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Chapter 44

Topological groups

Measure theory begins on the real line, which is of course a group; and one of the most fundamental properties of Lebesgue measure is its translation-invariance. Later we come to the standard measure on the unit circle, and counting measure on the integers is also translation-invariant, if we care to notice; moreover, Fourier series and transforms clearly depend utterly on the fact that shift operators don't disturb the measure-theoretic structures we are building. Yet another example appears in the usual measure on $\{0,1\}^I$, which is translation-invariant if we identify $\{0,1\}^I$ with the group $\mathbb{Z}_2^I$. Each of these examples is special in many other ways. But it turns out that a particular combination of properties which they share, all being locally compact Hausdorff spaces with group operations for which multiplication and inversion are continuous, is the basis of an extraordinarily powerful theory of invariant measures.

As usual, I have no choice but to move rather briskly through a wealth of ideas. The first step is to set out a suitably general existence theorem, assuring us that every locally compact Hausdorff topological group has non-trivial invariant Radon measures, that is, 'Haar measures'. As remarkable as the existence of Haar measures is their (essential) uniqueness; the algebra, topology and measure theory of a topological group are linked in so many ways that they form a peculiarly solid structure. I investigate a miscellany of facts about this structure in §443, including the basic theory of the modular functions linking left-invariant measures with right-invariant measures.

I have already mentioned that Fourier analysis depends on the translation-invariance of Lebesgue measure. It turns out that substantial parts of the abstract theory of Fourier series and transforms can be generalized to arbitrary locally compact groups. In particular, convolutions appear again, even in non-abelian groups. But for the central part of the theory, a transform relating functions on a group X to functions on its 'dual' group $\mathcal{X}$, we do need the group to be abelian. Actually I give only the foundation of this theory: if X is an abelian locally compact Hausdorff group, it is the dual of its dual. (In 'ordinary' Fourier theory, where we are dealing with the cases $X=\mathcal{X}=\mathbb{R}$ and $X=S^1$, $\mathcal{X}=\mathbb{Z}$, this duality is so straightforward that one hardly notices it.) But on the way to the duality theorem we necessarily see many of the themes of Chapter 28 in more abstract guises.

A further remarkable fact is that any Haar measure has a translation-invariant lifting. The proof demands a union between the ideas of the ordinary Lifting Theorem and some of the elaborate structure theory which has been developed for locally compact groups.

For the last two sections of the chapter, I look at groups which are not locally compact, and their actions on appropriate spaces. For a particularly important class of group actions, Borel measurable actions of Polish groups on Polish spaces, we have a natural necessary and sufficient condition for the existence of an invariant measure, complementing the result for locally compact spaces in 441C. In a slightly different direction, we can look at those groups, the 'amenable' groups, for which all actions (on compact Hausdorff spaces) have invariant measures. This again leads to some very remarkable ideas, which I sketch in §449.

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441 Invariant measures on locally compact spaces

I begin this chapter with the most important theorem on the existence of invariant measures: every locally compact Hausdorff group has left and right Haar measures (441E). I derive this as a corollary of a general result concerning invariant measures on locally compact spaces (441C), which has other interesting consequences (441H).

441A Group actions I repeat the definitions on which this chapter is based.

(a) If G is a group and X is a set, an **action** of G on X is a function $(a,x)\mapsto a\bullet x : G\times X\to X$ such that

$$(ab)\bullet x = a\bullet(b\bullet x) \text{ for all } a,\, b\in G,\, x\in X,$$

$$e\bullet x = x \text{ for every } x\in X$$

where e is the identity of G. In this context I write

$$a\bullet A = \{a\bullet x : x\in A\}$$

for $a\in G$, $A\subseteq X$. If f is a function defined on a subset of X, then $(a\bullet f)(x) = f(a^{-1}\bullet x)$ whenever $a\in G$ and $x\in X$ and $a^{-1}\bullet x\in \operatorname{dom} f$.

(b) If a group G acts on a set X, a measure μ on X is G**-invariant** if $\mu(a^{-1}\bullet E)$ is defined and equal to μE whenever $a\in G$ and μ measures E.

(c) If a group G acts on a set X and a measure μ on X is G-invariant, then $\int f(a\bullet x)\mu(dx)$ is defined and equal to $\int f\,d\mu$ whenever f is a virtually measurable $[-\infty,\infty]$-valued function defined on a conegligible subset of X and $\int f\,d\mu$ is defined in $[-\infty,\infty]$.

441B Lemma Let X be a topological space, G a group, and $\bullet$ an action of G on X such that $x\mapsto a\bullet x$ is continuous for every $a\in G$.

(a) If μ is a quasi-Radon measure on X such that $\mu(a\bullet U)\le \mu U$ for every open set $U\subseteq X$ and every $a\in G$, then μ is G-invariant.

(b) If μ is a Radon measure on X such that $\mu(a\bullet K)\le\mu K$ for every compact set $K\subseteq X$ and every $a\in G$, then μ is G-invariant.

441C Theorem Let X be a non-empty locally compact Hausdorff space and G a group acting on X. Suppose that

(i) $x\mapsto a\bullet x$ is continuous for every $a\in G$;

(ii) every orbit $\{a\bullet x : a\in G\}$ is dense;

(iii) whenever K and L are disjoint compact subsets of X there is a non-empty open subset U of X such that, for every $a\in G$, at most one of K, L meets $a\bullet U$.

Then there is a non-zero G-invariant Radon measure μ on X.

441D Definition If G is a topological group, a **left Haar measure** on G is a non-zero quasi-Radon measure μ on G which is invariant for the left action of G on itself.

Similarly, a **right Haar measure** is a non-zero quasi-Radon measure μ such that $\mu(Ea)=\mu E$ whenever $E\in\operatorname{dom}\mu$ and $a\in G$.

441E Theorem A locally compact Hausdorff topological group has left and right Haar measures, which are both Radon measures.

441F Definition If (X,ρ) is any metric space, its **isometry group** is the set of permutations $g : X\to X$ which are **isometries**, that is, $\rho(g(x),g(y))=\rho(x,y)$ for all x, $y\in X$.

441G The topology of an isometry group Let (X,ρ) be a metric space and G the isometry group of X.

(a) Give G the topology of pointwise convergence inherited from the product topology of X^X. Then G is a Hausdorff topological group and the action of G on X is continuous.

(b) If X is compact, so is G.

441H Theorem If (X,ρ) is a non-empty locally compact metric space with isometry group G, then there is a non-zero G-invariant Radon measure on X.

441J Proposition Let X be a set, G a group acting on X, and μ a G-invariant measure on X. If f is a real-valued function defined on a subset of X, and $a\in G$, then $\int f(x)\mu(dx)=\int f(a{\bullet}x)\mu(dx)$ if either integral is defined in $[-\infty,\infty]$.

441K Theorem Let X be a set, G a group acting on X, and μ a G-invariant measure on X with measure algebra $\mathfrak{A}$.

(a) We have an action of G on $\mathfrak{A}$ defined by setting $a{\bullet}E^{\bullet}=(a{\bullet}E)^{\bullet}$ whenever $a\in G$ and μ measures E.

(b) We have an action of G on $L^0=L^0(\mu)$ defined by setting $a{\bullet}f^{\bullet}=(a{\bullet}f)^{\bullet}$ for every $a\in G$, $f\in\mathcal{L}^0(\mu)$.

(c) For $1\le p\le\infty$ the formula of (b) defines actions of G on $L^p=L^p(\mu)$, and $\|a{\bullet}u\|_p=\|u\|_p$ whenever $u\in L^p$ and $a\in G$.

441L Proposition Let X be a locally compact Hausdorff space and G a group acting on X in such a way that $x\mapsto a{\bullet}x$ is continuous for every $a\in G$. If μ is a Radon measure on X, then μ is G-invariant iff $\int f(x)\mu(dx)=\int f(a{\bullet}x)\mu(dx)$ for every $a\in G$ and every continuous function $f:X\to\mathbb{R}$ with compact support.

Version of 21.3.07

442 Uniqueness of Haar measures

Haar measure has an extraordinary wealth of special properties, and it will be impossible for me to cover them all properly in this chapter. But surely the second thing to take on board, after the existence of Haar measures on locally compact Hausdorff groups, is the fact that they are, up to scalar multiples, unique. This is the content of 442B. We find also that while left and right Haar measures can be different, they are not only direct mirror images of each other (442C) – as is, I suppose, to be expected – but even more closely related (442F, 442H, 442L). Investigating this relation, we are led naturally to the 'modular function' of a group (442I).

442A Lemma Let X be a topological group and μ a left Haar measure on X.

(a) μ is strictly positive and locally finite.

(b) If $G\subseteq X$ is open and $\gamma<\mu G$, there are an open set H and an open neighbourhood U of the identity such that $HU\subseteq G$ and $\mu H\ge\gamma$.

(c) If X is locally compact and Hausdorff, μ is a Radon measure.

442B Theorem Let X be a topological group. If μ and ν are left Haar measures on X, they are multiples of each other.

442C Proposition Let X be a topological group and μ a left Haar measure on X. Setting $\nu E=\mu(E^{-1})$ whenever $E\subseteq X$ is such that $E^{-1}=\{x^{-1}:x\in E\}$ is measured by μ, ν is a right Haar measure on X.

442D Remark Clearly all the arguments of 442A-442C must be applicable to right Haar measures. Thus we may say that a topological group **carries Haar measures** if it has either a left or a right Haar measure.

442E Lemma Let X be a topological group, μ a left Haar measure on X and ν a right Haar measure on X. If G, $H\subseteq X$ are open, then

$$\mu G\cdot\nu H=\int_H\nu(xG^{-1})\mu(dx).$$

442F Domains of Haar measures: Proposition Let X be a topological group which carries Haar measures. If μ is a left Haar measure and ν is a right Haar measure on X, then they have the same domains and the same null ideals.

442G Corollary Let X be a topological group and μ a left Haar measure on X with domain Σ. Then, for $E \subseteq X$ and $a \in X$,

$$E \in \Sigma \iff E^{-1} \in \Sigma \iff Ea \in \Sigma,$$

$$\mu E = 0 \iff \mu E^{-1} = 0 \iff \mu(Ea) = 0.$$

442H Remark If X is any topological group which carries Haar measures, there is a distinguished σ-algebra Σ of subsets of X, which we may call the algebra of **Haar measurable sets**, which is the domain of any Haar measure on X. Similarly, there is a σ-ideal $\mathcal{N}$ of $\mathcal{P}X$, the ideal of **Haar negligible sets**[1], which is the null ideal for any Haar measure on X. Both Σ and $\mathcal{N}$ are translation-invariant and also invariant under the inversion operation $x \mapsto x^{-1}$.

If we form the quotient $\mathfrak{A} = \Sigma/\mathcal{N}$, then we have a fixed Dedekind complete Boolean algebra which is the **Haar measure algebra** of the group X in the sense that any Haar measure on X, whether left or right, has measure algebra based on $\mathfrak{A}$. If $a \in X$, the maps $x \mapsto ax$, $x \mapsto xa$, $x \mapsto x^{-1}$ give rise to Boolean automorphisms of $\mathfrak{A}$.

442I The modular function Let X be a topological group which carries Haar measures.

(a) There is a group homomorphism $\Delta : X \to]0, \infty[$ defined by the formula

$$\mu(Ex) = \Delta(x)\mu E \text{ whenever } \mu \text{ is a left Haar measure on } X \text{ and } E \in \operatorname{dom}\mu.$$

Δ is called the **left modular function** of X.

(b) We find now that $\nu(xE) = \Delta(x^{-1})\nu E$ whenever ν is a right Haar measure on X, $x \in X$ and $E \subseteq X$ is Haar measurable.

Thus we may call $x \mapsto \Delta(x^{-1}) = \frac{1}{\Delta(x)}$ the **right modular function** of X.

(c) If X is abelian, then $\Delta(x) = 1$ for every $x \in X$. Equally, if any left (or right) Haar measure μ on X is totally finite, then $\Delta(x) = 1$ for every $x \in X$. This will be the case for any compact Hausdorff topological group.

Groups in which $\Delta(x) = 1$ for every x are called **unimodular**.

(d) A topological group carrying Haar measures is unimodular iff every left Haar measure is a right Haar measure.

(e) In particular, if a group has any totally finite (left or right) Haar measure, its left and right Haar measures are the same, and it has a unique Haar probability measure, which we may call its **normalized Haar measure**.

In the other direction, any group with its discrete topology is unimodular, since counting measure is a two-sided Haar measure.

442J Proposition For any topological group carrying Haar measures, its left modular function is continuous.

442K Theorem Let X be a topological group and μ a left Haar measure on X. Let Δ be the left modular function of X.

(a) $\mu(E^{-1}) = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu$.

[1]**Warning!** do not confuse with the 'Haar null' sets described in 444Ye below.

(b)(i) $\int f(x^{-1})\mu(dx) = \int \Delta(x^{-1})f(x)\mu(dx)$ whenever f is a real-valued function such that either integral is defined in $[-\infty, \infty]$;

(ii) $\int f(x)\mu(dx) = \int \Delta(x^{-1})f(x^{-1})\mu(dx)$ whenever f is a real-valued function such that either integral is defined in $[-\infty, \infty]$.

(c) $\int f(xy)\mu(dx) = \Delta(y^{-1})\int f(x)\mu(dx)$ whenever $y \in X$ and f is a real-valued function such that either integral is defined in $[-\infty, \infty]$.

442L Corollary Let X be a group carrying Haar measures. If μ is a left Haar measure on X and ν is a right Haar measure, then each is an indefinite-integral measure over the other.

442Z Problem Let X be a compact Hausdorff space, and G the group of autohomeomorphisms of X. Suppose that G acts transitively on X. Does it follow that there is at most one G-invariant Radon probability measure on X?

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443 Further properties of Haar measure

I devote a section to filling in some details of the general theory of Haar measures before turning to the special topics dealt with in the rest of the chapter. The first question concerns the left and right shift operators acting on sets, on elements of the measure algebra, on measurable functions and on function spaces. All these operations can be regarded as group actions, and, if appropriate topologies are assigned, they are continuous actions (443C, 443G). As an immediate consequence of this I give an important result about product sets $\{ab : a \in A, b \in B\}$ in a topological group carrying Haar measures (443D).

The second part of the section revolves around a basic structure theorem: all the Haar measures considered here can be reduced to Haar measures on locally compact Hausdorff groups (443L). The argument involves two steps: the reduction to the Hausdorff case, which is elementary, and the completion of a Hausdorff topological group. Since a group carries more than one natural uniform structure we must take care to use the correct one, which in this context is the 'bilateral' uniformity (443H-443I, 443K). On the way I pick up an essential fact about the approximation of Haar measurable sets by Borel sets (443J). Finally, I give Halmos' theorem that Haar measures are completion regular (443M) and a note on the complementary nature of the meager and null ideals for atomless Haar measure (443O).

In the third part of the section I turn to the special properties of quotient groups of locally compact groups and the corresponding actions, following A.Weil. If X is a locally compact Hausdorff group and Y is a closed subgroup of X, then Y is again a locally compact Hausdorff group, so has Haar measures and a modular function; at the same time, we have a natural action of X on the set of left cosets of Y. It turns out that there is an invariant Radon measure for this action if and only if the modular function of Y matches that of X (443R). In this case we can express a left Haar measure of X as an integral of measures supported by the cosets of Y (443Q). When Y is a normal subgroup, so that X/Y is itself a locally compact Hausdorff group, we can relate the modular functions of X and X/Y (443T). We can apply these results whenever we have a continuous transitive action of a compact group on a compact space (443U).

443A Haar measurability Let X be a topological group carrying Haar measures.

(a) All Haar measures on X, whether left or right, have the same domain Σ, the algebra of 'Haar measurable' sets, and the same null ideal $\mathcal{N}$, the ideal of 'Haar negligible' sets. The corresponding quotient algebra $\mathfrak{A} = \Sigma/\mathcal{N}$, the 'Haar measure algebra', is the Boolean algebra underlying the measure algebra of any Haar measure. Σ_G is closed under Souslin's operation and $\mathfrak{A}$ is Dedekind complete. Recall that any semi-finite measure on $\mathfrak{A}$ gives rise to the same measure-algebra topology and uniformity on $\mathfrak{A}$, so we may speak of 'the' topology and uniformity of $\mathfrak{A}$.

$xE \in \Sigma$ whenever $E \in \Sigma$ and $x \in X$; $Ex \in \Sigma$ whenever $E \in \Sigma$ and $x \in X$. xE and Ex are Haar negligible whenever E is Haar negligible and $x \in X$. E^{-1} is Haar measurable or Haar negligible whenever E is.

Note that Σ and $\mathcal{N}$ are invariant in the strong sense that if $\phi : X \to X$ is any group automorphism which is also a homeomorphism, then $\Sigma = \{\phi[E] : E \in \Sigma\}$ and $\mathcal{N} = \{\phi[E] : E \in \mathcal{N}\}$.

(b) We have a symmetric notion of 'measurable envelope' in X: for any $A \subseteq X$, there is a Haar measurable set $E \supseteq A$ such that $\mu(E \cap F) = \mu^*(A \cap F)$ for any Haar measurable $F \subseteq X$ and any Haar measure μ on X. In this context I will call E a **Haar measurable envelope** of A.

(c) Similarly, we have a notion of **full outer Haar measure**: a subset A of X is of full outer Haar measure if X is a Haar measurable envelope of A.

(d) For any Haar measure μ on X, we can identify $L^\infty(\mu)$ with $L^\infty(\mathfrak{A})$ and $L^0(\mu)$ with $L^0(\mathfrak{A})$. Thus these constructions are independent of μ. The topology of convergence in measure of L^0 is independent of the particular Haar measure we may select. Of course the same is true of the norm of L^∞.

(e) I will use the phrases **Haar measurable function**, meaning a function measurable with respect to the σ-algebra of Haar measurable sets, and **Haar almost everywhere**, meaning 'on the complement of a Haar negligible set'. Note that we can identify $L^0(\mathfrak{A})$ with the set of equivalence classes in the space $\mathcal{L}^0$, where $\mathcal{L}^0$ is the space of Haar measurable real-valued functions defined Haar-a.e. in X, and $f \sim g$ if $f = g$ Haar-a.e.

(f) We have a canonical automorphism $a \mapsto \overset{\leftrightarrow}{a} : \mathfrak{A} \to \mathfrak{A}$ defined by writing $(E^\bullet)^\leftrightarrow = (E^{-1})^\bullet$ for every $E \in \Sigma$. Being an automorphism, this must be a homeomorphism for the measure-algebra topology of $\mathfrak{A}$. If $f \in \mathcal{L}^0$ then $\overset{\leftrightarrow}{f} \in \mathcal{L}^0$, where $\overset{\leftrightarrow}{f}(x) = f(x^{-1})$ whenever this is defined; and we can define an f-algebra automorphism $u \mapsto \overset{\leftrightarrow}{u} : L^0 \to L^0$ by saying that $(f^\bullet)^\leftrightarrow = (\overset{\leftrightarrow}{f})^\bullet$ for $f \in \mathcal{L}^0$. If we identify L^0 with $L^0(\mathfrak{A})$, then we can define the map $u \mapsto \overset{\leftrightarrow}{u}$ as the Riesz homomorphism associated with the Boolean homomorphism $a \mapsto \overset{\leftrightarrow}{a} : \mathfrak{A} \to \mathfrak{A}$.

(g) If X carries any totally finite Haar measure, it is unimodular, and has a unique, two-sided, Haar probability measure. For such groups we have L^p-spaces, for $1 \le p \le \infty$, defined by the group structure, with canonical norms.

443B Lemma Let X be a topological group and μ a left Haar measure on X. If $E \subseteq X$ is measurable and $\mu E < \infty$, then for any $\epsilon > 0$ there is a neighbourhood U of the identity e such that $\mu(E \triangle xEy) \le \epsilon$ whenever x, $y \in U$.

443C Theorem Let X be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra. Then we have continuous actions of X on $\mathfrak{A}$ defined by writing

$$x \bullet_l E^\bullet = (xE)^\bullet, \quad x \bullet_r E^\bullet = (Ex^{-1})^\bullet, \quad x \bullet_c E^\bullet = (xEx^{-1})^\bullet$$

for Haar measurable sets $E \subseteq X$ and $x \in X$.

443D Proposition Let X be a topological group carrying Haar measures. If $E \subseteq X$ is Haar measurable but not Haar negligible, and $A \subseteq X$ is not Haar negligible, then

(a) there are x, $y \in X$ such that $A \cap xE$, $A \cap Ey$ are not Haar negligible;

(b) EA and AE both have non-empty interior;

(c) $E^{-1}E$ and EE^{-1} are neighbourhoods of the identity.

443E Corollary Let X be a Hausdorff topological group carrying Haar measures. Then the following are equiveridical:

(i) X is locally compact;

(ii) every Haar measure on X is a Radon measure;

(iii) there is some compact subset of X which is not Haar negligible.

443F Lemma Let X be a topological group carrying Haar measures, and Y an open subgroup of X. If μ is a left Haar measure on X, then the subspace measure μ_Y is a left Haar measure on Y. Consequently a subset of Y is Haar measurable or Haar negligible, when regarded as a subset of the topological group Y, iff it is Haar measurable or Haar negligible when regarded as a subset of the topological group X.

443G Theorem Let X be a topological group with a left Haar measure μ. Let Σ be the domain of μ, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ the space of Σ-measurable real-valued functions defined almost everywhere in X, and $L^0 = L^0(\mu)$ the corresponding space of equivalence classes.

(a) $a{\bullet_l}f$, $a{\bullet_r}f$ and $a{\bullet_c}f$ belong to $\mathcal{L}^0$ for every $f \in \mathcal{L}^0$ and $a \in X$.

(b) If $a \in X$, then $\operatorname{ess\,sup}|a{\bullet_l}f| = \operatorname{ess\,sup}|a{\bullet_r}f| = \operatorname{ess\,sup}|f|$ for every $f \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$. For $1 \le p < \infty$, $\|a{\bullet_l}f\|_p = \|f\|_p$ and $\|a{\bullet_r}f\|_p = \Delta(a)^{-1/p}\|f\|_p$ for every $f \in \mathcal{L}^p = \mathcal{L}^p(\mu)$, where Δ is the left modular function of X.

(c) We have shift actions of X on L^0 defined by setting

$$a{\bullet_l}f^\bullet = (a{\bullet_l}f)^\bullet, \quad a{\bullet_r}f^\bullet = (a{\bullet_r}f)^\bullet, \quad a{\bullet_c}f^\bullet = (a{\bullet_c}f)^\bullet$$

for $a \in X$ and $f \in \mathcal{L}^0$. If $^\leftrightarrow$ is the reversal operator on L^0 defined in 443Af, we have

$$a{\bullet_l}\overset{\leftrightarrow}{u} = (a{\bullet_r}u)^\leftrightarrow, \quad a{\bullet_c}\overset{\leftrightarrow}{u} = (a{\bullet_c}u)^\leftrightarrow$$

for every $a \in X$ and $u \in L^0$.

(d) If we give L^0 its topology of convergence in measure these three actions, and also the reversal operator $^\leftrightarrow$, are continuous.

(e) For $1 \le p \le \infty$ the formulae of (c) define actions of X on $L^p = L^p(\mu)$, and $\|a{\bullet_l}u\|_p = \|u\|_p$ for every $u \in L^p$, $a \in X$; interpreting $\Delta(a)^{-1/\infty}$ as 1 if necessary, $\|a{\bullet_r}u\|_p = \Delta(a)^{-1/p}\|u\|_p$ whenever $u \in L^p$ and $a \in X$.

(f) For $1 \le p < \infty$ these actions are continuous.

443H Theorem Let X be a topological group carrying Haar measures. Then there is a neighbourhood of the identity which is totally bounded for the bilateral uniformity on X.

443I Corollary Let X be a topological group. If $A \subseteq X$ is totally bounded for the bilateral uniformity of X, it has finite outer measure for any (left or right) Haar measure on X.

443J Proposition Let X be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra.

(a) There is an open-and-closed subgroup Y of X such that, for any Haar measure μ on X, Y can be covered by countably many open sets of finite measure.

(b)(i) If $E \subseteq X$ is any Haar measurable set, there are an F_σ set $E' \subseteq E$ and a G_δ set $E'' \supseteq E$ such that $E'' \setminus E'$ is Haar negligible.

(ii) Every Haar negligible set is included in a Haar negligible Borel set, and for every Haar measurable set E there is a Borel set F such that $E \triangle F$ is Haar negligible.

(iii) The Haar measure algebra $\mathfrak{A}$ of X may be identified with $\mathcal{B}/\mathcal{I}$, where $\mathcal{B}$ is the Borel σ-algebra of X and $\mathcal{I}$ is the ideal of Haar negligible Borel sets.

(iv) Every member of $L^0(\mathfrak{A})$ can be identified with the equivalence class of some Borel measurable function from X to $\mathbb{R}$. Every member of $L^\infty(\mathfrak{A})$ can be identified with the equivalence class of a bounded Borel measurable function from X to $\mathbb{R}$.

443K Theorem Let X be a Hausdorff topological group carrying Haar measures. Then the completion $\widehat{X}$ of X under its bilateral uniformity is a locally compact Hausdorff group, and X is of full outer Haar measure in $\widehat{X}$. Any Haar measure on X is the subspace measure corresponding to a Haar measure on $\widehat{X}$.

443L Corollary Let X be any topological group with a Haar measure μ. Then we can find Z, λ and ϕ such that

(i) Z is a locally compact Hausdorff topological group;

(ii) λ is a Haar measure on Z;

(iii) $\phi : X \to Z$ is a continuous homomorphism, inverse-measure-preserving for μ and λ;

(iv) μ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$;

(v) if $E \subseteq X$ is Haar measurable, we can find a Haar measurable set $F \subseteq Z$ such that $\phi^{-1}[F] \subseteq E$ and $E \setminus \phi^{-1}[F]$ is Haar negligible;

(vi) a set $G \subseteq X$ is an open set in X iff it is of the form $\phi^{-1}[H]$ for some open set $H \subseteq Z$;

(vii) a set $G \subseteq X$ is a regular open set in X iff it is of the form $\phi^{-1}[H]$ for some regular open set $H \subseteq Z$;
(viii) a set $A \subseteq X$ is nowhere dense in X iff $\phi[A]$ is nowhere dense in Z.

443M Theorem Let X be a topological group and μ a Haar measure on X. Then μ is completion regular.

443N Proposition Let X be a topological group carrying Haar measures.
(i) Let G be a regular open subset of X. Then G is a cozero set.
(ii) Let F be a nowhere dense subset of X. Then F is included in a nowhere dense zero set.

443O Proposition Let X be a topological group and μ a left Haar measure on X. Then the following are equiveridical:
(i) μ is not purely atomic;
(ii) μ is atomless;
(iii) there is a non-negligible nowhere dense subset of X;
(iv) μ is inner regular with respect to the nowhere dense sets;
(v) there is a conegligible meager subset of X;
(vi) there is a negligible comeager subset of X.
If X is Hausdorff, we can add
(vii) the topology of X is not discrete.

443P Quotient spaces: Lemma Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology and $\pi : X \mapsto Z$ the canonical map, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a\bullet\pi x = \pi(ax)$ for a, $x \in X$. Let ν be a left Haar measure on Y and write $C_k(X)$, $C_k(Z)$ for the spaces of continuous real-valued functions with compact supports on X, Z respectively.

(a) We have a positive linear operator $T : C_k(X) \to C_k(Z)$ defined by writing

$$(Tf)(\pi x) = \int_Y f(xy)\nu(dy)$$

for every $f \in C_k(X)$ and $x \in X$. If $f > 0$ in $C_k(X)$ then $Tf > 0$ in $C_k(Z)$. If $h \ge 0$ in $C_k(Z)$ then there is an $f \ge 0$ in $C_k(X)$ such that $Tf = h$.

(b) If $a \in X$ and $f \in C_k(X)$, then $T(a\bullet_l f)(z) = (Tf)(a^{-1}\bullet z)$ for every $z \in Z$.

(c) Now suppose that a belongs to the normalizer of Y. In this case, we can define $\psi(a) \in]0, \infty[$ by the formula

$$\nu(aFa^{-1}) = \psi(a)\nu F \text{ for every } F \in \operatorname{dom}\nu,$$

and

$$T(a\bullet_r f)(\pi x) = \psi(a) \cdot (Tf)(\pi(xa))$$

for every $x \in X$ and $f \in C_k(X)$.

443Q Theorem Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology, and $\pi : X \to Z$ the canonical map, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a\bullet\pi x = \pi(ax)$ for a, $x \in X$. Let ν be a left Haar measure on Y. Suppose that λ is a non-zero X-invariant Radon measure on Z.

(a) For each $z \in Z$, we have a Radon measure ν_z on X defined by the formula

$$\nu_z E = \nu(Y \cap x^{-1}E)$$

whenever $\pi x = z$ and the right-hand side is defined. In this case, for a real-valued function f defined on a subset of X,

$$\int f\,d\nu_z = \int f(xy)\nu(dy)$$

whenever either side is defined in $[-\infty, \infty]$.

(b) We have a left Haar measure μ on X defined by the formulae

$$\int f\,d\mu = \iint f\,d\nu_z \lambda(dz)$$

for every $f \in C_k(X)$, and

$$\mu G = \int \nu_z G\,\lambda(dz)$$

for every open set $G \subseteq X$.

(c) If $D \subseteq Z$, then $D \in \operatorname{dom}\lambda$ iff $\pi^{-1}[D] \subseteq X$ is Haar measurable, and $\lambda D = 0$ iff $\pi^{-1}[D]$ is Haar negligible.

(d) If $\nu Y = 1$, then λ is the image measure $\mu\pi^{-1}$.

(e) Suppose now that X is σ-compact. Then $\mu E = \int \nu_z E\,\lambda(dz)$ for every Haar measurable set $E \subseteq X$. If $f \in \mathcal{L}^1(\mu)$, then $\int f\,d\mu = \iint f\,d\nu_z \lambda(dz)$.

(f) Still supposing that X is σ-compact, take $f \in \mathcal{L}^1(\mu)$, and for $a \in X$ set $f_a(y) = f(ay)$ whenever $y \in Y$ and $ay \in \operatorname{dom} f$. Then $Q_f = \{a : a \in X,\ f_a \in \mathcal{L}^1(\nu)\}$ is μ-conegligible, and the function $a \mapsto f_a^\bullet : Q_f \to L^1(\nu)$ is almost continuous.

443R Theorem Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a\bullet(xY) = axY$ for a, $x \in X$. Let Δ_X be the left modular function of X and Δ_Y the left modular function of Y. Then the following are equiveridical:

(i) there is a non-zero X-invariant Radon measure λ on Z;

(ii) Δ_Y is the restriction of Δ_X to Y.

443S Applications Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X.

(a) If Y is a normal subgroup of X, then $\Delta_Y = \Delta_X\restriction Y$.

Note that in this context any of the invariant measures λ of 443Q must be left Haar measures on the quotient group.

(b) If Y is compact, then $\Delta_Y = \Delta_X\restriction Y$. So we have an X-invariant Radon measure λ on X/Y. Since Y has a Haar probability measure, λ will be the image of a left Haar measure under the canonical map.

(c) If, in (b), Y is a normal subgroup, then $\Delta_{X/Y}\pi = \Delta_X$, writing $\pi : X \to X/Y$ for the canonical map.

(d) If Y is open, $\Delta_Y = \Delta_X\restriction Y$.

443T Theorem Let X be a locally compact Hausdorff topological group and Y a closed normal subgroup of X; let $Z = X/Y$ be the quotient group, and $\pi : X \to Z$ the canonical map. Write Δ_X, Δ_Z for the left modular functions of X, Z respectively. Define $\psi : X \to \,]0, \infty[$ by the formula

$$\nu(aFa^{-1}) = \psi(a)\nu F \text{ whenever } F \in \operatorname{dom}\nu \text{ and } a \in X,$$

where ν is a left Haar measure on Y. Then

$$\Delta_Z(\pi a) = \psi(a)\Delta_X(a)$$

for every $a \in X$.

443U Transitive actions: Theorem Let X be a compact Hausdorff topological group, Z a non-empty compact Hausdorff space, and $\bullet$ a transitive continuous action of X on Z. Write $\pi_z(x) = x\bullet z$ for $z \in Z$ and $x \in X$.

(a) For every $z \in Z$, $Y_z = \{x : x \in X,\ x\bullet z = z\}$ is a compact subgroup of X. If we give the set X/Y_z of left cosets of Y_z in X its quotient topology, we have a homeomorphism $\phi_z : X/Y_z \to Z$ defined by the formula $\phi_z(xY_z) = x\bullet z$ for every $x \in X$.

(b) Let μ be a Haar probability measure on X. Then the image measure $\mu\pi_z^{-1}$ is an X-invariant Radon probability measure on Z, and $\mu\pi_w^{-1}=\mu\pi_z^{-1}$ for all w, $z\in Z$.

(c) Every non-zero X-invariant Radon measure on Z is of the form $\mu\pi_z^{-1}$ for a Haar measure μ on X and some (therefore any) $z\in Z$.

(d) There is a strictly positive X-invariant Radon probability measure on Z, and any two non-zero X-invariant Radon measures on Z are scalar multiples of each other.

(e) Take any $z\in Z$, and let ν be the Haar probability measure of Y_z. If μ is a Haar measure on X, then

$$\mu E=\int\nu(Y_z\cap x^{-1}E)\mu(dx)$$

whenever $E\subseteq X$ is Haar measurable.

Version of 23.7.07/4.6.13

444 Convolutions

In this section, I look again at the ideas of §§255 and 257, seeking the appropriate generalizations to topological groups other than $\mathbb{R}$. Following Hewitt & Ross 63, I begin with convolutions of measures (444A-444E) before proceeding to convolutions of functions (444O-444V); in between, I mention the convolution of a function and a measure (444G-444M) and a general result concerning continuous group actions on quasi-Radon measure spaces (444F).

While I continue to give the results in terms of real-valued functions, the applications of the ideas here in the next section will be to complex-valued functions; so you may wish to keep the complex case in mind.

444A Convolution of measures: Proposition If X is a topological group and λ and ν are two totally finite quasi-Radon measures on X, we have a quasi-Radon measure $\lambda*\nu$ on X defined by saying that

$$\begin{aligned}(\lambda*\nu)(E)&=(\lambda\times\nu)\{(x,y):xy\in E\}\\&=\int\nu(x^{-1}E)\lambda(dx)=\int\lambda(Ey^{-1})\nu(dy)\end{aligned}$$

for every $E\in\mathrm{dom}(\lambda*\nu)$, where $\lambda\times\nu$ is the quasi-Radon product measure on $X\times X$.

444B Proposition If X is a topological group, $\lambda_1*(\lambda_2*\lambda_3)=(\lambda_1*\lambda_2)*\lambda_3$ for all totally finite quasi-Radon measures λ_1, λ_2 and λ_3 on X.

444C Theorem Let X be a topological group and λ, ν two totally finite quasi-Radon measures on X. Then

$$\int fd(\lambda*\nu)=\int f(xy)(\lambda\times\nu)d(x,y)=\iint f(xy)\lambda(dx)\nu(dy)=\iint f(xy)\nu(dy)\lambda(dx)$$

for any $(\lambda*\nu)$-integrable real-valued function f. In particular, $(\lambda*\nu)(X)=\lambda X\cdot\nu X$.

444D Proposition Let X be an abelian topological group. Then $\lambda*\nu=\nu*\lambda$ for all totally finite quasi-Radon measures λ, μ on X.

444E The Banach algebra of τ-additive measures (a) Let X be a topological group. $C_b(X)^\sim_\tau$ can be identified with the band M_τ of signed τ-additive Borel measures on X, that is, the set of those countably additive functionals ν defined on the Borel σ-algebra of X such that $|\nu|$ is τ-additive.

(b) For any τ-additive totally finite Borel measures λ, ν on X we can define their convolution $\lambda*\nu$ by

$$(\lambda*\nu)(E)=\int\nu(x^{-1}E)\lambda(dx)=\int\lambda(Ey^{-1})\nu(dy)$$

for any Borel set $E\subseteq X$. Now the map $*$ is bilinear in the sense that

$$(\lambda_1+\lambda_2)*\nu=\lambda_1*\nu+\lambda_2*\nu,$$

$$\lambda * (\nu_1 + \nu_2) = \lambda * \nu_1 + \lambda * \nu_2,$$

$$(\alpha\lambda) * \nu = \lambda * (\alpha\nu) = \alpha(\lambda * \nu)$$

for all totally finite τ-additive Borel measures λ, λ_1, λ_2, ν, ν_1, ν_2 and all $\alpha \ge 0$. Consequently we have a bilinear operator $* : M_\tau \times M_\tau \to M_\tau$ defined by saying that

$$(\lambda_1 - \lambda_2) * (\nu_1 - \nu_2) = \lambda_1 * \nu_1 - \lambda_1 * \nu_2 - \lambda_2 * \nu_1 + \lambda_2 * \nu_2$$

for all λ_1, λ_2, ν_1, $\nu_2 \in M_\tau^+$.

(c) $*$ is associative. $|\lambda * \nu| \le |\lambda| * |\nu|$ for any λ, $\nu \in M_\tau$.

(d) If λ, $\nu \in M_\tau^+$ then

$$\|\lambda * \nu\| = \|\lambda\|\|\nu\|.$$

Generally, for any λ, $\nu \in M_\tau$,

$$\|\lambda * \nu\| \le \|\lambda\|\|\nu\|.$$

M_τ is a Banach algebra under the operation $*$. If X is abelian then M_τ will be a commutative algebra.

444F Theorem Let X be a topological space, G a topological group and $\bullet$ a continuous action of G on X. For $A \subseteq X$, $a \in G$ write $a{\bullet}A = \{a{\bullet}x : x \in A\}$. Let ν be a measure on X.

(a) If $f : X \to [0, \infty]$ is lower semi-continuous, then $a \mapsto \int a{\bullet}f\,d\nu : G \to [0, \infty]$ is lower semi-continuous. In particular, if $V \subseteq X$ is open, then $a \mapsto \nu(a{\bullet}V) : G \to [0, \infty]$ is lower semi-continuous.

(b) If $f : X \to \mathbb{R}$ is continuous, then $a \mapsto (a{\bullet}f)^\bullet : G \to L^0$ is continuous, if $L^0 = L^0(\nu)$ is given the topology of convergence in measure.

(c) If ν is σ-finite and $E \subseteq X$ is a Borel set, then $a \mapsto (a{\bullet}E)^\bullet : G \to \mathfrak{A}$ is Borel measurable, if the measure algebra $\mathfrak{A}$ of ν is given its measure-algebra topology.

(d) If ν is σ-finite and $f : X \to \mathbb{R}$ is Borel measurable, then $a \mapsto (a{\bullet}f)^\bullet : G \to L^0$ is Borel measurable.

(e) If ν is σ-finite, then

(i) $a \mapsto \nu(a{\bullet}E) : G \to [0, \infty]$ is Borel measurable for any Borel set $E \subseteq X$;

(ii) if $f : X \to \mathbb{R}$ is Borel measurable, then $Q = \{a : \int a{\bullet}f\,d\nu$ is defined in $[-\infty, \infty]\}$ is a Borel set, and $a \mapsto \int a{\bullet}f\,d\nu : Q \to [-\infty, \infty]$ is Borel measurable.

444G Corollary Let X be a topological group and ν a σ-finite quasi-Radon measure on X.

(a) If $f : X \to \mathbb{R}$ is a Borel measurable function, then $\{x : \int f(y^{-1}x)\nu(dy)$ is defined in $[-\infty, \infty]\}$ is a Borel set in X and $x \mapsto \int f(y^{-1}x)\nu(dy)$ is Borel measurable.

(b) If f, $g : X \to \mathbb{R}$ are Borel measurable functions, then $\{x : \int f(xy^{-1})g(y)\nu(dy)$ is defined in $[-\infty, \infty]\}$ is a Borel set and $x \mapsto \int f(xy^{-1})g(y)\nu(dy)$ is Borel measurable.

(c) If ν is totally finite and $f : X \to \mathbb{R}$ is a bounded continuous function, then $x \mapsto \int f(y^{-1}x)\nu(dy) : X \to \mathbb{R}$ is continuous.

444H Convolutions of measures and functions Let X be a topological group. If f is a real-valued function defined on a subset of X, and ν is a measure on X, set

$$(\nu * f)(x) = \int f(y^{-1}x)\nu(dy)$$

whenever the integral is defined in $\mathbb{R}$.

444I Proposition Let X be a topological group and λ, ν two totally finite quasi-Radon measures on X.

(a) For any Borel measurable function $f : X \to \mathbb{R}$, $\nu * f$ is a Borel measurable function with a Borel domain.

(b) $\nu * f \in C_b(X)$ for every $f \in C_b(X)$.

(c) For any real-valued function f defined on a subset of X, $(\lambda * (\nu * f))(x) = ((\lambda * \nu) * f)(x)$ whenever the right-hand side is defined.

444J Convolutions of functions and measures Let X be a topological group carrying Haar measures; let Δ be its left modular function. If f is a real-valued function defined on a subset of X, and ν is a measure on X, set

$$(f * \nu)(x) = \int f(xy^{-1})\Delta(y^{-1})\nu(dy)$$

whenever the integral is defined in $\mathbb{R}$. If f is non-negative and ν-integrable, write $f\nu$ for the corresponding indefinite-integral measure over ν.

444K Proposition Let X be a topological group with a left Haar measure μ. Let ν be a totally finite quasi-Radon measure on X. Then for any non-negative μ-integrable real-valued function f, $f\mu$ is a quasi-Radon measure; moreover, $\nu * f$ and $f * \nu$ are μ-integrable, and we have

$$(\nu * f)\mu = \nu * f\mu, \quad (f * \nu)\mu = f\mu * \nu.$$

In particular, $\int \nu * f\,d\mu = \int f * \nu\,d\mu = \nu X \cdot \int f d\mu$.

444L Corollary Let X be a topological group carrying Haar measures. Suppose that ν is a non-zero quasi-Radon measure on X and $E \subseteq X$ is a Haar measurable set such that $\nu(xE) = 0$ for every $x \in X$. Then E is Haar negligible.

444M Proposition Let X be a topological group and μ a left Haar measure on X. Let ν be a quasi-Radon measure on X and $p \in [1, \infty]$.

(a) Suppose that $\nu X < \infty$. Then we have a bounded positive linear operator $u \mapsto \nu * u : L^p(\mu) \to L^p(\mu)$, of norm at most νX, defined by saying that $\nu * f^\bullet = (\nu * f)^\bullet$ for every $f \in \mathcal{L}^p(\mu)$.

(b) Set $\gamma = \int \Delta(y)^{(1-p)/p}\nu(dy)$ if $p < \infty$, $\int \Delta(y)^{-1}\nu(dy)$ if $p = \infty$, where Δ is the left modular function of X. Suppose that $\gamma < \infty$. Then we have a bounded positive linear operator $u \mapsto u * \nu : L^p(\mu) \to L^p(\mu)$, of norm at most γ, defined by saying that $f^\bullet * \nu = (f * \nu)^\bullet$ for every $f \in \mathcal{L}^p(\mu)$.

444N Lemma Let X be a topological group and μ a left Haar measure on X. Suppose that f, g, $h \in \mathcal{L}^0(\mu)$ are non-negative. Then, writing $\int \ldots d(x, y)$ to denote integration with respect to the quasi-Radon product measure $\mu \times \mu$,

$$\iint f(x)g(y)h(xy)dxdy = \iint f(x)g(y)h(xy)dydx = \int f(x)g(y)h(xy)d(x, y)$$

in $[0, \infty]$.

444O Convolutions of functions: Theorem Let X be a topological group and μ a left Haar measure on X. For f, $g \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$, write $(f * g)(x) = \int f(y)g(y^{-1}x)dy$ whenever this is defined in $\mathbb{R}$, taking the integral with respect to μ.

(a) Writing Δ for the left modular function of X,

$$\begin{aligned}(f * g)(x) &= \int f(y)g(y^{-1}x)dy = \int f(xy)g(y^{-1})dy \\ &= \int \Delta(y^{-1})f(y^{-1})g(yx)dy = \int \Delta(y^{-1})f(xy^{-1})g(y)dy\end{aligned}$$

whenever any of these integrals is defined in $\mathbb{R}$.

(b) If $f =_{\text{a.e.}} f_1$ and $g =_{\text{a.e.}} g_1$, then $f * g = f_1 * g_1$.

(c)(i) $|(f * g)(x)| \le (|f| * |g|)(x)$ whenever either is defined in $\mathbb{R}$.

(ii)

$$((f_1 + f_2) * g)(x) = (f_1 * g)(x) + (f_2 * g)(x),$$

$$(f * (g_1 + g_2))(x) = (f * g_1)(x) + (f * g_2)(x),$$

$$((\alpha f) * g)(x) = (f * (\alpha g))(x) = \alpha(f * g)(x)$$

whenever the right-hand expressions are defined in $\mathbb{R}$.

(d) If f, g and h belong to $\mathcal{L}^0$ and any of

$$\int(|f|*|g|)(x)|h|(x)dx, \quad \iint|f(x)g(y)h(xy)|dxdy,$$

$$\iint|f(x)g(y)h(xy)|dydx, \quad \int|f(x)g(y)h(xy)|d(x,y)$$

is defined in $[0,\infty[$ (writing $\int\ldots d(x,y)$ for integration with respect to the quasi-Radon product measure $\mu\times\mu$ on $X\times X$), then

$$\int(f*g)(x)h(x)dx, \quad \iint f(x)g(y)h(xy)dxdy,$$

$$\iint f(x)g(y)h(xy)dydx, \quad \int f(x)g(y)h(xy)d(x,y)$$

are all defined, finite and equal, provided that in the expression $(f*g)(x)h(x)$ we interpret the product as 0 when $h(x)=0$ and $(f*g)(x)$ is undefined.

(e) If f, g and h belong to $\mathcal{L}^0$, $f*g$ and $g*h$ are defined a.e. and $x\in X$ is such that either $(|f|*(|g|*|h|))(x)$ or $((|f|*|g|)*|h|)(x)$ is defined in $\mathbb{R}$, then $(f*(g*h))(x)$ and $((f*g)*h)(x)$ are defined and equal.

(f) If $a\in X$ and f, $g\in\mathcal{L}^0$,

$$a\bullet_l(f*g)=(a\bullet_l f)*g, \quad a\bullet_r(f*g)=f*(a\bullet_r g),$$

$$(a\bullet_r f)*g=\Delta(a^{-1})f*(a^{-1}\bullet_l g),$$

$$\overset{\leftrightarrow}{f}*\overset{\leftrightarrow}{g}=(g*f)^{\leftrightarrow}.$$

(g) If X is abelian then $f*g=g*f$ for all f and g.

444P Proposition Let X be a topological group and μ a left Haar measure on X.

(a) If $f\in\mathcal{L}^1(\mu)^+$ and $g\in\mathcal{L}^0(\mu)$ then $f*g$ is equal to $(f\mu)*g$.

(b) If $f\in\mathcal{L}^0(\mu)$ and $g\in\mathcal{L}^1(\mu)^+$ then $f*g=f*(g\mu)$.

444Q Proposition Let X be a topological group and μ a left Haar measure on X.

(a) Let f, g be non-negative μ-integrable functions. Then $f*g\in\mathcal{L}^1$ and

$$(f\mu)*(g\mu)=(f*g)\mu.$$

(b) For any f, $g\in\mathcal{L}^1$, $f*g\in\mathcal{L}^1$ and

$$\int f*g\,d\mu=\int fd\mu\int g\,d\mu, \quad \|f*g\|_1\le\|f\|_1\|g\|_1.$$

444R Proposition Let X be a topological group and μ a left Haar measure on X. Take any $p\in[1,\infty]$.

(a) If $f\in\mathcal{L}^1(\mu)$ and $g\in\mathcal{L}^p(\mu)$, then $f*g\in\mathcal{L}^p(\mu)$ and $\|f*g\|_p\le\|f\|_1\|g\|_p$.

(b) $f*\overset{\leftrightarrow}{g}=(g*\overset{\leftrightarrow}{f})^{\leftrightarrow}$ for all f, $g\in\mathcal{L}^0$. If X is unimodular then $\|\overset{\leftrightarrow}{f}\|_p=\|f\|_p$ for every $f\in\mathcal{L}^0$.

(c) Set $q=\infty$ if $p=1$, $p/(p-1)$ if $1<p<\infty$, 1 if $p=\infty$. If $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$, then $f*\overset{\leftrightarrow}{g}$ is defined everywhere in X and is continuous, and $\|f*\overset{\leftrightarrow}{g}\|_\infty\le\|f\|_p\|g\|_q$. If X is unimodular, then $f*g\in C_b(X)$ and $\|f*g\|_\infty\le\|f\|_p\|g\|_q$ for every $f\in\mathcal{L}^p(\mu)$, $g\in\mathcal{L}^q(\mu)$.

Remark In the formulae above, interpret $\|g\|_\infty$ as $\|g^\bullet\|_\infty=\operatorname{ess\,sup}|g|$ for $g\in\mathcal{L}^\infty=\mathcal{L}^\infty(\mu)$, and as ∞ for $g\in\mathcal{L}^0\setminus\mathcal{L}^\infty$.

444S Remarks Let X be a topological group and μ a left Haar measure on X.

(a) From 444Ob and 444Ra we see that we have a bilinear operator $(u,v)\mapsto u*v:L^1(\mu)\times L^p(\mu)\to L^p(\mu)$ defined by saying that $f^\bullet*g^\bullet=(f*g)^\bullet$ for every $f\in\mathcal{L}^1(\mu)$ and $g\in\mathcal{L}^p(\mu)$. $*$ can be regarded as a function from $L^1\times L^p$ to $\mathcal{L}^p$. Putting 443Ge together with 444Oe and 444Of, we have

$$u*(v*w)=(u*v)*w,$$

$$a\bullet_l(u*w)=(a\bullet_l u)*w, \quad a\bullet_r(u*w)=u*(a\bullet_r w),$$

$$(a\bullet_r u) * w = \Delta(a^{-1})u * (a^{-1}\bullet_l w)$$

whenever u, $v \in L^1$, $w \in L^p$ and $a \in X$.

Similarly, if the group is unimodular, and $\frac{1}{p} + \frac{1}{q} = 1$, the map $* : \mathcal{L}^p \times \mathcal{L}^q \to C_b(X)$ factors through a map from $L^p \times L^q$ to $C_b(X)$.

(b) In particular, $* : L^1 \times L^1 \to L^1$ is associative; evidently it is bilinear; and $\|u * v\|_1 \le \|u\|_1\|v\|_1$ for all u, $v \in L^1$. So L^1 is a Banach algebra. By 444Qb, $\int u * v = \int u \int v$ for all u, $v \in L^1$. L^1 is commutative if X is abelian.

(c) Let $\mathcal{B}$ be the Borel σ-algebra of X and M_τ the Banach algebra of signed τ-additive Borel measures on X, as in 444E. If, for $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $E \in \mathcal{B}$, we write $(f\mu\restriction\mathcal{B})(E) = \int_E f d\mu$, then $f\mu\restriction\mathcal{B} \in M_\tau$. For f, $g \in \mathcal{L}^1$, we have

$$f^\bullet = g^\bullet \text{ in } L^1 \Longrightarrow f =_{\text{a.e.}} g \Longrightarrow f\mu\restriction\mathcal{B} = g\mu\restriction\mathcal{B},$$

so we have an operator $T : L^1 \to M_\tau$ defined by setting $T(f^\bullet) = f\mu\restriction\mathcal{B}$ for $f \in \mathcal{L}^1$. T is a Riesz homomorphism; moreover, T is norm-preserving. $Tu * Tv = T(u * v)$ for all u, $v \in L^1$, and T is an embedding of L^1 as a subalgebra of M_τ.

444T Proposition Let X be a topological group and μ a left Haar measure on X. Then for any $p \in [1, \infty[$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$ there is a neighbourhood U of the identity e in X such that $\|\nu * f - f\|_p \le \epsilon$ and $\|f * \nu - f\|_p \le \epsilon$ whenever ν is a quasi-Radon measure on X such that $\nu U = \nu X = 1$.

444U Corollary Let X be a topological group and μ a left Haar measure on X. For any Haar measurable $E \subseteq X$ such that $0 < \mu E < \infty$, and any $f \in \bigcup_{1\le p<\infty} \mathcal{L}^p(\mu)$, write

$$f_E(x) = \frac{1}{\mu E}\int_{xE} f d\mu, \quad f'_E(x) = \frac{1}{\mu(Ex)}\int_{Ex} f d\mu$$

for $x \in X$. Then, for any $p \in [1, \infty[$, $f \in \mathcal{L}^p$ and $\epsilon > 0$, there is a neighbourhood U of the identity in X such that $\|f_E - f\|_p \le \epsilon$ and $\|f'_E - f\|_p \le \epsilon$ whenever $E \subseteq U$ is a non-negligible Haar measurable set.

444V Theorem Let X be a compact topological group and μ a left Haar measure on X.

(a) For any u, $v \in L^2 = L^2(\mu)$ we can interpret their convolution $u * v$ either as a member of the space $C(X)$ of continuous real-valued functions on X, or as a member of the space L^2.

(b) If $w \in L^2$, then $u \mapsto u * w$ is a compact linear operator whether regarded as a map from L^2 to $C(X)$ or as a map from L^2 to itself.

(c) If $w \in L^2$ and $w = \overleftrightarrow{w}$, then $u \mapsto u * w : L^2 \to L^2$ is a self-adjoint operator.

Version of 20.3.08

445 The duality theorem

In this section I present a proof of the Pontryagin-van Kampen duality theorem (445U). As in Chapter 28, and for the same reasons, we need to use complex-valued functions; the relevant formulae in §§443 and 444 apply unchanged, and I shall not repeat them here, but you may wish to re-read parts of those sections taking functions to be complex- rather than real-valued. (It *is* possible to avoid complex-valued measures, which I relegate to the exercises.) The duality theorem itself applies only to abelian locally compact Hausdorff groups, and it would be reasonable, on first reading, to take it for granted that all groups here are of this type, which simplifies some of the proofs a little.

My exposition is based on that of Rudin 67. I start with the definition of 'dual group', including a description of a topology on the dual (445A), and the simplest examples (445B), with a mention of Fourier-Stieltjes transforms of measures (445C-445D). The elementary special properties of dual groups of groups carrying Haar measures are in 445E-445G; in particular, in these cases, the bidual of a group begins to make sense, and we can start talking about Fourier transforms of functions.

Serious harmonic analysis begins with the identification of the dual group with the maximal ideal space of L^1 (445H-445K). The next idea is that of 'positive definite' function (445L-445M). Putting these together, we get the first result here which asserts that the dual group of an abelian group X carrying Haar measures is sufficiently large to effectively describe functions on X (Bochner's theorem, 445N). It is now easy to establish that X can be faithfully embedded in its bidual (445O). We also have most of the machinery necessary to describe the correctly normalized Haar measure of the dual group, with a first step towards identifying functions whose Fourier transforms will have inverse Fourier transforms (the Inversion Theorem, 445P). This leads directly to the Plancherel Theorem, identifying the L^2 spaces of X and its dual (445R). At this point it is clear that the bidual $\mathfrak{X}$ cannot be substantially larger than X, since they must have essentially the same L^2 spaces. A little manipulation of shifts and convolutions in L^2 (445S-445T) shows that X must be dense in $\mathfrak{X}$, and a final appeal to local compactness shows that X is closed in $\mathfrak{X}$.

445A Dual groups Let X be any topological group.

(a) A **character** on X is a continuous group homomorphism from X to $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$. It is easy to see that the set $\mathcal{X}$ of all characters on X is a subgroup of the group $(S^1)^X$. So $\mathcal{X}$ itself is an abelian group.

(b) Give $\mathcal{X}$ the topology of uniform convergence on subsets of X which are totally bounded for the bilateral uniformity on X. Then $\mathcal{X}$ is a Hausdorff topological group.

(c) Note that if X is locally compact, then the topology of $\mathcal{X}$ is the topology of uniform convergence on compact subsets of X.

(d) If X is compact, then $\mathcal{X}$ is discrete.

(e) If X is discrete then $\mathcal{X}$ is compact.

445B Examples (a) If $X = \mathbb{R}$ with addition, then $\mathcal{X}$ can also be identified with the additive group $\mathbb{R}$, if we write $\chi_y(x) = e^{iyx}$ for x, $y \in \mathbb{R}$.

(b) Let X be the group $\mathbb{Z}$ with its discrete topology. Then we may identify its dual group $\mathcal{X}$ with S^1 itself, writing $\chi_\zeta(n) = \zeta^n$ for $\zeta \in S^1$, $n \in \mathbb{Z}$.

(c) On the other hand, if $X = S^1$ with its usual topology, then we may identify its dual group $\mathcal{X}$ with $\mathbb{Z}$, writing $\chi_n(\zeta) = \zeta^n$ for $n \in \mathbb{Z}$, $\zeta \in S^1$.

(d) Let $\langle X_j \rangle_{j\in J}$ be any family of topological groups, and X their product. For each $j \in J$ let $\mathcal{X}_j$ be the dual group of X_j. Then the dual group of X can be identified with the subgroup $\mathcal{X}$ of $\prod_{j\in I} \mathcal{X}_j$ consisting of those $\chi \in \prod_{j\in J} \mathcal{X}_j$ such that $\{j : \chi(j)$ is not the identity$\}$ is finite; the action of $\mathcal{X}$ on X is defined by the formula

$$\chi{\bullet}x = \prod_{j\in J} \chi(j)(x(j)).$$

If I is finite, so that $\mathcal{X} = \prod_{j\in I} \mathcal{X}_j$, the topology of $\mathcal{X}$ is the product topology.

445C Fourier-Stieltjes transforms Let X be a topological group, and $\mathcal{X}$ its dual group. For any totally finite topological measure ν on X, we can form its 'characteristic function' or **Fourier-Stieltjes transform** $\hat\nu : \mathcal{X} \to \mathbb{C}$ by writing $\hat\nu(\chi) = \int \chi(x)\nu(dx)$.

445D Theorem Let X be a topological group, and $\mathcal{X}$ its dual group. If λ and ν are totally finite quasi-Radon measures on X, then $(\lambda * \nu)^\wedge = \hat\lambda \times \hat\nu$.

445E Proposition Let X be a topological group with a neighbourhood of the identity which is totally bounded for the bilateral uniformity on X, and $\mathcal{X}$ its dual group.

(a) The map $(\chi, x) \mapsto \chi(x) : \mathcal{X} \times X \to S^1$ is continuous.

(b) Let $\mathfrak{X}$ be the dual group of $\mathcal{X}$. Then we have a continuous homomorphism $x \mapsto \hat x : X \to \mathfrak{X}$ defined by setting $\hat x(\chi) = \chi(x)$ for $x \in X$ and $\chi \in \mathcal{X}$.

(c) For any totally finite quasi-Radon measure ν on X, its Fourier-Stieltjes transform $\hat\nu : \mathcal{X} \to \mathbb{C}$ is uniformly continuous.

445F Fourier transforms of functions Let X be a topological group with a left Haar measure μ. For any μ-integrable complex-valued function f, define its **Fourier transform** $\hat{f} : \mathcal{X} \to \mathbb{C}$ by setting $\hat{f}(\chi) = \int f(x)\chi(x)\mu(dx)$ for every character χ of X. $\hat{f} = \hat{g}$ whenever $f =_{\text{a.e.}} g$, so we can write $\hat{u}(\chi) = \hat{f}(\chi)$ whenever $u = f^\bullet$ in $L^1_{\mathbb{C}}(\mu)$.

445G Proposition Let X be a topological group with a left Haar measure μ. Then for any μ-integrable complex-valued functions f and g, $(f * g)^\wedge = \hat{f} \times \hat{g}$; $(u * v)^\wedge = \hat{u} \times \hat{v}$ for all u, $v \in L^1_{\mathbb{C}}(\mu)$.

445H Theorem Let X be a topological group with a left Haar measure μ; let $\mathcal{X}$ be its dual group and let Φ be the set of non-zero multiplicative linear functionals on the complex Banach algebra $L^1_{\mathbb{C}}(\mu)$. Then there is a one-to-one correspondence between $\mathcal{X}$ and Φ, defined by the formulae

$$\phi(f^\bullet) = \int f \times \chi \, d\mu = \hat{f}(\chi) \text{ for every } f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu),$$

$$\phi(a \bullet_l u) = \chi(a)\phi(u) \text{ for every } u \in L^1_{\mathbb{C}},\ a \in X,$$

for $\chi \in \mathcal{X}$ and $\phi \in \Phi$.

445I The topology of the dual group: Proposition Let X be a topological group with a left Haar measure μ, and $\mathcal{X}$ its dual group. For $\chi \in \mathcal{X}$, let $\chi^\bullet$ be its equivalence class in $L^0_{\mathbb{C}} = L^0_{\mathbb{C}}(\mu)$, and $\phi_\chi \in (L^1_{\mathbb{C}})^* = (L^1_{\mathbb{C}}(\mu))^*$ the multiplicative linear functional corresponding to χ. Then the maps $\chi \mapsto \chi^\bullet$ and $\chi \mapsto \phi_\chi$ are homeomorphisms between $\mathcal{X}$ and its images in $L^0_{\mathbb{C}}$ and $(L^1_{\mathbb{C}})^*$, if we give $L^0_{\mathbb{C}}$ the topology of convergence in measure and $(L^1_{\mathbb{C}})^*$ the weak* topology.

445J Corollary For any topological group X carrying Haar measures, its dual group $\mathcal{X}$ is locally compact and Hausdorff.

445K Proposition Let X be a topological group and μ a left Haar measure on X. Let $\mathcal{X}$ be the dual group of X, and write $C_0 = C_0(\mathcal{X};\mathbb{C})$ for the Banach algebra of continuous functions $h : \mathcal{X} \to \mathbb{C}$ such that $\{\chi : |h(\chi)| \ge \epsilon\}$ is compact for every $\epsilon > 0$.

(a) For any $u \in L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$, its Fourier transform $\hat{u}$ belongs to C_0.

(b) The map $u \mapsto \hat{u} : L^1_{\mathbb{C}} \to C_0$ is a multiplicative linear operator, of norm at most 1.

(c) Suppose that X is abelian. For $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$, set $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined. Then $\tilde{f} \in \mathcal{L}^1_{\mathbb{C}}$ and $\|\tilde{f}\|_1 = \|f\|_1$. For $u \in L^1_{\mathbb{C}}$, we may define $\tilde{u} \in L^1_{\mathbb{C}}$ by setting $\tilde{u} = \tilde{f}^\bullet$ whenever $u = f^\bullet$. Now $\hat{\tilde{u}}$ is the complex conjugate of $\hat{u}$, so $(u * \tilde{u})^\wedge = |\hat{u}|^2$.

(d) Still supposing that X is abelian, $\{\hat{u} : u \in L^1_{\mathbb{C}}\}$ is a norm-dense subalgebra of C_0, and $\|\hat{u}\|_\infty = r(u)$, the spectral radius of u, for every $u \in L^1_{\mathbb{C}}$.

445L Positive definite functions Let X be a group.

(a) A function $h : X \to \mathbb{C}$ is called **positive definite** if

$$\sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k h(x_k^{-1} x_j) \ge 0$$

for all $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $x_0, \dots, x_n \in X$.

(b) Suppose that $h : X \to \mathbb{C}$ is positive definite. Then, writing e for the identity of X,

(i) $|h(x)| \le h(e)$ for every $x \in X$;

(ii) $h(x^{-1}) = \overline{h(x)}$ for every $x \in X$.

(c) If $h : X \to \mathbb{C}$ is positive definite and $\chi : X \to S^1$ is a homomorphism, then $h \times \chi$ is positive definite.

(d) If X is an abelian topological group and μ a Haar measure on X, then for any $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ the convolution $f * \tilde{f} : X \to \mathbb{C}$ is continuous and positive definite, where $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined.

445M Proposition Let X be a topological group and ν a quasi-Radon measure on X. If $h : X \to \mathbb{C}$ is a continuous positive definite function, then $\iint h(y^{-1}x)f(x)\overline{f(y)}\nu(dx)\nu(dy) \ge 0$ for every ν-integrable function f.

445N Bochner's theorem Let X be an abelian topological group with a Haar measure μ, and $\mathcal{X}$ its dual group. Then for any continuous positive definite function $h : X \to \mathbb{C}$ there is a unique totally finite Radon measure ν on $\mathcal{X}$ such that

$$\int h \times f\,d\mu = \int \hat{f}\,d\nu \text{ for every } f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu),$$

$$h(x) = \int \chi(x)\nu(d\chi) \text{ for every } x \in X.$$

445O Proposition Let X be a Hausdorff abelian topological group carrying Haar measures. Then the map $x \mapsto \hat{x}$ from X to its bidual group $\mathfrak{X}$ is a homeomorphism between X and its image in $\mathfrak{X}$. In particular, the dual group $\mathcal{X}$ of X separates the points of X.

445P The Inversion Theorem Let X be an abelian topological group and μ a Haar measure on X. Then there is a unique Haar measure λ on the dual group $\mathcal{X}$ of X such that whenever $f : X \to \mathbb{C}$ is continuous, μ-integrable and positive definite, then $\hat{f} : \mathcal{X} \to \mathbb{C}$ is λ-integrable and

$$f(x) = \int \hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi)$$

for every $x \in X$.

445Q Remark If $h : X \to \mathbb{C}$ is μ-integrable, continuous and positive definite, then $\hat{\hat{h}}$ is non-negative and λ-integrable, and the Radon measure ν_h of 445N is just the indefinite-integral measure $\hat{\hat{h}}\lambda$.

λ is a Radon measure.

445R The Plancherel Theorem Let X be an abelian topological group with a Haar measure μ, and $\mathcal{X}$ its dual group. Let λ be the Haar measure on $\mathcal{X}$ corresponding to μ. Then there is a normed space isomorphism $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ defined by setting $T(f^\bullet) = \hat{f}^\bullet$ whenever $f \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$.

445S Proposition Let X be an abelian topological group with a Haar measure μ, $\mathcal{X}$ its dual group, λ the associated Haar measure on $\mathcal{X}$ and $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ the standard isometry. Suppose that f_0, $f_1 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and g_0, $g_1 \in \mathcal{L}^2_{\mathbb{C}}(\nu)$ are such that $Tf_0^\bullet = g_0^\bullet$ and $Tf_1^\bullet = g_1^\bullet$, and take any $\theta \in \mathcal{X}$. Then

(a) setting $f_2 = \bar{f}_0$, $g_2(\chi) = \overline{g_0(\chi^{-1})}$ whenever this is defined, $Tf_2^\bullet = g_2^\bullet$;

(b) setting $f_3 = f_1 \times \theta$, $g_3(\chi) = g_1(\theta\chi)$ whenever this is defined, $Tf_3^\bullet = g_3^\bullet$;

(c) setting $f_4 = f_0 \times f_1 \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, $\hat{f}_4(\theta) = (g_0 * g_1)(\theta)$.

445T Corollary Let X be an abelian topological group with a Haar measure μ, and λ the corresponding Haar measure on the dual group $\mathcal{X}$ of X. Then for any non-empty open set $H \subseteq \mathcal{X}$, there is an $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$ such that $\hat{f} \ne 0$ and $\hat{f}(\chi) = 0$ for $\chi \in \mathcal{X} \setminus H$.

445U The Duality Theorem Let X be a locally compact Hausdorff abelian topological group. Then the canonical map $x \mapsto \hat{x}$ from X to its bidual $\mathfrak{X}$ is an isomorphism between X and $\mathfrak{X}$ as topological groups.

Version of 8.10.13

446 The structure of locally compact groups

I develop those fragments of the structure theory of locally compact Hausdorff topological groups which are needed for the main theorem of the next section. Theorem 446B here is of independent interest, being both itself important and with a proof which uses the measure theory of this chapter in an interesting way; but the rest of the section, from 446D on, is starred. Note that in this section, unlike the last, groups are not expected to be abelian.

446A Finite-dimensional representations (a) Definitions (i) For any $r \in \mathbb{N}$, write $M_r = M_r(\mathbb{R})$ for the space of $r \times r$ real matrices. If we identify it with the space $\mathsf{B}(\mathbb{R}^r;\mathbb{R}^r)$, where $\mathbb{R}^r$ is given its Euclidean norm, then M_r becomes a unital Banach algebra, with identity I, the $r \times r$ identity matrix. Write $GL(r,\mathbb{R})$ for the group of invertible elements of M_r.

(ii) Let X be a topological group. A **finite-dimensional representation** of X is a continuous homomorphism from X to a group of the form $GL(r,\mathbb{R})$ for some $r \in \mathbb{N}$. If the homomorphism is injective the representation is called **faithful**.

(b) Observe that if X is any topological group and ϕ is a finite-dimensional representation with kernel Y, then X/Y has a faithful finite-dimensional representation ψ defined by writing $\psi(x^\bullet) = \phi(x)$ for every $x \in X$.

446B Theorem Let X be a compact Hausdorff topological group. Then for any $a \in X$, other than the identity, there is a finite-dimensional representation ϕ of X such that $\phi(a) \neq I$; and we can arrange that $\phi(x)$ is an orthogonal matrix for every $x \in X$.

446C Corollary Let X be a compact Hausdorff topological group. Then for any neighbourhood U of the identity of X there is a finite-dimensional representation of X with kernel included in U.

***446D Notation (a)** If X is a group and A is a subset of X I will write $A^0 = \{e\}$ and $A^{n+1} = AA^n$ for $n \in \mathbb{N}$.

(b) If X is a group with identity e, $e \in A \subseteq X$ and $n \in \mathbb{N}$, write $D_n(A) = \{x : x \in X,\ x^i \in A \text{ for every } i \le n\}$.

(i) $D_0(A) = X$.

(ii) $D_1(A) = A$.

(iii) $D_n(A) \subseteq D_m(A)$ whenever $m \le n$.

(iv) $D_{mn}(A) \subseteq D_m(D_n(A))$ for all m, $n \in \mathbb{N}$.

(v) If $r \in \mathbb{N}$ and $A^r \subseteq B$, then $D_n(A) \subseteq D_{nr}(B)$ for every $n \in \mathbb{N}$; in particular, $A \subseteq D_r(B)$.

(vi) If $A = A^{-1}$ then $D_n(A) = D_n(A)^{-1}$ for every $n \in \mathbb{N}$.

(vii) If $D_m(A) \subseteq B$ where $m \in \mathbb{N}$, then $D_{mn}(A) \subseteq D_n(B)$ for every $n \in \mathbb{N}$.

(c) In (b), if X is a topological group and A is closed, then every $D_n(A)$ is closed; if moreover A is compact, then $D_n(A)$ is compact for every $n \ge 1$. If A is a neighbourhood of e, then so is every $D_n(A)$.

***446E Lemma** Let X be a group, and $U \subseteq X$. Let $f : X \to [0,\infty[$ be a bounded function such that $f(x) = 0$ for $x \in X \setminus U$; set $\alpha = \sup_{x \in X} f(x)$. Let $A \subseteq X$ be a symmetric set containing e, and K a set including A^k, where $k \ge 1$. Define $g : X \to [0,\infty[$ by setting

$$g(x) = \frac{1}{k}\sum_{i=0}^{k-1} \sup\{f(yx) : y \in A^i\}$$

for $x \in X$. Then

(a) $f(x) \le g(x) \le \alpha$ for every $x \in X$, and $g(x) = 0$ if $x \notin K^{-1}U$.

(b) $|g(ax) - g(x)| \le \frac{j\alpha}{k}$ if $j \in \mathbb{N}$, $a \in A^j$ and $x \in X$.

(c) For any x, $z \in X$, $|g(x) - g(z)| \le \sup_{y \in K} |f(yx) - f(yz)|$.

***446F Lemma** Let X be a locally compact Hausdorff topological group and $\langle A_n \rangle_{n \in \mathbb{N}}$ a sequence of closed symmetric subsets of X all containing the identity e of X. Suppose that for every neighbourhood W of e there is an $n_0 \in \mathbb{N}$ such that $A_n \subseteq W$ for every $n \ge n_0$. Let U be a compact neighbourhood of e and suppose that for each $n \in \mathbb{N}$ we have $k(n) \in \mathbb{N}$ such that $A_n^{k(n)} \subseteq U$ and $A_n^{k(n)+1} \not\subseteq U$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and write Q for the limit $\lim_{n \to \mathcal{F}} A_n^{k(n)}$ in the space $\mathcal{C}$ of closed subsets of X with the Fell topology.

(i) If $Q^2 = Q$ then Q is a compact subgroup of X included in U and meeting the boundary of U.

(ii) If $Q^2 \neq Q$ then there are a neighbourhood W of e and an infinite set $I \subseteq \mathbb{N}$ such that for every $n \in I$ there are an $x \in A_n$ and an $i \leq k(n)$ such that $x^i \notin W$.

***446G 'Groups with no small subgroups' (a) Definition** Let X be a topological group. We say that X **has no small subgroups** if there is a neighbourhood U of the identity e of X such that the only subgroup of X included in U is $\{e\}$.

(b) If X is a Hausdorff topological group and U is a compact symmetric neighbourhood of the identity e such that the only subgroup of X included in U is $\{e\}$, then $\{D_n(U) : n \in \mathbb{N}\}$ is a base of neighbourhoods of e.

(c) In particular, a locally compact Hausdorff topological group with no small subgroups is metrizable.

***446H Lemma** Let X be a locally compact Hausdorff topological group. If $U \subseteq X$ is a compact symmetric neighbourhood of the identity which does not include any subgroup of X other than $\{e\}$, then there is an $r \geq 1$ such that $D_{rn}(U)^n \subseteq U$ for every $n \in \mathbb{N}$.

***446I Lemma** Let X be a locally compact Hausdorff topological group and U a compact symmetric neighbourhood of the identity in X such that U does not include any subgroup of X other than $\{e\}$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Suppose that $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \in D_n(U)$ for every $n \in \mathbb{N}$. Then we have a continuous homomorphism $q : \mathbb{R} \to X$ defined by setting $q(t) = \lim_{n\to\mathcal{F}} x_n^{i(n)}$ whenever $\langle i(n) \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n} = t$ in $\mathbb{R}$.

***446J Lemma** Let X be a locally compact Hausdorff topological group with no small subgroups. Then there is a neighbourhood V of the identity e such that $x = y$ whenever x, $y \in V$ and $x^2 = y^2$.

***446K Lemma** Let X be a locally compact Hausdorff topological group with no small subgroups. Then there is a compact symmetric neighbourhood U of the identity e such that whenever V is a neighbourhood of e there are an $n_0 \in \mathbb{N}$ and a neighbourhood W of e such that whenever $n \geq n_0$, $x \in D_n(U)$, $y \in D_n(U)$ and $x^n y^n \in W$, then $xy \in D_n(V)$.

***446L Definition** Let X be a topological group. A B**-sequence** in X is a non-increasing sequence $\langle V_n \rangle_{n\in\mathbb{N}}$ of closed neighbourhoods of the identity, constituting a base of neighbourhoods of the identity, such that there is some M such that for every $n \in \mathbb{N}$ the set $V_n V_n^{-1}$ can be covered by at most M left translates of V_n.

***446M Proposition** Let X be a locally compact Hausdorff topological group with no small subgroups. Then it has a B-sequence.

***446N Proposition** Let X be a locally compact Hausdorff topological group with a faithful finite-dimensional representation. Then it has a B-sequence.

***446O Theorem** Let X be a locally compact Hausdorff topological group. Then it has an open subgroup Y which has a compact normal subgroup Z such that Y/Z has no small subgroups.

***446P Corollary** Let X be a locally compact Hausdorff topological group. Then it has a chain $\langle X_\xi \rangle_{\xi\leq\kappa}$ of closed subgroups, where κ is an infinite cardinal, such that

(i) X_0 is open,

(ii) $X_{\xi+1}$ is a normal subgroup of X_ξ for every $\xi < \kappa$,

(iii) X_ξ is compact for $\xi \geq 1$,

(iv) $X_\xi = \bigcap_{\eta<\xi} X_\eta$ for non-zero limit ordinals $\xi \leq \kappa$,

(v) $X_\xi / X_{\xi+1}$ has a B-sequence for every $\xi < \kappa$,

(vi) $X_\kappa = \{e\}$, where e is the identity of X.

Version of 7.1.10

447 Translation-invariant liftings

I devote a section to the main theorem of Ionescu Tulcea & Ionescu Tulcea 67: a group carrying Haar measures has a translation-invariant lifting (447J). The argument uses an inductive construction of the same type as that used in §341 for the ordinary Lifting Theorem. It depends on the structure theory for locally compact groups described in §446. On the way I describe a Vitali theorem for certain metrizable groups (447C), with a corresponding density theorem (447D).

447A Liftings and lower densities Let X be a group carrying Haar measures, Σ its algebra of Haar measurable sets and $\mathfrak{A}$ its Haar measure algebra.

(a) Recall that a **lifting** of $\mathfrak{A}$ is either a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$, or a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that $E \triangle \phi E$ is Haar negligible for every $E \in \Sigma$ and $\phi E = \emptyset$ whenever E is Haar negligible. Such a lifting θ or ϕ is **left-translation-invariant** if $\theta((xE)^\bullet) = x(\theta E^\bullet)$ or $\phi(xE) = x(\phi E)$ for every $E \in \Sigma$ and $x \in X$.

(b) Now suppose that Σ_0 is a σ-subalgebra of Σ. In this case, a **partial lower density** on Σ_0 is a function $\underline{\phi} : \Sigma_0 \to \Sigma$ such that $\underline{\phi}E = \underline{\phi}F$ whenever E, $F \in \Sigma_0$ and $E \triangle F$ is negligible, $E \triangle \underline{\phi}E$ is negligible for every $E \in \Sigma_0$, $\underline{\phi}\emptyset = \emptyset$ and $\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all E, $F \in \Sigma_0$. As in (a), such a function is **left-translation-invariant** if $xE \in \Sigma_0$ and $\underline{\phi}(xE) = x(\underline{\phi}E)$ for every $x \in X$ and $E \in \Sigma_0$.

447B Lemma Let X be a group carrying Haar measures and Y a subgroup of X. Write Σ_Y for the algebra of Haar measurable subsets E of X such that $EY = E$, and suppose that $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ is a left-translation-invariant partial lower density. Then $G \subseteq \underline{\phi}(GY)$ for every open set $G \subseteq X$.

447C Vitali's theorem Let X be a topological group with a left Haar measure μ, and $\langle V_n \rangle_{n \in \mathbb{N}}$ a B-sequence in X. If $A \subseteq X$ is any set and K_x is an infinite subset of $\mathbb{N}$ for every $x \in A$, then there is a disjoint family $\mathcal{V}$ of sets such that $A \setminus \bigcup \mathcal{V}$ is negligible and every member of $\mathcal{V}$ is of the form xV_n for some $x \in A$ and $n \in K_x$.

447D Theorem Let X be a topological group with a left Haar measure μ, and $\langle V_n \rangle_{n \in \mathbb{N}}$ a B-sequence in X. Then for any Haar measurable set $E \subseteq X$,

$$\lim_{n \to \infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = \chi E(x)$$

for almost every $x \in X$.

447E Let X be a locally compact Hausdorff topological group, and Y a closed subgroup of X such that the modular function of Y is the restriction to Y of the modular function of X. Let μ be a left Haar measure on X and μ_Y a left Haar measure on Y.

(a) Writing $C_k(X)$ for the space of continuous real-valued functions on X with compact support, and X/Y for the set of left cosets of Y in X with the quotient topology, we have a linear operator $T : C_k(X) \to C_k(X/Y)$ defined by writing $(Tf)(x^\bullet) = \int_Y f(xy)\mu_Y(dy)$ whenever $x \in X$ and $f \in C_k(X)$; moreover, $T[C_k(X)^+] = C_k(X/Y)^+$, and we have an invariant Radon measure λ on X/Y such that $\int Tf d\lambda = \int f d\mu$ for every $f \in C_k(X)$. μ, μ_Y and λ here are related in exactly the same way as μ, ν and λ in 443Q. If Y is a normal subgroup of X, so that X/Y is the quotient group, λ is a left Haar measure. If Y is compact and μ_Y is the Haar probability measure on Y, then λ is the image measure $\mu\pi^{-1}$, where $\pi(x) = x^\bullet = xY$ for every $x \in X$.

(b) If $E \subseteq X$ and $EY = Y$, then E is Haar measurable iff $\tilde{E} = \{x^\bullet : x \in E\}$ belongs to the domain of λ, and E is Haar negligible iff $\tilde{E}$ is λ-negligible.

(c) Now suppose that X is σ-compact. Then for any Haar measurable $E \subseteq X$, $\mu E = \int g\,d\lambda$ in $[0,\infty]$, where $g(x^\bullet) = \mu_Y(Y \cap x^{-1}E)$ is defined for almost every $x \in X$. E is Haar negligible iff $\mu_Y(Y \cap x^{-1}E) = 0$ for almost every $x \in X$.

(d) Again suppose that X is σ-compact. Then we can extend the operator T of part (a) to an operator from $\mathcal{L}^1(\mu)$ to $\mathcal{L}^1(\lambda)$ by writing $(Tf)(x^\bullet) = \int f(xy)\mu_Y(dy)$ whenever $f \in \mathcal{L}^1(\mu)$, $x \in X$ and the integral is defined, and $\int Tf d\lambda = \int f d\mu$ for every $f \in \mathcal{L}^1(\mu)$. If $f \in \mathcal{L}^1(\mu)$, and we set $f_x(y) = f(xy)$ for all those $x \in X$, $y \in Y$ for which $xy \in \operatorname{dom} f$, then $Q = \{x : f_x \in \mathcal{L}^1(\mu_Y)\}$ is μ-conegligible, and $x \mapsto f_x^\bullet : Q \to L^1(\mu_Y)$ is almost continuous.

(e) If X is σ-compact, Y is compact and μ_Y is the Haar probability measure on Y, then

$$\iint f(xy)\mu_Y(dy)\mu(dx) = \int (Tf)(x^\bullet)\mu(dx) = \int Tf\,d\lambda = \int f d\mu$$

for every function f such that $\int f d\mu$ is defined in $[-\infty,\infty]$. In particular, $\mu E = \int \nu(Y \cap x^{-1}E)\mu(dx)$ for every Haar measurable set $E \subseteq X$.

447F Lemma Let X be a σ-compact locally compact Hausdorff topological group and Y a closed subgroup of X such that the modular function of Y is the restriction to Y of the modular function of X. Let Z be a compact normal subgroup of Y such that the quotient group Y/Z has a B-sequence. Let Σ_Y be the σ-algebra of those Haar measurable subsets E of X such that $EY = E$, and Σ_Z the algebra of Haar measurable sets $E \subseteq X$ such that $EZ = E$. Let $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ be a left-translation-invariant partial lower density. Then there is a left-translation-invariant partial lower density $\underline{\psi} : \Sigma_Z \to \Sigma_Z$ extending $\underline{\phi}$.

447G Lemma Let X be a σ-compact locally compact Hausdorff topological group, and $\langle Y_n\rangle_{n\in\mathbb{N}}$ a non-increasing sequence of compact subgroups of X with intersection Y. Let Σ be the algebra of Haar measurable subsets of X; set $\Sigma_{Y_n} = \{E : E \in \Sigma,\ EY_n = E\}$ for each n, and $\Sigma_Y = \{E : E \in \Sigma,\ EY = E\}$. Suppose that for each $n \in \mathbb{N}$ we are given a left-translation-invariant partial lower density $\underline{\phi}_n : \Sigma_{Y_n} \to \Sigma_{Y_n}$, and that $\underline{\phi}_{n+1}$ extends $\underline{\phi}_n$ for every n. Then there is a left-translation-invariant partial lower density $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ extending every $\underline{\phi}_n$.

447H Lemma Let X be a locally compact Hausdorff topological group, and Σ the algebra of Haar measurable sets in X. Then there is a left-translation-invariant lower density $\underline{\phi} : \Sigma \to \Sigma$.

447I Theorem Let X be a locally compact Hausdorff topological group. Then it has a left-translation-invariant lifting for its Haar measures.

447J Corollary Let X be any topological group carrying Haar measures. Then it has a left-translation-invariant lifting for its left Haar measures.

Version of 12.4.13

448 Polish group actions

I devote this section to two quite separate theorems. The first is an interesting result about measures on Polish spaces which are invariant under actions of Polish groups. In contrast to §441, we no longer have a strong general existence theorem for such measures, but instead have a natural necessary and sufficient condition in terms of countable dissections: there is an invariant probability measure on X if and only if there is no countable dissection of X into Borel sets which can be rearranged, by the action of the group, into two copies of X (448P).

The second theorem concerns the representation of group actions on measure algebras in terms of group actions on measure spaces. If we have a locally compact Polish group G (so that we do have Haar measures), and a Borel measurable action of G on the measure algebra of a Radon measure μ on a Polish space X, then it can be represented by a Borel measurable action of G on X (448S). The proof is mostly descriptive set theory based on §§423-424, but it also uses some interesting facts about L^0 spaces (448Q-448R).

448A Definitions Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For a, $b \in \mathfrak{A}$ I will say that an isomorphism $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ between the corresponding principal ideals belongs to the **countably full local semigroup generated by** G if there are a countable partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \pi_i\rangle_{i\in I}$ in G such that $\phi c = \pi_i c$ whenever $i \in I$ and $c \subseteq a_i$. If such an isomorphism exists I will say that a and b are **G-σ-equidecomposable**.

I write $a \preccurlyeq^\sigma_G b$ to mean that there is a $b' \subseteq b$ such that a and b' are G-σ-equidecomposable.

I will say that a function f with domain $\mathfrak{A}$ is **G-invariant** if $f(\pi a) = f(a)$ whenever $a \in \mathfrak{A}$ and $\pi \in G$.

448B Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_σ for the countably full local semigroup generated by G.

(a) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_σ, then $\phi^{-1} : \mathfrak{A}_b \to \mathfrak{A}_a$ also belongs to G^*_σ.

(b) Suppose that a, b, a', $b' \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_{a'}$, $\psi : \mathfrak{A}_b \to \mathfrak{A}_{b'}$ belong to G^*_σ. Then $\psi\phi \in G^*_\sigma$; its domain is $\mathfrak{A}_c$ where $c = \phi^{-1}(b \cap a')$, and its set of values is $\mathfrak{A}_{c'}$ where $c' = \psi(b \cap a')$.

(c) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_σ, then $\phi\restriction\mathfrak{A}_c \in G^*_\sigma$ for any $c \subseteq a$.

(d) Suppose that a, $b \in \mathfrak{A}$ and that $\psi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism such that there are a countable partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \phi_i\rangle_{i\in I}$ in G^*_σ such that $\psi c = \phi_i c$ whenever $i \in I$ and $c \subseteq a_i$. Then $\psi \in G^*_\sigma$.

448C Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_σ for the countably full local semigroup generated by G.

(a) For a, $b \in \mathfrak{A}$, $a \preccurlyeq^\sigma_G b$ iff there is a $\phi \in G^*_\sigma$ such that $a \in \operatorname{dom}\phi$ and $\phi a \subseteq b$.

(b)(i) $\preccurlyeq^\sigma_G$ is transitive and reflexive;

(ii) if $a \preccurlyeq^\sigma_G b$ and $b \preccurlyeq^\sigma_G a$ then a and b are G-σ-equidecomposable.

(c) G-σ-equidecomposability is an equivalence relation on $\mathfrak{A}$.

(d) If $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ are countable families in $\mathfrak{A}$, of which $\langle b_i\rangle_{i\in I}$ is disjoint, and $a_i \preccurlyeq^\sigma_G b_i$ for every $i \in I$, then $\sup_{i\in I} a_i \preccurlyeq^\sigma_G \sup_{i\in I} b_i$.

448D Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is an $a \neq 1$ such that a is G-σ-equidecomposable with 1;

(ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-σ-equidecomposable;

(iii) there are non-zero G-σ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) there are G-σ-equidecomposable a, $b \in \mathfrak{A}$ such that $a \subset b$.

448E Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. I will say that G is **countably non-paradoxical** if one of the following equiveridical statements is true:

(i) if a is G-σ-equidecomposable with 1 then $a = 1$;

(ii) there is no disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-σ-equidecomposable;

(iii) there are no non-zero G-σ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) if a, $b \in \mathfrak{A}$ are G-σ-equidecomposable and $a \subseteq b$ then $a = b$.

448F Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G.

(a) For any $a \in \mathfrak{A}$, $\operatorname{upr}(a, \mathfrak{C})$ is defined, and is given by the formula

$$\operatorname{upr}(a, \mathfrak{C}) = \sup\{\pi a : \pi \in G\}.$$

(b) If G^*_σ is the countably full local semigroup generated by G, then $\phi(c \cap a) = c \cap \phi a$ whenever $\phi \in G^*_\sigma$, $a \in \operatorname{dom}\phi$ and $c \in \mathfrak{C}$.

(c) $\operatorname{upr}(\phi a, \mathfrak{C}) = \operatorname{upr}(a, \mathfrak{C})$ whenever $\phi \in G^*_\sigma$ and $a \in \operatorname{dom}\phi$; $\operatorname{upr}(a, \mathfrak{C}) \subseteq \operatorname{upr}(b, \mathfrak{C})$ whenever $a \preccurlyeq^\sigma_G b$.

(d) If $a \preccurlyeq^\sigma_G b$ and $c \in \mathfrak{C}$ then $a \cap c \preccurlyeq^\sigma_G b \cap c$. So $a \cap c$ and $b \cap c$ are G-σ-equidecomposable whenever a and b are G-σ-equidecomposable and $c \in \mathfrak{C}$.

448G Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $\mathfrak{C}$ for the fixed-point subalgebra of G. Take any a, $b \in \mathfrak{A}$. Then $c_0 = \sup\{c : c \in \mathfrak{C},\ a \cap c \preccurlyeq^{\sigma}_{G} b\}$ is defined in $\mathfrak{A}$ and belongs to $\mathfrak{C}$; $a \cap c_0 \preccurlyeq^{\sigma}_{G} b$ and $b \setminus c_0 \preccurlyeq^{\sigma}_{G} a$.

448H Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Suppose that a, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a, \mathfrak{C}) = 1$. Then there are non-negative u, $v \in L^0(\mathfrak{C})$ such that

$$[\![u \ge n]\!] = \max\{c : c \in \mathfrak{C},\ \text{there is a disjoint family } \langle d_i\rangle_{i<n} \text{ such that } a \cap c \preccurlyeq^{\sigma}_{G} d_i \subseteq b \text{ for every } i < n\},$$

$$[\![v \le n]\!] = \max\{c : c \in \mathfrak{C},\ \text{there is a family } \langle d_i\rangle_{i<n} \text{ such that } d_i \preccurlyeq^{\sigma}_{G} a \text{ for every } i < n \text{ and } b \cap c \subseteq \sup_{i<n} d_i\}$$

for every $n \in \mathbb{N}$. Moreover, we have

(i) $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$,

(ii) $[\![v > 0]\!] = \operatorname{upr}(b, \mathfrak{C})$,

(iii) $u \le v \le u + \chi 1$.

448I Notation In the context of 448H, I will write $\lfloor b : a\rfloor$ for u, $\lceil b : a\rceil$ for v.

448J Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a, a_1, a_2, b, b_1, $b_2 \in \mathfrak{A}$ and that

$$\operatorname{upr}(a, \mathfrak{C}) = \operatorname{upr}(a_1, \mathfrak{C}) = \operatorname{upr}(a_2, \mathfrak{C}) = 1.$$

Then

(a) $\lfloor 0 : a\rfloor = \lceil 0 : a\rceil = 0$ and $\lfloor 1 : a\rfloor \ge \chi 1$.

(b) If $b_1 \preccurlyeq^{\sigma}_{G} b_2$ then $\lfloor b_1 : a\rfloor \le \lfloor b_2 : a\rfloor$ and $\lceil b_1 : a\rceil \le \lceil b_2 : a\rceil$.

(c) $\lceil b_1 \cup b_2 : a\rceil \le \lceil b_1 : a\rceil + \lceil b_2 : a\rceil$.

(d) If $b_1 \cap b_2 = 0$, then $\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor \le \lfloor b_1 \cup b_2 : a\rfloor$.

(e) If $c \in \mathfrak{C}$ is such that $a \cap c$ is a relative atom over $\mathfrak{C}$, then $c \subseteq [\![\lceil b : a\rceil - \lfloor b : a\rfloor = 0]\!]$.

(f) $\lfloor b : a_2\rfloor \ge \lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor$, $\lceil b : a_2\rceil \le \lceil b : a_1\rceil \times \lceil a_1 : a_2\rceil$.

448K Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. I will say that G has the **σ-refinement property** if for every $a \in \mathfrak{A}$ there is a $d \subseteq a$ such that $d \preccurlyeq^{\sigma}_{G} a \setminus d$ and $a' = a \setminus \operatorname{upr}(d, \mathfrak{C})$ is a relative atom over $\mathfrak{C}$.

448L Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with countable Maharam type. Then any countable subgroup of $\operatorname{Aut}\mathfrak{A}$ has the σ-refinement property.

448M Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. If G has the σ-refinement property, then for any $\epsilon > 0$ there is an $a^* \in \mathfrak{A}$ such that $\operatorname{upr}(a^*, \mathfrak{C}) = 1$ and $\lceil b : a^*\rceil \le \lfloor b : a^*\rfloor + \epsilon\lfloor 1 : a^*\rfloor$ for every $b \in \mathfrak{A}$.

448N Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that G has the σ-refinement property. Then there is a function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ such that

(i) θ is additive, non-negative and sequentially order-continuous;

(ii) $\theta a = 0$ iff $a = 0$, $\theta 1 = \chi 1$;

(iii) $\theta(a \cap c) = \theta a \times \chi c$ for every $a \in \mathfrak{A}$, $c \in \mathfrak{C}$; in particular, $\theta c = \chi c$ for every $c \in \mathfrak{C}$;

(iv) if a, $b \in \mathfrak{A}$ are G-σ-equidecomposable, then $\theta a = \theta b$; in particular, θ is G-invariant.

448O Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$ with the σ-refinement property. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Then the following are equiveridical:

(i) there are a Dedekind σ-complete Boolean algebra $\mathfrak{D}$, not $\{0\}$, and a G-invariant sequentially order-continuous non-negative additive function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ such that $\theta 1 = \chi 1$;

(ii) if $a \in \mathfrak{A}$ and $1 \preccurlyeq_G^\sigma a$, then $\operatorname{upr}(1 \setminus a, \mathfrak{C}) \ne 1$;

(iii) if $a \in \mathfrak{A}$ and $1 \preccurlyeq_G^\sigma a$, then $1 \not\preccurlyeq_G^\sigma 1 \setminus a$.

448P Theorem Let G be a Polish group acting on a non-empty Polish space $(X, \mathfrak{T})$ with a Borel measurable action $\bullet$. For Borel sets E, $F \subseteq X$ say that $E \preccurlyeq_G^\sigma F$ if there are a countable partition $\langle E_i\rangle_{i\in I}$ of E into Borel sets, and a family $\langle g_i\rangle_{i\in I}$ in G, such that $g_i{\bullet}E_i \subseteq F$ for every i and $\langle g_i{\bullet}E_i\rangle_{i\in I}$ is disjoint. Then the following are equiveridical:

(i) there is a G-invariant Radon probability measure μ on X;

(ii) if $F \subseteq X$ is a Borel set such that $X \preccurlyeq_G^\sigma F$, then $\bigcap_{n\in\mathbb{N}} g_n{\bullet}F \ne \emptyset$ for any sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in G;

(iii) there are no disjoint Borel sets E, $F \subseteq X$ such that $X \preccurlyeq_G^\sigma E$ and $X \preccurlyeq_G^\sigma F$.

448Q Lemma Let (X, Σ, μ) be a σ-finite measure space with countable Maharam type. Write $L^0(\Sigma)$ for the set of Σ-measurable functions from X to $\mathbb{R}$. Then there is a function $T : L^0(\mu) \to L^0(\Sigma)$ such that

(α) $u = (Tu)^\bullet$ for every $u \in L^0$,

(β) $(u, x) \mapsto (Tu)(x) : L^0 \times X \to \mathbb{R}$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable,

where $\mathcal{B} = \mathcal{B}(L^0)$ is the Borel σ-algebra of L^0 with its topology of convergence in measure.

448R Lemma Let (X, Σ, μ) be a σ-finite measure space with countable Maharam type.

(a) $L^0 = L^0(\mu)$, with its topology of convergence in measure, is a Polish space.

(b) Let $\mathfrak{A}$ be the measure algebra of μ, and $\mathfrak{A}^f$ the set $\{a : a \in \mathfrak{A},\ \bar\mu a < \infty\}$. Then the Borel σ-algebra $\mathcal{B} = \mathcal{B}(L^0)$ is the σ-algebra of subsets of L^0 generated by sets of the form $\{u : \bar\mu(a \cap [\![u \in F]\!]) > \alpha\}$, where $a \in \mathfrak{A}^f$, $F \subseteq \mathbb{R}$ is Borel, and $\alpha \in \mathbb{R}$.

448S Mackey's theorem (Mackey 62) Let G be a locally compact Polish group, (X, Σ) a standard Borel space and μ a σ-finite measure with domain Σ. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ with its measure-algebra topology. Let $\circ$ be a Borel measurable action of G on $\mathfrak{A}$ such that $a \mapsto g{\circ}a$ is a Boolean automorphism for every $g \in G$. Then we have a $(\mathcal{B}(G)\widehat{\otimes}\Sigma, \Sigma)$-measurable action $\bullet$ of G on X such that

$$g{\circ}E^\bullet = (g{\bullet}E)^\bullet$$

for every $g \in G$ and $E \in \Sigma$, writing $g{\bullet}E$ for $\{g{\bullet}x : x \in E\}$ as usual.

448T Corollary Let G be a σ-compact locally compact Hausdorff group, X a Polish space, μ a σ-finite Borel measure on X, and $(\mathfrak{A}, \bar\mu)$ the measure algebra of μ, with its measure-algebra topology. Let $\circ$ be a continuous action of G on $\mathfrak{A}$ such that $a \mapsto g{\circ}a$ is a Boolean automorphism for every $g \in G$. Then we have a Borel measurable action $\bullet$ of G on X such that

$$g{\circ}E^\bullet = (g{\bullet}E)^\bullet$$

for every $g \in G$ and $E \in \mathcal{B}(X)$.

Version of 13.6.13

449 Amenable groups

I end this chapter with a brief introduction to 'amenable' topological groups. I start with the definition (449A) and straightforward results assuring us that there are many amenable groups (449C). At a slightly deeper level we have a condition for a group to be amenable in terms of a universal object constructible from the group, not invoking 'all compact Hausdorff spaces' (449E). I give some notes on amenable locally compact groups, concentrating on a long list of properties equivalent to amenability (449J), and a version of Tarski's theorem characterizing amenable discrete groups (449M). I end with Banach's theorem on extending Lebesgue measure in one and two dimensions.

449A Definition A topological group G is **amenable** if whenever X is a non-empty compact Hausdorff space and $\bullet$ is a continuous action of G on X, then there is a G-invariant Radon probability measure on X.

449B Lemma Let G be a topological group, X a locally compact Hausdorff space, and $\bullet$ a continuous action of G on X.

(a) Writing C_0 for the Banach space of continuous real-valued functions on X vanishing at ∞, the map $a \mapsto a^{-1}\bullet f : C_0 \to C_0$ is uniformly continuous for the right uniformity on G and the norm uniformity of C_0, for any $f \in C_0$.

(b) If μ is a G-invariant Radon measure on X and $1 \le p < \infty$, then $a \mapsto a^{-1}\bullet u : G \to L^p$ is uniformly continuous for the right uniformity on G and the norm uniformity of $L^p = L^p(\mu)$, for any $u \in L^p$.

449C Theorem (a) Let G and H be topological groups such that there is a continuous surjective homomorphism from G onto H. If G is amenable, so is H.

(b) Let G be a topological group and suppose that there is a dense subset A of G such that every finite subset of A is included in an amenable subgroup of G. Then G is amenable.

(c) Let G be a topological group and H a normal subgroup of G. If H and G/H are both amenable, so is G.

(d) Let G be a topological group with two amenable subgroups H_0 and H_1 such that H_0 is normal and $H_0H_1 = G$. Then G is amenable.

(e) The product of any family of amenable topological groups is amenable.

(f) Any abelian topological group is amenable.

(g) Any compact Hausdorff topological group is amenable.

449D Theorem Let G be a topological group.

(a) Write U for the set of bounded real-valued functions on G which are uniformly continuous for the right uniformity of G. Then U is an M-space, and we have an action $\bullet_l$ of G on U defined by the formula $(a\bullet_l f)(y) = f(a^{-1}y)$ for $a, y \in G$ and $f \in U$.

(b) Let $Z \subseteq \mathbb{R}^U$ be the set of Riesz homomorphisms $z : U \to \mathbb{R}$ such that $z(\chi G) = 1$. Then Z is a compact Hausdorff space, and we have a continuous action of G on Z defined by the formula $(a\bullet z)(f) = z(a^{-1}\bullet_l f)$ for $a \in G$, $z \in Z$ and $f \in U$.

(c) Setting $\hat{a}(f) = f(a)$ for $a \in G$ and $f \in U$, the map $a \mapsto \hat{a} : G \to Z$ is a continuous function from G onto a dense subset of Z. If $a, b \in G$ then $a\bullet\hat{b} = \widehat{ab}$.

(d) Now suppose that X is a compact Hausdorff space, $(a, x) \mapsto a\bullet x$ is a continuous action of G on X, and $x_0 \in X$. Then there is a unique continuous function $\phi : Z \to X$ such that $\phi(\hat{e}) = x_0$ and $\phi(a\bullet z) = a\bullet\phi(z)$ for every $a \in G$ and $z \in Z$.

(e) If G is Hausdorff then the action of G on Z is faithful and the map $a \mapsto \hat{a}$ is a homeomorphism between G and its image in Z.

Definition The space Z, together with the canonical action of G on it and the map $a \mapsto \hat{a} : G \to Z$, is called the **greatest ambit** of the topological group G.

449E Corollary Let G be a topological group. Then the following are equiveridical:

(i) G is amenable;

(ii) there is a G-invariant Radon probability measure on the greatest ambit of G;

(iii) writing U for the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity, there is a positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ for every $f \in U$ and $a \in G$.

449F Corollary Let G be a topological group.

(a) If G is amenable, then

(i) every open subgroup of G is amenable;

(ii) every dense subgroup of G is amenable.

(b) Suppose that for every sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of the identity e of G there is a normal subgroup H of G such that $H \subseteq \bigcap_{n\in\mathbb{N}} V_n$ and G/H is amenable. Then G is amenable.

449G Example Let F_2 be the free group on two generators, with its discrete topology. Then F_2 is a σ-compact unimodular locally compact Polish group. But it is not amenable.

449H Lemma Let G be a locally compact Hausdorff topological group, and U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity. Let μ be a left Haar measure on G, and $*$ the corresponding convolution on $\mathcal{L}^0(\mu)$.

(a) If $h \in \mathcal{L}^1(\mu)$ and $f \in \mathcal{L}^\infty(\mu)$ then $h * f \in U$.

(b) Let $p : U \to \mathbb{R}$ be a positive linear functional such that $p(a\bullet_l f) = p(f)$ whenever $f \in U$ and $a \in G$. Then $p(h * f) = p(f) \int h \, d\mu$ for every $h \in \mathcal{L}^1(\mu)$ and $f \in U$.

449I Notation It will save repeated explanations if I say now that for the next two results, given a locally compact Hausdorff group G, Σ_G will be the algebra of Haar measurable subsets of G and $\mathcal{N}_G$ the ideal of Haar negligible subsets of G, while $\mathcal{B}_G$ will be the Borel σ-algebra of G. Recall that all three are left- and right-translation-invariant and inversion-invariant, and indeed autohomeomorphism-invariant, in that if $\gamma : G \to G$ is a function of any of the types

$$x \mapsto ax, \quad x \mapsto xa, \quad x \mapsto x^{-1}$$

or is a group automorphism which is also a homeomorphism, and $E \subseteq G$, then $\gamma[E]$ belongs to Σ_G, $\mathcal{N}_G$ or $\mathcal{B}_G$ iff E does.

449J Theorem Let G be a locally compact Hausdorff group; fix a left Haar measure μ on G. Write $\mathcal{L}^1$ for $\mathcal{L}^1(\mu)$ and L^∞ for $L^\infty(\mu)$, etc. Let C_{k1}^+ be the set of continuous functions $h : G \to [0, \infty[$ with compact supports such that $\int h \, d\mu = 1$, and suppose that $q \in [1, \infty[$. Then the following are equiveridical:

(i) G is amenable;

(ii) there is a positive linear functional $p : C_b(G) \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ for every $f \in C_b(G)$ and every $a \in G$;

(iii) there is a finitely additive functional $\phi : \mathcal{B}_G \to [0, 1]$ such that $\phi G = 1$, $\phi(aE) = \phi E$ for every $E \in \mathcal{B}_G$ and $a \in G$, and $\phi E = 0$ for every Haar negligible $E \in \mathcal{B}_G$;

(iv) there is a finitely additive functional $\phi : \Sigma_G \to [0, 1]$ such that $\phi G = 1$, $\phi(aE) = \phi(Ea) = \phi(E^{-1}) = \phi E$ for every $E \in \Sigma_G$ and $a \in G$, and $\phi E = 0$ for every $E \in \mathcal{N}_G$;

(v) there is a positive linear functional $\tilde{p} : L^\infty \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(a\bullet_l u) = \tilde{p}(a\bullet_r u) = \tilde{p}(a\bullet_c u) = \tilde{p}(\overset{\leftrightarrow}{u}) = \tilde{p}(u)$ for every $u \in L^\infty$ and every $a \in G$;

(vi) there is a positive linear functional $\tilde{p} : L^\infty \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(a\bullet_l u) = \tilde{p}(u)$ for every $u \in L^\infty$ and every $a \in G$;

(vii) there is a positive linear functional $\tilde{p} : L^\infty \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(\nu * u) = \nu G \cdot \tilde{p}(u)$ for every $u \in L^\infty$ and every totally finite Radon measure ν on G;

(viii) there is a positive linear functional $\tilde{p} : L^\infty \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(v * u) = \tilde{p}(u) \int v$ for every $v \in L^1$ and $u \in L^\infty$;

(ix) for every finite set $J \subseteq \mathcal{L}^1$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $\|g * h - (\int g \, d\mu)h\|_1 \le \epsilon$ for every $g \in J$;

(x) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $\|a\bullet_l h - h\|_1 \le \epsilon$ for every $a \in K$;

(xi) for any finite set $I \subseteq G$ and $\epsilon > 0$, there is a $u \in L^q$ such that $\|u\|_q = 1$ and $\|u - a\bullet_l u\|_q \le \epsilon$ for every $a \in I$;

(xii) for any finite set $I \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(L \triangle aL) \le \epsilon \mu L$ for every $a \in I$;

(xiii) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a symmetric compact neighbourhood L of the identity e in G such that $\mu(L \triangle aL) \le \epsilon \mu L$ for every $a \in K$;

(xiv) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(KL) \le (1 + \epsilon)\mu L$.

449K Proposition Let G be an amenable locally compact Hausdorff group, and H a subgroup of G. Then H is amenable.

449L Tarski's theorem Let G be a group acting on a non-empty set X. Then the following are equiveridical:

(i) there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu X = 1$ and $\nu(a{\bullet}A) = \nu A$ whenever $A \subseteq X$ and $a \in G$;

(ii) there are no $A_0, \dots , A_n$, $a_0, \dots , a_n$, $b_0, \dots , b_n$ such that $A_0, \dots , A_n$ are subsets of X covering X, $a_0, \dots , a_n, b_0, \dots , b_n$ belong to G, and $a_0{\bullet}A_0$, $b_0{\bullet}A_0$, $a_1{\bullet}A_1$, $b_1{\bullet}A_1, \dots , b_n{\bullet}A_n$ are all disjoint.

449M Corollary Let G be a group with its discrete topology. Then the following are equiveridical:

(i) G is amenable;

(ii) there are no $A_0, \dots , A_n$, $a_0, \dots , a_n$, $b_0, \dots , b_n$ such that $G = \bigcup_{i \le n} A_i$ $a_0, \dots , a_n, b_0, \dots , b_n$ belong to G, and a_0A_0, b_0A_0, a_1A_1, $b_1A_1, \dots , b_nA_n$ are disjoint.

449N Theorem Let G be a group which is amenable in its discrete topology, X a set, and $\bullet$ an action of G on X. Let $\mathcal{E}$ be a subring of $\mathcal{P}X$ and $\nu : \mathcal{E} \to [0, \infty[$ a finitely additive functional which is G-invariant in the sense that $g{\bullet}E \in \mathcal{E}$ and $\nu(g{\bullet}E) = \nu E$ whenever $E \in \mathcal{E}$ and $g \in G$. Then there is an extension of ν to a G-invariant non-negative finitely additive functional $\tilde{\nu}$ defined on the ideal $\mathcal{I}$ of subsets of X generated by $\mathcal{E}$.

449O Corollary If $r = 1$ or $r = 2$, there is a functional $\theta : \mathcal{P}\mathbb{R}^r \to [0, \infty]$ such that (i) $\theta(A \cup B) = \theta A + \theta B$ whenever A, $B \subseteq \mathbb{R}^r$ are disjoint (ii) θE is the Lebesgue measure of E whenever $E \subseteq \mathbb{R}^r$ is Lebesgue measurable (iii) $\theta(g[A]) = \theta A$ whenever $A \subseteq \mathbb{R}^r$ and $g : \mathbb{R}^r \to \mathbb{R}^r$ is an isometry.