simple estimate on the coefficients of the linear operators T_i. Let α_{irj} be such that $T_i(y) = \langle \sum_{j=0}^{\infty} \alpha_{irj} y(j) \rangle_{r<n}$ for $i \in I$, $r < n$ and $y \in \mathbb{R}^{\mathbb{N}}$. (Of course $\{j : \alpha_{irj} \neq 0\}$ is finite for each i and r.) Then

$$\int x(i,r)^2\mu(dx) = \int T_i(y)(r)^2\mu_G^{(\mathbb{N})}(dy) = \sum_{j=0}^{\infty}\alpha_{irj}^2,$$

so $|\alpha_{irj}| \le \epsilon$ whenever $i \in I_\epsilon$, $r < n$ and $j \in \mathbb{N}$.

(v) Now suppose that $H \subseteq \mathbb{R}^n$ is open, that $K \subseteq H$ is compact, and that $\mu_G^{(\mathbb{N})}\tilde{E}_K > 0$. Then $\mu_G^{(\mathbb{N})}\tilde{E}_H = 1$. **P** Let $\epsilon > 0$. Let $\langle\epsilon_j\rangle_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{j=0}^{\infty}\epsilon_j \le \frac{1}{2}\min(\epsilon, \mu_G^{(\mathbb{N})}\tilde{E}_K)$, and for each $j \in \mathbb{N}$ let $\gamma_j \ge 0$ be such that $\mu_G[-\gamma_j, \gamma_j] \ge 1 - \epsilon_j$. Then

$$\tilde{E} = \{y : y \in \tilde{E}_K,\ |y(j)| \le \gamma_j \text{ for every } j \in \mathbb{N}\}$$

has measure at least $\mu_G^{(\mathbb{N})}\tilde{E}_K - \sum_{j=0}^{\infty}\epsilon_j > 0$. By 254Sb, there are an $m \in \mathbb{N}$ and a set $\tilde{E}'$, of measure at least $1 - \frac{1}{2}\epsilon$, such that for every $y' \in \tilde{E}'$ there is a $y \in \tilde{E}$ such that $y(j) = y'(j)$ whenever $j > m$. Set $\tilde{E}'' = \{y : y \in \tilde{E}',\ |y(j)| \le \gamma_j \text{ for every } j \in \mathbb{N}\}$, so that $\mu_G^{(\mathbb{N})}\tilde{E}'' \ge 1 - \epsilon$.

Let $\delta > 0$ be such that $z' \in H$ whenever $z \in K$ and $\|z - z'\| \le 2\delta$. Let $\eta > 0$ be such that $2\eta\sqrt{n}\sum_{j=0}^{m}\gamma_j \le \delta$. If $y' \in \tilde{E}''$ and $i \in I_\eta$, there is a $y \in \tilde{E}$ such that $y(j) = y'(j)$ for $j > m$. Also $|y(j) - y'(j)| \le 2\gamma_j$ for $j \le m$, so

$$|T_i(y)(r) - T_i(y')(r)| \le \sum_{j=0}^{\infty}|\alpha_{irj}||y(j) - y'(j)| \le \sum_{j=0}^{m}2\eta\gamma_j \le \frac{\delta}{\sqrt{n}}$$

for every $r < n$, and $\|T_i(y) - T_i(y')\| \le \delta$.

Now $\tilde{F}_y \cap K \ne \emptyset$; take $z \in \tilde{F}_y \cap K$. For every $\zeta > 0$, there is an $i \in I_{\min(\eta,\zeta)}$ such that $\|z - T_i(y)\| \le \delta$, so that $\|z - T_i(y')\| \le 2\delta$. This means that $B(z, 2\delta) \cap \{T_i(y') : i \in I_\zeta\}$ is not empty. As $B(z, 2\delta)$ is compact, it must meet $\tilde{F}_{y'}$. But this means that $H \cap \tilde{F}_{y'} \ne \emptyset$, by the choice of δ. As y' is arbitrary, $\tilde{E}'' \subseteq \tilde{E}_H$, while $\mu_G^{(\mathbb{N})}\tilde{E}'' \ge 1 - \epsilon$.

This works for every $\epsilon > 0$. So $\tilde{E}_H$ is conegligible, as claimed. **Q**

(vi) If $H \subseteq \mathbb{R}^n$ is open and $\mu_G^{(\mathbb{N})}\tilde{E}_H > 0$, then (because H is σ-compact) there is a compact set $K \subseteq H$ such that $\mu_G^{(\mathbb{N})}\tilde{E}_K > 0$, and (e) tells us that $\mu_G^{(\mathbb{N})}\tilde{E}_H = 1$.

Set

$$\mathcal{V}_0 = \{V : V \in \mathcal{V},\ \mu_G^{(\mathbb{N})}\tilde{E}_V = 0\} = \{V : V \in \mathcal{V},\ \mu_G^{(\mathbb{N})}\tilde{E}_V < 1\}.$$

Then we see that

$$\mathcal{V}_0 = \{V : V \in \mathcal{V},\ \tilde{F}_y \cap V = \emptyset\}$$

for almost every $y \in \mathbb{R}^{\mathbb{N}}$, that is,

$$\mathcal{V}_0 = \{V : V \in \mathcal{V},\ F_x \cap V = \emptyset\}$$

for almost every $x \in \mathbb{R}^{I\times n}$. But as every F_x is closed, we have $F_x = \mathbb{R}^n \setminus \bigcup\mathcal{V}_0$ for almost every x. So we can set $F = \mathbb{R}^n \setminus \bigcup\mathcal{V}_0$.

(b)(i) Give $\mathbb{R}^{I\times 2n} \cong (\mathbb{R}^{I\times n})^2$ the measure λ corresponding to the product measure $\mu\times\mu$; by 456Be, this is a centered Gaussian distribution. For $(x_1, x_2) \in (\mathbb{R}^{I\times n})^2$, set

$$F'_{x_1x_2} = \bigcap_{\epsilon>0}\overline{\{(S_ix_1, S_ix_2) : i \in I_\epsilon\}}.$$

By (a), we have a closed set $F' \subseteq \mathbb{R}^{2n}$ such that $F' = F'_{x_1x_2}$ for almost all x_1, x_2. (Of course $I_\epsilon = \{i : \int|x_j(i,r)|^2\lambda(dx) \le \epsilon^2$ for every $j \in \{1,2\}$, $r < n\}$ whenever $\epsilon > 0$.)

(ii) Now $(z, 0) \in F'$. **P** Take $x_1 \in \mathbb{R}^{I\times n}$ such that $F = F_{x_1}$ and $E = \{x_2 : F' = F'_{x_1x_2}\}$ is conegligible. (Almost every point of $\mathbb{R}^{I\times n}$ has these properties.) For $k \in \mathbb{N}$, $z \in \overline{\{S_ix_1 : i \in I_{2^{-k}}\}}$; let $i_k \in I_{2^{-k}}$ be such that $\sum_{r<n}|z(r) - x_1(i_k, r)| \le 2^{-k}$. Next,

$$\sum_{k=0}^{\infty}\sum_{r=0}^{n-1}\int|x(i_k,r)|^2\mu(dx) \le \sum_{k=0}^{\infty}2^{-2k}n < \infty,$$

so $\sum_{k=0}^{\infty}\sum_{r=0}^{n-1}|x(i_k,r)|^2$ is finite for almost every $x \in \mathbb{R}^I$, and there must be an $x_2 \in E$ such that $\sum_{k=0}^{\infty}\sum_{r=0}^{n-1}|x_2(i_k,r)|^2$ is finite. But in this case $\lim_{k\to\infty}x_2(i_k,r) = 0$ for every r, while $\lim_{k\to\infty}x_1(i_k,r) = z(r)$. Accordingly $(z, 0) \in F'_{x_1x_2} = F'$. **Q**

(iii) Set $\beta=\sqrt{1-\alpha^2}$ and define $\tilde{T}:(\mathbb{R}^{I\times n})^2\to\mathbb{R}^{I\times n}$ by setting $\tilde{T}(x_1,x_2)=\alpha x_1+\beta x_2$ for x_1, $x_2\in\mathbb{R}^{I\times n}$. Then $\tilde{T}$ is a continuous linear operator, so the image measure $\lambda\tilde{T}^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^{I\times n}$ (456Ba). Moreover, it has the same covariance matrix as μ. **P** If i, $j\in I$ then

$$\begin{aligned}\int x(i)x(j)(\lambda\tilde{T}^{-1})(dx)&=\int\tilde{T}(x_1,x_2)(i)\tilde{T}(x_1,x_2)(j)\lambda(d(x_1,x_2))\\&=\int(\alpha x_1(i)+\beta x_2(i))(\alpha x_1(j)+\beta x_2(j))\lambda(d(x_1,x_2))\\&=\alpha^2\int x_1(i)x_1(j)\lambda(d(x_1,x_2))+\beta^2\int x_2(i)x_2(j)\lambda(d(x_1,x_2))\\&=(\alpha^2+\beta^2)\int x(i)x(j)\mu(dx)=\int x(i)x(j)\mu(dx).\ \mathbf{Q}\end{aligned}$$

So $\lambda\tilde{T}^{-1}=\mu$ (456Bb).

(iv) If x_1, $x_2\in\mathbb{R}^I$ are such that $(z,0)\in F'_{x_1x_2}$, then $\alpha z\in F_{\tilde{T}(x_1,x_2)}$. **P** For every $\epsilon>0$ there is an $i\in I_\epsilon$ such that $|z(r)-x_1(i,r)|\le\epsilon$ and $|x_2(i,r)|\le\epsilon$ for every $r<n$. But now $|\alpha z(r)-\tilde{T}(x_1,x_2)(r)|\le 2\epsilon$ for every $r<n$. **Q** So

$$\tilde{T}^{-1}[\{x:\alpha z\in F_x\}]=\{(x_1,x_2):\alpha z\in F_{\tilde{T}(x_1,x_2)}\}\supseteq\{(x_1,x_2):(z,0)\in F'_{x_1x_2}\}$$

is λ-conegligible, and $\alpha z\in F_x$ for μ-almost every x, that is, $\alpha z\in F$, as claimed.

(c) Suppose now that F is bounded.

(i) For $L\subseteq I$, $\alpha\ge 0$ set

$$Q(L,\alpha)=\bigcup_{i\in L,r<n}\{x:|x(i,r)|\ge\alpha\}.$$

By 456G, applied to the image of μ under the map $x\mapsto x{\restriction}L\times n:\mathbb{R}^{I\times n}\to\mathbb{R}^{L\times n}$,

$$\mu Q(L,\tfrac{1}{2}\alpha)\ge 2\mu Q(L,\alpha)(1-\mu Q(L,\alpha))^3$$

for every finite $L\subseteq I$ and every $\alpha\ge 0$.

Let $\beta>0$ be such that $\delta=2\beta(1-(n+1)\beta)^3-\beta>0$, and let $\alpha_0>0$ be such that $\alpha_0^2\beta\ge 1$ and $\|z\|<\frac{1}{2}\alpha_0$ for every $z\in F$, so that $\mu\{x:|x(i,r)|\ge\alpha_0\}\le\beta$ whenever $i\in I_1$ and $r<n$, and $\mu Q(\{i\},\alpha_0)\le n\beta$ for every $i\in I_1$. Set $K=\{z:z\in\mathbb{R}^n,\ \frac{1}{2}\alpha_0\le\max_{r<n}|z(r)|\le\alpha_0\}$, so that K is a compact set disjoint from F. For almost every x,

$$\emptyset=K\cap F=K\cap F_x=\bigcap_{k\ge 1}K\cap\overline{\{S_ix:i\in I_{1/k}\}},$$

so there is a $k\ge 1$ such that $S_ix\notin K$ for every $i\in I_{1/k}$. Since the sets

$$\{x:S_i(x)\in K\text{ for some }i\in I_{1/k}\}$$

form a non-increasing sequence of measurable sets with negligible intersection, there is a $k\ge 1$ such that

$$\mu\{x:S_i(x)\in K\text{ for some }i\in I_{1/k}\}<\delta.$$

(ii) ? Suppose, if possible, that

$$\mu Q(I_{1/k},\alpha_0)>\beta.$$

Let $L\subseteq I_{1/k}$ be a finite set of minimal size such that $\gamma=\mu Q(L,\alpha_0)\ge\beta$. Since $\mu Q(\{i\},\alpha_0)\le n\beta$ for any $i\in L$, and L is minimal, we must have

$$\beta\le\gamma\le(n+1)\beta.$$

Now this means that

$$\mu Q(L,\frac{1}{2}\alpha_0)\ge 2\gamma(1-\gamma)^3$$

(see (i) above)

$$\geq 2\gamma(1-(n+1)\beta)^3 = \gamma(1+\frac{\delta}{\beta}) \geq \gamma+\delta,$$

so that $\mu(Q(L,\frac{1}{2}\alpha_0)\setminus Q(L,\alpha_0)) \geq \delta$. But if $x \in Q(L,\frac{1}{2}\alpha_0)\setminus Q(L,\alpha_0)$ there is some $i \in L$ such that $\max_{r<n}|x(i,r)| \geq \frac{1}{2}\alpha_0$ while $\max_{r<n}|x(i,r)| < \alpha_0$, in which case $S_i(x) \in K$. So we get

$$\delta \leq \mu(Q(L,\frac{1}{2}\alpha_0)\setminus Q(L,\alpha_0)) \leq \mu\{x : S_i(x) \in K \text{ for some } i \in L\}$$
$$\leq \mu\{x : S_i(x) \in K \text{ for some } i \in I_{1/k}\} < \delta$$

which is absurd. **X**

(iii) Thus $\mu Q(I_{1/k},\alpha_0) \leq \beta$. For $\alpha \geq 0$, set $f(\alpha) = \mu Q(I_{1/k},\alpha)$; then f is non-increasing. Also $f(\frac{1}{2}\alpha) \geq 2f(\alpha)(1-f(\alpha))^3$ for every α. **P?** Otherwise, because

$$f(\alpha) = \sup\{\mu Q(L,\alpha) : L \subseteq I_{1/k} \text{ is finite}\},$$

there is a finite $L \subseteq I_{1/k}$ such that $f(\frac{1}{2}\alpha) < 2\gamma(1-\gamma)^3$, where $\gamma = \mu Q(L,\alpha)$. But in this case $\mu Q(L,\frac{1}{2}\alpha) \leq f(\frac{1}{2}\alpha) < 2\gamma(1-\gamma)^3$, which is impossible, as remarked in (i). **XQ**

Set $\zeta = \lim_{\alpha\to\infty} f(\alpha)$. Then

$$\zeta = \lim_{\alpha\to\infty} f(\tfrac{1}{2}\alpha) \geq 2\zeta(1-\zeta)^3.$$

But we also know, from (ii), that $\zeta \leq f(\alpha_0) \leq \beta$. So $(1-\zeta)^3 \geq (1-\beta)^3 > \frac{1}{2}$ and ζ must be 0.

What this means is that if we set $\epsilon = \frac{1}{k}$ then

$$\lim_{\alpha\to\infty} \mu\{x : \sup_{i\in I_\epsilon, r<n} |x(i,r)| > \alpha\} = 0,$$

that is, $\sup_{i\in I_\epsilon, r<n}|x(i,r)|$ is finite for almost every $x \in \mathbb{R}^{I\times n}$, as claimed.

456N Lemma Let J be a set and μ a centered Gaussian distribution on $\mathbb{R}^J$. Let M be the linear subspace of $L^2(\mu)$ generated by $\{\pi_j^\bullet : j \in J\}$, where $\pi_j(x) = x(j)$ for $x \in \mathbb{R}^J$ and $j \in J$. If M is separable (for the norm topology) then μ is τ-additive.

proof Suppose, if possible, otherwise.

(a) There is an upwards-directed family $\mathcal{G}$ of open Baire sets in $\mathbb{R}^J$ such that $W_0 = \bigcup\mathcal{G}$ is a Baire set and $\mu W_0 > \sup_{G\in\mathcal{G}}\mu G$. Let $\mathcal{G}_0 \subseteq \mathcal{G}$ be a countable upwards-directed set such that $\sup_{G\in\mathcal{G}_0}\mu G = \sup_{G\in\mathcal{G}}\mu G$, and set $W_1 = W_0 \setminus \bigcup\mathcal{G}_0$; then $\mu W_1 > 0$ and $\mu(W_1 \cap G) = 0$ for every $G \in \mathcal{G}$. Let W be a non-negligible zero set included in W_1.

For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be a countable base for the topology of $\mathbb{R}^n$ consisting of open balls. Let $\mathcal{G}_n^*$ be the family of open sets of $\mathbb{R}^J$ of the form $T^{-1}[V]$, where $T : \mathbb{R}^J \to \mathbb{R}^n$ is a continuous linear operator, $V \subseteq \mathbb{R}^n$ is open and $\mu(W \cap T^{-1}[V]) = 0$. Of course $\mathcal{G}_0^* = \emptyset$.

(b) For $n \geq 1$ and $V \in \mathcal{V}_n$, let $\mathcal{T}_{nV}$ be the family of continuous linear operators $T : \mathbb{R}^J \to \mathbb{R}^n$ such that $W \cap T^{-1}[V]$ is negligible, but not included in $\bigcup\mathcal{G}_{n-1}^*$. Index $\mathcal{T}_{nV}$ as $\langle T_i\rangle_{i\in I(n,V)}$; it will be convenient to do this in such a way that all the $I(n,V)$ are disjoint. Define f_{ir}, for $i \in I(n,V)$ and $r < n$, by saying that $T_i(x) = \langle f_{ir}(x)\rangle_{r<n}$ for $x \in \mathbb{R}^J$. Define $\phi_n : \bigcup_{V\in\mathcal{V}_n} I(n,V) \to M^n$ by setting $\phi_n(i) = \langle f_{ir}^\bullet\rangle_{r<n}$ for each $i \in \bigcup_{V\in\mathcal{V}_n} I(n,V)$. Because M is separable (in its norm topology), M^n is separable in its product topology (4A2P(a-v)). Fix a countable set $I'(n,V) \subseteq I(n,V)$ such that $\{\phi_n(i) : i \in I'(n,V)\}$ is dense in $\{\phi_n(i) : i \in I(n,V)\}$. Set $\rho_n(\langle u_r\rangle_{r<n}, \langle v_r\rangle_{r<n}) = \max_{r<n}\|u_r - v_r\|_2$ for $\langle u_r\rangle_{r<n}, \langle v_r\rangle_{r<n} \in M^n$, so that ρ_n is a metric defining the product topology of M^n.

(c) If $j \in I(n,V)$, then there is a $\delta > 0$ such that

$$\{T_i(x) : i \in I'(n,V),\ \rho_n(\phi_n(i),\phi_n(j)) \leq \delta\}$$

is bounded for almost every $x \in \mathbb{R}^J$. **P** Define $S : \mathbb{R}^J \to \mathbb{R}^{I'(n,V)\times n}$ by setting $(Sx)(i,r) = T_i(x,r) - T_j(x,r)$ for $x \in \mathbb{R}^J$, $i \in I'(n,V)$ and $r < n$. By 456Ba, the image measure $\lambda = \nu S^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^{I'(n,V)\times n}$. For $\delta > 0$, set

$$\begin{aligned} I'_\delta &= \{i : i \in I'(n,V),\ \int |y(i,r)|^2 \lambda(dy) \le \delta^2 \text{ for every } r < n\} \\ &= \{i : i \in I'(n,V),\ \int |Sx(i,r)|^2 \nu(dx) \le \delta^2 \text{ for every } r < n\} \\ &= \{i : i \in I'(n,V),\ \int |T_i(x)(r) - T_j(x)(r)|^2 \nu(dx) \le \delta^2 \text{ for every } r < n\} \\ &= \{i : i \in I'(n,V),\ \rho_n(\phi_n(i), \phi_n(j)) \le \delta\}. \end{aligned}$$

By 456Ma, there is a closed set $F \subseteq \mathbb{R}^n$ such that $F = \bigcap_{\delta>0} \overline{\{y(i,r) : i \in I'_\delta\}}$ for λ-almost every $y \in \mathbb{R}^{I'(n,V)\times n}$, so that $F = \bigcap_{\delta>0} \overline{\{Sx(i,r) : i \in I'_\delta\}}$ for ν-almost every $x \in \mathbb{R}^J$.

By 456Mb, $\alpha z \in F$ whenever $z \in F$ and $|\alpha| \le 1$. **?** If F is not bounded, then it must include a line L through 0. (The sets $\{\frac{1}{n}z : z \in F,\ \|z\| = n\}$, for $n \ge 1$, form a non-increasing sequence of non-empty compact sets, so there is a point z_0 belonging to them all; take L to be the set of multiples of z_0.) Let $D \subseteq L$ be a countable dense set. For $z \in D$ and $k \in \mathbb{N}$ we know that

for every $i \in I(n,V)$, $T_i(x) \notin V$ for almost every $x \in W$,

for almost every $x \in \mathbb{R}^J$ there is an $i \in I'(n,V)$ such that $\|T_i(x) - T_j(x) - z\| \le 2^{-k}$

and therefore

for almost every $x \in W$, $T_i(x) \notin V$ for every $i \in I'(n,V)$, but there is an $i \in I'(n,V)$ such that $\|T_i(x) - T_j(x) - z\| \le 2^{-k}$

so that

for almost every $x \in W$, the distance from $T_j(x) + z$ to the closed set $\mathbb{R}^n \setminus V$ is at most 2^{-k}.

This is true for every $k \in \mathbb{N}$, so we get

$T_j(x) + z \notin V$ for almost every $x \in W$.

And *this* is true for every $z \in D$, so we get

for almost every $x \in W$, $T_j(x) + z \notin V$ for every $z \in D$, so $T_j(x) \notin V + L$.

Let $S_0 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a linear operator with kernel L, and set $V' = S_0[V]$. Then $V' \subseteq \mathbb{R}^{n-1}$ is open, and $W \cap (S_0T_j)^{-1}[V'] = W \cap T_j^{-1}[V + L]$ is negligible. But this means that $T_j^{-1}[V] \subseteq (S_0T_j)^{-1}[V'] \in \mathcal{G}^*_{n-1}$ and $T_j^{-1}[V]$ is included in $\bigcup \mathcal{G}^*_{n-1}$; which contradicts the definition of $\mathcal{T}_{nV}$. **X**

So F is bounded. By 456Mc, there is some $\delta > 0$ such that $\sup_{i\in I'_\delta, r<n} |y(i,r)| < \infty$ for λ-almost every $y \in \mathbb{R}^{I'(n,V)\times n}$, in which case $\sup_{i\in I'_\delta, r<n} |Sx(i,r)| < \infty$ for ν-almost every $x \in \mathbb{R}^J$, that is, $\{T_i(x) - T_j(x) : i \in I'_\delta\}$ is bounded for ν-almost every x. Of course this means that $\{T_i(x) : i \in I(n,V),\ \rho_n(\phi_n(i), \phi_n(j)) \le \delta\}$ is bounded for almost every $x \in \mathbb{R}^J$. **Q**

(d) Accordingly $\phi_n[I(n,V)]$ is covered by the family $\mathcal{U}_{nV}$ of open sets $U \subseteq M^n$ such that $\{T_i(x) : i \in I'(n,V),\ \phi_n(i) \in U\}$ is bounded for almost every x. Because M^n is separable and metrizable, it is hereditarily Lindelöf (4A2P(a-iii)), so there is a sequence $\langle U_{nVk}\rangle_{k\in\mathbb{N}}$ in $\mathcal{U}_{nV}$ covering $\phi_n[I(n,V)]$. For each k, set $I_{nVk} = I'(n,V) \cap \phi_n^{-1}[U_{nVk}]$. Then $\{T_i(x) : i \in I_{nVk}\}$ is bounded for almost every x. Because $\phi_n[I'(n,V)]$ is dense in $\phi_n[I(n,V)]$, $\phi_n[I_{nVk}] = \phi_n[I'(n,V)] \cap U_{nVk}$ is dense in $\phi_n[I(n,V)] \cap U_{nVk}$. So for every $i \in I(n,V)$ there is a $k \in \mathbb{N}$ such that $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$.

(e) Recall that $W \subseteq \mathbb{R}^J$ is a non-negligible zero set included in

$$\bigcup_{n\ge1} \bigcup \mathcal{G}^*_n = \bigcup_{n\ge1} \bigcup_{V\in\mathcal{V}_n} \bigcup_{i\in I(n,V)} T_i^{-1}[V].$$

Let $J_0 \subseteq J$ be a countable set such that W is determined by coordinates in J_0.

Let $\langle \epsilon_j \rangle_{j\in J_0}$ and $\langle \epsilon'_{nVk} \rangle_{n\ge1, V\in\mathcal{V}_n, k\in\mathbb{N}}$ be families of strictly positive real numbers such that $\sum_{n=1}^\infty \sum_{V\in\mathcal{V}_n} \sum_{k=0}^\infty \epsilon'_{nVk}$ and $\sum_{j\in J_0} \epsilon_j$ are both at most $\frac{1}{3}\mu W$. Let $\langle \gamma_j \rangle_{j\in J_0}$ and $\langle \gamma'_{nVk} \rangle_{n\ge1, V\in\mathcal{V}_n, k\in\mathbb{N}}$ be such that

$$\mu\{x : x \in \mathbb{R}^J,\ |x(j)| \ge \gamma_j\} \le \epsilon_j \text{ for every } j \in J_0,$$

$$\mu\{x : x \in \mathbb{R}^J,\ \sup_{i\in I_{nVk}} \|T_i(x)\| \ge \gamma'_{nVk}\} \le \epsilon'_{nVk} \text{ for every } n \ge 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N}.$$

Set $W' = \{x : x \in W,\ |x(j)| \le \gamma_j \text{ for every } j \in J_0\}$; then $\mu W' \ge \frac{2}{3}\mu W$ and W' is of the form $C \times \mathbb{R}^{J\setminus J_0}$, where $C \subseteq \mathbb{R}^{J_0}$ is compact. Set

$$W'' = \{x : x \in W',\ \|T_i(x)\| \le \gamma'_{nVk} \text{ whenever } n \ge 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N} \text{ and } i \in I_{nVk}\};$$

then $\mu W'' \ge \frac{1}{3}\mu W$.

(f) Set $I = \bigcup_{n\ge1, V\in\mathcal{V}_n} I(n,V) \times n$. If $K \subseteq I$ is finite, then

$$W'_K = \{x : x \in W',\ \|f_{ir}(x)\| \le \gamma'_{nVk} \text{ whenever } n \ge 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N},\\ (i,r) \in K \text{ and } \phi_n(i) \in \overline{\phi_n[I_{nVk}]}\}$$

has measure at least $\frac{1}{3}\mu W$. **P** For each quintuple (i,r,n,V,k) with $n \ge 1$, $V \in \mathcal{V}_n$, $k \in \mathbb{N}$, $(i,r) \in K$ and $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$, there is a sequence $\langle i_m\rangle_{m\in\mathbb{N}}$ in I_{nVk} such that $\rho_n(\phi_n(i), \phi_n(i_m)) \le 2^{-m}$ for every m; so that $\|f_{ir} - f_{i_m r}\|_2 \le 2^{-m}$ for every m. But this means that $f_{ir}(x) = \lim_{m\to\infty} f_{i_m r}(x)$ for almost every $x \in \mathbb{R}^J$. Accordingly $|f_{ir}(x)| \le \gamma'_{nVk}$ for almost every $x \in W''$. Since there are only countably many such quintuples (i,r,n,V,k), we see that $W'' \setminus W'_K$ is negligible, so $\mu W'_K \ge \mu W'' \ge \frac{1}{3}\mu W$. **Q**

(g) For $x \in \mathbb{R}^J$, define $Tx \in \mathbb{R}^I$ by setting $(Tx)(i,r) = f_{ir}(x)$ for $i \in I(n,V)$ and $r < n$. Then $T : \mathbb{R}^J \to \mathbb{R}^I$ is a continuous linear operator. By 4A4H, $T[W']$ is closed.

For finite $K \subseteq I$, let $\mathcal{H}_K$ be the family of open subsets H of $\mathbb{R}^K$ such that $\mu\{x : x \in W, Tx{\restriction}K \in H\} = 0$. Then $\mathcal{H}_K$ is closed under countable unions so has a largest member H_K. Now there is a $K \in [I]^{<\omega}$ such that $Tx{\restriction}K \in H_K$ for every $x \in W'_K$. **P?** Otherwise, choose for each $K \in [I]^{<\omega}$ an $x_K \in W'_K$ such that $Tx_K{\restriction}K \notin H_K$. Let $\mathcal{F}$ be an ultrafilter on $[I]^{<\omega}$ containing $\{K : L \subseteq K \in [I]^{<\omega}\}$ for every finite $L \subseteq I$. If $(i,r) \in I$, there are $n \ge 1$, $V \in \mathcal{V}_n$ and $k \in \mathbb{N}$ such that $r < n$ and $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$, in which case $|f_{ir}(x_K)| \le \gamma'_{nVk}$ whenever $K \in [I]^{<\omega}$ contains (i,r). This means that $\lim_{K\to\mathcal{F}} f_{ir}(x_K)$ must be defined in $[-\gamma'_{nVk}, \gamma'_{nVk}]$; consequently $y^* = \lim_{K\to\mathcal{F}} Tx_K$ is defined in $\mathbb{R}^I$. Since $x_K \in W'$ for every K, $y^* \in \overline{T[W']} = T[W']$.

Let $x^* \in W'$ be such that $Tx^* = y^*$. Since $x^* \in W$, there are $n \ge 1$, $V \in \mathcal{V}_n$ and $i \in I(n,V)$ such that $T_i(x^*) \in V$. Set $L = \{(i,r) : r < n\}$, $H = \{z : z \in \mathbb{R}^L,\ \langle z(i,r)\rangle_{r<n} \in V\}$; then $\{x : Tx{\restriction}L \in H\} = T_i^{-1}[V]$. Since $y^*{\restriction}L = Tx^*{\restriction}L$ belongs to H, and H is open, there must be a $K \supseteq L$ such that $Tx_K{\restriction}L \in H$. But in this case $H' = \{z : z \in \mathbb{R}^K,\ z{\restriction}L \in H\}$ is an open subset of $\mathbb{R}^K$ and

$$\{x : Tx{\restriction}K \in H'\} = \{x : Tx{\restriction}L \in H\} = \{x : T_i(x) \in V\}$$

meets W in a negligible set, and $H' \subseteq H_K$. But this means that $Tx_K{\restriction}K \in H_K$, contrary to the choice of x_K. **XQ**

(h) Putting (f) and (g) together, we find ourselves trying to believe simultaneously that $\mu W'_K > 0$ and that $Tx{\restriction}K \in H_K$ for every $x \in W'_K$ and that $W'_K \subseteq W$ and that $\{x : x \in W, Tx{\restriction}K \in H_K\}$ is negligible. Faced with this we have to abandon the original supposition that μ is not τ-additive.

456O We now have all the ideas needed for the main theorem of this section.

Theorem (Talagrand 81) Every centered Gaussian distribution is τ-additive.

proof ? Suppose, if possible, that μ is a centered Gaussian distribution on a set $\mathbb{R}^I$ which is not τ-additive.

(a) By 456K, there are a set J and a universal centered Gaussian distribution ν on $\mathbb{R}^J$ and a continuous linear operator $T : \mathbb{R}^J \to \mathbb{R}^I$ which is inverse-measure-preserving for ν and μ. By 418Ha, ν is not τ-additive.

(b) As in part (a) of the proof of 456N, there are a non-negligible zero set $W \subseteq \mathbb{R}^J$ and a family $\mathcal{G}$ of open sets, covering W, such that $\nu(W \cap G) = 0$ for every $G \in \mathcal{G}$. Give J a Hilbert space structure such that $\int x(i)x(j)\nu(dx) = (i|j)$ for all $i, j \in J$. Let $K_0 \subseteq J$ be a countable set such that W is determined by coordinates in K_0, and let K be the closed linear subspace of J generated by K_0. Let $\mathcal{G}'$ be the family of open sets determined by coordinates in K which meet W in negligible sets. Then $W \subseteq \bigcup\mathcal{G}'$, by 456L.

Let λ be the centered Gaussian distribution on $\mathbb{R}^K$ for which the map $\tilde\pi_K = x \mapsto x{\restriction}K : \mathbb{R}^J \to \mathbb{R}^K$ is inverse-measure-preserving. Then $\tilde\pi_K[W]$ is a zero set in $\mathbb{R}^K$, $\lambda\tilde\pi_K[W] = \nu W > 0$, $\{\tilde\pi_K[G] : G \in \mathcal{G}'\}$ is a family of open sets in $\mathbb{R}^K$ covering $\tilde\pi_K[W]$, and $\lambda(\tilde\pi_K[W] \cap \tilde\pi_K[G]) = \nu(W \cap G) = 0$ for every $G \in \mathcal{G}'$;

so λ is not τ-additive. However, K, regarded as a normed space, is separable (see 4A4Bg); and if we set $\pi_j(y) = y(j)$ for $y \in \mathbb{R}^K$ and $j \in K$, then $\|\pi_i^\bullet - \pi_j^\bullet\|_2 = \|i - j\|$ for all i, $j \in K$. So $\{\pi_j^\bullet : j \in K\}$ is separable in $L^2(\lambda)$. And this is impossible, by 456N. **X**

Thus every centered Gaussian distribution must be τ-additive.

456P Corollary If μ is a centered Gaussian distribution on $\mathbb{R}^I$, there is a unique quasi-Radon measure $\tilde{\mu}$ on $\mathbb{R}^I$ extending μ. The support of μ as defined in 456H is the support of $\tilde{\mu}$ as defined in 411N.

proof By 415L, μ has a unique extension to a quasi-Radon measure $\tilde{\mu}$. Now the support Z of μ is a closed set, so $\tilde{\mu}Z = \mu^*Z$ (415L(i)). Also Z is self-supporting for μ. If $G \subseteq \mathbb{R}^I$ is an open set meeting Z, then there is a cozero set $H \subseteq G$ which also meets Z, and $\mu^*(Z \cap H) > 0$. It follows that $\mu^*(Z \setminus H) < 1$; as $\tilde{\mu}$ extends μ, $\tilde{\mu}(Z \setminus H) < 1$ and $\tilde{\mu}(Z \cap G) > 0$. This shows that Z is self-supporting for $\tilde{\mu}$, so must be the support of $\tilde{\mu}$ in the standard sense.

456Q Proposition Let I be a set and R the set of functions $\sigma : I \times I \to \mathbb{R}$ which are symmetric and positive semi-definite in the sense of 456C; give R the subspace topology induced by the usual topology of $\mathbb{R}^{I\times I}$. Let $P_{\mathrm{qR}}(\mathbb{R}^I)$ be the space of quasi-Radon probability measures on $\mathbb{R}^I$ with its narrow topology (437Jd). For $\sigma \in R$, let μ_σ be the centered Gaussian distribution on $\mathbb{R}^I$ with covariance matrix σ (456C), and $\tilde{\mu}_\sigma$ the quasi-Radon measure extending μ_σ (456P). Then R is a closed subset of $\mathbb{R}^{I\times I}$ and the function $\sigma \mapsto \tilde{\mu}_\sigma : R \to P_{\mathrm{qR}}(\mathbb{R}^I)$ is continuous.

proof (a) From 456C(iv) we see at once that R is closed. So the rest of this proof will be devoted to showing that $\sigma \mapsto \tilde{\mu}_\sigma$ is continuous.

(b) I had better begin with the one-dimensional case. If $I = \{j\}$ is a singleton, and we identify $\mathbb{R}^I$ with $\mathbb{R}$, then $\tilde{\mu}_\sigma$ is the ordinary normal distribution with mean 0 and variance $\sigma(j, j)$, counting the Dirac measure centered at 0 as a normal distribution with zero variance. If $H \subseteq \mathbb{R}$ is open and $\gamma \in \mathbb{R}$, set

$$G = \{\alpha : \alpha > 0, \frac{1}{\sqrt{2\pi\alpha}}\int_H e^{-t^2/\alpha}dt > \gamma\};$$

then G is open. If $0 \notin H$, then

$$\{\sigma : \tilde{\mu}_\sigma H > \gamma\} = \{\sigma : \sigma(i, i) \in G\}$$

is open. If $0 \in H$ and $\gamma \ge 1$, then $\{\sigma : \tilde{\mu}_\sigma H > \gamma\}$ is empty; if $0 \in H$ and $\gamma < 1$, then

$$\{\sigma : \tilde{\mu}_\sigma H > \gamma\} = \{\sigma : \sigma(i, i) \in G\} \cup \{0\}$$

is open because there is an $\eta > 0$ such that $[-\eta, \eta] \subseteq H$ and $\alpha \in G$ whenever $\alpha > 0$ and

$$\frac{1}{\sqrt{2\pi}}\int_{-\eta/\sqrt{\alpha}}^{\eta/\sqrt{\alpha}} e^{-t^2/2}dt > \gamma.$$

As H is arbitrary, $\sigma \mapsto \tilde{\mu}_\sigma$ is continuous.

(c) Now suppose that I is finite. Let $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ be a sequence in R with limit $\sigma \in R$. Let φ_n, φ be the characteristic functions of $\tilde{\mu}_{\sigma_n}$, $\tilde{\mu}_\sigma$ respectively (§285). If $y \in \mathbb{R}^I$, set $f(x) = x\,.\,y$ for $x \in \mathbb{R}^I$; then

$$\varphi(y) = \int e^{if(x)}\tilde{\mu}_\sigma(dx) = \int e^{it}(\tilde{\mu}_\sigma f^{-1})(dt),$$

writing $\tilde{\mu}_\sigma f^{-1}$ for the image Radon measure on $\mathbb{R}$. Now $\tilde{\mu}_\sigma f^{-1}$ is the one-dimensional Gaussian distribution with variance $\sum_{j,k\in I} \sigma(j, k)y(j)y(k)$ (see part (b) of the proof of 456B). But since

$$\sum_{j,k\in I} \sigma(j, k)y(j)y(k) = \lim_{n\to\infty} \sum_{j,k\in I} \sigma_n(j, k)y(j)y(k,$$

(a) tells us that $\tilde{\mu}_\sigma f^{-1} = \lim_{n\to\infty} \tilde{\mu}_{\sigma_n} f^{-1}$ for the narrow topology on $P_{\mathrm{qR}}(\mathbb{R})$, therefore also for the vague topology (437L), and $\varphi(y) = \lim_{n\to\infty} \varphi_n(y)$. By 285L, $\tilde{\mu}_\sigma = \lim_{n\to\infty} \tilde{\mu}_{\sigma_n}$ for the vague topology, therefore also for the narrow topology.

Thus $\sigma \mapsto \tilde{\mu}_\sigma$ is sequentially continuous. As I is countable, R is metrizable, and $\sigma \mapsto \tilde{\mu}_\sigma$ is continuous.

(d) For the general case, suppose that $H \subseteq \mathbb{R}^I$ is an open set and that $\gamma \in \mathbb{R}$. Set $G_{H\gamma} = \{\sigma : \sigma \in R, \tilde{\mu}_\sigma H > \gamma\}$.

(i) If H is determined by coordinates in a finite set $J \subseteq I$ then $G_{H\gamma}$ is open in R. **P** Let R_J be the set of symmetric positive semi-definite functions on $\mathbb{R}^{J\times J}$; write $h(\sigma) = \sigma\restriction J \times J$ for $\sigma \in R$, and $\tilde{\pi}_J(x) = x\restriction J$ for $x \in \mathbb{R}^I$. Of course $h(\sigma) \in R_J$ for $\sigma \in R$, and $h : R \to R_J$ is continuous. For $\sigma \in R$, we know that there is a centered Gaussian distribution ν on $\mathbb{R}^J$ such that $\tilde{\pi}_J$ is inverse-measure-preserving for μ_σ and ν, by 456Ba; the covariance matrix of ν is of course $h(\sigma)$, so we can call it $\mu_{h(\sigma)}$. Next, there is a quasi-Radon measure $\tilde{\nu}$ on $\mathbb{R}^J$ such that $\tilde{\pi}_J$ is inverse-measure-preserving for $\tilde{\mu}_\sigma$ and $\tilde{\nu}$ (418Hb); as $\tilde{\nu}$ must extend the Baire measure ν, it is the unique quasi-Radon measure extending ν, and we can call it $\tilde{\mu}_{h(\sigma)}$.

Because H is determined by coordinates in J, $H = \tilde{\pi}_J^{-1}[H']$ where $H' = \tilde{\pi}_J[H]$ is open in $\mathbb{R}^J$ (4A2B(f-i) again). So $G' = \{\tau : \tau \in R_J,\ \tilde{\mu}_\tau H' > \gamma\}$ is open in R_J, by (b), and

$$G_{H\gamma} = \{\sigma : (\tilde{\mu}_\sigma \tilde{\pi}_J^{-1})(H') > \gamma\} = \{\sigma : \tilde{\mu}_{h(\sigma)} H' > \gamma\} = h^{-1}[G']$$

is open in R. **Q**

(ii) In fact $G_{H\gamma}$ is open in R for any open set $H \subseteq \mathbb{R}^I$ and $\gamma \in \mathbb{R}$. **P** Take any $\sigma \in G_{H\gamma}$. Because $\tilde{\mu}_\sigma$ is τ-additive, and the family

$$\mathcal{V} = \{V : V \subseteq \mathbb{R}^I \text{ is open and determined by coordinates in a finite set}\}$$

is a base for the topology of $\mathbb{R}^I$ closed under finite unions, there is a $V \in \mathcal{V}$ such that $V \subseteq H$ and $\tilde{\mu}_\sigma V > \gamma$. Now $\sigma \in G_{V\gamma} \subseteq G_{H\gamma}$; by (i), $G_{V\gamma}$ is open, so $\sigma \in \operatorname{int} G_{H\gamma}$; as σ is arbitrary, $G_{H\gamma}$ is open. **Q** But this is just what we need to know to see that $\sigma \mapsto \tilde{\mu}_\sigma$ is continuous for the narrow topology on $P_{\mathrm{qR}}(\mathbb{R}^I)$, and the proof is complete.

456X Basic exercises (a) Let I be any set. (i) Show that if $y \in \ell^1(I)$ then $\int \sum_{i\in I} |y(i)x(i)|\mu_G^{(I)}(dx) = \frac{2}{\sqrt{2\pi}}\|y\|_1$. (*Hint*: start by evaluating $\mathbb{E}(|Z|)$ where Z is a standard normal random variable.) (ii) Show that if $y \in \ell^2(I)$ then $\int \sum_{i\in I} |y(i)x(i)|^2\mu_G^{(I)}(dx) = \|y\|_2^2$.

(b) Let $n \ge 1$ be an integer. (i) Show that $\mu_G^{(n)} T^{-1} = \mu_G^{(n)}$ for any orthogonal linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$. (ii) Set $p(x) = \frac{1}{\|x\|}x$ for $x \in \mathbb{R}^n \setminus \{0\}$; take $p(0)$ to be any point of S^{n-1}. Show that $\mu_G^{(n)} p^{-1}$ is a multiple of $(n-1)$-dimensional Hausdorff measure on S^{n-1}. (*Hint*: 443U.)

(c) Let G be a group, and $h : G \to \mathbb{R}$ a real positive definite function (definition: 445L). (i) Show that we have a centered Gaussian distribution μ on $\mathbb{R}^G$ with covariance matrix $\langle h(a^{-1}b)\rangle_{a,b\in G}$. (ii) Show that μ is invariant under the left shift action $\bullet_l$ of G on $\mathbb{R}^G$ (4A5Cc).

(d) Let I be a countable set, μ a centered Gaussian distribution on $\mathbb{R}^I$, and $\gamma \ge 0$. Set $\alpha = \mu\{x : \sup_{i\in I} |x(i)| \ge \gamma\}$. Show that $\mu\{x : \sup_{i\in I} |x(i)| \ge \frac{1}{2}\gamma\} \ge 2\alpha(1-\alpha)^3$.

(e) Let I be a set and $\langle \sigma_{ij}\rangle_{i,j\in I}$ a family of real numbers. Show that there is at most one inner product space structure on I for which $\sigma_{ij} = (i|j)$ for all i, $j \in I$.

(f) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent sequence of standard normal random variables, and $\langle \alpha_n\rangle_{n\in\mathbb{N}}$ a square-summable real sequence. (i) Show that for any permutation $\pi : \mathbb{N} \to \mathbb{N}$, $X = \sum_{n=0}^\infty \alpha_n X_n$ and $\sum_{n=0}^\infty \alpha_{\pi(n)} X_{\pi(n)}$ are finite and equal a.e. (*Hint*: 273B.) (ii) Show that X is normal, with mean 0 and variance $\sum_{n=0}^\infty \alpha_n^2$.

>(g) For any set I, I will say that a **centered Gaussian quasi-Radon measure** on $\mathbb{R}^I$ is a quasi-Radon measure μ on $\mathbb{R}^I$ such that every continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$ is either zero a.e. or is normally distributed with zero expectation. Show that

(i) there is a one-to-one correspondence between centered Gaussian quasi-Radon measures μ on $\mathbb{R}^I$ and centered Gaussian distributions ν on $\mathbb{R}^I$ obtained by matching μ with ν iff they agree on the zero sets of $\mathbb{R}^I$;

(ii) if μ, ν are centered Gaussian quasi-Radon measures on $\mathbb{R}^I$ and $\int x(i)x(j)\mu(dx) = \int x(i)x(j)\nu(dx)$ for all i, $j \in I$, then $\mu = \nu$;

(iii) the support of a centered Gaussian quasi-Radon measure on $\mathbb{R}^I$ is a linear subspace of $\mathbb{R}^I$;

(iv) if $\langle I_j\rangle_{j\in J}$ is a disjoint family of sets with union I, and μ_j is a centered Gaussian quasi-Radon measure on $\mathbb{R}^{I_j}$ for each $j\in J$, then the quasi-Radon product of $\langle\mu_j\rangle_{j\in J}$, regarded as a measure on $\mathbb{R}^I$, is a centered Gaussian quasi-Radon measure.

(h) Let I be a set, and let H be a Hilbert space with orthonormal basis $\langle e_i\rangle_{i\in I}$. For $i\in I$, $x\in\mathbb{R}^I$ set $f_i(x)=x(i)$. Show that there is a bounded linear operator $T:H\to L^1(\mu_G^{(I)})$ such that $Te_i=f_i^\bullet$ for every $i\in I$, and that $\|Tu\|_1=\frac{2}{\sqrt{2\pi}}\|u\|_2$ for every $u\in H$.

456Y Further exercises (a) Let (Ω,Σ,μ) be a probability space with measure algebra $(\mathfrak{A},\bar\mu)$, and $\langle u_i\rangle_{i\in I}$ a family in $L^2(\mu)\cong L^2(\mathfrak{A},\bar\mu)$ which is a centered Gaussian process in the sense that whenever $X_i\in\mathcal{L}^2(\mu)$ is such that $X_i^\bullet=u_i$ for every i, then $\langle X_i\rangle_{i\in I}$ is a centered Gaussian process. Suppose that $\gamma\ge 0$ and that $\alpha=\bar\mu(\sup_{i\in I}[\![|u_i|\ge\gamma]\!])$. Show that $\bar\mu(\sup_{i\in I}[\![|u_i|\ge\frac{1}{2}\gamma]\!])\ge 2\alpha(1-\alpha)^3$.

(b) Let U be a Hilbert space with an orthonormal basis $\langle u_j\rangle_{j\in J}$, and μ the universal centered Gaussian distribution on $\mathbb{R}^U$ with covariance matrix defined by the inner product of U. Show that there is a function $T:\mathbb{R}^J\to\mathbb{R}^U$, inverse-measure-preserving for $\mu_G^{(J)}$ and μ, such that whenever $\langle j_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of J and $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is a square-summable sequence in $\mathbb{R}$, then $(Tx)(\sum_{n=0}^\infty\alpha_n u_{j_n})=\sum_{n=0}^\infty\alpha_n x(j_n)$ for almost every $x\in\mathbb{R}^J$.

(c) Let U be an infinite-dimensional Hilbert space and μ the universal centered Gaussian distribution on $\mathbb{R}^U$ with covariance matrix defined by the inner product of U. Show that $\mu C=0$ for every compact set $C\subseteq\mathbb{R}^U$.

(d) Let I be a set and μ be a centered Gaussian distribution on $\mathbb{R}^I$. Show that the following are equiveridical: (i) μ has countable Maharam type; (ii) $L^2(\mu)$ is separable; (iii) I is separable under the pseudometric $(i,j)\mapsto\sqrt{\int(x(i)-x(j))^2\mu(dx)}$.

456 Notes and comments This section has aimed for a direct route to Talagrand's theorem 456O, leaving most of the real reasons for studying Gaussian processes (see Fernique 97) to one side. It should nevertheless be clear from such fragments as 252Xi, 456Bb, 456G and the exercises here that they are one of the many concepts of probability theory which are both significant and delightful. Very much the most important Gaussian processes are those associated with Brownian motion, which will be treated in §477 *et seq.*

You will of course have observed that the methods used here are entirely different from those in §455, even though one of the concerns of that section was a check for τ-additive distributions and corresponding quasi-Radon versions, as in 455K. However the results of §455 were based on the fact that in the most important cases the distributions there have extensions to Radon measures (455H). Gaussian distributions need not be like this at all, even when they have countable Maharam type; see 456Yc.

Version of 18.1.13

457 Simultaneous extension of measures

The questions addressed in §§451, 454 and 455 can all be regarded as special cases of a general class of problems: given a set X and a family $\langle\nu_i\rangle_{i\in I}$ of (probability) measures on X, when can we expect to find a measure on X extending every ν_i? An alternative formulation, superficially more general, is to ask: given a set X, a family $\langle(Y_i,\mathrm{T}_i,\nu_i)\rangle_{i\in I}$ of probability spaces, and functions $\phi_i:X\to Y_i$ for each i, when can we find a measure on X for which every ϕ_i is inverse-measure-preserving? Even the simplest non-trivial case, when $X=\prod_{i\in I}Y_i$ and every ϕ_i is the coordinate map, demands a significant construction (the product measures of Chapter 25). In this section I bring together a handful of important further cases which are accessible by the methods of this chapter. I begin with a discussion of extensions of finitely additive measures (457A-457D), which are much easier, before considering the problems associated with countably additive measures (457E-457G), with examples (457H-457J). In 457K-457M I look at a pair of optimisation problems.

457A It is helpful to start with a widely applicable result on common extensions of finitely additive measures.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\langle\mathfrak{B}_i\rangle_{i\in I}$ a non-empty family of subalgebras of $\mathfrak{A}$. For each $i\in I$, we may identify $L^\infty(\mathfrak{B}_i)$ with the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi b : b\in\mathfrak{B}_i\}$ (363Ga). Suppose that for each $i\in I$ we are given a finitely additive functional $\nu_i:\mathfrak{B}_i\to[0,1]$ such that $\nu_i 1=1$; write $⨍\ldots d\nu_i$ for the corresponding positive linear functional on $L^\infty(\mathfrak{B}_i)$ (363L). Then the following are equiveridical:

(i) there is an additive functional $\mu:\mathfrak{A}\to[0,1]$ extending every ν_i;

(ii) whenever $i_0,\ldots,i_n\in I$, $a_k\in\mathfrak{B}_{i_k}$ for $k\le n$, and $\sum_{k=0}^n\chi a_k\ge m\chi 1$ in $S(\mathfrak{A})$, where $m\in\mathbb{N}$, then $\sum_{k=0}^n\nu_{i_k}a_k\ge m$;

(iii) whenever $i_0,\ldots,i_n\in I$, $a_k\in\mathfrak{B}_{i_k}$ for $k\le n$, and $\sum_{k=0}^n\chi a_k\le m\chi 1$, where $m\in\mathbb{N}$, then $\sum_{k=0}^n\nu_{i_k}a_k\le m$;

(iv) whenever $i_0,\ldots,i_n\in I$ are distinct, $u_k\in L^\infty(\mathfrak{B}_{i_k})$ for every $k\le n$, and $\sum_{k=0}^n u_k\ge\chi 1$, then $\sum_{i=0}^n⨍u_k d\nu_{i_k}\ge 1$;

(v) whenever $i_0,\ldots,i_n\in I$ are distinct, $u_k\in L^\infty(\mathfrak{B}_{i_k})$ for every $k\le n$, and $\sum_{k=0}^n u_k\le\chi 1$, then $\sum_{i=0}^n⨍u_k d\nu_{i_k}\le 1$.

proof (a) It is elementary to check that if (i) is true then (ii)-(v) are all true, simply because we have a positive linear functional $⨍d\mu$ extending all the functionals $⨍d\nu_i$.

(b)(ii)⇒(iii) Given that $a_k\in\mathfrak{B}_{i_k}$ and $\sum_{k=0}^n\chi a_k\le m\chi 1$, then

$$\textstyle\sum_{k=0}^n\chi(1\setminus a_k)=(n+1)\chi 1-\sum_{k=0}^n\chi a_k\ge(n+1-m)\chi 1,$$

so

$$\textstyle\sum_{k=0}^n\nu_{i_k}a_k=n+1-\sum_{k=0}^n\nu_{i_k}(1\setminus a_k)\le n+1-(n+1-m)=m,$$

as required by (iii).

(c)(iii)⇒(i) Assume (iii). Set $\psi a=\sup\{\nu_i a : i\in I,\ a\in\mathfrak{B}_i\}$ for $a\in\mathfrak{A}$ (interpreting $\sup\emptyset$ as 0, as usual in such contexts). Then ψ satisfies the condition (ii) of 391F. **P?** Otherwise, there is a finite indexed family $\langle a_k\rangle_{k\in K}$ in $\mathfrak{A}$ such that $\inf_{k\in J}a_k=0$ whenever $J\subseteq K$ and $\#(J)\ge\sum_{k\in K}\psi a_i$. The general hypothesis of the lemma implies that $\mathfrak{A}\ne\{0\}$, so $\inf\emptyset=1\ne 0$ and K is non-empty. Taking K to be of minimal size, we get an example in which $\psi a_k>0$ for every $k\in K$. Set $m=\|\sum_{k\in K}\chi a_k\|_\infty$; then $m\in\mathbb{N}$ and $m<\sum_{k\in K}\psi a_k$, so we can find for each $k\in K$ an $i_k\in I$ such that $a_k\in\mathfrak{B}_{i_k}$ and $m<\sum_{k\in K}\nu_{i_k}a_k$. But this contradicts our hypothesis (iii). **XQ**

By 391F, there is a non-negative finitely additive functional μ such that $\mu 1=1$ and $\mu a\ge\psi a$ for every $a\in\mathfrak{A}$, that is, $\mu b\ge\nu_i b$ whenever $i\in I$ and $b\in\mathfrak{B}_i$. But observe now that, because $\mu 1=\nu_i 1$ and $\mu(1\setminus b)\ge\nu_i(1\setminus b)$, we actually have $\mu b=\nu_i b$ for every $b\in\mathfrak{B}_i$, so that μ extends ν_i, for every $i\in I$.

(d)(iv)⇒(ii) Suppose that (iv) is true, and that $i_0,\ldots,i_n\in I$, $a_k\in\mathfrak{B}_{i_k}$ for $k\le n$, and $\sum_{k=0}^n\chi a_k\ge m\chi 1$ in $S(\mathfrak{A})$, where $m\in\mathbb{N}$. If $m=0$ then of course $\sum_{k=0}^n\nu_{i_k}a_k\ge m$. Otherwise, set $J=\{i_k : k\le n\}$ and enumerate J as $\langle j_l\rangle_{l\le r}$. For $l\le r$ set $u_l=\frac{1}{m}\sum_{k\le n,i_k=j_l}\chi a_k$. Then $u_l\in S(\mathfrak{B}_{j_l})$ for each l, and

$$\textstyle\sum_{l=0}^r u_l=\frac{1}{m}\sum_{l=0}^r\sum_{k\le n,i_k=j_l}\chi a_k=\frac{1}{m}\sum_{k=0}^n\chi a_k\ge\chi 1.$$

As $j_0,\ldots,j_l$ are distinct,

$$\textstyle\sum_{l=0}^r⨍u_l d\nu_{j_l}=\frac{1}{m}\sum_{k=0}^n\nu_{i_k}a_k\ge 1.$$

So (ii) is true.

(e)(v)⇒(iii) Use the same argument as in (d) above.

457B Corollary Let X be a set and $\langle Y_i\rangle_{i\in I}$ a family of sets. Suppose that for each $i\in I$ we have an algebra $\mathcal{E}_i$ of subsets of Y_i, an additive functional $\nu_i:\mathcal{E}_i\to[0,1]$ such that $\nu_i Y_i=1$, and a function $f_i:X\to Y_i$. Then the following are equiveridical:

(i) there is an additive functional $\mu:\mathcal{P}X\to[0,1]$ such that $\mu f_i^{-1}[E]=\nu_i E$ whenever $i\in I$ and $E\in\mathcal{E}_i$;

(ii) whenever $i_0,\dots,i_n\in I$ and $E_k\in\mathcal{E}_{i_k}$ for $k\le n$, then there is an $x\in X$ such that $\sum_{k=0}^n\nu_{i_k}E_k\le\#(\{k:k\le n,\ f_{i_k}(x)\in E_k\})$.

proof (i)⇒(ii) is elementary; if $m=\lceil\sum_{k=0}^n\nu_{i_k}E_k\rceil-1$, then $\sum_{k=0}^n\mu f_{i_k}^{-1}[E_k]>m\mu X$, so $\sum_{k=0}^n\chi f_{i_k}^{-1}E_k\not\le m\chi X$, that is, there is an $x\in X$ such that

$$\#(\{k:f_{i_k}(x)\in E_k\})=\textstyle\sum_{k=0}^n(\chi f_{i_k}^{-1}[E_k])(x)\ge m+1\ge\sum_{k=0}^n\nu_{i_k}E_k.$$

(ii)⇒(i) Now suppose that (ii) is true. For $i\in I$ set $\mathfrak{B}_i=\{f_i^{-1}[E]:E\in\mathcal{E}_i\}$. Note that if $E\in\mathcal{E}_i$ and $\nu_iE>0$, then (applying (ii) with $n=0$, $i_0=i$ and $E_0=E$) $f_i^{-1}[E]$ cannot be empty; accordingly we have an additive functional $\nu'_i:\mathfrak{B}_i\to[0,1]$ defined by setting $\nu'_if^{-1}[E]=\nu_iE$ for every $E\in\mathcal{E}_i$, and $\nu'_iX=1$. If $i_0,\dots,i_n\in I$, $H_0\in\mathfrak{B}_{i_0},\dots,H_n\in\mathfrak{B}_{i_n}$ and $m\in\mathbb{N}$ are such that $\sum_{k=0}^n\chi H_k\le m\chi X$, express each H_k as $f_{i_k}^{-1}[E_k]$, where $E_k\in\mathcal{E}_{i_k}$; then there is an $x\in X$ such that

$$\textstyle\sum_{k=0}^n\nu'_{i_k}H_k=\sum_{k=0}^n\nu_{i_k}E_k\le\#(\{k:f_k(x)\in E_k\})=\sum_{k=0}^m\chi H_k(x)\le m.$$

But this means that the condition of 457A(iii) is satisfied, with $\mathfrak{A}=\mathcal{P}X$, so 457A(i) and (i) here are also true.

457C Corollary (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two subalgebras of $\mathfrak{A}$ with finitely additive functionals $\nu_i:\mathfrak{B}_i\to[0,1]$ such that $\nu_11=\nu_21=1$. Then the following are equiveridical:

(i) there is an additive functional $\mu:\mathfrak{A}\to[0,1]$ extending both the ν_i;

(ii) whenever $b_1\in\mathfrak{B}_1$, $b_2\in\mathfrak{B}_2$ and $b_1\cup b_2=1$, then $\nu_1b_1+\nu_2b_2\ge1$;

(iii) whenever $b_1\in\mathfrak{B}_1$, $b_2\in\mathfrak{B}_2$ and $b_1\cap b_2=0$, then $\nu_1b_1+\nu_2b_2\le1$.

(b) Let X, Y_1, Y_2 be sets, and for $i\in\{1,2\}$ let $\mathcal{E}_i$ be an algebra of subsets of Y_i, $\nu_i:\mathcal{E}_i\to[0,1]$ an additive functional such that $\nu_iY_i=1$, and $f_i:X\to Y_i$ a function. Then the following are equiveridical:

(i) there is an additive functional $\mu:\mathcal{P}X\to[0,1]$ such that $\mu f_i^{-1}[E]=\nu_iE$ whenever $i\in\{1,2\}$ and $E\in\mathcal{E}_i$;

(ii) $f_1^{-1}[E_1]\cap f_2^{-1}[E_2]\ne\emptyset$ whenever $E_1\in\mathcal{E}_1$, $E_2\in\mathcal{E}_2$ and $\nu_1E_1+\nu_2E_2>1$;

(iii) $\nu_1E_1\le\nu_2E_2$ whenever $E_1\in\mathcal{E}_1$, $E_2\in\mathcal{E}_2$ and $f_1^{-1}[E_1]\subseteq f_2^{-1}[E_2]$.

proof (a)(i)⇒(iii) is elementary (and is a special case of 457A(i)⇒457A(iii)).

(iii)⇒(ii) If (iii) is true, and $b_1\in\mathfrak{B}_1$, $b_2\in\mathfrak{B}_2$ are such that $b_1\cup b_2=1$, then $(1\setminus b_1)\cap(1\setminus b_2)=0$, so

$$\nu_1b_1+\nu_2b_2=2-\nu_1(1\setminus b_1)-\nu_2(1\setminus b_2)\ge1.$$

(ii)⇒(i) The point is that (ii) here implies (ii) of 457A. **P** Suppose that $i_0,\dots,i_n\in\{1,2\}$, $a_k\in\mathfrak{B}_{i_k}$ for $k\le n$ and $\sum_{k=0}^n\chi a_k\ge m\chi1$ in $S(\mathfrak{A})$, where $m\in\mathbb{N}$. Set $K_j=\{k:k\le n,\ i_k=j\}$ for each j, $u=\sum_{k\in K_1}\chi a_k\in S(\mathfrak{B}_1)$, $v=\sum_{k\in K_2}\chi a_k\in S(\mathfrak{B}_2)$. Then we can express u as $\sum_{j=0}^{m_1}\chi c_j$ where $c_j\in\mathfrak{B}_1$ for each $j\le m_1$ and $c_0\supseteq c_1\supseteq\dots\supseteq c_{m_1}$ (see the proof of 361Ec). Taking $c_j=0$ for $m_1<j\le m$ if necessary, we may suppose that $m_1\ge m$. Similarly, $v=\sum_{j=0}^{m_2}\chi d_j$ where $m_2\ge m$, $d_j\in\mathfrak{B}_2$ for each $j\le m_2$ and $d_0\supseteq\dots\supseteq d_{m_2}$.

For $j<m$, set $b_j=1\setminus(c_j\cup d_{m-j-1})$. Then, because $b_j\cap c_j=0$,

$$u\times\chi b_j=\textstyle\sum_{r=0}^{m_1}\chi(c_r\cap b_j)=\sum_{r=0}^{j-1}\chi(c_r\cap b_j)\le j\chi b_j,$$

and similarly $v\times\chi b_j\le(m-j-1)\chi b_j$, so

$$m\chi b_j\le(u+v)\times\chi b_j=u\times\chi b_j+v\times\chi b_j\le(m-1)\chi b_j,$$

and b_j must be 0.

Thus $c_j\cup d_{m-j-1}=1$ for every $j<m$. But this means that $\nu_1c_j+\nu_2d_{m-j-1}\ge1$ for every $j<m$, so that

$$\begin{aligned}\sum_{k=0}^n\nu_{i_k}a_k&=\sum_{k\in K_1}\nu_1a_k+\sum_{k\in K_2}\nu_2a_k=\int u\,d\nu_1+\int v\,d\nu_2\\&=\sum_{j=0}^{m_1}\nu_1c_j+\sum_{j=0}^{m_2}\nu_2d_j\ge\sum_{j=0}^{m-1}\nu_1c_j+\nu_2d_{m-1-j}\ge m,\end{aligned}$$

as required. **Q**

Because 457A(ii) implies 457A(i), we have the result.

(b) We can convert (i) and (ii) here into (a-i) and (a-iii) just above by the same translation as in 457B. So (i) and (ii) are equiveridical. As for (iii), this corresponds exactly to replacing E_2 by $Y_2 \setminus E_2$ in (ii).

***457D** The proof of 457A is based, at some remove, on the Hahn-Banach theorem, as applied in the proof of 391E-391F. An alternative proof uses the max-flow min-cut theorem of graph theory. To show the power of this method I apply it to an elaboration of 457C, as follows.

Proposition (Strassen 65) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two subalgebras of $\mathfrak{A}$. Suppose that $\nu_i : \mathfrak{B}_i \to [0,1]$ are finitely additive functionals such that $\nu_1 1 = \nu_2 1 = 1$, and $\theta : \mathfrak{A} \to [0,\infty[$ another additive functional. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0,\infty[$ extending both the ν_i, and such that $\mu a \le \theta a$ for every $a \in \mathfrak{A}$;

(ii) $\nu_1 b_1 + \nu_2 b_2 \le 1 + \theta(b_1 \cap b_2)$ whenever $b_1 \in \mathfrak{B}_1$ and $b_2 \in \mathfrak{B}_2$.

proof (a) As usual in this context, (i)$\Rightarrow$(ii) is elementary; if $\mu \le \theta$ extends both ν_j, and $b_j \in \mathfrak{B}_j$ for both j, then

$$\nu_1 b_1 + \nu_2 b_2 = \mu b_1 + \mu b_2 = \mu(b_1 \cup b_2) + \mu(b_1 \cap b_2) \le 1 + \theta(b_1 \cap b_2).$$

(b) For the reverse implication, suppose to begin with (down to the end of (d) below) that $\mathfrak{A}$ is finite. Let I, J and K be the sets of atoms of $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{A}$ respectively. Consider the transportation network (V, E, γ) where

$$V = \{(0,0)\} \cup \{(b,1) : b \in I\} \cup \{(d,2) : d \in K\} \cup \{(c,3) : c \in J\} \cup \{(1,4)\},$$

$$E = \{e^0_b : b \in I\} \cup \{e^1_d : d \in K\} \cup \{e^2_d : d \in K\} \cup \{e^3_c : c \in J\},$$

where

for $b \in I$, e^0_b runs from $(0,0)$ to $(b,1)$,
for $d \in K$, e^1_d runs from $(b,1)$ to $(d,2)$, where b is the member of I including d,
for $d \in K$, e^2_d runs from $(d,2)$ to $(c,3)$, where c is the member of J including d,
for $c \in J$, e^3_c runs from $(c,3)$ to $(1,4)$.

Define the capacity $\gamma(e)$ of each link by setting

$$\gamma(e^0_b) = \nu_1 b \text{ for } b \in I,$$

$$\gamma(e^1_d) = \gamma(e^2_d) = \theta d \text{ for } d \in K,$$

$$\gamma(e^3_c) = \nu_2 c \text{ for } c \in J.$$

By the max-flow min-cut theorem (4A4N), there are a flow ϕ and a cut X of the same value; that is, we have a function $\phi : E \to [0,\infty[$ and a set $X \subseteq E$ such that

$$\sum_{e \text{ starts from } v} \phi(e) = \sum_{e \text{ ends at } v} \phi(e)$$

for every $v \in V \setminus \{(0,0),(1,4)\}$,

$$\phi(e) \le \gamma(e)$$

for every $e \in E$,

$$\sum_{e \text{ starts from } (0,0)} \phi e = \sum_{e \text{ ends at } (1,4)} \phi e = \sum_{e \in X} \gamma(e),$$

and there is no path from $(0,0)$ to $(1,4)$ using only links in $E \setminus X$.

Now, for any $d \in K$, there is exactly one link e^1_d ending at d and exactly one link e^2_d starting from d. So $\phi(e^1_d) = \phi(e^2_d)$, and we may define an additive functional μ on $\mathfrak{A}$ by setting

$$\mu a = \sum_{d \in K, d \subseteq a} \phi(e^1_d) = \sum_{d \in K, d \subseteq a} \phi(e^2_d)$$

for every $a \in \mathfrak{A}$.

(c)(i) $\mu b \le \nu_1 b$ for every $b \in \mathfrak{B}_1$. **P** Because I is the set of atoms of the finite Boolean algebra $\mathfrak{B}_1$, it is enough to show that $\mu b \le \nu_1 b$ for every $b \in I$. Now, for such b,

$$\begin{aligned}\mu b &= \sum_{d \in K, d \subseteq b} \phi(e_d^1) = \sum_{e \text{ starts from } (b,1)} \phi(e)\\ &= \sum_{e \text{ ends at } (b,1)} \phi(e) = \phi(e_b^0) \le \gamma(e_b^0) = \nu_1 b,\end{aligned}$$

because the only link ending at $(b,1)$ is e_b^0. **Q**

(ii) Similarly, because the only link starting at $(c,3)$ has capacity $\nu_2 c$, $\mu c \le \nu_2 c$ for every $c \in J$. But this means that $\mu c \le \nu_2 c$ for every $c \in \mathfrak{B}_2$.

(iii) In third place, because

$$\mu d = \phi(e_d^1) \le \gamma(e_d^1) = \theta d$$

for every $d \in K$, $\mu a \le \theta a$ for every $a \in \mathfrak{A}$.

(d) (The key.) $\mu 1 \ge 1$. **P** We have

$$\begin{aligned}\mu 1 &= \sum_{d \in K} \mu d = \sum_{d \in K} \phi(e_d^1)\\ &= \sum_{b \in I} \sum_{d \in K, d \subseteq b} \phi(e_d^1) = \sum_{b \in I} \sum_{e \text{ starts from } (b,1)} \phi(e)\\ &= \sum_{b \in I} \sum_{e \text{ ends at } (b,1)} \phi(e) = \sum_{e \text{ starts from } (0,0)} \phi(e) = \sum_{e \in X} \gamma(e).\end{aligned}$$

Set

$$\begin{aligned}b^* &= \sup\{b : b \in I,\ e_b^0 \in X\} \in \mathfrak{B}_1,\\ a_1^* &= \sup\{d : d \in K,\ e_d^1 \in X\},\\ a_2^* &= \sup\{d : d \in K,\ e_d^2 \in X\},\\ c^* &= \sup\{c : c \in J,\ e_c^3 \in X\} \in \mathfrak{B}_2.\end{aligned}$$

For any $d \in K$, we have a four-link path $e_b^0, e_d^1, e_d^2, e_c^3$ from $(0,0)$ to $(1,4)$, where $b \in I$, $c \in J$ are the atoms of $\mathfrak{B}_1$, $\mathfrak{B}_2$ including d. At least one of the links in this path must belong to X, so that d is included in $b^* \cup a_1^* \cup a_2^* \cup c^*$. Thus, writing $a = (1 \setminus b^*) \cap (1 \setminus c^*)$, $a \subseteq a_1^* \cup a_2^*$ and $\theta a \le \theta a_1^* + \theta a_2^*$. But this means that

$$\begin{aligned}\mu 1 &= \sum_{e \in X} \gamma(e)\\ &= \sum_{b \in I, e_b^0 \in X} \gamma(e_b^0) + \sum_{d \in K, e_d^1 \in X} \gamma(e_d^1) + \sum_{d \in K, e_d^2 \in X} \gamma(e_d^2) + \sum_{c \in J, e_c^3 \in X} \gamma(e_c^3)\\ &= \sum_{b \in I, e_b^0 \in X} \nu_1 b + \sum_{d \in K, e_d^1 \in X} \theta d + \sum_{d \in K, e_d^2 \in X} \theta d + \sum_{c \in J, e_c^3 \in X} \nu_2 c\\ &= \nu_1 b^* + \theta a_1^* + \theta a_2^* + \nu_2 c^*\end{aligned}$$

(remember that θ is additive)

$$\ge \nu_1 b^* + \theta((1 \setminus b^*) \cap (1 \setminus c^*)) + \nu_2 c^* \ge \nu_1 b^* + \nu_1(1 \setminus b^*) + \nu_2(1 \setminus c^*) - 1 + \nu_2 c^*$$

(applying the hypothesis (ii))

$$= 1,$$

as claimed. **Q**

Since we already know that $\nu_1 1 = 1$ and that $\mu b \le \nu_1 b$ for every $b \in \mathfrak{B}_1$, we must have $\mu 1 = 1$ and $\mu b = \nu_1 b$ for every $b \in \mathfrak{B}$, so that μ extends ν_1. Similarly, μ extends ν_2.

(e) Thus the proposition is proved in the case in which $\mathfrak{A}$ is finite. In the general case, for each finite subset K of $\mathfrak{A}$ write $\mathfrak{A}_K$ for the subalgebra of $\mathfrak{A}$ generated by K. Then (b)-(d) tell us that there is a non-negative additive functional μ_K on $\mathfrak{A}_K$, dominated by θ on $\mathfrak{A}_K$, agreeing with ν_1 on $\mathfrak{A}_K \cap \mathfrak{B}_1$ and agreeing with ν_2 on $\mathfrak{A}_K \cap \mathfrak{B}_2$. Let μ be any cluster point of the μ_K in $[0,1]^{\mathfrak{A}}$ as K increases through the finite subsets of $\mathfrak{A}$; then μ will be a non-negative additive functional on $\mathfrak{A}$, dominated by θ, and extending ν_1 and ν_2.

This proves the result.

457E Proposition Let X be a non-empty set and $\langle \nu_i \rangle_{i \in I}$ a family of probability measures on X satisfying the conditions of Lemma 457A, taking $\mathfrak{A} = \mathcal{P}X$ and $\mathfrak{B}_i = \operatorname{dom} \nu_i$ for each i. Suppose that there is a countably compact class $\mathcal{K} \subseteq \mathcal{P}X$ such that every ν_i is inner regular with respect to $\mathcal{K}$. Then there is a probability measure μ on X extending every ν_i.

proof If $I = \emptyset$ this is trivial. Otherwise, by 457A, there is a finitely additive functional ν on $\mathcal{P}X$ extending every ν_i. Now 413Ua tells us that there is a complete measure μ on X such that $\mu X \le \nu X$ and $\mu K \ge \nu K$ for every $K \in \mathcal{K}$. In this case, for any $i \in I$ and $E \in \mathrm{T}_i = \operatorname{dom} \nu_i$, we must have

$$\begin{aligned} \mu_* E &\ge \sup_{K \in \mathcal{K}, K \subseteq E} \mu K \ge \sup_{K \in \mathcal{K} \cap \operatorname{dom} \nu_i, K \subseteq E} \mu K \\ &\ge \sup_{K \in \mathcal{K} \cap \operatorname{dom} \nu_i, K \subseteq E} \nu K = \sup_{K \in \mathcal{K} \cap \operatorname{dom} \nu_i, K \subseteq E} \nu_i K = \nu_i E. \end{aligned}$$

In particular, $\mu X \ge \nu_i X = 1$. Also $\mu X \le \nu X = 1$, so

$$\mu^* E = 1 - \mu_*(X \setminus E) \le 1 - \nu_i(X \setminus E) = \nu_i E$$

for any $E \in \mathrm{T}_i$; as μ is complete, μE is defined and equal to $\nu_i E$ for every $E \in \mathrm{T}_i$, and μ extends ν_i, as required.

457F Proposition (a) Let (X, Σ, μ) be a perfect probability space and (Y, T, ν) any probability space. Write $\Sigma \otimes \mathrm{T}$ for the algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma, F \in \mathrm{T}\}$. Suppose that $Z \subseteq X \times Y$ is such that

(i) Z is expressible as the intersection of a sequence in $\Sigma \otimes \mathrm{T}$,

(ii) $Z \cap (E \times F) \ne \emptyset$ whenever $E \in \Sigma$, $F \in \mathrm{T}$ are such that $\mu E + \nu F > 1$.

Then there is a probability measure λ on Z such that the maps $(x, y) \mapsto x : Z \to X$ and $(x, y) \mapsto y : Z \to Y$ are both inverse-measure-preserving.

(b) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of perfect probability spaces. Write $\bigotimes_{i \in I} \Sigma_i$ for the algebra of subsets of $X = \prod_{i \in I} X_i$ generated by $\{\{x : x \in X, x(i) \in E\} : i \in I, E \in \Sigma_i\}$. Suppose that $Z \subseteq X$ is such that

(i) Z is expressible as the intersection of a sequence in $\bigotimes_{i \in I} \Sigma_i$,

(ii) whenever $i_0, \dots, i_n \in I$ and $E_k \in \Sigma_{i_k}$ for $k \le n$, there is a $z \in Z$ such that $\#(\{k : k \le n, z(i_k) \in E_k\}) \ge \sum_{k=0}^n \mu_{i_k} E_k$.

Then there is a perfect probability measure λ on Z such that $z \mapsto z(i) : Z \to X_i$ is inverse-measure-preserving for every $i \in I$.

proof (a) Apply 457Cb to the coordinate maps $f_1 : Z \to X$ and $f_2 : Z \to Y$. The condition (ii) here shows that 457C(b-ii) is satisfied, so there is an additive functional $\theta : \mathcal{P}Z \to [0,1]$ such that $\theta f_1^{-1}[E] = \mu E$ for every $E \in \Sigma$ and $\theta f_2^{-1}[F] = \nu F$ for every $F \in \mathrm{T}$.

Define $\theta' : \Sigma \otimes \mathrm{T} \to [0,1]$ by setting $\theta' W = \theta(Z \cap W)$ for every $W \in \Sigma \otimes \mathrm{T}$. Then $\theta'(E \times Y) = \mu E$ for every $E \in \Sigma$ and $\theta'(X \times F) = \nu F$ for every $F \in \mathrm{T}$. Because μ is perfect, θ' has an extension to a measure $\tilde{\lambda}$ defined on $\Sigma \widehat{\otimes} \mathrm{T}$ (454C). Now Z is supposed to be expressible as $\bigcap_{n \in \mathbb{N}} W_n$ where $W_n \in \Sigma \otimes \mathrm{T}$ for every n; since

$$\tilde{\lambda} W_n = \theta' W_n = \theta(Z \cap W_n) = \theta Z = 1$$

for every n, $\tilde{\lambda} Z = 1$. So if we take λ to be the subspace measure on Z induced by $\tilde{\lambda}$, λ will be a probability measure on Z. If $E \in \Sigma$, then

$$\begin{aligned} \lambda(Z \cap (E \times Y)) &= \tilde{\lambda}(Z \cap (E \times Y)) = \tilde{\lambda}(E \times Y) \\ &= \theta'(E \times Y) = \theta(Z \cap (E \times Y)) = \mu E. \end{aligned}$$

So $f_1 : Z \to X$ is inverse-measure-preserving for λ and μ. Similarly, $f_2 : Z \to Y$ is inverse-measure-preserving for λ and ν.

(b) We use the same ideas, but appealing to 457B and 454D instead of 457Cb and 454C. Taking $f_i : X \to X_i$ to be the coordinate map for each $i \in I$, (ii) here, with 457B, tells us that there is an additive functional $\theta : \mathcal{P}Z \to [0,1]$ such that $\theta f_i^{-1}[E] = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$.

Define $\theta' : \bigotimes_{i\in I} \Sigma_i \to [0,1]$ by setting $\theta' W = \theta(Z \cap W)$ for every $W \in \bigotimes_{i\in I} \Sigma_i$. Then

$$\theta'\{x : x \in X,\, x(i) \in E\} = \theta\{z : z \in Z,\, z(i) \in E\} = \mu_i E$$

whenever $i \in I$ and $E \in \Sigma_i$. Because every μ_i is perfect, θ' has an extension to a perfect measure $\tilde\lambda$ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$ (454D). Now Z is supposed to be expressible as $\bigcap_{n\in\mathbb{N}} W_n$ where $W_n \in \bigotimes_{i\in I} \Sigma_i$ for every n; since

$$\tilde\lambda W_n = \theta' W_n = \theta(Z \cap W_n) = \theta Z = 1$$

for every n, $\tilde\lambda Z = 1$. So if we take λ to be the subspace measure on Z induced by $\tilde\lambda$, λ will be a probability measure on Z; by 451Dc, λ is perfect. If $i \in I$ and $E \in \Sigma_i$, then

$$\begin{aligned}\lambda\{z : z \in Z,\, z(i) \in E\} &= \tilde\lambda\{x : x \in X,\, x(i) \in E\} = \theta'\{x : x \in X,\, x(i) \in E\} \\ &= \theta\{z : z \in Z,\, z(i) \in E\} = \mu_i E.\end{aligned}$$

So $z \mapsto z(i) : Z \to X_i$ is inverse-measure-preserving for λ and μ_i for every $i \in I$, as required.

457G Theorem Let X be a set and $\langle\mu_i\rangle_{i\in I}$ a family of probability measures on X which is upwards-directed in the sense that for any i, $j \in I$ there is a $k \in I$ such that μ_k extends both μ_i and μ_j. Suppose that for any countable $J \subseteq I$ there is a measure on X extending μ_i for every $i \in J$. Then there is a measure on X extending μ_i for every $i \in I$.

proof Set $\Sigma_i = \operatorname{dom}\mu_i$ for each $i \in I$. Because $\langle\mu_i\rangle_{i\in I}$ is upwards-directed, $\mathrm{T} = \bigcup_{i\in I} \Sigma_i$ is an algebra of subsets of X, and we have a finitely additive functional $\nu : \mathrm{T} \to [0,1]$ defined by saying that $\nu E = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$. Now if $\langle E_n\rangle_{n\in\mathbb{N}}$ is any non-increasing sequence in T with empty intersection, there is a countable set $J \subseteq I$ such that $E_n \in \bigcup_{i\in J} \Sigma_i$ for every $n \in \mathbb{N}$. We are told that there is a measure λ on X extending μ_i for every $i \in J$; now $\nu E_n = \lambda E_n$ for every $n \in \mathbb{N}$, so $\lim_{n\to\infty} \nu E_n = 0$. By 413Lb, ν has an extension to a measure on X, which of course extends every μ_i.

457H Example Set $X = \{(x,y) : 0 \le x < y \le 1\} \subseteq [0,1]^2$. Write π_1, $\pi_2 : X \to \mathbb{R}$ for the coordinate maps, and μ_L for Lebesgue measure on $[0,1]$, with Σ_L its domain.

(a) There is a finitely additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu\pi_i^{-1}[E] = \mu_L E$ whenever $i \in \{1,2\}$ and $E \in \Sigma_L$. **P** If E_1, $E_2 \in \Sigma_L$ and $\mu_L E_1 + \mu_L E_2 > 1$, then neither is empty and $\inf E_1 < \sup E_2$, so there are $x \in E_1$, $y \in E_2$ such that $x < y$, and $(x,y) \in \pi_1^{-1}[E_1] \cap \pi_2^{-1}[E_2]$. So the result follows by 457Cb. **Q**

(b) However, there is no measure μ on X for which both π_1 and π_2 are inverse-measure-preserving. **P?** If there were,

$$\int \pi_1(x,y)\mu(d(x,y)) = \int x\mu_L(dx) = \int y\mu_L(dy) = \int \pi_2(x,y)\mu(d(x,y))$$

by 235G; but $\pi_1(x,y) < \pi_2(x,y)$ for every $(x,y) \in X$, so this is impossible. **XQ**

(c) If we write $\mathrm{T}_i = \{\pi_i^{-1}[E] : E \subseteq [0,1] \text{ is Borel}\}$ for each i, then we have a measure ν_i with domain T_i defined by setting $\nu_i\pi_i^{-1}[E] = \mu_L E$ for each Borel set $E \subseteq [0,1]$. Now ν_1 and ν_2 have no common extension to a Borel measure on X, even though X is a Polish space and each ν_i is a compact measure, being inner regular with respect to the compact class $\mathcal{K}_i = \{\pi_i^{-1}[K] : K \subseteq \,]0,1[\, \text{ is compact}\}$. (The trouble is that $\mathcal{K}_1 \cup \mathcal{K}_2$ is *not* compact, so we cannot apply 457E.)

457I Example Let μ_L be Lebesgue measure on $[0,1]$ and Σ_L its domain. Set

$$X = \{(\xi_1,\xi_2,\xi_3) : 0 \le \xi_i \le 1 \text{ for each } i,\ \sum_{i=1}^3 \xi_i \le \tfrac{3}{2},\ \sum_{i=1}^3 \xi_i^2 \le 1\}.$$

For $1 \le i \le 3$ set $\pi_i(x) = \xi_i$ for $x = (\xi_1,\xi_2,\xi_3) \in X$.

(a) If $E_i \in \Sigma_L$ for $i \le 3$, then there is an $x \in X$ such that $\#(\{i : \pi_i(x) \in E_i\}) \ge \sum_{i=1}^3 \mu_L E_i$. **P** Set $\alpha_i = \inf(E_i \cup \{1\})$ for each i, and set

$$m = \lceil \sum_{i=1}^3 \mu_L E_i \rceil \le \lceil \sum_{i=1}^3 1 - \alpha_i \rceil = 3 - \lfloor \sum_{i=1}^3 \alpha_i \rfloor,$$

so that $\sum_{i=1}^3 \alpha_i < 4 - m$. Take $\xi_i \in E_i \cup \{1\}$ such that $\sum_{i=1}^3 \xi_i < 4 - m$. It will be enough to consider the case in which $\xi_1 \le \xi_2 \le \xi_3$.

(i) If $m = 1$, then $\sum_{i=1}^3 \xi_i < 3$ so $\xi_1 < 1$ and $\xi_1 \in E_1$. Set $x = (\xi_1, 0, 0)$; then $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) \ge 1 \ge \sum_{i=1}^3 \mu_L E_i.$$

(ii) If $m = 2$, then $\sum_{i=1}^3 \xi_i < 2$ so $\xi_2 < 1$ and $\xi_1 \in E_1$, $\xi_2 \in E_2$. Set $x = (\xi_1, \xi_2, 0)$. We have $\xi_1 + \xi_2 \le \frac{4}{3} \le \frac{3}{2}$. Also

$$\xi_2 \le \frac{1}{2}(\xi_2 + \xi_3) \le 1 - \frac{1}{2}\xi_1,$$

so

$$\xi_1^2 + \xi_2^2 \le \xi_1^2 + (1 - \frac{1}{2}\xi_1)^2 = 1 - \xi_1 + \frac{5}{4}\xi_1^2 \le 1$$

because $\xi_1 \le \frac{2}{3} \le \frac{4}{5}$. So $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) \ge 2 \ge \sum_{i=1}^3 \mu_L E_i.$$

(iii) If $m = 3$ then $\sum_{i=1}^3 \xi_i < 1$ so $\xi_i \in E_i$ for every i; set $x = (\xi_1, \xi_2, \xi_3)$. Since $\sum_{i=1}^3 \xi_i^2 \le \sum_{i=1}^3 \xi_i \le 1$, $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) = 3 \ge \sum_{i=1}^3 \mu_L E_i.$$

Putting these together, we have the result. **Q**

(b) There is no finitely additive functional ν on X such that $\nu\pi_i^{-1}[E] = \mu_L E$ for each i and every $E \in \Sigma_L$. **P?** Suppose there were. Set $\mathrm{T}_i = \{\pi_i^{-1}[E] : E \in \Sigma_L\}$ and $\nu_i = \nu \restriction \mathrm{T}_i$ for each i. Then ν_i is a probability measure on X; moreover, because X is compact, $\pi_i^{-1}[K]$ is compact for every compact $K \subseteq [0,1]$, so ν_i is inner regular with respect to the compact subsets of X. By 457E, the ν_i have a common extension to a countably additive measure μ. Now

$$\int_X \xi_1 + \xi_2 + \xi_3 \, \mu(dx) = 3\int_0^1 t \, dt = \frac{3}{2},$$

so we must have $\xi_1 + \xi_2 + \xi_3 = \frac{3}{2}$ for μ-almost every x; similarly,

$$\int_X \xi_1^2 + \xi_2^2 + \xi_3^2 \, \mu(dx) = 3\int_0^1 t^2 \, dt = 1,$$

so we must have $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$ for μ-almost every x. Since

$$(\frac{3}{2} - \xi_3)^2 = (\xi_1 + \xi_2)^2 \le 2(\xi_1^2 + \xi_2^2) \le 2(1 - \xi_3^2)$$

for almost every x, $\xi_3 - \xi_3^2 \ge \frac{1}{12}$ for almost every x, which is impossible, since $\mu\{x : \xi_3 \le \frac{1}{2} - \frac{1}{\sqrt{6}}\} > 0$. **XQ**

457J Example There are a set X and a family $\langle \mu_i \rangle_{i \in I}$ of probability measures on X such that (i) for every countable set $J \subseteq I$ there is a measure on X extending μ_i for every $i \in J$ (ii) there is no measure on X extending μ_i for every $i \in I$.

proof By 439Fc, there is an uncountable universally negligible subset of $[0,1]$. Because $[0,1]$ and $\mathcal{P}\mathbb{N}$ are uncountable Polish spaces, they have isomorphic Borel structures (424Cb), so there is an uncountable universally negligible set $X_0 \subseteq \mathcal{P}\mathbb{N}$. The map $a \mapsto \mathbb{N} \setminus a$ is an autohomeomorphism of $\mathcal{P}\mathbb{N}$, so $X_1 = \{\mathbb{N} \setminus a : a \in X_0\}$ is universally negligible, and $X = X_0 \cup X_1$ is universally negligible (439Cb).

For $n \in \mathbb{N}$, set $E_n = \{a : n \in a \in X\}$ and $\Sigma_n = \{\emptyset, E_n, X \setminus E_n, X\}$; note that, because X is closed under complementation, neither E_n nor $X \setminus E_n$ is empty, and we have a probability measure μ_n with domain Σ_n defined by setting $\mu_n E_n = \mu_n(X \setminus E_n) = \frac{1}{2}$. Next, for $a \in X$, set $\Sigma'_a = \{\emptyset, \{a\}, X \setminus \{a\}, X)$, and let μ'_a be the probability measure with domain Σ'_a defined by setting $\mu'_a\{a\} = 0$.

If $J \subseteq X$ is countable, then there is a probability measure on X extending μ_n for every $n \in \mathbb{N}$ and μ'_a for every $a \in J$. **P** Because X_0 is uncountable, there is a $b \in X_0$ such that neither b nor $b' = \mathcal{P}\mathbb{N} \setminus b$ belongs to J. Let μ be the probability measure with domain $\mathcal{P}X$ defined by setting $\mu\{b\} = \mu\{b'\} = \frac{1}{2}$; this extends all the μ_n and all the μ'_a for $a \in J$. **Q**

? Suppose, if possible, that μ is a measure on X extending every μ_n and every μ'_a. In this case, because μ extends every μ_n, its domain includes the Borel σ-algebra $\mathcal{B}$ of X, and $\mu\restriction\mathcal{B}$ is a Borel probability measure on X. Since X is universally negligible, there is a point $a \in X$ such that $\mu\{a\} > 0$; in which case μ cannot extend μ'_a. **X**

Thus the μ_n, μ'_a constitute a family of the kind required.

457K In addition to existence, we can ask for solutions to simultaneous-extension problems which are optimal in some sense; some transportation problems can be interpreted as questions of this kind. In this direction I give just one result, which is also connected to the ideas of §437.[22]

Definition (Bogachev 07, §8.10(viii)) Let (X, ρ) be a metric space. For quasi-Radon probability measures μ, ν on X, set

$$\rho_{\mathrm{W}}(\mu, \nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to \mathbb{R} \text{ is bounded and 1-Lipschitz}\}.$$

(Compare the metric ρ_{KR} of 437Qb. ρ_{W} is sometimes called the 'Wasserstein metric'.)

457L Theorem Let (X, ρ) be a metric space and P_{qR} the set of quasi-Radon probability measures on X; define ρ_{W} as in 457K.

(a) For all μ, ν and λ in P_{qR},

$$\rho_{\mathrm{W}}(\mu, \nu) = \rho_{\mathrm{W}}(\nu, \mu), \quad \rho_{\mathrm{W}}(\mu, \lambda) \le \rho_{\mathrm{W}}(\mu, \nu) + \rho_{\mathrm{W}}(\nu, \lambda),$$

$$\rho_{\mathrm{W}}(\mu, \nu) = 0 \text{ iff } \mu = \nu.$$

(b) (cf. Vasershtein 69) If μ, $\nu \in P_{\mathrm{qR}}$, then $\rho_{\mathrm{W}}(\mu, \nu) = \inf_{\lambda \in Q(\mu,\nu)} \int \rho(x, y)\lambda(d(x, y))$, where $Q(\mu, \nu)$ is the set of quasi-Radon probability measures on $X \times X$ with marginal measures μ and ν.

(c) In (b), if μ and ν are Radon measures, $Q(\mu, \nu)$ is included in $P_{\mathrm{R}}(X \times X)$, the space of Radon probability measures on $X \times X$, and is compact for the narrow topology on $P_{\mathrm{R}}(X \times X)$; and there is a $\lambda \in Q(\mu, \nu)$ such that $\rho_{\mathrm{W}}(\mu, \nu) = \int \rho(x, y)\lambda(d(x, y))$.

(d) If ρ is bounded, then ρ_{W} is a metric on P_{qR} inducing the narrow topology (definition: 437Jd).

proof (a) The first two clauses are immediate from the definition. For the third, observe that if $\mu \ne \nu$ then $\rho_{\mathrm{W}}(\mu, \nu) \ge \rho_{\mathrm{KR}}(\mu, \nu) > 0$ by 437R.

(b) Write $\zeta \in [0, \infty]$ for $\rho_{\mathrm{W}}(\mu, \nu)$, $\mathcal{L}^\infty_{\operatorname{dom}\mu}$ for the space of bounded $\operatorname{dom}\mu$-measurable functions from X to $\mathbb{R}$ and $\mathcal{L}^\infty_{\operatorname{dom}\nu}$ for the space of bounded $\operatorname{dom}\nu$-measurable functions from X to $\mathbb{R}$.

(i) We have

$$\zeta = \sup\{\int u\,d\mu + \int v\,d\nu : u \in \mathcal{L}^\infty_{\operatorname{dom}\mu}, v \in \mathcal{L}^\infty_{\operatorname{dom}\nu},$$
$$u(x) + v(y) \le \rho(x, y) \text{ for all } x, y \in X\}.$$

P(α) Suppose that $u \in \mathcal{L}^\infty_{\operatorname{dom}\mu}$, $v \in \mathcal{L}^\infty_{\operatorname{dom}\nu}$ and $u(x) + v(y) \le \rho(x, y)$ for all x, $y \in X$. Set

$$w(x) = \inf_{y \in X} \rho(x, y) - v(y)$$

for $x \in X$. Then $u(x) \le w(x)$ and $w(x) + v(x) \le 0$ for every x, so $u \le w \le -v$ and w is bounded; also w is 1-Lipschitz, because if x, $x' \in X$ then

[22] I am indebted to J.Pachl for leading me to this material.

$$w(x) - \rho(x, x') = \inf_{y \in X} \rho(x, y) - v(y) - \rho(x, x') \le \inf_{y \in X} \rho(x', y) - v(y) = w(x').$$

Accordingly

$$\int u\,d\mu + \int v\,d\nu \le \int w\,d\mu - \int w\,d\nu \le \zeta.$$

(β) In the other direction, given $\gamma < \zeta$, there is a bounded 1-Lipschitz function $u : X \to \mathbb{R}$ such that $|\int u\,d\mu - \int u\,d\nu| \ge \gamma$. Replacing u by $-u$ if necessary, we can arrange that $\int u\,d\mu - \int u\,d\nu \ge \gamma$. Now set $v = -u$; then $u(x) + v(y) \le \rho(x, y)$ for all x, y, and $\int u\,d\mu + \int v\,d\nu \ge \gamma$. **Q**

It follows that if $u \in \mathcal{L}^\infty_{\mathrm{dom}\,\mu}$, $v \in \mathcal{L}^\infty_{\mathrm{dom}\,\nu}$ and $u(x) + v(y) \le \beta\rho(x, y)$ for all x, $y \in X$, where $\beta > 0$, then

$$\int u\,d\mu + \int v\,d\nu = \beta(\int \frac{1}{\beta} u\,d\mu + \int \frac{1}{\beta} v\,d\nu) \le \beta\zeta.$$

(ii) $\int \rho\,d\lambda \ge \zeta$ for every $\lambda \in Q(\mu, \nu)$. **P** If $u \in \mathcal{L}^\infty_{\mathrm{dom}\,\mu}$, $v \in \mathcal{L}^\infty_{\mathrm{dom}\,\nu}$ and $u(x) + v(y) \le \rho(x, y)$ for all x, $y \in X$, then

$$\int u\,d\mu + \int v\,d\nu = \int u(x)\lambda(d(x, y)) + \int v(y)\lambda(d(x, y))$$

(235G)

$$\le \int \rho\,d\lambda$$

so (i) gives us the result. **Q**

If $\zeta = \infty$, we can stop; so henceforth suppose that ζ is finite.

(iii) Define $p : \ell^\infty(X \times X) \to [0, \infty[$ by setting

$$p(w) = \inf\{\alpha + \beta\zeta : \alpha,\ \beta > 0,\ w(x, y) \le \alpha + \beta\rho(x, y) \text{ for all } x,\ y \in X\}.$$

Then $p(w + w') \le p(w) + p(w')$ and $p(\alpha w) = \alpha p(w)$ whenever w, $w' \in \ell^\infty(X \times X)$ and $\alpha \in [0, \infty[$. For u, $v \in \mathbb{R}^X$ define $u \otimes v \in \mathbb{R}^{X \times X}$ by setting $(u \otimes v)(x, y) = u(x)v(y)$ for all x, $y \in X$ (cf. 253B); set

$$V = \{(u \otimes \chi X) + (\chi X \otimes v) : u \in \mathcal{L}^\infty_{\mathrm{dom}\,\mu},\ v \in \mathcal{L}^\infty_{\mathrm{dom}\,\nu}\}.$$

Let $\mu \times \nu$ be the quasi-Radon product measure on $X \times X$ (417R). Then we have a linear functional $h_0 : V \to \mathbb{R}$ defined by saying that $h_0(w) = \int w\,d(\mu \times \nu)$ for $w \in V$. The point is that $h_0(w) \le p(w)$ for every $w \in V$. **P** We have $u \in \mathcal{L}^\infty_{\mathrm{dom}\,\mu}$, $v \in \mathcal{L}^\infty_{\mathrm{dom}\,\nu}$ such that $w(x, y) = u(x) + v(y)$ for all x, $y \in X$. If α, $\beta > 0$ are such that $w(x, y) \le \alpha + \beta\rho(x, y)$ for all x, $y \in X$, set $u_0(x) = u(x) - \alpha$ for every x; then $u_0(x) + v(y) \le \beta\rho(x, y)$ for all x and y, so

$$h_0(w) = \int u \otimes \chi X\,d(\mu \times \nu) + \int \chi X \otimes v\,d(\mu \times \nu) = \int u\,d\mu + \int v\,d\nu$$
$$= \alpha + \int u_0\,d\mu + \int v\,d\nu \le \alpha + \beta\zeta$$

by the last remark in (i). As α and β are arbitrary, $h_0(w) \le p(w)$. **Q**

(iv) By the Hahn-Banach theorem (3A5Aa), there is a linear functional $h : \ell^\infty(X \times X) \to \mathbb{R}$, extending h_0, such that $h(w) \le p(w)$ for every $w \in \ell^\infty(X \times X)$. In this case, h must be a positive linear functional, because if $w \ge 0$ then $p(-w) = 0$, so $h(-w) \le 0$. Since also

$$h(\chi(X \times X)) = h_0(\chi(X \times X)) = (\mu \times \nu)(X \times X) = 1,$$

$\|h\| = 1$ in $\ell^\infty(X \times X)^*$. If u, $v \in C_b(X)$ then

$$h(u \otimes \chi X) = h_0(u \otimes \chi X) = \int u\,d\mu, \quad h(\chi X \otimes v) = h_0(\chi X \otimes v) = \int v\,d\nu.$$

Let $\theta : \mathcal{P}(X \times X) \to [0, 1]$ be the additive functional defined by setting $\theta W = h(\chi W)$ for $W \subseteq X \times X$. Observe that $\theta(E \times X) = \mu E$ for every $E \in \mathrm{dom}\,\mu$ and $\theta(X \times E) = \nu E$ for every $E \in \mathrm{dom}\,\nu$.

(v) Because both μ and ν are inner regular with respect to the totally bounded sets (434L), there is a separable subset Y of X such that $\mu Y = \nu Y = 1$, and we can take Y to be a Borel set. Now let $\epsilon > 0$.

Then we have a countable partition $\langle E_i\rangle_{i\in I}$ of Y into non-empty Borel sets of diameter at most ϵ. For i, $j\in I$, set

$$\begin{aligned}\alpha_{ij}&=\frac{\theta(E_i\times E_j)}{\mu E_i\,\nu E_j}\text{ if }\mu E_i\cdot\nu E_j>0,\\&=0\text{ otherwise.}\end{aligned}$$

Since $\theta(E_i\times E_j)\le\min(\mu E_i,\nu E_j)$, $\theta(E_i\times E_j)=\alpha_{ij}\mu E_i\nu E_j$. If $i\in I$ is such that $\mu E_i>0$, then $\sum_{j\in I}\alpha_{ij}\nu E_j=1$. **P** For any $\eta>0$ there is a finite $K_0\subseteq I$ such that $\nu(X\setminus\bigcup_{j\in K_0}E_j)\le\eta$. Now

$$\begin{aligned}|1-\sum_{j\in K}\alpha_{ij}\nu E_j|\mu E_i&=|\mu E_i-\sum_{j\in K}\theta(E_i\times E_j)|=|\theta(E_i\times X)-\theta(E_i\times\bigcup_{j\in K}E_j)|\\&=\theta(E_i\times(X\setminus\bigcup_{j\in K}E_j))\le\theta(X\times(X\setminus\bigcup_{j\in K}E_j))\\&=\nu(X\setminus\bigcup_{j\in K}E_j)\le\eta\end{aligned}$$

whenever K is a finite subset of I including K_0; as η is arbitrary, $\mu E_i\cdot\sum_{j\in I}\alpha_{ij}\nu E_j=\mu E_i$ and $\sum_{j\in I}\alpha_{ij}\nu E_j=1$. **Q** Similarly, $\sum_{i\in I}\alpha_{ij}\mu E_i=1$ whenever $\nu E_j>0$.

(vi) Define a Borel measurable function $w_0:X\times X\to[0,\infty[$ by setting

$$\begin{aligned}w_0(x,y)&=\alpha_{ij}\text{ if }i,\,j\in I,\,x\in E_i\text{ and }y\in E_j,\\&=0\text{ if }(x,y)\in(X\times X)\setminus(Y\times Y).\end{aligned}$$

Let λ be the indefinite-integral measure over $\mu\times\nu$ defined by w_0; then λ is a quasi-Radon probability measure with marginals μ, ν. **P** If $E\in\operatorname{dom}\mu$, then

$$\begin{aligned}\lambda(E\times X)&=\int_{E\times X}w_0d(\mu\times\nu)=\sum_{i,j\in I}\int_{(E\cap E_i)\times E_j}w_0d(\mu\times\nu)\\&=\sum_{i,j\in I}\alpha_{ij}\mu(E\cap E_i)\cdot\nu E_j=\sum_{i\in I}\mu(E\cap E_i)\end{aligned}$$

(because $\sum_{j\in I}\alpha_{ij}\nu E_j=1$ whenever $\mu E_i>0$)

$$=\mu E.$$

In particular, $\lambda(X\times X)=1$, so λ is a probability measure, and is quasi-Radon by 415Ob; and the coordinate projection $(x,y)\mapsto x$ is inverse-measure-preserving for λ and μ. To see that μ is exactly the image measure, observe that if $E\subseteq X$ is such that $\lambda(E\times X)$ is defined, then $(E\cap E_i)\times E_j$ must be measured by $\mu\times\nu$ whenever $\alpha_{ij}>0$. For any $i\in I$ such that $\mu E_i>0$, there is surely some j such that $\alpha_{ij}>0$, in which case $E\cap E_i\in\operatorname{dom}\mu$; since $\bigcup_{i\in I}E_i$ is μ-conegligible (and μ is complete and I is countable), $E\in\operatorname{dom}\mu$. Thus μ is the marginal of λ on the first coordinate. Similarly, ν is the marginal of λ on the second coordinate. **Q**

For i, $j\in I$ we have

$$\lambda(E_i\times E_j)=\alpha_{ij}\mu E_i\cdot\nu E_j=\theta(E_i\times E_j).$$

(vii) $\int\rho\,d\lambda\le\zeta+2\epsilon$. **P** For i, $j\in I$, set

$$\beta_{ij}=\inf\nolimits_{x\in E_i,y\in E_j}\rho(x,y);$$

set

$$\begin{aligned}w(x,y)&=\beta_{ij}\text{ if }i,\,j\in I,\,x\in E_i\text{ and }y\in E_j,\\&=0\text{ if }(x,y)\in(X\times X)\setminus(Y\times Y).\end{aligned}$$

Then

$$w \le \rho \times \chi(Y \times Y) \le w + 2\epsilon\chi(X \times X),$$

so

$$\int \rho\,d\lambda = \int_{Y\times Y} \rho\,d\lambda \le 2\epsilon + \int w\,d\lambda$$
$$= 2\epsilon + \sum_{i,j\in I} \beta_{ij}\lambda(E_i \times E_j) = 2\epsilon + \sum_{i,j\in I} \beta_{ij}\theta(E_i \times E_j).$$

Now, for any finite $K \subseteq I$,

$$\sum_{i,j\in K} \beta_{ij}\theta(E_i \times E_j) = h(w \times \chi(\bigcup_{i,j\in K} E_i \times E_j)) \le h(\rho) \le p(\rho) \le \zeta$$

by the definition of p. So $\int \rho\,d\lambda \le 2\epsilon + \zeta$, as claimed. **Q**

(viii) As ϵ is arbitrary,

$$\inf_{\lambda\in Q(\mu,\nu)} \int \rho\,d\lambda \le \zeta.$$

With (ii), this completes the proof of (b).

(c) For every $\epsilon > 0$, there is a compact set $K \subseteq X$ such that $\mu(X \setminus K) + \nu(X \setminus K) \le \epsilon$. In this case $\lambda((X\times X)\setminus(K\times K)) \le \epsilon$ for every $\lambda \in Q(\mu,\nu)$. In the first place, this shows that if $\lambda \in Q(\mu,\nu)$, then λ is a Radon measure, by 416C(iv). Thus $Q(\mu,\nu) \subseteq P_{\mathrm{R}}(X \times X)$. Next, we see also that $Q(\mu,\nu)$ is uniformly tight (437O), therefore relatively compact in the space $M^+_{\mathrm{R}}(X \times X)$ of totally finite Radon measures on $X \times X$ (437P).

Writing π_1, π_2 for the coordinate projections from $X \times X$ to X, we see that

$$Q(\mu,\nu) = \{\lambda : \lambda \in M^+_{\mathrm{R}}(X \times X),\ \lambda\pi_1^{-1} = \mu \text{ and } \lambda\pi_2^{-1} = \nu\}.$$

Since the functions $\lambda \mapsto \lambda\pi_1^{-1}$ and $\lambda \mapsto \lambda\pi_2^{-1}$ from $M^+_{\mathrm{R}}(X \times X)$ to $M^+_{\mathrm{R}}(X)$ are continuous (437N), and $M^+_{\mathrm{R}}(X)$ is Hausdorff in its narrow topology (437R(a-ii)), $Q(\mu,\nu)$ is closed in $M^+_{\mathrm{R}}(X\times X)$, therefore compact.

Finally, the function $\lambda \mapsto \int \rho\,d\lambda$ from $M^+_{\mathrm{R}}(X \times X)$ to $[0,\infty]$ is lower semi-continuous (437Jg), and must attain its infimum on the compact set $Q(\mu,\nu)$ (4A2B(d-viii)). But (b) tells us that this infimum is just $\rho_{\mathrm{W}}(\mu,\nu)$.

(d)(i) Suppose first that $\rho(x,y) \le 2$ for all x, $y \in X$. Then $\rho_{\mathrm{W}} = \rho_{\mathrm{KR}}\restriction P_{\mathrm{qR}} \times P_{\mathrm{qR}}$. **P** As already noted in (a), $\rho_{\mathrm{W}}(\mu,\nu) \ge \rho_{\mathrm{KR}}(\mu,\nu)$ for all μ, $\nu \in P_{\mathrm{qR}}$. In the other direction, if μ, $\nu \in P_{\mathrm{qR}}$ and $u : X \to \mathbb{R}$ is 1-Lipschitz, then $|u(x) - u(y)| \le 2$ for all x, $y \in X$, so there is an $\alpha \in \mathbb{R}$ such that $|u(x) - \alpha| \le 1$ for all $x \in X$. Set $v(x) = u(x) - \alpha$ for every x; then $v : X \to [-1,1]$ is 1-Lipschitz, so

$$|\int u\,d\mu - \int u\,d\nu| = |\int v\,d\mu - \int v\,d\nu|$$

(because $\mu X = \nu X$)

$$\le \rho_{\mathrm{KR}}(\mu,\nu).$$

As u is arbitrary, $\rho_{\mathrm{W}}(\mu,\nu) \le \rho_{\mathrm{KR}}(\mu,\nu)$ and the two metrics are equal. **Q**

(ii) In general, take $\gamma > 0$ such that $\rho(x,y) \le 2\gamma$ for all x, $y \in X$. Set $\sigma = \frac{1}{\gamma}\rho$, so that σ is a metric on X equivalent to ρ. Now σ_{KR} defines the narrow topology on P_{qR}, by 437R(g-i), so $\rho_{\mathrm{W}} = \gamma\sigma_{\mathrm{W}} = \gamma\sigma_{\mathrm{KR}}\restriction P_{\mathrm{qR}}\times P_{\mathrm{qR}}$ also does.

457M If we relax our demands, and look for measures dominated by each measure in a family rather than extending them, similar methods give further results.

Theorem (see KELLERER 84) Let X be a Hausdorff space and $\langle\nu_i\rangle_{i\in I}$ a non-empty finite family of locally finite measures on X all inner regular with respect to the closed sets.

(a) For $A \subseteq X \times [0,\infty[$, set

$$c(A) = \inf\{\sum_{i\in I}\int h_i d\nu_i : h_i : X \to [0,\infty] \text{ is } \operatorname{dom}\nu_i\text{-measurable for each } i \in I,$$

$$\alpha \le \sum_{i\in I} h_i(x) \text{ whenever } (x,\alpha) \in A\}.$$

(i) c is a Choquet capacity (definition: 432J).

(ii) For every $A \subseteq X \times [0,\infty[$, the infimum in the definition of $c(A)$ is attained.

(b) Let $f : X \to [0,\infty[$ be a function such that $\{x : f(x) \ge \alpha\}$ is K-analytic for every $\alpha > 0$. Then

$$\inf\{\sum_{i\in I}\int h_i d\nu_i : h_i : X \to [0,\infty] \text{ is } \operatorname{dom}\nu_i\text{-measurable for each } i \in I,\ f \le \sum_{i\in I} h_i\}$$

$$= \sup\{\int f\,d\mu : \mu \text{ is a Radon measure on } X \text{ and } \mu \le \nu_i \text{ for every } i \in I\},$$

where '$\mu \le \nu_i$' here is to be interpreted in the sense of 234P.

proof (a)(i)($\boldsymbol{\alpha}$) For $f : X \to [0,\infty]$ set

$$\Omega_f = \{(x,\alpha) : x \in X,\ \alpha \le f(x)\}, \quad \Omega'_f = \{(x,\alpha) : x \in X,\ \alpha < f(x)\}$$

as in 252N.

It will be convenient to amalgamate the ν_i into a single measure, as follows. Let (Y, T, ν) be the direct sum of the family $\langle (X_i, \nu_i)\rangle_{i\in I}$ in the sense of 214L, so that $Y = X \times I$ and $\nu E = \sum_{i\in I} \nu_i\{x : (x,i) \in E\}$ for those $E \subseteq Y$ for which the sum is defined. Give Y its disjoint-union topology, that is, the product topology if I is given the discrete topology; then it is easy to check that ν is locally finite (see 411Xh) and inner regular with respect to the closed sets (see 412Xp). For $h \in [0,\infty]^Y$ and $x \in X$ set $(Th)(x) = \sum_{i\in I} h(x,i)$; observe that $T(h + h') = Th + Th'$ and $T(\alpha h) = \alpha Th$ for all h, $h' : Y \to [0,\infty]$ and $\alpha \ge 0$. Now, for any $A \subseteq X \times [0,\infty[$, we have

$$c(A) = \inf\{\int h\,d\nu : h : Y \to [0,\infty] \text{ is T-measurable},$$

$$\alpha \le Th(x) \text{ whenever } (x,\alpha) \in A\}$$

(because $\int h\,d\nu = \sum_{i\in I}\int h(x,i)\nu_i(dx)$ for non-negative h, by 214M)

$$= \inf\{\int h\,d\nu : h : Y \to [0,\infty] \text{ is T-measurable, } A \subseteq \Omega_{Th}\}.$$

($\boldsymbol{\beta}$) Of course $c : \mathcal{P}(X \times [0,\infty[) \to [0,\infty]$ is non-decreasing. To see that it is sequentially order-continuous on the left, I show in fact that if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of $X \times [0,\infty]$ with union A, and $\gamma = \sup_{n\in\mathbb{N}} c(A_n)$ is finite, then there is a T-measurable $h : Y \to [0,\infty]$ such that $\alpha \le Th(x)$ whenever $(x,\alpha) \in A$ and $\int h\,d\nu = \gamma$. **P** Surely $c(A) \ge \gamma$. For each $n \in \mathbb{N}$ we have a T-measurable $h_n : Y \to [0,\infty]$ such that $\int h_n d\nu \le \gamma + 2^{-n}$ and $A_n \subseteq \Omega_{Th_n}$. By Komlós's theorem (276H), there is a strictly increasing sequence $\langle n(k)\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\lim_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^m h_{n(k)}$ is defined ν-a.e.; set $h = \limsup_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^m h_{n(k)}$. Then $h : Y \to [0,\infty]$ is T-measurable, and $h =_{\text{a.e.}} \liminf_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^m h_{n(k)}$. By Fatou's Lemma,

$$\int h\,d\nu \le \liminf_{m\to\infty}\frac{1}{m+1}\sum_{k=0}^m \int h_{n(k)}d\nu \le \gamma,$$

while if $j \in \mathbb{N}$ and $(x,\alpha) \in A_j$. $\alpha \le Th_{n(k)}(x)$ for every $k \ge j$, so

$$\alpha \le \liminf_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} Th_{n(k)}(x) \le \limsup_{m\to\infty} \sum_{i\in I} \frac{1}{m+1} \sum_{k=0}^{m} h_{n(k)}(x,i)$$

$$\le \sum_{i\in I} \limsup_{m\to\infty} \frac{1}{m+1} \sum_{k=0}^{m} h_{n(k)}(x,i)$$

(because I is finite)

$$= \sum_{i\in I} h(x,i) = Th(x).$$

Thus $A \subseteq \Omega_{Th}$, so

$$c(A) \le \int h\,d\nu \le \gamma \le c(A)$$

and we have equality. **Q**

(γ) Now suppose that $K \subseteq X\times[0,\infty[$ is compact, and $\epsilon > 0$. Set $L = \pi_1[K]$, where $\pi_1 : X\times[0,\infty[\to X$ is the canonical map; then $L \subseteq X$ and $L \times I \subseteq Y$ are compact. Because ν is locally finite, there is an open set $H \subseteq Y$ such that $L \times I \subseteq H \in \mathrm{T}$ and νH is finite (see 411Ga). Let ν_H be the subspace measure induced by ν on H, and T_H its domain; then ν_H is totally finite and inner regular with respect to the closed sets (412Pc), therefore outer regular with respect to the open sets (411D). Let $h : Y \to [0,\infty]$ be a T-measurable function such that $A \subseteq \Omega_{Th}$ and $\int h\,d\nu \le c(K) + \epsilon$. Set $h_1(y) = h(y) + \frac{\epsilon}{\nu H}$ for $y \in H$; then $\int_H h_1 d\nu_H \le c(K) + 2\epsilon$. By 412Wa, there is a lower semi-continuous T_H-measurable $g_1 : H \to [0,\infty]$ such that $h_1 \le g_1$ and $\int_H g_1 d\nu_H \le c(K) + 3\epsilon$. Extend g_1 to a function $g : Y \to [0,\infty]$ by setting $g(y) = 0$ for $y \in Y \setminus H$; then g is T-measurable and lower semi-continuous and $\int g\,d\nu \le c(K) + 3\epsilon$. Moreover, if $(x,\alpha) \in K$, then

$$Tg(x) > T(h \times \chi H)(x) = Th(x)$$

(because $\{x\} \times I \subseteq H$)

$$\ge \alpha,$$

so $K \subseteq \Omega'_{Tg}$.

The point is that Ω'_{Tg} is open in $X \times [0,\infty[$. **P** If $x \in X$ and $0 \le \alpha < Tg(x) = \sum_{i\in I} g(x,i)$, let $\langle\alpha_i\rangle_{i\in I}$ be such that $0 \le \alpha_i < g(x,i)$ for each $i \in I$ and $\sum_{i\in I} \alpha_i = \alpha' > \alpha$. Set $G = \bigcap_{i\in I}\{z : z \in X,\ g(z,i) > \alpha_i\}$; then G is an open subset of X, and $(x,\alpha) \in G \times [0,\alpha'[\subseteq \Omega'_{Tg}$. Thus $(x,\alpha) \in \operatorname{int}\Omega'_g$; as (x,α) is arbitrary, Ω'_{Tg} is open. **Q**

Since $c(\Omega'_{Tg})$ is surely less than or equal to $\int g\,d\nu$, and ϵ is arbitrary, we have

$$c(K) = \inf\{c(U) : U \subseteq X \times [0,\infty[\text{ is open and } K \subseteq U\}.$$

Thus all the conditions of 432Ja are satisfied, and c is a Choquet capacity.

(ii) We need consider only the case $c(A) < \infty$, which is dealt with in (i-β) above, if we take $A_n = A$ for every n.

(b)(i) For $g : X \to [-\infty,\infty]$, set

$$p(g) = \inf\{\sum_{i\in I} \int h_i d\nu_i : h_i \in [0,\infty]^X \text{ is } \operatorname{dom}\nu_i\text{-measurable for each } i \in I,$$

$$|g| \le \sum_{i\in I} h_i\}$$

$$= \inf\{\int h\,d\nu : h : Y \to [0,\infty] \text{ is T-measurable, } \Omega_{|g|} \subseteq \Omega_{Th}\} = c(\Omega_{|g|}).$$

Then $p(\alpha g) = |\alpha| p(g)$ whenever $g \in [-\infty,\infty]^X$ and $\alpha \in \mathbb{R}$, $p(g_1) \le p(g_2)$ whenever $|g_1| \le |g_2|$, and $p(g_1 + g_2) \le p(g_1) + p(g_2)$ for all g_1, $g_2 : X \to [-\infty,\infty]$; so if we set $V = \{g : g \in \mathbb{R}^X,\ p(g) < \infty\}$, V is a solid linear subspace of $\mathbb{R}^X$ and $p \restriction V$ is a seminorm.

Suppose that μ is a Radon measure on X and $\mu \le \nu_i$ for every $i \in I$. Then $\int f d\mu \le p(f)$. **P** Because μ measures every K-analytic set (432A), $\int f d\mu$ is defined. If $p(f) = \infty$ then of course $\int f d\mu \le p(f)$. Otherwise, for any $\gamma > p(f)$, we have $\operatorname{dom}\nu_i$-measurable functions $h_i : X \to [0,\infty]$ such that $f \le \sum_{i \in I} h_i$ and $\sum_{i\in I} \int h_i d\nu_i \le \gamma$. But now $\int h_i d\mu$ is defined and less than or equal to $\int h_i d\nu_i$ for each i (234Qc), so $\int f d\mu \le \sum_{i\in I} \int h_i d\mu \le \gamma$. As γ is arbitrary, $\int f d\mu \le p(f)$. **Q**

(ii)(α) In the other direction, suppose that $\gamma < p(f)$, and set $A = \{(x,\alpha) : 0 < \alpha < f(x)\}$; then

$$A = \bigcup_{q\in\mathbb{Q}} \{(x,\alpha) : f(x) \ge q > \alpha > 0\}$$

is K-analytic (422Ge, 422Hc, 423Bb, 423C). On the other hand, for any $h : Y \to [0,\infty]$, $A \subseteq \Omega_{Th}$ iff $\Omega_f \subseteq \Omega_{Th}$. So

$$c(A) = c(\Omega_f) = p(f) > \gamma.$$

By Choquet's theorem 432K, there is a compact set $K \subseteq A$ such that $c(K) > \gamma$. Set

$$f_1(x) = \sup(\{0\} \cup K[\{x\}])$$

for $x \in X$. As in (a-i-γ) above, we have for any $i \in I$ an open set G including $L_0 = \pi_1[K]$ such that $\nu_i G$ is defined and finite, so χL_0 and f_1 belong to V. By the Hahn-Banach theorem (4A4Da), there is a linear functional $\theta : V \to \mathbb{R}$ such that $|\theta(g)| \le p(g)$ for every $g \in V$ and $\theta(f_1) = p(f_1)$. Since $|\theta(g)| \le p(g_0)$ whenever $|g| \le g_0$, θ is order-bounded, and if θ^+ is its positive part (355Eb), we shall still have $\theta^+(g) \le p(g)$ for every $g \in V$ and $\theta^+(f_1) = p(f_1)$.

(β) Set $\mu_0 C = \theta^+(\chi(C \cap L_0))$ for $C \subseteq X$. Then $\mu_0 : \mathcal{P}X \to [0,\infty[$ is additive. By 416K, there is a Radon measure μ on X such that $\mu L \ge \mu_0 L$ for every compact $L \subseteq X$ and $\mu G \le \mu_0 G$ for every open $G \subseteq X$. Now $\operatorname{dom}\nu_i \subseteq \operatorname{dom}\mu$ for every $i \in I$. **P** Suppose that $E \in \operatorname{dom}\nu_i$. Let $L \subseteq X$ be compact. Then there is an open set $G_0 \supseteq L$ such that $\nu_i G_0$ is defined and finite. Take any $\delta > 0$. Because the subspace measure induced by ν_i on G_0 is totally finite and inner regular with respect to the closed sets, there are a closed set F and an open set G, both measured by ν_i, such that $F \subseteq E \cap G_0 \subseteq G$ and $\nu_i(G \setminus F) \le \delta$. In this case

$$\mu(G \setminus F) \le \mu_0(G \setminus F) = \theta^+(\chi(L_0 \cap G \setminus F)) \le p(\chi(G \setminus F)) \le \int \chi(G \setminus F) d\nu_i \le \delta.$$

So

$$\mu^*(E \cap G_0) \le \mu G \le \mu F + \delta \le \mu_*(E \cap G_0) + \delta;$$

as δ is arbitrary, $\mu^*(E \cap G_0) = \mu_*(E \cap G_0)$ and μ measures $E \cap G_0$ (Ef), and therefore also measures $E \cap L = E \cap G_0 \cap L$. As L is arbitrary, μ measures E (412Ja). **Q**

In fact, $\mu \le \nu_i$. **P** If ν_i measures E and $L \subseteq X$ is compact, the arguments just above show that for any $\delta > 0$ there is an open set $G \supseteq E \cap L$ such that $\nu_i G \le \nu_i E + \delta$, so that

$$\mu(E \cap L) \le \mu G \le \mu_0 G = \theta^+(\chi(G \cap L_0)) \le p(\chi(G \cap L_0)) \le \nu_i G \le \nu_i E + \delta.$$

As L and δ are arbitrary, $\mu E \le \nu_i E$. **Q**

(γ) To estimate $\int f d\mu$, recall that $\theta^+(f_1) > \gamma$, while $\theta^+(\chi L_0)$ is finite. There is therefore an $\eta > 0$ such that $\eta\theta^+(\chi L_0) \le \theta^+(f_1) - \gamma$ and $\theta^+(f_2) \ge \gamma$, where $f_2 = (f_1 - \eta\chi L_0)^+ = (f_1 - \eta\chi X)^+$. For $k \in \mathbb{N}$ set $F_k = \pi_1[K \cap [(k+1)\eta,\infty[]$, so that each F_k is a compact subset of L_0 and $f_2 \le \sum_{k=0}^m \eta\chi F_k \le f$, where $m \in \mathbb{N}$ is such that $K \subseteq X \times [0, m\eta]$. Now

$$\begin{aligned}\gamma \le \theta^+(f_2) \le \theta^+(\sum_{k=0}^m \eta\chi F_k) &= \eta\sum_{k=0}^m \theta^+(\chi F_k)\\ &= \eta\sum_{k=0}^m \mu_0 F_k \le \eta\sum_{k=0}^m \mu F_k \le \int f d\mu.\end{aligned}$$

(δ) As γ is arbitrary,

$$\sup\{\int f d\mu : \mu \text{ is a Radon measure on } X,\ \mu \le \nu_i \text{ for every } i \in I\} \ge p(f)$$

and we must have equality. This completes the proof.

457N Remarks It may not be quite obvious how close the domination requirement '$\mu \le \nu_i$ for every $i \in I$' is to the marginal requirement '$\nu_i = \mu\pi_i^{-1}$ for every $i \in I$', so I spell out the correspondence. Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, $X = \prod_{i\in I} X_i$, and $\pi_i : X \to X_i$ the canonical map for each i.

(a) For each $i \in I$ we have a (unique) pull-back probability measure ν_i on X with domain $\{\pi_i^{-1}[E] : E \in \Sigma_i\}$ such that the image measure $\nu_i\pi_i^{-1}$ is μ_i (see 234F). Now it is elementary to check that, for a measure μ on X, $\mu \le \nu_i$ iff $\mu\pi_i^{-1} \le \mu_i$; and if μ is required to be a probability measure, then $\mu \le \nu_i$ iff μ extends ν_i iff $\mu\pi_i^{-1}$ extends μ_i.

(b) We find also that if $\mu \le \nu_i$ for every i, then there is a probability measure μ' on X such that $\mu \le \mu'$ and μ' extends ν_i for every i. **P** Set $\gamma = \mu X$. If $\gamma = 1$, set $\mu' = \mu$. Otherwise, for each $i \in I$, set $\lambda_i E = \frac{1}{1-\gamma}(\mu_i E - \mu\pi_i^{-1}[E])$ for $E \in \Sigma_i$. Then λ_i is a probability measure on X_i; let $\lambda = \prod_{i\in I} \lambda_i$ be the product measure, and set $\mu' = \mu + (1-\gamma)\lambda$. Then

$$\mu'\pi_i^{-1} = \mu\pi_i^{-1} + (1-\gamma)\lambda\pi_i^{-1} = \mu\pi_i^{-1} + (1-\gamma)\lambda_i = \mu_i$$

and μ' extends ν_i for each i. **Q**

(c) In the simplest intended applications, therefore, in which we have two Radon probability spaces (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) and a profit function $f : X \to [0, \infty[$, and we are looking for a Radon probability measure μ on $X = X_1 \times X_2$, with marginals μ_1 and μ_2, maximising $\int f d\mu$, then we can seek to apply 457Mb with the pull-back measures ν_1 and ν_2 of (a) here to see that the optimum is

$$\inf\{\int h_1 d\mu_1 + \int h_2 d\mu_2 : f(x_1, x_2) \le h_1(x_1) + h_2(x_2)\ \forall\, x_1 \in X_1,\ x_2 \in X_2\}.$$

If the process of part (b-ii) of the proof of 457M leads to a more or less optimal measure μ which is not itself a probability measure, we can increase it to μ' with $\mu'\pi_i^{-1}$ extending μ_i for each i; and in this case we shall have $\mu'\pi_i^{-1} = \mu_i$ for each i, by 418I and 416E, as usual. Of course we shall need to confirm that $\int f d\mu'$ is defined, but in the context of 457Mb, this will automatically be so.

(d) There is an obvious parallel between the formulae of 457M and that in part (b-i) of the proof of 457L. Allowing for the change of direction, where an infimum in 457L corresponds to a supremum in 457M, the pattern of the duality is the same in both cases, and there is some overlap (457Xq). But the arguments of the two theorems – in particular, the proofs that we can get countably additive measures from the finitely additive measures provided by the Hahn-Banach theorem – are rather different.

457X Basic exercises (a) Let X be a non-empty set and $\langle\nu_i\rangle_{i\in I}$ a family of probability measures on X satisfying the conditions of Lemma 457A, taking $\mathfrak{A} = \mathcal{P}X$ and $\mathfrak{B}_i = \operatorname{dom}\nu_i$ for each i. Suppose that there is a totally finite measure θ on X such that θE is defined and greater than or equal to $\nu_i E$ whenever $i \in I$ and ν_i measures E. Show that there is a measure on X extending every ν_i. (*Hint*: 391E.)

(b) Find a set X and non-negative additive functionals μ_1, μ_2 defined on subalgebras of $\mathcal{P}X$ which agree on $\operatorname{dom}\mu_1 \cap \operatorname{dom}\mu_2$ but have no common extension to a non-negative additive functional. (*Hint*: take $\#(X) = 3$.)

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\langle\nu_i\rangle_{i\in I}$ a family of non-negative finitely additive functionals, each ν_i being defined on a subalgebra $\mathfrak{B}_i$ of $\mathfrak{A}$. Show that if any finite number of the ν_i have a common extension to an additive functional on a subalgebra of $\mathfrak{A}$, then the whole family has a common extension to an additive functional on the whole algebra $\mathfrak{A}$.

(d) Set $X = \{0, 1, 2\}$ and in the algebra $\mathcal{P}X$ let $\mathfrak{B}_i$ be the subalgebra $\{\emptyset, \{i\}, X \setminus \{i\}, X\}$ for each i. Let $\nu_i : \mathfrak{B}_i \to [0, 1]$ be the additive functional such that $\nu_i\{i\} = \frac{1}{2}$, $\nu_i X = 1$. Show that any pair of ν_0, ν_1, ν_2 have a common extension to an additive functional on $\mathcal{P}X$, but that the three together have no such extension.

(e) Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\nu : \mathfrak{B} \to [0,\infty[$, $\theta : \mathfrak{A} \to [0,\infty[$ additive functionals such that $\nu b \le \theta b$ for every $b \in \mathfrak{B}$. Show directly, without using either 457D or 391F, that there is an additive functional $\mu : \mathfrak{A} \to [0,\infty[$, extending ν, such that $\mu a \le \theta a$ for every $a \in \mathfrak{A}$. (*Hint*: first consider the case in which $\mathfrak{A}$ is the algebra generated by $\mathfrak{B} \cup \{c\}$.)

>(f) Let $(Y_1, \mathfrak{S}_1, \mathrm{T}_1, \nu_1)$ and $(Y_2, \mathfrak{S}_2, \mathrm{T}_2, \nu_2)$ be Radon probability spaces and $X \subseteq Y_1 \times Y_2$ a closed set. Show that the following are equiveridical: (i) there is a measure on X such that the coordinate map from X to Y_i is inverse-measure-preserving for both i; (ii) there is a Radon measure on X such that the coordinate map from X to Y_i is inverse-measure-preserving for both i; (iii) for every compact $K \subseteq Y_1$, $\nu_1 K \le \nu_2^*(X[K])$. (*Hint*: for (iii)$\Rightarrow$(ii), use 457C to show that there is a finitely additive functional ν on $\mathcal{P}X$ of the required type; now observe that ν must give large mass to compact subsets of X, and apply 413U.)

>(g) Suppose that $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $I \subseteq \mathfrak{A}$ a finite set; let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by $I \cup \mathfrak{B}$ and $\nu : \mathfrak{C} \to [0,\infty[$ a finitely additive functional. (i) Show that if $\nu{\restriction}\mathfrak{B}$ is completely additive then ν is completely additive. (ii) Show that if $\mathfrak{A}$ is Dedekind σ-complete, $\mathfrak{B}$ is a σ-subalgebra and $\nu{\restriction}\mathfrak{B}$ is countably additive then ν is countably additive.

(h) Let (X, Σ, μ) be a probability space, $\mathcal{A}$ a finite family of subsets of X and T the subalgebra of $\mathcal{P}X$ generated by $\Sigma \cup \mathcal{A}$. Show that if $\nu : \mathrm{T} \to [0,1]$ is a finitely additive functional extending μ, then ν is countably additive.

(i) Let (X, Σ, μ) be a probability space, $\langle A_i \rangle_{i \in I}$ a partition of X and $\langle \alpha_i \rangle_{i \in I}$ a family in $[0,1]$ summing to 1. Show that the following are equiveridical: (i) there is a measure ν on X, extending μ, such that $\nu A_i = \alpha_i$ for every $i \in I$; (ii) there is a finitely additive functional $\nu : \mathcal{P}X \to [0,1]$, extending μ, such that $\nu A_i = \alpha_i$ for every $i \in I$; (iii) $\mu_*(\bigcup_{i \in J} A_i) \le \sum_{i \in J} \alpha_i$ for every $J \subseteq I$; (iv) $\mu^*(\bigcup_{i \in J} A_i) \ge \sum_{i \in J} \alpha_i$ for every finite $J \subseteq I$. (*Hint*: for (ii)$\Rightarrow$(i) use 457Xh.)

(j) Let $X \subseteq [0,1]^2$ be a Lebesgue measurable set such that $X \cap (E \times F)$ is not negligible for any non-negligible sets E, $F \subseteq [0,1]$. (For the construction of such sets, see the notes to §325.) Show that there is a Radon measure on X such that both the coordinate projections from X to $[0,1]$ are inverse-measure-preserving, where $[0,1]$ is given Lebesgue measure. (*Hint*: show that there is a measure-preserving bijection ϕ between conegligible subsets of $[0,1]$ which is covered by X; ϕ can be taken to be of the form $\phi(x) = x - \alpha_n$ for $x \in E_n$.)

(k) Set $X = \{(t, 2t) : 0 \le t \le \frac{1}{2}\} \cup \{(t, 2t-1) : \frac{1}{2} \le t \le 1\}$. Show that there is a Radon measure on X for which both the coordinate maps onto $[0,1]$ are inverse-measure-preserving, but that X does not include the graph of any measure-preserving bijection between conegligible subsets of $[0,1]$.

(l) Let X be the eighth-sphere $\{x : x \in [0,1]^3, \|x\| = 1\}$. Show that there is a measure on X such that all three coordinate maps from X onto $[0,1]$ are inverse-measure-preserving. (*Hint*: 265Xe.)

(m) Set $X = \{x : x \in [0,1]^3, \xi_1 + \xi_2 + \xi_3 = \frac{3}{2}\}$. Show that there is a measure on X such that all the coordinate maps from X onto $[0,1]$ are inverse-measure-preserving. (*Hint*: note that X is a regular hexagon; try one-dimensional Hausdorff measure on its boundary.)

(n) Explain how to adapt the example in 457J to provide a family $\langle \mu_i \rangle_{i \in I}$ of probability measures on a set X such that (i) $\langle \mu_i \rangle_{i \in I}$ is upwards-directed, in the sense of 457G (iii) there is no measure on X extending μ_i for every $i \in I$.

(o) Let X be a topological space and P_{qR} the set of quasi-Radon probability measures on X. For μ, $\nu \in P_{\mathrm{qR}}$, write $Q(\mu, \nu)$ for the set of quasi-Radon probability measures on $X \times X$ which have marginal measures μ on the first copy of X, ν on the second. (i) For a bounded continuous pseudometric ρ on X, set $\rho_{\mathrm{W}}(\mu, \nu) = \inf\{\int \rho(x,y)\lambda(d(x,y)) : \lambda \in Q(\mu, \nu)\}$. Show that ρ_{W} is a pseudometric on P_{qR}. (ii) Show that if X is completely regular and P is a family of bounded pseudometrics defining the topology of X, then $\{\rho_{\mathrm{W}} : \rho \in \mathrm{P}\}$ defines the narrow topology of P_{qR}.

(p) Suppose that X, $\langle\nu_i\rangle_{i\in I}$ and $c:\mathcal{P}(X\times[0,\infty[)\to[0,\infty]$ are as in 457M. (i) Show that c is a submeasure. (ii) Show that if every ν_i is outer regular with respect to the open sets, then c is an outer regular Choquet capacity.

(q) Show that if the metric ρ is bounded, then 457Lc can be deduced from 457Mb and part (b-i) of the proof of 457L.

(r) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\le n}$ be a finite family of Radon probability spaces, $X=\prod_{i\in I}X_i$, and $f:X\to\mathbb{R}$ a bounded Baire measurable function. Show that

$$\inf\{\int f d\mu : \mu \text{ is a Radon measure on } X \text{ with marginal measure } \mu_i \text{ on each } X_i\}$$
$$=\sup\{\sum_{i=0}^n\int h_i d\mu_i : h_i\in\ell^\infty(X_i) \text{ is } \Sigma_i\text{-measurable for each } i,$$
$$\sum_{i=0}^n h_i(\xi_i)\le f(x) \text{ whenever } x=(\xi_0,\dots,\xi_n)\in X\}.$$

(*Hint*: reduce to the case in which every X_i is K_σ.)

457Y Further exercises (a) Show that for any $n\ge 2$ there are a finite set X and a family $\langle\mu_i\rangle_{i\le n}$ of measures on X such that $\{\mu_i : i\le n,\, i\ne j\}$ have a common extension to a measure on X for every $j\le n$, but the whole family $\{\mu_i : i\le n\}$ has no such extension.

(b) Show that the example in 457H has the property: if f_i is a ν_i-integrable real-valued function for each i, and $\int f_1 d\nu_1+\int f_2 d\nu_2<1$, then there is an $(x,y)\in\operatorname{dom} f_1\cap\operatorname{dom} f_2$ such that $f_1(x,y)+f_2(x,y)<1$.

(c) Suppose we replace the set X in 457H with $X'=X\cup\{(x,x):x\in[0,\frac{1}{2}]\}$, and write ν_i' for the measures on X' defined by the coordinate projections. Show that (i) if f_i is a ν_i'-integrable real-valued function on X' for each i, and $\int f_1 d\nu_1'+\int f_2 d\nu_2'\le 1$, then there is an $(x,y)\in\operatorname{dom} f_1\cap\operatorname{dom} f_2$ such that $f_1(x,y)+f_2(x,y)\le 1$ (ii) there is no measure on X' extending both ν_i'.

(d) In 457Xm, show that there are many Radon measures on X such that all the coordinate maps from X onto $[0,1]$ are inverse-measure-preserving.

(e) Give an example of a compact Hausdorff space X, a sequence $\langle\nu_n\rangle_{n\in\mathbb{N}}$ of tight probability measures on X, and a K_σ set $E\subseteq X$ such that

$$\inf\{\textstyle\sum_{n=0}^\infty\int h_n d\nu_n : \chi E\le\sum_{n=0}^\infty h_n\}=1,$$

$$\sup\{\mu E : \mu \text{ is a Radon measure on } X \text{ and } \mu\le\nu_n \text{ for every } n\in\mathbb{N}\}\le\frac{1}{2}.$$

457Z Problems Give $[0,1]$ Lebesgue measure.

(a) Characterize the sets $X\subseteq[0,1]^2$ for which there is a measure on X such that both the projections from X to $[0,1]$ are inverse-measure-preserving.

(b) Set $X=\{x:x\in[0,1]^3,\ \|x\|=1\}$. Is there more than one Radon measure on X for which all three coordinate maps from X onto $[0,1]$ are inverse-measure-preserving? (See 457Xl, 457Yd.)

457 Notes and comments In the context of this section, as elsewhere (compare 391E-391G and 391J), finitely additive extensions, as in 457A-457D, generally present easier problems than countably additive extensions. So techniques for turning additive functionals into measures (391D, 413L, 413U, 416K, 454C, 454D, 457E, 457G, 457Lb, 457Mb, 457Xi) are very valuable. Note that 457D offers possibilities in this direction: if θ there is countably additive, μ also will be (457Xa).

457H and 457J demonstrate obstacles which can arise when seeking countably additive extensions even when finitely additive extensions give no difficulty. For finitely additive extensions a problem can arise at any finite number of measures (see 457Ya), but there is no further obstruction with infinite families (457Xc). For countably additive measures we have a positive result (457G) only under very restricted circumstances; relaxing any of the hypotheses can lead to failure (457J, 457Xn). Even in the apparently concrete case in which we have an open or closed set $X \subseteq [0,1]^2$ and we are seeking a measure on X with prescribed image measures on each coordinate, there can be surprises (457H, 457Xj, 457Xk), and I know of no useful description of the sets for which such a measure can be found (457Za).

The two-dimensional case has a special feature: when verifying the conditions (ii) or (iii) in 457A, or the condition (ii) of 457B, it is enough to consider only one set associated with each coordinate (457C). Put another way, in conditions (iv) and (v) of 457A it is enough to examine indicator functions. This is not the case as soon as we have three coordinates (457I). Compare 457A(ii)-(iii) with the definition of 'intersection number' of an indexed family in a Boolean algebra (391H), where we had to allow repetitions for essentially the same reason.

In 457K-457L, we can of course work with τ-additive Borel measures in place of quasi-Radon measures, as in 437M. The essential content of 457L is already displayed in the case of separable X, in which case all Borel measures are τ-additive, and we can fractionally simplify our hypotheses; indeed this is true whenever X has measure-free weight (438J).

The functional ρ_{W} of 457K-457L is a kind of $[0,\infty]$-valued metric; see 4A2T for another occasion on which it would have saved explanation if the definition of 'metric' allowed infinite distances. In 457Lb we think of the metric ρ as representing a cost to be minimised, and in 457Mb we think of f as a profit to be maximised; since both arguments rely on the functions being non-negative, they cannot be simply inverted unless ρ or f is bounded above (as in 457Xq), and there is a further complication from the asymmetric nature of the condition '$\{x : f(x) \ge \alpha\}$ is K-analytic' in 457M. However, for the primary applications, as in 457Xr, this is not a problem. Observe that the same pattern has already appeared in 457A(iv)-(v).

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458 Relative independence and relative products

Stochastic independence is one of the central concepts of probability theory, and pervades measure theory. I come now to a generalization of great importance. If X_1, X_2 and Y are random variables, we may find that X_1 and X_2 are 'relatively independent over Y', or 'independent when conditioned on Y', in the sense that if we know the value of Y, then we learn nothing further about one of the X_i if we are told the value of the other. For any stochastic process, where information comes to us piecemeal, this idea is likely to be fundamental. In this section I set out a general framework for discussion of relative independence (458A), introducing relative distributions (458I) and relative independence in measure algebras (458L-458M). In the last third of the section I look at 'relative product measures' (458N, 458Q), giving the basic existence theorems (458O, 458S, 458T).

458A Relative independence Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ.

(a) I say that a family $\langle E_i\rangle_{i\in I}$ in Σ is **relatively (stochastically) independent** over T if whenever $J \subseteq I$ is finite and not empty, and g_i is a conditional expectation of χE_i on T for each $i \in J$, then $\mu(F\cap\bigcap_{i\in J} E_i) = \int_F \prod_{i\in J} g_i d\mu$ for every $F \in \mathrm{T}$; that is, $\prod_{i\in J} g_i$ is a conditional expectation of $\chi(\bigcap_{i\in J} E_i)$ on T. (Note that this does not depend on which conditional expectations g_i we take, since any two conditional expectations of χE_i must be equal almost everywhere.) A family $\langle\Sigma_i\rangle_{i\in I}$ of subalgebras of Σ is **relatively independent** over T if $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i \in \Sigma_i$ for every $i \in I$.

(b) I say that a family $\langle f_i\rangle_{i\in I}$ in $\mathcal{L}^0(\mu)$ (the space of almost-everywhere-defined virtually measurable real-valued functions, or 'random variables') is **relatively independent** over T if $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T with respect to the completion of μ, where Σ_i is the σ-algebra defined by f_i in the sense of 272C, that is, the σ-algebra generated by $\{f_i^{-1}[F] : F \subseteq \mathbb{R}$ is a Borel set$\}$.

(c) I remark at once that a family of subalgebras or random variables is relatively independent iff every finite subfamily is (cf. 272Bb).

(d) It will be convenient to have a shorthand referring to lattices of σ-algebras of sets. If Σ, T are algebras of subsets of a set X, I will write $\Sigma \vee \mathrm{T}$ for the σ-algebra of subsets of X generated by $\Sigma \cup \mathrm{T}$; similarly, if $\langle \Sigma_i \rangle_{i \in I}$ is a family of algebras of subsets of X, then $\bigvee_{i \in I} \Sigma_i$ will be the σ-algebra generated by $\bigcup_{i \in I} \Sigma_i$. Note that the functions $\vee$, $\bigvee$ here are always supposed to yield σ-algebras, even if we start with algebras which are not closed under countable unions, so that $\Sigma \vee \Sigma$ could in principle be strictly larger than Σ. As will become evident in 458D and 458G, the difference between a σ-algebra and a simple algebra of sets is relatively unimportant just here.

458B There are some surprising results at the very beginning of the theory of relative independence; see 458Xa, for instance. On the positive side, we have the following facts.

Lemma Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle \Sigma_i \rangle_{i \in I}$ a family of subalgebras of Σ such that $\mathrm{T} \subseteq \bigcup_{i \in I} \Sigma_i$. Suppose that whenever $J \subseteq I$ is finite and not empty, $E_i \in \Sigma_i$ and g_i is a conditional expectation of χE_i on T for each $i \in J$, then $\mu(\bigcap_{i \in J} E_i) = \int \prod_{i \in J} g_i d\mu$. Then $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T.

proof Take $F \in \mathrm{T}$, a finite non-empty $J \subseteq I$ and $E_i \in \Sigma_i$ for $i \in J$. Let $j \in I$ be such that $F \in \Sigma_j$. Set $K = J \cup \{j\}$; if $j \notin J$, set $E_j = X$. Now set $E'_j = E_j \cap F$ and $E'_i = E_i$ for $i \in K \setminus \{j\}$.

For $i \in K$, let g_i be a conditional expectation of χE_i on T. Set $g'_j = g_j \times \chi F$ and $g'_i = g_i$ for $i \in K \setminus \{j\}$; then g'_i is a conditional expectation of $\chi E'_i$ for each $i \in K$. So we have

$$\mu(F \cap \bigcap_{i \in J} E_i) = \mu(\bigcap_{i \in K} E'_i) = \int \prod_{i \in K} g'_i d\mu = \int_F \prod_{i \in J} g_i d\mu.$$

As F and $\langle E_i \rangle_{i \in J}$ are arbitrary, $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T.

458C Proposition Let (X, Σ, μ) be a probability space, $\mathbb{T}$ a non-empty upwards-directed family of subalgebras of Σ, and $\langle \Sigma_i \rangle_{i \in I}$ a family of σ-subalgebras of Σ which is relatively independent over T for every $\mathrm{T} \in \mathbb{T}$. Then $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\bigvee \mathbb{T}$.

proof (a) Suppose first that $\mathbb{T}$ is countable; because it is upwards-directed, there is a non-decreasing sequence $\langle \mathrm{T}_n \rangle_{n \in \mathbb{N}}$ in $\mathbb{T}$ such that $\bigcup \mathbb{T} = \bigcup_{n \in \mathbb{N}} \mathrm{T}_n$ and $\bigvee \mathbb{T} = \bigvee_{n \in \mathbb{N}} \mathrm{T}_n$. Take a non-empty finite set $J \subseteq I$ and $E_i \in \Sigma_i$ for $i \in J$; set $E = \bigcap_{i \in J} E_i$. For $i \in J$, let g_{ni} be a conditional expectation of χE_i on T_n for each n; then $g_i = \lim_{n \to \infty} g_{ni}$ is a conditional expectation of χE_i on $\bigvee \mathbb{T}$ (275I). Similarly, if h_n is a conditional expectation of χE on T_n for each n, $h = \lim_{n \to \infty} h_n$ is a conditional expectation of χE on $\bigvee \mathbb{T}$. Since $\langle E_i \rangle_{i \in J}$ is relatively independent over T_n, $h_n =_{\text{a.e.}} \prod_{i \in J} g_{ni}$ for each n; accordingly $h =_{\text{a.e.}} \prod_{i \in J} h_i$, and $\prod_{i \in J} h_i$ is a conditional expectation of χE on $\bigvee \mathbb{T}$. As $\langle E_i \rangle_{i \in J}$ is arbitrary, $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\bigvee \mathbb{T}$.

(b) For the general case, take a non-empty finite $J \subseteq I$ and $E_i \in \Sigma_i$ for $i \in J$; set $E = \bigcap_{i \in I} E_i$. For each $i \in J$, let $g_i : X \to [0,1]$ be a $\bigvee \mathbb{T}$-measurable conditional expectation of χE_i on $\bigvee \mathbb{T}$, and $g : X \to [0,1]$ a $\bigvee \mathbb{T}$-measurable conditional expectation of χE on $\bigvee \mathbb{T}$. Then for every $i \in J$ and $q \in \mathbb{Q}$ there is a countable set $\mathbb{T}_{iq} \subseteq \mathbb{T}$ such that $\{x : g_i(x) \ge q\} \in \bigvee \mathbb{T}_{iq}$; similarly, there is for each $q \in \mathbb{Q}$ a countable set $\mathbb{T}'_q \subseteq \mathbb{T}$ such that $\{x : g(x) \ge q\} \in \bigvee \mathbb{T}'_q$. Let $\tilde{\mathbb{T}}$ be a countable upwards-directed subset of $\mathbb{T}$ including $\bigcup_{i \in J, q \in \mathbb{Q}} \mathbb{T}_{iq} \cup \bigcup_{q \in \mathbb{Q}} \mathbb{T}'_q$. Then every g_i is $\bigvee \tilde{\mathbb{T}}$-measurable, so is a conditional expectation of χE_i on $\bigvee \tilde{\mathbb{T}}$; similarly, g is a conditional expectation of χE on $\bigvee \tilde{\mathbb{T}}$. By (i), $g =_{\text{a.e.}} \prod_{i \in J} g_i$, so $\prod_{i \in J} g_i$ is a conditional expectation of χE on $\bigvee \mathbb{T}$. As $\langle E_i \rangle_{i \in J}$ is arbitrary, $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\bigvee \mathbb{T}$, as claimed.

458D Proposition Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ and $\langle \Sigma_i \rangle_{i \in I}$ a family of subalgebras of Σ which is relatively independent over T.

(a) If $J \subseteq I$ and Σ'_i is a subalgebra of Σ_i for $i \in J$, then $\langle \Sigma'_j \rangle_{i \in J}$ is relatively independent over T.

(b) Set $\Sigma^*_i = \Sigma_i \vee \mathrm{T}$ for $i \in I$. Then $\langle \Sigma^*_i \rangle_{i \in I}$ is relatively independent over T.

(c) If $\mathcal{E} \subseteq \bigcup_{i \in I} \Sigma_i$, then $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over the σ-algebra generated by $\mathrm{T} \cup \mathcal{E}$.

proof (a) Immediate from the definition in 458Aa.

(b)(i) Suppose that $F_0\in\mathrm{T}$ and that Σ'_i is the algebra generated by $\Sigma_i\cup\{F_0\}$ for each $i\in I$. Then $\langle\Sigma'_i\rangle_{i\in I}$ is relatively independent over T. **P** Suppose that $J\subseteq I$ is finite and not empty, and that $E_i\in\Sigma'_i$ for each $i\in J$. For $i\in I$, we can express E_i as $(G_i\cap F_0)\cup(H_i\setminus F_0)$, where G_i, $H_i\in\Sigma_i$. Let g_i, h_i be conditional expectations of χG_i, χH_i on T; then $f_i=g_i\times\chi F_0+h_i\times\chi(X\setminus F_0)$ is a conditional expectation of χE on T. Now, for any $F\in\mathrm{T}$, we have

$$\int_F\prod_{i\in J}f_i=\int_F\prod_{i\in J}g_i\times\chi F_0+\prod_{i\in J}h_i\times\chi(X\setminus F_0)$$
$$=\int_{F\cap F_0}\prod_{i\in J}g_i+\int_{F\setminus F_0}\prod_{i\in J}h_i=\mu(F\cap\bigcap_{i\in J}G_i\cap F_0)+\mu(F\cap\bigcap_{i\in J}H_i\setminus F_0)$$

(because the families $\langle G_i\rangle_{i\in J}$ and $\langle H_i\rangle_{i\in J}$ are both relatively independent over T)

$$=\mu(F\cap\bigcap_{i\in J}E_i).$$

As $\langle E_i\rangle_{i\in J}$ is arbitrary, $\langle\Sigma'_i\rangle_{i\in I}$ is relatively independent over T. **Q**

(ii) Suppose that $\mathcal{E}\subseteq\mathrm{T}$ is finite, and that Σ'_i is the algebra generated by $\Sigma_i\cup\mathcal{E}$ for each i. Then $\langle\Sigma'_i\rangle_{i\in I}$ is relatively independent over T. **P** Induce on $\#(\mathcal{E})$, using (i) for the inductive step. **Q**

(iii) Suppose that Σ'_i is the algebra generated by $\Sigma_i\cup\mathrm{T}$ for each $i\in I$. Then $\langle\Sigma'_i\rangle_{i\in I}$ is relatively independent over T. **P** If $J\subseteq I$ is finite and not empty, and $E_i\in\Sigma'_i$ for each $i\in J$, then there is a finite set $\mathcal{E}\subseteq\mathrm{T}$ such that E_i belongs to the algebra Σ''_i generated by $E_i\cup\mathcal{E}$ for every $i\in J$. By (ii), $\langle\Sigma''_i\rangle_{i\in I}$ is relatively independent over T, so $\langle E_i\rangle_{i\in J}$ is relatively independent over T; as $\langle E_i\rangle_{i\in J}$ is arbitrary, $\langle\Sigma'_i\rangle_{i\in I}$ is relatively independent over T. **Q**

(iv) Finally, suppose that $J\subseteq I$ is finite and not empty, that $E_i\in\Sigma^*_i$ for each $i\in J$, that $F\in\mathrm{T}$ and that $\epsilon>0$. For $i\in J$, let Σ'_i be the algebra generated by $\Sigma_i\cup\mathrm{T}$; then there is an $E'_i\in\Sigma'_i$ such that $\mu(E'_i\triangle E_i)\le\epsilon$ (136H). Let g_i, g'_i be conditional expectations of χE_i, $\chi E'_i$ on T; we can arrange that they are all defined on the whole of X and take values in $[0,1]$. Then

$$|\mu(F\cap\bigcap_{i\in J}E_i)-\int_F\prod_{i\in J}g_i|\le\sum_{i\in J}\mu(E_i\triangle E'_i)+|\mu(F\cap\bigcap_{i\in J}E'_i)-\int_F\prod_{i\in J}g'_i|$$
$$+\int_F|\prod_{i\in J}g'_i-\prod_{i\in J}g_i|$$
$$\le\epsilon\#(J)+0+\int_F\sum_{i\in J}|g'_i-g_i|$$

((iii) above and 285O)

$$\le\epsilon\#(J)+\sum_{i\in J}\int|g'_i-g_i|$$
$$\le\epsilon\#(J)+\sum_{i\in J}\int|\chi E'_i-\chi E_i|$$

(233J or 242Je)

$$=\epsilon\#(J)+\sum_{i\in J}\mu(E'_i\triangle E_i)\le 2\epsilon\#(J).$$

As ϵ is arbitrary,

$$\mu(F\cap\bigcap_{i\in J}E_i)=\int_F\prod_{i\in J}g_i.$$

As $\langle E_i\rangle_{i\in J}$ and F are arbitrary, $\langle\Sigma^*_i\rangle_{i\in I}$ is relatively independent.

(c) For any $\mathcal{E}\subseteq\bigcup_{i\in I}\Sigma_i$, write $\mathrm{T}_{\mathcal{E}}$ for the σ-algebra generated by $\mathrm{T}\cup\mathcal{E}$.

(i) Suppose that i, $j \in I$ are distinct, $E \in \Sigma_i$, g is a conditional expectation of χE on T, and $H \in \Sigma_j$. Then g is a conditional expectation of χE on $\mathrm{T}_{\{H\}}$. **P** Let h be a conditional expectation of χH on T. If $F \in \mathrm{T}$, then

$$\mu(F \cap H \cap E_i) = \int_F g \times h$$

(because Σ_j and Σ_i are relatively independent over T)

$$= \int_F g \times \chi H$$

(because $g \times h$ is a conditional expectation of $g \times \chi H$ on T, see 233Eg)

$$= \int_{F \cap H} g.$$

Similarly, $\mu(F \cap E_i \setminus H) = \int_{F \setminus H} g$. Now any $G \in \mathrm{T}_{\{H\}}$ is expressible as $(F_1 \cap H) \cup (F_2 \setminus H)$ where F_1, $F_2 \in \mathrm{T}$, so that

$$\mu(G \cap E) = \mu(F_1 \cap E \cap H) + \mu(F_2 \cap E \setminus H) = \int_{F_1 \cap H} g + \int_{F_2 \setminus H} g = \int_G g,$$

as required. **Q**

(ii) If $j \in I$ and $H \in \Sigma_j$, $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\mathrm{T}_{\{H\}}$.

P **(α)** Let $J \subseteq I$ be a non-empty finite set containing j, and $\langle E_i \rangle_{i \in J}$ a family such that $E_i \in \Sigma_i$ for $i \in J$. Set $K = J \setminus \{j\}$. For $i \in K$, let $g_i : X \to [0,1]$ be a T-measurable conditional expectation of χE_i on T. Then g_i is a conditional expectation of χE_i on $\mathrm{T}_{\{H\}}$, by (i). Let g_j be a conditional expectation of χE_j on $\mathrm{T}_{\{H\}}$, and g'_j a conditional expectation of $\chi(E_j \cap H)$ on T. Then, for any $F \in \mathrm{T}$,

$$\mu(F \cap H \cap \bigcap_{i \in J} E_i) = \mu(F \cap (E_j \cap H) \cap \bigcap_{i \in K} E_i) = \int_F g'_j \times \prod_{i \in K} g_i$$

(because $\langle \Sigma_i \rangle_{i \in J}$ is relatively independent over T)

$$= \int_F \chi(E_j \cap H) \times \prod_{i \in K} g_i$$

(233Eg again, because $\prod_{i \in K} g_i$ is bounded and T-measurable)

$$= \int_{F \cap H} \chi E_j \times \prod_{i \in K} g_i = \int_{F \cap H} g_j \times \prod_{i \in K} g_i$$

(because $\prod_{i \in K} g_i$ is bounded and $\mathrm{T}_{\{H\}}$-measurable). Similarly,

$$\mu(F \cap \bigcap_{i \in J} E_i \setminus H) = \int_{F \setminus H} g_j \times \prod_{i \in K} g_i = \int_{F \setminus H} \prod_{i \in J} g_i$$

for every $F \in \mathrm{T}$; putting these together, as in (i),

$$\mu(G \cap \bigcap_{i \in J} E_i) = \int_G \prod_{i \in J} g_i$$

for every $G \in \mathrm{T}_{\{H\}}$, and $\prod_{i \in J} g_i$ is a conditional expectation of $\chi(\bigcap_{i \in J} E_i)$ on $\mathrm{T}_{\{H\}}$.

(β) This is not exactly the formula demanded by the definition in 458Aa, because I supposed that $j \in J$; but if we have a non-empty finite $J \subseteq I \setminus \{j\}$ and $\langle E_j \rangle_{i \in J} \in \prod_{i \in J} \Sigma_j$, set $J' = J \cup \{j\}$ and $E_j = X$ to see that there is a family $\langle g_i \rangle_{i \in J'}$ such that g_i is a conditional expectation of χE_i on $\mathrm{T}_{\{H\}}$ for every i, and

$$\mu(G \cap \bigcap_{i \in J} E_i) = \mu(G \cap \bigcap_{i \in J'} E_i) = \int_G \prod_{i \in J'} g_i = \int_G \prod_{i \in J} g_i$$

for every $G \in \mathrm{T}_{\{H\}}$. So $\langle \Sigma_i \rangle_{i \in I}$ really is relatively independent over $\mathrm{T}_{\{H\}}$. **Q**

(iii) Inducing on $\#(\mathcal{E})$, we see that $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\mathrm{T}_{\mathcal{E}}$ whenever $\mathcal{E} \subseteq \bigcup_{i \in I} \Sigma_i$ is finite. By 458C, $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\mathrm{T}_{\mathcal{E}}$ for every $\mathcal{E} \subseteq \bigcup_{i \in I} \Sigma_i$.

Remark Putting (a) and (b) above together, we see that if $\langle\Sigma_i\rangle_{i\in I}$ is a family of subalgebras of Σ, and $\hat\Sigma_i$ is the σ-algebra generated by Σ_i for each i, then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\langle\hat\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

458E Example The simplest examples of relatively independent σ-algebras arise as follows. Let (X,Σ,μ) be a probability space, $\langle\mathrm{T}_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ, as in 272Ab, and T a σ-subalgebra of Σ which is independent of $\bigvee_{i\in I}\mathrm{T}_i$. For each $i\in I$, let Σ_i be $\mathrm{T}\vee\mathrm{T}_i$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

proof In view of 458Db, it is enough to show that $\langle\mathrm{T}_i\rangle_{i\in I}$ is relatively independent over T. But if we have a non-empty finite $J\subseteq I$ and $E_i\in\mathrm{T}_i$ for $i\in I$, then $\mu(E_i\cap F)=\mu E_i\cdot\mu F$ for $F\in\mathrm{T}$, so $f_i=\mu E_i\cdot\chi X$ is a conditional expectation of χE_i on T, for each i. Similarly, setting $E=\bigcap_{i\in J}E_i$, $\mu E\cdot\chi X$ is a conditional expectation of χE on T. Since $\mu E=\prod_{i\in J}\mu E_j$, $\prod_{i\in J}f_i$ is a conditional expectation of χE on T, which is what we need to know.

458F The following facts are elementary but occasionally useful.

Proposition Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ.

(a) Let $\langle f_i\rangle_{i\in I}$ be a family of non-negative μ-integrable functions on X which is relatively independent over T. For each $i\in I$ let g_i be a conditional expectation of f_i on T. Then for any $F\in\mathrm{T}$ and $i_0,\dots,i_n\in I$,

$$\int_F\prod_{j=0}^n g_{i_j}\le\int_F\prod_{j=0}^n f_{i_j}$$

with equality if all the i_j are distinct.

(b) Suppose that Σ_1, Σ_2 are σ-subalgebras of Σ which are relatively independent over T, and that $f\in\mathcal{L}^1(\mu\restriction\Sigma_1)$. If g is a conditional expectation of f on T, then it is a conditional expectation of f on $\mathrm{T}\vee\Sigma_2$.

proof (a) Let Σ_i be the σ-algebra generated by f_i for each i, so that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent (with respect to the completion of μ) over T.

(i) To begin with, suppose that $i_0,\dots,i_n$ are all different.

(α) If $f_i=\chi E_i$ for each $i\in I$, where $E_i\in\Sigma_i$, the result is just the definition of 'relative independence' in 458Aa.

(β) Because both sides of the desired equality are multilinear expressions of the inputs, and conditional expectation is an essentially linear operation, the same is true if all the f_i are simple functions.

(γ) For general non-negative integrable random variables f_i, let $\langle f_{ik}\rangle_{k\in\mathbb{N}}$ be a non-decreasing sequence of non-negative Σ_i-simple functions converging almost everywhere to f_i for each i, and g_{ik} a conditional expectation of f_{ik} for all i and k. Then $\langle g_{ik}\rangle_{k\in\mathbb{N}}$ is non-decreasing almost everywhere and converges a.e. to the given conditional expectation g_i of f_i. So

$$\int_F\prod_{j=0}^n g_{i_j}=\lim_{k\to\infty}\int_F\prod_{j=0}^n g_{i_jk}=\lim_{k\to\infty}\int_F\prod_{j=0}^n f_{i_jk}=\int_F\prod_{j=0}^n f_{i_j},$$

as required.

(ii)(α) Now suppose that the $i_0,\dots,i_n$ are not all distinct, but that all the f_{i_j} are bounded. Let $l_0,\dots,l_m$ enumerate $\{i_0,\dots,i_n\}$ and for $j\le m$ set $k_j=\#(\{r:i_r=l_j\})$. For each $j\le m$, let h_j be a conditional expectation of $f_{l_j}^{k_j}=|f_{l_j}|^{k_j}$. Because $t\mapsto|t|^{k_j}$ is convex, $g_{l_j}^{k_j}\le_{\text{a.e.}}h_j$ (233J). So

$$\int_F\prod_{j=0}^n g_{i_j}=\int_F\prod_{j=0}^m g_{l_j}^{k_j}\le\int_F\prod_{j=0}^m h_j=\int_F\prod_{j=0}^m f_{l_j}^{k_j}$$

(by part (i), because each $f_{l_j}^{k_j}$ is Σ_{l_j}-measurable)

$$=\int_F\prod_{j=0}^n f_{i_j},$$

as required.

(β) Finally, for the general case, take simple functions f_{ik} and conditional expectations g_{ik} as in (a-iii) above. Then

$$\int_F \prod_{j=0}^n g_{i_j} = \lim_{k\to\infty} \int_F \prod_{j=0}^n g_{i_j k} \le \lim_{k\to\infty} \prod_{j=0}^n \int_F f_{i_j k} = \int_F \prod_{j=0}^n f_{i_j},$$

and the proof is complete.

(b) Adjusting f on a negligible set if necessary, we may suppose that f is Σ_1-measurable. Take any $F \in \Sigma_2 \vee \mathrm{T}$, and let $h \ge 0$ be a conditional expectation of χF on T. By 458Db and 458Da, Σ_1 and $\Sigma_2 \vee \mathrm{T}$ are relatively independent over T, so f and χF are relatively independent over T. Accordingly

$$\int_F f = \int f \times \chi F = \int g \times h$$

(applying (a) to the positive and negative parts of f)

$$= \int g \times \chi F$$

(233K)

$$= \int_F g.$$

As F is arbitrary and

$$g \in \mathcal{L}^1(\mu\restriction\mathrm{T}) \subseteq \mathcal{L}^1(\mu\restriction\Sigma_2 \vee \mathrm{T}),$$

g is a conditional expectation of f on $\Sigma_2 \vee \mathrm{T}$.

Remark In (a), I have avoided speaking of conditional expectations of products $\prod_{j=0}^n f_{i_j}$ because these need not be integrable functions. But when $\prod_{j=0}^n f_{i_j}$ is integrable and has a conditional expectation g, then we must have $\prod_{j=0}^n g_{i_j} \le_{\text{a.e.}} g$, with equality almost everywhere when the i_j are distinct.

***458G** It is sometimes useful to know that 'relative independence' can be defined without using the apparatus of conditional expectations; indeed, we have a formulation which can be used with finitely additive functionals rather than measures.

Lemma Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ. Let $\mathbb{T}$ be the family of finite subalgebras of T. For $\Lambda \in \mathbb{T}$ write $\mathcal{A}_\Lambda$ for the set of non-negligible atoms in Λ. For non-empty finite $J \subseteq I$, $\langle E_i\rangle_{i\in J} \in \prod_{i\in J} \Sigma_i$ and $F \in \mathrm{T}$, set

$$\phi_\Lambda(F, \langle E_i\rangle_{i\in J}) = \sum_{H\in\mathcal{A}_\Lambda} \mu(H \cap F) \cdot \prod_{i\in J} \frac{\mu(E_i\cap H)}{\mu H}.$$

Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\lim_{\Lambda\in\mathbb{T},\Lambda\uparrow} \phi_\Lambda(F, \langle E_i\rangle_{i\in J}) = \mu(F \cap \bigcap_{i\in J} E_i)$ whenever $J \subseteq I$ is finite and not empty, $E_i \in \Sigma_i$ for every $i \in J$ and $F \in \mathrm{T}$.

proof (a) The point is just that if $J \subseteq I$ is finite and not empty, $E_i \in \Sigma_i$ for $i \in J$, g_i is a conditional expectation of χE_i on T for each i, and $F \in \mathrm{T}$, then $\int_F \prod_{i\in J} g_i d\mu = \lim_{\Lambda\uparrow} \phi_\Lambda(F, \langle E_i\rangle_{i\in J})$. **P** Adjusting each g_i on a negligible set if necessary, we may suppose that it is T-measurable, defined everywhere on X and takes values between 0 and 1.

Fix $n \in \mathbb{N}$ for the moment. Let Λ_n be the finite subalgebra of T generated by sets of the form $\{x : g_i(x) \le 2^{-n}k\}$ for $i \in J$ and $k \le 2^n$, and Λ any finite subalgebra of Σ_0 including Λ_n. If $H \in \mathcal{A}_\Lambda$ then there are integers k_{iH}, for $i \in J$, such that $2^{-n}k_{iH} \le g_i(x) < 2^{-n}(k_{iH} + 1)$ for every $i \in J$ and $x \in H$. So

$$2^{-n}k_{iH} \le \frac{\mu_i(E\cap H)}{\mu H} < 2^{-n}(k_{iH} + 1)$$

for each i. Accordingly

$$\sum_{H\in\mathcal{A}_\Lambda}\mu(H\cap F)\cdot\prod_{i\in J}2^{-n}k_{iH}\le\phi_\Lambda(F,\langle E_i\rangle_{i\in J})$$
$$\le\sum_{H\in\mathcal{A}_\Lambda}\mu(H\cap F)\cdot\prod_{i\in J}\min(1,2^{-n}(k_{iH}+1)),$$

that is,

$$\int_F\prod_{i\in J}g'_{in}d\mu\le\phi_\Lambda(F,\langle E_i\rangle_{i\in J})\le\int_F\prod_{i\in J}g''_{in}d\mu,$$

where $g'_{in}(x)=2^{-n}k$, $g''_{in}(x)=\min(1,2^{-n}(k+1))$ when $2^{-n}k\le g_i(x)<2^{-n}(k+1)$. But this means that

$$|\phi_\Lambda(F,\langle E_i\rangle_{i\in J})-\int_F\prod_{i\in J}g_id\mu|\le\int\prod_{i\in J}g''_{in}-\prod_{i\in J}g'_id\mu\le\int\sum_{i\in J}g''_{in}-g'_id\mu$$

(because all the g'_{in}, g''_{in} take values in $[0,1]$)

$$\le 2^{-n}\#(J).$$

Since this is true for every $\Lambda\supseteq\Lambda_n$ and every $n\in\mathbb{N}$, $\lim_{\Lambda\uparrow}\phi_\Lambda(F,\langle E_i\rangle_{i\in J})=\int_F\prod_{i\in J}g_id\mu$. **Q**

(b) Accordingly the condition given exactly matches the definition in 458A.

458H All the fundamental theorems concerning stochastic independence have relativized forms. A simple one is the following.

Proposition (Compare 272K.) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle\Sigma_i\rangle_{i\in I}$ be a family of σ-subalgebras of Σ which is relatively independent over T. Let $\langle I_j\rangle_{j\in J}$ be a partition of I, and for each $j\in J$ let $\tilde\Sigma_j$ be $\bigvee_{i\in I_j}\Sigma_i$.

(a) If $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T, then $\langle\tilde\Sigma_j\rangle_{j\in J}$ is relatively independent over T.

(b) Suppose that $\langle\tilde\Sigma_j\rangle_{j\in J}$ is relatively independent over T and that $\langle\Sigma_i\rangle_{i\in I_j}$ is relatively independent over T for every $j\in J$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

proof For each $E\in\Sigma$ let f_E be a conditional expectation of χE on T.

(a) Take any finite $K\subseteq J$, and let $\boldsymbol{W}$ be the set of families $\langle W_j\rangle_{j\in K}$ such that $W_j\in\tilde\Sigma_j$ for each $j\in K$ and $\mu(F\cap\bigcap_{j\in K}W_j)=\int_F\prod_{j\in K}f_{W_j}d\mu$ for every $F\in\mathrm{T}$. For each $j\in K$, let $\mathcal{C}_j$ be the family of measurable cylinders expressible as $W=X\cap\bigcap_{i\in L}E_i$ where $L\subseteq I_j$ is finite and $E_i\in\Sigma_i$ for $i\in L$. Note that in this case

$$\mu(F\cap W)=\mu(F\cap\bigcap_{i\in L}E_i)=\int_F\prod_{i\in L}f_{E_i}d\mu$$

for every $F\in\mathrm{T}$, so $f_W=_{\text{a.e.}}\prod_{i\in L}f_{E_i}$, taking the product to be χX if L is empty.

If $W_j\in\mathcal{C}_j$ for each $j\in K$, then $\langle W_j\rangle_{j\in K}\in\boldsymbol{W}$. **P** Express W_j as $X\cap\bigcap_{i\in L_j}E_i$ where $L_j\subseteq I_j$ is finite and $E_i\in\Sigma_i$ whenever $j\in K$ and $i\in L_j$. Then, setting $L=\bigcup_{j\in K}L_j$,

$$\mu(F\cap\bigcap_{j\in K}W_j)=\mu(F\cap\bigcap_{i\in L}E_i)=\int_F\prod_{i\in L}f_{E_i}d\mu$$

(because $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent)

$$=\int_F\prod_{j\in K}\prod_{i\in L_j}f_{E_i}d\mu=\int_F\prod_{j\in K}f_{W_i}d\mu$$

for every $F\in\mathrm{T}$. **Q**

Observe next that if we fix $k\in K$, and a family $\langle W_j\rangle_{j\in K\setminus\{k\}}$, then the set of those $W_k\in\tilde\Sigma_k$ such that $\langle W_j\rangle_{j\in K}\in\boldsymbol{W}$ is a Dynkin class, so if it includes $\mathcal{C}_k$ it must include the σ-algebra generated by $\mathcal{C}_k$, viz., $\tilde\Sigma_k$. Now an easy induction on n shows that if $\langle W_j\rangle_{j\in K}\in\prod_{j\in K}\tilde\Sigma_j$ and $\#(\{j:W_j\notin\mathcal{C}_j\})=n$, then $\langle W_j\rangle_{j\in K}\in\boldsymbol{W}$. Taking $n=\#(K)$ we see that $\prod_{j\in K}\tilde\Sigma_j\subseteq\boldsymbol{W}$.

As this is true for every finite $K \subseteq J$, $\langle \tilde{\Sigma}_j \rangle_{j \in J}$ is relatively independent over T, as claimed.

(b) This time, let $K \subseteq I$ be a non-empty finite set, and $E_i \in \Sigma_i$ for $i \in K$. Set $L = \{j : j \in J, K \cap I_j \ne \emptyset\}$, and for $j \in L$ set $G_j = \bigcap_{i \in K \cap I_j} E_i$; set $E = \bigcap_{i \in J} E_i = \bigcap_{j \in L} G_j$. Because $\langle \Sigma_i \rangle_{i \in I_j}$ is relatively independent over T, $f_{G_j} =_{\text{a.e.}} \prod_{i \in K \cap I_j} f_{E_i}$. Because $\langle \tilde{\Sigma}_j \rangle_{j \in J}$ is relatively independent over T,

$$f_E =_{\text{a.e.}} \prod_{j \in L} f_{G_j} =_{\text{a.e.}} \prod_{i \in K} f_{E_i}.$$

As $\langle E_i \rangle_{i \in K}$ is arbitrary, we have the result.

458I For the next result, we need a concept of 'relative probability distribution', as follows.

Definition Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $f \in \mathcal{L}^0(\mu)$. Then a **relative distribution** of f over T will be a family $\langle \nu_x \rangle_{x \in X}$ of Radon probability measures on $\mathbb{R}$ such that $x \mapsto \nu_x H$ is a conditional expectation of $\chi f^{-1}[H]$ on T for every Borel set $H \subseteq \mathbb{R}$.

458J Theorem Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $f \in \mathcal{L}^0(\mu)$. Then there is a relative distribution of f over T, which is essentially unique in the sense that if $\langle \nu_x \rangle_{x \in X}$ and $\langle \nu'_x \rangle_{x \in X}$ are two such relative distributions, then $\nu_x = \nu'_x$ for $\mu{\restriction}\mathrm{T}$-almost every x.

proof (a) Write μ_0 for the restriction of μ to T, $\hat{\mu}$ for the completion of μ, $\hat{\Sigma}$ for the domain of $\hat{\mu}$, and $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$. Then the function $x \mapsto (x, f(x)) : \operatorname{dom} f \to X \times \mathbb{R}$ is $(\hat{\Sigma}, \mathrm{T} \widehat{\otimes} \mathcal{B})$-measurable, just because $F \cap f^{-1}[H] \in \hat{\Sigma}$ for every $F \in \mathrm{T}$ and $H \in \mathcal{B}$. So we have a probability measure ν on $X \times \mathbb{R}$ defined by setting $\nu W = \hat{\mu}\{x : (x, f(x)) \in W\}$ for every $W \in \mathrm{T} \widehat{\otimes} \mathcal{B}$. The marginal measure on $\mathbb{R}$ is tight just because it is a Borel probability measure (433Ca). By 452M, we have a family $\langle \nu_x \rangle_{x \in X}$ of Radon probability measures on $\mathbb{R}$ such that $\nu W = \int \nu_x W[\{x\}] \mu_0(dx)$ for every $W \in \mathrm{T} \widehat{\otimes} \mathcal{B}$. In particular, if $H \in \mathcal{B}$,

$$\int_F \chi f^{-1}[H] d\mu = \hat{\mu}(F \cap f^{-1}[H]) = \nu(F \times H) = \int_F \nu_x H \, \mu(dx)$$

for every $F \in \mathrm{T}$, and $x \mapsto \nu_x H$ is a conditional expectation of $\chi f^{-1}[H]$ on T, that is, $\langle \nu_x \rangle_{x \in X}$ is a relative distribution of f over T.

(b) Now suppose that $\langle \nu'_x \rangle_{x \in X}$ is another relative distribution of f over T. Then for each $H \in \mathcal{B}$ we have $\int_F \nu_x H \mu(dx) = \int_F \nu'_x H \mu(dx)$ for every $F \in \mathrm{T}$, so that $\nu_x H = \nu'_x H$ for μ_0-almost every x. But this means that for μ_0-almost every x, we have $\nu_x H = \nu'_x H$ for every interval H with rational endpoints; and for such x we must have $\nu_x = \nu'_x$ (415H(v)).

458K Now we can state and prove a result corresponding to 272G.

Theorem Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_i \rangle_{i \in I}$ a family in $\mathcal{L}^0(\mu)$. For each $i \in I$, let $\langle \nu_{ix} \rangle_{x \in X}$ be a relative distribution of f_i over T, and $\tilde{f}_i : X \to \mathbb{R}$ an arbitrary extension of f_i to the whole of X. Then the following are equiveridical:

(i) $\langle f_i \rangle_{i \in I}$ is relatively independent over T;

(ii) for any Baire set $W \subseteq \mathbb{R}^I$ and any $F \in \mathrm{T}$,

$$\hat{\mu}(F \cap \boldsymbol{f}^{-1}[W]) = \int_F \lambda_x W \mu(dx),$$

where $\hat{\mu}$ is the completion of μ, $\boldsymbol{f}(x) = \langle \tilde{f}_i(x) \rangle_{i \in I}$ for $x \in X$, and λ_x is the product of $\langle \nu_{ix} \rangle_{i \in I}$ for each x;

(iii) for any non-negative Baire measurable function $h : \mathbb{R}^I \to \mathbb{R}$ and any $F \in \mathrm{T}$,

$$\int_F h\boldsymbol{f} d\mu = \int_F \int h \, d\lambda_x \mu(dx).$$

proof (a) Note first that if $i \in I$ and $H \subseteq \mathbb{R}$ is a Borel set, then $x \mapsto \nu_{ix} H$ is a conditional expectation of $\chi f_i^{-1}[H]$ on T, so that $\int_F \nu_{ix} H \mu(dx) = \hat{\mu}(F \cap f_i^{-1}[H])$ for every $F \in \mathrm{T}$.

Suppose that $\langle f_i \rangle_{i \in I}$ is relatively independent, and $F \in \mathrm{T}$. Let $\mathcal{C}$ be the family of Baire measurable cylinders of $\mathbb{R}^I$ expressible in the form $C = \{z : z \in \mathbb{R}^I, z(i) \in H_i$ for every $i \in J\}$ where $J \subseteq I$ is finite and $H_i \subseteq \mathbb{R}$ is a Borel set for each $i \in J$. For such a set C,

$$\hat{\mu}(F \cap \boldsymbol{f}^{-1}[C]) = \hat{\mu}(F \cap \bigcap_{i \in J} \tilde{f}_i^{-1}[H_i]) = \hat{\mu}(F \cap \bigcap_{i \in J} f_i^{-1}[H_i]) = \int_F \prod_{i \in J} \nu_{ix} H_i \mu(dx)$$

(interpreting an empty product as χX)

$$= \int_F \lambda_x C \mu(dx).$$

So

$$\mathcal{W} = \{W : W \subseteq \mathbb{R}^I,\ \hat{\mu}(F \cap \boldsymbol{f}^{-1}[W]) = \int \lambda_x W \mu(dx)\}$$

includes $\mathcal{C}$; since it is a Dynkin class, it contains every Baire subset of $\mathbb{R}^I$ (by the Monotone Class Theorem, 136B), and (ii) is true.

(b) Now suppose that (ii) is true. Let Σ_i be the σ-algebra defined by f_i for each i. If $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in J$, then there are Borel sets $H_i \subseteq \mathbb{R}$ such that $E_i \triangle f_i^{-1}[H_i]$ is negligible for each i, so that $x \mapsto \nu_{ix} H_i$ is a conditional expectation of χE_i on T. Now by the same equations as before, in the opposite direction,

$$\int_F \prod_{i \in J} \nu_{ix} H_i \mu(dx) = \int_F \lambda_x C \mu(dx)$$

(where $C = \{z : z(i) \in H_i \text{ for } i \in J\}$)

$$= \hat{\mu}(F \cap \boldsymbol{f}^{-1}[C]) = \hat{\mu}(F \cap \bigcap_{i \in J} f_i^{-1}[H_i]) = \hat{\mu}(F \cap \bigcap_{i \in J} E_i)$$

for every $F \in \mathrm{T}$. As $\langle E_i \rangle_{i \in J}$ is arbitrary, $\langle \Sigma_i \rangle_{i \in I}$ and $\langle f_i \rangle_{i \in I}$ are relatively independent.

(c) Thus (i)$\Leftrightarrow$(ii). For (ii)$\Rightarrow$(iii), observe that (ii) covers the case in which h is an indicator function χW; for the general case, express h as the supremum of a non-decreasing sequence of linear combinations of indicator functions, as usual. And (iii)$\Rightarrow$(ii) is trivial.

Remarks Of course the ungainly shift to $\tilde{f}_i$ is unnecessary if I is countable; but for uncountable I the intersection $\bigcap_{i \in I} \operatorname{dom} f_i$, which is the only suitable domain for $\boldsymbol{f}$, may not be conegligible.

I said that λ_x should be 'the product of $\langle \nu_{ix} \rangle_{i \in I}$'. Since the ν_{ix} are Radon probability measures, we have two possible interpretations of this: either the 'ordinary' product measure of §254 or the 'quasi-Radon' product measure of §417. But as we are interested only in the values of $\lambda_x W$ for Baire sets W, it makes no difference which we use.

458L Measure algebras We can look at the same ideas in the context of measure algebras. Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$.

(a) If $a \in \mathfrak{A}$, then we can say that $u \in L^\infty(\mathfrak{C})$ is the conditional expectation of χa on $\mathfrak{C}$ if $\int_c u = \bar{\mu}(c \cap a)$ for every $c \in \mathfrak{C}$ (365Q[23]). Now we can say that a family $\langle b_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\bar{\mu}(c \cap \inf_{i \in J} b_i) = \int_c \prod_{i \in J} u_i$ whenever $J \subseteq I$ is a non-empty finite set and u_i is the conditional expectation of χb_i on $\mathfrak{C}$ for every $i \in J$; while a family $\langle \mathfrak{B}_i \rangle_{i \in I}$ of subalgebras of $\mathfrak{A}$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\langle b_i \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$ whenever $b_i \in \mathfrak{B}_i$ for every $i \in I$.

Corresponding to 458Ab, we can say that a family $\langle w_i \rangle_{i \in I}$ in $L^0(\mathfrak{A})$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\langle \mathfrak{B}_i \rangle_{i \in I}$ is relatively stochastically independent, where $\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{[\![w_i > \alpha]\!] : \alpha \in \mathbb{R}\}$ for each i.

Returning to the original form of these ideas, we say that a family $\langle b_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **(stochastically) independent** if it is relatively independent over $\{0, 1\}$, that is, if $\bar{\mu}(\inf_{i \in J} b_i) = \prod_{i \in J} \bar{\mu} b_i$ whenever $J \subseteq I$ is finite. Similarly, a family $\langle \mathfrak{B}_i \rangle_{i \in I}$ of subalgebras of $\mathfrak{A}$ is (stochastically) independent, in the sense of 325L, iff it is relatively independent over $\{0, 1\}$ in the sense here.

[23] Formerly 365R.

(b) Let (X,Σ,μ) be a probability space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Let $\langle E_i\rangle_{i\in I}$, $\langle\Sigma_i\rangle_{i\in I}$ and $\langle f_i\rangle_{i\in I}$ be, respectively, a family in Σ, a family of subalgebras of Σ, and a family of μ-virtually measurable real-valued functions defined almost everywhere on X; let T be a σ-subalgebra of Σ. For $i\in I$, set $a_i=E_i^\bullet\in\mathfrak{A}$, $\mathfrak{B}_i=\{E^\bullet:E\in\Sigma_i\}$, and $w_i=f_i^\bullet\in L^0(\mathfrak{A})$, identified with $L^0(\mu)$ (364Ic). Set $\mathfrak{C}=\{F^\bullet:F\in\mathrm{T}\}$. Then

$\langle a_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle E_i\rangle_{i\in I}$ is relatively independent over T,

$\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T,

$\langle w_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ iff $\langle f_i\rangle_{i\in I}$ is relatively independent over T.

P The point is that if $f\in\mathcal{L}^1(\mu)$ (in particular, if $f=\chi E$ for some $E\in\Sigma$), and $g\in\mathcal{L}^1(\mu\restriction\mathrm{T})\subseteq\mathcal{L}^1(\mu)$ is a conditional expectation of f on T, then $g^\bullet$ is a conditional expectation of $f^\bullet$ on $\mathfrak{C}$; see 242J and 365Q. **Q**

(c) Corresponding to 458B, we see that if $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ such that $\mathfrak{C}\subseteq\bigcup_{i\in I}\mathfrak{A}_i$, and $\int\prod_{i\in J}u_i d\bar\mu=\bar\mu(\inf_{i\in J}a_i)$ whenever $J\subseteq I$ is finite and not empty and $a_i\in\mathfrak{A}_i$ and $u_i\in L^\infty(\mathfrak{C})$ is a conditional expectation of χa_i on $\mathfrak{C}$ for each $i\in J$, then $\langle\mathfrak{A}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$.

(d) Corresponding to 458Db, we see that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, and $\mathfrak{B}_i^*$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}_i\cup\mathfrak{C}$ for each i, then $\langle\mathfrak{B}_i^*\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$. The most natural proof, from where we are now standing, is to express $(\mathfrak{A},\bar\mu)$ as the measure algebra of a probability space (X,Σ,μ), set $\mathrm{T}=\{F:F^\bullet\in\mathfrak{C}\}$ and $\Sigma_i=\{E:E^\bullet\in\mathfrak{B}_i\}$ for each $i\in I$, and use 458D.

Corresponding to 458Dc, we see that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, $D_i\subseteq\mathfrak{B}_i$ for every $i\in I$, and $\mathfrak{D}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\bigcup_{i\in I}D_i$, then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$.

(e) Following 458H, we have the result that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$, and $\langle I_j\rangle_{j\in J}$ is a partition of I, and $\tilde{\mathfrak{B}}_j$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I_j}\mathfrak{B}_i$ for every $j\in J$, then $\langle\tilde{\mathfrak{B}}_j\rangle_{j\in J}$ is relatively independent over $\mathfrak{C}$.

(f) Note that if $a\in\mathfrak{A}$ and u is the conditional expectation of χa on $\mathfrak{C}$, then $[\![u>0]\!]=\mathrm{upr}(a,\mathfrak{C})$, by 365Qc. So if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, and $J\subseteq I$ is finite, and $b_i\in\mathfrak{B}_i$ for each $i\in J$, then $\inf_{i\in J}b_i=0$ iff $\inf_{i\in J}\mathrm{upr}(b_i,\mathfrak{C})=0$. (If u_i is a conditional expectation of χb_i on $\mathfrak{C}$ for each i, then

$$\inf_{i\in J}\mathrm{upr}(b_i,\mathfrak{C})=\inf_{i\in J}[\![u_i>0]\!]=[\![\textstyle\prod_{i\in J}u_i>0]\!]$$

is zero iff $\bar\mu(\inf_{i\in J}b_i)=\int\prod_{i\in J}u_i=0$.)

(g) We have a straightforward version of 458E, as follows. If $\langle\mathfrak{C}_i\rangle_{i\in I}$ is a stochastically independent family of closed subalgebras of $\mathfrak{A}$, $\mathfrak{C}$ is independent of the algebra generated by $\bigcup_{i\in I}\mathfrak{C}_i$, and $\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\mathfrak{C}_i$ for each i, then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$. (Either repeat the proof of 458E, looking at $\mathfrak{B}_{i0}=\mathfrak{C}_i$ and $\mathfrak{B}_{i1}=\mathfrak{C}$ for each i, or move to a measure space representing $\mathfrak{A}$ and quote 458E.)

(h) Similarly, we can translate 458F into this language. Let $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{C},\bar\mu\restriction\mathfrak{C})\subseteq L^1(\mathfrak{A},\bar\mu)$ be the conditional expectation operator associated with $\mathfrak{C}$ (365Q again). Suppose that $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of closed subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$. Then

$$\int_c\prod_{j=0}^n Pu_j\le\int_c\prod_{j=0}^n u_j$$

whenever $c\in\mathfrak{C}$, $i_0,\dots,i_n\in I$ and $u_j\in L^1(\mathfrak{B}_{i_j},\bar\mu\restriction\mathfrak{B}_{i_j})^+$ for each $j\le n$, with equality if $i_0,\dots,i_n$ are distinct.

458M Proposition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{B}$, $\mathfrak{C}$ closed subalgebras of $\mathfrak{A}$. Write $P_{\mathfrak{B}}$, $P_{\mathfrak{C}}$ and $P_{\mathfrak{B}\cap\mathfrak{C}}$ for the conditional expectation operators associated with $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{B}\cap\mathfrak{C}$. Then the following are equiveridical:

(i) $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{B}\cap\mathfrak{C}$;

(ii) $P_{\mathfrak{B}\cap\mathfrak{C}}(v\times w)=P_{\mathfrak{B}\cap\mathfrak{C}}v\times P_{\mathfrak{B}\cap\mathfrak{C}}w$ whenever $v\in L^\infty(\mathfrak{B})$ and $w\in L^\infty(\mathfrak{C})$;

(iii) $P_{\mathfrak{B}}P_{\mathfrak{C}}=P_{\mathfrak{B}\cap\mathfrak{C}}$;

(iv) $P_{\mathfrak{B}}P_{\mathfrak{C}} = P_{\mathfrak{C}}P_{\mathfrak{B}}$;
(v) $P_{\mathfrak{B}}u \in L^0(\mathfrak{C})$ for every $u \in L^1(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C})$.

proof Write P for $P_{\mathfrak{B}\cap\mathfrak{C}}$.

(i)⇒(ii) If (i) is true, $v \in L^\infty(\mathfrak{B})$ and $w \in L^\infty(\mathfrak{C})$, then $Pv \times Pw$ certainly belongs to $L^\infty(\mathfrak{B}\cap\mathfrak{C})$, and if $d \in \mathfrak{B}\cap\mathfrak{C}$, $\int_d Pv \times Pw = \int_d v \times w$ by 458Lh. So $Pv \times Pw = P(v \times w)$.

(ii)⇒(i) If (ii) is true, $b \in \mathfrak{B}$, $c \in \mathfrak{C}$ and $d \in \mathfrak{B}\cap\mathfrak{C}$, then

$$\bar{\mu}(d \cap b \cap c) = \int_d \chi b \times \chi c = \int_d P(\chi b \times \chi c) = \int_d P(\chi b) \times P(\chi c)$$

as required by the definition in 458La.

(ii)⇒(iii) Suppose that (ii) is true. First note that if $w \in L^\infty(\mathfrak{C})$ then $Pw = P_{\mathfrak{B}}w$. **P** Of course $Pw \in L^\infty(\mathfrak{B}\cap\mathfrak{C}) \subseteq L^\infty(\mathfrak{B})$. If $b \in \mathfrak{B}$, then

$$\int_b w = \int \chi b \times w = \int P(\chi b \times w) = \int P\chi b \times Pw = \int \chi b \times Pw$$

(365Qa)

$$= \int_b Pw,$$

so that Pw possesses the defining properties of $P_{\mathfrak{B}}w$. **Q**

But this means that if $u \in L^\infty(\mathfrak{A})$, $P_{\mathfrak{B}}P_{\mathfrak{C}}u = PP_{\mathfrak{C}}u$, which in turn is equal to Pu just because $\mathfrak{B}\cap\mathfrak{C} \subseteq \mathfrak{C}$ (see 233Eh). As u is arbitrary, $P_{\mathfrak{B}}P_{\mathfrak{C}}$ agrees with P on $L^\infty(\mathfrak{A})$; but $L^\infty(\mathfrak{A})$ is $\| \|_1$-dense in $L^1(\mathfrak{A}, \bar{\mu})$, and $P_{\mathfrak{B}}P_{\mathfrak{C}}$ and P are both $\| \|_1$-continuous, so they agree everywhere on $L^1(\mathfrak{A}, \bar{\mu})$ and are equal, as required by (iii).

(iii)⇒(ii) Suppose that (iii) is true, and that $v \in L^\infty(\mathfrak{B})$, $w \in L^\infty(\mathfrak{C})$ and $d \in \mathfrak{B}\cap\mathfrak{C}$. Then

$$\int_d Pv \times Pw = \int \chi d \times Pv \times Pw = \int \chi d \times v \times Pw$$

(because $\chi d \times Pw \in L^\infty(\mathfrak{B}\cap\mathfrak{C})$)

$$= \int \chi d \times v \times P_{\mathfrak{B}}w$$

(because $P_{\mathfrak{C}}w = w$)

$$= \int \chi d \times v \times w$$

(because $\chi d \times v \in L^\infty(\mathfrak{B})$)

$$= \int_d v \times w.$$

As d is arbitrary and $Pv \times Pw \in L^\infty(\mathfrak{B}\cap\mathfrak{C})$, $Pv \times Pw = P(v \times w)$.

(i)⇒(iv) follows immediately from (i)⇒(iii) and the symmetry of the relation '$\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{B}\cap\mathfrak{C}$'.

(iv)⇒(v) If (iv) is true and $u \in L^1(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C})$, then

$$P_{\mathfrak{B}}u = P_{\mathfrak{B}}P_{\mathfrak{C}}u = P_{\mathfrak{C}}P_{\mathfrak{B}}u \in L^0(\mathfrak{C}),$$

so (v) is true.

(v)⇒(iii) If (v) is true, and $u \in L^1_{\bar{\mu}}$, then $P_{\mathfrak{C}}u \in L^1(\mathfrak{C}, \bar{\mu}{\restriction}\mathfrak{C})$, so $P_{\mathfrak{B}}P_{\mathfrak{C}}u$ belongs to $L^0(\mathfrak{C}) \cap L^0(\mathfrak{B}) = L^0(\mathfrak{B}\cap\mathfrak{C})$, and of course

$$\int_d P_{\mathfrak{B}}P_{\mathfrak{C}}u = \int_d P_{\mathfrak{C}}u = \int_d u$$

for every $d \in \mathfrak{B}\cap\mathfrak{C}$. So $P_{\mathfrak{B}}P_{\mathfrak{C}}u = Pu$. As u is arbitrary, (iii) is true.

458N Relative free products of probability algebras: Definition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C},\bar\nu)$ a probability algebra, and suppose that we are given a measure-preserving Boolean homomomorphism $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ for each $i\in I$. A **relative free product** of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$ is a probability algebra $(\mathfrak{A},\bar\mu)$, together with a measure-preserving Boolean homomorphism $\phi_i:\mathfrak{A}_i\to\mathfrak{A}$ for each $i\in I$, such that

$\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{i\in I}\phi_i[\mathfrak{A}_i]$,

$\phi_i\pi_i=\phi_j\pi_j:\mathfrak{C}\to\mathfrak{A}$ for all i, $j\in I$,

writing $\mathfrak{D}$ for the common value of the $\phi_i[\pi_i[\mathfrak{C}]]$, $\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$.

Remark The homomorphisms π_i and ϕ_i are essential for the formal content of this definition, and will necessarily appear in the basic result 458O. But conceptually they are a nuisance; we should much prefer to think of every $\mathfrak{A}_i$ as a subalgebra of $\mathfrak{A}$, and of $\mathfrak{C}$ as actually equal to $\bigcap_{i\in I}\mathfrak{A}_i=\mathfrak{D}$. It may help if I spell out the key condition '$\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$' in terms of $\mathfrak{C}$ and the π_i.

The common value π of the $\phi_i\pi_i$ is a measure-preserving isomorphism between $\mathfrak{C}$ and $\mathfrak{D}$, so gives rise to an f-algebra isomorphism $S:L^0(\mathfrak{C})\to L^0(\mathfrak{D})$ such that $S(\chi c)=\chi(\pi c)$ for every $c\in\mathfrak{C}$ (364P); note that $S[L^\infty(\mathfrak{C})]=L^\infty(\mathfrak{D})$ and $\int Su\,d\bar\mu=\int u\,d\bar\nu$ for every $u\in L^1(\mathfrak{C})$ (365N[24]). If $u\in L^\infty(\mathfrak{C})$ and $d\in\mathfrak{D}$, then

$$\begin{aligned}\int_d Su\,d\bar\mu&=\int Su\times\chi d\,d\bar\mu=\int Su\times S(\chi(\pi^{-1}d))d\bar\mu\\&=\int S(u\times\chi(\pi^{-1}d))d\bar\mu=\int u\times\chi(\pi^{-1}d)\,d\bar\nu=\int_{\pi^{-1}d}u\,d\bar\nu.\end{aligned}$$

Next, for $i\in I$ and $a\in\mathfrak{A}_i$, we have a completely additive functional $c\mapsto\bar\mu_i(a\cap\pi_ic):\mathfrak{C}\to[0,1]$; let $u_{ia}\in L^\infty(\mathfrak{C})$ be a corresponding Radon-Nikodým derivative, so that $\int_c u_{ia}d\bar\nu=\bar\mu_i(a\cap\pi_ic)$ for every $c\in\mathfrak{C}$ (365E). (Thus $u_{ia}\in L^\infty(\mathfrak{C})$ corresponds to the conditional expectation of χa on the algebra $\pi_i[\mathfrak{C}]\subseteq\mathfrak{A}_i$.) The image Su_{ia} in $L^\infty(\mathfrak{D})$ is defined by the property

$$\int_d Su_{ia}d\bar\mu=\int_{\pi^{-1}d}u_{ia}d\bar\nu=\bar\mu_i(a\cap\pi_i(\phi_i\pi_i)^{-1}d)=\bar\mu_i(a\cap\phi_i^{-1}d)=\bar\mu(\phi_ia\cap d)$$

for every $d\in\mathfrak{D}$; that is, Su_{ia} is the conditional expectation of $\chi(\phi_ia)$ on $\mathfrak{D}$.

Note also that $\mathfrak{D}\subseteq\phi_i[\mathfrak{A}_i]$ for every $i\in I$. So we can use the criterion of 458B/458Lc to see that

$\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$

iff $\bar\mu(\inf_{i\in J}\phi_ia_i)=\int\prod_{i\in J}Su_{i,a_i}d\bar\mu$

whenever $J\subseteq I$ is finite and not empty and $a_i\in\mathfrak{A}_i$ for $i\in J$

iff $\bar\mu(\inf_{i\in J}\phi_ia_i)=\int\prod_{i\in J}u_{i,a_i}d\bar\nu$

whenever $J\subseteq I$ is finite and not empty and $a_i\in\mathfrak{A}_i$ for $i\in J$

because S is multiplicative, so we always have

$$\int\prod_{i\in J}Su_{i,a_i}d\bar\mu=\int S(\prod_{i\in J}u_{i,a_i})d\bar\mu=\int\prod_{i\in J}u_{i,a_i}d\bar\nu.$$

458O Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, $(\mathfrak{C},\bar\nu)$ a probability algebra and $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ a measure-preserving Boolean homomomorphism for each $i\in I$. Then $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ has an essentially unique relative free product over $(\mathfrak{C},\bar\nu)$.

proof (a)(i) Let $\mathfrak{B}$ be the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$ (315I); write $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{B}$ for the canonical embedding of $\mathfrak{A}_i$ in $\mathfrak{B}$. For each $i\in I$, $a\in\mathfrak{A}_i$ let $u_{ia}\in L^\infty(\mathfrak{C})$ be such that $\int_c u_{ia}d\bar\nu=\bar\mu_i(a\cap\pi_ic)$ for every $c\in\mathfrak{C}$ (365E again).

Because the map $a\mapsto u_{ia}:\mathfrak{A}_i\to L^\infty(\mathfrak{C})$ is additive for each i, 326E tells us that there is a unique additive functional $\lambda:\mathfrak{B}\to[0,1]$ such that

[24] Formerly 365O.

$$\lambda(\inf_{i\in J}\varepsilon_i a_i)=\int\prod_{i\in J}u_{i,a_i}d\bar\nu$$

whenever $J\subseteq I$ is a non-empty finite set and $a_i\in\mathfrak{A}_i$ for every $i\in J$. Of course $u_{i1}=\chi 1$ in $L^\infty(\mathfrak{C})$ for every $i\in I$, interpreting the '1' in u_{i1} in the Boolean algebra $\mathfrak{A}_i$, and the '1' in $\chi 1$ in the Boolean algebra $\mathfrak{C}$; so (this time interpreting '1' in $\mathfrak{B}$) $\lambda 1=1$ (the final '1' being a real number).

We see also that $u_{i,\pi_i c}=\chi c$ whenever $i\in I$ and $c\in\mathfrak{C}$. **P**

$$\int_d u_{i,\pi_i c}d\bar\nu=\bar\mu_i(\pi_i c\cap\pi_i d)=\bar\nu(c\cap d)=\int_d\chi c\,d\bar\nu$$

for every $d\in\mathfrak{D}$. **Q**

(ii) By 392I, there are a probability algebra $(\mathfrak{A},\bar\mu)$ and a Boolean homomorphism $\phi:\mathfrak{B}\to\mathfrak{A}$ such that $\lambda=\bar\mu\phi$. We can of course suppose that $\mathfrak{A}$ is the order-closed subalgebra of itself generated by $\phi[\mathfrak{B}]$ (which is in fact automatically the case if we use the construction in the proof of 392I).

For each $i\in I$, set $\phi_i=\phi\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$. This is a Boolean homomorphism because ϕ and ε_i are. If $a\in\mathfrak{A}_i$, then

$$\bar\mu\phi_i a=\bar\mu(\phi\varepsilon_i a)=\lambda\varepsilon_i a=\int u_{ia}d\bar\nu=\bar\mu_i(a\cap\pi_i 1)=\bar\mu_i a,$$

so ϕ_i is measure-preserving.

If i, $j\in I$ and $c\in\mathfrak{C}$ then

$$\begin{aligned}\bar\mu(\phi_i\pi_i c\,\triangle\,\phi_j\pi_j c)&=\lambda(\varepsilon_i\pi_i c\,\triangle\,\varepsilon_j\pi_j c)\\&=\lambda(\varepsilon_i\pi_i c)+\lambda(\varepsilon_j\pi_j c)-2\lambda(\varepsilon_i\pi_i c\cap\varepsilon_j\pi_j c)\\&=\int u_{i,\pi_i c}d\bar\nu+\int u_{j,\pi_j c}d\bar\nu-2\int u_{i,\pi_i c}\times u_{j,\pi_j c}d\bar\nu\\&=\int\chi c\,d\bar\nu+\int\chi c\,d\bar\nu-2\int\chi c\times\chi c\,d\bar\nu=0.\end{aligned}$$

So $\phi_i\pi_i=\phi_j\pi_j$. Let $\mathfrak{D}$ be the common value of $\phi_i[\pi_i[\mathfrak{C}]]$. (In the trivial case $I=\emptyset$, take $\mathfrak{D}=\mathfrak{A}=\{0,1\}$.)

(iii) Suppose that $J\subseteq I$ is finite and not empty and that $a_i\in\mathfrak{A}_i$ for each $i\in J$. Then

$$\bar\mu(\inf_{i\in J}\phi_i a_i)=\bar\mu(\inf_{i\in J}\phi\varepsilon_i a_i)=\bar\mu\phi(\inf_{i\in J}\varepsilon_i a_i)=\lambda(\inf_{i\in J}\varepsilon_i a_i)=\int\prod_{i\in J}u_{i,a_i}d\bar\nu.$$

But this is precisely the condition described in 458N, so $\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$, and $(\mathfrak{A},\bar\mu,\langle\phi_i\rangle_{i\in I})$ is a relative free product of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$.

(b) Now suppose that $(\mathfrak{A}',\bar\mu',\langle\phi'_i\rangle_{i\in I})$ is another relative free product of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$. Then we have a Boolean homomorphism $\psi:\mathfrak{B}\to\mathfrak{A}'$ such that $\phi'_i=\psi\varepsilon_i$ for every $i\in I$ (315J). In this case, $\bar\mu'\psi=\lambda$. **P** Let π' be the common value of $\phi'_i\pi_i$ for $i\in I$ and set $\mathfrak{D}'=\pi'[\mathfrak{C}]$ If $J\subseteq I$ is finite and not empty, and $a_i\in\mathfrak{A}_i$ for $i\in J$, then

$$\bar\mu'(\psi(\inf_{i\in J}\varepsilon_i a_i))=\bar\mu'(\inf_{i\in J}\phi'_i a_i)=\int\prod_{i\in J}u_{i,a_i}d\bar\nu=\lambda(\inf_{i\in J}\varepsilon_i a_i).$$

Because λ is the only additive functional on $\mathfrak{B}$ taking the right values on elements of this form, $\bar\mu'\psi=\lambda$. **Q**

In particular, $\psi b=0$ whenever $b\in\mathfrak{B}$ and $\lambda b=0$. It follows that $\psi b=\psi b'$ whenever b, $b'\in\mathfrak{B}$ and $\phi b=\phi b'$, since in this case $\lambda(b\,\triangle\,b')=\bar\mu(\phi b\,\triangle\,\phi b')$ is zero. So we have a function $\theta:\phi[\mathfrak{B}]\to\mathfrak{A}'$ defined by setting $\theta(\phi b)=\psi b$ for every $b\in\mathfrak{B}$, and of course θ is a Boolean homomorphism; moreover,

$$\bar\mu'\theta(\phi b)=\bar\mu'\psi b=\lambda b=\bar\mu\phi b$$

for every b, so θ is measure-preserving and an isometry for the measure metrics of $\mathfrak{A}$ and $\mathfrak{A}'$. If $i\in I$ and $a\in\mathfrak{A}$, then

$$\theta\phi_i a=\theta\phi\varepsilon_i a=\psi\varepsilon_i a=\phi'_i a,$$

so $\theta\phi_i=\phi'_i$ for every i. Because $\mathfrak{A}$ and $\mathfrak{A}'$ are the closed subalgebras generated by $\bigcup_{i\in I}\phi_i[\mathfrak{A}_i]$ and $\bigcup_{i\in I}\phi'_i[\mathfrak{A}_i]$ respectively, $\phi[\mathfrak{B}]$ and $\psi[\mathfrak{B}]$ are dense (323J). The isometry θ therefore extends uniquely to a measure algebra isomorphism $\hat\theta:\mathfrak{A}\to\mathfrak{A}'$ which must be the unique isomorphism such that $\hat\theta\phi_i=\phi'_i$ for every i. Thus $(\mathfrak{A},\bar\mu,\langle\phi_i\rangle_{i\in I})$ and $(\mathfrak{A}',\bar\mu',\langle\phi'_i\rangle_{i\in I})$ are isomorphic, and the relative free product is essentially unique.

458P Developing the argument of the last part of the proof of 458O, we have the following.

Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$, $\langle(\mathfrak{A}'_i,\bar\mu'_i)\rangle_{i\in I}$ be two families of probability algebras, and $\psi_i:\mathfrak{A}_i\to\mathfrak{A}'_i$ a measure-preserving Boolean homomorphism for each i. Let $(\mathfrak{C},\bar\nu)$, $(\mathfrak{C}',\bar\nu')$ be probability algebras and $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$, $\pi'_i:\mathfrak{C}'\to\mathfrak{A}'_i$ measure-preserving Boolean homomomorphisms for each $i\in I$; suppose that we have a measure-preserving isomorphism $\psi:\mathfrak{C}\to\mathfrak{C}'$ such that $\pi'_i\psi=\psi_i\pi_i:\mathfrak{C}\to\mathfrak{A}'_i$ for each i. Let $(\mathfrak{A},\bar\mu,\langle\phi_i\rangle_{i\in I})$ and $(\mathfrak{A}',\bar\mu',\langle\phi'_i\rangle_{i\in I})$ be relative free products of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$, $\langle(\mathfrak{A}'_i,\bar\mu'_i,\pi'_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$, $(\mathfrak{C}',\bar\nu')$ respectively. Then there is a unique measure-preserving Boolean homomorphism $\hat\psi:\mathfrak{A}\to\mathfrak{A}'$ such that $\hat\psi\phi_i=\phi'_i\pi_i:\mathfrak{A}_i\to\mathfrak{A}'$ for every $i\in I$.

proof By the uniqueness assertion of 458O, we may suppose that $(\mathfrak{A},\bar\mu,\langle\phi_i\rangle_{i\in I})$ has been constructed by the method of part (a) of the proof of 458O.

(a) For $i\in A$, $a\in\mathfrak{A}$ and $a'\in\mathfrak{A}'$ let $u_{ia}\in L^\infty(\mathfrak{C})$, $u'_{ia'}\in L^\infty(\mathfrak{C}')$ be defined as in (a-i) of the proof of 458O, so that

$$\int_c u_{ia}d\bar\nu=\bar\mu_i(a\cap\pi_ic),\qquad \int_{c'}u'_{ia'}d\bar\nu'=\bar\mu'_i(a'\cap\pi'_ic')$$

whenever $c\in\mathfrak{C}$ and $c'\in\mathfrak{C}'$. Let $T:L^0(\mathfrak{C})\to L^0(\mathfrak{C}')$ be the f-algebra isomorphism such that $T(\chi c)=\chi(\psi c)$ for every $c\in\mathfrak{C}$. Now $u'_{i,\psi_ia}=Tu_{ia}$ whenever $i\in I$ and $a\in\mathfrak{A}_i$. **P** If $c\in\mathfrak{C}$, then

$$\begin{aligned}\int_{\psi c}Tu_{ia}d\bar\nu'&=\int_c u_{ia}d\bar\nu=\bar\mu_i(a\cap\pi_ic)\\&=\bar\mu'_i\psi_i(a\cap\pi_ic)=\bar\mu'_i(\psi_ia\cap\pi'_i\psi c)=\int_{\psi c}u'_{i,\psi_ia}d\bar\nu'.\end{aligned}$$

Because ψ is surjective, it follows that $Tu_{ia}=u'_{i,\psi_ia}$. **Q**

(b) Let $\mathfrak{B}$ be the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$ and $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{B}$ the canonical embedding for each i; let λ be the functional on $\mathfrak{B}$ defined by the process of (a-i) in the proof of 458O. By 315J, there is a Boolean homomorphism $\theta:\mathfrak{B}\to\mathfrak{A}'$ such that $\theta\varepsilon_i=\phi'_i\psi_i:\mathfrak{A}_i\to\mathfrak{A}'$ for every i. Now $\bar\mu'\theta=\lambda$. **P** If $J\subseteq I$ is finite and $a_i\in\mathfrak{A}_i$ for every $i\in J$, then

$$\begin{aligned}\bar\mu'\theta(\inf_{i\in J}\varepsilon_ia_i)&=\bar\mu'(\inf_{i\in J}\phi'_i\psi_ia_i)=\int\prod_{i\in J}u'_{i,\psi_ia_i}d\bar\nu'\\&=\int\prod_{i\in J}Tu_{i,a_i}d\bar\nu'=\int T(\prod_{i\in J}u_{i,a_i})d\bar\nu'\\&=\int\prod_{i\in J}u_{i,a_i}d\bar\nu=\lambda(\inf_{i\in J}\varepsilon_ia_i).\end{aligned}$$

As λ, θ and $\bar\nu$ are all additive, $\lambda=\bar\mu'\theta$ (using 315Kb). **Q**

(c) Let $\phi:\mathfrak{B}\to\mathfrak{A}$ be the map described in (a-ii) of the proof of 458O. Then $\bar\mu(\phi b)=\lambda b=\bar\mu'(\theta b)$ for every $b\in\mathfrak{B}$; in particular,

$$\phi b=0\implies\bar\mu(\phi b)=0\implies\bar\mu'(\theta b)=0\implies\theta b=0.$$

There is therefore a Boolean homomorphism $\tilde\theta:\phi[\mathfrak{B}]\to\mathfrak{A}'$ such that $\tilde\theta\phi=\theta$, and $\tilde\theta$ is measure-preserving on $\phi[\mathfrak{B}]$. Since $\phi[\mathfrak{B}]$ is topologically dense in $\mathfrak{A}$ (use 323J), $\tilde\theta$ has an extension to a measure-preserving Boolean homomorphism $\hat\psi:\mathfrak{A}\to\mathfrak{A}'$ (324O). Now, for $i\in I$ and $a\in\mathfrak{A}_i$,

$$\hat\psi\phi_ia=\hat\psi\phi\varepsilon_ia=\tilde\theta\phi\varepsilon_ia=\theta\varepsilon_ia=\phi'_i\psi_ia,$$

as required.

(d) To see that $\hat\psi$ is unique, we need observe only that the given formula defines it on the subalgebra $\phi[\mathfrak{B}]$ and that this is topologically dense in $\mathfrak{A}$, while $\hat\psi$, being measure-preserving, must be continuous.

458Q Relative product measures: Definitions (a) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, Y a set, and $\pi_i : X_i \to Y$ a function for each $i \in I$. The **fiber product** of $\langle (X_i, \pi_i)\rangle_{i\in I}$ is the set $\Delta = \{x : x \in \prod_{i\in I} X_i$, $\pi_i x(i) = \pi_j x(j)$ for all i, $j \in I\}$.

(b) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces and (Y, T, ν) a probability space, and suppose that we are given an inverse-measure-preserving function $\pi_i : X_i \to Y$ for each $i \in I$; let Δ be the fiber product of $\langle (X_i, \pi_i)\rangle_{i\in I}$. A **relative product measure** on Δ is a probability measure μ on Δ such that

(†) whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for $i \in J$, and g_i is a Radon-Nikodým derivative with respect to ν of the functional $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0, 1]$ for each $i \in J$, then $\mu\{x : x \in \Delta,\ x(i) \in E_i$ for every $i \in J\}$ is defined and equal to $\int \prod_{i\in J} g_i d\nu$;

(‡) for every $W \in \Sigma$ there is a W' in the σ-algebra generated by $\{\{x : x \in \Delta,\ x(i) \in E\} : i \in I$, $E \in \Sigma_i\}$ such that $\mu(W\triangle W') = 0$.

Remark If μ is a relative product measure of $\langle (\mu_i, \pi_i)\rangle_{i\in I}$ over ν, then all the functions $x \mapsto x(i) : \Delta \to X_i$ are inverse-measure-preserving. **P** The condition (†) tells us that if $E \in \Sigma_i$ and g is any Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F])$, then

$$\mu\{x : x(i) \in E\} = \int g\, d\nu = \mu_i E. \ \mathbf{Q}$$

It follows that if I is not empty then we have an inverse-measure-preserving function $\pi : \Delta \to Y$ defined by setting $\pi x = \pi_i x(i)$ whenever $x \in \Delta$ and $i \in I$.

Note that when verifying (†) we need check the equality $\mu\{x : x \in \Delta,\ x(i) \in E_i$ for every $i \in J\} = \int \prod_{i\in J} g_i d\nu$ for only one representative family $\langle g_i\rangle_{i\in J}$ of Radon-Nikodým derivatives for any given $\langle E_i\rangle_{i\in J}$.

458R Proposition Suppose that $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ is a family of probability spaces, (Y, T, ν) a probability space, $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$, Δ the fiber product of $\langle (X_i, \pi_i)\rangle_{i\in I}$ and μ a relative product measure of $\langle (\mu_i, \pi_i)\rangle_{i\in I}$. Let $(\mathfrak{A}_i, \bar\mu_i)$, $(\mathfrak{C}, \bar\nu)$ and $(\mathfrak{A}, \bar\mu)$ be the measure algebras of μ_i, ν and μ respectively, and for $i \in I$ define $\bar\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ and $\bar\phi_i : \mathfrak{A}_i \to \mathfrak{A}$ by setting $\bar\pi_i F^\bullet = \pi_i^{-1}[F]^\bullet$, $\bar\phi_i E^\bullet = \{x : x \in \Delta,\ x(i) \in E\}^\bullet$ whenever $F \in \mathrm{T}$ and $E \in \Sigma_i$. Then $(\mathfrak{A}, \bar\mu, \langle\bar\phi_i\rangle_{i\in I})$ is a relative free product of $\langle (\mathfrak{A}_i, \bar\mu_i, \bar\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C}, \bar\nu)$.

proof The case $I = \emptyset$ is trivial (if you care to follow through the definitions to the letter, $\Delta = \prod_{i\in I} X_i = \{\emptyset\}$ and $\mathfrak{A}$ is the two-point algebra). So I will take it that I is not empty. For $i \in I$ define $\phi_i : \Delta \to X_i$ by setting $\phi_i(x) = x(i)$ for $x \in \Delta$.

(a) Of course we have to check that all the $\bar\pi_i$ and $\bar\phi_i$ are measure-preserving Boolean homomorphisms between the appropriate algebras, but in view of the remark following the definition 458Q, this is elementary. The condition that $\mathfrak{A}$ should be the closed subalgebra generated by $\bigcup_{i\in I}\bar\phi_i[\mathfrak{A}_i]$ is just a translation of the condition (‡).

(b) As I is not empty, we have a well-defined inverse-measure-preserving map $\pi : \Delta \to Y$ given by the formula $\pi(x) = \pi_i x(i)$ whenever $x \in \Delta$ and $i \in I$. Let $\bar\pi : \mathfrak{C} \to \mathfrak{A}$ be the corresponding measure-preserving homomorphism, so that $\bar\pi = \bar\phi_i\bar\pi_i$ for every i. Set $\mathfrak{D} = \bar\pi[\mathfrak{C}] \subseteq \mathfrak{A}$, and let $T : L^\infty(\mathfrak{C}) \to L^\infty(\mathfrak{D})$ be the f-algebra isomorphism corresponding to $\bar\pi$ (363F). For $i \in I$ and $E \in \Sigma_i$, let g_{iE} be a Radon-Nikodým derivative with respect to ν of the functional $F \mapsto \mu_i(E\cap\pi_i^{-1}[F])$, and set $u_{iE} = Tg_{iE}^\bullet \in L^\infty(\mathfrak{D})$. Then u_{iE} is the conditional expectation of $\chi\{x : x(i) \in E\}^\bullet$ on $\mathfrak{D}$. **P** If $d \in \mathfrak{D}$, it is of the form $\bar\pi F^\bullet = \bar\phi_i\bar\pi_i F^\bullet$ where $F \in \mathrm{T}$, so that $\chi d = T(\chi F)^\bullet$ and

$$\begin{aligned}\int_d u_{iE} d\bar\mu &= \int u_{iE} \times \chi d\, d\bar\mu = \int T(g_{iE}^\bullet \times \chi F^\bullet) d\bar\mu \\ &= \int g_{iE}^\bullet \times \chi F^\bullet d\bar\nu = \int_F g_{iE} d\nu \\ &= \mu_i(E \cap \pi_i^{-1}[F]) = \mu(\phi_i^{-1}[E] \cap \phi_i^{-1}[\pi_i^{-1}[F]]) = \bar\mu(d \cap \phi_i^{-1}[E]^\bullet).\end{aligned}$$

As d is arbitrary, we have the result. **Q**

(c) It follows that if $J \subseteq I$ is finite and not empty, and $a_i \in \bar\phi_i[\mathfrak{A}_i]$ and v_i is the conditional expectation of χa_i on $\mathfrak{D}$ for each $i \in J$, then $\bar\mu(\inf_{i\in J} a_i) = \int \prod_{i\in J} v_i d\bar\mu$. **P** Express a_i as $\phi_i^{-1}[E_i]^\bullet$, where $E_i \in \Sigma_i$, so that $v_i = u_{i,E_i}$ for each i. Then

$$\int \prod_{i\in J} v_i d\bar\mu = \int \prod_{i\in J} Tg^\bullet_{i,E_i} d\bar\mu = \int T(\prod_{i\in J} g^\bullet_{i,E_i}) d\bar\mu = \int \prod_{i\in J} g^\bullet_{i,E_i} d\bar\nu$$
$$= \int \prod_{i\in J} g_{i,E_i} d\nu = \mu(\bigcap_{i\in J} \phi_i^{-1}[E_i]) = \bar\mu(\inf_{i\in J} a_i). \text{ } \mathbf{Q}$$

But this is exactly what we need to know to see that $\langle\bar\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$, completing the proof that $(\mathfrak{A}, \bar\mu, \langle\bar\phi_i\rangle_{i\in I})$ is a relative free product of $\langle(\mathfrak{A}_i, \bar\mu_i, \bar\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C}, \bar\nu)$.

458S There is no general result on relative product measures to match 458O (see 458Xj-458Xm). The general question of when we can expect relative product measures to exist seems interesting (458Yf, 458Yg). Here I give a couple of sample results dealing with important special cases.

Proposition Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each i. Suppose that for each i we have a disintegration $\langle\mu_{iy}\rangle_{y\in Y}$ of μ_i such that $\mu^*_{iy}\pi_i^{-1}[\{y\}] = \mu_{iy}X_i = 1$ for every $y \in Y$. Let $\Delta \subseteq \prod_{i\in I} X_i$ be the fiber product of $\langle(X_i, \pi_i)\rangle_{i\in I}$, and Υ the subspace σ-algebra on Δ induced by $\widehat{\bigotimes}_{i\in I}\Sigma_i$. For $y \in Y$, let λ_y be the product of $\langle\mu_{iy}\rangle_{i\in I}$, $(\lambda_y)_\Delta$ the subspace measure on Δ and λ'_y its restriction to Υ. Then $\mu W = \int \lambda'_y W \nu(dy)$ is defined for every $W \in \Upsilon$, and μ is a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ over ν.

proof If $y \in Y$, then

$$\lambda'_y\Delta = \lambda^*_y\Delta = \lambda^*_y(\textstyle\prod_{i\in I} \pi_i^{-1}[\{y\}]) = 1$$

(254Lb). For $i \in I$ and $E \in \Sigma_i$ set $g_{iE}(y) = \mu_{iy}E$ when this is defined; then g_{iE} is a Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0,1]$ (452Qa). Write X for $\prod_{i\in I} X_i$; for $i \in I$ and $x \in X$ set $\phi_i(x) = x(i)$. If $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in J$, then

$$\int \lambda'_y(\Delta \cap \bigcap_{i\in J} \phi_i^{-1}[E_i])\nu(dy) = \int (\lambda_y)_\Delta(\Delta \cap \bigcap_{i\in J} \phi_i^{-1}[E_i])\nu(dy)$$
$$= \int \lambda_y(X \cap \bigcap_{i\in J} \phi_i^{-1}[E_i])\nu(dy)$$

(because $\lambda^*_y\Delta = 1$ and λ_y measures every $\phi_i^{-1}[E_i]$ for almost every y)

$$= \int \prod_{i\in J} \mu_{iy}E_i \nu(dy) = \int \prod_{i\in J} g_{i,E_i} d\nu.$$

In particular, $\int \lambda'_y(\Delta \cap \bigcap_{i\in J} \phi_i^{-1}[E_i])\nu(dy)$ is defined.

The set $\{W : W \subseteq X, \int \lambda'_y(W \cap \Delta)\nu(dy)$ is defined$\}$ is a Dynkin class of subsets of X containing $\bigcap_{i\in J} \phi_i^{-1}[E_i]$ whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for each $i \in J$; by the Monotone Class Theorem, it includes $\widehat{\bigotimes}_{i\in I}\Sigma_i$. So $\mu W = \int \lambda'_y W \nu(dy)$ is defined for every $W \in \Upsilon$. Moreover, the formula displayed above tells us that $\mu(\Delta \cap \bigcap_{i\in I} \phi_i^{-1}[E_i]) = \int \prod_{i\in J} g_{i,E_i} d\nu$ whenever $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in I$. Thus (†) of 458Q is satisfied. And (‡) is true by the choice of Υ.

458T The latitude I have permitted in the definition of 'relative product' makes it possible to look for relative product measures with further properties, as in the following.

Proposition Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of compact Radon probability spaces, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a Radon probability space, and $\pi_i : X_i \to Y$ a continuous inverse-measure-preserving function for each i. Then $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure μ over ν which is a Radon measure for the topology on the fiber product of $\langle(X_i, \pi_i)\rangle_{i\in I}$ induced by the product topology on $\prod_{i\in I} X_i$.

proof (a) For $i \in I$ and $E \in \Sigma_i$ let g_{iE} be a Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0,1]$. Let $\mathcal{C}$ be the family of measurable cylinders in $X = \prod_{i\in I} X_i$; write $\phi_i(x) = x(i)$ for $x \in X$ and $i \in I$. We have a functional $\lambda_0 : \mathcal{C} \to [0,1]$ defined by setting

$$\lambda_0(\bigcap_{i\in J}\phi_i^{-1}[E_i])=\int\prod_{i\in J}g_{i,E_i}d\nu$$

whenever $J\subseteq I$ is finite and not empty and $E_i\in\Sigma_i$ for every $i\in I$. It is easy to check that λ_0 is additive in the sense required by 454E so (because every μ_i is perfect, by 416Wa) it has an extension to a measure λ on X with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$. By 454Aa, with $\mathcal{K}$ the family of compact subsets of X, λ is inner regular with respect to the compact sets. By 416N, there is a Radon measure $\tilde\lambda$ on X extending λ.

(b) Let Δ be the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$. Now the point is that Δ is $\tilde\lambda$-conegligible. **P** Because every π_i is continuous, Δ is closed. **?** If it is not conegligible, then, because $\tilde\lambda$ is τ-additive, there must be a basic open set of non-zero measure disjoint from Δ; express such a set as $W=\prod_{i\in J}\phi_i^{-1}[G_i]$ where $J\subseteq I$ is finite and $G_i\subseteq X_i$ is open for each $i\in J$. Because $\tilde\lambda$ is inner regular with respect to the compact sets, there is a compact set $K\subseteq W$ such that $\tilde\lambda K>0$; setting $K_i=\phi_i[K]$, $K_i\subseteq G_i$ is compact for each i and $W'=\bigcap_{i\in J}\phi_i^{-1}[K_i]$ is non-negligible. Now we have

$$0<\tilde\lambda(\prod_{i\in J}\phi_i^{-1}[K_i])=\lambda_0(\prod_{i\in J}\phi_i^{-1}[K_i])=\int\prod_{i\in J}g_{i,K_i}d\nu,$$

so $F=\{y:y\in Y,\ g_{i,K_i}(y)>0\text{ for every }i\in J\}$ is non-negligible. On the other hand, for each $i\in J$ we have

$$\int_{Y\setminus\pi_i[K_i]}g_{i,K_i}d\nu=\mu_i(K_i\cap\pi_i^{-1}[Y\setminus K_i])=0$$

so that $F\setminus\pi_i[K_i]$ is negligible. Accordingly $\bigcap_{i\in J}\pi_i[K_i]$ is non-negligible, and must meet the support Y_0 of Y; let y be any point of the intersection. For $i\in J$, choose $x(i)\in K_i$ such that $\pi_ix(i)=y$. For $i\in I\setminus J$, $\pi_i[X_i]$ is a compact subset of Y, and $\nu\pi_i[X_i]=\mu_i\pi_i^{-1}[\pi_i[X_i]]=1$, so $Y_0\subseteq\pi_i[X_i]$ and we can therefore choose $x(i)\in X_i$ with $\pi_ix(i)=y$. This defines $x\in\Delta$. But as $x(i)\in K_i$ for $i\in J$, we also have

$$x\in\bigcap_{i\in J}\phi_i^{-1}[K_i]\subseteq\bigcap_{i\in J}\phi_i^{-1}[G_i]\subseteq X\setminus\Delta,$$

which is impossible. **X**

Thus Δ is $\tilde\lambda$-conegligible, as claimed. **Q**

(c) Let μ be the subspace measure on Δ induced by $\tilde\lambda$, and Σ its domain, so that μ is a Radon probability measure on Δ with its subspace topology (416Rb). Concerning (†) of 458Q, if $J\subseteq I$ is finite and not empty and $E_i\in\Sigma_i$ for $i\in J$, then

$$\mu(\Delta\cap\bigcap_{i\in J}\phi_i^{-1}[E_i])=\tilde\lambda(\bigcap_{i\in J}\phi_i^{-1}[E_i])=\lambda_0(\bigcap_{i\in J}\phi_i^{-1}[E_i])=\int\prod_{i\in J}g_{i,E_i}d\nu,$$

as required. Finally, for (‡), the σ-algebra Υ of subsets of Δ generated by $\{\Delta\cap\phi_i^{-1}[E]:i\in I,\ E\in\Sigma_i\}$ is just the subspace σ-algebra induced by $\widehat{\bigotimes}_{i\in I}\Sigma_i$. Let $\mathfrak{A}$ be the measure algebra of $\tilde\lambda$ and $\mathfrak{B}\subseteq\mathfrak{A}$ the set $\{W^\bullet:W\in\widehat{\bigotimes}_{i\in I}\Sigma_i\}$. Then $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$. If $W\subseteq X$ is open, then for every $\epsilon>0$ there is a $W_0\in\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $W_0\subseteq W$ and $\tilde\lambda(W\setminus W_0)\le\epsilon$, so $W^\bullet\in\mathfrak{B}$; accordingly $\{W:W^\bullet\in\mathfrak{B}\}$ contains every open set and every Borel set and must be the whole of $\operatorname{dom}\tilde\lambda$. Returning to the measure μ, we see that if $W\in\Sigma$ there must be a $W_0\in\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $\tilde\lambda(W\triangle W_0)=0$; now $W_0\cap\Delta\in\Upsilon$ and $\mu(W\triangle(W_0\cap\Delta))=0$. So (‡) also is true, and we have a relative product measure of the declared type.

458U We can of course make a general search through theorems about product measures, looking for ways of re-presenting them as theorems about relative product measures. There is an associative law, for instance (458Xr). To give an idea of what is to be expected, I offer a result corresponding to 253D.

Proposition Let (X_1,Σ_1,μ_1), (X_2,Σ_2,μ_2) and (Y,T,ν) be probability spaces, and $\pi_1:X_1\to Y$, $\pi_2:X_2\to Y$ inverse-measure-preserving functions. Let Δ be the fiber product of (X_1,π_1) and (X_2,π_2), and suppose that μ is a relative product measure of (μ_1,π_1) and (μ_2,π_2) over ν; set $\pi x=\pi_1x(1)=\pi_2x(2)$ for $x\in\Delta$. Take $f_1\in\mathcal{L}^1(\mu_1)$ and $f_2\in\mathcal{L}^2(\mu_2)$, and set $(f_1\otimes f_2)(x)=f_1(x(1))f_2(x(2))$ when $x\in\Delta\cap(\operatorname{dom}f_1\times\operatorname{dom}f_2)$. For $i=1$, 2 let $g_i\in\mathcal{L}^1(\nu)$ be a Radon-Nikodým derivative of $H\mapsto\int_{\pi_i^{-1}[H]}f_id\mu_i:\mathrm{T}\to\mathbb{R}$. Then $\int_F g_1\times g_2d\nu=\int_{\pi^{-1}[F]}f_1\otimes f_2d\mu$ for every $F\in\mathrm{T}$.

proof (a) Suppose first that $f_1=\chi E_1$ and $f_2=\chi E_2$ where $E_1\in\Sigma_1$ and $E_2\in\Sigma_2$. For each i, set $f_i'=\chi(E_i\cap\pi_i^{-1}[F])$ and $g_i'=g_i\times\chi F$; then

$$\int_H g_i' d\nu = \int_{H\cap F} g_i d\nu = \int_{\pi_i^{-1}[H\cap F]} f_i d\mu_i = \int_{\pi_i^{-1}[H]} f_i' d\mu_i$$

for every $H \in \mathrm{T}$. Now

$$\int_F g_1 \times g_2 \, d\nu = \int g_1' \times g_2' \, d\nu = \mu\{x : x \in \Delta, \, x(i) \in E_i' \text{ for both } i\}$$

(($\dagger$) of the definition 458Qb)

$$= \mu\{x : x \in \pi^{-1}[F], \, x(i) \in E_i \text{ for both } i\} = \int_{\pi^{-1}[F]} f_1 \times f_2 d\mu.$$

So we have the result in this case. (Cf. 458B.)

the functional $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0,1]$ **(b)** Generally, the formula for g_i corresponds to a linear operator from $L^1(\mu_i)$ to $L^1(\nu)$, so the result is true for simple functions f_1 and f_2. If f_1 and f_2 are almost everywhere limits of non-decreasing sequences $\langle f_{1n}\rangle_{n\in\mathbb{N}}$, $\langle f_{2n}\rangle_{n\in\mathbb{N}}$ of non-negative simple functions, then the corresponding sequences $\langle g_{1n}\rangle_{n\in\mathbb{N}}$, $\langle g_{2n}\rangle_{n\in\mathbb{N}}$ will also be non-decreasing and non-negative and convergent to g_1, g_2 ν-a.e.; moreover, because $x \mapsto x(1)$ and $x \mapsto x(2)$ are inverse-measure-preserving, $f_1 \otimes f_2 = \lim_{n\to\infty} f_{1n} \otimes f_{2n}$ μ-a.e. So in this case we shall have

$$\begin{aligned}\int_{\pi^{-1}[F]} f_1 \otimes f_2 d\mu &= \lim_{n\to\infty} \int_{\pi^{-1}[F]} f_{1n} \otimes f_{2n} d\mu \\ &= \lim_{n\to\infty} \int_F g_{1n} \times g_{2n} d\nu = \int_F g_1 \times g_2 d\nu\end{aligned}$$

for every $F \in \mathrm{T}$. Finally, considering positive and negative parts, we can extend the result to general integrable f_1 and f_2.

458X Basic exercises >(a) Find an example of a probability space (X,Σ,μ) with σ-subalgebras Σ_1, Σ_2 and T of Σ such that Σ_1 and Σ_2 are independent but are not relatively independent over T.

(b) Let (X,Σ,μ) be a probability space and T, Σ_1 and Σ_2 σ-subalgebras of Σ. Show that if $\Sigma_1 \subseteq \mathrm{T}$ then Σ_1 and Σ_2 are relatively independent over T.

>(c) Let (X,Σ,μ) be a probability space and T a subalgebra of Σ. Let $\langle\mathcal{E}_i\rangle_{i\in I}$ be a family of subsets of Σ such that (i) each $\mathcal{E}_i$ is closed under finite intersections (ii) $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i \in \mathcal{E}_i$ for every i. For each $i \in I$, let Σ_i be the σ-subalgebra of Σ generated by $\mathcal{E}_i$. Show that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

>(d) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let f_1, f_2 be μ-integrable real-valued functions which are relatively independent over T, and suppose that $f_1 \times f_2$ is integrable. Let g_1, g_2 be conditional expectations of f_1, f_2 on T. Show that $g_1 \times g_2$ is a conditional expectation of $f_1 \times f_2$ on T.

(e) In 458I, show that (writing $\hat\mu$ for the completion of μ) $\hat\mu(F \cap f^{-1}[H]) = \int_F \nu_x H \mu(dx)$ for every $F \in \mathrm{T}$ and every universally measurable $H \subseteq \mathbb{R}$.

(f) Let (X,Σ,μ) be a probability space and Σ_1, Σ_2 and T σ-subalgebras of Σ. Show that the following are equiveridical: (i) Σ_1 and Σ_2 are relatively independent over T; (ii) whenever $f \in \mathcal{L}^1(\mu{\restriction}\Sigma_1)$ and g is a conditional expectation of f on T, then g is a conditional expectation of f on $\Sigma_2 \vee \mathrm{T}$.

(g) Prove 458Ld directly from 313G, without appealing to 458D.

(h) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras. Show that their probability algebra free product (325K) can be identified with their relative free product over $(\mathfrak{C},\bar\nu)$ if $\mathfrak{C}$ is the two-element Boolean algebra, $\bar\nu$ its unique probability measure, and $\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ the trivial Boolean homomorphism for every i.

>(i) Let Y be a set, $\langle Z_i\rangle_{i\in I}$ a family of sets, and $\pi_i : Y\times Z_i\to Y$ the canonical map for each i. Show that the fiber product of $\langle(Y\times Z_i,\pi_i)\rangle_{i\in I}$ can be identified with $Y\times\prod_{i\in I}Z_i$.

(j) Let ν be Lebesgue measure on $[0,1]$, and X_1, $X_2\subseteq[0,1]$ disjoint sets with outer measure 1. For each $i\in\{1,2\}$ let μ_i be the subspace measure on X_i and $\pi_i : X_i\to[0,1]$ the identity map. Show that (μ_1,π_1) and (μ_2,π_2) have no relative product measure over ν.

(k) Let ν be the usual measure on the split interval $I^{\parallel}$ (343J), and μ Lebesgue measure on $[0,1]$. Set $\pi_1(t)=t^+$, $\pi_2(t)=t^-$ for $t\in[0,1]$. Show that (μ,π_1) and (μ,π_2) have no relative product measure over ν.

(l) Let ν be Lebesgue measure on $[0,1]$. For each $t\in[0,1]$, set $X_t=[0,1]\setminus\{t\}$; let μ_t be the subspace measure on X_t and $\pi_t : X_t\to[0,1]$ the identity map. Show that $\langle(\mu_t,\pi_t)\rangle_{t\in[0,1]}$ has no relative product measure over ν.

(m)(i) Show that there is a set $X\subseteq[0,1]^2$ with outer planar Lebesgue measure 1 and just one point in each vertical section. (*Hint*: 419H-419I.) (ii) Set $X_1=X_2=X$ and $\mu_1=\mu_2$ the subspace measure on X; let (Y,T,ν) be $[0,1]$ with Lebesgue measure, and $\pi_1=\pi_2$ the first-coordinate projection from X to Y. Show that (μ_1,π_1) and (μ_2,π_2) have no relative product measure over ν.

(n) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ, all including T. Set $\pi_i(x)=x$ for every $x\in X$. Show that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\mu\restriction\Sigma^*$ is a relative product measure of $\langle(\mu\restriction\Sigma_i,\pi_i)\rangle_{i\in I}$ over $\mu\restriction\mathrm{T}$, where $\Sigma^*=\bigvee_{i\in I}\Sigma_i$.

(o)(i) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces and (X,Σ,μ) their ordinary probability space product. Show that μ is a relative product measure of $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ over ν where Y is a singleton set, ν its unique probability measure, and $\pi_i : X_i\to Y$ the unique function for each i. (ii) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces and $(X,\mathfrak{T},\Sigma,\mu)$ their quasi-Radon probability space product (417R). Show that μ is a relative product measure of $\langle\mu_i\rangle_{i\in I}$ in the same sense as in (i).

(p) Suppose that $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is a family of probability spaces and (Y,T,ν) is a probability space, and that for each $i\in I$ we are given an inverse-measure-preserving function $\pi_i : X_i\to Y$. Write $\hat\mu_i$ and $\hat\nu$ for the completions of μ_i, ν respectively. Show that $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν iff $\langle(\hat\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over $\hat\nu$.

(q) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y,T,ν) a probability space, and $\pi_i : X_i\to Y$ an inverse-measure-preserving function for each $i\in I$. Show that if $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν, so does $\langle(\mu_i,\pi_i)\rangle_{i\in J}$ for any $J\subseteq I$.

(r) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y,T,ν) a probability space, and $\pi_i : X_i\to Y$ an inverse-measure-preserving function for each $i\in I$. Let $\langle J_k\rangle_{k\in K}$ be a partition of I into non-empty sets. For each $k\in K$, let Δ_k be the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in J_k}$; suppose that $\tilde\mu_k$ is a relative product measure of $\langle(\mu_i,\pi_i)\rangle_{i\in J_k}$. Define $\tilde\pi_k : \Delta_k\to Y$ by setting $\tilde\pi_k(x)=\pi_ix(i)$ whenever $x\in\Delta_k$ and $i\in J_k$, so that $\tilde\pi_k$ is inverse-measure-preserving. Suppose that μ is a relative product measure of $\langle(\tilde\mu_k,\tilde\pi_k)\rangle_{k\in K}$ over ν. Show that μ can be regarded as a relative product measure of $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ over ν.

(s) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle(X'_i,\Sigma'_i,\mu'_i)\rangle_{i\in I}$ be two families of probability spaces, (Y,T,ν) a probability spaces, and $f_i : X_i\to X'_i$, $\pi_i : X'_i\to Y'$ inverse-measure-preserving functions for each i. Show that if there is a relative product measure of $\langle(\mu_i,\pi_if_i)\rangle_{i\in I}$ over ν, then there is a relative product measure of $\langle(\mu'_i,\pi_i)\rangle_{i\in I}$ over ν.

(t) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of probability spaces, (Y,T,ν) a probability space, and $\pi_i : X_i\to Y$ an inverse-measure-preserving function for each i. Suppose that for each i we have a disintegration of μ_i over ν which is strongly consistent with π_i. Show that $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν.

(u) Let (X,Σ,μ), (X',Σ',μ') and (Y,T,ν) be probability spaces. Suppose that $\pi : X\to Y$ and $\pi' : X'\to Y$ are inverse-measure-preserving functions, and that μ' has a disintegration $\langle\mu'_y\rangle_{y\in Y}$ over (Y,T,ν) which is strongly consistent with π'. Show that (μ,π) and (μ',π') have a relative product measure over ν. (*Hint*: set $\lambda W=\int\mu'_{\pi(x)}W[\{x\}]\mu(dx)$ for every $W\in\Sigma\widehat{\otimes}\Sigma'$.)

>(v) Let Y be a Hausdorff space, $\langle Z_i\rangle_{i\in I}$ a family of Hausdorff spaces, μ_i a Radon probability measure on $Z_i\times Y$ and $\pi_i : Y\times Z_i\to Y$ the canonical map for each i. Suppose that all the image measures $\mu_i\pi_i^{-1}$ on Y are the same, and that all but countably many of the Z_i are compact. Show that there is a Radon probability measure μ on $Y\times\prod_{i\in I}Z_i$ such that $\mu_i=\mu\phi_i^{-1}$ for each i, where $\phi_i(y,z)=(y,z(i))$ for $y\in Y$, $z\in\prod_{j\in I}Z_j$.

(w) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of Radon probability spaces, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a Radon probability space, and $\pi_i : X_i\to Y$ an almost continuous inverse-measure-preserving function for each i. Show that $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν which is a Radon measure for the topology on the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$ induced by the product topology on $\prod_{i\in I}X_i$. Discuss the relation of this result to 418Q.

458Y Further exercises (a)(i) Let (X,Σ,μ) be a probability space, $\langle\mathrm{T}_n\rangle_{n\in\mathbb{N}}$ a non-increasing sequence of σ-subalgebras of Σ with intersection T, and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ. Suppose that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T_n for every n. Show that it is relatively independent over T. (*Hint*: 275K.) (ii) Give an example of a probability space (X,Σ,μ), a downwards-directed family $\mathbb{T}$ of σ-subalgebras of Σ, and a family $\langle E_i\rangle_{i\in I}$ in Σ which is relatively independent over T for every $\mathrm{T}\in\mathbb{T}$, but not over $\bigcap\mathbb{T}$.

(b) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle\mathcal{E}_i\rangle_{i\in I}$ be a family of subsets of Σ such that (i) $E\cap F\in\mathcal{E}_i$ whenever $i\in I$ and E, $F\in\mathcal{E}_i$ (ii) $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i\in\mathcal{E}_i$ for every $i\in I$. For each $i\in I$, let Σ_i be the σ-algebra generated by $\mathcal{E}_i$. Show that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

(c) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$ a sequence of σ-subalgebras of Σ which is relatively independent over T. Show that for every $E\in\bigcap_{n\in\mathbb{N}}\bigvee_{m\ge n}\Sigma_n$ there is an $F\in\mathrm{T}$ such that $E\triangle F$ is negligible. (Compare 272O.)

(d) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and f, $g\in\mathcal{L}^0(\mu)$ relatively independent over T; suppose that $\langle\nu_x\rangle_{x\in X}$ and $\langle\nu'_x\rangle_{x\in X}$ are relative distributions of f and g over T. Show that $\langle\nu_x*\nu'_x\rangle_{x\in X}$ is a relative distribution of $f+g$ over T. (Compare 272T.)

(e) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^2(\mu)$ such that $\langle f_n\rangle_{n\in\mathbb{N}}$ is relatively independent over T and $\int_F f_n d\mu=0$ for every $n\in\mathbb{N}$ and every $F\in\mathrm{T}$. (i) Suppose that $\langle\beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0,\infty[$, diverging to ∞, such that $\sum_{n=0}^{\infty}\frac{1}{\beta_n^2}\|f_n\|_2^2<\infty$. Show that $\lim_{n\to\infty}\frac{1}{\beta_n}\sum_{i=0}^{n}f_i=0$ a.e. (ii) Suppose that $\sup_{n\in\mathbb{N}}\|f_n\|_2<\infty$. Show that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}f_i=0$ a.e. (Compare 273D.)

(f) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y,T,ν) a probability space, and $\pi_i : X_i\to Y$ an inverse-measure-preserving function for each $i\in I$. For $i\in I$ and $E\in\Sigma_i$ let g_{iE} be a Radon-Nikodým derivative of the functional $F\mapsto\mu_i(E\cap\pi_i^{-1}[F])$. Let $\mathcal{C}$ be the family of measurable cylinders in $X=\prod_{i\in I}X_i$. If $C=\{x : x\in X,\ x(i)\in E_i$ for every $i\in J\}$ where $J\subseteq I$ is finite and not empty and $E_i\in\Sigma_i$ for $i\in J$, set $\lambda_0C=\int\prod_{i\in J}g_{i,E_i}d\nu$. Let $\Delta\subseteq X$ be the fiber product of $\langle(X_i,\pi_i)\rangle_{i\in I}$. Show that the following are equiveridical: (i) $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν (ii) whenever $\langle C_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}$ covering Δ, $\sum_{n=0}^{\infty}\lambda_0C_n\ge 1$.

(g) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y,T,ν) a probability space, and $\pi_i : X_i\to Y$ a surjective inverse-measure-preserving function for each $i\in I$. Suppose that $\langle(\mu_i,\pi_i)\rangle_{i\in J}$ has a relative product measure over ν for every countable $J\subseteq I$. Show that $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν.

(h) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of perfect probability spaces, (Y,T,ν) a countably separated probability space, and $\pi_i : X_i\to Y$ an inverse-measure-preserving function for each $i\in I$. Show that $\langle(\mu_i,\pi_i)\rangle_{i\in I}$ has a relative product measure over ν.

(i) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Let $\mathfrak{C}_0 \subseteq \mathfrak{C}$ be the core subalgebra of countable Maharam type described in the canonical form of such structures given in 333N. Show that there is a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, including $\mathfrak{C}_0$, such that $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{C}_0$, and $\mathfrak{A}$ is the closed subalgebra of itself generated by $\mathfrak{B} \cup \mathfrak{C}$.

458 Notes and comments The elementary theory of relative independence has two aspects. First, there is the matter of systematically formulating and verifying appropriate variations on standard results on stochastic independence; 458F, 458H, 458J, 458K, 458Xd, 458Yc-458Ye come under this heading. More interestingly, we study the new phenomena associated with changes in the core σ-algebras, as in 458C, 458D and 458Xa.

At a couple of points in Volume 3 (Dye's theorem, in §388, and Kawada's theorem, in §395) I took the trouble to generalize standard theorems to 'non-ergodic' forms. In both 388L and 395P the results are complicated by potentially non-trivial closed subalgebras of the probability algebra we are studying. I remarked on both occasions that the generalization is only a matter of technique, but I do not suppose that it was obvious just why this must be so. It is however a fundamental theorem of the topic of 'random reals' in the theory of forcing that *any* theorem about probability algebras must have a relativized form as a theorem about probability algebras with arbitrary closed subalgebras. The concept of 'relative Maharam type' from §333, for instance, is what matches 'Maharam type' for simple algebras; the concept of 'exchangeable' sequence (definition: 459C) is what matches 'independent identically distributed' sequence. (In probability theory, the keyword is 'mixture'.) In this section I present another example in the idea of 'relatively independent' closed subalgebras (458L-458M). I should emphasize that the forcing method, when we eventually come to it in §556 in Volume 5, will not as a rule apply directly to measure spaces; it deals with measure algebras. But of course the ideas generated by this theory can often be profitably applied to constructions in measure spaces, and this is what I am seeking to do with relatively independent σ-algebras and relative product measures.

Just as independent σ-algebras are associated with product spaces (272J), relatively independent algebras are associated with relative products (458Xn). The archetype of a relative product measure is 458S; it is a kind of disintegrated product. It is frequently profitable to express the 'relative' concepts of measure theory in terms of disintegrations.

I introduce 'relative free products' of probability algebras before proceeding to measure spaces because the uniqueness property proved in 458O shows that we have an unambiguous definition. For measure spaces it seems for the moment better to leave ourselves a bit of freedom, not (for instance) favouring one product construction over another (458Xo). The requirement that a relative product measure be carried by the fiber product is seriously limiting (458Xj-458Xl, 458Yf), and forces us to seek strongly consistent disintegrations (458S), at least for uncountable products (see 458Xt). However, as we might hope, the special case of compact spaces with Radon measures and continuous functions is amenable to a different approach (458T); and we have a one-sided method for the product of two spaces (458Xu) which is reminiscent of 454C and 457F.

There are corresponding complications when we come to look at maps between different relative products. For measure algebras, we have a natural theorem (458P), based on the same algebraic considerations as the corresponding theorems in §§315 and 325; the only possibly surprising feature is the need to assume that $\psi : \mathfrak{C} \to \mathfrak{C}'$ is actually an isomorphism. For measure spaces there is a similar result (458Xs).

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459 Symmetric measures and exchangeable random variables

Among the relatively independent families of random variables discussed in 458K, it is natural to give extra attention to those which are 'relatively identically distributed'. It turns out that these have a particularly appealing characterization as the 'exchangeable' families (459C). In the same way, among the measures on a product space X^I there is a special place for those which are invariant under permutations of coordinates (459E, 459H). A more abstract kind of permutation-invariance is examined in 495L-495M.

459A The following elementary fact seems to have gone unmentioned so far.

Lemma Let (X,Σ,μ) and (Y,T,ν) be probability spaces and $\phi:X\to Y$ an inverse-measure-preserving function; set $\Sigma_0=\{\phi^{-1}[F]:F\in\mathrm{T}\}$. Let T_1 be a σ-subalgebra of T and $\Sigma_1=\{f^{-1}[F]:F\in\mathrm{T}_1\}$. If $g\in\mathcal{L}^1(\nu)$ and h is a conditional expectation of g on T_1, then $h\phi$ is a conditional expectation of $g\phi$ on Σ_1.

proof h is $\nu\restriction\mathrm{T}_1$-integrable and ϕ is inverse-measure-preserving for $\mu\restriction\Sigma_1$ and $\nu\restriction\mathrm{T}_1$, so $h\phi$ is $\mu\restriction\Sigma_1$-integrable (235G). If $E\in\Sigma_1$ then there is an $F\in\mathrm{T}_1$ such that $E=\phi^{-1}[F]$, and now

$$\int_E g\phi\,d\mu=\int_{f^{-1}[F]}g\phi\,d\mu=\int_F g\,d\nu=\int_F h\,d\nu=\int_E h\phi\,d\mu.$$

As E is arbitrary, $h\phi$ is a conditional expectation of $g\phi$ on Σ_1.

459B Theorem Let (X,Σ,μ) be a probability space, Z a set, Υ a σ-algebra of subsets of Z and $\langle f_i\rangle_{i\in I}$ an infinite family of (Σ,Υ)-measurable functions from X to Z. For each $i\in I$, set $\Sigma_i=\{f_i^{-1}[H]:H\in\Upsilon\}$. Then the following are equiveridical:

(i) whenever $i_0,\dots,i_r\in I$ are distinct, $j_0,\dots,j_r\in I$ are distinct, and $H_k\in\Upsilon$ for each $k\le r$, then $\mu(\bigcap_{k\le r}f_{i_k}^{-1}[H_k])=\mu(\bigcap_{k\le r}f_{j_k}^{-1}[H_k])$;

(ii) there is a σ-subalgebra T of Σ such that

(α) $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T,

(β) whenever i, $j\in I$, $H\in\Upsilon$ and $F\in\mathrm{T}$, then $\mu(F\cap f_i^{-1}[H])=\mu(F\cap f_j^{-1}[H])$.

Moreover, if I is totally ordered by $\le$, we can add

(iii) whenever $i_0<\ldots<i_r\in I$, $j_0<\ldots<j_r\in I$ and $H_k\in\Upsilon$ for each $k\le r$, then $\mu(\bigcap_{k\le r}f_{i_k}^{-1}[H_k])=\mu(\bigcap_{k\le r}f_{j_k}^{-1}[H_k])$.

proof (a) Since there is always some total order on I, we may assume that we have one from the start. Of course (i)$\Rightarrow$(iii). Also (ii)$\Rightarrow$(i). **P** Suppose that (ii) is true. Then (ii-β) tells us that for each $H\in\Upsilon$ there is a T-measurable function $g_H:X\to[0,1]$ which is a conditional expectation of $\chi(f_i^{-1}[H])$ on T for every $i\in I$. Now suppose that $i_0,\dots,i_r\in I$ are distinct, $j_0,\dots,j_r\in I$ are distinct, and $H_k\in\Upsilon$ for each $k\le r$. Then

$$\mu(\bigcap_{k\le r}f_{i_k}^{-1}[H_k])=\int(\prod_{k=0}^r g_{H_k})d\mu=\mu(\bigcap_{k\le r}f_{j_k}^{-1}[H_k])$$

because $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T. So (i) is true. **Q**

So henceforth I will suppose that (iii) is true and seek to prove (ii).

(b) Suppose first that $I=\mathbb{N}$ with its usual ordering.

(α) For each n, $r\in\mathbb{N}$, let Σ_{nr} be the σ-subalgebra of Σ generated by $\bigcup_{n\le i\le n+r}\Sigma_i$; let T_n be the σ-algebra generated by $\bigcup_{r\in\mathbb{N}}\Sigma_{nr}$, and $\mathrm{T}=\bigcap_{n\in\mathbb{N}}\mathrm{T}_n$. For $n\in\mathbb{N}$ and $H\in\Upsilon$, let $g_{nH}:X\to\mathbb{R}$ be a T-measurable function which is a conditional expectation of $\chi f_n^{-1}[H]$ on T.

(β) (The key.) For any $n\in\mathbb{N}$ and Borel set $H\subseteq\mathbb{R}$, g_{nH} is a conditional expectation of $\chi f_n^{-1}[H]$ on T_{n+1}. **P** For m, $r\in\mathbb{N}$, let $h_{mr}:X\to[0,1]$ be a Σ_{mr}-measurable function which is a conditional expectation of $\chi f_n^{-1}[H]$ on Σ_{mr}; for $m\in\mathbb{N}$, set $h_m=\lim_{r\to\infty}h_{mr}$ where this is defined. By Lévy's martingale theorem (275I) h_m is defined almost everywhere and is a conditional expectation of $\chi f_n^{-1}[H]$ on T_m.

For m, $r\in\mathbb{N}$, define $F_{mr}:X\to Z^{r+2}$ by setting $F_{mr}(x)=(f_n(x),f_m(x),f_{m+1}(x),\dots,f_{m+r}(x))$ for $x\in X$. At this point, examine the hypothesis (iii). This implies that if $m>n$ and $r\in\mathbb{N}$ then

$$\begin{aligned}\mu(F_{mr}^{-1}[H'\times H_0\times\ldots\times H_r])&=\mu(f_n^{-1}[H']\cap\bigcap_{k\le r}f_{m+k}^{-1}[H_k])\\&=\mu(f_n^{-1}[H']\cap\bigcap_{k\le r}f_{m+1+k}^{-1}[H_k])\\&=\mu(F_{m+1,r}^{-1}[H'\times H_0\times\ldots\times H_r])\end{aligned}$$

for all H', $H_0,\dots,H_r\in\Upsilon$. By the Monotone Class Theorem (136C), the image measures μF_{mr}^{-1} and $\mu F_{m+1,r}^{-1}$ agree on the σ-algebra $\widehat{\bigotimes}_{r+2}\Upsilon$ of subsets of Z^{r+2} generated by measurable cylinders; set

$$\lambda = \mu F_{mr}^{-1} \restriction \widehat{\bigotimes}_{r+2} \Upsilon = \mu F_{m+1,r}^{-1} \restriction \widehat{\bigotimes}_{r+2} \Upsilon.$$

Let Λ be the σ-subalgebra of $\widehat{\bigotimes}_{r+2} \Upsilon$ generated by sets of the form $Z \times H_0 \times \ldots \times H_r$ where $H_0, \ldots, H_r \in \Upsilon$, and let h be a conditional expectation of $\chi(H \times Z^{r+1})$ on Λ with respect to λ. Then 459A tells us that hF_{mr} is a conditional expectation of $\chi(f_n^{-1}[H])$ on Σ_{mr}, and is therefore equal almost everywhere to h_{mr}. Similarly, $hF_{m+1,r} =_{\text{a.e.}} h_{m+1,r}$, and this is true for every $r \in \mathbb{N}$. But as F_{mr} and $F_{m+1,r}$ are both inverse-measure-preserving for μ and λ, this means that h_{mr}, h and $h_{m+1,r}$ all have the same distribution. In particular, $\int h_{mr}^2 d\mu = \int h_{m+1,r}^2 d\mu$. Now $\langle h_{mr} \rangle_{r \in \mathbb{N}}$ and $\langle h_{m+1,r} \rangle_{r \in \mathbb{N}}$ converge almost everywhere to h_m and h_{m+1} respectively, so

$$\int h_m^2 d\mu = \lim_{r \to \infty} \int h_{mr}^2 d\mu = \lim_{r \to \infty} \int h_{m+1,r}^2 d\mu = \int h_{m+1}^2 d\mu.$$

On the other hand, $\mathrm{T}_{m+1} \subseteq \mathrm{T}_m$, so h_{m+1} is a conditional expectation of h_m on T_{m+1} (233Eh). This means that

$$\int h_m \times h_{m+1} d\mu = \int h_{m+1} \times h_{m+1} d\mu$$

(233Eg). A direct calculation tells us that $\int (h_m - h_{m+1})^2 d\mu = 0$, so that $h_m =_{\text{a.e.}} h_{m+1}$. Inducing on r, we see that $h_m =_{\text{a.e.}} h_r$ whenever $n < m \le r$.

Now the reverse martingale theorem (275K) tells us that $\lim_{m \to \infty} h_m$ is defined almost everywhere and is a conditional expectation of $\chi f_n^{-1}[H]$ on T, that is, is equal almost everywhere to g_{nH}. Since the h_m, for $m > n$, are equal almost everywhere, they are all equal to g_{nH} a.e. In particular, g_{nH} is equal a.e. to h_{n+1}, and is a conditional expectation of $\chi f_n^{-1}[H]$ on T_{n+1}. **Q**

(γ) If $n \in \mathbb{N}$ and $H_0, \ldots, H_r \in \Upsilon$, then $\prod_{i=0}^r g_{n+i,H_i}$ is a conditional expectation of $\chi(\bigcap_{i \le r} f_{n+i}^{-1}[H_i])$ on T. **P** Induce on r. For $r = 0$ this is just the definition of g_{nH_0}. For the inductive step to $r \ge 1$, observe that $g_{nH_0} \times \prod_{i=1}^r \chi f_{n+i}^{-1}[H_i]$ is a conditional expectation of $\prod_{i=0}^r \chi f_{n+i}^{-1}[H_i]$ on T_{n+1}, by 233Eg or 233K, because g_{nH_0} is a conditional expectation of $\chi f_n^{-1}[H_0]$ on T_{n+1} and $\prod_{i=1}^r \chi f_{n+i}^{-1}[H_i]$ is T_{n+1}-measurable. But as (by the inductive hypothesis) $\prod_{i=1}^r g_{n+i,H_i}$ is a conditional expectation of $\prod_{i=1}^r \chi f_{n+i}^{-1}[H_i]$ on T, while g_{nH_0} is T-measurable, $\prod_{i=0}^r g_{n+i,H_i}$ is a conditional expectation of $\prod_{i=0}^r \chi f_{n+i}^{-1}[H_i]$ on T, by 233Eg/233K again. **Q**

(δ) In particular, $\prod_{i=0}^r g_{iH_i}$ is a conditional expectation of $\chi(\bigcap_{i \le r} f_i^{-1}[H_i])$ on T for every $r \in \mathbb{N}$ and $H_0, \ldots, H_r \in \Upsilon$. This shows that $\langle \Sigma_n \rangle_{n \in \mathbb{N}}$ is relatively independent over T.

(ϵ) Now consider part (β) of the condition (ii). For this, observe that if $m > 0$, $H \in \Upsilon$ and $H_i \in \Upsilon$ for $i \le r$, then

$$\mu(f_0^{-1}[H] \cap \bigcap_{i \le r} f_{m+i+1}^{-1}[H_i]) = \mu(f_m^{-1}[H] \cap \bigcap_{i \le r} f_{m+i+1}^{-1}[H_i]).$$

By the Monotone Class Theorem,

$$\mu(F \cap f_0^{-1}[H]) = \mu(F \cap f_m^{-1}[H])$$

for any $F \in \mathrm{T}_{m+1}$ and in particular for any $F \in \mathrm{T}$. Thus (ii) is true.

(c) Now suppose that there is a strictly increasing sequence $\langle j_k \rangle_{k \in \mathbb{N}}$ in I. For each n, let T_n be the σ-algebra generated by $\bigcup_{k \ge n} \Sigma_{j_k}$, and set $\mathrm{T} = \bigcap_{n \in \mathbb{N}} \mathrm{T}_n$. Then (b), applied to $\langle f_{j_k} \rangle_{k \in \mathbb{N}}$, tells us that $\langle \Sigma_{j_k} \rangle_{k \in \mathbb{N}}$ is relatively independent over T and that for each $H \in \Upsilon$ there is a function g_H which is a conditional expectation of $\chi(f_{j_k}^{-1}[H])$ on T for every $k \in \mathbb{N}$.

(α) If $i_0, \ldots, i_r \in I$ are distinct and $H_0, \ldots, H_r \in \Upsilon$, then $\mu(\bigcap_{k \le r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k \le r} f_{j_k}^{-1}[H_k])$. **P** Let ρ be the permutation of $\{0, \ldots, r\}$ such that $i_{\rho(0)} < i_{\rho(1)} < \ldots < i_{\rho(r)}$. Then

$$\begin{aligned}\mu(\bigcap_{k \le r} f_{i_k}^{-1}[H_k]) &= \mu(\bigcap_{k \le r} f_{i_{\rho(k)}}^{-1}[H_{\rho(k)}]) = \mu(\bigcap_{k \le r} f_{j_k}^{-1}[H_{\rho(k)}]) \\ &= \int (\prod_{k=0}^r g_{H_{\rho(k)}}) d\mu = \int (\prod_{k=0}^r g_{H_k}) d\mu = \mu(\bigcap_{k \le r} f_{j_k}^{-1}[H_k]). \quad \mathbf{Q}\end{aligned}$$

(β) Now suppose that $i_0, \ldots, i_r \in I$ are distinct. Then there is some $m \in \mathbb{N}$ such that $j_k \ne i_l$ for any $l \le r$ and $k \ge m$. In this case, consider the sequence $f_{i_0}, \ldots, f_{i_r}, f_{j_m}, f_{j_{m+1}}, \ldots$. By ($\alpha$) here, this sequence satisfies the condition (iii). We can therefore apply the construction of (b). But observe that the tail σ-algebra obtained from $f_{i_0},\dots,f_{i_r},f_{j_m},\dots$ is precisely T, as defined from $\langle f_{j_k}\rangle_{k\in\mathbb{N}}$ just above. So $\langle\Sigma_{i_k}\rangle_{k\le r}$ is relatively independent over T. As $i_0,\dots,i_r$ are arbitrary, $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T. At the same time we see that if $H\in\Upsilon$ then all the $\chi(f_{i_k}^{-1}[H])$ have the same conditional expectations over T as $\chi(f_{j_m}^{-1}[H])$. So (ii-β) is satisfied.

(d) Finally, if there is no strictly increasing sequence in I, then $(I,\ge)$ is well-ordered; since I is infinite, the well-ordering starts with an initial segment of order type ω, that is, a sequence $\langle j_k\rangle_{k\in\mathbb{N}}$ such that $j_0>j_1>\dots$. But note now that (iii) tells us that $\mu(\bigcap_{k\le r}f_{i_k}^{-1}[H_k])=\mu(\bigcap_{k\le r}f_{j_k}^{-1}[H_k])$ whenever $i_0>\dots>i_r$ and $j_0>\dots>j_r$ and $H_k\in\Upsilon$ for every k. So we can apply (c) to $(I,\ge)$ to get the result in this case also.

459C Exchangeable random variables I spell out the leading special case of this theorem.

De Finetti's theorem Let (X,Σ,μ) be a probability space, and $\langle f_i\rangle_{i\in I}$ an infinite family in $\mathcal{L}^0(\mu)$. Then the following are equiveridical:

(i) the joint distribution of $(f_{i_0},f_{i_1},\dots,f_{i_r})$ is the same as the joint distribution of $(f_{j_0},f_{j_1},\dots,f_{j_r})$ whenever $i_0,\dots,i_r\in I$ are distinct and $j_0,\dots,j_r\in I$ are distinct;

(ii) there is a σ-subalgebra T of Σ such that $\langle f_i\rangle_{i\in I}$ is relatively independent over T and all the f_i have the same relative distribution over T.

Moreover, if I is totally ordered by $\le$, we can add

(iii) the joint distribution of $(f_{i_0},f_{i_1},\dots,f_{i_r})$ is the same as the joint distribution of $(f_{j_0},f_{j_1},\dots,f_{j_r})$ whenever $i_0<\dots<i_r$ and $j_0<\dots<j_r$ in I.

Remark Families of random variables satisfying the condition in (i) are called **exchangeable**. The equivalence of (i) and (ii) can be expressed by saying that 'an exchangeable family of random variables is a mixture of independent identically distributed families'.

proof Changing each f_i on a negligible set will not change either their joint distributions (271De) or their relative distributions over T or their relative independence; so we may suppose that every f_i is a Σ-measurable function from X to $\mathbb{R}$. Now look at 459B, taking (Z,Υ) to be $\mathbb{R}$ with its Borel σ-algebra. The condition 459B(i) reads

whenever $i_0,\dots,i_r\in I$ are distinct, $j_0,\dots,j_r\in I$ are distinct, and $H_k\in\Upsilon$ for each $k\le r$, then $\mu(\bigcap_{k\le r}f_{i_k}^{-1}[H_k])=\mu(\bigcap_{k\le r}f_{j_k}^{-1}[H_k])$,

matching (i) here, by 271B; similarly, (iii) of 459B matches (iii) here. Equally, condition (ii) here is just a re-phrasing of 459B(ii) in the language of 458A and 458I-458J. So 459B gives the result.

459D Specializing 459B in another direction, we have the case in which X is actually the product Z^I. In this case, the condition 459B(i) corresponds to a strong kind of symmetry in the measure μ. It now makes sense to look for subsets of $X=Z^I$ which are essentially invariant under permutations, and we have the following result.

Proposition Let Z be a set, Υ a σ-algebra of subsets of Z, I an infinite set and μ a measure on Z^I with domain the σ-algebra $\widehat{\bigotimes}_I\Upsilon$ generated by $\{\pi_i^{-1}[H]:i\in I,\,H\in\Upsilon\}$, taking $\pi_i(x)=x(i)$ for $x\in Z^I$ and $i\in I$. For each permutation ρ of I, define $\hat\rho:Z^I\to Z^I$ by setting $\hat\rho(x)=x\rho$ for $x\in Z^I$. Suppose that $\mu=\mu\hat\rho^{-1}$ for every ρ. Let $\mathcal{E}$ be the family of those sets $E\in\widehat{\bigotimes}_I\Upsilon$ such that $\mu(E\triangle\hat\rho^{-1}[E])=0$ for every permutation ρ of I, and $\mathcal{V}$ the family of those sets $V\in\widehat{\bigotimes}_I\Upsilon$ such that V is determined by coordinates in $I\setminus\{i\}$ for every $i\in I$.

(a) $\mathcal{E}$ is a σ-subalgebra of $\widehat{\bigotimes}_I\Upsilon$.

(b) $\mathcal{V}$ is a σ-subalgebra of $\mathcal{E}$.

(c) If $E\in\mathcal{E}$ and $J\subseteq I$ is infinite, then there is a $V\in\mathcal{V}$, determined by coordinates in J, such that $\mu(E\triangle V)=0$.

(d) Setting $\Sigma_i=\{\pi_i^{-1}[H]:H\in\Upsilon\}$ for each $i\in I$,

(α) $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over $\mathcal{E}$,

(β) for every $H\in\Upsilon$ there is an $\mathcal{E}$-measurable function $g_H:Z^I\to[0,1]$ which is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on $\mathcal{E}$ for every $i\in I$.

proof (a) is elementary.

(b) Let $V \in \mathcal{V}$. Suppose that $\rho : I \to I$ is a permutation, $J \subseteq I$ is finite and $H_j \in \Upsilon$ for every $j \in J$. Then there is a permutation $\sigma : I \to I$ such that $\sigma(j) = \rho(j)$ for every $j \in J$ and $J' = \{i : \sigma(i) \neq i\}$ is finite. By 254Ta, V is determined by coordinates in $I \setminus J'$, so $\hat{\sigma}^{-1}[V] = V$. Now

$$\mu(\hat{\rho}^{-1}[V] \cap \bigcap_{j \in J} \pi_j^{-1}[H_j]) = \mu(V \cap \bigcap_{j \in J} \pi_{\rho(j)}^{-1}[H_j]) = \mu(V \cap \bigcap_{j \in J} \pi_{\sigma(j)}^{-1}[H_j])$$
$$= \mu(\hat{\sigma}^{-1}[V] \cap \bigcap_{j \in J} \pi_j^{-1}[H_j]) = \mu(V \cap \bigcap_{j \in J} \pi_j^{-1}[H_j]).$$

By the Monotone Class Theorem, as usual, $\mu(E \cap \hat{\rho}^{-1}[V]) = \mu(E \cap V)$ for every $E \in \widehat{\bigotimes}_I \Upsilon$. In particular, taking $E = V$ and $E = Z^I \setminus V$, we see that $V \triangle \hat{\rho}^{-1}[V]$ is negligible. As ρ is arbitrary, $V \in \mathcal{E}$.

This shows that $\mathcal{V} \subseteq \mathcal{E}$. Of course $\mathcal{V}$ is a σ-algebra, since it is just the intersection of the σ-algebras $\{V : V \in \widehat{\bigotimes}_I \Sigma,\ V$ is determined by coordinates in $I \setminus \{i\}\}$.

(c) For each $n \in \mathbb{N}$, there is a set $E_n \in \widehat{\bigotimes}_I \Sigma$, determined by a finite set J_n of coordinates, such that $\mu(E \triangle E_n) \le 2^{-n}$. Choose permutations ρ_n of I such that $\langle \rho_n[J_n] \rangle_{n \in \mathbb{N}}$ is a disjoint sequence of subsets of J. Set $F_n = \hat{\rho}_n^{-1}[E_n]$; then F_n is determined by coordinates in $\rho_n[J_n]$ for each $n \in \mathbb{N}$, so $V = \bigcap_{n \in \mathbb{N}} \bigcup_{m \ge n} F_m$ belongs to $\mathcal{V}$ and is determined by coordinates in J. Also

$$\mu(E \triangle F_n) = \mu(\hat{\rho}[E] \triangle E_n) = \mu(E \triangle E_n) \le 2^{-n}$$

for each n, so $\mu(E \triangle V) = 0$, as required.

(d) Let $\langle j_n \rangle_{n \in \mathbb{N}}$ be any sequence of distinct points of I. Set $J = \{j_n : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$ let T_n be the σ-algebra generated by $\bigcup_{k \ge n} \Sigma_{j_k}$, and set $\mathrm{T} = \bigcap_{n \in \mathbb{N}} \mathrm{T}_n$, so that $\mathrm{T} = \{V : V \in \mathcal{V},\ V$ is determined by coordinates in $J\}$. **P** Of course $\mathrm{T} \subseteq \mathcal{V}$ and every member of T is determined by coordinates in J, because every member of T_0 is. On the other hand, if $V \in \mathcal{V}$ is determined by coordinates in J, then fix some $w \in Z^{I \setminus J}$. In this case, identifying Z^I with $Z^J \times Z^{I \setminus J}$, the set $V_1 = \{z : z \in Z^J,\ (z, w) \in V\}$ must belong to $\widehat{\bigotimes}_J \Upsilon$, so $V = V_1 \times Z^{I \setminus J}$ belongs to T_0. Applying the same idea to $J \setminus \{j_k : k < n\}$, we see that $V \in \mathrm{T}_n$ for every n, so that $V \in \mathrm{T}$. **Q**

Part (c) of the proof of 459B tells us that $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T and that for every $H \in \Upsilon$ there is a T-measurable g_H which is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on T for every $i \in I$. Now (c) here tells us that g_H is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on $\mathcal{E}$; and examining the definition in 458Aa, we see that $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\mathcal{E}$, as claimed.

459E If μ is countably compact, we have a strong disintegration theorem, as follows.

Theorem Let Z be a set, Υ a σ-algebra of subsets of Z, I an infinite set, and μ a countably compact probability measure on Z^I with domain the σ-algebra $\widehat{\bigotimes}_I \Upsilon$ generated by $\{\pi_i^{-1}[H] : i \in I,\ H \in \Upsilon\}$, taking $\pi_i(x) = x(i)$ for $x \in Z^I$ and $i \in I$. Then the following are equiveridical:

(i) for every permutation ρ of I, $x \mapsto x\rho : Z^I \to Z^I$ is inverse-measure-preserving for μ;

(ii) for every transposition ρ of two elements of I, $x \mapsto x\rho : Z^I \to Z^I$ is inverse-measure-preserving for μ;

(iii) for each $n \in \mathbb{N}$ and any two injective functions p, $q : n \to I$ the maps $x \mapsto xp : Z^I \to Z^n$, $x \mapsto xq : Z^I \to Z^n$ induce the same measure on Z^n;

(iv) there are a probability space (Y, T, ν) and a family $\langle \lambda_y \rangle_{y \in Y}$ of probability measures on Z such that $\langle \lambda_y^I \rangle_{y \in Y}$ is a disintegration of μ over ν, writing λ_y^I for the product of copies of λ_y.

Moreover, if I is totally ordered, we can add

(v) for each $n \in \mathbb{N}$ and any two strictly increasing functions p, $q : n \to I$ the maps $x \mapsto xp : Z^I \to Z^n$, $x \mapsto xq : Z^I \to Z^n$ induce the same measure on Z^n.

If the conditions (i)-(v) are satisfied, then there is a countably compact measure λ, with domain Υ, which is the common marginal measure of μ on every coordinate; and if $\mathcal{K}$ is a countably compact class of subsets of Z, closed under finite unions and countable intersections, such that λ is inner regular with respect to $\mathcal{K}$, then (iv)′ there are a probability space (Y, T, ν) and a family $\langle\lambda_y\rangle_{y\in Y}$ of complete probability measures on Z, all with domains including $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that $\langle\lambda_y^I\rangle_{y\in Y}$ is a disintegration of μ over ν.

proof (a) Since any set I can be totally ordered, we may suppose from the outset that we have been given a total ordering $\le$ of I. I start with the easy bits.

(iv)′⇒(iv) is trivial, at least if there is a common countably compact marginal measure on Z.

(iv)⇒(i) If (iv) is true and $\rho : I \to I$ is a permutation, take any $E \in \widehat{\bigotimes}_I\Sigma$ and set $E' = \{x : x \in Z^I, x\rho \in E\}$. For any $y \in Y$, $x \mapsto x\rho$ is an isomorphism of the measure space (Z^I, λ_y^I), so

$$\mu E' = \int \lambda_y^I E'\,\nu(dy) = \int \lambda_y^I E\,\nu(dy) = \mu E.$$

As E is arbitrary, (i) is true.

(i)⇒(ii) is trivial.

(ii)⇒(iii) There is a permutation ρ of I such that $q = \rho p$ and ρ moves only finitely many points of I, that is, ρ is a product of transpositions. By (ii), $x \mapsto x\rho$ and $x \mapsto x\rho^{-1}$ are inverse-measure-preserving for μ, that is, are isomorphisms of (Z^I, μ). But this means that $x \mapsto xp$ and $x \mapsto x\rho p = xq$ must induce the same measure on Z^n.

(iii)⇒(v) is trivial.

(b) So for the rest of the proof I assume that (v) is true. Taking $n = 1$ in the statement of (v), we see that there is a common image measure $\lambda = \mu\pi_i^{-1}$ for every $i \in I$. By 452R, λ is countably compact. Let $\mathcal{K} \subseteq \mathcal{P}Z$ be a countably compact class, closed under finite unions and countable intersections, such that λ is inner regular with respect to $\mathcal{K}$.

In 459B, set $X = Z^I$ and $\Sigma = \widehat{\bigotimes}_I\Upsilon$ and $f_i = \pi_i : X \to Z$ for $i \in I$. Then (v) here corresponds to (iii) of 459B, so (translating (ii) of 459B) we have a σ-subalgebra T of $\widehat{\bigotimes}_I\Upsilon$ and a family $\langle g_H\rangle_{H\in\Upsilon}$ of T-measurable functions from Z^I to $[0,1]$ such that

$$\mu(\bigcap_{i\in J}\pi_i^{-1}[H_i]) = \int(\prod_{i\in J} g_{H_i})d\mu$$

whenever $J \subseteq I$ is finite and not empty and $H_i \in \Upsilon$ for $i \in J$. In particular, g_H is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on T whenever $H \in \Upsilon$ and $i \in I$.

Fix $i^* \in I$ for the moment. Set $\nu = \mu\restriction\mathrm{T}$. The inverse-measure-preserving function π_{i^*} from (X, μ) to (Z, λ) gives us an integral-preserving Riesz homomorphism $T_0 : L^\infty(\lambda) \to L^\infty(\mu)$ defined by setting $T_0h^\bullet = (h\pi_{i^*})^\bullet$ for every $h \in \mathcal{L}^\infty(\lambda)$. Let $P : L^1(\mu) \to L^1(\nu)$ be the conditional expectation operator; then $T = PT_0 : L^\infty(\lambda) \to L^\infty(\nu)$ is an integral-preserving positive linear operator, and $T(\chi Z^\bullet) = \chi X^\bullet$.

By 452H, we have a family $\langle\lambda_x\rangle_{x\in X}$ of complete probability measures on Z, all with domains including $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that $\int_F h\pi_{i^*}d\mu = \int_F\int_Z h\,d\lambda_x\nu(dx)$ for every $h \in \mathcal{L}^\infty(\lambda)$ and $F \in \mathrm{T}$. In particular, setting $g'_H(x) = \lambda_x H$ whenever $H \in \Upsilon$ and $x \in X$ are such that $H \in \operatorname{dom}\lambda_x$, then g'_H will be a conditional expectation of $\chi\pi_{i^*}^{-1}[H]$ on T, and will be equal ν-almost everywhere to g_H.

This means that if $J \subseteq I$ is finite and not empty and $H_i \in \Upsilon$ for $i \in J$,

$$\begin{aligned}\int_X \lambda_x^I(\bigcap_{i\in J}\pi_i^{-1}[H_i])\nu(dx) &= \int_X\prod_{i\in J}\lambda_x H_i\,\nu(dx) = \int_X\prod_{i\in J} g'_{H_i}d\nu\\ &= \int_X\prod_{i\in J} g_{H_i}d\nu = \int_X\prod_{i\in J} g_{H_i}d\mu = \mu(\bigcap_{i\in J}\pi_i^{-1}[H_i]).\end{aligned}$$

Thus the family $\mathcal{W}$ of sets $E \subseteq X$ such that $\int\lambda_x^I E\,\nu(dx)$ and μE are defined and equal contains all measurable cylinders. As $\mathcal{W}$ is a Dynkin class it includes $\widehat{\bigotimes}_I\Upsilon$. But this says exactly that $\langle\lambda_x^I\rangle_{x\in X}$ is a disintegration of μ over ν, as required by (iv)′.

Thus (v)⇒(iv)′ and the proof is complete.

459F Lemma Let X be a Hausdorff space and $P_{\mathrm{R}}(X)$ the space of Radon probability measures on X with its narrow topology (definition: 437Jd). If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of compact subsets of X, then $A = \{\mu : \mu \in P_R(X), \mu(\bigcup_{n\in\mathbb{N}} K_n) = 1\}$ is a K-analytic subset of $P_R(X)$.

proof (Recall that $P_{\mathrm{R}}(X)$ is Hausdorff, by 437R(a-ii).) For each $n \in \mathbb{N}$, let C_n be the set of Radon measures on K_n with magnitude at most 1; by 437R(f-ii), C_n is compact in its narrow topology. Let C be the compact space $\prod_{n\in\mathbb{N}} C_n$; for the rest of this proof, I will use the formula $\boldsymbol{\mu} = \langle\mu_n\rangle_{n\in\mathbb{N}}$ to describe the coordinates of members of C. Define $\psi : C \to [0,1]^{\mathbb{N}}$ by setting $\psi(\boldsymbol{\mu})(n) = \mu_n K_n$ for $n \in \mathbb{N}$ and $\boldsymbol{\mu} \in C$. Then ψ is continuous. Since $B = \{\langle\alpha_n\rangle_{n\in\mathbb{N}} : \sum_{n=0}^{\infty} \alpha_n = 1\}$ is a Borel subset of $[0,1]^{\mathbb{N}}$, therefore a Baire set (4A3Kb), $D = \psi^{-1}[B]$ is a Baire subset of C (4A3Kc), therefore Souslin-F (421L) and K-analytic (422Hb).

For $\boldsymbol{\mu} \in D$, define a function $\phi(\boldsymbol{\mu})$ by saying that

$$\phi(\boldsymbol{\mu})(E) = \sum_{n=0}^{\infty} \mu_n(E \cap K_n) \text{ if } E \subseteq X \text{ and } \mu_n \text{ measures } E \cap K_n \text{ for every } n$$

and is undefined otherwise. It is easy to check that $\phi(\boldsymbol{\mu}) \in P_{\mathrm{R}}(X)$. Also $\phi : D \to P_{\mathrm{R}}(X)$ is continuous. **P** If $G \subseteq X$ is open, then $\nu \mapsto \nu(G \cap K_n) : C_n \to [0,1]$ and therefore $\boldsymbol{\mu} \mapsto \mu_n(G \cap K_n) : D \to [0,1]$ are lower semi-continuous for each n (4A2B(d-ii)), so $\boldsymbol{\mu} \mapsto \phi(\boldsymbol{\mu})(G)$ is lower semi-continuous (4A2B(d-iii), 4A2B(d-v)), and $\{\boldsymbol{\mu} : \phi(\boldsymbol{\mu})(G) > \alpha\}$ is open for every α; by 4A2B(a-iii), ϕ is continuous. **Q**

Consequently $A = \phi[D]$ is K-analytic (422Gd).

459G Lemma Let X be a topological space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a totally finite quasi-Radon measure space, $y \mapsto \mu_y$ a continuous function from Y to the space $M_{\mathrm{qR}}^{+}(X)$ of totally finite quasi-Radon measures on X with its narrow topology, and $\mathcal{U}$ a base for the topology of X, containing X and closed under finite intersections. If $\mu \in M_{\mathrm{qR}}^{+}(X)$ is such that $\mu U = \int \mu_y U\,\nu(dy)$ for every $U \in \mathcal{U}$, then $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν.

proof (a) Let $\mathcal{E}$ be the family of subsets E of X such that μE and $\int \mu_y E\,\nu(dy)$ are defined and equal. Because $X \in \mathcal{E}$, $\mathcal{E}$ is a Dynkin class; as $\mathcal{U}$ is included in $\mathcal{E}$ and is closed under finite intersections, the σ-algebra of sets generated by $\mathcal{U}$ is included in $\mathcal{E}$, and in particular any finite union of members of $\mathcal{U}$ belongs to $\mathcal{E}$.

(b) In fact every open subset of X belongs to $\mathcal{E}$. **P** If $G \subseteq X$ is open, set $\mathcal{H} = \{H : H \subseteq G$ is a finite union of members of $\mathcal{U}\}$. Then $\mathcal{H}$ is upwards-directed and has union G. Set $f_H(y) = \mu_y H$ for $y \in Y$ and $H \in \mathcal{H}$. Since $\lambda \mapsto \lambda H : M_{\mathrm{qR}}^{+}(X) \to \mathbb{R}$ is lower semi-continuous (by the definition of the narrow topology) and $y \mapsto \mu_y$ is continuous, $f_H : Y \to \mathbb{R}$ is lower semi-continuous (4A2B(d-ii) again). Moreover, $\{f_H : H \in \mathcal{H}\}$ is an upwards-directed family of functions with supremum f_G, where $f_G(y) = \mu_y G$ for each y, because every μ_y is τ-additive. Now

$$\mu G = \sup_{H\in\mathcal{H}} \mu H = \sup_{H\in\mathcal{H}} \int f_H d\nu = \int f_G d\nu$$

(414Ba)

$$= \int \mu_y G\,\nu(dy)$$

and $G \in \mathcal{E}$. **Q**

(c) It follows that every Borel subset of X belongs to $\mathcal{E}$, that is, that $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of the restriction $\mu_{\mathcal{B}}$ to the Borel σ-algebra of X. Since every μ_y is complete, $\langle\mu_y\rangle_{y\in Y}$ is also a disintegration over ν of the completion of $\mu_{\mathcal{B}}$ (452B(a-ii)), which is μ.

459H Theorem Let Z be a Hausdorff space, I an infinite set, and $\tilde{\mu}$ a quasi-Radon probability measure on Z^I such that the marginal measures on each copy of Z are Radon measures. Write $P_{\mathrm{R}}(Z)$ for the set of Radon probability measures on Z with its narrow topology. Then the following are equiveridical:

(i) for every permutation ρ of I, $w \mapsto w\rho : Z^I \to Z^I$ is inverse-measure-preserving for $\tilde{\mu}$;

(ii) for every transposition ρ of two elements of I, $w \mapsto w\rho : Z^I \to Z^I$ is inverse-measure-preserving for $\tilde{\mu}$;

(iii) for each $n \in \mathbb{N}$ and any two injective functions p, $q : n \to I$ the maps $w \mapsto wp : Z^I \to Z^n$ and $w \mapsto wq : Z^I \to Z^n$ induce the same measure on Z^n;

(iv) there are a probability space (Y, T, ν) and a family $\langle\mu_y\rangle_{y\in Y}$ of τ-additive Borel probability measures on Z such that $\langle\tilde{\mu}_y^I\rangle_{y\in Y}$ is a disintegration of $\tilde{\mu}$ over ν, writing $\tilde{\mu}_y^I$ for the τ-additive product of copies of μ_y;

(v) there is a Radon probability measure $\tilde{\nu}$ on $P_{\mathrm{R}}(Z)$ such that $\langle\tilde{\theta}^I\rangle_{\theta\in P_{\mathrm{R}}(Z)}$ is disintegration of $\tilde{\mu}$ over $\tilde{\nu}$, writing $\tilde{\theta}^I$ for the quasi-Radon product of copies of θ.

Moreover, if I is totally ordered, we can add

(vi) for each $n\in\mathbb{N}$ and any two strictly increasing functions p, $q : n \to I$ the maps $w \mapsto wp : Z^I \to Z^n$ and $w \mapsto wq : Z^I \to Z^n$ induce the same measure on Z^n.

proof (a) As in 459E, we need consider only the case in which I is totally ordered, and the implications

$$\text{(v)}\Rightarrow\text{(iv)}\Rightarrow\text{(i)}\Rightarrow\text{(ii)}\Rightarrow\text{(iii)}\Rightarrow\text{(vi)}$$

are elementary. So henceforth I will suppose that (vi) is true and seek to prove (v).

(b) We are going to need a second topology on the set Z, so I will call the original topology $\mathfrak{T}$, and for the rest of this proof I will declare the topology on which each topological concept or construction is based. Write μ for $\tilde{\mu}\restriction\widehat{\bigotimes}_I\mathcal{B}(Z,\mathfrak{T})$, where $\mathcal{B}(Z,\mathfrak{T})$ is the Borel σ-algebra of Z for the topology $\mathfrak{T}$. Then (vi) is also true of μ. (Strictly speaking, we ought to check that the different images of μ all have the same domain. But this is true, because the image of μ corresponding to a strictly increasing function $p : r \to I$ has domain $\widehat{\bigotimes}_r\mathcal{B}(Z,\mathfrak{T})$.) The (unique) marginal measure λ of μ is the restriction to $\mathcal{B}(Z,\mathfrak{T})$ of the $\mathfrak{T}$-Radon measure $\tilde{\lambda}$ which is the marginal of $\tilde{\mu}$, so is a $\mathfrak{T}$-tight $\mathfrak{T}$-Borel measure, therefore countably compact. By 454A(b-ii), μ is countably compact. So 459E, with $\mathcal{K}$ the family of $\mathfrak{T}$-compact subsets of Z, tells us that there are a probability space $(Y_0, \mathrm{T}_0, \nu_0)$ and a family $\langle\mu_y\rangle_{y\in Y_0}$ in $P_{\mathrm{R}}(Z,\mathfrak{T})$ such that $\langle\mu_y^I\rangle_{y\in Y_0}$ is a disintegration of μ over ν_0, writing μ_y^I for the ordinary product of copies of μ_y. We can of course suppose that ν_0 is complete. Note also that $\langle\mu_y\rangle_{y\in Y_0}$ is a disintegration of λ; this is clearly achieved by the proof of 459E, and it is necessarily true if $\langle\mu_y^I\rangle_{y\in Y_0}$ is to be a disintegration of μ. Because every μ_y is complete, $\langle\mu_y\rangle_{y\in Y_0}$ is also a disintegration of the completion $\tilde{\lambda}$ of λ.

(c) Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of $\mathfrak{T}$-compact subsets of Z such that $\sum_{n=0}^{\infty}\tilde{\lambda}K_n = 1$ (412Aa). Let $\mathfrak{S}$ be

$$\{H : H \subseteq Z,\, Z\setminus(H\cap K_n)\in\mathfrak{T} \text{ for every } n\in\mathbb{N}\}.$$

Then $\mathfrak{S}$ is a locally compact topology on Z finer than $\mathfrak{T}$. (If you like, $\mathfrak{S}$ is the disjoint union topology corresponding to the partition $\{K_n : n\in\mathbb{N}\}\cup\{\{z\} : z\in Z\setminus\bigcup_{n\in\mathbb{N}}K_n\}$.) Note that the subspace topologies on any K_n induced by $\mathfrak{S}$ and $\mathfrak{T}$ are the same, so that a $\mathfrak{T}$-compact subset of K_n is $\mathfrak{S}$-compact. Because $\mathfrak{S}$ is finer than $\mathfrak{T}$, $P_{\mathrm{R}}(Z,\mathfrak{S})\subseteq P_{\mathrm{R}}(Z,\mathfrak{T})$ (use 418I). If $\theta\in P_{\mathrm{R}}(Z,\mathfrak{T})$ and $\theta(\bigcup_{n\in\mathbb{N}}K_n) = 1$, then, from the standpoint of the topology $\mathfrak{S}$, θ is a complete topological probability measure inner regular with respect to the compact sets, so belongs to $P_{\mathrm{R}}(Z,\mathfrak{S})$. In particular, $\tilde{\lambda}\in P_{\mathrm{R}}(Z,\mathfrak{S})$.

We shall need to know that the family $\mathcal{V}$ of $\mathfrak{T}$-Borel $\mathfrak{S}$-cozero subsets of Z is a base for $\mathfrak{S}$. **P** If $z\in H\in\mathfrak{S}$, then if $z\notin\bigcup_{n\in\mathbb{N}}K_n$ the singleton $\{z\}$ belongs to $\mathcal{V}$. If $n\in\mathbb{N}$ and $z\in K_n$, then $H\cap K_n\in\mathfrak{S}$; as $\mathfrak{S}$ is locally compact, there is an $\mathfrak{S}$-cozero set G such that $z\in G\subseteq H\cap K_n$, and now G is $\mathfrak{T}$-relatively open in the $\mathfrak{T}$-compact set K_n, so G is $\mathfrak{T}$-Borel. **Q**

(d) We know that

$$\int\mu_y(\bigcup_{n\in\mathbb{N}}K_n)\nu_0(dy) = \tilde{\lambda}(\bigcup_{n\in\mathbb{N}}K_n) = 1;$$

since $\mu_yZ = 1$ for every y, the set $Y = \{y : y\in Y_0,\, \mu_y(\bigcup_{n\in\mathbb{N}}K_n) = 1\}$ must be ν_0-conegligible. Let ν be the subspace measure induced by ν_0 on Y. Then $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of $\tilde{\lambda}$ over ν, and $\mu_y\in P_{\mathrm{R}}(Z,\mathfrak{S})$ for every $y\in Y$, by (c).

(e) By 459F, the set

$$A = \{\theta : \theta\in P_{\mathrm{R}}(Z,\mathfrak{S}),\, \theta(\bigcup_{n\in\mathbb{N}}K_n = 1\}$$

is K-analytic in its narrow topology, while $\mu_y\in A$ for every $y\in Y$. If $G\in\mathcal{V}$ and $\alpha > 0$, $\{y : y\in Y_0,\, \mu_yG > \alpha\}\in\mathrm{T}_0$, so $\{y : y\in Y,\, \mu_yG > \alpha\}$ is measured by ν. By 432I, applied to the map $y\mapsto\mu_y : Y\to A$, there is a Radon probability measure $\tilde{\nu}_A$ on A such that

$$\int h\,d\tilde{\nu}_A = \int h(\mu_y)\nu(dy)$$

for every bounded continuous $h : A \to \mathbb{R}$.

(f) Now suppose that $f : Z \to \mathbb{R}$ is bounded and $\mathfrak{S}$-continuous. Then $\theta \mapsto \int f d\theta : P_{\mathrm{R}}(Z, \mathfrak{S}) \to \mathbb{R}$ is continuous (437K), so that

$$\iint f d\theta\, \tilde{\nu}_A(d\theta) = \iint f d\mu_y \nu(dy).$$

If $G \in \mathcal{V}$, there is a non-decreasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative $\mathfrak{S}$-continuous functions with supremum χG, so

$$\begin{aligned}\int \theta G\, \tilde{\nu}_A(d\theta) &= \sup_{n\in\mathbb{N}} \iint f_n d\theta\, \tilde{\nu}_A(d\theta) = \sup_{n\in\mathbb{N}} \iint f_n d\mu_y \nu(dy) \\ &= \int \mu_y G\, \nu(dy) = \lambda G = \tilde{\lambda} G.\end{aligned}$$

So we can apply 459G to the identity map from A to itself and the family $\langle \theta \rangle_{\theta\in A}$ to see that $\langle \theta \rangle_{\theta\in A}$ is a disintegration of $\tilde{\lambda}$ over $\tilde{\nu}_A$.

It follows that if $E \subseteq Z$ is $\tilde{\lambda}$-negligible, then $\theta E = 0$ for $\tilde{\nu}_A$-almost every θ. Moreover, since $\langle \mu_y \rangle_{y\in Y}$ is a disintegration of $\tilde{\lambda}$ over ν, $\mu_y E = 0$ for ν-almost every y.

(g) If $J \subseteq I$ is finite, $G_j \in \mathfrak{T}$ for $j \in J$, and $W = \{w : w \in Z^I,\ w(j) \in G_j \text{ for } j \in J\}$, then

$$\tilde{\mu} W = \int \theta^I W \tilde{\nu}_A(d\theta).$$

P Because $\mathcal{V}$ is a base for $\mathfrak{S}$ closed under countable unions, and $\tilde{\lambda}$ is $\mathfrak{S}$-Radon, there is for each $j \in J$ a $G'_j \in \mathcal{V}$, included in G_j, such that $\tilde{\lambda} G'_J = \tilde{\lambda} G_j$. Set $W' = \{w : w \in Z^I,\ w(j) \in G'_j \text{ for } j \in J\}$. We have

$$W \setminus W' \subseteq \bigcup_{j\in J} \{w : w(j) \in G_j \setminus G'_j\},$$

while

$$\tilde{\mu}\{w : w(j) \in G_j \setminus G'_j\} = \tilde{\lambda}(G_j \setminus G'_j) = 0$$

for each j, so $\tilde{\mu} W'$ is defined and equal to $\tilde{\mu} W = \mu W$. Note that the same calculation shows that $\theta^I W = \theta^I W'$ whenever $\theta \in A$ is such that $\theta G'_j = \theta G_j$ for every j, that is, for $\tilde{\nu}_A$-almost every θ. Now, for each $j \in J$, we have a non-decreasing sequence $\langle f_{jn} \rangle_{n\in\mathbb{N}}$ of non-negative $\mathfrak{S}$-continuous real-valued functions with supremum $\chi G'_j$. Set $g_n(w) = \prod_{j\in J} f_{jn}(w(j))$ for $w \in Z^I$ and $n \in \mathbb{N}$. (I suppose you should take $g_n(w) = 1$ if J is empty.) Then each g_n is $\mathfrak{S}^I$-continuous, so if we set $h_n(\theta) = \int g_n d\theta^I$ for $\theta \in A$, h_n is continuous (put 437Mb and 437Kb together). Also $\langle g_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum $\chi W'$, so

$$\theta^I W' = \sup_{n\in\mathbb{N}} \int g_n d\theta^I = \sup_{n\in\mathbb{N}} h_n(\theta)$$

for $\theta \in A$. Accordingly

$$\begin{aligned}\tilde{\mu} W = \tilde{\mu} W' &= \int \mu_y^I W' \nu(dy) = \sup_{n\in\mathbb{N}} \int h_n(\mu_y)\nu(dy) \\ &= \sup_{n\in\mathbb{N}} \int h_n d\tilde{\nu}_A = \sup_{n\in\mathbb{N}} \iint g_n d\theta^I\, \tilde{\nu}_A(d\theta) \\ &= \int \theta^I W'\, \tilde{\nu}_A(d\theta) = \int \theta^I W\, \tilde{\nu}_A(d\theta),\end{aligned}$$

as required. **Q**

(h) We are nearly ready to dispense with the topology $\mathfrak{S}$. Since the embeddings $A \subsetneq P_{\mathrm{R}}(Z, \mathfrak{S}) \subsetneq P_{\mathrm{R}}(Z, \mathfrak{T})$ are continuous (437Jh), we have an image Radon probability measure $\tilde{\nu}$ on $P_{\mathrm{R}}(Z, \mathfrak{T})$, and

$$\int_{P_{\mathrm{R}}(Z,\mathfrak{T})} h\, d\tilde{\nu} = \int_A h\, d\tilde{\nu}_A$$

for every $h : P_{\mathrm{R}}(Z, \mathfrak{T}) \to \mathbb{R}$ such that $\int_A h\, d\tilde{\nu}_A$ is defined.

In particular, if we take $\mathcal{W}$ to be the family of $\mathfrak{T}^I$-open cylinder sets expressible as $\{w : w \in Z^I,\ w(j) \in G_j$ for $j \in J\}$ where $J \subseteq I$ is finite and $G_j \in \mathfrak{T}$ for each j, (g) tells us that

$$\tilde{\mu}W = \int \theta^I W \,\tilde{\nu}_A(d\theta) = \int \theta^I W \,\tilde{\nu}(d\theta) = \int \tilde{\theta}^I W \,\tilde{\nu}(d\theta)$$

for every $W \in \mathcal{W}$, where I now write $\mathfrak{T}^I$ for the product topology on Z^I corresponding to the topology $\mathfrak{T}$ on Z, and $\tilde{\theta}^I$ for the $\mathfrak{T}^I$-quasi-Radon product measure on Z^I corresponding to the $\mathfrak{T}$-Radon measure θ (417R). Now turn again to 437Mb and 459G; $\theta \mapsto \tilde{\theta}^I$ is a continuous function from $P_{\mathrm{R}}(Z,\mathfrak{T})$ to the space $P_{\mathrm{qR}}(Z^I,\mathfrak{T}^I)$ of $\mathfrak{T}^I$-quasi-Radon probability measures on Z^I, and $\mathcal{W}$ is a base for the topology $\mathfrak{T}^I$, so $\langle\tilde{\theta}^I\rangle_{\theta\in P_{\mathrm{R}}(Z,\mathfrak{T})}$ is a disintegration of $\tilde{\mu}$ over $\tilde{\nu}$, which is what I set out to prove.

459I I come now to a lemma based on ideas in Tao 07. It is in a form more elaborate than is required for the elementary application here (459J), but which will be needed in §497.

Lemma Let (X,Σ,μ) be a probability space and I a set. For a family $\mathbb{T}$ of subalgebras of $\mathcal{P}X$, write $\bigvee\mathbb{T}$ for the σ-algebra generated by $\bigcup\mathbb{T}$, as in 458Ad. Let G be the group of permutations ϕ of I such that $\{i : \phi(i) \ne i\}$ is finite. Suppose that $\bullet$ is an action of G on X such that $x \mapsto \phi\bullet x$ is inverse-measure-preserving for each $\phi \in G$; set $\phi\bullet A = \{\phi\bullet x : x \in A\}$ for $\phi \in G$ and $A \subseteq X$, as in 441Aa and 4A5Bc. Let $\langle\Sigma_J\rangle_{J\subseteq I}$ be a family of σ-subalgebras of Σ such that

(i) for every $J \subseteq I$, Σ_J is the σ-algebra generated by $\bigcup_{K\subseteq J\text{ is finite}}\Sigma_K$;

(ii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$, then $\phi\bullet E \in \Sigma_{\phi[J]}$;

(iii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$ is such that $\phi(i) = i$ for every $i \in J$, then $\phi\bullet E = E$.

Suppose that $\mathcal{J}^*$ is a filter on I not containing any infinite set, and that $K \subseteq I$, $\mathcal{K} \subseteq \mathcal{P}I$ and $\mathcal{J} \subseteq \mathcal{J}^*$ are such that for every $K' \in \mathcal{K}$ there is a $J \in \mathcal{J}$ such that $K \cap K' \subseteq J$. Then Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$.

proof (a)(i) Let us note straight away that condition (i) above implies that $\Sigma_K \subseteq \Sigma_J$ whenever $K \subseteq J \subseteq I$.

(ii) For any σ-subalgebra T of Σ, I will (slightly abusing notation, as in 242Jh) write $L^2(\mu\restriction\mathrm{T})$ for the $\| \|_2$-closed linear subspace of $L^2(\mu)$ consisting of equivalence classes of μ-square-integrable T-measurable real-valued functions defined on X, and $P_{\mathrm{T}} : L^2(\mu) \to L^2(\mu\restriction\mathrm{T})$ for the corresponding conditional-expectation operator (244M). Note that P_{T} is an orthogonal projection (244Nb).

(iii) We have an action of G on $L^2(\mu)$, defined by saying that

$$(\phi\bullet f)(x) = f(\phi^{-1}\bullet x) \text{ for } \phi \in G,\ x \in X \text{ and } f \in \mathbb{R}^X$$

(4A5C(c-i)),

$$\phi\bullet f^\bullet = (\phi\bullet f)^\bullet \text{ for } \phi \in G \text{ and } f \in \mathcal{L}^2(\mu) \cap \mathbb{R}^X$$

(441Kc).

(iv) If $\mathbb{T}$ is the family of σ-algebras of subsets of X, we have an action of G on $\mathbb{T}$ defined by setting

$$\phi\bullet\mathrm{T} = \{\phi\bullet E : E \in \mathrm{T}\}$$

for $\mathrm{T} \in \mathbb{T}$ and $\phi \in G$. If $\langle\mathrm{T}_\gamma\rangle_{\gamma\in\Gamma}$ is a family in $\mathbb{T}$, then $\phi\bullet\bigvee_{\gamma\in\Gamma}\mathrm{T}_\gamma = \bigvee_{\gamma\in\Gamma}\phi\bullet\mathrm{T}_\gamma$ for every $\phi \in G$, just because $E \mapsto \phi\bullet E$ is an automorphism of the Boolean algebra $\mathcal{P}X$.

(v) If $\phi \in G$ and $L \subseteq I$, then $\phi\bullet\Sigma_L = \Sigma_{\phi[L]}$. **P** Condition (ii) of this lemma says just that $\phi\bullet\Sigma_L = \{\phi\bullet E : E \in \Sigma_L\}$ is included in $\Sigma_{\phi[L]}$; and now of course

$$\Sigma_{\phi[L]} = \phi\bullet\phi^{-1}\bullet\Sigma_L \subseteq \phi\bullet\Sigma_{\phi^{-1}[\phi[L]]} = \phi\bullet\Sigma_L. \textbf{ Q}$$

(vi) If $\phi \in G$ and T is a σ-subalgebra of Σ, then $\phi\bullet(P_{\mathrm{T}}u) = P_{\phi\bullet\mathrm{T}}(\phi\bullet u)$ for every $u \in L^2(\mu)$. **P** I should of course note that $\phi\bullet\Sigma = \Sigma$ because $x \mapsto \phi\bullet x$ is an automorphism of (X,Σ,μ), so $\phi\bullet\mathrm{T} \subseteq \Sigma$ and we can speak of $P_{\phi\bullet\mathrm{T}}$. Let $f : X \to \mathbb{R}$ be a Σ-measurable function such that $f^\bullet = u$, and $g : X \to \mathbb{R}$ a T-measurable function which is a conditional expectation of f on T. In this case, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\{x : (\phi\bullet g)(x) > \alpha\} &= \{x : g(\phi^{-1}\bullet x) > \alpha\} = \{\phi\bullet x : g(x) > \alpha\}\\ &= \phi\bullet\{x : g(x) > \alpha\} \in \phi\bullet\mathrm{T},\end{aligned}$$

so $\phi\bullet g$ is $(\phi\bullet\mathrm{T})$-measurable. Next, for any $F \in \phi\bullet\mathrm{T}$,

$$\int_F \phi{\bullet}g\,d\mu = \int_F g(\phi^{-1}{\bullet}x)\mu(dx) = \int_{\phi^{-1}{\bullet}F} g(x)\mu(dx)$$

(applying 235G to the inverse-measure-preserving function $x \mapsto \phi{\bullet}x : X \to X$ and the integrable function $x \mapsto g(\phi^{-1}{\bullet}x)$)

$$= \int_{\phi^{-1}{\bullet}F} f\,d\mu$$

(because $\phi^{-1}{\bullet}F \in \mathrm{T}$)

$$= \int_F \phi{\bullet}f d\mu.$$

As F is arbitrary, $\phi{\bullet}g$ is a conditional expectation of $\phi{\bullet}f$ on $\phi{\bullet}\mathrm{T}$, and

$$\phi{\bullet}(P_{\mathrm{T}}u) = \phi{\bullet}g^{\bullet} = (\phi{\bullet}g)^{\bullet} = P_{\phi{\bullet}\mathrm{T}}(\phi{\bullet}f)^{\bullet} = P_{\phi{\bullet}\mathrm{T}}(\phi{\bullet}u). \text{ **Q**}$$

(b)(i) Let $\langle J_\gamma\rangle_{\gamma\in\Gamma}$ be a non-empty finite family of subsets of I with infinite intersection, and set $\Lambda = \bigvee_{\gamma\in\Gamma}\Sigma_{J_\gamma}$. Suppose that K, $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ are such that

$$K \in [I]^{<\omega}, \quad K_\gamma \in [I]^{<\omega} \text{ and } K \cap K_\gamma \subseteq J_\gamma \text{ for every } \gamma \in \Gamma.$$

Take $E \in \Sigma_K$ and $F_\gamma \in \Sigma_{K_\gamma}$ for every $\gamma \in \Gamma$, and set $F = \bigcap_{\gamma\in\Gamma} F_\gamma$. Let g, $h : X \to [0,1]$ be Λ-measurable functions which are conditional expectations of χE, χF respectively on Λ. Let $\epsilon > 0$.

(ii) For $L \subseteq I$ set $\Lambda_L = \bigvee_{\gamma\in\Gamma}\Sigma_{J_\gamma\cap L} \subseteq \Lambda$. For any $u \in L^2(\mu)$ there is a finite $L \subseteq I$ such that $\|P_{\mathrm{T}}u - P_\Lambda u\|_2 \le \epsilon$ whenever T is a σ-subalgebra of Λ including Λ_L. **P** By condition (i) of this lemma, Λ is the σ-algebra generated by

$$\bigcup_{L\subseteq I \text{ is finite}} \bigcup_{\gamma\in\Gamma} \Sigma_{J_\gamma\cap L},$$

so $\{\Lambda_L : L \in [I]^{<\omega}\}$ is an upwards-directed family of σ-algebras whose union σ-generates Λ, and $\bigcup_{L\subseteq I \text{ is finite}} L^2(\mu{\restriction}\Lambda_L)$ is norm-dense in $L^2(\mu{\restriction}\Lambda)$. There are therefore a finite $L \subseteq I$ and a $v \in L^2(\mu{\restriction}\Lambda_L)$ such that $\|v - P_\Lambda u\|_2 \le \epsilon$. If now $\Lambda_L \subseteq \mathrm{T} \subseteq \Lambda$, $v \in L^2(\mu{\restriction}\mathrm{T})$, while P_{T} is the orthogonal projection onto $L^2(\mu{\restriction}\mathrm{T})$, so

$$\|P_{\mathrm{T}}u - P_\Lambda u\|_2 = \|P_{\mathrm{T}}P_\Lambda u - P_\Lambda u\|_2 \le \|v - P_\Lambda u\|_2 \le \epsilon. \text{ **Q**}$$

(iii Set $u = \chi E^{\bullet}$ and $v = \chi F^{\bullet}$, so that $g^{\bullet} = P_\Lambda u$ and $h^{\bullet} = P_\Lambda v$. By (b), there is an $L_0 \in [I]^{<\omega}$ such that

$$\|P_{\mathrm{T}}u - P_\Lambda u\|_2 \le \epsilon, \quad \|P_{\mathrm{T}}v - P_\Lambda v\|_2 \le \epsilon, \quad \|P_{\mathrm{T}}(u\times v) - P_\Lambda(u\times v)\|_2 \le \epsilon$$

whenever T is a σ-subalgebra of Λ including Λ_{L_0}. We can suppose that $L_0 \supseteq K \cup \bigcup_{\gamma\in\Gamma} K_\gamma$. Write T_0 for Λ_{L_0}. We have

$$\begin{aligned}
&\|P_\Lambda u \times P_\Lambda v - P_{\mathrm{T}_0}u \times P_{\mathrm{T}_0}v\|_2 \\
&\qquad\le \|P_\Lambda u \times (P_\Lambda v - P_{\mathrm{T}_0}v)\|_2 + \|(P_\Lambda u - P_{\mathrm{T}_0}u) \times P_{\mathrm{T}_0}v\|_2 \\
&\qquad\le \|P_\Lambda v - P_{\mathrm{T}_0}v\|_2 + \|P_\Lambda u - P_{\mathrm{T}_0}u\|_2
\end{aligned}$$

(because $\|P_\Lambda u\|_\infty$ and $\|P_{\mathrm{T}_0}v\|_\infty$ are both at most 1)

$$\le 2\epsilon.$$

(iv) Let $L_1 \subseteq \bigcap_{\gamma\in\Gamma} J_\gamma \setminus L_0$ be a set of size $\#(L_0 \setminus K)$; let $\phi \in G$ be such that $\phi[L_0 \setminus K] = L_1$, ϕ^2 is the identity and $\phi(i) = i$ for $i \in I \setminus (L_1 \cup (L_0 \setminus K))$. In this case, $\phi(i) = i$ for $i \in K$, so $\phi[L] \subseteq (L \cap K) \cup \bigcap_{\gamma\in\Gamma} J_\gamma$ for every $L \subseteq L_0$. Setting $M_\gamma = (L_0 \cap J_\gamma) \cup \phi[L_0 \cap J_\gamma]$, we have

$$L_0 \cap J_\gamma \subseteq M_\gamma = \phi[M_\gamma] \subseteq J_\gamma, \quad \phi[K_\gamma] \subseteq J_\gamma$$

for each $\gamma \in \Gamma$. (This is where we need to know that $K \cap K_\gamma \subseteq J_\gamma$.)

Now

$$\phi\bullet u=\phi\bullet(\chi E^\bullet)=\chi(\phi\bullet E)^\bullet=\chi E^\bullet=u$$

by condition (iii) of this lemma; also

$$\|\phi\bullet(P_{\mathrm{T}_0}u)-P_\Lambda u\|_2\le 3\epsilon.$$

P By (a-iv) and (a-v),

$$\begin{aligned}\phi\bullet\mathrm{T}_0=\phi\bullet\bigvee_{\gamma\in\Gamma}\Sigma_{L_0\cap J_\gamma}&=\bigvee_{\gamma\in\Gamma}\phi\bullet\Sigma_{L_0\cap J_\gamma}\\&=\bigvee_{\gamma\in\Gamma}\Sigma_{\phi[L_0\cap J_\gamma]}\subseteq\bigvee_{\gamma\in\Gamma}\Sigma_{M_\gamma}\subseteq\bigvee_{\gamma\in\Gamma}\Sigma_{J_\gamma}=\Lambda.\end{aligned}$$

Set $\mathrm{T}=\mathrm{T}_0\vee\phi\bullet\mathrm{T}_0$; then $\mathrm{T}_0\subseteq\mathrm{T}=\phi[\mathrm{T}]\subseteq\Lambda$. But now

$$\phi\bullet(P_{\mathrm{T}}u)=P_{\phi\bullet\mathrm{T}}(\phi\bullet u)=P_{\mathrm{T}}u$$

(see (a-vi)), so

$$\begin{aligned}\|\phi\bullet(P_{\mathrm{T}_0}u)-P_\Lambda u\|_2&\le\|\phi\bullet(P_{\mathrm{T}_0}u)-\phi\bullet(P_{\mathrm{T}}u)\|_2+\|P_{\mathrm{T}}u-P_\Lambda u\|_2\\&=\|P_{\mathrm{T}}u-P_{\mathrm{T}_0}u\|_2+\|P_{\mathrm{T}}u-P_\Lambda u\|_2\\&\le\|P_\Lambda u-P_{\mathrm{T}_0}u\|_2+2\|P_{\mathrm{T}}u-P_\Lambda u\|_2\le 3\epsilon.\ \mathbf{Q}\end{aligned}$$

(v) Set

$$\mathrm{T}^*=\textstyle\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma\cup M_\gamma}.$$

Because $L_0\cap J_\gamma\subseteq M_\gamma$ for every γ, T^* and

$$\phi\bullet\mathrm{T}^*=\textstyle\bigvee_{\gamma\in\Gamma}\Sigma_{\phi[K_\gamma]\cup M_\gamma}$$

include $\Lambda_{L_0}=\mathrm{T}_0$, while $\phi\bullet\mathrm{T}^*\subseteq\Lambda$ because $\phi[K_\gamma]\cup M_\gamma\subseteq J_\gamma$ for every γ. Also $F\in\mathrm{T}^*$, because $F_\gamma\in\Sigma_{K_\gamma}\subseteq\mathrm{T}^*$ for every γ. Now
and

$$\|P_{\mathrm{T}_0}(u\times v)-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2=\|P_{\mathrm{T}_0}P_{\mathrm{T}^*}(u\times v)-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2$$

(because $T_0\subseteq T^*$)

$$=\|P_{\mathrm{T}_0}(v\times P_{\mathrm{T}^*}u)-P_{\mathrm{T}_0}(v\times P_{\mathrm{T}_0}u)\|_2$$

(because $v\in L^2(\mu\restriction\mathrm{T}^*)$ and $P_{\mathrm{T}_0}u\in L^2(\mu\restriction\mathrm{T}_0)$, see 242L)

$$\le\|v\times P_{\mathrm{T}^*}u-v\times P_{\mathrm{T}_0}u\|_2\le\|P_{\mathrm{T}^*}u-P_{\mathrm{T}_0}u\|_2$$

(because $\|v\|_\infty\le 1$)

$$\begin{aligned}&=\|\phi\bullet(P_{\mathrm{T}^*}u)-\phi\bullet(P_{\mathrm{T}_0}u)\|_2\\&=\|P_{\phi\bullet\mathrm{T}^*}(\phi\bullet u)-\phi\bullet(P_{\mathrm{T}_0}u)\|_2\\&\le\|P_{\phi\bullet\mathrm{T}^*}u-P_\Lambda u\|_2+\|P_\Lambda u-\phi\bullet(P_{\mathrm{T}_0}u)\|_2\\&\le\epsilon+3\epsilon=4\epsilon.\end{aligned}$$

(vi) Putting these together,

$$\begin{aligned}\|P_\Lambda(u\times v)-P_\Lambda u\times P_\Lambda v\|_2&\le\|P_\Lambda(u\times v)-P_{\mathrm{T}_0}(u\times v)\|_2\\&\qquad+\|P_{\mathrm{T}_0}(u\times v)-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2\\&\qquad+\|P_\Lambda u\times P_\Lambda v-P_{\mathrm{T}_0}u\times P_{\mathrm{T}_0}v\|_2\\&\le\epsilon+4\epsilon+2\epsilon=7\epsilon.\end{aligned}$$

(vii) As ϵ is arbitrary, $P_\Lambda(u\times v)=P_\Lambda u\times P_\Lambda v$, that is, $g\times h$ is a conditional expectation of $\chi(E\cap F)$ on Λ, and E and F are relatively independent over Λ.

(c) It follows that Σ_K and $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ are relatively independent over Λ. **P** Suppose that $E\in\Sigma_K$, and consider the set

$$\mathcal{E}=\{F:F\in\Sigma,\,P_\Lambda\chi(E\cap F)^\bullet=P_\Lambda(\chi E^\bullet)\times P_\Lambda(\chi F^\bullet)\}.$$

Then $\mathcal{E}$ is a Dynkin class, and by (b) above it contains

$$\mathcal{E}_0=\{\bigcap_{\gamma\in\Gamma}F_\gamma:F_\gamma\in\Sigma_{K_\gamma}\text{ for every }\gamma\in\Gamma\},$$

which is closed under $\cap$. Accordingly $\mathcal{E}$ includes the σ-algebra generated by $\mathcal{E}_0$, which is $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$. Thus

$$P_\Lambda\chi(E\cap F)^\bullet=P_\Lambda(\chi E^\bullet)\times P_\Lambda(\chi F^\bullet)$$

for every $E\in\Sigma_K$ and $F\in\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$, and Σ_K and $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ are relatively independent over Λ. **Q**

(d) Now suppose that $\langle J_\gamma\rangle_{\gamma\in\Gamma}$ is a non-empty finite family of subsets of I with infinite intersection. As before, write Λ for $\bigvee_{\gamma\in\Gamma}\Sigma_{J_\gamma}$. Suppose that $K\subseteq I$ and that $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K\cap K_\gamma\subseteq J_\gamma$ for every $\gamma\in\Gamma$. Then Σ_K and $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ are relatively independent over Λ. **P** Set $\mathrm{T}=\bigcup\{\Sigma_L:L\in[K]^{<\omega}\}$ and for $\gamma\in\Gamma$ set $\mathrm{T}_\gamma=\bigcup\{\Sigma_L:L\in[K_\gamma]^{<\omega}\}$. Then (b)-(c) tell us that T and the algebra T' σ-generated by $\bigcup_{\gamma\in\Gamma}\mathrm{T}_\gamma$ are relatively independent over Λ. Since Σ_K is the σ-algebra generated by T, while $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ is the σ-algebra generated by T', 458Da-458Db tell us that Σ_K and $\bigvee_{\gamma\in\Gamma}\Sigma_{K_\gamma}$ are relatively independent over Λ. **Q**

(e) At last we are ready to approach the sets K, $\mathcal{K}$ and $\mathcal{J}$ of the statement of this lemma. The case $\mathcal{J}=\emptyset$ is trivial (as then $\mathcal{K}$ must also be empty), so suppose that $\mathcal{J}$ is non-empty.

(i) To begin with, suppose that $\mathcal{J}$ and $\mathcal{K}$ are finite. In this case, we can find finite families $\langle J_\gamma\rangle_{\gamma\in\Gamma}$ and $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ running over $\mathcal{J}$, $\mathcal{K}\cup\{\emptyset\}$ respectively such that $K\cap K_\gamma\subseteq J_\gamma$ for every γ. So (d) tells us that Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}\vee\Sigma_\emptyset$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$.

(ii) If $\mathcal{K}$ is finite but $\mathcal{J}$ is infinite, then let $\mathcal{J}_0\subseteq\mathcal{J}$ be a finite set such that for every $K'\in\mathcal{K}$ there is a $J\in\mathcal{J}_0$ including $K\cap K'$. Then for any finite $\mathcal{J}'\subseteq\mathcal{J}$ including $\mathcal{J}_0$, Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}'}\Sigma_J$. Since

$$\{\bigvee_{J\in\mathcal{J}'}\Sigma_J:\mathcal{J}_0\subseteq\mathcal{J}'\in[\mathcal{J}]^{<\omega}\}$$

is an upwards-directed family of σ-algebras whose union σ-generates $\bigvee_{J\in\mathcal{J}}\Sigma_J$, 458C tells us that Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$.

(iii) Finally, for the general case, (ii) tells us that Σ_K and $\bigvee_{K'\in\mathcal{K}'}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$ for every finite $\mathcal{K}'\subseteq\mathcal{K}$, so Σ_K and $\bigvee_{K'\in\mathcal{K}}\Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}}\Sigma_J$, by 458D again.

459J Corollary Let (X,Σ,μ) be a probability space and I a set. Let G be the group of permutations ϕ of I such that $\{i:\phi(i)\ne i\}$ is finite. Suppose that $\bullet$ is an action of G on X such that $x\mapsto\phi\bullet x$ is inverse-measure-preserving for each $\phi\in G$. Let $\langle\Sigma_J\rangle_{J\subseteq I}$ be a family of σ-subalgebras of Σ such that

(i) for every $J\subseteq I$, Σ_J is the σ-algebra generated by $\bigcup_{K\subseteq J\text{ is finite}}\Sigma_K$;

(ii) if $J\subseteq I$, $E\in\Sigma_J$ and $\phi\in G$, then $\phi\bullet E\in\Sigma_{\phi[J]}$;

(iii) if $J\subseteq I$, $E\in\Sigma_J$ and $\phi\in G$ is such that $\phi(i)=i$ for every $i\in J$, then $\phi\bullet E=E$.

Then if $J\subseteq I$ is infinite and $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K_\gamma\cap K_\delta\subseteq J$ for all distinct γ, $\delta\in\Gamma$, $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J.

proof By 459I, Σ_{K_γ} and $\bigvee_{\delta\in\Delta}\Sigma_{K_\delta}$ are relatively independent over Σ_J whenever $\Delta\subseteq\Gamma$ and $\gamma\in\Gamma\setminus\Delta$. Now 458Hb tells us that we can induce on $\#(\Delta)$ to see that $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Delta}$ is relatively independent over Σ_J for every finite $\Delta\subseteq\Gamma$, and it follows at once that $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J, as remarked in 458Ac.

proof Note first that if G is the group of permutations ϕ of I such that $\{i:\phi(i)\ne i\}$ is finite, then any $\phi\in G$ is expressible as the product of finitely many transpositions, so $w\mapsto w\phi$ is an automorphism of

(X^I,μ). Let $\bullet$ be the action of G on X^I defined by saying that $\phi\bullet w = w\phi^{-1}$ for $x \in X^I$ and $\phi \in G$. Then $w \mapsto \phi\bullet w$ is inverse-measure-preserving for every ϕ.

If $L \subseteq I$ then Σ_L is the σ-algebra of subsets of X^I generated by sets of the form $\{x : x(i) \in E\}$ where $i \in L$ and $E \in \Sigma$. So Σ_L is the σ-algebra generated by $\bigcup\{\Sigma_K : K \in [L]^{<\omega}\}$.

If $i \in I$, $E \in \Sigma$ and $\phi \in G$, then

$$\phi\bullet\{x : x(i) \in E\} = \{\phi\bullet x : x(i) \in E\} = \{x : (\phi^{-1}\bullet x)(i) \in E\} = \{x : x(\phi(i)) \in E\}.$$

So if $L \subseteq I$ and $\phi \in G$, $\{W : \phi\bullet W \in \Sigma_{\phi[L]}\}$ is a σ-algebra of subsets of X^I containing $\{x : x(i) \in E\}$ whenever $i \in L$ and $E \in \Sigma$, therefore including Σ_L; that is, $\phi\bullet W \in \Sigma_{\phi[L]}$ whenever $W \in \Sigma_L$.

If $L \subseteq I$ and $\phi \in G$ is such that $\phi(i) = i$ for every $i \in L$, then $\{W : \phi\bullet W = W\}$ is a σ-algebra of subsets of X^I containing $\{x : x(i) \in E\}$ whenever $i \in L$ and $E \in \Sigma$, so $\phi\bullet W = W$ for every $W \in \Sigma_L$.

Thus the conditions of 459I are satisfied, and the result follows at once.

459K Following the results of §452 (especially 452Ye), we do not generally expect to find disintegrations of measures which are not countably compact. It may however illuminate the constructions here if I give a specific example related to the contexts of 459E and 459H.

Example (Dubins & Freedman 79) There are a separable metrizable space Z and a quasi-Radon measure on $Z^{\mathbb{N}}$, invariant under permutations of coordinates, which cannot be disintegrated into powers of measures on Z.

proof (a) Let λ be Lebesgue measure on $[0,1]$. $Q = [0,1] \times [0,1]^{\mathbb{N}}$, with its usual topology, is a compact metrizable space, so has just $\mathfrak{c}$ Borel sets (4A3F). Let $\langle W_\xi\rangle_{\xi<\mathfrak{c}}$ enumerate the Borel subsets of Q with non-zero measure for the product measure $\lambda \times \lambda^{\mathbb{N}}$. (Remember that $\lambda \times \lambda^{\mathbb{N}}$ is a Radon measure, by 416U.) For each ξ, we have $0 < (\lambda \times \lambda^{\mathbb{N}})(W_\xi) = \int \lambda^{\mathbb{N}}(W_\xi[\{t\}])\lambda(dt)$, so $A_\xi = \{t : W_\xi[\{t\}] \ne \emptyset\}$ has cardinal $\mathfrak{c}$ (419H); we can therefore choose $\langle t_\xi\rangle_{\xi<\mathfrak{c}}$ in $[0,1]$ such that $t_\xi \in A_\xi \setminus \{t_\eta : \eta < \xi\}$ for every $\xi < \mathfrak{c}$. Now choose $t_{\xi n}$, for $\xi < \mathfrak{c}$ and $n \in \mathbb{N}$, such that $(t_\xi, \langle t_{\xi n}\rangle_{n\in\mathbb{N}}) \in W_\xi$. Set $Z = \{(t_\xi, t_{\xi n}) : \xi < \mathfrak{c},\ n \in \mathbb{N}\} \subseteq [0,1]^2$.

(b) Set $X = ([0,1]^2)^{\mathbb{N}}$ and define $\phi : Q \to X$ by setting $\phi(t, \langle t_n\rangle_{n\in\mathbb{N}}) = \langle (t, t_n)\rangle_{n\in\mathbb{N}}$ for t, $t_n \in [0,1]$. Then ϕ is a homeomorphism between Q and $\phi[Q]$, so there is a unique Radon measure $\mu^\#$ on X such that ϕ is inverse-measure-preserving for $\lambda \times \lambda^{\mathbb{N}}$ and $\mu^\#$. Now $\mu^\#$ is invariant under permutations of coordinates, because if $\rho : \mathbb{N} \to \mathbb{N}$ is a permutation and $\hat\rho(x) = x\rho$ for $x \in X$, then $\hat\rho\phi = \phi\bar\rho$, where $\bar\rho(t, \langle t_n\rangle_{n\in\mathbb{N}}) = (t, \langle t_{\rho(n)}\rangle_{n\in\mathbb{N}})$; and as $\bar\rho : Q \to Q$ is inverse-measure-preserving, so is $\hat\rho : X \to X$.

Also $Z^{\mathbb{N}}$ has full outer measure for $\mu^\#$. **P** If $\mu^\# W > 0$, then $(\lambda \times \lambda^{\mathbb{N}})\phi^{-1}[W] > 0$, so there is some $\xi < \mathfrak{c}$ such that $W_\xi \subseteq \phi^{-1}[W]$. Now $\langle (t_\xi, t_{\xi n})\rangle_{n\in\mathbb{N}} \in Z^{\mathbb{N}} \cap W$. **Q** Accordingly the subspace measure $\tilde\mu$ on Z is a probability measure. Because $\mu^\#$ is invariant under permutations of coordinates, so is $\tilde\mu$; because $\mu^\#$ is a Radon measure, $\tilde\mu$ is a quasi-Radon measure (416Ra).

(c) ? Suppose, if possible, that there are a probability space (Y, T, ν) and a family $\langle \mu_y\rangle_{y\in Y}$ of probability measures on Z such that $\tilde\mu E = \int \mu_y^{\mathbb{N}} E\,\nu(dy)$ for every Borel set $E \subseteq Z^{\mathbb{N}}$. (The argument to follow will not depend on which product measure is used in forming the $\mu_y^{\mathbb{N}}$.) Looking at sets of the form $(Z \cap H) \times Z \times Z \times \ldots$, where $H \subseteq [0,1]^2$ is a Borel set, we see that $\mu_y(Z \cap H)$ must be defined for almost every y; as Z is second-countable, μ_y must be a topological measure for almost every y. Looking at sets of the form $(Z \cap (G_0 \times [0,1])) \times (Z \cap (G_1 \times [0,1])) \times Z \times \ldots$, where G_0 and G_1 are disjoint Borel subsets of $[0,1]$, we see that $\mu_y(Z \cap (G_0 \times [0,1])) \cdot \mu_y(Z \cap (G_1 \times [0,1])) = 0$ for almost every y; as $[0,1]$ is second-countable and Hausdorff, there must be, for almost every $y \in Y$, an $s_y \in [0,1]$ such that $\mu_y(Z \cap (\{s_y\} \times [0,1])) = 1$.

Next, if $G \subseteq [0,1]$ is a Borel set, then $\mu_y(Z \cap ([0,1] \times G)) = \lambda G$ for almost every y. **P**

$$h(y) = \mu_y^{\mathbb{N}}((Z \cap ([0,1] \times G)) \times Z \times \ldots) = \mu_y(Z \cap ([0,1] \times G))$$

is defined for almost every y, and h is ν-integrable, with

$$\int h\,d\nu = \tilde\mu((Z \cap ([0,1] \times G)) \times Z \times \ldots) = \mu^\#(([0,1] \times G) \times [0,1]^2 \times \ldots) = \lambda G.$$

At the same time,

$$\begin{aligned}\int h(y)(1-h(y))\nu(dy) &= \tilde\mu((Z\cap([0,1]\times G))\times(Z\cap([0,1]\times([0,1]\setminus G)))\times Z\times\ldots)\\ &= \mu^{\#}(([0,1]\times G)\times([0,1]\times([0,1]\setminus G))\times[0,1]^2\times\ldots)\\ &= \lambda G(1-\lambda G).\end{aligned}$$

Rearranging, we see that $\int h^2 d\nu = (\int h)^2$. But this means that $\int (h(y)-\int h)^2\nu(dy) = 0$ and $h(y) = \lambda G$ for almost every y. **Q**

It follows that, for at least some y, $\mu_y(Z\cap(\{s_y\}\times G)) = \lambda G$ for every interval $G\subseteq[0,1]$ with rational endpoints. But this is impossible, because all the vertical sections of Z are countable. **X**

Thus there is no such disintegration, as claimed.

459X Basic exercises >(a) Let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an exchangeable sequence of real-valued random variables on X all with finite expectation. Use 459C and 273I to show that $\langle\frac{1}{n+1}\sum_{i=0}^n f_i\rangle_{n\in\mathbb{N}}$ converges a.e. (Compare 276Xg[25].)

(b) Let (X,Σ,μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an exchangeable sequence of real-valued random variables on X all with finite variance, such that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f_i = 0$ a.e. Show that $\langle\Pr(\sum_{i=0}^n f_i\ge \alpha\sqrt{n+1})\rangle_{n\in\mathbb{N}}$ is convergent for every $\alpha\in\mathbb{R}$. (*Hint*: 274I.)

(c) Let X be a completely regular topological space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a totally finite quasi-Radon measure space, and $y\mapsto\mu_y$ a continuous function from Y to the space $M_{\mathrm{qR}}^+(X)$ of totally finite quasi-Radon measures on X with its narrow topology. Show that if $\mu\in M_{\mathrm{qR}}^+(X)$ is such that $\int f d\mu = \iint f d\mu_y\,\nu(dy)$ for every $f\in C_b(X)$, then $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν.

>(d) (Diaconis & Freedman 80) Let Z be a non-empty compact Hausdorff space and I an infinite set including $\mathbb{N}$. Let $\tilde\mu$ be a Radon probability measure on Z^I invariant under permutations of I. For $k\le n$ let $D_{nk}\subseteq n^k$ be the set of injective functions from k to n and Ω_{nk} the set $Z^I\times n^k\times D_{nk}$, endowed with the product λ_{nk} of $\tilde\mu$ and the uniform probability measures on the finite sets n^k and D_{nk}. Define $\phi_{nk}:\Omega_{nk}\to Z^k$ and $\psi_{nk}:\Omega_{nk}\to Z^k$ by setting

$$\begin{aligned}\phi_{nk}(w,p,q) &= wp,\\ \psi_{nk}(w,p,q) &= wp \text{ if } p\in D_{nk},\\ &= wq \text{ otherwise.}\end{aligned}$$

(i) Show that there is a disintegration $\langle\mu^k_{nw}\rangle_{w\in Z^I}$ of the image measure $\lambda_{nk}\phi_{nk}^{-1}$ over $\tilde\mu$ where each μ_{nw} is a suitable point-supported measure on Z^k. (ii) Show that the image measure $\lambda_{nk}\psi_{nk}^{-1}$ is the image measure $\tilde\mu_k = \tilde\mu\tilde\pi_k^{-1}$, where $\tilde\pi_k(w) = w{\restriction}k$ for $w\in Z^I$. (iii) Show that if $n>0$ then $|\tilde\mu_k W - \int\mu^k_{nw}W\,\tilde\mu(dw)|\le\frac{k(k-1)}{2n}$ for every Baire set $W\subseteq Z^k$. (iv) Show that there is a Radon probability measure $\tilde\nu_n$ on $P_{\mathrm{R}}(Z)$ for which $w\mapsto\mu_{nw}$ is inverse-measure-preserving. (v) Show that if $\tilde\nu$ is any cluster point of $\langle\tilde\nu_n\rangle_{n\in\mathbb{N}}$ in $P_{\mathrm{R}}(Z)$ then $\langle\tilde\theta^I\rangle_{\theta\in P_{\mathrm{R}}(Z)}$ is a disintegration of $\tilde\mu$ over $\tilde\nu$, writing $\tilde\theta^I$ for the Radon product of copies of any $\theta\in P_{\mathrm{R}}(Z)$.

>(e) (Hewitt & Savage 55) Let X be a non-empty compact Hausdorff space and I an infinite set. Let Q be the set of Radon probability measures on X^I which are invariant under permutations of I. Show that (i) Q is a closed convex subset of the set $P_{\mathrm{R}}(X^I)$ of all Radon probability measures on X^I with its narrow topology; (ii) Q is isomorphic, as topological convex structure, to $P_{\mathrm{R}}(P_{\mathrm{R}}(X))$; (iii) the extreme points of Q are just the powers of Radon probability measures on X.

(f) Let X, I be sets, Σ a σ-algebra of subsets of X and μ a probability measure with domain $\widehat{\bigotimes}_I\Sigma$ which is transposition-invariant in the sense that for every transposition $\tau:I\to I$ the function $x\mapsto x\tau:X^I\to X^I$ is inverse-measure-preserving. For $J\subseteq I$, let Σ_J be the σ-algebra

[25] Formerly 276Xe.

$$W : W \in \widehat{\bigotimes}_I \Sigma,\ W \text{ is determined by coordinates in } J\}.$$

Show that if $J \subseteq I$ is infinite and $\langle K_\gamma \rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K_\gamma \cap K_\delta \subseteq J$ for all distinct γ, $\delta \in \Gamma$, $\langle \Sigma_{K_\gamma} \rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J (i) using 459D (ii) using 459J.

459Y Further exercises (a) Let X be a topological space and I an infinite set. Write $P_\tau(X)$, $P_\tau(X^I)$ and $P_\tau(P_\tau(X))$ for the spaces of τ-additive Borel probability measures in X, X^I and $P_\tau(X)$ respectively, with their narrow topologies. (i) For $\theta \in P_\tau(X)$ write $\tilde\theta^I$ for the τ-additive Borel measure on X^I corresponding to θ, that is, the restriction to the Borel σ-algebra of X^I of the τ-additive product measure described in 417F. Show that $\theta \mapsto \tilde\theta^I : P_\tau(X) \mapsto P_\tau(X^I)$ is continuous. (ii) Show that if $\nu \in P_\tau(P_\tau(X))$ there is a unique $\mu_\nu \in P_\tau(X^I)$ such that $\langle \tilde\theta^I \rangle_{\theta\in P_\tau(X)}$ is a disintegration of μ_ν over ν, where $\tilde\theta^I$ is the τ-additive Borel product measure on X^I corresponding to $\theta \in P_\tau(X)$. (iii) Show that $\nu \mapsto \mu_\nu$ is a homeomorphism between $P_\tau(P_\tau(X))$ and its image in $P_\tau(X^I)$.

(b) Discuss the problems which arise in 459B, 459C, 459E and 459H if the index set I is finite.

459 Notes and comments As I have presented this material, the centre of the argument of 459A-459H lies in the martingales in part (b-β) of the proof of 459B. We are trying to resolve the functions f_i into 'common' and 'independent' parts. The 'common' part is given by the conditional expectations of the f_i over an appropriate σ-algebra T, and we approach these by looking at the conditional expectations of each f_i on σ-algebras T_n generated by 'distant' f_j. All the most important ideas are already exhibited when the index set I is equal to $\mathbb{N}$. Note in particular that in the basic hypothesis that all finite strings $(f_{i_0}, \dots , f_{i_r})$ have the same joint distribution, it is enough to look at increasing strings. But there is a striking phenomenon which appears in sharper relief with uncountable sets I: any sequence $\langle j_k \rangle_{k\in\mathbb{N}}$ of distinct elements of I can be used to generate an adequate σ-algebra, because while the tail σ-algebra of sets depends on the choice of the j_k, they all lead to the same closed subalgebra of the measure algebra (459D).

Perhaps I should emphasize at this point that I really does have to be infinite, though for large finite I there are approximations to the results here.

The proof of 459B is one of the standard proofs of De Finetti's theorem, with trifling modifications. In the case of real-valued random variables we have a notion of relative distribution (458I) which gives a quick way of saying that all the f_i have the same conditional expectations over T, as in 459C(ii). For variables taking values in other spaces the situation may be different (459K), unless (as in §452) we have a countably compact measure (459E).

Specializing to the case $X = Z^I$ in 459B, we find ourselves examining symmetric measures on infinite product spaces, which are of great interest in themselves. Note that while in the hypothesis of 459E I have asked for the measure μ on the product space Z^I to be countably compact, what is actually necessary is that the marginal measure on Z should be countably compact. By 454Ab, this comes to the same thing.

As in 452O, we can look for a disintegration consisting of Radon measures, provided of course that the marginal measure is a Radon measure. What we have to work harder for is a direct expression in terms of an integral $\int \tilde\theta^I \tilde\nu(d\theta)$ where $\tilde\nu$ is itself a Radon probability measure on the space of Radon probability measures θ (459H). But most of the extra work consists of finding the correct reduction to the case of locally compact spaces. For compact spaces we can approach by a completely different route (459Xd). I will not go farther with this idea here, but I note that the method can be used in a wide variety of problems involving symmetric structures.

Lemma 459I is entirely different. I include it here because it gives another approach to relative independence and looks at permutation-invariant measures, though in a more abstract setting which does not bind us to the product spaces which are their most natural expressions. Its power lies precisely in the fact that in its hypotheses we do *not* suppose that $\Sigma_{J\cup K} = \Sigma_J \vee \Sigma_K$ for J, $K \subseteq I$, so the σ-algebras $\bigvee_{K'\in\mathcal{K}} \Sigma_{K'}$ and $\bigvee_{J\in\mathcal{J}} \Sigma_J$ have to be handled with special care.

Version of 27.2.04

Concordance for Volume 4

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

452I In Fremlin 00 I quote Pachl's result that if (X,Σ,μ) is countably compact, (Y,T,ν) is strictly localizable and $f:X\to Y$ is inverse-measure-preserving, then ν is countably compact; this is now in 452R.

455D The material on Brownian motion in §455, mentioned in König 04 and König 06, has been moved to §477.

458Yd This exercise (on the strong law of large numbers for relatively independent sequences), referred to in the 2008 and 2015 printings of Volume 5, is now 458Ye.

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