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# Chapter 44

## Topological groups

Measure theory begins on the real line, which is of course a group; and one of the most fundamental properties of Lebesgue measure is its translation-invariance (134A). Later we come to the standard measure on the unit circle (255M), and counting measure on the integers is also translation-invariant, if we care to notice; moreover, Fourier series and transforms clearly depend utterly on the fact that shift operators don't disturb the measure-theoretic structures we are building. Yet another example appears in the usual measure on $\{0,1\}^I$, which is translation-invariant if we identify $\{0,1\}^I$ with the group $\mathbb{Z}_2^I$ (345Ab). Each of these examples is special in many other ways. But it turns out that a particular combination of properties which they share, all being locally compact Hausdorff spaces with group operations for which multiplication and inversion are continuous, is the basis of an extraordinarily powerful theory of invariant measures.

As usual, I have no choice but to move rather briskly through a wealth of ideas. The first step is to set out a suitably general existence theorem, assuring us that every locally compact Hausdorff topological group has non-trivial invariant Radon measures, that is, 'Haar measures' (441E). As remarkable as the existence of Haar measures is their (essential) uniqueness (442B); the algebra, topology and measure theory of a topological group are linked in so many ways that they form a peculiarly solid structure. I investigate a miscellany of facts about this structure in §443, including the basic theory of the modular functions linking left-invariant measures with right-invariant measures.

I have already mentioned that Fourier analysis depends on the translation-invariance of Lebesgue measure. It turns out that substantial parts of the abstract theory of Fourier series and transforms can be generalized to arbitrary locally compact groups. In particular, convolutions (§255) appear again, even in non-abelian groups (§444). But for the central part of the theory, a transform relating functions on a group $X$ to functions on its 'dual' group $\mathcal{X}$, we do need the group to be abelian. Actually I give only the foundation of this theory: if $X$ is an abelian locally compact Hausdorff group, it is the dual of its dual (445U). (In 'ordinary' Fourier theory, where we are dealing with the cases $X=\mathcal{X}=\mathbb{R}$ and $X=S^1$, $\mathcal{X}=\mathbb{Z}$, this duality is so straightforward that one hardly notices it.) But on the way to the duality theorem we necessarily see many of the themes of Chapter 28 in more abstract guises.

A further remarkable fact is that any Haar measure has a translation-invariant lifting (447J). The proof demands a union between the ideas of the ordinary Lifting Theorem (§341) and some of the elaborate structure theory which has been developed for locally compact groups (§446).

For the last two sections of the chapter, I look at groups which are not locally compact, and their actions on appropriate spaces. For a particularly important class of group actions, Borel measurable actions of Polish groups on Polish spaces, we have a natural necessary and sufficient condition for the existence of an invariant measure (448P), complementing the result for locally compact spaces in 441C. In a slightly different direction, we can look at those groups, the 'amenable' groups, for which all actions (on compact Hausdorff spaces) have invariant measures. This again leads to some very remarkable ideas, which I sketch in §449.

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### 441 Invariant measures on locally compact spaces

I begin this chapter with the most important theorem on the existence of invariant measures: every locally compact Hausdorff group has left and right Haar measures (441E). I derive this as a corollary of a general result concerning invariant measures on locally compact spaces (441C), which has other interesting consequences (441H).

**441A Group actions** I repeat the definitions from §4A5 on which this chapter is based.

**(a)** If $G$ is a group and $X$ is a set, an **action** of $G$ on $X$ is a function $(a,x) \mapsto a\bullet x : G \times X \to X$ such that

$$(ab)\bullet x = a\bullet(b\bullet x) \text{ for all } a,\, b \in G,\, x \in X,$$

$$e\bullet x = x \text{ for every } x \in X$$

where $e$ is the identity of $G$ (4A5Ba). In this context I write

$$a\bullet A = \{a\bullet x : x \in A\}$$

for $a \in G$, $A \subseteq X$ (4A5Bc). If $f$ is a function defined on a subset of $X$, then $(a\bullet f)(x) = f(a^{-1}\bullet x)$ whenever $a \in G$ and $x \in X$ and $a^{-1}\bullet x \in \operatorname{dom} f$ (4A5C(c-i)).

**(b)** If a group $G$ acts on a set $X$, a measure $\mu$ on $X$ is $G$**-invariant** if $\mu(a^{-1}\bullet E)$ is defined and equal to $\mu E$ whenever $a \in G$ and $\mu$ measures $E$.

(Of course this is the same thing as saying that $\mu(a\bullet E) = \mu E$ for every $a \in G$ and measurable set $E$; I use the formula with $a^{-1}$ so as to match my standard practice when $a$ is actually a function from $X$ to $X$.)

**(c)** If a group $G$ acts on a set $X$ and a measure $\mu$ on $X$ is $G$-invariant, then $\int f(a\bullet x)\mu(dx)$ is defined and equal to $\int f\,d\mu$ whenever $f$ is a virtually measurable $[-\infty,\infty]$-valued function defined on a conegligible subset of $X$ and $\int f\,d\mu$ is defined in $[-\infty,\infty]$. **P** Set $\phi(x) = a\bullet x$ for $x \in X$. Then $\phi^{-1}[E] = \{x : a\bullet x \in E\} = a^{-1}\bullet E$ for every $E \subseteq X$, so $\phi : X \to X$ is inverse-measure-preserving. Now $\int f(a\bullet x)\mu(dx) = \int f\phi\,d\mu = \int f d\mu$ by 235Gb. **Q**

**441B** It will be useful later to be able to quote the following elementary results.

**Lemma** Let $X$ be a topological space, $G$ a group, and $\bullet$ an action of $G$ on $X$ such that $x \mapsto a\bullet x$ is continuous for every $a \in G$.

(a) If $\mu$ is a quasi-Radon measure on $X$ such that $\mu(a\bullet U) \le \mu U$ for every open set $U \subseteq X$ and every $a \in G$, then $\mu$ is $G$-invariant.

(b) If $\mu$ is a Radon measure on $X$ such that $\mu(a\bullet K) \le \mu K$ for every compact set $K \subseteq X$ and every $a \in G$, then $\mu$ is $G$-invariant.

**proof** Note first that the maps $x \mapsto a\bullet x$ are actually homeomorphisms (4A5Bd), so that $a\bullet U$ and $a\bullet K$ will be open, or compact, as $U$ and $K$ are. Next, the inequality $\le$ in the hypotheses is an insignificant refinement; since we must also have

$$\mu U = \mu(a^{-1}\bullet a\bullet U) \le \mu(a\bullet U)$$

in (a),

$$\mu K = \mu(a^{-1}\bullet a\bullet K) \le \mu(a\bullet K)$$

in (b), we always have equality here.

Now fix $a \in G$, and set $T_a(x) = a\bullet x$ for $x \in X$. Then $T_a$ is a homeomorphism, so the image measure $\mu T_a^{-1}$ will be quasi-Radon, or Radon, if $\mu$ is. In (a), we are told that $\mu T_a^{-1}$ agrees with $\mu$ on the open sets, while in (b) we are told that they agree on the compact sets; so in both cases we have $\mu = \mu T_a^{-1}$, by 415H(iii) or 416E(b-ii). Consequently we have $\mu T_a^{-1}[E] = \mu E$ whenever $\mu$ measures $E$. As $a$ is arbitrary, $\mu$ is $G$-invariant.

**441C Theorem** (Steinlage 75) Let $X$ be a non-empty locally compact Hausdorff space and $G$ a group acting on $X$. Suppose that

(i) $x \mapsto a\bullet x$ is continuous for every $a \in G$;

(ii) every orbit $\{a\bullet x : a \in G\}$ is dense;

(iii) whenever $K$ and $L$ are disjoint compact subsets of $X$ there is a non-empty open subset $U$ of $X$ such that, for every $a \in G$, at most one of $K$, $L$ meets $a\bullet U$.

Then there is a non-zero $G$-invariant Radon measure $\mu$ on $X$.

**proof (a)** $\bigcup_{a\in G} a\bullet U = X$ for every non-empty open $U \subseteq X$. **P** If $x \in X$, then the orbit of $x$ must meet $U$, so there is a $a \in G$ such that $a\bullet x \in U$; but this means that $x \in a^{-1}\bullet U$. **Q**

Fix some point $z_0$ of $X$ and write $\mathcal{V}$ for the set of open sets containing $z_0$. Then if $K$, $L$ are disjoint compact subsets of $X$ there is a $U \in \mathcal{V}$ such that, for every $a \in G$, at most one of $K$, $L$ meets $a\bullet U$. **P** By hypothesis, there is a non-empty open set $V$ such that, for every $a \in G$, at most one of $K$, $L$ meets $a\bullet V$. Now there is an $b \in G$ such that $b\bullet z_0 \in V$; set $U = b^{-1}\bullet V$. Because $b^{-1}$ acts on $X$ as a homeomorphism, $U \in \mathcal{V}$; and if $a \in G$, then $a\bullet U = (ab^{-1})\bullet V$ can meet at most one of $K$ and $L$. **Q**

**(b)** If $U \in \mathcal{V}$ and $A \subseteq X$ is any relatively compact set, then $\{a\bullet U : a \in G\}$ is an open cover of $X$, so there is a finite set $I \subseteq G$ such that $\overline{A} \subseteq \bigcup_{a\in I} a\bullet U$. Write $\lceil A : U\rceil$ for $\min\{\#(I) : I \subseteq G,\, A \subseteq \bigcup_{a\in I} a\bullet U\}$.

**(c)** The following facts are now elementary.

**(i)** If $U \in \mathcal{V}$ and $A$, $B \subseteq X$ are relatively compact, then

$$0 \le \lceil A : U\rceil \le \lceil A \cup B : U\rceil \le \lceil A : U\rceil + \lceil B : U\rceil,$$

and $\lceil A : U\rceil = 0$ iff $A = \emptyset$.

**(ii)** If $U$, $V \in \mathcal{V}$ and $V$ is relatively compact, and $A \subseteq X$ also is relatively compact, then

$$\lceil A : U\rceil \le \lceil A : V\rceil \lceil V : U\rceil.$$

**P** If $A \subseteq \bigcup_{a\in I} a\bullet V$ and $V \subseteq \bigcup_{b\in J} b\bullet U$, then $A \subseteq \bigcup_{a\in I,b\in J}(ab)\bullet U$. **Q**

**(iii)** If $U \in \mathcal{V}$, $A \subseteq X$ is relatively compact and $b \in G$, then $\lceil b\bullet A : U\rceil = \lceil A : U\rceil$. **P** If $I \subseteq G$ and $A \subseteq \bigcup_{a\in I} a\bullet U$, then $b\bullet A \subseteq \bigcup_{a\in I}(ba)\bullet U$, so $\lceil b\bullet A : U\rceil \le \#(I)$; as $I$ is arbitrary, $\lceil b\bullet A : U\rceil \le \lceil A : U\rceil$. On the other hand, the same argument shows that

$$\lceil A : U\rceil = \lceil b^{-1}\bullet b\bullet A : U\rceil \le \lceil b\bullet A : U\rceil,$$

so we must have equality. **Q**

**(d)** Fix a relatively compact $V_0 \in \mathcal{V}$. (This is the first place where we use the hypothesis that $X$ is locally compact.) For every $U \in \mathcal{V}$ and every relatively compact set $A \subseteq X$ write

$$\lambda_U A = \frac{\lceil A : U\rceil}{\lceil V_0 : U\rceil}.$$

Then (c) tells us immediately that

(i) if $A$, $B \subseteq X$ are relatively compact,

$$0 \le \lambda_U A \le \lambda_U(A \cup B) \le \lambda_U A + \lambda_U B;$$

(ii) $\lambda_U A \le \lceil A : V_0\rceil$ for every relatively compact $A \subseteq X$;
(iii) $\lambda_U(b\bullet A) = \lambda_U A$ for every relatively compact $A \subseteq X$ and every $b \in G$;
(iv) $\lambda_U V_0 = 1$.

**(e)** Now for the point of the hypothesis (iii) of the theorem. If $K$, $L$ are disjoint compact subsets of $X$, there is a $V \in \mathcal{V}$ such that $\lambda_U(K \cup L) = \lambda_U K + \lambda_U L$ whenever $U \in \mathcal{V}$ and $U \subseteq V$. **P** By (a), there is a $V \in \mathcal{V}$ such that any translate $a\bullet V$ can meet at most one of $K$ and $L$. Take any $U \in \mathcal{V}$ included in $V$. Let $I \subseteq G$ be such that $\bigcup_{a\in I} a\bullet U \supseteq K \cup L$ and $\#(I) = \lceil K \cup L : U\rceil$. Then

$$I' = \{a : a \in I,\, K \cap a\bullet U \ne \emptyset\}, \quad I'' = \{a : a \in I,\, L \cap a\bullet U \ne \emptyset\}$$

are disjoint. $K \subseteq \bigcup_{a\in I'} a\bullet U$, so $\lceil K : U\rceil \le \#(I')$, and similarly $\lceil L : U\rceil \le \#(I'')$. But this means that

$$\lceil K : U\rceil + \lceil L : U\rceil \le \#(I') + \#(I'') \le \#(I) = \lceil K \cup L : U\rceil,$$

$$\lambda_U K + \lambda_U L \le \lambda_U(K \cup L).$$

As we already know that $\lambda_U(K \cup L) \le \lambda_U K + \lambda_U L$, we must have equality, as claimed. **Q**

**(f)** Let $\mathcal{F}$ be an ultrafilter on $\mathcal{V}$ containing all sets of the form $\{U : U \in \mathcal{V}, U \subseteq V\}$ for $V \in \mathcal{V}$. If $A \subseteq X$ is relatively compact, $0 \le \lambda_U A \le \lceil A : V_0\rceil$ for every $U \in \mathcal{V}$, so $\lambda A = \lim_{U\to\mathcal{F}} \lambda_U A$ is defined in $[0, \lceil A : V_0\rceil]$. From (d-i) and (d-iii) we see that

$$0 \leq \lambda(b{\bullet}A) = \lambda A \leq \lambda(A \cup B) \leq \lambda A + \lambda B$$

for all relatively compact $A$, $B \subseteq X$ and $b \in G$. From (d-iv) we see that $\lambda V_0 = 1$. Moreover, from (e) we see that if $K$, $L \subseteq X$ are disjoint compact sets,

$$\{U : U \in \mathcal{V},\ \lambda_U(K \cup L) = \lambda_U K + \lambda_U L\} \in \mathcal{F},$$

so $\lambda(K \cup L) = \lambda K + \lambda L$.

**(g)** By 416M, there is a Radon measure $\mu$ on $X$ such that

$$\mu K = \inf\{\lambda L : L \subseteq X \text{ is compact, } K \subseteq \operatorname{int} L\}$$

for every compact set $K \subseteq X$. Now $\mu$ is $G$-invariant. **P** Take $b \in G$. If $K$, $L \subseteq X$ are compact and $K \subseteq \operatorname{int} L$, then $b{\bullet}K \subseteq \operatorname{int} b{\bullet}L$, because $x \mapsto b{\bullet}x$ is a homeomorphism; so

$$\mu(b{\bullet}K) \leq \lambda(b{\bullet}L) = \lambda L.$$

As $L$ is arbitrary, $\mu(b{\bullet}K) \leq \mu K$. As $b$ and $K$ are arbitrary, $\mu$ is $G$-invariant, by 441Bb. **Q**

**(h)** Finally, $\mu V_0 \geq \lambda V_0 \geq 1$, so $\mu$ is non-zero.

**441D** The hypotheses of 441C are deliberately drawn as widely as possible. The principal application is the one for which the chapter is named.

**Definition** If $G$ is a topological group, a **left Haar measure** on $G$ is a non-zero quasi-Radon measure $\mu$ on $G$ which is invariant for the left action of $G$ on itself, that is, $\mu(aE) = \mu E$ whenever $\mu$ measures $E$ and $a \in G$.

Similarly, a **right Haar measure** is a non-zero quasi-Radon measure $\mu$ such that $\mu(Ea) = \mu E$ whenever $E \in \operatorname{dom}\mu$ and $a \in G$.

(My reasons for requiring 'quasi-Radon' here will appear in §§442 and 443.)

**441E Theorem** A locally compact Hausdorff topological group has left and right Haar measures, which are both Radon measures.

**proof** Both the left and right actions of $G$ on itself satisfy the conditions of 441C. **P** In both cases, condition (i) is just the (separate) continuity of multiplication, and (ii) is trivial, as every orbit is the whole of $G$. As for (iii), let us take the left action first. Given disjoint compact subsets $K$, $L$ of $G$, then $M = \{y^{-1}z : y \in K,\ z \in L\}$ is a compact subset of $G$ not containing the identity $e$. Because the topology is Hausdorff, $M$ is closed and $X \setminus M$ is a neighbourhood of $e$. Because multiplication and inversion are continuous, there are open neighbourhoods $V$, $V'$ of $e$ such that $u^{-1}v \in G \setminus M$ whenever $u \in V$ and $v \in V'$. Set $U = V \cap V'$; then $U$ is a non-empty open set in $G$.

**?** Suppose, if possible, that there is a $a \in G$ such that $a{\bullet}U = aU$ meets both $K$ and $L$. Take $y \in K \cap aU$ and $z \in L \cap aU$. Then $a^{-1}y \in U \subseteq V$ and $a^{-1}z \in U \subseteq V'$, so

$$y^{-1}z = (a^{-1}y)^{-1}a^{-1}z \in G \setminus M;$$

but also $y^{-1}z \in M$. **X**

Thus $aU$ meets at most one of $K$, $L$ for any $a \in G$. As $K$ and $L$ are arbitrary, condition (iii) of 441C is satisfied.

For the right action, we use the same ideas, but vary the formulae. Set $M = \{yz^{-1} : y \in K,\ z \in L\}$, and choose $V$ and $V'$ such that $uv^{-1} \in G \setminus M$ for $u \in V$, $v \in V'$. Then if $a \in G$, $y \in K$ and $z \in L$, $za(ya)^{-1} \in M$ and one of $za$, $ya$ does not belong to $U = V \cap V'$, that is, one of $z$, $y$ does not belong to $Ua^{-1} = a{\bullet}U$. **Q**

Then 441C provides us with non-zero left and right Haar measures on $G$, and also tells us that they are Radon measures.

**441F** A different type of example is provided by locally compact metric spaces.

**Definition** If $(X, \rho)$ is any metric space, its **isometry group** is the set of permutations $g : X \to X$ which are **isometries**, that is, $\rho(g(x), g(y)) = \rho(x, y)$ for all $x$, $y \in X$.

**441G The topology of an isometry group** Let $(X, \rho)$ be a metric space and $G$ the isometry group of $X$.

**(a)** Give $G$ the topology of pointwise convergence inherited from the product topology of $X^X$. Then $G$ is a Hausdorff topological group and the action of $G$ on $X$ is continuous. **P** If $x \in X$, $g_0$, $h_0 \in G$ and $\epsilon > 0$, then $V = \{g : \rho(gh_0(x), g_0h_0(x)) \le \frac{1}{2}\epsilon\}$ is a neighbourhood of $g_0$ and $V' = \{h : \rho(h(x), h_0(x)) \le \frac{1}{2}\epsilon\}$ is a neighbourhood of $h_0$. If $g \in V$ and $h \in V'$ then

$$\begin{aligned}\rho(gh(x), g_0h_0(x)) &\le \rho(gh(x), gh_0(x)) + \rho(gh_0(x), g_0h_0(x))\\ &\le \rho(h(x), h_0(x)) + \frac{1}{2}\epsilon \le \epsilon.\end{aligned}$$

As $g_0$, $h_0$ and $\epsilon$ are arbitrary, the function $(g, h) \mapsto gh(x)$ is continuous; as $x$ is arbitrary, multiplication on $G$ is continuous. As for inversion, suppose that $g_0 \in G$, $\epsilon > 0$ and $x \in X$. Then $V = \{g : \rho(gg_0^{-1}(x), x) \le \epsilon\}$ is a neighbourhood of $g_0$, and if $g \in V$ then

$$\rho(g^{-1}(x), g_0^{-1}(x)) = \rho(x, gg_0^{-1}(x)) \le \epsilon.$$

Because $g_0$, $\epsilon$ and $x$ are arbitrary, inversion on $G$ is continuous, and $G$ is a topological group. Because $X$ is Hausdorff, so is $G$.

To see that the action is continuous, take $g_0 \in G$, $x_0 \in X$ and $\epsilon > 0$. Then $V = \{g : g \in G,\ \rho(g(x_0), g_0(x_0)) \le \frac{1}{2}\epsilon\}$ is a neighbourhood of $g_0$. If $g \in V$ and $x \in U(x_0, \frac{1}{2}\epsilon)$, then

$$\rho(g(x), g_0(x_0)) \le \rho(g(x), g(x_0)) + \rho(g(x_0), g_0(x_0)) \le \rho(x, x_0) + \frac{1}{2}\epsilon \le \epsilon.$$

As $g_0$, $x_0$ and $\epsilon$ are arbitrary, $(g, x) \mapsto g(x) : G \times X \to X$ is continuous. **Q**

**(b)** If $X$ is compact, so is $G$. **P** By Tychonoff's theorem (3A3J), $X^X$ is compact. Suppose that $g \in X^X$ belongs to the closure of $G$ in $X^X$. For any $x$, $y \in X$, the set $\{f : f \in X^X,\ \rho(f(x), f(y)) = \rho(x, y)\}$ is closed and includes $G$, so contains $g$; thus $g$ is an isometry. **?** If $g[X] \ne X$, take $x \in X \setminus g[X]$ and set $x_n = g^n(x)$ for every $n \in \mathbb{N}$. Because $g$ is continuous and $X$ is compact, $g[X]$ is closed and there is some $\delta > 0$ such that $U(x, \delta) \cap g[X] = \emptyset$. But this means that

$$\rho(x_m, x_n) = \rho(g^m(x), g^m(x_{n-m})) = \rho(x, x_{n-m}) \ge \delta$$

whenever $m < n$, so that $\langle x_n \rangle_{n \in \mathbb{N}}$ can have no cluster point in $X$; which is impossible, because $X$ is supposed to be compact. **X** This shows that $g$ is surjective and belongs to $G$. As $g$ is arbitrary, $G$ is closed in $X^X$, therefore compact. **Q**

**441H Theorem** If $(X, \rho)$ is a non-empty locally compact metric space with isometry group $G$, then there is a non-zero $G$-invariant Radon measure on $X$.

**proof (a)** Fix any $x_0 \in X$, and set $Z = \overline{\{g(x_0) : g \in G\}}$; then $Z$ is a closed subset of $X$, so is in itself locally compact. Let $H$ be the isometry group of $Z$.

**(b)** We need to know that $g{\restriction}Z \in H$ for every $g \in G$. **P** Because $g : X \to X$ is a homeomorphism,

$$g[Z] = \overline{\{gg'(x_0) : g' \in G\}} = Z,$$

so $g{\restriction}Z$ is a permutation of $Z$, and of course it is an isometry, that is, belongs to $H$. **Q**

**(c)** Now $Z$ and $H$ satisfy the conditions of 441C.

**P (i)** is true just because all isometries are continuous.

**(ii)** Take $z \in Z$ and let $U$ be a non-empty relatively open subset of $Z$. Then $U = Z \cap V$ for some open set $V \subseteq X$; as $Z \cap V \ne \emptyset$, there must be a $g_0 \in G$ such that $g_0(x_0) \in V$. At the same time, there is a sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ in $G$ such that $z = \lim_{n \to \infty} h_n(x_0)$. Now

$$\rho(g_0(x_0), g_0h_n^{-1}(z)) = \rho(h_n(x_0), z) \to 0$$

as $n \to \infty$, so there is some $n$ such that $g_0 h_n^{-1}(z) \in V$; of course $g_0 h_n^{-1}(z)$ also belongs to $U$, while $g_0 h_n^{-1} \restriction Z$ belongs to $H$, by (b) above. As $U$ is arbitrary, the $H$-orbit of $z$ is dense in $Z$; as $z$ is arbitrary, $H$ satisfies condition (ii) of 441C.

**(iii)** Given that $K$ and $L$ are disjoint compact subsets of $Z$, there must be a $\delta > 0$ such that $\rho(y, z) \geq \delta$ for every $y \in K$, $z \in L$. Let $U$ be the relatively open ball $\{z : z \in Z, \rho(z, x_0) < \frac{1}{2}\delta\}$. Then for any $h \in H$, $\rho(y, z) < \delta$ for any $y$, $z \in h[U]$, so $h[U]$ cannot meet both $K$ and $L$. **Q**

**(d)** 441C therefore provides us with a non-zero $H$-invariant Radon measure $\nu$ on $Z$. Setting $\mu E = \nu(E \cap Z)$ whenever $E \subseteq X$ and $E \cap Z \in \operatorname{dom} \nu$, it is easy to check that $\mu$ is $G$-invariant (using (b) again) and is a Radon measure on $X$.

**441I Remarks (a)** Evidently there is a degree of overlap between the cases above. In an abelian group, for instance, the left and right group actions necessarily give rise to the same invariant measures. If we take $X = \mathbb{R}^2$, it has a group structure (addition) for which we have invariant measures (e.g., Lebesgue measure); these are just the translation-invariant measures. But 441H tells us that we also have measures which are invariant under all isometries (rotations and reflections as well as translations); from where we now stand, there is no surprise remaining in the fact that Lebesgue measure is invariant under this much larger group. (Though if you look back at Chapter 26, you will see that a bare-handed proof of this takes a certain amount of effort.) If we turn next to the unit sphere $\{x : \|x\| = 1\}$ in $\mathbb{R}^3$, we find that there is no useful group structure, but it is a compact metric space, so carries invariant measures, e.g., two-dimensional Hausdorff measure.

**(b)** The arguments of 441C leave open the question of how far the invariant measures constructed there are unique. Of course any scalar multiple of an invariant measure will again be invariant. It is natural to give a special place to invariant probability measures, and call them 'normalized'; whenever we have a non-zero totally finite invariant measure we shall have an invariant probability measure. Counting measure on any set will be invariant under any action of any group, and it is natural to say that these measures also are 'normalized'; when faced with a finite set with two or more elements, we have to choose which normalization seems most reasonable in the context.

**(c)** We shall see in 442B that Haar measures (with a given handedness) are necessarily scalar multiples of each other. In 442Ya, 443Ud and 443Xy we have further situations in which invariant measures are essentially unique. If, in 441C, there are non-trivial $G$-invariant subsets of $X$, we do not expect such a result. But there are interesting cases in which the question seems to be open.

**441J** Of course we shall be much concerned with integration with respect to invariant measures. The results we need are elementary corollaries of theorems already dealt with at length, but it will be useful to have them spelt out.

**Proposition** Let $X$ be a set, $G$ a group acting on $X$, and $\mu$ a $G$-invariant measure on $X$. If $f$ is a real-valued function defined on a subset of $X$, and $a \in G$, then $\int f(x)\mu(dx) = \int f(a{\bullet}x)\mu(dx)$ if either integral is defined in $[-\infty, \infty]$.

**proof** Apply 235G to the inverse-measure-preserving functions $x \mapsto a{\bullet}x$ and $x \mapsto a^{-1}{\bullet}x$.

**441K Theorem** Let $X$ be a set, $G$ a group acting on $X$, and $\mu$ a $G$-invariant measure on $X$ with measure algebra $\mathfrak{A}$.

(a) We have an action of $G$ on $\mathfrak{A}$ defined by setting $a{\bullet}E^\bullet = (a{\bullet}E)^\bullet$ whenever $a \in G$ and $\mu$ measures $E$.

(b) We have an action of $G$ on $L^0 = L^0(\mu)$ defined by setting $a{\bullet}f^\bullet = (a{\bullet}f)^\bullet$ for every $a \in G$, $f \in \mathcal{L}^0(\mu)$.

(c) For $1 \leq p \leq \infty$ the formula of (b) defines actions of $G$ on $L^p = L^p(\mu)$, and $\|a{\bullet}u\|_p = \|u\|_p$ whenever $u \in L^p$ and $a \in G$.

**proof (a)** If $E$, $F \in \operatorname{dom} \mu$ and $E^\bullet = F^\bullet$, then (because $x \mapsto a^{-1}{\bullet}x$ is inverse-measure-preserving) $(a{\bullet}E)^\bullet = (a{\bullet}F)^\bullet$ for every $a \in G$. So the given formula does define a function from $G \times \mathfrak{A}$ to $\mathfrak{A}$. It is now easy to check that it is an action.

**(b)** Take $f \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$ and $a \in G$. Set $\phi_a(x) = a^{-1}\bullet x$ for $x \in X$, so that $\phi_a : X \to X$ is inverse-measure-preserving. Then $a\bullet f = f\phi_a$ belongs to $\mathcal{L}^0$. If $f,\ g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then $f\phi_a =_{\text{a.e.}} g\phi_a$, so $(a\bullet f)^\bullet = (a\bullet g)^\bullet$. This shows that the given formula defines a function from $G \times L^0$ to $L^0$, and again it is easy to see that it is an action.

**(c)** If $f \in \mathcal{L}^p(\mu)$ then

$$\int |a\bullet f|^p d\mu = \int |f(a^{-1}\bullet x)|^p d\mu = \int |f|^p d\mu$$

by 441J. So $a\bullet f \in \mathcal{L}^p(\mu)$ and $\|a\bullet f^\bullet\|_p = \|f^\bullet\|_p$. Thus we have a function from $G \times L^p$ to $L^p$, and once more it must be an action.

**441L Proposition** Let $X$ be a locally compact Hausdorff space and $G$ a group acting on $X$ in such a way that $x \mapsto a\bullet x$ is continuous for every $a \in G$. If $\mu$ is a Radon measure on $X$, then $\mu$ is $G$-invariant iff $\int f(x)\mu(dx) = \int f(a\bullet x)\mu(dx)$ for every $a \in G$ and every continuous function $f : X \to \mathbb{R}$ with compact support.

**proof** For $a \in G$, set $T_a(x) = a\bullet x$ for every $x \in X$. Then $\nu_a = \mu T_a^{-1}$ is a Radon measure on $X$. If $f \in C_k(X)$, then

$$\int f d\nu_a = \int f T_a d\mu$$

by 235G. Now

$$\begin{aligned}
\mu \text{ is } G\text{-invariant} &\iff \nu_a = \mu \text{ for every } a \in G \\
&\iff \int f d\nu_a = \int f d\mu \text{ for every } a \in G,\ f \in C_k(X)
\end{aligned}$$

(416E(b-v))

$$\iff \int f T_a d\mu = \int f d\mu \text{ for every } a \in G,\ f \in C_k(X)$$

as claimed.

**441X Basic exercises >(a)** Let $X$ be a set. (i) Show that there is a one-to-one correspondence between actions $\bullet$ of the group $\mathbb{Z}$ on $X$ and permutations $f : X \to X$ defined by the formula $n\bullet x = f^n(x)$. (ii) Show that if $f : X \to X$ is a permutation, a measure $\mu$ on $X$ is $\mathbb{Z}$-invariant for the corresponding action iff $f$ and $f^{-1}$ are both inverse-measure-preserving. (iii) Show that if $X$ is a compact Hausdorff space and $\bullet$ is a continuous action of $\mathbb{Z}$ on $X$, then there is a $\mathbb{Z}$-invariant Radon probability measure on $X$. (*Hint*: 437T.)

**(b)** Let $(X, \mathrm{T}, \nu)$ be a measure space and $G$ a group acting on $X$. Set $\Sigma = \{E : E \subseteq X,\ g\bullet E \in \mathrm{T}$ for every $g \in G\}$, and for $E \in \Sigma$ set

$$\mu E = \sup\{\sum_{i=0}^{n} \nu(g_i\bullet F_i) : n \in \mathbb{N}, F_0, \ldots, F_n \text{ are disjoint subsets of } E$$
$$\text{belonging to } \Sigma,\ g_i \in G \text{ for each } i \le n\}$$

(cf. 112Yd, 234Xl). Show that $\mu$ is a $G$-invariant measure on $X$.

**(c)** Let $r \ge 1$ be an integer, and $X = [0, 1[^r$. Let $G$ be the set of $r \times r$ matrices with integer coefficients and determinant $\pm 1$, and for $A \in G$, $x \in X$ say that $A\bullet x = \begin{pmatrix} <\eta_1> \\ \ldots \\ <\eta_r> \end{pmatrix}$ where $\begin{pmatrix} \eta_1 \\ \ldots \\ \eta_r \end{pmatrix} = Ax$ and $<\alpha>$ is the fractional part of $\alpha$ for each $\alpha \in \mathbb{R}$. (i) Show that $\bullet$ is an action of $G$ on $X$, and that Lebesgue measure on $X$ is $G$-invariant. (ii) Show that if $X$ is given the compact Hausdorff topology corresponding to the bijection $\alpha \mapsto (\cos 2\pi\alpha, \sin 2\pi\alpha)$ from $X$ to the unit circle in $\mathbb{R}^2$, and $G$ is given its discrete topology, the action is continuous.

**(d)** Let $X$ be a topological space and $G$ a group acting on $X$ such that $(\alpha)$ all the maps $x \mapsto a{\bullet}x$ are continuous $(\beta)$ all the orbits of $G$ are dense. Show that any non-zero $G$-invariant quasi-Radon measure on $X$ is strictly positive.

**>(e)** Let $G$ be a compact Hausdorff topological group. (i) Show that its conjugacy classes are closed. (ii) Show that if $K$, $L \subseteq G$ are disjoint compact sets then $\{ac^{-1}da^{-1} : a \in G, c \in K, d \in L\}$ is a compact set not containing $e$, so that there is a neighbourhood $U$ of $e$ such that whenever $c^{-1}d \in U$ and $a \in G$ then either $aca^{-1} \notin K$ or $ada^{-1} \notin L$. (iii) Show that every conjugacy class of $G$ carries a Radon probability measure which is invariant under the conjugacy action of $G$.

**(f)** Let $(G, \cdot)$ be a topological group. (i) On $G$ define a binary operation $\diamond$ by saying that $x \diamond y = y \cdot x$ for all $x$, $y \in G$. Show that $(G, \diamond)$ is a topological group isomorphic to $(G, \cdot)$, and that any element of $G$ has the same inverse for either group operation. (ii) Suppose that $\mu$ is a left Haar measure on $(G, \cdot)$. Show that $\mu$ is a right Haar measure on $(G, \diamond)$. (iii) Set $\phi(a) = a^{-1}$ for $a \in G$. Show that if $\mu$ is a left Haar measure on $(G, \cdot)$ then the image measure $\mu\phi^{-1}$ is a right Haar measure on $(G, \cdot)$. (iv) Show that $(G, \cdot)$ has a left Haar measure iff it has a right Haar measure. (v) Show that $(G, \cdot)$ has a left Haar probability measure iff it has a totally finite left Haar measure iff it has a right Haar probability measure. (iv) Show that $(G, \cdot)$ has a $\sigma$-finite left Haar measure iff it has a $\sigma$-finite right Haar measure.

**>(g)**(i) For Lebesgue measurable $E \subseteq \mathbb{R} \setminus \{0\}$, set $\nu E = \int_E \frac{1}{|x|}dx$. Show that $\nu$ is a (two-sided) Haar measure if $\mathbb{R} \setminus \{0\}$ is given the group operation of multiplication. (ii) For Lebesgue measurable $E \subseteq \mathbb{C} \setminus \{0\}$, identified with $\mathbb{R}^2 \setminus \{0\}$, set $\nu E = \int_E \frac{1}{|z|^2}\mu(dz)$, where $\mu$ is two-dimensional Lebesgue measure. Show that $\nu$ is a (two-sided) Haar measure on $\mathbb{C} \setminus \{0\}$ if we take complex multiplication for the group operation. (*Hint*: 263D.)

**>(h)**(i) Show that Lebesgue measure on $\mathbb{R}^r$ is a (two-sided) Haar measure, for any $r \geq 1$, if we take addition for the group operation. (ii) Show that the usual measure on $\{0,1\}^I$ is a two-sided Haar measure, for any set $I$, if we give $\{0,1\}^I$ the group operation corresponding to its identification with $\mathbb{Z}_2^I$. (iii) Describe the corresponding Haar measure on $\mathcal{P}I$ when $\mathcal{P}I$ is given the group operation $\triangle$.

**(i)** Let $G$ be a locally compact Hausdorff topological group. (i) Show that any (left) Haar measure on $G$ must be strictly positive. (ii) Show that $G$ has a totally finite (left) Haar measure iff it is compact. (iii) Show that $G$ has a $\sigma$-finite (left) Haar measure iff it is $\sigma$-compact.

**>(j)**(i) Let $G$ and $H$ be topological groups with left Haar measures $\mu$ and $\nu$. Show that the quasi-Radon product measure on $G \times H$ (417N) is a left Haar measure on $G \times H$. (ii) Let $\langle G_i \rangle_{i \in I}$ be a family of topological groups, and suppose that each $G_i$ has a left Haar probability measure (as happens, for instance, if each $G_i$ is compact). Show that the quasi-Radon product measure on $\prod_{i \in I} G_i$ (417O) is a left Haar measure on $\prod_{i \in I} G_i$.

**(k)**(i) Show that any (left) Haar measure on a topological group, as defined in 441D, must be locally finite. (ii) Show that any (left) Haar measure on a locally compact Hausdorff group must be a Radon measure.

**(l)** Let $r \geq 1$ be an integer, and set $X = \{x : x \in \mathbb{R}^r, \|x\| = 1\}$. Let $G$ be the group of orthogonal $r \times r$ real matrices, so that $G$ acts transitively on $X$. Show that (when given its natural topology as a subset of $\mathbb{R}^{r^2}$) $G$ is a compact Hausdorff topological group. Let $\mu$ be a left Haar measure on $G$, and $x$ any point of $X$; set $\phi_x(T) = Tx$ for $T \in G$. Show that the image measure $\mu\phi_x^{-1}$ is a $G$-invariant measure on $X$, independent of the choice of $x$.

**(m)** Let $X$ be a non-abelian Hausdorff topological group with a left Haar probability measure $\mu$. Let $\lambda$ be the quasi-Radon product measure on $X^2$. Show that $\lambda\{(x, y) : xy = yx\} \leq \frac{5}{8}$. (*Hint*: if $Z$ is the centre of $X$, $X/Z$ is not cyclic, so $\mu Z \leq \frac{1}{4}$.)

**(n)** Let $X$ be a compact metric space, and $g : X \to X$ any isometry. Show that $g$ is surjective. (*Hint*: if $x \in X$, then $\rho(g^m x, g^n x) \geq \rho(x, g[X])$ for any $m < n$.)

**(o)** Let $(X,\rho)$ be a locally compact metric space, and $\mathcal{C}$ the set of closed subsets of $X$ with its Fell topology (4A2T). Show that if $G$ is the isometry group of $X$ with its topology of pointwise convergence, then $(g,F)\mapsto g[F]$ is a continuous action of $G$ on $\mathcal{C}$.

**(p)** Let $(X,\rho)$ be a metric space and $\mathcal{K}$ the family of compact subsets of $X$ with the topology induced by the Vietoris topology on the space of closed subsets of $X$ (4A2T). Show that if $G$ is the isometry group of $X$ with its topology of pointwise convergence, then $(g,K)\mapsto g[K]$ is a continuous action of $G$ on $\mathcal{K}$.

**>(q)** Let $(X,\rho)$ be a metric space, and $G$ its isometry group with the topology $\mathfrak{T}$ of pointwise convergence. (i) Show that if $X$ is compact, $\mathfrak{T}$ can be defined by the metric $(g,h)\mapsto\max_{x\in X}\rho(g(x),h(x))$. (ii) Show that if $\{y:\rho(y,x)\le\gamma\}$ is compact for every $x\in X$ and $\gamma>0$, then $G$ is locally compact. (iii) Show that if $X$ is separable then $G$ is metrizable. (iv) Show that if $(X,\rho)$ is complete then $G$ is complete under its bilateral uniformity. (v) Show that if $X$ is separable and $(X,\rho)$ is complete then $G$ is Polish.

**>(r)** Give $\mathbb{N}$ the zero-one metric $\rho$. Let $G$ be the isometry group of $\mathbb{N}$ (that is, the group of all permutations of $\mathbb{N}$) with its topology of pointwise convergence. (i) Show that $G$ is a $\mathrm{G}_\delta$ subset of $\mathbb{N}^{\mathbb{N}}$, so is a Polish group. (ii) Show that if we set $\Delta(g,h)=\min\{n:n\in\mathbb{N},\ g(n)\ne h(n)\}$ and $\sigma(g,h)=1/(1+\Delta(g^{-1},h^{-1}))$ for distinct $g$, $h\in G$, then $\sigma$ is a right-translation-invariant metric on $G$ inducing its topology. (iii) Show that there is no complete right-translation-invariant metric on $G$ inducing its topology. (*Hint*: any such metric must have the same Cauchy sequences as $\sigma$.) (iv) Show that $G$ is not locally compact.

**>(s)** Let $r\ge1$ be an integer, and $S_{r-1}$ the sphere $\{x:x\in\mathbb{R}^r,\ \|x\|=1\}$. (i) Show that every isometry $\phi$ from $S_{r-1}$ to itself corresponds to an orthogonal $r\times r$ matrix $T$. (*Hint*: $T=\langle\phi(e_i)\,\boldsymbol{.}\,e_j\rangle_{i,j<r}$.) (ii) Show that the topology of pointwise convergence on the isometry group of $S_{r-1}$ corresponds to the topology on the set of $r\times r$ matrices regarded as a subset of $\mathbb{R}^{r^2}$.

**(t)** Let $X$ be a locally compact metric space and $G$ a subgroup of the isometry group of $X$. Show that for every $x\in X$ there is a non-zero $G$-invariant Radon measure on $\overline{\{g(x):g\in G\}}$.

**(u)** Let $X$ be a set with its zero-one metric and $G$ the group of permutations of $X$ with its topology of pointwise convergence. Let $\mathcal{W}\subseteq\mathcal{P}(X^2)$ be the set of total orderings of $X$. Show that $\mathcal{W}$ is compact for the usual topology of $\mathcal{P}(X^2)$. For $g\in G$ and $W\in\mathcal{W}$ write $g\bullet W=\{(g(x),g(y)):(x,y)\in W\}$; show that $\bullet$ is a continuous action of $G$ on $\mathcal{W}$. Show that there is a unique $G$-invariant Radon probability measure $\mu$ on $\mathcal{W}$ such that $\mu\{W:(x_i,x_{i+1})\in W\text{ for every }i<n\}=\frac{1}{(n+1)!}$ whenever $x_0,\dots,x_n\in X$ are distinct.

**441Y Further exercises (a)** Let $(X,\rho)$ be a metric space, and $\mathcal{C}$ the family of non-empty closed subsets of $X$, with its Hausdorff metric $\tilde\rho$ (4A2T). Show that if $G$ is the isometry group of $X$, $(g,F)\mapsto g[F]$ is an action of $G$ on $\mathcal{C}$.

**(b)** (M.Elekes & T.Keleti) Let $X$ be a set, $G$ a group acting on $X$, $\Sigma$ a $\sigma$-algebra of subsets of $X$ such that $g\bullet E\in\Sigma$ whenever $E\in\Sigma$ and $g\in G$, and $H$ an element of $\Sigma$. Suppose that $\mu$ is a measure, with domain the subspace $\sigma$-algebra $\Sigma_H$, such that $\mu(g\bullet E)=\mu E$ whenever $E\in\Sigma_H$ and $g\in G$ are such that $g\bullet E\subseteq H$. (i) Show that $\sum_{n=0}^\infty\mu E_n=\sum_{n=0}^\infty\mu E'_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ and $\langle E'_n\rangle_{n\in\mathbb{N}}$ are sequences in $\Sigma_H$ for which there are sequences $\langle g_n\rangle_{n\in\mathbb{N}}$, $\langle g'_n\rangle_{n\in\mathbb{N}}$ in $G$ such that $\langle g_n\bullet E_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n\bullet E'_n\rangle_{n\in\mathbb{N}}$ are partitions of the same subset of $X$. (ii) Show that there is a $G$-invariant measure with domain $\Sigma$ which extends $\mu$.

**(c)** Take $1\le s\le r\in\mathbb{N}$. Let $\mathcal{C}$ be the set of closed subsets of $\mathbb{R}^r$ with its Fell topology. Let $\mathcal{C}_s\subseteq\mathcal{C}$ be the set of $s$-dimensional linear subspaces of $\mathbb{R}^r$. Show that $\mathcal{C}_s$ is a closed subset of $\mathcal{C}$, therefore a compact metrizable space in its own right, and that the group $G$ of orthogonal $r\times r$ matrices acts transitively on $\mathcal{C}_s$, so that there is a $G$-invariant Radon probability measure on $\mathcal{C}_s$.

**(d)** For $w$, $z\in\mathbb{C}\setminus\{0\}$ set $\rho(w,z)=|\operatorname{Ln}(\frac{w}{z})|$, where $\operatorname{Ln}v=\ln|v|+i\arg v$ for non-zero complex numbers $v$. (i) Show that $\rho$ is a metric on $\mathbb{C}\setminus\{0\}$. (ii) Show that the 2-dimensional Hausdorff measure $\mu_{H2}^{(\rho)}$ derived from $\rho$ (264K, 471A) is a Haar measure for the multiplicative group $\mathbb{C}\setminus\{0\}$. (iii) Show that $\mu_{H2}^{(\rho)}=\frac{4}{\pi}\nu$, where $\nu$ is the measure of 441Xg(ii).

**(e)** Let $X$ be the group of real $r \times r$ orthogonal matrices, where $r \geq 2$ is an integer. Give $X$ the Euclidean metric, regarding it as a subset of $\mathbb{R}^{r^2}$. (i) Show that the left and right actions of $X$ on itself are distance-preserving. (ii) Show that $\frac{r(r-1)}{2}$-dimensional Hausdorff measure on $X$ is a two-sided Haar measure.

**(f)** Let $X = \mathrm{SO}(3)$ be the set of real $3 \times 3$ orthogonal matrices with determinant 1. Give $X$ the metric corresponding to its embedding in 9-dimensional Euclidean space. (i) Show that $X$ can be parametrized as the set of matrices

$$\phi\begin{pmatrix} z \\ \alpha \\ \theta \end{pmatrix} = \begin{pmatrix} z & -\cos\theta\sqrt{1-z^2} & \sin\theta\sqrt{1-z^2} \\ \cos\alpha\sqrt{1-z^2} & z\cos\alpha\cos\theta - \sin\alpha\sin\theta & -z\cos\alpha\sin\theta - \sin\alpha\cos\theta \\ \sin\alpha\sqrt{1-z^2} & z\sin\alpha\cos\theta + \cos\alpha\sin\theta & -z\sin\alpha\sin\theta + \cos\alpha\cos\theta \end{pmatrix}$$

with $z \in [-1,1]$, $\alpha \in [-\pi,\pi]$ and $\theta \in [-\pi,\pi]$. (ii) Show that if $T$ is the $9 \times 3$ matrix which is the derivative of $\phi$ at $\begin{pmatrix} z \\ \alpha \\ \theta \end{pmatrix}$, then $T^\top T = \begin{pmatrix} \frac{2}{1-z^2} & 0 & 0 \\ 0 & 2 & 2z \\ 0 & 2z & 2 \end{pmatrix}$ has constant determinant, so that if $\mu$ is Lebesgue measure on $[-1,1] \times [-\pi,\pi]^2$ then the image measure $\mu\phi^{-1}$ is a Haar measure on $X$. (*Hint*: 441Ye, 265E.) (iii) Show that if $A \in X$ corresponds to a rotation through an angle $\gamma(A) \in [0,\pi]$ then its trace $\mathrm{tr}(A)$ (that is, the sum of its diagonal entries) is $1 + 2\cos\gamma(A)$. (*Hint*: $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any square matrices $A$ and $B$ of the same size.) (iv) Show that if $X$ is given its Haar probability measure then $\cos\gamma(A)$ has expectation $-\frac{1}{2}$.

**(g)** Let $\mathbb{H}$ be the division ring of the **quaternions**, that is, $\mathbb{R}^4$ with its usual addition and with multiplication defined by the rule

$$\begin{aligned}(\xi_0,\xi_1,\xi_2,\xi_3) \times (\eta_0,\eta_1,\eta_2,\eta_3) = (&\xi_0\eta_0 - \xi_1\eta_1 - \xi_2\eta_2 - \xi_3\eta_3, \xi_0\eta_1 + \xi_1\eta_0 + \xi_2\eta_3 - \xi_3\eta_2, \\ &\xi_0\eta_2 - \xi_1\eta_3 + \xi_2\eta_0 + \xi_4\eta_1, \xi_0\eta_3 + \xi_1\eta_2 - \xi_2\eta_1 + \xi_3\eta_0).\end{aligned}$$

For Lebesgue measurable $E \subseteq \mathbb{H} \setminus \{0\}$, set $\nu E = \int_E \frac{1}{\|x\|^4} dx$. Show that (i) $\|x \times y\| = \|x\|\|y\|$ for all $x$, $y \in \mathbb{H}$ (ii) $\mathbb{H} \setminus \{0\}$ is a group (iii) $\nu$ is a (two-sided) Haar measure on $\mathbb{H} \setminus \{0\}$.

**(h)** In 441Xl, show that $\mu\phi_x^{-1}$ is a scalar multiple of Hausdorff $(r-1)$-dimensional measure on $X$.

**(i)** For any topological spaces $X$ and $Y$, and any set $G$ of functions from $X$ to $Y$, the **compact-open** topology on $G$ is the topology generated by sets of the form $\{g : g \in G, g[K] \subseteq H\}$, where $K \subseteq X$ is compact and $H \subseteq Y$ is open. Show that if $(X,\rho)$ is a metric space and $G$ is the isometry group of $X$, then the topology of pointwise convergence on $G$ is its compact-open topology.

**(j)** Let $X$ be a compact Hausdorff space and $G$ the group of all homeomorphisms from $X$ to itself. (i) Let $\mathrm{P}$ be the family of all continuous pseudometrics on $X$ (see 4A2Jg). For $\rho \in \mathrm{P}$ and $g$, $h \in G$, set $\tau_\rho(g,h) = \max_{x\in X}\rho(g(x),h(x))$. Show that every $\tau_\rho$ is a right-translation-invariant pseudometric on $G$, and that $G$ with the topology generated by $\{\tau_\rho : \rho \in \mathrm{P}\}$ is a topological group. (ii) Show that this is the compact-open topology as defined in 441Yi. (iii) Show that if $X$ is metrizable then $G$ is Polish.

**(k)** Let $(X,\mathcal{W})$ be a locally compact Hausdorff uniform space, and suppose that $G$ is a group acting on $X$ 'uniformly equicontinuously'; that is, for every $W \in \mathcal{W}$ there is a $V \in \mathcal{W}$ such that $(a\bullet x, a\bullet y) \in W$ whenever $(x,y) \in V$ and $a \in G$. Show that there is a non-zero $G$-invariant Radon measure on $X$.

**(l)** Let $X$ be a compact metric space and $G$ the isometry group of $X$. Show that every $G$-orbit in $X$ is closed.

**(m)** Let $T$ be any set, and $\rho$ the $\{0,1\}$-valued metric on $T$. Let $X$ be the set of partial orders on $T$, regarded as a subset of $\mathcal{P}(T \times T)$. Show that $X$ is compact (cf. 418Xy). Let $G$ be the group of isometries of $T$ with its topology of pointwise convergence. Set $\pi\bullet x = \{(t,u) : (\pi^{-1}(t),\pi^{-1}(u)) \in x\}$ for $\pi \in G$ and $x \in X$. Show that $\bullet$ is a continuous action of $G$ on $X$. Show that there is a strictly positive $G$-invariant Radon probability measure $\mu$ on $X$.

**(n)** Let $X$ be a set, $G$ a group acting on $X$, and $\mu$ a $G$-invariant measure on $X$ with measure algebra $\mathfrak{A}$. Show that if $\tau$ is any rearrangement-invariant extended Fatou norm on $L^0(\mathfrak{A})$ then the formula of 441Kb defines a norm-preserving action of $G$ on the Banach space $L^\tau$.

**(o)** Let $\bullet_X$ be an action of a group $G$ on a set $X$, $\mu$ a $G$-invariant measure on $X$, $(\mathfrak{A},\bar\mu)$ its measure algebra and $\bullet_{\mathfrak{A}}$ the induced action on $\mathfrak{A}$. Set $Z=X^G$; define $\phi:X\to Z$ by setting $\phi(x)=\langle g^{-1}\bullet x\rangle_{g\in G}$ for $x\in X$; let $\nu$ be the image measure $\mu\phi^{-1}$, and $(\mathfrak{B},\bar\nu)$ its measure algebra. Let $\bullet_Z$ be the left shift action of $G$ on $Z$; show that $\nu$ is $\bullet_Z$-invariant, so that there is an induced action $\bullet_{\mathfrak{B}}$ on $\mathfrak{B}$. Show that $(\mathfrak{A},\bar\mu,\bullet_{\mathfrak{A}})$ and $(\mathfrak{B},\bar\nu,\bullet_{\mathfrak{B}})$ are isomorphic.

**(p)** Let $X$ be a topological space, $G$ a topological group and $\bullet$ a continuous action of $G$ on $X$. Let $M^+_{\mathrm{qR}}$ be the set of totally finite quasi-Radon measures on $X$. (i) Show that we have an action $\bullet$ of $G$ on $M^+_{\mathrm{qR}}$ defined by saying that $(a\bullet\nu)(E)=\nu(a^{-1}\bullet E)$ whenever $a\in G$, $\nu\in M^+_{\mathrm{qR}}$ and $E\subseteq X$ are such that $\nu$ measures $a^{-1}\bullet E$. (ii) Show that this action is continuous if we give $M^+_{\mathrm{qR}}$ its narrow topology. (iii) Show that if $\nu\in M^+_{\mathrm{qR}}$, $f\in\mathcal{L}^1(\nu)$ is non-negative and $f\nu$ is the corresponding indefinite-integral measure, then $a\bullet(f\nu)$ is the indefinite-integral measure $(a\bullet f)(a\bullet\nu)$ for every $a\in G$.

**(q)** Let $X$ be a set, $G$ a group acting on $X$ and $\mu$ a totally finite $G$-invariant measure on $X$ with domain $\Sigma$. Suppose there is a probability measure $\nu$ on $G$, with domain $\mathrm{T}$, such that $(a,x)\mapsto a^{-1}\bullet x:G\times X\to X$ is $(\mathrm{T}\widehat{\otimes}\Sigma,\Sigma)$-measurable and $\nu$ is invariant under the left action of $G$ on itself. Let $u\in L^0(\mu)$ be such that $a\bullet u=u$ for every $a\in G$. Show that there is an $f\in\mathcal{L}^0(\mu)$ such that $f^\bullet=u$ and $a\bullet f=f$ for every $a\in G$. (*Hint*: if $u=g^\bullet$ where $g:X\to\mathbb{R}$ is $\Sigma$-measurable and $\mu$-integrable, try $f(x)=\int g(a^{-1}\bullet x)\nu(da)$ when this is defined.)

**(r)** Let $X$ be a topological space, $G$ a compact Hausdorff group, $\bullet$ a continuous action of $G$ on $X$, and $\mu$ a $G$-invariant quasi-Radon measure on $X$. Let $u\in L^0(\mu)$ be such that $a\bullet u=u$ for every $a\in G$. Show that there is an $f\in\mathcal{L}^0(\mu)$ such that $f^\bullet=u$ and $a\bullet f=f$ for every $a\in G$.

**441 Notes and comments** The proof I give of 441C is essentially the same as the proof of 441E in Halmos 50, §58.

In part (f) of the proof of 441C I use an ultrafilter, relying on a fairly strong consequence of the Axiom of Choice. In this volume, as in the last, I generally employ the axiom of choice without stopping to consider whether it is really needed. But Haar measure, at least, is so important that I point out now that it can be built with much weaker principles. For a construction of Haar measure not dependent on choice, see 561G in Volume 5. I ought to remark that the argument there leads us to a translation-invariant linear functional rather than a measure, and that while there is still a version of the Riesz representation theorem (564I), we may get something less than a proper countably additive measure if we do not have countable choice. Moreover, in the absence of the full axiom of choice, we may find that we have fewer locally compact topological groups than we expect.

While Haar measure is surely the pre-eminent application of the theory here, I think that some of the other consequences of 441C (441H, 441Xe, 441Xl, 441Yc, 441Yk) are sufficiently striking to justify the trouble involved in the extra generality. I ought to remark that there are important examples of invariant measures which have nothing to do with 441C. Some of these will appear in §449; for the moment I note only 441Xa.

Federer 69, §2.7, develops a general theory of 'covariant' measures $\mu$ ('relatively invariant' in Halmos 50) such that $\mu(a\bullet E)=\psi(a)\mu E$ for appropriate sets $E\subseteq X$ and $a\in G$, where $\psi:G\to\,]0,\infty[$ is a homomorphism; for instance, taking $\mu$ to be Lebesgue measure on $\mathbb{R}^r$, we have $\mu T[E]=|\det T|\mu E$ for every linear space isomorphism $T:\mathbb{R}^r\to\mathbb{R}^r$ and every measurable set $E$ (263A). The theory I have described here can deal only with the subgroup of isometric linear isomorphisms (that is, the orthogonal group). Covariant measures arise naturally in many other contexts, such as 443T below.

Hausdorff measures, being defined in terms of metrics, are necessarily invariant under isometries, so appear naturally in this context, starting with 264I. There are interesting challenges both in finding suitable metrics and in establishing exact constants, as in 441Yd-441Ye.

It is worth pausing over the topology of an isometry group, as described in 441G. It is quite surprising that such an elementary idea should give us a topological group at all. I offer some exercises (441Xq-441Xs, 441Yi) to help you relate the construction to material which may be more familiar. It is of course a 'weak' topology, except when the underlying space is compact (441Xq(i)). See also 441Yj. These groups are rarely locally compact, and you may find them pushed out of your mind by the extraordinary theory which you will see developed in the next hundred pages; but in the last two sections of this chapter, and in §493, they will become leading examples.

Version of 21.3.07

## 442 Uniqueness of Haar measures

Haar measure has an extraordinary wealth of special properties, and it will be impossible for me to cover them all properly in this chapter. But surely the second thing to take on board, after the existence of Haar measures on locally compact Hausdorff groups (441E), is the fact that they are, up to scalar multiples, unique. This is the content of 442B. We find also that while left and right Haar measures can be different (442Xf), they are not only direct mirror images of each other (442C) – as is, I suppose, to be expected – but even more closely related (442F, 442H, 442L). Investigating this relation, we are led naturally to the 'modular function' of a group (442I).

**442A Lemma** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$.

(a) $\mu$ is strictly positive and locally finite.

(b) If $G \subseteq X$ is open and $\gamma < \mu G$, there are an open set $H$ and an open neighbourhood $U$ of the identity such that $HU \subseteq G$ (writing $HU$ for $\{xy : x \in H,\, y \in U\}$) and $\mu H \ge \gamma$.

(c) If $X$ is locally compact and Hausdorff, $\mu$ is a Radon measure.

**proof (a)(i) ?** If $G \subseteq X$ were a non-empty open set such that $\mu G = 0$, then we should have $\mu(xG) = 0$ for every $x \in X$, so that $X$ would be covered by negligible open sets; but as $\mu$ is supposed to be $\tau$-additive, $\mu X = 0$. **X**

**(ii)** Because $\mu$ is effectively locally finite, there is some non-empty open set $G$ such that $\mu G < \infty$; but now $\{xG : x \in X\}$ is a cover of $X$ by open sets of finite measure.

**(b)** Let $\mathcal{U}$ be the family of open sets containing the identity $e$, and $\mathcal{H}$ the family of open sets $H$ such that $HU \subseteq G$ for some $U \in \mathcal{U}$. Because $\mathcal{U}$ is downwards-directed, $\mathcal{H}$ is upwards-directed; because $e \in U$ for every $U \in \mathcal{U}$, $\bigcup \mathcal{H} \subseteq G$. If $x \in G$, then $x^{-1}G \in \mathcal{U}$, and there is a $U \in \mathcal{U}$ such that $UU \subseteq x^{-1}G$; but now $xUU \subseteq G$, so $xU \in \mathcal{H}$. Thus $\bigcup \mathcal{H} = G$. Because $\mu$ is $\tau$-additive, there is an $H \in \mathcal{H}$ such that $\mu H \ge \gamma$.

**(c)** Use (a) and 416G.

**442B Theorem** Let $X$ be a topological group. If $\mu$ and $\nu$ are left Haar measures on $X$, they are multiples of each other.

**proof (a)** Let $\mathcal{G}$ be the family of non-empty open sets $G$ such that $\mu G$ and $\nu G$ are both finite; because $\mu$ and $\nu$ are locally finite (442Aa), $\mathcal{G}$ is a base for the topology of $X$. Note that $G \cup H \in \mathcal{G}$ for all $G$, $H \in \mathcal{G}$. Set $\mathcal{U} = \{U : U \in \mathcal{G},\, U = U^{-1},\, e \in U\}$, where $e$ is the identity of $G$; then $\mathcal{U}$ is a base of neighbourhoods of $e$ (4A5Ec). Let $\mathcal{F}$ be the filter on $\mathcal{U}$ generated by the sets $\{U : U \in \mathcal{U},\, U \subseteq V\}$ as $V$ runs over $\mathcal{U}$.

**(b)** (The key.) If $G \in \mathcal{G}$ and $0 < \epsilon < 1$, there is a $V_1 \in \mathcal{U}$ such that

$$(1-\epsilon)\frac{\mu G}{\nu G} \le \frac{\mu U}{\nu U}$$

whenever $U \in \mathcal{U}$ and $U \subseteq V_1$. **P** By 442Aa, $\mu G$ and $\nu G$ are both non-zero. By 442Ab, there are an open set $H$ and a neighbourhood $V_1$ of $e$ such that $HV_1 \subseteq G$ and $\mu H \ge (1-\epsilon)\mu G$; shrinking $V_1$ if need be, we may suppose that $V_1 \in \mathcal{U}$. Take any $U \in \mathcal{U}$ such that $U \subseteq V_1$, so that $HU \subseteq G$. Consider the product quasi-Radon measure $\lambda$ of $\mu$ and $\nu$ on $X\times X$ (417R), and the set $W=\{(x,y):x,\,y\in G,\,x^{-1}y\in U\}$. Because the function $(x,y)\mapsto x^{-1}y$ is continuous (4A5Eb), $W$ is open. Consequently

$$\int\mu\{x:(x,y)\in W\}\nu(dy)=\lambda W=\int\nu\{y:(x,y)\in W\}\mu(dx)$$

(417C(b-v-$\beta$)). But we see that if $x\in H$ then $xU\subseteq G$, so $(x,y)\in W$ whenever $y\in xU$, and

$$\int\nu\{y:(x,y)\in W\}\mu(dx)\ge\int_H\nu(xU)\mu(dx)=\mu H\cdot\nu U.$$

On the other hand,

$$\int\mu\{x:(x,y)\in W\}\nu(dy)\le\int_G\mu\{x:x^{-1}y\in U\}\nu(dy)=\int_G\mu\{x:y^{-1}x\in U\}\nu(dy)$$
$$=\int_G\mu(yU)\nu(dy)=\mu U\cdot\nu G,$$

so that

$$(1-\epsilon)\mu G\cdot\nu U\le\mu H\cdot\nu U\le\lambda W\le\mu U\cdot\nu G.$$

Dividing both sides by $\nu U\cdot\nu G$, we have the result. **Q**

**(c)** In the same way, there is a $V_2\in\mathcal{V}$ such that

$$(1-\epsilon)\frac{\nu G}{\mu G}\le\frac{\nu U}{\mu U}$$

whenever $U\in\mathcal{U}$ and $U\subseteq V_2$. So if $U\in\mathcal{U}$ and $U\subseteq V_1\cap V_2$, we have

$$(1-\epsilon)\frac{\mu G}{\nu G}\le\frac{\mu U}{\nu U}\le\frac{1}{1-\epsilon}\frac{\mu G}{\nu G}.$$

As $\epsilon$ is arbitrary,

$$\lim_{U\to\mathcal{F}}\frac{\mu U}{\nu U}=\frac{\mu G}{\nu G}.$$

And this is true for every $G\in\mathcal{G}$.

**(d)** So if we set $\alpha=\lim_{U\to\mathcal{F}}\frac{\mu U}{\nu U}$, we shall have $\mu G=\alpha\nu G$ for every $G\in\mathcal{G}$. Now $\mu$ and $\alpha\nu$ are quasi-Radon measures agreeing on the base $\mathcal{G}$, which is closed under finite unions, so are identical, by 415H(iv).

**442C Proposition** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Setting $\nu E=\mu(E^{-1})$ whenever $E\subseteq X$ is such that $E^{-1}=\{x^{-1}:x\in E\}$ is measured by $\mu$, $\nu$ is a right Haar measure on $X$.

**proof** Set $\phi(x)=x^{-1}$ for $x\in X$. Then $\phi$ is a homeomorphism, so the image measure $\nu=\mu\phi^{-1}$ is a quasi-Radon measure. It is non-zero because $\nu X=\mu X$. If $E\in\operatorname{dom}\nu$ and $x\in X$, then

$$\nu(Ex)=\mu(x^{-1}E^{-1})=\mu E^{-1}=\nu E.$$

So $\nu$ is a right Haar measure.

**442D Remark** Clearly all the arguments of 442A-442C must be applicable to right Haar measures; that is, any right Haar measure must be locally finite and strictly positive; two right Haar measures on the same group must be multiples of each other; and if $X$ carries a right Haar measure $\nu$ then $E\mapsto\nu E^{-1}$ will be a left Haar measure on $X$. (If you are unhappy with such a bold appeal to the symmetry between 'left' and 'right' in topological groups, write the reflected version of 442C out in full, and use it to reflect 442A-442B.)

Thus we may say that a topological group **carries Haar measures** if it has either a left or a right Haar measure. These can, of course, be the same; in fact it takes a certain amount of exploration to find a group in which they are different (e.g., 442Xf).

**442E Lemma** Let $X$ be a topological group, $\mu$ a left Haar measure on $X$ and $\nu$ a right Haar measure on $X$. If $G$, $H\subseteq X$ are open, then

$$\mu G\cdot\nu H=\int_H\nu(xG^{-1})\mu(dx).$$

**proof** Let $\lambda$ be the quasi-Radon product measure of $\mu$ and $\nu$ on $X \times X$. The sets $W_1 = \{(x,y) : y^{-1}x \in G,\, x \in H\}$ and $W_2 = \{(x,y) : x \in G,\, yx \in H\}$ are both open, so 417C(b-v-$\beta$) tells us that

$$\begin{aligned}
\int_H \nu(xG^{-1})\mu(dx) &= \int \nu\{y : x \in H,\, y^{-1}x \in G\}\mu(dx) = \lambda W_1 \\
&= \int \mu\{x : x \in H,\, y^{-1}x \in G\}\nu(dy) = \int \mu(H \cap yG)\nu(dy) \\
&= \int \mu(y^{-1}H \cap G)\nu(dy) = \int \mu\{x : x \in G,\, yx \in H\}\nu(dy) \\
&= \lambda W_2 = \int_G \nu\{y : yx \in H\}\mu(dx) \\
&= \int_G \nu(Hx^{-1})\mu(dx) = \mu G \cdot \nu H.
\end{aligned}$$

**442F Domains of Haar measures** 442B tells us, in part, that any two left Haar measures on a topological group must have the same domain and the same negligible sets; similarly, any two right Haar measures have the same domain and the same negligible sets. In fact left and right Haar measures agree on both.

**Proposition** Let $X$ be a topological group which carries Haar measures. If $\mu$ is a left Haar measure and $\nu$ is a right Haar measure on $X$, then they have the same domains and the same null ideals.

**proof (a)** Suppose that $F \subseteq X$ is a closed set such that $\mu F = 0$. Then $\nu F = 0$. **P?** Otherwise, there is an open set $H$ such that $\nu H < \infty$ and $\nu(F \cap H) > 0$. Let $G$ be any open set such that $0 < \mu G < \infty$. By 442E,

$$\mu G \cdot \nu H = \int_H \nu(xG^{-1})\mu(dx).$$

But also

$$\mu G \cdot \nu(H \setminus F) = \int_{H \setminus F} \nu(xG^{-1})\mu(dx) = \int_H \nu(xG^{-1})\mu(dx) = \mu G \cdot \nu H,$$

so $\mu G \cdot \nu(H \cap F) = 0$. **XQ**

**(b)** It follows that

$$\nu E = \sup_{F \subseteq E \text{ is closed}} \nu F = 0$$

whenever $E$ is a Borel set such that $\mu E = 0$. Now take any $E \in \operatorname{dom} \mu$. Set

$$\mathcal{G} = \{G : G \subseteq X \text{ is open},\, \mu G < \infty,\, \nu G < \infty\}.$$

Because both $\mu$ and $\nu$ are locally finite, $\mathcal{G}$ covers $X$. If $G \in \mathcal{G}$, there are Borel sets $E'$, $E''$ such that $E' \subseteq E \cap G \subseteq E''$ and $\mu(E'' \setminus E') = 0$. In this case $\nu(E'' \setminus E') = 0$ so $E \cap G \in \operatorname{dom} \nu$. Because $\nu$ is complete, locally determined and $\tau$-additive, $E \in \operatorname{dom} \nu$ (414I). If $\mu E = 0$, it follows that

$$\nu E = \sup_{F \subseteq E \text{ is closed}} \nu F = 0$$

just as above.

**(c)** Thus $\nu$ measures $E$ whenever $\mu$ measures $E$, and $E$ is $\nu$-negligible whenever it is $\mu$-negligible. I am sure you will have no difficulty in believing that all the arguments above, in particular that of 442E, can be re-cast to show that $\operatorname{dom} \nu \subseteq \operatorname{dom} \mu$; alternatively, apply the result in the form just demonstrated to the left Haar measure $\nu'$ and the right Haar measure $\mu'$, where

$$\nu' E = \nu E^{-1}, \quad \mu' E = \mu E^{-1}$$

as in 442C.

**442G Corollary** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$ with domain $\Sigma$. Then, for $E \subseteq X$ and $a \in X$,

$$E \in \Sigma \iff E^{-1} \in \Sigma \iff Ea \in \Sigma,$$

$$\mu E = 0 \iff \mu E^{-1} = 0 \iff \mu(Ea) = 0.$$

**proof** Apply 442F with $\nu E = \mu E^{-1}$.

**442H Remark** From 442F-442G we see that if $X$ is any topological group which carries Haar measures, there is a distinguished $\sigma$-algebra $\Sigma$ of subsets of $X$, which we may call the algebra of **Haar measurable sets**, which is the domain of any Haar measure on $X$. Similarly, there is a $\sigma$-ideal $\mathcal{N}$ of $\mathcal{P}X$, the ideal of **Haar negligible sets**[1], which is the null ideal for any Haar measure on $X$. Both $\Sigma$ and $\mathcal{N}$ are translation-invariant and also invariant under the inversion operation $x \mapsto x^{-1}$.

If we form the quotient $\mathfrak{A} = \Sigma/\mathcal{N}$, then we have a fixed Dedekind complete Boolean algebra which is the **Haar measure algebra** of the group $X$ in the sense that any Haar measure on $X$, whether left or right, has measure algebra based on $\mathfrak{A}$. If $a \in X$, the maps $x \mapsto ax$, $x \mapsto xa$, $x \mapsto x^{-1}$ give rise to Boolean automorphisms of $\mathfrak{A}$.

For a member of $\Sigma$, we have a notion of '$\sigma$-finite' which is symmetric between left and right (442Xd). We do not in general have a corresponding two-sided notion of 'finite measure' (442Xg(i)); but of course we can if we wish speak of a set as having 'finite left Haar measure' or 'finite right Haar measure' without declaring which Haar measure we are thinking of. It is the case, however, that if the group $X$ itself has finite left Haar measure, it also has finite right Haar measure; see 442Ic-d below.

**442I The modular function** Let $X$ be a topological group which carries Haar measures.

**(a)** There is a group homomorphism $\Delta : X \to \,]0,\infty[$ defined by the formula

$$\mu(Ex) = \Delta(x)\mu E \text{ whenever } \mu \text{ is a left Haar measure on } X \text{ and } E \in \operatorname{dom}\mu.$$

**P** Fix on a left Haar measure $\tilde{\mu}$ on $X$. For $x \in X$, let $\mu_x$ be the function defined by saying

$$\mu_x E = \tilde{\mu}(Ex) \text{ whenever } E \subseteq X,\ Ex \in \operatorname{dom}\mu,$$

that is, for every Haar measurable set $E \subseteq X$. Because the function $\phi_x : X \to X$ defined by setting $\phi_x(y) = yx^{-1}$ is a homeomorphism, $\mu_x = \tilde{\mu}\phi_x^{-1}$ is a quasi-Radon measure on $X$; and

$$\mu_x(yE) = \tilde{\mu}(yEx) = \tilde{\mu}(Ex) = \mu_x E$$

whenever $\mu_x$ measures $E$, so $\mu_x$ is a left Haar measure on $X$. By 442B, there is a $\Delta(x) \in \,]0,\infty[$ such that $\mu_x = \Delta(x)\tilde{\mu}$; because $\tilde{\mu}$ surely takes at least one value in $]0,\infty[$, $\Delta(x)$ is uniquely defined.

If $\mu$ is any other left Haar measure on $X$, then $\mu = \alpha\tilde{\mu}$ for some $\alpha > 0$, so that

$$\mu(Ex) = \alpha\tilde{\mu}(Ex) = \alpha\Delta(x)\tilde{\mu}E = \Delta(x)\mu E.$$

Thus $\Delta : X \to \,]0,\infty[$ has the property asserted in the formula offered.

To see that $\Delta$ is a group homomorphism, take any $x$, $y \in X$ and a Haar measurable set $E$ such that $0 < \tilde{\mu}E < \infty$, and observe that

$$\Delta(xy)\tilde{\mu}E = \tilde{\mu}(Exy) = \Delta(y)\tilde{\mu}(Ex) = \Delta(y)\Delta(x)\tilde{\mu}E,$$

so that $\Delta(xy) = \Delta(y)\Delta(x) = \Delta(x)\Delta(y)$. **Q**

$\Delta$ is called the **left modular function** of $X$.

**(b)** We find now that $\nu(xE) = \Delta(x^{-1})\nu E$ whenever $\nu$ is a right Haar measure on $X$, $x \in X$ and $E \subseteq X$ is Haar measurable. **P** Let $\mu$ be the left Haar measure derived from $\nu$, so that $\mu E = \nu E^{-1}$ whenever $E$ is Haar measurable. If $x \in X$ and $E \in \operatorname{dom}\nu$, then

$$\nu(xE) = \mu(E^{-1}x^{-1}) = \Delta(x^{-1})\mu E^{-1} = \Delta(x^{-1})\nu E. \ \mathbf{Q}$$

Thus we may call $x \mapsto \Delta(x^{-1}) = \frac{1}{\Delta(x)}$ the **right modular function** of $X$.

[1] **Warning!** do not confuse with the 'Haar null' sets described in 444Ye below.

**(c)** If $X$ is abelian, then obviously $\Delta(x) = 1$ for every $x \in X$, because $\mu(Ex) = \mu(xE) = \mu E$ whenever $x \in X$, $\mu$ is a left Haar measure on $X$ and $E$ is Haar measurable. Equally, if any (therefore every) left (or right) Haar measure $\mu$ on $X$ is totally finite, then $\mu(Xx) = \mu(xX) = \mu X$, so again $\Delta(x) = 1$ for every $x \in X$. This will be the case, in particular, for any compact Hausdorff topological group (recall that by 441E any such group carries Haar measures), because its Haar measures are locally finite, therefore totally finite.

Groups in which $\Delta(x) = 1$ for every $x$ are called **unimodular**.

**(d)** From the definition of $\Delta$, we see that a topological group carrying Haar measures is unimodular iff every left Haar measure is a right Haar measure.

**(e)** In particular, if a group has any totally finite (left or right) Haar measure, its left and right Haar measures are the same, and it has a unique Haar probability measure, which we may call its **normalized Haar measure**.

In the other direction, any group with its discrete topology is unimodular, since counting measure is a two-sided Haar measure.

**442J Proposition** For any topological group carrying Haar measures, its left modular function is continuous.

**proof** Let $X$ be a topological group carrying Haar measures, with left modular function $\Delta$.

**(a)** If $\epsilon > 0$, there is an open set $U_\epsilon$ containing the identity $e$ of $X$ such that $\Delta(x) \le 1+\epsilon$ for every $x \in U$. **P** Take any left Haar measure $\mu$ on $X$, and an open set $G$ such that $0 < \mu G < \infty$. By 442Ab, there are an open set $H$ and a neighbourhood $U_\epsilon$ of the identity such that $HU_\epsilon \subseteq G$ and $\mu G \le (1+\epsilon)\mu H$. If $x \in U_\epsilon$, then $Hx \subseteq G$, so $\Delta(x) = \frac{\mu(Hx)}{\mu H} \le 1+\epsilon$. **Q**

**(b)** Now, given $x_0 \in X$ and $\epsilon > 0$, $V = \{x : x^{-1}x_0 \in U_\epsilon,\ x_0^{-1}x \in U_\epsilon\}$ is an open set containing $x_0$. If $x \in V$, then

$$\Delta(x) = \Delta(x_0)\Delta(x_0^{-1}x) \le (1+\epsilon)\Delta(x_0),$$

$$\Delta(x_0) = \Delta(x)\Delta(x^{-1}x_0) \le (1+\epsilon)\Delta(x),$$

so

$$\tfrac{1}{1+\epsilon}\Delta(x_0) \le \Delta(x) \le (1+\epsilon)\Delta(x).$$

As $\epsilon$ is arbitrary, $\Delta$ is continuous at $x_0$; as $x_0$ is arbitrary, $\Delta$ is continuous.

**442K Theorem** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Let $\Delta$ be the left modular function of $X$.

(a) $\mu(E^{-1}) = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu$.

(b)(i) $\int f(x^{-1})\mu(dx) = \int \Delta(x^{-1})f(x)\mu(dx)$ whenever $f$ is a real-valued function such that either integral is defined in $[-\infty, \infty]$;

(ii) $\int f(x)\mu(dx) = \int \Delta(x^{-1})f(x^{-1})\mu(dx)$ whenever $f$ is a real-valued function such that either integral is defined in $[-\infty, \infty]$.

(c) $\int f(xy)\mu(dx) = \Delta(y^{-1}) \int f(x)\mu(dx)$ whenever $y \in X$ and $f$ is a real-valued function such that either integral is defined in $[-\infty, \infty]$.

**proof (a)(i)** Setting $\nu_1 E = \mu E^{-1}$ for Haar measurable sets $E \subseteq X$, we know that $\nu_1$ is a right Haar measure, so 442E tells us that

$$\mu G \cdot \nu_1 H = \int_H \nu_1(xG^{-1})\mu(dx) = \int_H \mu(Gx^{-1})\mu(dx) = \mu G \int_H \Delta(x^{-1})\mu(dx)$$

for all open sets $G$, $H \subseteq X$. Since there is an open set $G$ such that $0 < \mu G < \infty$, $\mu H^{-1} = \int_H \Delta(x^{-1})\mu(dx)$ for every open set $H \subseteq X$.

**(ii)** Now let $\nu_2$ be the indefinite-integral measure defined by setting $\nu_2 E = \int \Delta(x^{-1})\chi E(x)\mu(dx)$ whenever this is defined in $[0, \infty]$ (234J[2]). Then $\nu_2$ is effectively locally finite. **P** If $\nu_2 E > 0$, then $\mu E > 0$,

[2]Formerly 234B.

so there is an $n \in \mathbb{N}$ such that $\mu(E \cap H) > 0$, where $H$ is the open set $\{x : \Delta(x^{-1}) < n\}$. Now there is an open set $G \subseteq H$ such that $\mu G < \infty$ and $\mu(E \cap G) > 0$, in which case $\nu_2 G \le n\mu G < \infty$ and $\nu_2(E \cap G) > 0$. **Q**

Accordingly $\nu_2$ is a quasi-Radon measure (415Ob). Since it agrees with the quasi-Radon measure $\nu_1$ on open sets, by (i), the two are equal; that is, $\mu E^{-1} = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu$.

**(b)(i)** Apply 235E with $X = Y$, $\Sigma = \mathrm{T} = \operatorname{dom}\mu$, $\mu = \nu$ and $\phi(x) = x^{-1}$, $J(x) = \Delta(x^{-1})$, $g(x) = \Delta(x^{-1})f(x)$ for $x \in X$. From (a) we have

$$\int J \times \chi(\phi^{-1}[F])d\mu = \int_{F^{-1}} \Delta(x^{-1})\mu(dx) = \mu F = \nu F$$

for every $F \in \mathrm{T}$ (using 442G to see that $F^{-1} \in \Sigma$). So we get

$$\int f(x^{-1})\mu(dx) = \int \Delta(x^{-1})g(x^{-1})\mu(dx) = \int J \times g\phi\, d\mu$$
$$= \int g\, d\nu = \int \Delta(x^{-1})f(x)\mu(dx)$$

if any of the integrals is defined in $[-\infty, \infty]$.

**(ii)** Set $\overleftrightarrow{f}(x) = f(x^{-1})$ whenever this is defined (4A5C(c-ii)), and apply (i) to $\overleftrightarrow{f}$.

**(c)** Similarly, apply 235E with $\mu = \nu$, $\phi(x) = xy$, $J(x) = \Delta(y)$ for every $x \in X$; then

$$\int J \times \chi(\phi^{-1}[F])d\mu = \Delta(y)\mu(Fy^{-1}) = \mu F$$

for every $F \in \operatorname{dom}\mu$, so

$$\int f(xy)\mu(dx) = \Delta(y^{-1})\int J \times f\phi\, d\mu = \Delta(y^{-1})\int f(x)\mu(dx).$$

**442L Corollary** Let $X$ be a group carrying Haar measures. If $\mu$ is a left Haar measure on $X$ and $\nu$ is a right Haar measure, then each is an indefinite-integral measure over the other.

**proof** Let $\overleftrightarrow{\mu}$ be the right Haar measure defined by setting $\overleftrightarrow{\mu}E = \mu E^{-1}$ for every Haar measurable $E \subseteq X$. Then $\overleftrightarrow{\mu}E = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu = \operatorname{dom}\overleftrightarrow{\mu}$, so $\overleftrightarrow{\mu}$ is an indefinite-integral measure over $\mu$; because $\nu$ is a multiple of $\overleftrightarrow{\mu}$, it also is an indefinite-integral measure over $\mu$. Similarly, or because $\Delta$ is strictly positive, $\mu$ is an indefinite-integral measure over $\nu$.

**442X Basic exercises >(a)** Let $X$ and $Y$ be topological groups with (left) Haar probability measures $\mu$ and $\nu$, and $\phi : X \to Y$ a continuous surjective group homomorphism. Show that $\phi$ is inverse-measure-preserving for $\mu$ and $\nu$.

**(b)** (i) Let $X$ and $Y$ be two topological groups carrying Haar measures. Show that the product topological group $X \times Y$ (4A5G) carries Haar measures. (ii) Let $\langle X_i\rangle_{i\in I}$ be any family of topological groups carrying totally finite Haar measures. Show that the product group $\prod_{i\in I} X_i$ carries a totally finite Haar measure. (*Hint*: 417O.)

**(c)** Let $X$ be a subgroup of the group $(\mathbb{R}, +)$. Show that $X$ carries Haar measures iff it is either discrete (so that counting measure is a Haar measure on $X$) or of full outer Lebesgue measure (so that the subspace measure on $X$ is a Haar measure). (*Hint*: if $G$ has a Haar measure $\nu$ and is not discrete, then $\nu(G \cap [\alpha, \beta]) = (\beta - \alpha)\nu(G \cap [0, 1])$ whenever $\alpha \le \beta$.) In particular, $\mathbb{Q}$ does not carry Haar measures.

**(d)** Let $X$ be a topological group carrying Haar measures; let $\Sigma$ be the algebra of Haar measurable subsets of $X$. Let $\mu$ and $\nu$ be any Haar measures on $X$ (either left or right). Show that a set $E \in \Sigma$ can be covered by a sequence of sets of finite measure for $\mu$ iff it can be covered by a sequence of sets of finite measure for $\nu$.

**(e)** Let $X$ be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra (in the sense of 442H). Show that we have left, right and conjugacy actions of $X$ on $\mathfrak{A}$ given by the formulae $z{\bullet}E^\bullet = (zE)^\bullet$, $z{\bullet}E^\bullet = (Ez^{-1})^\bullet$ and $z{\bullet}E^\bullet = (zEz^{-1})^\bullet$ for every Haar measurable $E \subseteq X$ and every $z \in X$.

**>(f)** On $\mathbb{R}^2$ define a binary operation $*$ by setting $(\xi_1,\xi_2)*(\eta_1,\eta_2)=(\xi_1+\eta_1,\xi_2+e^{\xi_1}\eta_2)$. (i) Show that $*$ is a group operation under which $\mathbb{R}^2$ is a locally compact topological group. (ii) Show that Lebesgue measure $\mu$ is a right Haar measure for $*$. (iii) Let $\nu$ be the indefinite-integral measure on $\mathbb{R}^2$ defined by setting $\nu E=\int_E e^{-\xi_1}d\xi_1 d\xi_2$ for Lebesgue measurable sets $E\subseteq\mathbb{R}^2$. Show that $\nu$ is a left Haar measure for $*$. (*Hint*: 263D.) (iv) Thus $(\mathbb{R}^2,*)$ is not unimodular. (v) Show that the left modular function of $(\mathbb{R}^2,*)$ is $(\xi_1,\xi_2)\mapsto e^{-\xi_1}$.

**>(g)** Let $X$ be any topological group which is not unimodular. (i) Show that there is an open subset of $X$ which is of finite measure for all left Haar measures on $X$ and of infinite measure for all right Haar measures. (*Hint*: the modular function is unbounded.) (ii) Let $\mu$ be a left Haar measure on $X$ and $\nu$ a right Haar measure. Show that $L^0(\mu)=L^0(\nu)$ and $L^\infty(\mu)=L^\infty(\mu)$, but that $L^p(\mu)\ne L^p(\nu)$ for any $p\in[1,\infty[$.

**(h)** Let $X$ and $Y$ be topological groups carrying Haar measures, with left modular functions $\Delta_X$ and $\Delta_Y$ respectively. Show that the left modular function of $X\times Y$ is $(x,y)\mapsto\Delta_X(x)\Delta_Y(y)$.

**(i)** Let $X$ be any topological group and $\Delta:X\to\,]0,\infty[$ a group homomorphism such that $\{x:\Delta(x)\le 1+\epsilon\}$ is a neighbourhood of the identity in $X$ for every $\epsilon>0$. Show that $\Delta$ is continuous.

**(j)** Let $X$ be a topological group with a right Haar measure $\nu$ and left modular function $\Delta$. Show that $\nu E^{-1}=\int_E\Delta(x)\nu(dx)$ for every Haar measurable set $E\subseteq X$.

**442Y Further exercises (a)** In 441Yc, show that the only $G$-invariant Radon measures on $\mathcal{C}_s$ are multiples of Hausdorff $s(r-s)$-dimensional measure on $\mathcal{C}_s$. (*Hint*: $G$ itself is $\frac{r(r-1)}{2}$-dimensional (cf. 441Ye), and for any $C\in\mathcal{C}_s$ the stabilizer of $C$ is $\frac{s(s-1)}{2}+\frac{(r-s)(r-s-1)}{2}$-dimensional. See Federer 69, 3.2.28.)

**(b)** Let $r\ge 1$, and let $X$ be the group of non-singular $r\times r$ real matrices. Regarding $X$ as an open subset of $\mathbb{R}^{r^2}$, show that a two-sided Haar measure $\mu$ can be defined on $X$ by setting $\mu E=\int_E\frac{1}{|\det A|^r}\mu_L(dA)$, where $\mu_L$ is Lebesgue measure on $\mathbb{R}^{r^2}$; so that $X$ is unimodular.

**(c)** Show that there is a set $A\subseteq[0,1]$, of Lebesgue outer measure 1, such that no countable set of translates of $A$ covers any set of Lebesgue measure greater than 0. (*Hint*: let $\langle F_\xi\rangle_{\xi<\mathfrak{c}}$ run over the uncountable closed subsets of $[0,1]$ with cofinal repetitions (4A3Fa), and enumerate the countable subsets of $\mathbb{R}$ as $\langle I_\xi\rangle_{\xi<\mathfrak{c}}$. Choose inductively $x_\xi$, $x'_\xi\in F_\xi$ such that $x_\xi\notin\bigcup_{\eta,\zeta<\xi}x'_\eta-I_\zeta$, $x'_\xi\notin\bigcup_{\eta,\zeta\le\xi}x_\eta+I_\zeta$; set $A=\{x_\xi:\xi<\mathfrak{c}\}$.) Show that we can extend Lebesgue measure on $\mathbb{R}$ to a translation-invariant measure for which $A$ is negligible. (*Hint*: 417A.)

**(d)** Let $(X,\rho)$ be a metric space, and $\mu$, $\nu$ two non-zero quasi-Radon measures on $X$ such that $\mu B(x,\delta)=\mu B(y,\delta)$ and $\nu B(x,\delta)=\nu B(y,\delta)$ for all $\delta>0$ and $x$, $y\in X$. Show that $\mu$ is a multiple of $\nu$.

**442Z Problem** Let $X$ be a compact Hausdorff space, and $G$ the group of autohomeomorphisms of $X$. Suppose that $G$ acts transitively on $X$. Does it follow that there is at most one $G$-invariant Radon probability measure on $X$?

**442 Notes and comments** Haar measure dominates the theory of locally compact topological groups for two reasons: it is ubiquitous (the existence theorem, 441E) and essentially defined by the group structure (the uniqueness theorem, 442B). I have tried to show that these are rather different results by setting the theorems out with different hypotheses. I presented the existence theorem as a special case of 441C, which demands a locally compact space and a group, but allows them to be different. In the uniqueness theorem (roughly following Halmos 50) I demand a group with an invariant quasi-Radon measure, but do not (at this point) ask for any hypothesis of compactness. In fact it will become apparent in the next section that this is a somewhat spurious generality; 442B and 442I here can be deduced from the traditional forms in which the group is assumed to be locally compact and Hausdorff. From the point of view of the topological measure theory to which this volume is devoted, however, I think the small extra labour involved in tracing through the arguments without relying on the Riesz Representation Theorem is instructive. For instance, it emphasizes interesting features of the domains and null ideals of Haar measures (442H).

There is however a more serious question concerning the uniqueness theorem. I do not know whether it really belongs to the theory of topological groups, as described here, or to the theory of group actions along with 441C. The trouble is that I know of no example of a Hausdorff space $X$ and a transitive group $G$ of homeomorphisms of $X$ such that $X$ carries $G$-invariant Radon measures which are not multiples of each other (see 442Z). 443U and 443Xy below eliminate the simplest possibilities. We do need to put some restriction on the measures; for instance, counting measure on $\mathbb{R}$ is translation-invariant, but has nothing to do with Lebesgue measure. There are also proper translation-invariant extensions of Lebesgue measure (442Yc); for far-reaching elaborations of this idea see HEWITT & ROSS 63, §16.

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## 443 Further properties of Haar measure

I devote a section to filling in some details of the general theory of Haar measures before turning to the special topics dealt with in the rest of the chapter. The first question concerns the left and right shift operators acting on sets, on elements of the measure algebra, on measurable functions and on function spaces. All these operations can be regarded as group actions, and, if appropriate topologies are assigned, they are continuous actions (443C, 443G). As an immediate consequence of this I give an important result about product sets $\{ab : a \in A,\, b \in B\}$ in a topological group carrying Haar measures (443D).

The second part of the section revolves around a basic structure theorem: all the Haar measures considered here can be reduced to Haar measures on locally compact Hausdorff groups (443L). The argument involves two steps: the reduction to the Hausdorff case, which is elementary, and the completion of a Hausdorff topological group. Since a group carries more than one natural uniform structure we must take care to use the correct one, which in this context is the 'bilateral' uniformity (443H-443I, 443K). On the way I pick up an essential fact about the approximation of Haar measurable sets by Borel sets (443J). Finally, I give Halmos' theorem that Haar measures are completion regular (443M) and a note on the complementary nature of the meager and null ideals for atomless Haar measure (443O).

In the third part of the section I turn to the special properties of quotient groups of locally compact groups and the corresponding actions, following A.Weil. If $X$ is a locally compact Hausdorff group and $Y$ is a closed subgroup of $X$, then $Y$ is again a locally compact Hausdorff group, so has Haar measures and a modular function; at the same time, we have a natural action of $X$ on the set of left cosets of $Y$. It turns out that there is an invariant Radon measure for this action if and only if the modular function of $Y$ matches that of $X$ (443R). In this case we can express a left Haar measure of $X$ as an integral of measures supported by the cosets of $Y$ (443Q). When $Y$ is a normal subgroup, so that $X/Y$ is itself a locally compact Hausdorff group, we can relate the modular functions of $X$ and $X/Y$ (443T). We can apply these results whenever we have a continuous transitive action of a compact group on a compact space (443U).

**443A Haar measurability** I recall and expand on some facts already noted in 442H. Let $X$ be a topological group carrying Haar measures.

**(a)** All Haar measures on $X$, whether left or right, have the same domain $\Sigma$, which I call the algebra of 'Haar measurable' sets, and the same null ideal $\mathcal{N}$, which I call the ideal of 'Haar negligible' sets. The corresponding quotient algebra $\mathfrak{A} = \Sigma/\mathcal{N}$, the 'Haar measure algebra', is the Boolean algebra underlying the measure algebra of any Haar measure. Because Haar measures are (by the definition in 441D) quasi-Radon, therefore complete and strictly localizable (415A), $\Sigma_G$ is closed under Souslin's operation (431A) and $\mathfrak{A}$ is Dedekind complete (322Be). Recall that any semi-finite measure on $\mathfrak{A}$, and in particular any Haar measure on $X$, gives rise to the same measure-algebra topology and uniformity on $\mathfrak{A}$ (324H), so we may speak of 'the' topology and uniformity of $\mathfrak{A}$.

Because $\Sigma$ is the domain of a left Haar measure, $xE \in \Sigma$ whenever $E \in \Sigma$ and $x \in X$; because $\Sigma$ is the domain of a right Haar measure, $Ex \in \Sigma$ whenever $E \in \Sigma$ and $x \in X$. Similarly, $xE$ and $Ex$ are Haar negligible whenever $E$ is Haar negligible and $x \in X$. Moreover, $E^{-1} = \{x^{-1} : x \in E\}$ is Haar measurable or Haar negligible whenever $E$ is.

Note that $\Sigma$ and $\mathcal{N}$ are invariant in the strong sense that if $\phi : X \to X$ is any group automorphism which is also a homeomorphism, then $\Sigma = \{\phi[E] : E \in \Sigma\}$ and $\mathcal{N} = \{\phi[E] : E \in \mathcal{N}\}$. **P** If $\mu$ is a left Haar measure on $X$, let $\nu$ be the image measure $\mu\phi^{-1}$. Because $\phi$ is a homeomorphism, $\nu$ is a non-zero quasi-Radon measure. If $\nu$ measures $E$ and $a \in X$, then

$$\nu(aE) = \mu\phi^{-1}[aE] = \mu((\phi^{-1}a)(\phi^{-1}[E])) = \mu\phi^{-1}[E] = \nu E.$$

So $\nu$ is again a left Haar measure, and has domain $\Sigma$ and null ideal $\mathcal{N}$. But $\operatorname{dom}\nu = \{E : \phi^{-1}[E] \in \Sigma\} = \{\phi[E] : E \in \Sigma\}$ and $\nu^{-1}[\{0\}] = \{E : \phi^{-1}[E] \in \mathcal{N}\} = \{\phi[E] : E \in \mathcal{N}$. **Q**

**(b)** We even have a symmetric notion of 'measurable envelope' in $X$: for any $A \subseteq X$, there is a Haar measurable set $E \supseteq A$ such that $\mu(E \cap F) = \mu^*(A \cap F)$ for any Haar measurable $F \subseteq X$ and any Haar measure $\mu$ on $X$. **P** Start with a fixed Haar measure $\mu_0$. Then $A$ has a measurable envelope $E$ for $\mu_0$, by 213J and 213L. Now to say that '$E$ is a measurable envelope for $A$' means just that (i) $A \subseteq E \in \Sigma$ (ii) if $F \in \Sigma$ and $F \subseteq E \setminus A$ then $F \in \mathcal{N}$, so $E$ is also a measurable envelope for $A$ for any other Haar measure on $X$. **Q**

In this context I will call $E$ a **Haar measurable envelope** of $A$.

**(c)** Similarly, we have a notion of **full outer Haar measure**: a subset $A$ of $X$ is of full outer Haar measure if $X$ is a Haar measurable envelope of $A$, that is, $A \cap E \neq \emptyset$ whenever $E$ is a Haar measurable set which is not Haar negligible, that is, $A$ is of full outer measure for any (left or right) Haar measure on $X$.

**(d)** For any Haar measure $\mu$ on $X$, we can identify $L^\infty(\mu)$ with $L^\infty(\mathfrak{A})$ (363I) and $L^0(\mu)$ with $L^0(\mathfrak{A})$ (364Ic). Thus these constructions are independent of $\mu$. The topology of convergence in measure of $L^0$ is determined by its Riesz space structure (367T); so that this also is independent of the particular Haar measure we may select. Of course the same is true of the norm of $L^\infty$. Note however that the spaces $L^p$, for $1 \le p < \infty$, are different for left and right Haar measures on any group which is not unimodular (442Xg), and that even for left Haar measures $\mu$ the norm on $L^p(\mu)$ changes if $\mu$ is re-normalized.

**(e)** When it seems appropriate, I will use the phrases **Haar measurable function**, meaning a function measurable with respect to the $\sigma$-algebra of Haar measurable sets, and **Haar almost everywhere**, meaning 'on the complement of a Haar negligible set'. Note that we can identify $L^0(\mathfrak{A})$ with the set of equivalence classes in the space $\mathcal{L}^0$, where $\mathcal{L}^0$ is the space of Haar measurable real-valued functions defined Haar-a.e. in $X$, and $f \sim g$ if $f = g$ Haar-a.e. In the language of §241, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ for any Haar measure $\mu$ on $X$.

**(f)** Because $E^{-1} \in \Sigma$ for every $E \in \Sigma$, and $E^{-1} \in \mathcal{N}$ whenever $E \in \mathcal{N}$, we have a canonical automorphism $a \mapsto \overset{\leftrightarrow}{a} : \mathfrak{A} \to \mathfrak{A}$ defined by writing $(E^\bullet)^\leftrightarrow = (E^{-1})^\bullet$ for every $E \in \Sigma$. Being an automorphism, this must be a homeomorphism for the measure-algebra topology of $\mathfrak{A}$ (324G). In the same way, if $f \in \mathcal{L}^0$ then $\overset{\leftrightarrow}{f} \in \mathcal{L}^0$, where $\overset{\leftrightarrow}{f}(x) = f(x^{-1})$ whenever this is defined (4A5C(c-ii)), since $\{x : x \in \operatorname{dom}\overset{\leftrightarrow}{f},\ \overset{\leftrightarrow}{f}(x) > \alpha\} = \{x : x \in \operatorname{dom} f,\ f(x) > \alpha\}^{-1}$ belongs to $\Sigma$ for every $\alpha$; and we can define an $f$-algebra automorphism $u \mapsto \overset{\leftrightarrow}{u} : L^0 \to L^0$ by saying that $(f^\bullet)^\leftrightarrow = (\overset{\leftrightarrow}{f})^\bullet$ for $f \in \mathcal{L}^0$. If we identify $L^0$ with $L^0(\mathfrak{A})$ rather than with a set of equivalence classes in $\mathcal{L}^0$, then we can define the map $u \mapsto \overset{\leftrightarrow}{u}$ as the Riesz homomorphism associated with the Boolean homomorphism $a \mapsto \overset{\leftrightarrow}{a} : \mathfrak{A} \to \mathfrak{A}$, as in 364P. Note that $\|\overset{\leftrightarrow}{u}\|_\infty = \|u\|_\infty$ for every $u \in L^\infty$, but that (unless $X$ is unimodular) other $L^p$ spaces are not invariant under the involution $^\leftrightarrow$ (442Xg again).

**(g)** If $X$ carries any totally finite (left or right) Haar measure, it is unimodular, and has a unique, two-sided, Haar probability measure (442Ie). (In particular, this is the case whenever $X$ is compact.) For such groups we have $L^p$-spaces, for $1 \le p \le \infty$, defined by the group structure, with canonical norms.

**443B Lemma** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. If $E \subseteq X$ is measurable and $\mu E < \infty$, then for any $\epsilon > 0$ there is a neighbourhood $U$ of the identity $e$ such that $\mu(E \triangle xEy) \le \epsilon$ whenever $x, y \in U$.

**proof** Set $\delta = \frac{\min(1,\epsilon)}{10+3\mu E} > 0$. Write $\mathcal{U}$ for the family of open neighbourhoods of $e$. Because $\mu$ is effectively locally finite, there is an open set $G_0$ of finite measure such that $\mu(E \setminus G_0) \le \delta$. Let $F \subseteq G_0 \setminus E$ be a closed set such that $\mu F \ge \mu(G_0 \setminus E) - \delta$, and set $G = G_0 \setminus F$, so that $\mu(G \setminus E) \le \delta$ and $\mu(E \setminus G) \le \delta$. For $U \in \mathcal{U}$ set $H_U = \operatorname{int}\{x : UxU \subseteq G\}$. Then $\mathcal{H} = \{H_U : U \in \mathcal{U}\}$ is upwards-directed, and has union $G$, because if $x \in G$ there is a $U \in \mathcal{U}$ such that $UxUU \subseteq G$, so that $x \in H_U$. So there is a $V \in \mathcal{U}$ such that $\mu(G \setminus H_V) \le \delta$. Recall that the left modular function $\Delta$ of $X$ is continuous (442J). So there is a $U \in \mathcal{U}$ such that $U \subseteq V$ and $|\Delta(y) - 1| \le \delta$ for every $y \in U$.

Now suppose that $x$, $y \in U$. Set $E_1 = E \cap H_V$. Then $xE_1y \subseteq G$, so

$$\mu(E_1 \cup xE_1y) \le \mu G \le \mu E + \delta.$$

On the other hand,

$$\mu E_1 \ge \mu E - \mu(E \setminus G) - \mu(G \setminus H_V) \ge \mu E - 2\delta,$$

$$\mu(xE_1y) = \Delta(y)\mu E_1 \ge (1 - \delta)(\mu E - 2\delta) \ge \mu E - (2 + \mu E)\delta.$$

So

$$\mu(E \cap xEy) \ge \mu(E_1 \cap xE_1y) = \mu E_1 + \mu(xE_1y) - \mu(E_1 \cup xE_1y) \ge \mu E - (5 + \mu E)\delta.$$

At the same time,

$$\mu(xEy) = \Delta(y)\mu E \le (1 + \delta)\mu E.$$

So

$$\mu(E \triangle xEy) = \mu E + \mu(xEy) - 2\mu(E \cap xEy) \le (10 + 3\mu E)\delta \le \epsilon,$$

as required.

**443C Theorem** Let $X$ be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra. Then we have continuous actions of $X$ on $\mathfrak{A}$ defined by writing

$$x \bullet_l E^\bullet = (xE)^\bullet, \quad x \bullet_r E^\bullet = (Ex^{-1})^\bullet, \quad x \bullet_c E^\bullet = (xEx^{-1})^\bullet$$

for Haar measurable sets $E \subseteq X$ and $x \in X$.

**proof (a)** The functions $\bullet_l$, $\bullet_r$ and $\bullet_c$ are all well defined because the maps $E \mapsto xE$, $E \mapsto Ex^{-1}$ and $E \mapsto xEx^{-1}$ are all Boolean automorphisms of the algebra $\Sigma$ of Haar measurable sets preserving the ideal of Haar negligible sets (442G). It is elementary to check that they are actions of $X$ on $\mathfrak{A}$.

Fix a left Haar measure $\mu$ on $X$ and let $\bar{\mu}$ be the corresponding measure on $\mathfrak{A}$. Then the topology of $\mathfrak{A}$ is defined by the pseudometrics $\rho_a$, for $\bar{\mu}a < \infty$, where $\rho_a(b, c) = \bar{\mu}(a \cap (b \triangle c))$.

**(b)** Now suppose that $x_0 \in X$, $b_0 \in \mathfrak{A}$, $\bar{\mu}a < \infty$ and $\epsilon > 0$. Let $E$, $F_0 \in \Sigma$ be such that $E^\bullet = a$ and $F_0^\bullet = b_0$; set $\delta = \frac{1}{4}\epsilon > 0$. Note that

$$\mu(x_0^{-1}E \cap F_0) \le \mu(x_0^{-1}E) = \mu E < \infty.$$

Let $U$ be a neighbourhood of the identity $e$ such that

$$\mu(E \triangle yE) \le \delta, \quad \mu((x_0^{-1}E \cap F_0) \triangle y(x_0^{-1}E \cap F_0)) \le \delta$$

whenever $y \in U$ (443B). Set

$$a' = x_0^{-1} \bullet_l a = (x_0^{-1}E)^\bullet.$$

Now suppose that $x \in Ux_0 \cap x_0U^{-1}$ and that $\rho_{a'}(b, b_0) \le \delta$. Then $\rho_a(x \bullet_l b, x_0 \bullet_l b_0) \le \epsilon$. **P** Let $F \in \Sigma$ be such that $F^\bullet = b$. Then $xx_0^{-1}$ and $x^{-1}x_0$ both belong to $U$, so

$$
\begin{aligned}
\rho_a(x\bullet_l b, x_0\bullet_l b_0) &= \mu(E\cap(xF\triangle x_0F_0))\\
&= \mu(E\cap xF)+\mu(E\cap x_0F_0)-2\mu(E\cap xF\cap x_0F_0)\\
&= \mu(x^{-1}E\cap F)+\mu(E\cap x_0F_0)-2\mu(x^{-1}x_0(x_0^{-1}E\cap F_0)\cap F)\\
&\le \mu(x_0^{-1}E\cap F)+\mu(x^{-1}E\triangle x_0^{-1}E)+\mu(x_0^{-1}E\cap F_0)\\
&\qquad -2\mu(x_0^{-1}E\cap F_0\cap F)+2\mu(x^{-1}x_0(x_0^{-1}E\cap F_0)\triangle(x_0^{-1}E\cap F_0))\\
&\le \mu(x_0^{-1}E\cap(F\triangle F_0))+\mu(E\triangle xx_0^{-1}E)+2\delta\\
&\le \delta+\delta+2\delta=4\delta=\epsilon. \quad \mathbf{Q}
\end{aligned}
$$

As $x_0$, $b_0$, $a$ and $\epsilon$ are arbitrary, $\bullet_l$ is continuous.

**(c)** The same arguments, using a right Haar measure to provide pseudometrics defining the topology of $\mathfrak{A}$, show that $\bullet_r$ is continuous. (Or use the method of 443X(d-ii).)

**(d)** Accordingly the map $(x,y,a)\mapsto x\bullet_l(y\bullet_r a)$ is continuous. So

$$(x,a)\mapsto x\bullet_l(x\bullet_r a)=x\bullet_c a$$

is continuous.

**443D Proposition** Let $X$ be a topological group carrying Haar measures. If $E\subseteq X$ is Haar measurable but not Haar negligible, and $A\subseteq X$ is not Haar negligible, then

(a) there are $x$, $y\in X$ such that $A\cap xE$, $A\cap Ey$ are not Haar negligible;

(b) $EA$ and $AE$ both have non-empty interior;

(c) $E^{-1}E$ and $EE^{-1}$ are neighbourhoods of the identity.

**proof (a)(i)** Let $\mu$ be any left Haar measure on $X$, and for Borel sets $F\subseteq X$ set

$$\nu F=\sup\{\mu(F\cap IE): I\subseteq X \text{ is finite}\}.$$

It is easy to check that $\nu$ is an effectively locally finite $\tau$-additive Borel measure, inner regular with respect to the closed sets, because $\{IE: I\in[X]^{<\omega}\}$ is upwards-directed and $\mu$ is quasi-Radon. Moreover,

$$\nu(xF)=\sup_{I\in[X]^{<\omega}}\mu(xF\cap IE)=\sup_{I\in[X]^{<\omega}}\mu(F\cap x^{-1}IE)=\nu F$$

for every Borel set $F\subseteq X$ and every $x\in X$. Accordingly the c.l.d. version $\tilde\nu$ of $\nu$ is a left-translation-invariant quasi-Radon measure on $X$ (415Cb); since $\nu E>0$, $\tilde\nu$ is non-zero and is itself a left Haar measure. Consequently $A$ is not $\tilde\nu$-negligible. Let $H$ be a measurable envelope of $A$ for Haar measure (443Ab). Then $H$ is not Haar negligible, so there is a closed set $F\subseteq H$ which is not Haar negligible, and $\nu F=\tilde\nu F>0$. Thus there is an $x\in X$ such that

$$0<\mu(F\cap xE)\le\mu(H\cap xE)=\mu^*(A\cap xE),$$

and $A\cap xE$ is not Haar negligible.

**(ii)** Applying the same arguments, but starting with a right Haar measure $\mu$, we see that there is a $y\in X$ such that $A\cap Ey$ is not Haar negligible.

**(b)** Let $\mu$ be a left Haar measure on $X$, and $F$ a Haar measurable envelope of $A$. The function $x\mapsto(xE^{-1})^\bullet: X\to\mathfrak{A}$ is continuous, where $\mathfrak{A}$ is the Haar measure algebra of $X$ (443C), so

$$H=\{x:\mu^*(A\cap xE^{-1})>0\}=\{x:F^\bullet\cap(xE^{-1})^\bullet\ne0\}$$

is open. Now

$$H\subseteq\{x:A\cap xE^{-1}\ne\emptyset\}=AE,$$

so $H\subseteq\operatorname{int}AE$; and $E^{-1}$ is Haar measurable and not Haar negligible, so $H\ne\emptyset$, by (a). Thus $\operatorname{int}AE\ne\emptyset$.

Similarly, using a right Haar measure (or observing that $EA=(A^{-1}E^{-1})^{-1}$), we see that $EA$ has non-empty interior.

**(c)** Again taking a left Haar measure $\mu$, $\mu$ is semi-finite, so there is an $F \subseteq E$ such that $0 < \mu F < \infty$. By 443B, there is a neighbourhood $U$ of the identity such that $\mu(F \triangle xFy) < \mu F$ for all $x, y \in U$. In particular, if $x \in U$, $\mu(F \setminus xF) < \mu F$, so $F \cap xF \neq \emptyset$ and $x \in FF^{-1} \subseteq EE^{-1}$; at the same time, $\mu(F \setminus Fx) < \mu F$, $F \cap Fx \neq \emptyset$ and $x \in F^{-1}F \subseteq E^{-1}E$. So $E^{-1}E$ and $EE^{-1}$ both include $U$ and are neighbourhoods of the identity.

**443E Corollary** Let $X$ be a Hausdorff topological group carrying Haar measures. Then the following are equiveridical:

(i) $X$ is locally compact;

(ii) every Haar measure on $X$ is a Radon measure;

(iii) there is some compact subset of $X$ which is not Haar negligible.

**proof (i)$\Rightarrow$(ii)** Haar measures are locally finite quasi-Radon measures (441D, 442Aa), so on locally compact Hausdorff spaces must be Radon measures (416G).

**(ii)$\Rightarrow$(iii)** is obvious, just because Haar measures are non-zero and any Radon measure is tight (that is, inner regular with respect to the closed compact sets).

**(iii)$\Rightarrow$(i)** If $K \subseteq X$ is a compact set which is not Haar negligible, then $KK$ is a compact set with non-empty interior, so $X$ is locally compact (4A5Eg).

**443F** Later in the chapter we shall need the following straightforward fact.

**Lemma** Let $X$ be a topological group carrying Haar measures, and $Y$ an open subgroup of $X$. If $\mu$ is a left Haar measure on $X$, then the subspace measure $\mu_Y$ is a left Haar measure on $Y$. Consequently a subset of $Y$ is Haar measurable or Haar negligible, when regarded as a subset of the topological group $Y$, iff it is Haar measurable or Haar negligible when regarded as a subset of the topological group $X$.

**proof** By 415B, $\mu_Y$ is a quasi-Radon measure; because $\mu$ is strictly positive, $\mu_Y$ is non-zero, and it is easy to check that it is left-translation-invariant. So it is a Haar measure on $Y$. The rest follows at once from 442H/443A.

**443G** We can repeat the ideas of 443C in terms of function spaces, as follows.

**Theorem** Let $X$ be a topological group with a left Haar measure $\mu$. Let $\Sigma$ be the domain of $\mu$, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ the space of $\Sigma$-measurable real-valued functions defined almost everywhere in $X$, and $L^0 = L^0(\mu)$ the corresponding space of equivalence classes (§241).

(a) $a\bullet_l f$, $a\bullet_r f$ and $a\bullet_c f$ (definitions: 4A5C(c-ii)) belong to $\mathcal{L}^0$ for every $f \in \mathcal{L}^0$ and $a \in X$.

(b) If $a \in X$, then $\operatorname{ess\,sup}|a\bullet_l f| = \operatorname{ess\,sup}|a\bullet_r f| = \operatorname{ess\,sup}|f|$ for every $f \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$, where $\operatorname{ess\,sup}|f|$ is the essential supremum of $|f|$ (243Da). For $1 \le p < \infty$, $\|a\bullet_l f\|_p = \|f\|_p$ and $\|a\bullet_r f\|_p = \Delta(a)^{-1/p}\|f\|_p$ for every $f \in \mathcal{L}^p = \mathcal{L}^p(\mu)$, where $\Delta$ is the left modular function of $X$.

(c) We have shift actions of $X$ on $L^0$ defined by setting

$$a\bullet_l f^\bullet = (a\bullet_l f)^\bullet, \quad a\bullet_r f^\bullet = (a\bullet_r f)^\bullet, \quad a\bullet_c f^\bullet = (a\bullet_c f)^\bullet$$

for $a \in X$ and $f \in \mathcal{L}^0$. If $^\leftrightarrow$ is the reversal operator on $L^0$ defined in 443Af, we have

$$a\bullet_l \overset{\leftrightarrow}{u} = (a\bullet_r u)^\leftrightarrow, \quad a\bullet_c \overset{\leftrightarrow}{u} = (a\bullet_c u)^\leftrightarrow$$

for every $a \in X$ and $u \in L^0$.

(d) If we give $L^0$ its topology of convergence in measure these three actions, and also the reversal operator $^\leftrightarrow$, are continuous.

(e) For $1 \le p \le \infty$ the formulae of (c) define actions of $X$ on $L^p = L^p(\mu)$, and $\|a\bullet_l u\|_p = \|u\|_p$ for every $u \in L^p$, $a \in X$; interpreting $\Delta(a)^{-1/\infty}$ as 1 if necessary, $\|a\bullet_r u\|_p = \Delta(a)^{-1/p}\|u\|_p$ whenever $u \in L^p$ and $a \in X$.

(f) For $1 \le p < \infty$ these actions are continuous.

**proof (a)** Let $f \in \mathcal{L}^0$. Then $F = \operatorname{dom} f$ is conegligible, so $aF = \operatorname{dom} a\bullet_l f$ and $Fa^{-1} = \operatorname{dom} a\bullet_r f$ are conegligible (442G). For any $\alpha \in \mathbb{R}$, set $E_\alpha = \{x : x \in F,\ f(x) < \alpha\}$; then $\{x : (a\bullet_l f)(x) < \alpha\} = aE_\alpha$ and

$\{x : (a\bullet_r f)(x) < \alpha\} = E_\alpha a^{-1}$ are measurable, so $a\bullet_l f$ and $a\bullet_r f$ are measurable. Thus $a\bullet_l f$ and $a\bullet_r f$ belong to $\mathcal{L}^0$. It follows at once that $a\bullet_c f = a\bullet_l(a\bullet_r f)$ belongs to $\mathcal{L}^0$.

**(b)(i)** For $\alpha \ge 0$,

$$\begin{aligned}\operatorname{ess\,sup}|f| \le \alpha &\iff |f(x)| \le \alpha \text{ for almost all } x\\ &\iff |(a\bullet_l f)(x)| \le \alpha \text{ for almost all } x\\ &\iff |(a\bullet_r f)(x)| \le \alpha \text{ for almost all } x\end{aligned}$$

because the null ideal of $\mu$ is invariant under both left and right translations. So $\operatorname{ess\,sup}|f| = \operatorname{ess\,sup}|a\bullet_l f| = \operatorname{ess\,sup}|a\bullet_r f|$.

**(ii)** For $1 \le p < \infty$,

$$\|a\bullet_l f\|_p^p = \int |(a\bullet_l f)(x)|^p\mu(dx) = \int |f(a^{-1}x)|^p\mu(dx) = \int |f(x)|^p\mu(dx)$$

(441J)

$$= \|f\|_p^p,$$

$$\|a\bullet_r f\|_p^p = \int |(a\bullet_r f)(x)|^p\mu(dx) = \int |f(xa)|^p\mu(dx) = \Delta(a^{-1})\int |f(x)|^p\mu(dx)$$

(442Kc)

$$= (\Delta(a)^{-1/p}\|f\|_p)^p.$$

**(c)(i)** I have already checked that $a\bullet_l f$, $a\bullet_r f$ and $a\bullet_c f$ belong to $\mathcal{L}^0$ whenever $f \in \mathcal{L}^0$ and $a \in X$. If $f$, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, let $E$ be the conegligible set $\{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(x) = g(x)\}$; then $aE$ and $Ea^{-1}$ and $aEa^{-1}$ are conegligible and

$$(a\bullet_l f)(x) = (a\bullet_l g)(x) \text{ for every } x \in aE, \quad (a\bullet_r f)(x) = (a\bullet_r g)(x) \text{ for every } x \in Ea^{-1},$$

$$(a\bullet_c f)(x) = (a\bullet_c g)(x) \text{ for every } x \in aEa^{-1},$$

so $a\bullet_l f =_{\text{a.e.}} a\bullet_l g$, $a\bullet_r f =_{\text{a.e.}} a\bullet_r g$ and $a\bullet_c f =_{\text{a.e.}} a\bullet_c g$. Accordingly the formulae given define functions $\bullet_l$, $\bullet_r$ and $\bullet_c$ from $X \times L^0$ to $L^0$. They are actions just because the original $\bullet_l$, $\bullet_r$ and $\bullet_c$ are actions of $X$ on $\mathcal{L}^0$ (4A5Cc-4A5Cd).

**(ii)** If $f \in \mathcal{L}^0$, then

$$(a\bullet_l \overset{\leftrightarrow}{f})(x) = \overset{\leftrightarrow}{f}(a^{-1}x) = f(x^{-1}a) = (a\bullet_r f)(x^{-1}) = (a\bullet_r f)^\leftrightarrow(x)$$

when any of these is defined, which is almost everywhere, so $a\bullet_l u = (a\bullet_r u)^\leftrightarrow$ for every $u \in L^0$. Similarly,

$$(a\bullet_c \overset{\leftrightarrow}{f})(x) = \overset{\leftrightarrow}{f}(a^{-1}xa) = f(a^{-1}x^{-1}a) = (a\bullet_c f)(x^{-1}) = (a\bullet_c f)^\leftrightarrow(x)$$

and $a\bullet_c \overset{\leftrightarrow}{u} = (a\bullet_c u)^\leftrightarrow$.

**(d)(i)** In 367T there is a description of convergence in measure on $L^0$ in terms of its Riesz space structure. As $^\leftrightarrow$ is a Riesz space automorphism of $L^0$, it must also be a homeomorphism for the topology of convergence in measure.

**(ii)** To see that $\bullet_l$ is continuous, it will be convenient to work with the space $\mathcal{L}^0_\Sigma$ of Haar measurable real-valued functions defined on the whole of $X$. I will use a characterization of convergence in measure from 245F: a subset $W$ of $L^0$ is open iff whenever $f_0^\bullet \in W$ there are a set $E$ of finite measure and an $\epsilon > 0$ such that $f^\bullet \in W$ whenever $\mu\{x : x \in E,\ |f(x) - f_0(x)| > \epsilon\} \le \epsilon$. Now if $E$ is a measurable set of finite measure, $f \in \mathcal{L}^0_\Sigma$ and $\epsilon > 0$, there is a neighbourhood $U$ of the identity $e$ of $X$ such that $\mu\{x : x \in E,\ |f(ax) - f(x)| \ge \epsilon\} \le \epsilon$ for every $a \in U$. **P** Let $m \ge 1$ be such that $\mu\{x : x \in E,\ |f(x)| \ge m\epsilon\} \le \frac{1}{2}\epsilon$. For $-m \le k < m$, set $E_k = \{x : x \in E,\ k\epsilon \le f(x) < (k+1)\epsilon\}$. By 443B, there is a neighbourhood $U$ of $e$ such that

$$\mu(E_k \triangle a^{-1}E_k) \le \frac{\epsilon}{4m}$$

whenever $a \in U$ and $-m \le k < m$. Now, for $a \in U$,

$$\{x : x \in E,\, |f(ax) - f(x)| \ge \epsilon\} \subseteq \{x : x \in E,\, |f(x)| \ge m\epsilon\} \cup \bigcup_{k=-m}^{m-1} (E_k \triangle a^{-1}E_k)$$

has measure at most

$$\frac{\epsilon}{2} + 2m\frac{\epsilon}{4m} = \epsilon. \ \mathbf{Q}$$

**(iii)** Let $E$ be a measurable set of finite measure, $a_0 \in X$, $f_0 \in \mathcal{L}^0_\Sigma$ and $\epsilon > 0$. Set $\delta = \epsilon/3 > 0$. Note that $\mu(a_0^{-1}E) = \mu E$ is finite. Let $U$ be a neighbourhood of $e$ such that

$$\mu\{x : x \in a_0^{-1}E,\, |f_0(yx) - f_0(x)| \ge \delta\} \le \delta, \quad \mu(yE \triangle E) \le \delta$$

whenever $y \in U$.

Now suppose that $a \in Ua_0 \cap a_0U^{-1}$ and that $f \in \mathcal{L}^0_\Sigma$ is such that $\mu\{x : x \in a_0^{-1}E,\, |f(x) - f_0(x)| \ge \delta\} \le \delta$. In this case,

$$\begin{aligned}
&\{x : x \in E,\, |f(a^{-1}x) - f_0(a_0^{-1}x)| \ge \epsilon\}\\
&\quad\subseteq \{x : x \in E,\, |f(a^{-1}x) - f_0(a^{-1}x)| \ge \delta\}\\
&\qquad\qquad \cup \{x : x \in E,\, |f_0(a^{-1}x) - f_0(a_0^{-1}x)| \ge \delta\}\\
&\quad\subseteq (E \triangle aa_0^{-1}E) \cup \{x : x \in aa_0^{-1}E,\, |f(a^{-1}x) - f_0(a^{-1}x)| \ge \delta\}\\
&\qquad\qquad \cup a_0\{w : w \in a_0^{-1}E,\, |f_0(a^{-1}a_0w) - f_0(w)| \ge \delta\}\\
&\quad\subseteq (E \triangle aa_0^{-1}E) \cup a\{w : w \in a_0^{-1}E,\, |f(w) - f_0(w)| \ge \delta\}\\
&\qquad\qquad \cup a_0\{w : w \in a_0^{-1}E,\, |f_0(a^{-1}a_0w) - f_0(w)| \ge \delta\}
\end{aligned}$$

has measure at most $3\delta = \epsilon$ because $aa_0^{-1}$, $e$ and $a^{-1}a_0$ all belong to $U$. Because $E$ and $\epsilon$ are arbitrary, the function $(a, u) \mapsto a\bullet_l u$ is continuous at $(a_0, f_0^\bullet)$; as $a_0$ and $f_0$ are arbitrary, $\bullet_l$ is continuous.

**(iv)** Now

$$(a, u) \mapsto a\bullet_r u = (a\bullet_l \overset{\leftrightarrow}{u})^{\leftrightarrow}$$

must also be continuous. It follows at once that $\bullet_c$ is continuous, since $a\bullet_c u = a\bullet_l(a\bullet_r u)$.

**(e)** follows at once from (b) and (c).

**(f)** Fix $p \in [1, \infty[$.

**(i)** If $u \in L^p$ and $\epsilon > 0$, there is a neighbourhood $U$ of $e$ such that $\|u - y\bullet_l(z\bullet_r u)\|_p \le \epsilon$ whenever $y$, $z \in U$. $\mathbf{P}$ When $u$ is of the form $(\chi E)^\bullet$, where $\mu E < \infty$, we have

$$y\bullet_l(z\bullet_r u) = \chi(yEz^{-1})^\bullet, \quad \|u - y\bullet_l(z\bullet_r u)\|_p = \mu(E \triangle yEz^{-1})^{1/p},$$

so the result is immediate from 443B. If $u = \sum_{i=0}^n \alpha_i(\chi E_i)^\bullet$, where every $E_i$ has finite measure, then, setting $u_i = (\chi E_i)^\bullet$ for each $i$,

$$\|u - y\bullet_l(z\bullet_r u)\|_p \le \sum_{i=0}^n |\alpha_i| \|u_i - y\bullet_l(z\bullet_r u_i)\|_p \le \epsilon$$

whenever $y$ and $z$ are close enough to $e$. In general, there is a $v$ of this form such that $\|u - v\|_p \le \frac{1}{4}\epsilon$. If we take a neighbourhood $U$ of $e$ such that $\|v - y\bullet_l(z\bullet_r v)\|_p \le \frac{1}{4}\epsilon$ and $\Delta(z)^{-1/p} \le 2$ whenever $y$, $z \in U$, then

$$\|y\bullet_l(z\bullet_r u) - y\bullet_l(z\bullet_r v)\|_p = \Delta(z)^{-1/p}\|u - v\|_p \le \frac{1}{2}\epsilon$$

whenever $z \in U$, so

$$\|u - y\bullet_l(z\bullet_r u)\|_p \le \|u - v\|_p + \|v - y\bullet_l(z\bullet_r v)\|_p + \|y\bullet_l(z\bullet_r v) - y\bullet_l(z\bullet_r u)\|_p \le \epsilon$$

whenever $y$, $z \in U$. $\mathbf{Q}$

**(ii)** Now suppose that $u_0 \in L^p$, $a_0$, $b_0 \in X$ and $\epsilon > 0$. Set $v_0 = a_0\bullet_l(b_0\bullet_r u_0)$ and $\delta = \epsilon/(1 + 2\Delta(b_0)^{-1/p}) > 0$. Let $U$ be a neighbourhood of $e$ such that $\Delta(y)^{-1/p} \le 2$ and $\|v_0 - y\bullet_l(z\bullet_r v_0)\|_p \le \delta$ whenever $y$, $z \in U$. If $a \in Ua_0$, $b \in Ub_0$ and $\|u - u_0\|_p \le \delta$, then

$$\begin{aligned}\|a\bullet_l(b\bullet_r u)-v_0\|_p &\le \|a\bullet_l(b\bullet_r u)-a\bullet_l(b\bullet_r u_0)\|_p+\|a\bullet_l(b\bullet_r u_0)-v_0\|_p\\ &=\Delta(b)^{-1/p}\|u-u_0\|_p+\|aa_0^{-1}\bullet_l(bb_0^{-1}\bullet_r v_0)-v_0\|_p\\ &\le \Delta(bb_0^{-1})^{-1/p}\Delta(b_0)^{-1/p}\delta+\delta\le\delta(1+2\Delta(b_0)^{-1/p})=\epsilon.\end{aligned}$$

As $\epsilon$ is arbitrary, $(a,b,u)\mapsto a\bullet_l(b\bullet_r u)$ is continuous at $(a_0,b_0,u_0)$.

As in (c), this is enough to show that $\bullet_l$, $\bullet_r$ and $\bullet_c$ are all continuous actions.

**Remark** I have written this out for a left Haar measure $\mu$, since the spaces $L^p(\mu)$ depend on this; if $\nu$ is a right Haar measure, and $X$ is not unimodular, then $L^p(\mu)\ne L^p(\nu)$ for $1\le p<\infty$. But recall that the topology of convergence in measure on $L^0$ is the same for all Haar measures (443Ad), so (c) above, and the case $p=\infty$ of (b) and (d), are two-sided; they belong to the theory of the Haar measure algebra.

**443H Theorem** Let $X$ be a topological group carrying Haar measures. Then there is a neighbourhood of the identity which is totally bounded for the bilateral uniformity on $X$.

**proof** Let $\mu$ be a left Haar measure on $X$. Let $V_0$ be a neighbourhood of the identity $e$ such that $\mu V_0<\infty$ (442Aa). Let $V$ be a neighbourhood of $e$ such that $VV\subseteq V_0$ and $V^{-1}=V$.

**?** Suppose, if possible, that $V$ is not totally bounded for the bilateral uniformity on $X$. By 4A5Oa, one of the following must occur:

**case 1** There is an open neighbourhood $U$ of $e$ such that $V\not\subseteq IU$ for any finite set $I\subseteq X$. In this case, we may choose a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $V$ inductively such that $x_n\notin x_iU$ whenever $i<n$. Let $U_1$ be an open neighbourhood of $e$ such that $U_1\subseteq V$ and $U_1U_1^{-1}\subseteq U$; then $\langle x_nU_1\rangle_{n\in\mathbb{N}}$ is disjoint. Since $\mu(x_nU_1)=\mu U_1>0$ for every $n$ (by the other clause in 442Aa), $\mu(\bigcup_{n\in\mathbb{N}}x_nU_1)=\infty$; but $x_nU_1\subseteq V_0$ for every $n$, so this is impossible.

**case 2** There is an open neighbourhood $U$ of $e$ such that $V\not\subseteq UI$ for any finite set $I\subseteq X$. So we may choose a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $V$ inductively such that $x_n\notin Ux_i$ whenever $i<n$. Let $U_1$ be an open neighbourhood of $e$ such that $U_1\subseteq V$ and $U_1U_1^{-1}\subseteq U$; then $\langle U_1^{-1}x_n\rangle_{n\in\mathbb{N}}$ is disjoint, so $\langle x_n^{-1}U_1\rangle_{n\in\mathbb{N}}$ is also disjoint. Since $\mu(x_n^{-1}U_1)=\mu U_1>0$ for every $n$, $\mu(\bigcup_{n\in\mathbb{N}}x_n^{-1}U_1)=\infty$; but $x_n^{-1}U_1\subseteq V_0$ for every $n$, so this also is impossible. **X**

Thus $V$ is totally bounded for the bilateral uniformity on $X$, and we have the required totally bounded neighbourhood of $e$.

**443I Corollary** Let $X$ be a topological group. If $A\subseteq X$ is totally bounded for the bilateral uniformity of $X$, it has finite outer measure for any (left or right) Haar measure on $X$.

**proof** If $\mu$ is a Haar measure on $X$, let $U$ be an open neighbourhood of the identity $e$ of finite measure. There is a finite set $I$ such that $A\subseteq IU\cap UI$ (4A5Oa again), so that $\mu^*A\le\#(I)\mu U$ is finite.

**443J Proposition** Let $X$ be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra.

(a) There is an open-and-closed subgroup $Y$ of $X$ such that, for any Haar measure $\mu$ on $X$, $Y$ can be covered by countably many open sets of finite measure.

(b)(i) If $E\subseteq X$ is any Haar measurable set, there are an $\mathrm{F}_\sigma$ set $E'\subseteq E$ and a $\mathrm{G}_\delta$ set $E''\supseteq E$ such that $E''\setminus E'$ is Haar negligible.

(ii) Every Haar negligible set is included in a Haar negligible Borel set, and for every Haar measurable set $E$ there is a Borel set $F$ such that $E\triangle F$ is Haar negligible.

(iii) The Haar measure algebra $\mathfrak{A}$ of $X$ may be identified with $\mathcal{B}/\mathcal{I}$, where $\mathcal{B}$ is the Borel $\sigma$-algebra of $X$ and $\mathcal{I}$ is the ideal of Haar negligible Borel sets.

(iv) Every member of $L^0(\mathfrak{A})$ can be identified with the equivalence class of some Borel measurable function from $X$ to $\mathbb{R}$. Every member of $L^\infty(\mathfrak{A})$ can be identified with the equivalence class of a bounded Borel measurable function from $X$ to $\mathbb{R}$.

**proof (a)** Let $V$ be an open neighbourhood of the identity which is totally bounded for the bilateral uniformity of $X$ (443H); we may suppose that $V^{-1}=V$. Set $Y=V\cup VV\cup VVV\cup VVVV\cup\ldots$. Then $Y$

is an open subgroup of $X$, therefore also closed (4A5Ek). By 4A5Ob, every power of $V$ is totally bounded, so $Y$ is a countable union of totally bounded sets. If $\mu$ is any left Haar measure on $X$, then any totally bounded set has finite outer measure for $\mu$ (443I). Thus $Y$ is a countable union of sets of finite measure for $\mu$. The same argument applies to right Haar measures, so $Y$ is a subgroup of the required form.

**(b)(i)** Let $E \subseteq X$ be a Haar measurable set, and fix a left Haar measure $\mu$ on $X$. Take the open subgroup $Y$ of (a), and index the set of its left cosets as $\langle Y_i\rangle_{i\in I}$; because any translate of a totally bounded set is totally bounded (4A5Ob again), each $Y_i$ is an open set expressible as $\bigcup_{n\in\mathbb{N}} H_{in}$, where every $H_{in}$ is a totally bounded open set, so that $\mu H_{in}$ is finite.

For $i \in I$ and $m$, $n \in \mathbb{N}$ there is a closed set $F_{imn} \subseteq E \cap H_{im}$ such that $\mu F_{imn} \ge \mu(E \cap H_{im}) - 2^{-n}$. Set $F_{mn} = \bigcup_{i\in I} F_{imn}$ for each $m$, $n \in \mathbb{N}$; then $F_{mn}$ is closed (4A2Bb). So $E' = \bigcup_{m,n\in\mathbb{N}} F_{mn}$ is $\mathrm{F}_\sigma$. For each $i \in I$,

$$(E \setminus E') \cap Y_i \subseteq \bigcup_{m\in\mathbb{N}}\bigl(E \cap H_{im} \setminus \bigcup_{n\in\mathbb{N}} F_{imn}\bigr)$$

is negligible; thus $\{G : G \subseteq X \text{ is open}, \mu(G \cap E \setminus E') = 0\}$ covers $X$ and $E \setminus E'$ must be negligible (414Ea).

In the same way, there is an $\mathrm{F}_\sigma$ set $F^* \subseteq X \setminus E$ such that $(X \setminus E) \setminus F^*$ is negligible; now $E'' = X \setminus F^*$ is $\mathrm{G}_\delta$ and $E'' \setminus E$ is negligible, so $E'' \setminus E'$ also is. (I am speaking here as if 'negligible' meant '$\mu$-negligible'. But of course this is the same thing as the 'Haar negligible' of the statement of the proposition.)

**(ii), (iii), (iv)** follow at once.

**443K Theorem** Let $X$ be a Hausdorff topological group carrying Haar measures. Then the completion $\widehat{X}$ of $X$ under its bilateral uniformity is a locally compact Hausdorff group, and $X$ is of full outer Haar measure in $\widehat{X}$. Any (left or right) Haar measure on $X$ is the subspace measure corresponding to a Haar measure (of the same chirality) on $\widehat{X}$.

**proof (a)** By 443H and 4A5N, $\widehat{X}$ is a locally compact Hausdorff group in which $X$ is embedded as a dense subgroup.

**(b)** Let $\mu$ be a left Haar measure on $X$. Then there is a Radon measure $\lambda$ on $\widehat{X}$ such that $\mu$ is the subspace measure $\lambda_X$. **P** For Borel sets $E \subseteq \widehat{X}$, set $\nu E = \mu(X \cap E)$. Then $\nu$ is a Borel measure, and it is $\tau$-additive because $\mu$ is. Any point of $\widehat{X}$ has a compact neighbourhood $V$ in $\widehat{X}$; now $V$ must be totally bounded for the bilateral uniformity of $\widehat{X}$ (4A2Je), so $V \cap X$ is totally bounded for the bilateral uniformity of $X$ (4A5Ma), and $\nu V = \mu(V \cap X)$ is finite (443I). Thus $\nu$ is locally finite. If $\nu E > 0$, there is an open set $H \subseteq X$ such that $\mu H < \infty$ and $\mu(H \cap X \cap E) > 0$, because $\mu$ is effectively locally finite; now there is an open set $G \subseteq \widehat{X}$ such that $H = X \cap G$, so that $\nu G < \infty$ and $\nu(G \cap E) > 0$. Thus $\nu$ is effectively locally finite.

By 416H, the c.l.d. version $\lambda$ of $\nu$ is a Radon measure on $\widehat{X}$. Since $\lambda K = \nu K = 0$ whenever $K \subseteq \widehat{X} \setminus X$ is compact, $X$ is of full outer measure for $\lambda$. Accordingly

$$\lambda_X(G \cap X) = \lambda G = \nu G = \mu(G \cap X)$$

for every open set $G \subseteq \widehat{X}$, and $\lambda_X = \mu$, because they are quasi-Radon measures agreeing on the open sets (415B, 415H(iii)). **Q**

**(c)** Continuing the argument of (b), $\lambda$ is a left Haar measure on $\widehat{X}$. **P** Let $G \subseteq \widehat{X}$ be open, and $z \in \widehat{X}$. If $K \subseteq zG$ is compact, then $z^{-1}K \subseteq G$, and $\{w : w \in \widehat{X}, wK \subseteq G\}$ is a non-empty open set (4A5Ei), so meets $X$. Take $x \in X$ such that $xK \subseteq G$; then

$$\lambda G = \mu(X \cap G) \ge \mu(X \cap xK) = \mu(x(X \cap K)) = \mu(X \cap K) = \lambda K.$$

As $K$ is arbitrary, $\lambda(zG) \le \lambda G$. By 441Ba, $\lambda$ is invariant under the left action of $\widehat{X}$ on itself, that is, is a left Haar measure. **Q**

We know that $X$ is of full outer measure for $\lambda$, so this shows that it has full outer Haar measure in $\widehat{X}$.

**(d)** The same arguments, looking at $Gz$ and $Kz^{-1}$ in (c), show that if $\mu$ is a right Haar measure on $X$ it is the subspace measure $\lambda_X$ for a right Haar measure $\lambda$ on $\widehat{X}$.

**443L Corollary** Let $X$ be any topological group with a Haar measure $\mu$. Then we can find $Z$, $\lambda$ and $\phi$ such that

(i) $Z$ is a locally compact Hausdorff topological group;

(ii) $\lambda$ is a Haar measure on $Z$;

(iii) $\phi : X \to Z$ is a continuous homomorphism, inverse-measure-preserving for $\mu$ and $\lambda$;

(iv) $\mu$ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$;

(v) if $E \subseteq X$ is Haar measurable, we can find a Haar measurable set $F \subseteq Z$ such that $\phi^{-1}[F] \subseteq E$ and $E \setminus \phi^{-1}[F]$ is Haar negligible;

(vi) a set $G \subseteq X$ is an open set in $X$ iff it is of the form $\phi^{-1}[H]$ for some open set $H \subseteq Z$;

(vii) a set $G \subseteq X$ is a regular open set in $X$ iff it is of the form $\phi^{-1}[H]$ for some regular open set $H \subseteq Z$;

(viii) a set $A \subseteq X$ is nowhere dense in $X$ iff $\phi[A]$ is nowhere dense in $Z$.

**proof (a)** Let $Y \subseteq X$ be the closure of $\{e\}$, where $e$ is the identity of $X$. Then $Y$ is a closed normal subgroup of $X$, and if $\phi : X \to X/Y$ is the quotient map, every open (or closed) subset of $X$ is of the form $\phi^{-1}[H]$ for some open (or closed) set $H \subseteq Y$ (4A5Kb).

Consider the image measure $\nu = \mu\phi^{-1}$ on $X/Y$. This is quasi-Radon. **P** Because $\mu$ is a complete $\tau$-additive topological measure, so is $\nu$. If $F \in \operatorname{dom}\nu$ and $\nu F > 0$, there is an open set $G \subseteq X$ such that $\mu G < \infty$ and $\mu(G \cap \phi^{-1}[F]) > 0$; now $G = \phi^{-1}[H]$ for some open set $H \subseteq X/Y$, and $\nu H = \mu G$ is finite, while $\nu(H \cap F) = \mu(G \cap \phi^{-1}[F]) > 0$. Thus $\nu$ is effectively locally finite (therefore semi-finite). Again, if $F \in \operatorname{dom}\nu$ and $\nu F > \gamma$, there is a closed set $E \subseteq \phi^{-1}[F]$ such that $\mu E \geq \gamma$; now $E$ is expressible as $\phi^{-1}[H]$ for some closed set $H \subseteq X/Y$; because $\phi$ is surjective, $H \subseteq F$, and $\nu H = \mu E \geq \gamma$. Thus $\nu$ is inner regular with respect to the closed sets. Finally, suppose that $F \subseteq X/Y$ is such that $F \cap F' \in \operatorname{dom}\nu$ whenever $\nu F' < \infty$. If $E \subseteq X$ is a closed set of finite measure, it is of the form $\phi^{-1}[F']$ where $\nu F' = \mu E < \infty$, so $F' \cap F \in \operatorname{dom}\nu$ and $E \cap \phi^{-1}[F] \in \operatorname{dom}\mu$; by 412Ja, we can conclude that $\phi^{-1}[F] \in \operatorname{dom}\mu$ and $F \in \operatorname{dom}\nu$. Thus $\nu$ is locally determined and is a quasi-Radon measure. **Q**

We find also that $\nu$ is a left Haar measure. **P** If $z \in X/Y$ and $F \in \operatorname{dom}\nu$, express $z$ as $\phi(x)$ where $x \in X$; then $\phi^{-1}[zF] = x\phi^{-1}[F]$, so

$$\nu(zF) = \mu(x\phi^{-1}[F]) = \mu\phi^{-1}[F] = \nu F. \text{ **Q**}$$

**(b)** Thus $X/Y$ is a topological group with a left Haar measure $\nu$. Because $Y$ is closed, $X/Y$ is Hausdorff (4A5J(b-ii-$\alpha$)). We can therefore form its completion $Z = \widehat{X/Y}$, a locally compact Hausdorff group, and find a left Haar measure $\lambda$ on $Z$ such that $\nu$ is the corresponding subspace measure on $X/Y$, which is of full outer measure for $\lambda$ (443K). The embedding $X/Y \subsetneqq Z$ is therefore inverse-measure-preserving for $\nu$ and $\lambda$, so that $\phi$, regarded as a map from $X$ to $Z$, is inverse-measure-preserving for $\mu$ and $\lambda$. Also, of course, $\phi : X \to Z$ is a continuous homomorphism.

If $E \subseteq X$ and $\mu E > \gamma$, there is a closed set $E' \subseteq E$ such that $\mu E' > \gamma$. Now $E'$ is of the form $\phi^{-1}[F]$ where $F \subseteq X/Y$ is closed and $\nu F = \mu E' > \gamma$. Next, $F$ is of the form $(X/Y) \cap F'$ where $F' \subseteq Z$ is closed and $\lambda F' = \nu F > \gamma$. So there is a compact set $K \subseteq F'$ such that $\lambda K \geq \gamma$, and we have

$$\phi^{-1}[K] \subseteq \phi^{-1}[F'] = \phi^{-1}[F] \subseteq E, \quad \mu\phi^{-1}[K] = \nu(K \cap (X/Y)) = \lambda K \geq \gamma.$$

As $E$ and $\gamma$ are arbitrary, $\mu$ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$. So (i)-(iv) are true.

**(c)** If $E \subseteq X$ is Haar measurable, then by 443J(b-i) there is an $\mathrm{F}_\sigma$ set $E' \subseteq E$ such that $E \setminus E'$ is Haar negligible. Now there are an $\mathrm{F}_\sigma$ set $H \subseteq X/Y$ such that $E' = \phi^{-1}[H]$ and an $\mathrm{F}_\sigma$ set $F \subseteq Z$ such that $H = (X/Y) \cap F$, in which case $E' = \phi^{-1}[F]$, and $E \setminus \phi^{-1}[F]$ is Haar negligible. This deals with (v).

**(d)** Concerning (vi)-(viii), we just have to put 4A2B and 4A5Kb together. If $G$, $A \subseteq X$, then

$$\begin{aligned} G \text{ is open in } X \iff & \text{ there is an open } V \subseteq X/Y \text{ such that } G = \phi^{-1}[V] \\ \iff & \text{ there are a } V \subseteq X/Y \text{ and an open } H \subseteq Z \\ & \qquad \text{ such that } G = \phi^{-1}[V] \text{ and } V = (X/Y) \cap H \\ \iff & \text{ there is an open } H \subseteq Z \text{ such that } G = \phi^{-1}[H]; \end{aligned}$$

$$G \text{ is a regular open set in } X \iff \phi[G] \text{ is a regular open subset of } X/Y$$

(4A5K(b-iii), 4A2B(f-iii))

$$\iff \text{there is a regular open } H \subseteq Z \text{ such that } \phi[G] = (X/Y) \cap H$$

(4A2B(j-ii))

$$\iff \text{there is a regular open } H \subseteq Z \text{ such that } G = \phi^{-1}[H];$$

$$A \text{ is nowhere dense in } X \iff \phi[A] \text{ is nowhere dense in } X/Y$$

(4A5K(iv)-(v))

$$\iff \phi[A] \text{ is nowhere dense in } Z$$

(4A2B(j-i)).

**443M Theorem** (Halmos 50) Let $X$ be a topological group and $\mu$ a Haar measure on $X$. Then $\mu$ is completion regular.

**proof (a)** Suppose first that $\mu$ is a left Haar measure and that $X$ is locally compact and Hausdorff. In this case any self-supporting compact set $K \subseteq X$ is a zero set. **P** For each $n \in \mathbb{N}$, there is an open neighbourhood $U_n$ of the identity such that $\mu(K \triangle xK) \le 2^{-n}$ for every $x \in U_n$ (443B); we may suppose that $\overline{U}_{n+1} \subseteq U_n$ for each $n$. Each set $U_n K$ is open (4A5Ed), so $\bigcap_{n\in\mathbb{N}} U_n K$ is a $G_\delta$ set. **?** If $K \ne \bigcap_{n\in\mathbb{N}} U_n K$, there is an $x \in \bigcap_{n\in\mathbb{N}} U_n K \setminus K$. For each $n \in \mathbb{N}$, there are $y_n \in U_n$, $z_n \in K$ such that $x = y_n z_n$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Then $z = \lim_{n\to\mathcal{F}} z_n$ is defined in $K$, so

$$xz^{-1} = \lim_{n\to\mathcal{F}} xz_n^{-1} = \lim_{n\to\mathcal{F}} y_n$$

is defined in $X$; because $y_n \in \overline{U}_i$ for every $i \le n$,

$$xz^{-1} \in \bigcap_{i\in\mathbb{N}} \overline{U}_i = \bigcap_{i\in\mathbb{N}} U_i.$$

Consequently $\mu(xz^{-1}K \triangle K) = 0$; because $\mu$ is left-translation-invariant, $\mu(K \setminus zx^{-1}K) = 0$. But as $x \notin K$, $z \in K \setminus zx^{-1}K$ and $K \cap (X \setminus zx^{-1}K)$ is non-empty. And $zx^{-1}K$ is closed, so $X \setminus zx^{-1}K$ is open and $K$ is not self-supporting, contrary to hypothesis. **X**

Thus $K = \bigcap_{n\in\mathbb{N}} U_n K$ is a $G_\delta$ set. Being a compact $G_\delta$ set in a completely regular Hausdorff space, it is a zero set (4A2F(h-v)). **Q**

Since $\mu$ is surely inner regular with respect to the compact self-supporting sets (414F), it is inner regular with respect to the zero sets, and is completion regular.

**(b)** Now suppose that $\mu$ is a left Haar measure on an arbitrary topological group $X$. By 443L, we can find a locally compact Hausdorff topological group $Z$, a continuous homomorphism $\phi : X \to Z$ and a left Haar measure $\lambda$ on $Z$ such that $\phi$ is inverse-measure-preserving for $\mu$ and $\lambda$ and $\mu$ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$. Now if $E \in \operatorname{dom}\mu$ and $\gamma < \mu E$, there is a compact set $K \subseteq Z$ such that $\phi^{-1}[K] \subseteq E$ and $\nu K > \gamma$. Next, there is a zero set $L \subseteq K$ such that $\nu L \ge \gamma$; in which case $\phi^{-1}[L] \subseteq E$ is a zero set and $\mu\phi^{-1}[L] \ge \gamma$. Thus $\mu$ is inner regular with respect to the zero sets and is completion regular.

**(c)** Finally, if $\mu$ is a right Haar measure on a topological group $X$, let $\overleftrightarrow{\mu}$ be the corresponding left Haar measure, setting $\overleftrightarrow{\mu}E = \mu E^{-1}$ for Haar measurable sets $E$. Then $\overleftrightarrow{\mu}$ is inner regular with respect to the zero sets; because $x \mapsto x^{-1} : X \to X$ is a homeomorphism, so is $\mu$.

**443N** I give a simple result showing how the measure-theoretic properties of groups carrying Haar measures have topological consequences which might not be expected.

**Proposition** Let $X$ be a topological group carrying Haar measures (for instance, $X$ might be any locally compact Hausdorff group).

(i) Let $G$ be a regular open subset of $X$. Then $G$ is a cozero set.

(ii) Let $F$ be a nowhere dense subset of $X$. Then $F$ is included in a nowhere dense zero set.

**proof (a)** Suppose to begin with that $X$ is locally compact, $\sigma$-compact and Hausdorff. Let $\mu$ be a left Haar measure on $X$; then $\mu$ is $\sigma$-finite, because $X$ is covered by a sequence of compact sets, which must all have finite measure.

**(i)** Write $\mathcal{U}$ for the family of open neighbourhoods of the identity $e$ of $X$. For each $U \in \mathcal{U}$, set $H_U = \operatorname{int}\{x : xU \subseteq G\}$; then $\{H_U : U \in \mathcal{U}\}$ is an upwards-directed family of open sets with union $G$, as in the proofs of 442Ab and 442B, so $G^\bullet = \sup_{U\in\mathcal{U}} H_U^\bullet$ in the measure algebra $\mathfrak{A}$ of $\mu$. Because $\mu$ is $\sigma$-finite, $\mathfrak{A}$ is ccc (322G) and there is a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that $G^\bullet = \sup_{n\in\mathbb{N}} H_{U_n}^\bullet$ (316E). In this case, $G \setminus \bigcup_{n\in\mathbb{N}} H_{U_n}$ is negligible, so must have empty interior.

By 4A5S, there is a closed normal subgroup $Y$ of $X$, included in $\bigcap_{n\in\mathbb{N}} U_n$, such that $X/Y$ is metrizable. Let $\pi : X \to X/Y$ be the canonical map.

For each $n \in \mathbb{N}$, $Q_n = \pi[H_{U_n}]$ is open (4A5J(a-i)), and

$$H_{U_n} \subseteq \pi^{-1}[Q_n] = H_{U_n}Y \subseteq H_{U_n}U_n \subseteq G.$$

So

$$\overline{G} = \overline{\bigcup_{n\in\mathbb{N}} H_{U_n}} = \overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}[Q_n]}.$$

Setting $Q = \operatorname{int}\overline{\bigcup_{n\in\mathbb{N}} Q_n}$, and using 4A5J(a-i) and 4A2B(f-ii), we see that

$$\pi^{-1}[Q] = \operatorname{int}\overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}[Q_n]} = \operatorname{int}\overline{G} = G$$

(this is where I use the hypothesis that $G$ is a regular open set). But $Q$, being an open set in a metrizable space, is a cozero set (4A2Lc), so $G = \pi^{-1}[Q]$ is a cozero set (4A2C(b-iv)), as required by (i).

**(ii)** Now consider the nowhere dense set $F \subseteq X$. This time, let $\mathcal{G}$ be a maximal disjoint family of cozero subsets of $X \setminus F$. Then $\mathcal{G}$ is countable, again because $\mu$ is $\sigma$-finite, and $\bigcup\mathcal{G}$ is dense, because the topology of $X$ is completely regular. So $X \setminus \bigcup\mathcal{G}$ is a nowhere dense zero set including $F$.

**(b)** Next, suppose that $X$ is any locally compact Hausdorff topological group. Then $X$ has a $\sigma$-compact open subgroup $X_0$ (4A5El). By (a), any regular open set in $X_0$ is a cozero set in $X_0$. The same applies to all the (left) cosets of $X_0$, because these are homeomorphic to $X_0$.

If $C$ is any coset of $X_0$, then $G \cap C$ is a regular open set in $C$, so is a cozero set in $C$. But as the left cosets of $X_0$ form a partition of $X$ into open sets, $G$ is also a cozero set in $X$ (4A2C(b-vii)).

Similarly, $F \cap C$ is nowhere dense in $C$ for every left coset $C$ of $X_0$, so is included in a nowhere dense zero set in $C$, and the union of these will be a nowhere dense zero set in $X$ including $F$.

**(c)** Now suppose that $X$ is any group carrying Haar measures. Let $Z$, $\lambda$ and $\phi : X \to Z$ be as in 443L. Then $G$ is expressible as $\phi^{-1}[H]$ for some regular open set $H \subseteq Z$ (443L(vii)); by (b), $H$ is a cozero set, so $G$ also is (4A2C(b-iv) again). As for $F$, $\phi[F]$ is nowhere dense in $Z$, by 443L(viii). Let $F' \supseteq \phi[F]$ be a nowhere dense zero set; then $\phi^{-1}[F'] \supseteq F$ is a zero set, and $\phi[\phi^{-1}[F']] \subseteq F'$ is nowhere dense, so $\phi^{-1}[F']$ is nowhere dense.

**443O** An expected result, well known for Lebesgue measure, but which seems to need a little attention for the non-metrizable case, is the following.

**Proposition** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Then the following are equiveridical:

(i) $\mu$ is not purely atomic;

(ii) $\mu$ is atomless;

(iii) there is a non-negligible nowhere dense subset of $X$;

(iv) $\mu$ is inner regular with respect to the nowhere dense sets;

(v) there is a conegligible meager subset of $X$;

(vi) there is a negligible comeager subset of $X$.

If $X$ is Hausdorff, we can add

(vii) the topology of $X$ is not discrete.

**proof** Write $\Sigma$ for the domain of $\mu$.

**(a)(i)⇒(ii)** If $\mu$ is not purely atomic, let $E \in \Sigma$ be a non-negligible set not including any atom. If $F \in \Sigma$ is any other non-negligible set, then there is an $x \in X$ such that $F \cap xE$ is not negligible, by 443Da. Now the subspace measures on $F \cap xE$ and $x^{-1}F \cap E$ are isomorphic, and the latter is atomless, so $F \cap xE$ is not an atom and $F$ is not an atom. As $F$ is arbitrary, $\mu$ is atomless.

**(b)(iii)⇒(iv)** The argument is similar. Suppose that $A$ is a non-negligible nowhere dense subset of $X$; then $E = \overline{A}$ is a non-negligible closed nowhere dense set. If $F \in \Sigma$ is non-negligible, there is an $x \in X$ such that $F \cap xE$ is non-negligible; as $y \mapsto xy : X \to X$ is a homeomorphism, $xE$ and $F \cap xE$ are nowhere dense. Thus every non-negligible measurable set includes a nowhere dense non-negligible measurable set; as the family of nowhere dense sets is an ideal, $\mu$ is inner regular with respect to the nowhere dense sets (412Aa).

**(c)(iv)⇒(i)** Suppose that $\mu$ is inner regular with respect to the nowhere dense sets. Let $U$ be an open neighbourhood of the identity $e$ of $X$ with finite measure (442Aa once more), and $V$ an open neighbourhood of $e$ such that $VV^{-1}V \subseteq U$. Then $\mu V > 0$, by the other half of 442Aa.

**?** If there is a $\mu$-atom $E \subseteq V$, let $F_0 \subseteq E$ be a non-negligible measurable nowhere dense set, $F_1 \subseteq F_0$ a non-negligible closed set and $F \subseteq F_1$ a non-empty self-supporting closed set (414F). Because $y \mapsto xy$ is a measure-preserving automorphism, $xF$ is a self-supporting closed set, and an atom for $\mu$, for every $x \in X$. So if $x$, $y \in X$ and $xF \cap yF$ is non-negligible, then $xF \setminus yF$ is negligible and $xF \subseteq yF$; similarly, $yF \subseteq xF$, so $xF = yF$. Now no finite number of translates of the nowhere dense set $F$ can cover the non-empty open set $V$, while $V \subseteq \bigcup_{x \in X} xF$, so we must have a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in $X$ such that $x_nF \cap V \setminus \bigcup_{i<n} x_iF \neq \emptyset$ for every $n \in \mathbb{N}$. In this case, $x_iF \cap x_nF$ is negligible whenever $i < n$, so

$$\mu(\bigcup_{n \in \mathbb{N}} x_nF) = \sum_{n \in \mathbb{N}} \mu(x_nF) = \infty.$$

However $V \cap x_nF \neq \emptyset$, so $x_n \in VF^{-1} \subseteq VV^{-1}$ and $x_nF \subseteq VV^{-1}V \subseteq U$ for every $n \in \mathbb{N}$; and $U$ is supposed to have finite measure. **X**

Accordingly $V$ does not include any atom and $\mu$ cannot be purely atomic.

**(d)(iv)⇒(v)** Suppose that $\mu$ is inner regular with respect to the nowhere dense sets. Let $\mathcal{G}$ be a maximal disjoint family of open sets of non-zero finite measure. Then $\operatorname{int}(X \setminus \bigcup \mathcal{G})$ is negligible, because $\mu$ is effectively locally finite, so must be empty, and $X \setminus \bigcup \mathcal{G}$ is nowhere dense. For each $G \in \mathcal{G}$, let $\langle F_{Gn} \rangle_{n \in \mathbb{N}}$ be a sequence of nowhere dense measurable subsets of $G$ such that $\mu G = \lim_{n \to \infty} \mu F_{Gn}$. For $n \in \mathbb{N}$, set $A_n = \bigcup_{G \in \mathcal{G}} F_{Gn}$. Then $A_n$ is nowhere dense. **P** If $H \subseteq X$ is open and not empty, either $H \cap A_n = \emptyset$ or there is a $G \in \mathcal{G}$ such that $H \cap G \neq \emptyset$, in which case $H \cap G \setminus \overline{F}_{Gn} \subseteq H \setminus A_n$ is open and non-empty. **Q** So $D = (X \setminus \bigcup \mathcal{G}) \cup \bigcup_{n \in \mathbb{N}} \overline{A}_n$ is meager. **?** If $D$ is not conegligible, let $H$ be an open set of finite measure such that $H \setminus D$ is non-negligible. As $H \setminus D \subseteq \bigcup \mathcal{G}$, there is a $G \in \mathcal{G}$ such that $\mu(H \cap G \setminus D) > 0$; but $H \cap G \setminus D \subseteq G \setminus \bigcup_{n \in \mathbb{N}} F_{Gn}$. **X**

Thus $D$ is a conegligible meager set.

**(e)(v)⇔(vi)** The complement of a witness for (v) witnesses (vi), and conversely.

**(f)(v)⇒(iii)** is elementary, since $\mu$ is non-zero.

**(g)(ii)⇒(iii)** Suppose that $\mu$ is atomless. I proceed through an expanding series of special cases, as in 443N.

**($\alpha$)** Suppose to begin with that $X$ is locally compact, Hausdorff and $\sigma$-compact. In this case $X$ has a closed negligible normal subgroup $Y$ such that $X/Y$ is separable and metrizable. **P** Since $\mu$ is atomless, $\{e\}$ must be negligible. Since $\mu$ is locally finite and inner regular with respect to the closed sets, there must be a sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of open neighbourhoods of $e$ such that $\inf_{n \in \mathbb{N}} \mu U_n = 0$. By 4A5S again, there is a closed normal subgroup $Y \subseteq \bigcap_{n \in \mathbb{N}} U_n$ such that $X/Y$ is metrizable, and of course $Y$ is negligible. Since the canonical map from $X$ onto $X/Y$ is continuous (4A5J(a-i) again), $X/Y$ is $\sigma$-compact, therefore separable (4A2Hd, 4A2Pd). **Q**

Write $\pi : X \to X/Y$ for the canonical map. Let $D \subseteq X/Y$ be a countable dense set, and consider $\pi^{-1}[D]$. This is a countable union of translates of $Y$, so is negligible; let $F \subseteq X \setminus \pi^{-1}[D]$ be a closed non-negligible set. Then $\pi[F]$ does not meet $D$. Because $\pi$ is an open mapping (4A5Ja once more), $\operatorname{int} F = \emptyset$ and $F$ is nowhere dense.

Thus in this case we have a non-negligible nowhere dense set, as required.

**($\beta$)** Now suppose just that $X$ is locally compact and Hausdorff. In this case it has an open $\sigma$-compact subgroup $X_0$ say. The subspace measure $\mu_{X_0}$ on $X_0$ is a left Haar measure on $X_0$ (443F), and is atomless; by ($\alpha$), there is a nowhere dense set $F \subseteq X_0$ such that $0 < \mu_{X_0} F = \mu F$. So in this case too we have a non-negligible nowhere dense set.

**($\gamma$)** For the general case, let $Z$, $\lambda$ and $\phi : X \to Z$ be as in 443L. Then $\lambda$ is atomless. **P** 443L(v) implies that the measure-preserving Boolean homomorphism from the measure algebra of $\lambda$ to the measure algebra of $\mu$ induced by $\phi$ is surjective, therefore an isomorphism; so both measure algebras are atomless and $\lambda$ is atomless (322Bg). **Q**

By ($\beta$), there is a nowhere dense non-negligible subset $H$ of $Z$; replacing $H$ by its closure, if necessary, we may suppose that $H$ is closed. Set $F = \phi^{-1}[H]$; then $F \subseteq X$ is closed and non-negligible because $\phi$ is continuous and inverse-measure-preserving. Since $\phi[F] \subseteq H$ is nowhere dense, so is $F$ (443L(viii)). Thus we have the required non-negligible nowhere dense set in the general case also.

**(h)** Now suppose that $X$ is Hausdorff. If $\mu$ is atomless, then $\mu\{x\} = 0$ for every $x$, so $\{x\}$ is never open and the topology is not discrete. If $\mu$ has an atom $E$, let $F \subseteq E$ be a closed self-supporting set of non-zero measure; then $F$ also is an atom, so cannot have two disjoint non-empty relatively open sets, and must be a singleton. Thus we have an $x_0$ such that $\mu\{x_0\} > 0$; as $\mu$ is left-translation-invariant, $\mu\{x\} = \mu\{x_0\}$ for every $x \in X$. We know also that there is an open set $G$ of non-zero finite measure, which must be finite; so every singleton subset of $G$ is open. It follows that every singleton subset of $X$ is open, and $X$ has its discrete topology. Thus (ii)$\Leftrightarrow$(vii) when $X$ is Hausdorff.

**443P Quotient spaces** I come now to the relationship between the modular functions of §442, normal subgroups and quotient spaces.

**Lemma** Let $X$ be a locally compact Hausdorff topological group and $Y$ a closed subgroup of $X$. Let $Z = X/Y$ be the set of left cosets of $Y$ in $X$ with the quotient topology and $\pi : X \mapsto Z$ the canonical map, so that $Z$ is a locally compact Hausdorff space and we have a continuous action of $X$ on $Z$ defined by writing $a \bullet \pi x = \pi(ax)$ for $a$, $x \in X$ (4A5J(b-iii)). Let $\nu$ be a left Haar measure on $Y$ and write $C_k(X)$, $C_k(Z)$ for the spaces of continuous real-valued functions with compact supports on $X$, $Z$ respectively.

(a) We have a positive linear operator $T : C_k(X) \to C_k(Z)$ defined by writing

$$(Tf)(\pi x) = \int_Y f(xy)\nu(dy)$$

for every $f \in C_k(X)$ and $x \in X$. If $f > 0$ in $C_k(X)$ then $Tf > 0$ in $C_k(Z)$. If $h \geq 0$ in $C_k(Z)$ then there is an $f \geq 0$ in $C_k(X)$ such that $Tf = h$.

(b) If $a \in X$ and $f \in C_k(X)$, then $T(a \bullet_l f)(z) = (Tf)(a^{-1} \bullet z)$ for every $z \in Z$.

(c) Now suppose that $a$ belongs to the normalizer of $Y$ (that is, $aYa^{-1} = Y$). In this case, we can define $\psi(a) \in ]0, \infty[$ by the formula

$$\nu(aFa^{-1}) = \psi(a)\nu F \text{ for every } F \in \operatorname{dom} \nu,$$

and

$$T(a \bullet_r f)(\pi x) = \psi(a) \cdot (Tf)(\pi(xa))$$

for every $x \in X$ and $f \in C_k(X)$.

**proof** I should begin by remarking that because $Y$ is a closed subgroup of a locally compact Hausdorff group, it is itself a locally compact Hausdorff group, so does have a left Haar measure, which is a Radon measure.

**(a)(i)** The first thing to check is that if $f \in C_k(X)$ then $Tf$ is well-defined as a member of $\mathbb{R}^Z$. **P** ($\alpha$) If $x \in X$, then $y \mapsto f(xy) : Y \to \mathbb{R}$ is a continuous function with compact support, so $\int f(xy)\nu(dy)$ is defined in $\mathbb{R}$. ($\beta$) If $x_1$, $x_2 \in X$ and $\pi x_1 = \pi x_2$, then $x_1^{-1}x_2 \in Y$, and

$$\int f(x_2y)\nu(dy) = \int f(x_1(x_1^{-1}x_2y))\nu(dy) = \int f(x_1y)\nu(dy),$$

applying 441J to the function $y \mapsto f(x_1y)$ and the left action of $Y$ on itself. Thus we can safely write $(Tf)(\pi x) = \int f(xy)\nu(dy)$ for every $x \in X$, and $Tf$ will be a real-valued function on $Z$. **Q**

**(ii)** Now $Tf$ is continuous for every $f \in C_k(X)$. **P** Given $z_0 \in Z$, take $x_0 \in X$ such that $z = \pi x_0$. We have an $h \in C_k(X)^+$ such that for every $\epsilon > 0$ there is an open set $U_\epsilon$ containing $x_0$ such that $|f(x_0y) - f(xy)| \le \epsilon h(y)$ whenever $x \in U_\epsilon$ and $y \in X$ (4A5Pb). In this case,

$$|(Tf)(\pi x) - (Tf)(\pi x_0)| = |\int f(xy) - f(x_0y)\nu(dy)| \le \epsilon\int_Y h(y)\nu(dy)$$

for every $x \in U_\epsilon$, so that $|(Tf)(z) - (Tf)(z_0)| \le \epsilon\int_Y h\,d\nu$ for every $z \in \pi[U_\epsilon]$. Since each $\pi[U_\epsilon]$ is an open neighbourhood of $z_0$ (4A5J(a-i), as always), $Tf$ is continuous at $z_0$; as $z_0$ is arbitrary, $Tf$ is continuous. **Q**

**(iii)** Since

$$\begin{aligned}\{z : (Tf)(z) \ne 0\} &= \{\pi x : \int f(xy)\nu(dy) \ne 0\} \subseteq \{\pi x : f(xy) \ne 0 \text{ for some } y \in Y\}\\ &= \{\pi x : f(x) \ne 0\} \subseteq \pi[\overline{\{x : f(x) \ne 0\}}]\end{aligned}$$

is relatively compact, $Tf \in C_k(Z)$ for every $f \in C_k(X)$.

**(iv)** The formula for $Tf$ makes it plain that $T : C_k(X) \to C_k(Z)$ is a positive linear operator.

**(v)** If $f \in C_k(X)^+$ and $x \in X$ are such that $f(x) > 0$, then $\{y : y \in Y,\ f(xy) > 0\}$ is a non-empty open set in $Y$; because $\nu$ is strictly positive,

$$(Tf)(\pi x) = \int f(xy)\nu(dy) > 0.$$

In particular, $Tf > 0$ if $f > 0$. Moreover, if $z \in Z$ there is an $f \in C_k(X)^+$ such that $(Tf)(z) > 0$. Now

$$\{\{z : (Tf)(z) > 0\} : f \in C_k(X)^+\}$$

is an upwards-directed family of open subsets of $Z$, so if $L \subseteq Z$ is any compact set there is an $f \in C_k(X)^+$ such that $(Tf)(z) > 0$ for every $z \in L$.

**(vi)** Now suppose that $h \in C_k(Z)^+$. By (v), there is an $f_0 \in C_k(X)^+$ such that $(Tf_0)(z) > 0$ whenever $z \in \overline{\{w : h(w) \ne 0\}}$. Setting $h'(z) = h(z)/(Tf_0)(z)$ when $h(z) \ne 0$, 0 for other $z \in Z$, we have $h' \in C_k(Z)$ and $h = h' \times Tf_0$. Set

$$f(x) = f_0(x)h'(\pi x) \ge 0$$

for every $x \in X$. Because $h'$ and $\pi$ are continuous, $f \in C_k(X)$. For any $x \in X$,

$$\begin{aligned}(Tf)(\pi x) &= \int f_0(xy)h'(\pi(xy))\nu(dy) = h'(\pi x)\int f_0(xy)\nu(dy)\\ &= h'(\pi x)(Tf_0)(\pi x) = h(\pi x).\end{aligned}$$

Thus $Tf = h$.

**(b)** If $z = \pi x$, then $a^{-1}\bullet z = \pi(a^{-1}x)$, so

$$(Tf)(a^{-1}\bullet z) = \int f(a^{-1}xy)\nu(dy) = \int (a\bullet_l f)(xy)\nu(dy) = T(a\bullet_l f)(z).$$

**(c)** Define $\phi : Y \to Y$ by writing $\phi(y) = a^{-1}ya$ for $y \in Y$. Because $\phi$ is a homeomorphism, the image measure $\nu\phi^{-1}$ is a Radon measure on $Y$; because $\phi$ is a group automorphism, $\nu\phi^{-1}$ is a left Haar measure. (If $F \in \operatorname{dom}\nu\phi^{-1}$ and $y \in Y$, then

$$\nu\phi^{-1}[yF] = \nu(ayFa^{-1}) = \nu(Fa^{-1}) = \nu(aFa^{-1}) = \nu\phi^{-1}[F].)$$

$\nu\phi^{-1}$ must therefore be a multiple of $\nu$; say $\nu\phi^{-1} = \psi(a)\nu$.

If $g \in C_k(Y)$, then

$$\int g(a^{-1}ya)\nu(dy) = \int g\phi\,d\nu = \int g\,d(\nu\phi^{-1}) = \psi(a)\int g\,d\nu.$$

Now take $f \in C_k(X)$. Then

$$T(a\bullet_r f)(\pi x) = \int (a\bullet_r f)(xy)\nu(dy) = \int f(xya)\nu(dy)$$
$$= \int f(xa(a^{-1}ya))\nu(dy) = \psi(a)\int f(xay)\nu(dy)$$

(using the remark above with $g(y) = f(xay)$)

$$= \psi(a)(Tf)(\pi(xa))$$

for every $x \in X$, as claimed.

**443Q Theorem** Let $X$ be a locally compact Hausdorff topological group and $Y$ a closed subgroup of $X$. Let $Z = X/Y$ be the set of left cosets of $Y$ in $X$ with the quotient topology, and $\pi : X \to Z$ the canonical map, so that $Z$ is a locally compact Hausdorff space and we have a continuous action of $X$ on $Z$ defined by writing $a\bullet\pi x = \pi(ax)$ for $a$, $x \in X$. Let $\nu$ be a left Haar measure on $Y$. Suppose that $\lambda$ is a non-zero $X$-invariant Radon measure on $Z$.

(a) For each $z \in Z$, we have a Radon measure $\nu_z$ on $X$ defined by the formula

$$\nu_z E = \nu(Y \cap x^{-1}E)$$

whenever $\pi x = z$ and the right-hand side is defined. In this case, for a real-valued function $f$ defined on a subset of $X$,

$$\int f\,d\nu_z = \int f(xy)\nu(dy)$$

whenever either side is defined in $[-\infty,\infty]$.

(b) We have a left Haar measure $\mu$ on $X$ defined by the formulae

$$\int f\,d\mu = \iint f\,d\nu_z\lambda(dz)$$

for every $f \in C_k(X)$, and

$$\mu G = \int \nu_z G\,\lambda(dz)$$

for every open set $G \subseteq X$.

(c) If $D \subseteq Z$, then $D \in \operatorname{dom}\lambda$ iff $\pi^{-1}[D] \subseteq X$ is Haar measurable, and $\lambda D = 0$ iff $\pi^{-1}[D]$ is Haar negligible.

(d) If $\nu Y = 1$, then $\lambda$ is the image measure $\mu\pi^{-1}$.

(e) Suppose now that $X$ is $\sigma$-compact. Then $\mu E = \int \nu_z E\,\lambda(dz)$ for every Haar measurable set $E \subseteq X$. If $f \in \mathcal{L}^1(\mu)$, then $\int f\,d\mu = \iint f\,d\nu_z\lambda(dz)$.

(f) Still supposing that $X$ is $\sigma$-compact, take $f \in \mathcal{L}^1(\mu)$, and for $a \in X$ set $f_a(y) = f(ay)$ whenever $y \in Y$ and $ay \in \operatorname{dom} f$. Then $Q_f = \{a : a \in X,\ f_a \in \mathcal{L}^1(\nu)\}$ is $\mu$-conegligible, and the function $a \mapsto f_a^\bullet : Q_f \to L^1(\nu)$ is almost continuous.

**proof (a)** First, we do have a function $\nu_z$ depending only on $z$, because if $z = \pi x_1 = \pi x_2$ then $x_2^{-1}x_1 \in Y$, so

$$\nu(Y \cap x_1^{-1}E) = \nu(x_2^{-1}x_1(Y \cap x_1^{-1}E)) = \nu(Y \cap x_2^{-1}E)$$

whenever either side is defined. Of course $\nu_z$, being the image of the Radon measure $\nu$ under the continuous map $y \mapsto xy : Y \to X$ whenever $\pi x = z$, is always a Radon measure on $X$ (418I). We also have

$$\int_Y f(xy)\nu(dy) = \int_X f\,d\nu_{\pi x}$$

whenever $x \in X$ and $f$ is a real-valued function such that either side is defined in $[-\infty,\infty]$, by 235J.

I remark here that if $z \in Z$ then the coset $C = \pi^{-1}[\{z\}]$ is $\nu_z$-conegligible, because if $\pi x = z$ then $Y = Y \cap x^{-1}C$.

**(b)(i)** Let $T : C_k(X) \to C_k(Z)$ be the positive linear operator of 443P; that is,

$$(Tf)(z) = \int f(xy)\nu(dy) = \int f\,d\nu_z$$

whenever $f \in C_k(X)$, $x \in X$ and $z = \pi x$. Then we have a positive linear functional $\theta : C_k(X) \to \mathbb{R}$ defined by setting $\theta(f) = \int Tf\,d\lambda$ for every $f \in C_k(X)$. By the Riesz Representation Theorem (436J), there is a Radon measure $\mu$ on $X$ defined by saying that $\int f\,d\mu = \theta(f)$ for every $f \in C_k(X)$. Note that $\mu$ is non-zero. **P** Because $\lambda$ is non-zero, there is some $h \in C_k(Z)^+$ such that $\int h\,d\lambda \neq 0$; now there is some $f \in C_k(X)$ such that $Tf = h$, by 443Pa, and $\int f\,d\mu \neq 0$. **Q**

**(ii)** $\mu$ is a left Haar measure. **P** If $f \in C_k(X)$ and $a \in X$, then we have $T(a\bullet_l f)(z) = (Tf)(a^{-1}\bullet z)$ for every $z \in Z$, by 443Pb. So

$$\int a\bullet_l f d\mu = \int T(a\bullet_l f)d\lambda = \int Tf(a^{-1}\bullet z)\lambda(dz) = \int Tf(z)\lambda(dz)$$

(by 441J or 441L, because $\lambda$ is $X$-invariant)

$$= \int f\,d\mu.$$

By 441L in the other direction, $\mu$ is invariant under the left action of $X$ on itself, that is, is a left Haar measure. **Q**

**(iii)** If $G \subseteq X$ is open then $\mu G = \int \nu_z G\,\lambda(dz)$. **P** Set $A = \{f : f \in C_k(X),\ 0 \le f \le \chi G\}$. Then $\mu G = \sup_{f\in A} \int f\,d\mu$ and

$$\nu_z G = \sup_{f\in A} \int f\,d\nu_z = \sup_{f\in A}(Tf)(z)$$

for every $z \in Z$, by 414Ba, because $\nu_z$ is $\tau$-additive. But as $Tf$ is continuous for every $f \in A$, and $\lambda$ is $\tau$-additive, we also have

$$\int \nu_z G\,\lambda(dz) = \sup_{f\in A} \int Tf\,d\lambda = \sup_{f\in A} \int f\,d\mu = \mu G. \textbf{ Q}$$

**(c)(i)** Let $\mathcal{A}$ be the family of those sets $A \subseteq X$ such that $\mu A$ and $\int \nu_z(A)\lambda(dz)$ are defined in $[0,\infty]$ and equal. Then $\bigcup_{n\in\mathbb{N}} A_n$ belongs to $\mathcal{A}$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$, and $A \setminus B \in \mathcal{A}$ whenever $A$, $B \in \mathcal{A}$, $B \subseteq A$ and $\mu A < \infty$. Moreover, if $A \in \mathcal{A}$ and $\mu A = 0$, then every subset of $A$ belongs to $\mathcal{A}$, since $A$ must be $\nu_z$-negligible for $\lambda$-almost every $z$. We also know from (b) that every open set belongs to $\mathcal{A}$.

Applying the Monotone Class Theorem (136B) to $\{A : A \in \mathcal{A},\ A \subseteq G\}$, we see that if $E \subseteq X$ is a Borel set included in an open set $G$ of finite measure, then $E \in \mathcal{A}$. So if $E$ is a relatively compact Haar measurable set, $E \in \mathcal{A}$ (using 443J(b-i), or otherwise).

**(ii)** If $D \in \operatorname{dom}\lambda$ then $\pi^{-1}[D] \in \operatorname{dom}\mu$. **P** Let $K \subseteq X$ be compact. Then $\pi[K] \subseteq Z$ is compact, so there are Borel sets $F_1$, $F_2 \subseteq Z$ such that $F_1 \subseteq D \cap \pi[K] \subseteq F_2$ and $F_2 \setminus F_1$ is $\lambda$-negligible. Now $\nu_z(K \cap \pi^{-1}[F_2 \setminus F_1]) = 0$ whenever $z \notin F_2 \setminus F_1$, by the remark added to the proof of (a) above, so

$$\mu(K \cap \pi^{-1}[F_2 \setminus F_1]) = \int \nu_z(K \cap \pi^{-1}[F_2 \setminus F_1])\lambda(dz) = 0.$$

Since

$$K \cap \pi^{-1}[F_1] \subseteq K \cap \pi^{-1}[D] \subseteq K \cap \pi^{-1}[F_2],$$

$K \cap \pi^{-1}[D] \in \operatorname{dom}\mu$. As $K$ is arbitrary, $\pi^{-1}[D]$ is measured by $\mu$, so is Haar measurable. **Q**

If $\lambda D = 0$ then the same arguments show that $\mu(K \cap \pi^{-1}[D]) = 0$ for every compact $K \subseteq X$, so that $\mu\pi^{-1}[D] = 0$ and $\pi^{-1}[D]$ is Haar negligible.

**(iii)** Now suppose that $D \subseteq Z$ is such that $\pi^{-1}[D] \in \operatorname{dom}\mu$. Let $L \subseteq Z$ be compact. Then there is a relatively compact open set $G \subseteq X$ such that $\pi[G] \supseteq L$ (because $\{\pi[G] : G \subseteq X$ is open and relatively compact$\}$ is an upwards-directed family of open sets covering $Z$). In this case,

$$\int \nu_z(G \cap \pi^{-1}[D \cap L])\lambda(dz) = \mu(G \cap \pi^{-1}[D] \cap \pi^{-1}[L])$$

is well-defined, by (i). But if $z = \pi x$ then

$$\begin{aligned}\nu_z(G \cap \pi^{-1}[D \cap L]) &= 0 \text{ if } z \notin D \cap L,\\ &= \nu_z G = \nu(Y \cap x^{-1}G) > 0 \text{ if } z \in D \cap L,\end{aligned}$$

because if $z \in L$ then $\pi x \in \pi[G]$ and $Y \cap x^{-1}G \neq \emptyset$. So

$$D \cap L = \{z : \nu_z(G \cap \pi^{-1}[D \cap L]) > 0\}$$

is measured by $\lambda$. As $L$ is arbitrary, and $\lambda$ is a Radon measure, $D \in \operatorname{dom} \lambda$.

**(iv)** If $\pi^{-1}[D]$ is Haar negligible, then, in (iii) above, we shall have $\int \nu_z(G \cap \pi^{-1}[D \cap L])\lambda(dz) = 0$, so that $\lambda(D \cap L) = 0$; as $L$ is arbitrary, $\lambda D = 0$, by 412Ib or 412Jc.

**(d)** If $\nu Y = 1$, then, for any open set $H \subseteq Z$, $\nu_z \pi^{-1}[H] = 1$ if $z \in H$, 0 otherwise. So

$$\mu\pi^{-1}[H] = \int \nu_z(\pi^{-1}[H])\lambda(dz) = \lambda H.$$

Thus $\lambda$ and the image measure $\mu\pi^{-1}$ agree on the open sets and, being Radon measures (418I again), must be equal (416E(b-iii)).

**(e)** If $X$ is actually $\sigma$-compact, then (c)(i) of this proof tells us that $\mu E = \int \nu_z E\, \lambda(dz)$ for every Haar measurable set $E \subseteq X$, since $E$ is the union of an increasing sequence of relatively compact measurable sets. Consequently $\int f\, d\mu = \iint f\, d\nu_z \lambda(dz)$ for every $\mu$-simple function $f$. Now suppose that $f$ is a non-negative $\mu$-integrable function. Then there is a non-decreasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative $\mu$-simple functions converging to $f$ everywhere in $\operatorname{dom} f$. If we set $A = \{z : \nu_z^*(X \setminus \operatorname{dom} f) > 0\}$, then

$$\int \nu_z(X \setminus \operatorname{dom} f)\lambda(dz) = \mu(X \setminus \operatorname{dom} f) = 0,$$

so $\lambda A = 0$. Since $\int f\, d\nu_z = \lim_{n\to\infty} \int f_n d\nu_z$ for every $z \in Z \setminus A$,

$$\begin{aligned}\int f\, d\mu &= \lim_{n\to\infty} \int f_n\, d\mu \\ &= \lim_{n\to\infty} \iint f_n d\nu_z \lambda(dz) = \iint f\, d\nu_z \lambda(dz).\end{aligned}$$

Applying this to the positive and negative parts of $f$, we see that the same formula is valid for any $\mu$-integrable function $f$.

**(f)(i)** If $f \in \mathcal{L}^1(\mu)$ and $a \in X$, then

$$\int f_a d\nu = \int f(ay)\nu(dy) = \int f\, d\nu_{\pi a}$$

if any of these are defined. So if $f$, $g \in \mathcal{L}^1(\mu)$, $\|f_a - g_a\|_1 = \int |f - g| d\nu_{\pi a}$ if either is defined.

**(ii)** Let $\Phi$ be the set of all almost continuous functions from $\mu$-conegligible subsets of $X$ to $L^1(\nu)$, where $L^1(\nu)$ is given its norm topology. (In terms of the definition in 411M, a member $\phi$ of $\Phi$ is to be almost continuous with respect to the subspace measure on $\operatorname{dom} \phi$.) If $\phi \in \Phi$ and $\psi$ is a function from a conegligible subset of $X$ to $L^1(\nu)$ which is equal almost everywhere to $\phi$, then $\psi \in \Phi$. If $\langle \phi_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\Phi$ converging $\mu$-almost everywhere to $\psi$, then $\psi \in \Phi$ (418F).

**(iii)** For $f \in \mathcal{L}^1(\mu)$, set $\phi_f(a) = f_a^\bullet$ whenever this is defined in $L^1(\nu)$. Set $M = \{f : f \in \mathcal{L}^1(\mu),\ \phi_f \in \Phi\}$. If $\langle f^{(n)} \rangle_{n\in\mathbb{N}}$ is a sequence in $M$, $f \in \mathcal{L}^1(\mu)$ and $\|f^{(n)} - f\|_1 \leq 4^{-n}$ for every $n$, then $f \in M$. **P** Set $g = \sum_{n=0}^\infty 2^n |f^{(n)} - f|$, defined on

$$\{x : x \in \operatorname{dom} f \cap \bigcap_{n\in\mathbb{N}} \operatorname{dom} f^{(n)},\ \sum_{n=0}^\infty 2^n |f^{(n)}(x) - f(x)| < \infty\};$$

then $g \in \mathcal{L}^1(\mu)$. Now

$$D = \{z : z \in Z,\ g \text{ is } \nu_z\text{-integrable}\}$$

is $\lambda$-conegligible, by (e), and $E = \{a : a \in X,\ \pi a \in D\}$ is $\mu$-conegligible, by (c).

If $a \in E$, then

$$|f_a^{(n)} - f_a| \leq 2^{-n} g\ \nu_{\pi a}\text{-a.e.}$$

for every $n \geq 1$. So

$$\|\phi_f(a) - \phi_{f^{(n)}}(a)\|_1 = \int_Y |f_a - f_a^{(n)}|d\nu = \int_X |f - f^{(n)}|d\nu_{\pi a}$$
$$\leq 2^{-n} \int g\, d\nu_{\pi a} \to 0$$

as $n \to \infty$. Thus $\phi_f = \lim_{n\to\infty} \phi_{f^{(n)}}$ almost everywhere, and $\phi \in \Phi$, by (ii). **Q**

**(iv)** $C_k(X) \subseteq M$. **P** If $f \in C_k(X)$ and $a_0 \in X$, then there is an $h \in C_k(X)^+$ such that for every $\epsilon > 0$ there is an open set $G_\epsilon$ containing $a_0$ such that $|f(a_0y) - f(ay)| \leq \epsilon h(y)$ whenever $a \in G_\epsilon$ and $y \in X$ (4A5Pb again). In this case,

$$\|\phi_f(a_0) - \phi_f(a)\|_1 = \int |f(a_0y) - f(ay)|\nu(dy) \leq \epsilon \int_Y h\, d\nu$$

whenever $a \in G_\epsilon$. As $\epsilon$ is arbitrary, $\phi_f$ is continuous at $a_0$; as $a_0$ is arbitrary, $\phi_f$ is continuous, and $f \in M$. **Q**

**(v)** Now take any $f \in \mathcal{L}^1(\mu)$. Then for each $n \in \mathbb{N}$ we can find $f^{(n)} \in C_k(X)$ such that $\|f^{(n)} - f\|_1 \leq 4^{-n}$ (416I), so $f \in M$, by (iii). This completes the proof.

**443R Theorem** Let $X$ be a locally compact Hausdorff topological group and $Y$ a closed subgroup of $X$. Let $Z = X/Y$ be the set of left cosets of $Y$ in $X$ with the quotient topology, so that $Z$ is a locally compact Hausdorff space and we have a continuous action of $X$ on $Z$ defined by writing $a{\bullet}(xY) = axY$ for $a$, $x \in X$. Let $\Delta_X$ be the left modular function of $X$ and $\Delta_Y$ the left modular function of $Y$. Then the following are equiveridical:

(i) there is a non-zero $X$-invariant Radon measure $\lambda$ on $Z$;

(ii) $\Delta_Y$ is the restriction of $\Delta_X$ to $Y$.

**proof** Fix a left Haar measure $\nu$ on $Y$, and let $T : C_k(X) \to C_k(Z)$ be the corresponding linear operator as defined in 443Pa.

**(a)(i)⇒(ii)** Suppose that $\lambda$ is a non-zero $X$-invariant Radon measure on $Z$. Construct a left Haar measure $\mu$ on $X$ as in 443Q. In the notation of part (b-i) of the proof of 443Q, we have

$$\int f d\mu = \iint f d\nu_z \lambda(dz) = \int Tf d\lambda$$

for every $f \in C_k(X)$.

Suppose that $a \in Y$. In this case, $a$ surely belongs to the normalizer of $Y$, and, in the language of 443Pc, we have $\nu(aFa^{-1}) = \psi(a)\nu F$ for every $F \in \operatorname{dom}\nu$. But as

$$\nu(aFa^{-1}) = \nu(Fa^{-1}) = \Delta_Y(a^{-1})\nu F,$$

we must have $\psi(a) = \Delta_Y(a^{-1})$.

Fix some $f > 0$ in $C_k(X)$. We have

$$T(a{\bullet_r}f)(\pi x) = \psi(a) \cdot (Tf)(\pi(xa)) = \psi(a) \int f(xay)\nu(dy) = \psi(a) \int f(xy)\nu(dy)$$

(because $a \in Y$)

$$= \psi(a) \cdot (Tf)(\pi x)$$

for every $x$, so that (using 442Kc) we have

$$\Delta_X(a^{-1}) \int f\, d\mu = \int a{\bullet_r} f d\mu = \int T(a{\bullet_r} f)d\lambda$$
$$= \psi(a) \int Tf\, d\lambda = \psi(a) \int f\, d\mu = \Delta_Y(a^{-1}) \int f\, d\mu.$$

As $\int f\, d\mu > 0$, we must have $\Delta_X(a^{-1}) = \Delta_Y(a^{-1})$; as $a$ is arbitrary, $\Delta_Y = \Delta_X \restriction Y$, as required by (ii).

**(b)(ii)⇒(i)** Now suppose that $\Delta_Y = \Delta_X \restriction Y$. This time, start with a left Haar measure $\mu$ on $X$.

**($\alpha$)** (The key.) If $f \in C_k(X)$ is such that $Tf \ge 0$ in $C_k(Z)$, then $\int f\,d\mu \ge 0$. **P** There is an $h \in C_k(Z)^+$ such that $h(z) = 1$ whenever $z \in Z$ and $(Tf)(z) \ne 0$; by 443Pa, we can find a $g \in C_k(X)^+$ such that $Tg = h$. Now observe that $x \mapsto (Tf)(\pi x)$ is a non-negative continuous real-valued function on $X$, so

$$0 \le \int_X g(x)(Tf)(\pi x)\mu(dx)$$

$$= \int_X g(x)\int_Y f(xy)\nu(dy)\mu(dx) = \int_Y\int_X g(x)f(xy)\mu(dx)\nu(dy)$$

(by 417Ga or 417Gb, because $(x, y) \mapsto g(x)f(xy) : X \times Y \to \mathbb{R}$ is a continuous function with compact support)

$$= \int_Y \Delta_X(y^{-1})\int_X g(xy^{-1})f(x)\mu(dx)\nu(dy)$$

(applying 442Kc to the function $x \mapsto g(xy^{-1})f(x)$)

$$= \int_X f(x)\int_Y \Delta_X(y^{-1})g(xy^{-1})\nu(dy)\mu(dx)$$

(because $(x, y) \mapsto \Delta_X(y^{-1})g(xy^{-1})f(x)$ is continuous and has compact support)

$$= \int_X f(x)\int_Y \Delta_Y(y^{-1})g(xy^{-1})\nu(dy)\mu(dx)$$

(because $\Delta_X{\restriction}Y = \Delta_Y$, by hypothesis)

$$= \int_X f(x)\int_Y g(xy)\nu(dy)\mu(dx)$$

(applying 442K(b-ii) to the function $y \mapsto g(xy)$)

$$= \int_X f(x)(Tg)(\pi x)\mu(dx) = \int_X f(x)\mu(dx)$$

because $(Tg)(\pi x) = 1$ whenever $f(x) \ne 0$. **Q**

**($\beta$)** Applying this to $f$ and $-f$, we see that $\int f\,d\mu = 0$ whenever $Tf = 0$, so that $\int f\,d\mu = \int g\,d\mu$ whenever $f, g \in C_k(X)$ and $Tf = Tg$. Accordingly (because $T$ is surjective) we have a functional $\theta : C_k(Z) \to \mathbb{R}$ defined by saying that $\theta(Tf) = \int f\,d\mu$ whenever $f \in C_k(X)$, and $\theta$ is positive and linear. By the Riesz Representation Theorem again, there is a Radon measure $\lambda$ on $Z$ such that $\theta(h) = \int h\,d\lambda$ for every $h \in C_k(Z)$.

If $a \in X$ and $h \in C_k(Z)$ take $f \in C_k(X)$ such that $Tf = h$. Then, for any $x \in X$,

$$(Tf)(a{\bullet}\pi x) = (Tf)(\pi(ax)) = \int f(axy)\nu(dy) = \int (a^{-1}{\bullet_l}f)(xy)\nu(dy) = T(a^{-1}{\bullet_l}f)(\pi x).$$

So

$$\int h(a{\bullet}z)\lambda(dz) = \int (Tf)(a{\bullet}z)\lambda(dz) = \int T(a^{-1}{\bullet_l}f)(z)\lambda(dz)$$

$$= \int a^{-1}{\bullet_l}f d\mu = \int f\,d\mu = \int h(z)\lambda(dz).$$

By 441L again, $\lambda$ is $X$-invariant. Also $\lambda$ is non-zero because there is surely some $f$ such that $\int f\,d\mu \ne 0$. So we have the required non-zero $X$-invariant Radon measure on $Z$.

**443S Applications** This theorem applies in a variety of cases. Let $X$ be a locally compact Hausdorff topological group and $Y$ a closed subgroup of $X$.

**(a)** If $Y$ is a normal subgroup of $X$, then $\Delta_Y = \Delta_X{\restriction}Y$. **P** $X/Y$ has a group structure under which it is a locally compact Hausdorff group (4A5J(b-ii)). It therefore has a left Haar measure, which is surely $X$-invariant in the sense of 443R. **Q**

Note that in this context any of the invariant measures $\lambda$ of 443Q must be left Haar measures on the quotient group.

**(b)** If $Y$ is compact, then $\Delta_Y = \Delta_X \restriction Y$. **P** Both $\Delta_Y$ and $\Delta_X \restriction Y$ are continuous homomorphisms from $Y$ to $]0, \infty[$; since the only compact subgroup of $]0, \infty[$ is $\{1\}$, they are both constant with value 1. **Q** So 443R tells us that we have an $X$-invariant Radon measure $\lambda$ on $X/Y$. Since $Y$ has a Haar probability measure, $\lambda$ will be the image of a left Haar measure under the canonical map (443Qd).

**(c)** If, in (b), $Y$ is a normal subgroup, then we find that, for $W \in \operatorname{dom} \lambda$ and $x \in X$,

$$\lambda(W \cdot \pi x) = \mu(\pi^{-1}[W] \cdot x) = \Delta_X(x)\lambda W,$$

so that $\Delta_{X/Y}\pi = \Delta_X$, writing $\pi : X \to X/Y$ for the canonical map. This is a special case of 443T below, because (in the terminology there) $\psi(a) = \nu Y/\nu Y = 1$ for every $a \in X$.

**(d)** If $Y$ is open, $\Delta_Y = \Delta_X \restriction Y$. **P** If $\mu$ is a left Haar measure on $X$, then the subspace measure $\mu_Y$ is a left Haar measure on $Y$ (443F). There is an open set $G \subseteq Y$ such that $0 < \mu G < \infty$, and now $\mu(Gy) = \Delta_X(y)\mu G = \Delta_Y(y)\mu G$ for every $y \in Y$. **Q** This time, $X/Y$ is discrete, so counting measure is an $X$-invariant Radon measure on $X/Y$.

**443T Theorem** Let $X$ be a locally compact Hausdorff topological group and $Y$ a closed normal subgroup of $X$; let $Z = X/Y$ be the quotient group, and $\pi : X \to Z$ the canonical map. Write $\Delta_X$, $\Delta_Z$ for the left modular functions of $X$, $Z$ respectively. Define $\psi : X \to ]0, \infty[$ by the formula

$$\nu(aFa^{-1}) = \psi(a)\nu F \text{ whenever } F \in \operatorname{dom} \nu \text{ and } a \in X,$$

where $\nu$ is a left Haar measure on $Y$ (cf. 443Pc). Then

$$\Delta_Z(\pi a) = \psi(a)\Delta_X(a)$$

for every $a \in X$.

**proof** Let $T : C_k(X) \to C_k(Z)$ be the map defined in 443P, and $\lambda$ a left Haar measure on $Z$; then, as in 443Qb, we have a left Haar measure $\mu$ on $X$ defined by the formula $\int f\,d\mu = \int Tf\,d\lambda$ for every $f \in C_k(X)$. Fix on some $f > 0$ in $C_k(X)$ and $a \in X$, and set $w = \pi a$. By 443Pc, we have

$$T(a \bullet_r f)(z) = \psi(a)(Tf)(\pi(xa)) = \psi(a)(Tf)(zw)$$

whenever $\pi x = z$. So

$$\Delta_X(a^{-1}) \int f\,d\mu = \int a \bullet_r f d\mu$$

(442Kc once more)

$$= \int T(a \bullet_r f) d\lambda = \psi(a) \int (Tf)(zw)\lambda(dz)$$
$$= \psi(a)\Delta_Z(w^{-1}) \int Tf(z)\lambda(dz) = \psi(a)\Delta_Z(w^{-1}) \int f\,d\mu.$$

Thus $\Delta_X(a^{-1}) = \psi(a)\Delta_Z(w^{-1})$; because both $\Delta_X$ and $\Delta_Z$ are multiplicative,

$$\Delta_Z(\pi a) = \Delta_Z(w) = \psi(a)\Delta_X(a).$$

**443U Transitive actions** All the results from 443P onwards have been expressed in terms of groups acting on quotient groups. But the same structures can appear if we start from a group action. To simplify the hypotheses, I give the following result for compact groups only.

**Theorem** Let $X$ be a compact Hausdorff topological group, $Z$ a non-empty compact Hausdorff space, and $\bullet$ a transitive continuous action of $X$ on $Z$. Write $\pi_z(x) = x \bullet z$ for $z \in Z$ and $x \in X$.

(a) For every $z \in Z$, $Y_z = \{x : x \in X, x \bullet z = z\}$ is a compact subgroup of $X$. If we give the set $X/Y_z$ of left cosets of $Y_z$ in $X$ its quotient topology, we have a homeomorphism $\phi_z : X/Y_z \to Z$ defined by the formula $\phi_z(xY_z) = x \bullet z$ for every $x \in X$.

(b) Let $\mu$ be a Haar probability measure on $X$. Then the image measure $\mu\pi_z^{-1}$ is an $X$-invariant Radon probability measure on $Z$, and $\mu\pi_w^{-1} = \mu\pi_z^{-1}$ for all $w$, $z \in Z$.

(c) Every non-zero $X$-invariant Radon measure on $Z$ is of the form $\mu\pi_z^{-1}$ for a Haar measure $\mu$ on $X$ and some (therefore any) $z \in Z$.

(d) There is a strictly positive $X$-invariant Radon probability measure on $Z$, and any two non-zero $X$-invariant Radon measures on $Z$ are scalar multiples of each other.

(e) Take any $z \in Z$, and let $\nu$ be the Haar probability measure of $Y_z$. If $\mu$ is a Haar measure on $X$, then

$$\mu E = \int \nu(Y_z \cap x^{-1}E)\mu(dx)$$

whenever $E \subseteq X$ is Haar measurable.

**proof (a)** Because $\bullet$ is an action of $X$ on $Z$, $Y_z$ is always a subgroup; because $\bullet$ is continuous, $Y_z$ is closed, therefore compact. Given $z \in Z$, then for $x$, $y \in X$ we have

$$x{\bullet}z = y{\bullet}z \iff x^{-1}y \in Y_z \iff xY_z = yY_z.$$

So the formula given for $\phi_z$ defines an injection from $Z/Y_z$ to $Z$, which is surjective because $\bullet$ is transitive. To see that $\phi_z$ is continuous, take any open set $H \subseteq Z$. Then

$$\{x : xY_z \in \phi_z^{-1}[H]\} = \{x : x{\bullet}z \in H\} = \pi_z^{-1}[H]$$

is open in $X$ (because $\bullet$ is continuous), so $\phi_z^{-1}[H]$ is open in $X/Y_z$. Because $X/Y_z$ is compact and $\phi_z$ is a bijection, $\phi_z$ is a homeomorphism (3A3Dd).

**(b)** Because $X$ is compact, therefore unimodular (442Ic), we can speak of 'Haar measures' on $X$ without specifying 'left' or 'right'. If $\mu$ is the Haar probability measure on $X$, then the image measure $\mu\pi_z^{-1}$ is a Radon probability measure on $Z$ (418I once more). To see that the measures $\mu\pi_z^{-1}$ are $X$-invariant, take any Borel set $H \subseteq Z$ and $y \in X$, and consider

$$\begin{aligned}(\mu\pi_z^{-1})(y^{-1}{\bullet}H) &= \mu\{x : x{\bullet}z \in y^{-1}{\bullet}H\} = \mu\{x : yx{\bullet}z \in H\} = \mu\{x : yx \in \pi_z^{-1}[H]\}\\ &= \mu(y^{-1}\pi_z^{-1}[H]) = \mu(\pi_z^{-1}[H]) = (\mu\pi_z^{-1})(H).\end{aligned}$$

By 441B, this is enough to ensure that $\mu\pi_z^{-1}$ is invariant.

If $w$, $z \in Z$ and $H \subseteq Z$ is a Borel set, then there is a $y \in X$ such that $y{\bullet}w = z$, and now

$$\pi_w^{-1}[H] = \{x : x{\bullet}w \in H\} = \{x : (xy^{-1}){\bullet}z \in H\} = \{x : xy^{-1} \in \pi_z^{-1}[H]\} = (\pi_z^{-1}[H])y.$$

But $\mu$ is a two-sided Haar measure, so

$$(\mu\pi_w^{-1})(H) = \mu(\pi_w^{-1}[H]) = \mu((\pi_z^{-1}[H])y) = \mu(\pi_z^{-1}[H]) = (\mu\pi_z^{-1})(H).$$

Thus $\mu\pi_w^{-1}$ and $\mu\pi_z^{-1}$ agree on the Borel sets and must be equal (416Eb).

**(c)** Now we come to the interesting bit. Suppose that $\lambda$ is a non-zero $X$-invariant Radon measure on $Z$. Take any $z \in Z$ and consider the Radon measure $\lambda'$ on $X/Y_z$ got by setting $\lambda' H = \lambda\phi_z[H]$ whenever $H \subseteq X/Y_z$ and $\phi_z[H]$ is measured by $\lambda$. In this case, if $x \in X$ and $H \subseteq X/Y_z$ is measured by $\lambda'$,

$$\begin{aligned}\lambda'(x{\bullet}H) &= \lambda\{\phi_z(x{\bullet}w) : w \in H\} = \lambda\{\phi_z(x{\bullet}yY_z) : y \in X,\, yY_z \in H\}\\ &= \lambda\{\phi_z(xyY_z) : y \in X,\, yY_z \in H\} = \lambda\{xy{\bullet}z : y \in X,\, yY_z \in H\}\\ &= \lambda\{x{\bullet}(y{\bullet}z) : y \in X,\, yY_z \in H\} = \lambda(x{\bullet}\phi_z[H]) = \lambda\phi_z[H]\end{aligned}$$

(because $\lambda$ is $X$-invariant)

$$= \lambda' H.$$

So $\lambda'$ is $X$-invariant.

Now let $\nu$ be the Haar probability measure on the compact Hausdorff group $Y_z$. By 443Qb, we have a (left) Haar measure $\mu$ on $X$ defined by the formula $\mu G = \int \nu_w G\,\lambda'(dw)$ for every open $G \subseteq X$, where $\nu_{xY_z} G = \nu(Y_z \cap x^{-1}G)$ for every $y \in X$ and every open $G \subseteq X$. Let $H \subseteq Z$ be an open set. Then for any $x \in X$,

$$\begin{aligned}\nu_{xY_z}(\pi_z^{-1}[H]) &= \nu\{y : y \in Y_z,\, xy \in \pi_z^{-1}[H]\} = \nu\{y : y \in Y_z,\, xy{\bullet}z \in H\}\\ &= \nu\{y : y \in Y_z,\, x{\bullet}(y{\bullet}z) \in H\} = \nu\{y : y \in Y_z,\, x{\bullet}z \in H\}\\ &= \nu Y_z = 1 \text{ if } x{\bullet}z \in H,\\ &= \nu\emptyset = 0 \text{ otherwise.}\end{aligned}$$

So

$$\mu(\pi_z^{-1}[H]) = \lambda'\{xY_z : x \in X,\, x{\bullet}z \in H\} = \lambda'\phi_z^{-1}[H] = \lambda H.$$

As $H$ is arbitrary, the image measure $\mu\pi_z^{-1}$ agrees with $\lambda$ on the open subsets of $Z$; as they are both Radon measures, $\mu\pi_z^{-1} = \lambda$, as required.

**(d)** This is now easy. $X$ carries a non-zero Haar measure, so by (b) there is an $X$-invariant Radon probability measure on $Z$. If $\lambda_1$ and $\lambda_2$ are non-zero $X$-invariant Radon measures on $Z$, then they are of the form $\mu_1\pi_w^{-1}$ and $\mu_2\pi_z^{-1}$ where $\mu_1$ and $\mu_2$ are Haar measures on $X$ and $w$, $z \in Z$. By (b) again, $\mu_1\pi_w^{-1} = \mu_1\pi_z^{-1}$, and since $\mu_1$ and $\mu_2$ are multiples of each other (442B), so are $\lambda_1$ and $\lambda_2$.

To see that the invariant probability measure $\lambda$ on $X$ is strictly positive, take any non-empty open set $H \subseteq Z$. Then $Z$ is covered by the open sets $x{\bullet}H$, as $x$ runs over $X$. Because $Z$ is compact, it is covered by finitely many of these, so at least one of them has non-zero measure. But they all have the same measure as $H$, so $\lambda H > 0$.

**(e)** Write $\theta : X \to X/Y_z$ for the canonical map. For $w \in X/Y_z$ we have a Radon measure $\nu_w$ on $X$ defined by setting $\nu_w E = \nu(Y_z \cap x^{-1}E)$ whenever $\theta x = w$ and the right-hand side is defined (443Qa). By (a)-(b) above, or otherwise, there is a non-zero $X$-invariant Radon measure $\lambda$ on $X/Y_z$; re-scaling if necessary, we may suppose that $\lambda(X/Y_z) = \mu X$. By 443Qe, we have a Haar measure $\mu'$ on $X$ defined by setting $\mu' E = \int \nu_w E\,\lambda(dw)$ for every Haar measurable $E$; since

$$\mu' X = \int \nu_w X\,\lambda(dw) = \int \nu Y_z\,\lambda(dw) = \lambda(X/Y_z) = \mu X,$$

$\mu' = \mu$. Moreover, $\lambda = \mu'\theta^{-1}$ (443Qd). So

$$\begin{aligned}\mu E = \mu' E &= \int \nu_w E\,\lambda(dw)\\ &= \int \nu_{\theta x} E\,\mu'(dx) = \int \nu(Y_z \cap x^{-1}E)\mu(dx)\end{aligned}$$

for every Haar measurable $E \subseteq X$.

**443X Basic exercises >(a)** Let $X$ be a topological group, $\mu$ a left Haar measure on $X$ and $\lambda$ the corresponding quasi-Radon product measure on $X \times X$. (i) Show that the maps $(x, y) \mapsto (y, x)$, $(x, y) \mapsto (x, xy)$, $(x, y) \mapsto (y^{-1}x, y)$ are automorphisms of the measure space $(X \times X, \lambda)$. (*Hint*: use 417C(b-v-$\beta$) to show that they preserve the measures of open sets.) (ii) Show that the maps $(x, y) \mapsto (y^{-1}, xy)$, $(x, y) \mapsto (yx, x^{-1})$, $(x, y) \mapsto (y^3x, x^{-1}y^{-2})$ are automorphisms of $(X \times X, \lambda)$. (*Hint*: express them as compositions of maps of the forms in (ii).)

**(b)** Let $X$ be a topological group carrying Haar measures and $A \subseteq X$. (i) Show that $A$ is self-supporting (definition: 411Na) for one Haar measure on $X$ iff it is self-supporting for every Haar measure on $X$. (ii) Show that $A$ has non-zero inner measure for one Haar measure on $X$ iff it has non-zero inner measure for every Haar measure on $X$.

**(c)** Let $X$ be a topological group, $\mu$ a Haar measure on $X$, and $E$, $F$ measurable subsets of $X$. Show that $(x, y, w, z) \mapsto \mu(xEy \cap wFz) : X^4 \to [0, \infty]$ is lower semi-continuous.

**(d)** Let $X$ be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra. (i) Show that we have a continuous action of $X \times X$ on $\mathfrak{A}$ defined by the formula $(x, y){\bullet}E^\bullet = (xEy^{-1})^\bullet$ for $x$, $y \in X$ and Haar measurable sets $E \subseteq X$. (ii) Show that if $x \in X$ and $a \in \mathfrak{A}$ then $x{\bullet}_r a = (x{\bullet}_l \overleftrightarrow{a})^{\leftrightarrow}$, where $\overleftrightarrow{a}$ is as defined in 443Af.

**(e)** Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Show that it is isomorphic to the measure algebra of a topological group with a Haar measure iff it is localizable and quasi-homogeneous in the sense of 374G-374H.

**(f)** Let $X$ be a topological group with a left Haar measure $\mu$. (i) Show that if $Y$ is a subgroup of $X$ such that $\mu_*Y > 0$, then $Y$ is open. In particular, any non-negligible closed subgroup of $X$ is open. (ii) Let $Y$ be any subgroup of $X$ which is not Haar negligible. Show that the subspace measure $\mu_Y$ is a left Haar measure on $Y$. Show that $\overline{Y}$ is a Haar measurable envelope of $Y$. (*Hint*: apply 443Db inside the topological group $\overline{Y}$.)

**(g)** Write out a version of 443G for right Haar measures.

**(h)** Let $X$ be a topological group carrying Haar measures, and $L^0$ the space of equivalence classes of Haar measurable functions, as in 443A; let $u \mapsto \overleftrightarrow{u} : L^0 \to L^0$ be the operator of 443Af. Show that if $\mu$ is a left Haar measure on $X$ and $\nu$ is a right Haar measure, $p \in [1,\infty]$ and $u \in L^0$, then $\overleftrightarrow{u} \in L^p(\nu)$ iff $u \in L^p(\mu)$.

**(i)** Let $X$ be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra. Show that, in the language of 443C and 443G, $\chi(x\bullet_l a) = x\bullet_l \chi a$ and $\chi(x\bullet_r a) = x\bullet_r \chi a$ for every $x \in X$ and $a \in \mathfrak{A}$.

**(j)** Let $X$ be a topological group carrying Haar measures. Show that $X$ is totally bounded for its bilateral uniformity iff $X$ is totally bounded for its right uniformity (definition: 4A5Ha) iff its Haar measures are totally finite.

**(k)** Let $X$ be a topological group, $\mu$ a left Haar measure on $X$, and $A \subseteq X$ a set which is self-supporting for $\mu$. Show that the following are equiveridical: (i) for every neighbourhood $U$ of the identity $e$, there is a countable set $I \subseteq X$ such that $A \subseteq UI$; (ii) for every neighbourhood $U$ of $e$, there is a countable set $I \subseteq X$ such that $A \subseteq IU$; (iii) for every neighbourhood $U$ of $e$, there is a countable set $I \subseteq X$ such that $A \subseteq IUI$; (iv) $A$ can be covered by countably many sets of finite measure for $\mu$; (v) $A$ can be covered by countably many open sets of finite measure for $\mu$; (vi) $A$ can be covered by countably many sets which are totally bounded for the bilateral uniformity on $X$.

**>(l)** Let $X$ be a topological group carrying Haar measures. (i) Show that the following are equiveridical: ($\alpha$) $X$ is ccc; ($\beta$) $X$ has a $\sigma$-finite Haar measure; ($\gamma$) every Haar measure on $X$ is $\sigma$-finite. (ii) Show that if $X$ is locally compact and Hausdorff, we can add ($\delta$) $X$ is $\sigma$-compact.

**(m)** Let $X$ be a topological group carrying Haar measures. Show that every subset of $X$ has a Haar measurable envelope which is a Borel set.

**(n)** In 443L, show that (i) $\phi[A]$ is Haar negligible in $Z$ whenever $A$ is Haar negligible in $X$ (ii) $\Delta_X = \Delta_Z\phi$, where $\Delta_X$, $\Delta_Z$ are the left modular functions of $X$, $Z$ respectively (iii) $\phi[X]$ is dense in $Z$ (iv) $Z$ is unimodular iff $X$ is unimodular.

**>(o)** Let $X$ and $Y$ be topological groups with left Haar measures $\mu$ and $\nu$. Show that the c.l.d. and quasi-Radon product measures of $\mu$ and $\nu$ on $X \times Y$ coincide. (*Hint*: start with locally compact Hausdorff spaces, and show that a compact $\mathrm{G}_\delta$ set in $X \times Y$ belongs to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, where $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are the Borel $\sigma$-algebras of $X$ and $Y$; now use 441Xj and 443L.)

**(p)** Let $\langle X_i\rangle_{i\in I}$ be a family of topological groups and $X = \prod_{i\in I} X_i$ their product. Suppose that each $X_i$ has a Haar probability measure $\mu_i$. Show that the ordinary and quasi-Radon product measures on $X$ coincide.

**(q)** Let $X$ be a locally compact Hausdorff group and $Y$ a closed subgroup of $X$; write $X/Y$ for the space of left cosets of $Y$ in $X$, with its quotient topology. Show that if $\lambda_1$ and $\lambda_2$ are non-zero $X$-invariant Radon measures on $X/Y$, then each is a multiple of the other. (*Hint*: look at the Haar measures they define on $X$.)

**>(r)** Write $S^1$ for the circle group $\{s : s \in \mathbb{C}, |s| = 1\}$, and set $X = S^1 \times S^1$, where the first copy of $S^1$ is given its usual topology and the second copy is given its discrete topology, so that $X$ is an abelian locally compact Hausdorff group. Set $E=\{(s,s):s\in S^1\}$. (i) Show that $E$ is a closed Haar negligible subset of $X$. (ii) Set $Y=\{(1,s):s\in S^1\}$; check that $Y$ is a closed normal subgroup of $X$, and that the quotient group $X/Y$ can be identified with $S^1$ with its usual topology; let $\lambda$ be the Haar probability measure of $X/Y$. Let $\nu$ be counting measure on $Y$. Show that, in the language of 443Q, $\nu_zE=1$ for every $z\in X/Y$, so that $\mu E\ne\int\nu_zE\,\lambda(dz)$. (iii) Setting $f=\chi E$, show that the map $a\mapsto f_a^\bullet$ described in 443Qf is not almost continuous.

**(s)**(i) In 443P, suppose that $G\subseteq X$ is an open set such that $GY=X$. Show that for every $h\in C_k(Z)^+$ there is an $f\in C_k(X)^+$ such that $Tf=h$ and $\overline{\{x:f(x)>0\}}\subseteq G$. (ii) In 443Pc, show that $\psi$ is multiplicative. (iii) In 443R, suppose that there is an open set $G\subseteq X$ such that $GY$ has finite measure for the left Haar measures of $X$. Show that $Z$ has an $X$-invariant Radon probability measure. (*Hint*: $Y$ is totally bounded for its right uniformity.)

**(t)** Let $X$ be a locally compact Hausdorff group. Show that it has a closed normal subgroup $Y$ such that $Y$ and $X/Y$ are both unimodular. (*Hint*: take $Y=\{x:\Delta(x)=1\}$.)

**>(u)** Let $X=\mathbb{R}^2$ be the example of 442Xf. (i) Let $Y_1$ be the subgroup $\{(\xi,0):\xi\in\mathbb{R}\}$. Describe the left cosets of $Y_1$ in $X$. Show that there is no non-trivial $X$-invariant Radon measure on the set $X/Y_1$ of these left cosets. Find a base $\mathcal{U}$ for the topology of $X/Y_1$ such that you can identify the sets $x\bullet U$, where $x\in X$ and $U\in\mathcal{U}$, with sufficient precision to explain why the hypothesis (iii) of 441C is not satisfied. (ii) Let $Y_2$ be the normal subgroup $\{(0,\xi):\xi\in\mathbb{R}\}$. Find the associated function $\psi:X\to\,]0,\infty[$ as described in 443Pc and 443T.

**(v)** Let $X$ be a locally compact Hausdorff group and $Y$ a compact normal subgroup of $X$. Show that $X$ is unimodular iff the quotient group $X/Y$ is unimodular. (*Hint*: the function $\psi$ of 443T must be constant.)

**(w)** Take any integer $r\ge1$, and let $G$ be the isometry group of $\mathbb{R}^r$ with its topology of pointwise convergence (441G). (i) Show that $G$ is metrizable and locally compact. (*Hint*: 441Xq.) (ii) Let $H\subseteq G$ be the set of translations. Show that $H$ is an abelian closed normal subgroup of $G$, and that Lebesgue measure on $\mathbb{R}^r$ can be regarded as a Haar measure on $H$. (iii) Show that the quotient group $G/H$ is compact. (iv) Show that $G$ is unimodular. (*Hint*: the function $\psi$ of 443T is constant.)

**>(x)** Set $X=\mathbb{R}^3$ with the operation

$$(\xi_1,\xi_2,\xi_3)*(\eta_1,\eta_2,\eta_3)=(\xi_1+\eta_1,\xi_2+e^{\xi_1}\eta_2,\xi_3+e^{-\xi_1}\eta_3).$$

(i) Show that (with the usual topology of $\mathbb{R}^3$) $X$ is a topological group. (ii) Show that it is unimodular. (*Hint*: Lebesgue measure is a two-sided Haar measure.) (iii) Show that $X$ has both a closed subgroup and a Hausdorff quotient group which are not unimodular.

**>(y)** Let $(X,\rho)$ be a non-empty compact metric space such that the group $G$ of isometries of $X$ is transitive. Show that any two non-zero $G$-invariant Radon measures on $X$ must be multiples of each other. (*Hint*: 441Gb, 443U.)

**>(z)** Show that 443G is equally valid if we take functions to be complex-valued rather than real-valued, and work with $L^p_{\mathbb{C}}$ rather than $L^p$.

**443Y Further exercises (a)** Let $X$ be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra. Show that two principal ideals of $\mathfrak{A}$ are isomorphic (as Boolean algebras) iff they have the same cellularity.

**(b)** Let $X$ be a topological group carrying Haar measures, and $E\subseteq X$ a Haar measurable set such that $E\cap U$ is not Haar negligible for any neighbourhood $U$ of the identity. Show that for any $A\subseteq X$ the set $A'=\{x:x\in A,\ A\cap xE$ is Haar negligible$\}$ is Haar negligible.

**(c)** Let $X$ be a locally compact Hausdorff group. Show that we have continuous shift actions $\bullet_l$, $\bullet_r$ and $\bullet_c$ of $X$ on the Banach space $C_0(X)$ defined by formulae corresponding to those of 443G.

**(d)** Let $X$ be a compact Hausdorff topological group and $\mathfrak{A}$ its Haar measure algebra. Let $Y$ be a subgroup of $X$; for $y \in Y$, define $\hat{y} \in \operatorname{Aut}\mathfrak{A}$ by setting $\hat{y}(a) = y \bullet_l a$ for $a \in \mathfrak{A}$. Show that $\{\hat{y} : y \in Y\}$ is ergodic (definition: 395Ge) iff $Y$ is dense in $X$.

**(e)** Let $\mathfrak{A}$ be a Boolean algebra, $G$ a group, and $\bullet$ an action of $G$ on $\mathfrak{A}$ such that $a \mapsto g\bullet a$ is a Boolean automorphism for every $g \in G$. (i) Show that we have a corresponding action of $G$ on $L^\infty = L^\infty(\mathfrak{A})$ defined by saying that, for every $g \in G$, $g\bullet\chi a = \chi(g\bullet a)$ for $a \in \mathfrak{A}$ and $u \mapsto g\bullet u$ is a positive linear operator on $L^\infty$. (ii) Show that if $\mathfrak{A}$ is Dedekind $\sigma$-complete, this action on $L^\infty$ extends to an action on $L^0 = L^0(\mathfrak{A})$ defined by saying that $[\![g\bullet u > \alpha]\!] = g\bullet[\![u > \alpha]\!]$ for $g \in G$, $u \in L^0$ and $\alpha \in \mathbb{R}$.

**(f)** Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, $G$ a topological group, and $\bullet$ a continuous action of $G$ on $\mathfrak{A}$ (when $\mathfrak{A}$ is given its measure-algebra topology) such that $a \mapsto g\bullet a$ is a measure-preserving Boolean automorphism for every $g \in G$. (i) Show that the corresponding action of $G$ on $L^0 = L^0(\mathfrak{A})$, as defined in 443Ye, is continuous when $L^0$ is given the topology of convergence in measure, and induces continuous actions of $G$ on $L^p = L^p(\mathfrak{A}, \bar{\mu})$ for $1 \le p < \infty$. (ii) Show that if we give the unit ball $B$ of $L^\infty = L^\infty(\mathfrak{A})$ the topology induced by $\mathfrak{T}_s(L^\infty, L^1)$, then the action of $G$ on $L^0$ induces a continuous action of $G$ on $B$.

**(g)** Let $X$ be a topological group with a left Haar measure $\mu$, and $A \subseteq X$. Show that the following are equiveridical: (i) $A$ is totally bounded for the bilateral uniformity of $X$ (ii) there are non-empty open sets $G$, $H \subseteq X$ such that $\mu(AG)$, $\mu(A^{-1}H)$ are both finite.

**(h)** Give an example of a locally compact Hausdorff group, with left Haar measure $\mu$, such that no open normal subgroup can be covered by a sequence of sets of finite measure for $\mu$.

**(i)** Let $X$ be a topological group. Let $\Sigma$ be the family of subsets of $X$ expressible in the form $\phi^{-1}[F]$ for some Borel subset $F$ of a separable metrizable topological group $Y$ and some continuous homomorphism $\phi : X \to Y$. Show that $\Sigma$ is a $\sigma$-algebra of subsets of $X$ and that multiplication, regarded as a function from $X \times X$ to $X$, is $(\Sigma\widehat{\otimes}\Sigma, \Sigma)$-measurable. Show that any compact $\mathrm{G}_\delta$ set belongs to $\Sigma$. Show that if $X$ is $\sigma$-compact, then $\Sigma$ is the Baire $\sigma$-algebra of $X$.

**(j)** Let $X$ be any Hausdorff topological group with cardinal greater than $\mathfrak{c}$. Let $\mathcal{B}$ be the Borel $\sigma$-algebra of $X$. Show that $(x, y) \mapsto xy$ is not $(\mathcal{P}X\widehat{\otimes}\mathcal{P}X, \mathcal{B})$-measurable.

**(k)** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Show that $\mu$ is inner regular with respect to the family of closed sets $F \subseteq X$ such that $F = \bigcap_{n\in\mathbb{N}} FU_n$ for some sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of the identity.

**(l)** Let $X$ be a topological group carrying Haar measures. Let $E \subseteq X$ be a Haar measurable set such that $E \cap U$ is not Haar negligible for any neighbourhood $U$ of the identity. Show that there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $E$ such that $x_{i_0}x_{i_1}\ldots x_{i_n} \in E$ whenever $n \in \mathbb{N}$ and $i_0 < i_1 < \ldots < i_n$ in $\mathbb{N}$. (See Plewik & Voigt 91.)

**(m)** Let $X$ be a topological group and $\mu$ a Haar measure on $X$. Show that any closed self-supporting subset of $X$ is a zero set.

**(n)** Find a compact Hausdorff space $X$ with a strictly positive Radon measure such that there is a regular open set $G \subseteq X$ which is not a cozero set.

**(o)** Let $X$ be a locally compact Hausdorff topological group which is not discrete (as topological space). (i) Show that there is a Haar negligible zero set containing the identity of $X$. (ii) Show that if $X$ is $\sigma$-compact, it has a Haar negligible compact normal subgroup $Y$ which is a zero set in $X$, so that $X/Y$ is metrizable. (iii) Show that there is a Haar negligible set $A \subseteq X$ such that $AA$ is not Haar measurable.

**(p)** Find a non-discrete locally compact Hausdorff topological group $X$ such that if $Y$ is a normal subgroup of $X$ which is a zero set in $X$ then $Y$ is open.

**(q)** Show that there is a subgroup $X$ of the additive group $\mathbb{R}^2$ such that $X$ has full outer Lebesgue measure but $\{\xi : (\xi, 0) \in X\} = \mathbb{Q}$. Show that $X$ carries Haar measures, but that its closed subgroup $X \cap (\mathbb{R} \times \{0\})$ does not.

**(r)** Show that if $X$ is a locally compact Hausdorff group, $Y$ a compact subgroup of $X$, $Z = X/Y$ the set of left cosets of $Y$ and $\pi : X \to Z$ the canonical map, and $Z$ is given its quotient topology, then $R = \{(\pi x, x) : x \in X\}$ is an usco-compact relation in $Z \times X$ and $R[L] = \pi^{-1}[L]$ is compact for every compact $L \subseteq Z$.

**(s)** Let $G$ be the isometry group of $\mathbb{R}^r$, as in 443Xw. (i) Show that if we set $\rho(g, h) = \sup_{\|x\| \le 1} \|g(x) - h(x)\|$, then $\rho$ is a metric on $G$ defining its topology. (ii) Describe Haar measures on $G$ ($\alpha$) in terms of Hausdorff measure of an appropriate dimension for the metric $\rho$ ($\beta$) in terms of a parametrization of $G$ and Lebesgue measure on a suitable Euclidean space.

**(t)** Let $r \ge 1$ be an integer, and $G$ the group of invertible affine transformations of $\mathbb{R}^r$, with the topology of pointwise convergence inherited from $(\mathbb{R}^r)^{\mathbb{R}^r}$. (i) Show that $G$ is a locally compact Hausdorff topological group. (ii) Show that $G$ is not unimodular, and find its modular functions.

**(u)** Let $X$ be a topological group with a left Haar measure $\mu$ and left modular function $\Delta$; suppose that $X$ is not unimodular. Show that $\mu\{x : \alpha < \Delta(x) < \beta\} = \infty$ whenever $\alpha < \beta$ and $\{x : \alpha < \Delta(x) < \beta\}$ is non-empty.

**443 Notes and comments** Most of us, by the time we come to study measures on general topological groups, have come to trust our intuition concerning the behaviour of Lebesgue measure on $\mathbb{R}$, and the principal discipline imposed by the subject is the search for the true path between hopelessness and over-confidence when extending this intuition to the general setting. The biggest step is the loss of commutativity, especially as the non-abelian groups of elementary courses in group theory are mostly finite, and are therefore untrustworthy guides to the concerns of this chapter. Accordingly we find ourselves going rather slowly and carefully through the calculations in such results as 443C. Note, for instance, that in an abelian group the actions I call $\bullet_l$ and $\bullet_r$ are still different; $x \bullet_l E$ corresponds to $x + E = E + x$, but $x \bullet_r E$ corresponds to $E - x$. (The action $\bullet_c$ becomes trivial, of course.) When we come to translate the formulae of Fourier analysis in the next section, manoeuvres of this kind will often be necessary. In the present section, I have done my best to give results in 'symmetric' forms; you may therefore take it that when the words 'left' or 'right' appear in the statement of a proposition, there is some real need to break the symmetry. Subject to these remarks, such results as 443C and 443G are just a matter of careful conventional analysis. I see that I have used slightly different techniques in the two cases. Of course 443C can be thought of as a special case of 443Gc-443Gd, if we remember that $\chi : \mathfrak{A} \to L^0(\mathfrak{A})$ embeds $\mathfrak{A}$ topologically as a subspace of $L^0$ (367R).

443B, 443D and 443E belong to a different family; they deal with actual sets rather than with members of a measure algebra or a function space. I suppose it is 443D which will most often be quoted. Its corollary 443E deals with the obvious question of when Haar measures are tight.

Readers who have previously encountered the theory of Haar measures on locally compact groups will have been struck by how closely the more general theory here is able to follow it. The explanation lies in 443K-443L; my general Haar measures are really just subspace measures on subgroups of full Haar measure in locally compact groups. Knowing this, it is easy to derive all the results above from the locally compact theory. I use this method only in 443M-443O, because (following Halmos 50) I feel that from the point of view of pure measure theory the methods show themselves more clearly if we do not use ideas depending on compactness, but instead rely directly on $\tau$-additivity. But I note that 443Xo and 443Yi also go faster with the aid of 443L.

Halmos 50 goes a little farther, with a theory of groups carrying translation-invariant measures for which the operation $(x, y) \mapsto xy$ is $(\Sigma \widehat{\otimes} \Sigma, \Sigma)$-measurable, where $\Sigma$ is the domain of the measure. The essence of the theory here, and my reason for insisting on $\tau$-additive measures, is that for these we have a suitable theory of product measures. If we start with quasi-Radon measures and use the quasi-Radon product measure, then multiplication is measurable just because it is continuous. In order to achieve similar results without either assuming metrizability or using the theory of $\tau$-additive product measures, we have to restrict the measure to something smaller than the Borel $\sigma$-algebra, as in 443Yg. (See 443Yj.) 443J and 443Xo show that certain disconcerting features of general Radon measures (419C, 419E) cannot arise in the case of Haar measures.

When I come to look at subgroups and quotient groups, I do specialize to the locally compact case. One obstacle to generalization is the fact that a closed subgroup of a group carrying Haar measures need not itself carry Haar measures (443Yq). 443P and 443R are taken from FEDERER 69, who goes farther, with many other applications. I give a fairly detailed analysis of the relationship between a Haar measure on a group $X$ and a corresponding $X$-invariant measure on a family of left cosets (443Q) for the sake of applications in §447. One of the challenges here is to distinguish clearly those results which apply to all locally compact groups from those which rely on $\sigma$-compactness or some such limitation. There is a standard example (443Xr) which provides a useful test in such questions. You may recognise this as a version of a fundamental example related to Fubini's theorem, given in 252K.

Most of the results of this section begin with a topological group $X$. But starting from §441 it is equally natural to start with a group action. If we have a topological group $X$ acting continuously on a topological space $Z$, and $X$ carries Haar measures, then we have a good chance of finding an invariant measure on $Z$. In the simplest case, in which $X$ and $Z$ are both compact and the action is transitive, there is a unique invariant Radon probability measure on $Z$ (443U).

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## 444 Convolutions

In this section, I look again at the ideas of §§255 and 257, seeking the appropriate generalizations to topological groups other than $\mathbb{R}$. Following HEWITT & ROSS 63, I begin with convolutions of measures (444A-444E) before proceeding to convolutions of functions (444O-444V); in between, I mention the convolution of a function and a measure (444G-444M) and a general result concerning continuous group actions on quasi-Radon measure spaces (444F).

While I continue to give the results in terms of real-valued functions, the applications of the ideas here in the next section will be to complex-valued functions; so you may wish to keep the complex case in mind (444Xx).

**444A Convolution of measures: Proposition** If $X$ is a topological group and $\lambda$ and $\nu$ are two totally finite quasi-Radon measures on $X$, we have a quasi-Radon measure $\lambda * \nu$ on $X$ defined by saying that

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)\{(x, y) : xy \in E\}\\ &= \int \nu(x^{-1}E)\lambda(dx) = \int \lambda(Ey^{-1})\nu(dy)\end{aligned}$$

for every $E \in \operatorname{dom}(\lambda * \nu)$, where $\lambda \times \nu$ is the quasi-Radon product measure on $X \times X$.

**proof** Set $\phi(x, y) = xy$ for $x$, $y \in X$. Then $\phi$ is continuous, while $X$, being a topological group, is regular (4A5Ha); so so 418Hb tells us that there is a unique quasi-Radon measure $\lambda*\nu$ on $X$ such that $\phi$ is inverse-measure-preserving for $\lambda\times\nu$ and $\lambda*\nu$, that is, $(\lambda*\nu)(E) = (\lambda\times\nu)\{(x, y) : xy \in E\}$ whenever $E$ is measured by $\lambda * \nu$.

As for the other formulae, Fubini's theorem (417Ga) tells us that

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)\phi^{-1}[E]\\ &= \int \nu(\phi^{-1}[E])[\{x\}]\lambda(dx) = \int \nu(x^{-1}E)\lambda(dx)\\ &= \int \lambda(\phi^{-1}[E])^{-1}[\{y\}]\nu(dy) = \int \lambda(Ey^{-1})\nu(dy)\end{aligned}$$

for any $E \in \operatorname{dom}(\lambda * \nu)$.

**444B Proposition** If $X$ is a topological group, $\lambda_1 * (\lambda_2 * \lambda_3) = (\lambda_1 * \lambda_2) * \lambda_3$ for all totally finite quasi-Radon measures $\lambda_1$, $\lambda_2$ and $\lambda_3$ on $X$.

**proof** If $E \subseteq X$ is Borel, then

$$\begin{aligned}(\lambda_1 * (\lambda_2 * \lambda_3))(E) &= \int (\lambda_2 * \lambda_3)(x^{-1}E)\lambda_1(dx)\\ &= \int (\lambda_2 \times \lambda_3)\{(y,z) : xyz \in E\}\lambda_1(dx)\\ &= (\lambda_1 \times (\lambda_2 \times \lambda_3))\{(x,(y,z)) : xyz \in E\}\\ &= ((\lambda_1 \times \lambda_2) \times \lambda_3)\{((x,y),z) : xyz \in E\}\\ &= \int (\lambda_1 \times \lambda_2)\{(x,y) : xyz \in E\}\lambda_3(dz)\\ &= ((\lambda_1 * \lambda_2) * \lambda_3)(E).\end{aligned}$$

(For the central identification between $\lambda_1\times(\lambda_2\times\lambda_3)$ and $(\lambda_1\times\lambda_2)\times\lambda_3$, observe that as both are quasi-Radon measures it is enough to check that they agree on sets of the type $G_1 \times (G_2 \times G_3) \cong (G_1 \times G_2) \times G_3$ where $G_1$, $G_2$ and $G_3$ are open, as in 417J.)

So $\lambda_1 * (\lambda_2 * \lambda_3)$ and $(\lambda_1 * \lambda_2) * \lambda_3$ agree on the Borel sets and must be identical.

**444C Theorem** Let $X$ be a topological group and $\lambda$, $\nu$ two totally finite quasi-Radon measures on $X$. Then

$$\int f d(\lambda * \nu) = \int f(xy)(\lambda \times \nu)d(x,y) = \iint f(xy)\lambda(dx)\nu(dy) = \iint f(xy)\nu(dy)\lambda(dx)$$

for any $(\lambda * \nu)$-integrable real-valued function $f$. In particular, $(\lambda * \nu)(X) = \lambda X \cdot \nu X$.

**proof** If $f$ is of the form $\chi E$, so that

$$\int f d(\lambda * \nu) = (\lambda * \nu)(E), \quad \int f(xy)(\lambda \times \nu)d(x,y) = (\lambda \times \nu)\{(x,y) : xy \in E\},$$

$$\iint f(xy)\lambda(dx)\nu(dy) = \int \lambda(Ey^{-1})\nu(dy), \quad \iint f(xy)\nu(dy)\lambda(dx) = \int \nu(x^{-1}E)\lambda(dx)$$

this is covered by the result in 444A. Now it is easy to run through the standard progression to the cases of (i) simple functions (ii) non-negative Borel measurable functions defined everywhere (iii) functions defined, and zero, almost everywhere (iv) non-negative integrable functions and (v) arbitrary integrable functions.

**444D Proposition** Let $X$ be an abelian topological group. Then $\lambda * \nu = \nu * \lambda$ for all totally finite quasi-Radon measures $\lambda$, $\mu$ on $X$.

**proof** For any Borel set $E \subseteq X$,

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)\{(x,y) : xy \in E\} = (\nu \times \lambda)\{(y,x) : xy \in E\}\\ &= (\nu \times \lambda)\{(y,x) : yx \in E\} = (\nu * \lambda)(E).\end{aligned}$$

**444E The Banach algebra of $\tau$-additive measures (a)** Let $X$ be a topological group. Recall from 437Ab that we have a band $C_b(X)^\sim_\tau$ in the $L$-space $C_b(X)^\sim$ consisting of those order-bounded linear functionals $f : C_b(X) \to \mathbb{R}$ such that $|f|$ is smooth (equivalently, $f^+$ and $f^-$ are both smooth); that is, such that $|f|$, $f^+$ and $f^-$ can be represented by totally finite quasi-Radon measures on $X$. Because $X$ is completely regular (4A5Ha again), $C_b(X)^\sim_\tau$ can be identified with the band $M_\tau$ of signed $\tau$-additive Borel measures on $X$, that is, the set of those countably additive functionals $\nu$ defined on the Borel $\sigma$-algebra of $X$ such that $|\nu|$ is $\tau$-additive (437G).

**(b)** For any $\tau$-additive totally finite Borel measures $\lambda$, $\nu$ on $X$ we can define their convolution $\lambda * \nu$ by the formulae of 444A, that is,

$$(\lambda * \nu)(E) = \int \nu(x^{-1}E)\lambda(dx) = \int \lambda(Ey^{-1})\nu(dy)$$

for any Borel set $E \subseteq X$, if we note that the completions $\hat{\lambda}$, $\hat{\nu}$ of $\lambda$ and $\nu$ are quasi-Radon measures (415Cb), so that $\lambda * \nu$, as defined by these formulae, is just the restriction of $\hat{\lambda} * \hat{\nu}$, as defined in 444A, to the Borel $\sigma$-algebra. Now the formulae make it obvious that the map $*$ is bilinear in the sense that

$$(\lambda_1 + \lambda_2) * \nu = \lambda_1 * \nu + \lambda_2 * \nu,$$

$$\lambda * (\nu_1 + \nu_2) = \lambda * \nu_1 + \lambda * \nu_2,$$

$$(\alpha\lambda) * \nu = \lambda * (\alpha\nu) = \alpha(\lambda * \nu)$$

for all totally finite $\tau$-additive Borel measures $\lambda$, $\lambda_1$, $\lambda_2$, $\nu$, $\nu_1$, $\nu_2$ and all $\alpha \geq 0$. Consequently, regarding elements of $M_\tau$ as functionals on the Borel $\sigma$-algebra, we have a bilinear operator $* : M_\tau \times M_\tau \to M_\tau$ defined by saying that

$$(\lambda_1 - \lambda_2) * (\nu_1 - \nu_2) = \lambda_1 * \nu_1 - \lambda_1 * \nu_2 - \lambda_2 * \nu_1 + \lambda_2 * \nu_2$$

for all $\lambda_1$, $\lambda_2$, $\nu_1$, $\nu_2 \in M_\tau^+$.

**(c)** We see from 444B that $*$ is associative on $M_\tau^+$, so it will be associative on the whole of $M_\tau$. Observe that $*$ is positive in the sense that $\lambda * \nu \geq 0$ if $\lambda$, $\nu \geq 0$; so that

$$\begin{aligned}|\lambda * \nu| &= |\lambda^+ * \nu^+ - \lambda^+ * \nu^- - \lambda^- * \nu^+ + \lambda^- * \nu^-| \\ &\leq \lambda^+ * \nu^+ + \lambda^+ * \nu^- + \lambda^- * \nu^+ + \lambda^- * \nu^- \\ &= |\lambda| * |\nu|\end{aligned}$$

for any $\lambda$, $\nu \in M_\tau$.

**(d)** If $\lambda$, $\nu \in M_\tau^+$ then $\|\lambda\| = \lambda X$ and $\|\nu\| = \nu X$ (362Ba), so

$$\|\lambda * \nu\| = (\lambda * \nu)(X) = (\lambda \times \nu)(X \times X) = \lambda X \cdot \nu X = \|\lambda\|\|\nu\|.$$

Generally, for any $\lambda$, $\nu \in M_\tau$,

$$\|\lambda * \nu\| = \||\lambda * \nu|\| \leq \||\lambda| * |\nu|\| = \||\lambda|\|\||\nu|\| = \|\lambda\|\|\nu\|.$$

Thus $M_\tau$ is a Banach algebra under the operation $*$, as well as being an $L$-space. If $X$ is abelian then $M_\tau$ will be a commutative algebra, by 444D.

**444F** In preparation for the next construction I give a general result extending ideas already touched on in 443C and 443G.

**Theorem** Let $X$ be a topological space, $G$ a topological group and $\bullet$ a continuous action of $G$ on $X$. For $A \subseteq X$, $a \in G$ write $a\bullet A = \{a\bullet x : x \in A\}$. Let $\nu$ be a measure on $X$.

(a) If $f : X \to [0, \infty]$ is lower semi-continuous, then $a \mapsto \int a\bullet f \, d\nu : G \to [0, \infty]$ is lower semi-continuous. (See 4A5Cc for the definition of $a\bullet f$.) In particular, if $V \subseteq X$ is open, then $a \mapsto \nu(a\bullet V) : G \to [0, \infty]$ is lower semi-continuous.

(b) If $f : X \to \mathbb{R}$ is continuous, then $a \mapsto (a\bullet f)^\bullet : G \to L^0$ is continuous, if $L^0 = L^0(\nu)$ is given the topology of convergence in measure.

(c) If $\nu$ is $\sigma$-finite and $E \subseteq X$ is a Borel set, then $a \mapsto (a\bullet E)^\bullet : G \to \mathfrak{A}$ is Borel measurable, if the measure algebra $\mathfrak{A}$ of $\nu$ is given its measure-algebra topology.

(d) If $\nu$ is $\sigma$-finite and $f : X \to \mathbb{R}$ is Borel measurable, then $a \mapsto (a\bullet f)^\bullet : G \to L^0$ is Borel measurable.

(e) If $\nu$ is $\sigma$-finite, then

(i) $a \mapsto \nu(a\bullet E) : G \to [0, \infty]$ is Borel measurable for any Borel set $E \subseteq X$;

(ii) if $f : X \to \mathbb{R}$ is Borel measurable, then $Q = \{a : \int a\bullet f \, d\nu$ is defined in $[-\infty, \infty]\}$ is a Borel set, and $a \mapsto \int a\bullet f \, d\nu : Q \to [-\infty, \infty]$ is Borel measurable.

**proof (a)(i)** Note first that if $f : X \to [0, \infty]$ is Borel measurable, then, for each $a \in G$, $a\bullet f$ is the composition of $f$ with the continuous function $x \mapsto a^{-1}\bullet x$, so is Borel measurable, and if $f$ is finite-valued then $(a\bullet f)^\bullet$ is defined in $L^0 = L^0(\nu)$.

**(ii)** Suppose that $f : X \to [0, \infty]$ is lower semi-continuous, $\gamma \in [0, \infty[$, $a \in G$ and $\int a\bullet f > \gamma$. Let $\mathcal{U}$ be the set of open neighbourhoods of $a$ in $G$. For $U \in \mathcal{U}$, $x \in X$ set

$$\phi_U(x) = \sup\{\inf_{b \in U, y \in V} (b\bullet f)(y) : V \text{ is an open neighbourhood of } x \text{ in } X\}.$$

Then $\phi_U$ is lower semi-continuous. **P** If $\phi_U(x) > \alpha$, there is an open neighbourhood $V$ of $x$ such that $\inf_{b\in U,y\in V}(b\bullet f)(y) > \alpha$; now $\phi_U(y) > \alpha$ for every $y \in V$. **Q** If $U' \subseteq U$ then $\phi_{U'} \geq \phi_U$, so $\{\phi_U : U \in \mathcal{U}\}$ is upwards-directed. Also $\sup_{U\in\mathcal{U}} \phi_U = a\bullet f$ in $[0,\infty]^X$. **P** Of course $\phi_U(x) \leq (a\bullet f)(x)$ for every $x$. If $x \in X$ and $(a\bullet f)(x) > \alpha$, then $\{y : f(y) > \alpha\}$ is an open set containing $a^{-1}\bullet x$, so (because $\bullet$ is continuous) there are a $U \in \mathcal{U}$ and an open neighbourhood $V$ of $x$ such that $f(b^{-1}\bullet y) > \alpha$ whenever $b \in U$ and $y \in V$; in which case $\phi_U(x) \geq \alpha$. As $\alpha$ is arbitrary, $\sup_{U\in\mathcal{U}} \phi_U(x) = (a\bullet f)(x)$. **Q**

By 414Ba, $\int a\bullet f\,d\nu = \sup_{U\in\mathcal{U}} \int \phi_U d\nu$, and there is a $U \in \mathcal{U}$ such that $\int \phi_U d\nu > \gamma$. Now suppose that $b \in U$; then $\phi_U(x) \leq (b\bullet f)(x)$ for every $x$, so $\int b\bullet f\,d\nu > \gamma$. This shows that $\{a : \int a\bullet f\,d\nu > \gamma\}$ is an open set in $G$, so that $a \mapsto \int a\bullet f\,d\nu$ is lower semi-continuous.

**(iii)** If $V \subseteq X$ is open, then $\chi V$ is lower semi-continuous, and $\chi(a\bullet V) = a\bullet(\chi V)$ for every $a \in G$. So $a \mapsto \nu(a\bullet V) = \int a\bullet(\chi V)d\nu$ is lower semi-continuous.

Thus (a) is true.

**(b)** Take any $a \in G$, $E \in \operatorname{dom}\nu$ such that $\nu E < \infty$ and $\epsilon > 0$. Let $\mathcal{U}$ be the family of open neighbourhoods of $a$ in $G$, and for $U \in \mathcal{U}$ set

$$H_U = \operatorname{int}\{x : |(b\bullet f)(x) - (a\bullet f)(x)| \leq \epsilon \text{ whenever } b \in U\}.$$

Then $\{H_U : U \in \mathcal{U}\}$ is upwards-directed. Also, it has union $X$. **P** If $x \in X$ then, because $(b,y) \mapsto f(b^{-1}\bullet y)$ is continuous, there are a $U \in \mathcal{U}$ and an open neighbourhood $V$ of $x$ such that $|(b\bullet f)(y) - (a\bullet f)(x)| \leq \frac{1}{2}\epsilon$ whenever $b \in U$ and $y \in V$. But now $|(b\bullet f)(y) - (a\bullet f)(y)| \leq \epsilon$ whenever $b \in U$ and $y \in V$, so that $H_U$ includes $V$, which contains $x$. **Q**

So there is a $U \in \mathcal{U}$ such that $\nu(E \setminus H_U) \leq \epsilon$ (414Ea). In this case, for any $b \in U$, we must have

$$\int_E \min(1, |b\bullet f - a\bullet f|)d\nu \leq \epsilon(1 + \nu E).$$

As $E$ and $\epsilon$ are arbitrary, $b \mapsto (b\bullet f)^\bullet$ is continuous at $a$; as $a$ is arbitrary, it is continuous everywhere. Thus (b) is true.

**(c)(i)** Let us start by supposing that $E$ is an open set and that $\nu$ is totally finite. In this case the function $a \mapsto \nu(a\bullet E)$ is lower semi-continuous, by (a) above, therefore Borel measurable. Now let $W \subseteq \mathfrak{A}$ be an open set, and write $H = \{a : a \in G,\ (a\bullet E)^\bullet \in W\}$. For $m$, $k \in \mathbb{N}$ set $H_{mk} = \{a : 2^{-m}k \leq \nu(a\bullet E) < 2^{-m}(k+1)\}$, so that $H_{mk}$ is Borel, and $U_{mk} = G\setminus\overline{H_{mk} \setminus H}$, so that $U_{mk}$ is open. Let $H'$ be $\bigcup_{m,k\in\mathbb{N}} H_{mk} \cap U_{mk}$; then $H'$ is Borel and $H' \subseteq H$. In fact $H' = H$. **P** If $a \in H$, then $W$ is an open set containing $(a\bullet E)^\bullet$. Let $\delta > 0$ be such that $a \in W$ whenever $\bar\nu(a \triangle (a\bullet E)^\bullet) \leq \delta$, where $\bar\nu$ is the measure on $\mathfrak{A}$; let $m$, $k \in \mathbb{N}$ be such that $2^{-m} \leq \frac{1}{4}\delta$ and $2^{-m}k \leq \nu(a\bullet E) < 2^{-m}(k+1)$. If we take $\nu'$ to be the indefinite-integral measure over $\nu$ defined by $\chi(a\bullet E)$, then $\nu'$ is a quasi-Radon measure (415Ob), so (by (a) again) $U = \{b : \nu'(b\bullet E) > 2^{-m}(k-1)\}$ is an open set, and of course it contains $a$. If $b \in U \cap H_{mk}$, then

$$\begin{aligned}\nu((b\bullet E)\triangle(a\bullet E)) &= \nu(b\bullet E) + \nu(a\bullet E) - 2\nu((b\bullet E) \cap (a\bullet E))\\ &\leq 2^{-m}(k+1) + 2^{-m}(k+1) - 2\cdot 2^{-m}(k-1) \leq 4\cdot 2^{-m} \leq \delta,\end{aligned}$$

so $(b\bullet E)^\bullet \in W$ and $b \in H$. This shows that $U \cap (H_{mk} \setminus H) = \emptyset$ and $U \subseteq U_{mk}$ and $a \in U \cap H_{mk} \subseteq H'$. As $a$ is arbitrary, $H = H'$. **Q** Thus $H$ is a Borel subset of $G$. As $W$ is arbitrary, the map $a \mapsto (a\bullet E)^\bullet$ is Borel measurable.

**(ii)** To extend this to a general $\sigma$-finite quasi-Radon measure $\nu$, still supposing that $E$ is open, let $h : X \to \mathbb{R}$ be a strictly positive integrable function (215B(viii)) and $\nu'$ the corresponding indefinite-integral measure. As in (i), this $\nu'$ also is a quasi-Radon measure. Since $\nu$ and $\nu'$ have the same domains and the same null ideals, the Boolean algebra $\mathfrak{A}$ is still the underlying algebra of the measure algebra of $\nu'$; by 324H, the topologies on $\mathfrak{A}$ induced by the measures $\bar\nu$, $\bar\nu'$ are the same. So we can apply (i) to the measure $\nu'$ to see that $a \mapsto (a\bullet E)^\bullet : G \to \mathfrak{A}$ is still Borel measurable.

**(iii)** Next, suppose that $E$ is expressible as $\bigcup_{i\leq n} V_{2i} \setminus V_{2i+1}$ where each $V_i$ is open. Then $a \mapsto (a\bullet E)^\bullet$ is Borel measurable. **P** Set $X' = X \times \{0,\dots,2n+1\}$, with the product topology (giving $\{0,\dots,2n+1\}$ its discrete topology) and define a measure $\nu'$ on $X$ and an action of $G$ on $X'$ by setting

$$\nu' F = \sum_{i=0}^{2n+1} \nu\{x : (x,i) \in F\}$$

whenever $F \subseteq X'$ is such that $\{x : (x, i) \in F\} \in \operatorname{dom}\nu$ for every $i \le 2n+1$,

$$a \bullet (x, i) = (a \bullet x, i)$$

whenever $a \in G$, $x \in X$ and $i \le 2n+1$. Then $V = \{(x, i) : i \le 2n+1, \, x \in V_i\}$ is an open set in $X'$, while $\nu'$ is a $\sigma$-finite quasi-Radon measure, as is easily checked; so, by (ii), the map $a \mapsto (a \bullet V)^\bullet : G \to \mathfrak{A}'$ is Borel measurable, where $\mathfrak{A}'$ is the measure algebra of $\nu'$. On the other hand, we can identify $\mathfrak{A}'$ with the simple power $\mathfrak{A}^{2n+2}$ (322Lb), and the map

$$\langle c_i \rangle_{i \le 2n+1} \mapsto \sup_{i \le n} c_{2i} \setminus c_{2i+1} : \mathfrak{A}^{2n+2} \to \mathfrak{A}$$

is continuous, by 323B. So the map

$$a \mapsto (a \bullet E)^\bullet = \sup_{i \le n} (a \bullet V_{2i})^\bullet \setminus (a \bullet V_{2i+1})^\bullet$$

is the composition of a Borel measurable function with a continuous function, and is Borel measurable. **Q**

**(iv)** Now the family $\mathcal{E}$ of all those Borel sets $E \subseteq X$ such that $a \mapsto (a \bullet E)^\bullet$ is Borel measurable is closed under unions and intersections of monotonic sequences. **P** $(\alpha)$ If $\langle E_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathcal{E}$ with union $E$, then

$$(a \bullet E)^\bullet = \sup_{n \in \mathbb{N}} (a \bullet E_n)^\bullet = \lim_{n \to \infty} (a \bullet E_n)^\bullet$$

(323Ea) for every $a \in G$. So $a \mapsto (a \bullet E)^\bullet$ is the pointwise limit of a sequence of Borel measurable functions into a metrizable space (323Gb, because $(\mathfrak{A}, \bar{\nu})$ is $\sigma$-finite), and is Borel measurable, by 418Ba. Thus $E \in \mathcal{E}$. $(\beta)$ If $\langle E_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{E}$ the same argument applies, since this time

$$(a \bullet E)^\bullet = \inf_{n \in \mathbb{N}} (a \bullet E_n)^\bullet = \lim_{n \to \infty} (a \bullet E_n)^\bullet$$

(323Eb) for every $a \in G$. **Q**

Since $\mathcal{E}$ contains all sets of the form $\bigcup_{i \le n} V_i \cap F_i$ where every $V_i$ is open and every $F_i$ is closed, by (iii), $\mathcal{E}$ must be the whole Borel $\sigma$-algebra, by 4A3C(g-ii).

This completes the proof of (c).

**(d)(i)** We need the following extension of (c): if $\langle E_n \rangle_{n \in \mathbb{N}}$ is any sequence of Borel sets in $X$, then $a \mapsto \langle (a \bullet E_n)^\bullet \rangle_{n \in \mathbb{N}} : G \mapsto \mathfrak{A}^{\mathbb{N}}$ is Borel measurable. **P** I repeat the idea of (c-iii) above. On $X' = X \times \mathbb{N}$ define a measure $\nu'$ by setting

$$\nu' F = \sum_{n=0}^{\infty} \nu\{x : (x, n) \in F\}$$

whenever $F \subseteq X'$ is such that $\{x : (x, n) \in F\} \in \operatorname{dom}\nu$ for every $n \in \mathbb{N}$. As before, it is easy to check that $\nu'$ is a $\sigma$-finite quasi-Radon measure, if we give $\mathbb{N}$ its discrete topology and $X'$ the product topology. As before, set $a \bullet (x, n) = (a \bullet x, n)$ for $a \in G$, $x \in X$ and $n \in \mathbb{N}$, to obtain a continuous action of $G$ on $X'$. Applying (c) to this action, the map $a \mapsto (a \bullet E)^\bullet : G \to \mathfrak{A}'$ is Borel measurable, where $\mathfrak{A}'$ is the measure algebra of $\nu'$ and $E = \{(x, n) : n \in \mathbb{N}, \, x \in E_n\}$. But we can identify $\mathfrak{A}'$ (as Boolean algebra) with $\mathfrak{A}^{\mathbb{N}}$, by 322L, as before; so that if we re-interpret $a \mapsto (a \bullet E)^\bullet : G \to \mathfrak{A}'$ as $a \mapsto \langle (a \bullet E_n)^\bullet \rangle_{n \in \mathbb{N}} : G \to \mathfrak{A}^{\mathbb{N}}$ it is still Borel measurable. (As in (c-ii), this time using 323L, the measure-algebra topology of $\mathfrak{A}'$ matches the product topology on $\mathfrak{A}^{\mathbb{N}}$.) **Q**

**(ii)** Now suppose that $f : X \to [0, 1]$ is Borel measurable. Define $\langle E_n \rangle_{n \in \mathbb{N}}$ inductively by the formula

$$E_n = \{x : x \in X, \, (f - \sum_{i<n} 2^{-i-1} \chi E_i)(x) \ge 2^{-n-1}\}.$$

Then every $E_n$ is a Borel set and $f = \sum_{n=0}^{\infty} 2^{-n-1} \chi E_n$. Next, observe that the function

$$\langle c_n \rangle_{n \in \mathbb{N}} \mapsto \sum_{n=0}^{\infty} 2^{-n-1} \chi c_n : \mathfrak{A}^{\mathbb{N}} \to L^0$$

is continuous, because each of the maps $c \mapsto 2^{-n-1} \chi c$ is continuous (367R), addition is continuous (245Da, 367Ma) and the series is uniformly summable. Accordingly we may think of the map $a \mapsto (a \bullet f)^\bullet$ as the composition of the continuous function $\langle c_n \rangle_{n \in \mathbb{N}} \mapsto \sum_{n=0}^{\infty} 2^{-n-1} \chi c_i$ with the Borel measurable function $a \mapsto \langle (a \bullet E_n)^\bullet \rangle_{n \in \mathbb{N}}$, and it is Borel measurable.

**(iii)** For general Borel measurable $f : X \to \mathbb{R}$, set $q(t) = \frac{1}{2}(1 + \frac{t}{|t|+1})$, so that $q : \mathbb{R} \to ]0, 1[$ is a homeomorphism, and set $p = q^{-1} : ]0, 1[ \to \mathbb{R}$. Then the function $\bar{p} : P \to L^0$ is continuous, where $P = \{u : u \in L^0, \, [\![ u \in ]0, 1[ \, ]\!] = 1\}$ (367S). But now

$$a\bullet f = p\circ q\circ(a\bullet f) = p\circ(a\bullet(q\circ f))$$

for every $a$, so that $a \mapsto (a\bullet f)^\bullet$ is the composition of the Borel map $a \mapsto (a\bullet(q\circ f))^\bullet$ with the continuous map $\bar p$, and is Borel measurable.

**(e)(i)** We need only recall that $\bar\nu : \mathfrak{A} \to \mathbb{R}$ is lower semi-continuous (323Cb), so that (applying (c) above)

$$a \mapsto \nu(a\bullet E) = \bar\nu(a\bullet E)^\bullet$$

is a composition of Borel measurable functions and is Borel measurable. (Of course there are much more direct arguments, using fragments of the proof above.)

**(ii)** The point is that the maps $u \mapsto u^+$, $u \mapsto u^- : L^0 \to (L^0)^+$ are continuous (245Db, 367M), while $u \mapsto \int u : (L^0)^+ \to [0,\infty]$ is lower semi-continuous (369Mb), therefore Borel measurable. Accordingly $a \mapsto \int(a\bullet f)^+ = \int((a\bullet f)^\bullet)^+$ and $a \mapsto \int(a\bullet f)^-$ are Borel measurable functions from $G$ to $[0,\infty]$, so that

$$Q = \{a : \min(\int(a\bullet f)^+, \int(a\bullet f)^-) < \infty\}$$

is a Borel set, and

$$a \mapsto \int a\bullet f = \int(a\bullet f)^+ - \int(a\bullet f)^- : Q \to [-\infty,\infty]$$

is Borel measurable.

**444G Corollary** Let $X$ be a topological group and $\nu$ a $\sigma$-finite quasi-Radon measure on $X$.

(a) If $f : X \to \mathbb{R}$ is a Borel measurable function, then $\{x : \int f(y^{-1}x)\nu(dy)$ is defined in $[-\infty,\infty]\}$ is a Borel set in $X$ and $x \mapsto \int f(y^{-1}x)\nu(dy)$ is Borel measurable.

(b) If $f$, $g : X \to \mathbb{R}$ are Borel measurable functions, then $\{x : \int f(xy^{-1})g(y)\nu(dy)$ is defined in $[-\infty,\infty]\}$ is a Borel set and $x \mapsto \int f(xy^{-1})g(y)\nu(dy)$ is Borel measurable.

(c) If $\nu$ is totally finite and $f : X \to \mathbb{R}$ is a bounded continuous function, then $x \mapsto \int f(y^{-1}x)\nu(dy) : X \to \mathbb{R}$ is continuous.

**proof (a)** Set $\overleftrightarrow{f}(x) = f(x^{-1})$ for $x \in X$ (4A5C(c-ii)); then $\overleftrightarrow{f}$ is Borel measurable. Let $\bullet_l$ be the left action of $X$ on itself. Then, in the language of 444Fe, $Q = \{x : \int x\bullet_l\overleftrightarrow{f}\,d\nu$ is defined in $[-\infty,\infty]\}$ is a Borel set, and $x \mapsto \int x\bullet_l\overleftrightarrow{f}\,d\nu$ is Borel measurable. But

$$(x\bullet_l\overleftrightarrow{f})(y) = \overleftrightarrow{f}(x^{-1}y) = f(y^{-1}x)$$

for all $x$, $y$, so $\int x\bullet_l\overleftrightarrow{f}\,d\nu = \int f(y^{-1}x)\nu(dy)$ if either integral is defined.

**(b)(i)** Set $\overleftrightarrow{\nu}E = \nu E^{-1}$ when this is defined, writing $E^{-1} = \{x^{-1} : x \in E\}$ for $E \subseteq X$; that is, $\overleftrightarrow{\nu}$ is the image measure $\nu\phi^{-1}$, where $\phi(x) = x^{-1}$ for $x \in X$. Because $\phi$ is a homeomorphism, $\overleftrightarrow{\nu}$ is a quasi-Radon measure. By 235J, $\int h\,d\overleftrightarrow{\nu} = \int h(x^{-1})\nu(dx)$ for any real-valued function $h$ for which either integral is defined in $[-\infty,\infty]$.

**(ii)** Now suppose that $f$ and $g$ are non-negative Borel measurable functions from $X$ to $\mathbb{R}$. Then $\overleftrightarrow{g}$ also is a non-negative Borel measurable function. We know from 444Fd that $x \mapsto (x^{-1}\bullet_l f)^\bullet : X \to L^0(\overleftrightarrow{\nu})$ is a Borel measurable function; now multiplication in $L^0$ is continuous (367Mb), so the map $x \mapsto ((x^{-1}\bullet_l f)\times\overleftrightarrow{g})^\bullet$ is Borel measurable; since integration is lower semi-continuous on $(L^0)^+$, $x \mapsto \int(x^{-1}\bullet_l f)\times\overleftrightarrow{g}\,d\overleftrightarrow{\nu} : X \to [0,\infty]$ is Borel measurable. But

$$\int(x^{-1}\bullet_l f)\times\overleftrightarrow{g}\,d\overleftrightarrow{\nu} = \int f(xy)g(y^{-1})\overleftrightarrow{\nu}(dy) = \int f(xy^{-1})g(y)\nu(dy)$$

whenever any of the integrals is defined in $[-\infty,\infty]$, so this is the function we needed to know about.

**(iii)** For general Borel measurable functions $f$ and $g$, we have

$$\begin{aligned}\int f(xy^{-1})g(y)\nu(dy) = \Bigl(\int f^+(xy^{-1})g^+(y)\nu(dy) + \int f^-(xy^{-1})g^-(y)\nu(dy)\Bigr)\\ - \Bigl(\int f^+(xy^{-1})g^-(y)\nu(dy) + \int f^-(xy^{-1})g^+(y)\nu(dy)\Bigr)\end{aligned}$$

exactly when the subtraction can be done in $[-\infty,\infty]$, so that $x\mapsto\int f(xy^{-1})g(y)\nu(dy)$ is a difference of Borel measurable functions and is Borel measurable, with Borel measurable domain.

**(c)** Continuing the argument of (a), if $f$ is bounded and continuous and $\nu$ is totally finite then all the functions $x\bullet_l\overset{\leftrightarrow}{f}$ are bounded and continuous, so $\int x\bullet_l\overset{\leftrightarrow}{f}\,d\nu$ is defined and finite for every $x$. Next, the function $x\mapsto(x\bullet_l\overset{\leftrightarrow}{f})^\bullet:X\to L^0(\nu)$ is continuous for the topology of convergence in measure (444Fb), which agrees with the norm topology of $L^1(\nu)$ on $\|\,\|_\infty$-bounded sets (246Jb). It follows that

$$x\mapsto\int(x\bullet_l\overset{\leftrightarrow}{f})^\bullet=\int f(y^{-1}x)\nu(dy)$$

is continuous.

**444H Convolutions of measures and functions** I introduce some notation which I shall use for the rest of the section. Let $X$ be a topological group. If $f$ is a real-valued function defined on a subset of $X$, and $\nu$ is a measure on $X$, set

$$(\nu*f)(x)=\int f(y^{-1}x)\nu(dy)$$

whenever the integral is defined in $\mathbb{R}$.

**444I Proposition** Let $X$ be a topological group and $\lambda$, $\nu$ two totally finite quasi-Radon measures on $X$.

(a) For any Borel measurable function $f:X\to\mathbb{R}$, $\nu*f$ is a Borel measurable function with a Borel domain.

(b) $\nu*f\in C_b(X)$ for every $f\in C_b(X)$.

(c) For any real-valued function $f$ defined on a subset of $X$, $(\lambda*(\nu*f))(x)=((\lambda*\nu)*f)(x)$ whenever the right-hand side is defined.

**proof (a)** This follows at once from 444Ga.

**(b)** This is just a restatement of 444Gc.

**(c)** If $((\lambda*\nu)*f)(x)$ is defined, then

$$\begin{aligned}((\lambda*\nu)*f)(x)&=\int f(t^{-1}x)(\lambda*\nu)(dt)=\iint f((yz)^{-1}x)\nu(dz)\lambda(dy)\\&\qquad\text{(444C)}\\&=\iint f(z^{-1}y^{-1}x)\nu(dz)\lambda(dy)\\&=\int(\nu*f)(y^{-1}x)\lambda(dy)=(\lambda*(\nu*f))(x).\end{aligned}$$

**444J Convolutions of functions and measures** Let $X$ be a topological group carrying Haar measures; let $\Delta$ be its left modular function (442I). If $f$ is a real-valued function defined on a subset of $X$, and $\nu$ is a measure on $X$, set

$$(f*\nu)(x)=\int f(xy^{-1})\Delta(y^{-1})\nu(dy)$$

whenever the integral is defined in $\mathbb{R}$. From 444Gb we see that if $\nu$ is a $\sigma$-finite quasi-Radon measure and $f$ is Borel measurable, then $f*\nu$ is a Borel measurable function with a Borel domain. If $f$ is non-negative and $\nu$-integrable, write $f\nu$ for the corresponding indefinite-integral measure over $\nu$ (234J[3]).

**444K Proposition** Let $X$ be a topological group with a left Haar measure $\mu$. Let $\nu$ be a totally finite quasi-Radon measure on $X$. Then for any non-negative $\mu$-integrable real-valued function $f$, $f\mu$ is a quasi-Radon measure; moreover, $\nu*f$ and $f*\nu$ are $\mu$-integrable, and we have

[3]Formerly 234B.

$$(\nu * f)\mu = \nu * f\mu, \quad (f * \nu)\mu = f\mu * \nu.$$

In particular, $\int \nu * f \, d\mu = \int f * \nu \, d\mu = \nu X \cdot \int f d\mu$.

**proof (a)** $f\mu$ is a quasi-Radon measure by 415Ob.

**(b)** Suppose first that $f$ is Borel measurable and defined everywhere on $X$, as well as being non-negative and $\mu$-integrable.

**(i)** Let $E \subseteq X$ be a Borel set such that $\mu E < \infty$. The function $(x, y) \mapsto f(x^{-1}y)\chi E(y) : X \times X \to [0, \infty[$ is Borel measurable, so

$$(\nu * f\mu)(E) = \int (f\mu)(x^{-1}E)\nu(dx)$$

(444A)

$$= \iint f(y)\chi(x^{-1}E)(y)\mu(dy)\nu(dx)$$

(by the definition of $f\mu$, 234I[4])

$$= \iint f(x^{-1}y)\chi(x^{-1}E)(x^{-1}y)\mu(dy)\nu(dx)$$

(441J)

$$= \iint f(x^{-1}y)\chi E(y)\mu(dy)\nu(dx) = \iint f(x^{-1}y)\chi E(y)\nu(dx)\mu(dy)$$

(by Fubini's theorem, 417Ga, because $(x, y) \mapsto f(x^{-1}y)\chi E(y)$ is non-negative and Borel measurable and zero outside $X \times E$). So $\int f(x^{-1}y)\nu(dx)$ must be finite for $\mu$-almost every $y \in E$. Because $\mu$ is complete and locally determined and inner regular with respect to the Borel sets of finite measure, $(\nu * f)(y) = \int f(x^{-1}y)\nu(dx)$ is defined in $\mathbb{R}$ for $\mu$-almost every $y \in X$. So we have an indefinite-integral measure $(\nu * f)\mu$. Next, we have

$$\infty > \nu X \int f d\mu = (\nu * f\mu)(X) \geq (\nu * f\mu)(E)$$
$$= \iint f(x^{-1}y)\chi E(y)\nu(dx)\mu(dy) = \iint (\nu * f)(y)\chi E(y)\mu(dy) = ((\nu * f)\mu)(E)$$

for every Borel set $E$ such that $\mu E$ is finite. Again because $\mu$ is inner regular with respect to the Borel sets of finite measure, $\nu * f$ is $\mu$-integrable and $(\nu * f)\mu$ is totally finite. Since $\nu * f\mu$ and $(\nu * f)(\mu)$ are totally finite quasi-Radon measures agreeing on open sets of finite measure for $\mu$, and $\mu$ is locally finite (442Aa), 415H(iv) assures us that $\nu * f\mu = (\nu * f)(\mu)$.

**(ii)** Now consider $f * \nu$. This time, if $E \subseteq X$ is Borel and $\mu E < \infty$,

$$(f\mu * \nu)(E) = \int (f\mu)(Ey^{-1})\nu(dy) = \iint_{Ey^{-1}} f(x)\mu(dx)\nu(dy)$$
$$= \int \Delta(y^{-1}) \int_E f(xy^{-1})\mu(dx)\nu(dy)$$

(by 442Kc)

$$= \int_E \int \Delta(y^{-1})f(xy^{-1})\nu(dy)\mu(dx).$$

Once again, we see that $\int \Delta(y^{-1})f(xy^{-1})\nu(dy)$ is defined for $\mu$-almost every $x \in E$; as $E$ is arbitrary, $(f * \nu)(x)$ is defined in $\mathbb{R}$ for $\mu$-almost every $x \in X$; and 444Gb tells us that $f * \nu$ is Borel measurable. As before,

[4] Formerly 234A.

$$\int_E f*\nu\,d\mu \le (f\mu*\nu)(X) = \int f d\mu\cdot\nu X < \infty$$

for Borel sets $E$ with $\mu E < \infty$, so that $f*\nu$ is $\mu$-integrable; as before, the quasi-Radon measures $(f*\nu)\mu$ and $f\mu*\nu$ agree on open sets of finite $\mu$-measure, so coincide.

**(c)** Next, suppose that $f$ is defined and zero $\mu$-a.e. In this case there is a Borel set $E$ such that $\mu E = 0$ and $f(x)$ is defined and equal to zero for every $x \in X\setminus E$ (443J(b-ii)). Set $g = \chi E$. Then $g\mu$ is the zero measure, so $(\nu*g)\mu = \nu*g\mu$, $(g*\nu)\mu = g\mu*\nu$ are all zero; that is, there is some $\mu$-conegligible set $F$ such that

$$0 = (\nu*g)(x) = \int \chi E(y^{-1}x)\nu(dy) = \nu(xE^{-1}),$$

$$0 = (g*\nu)(x) = \int \chi E(xy^{-1})\Delta(y^{-1})\nu(dy) = \int_{E^{-1}x}\Delta(y^{-1})\nu(dy)$$

for every $x \in F$. But now, for $x \in F$, we must have $\nu(xE^{-1}) = \nu(E^{-1}x) = 0$ (since $\Delta$ is strictly positive), so that

$$(\nu*f)(x) = \int f(y^{-1}x)\nu(dy) = \int_{xE^{-1}} f(y^{-1}x)\nu(dy) = 0,$$

because if $y \notin xE^{-1}$ then $y^{-1}x \notin E$ and $f(y^{-1}x) = 0$. Similarly,

$$(f*\nu)(x) = \int f(xy^{-1})\Delta(y^{-1})\nu(dy) = \int_{E^{-1}x} f(xy^{-1})\Delta(y^{-1})\nu(dy) = 0.$$

Thus $\nu*f$ and $f*\nu$ are defined, and zero, $\mu$-almost everywhere.

**(d)** For an arbitrary non-negative $\mu$-integrable function $f$, we can express it in the form $g+h$ where $g$ is a non-negative $\mu$-integrable Borel measurable function defined everywhere, and $h$ is zero almost everywhere. In this case, $\nu*h^+$, $\nu*h^-$, $h^+*\nu$ and $h^-*\nu$ are defined, and zero, $\mu$-a.e., so $\nu*f =_{\text{a.e.}} \nu*g$ and $f*\nu =_{\text{a.e.}} g*\nu$. We therefore have

$$(\nu*f)\mu = (\nu*g)\mu = \nu*g\mu = \nu*f\mu, \quad (f*\nu)\mu = (g*\nu)\mu = g\mu*\nu = f\mu*\nu,$$

as required.

**(e)** Finally, we have

$$\int \nu*f d\mu = ((\nu*f)\mu)(X) = (\nu*f\mu)(X) = \nu X\cdot(f\mu)(X) = \nu X\cdot\int f d\mu,$$

$$\int f*\nu\,d\mu = ((f*\nu)\mu)(X) = (f\mu*\nu)(X) = (f\mu)(X)\cdot\nu X = \nu X\cdot\int f d\mu.$$

**444L Corollary** Let $X$ be a topological group carrying Haar measures. Suppose that $\nu$ is a non-zero quasi-Radon measure on $X$ and $E \subseteq X$ is a Haar measurable set such that $\nu(xE) = 0$ for every $x \in X$. Then $E$ is Haar negligible.

**proof** Let $\mu$ be a left Haar measure on $X$. There is a non-zero totally finite quasi-Radon measure $\nu'$ on $X$ such that $\nu'(xE) = 0$ for every $x \in X$. **P** Take any $F$ such that $0 < \nu F < \infty$, and set $\nu' H = \nu(H\cap F)$ whenever this is defined. **Q** Let $G$ be any Borel set such that $\mu G < \infty$, and set $f = \chi(G\cap E^{-1})$. Then $f$ is $\mu$-integrable, and

$$(\nu'*f)(x) = \int \chi(G\cap E^{-1})(y^{-1}x)\nu'(dy) = \nu'(xG^{-1}\cap xE) = 0$$

for every $x \in X$. By 444K, $\nu'*f\mu = (\nu'*f)\mu$ is the zero measure, and $(f\mu)(X) = 0$, that is, $\mu(G\cap E^{-1}) = 0$. As $G$ is arbitrary, $\mu E^{-1} = 0$ and $E$ is Haar negligible (442H).

**444M Proposition** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Let $\nu$ be a quasi-Radon measure on $X$ and $p \in [1,\infty]$.

(a) Suppose that $\nu X < \infty$. Then we have a bounded positive linear operator $u \mapsto \nu*u : L^p(\mu) \to L^p(\mu)$, of norm at most $\nu X$, defined by saying that $\nu*f^\bullet = (\nu*f)^\bullet$ for every $f \in \mathcal{L}^p(\mu)$.

(b) Set $\gamma = \int \Delta(y)^{(1-p)/p}\nu(dy)$ if $p < \infty$, $\int \Delta(y)^{-1}\nu(dy)$ if $p = \infty$, where $\Delta$ is the left modular function of $X$. Suppose that $\gamma < \infty$. Then we have a bounded positive linear operator $u \mapsto u*\nu : L^p(\mu) \to L^p(\mu)$, of norm at most $\gamma$, defined by saying that $f^\bullet*\nu = (f*\nu)^\bullet$ for every $f \in \mathcal{L}^p(\mu)$.

**proof** I will write $\mathcal{L}^p$, $L^p$ for $\mathcal{L}^p(\mu)$, $L^p(\mu)$. Note that if $f_1$, $f_2 \in \mathcal{L}^0(\mu)$ and $f_1 = f_2$ $\mu$-a.e., then 444L tells us that $\nu * |f_1 - f_2|$ and $|f_1 - f_2| * \nu$ are both zero $\mu$-almost everywhere, so that $\nu * f_1 =_{\text{a.e.}} \nu * f_2$ and $f_1 * \nu =_{\text{a.e.}} f_2 * \nu$, in the sense that there is a $\mu$-conegligible set $F$ such that $(\nu * f_1)\restriction F = (\nu * f_2)\restriction F$ and $(f_1 * \nu)\restriction F = (f_2 * \nu)\restriction F$. In particular, if we are told that $\nu * f_1$ belongs to $\mathcal{L}^p$, and that $f_1^\bullet = f_2^\bullet$ in $L^0(\mu)$, then we can be sure that $\nu * f_2 \in \mathcal{L}^p$ and $(\nu * f_2)^\bullet = (\nu * f_1)^\bullet$; and similarly for $f_1 * \nu$, $f_2 * \nu$.

**(a)** If $\nu X = 0$ the result is trivial. Multiplying $\nu$ by a positive scalar does not affect the inequalities we need, so we may suppose that $\nu X = 1$. If $f \geq 0$ is $\mu$-integrable, then 444K tells us that $\nu * f$ is $\mu$-integrable and that

$$\begin{aligned}\|\nu * f\|_1 &= \int_X (\nu * f)(x)\mu(dx) = ((\nu * f)\mu)(X) \\ &= (\nu * f\mu)(X) = \nu X \cdot (f\mu)(X) = \|f\|_1,\end{aligned}$$

using 444C or 444A for the penultimate equality. Since evidently $\nu * (f + g) = \nu * f + \nu * g$, $\nu * (\alpha f) = \alpha \nu * f$ at any point where the right-hand sides of the equations are defined in $\mathbb{R}$, we have a positive linear operator $T_1 : L^1 \to L^1$ defined by saying that $T_1 g^\bullet = (\nu * g)^\bullet$ for every $\mu$-integrable Borel measurable function $g$, with $\|T_1\| = 1$.

Similarly, if $h : X \to \mathbb{R}$ is a bounded Borel measurable function, then $\nu * h$ also is a Borel measurable function, by 444Ga. Of course it is bounded, since

$$|(\nu * h)(x)| = |\int h(y^{-1}x)\nu(dy)| \leq \sup_{y \in X} |h(y)|$$

for every $x$. So we have a positive linear operator $T_\infty : L^\infty \to L^\infty$ defined by saying that $T_\infty h^\bullet = (\nu * h)^\bullet$ for every bounded Borel measurable function $h$. Moreover, if $u \in L^\infty$, there is a Borel measurable $h$ such that $h^\bullet = u$ and $\sup_{y \in X} |h(y)| = \|u\|_\infty$, so that

$$\|T_\infty u\|_\infty \leq \sup_{x \in X} |(\nu * h)(x)| \leq \|u\|_\infty;$$

thus $\|T_\infty\| \leq 1$.

Since $T_1$ and $T_\infty$ agree on $L^1 \cap L^\infty$, they have a common extension to a linear operator $T : L^1 + L^\infty \to L^1 + L^\infty$. By 371Gd, $\|Tu\|_p \leq \|u\|_p$ whenever $p \in [1, \infty]$ and $u \in L^p$. (Strictly speaking, I am relying on the standard identifications of $L^1$, $L^\infty$ and $L^p$ with the corresponding subspaces of $L^0(\mathfrak{A})$, where $(\mathfrak{A}, \bar\mu)$ is the measure algebra of $\mu$. Of course the argument for 371Gd applies equally well in $L^0(\mu)$.) Now suppose that $f \in \mathcal{L}^p$. Then it is expressible as $g + h$ where $g \in \mathcal{L}^1$ and $h : X \to \mathbb{R}$ is a bounded Borel measurable function, so we shall have

$$\nu * f = \nu * g + \nu * h \text{ wherever the right-hand side is defined;}$$

accordingly $\nu * f$ is defined $\mu$-a.e. and is measurable, and

$$\begin{aligned}\|\nu * f\|_p &= \|(\nu * g)^\bullet + (\nu * h)^\bullet\|_p = \|T_1 g^\bullet + T_\infty h^\bullet\|_p \\ &= \|Tf^\bullet\|_p \leq \|f^\bullet\|_p = \|f\|_p,\end{aligned}$$

as required.

**(b)(i)** As in (a), the case $\nu X = 0$ is trivial. Otherwise, because $\Delta$ is strictly positive, $\gamma > 0$; again considering a scalar multiple of $\nu$ if necessary, we may suppose that $\gamma = 1$. Note that $\nu$ is surely $\sigma$-finite.

**(ii)** If $p = 1$, then $\nu X = \gamma = 1$. If $f \in \mathcal{L}^1$ is non-negative, then, by 444K, as in (a) above,

$$\|f * \nu\|_1 = ((f * \nu)\mu)(X) = (f\mu * \nu)(X) = (f\mu)(X) \cdot \nu X = \|f\|_1.$$

For general $\mu$-integrable $f$,

$$\|f * \nu\|_1 \leq \|f^+ * \nu\|_1 + \|f^- * \nu\|_1 = \|f^+\|_1 + \|f^-\|_1 = \|f\|_1.$$

**(iii)** If $p = \infty$, then directly from the formula $(f * \nu)(x) = \int f(xy^{-1})\Delta(y^{-1})\nu(dy)$ we see that if $f : X \to \mathbb{R}$ is a bounded Borel measurable function then

$$|(f * \nu)(x)| \leq \int \Delta(y)^{-1}\nu(dy) \cdot \sup_{y \in X} |f(y)| = \sup_{y \in X} |f(y)|$$

for every $x$. Since changing $f$ on a $\mu$-negligible set changes $f*\nu$ on a $\mu$-negligible set, we can argue as in (a) above to see that $f^\bullet \mapsto (f*\nu)^\bullet$ defines a linear operator from $L^\infty$ to itself of norm at most 1.

**(iv)** Now suppose that $1<p<\infty$. Set $q=\frac{p}{p-1}$, so that $\int \Delta(y^{-1})^{1/q}\nu(dy)=\gamma=1$. Suppose for the moment that $f\in\mathcal{L}^p$ is a non-negative Borel measurable function, and let $h:X\to\mathbb{R}$ be another non-negative Borel measurable function such that $\int h^q d\mu\le 1$. In this case

$$\int (f*\nu)\times h\,d\mu=\iint h(x)f(xy^{-1})\Delta(y^{-1})\nu(dy)\mu(dx)$$
$$=\iint h(x)f(xy^{-1})\Delta(y^{-1})\mu(dx)\nu(dy)$$

(by 417Ga, because $(x,y)\mapsto h(x)f(xy^{-1})\Delta(y^{-1})$ is Borel measurable and $\{x:h(x)\ne 0\}$ is a countable union of sets of finite measure for $\mu$, while $\nu$ is $\sigma$-finite)

$$=\iint h(xy)f(x)\mu(dx)\nu(dy)$$

by 442Kc, as usual, at least if the last integral is finite. But, for any $y\in X$,

$$\int h(xy)f(x)\mu(dx)\le\|f\|_p\bigl(\int|h(xy)|^q\mu(dx)\bigr)^{1/q}$$
$$=\|f\|_p\bigl(\Delta(y^{-1})\int|h(x)|^q\mu(dx)\bigr)^{1/q}\le\|f\|_p\Delta(y^{-1})^{1/q}.$$

So

$$\int(f*\nu)\times h\,d\mu=\iint h(xy)f(x)\mu(dx)\nu(dy)\le\int\|f\|_p\Delta(y^{-1})^{1/q}\nu(dy)=\|f\|_p.$$

Because $\mu$ (being a quasi-Radon measure) is semi-finite, this means that $f*\nu\in\mathcal{L}^p$ and that $\|f*\nu\|_p\le\|f\|_p$ (366D-366E, or 244Xe and 244Fa). (Once again, we need to know that every member of $L^q$ can be represented by a Borel measurable function; this is a consequence of 443J or 412Xe.)

For general Borel measurable $f:X\to\mathbb{R}$ such that $\int|f|^pd\mu<\infty$, we know that from 444G that $f*\nu$ is Borel measurable, while $|f*\nu|\le|f|*\nu$ (and $f*\nu$ is defined wherever $|f|*\nu$ is finite), so that

$$\|f*\nu\|_p\le\||f|*\nu\|_p\le\||f|\|_p=\|f\|_p.$$

Finally, if $f\in\mathcal{L}^p$ is arbitrary, then there is a Borel measurable $g:X\to\mathbb{R}$ such that $f=_{\text{a.e.}}g$, so that $f*\nu=_{\text{a.e.}}g*\nu$ and

$$\|f*\nu\|_p=\|g*\nu\|_p\le\|g\|_p=\|f\|_p.$$

It follows at once that we have a bounded linear operator $f^\bullet\mapsto(f*\nu)^\bullet:L^p\to L^p$, of norm at most $1=\gamma$.

**444N** The following lemma on exchanging the order of repeated integrals will be fundamental to the formulae in the rest of the section.

**Lemma** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Suppose that $f$, $g$, $h\in\mathcal{L}^0(\mu)$ (the space of measurable real-valued functions defined $\mu$-a.e. in $X$) are non-negative. Then, writing $\int\ldots d(x,y)$ to denote integration with respect to the quasi-Radon product measure $\mu\times\mu$,

$$\iint f(x)g(y)h(xy)dxdy=\iint f(x)g(y)h(xy)dydx=\int f(x)g(y)h(xy)d(x,y)$$

in $[0,\infty]$.

**proof** Following the standard pattern in results of this type, I deal with successively more complicated functions $f$, $g$ and $h$. Evidently the situation is symmetric, so that it will be enough if I can show that $\iint f(x)g(y)h(xy)dxdy=\int f(x)g(y)h(xy)d(x,y)$.

**(a)** Suppose first that $f=\chi F$, $g=\chi G$ and $h=\chi H$, where $F$, $G$, $H$ are Borel subsets of $X$. In this case

$$\iint f(x)g(y)h(xy)dxdy=\sup_{U,V\in\Sigma^f}\int_V\int_U f(x)g(y)h(xy)dxdy,$$

where $\Sigma^f$ is the ideal of measurable sets of finite measure for $\mu$. **P** For $y \in X$, $n \in \mathbb{N}$ and $U \in \Sigma^f$ write

$$q(y) = \int f(x)h(xy)dx = \mu(F \cap Hy^{-1}), \quad q_U(y) = \int_U f(x)h(xy)dx = \mu(U \cap F \cap Hy^{-1}),$$

$$q^{(n)}(y) = \min(n, q(y)), \quad q_U^{(n)}(y) = \min(n, q_U(y)).$$

Then every $q_U$ is continuous, by 443C (with a little help from 323Cc), while $\sup_{U \in \Sigma^f} q_U(y) = q(y)$ for every $y$, because $\mu$ is semi-finite. Because $\mu$ is $\tau$-additive and effectively locally finite, $(q^{(n)})^\bullet = \sup_{U \in \Sigma^f}(q_U^{(n)})^\bullet$ in $L^0(\mu)$ for every $n$ (414Ab); because $\Sigma^f$ is upwards-directed,

$$\begin{aligned}\int q(y)g(y)dy &= \sup_{n \in \mathbb{N}} \int q^{(n)}(y)g(y)dy \\ &= \sup_{n \in \mathbb{N}, U \in \Sigma^f} \int q_U^{(n)}(y)g(y)dy = \sup_{U \in \Sigma^f} \int q_U(y)g(y)dy,\end{aligned}$$

that is,

$$\iint f(x)g(y)h(xy)dxdy = \sup_{U \in \Sigma_f} \int \int_U f(x)g(y)h(xy)dxdy.$$

On the other hand, for any $U \in \Sigma^f$, we surely have

$$\int \int_U f(x)g(y)h(xy)dxdy = \sup_{V \in \Sigma^f} \int_V \int_U f(x)g(y)h(xy)dxdy,$$

again because $\mu$ is semi-finite. Putting these together, we have the result. **Q**

Looking at the other side of the equation, $\int f(x)g(y)h(xy)d(x,y) = (\mu \times \mu)W$, where $W = (F \times G) \cap \{(x,y) : xy \in H\}$ is a Borel set; so that

$$\begin{aligned}\iint f(x)g(y)h(xy)dxdy &= \sup_{U,V \in \Sigma^f} (\mu \times \mu)((U \times V) \cap W) \\ &= \sup_{U,V \in \Sigma^f} \int_{U \times V} f(x)g(y)h(xy)d(x,y)\end{aligned}$$

(417C(b-iii)). But now we can apply 417Ga to see that, for any $U$, $V \in \Sigma^f$,

$$\int_{U \times V} f(x)g(y)h(xy)d(x,y) = \int_V \int_U f(x)g(y)h(xy)dxdy.$$

Taking the supremum over $U$ and $V$, we get

$$\int f(x)g(y)h(xy)d(x,y) = \iint f(x)g(y)h(xy)dxdy.$$

**(b)** Clearly both sides of our equation

$$\int f(x)g(y)h(xy)d(x,y) = \iint f(x)g(y)h(xy)dxdy$$

are additive in $f$, $g$ and $h$ separately (subtraction, of course, will be another matter, as I am allowing $\infty$ to appear without restriction); and also behave identically if $f$ or $g$ or $h$ is multiplied by a non-negative scalar. So the identity will be valid if $f$, $g$ and $h$ are all finite sums of non-negative multiples of indicator functions of Borel sets. Moreover, by repeated use of B.Levi's theorem, we see that if $\langle f_n \rangle_{n \in \mathbb{N}}$, $\langle g_n \rangle_{n \in \mathbb{N}}$ and $\langle h_n \rangle_{n \in \mathbb{N}}$ are non-decreasing sequences of such functions with suprema $f$, $g$ and $h$, then

$$\begin{aligned}\int f(x)g(y)h(xy)d(x,y) &= \sup_{n \in \mathbb{N}} \int f_n(x)g_n(y)h_n(xy)d(x,y) \\ &= \sup_{n \in \mathbb{N}} \iint f_n(x)g_n(y)h_n(xy)dxdy = \iint f(x)g(y)h(xy)dxdy.\end{aligned}$$

So the identity is valid for all non-negative Borel measurable functions $f$, $g$ and $h$.

**(c)** Finally, suppose only that $f$, $g$ and $h$ are non-negative, measurable and defined almost everywhere. In this case, by 443J(b-iv), there are Borel measurable functions $f_0$, $g_0$ and $h_0$, non-negative, defined everywhere on $X$ and equal almost everywhere to $f$, $g$ and $h$ respectively. Let $E$ be the conegligible set

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} h \cap \operatorname{dom} g,\ f(x) = f_0(x),\ g(x) = g_0(x),\ h(x) = h_0(x)\}.$$

We find that $\int f(x)h(xy)dx = \int f_0(x)h_0(xy)dx$ for every $y \in X$. **P** $E \cap Ey^{-1}$ is conegligible (see 443A), and $f(x)h(xy) = f_0(x)h_0(xy)$ for every $x \in E \cap Ey^{-1}$. **Q** Consequently

$$\iint f(x)g(y)h(xy)dxdy = \iint f_0(x)g_0(y)h_0(xy)dxdy.$$

Secondly, $f(x)g(y)h(xy) = f_0(x)g_0(y)h_0(xy)$ $(\mu \times \mu)$-a.e. **P** Set $W = \{(x,y) : x \in E,\ y \in E,\ xy \in E\}$. Fubini's theorem, applied to $(U \times V) \setminus W$ where $U$, $V \in \Sigma^f$, shows that $W$ is conegligible; but of course $f(x)g(y)h(xy) = f_0(x)g_0(y)h_0(xy)$ whenever $(x,y) \in W$. **Q** Accordingly

$$\int f(x)g(y)h(xy)d(x,y) = \int f_0(x)g_0(y)h_0(xy)d(x,y).$$

Combining this with the result of (b), applied to $f_0$, $g_0$ and $h_0$, we see that once again

$$\int f(x)g(y)h(xy)d(x,y) = \iint f(x)g(y)h(xy)dxdy,$$

as required.

**444O Convolutions of functions: Theorem** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. For $f$, $g \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$, write $(f * g)(x) = \int f(y)g(y^{-1}x)dy$ whenever this is defined in $\mathbb{R}$, taking the integral with respect to $\mu$.

(a) Writing $\Delta$ for the left modular function of $X$,

$$\begin{aligned}(f * g)(x) &= \int f(y)g(y^{-1}x)dy = \int f(xy)g(y^{-1})dy \\ &= \int \Delta(y^{-1})f(y^{-1})g(yx)dy = \int \Delta(y^{-1})f(xy^{-1})g(y)dy\end{aligned}$$

whenever any of these integrals is defined in $\mathbb{R}$.

(b) If $f =_{\text{a.e.}} f_1$ and $g =_{\text{a.e.}} g_1$, then $f * g = f_1 * g_1$.

(c)(i) $|(f * g)(x)| \le (|f| * |g|)(x)$ whenever either is defined in $\mathbb{R}$.

(ii)

$$((f_1 + f_2) * g)(x) = (f_1 * g)(x) + (f_2 * g)(x),$$

$$(f * (g_1 + g_2))(x) = (f * g_1)(x) + (f * g_2)(x),$$

$$((\alpha f) * g)(x) = (f * (\alpha g))(x) = \alpha(f * g)(x)$$

whenever the right-hand expressions are defined in $\mathbb{R}$.

(d) If $f$, $g$ and $h$ belong to $\mathcal{L}^0$ and any of

$$\int (|f| * |g|)(x)|h|(x)dx, \quad \iint |f(x)g(y)h(xy)|dxdy,$$

$$\iint |f(x)g(y)h(xy)|dydx, \quad \int |f(x)g(y)h(xy)|d(x,y)$$

is defined in $[0, \infty[$ (writing $\int \ldots d(x,y)$ for integration with respect to the quasi-Radon product measure $\mu \times \mu$ on $X \times X$), then

$$\int (f * g)(x)h(x)dx, \quad \iint f(x)g(y)h(xy)dxdy,$$

$$\iint f(x)g(y)h(xy)dydx, \quad \int f(x)g(y)h(xy)d(x,y)$$

are all defined, finite and equal, provided that in the expression $(f * g)(x)h(x)$ we interpret the product as 0 when $h(x) = 0$ and $(f * g)(x)$ is undefined.

(e) If $f$, $g$ and $h$ belong to $\mathcal{L}^0$, $f*g$ and $g*h$ are defined a.e. and $x \in X$ is such that either $(|f|*(|g|*|h|))(x)$ or $((|f| * |g|) * |h|)(x)$ is defined in $\mathbb{R}$, then $(f * (g * h))(x)$ and $((f * g) * h)(x)$ are defined and equal.

(f) If $a \in X$ and $f$, $g \in \mathcal{L}^0$,

$$a\bullet_l(f * g) = (a\bullet_l f) * g, \quad a\bullet_r(f * g) = f * (a\bullet_r g),$$

$$(a\bullet_r f) * g = \Delta(a^{-1})f * (a^{-1}\bullet_l g),$$

$$\overset{\leftrightarrow}{f} * \overset{\leftrightarrow}{g} = (g * f)^{\leftrightarrow}.$$

(g) If $X$ is abelian then $f * g = g * f$ for all $f$ and $g$.

**proof (a)** Use 441J and 442Kb to see that the four formulae for $f * g$ coincide.

**(b)** Setting

$$E = \{y : y \in \operatorname{dom} f \cap \operatorname{dom} f_1 \cap \operatorname{dom} g \cap \operatorname{dom} g_1,\ f(y) = f_1(y),\ g(y) = g_1(y)\},$$

$E$ is conegligible. If $x \in X$, then $f(y)g(y^{-1}x) = f_1(y)g_1(y^{-1}x)$ for every $y \in E \cap xE^{-1}$, which is also conegligible, by 443A; so $(f * g)(x) = (f_1 * g_1)(x)$ if either is defined.

**(c)** These are all elementary.

**(d)** First consider non-negative $f$, $g$ and $h$. The point is that, if any of the integrals is defined and finite,

$$\begin{aligned}\int (f * g)(x)h(x)dx &= \iint \Delta(x^{-1})f(x^{-1})g(xy)h(y)dxdy \\ &= \iint \Delta(x^{-1})f(x^{-1})g(xy)h(y)dydx\end{aligned}$$

(by 444N, recalling that $x \mapsto \Delta(x^{-1})f(x^{-1})$ will belong to $\mathcal{L}^0$ if $f$ does, by 442J and 442H)

$$= \iint f(x)g(x^{-1}y)h(y)dydx = \iint f(x)g(y)h(xy)dydx$$

(substituting $xy$ for $y$ in the inner integral, as permitted by 441J). (The 'and finite' at the beginning of the last sentence is there because I have changed the rules since the last paragraph, and $f * g$ is not allowed to take the value $\infty$. So we have to take care that

$$\{y : h(y) > 0,\ \textstyle\int f(x)g(x^{-1}y)dx = \infty\}$$

is negligible.) Now applying 444N again, we get

$$\begin{aligned}\int (f * g)(x)h(x)dx &= \iint f(x)g(y)h(xy)dxdy \\ &= \iint f(x)g(y)h(xy)dydx = \int f(x)g(y)h(xy)d(x,y)\end{aligned}$$

if any of these integrals is finite.

For the general case, the hypothesis on $|f|$, $|g|$ and $|h|$ is sufficient to ensure that the four expressions are equal for any combination of $f^{\pm}$, $g^{\pm}$ and $h^{\pm}$; adding and subtracting these combinations appropriately, we get the result.

**(e)** The point is that, for non-negative $f$, $g$ and $h$,

$$((f * g) * h)(x) = \int (f * g)(z)h(z^{-1}x)dz = \int (f * g)(z)h'(z)dz$$

(setting $h'(z) = h(z^{-1}x)$)

$$= \iint f(y)g(z)h'(yz)dzdy$$

(using (d); to see that $h'$ is measurable, refer to 443A as usual)

$$\begin{aligned}&= \iint f(y)g(z)h(z^{-1}y^{-1}x)dzdy \\ &= \int f(y)(g * h)(y^{-1}x)dy = (f * (g * h))(x)\end{aligned}$$

at least as long as one of the expressions here is finite. (Note that, as in 255J, we need to suppose that $f * g$ and $g * h$ are defined a.e. when moving from $\int (f * g)(z)h(z^{-1}x)dz$ to $\iint f(y)g(z)h(z^{-1}y^{-1})dydz$ and from

$\iint f(y)g(z)h(z^{-1}y^{-1}x)dzdy$ to $\int f(y)(g*h)(y^{-1}x)dy$, since in part (d) I am more tolerant of infinities in the repeated integrals than I was in the definition of $f*g$.) Once again, subject to the inner integrals implicit in the formulae $f*(g*h)$ and $(f*g)*h$ being adequately defined, we can use addition and subtraction to obtain the result for general $f$, $g$ and $h$.

**(f)** These are immediate from the formulae in (a), using 442K if necessary.

**(g)** If $X$ is abelian, then $\Delta(y)=1$ for every $y$, so

$$(g*f)(x)=\int g(y)f(y^{-1}x)dy=\int g(y)\Delta(y^{-1})f(xy^{-1})dy=(f*g)(x)$$

if either $(f*g)(x)$ or $(g*f)(x)$ is defined in $\mathbb{R}$.

**444P Proposition** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$.

(a) If $f\in\mathcal{L}^1(\mu)^+$ and $g\in\mathcal{L}^0(\mu)$ then $f*g$, as defined in 444O, is equal to $(f\mu)*g$ as defined in 444H.

(b) If $f\in\mathcal{L}^0(\mu)$ and $g\in\mathcal{L}^1(\mu)^+$ then $f*g=f*(g\mu)$.

**proof** Again, these are immediate from the formulae above:

$$(f*g)(x)=\int g(y^{-1}x)f(y)\mu(dy)=\int g(y^{-1}x)(f\mu)(dy)=(f\mu*g)(x),$$

$$(f*g)(x)=\int f(xy^{-1})\Delta(y^{-1})g(y)\mu(dy)=\int f(xy^{-1})\Delta(y^{-1})(g\mu)(dy)=(f*g\mu)(x)$$

whenever these are defined, using 235K, as usual, to calculate $\int\ldots d(f\mu)$, $\int\ldots d(g\mu)$. (Note that as we assume throughout that $f$ and $g$ are defined $\mu$-almost everywhere, all the functions $y\mapsto g(y^{-1}x)$, $y\mapsto f(xy^{-1})$ are also defined $\mu$-a.e., by the results set out in 443A.)

**444Q Proposition** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$.

(a) Let $f$, $g$ be non-negative $\mu$-integrable functions. Then, defining $f*g$ as in 444O, we have $f*g\in\mathcal{L}^1=\mathcal{L}^1(\mu)$ and

$$(f\mu)*(g\mu)=(f*g)\mu.$$

(b) For any $f$, $g\in\mathcal{L}^1$, $f*g\in\mathcal{L}^1$ and

$$\int f*g\,d\mu=\int fd\mu\int g\,d\mu,\quad \|f*g\|_1\le\|f\|_1\|g\|_1.$$

**proof (a)** Putting 444K and 444P together, $f\mu*g\mu=(f\mu*g)\mu$, so that $f*g=f\mu*g$ is $\mu$-integrable, and

$$(f*g)\mu=(f\mu*g)\mu=f\mu*g\mu.$$

**(b)** Taking $h=\chi X$ in 444Od, we get $\int f*g\,d\mu=\int fd\mu\int g\,d\mu$. Now

$$\|f*g\|_1=\int|f*g|\le\int|f|*|g|=\int|f|\int|g|=\|f\|_1\|g\|_1.$$

**444R Proposition** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Take any $p\in[1,\infty]$.

(a) If $f\in\mathcal{L}^1(\mu)$ and $g\in\mathcal{L}^p(\mu)$, then $f*g\in\mathcal{L}^p(\mu)$ and $\|f*g\|_p\le\|f\|_1\|g\|_p$.

(b) $f*\overset{\leftrightarrow}{g}=(g*\overset{\leftrightarrow}{f})^{\leftrightarrow}$ for all $f$, $g\in\mathcal{L}^0$. If $X$ is unimodular then $\|\overset{\leftrightarrow}{f}\|_p=\|f\|_p$ for every $f\in\mathcal{L}^0$.

(c) Set $q=\infty$ if $p=1$, $p/(p-1)$ if $1<p<\infty$, $1$ if $p=\infty$. If $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$, then $f*\overset{\leftrightarrow}{g}$ is defined everywhere in $X$ and is continuous, and $\|f*\overset{\leftrightarrow}{g}\|_\infty\le\|f\|_p\|g\|_q$. If $X$ is unimodular, then $f*g\in C_b(X)$ and $\|f*g\|_\infty\le\|f\|_p\|g\|_q$ for every $f\in\mathcal{L}^p(\mu)$, $g\in\mathcal{L}^q(\mu)$.

**Remark** In the formulae above, interpret $\|g\|_\infty$ as $\|g^\bullet\|_\infty=\operatorname{ess\,sup}|g|$ for $g\in\mathcal{L}^\infty=\mathcal{L}^\infty(\mu)$, and as $\infty$ for $g\in\mathcal{L}^0\setminus\mathcal{L}^\infty$. Because $\mu$ is strictly positive, this agrees with the usual definition $\|g\|_\infty=\sup_{x\in X}|g(x)|$ when $g$ is continuous and defined everywhere on $X$.

**proof (a)** If $f\ge 0$, then $f*g=(f\mu)*g$ belongs to $\mathcal{L}^p(\mu)$, and

$$\|f*g\|_p=\|f\mu*g\|_p\le(f\mu)(X)\|g\|_p=\|f\|_1\|g\|_p,$$

by 444Pa and 444Ma. Generally, $f*g=_{\text{a.e.}}f^+*g-f^-*g$ belongs to $\mathcal{L}^p(\mu)$ and

$$\|f*g\|_p \le \|f^+*g\|_p + \|f^-*g\|_p \le (\|f^+\|_1 + \|f^-\|_1)\|g\|_p = \|f\|_1\|g\|_p.$$

**(b)(i)** By 443A once more, $\overset{\leftrightarrow}{f} \in \mathcal{L}^0$ whenever $f \in \mathcal{L}^0$. For $x \in X$,

$$(f*\overset{\leftrightarrow}{g})(x) = \int f(y)\overset{\leftrightarrow}{g}(y^{-1}x)dy = \int \overset{\leftrightarrow}{f}(y^{-1})g(x^{-1}y)dy = (g*\overset{\leftrightarrow}{f})(x^{-1}) = (g*\overset{\leftrightarrow}{f})^{\leftrightarrow}(x)$$

if any of these are defined.

**(ii)** If $X$ is unimodular, then, for any $f \in \mathcal{L}^0$,

$$\|\overset{\leftrightarrow}{f}\|_p^p = \int |f(x^{-1})|^p dx = \int \Delta(x^{-1})|f(x)|^p dx = \|f\|_p^p;$$

while $\operatorname{ess\,sup}|\overset{\leftrightarrow}{f}| = \operatorname{ess\,sup}|f|$ because $E^{-1}$ is conegligible whenever $E \subseteq X$ is conegligible.

**(c)(i)** For any $x \in X$,

$$(f*\overset{\leftrightarrow}{g})(x) = \int f(y)g(x^{-1}y)dy = \int f \times (x \bullet_l g)$$

in the language of 443G and 444O. By 443Gb, $x\bullet_l g \in \mathcal{L}^q$, so $(f*\overset{\leftrightarrow}{g})(x) = \int f \times (x\bullet_l g)$ is defined.

**(ii)** If $p > 1$, so that $q < \infty$, then $x \mapsto (x\bullet_l g)^\bullet : X \to L^q$ is continuous (443Gf), so

$$x \mapsto (f*\overset{\leftrightarrow}{g})(x) = \int f^\bullet \times (x\bullet_l g)^\bullet$$

is continuous, because $f^\bullet \in L^p \cong (L^q)^*$. If $p = 1$, then

$$(f*\overset{\leftrightarrow}{g})(x) = \int f(xy)g(y)dy = \int (x^{-1}\bullet_l f) \times g$$

for every $x$; since $x \mapsto (x^{-1}\bullet_l f)^\bullet : X \to L^1$ is continuous, so is $f*\overset{\leftrightarrow}{g}$.

**(iii)** If $X$ is unimodular then $f*g = f*\tilde{\overset{\leftrightarrow}{g}}$ is continuous, because $\overset{\leftrightarrow}{g} \in \mathcal{L}^q$ by (b), and $\|f*g\|_\infty \le \|f\|_p\|\overset{\leftrightarrow}{g}\|_q = \|f\|_p\|g\|_q$.

**444S Remarks** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$.

**(a)** From 444Ob and 444Ra we see that we have a bilinear operator $(u,v) \mapsto u*v : L^1(\mu)\times L^p(\mu) \to L^p(\mu)$ defined by saying that $f^\bullet * g^\bullet = (f*g)^\bullet$ for every $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^p(\mu)$. Indeed, 444Ob tells us that $*$ can actually be regarded as a function from $L^1 \times L^p$ to $\mathcal{L}^p$. Putting 443Ge together with 444Oe and 444Of, we have

$$u*(v*w) = (u*v)*w,$$

$$a\bullet_l(u*w) = (a\bullet_l u)*w, \quad a\bullet_r(u*w) = u*(a\bullet_r w),$$

$$(a\bullet_r u)*w = \Delta(a^{-1})u*(a^{-1}\bullet_l w)$$

whenever $u$, $v \in L^1$, $w \in L^p$ and $a \in X$.

Similarly, if the group is unimodular, and $\frac{1}{p}+\frac{1}{q} = 1$, the map $* : \mathcal{L}^p \times \mathcal{L}^q \to C_b(X)$ (444Rc) factors through a map from $L^p \times L^q$ to $C_b(X)$.

**(b)** In particular, $* : L^1 \times L^1 \to L^1$ is associative; evidently it is bilinear; and $\|u*v\|_1 \le \|u\|_1\|v\|_1$ for all $u$, $v \in L^1$. So $L^1$ is a normed algebra; since $L^1$ is $\| \|_1$-complete, it is a Banach algebra. By 444Qb, $\int u*v = \int u \int v$ for all $u$, $v \in L^1$. $L^1$ is commutative if $X$ is abelian (444Og).

**(c)** Let $\mathcal{B}$ be the Borel $\sigma$-algebra of $X$ and $M_\tau$ the Banach algebra of signed $\tau$-additive Borel measures on $X$, as in 444E. If, for $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $E \in \mathcal{B}$, we write $(f\mu\restriction\mathcal{B})(E) = \int_E f d\mu$, then $f\mu\restriction\mathcal{B} \in M_\tau$; for $f \ge 0$, this is because the indefinite-integral measure $f\mu$ is a quasi-Radon measure, and in general it is because $f\mu\restriction\mathcal{B} = f^+\mu\restriction\mathcal{B} - f^-\mu\restriction\mathcal{B}$. For $f$, $g \in \mathcal{L}^1$, we have

$$f^\bullet = g^\bullet \text{ in } L^1 \implies f =_{\text{a.e.}} g \implies f\mu\restriction\mathcal{B} = g\mu\restriction\mathcal{B},$$

so we have an operator $T : L^1 \to M_\tau$ defined by setting $T(f^\bullet) = f\mu\restriction\mathcal{B}$ for $f \in \mathcal{L}^1$. It is easy to check that

$$(f+g)\mu\restriction\mathcal{B} = f\mu\restriction\mathcal{B} + g\mu\restriction\mathcal{B}, \quad (\alpha f)\mu\restriction\mathcal{B} = \alpha(f\mu\restriction\mathcal{B}),$$

$$(|f|\mu\restriction\mathcal{B})(E)=\int_E |f| = \sup_{F\in\mathcal{B},F\subseteq E}\int_F f-\int_{E\setminus F} f = |f\mu\restriction\mathcal{B}|(E)$$

for $f$, $g\in\mathcal{L}^1$, $\alpha\in\mathbb{R}$ and $E\in\mathcal{B}$, so that $T$ is a Riesz homomorphism; moreover,

$$\|f\mu\restriction\mathcal{B}\| = |f\mu\restriction\mathcal{B}|(X) = (|f|\mu\restriction\mathcal{B})(X) = \int |f|d\mu = \|f^\bullet\|_1$$

for $f\in\mathcal{L}^1$, so that $T$ is norm-preserving. Finally, for non-negative $f$, $g\in\mathcal{L}^1$, we have

$$Tf^\bullet * Tg^\bullet = (f\mu\restriction\mathcal{B})*(g\mu\restriction\mathcal{B}) = (f\mu * g\mu)\restriction\mathcal{B}$$

(444Eb, since the completions of $f\mu\restriction\mathcal{B}$, $g\mu\restriction\mathcal{B}$ are the quasi-Radon indefinite-integral measures $f\mu$, $g\mu$)

$$= (f*g)\mu\restriction\mathcal{B}$$

(444Qa)

$$= T(f*g)^\bullet = T(f^\bullet * g^\bullet).$$

Thus $Tu*Tv=T(u*v)$ for non-negative $u$, $v\in L^1$; by linearity, $Tu*Tv=T(u*v)$ for all $u$, $v\in L^1$, and $T$ is an embedding of $L^1$ as a subalgebra of $M_\tau$.

**444T Proposition** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Then for any $p\in[1,\infty[$, $f\in\mathcal{L}^p(\mu)$ and $\epsilon>0$ there is a neighbourhood $U$ of the identity $e$ in $X$ such that $\|\nu*f-f\|_p\le\epsilon$ and $\|f*\nu-f\|_p\le\epsilon$ whenever $\nu$ is a quasi-Radon measure on $X$ such that $\nu U=\nu X=1$.

**proof (a)** To begin with, suppose that $f$ is non-negative, continuous and bounded, and that $G=\{x: f(x)>0\}$ has finite measure; set $M=\sup_{x\in X}f(x)$. Write $\mathcal{U}$ for the family of neighbourhoods of $e$. Take $\delta>0$, $\eta\in\,]0,1]$ such that

$$(2\delta+(1+\delta)^p-1)^{1/p}\|f\|_p\le\epsilon,$$

$$(1-\eta)^p\int((f-\eta\chi X)^+)^p d\mu - M^p\eta \ge (1-\delta)\int f^p d\mu,\quad (1-\eta)^{(1-p)/p}\le 1+\delta.$$

For each $U\in\mathcal{U}$, set

$$H_U=\operatorname{int}\{x: f(y)\ge f(x)-\eta \text{ for every } y\in xU^{-1}\cup U^{-1}x\}.$$

Then $H_U$ is open and for every $x\in X$ there is a $U\in\mathcal{U}$ such that $|f(y)-f(x)|\le\frac{1}{2}\eta$ whenever $y\in xUU^{-1}\cup U^{-1}xU$, so that $x\in\operatorname{int}xU\subseteq H_U$. Thus $\{H_U:U\in\mathcal{U}\}$ is an upwards-directed family of open sets with union $X$, and there is a $U\in\mathcal{U}$ such that $\mu(G\setminus H_U)\le\eta$; moreover, because $\Delta$ is continuous, we can suppose that $\Delta(y^{-1})\ge 1-\eta$ for every $y\in U$.

Now suppose that $\nu$ is a quasi-Radon measure on $X$ such that $\nu U=\nu X=1$. Then, for any $x\in H_U$,

$$(\nu*f)(x)=\int f(y^{-1}x)\nu(dy)=\int_U f(y^{-1}x)\nu(dy)\ge f(x)-\eta$$

because $x\in H_U$ and $y^{-1}x\in U^{-1}x$ whenever $y\in U$. Similarly,

$$(f*\nu)(x)=\int_U f(xy^{-1})\Delta(y^{-1})\nu(dy)\ge(f(x)-\eta)(1-\eta)$$

for every $x\in H_U$. Now this means that, setting $h_1=\nu*f$, $h_2=f*\nu$ we have $(f\wedge h_i)(x)\ge(f(x)-\eta)(1-\eta)$ for every $x\in H_U$ and both $i$. Accordingly

$$\begin{aligned}\int(f\wedge h_i)^p d\mu &\ge (1-\eta)^p\int_{G\cap H_U}((f-\eta\chi X)^+)^p d\mu\\ &\ge (1-\eta)^p\int((f-\eta\chi X)^+)^p d\mu - \int_{G\setminus H_U} f^p\\ &\ge (1-\eta)^p\int((f-\eta\chi X)^+)^p d\mu - M^p\eta \ge (1-\delta)\int f^p d\mu.\end{aligned}$$

Now, just because $f$ and $h_i$ are non-negative, and $p\ge 1$,

$$|f-h_i|^p+2(f\wedge h_i)^p\le f^p+h_i^p.$$

Also, writing

$$\gamma = \int \Delta(y)^{(1-p)/p}\nu(dy) = \int_U \Delta(y)^{(1-p)/p}\nu(dy) \le (1-\eta)^{(1-p)/p} \le 1+\delta,$$

we have

$$\|h_1\|_p = \|\nu * f\|_p \le \|f\|_p, \quad \|h_2\|_p = \|f * \nu\|_p \le \gamma\|f\|_p$$

(444M), so that $\int h_i^p d\nu \le (1+\delta)^p \int f^p d\mu$ for both $i$, and

$$\int |f-h_i|^p d\mu \le \int f^p d\mu + \int h_i^p d\mu - 2\int (f \wedge h_i)^p d\mu \le (2\delta + (1+\delta)^p - 1)\int f^p d\mu$$

for both $i$. But this means that

$$\max(\|f - f * \nu\|_p, \|f - \nu * f\|_p) \le (2\delta + (1+\delta)^p - 1)^{1/p}\|f\|_p \le \epsilon.$$

As $\nu$ is arbitrary, we have found a suitable $U$.

**(b)** For any continuous bounded function $f$ such that $\mu\{x : f(x) \ne 0\} < \infty$, we can find neighbourhoods $U_1$, $U_2$ of $e$ such that

$$\|f^+ - \nu * f^+\|_p \le \tfrac{1}{2}\epsilon, \quad \|f^+ - f^+ * \nu\|_p \le \tfrac{1}{2}\epsilon$$

whenever $\nu U_1 = \nu X = 1$,

$$\|f^- - \nu * f^-\|_p \le \tfrac{1}{2}\epsilon, \quad \|f^- - f^- * \nu\|_p \le \tfrac{1}{2}\epsilon$$

whenever $\nu U_2 = \nu X = 1$. So we shall have

$$\|f - \nu * f\|_p \le \epsilon, \quad \|f - f * \nu\|_p \le \epsilon$$

whenever $\nu(U_1 \cap U_2) = \nu X = 1$.

**(c)** For general $f \in \mathcal{L}^p(\mu)$, there is a bounded continuous function $g : X \to \mathbb{R}$ such that $\mu\{x : g(x) \ne 0\} < \infty$ and $\|f - g\|_p \le \frac{1}{4}\epsilon$ (415Pa). Now there is a neighbourhood $U_1$ of $e$ such that

$$\|g - \nu * g\|_p \le \tfrac{1}{4}\epsilon, \quad \|g - g * \nu\|_p \le \tfrac{1}{4}\epsilon$$

whenever $\nu U_1 = \nu X = 1$. There is also a neighbourhood $U_2$ of $e$ such that $\Delta(y^{-1})^{(1-p)/p} \le 2$ for every $y \in U_2$, so that

$$\|g * \nu - f * \nu\|_p \le 2\|g - f\|_p \le \tfrac{1}{2}\epsilon$$

whenever $\nu U_2 = \nu X = 1$. Since we have

$$\|\nu * g - \nu * f\|_p \le \|g - f\|_p \le \tfrac{1}{4}\epsilon$$

whenever $\nu X = 1$, we get $\|f - \nu * f\|_p \le \epsilon$, $\|f = f * \nu\|_p \le \epsilon$ whenever $\nu(U_1 \cap U_2) = \nu X = 1$.

This completes the proof.

**444U Corollary** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. For any Haar measurable $E \subseteq X$ such that $0 < \mu E < \infty$, and any $f \in \bigcup_{1 \le p < \infty} \mathcal{L}^p(\mu)$, write

$$f_E(x) = \frac{1}{\mu E}\int_{xE} f d\mu, \quad f'_E(x) = \frac{1}{\mu(Ex)}\int_{Ex} f d\mu$$

for $x \in X$. Then, for any $p \in [1, \infty[$, $f \in \mathcal{L}^p$ and $\epsilon > 0$, there is a neighbourhood $U$ of the identity in $X$ such that $\|f_E - f\|_p \le \epsilon$ and $\|f'_E - f\|_p \le \epsilon$ whenever $E \subseteq U$ is a non-negligible Haar measurable set.

**proof** Take $\delta \in ]0, 1[$ such that $\delta(1-\delta)^{(1-p)/p}\|f\|_p \le \frac{1}{2}\epsilon$. By 444T, there is a neighbourhood $U$ of the identity such that $\|f - f * \nu\|_p \le \frac{1}{2}\epsilon$, $\|f - \nu * f\|_p \le \epsilon$ whenever $\nu$ is a quasi-Radon measure on $X$ such that $\nu U = \nu X = 1$. Shrinking $U$ if necessary, we may suppose also that $U = U^{-1}$, that $\mu U < \infty$ and that $|\Delta(x) - 1| \le \delta$ for every $x \in U$, where $\Delta$ is the left modular function of $X$. If $E \subseteq U$ and $\mu E > 0$, consider the quasi-Radon measures $\nu$, $\nu'$, $\overleftrightarrow{\nu}$ and $\overleftrightarrow{\nu}'$ on $X$ defined by setting

$$\nu F = \frac{1}{\mu E^{-1}}\int_{E \cap F} \Delta(x^{-1})\mu(dx), \quad \nu' F = \frac{\mu(E \cap F)}{\mu E}, \quad \overleftrightarrow{\nu} F = \nu F^{-1}, \quad \overleftrightarrow{\nu}' F = \nu' F^{-1}$$

whenever these are defined. (They are quasi-Radon measures because $\nu$ and $\nu'$ are totally finite indefinite-integral measures over $\mu$ and the map $x \mapsto x^{-1}$ is a homeomorphism.) Because $E \subseteq U = U^{-1}$, we have

$$\overleftrightarrow{\nu}U=\nu U^{-1}=\frac{1}{\mu E^{-1}}\int_E\Delta(x^{-1})\mu(dx)=1=\overleftrightarrow{\nu}X$$

by 442Ka, while

$$\overleftrightarrow{\nu}'X=\overleftrightarrow{\nu}'U=\nu'U^{-1}=1.$$

Now consider $f*\overleftrightarrow{\nu}$ and $\overleftrightarrow{\nu}'*f$. For any $x\in X$,

$$(f*\overleftrightarrow{\nu})(x)=\int f(xy^{-1})\Delta(y^{-1})\overleftrightarrow{\nu}(dy)=\int f(xy)\Delta(y)\nu(dy)$$

(because $\overleftrightarrow{\nu}$ is the image of $\nu$ under the map $y\mapsto y^{-1}$)

$$=\frac{1}{\mu E^{-1}}\int\chi E(y)\Delta(y^{-1})f(xy)\Delta(y)\mu(dy)$$

(noting that $\nu$ is an indefinite-integral measure over $\mu$, and using 235K)

$$=\frac{1}{\mu E^{-1}}\int\chi E(x^{-1}y)f(y)\mu(dy)=\frac{1}{\mu E^{-1}}\int_{xE}f(y)\mu(dy)=\frac{\mu E}{\mu E^{-1}}f_E(x),$$

$$(\overleftrightarrow{\nu}'*f)(x)=\int f(y^{-1}x)\overleftrightarrow{\nu}'(dy)=\int f(yx)\nu'(dy)$$

(because $\overleftrightarrow{\nu}'$ is the image of $\nu'$ under the map $y\mapsto y^{-1}$)

$$=\frac{1}{\mu E}\int\chi E(y)f(yx)\mu(dy)=\frac{\Delta(x^{-1})}{\mu E}\int\chi E(yx^{-1})f(y)\mu(dy)$$

(by 442Kc)

$$=\frac{1}{\mu(Ex)}\int_{Ex}f(y)\mu(dy)=f'_E(x).$$

So

$$\|f-f_E\|_p\le\|f-f*\overleftrightarrow{\nu}\|_p+\Bigl|\frac{\mu E^{-1}}{\mu E}-1\Bigr|\|f*\overleftrightarrow{\nu}\|_p.$$

Now $\mu E^{-1}=\int_E\Delta(y^{-1})\mu(dy)$, so that

$$|\mu E-\mu E^{-1}|\le\int_E|\Delta(y^{-1})-1|\mu(dy)\le\delta\mu E,\quad\Bigl|\frac{\mu E^{-1}}{\mu E}-1\Bigr|\le\delta;$$

also

$$\|f*\overleftrightarrow{\nu}\|_p\le\|f\|_p\int\Delta(y)^{(1-p)/p}\nu(dy)\le\|f\|_p(1-\delta)^{(1-p)/p}$$

by 444Mb, so

$$\Bigl|\frac{\mu E^{-1}}{\mu E}-1\Bigr|\|f*\overleftrightarrow{\nu}\|_p\le\delta(1-\delta)^{(1-p)/p}\|f\|_p\le\frac{1}{2}\epsilon,$$

and $\|f-f_E\|_p\le\epsilon$.

On the other hand,

$$\|f-f'_E\|_p=\|f-\overleftrightarrow{\nu}'*f\|_p\le\epsilon;$$

as $E$ is arbitrary, we have found a suitable $U$.

**444V** So far I have not emphasized the special properties of compact groups. But of course they are the centre of the subject, and for the sake of a fundamental theorem in §446 I give the following result.

**Theorem** Let $X$ be a compact topological group and $\mu$ a left Haar measure on $X$.

(a) For any $u$, $v\in L^2=L^2(\mu)$ we can interpret their convolution $u*v$ either as a member of the space $C(X)$ of continuous real-valued functions on $X$, or as a member of the space $L^2$.

(b) If $w\in L^2$, then $u\mapsto u*w$ is a compact linear operator whether regarded as a map from $L^2$ to $C(X)$ or as a map from $L^2$ to itself.

(c) If $w\in L^2$ and $w=\overleftrightarrow{w}$ (as defined in 443Af), then $u\mapsto u*w:L^2\to L^2$ is a self-adjoint operator.

**proof (a)** Being compact, $X$ is unimodular (442Ic). As noted in 444Sa, $*$ can be regarded as a bilinear operator from $L^2 \times L^2$ to $C_b(X) = C(X)$. Because $\mu X$ must be finite, we now have a natural map $f \mapsto f^\bullet$ from $C(X)$ to $L^2$, so that we can think of $u * v$ as a member of $L^2$ for $u$, $v \in L^2$.

**(b)(i)** Evidently $u \mapsto u * w : L^2 \to C(X)$ is linear, for any $w \in L^2$.

**(ii)** Let $B$ be the unit ball of $L^2$, and give it the topology induced by the weak topology $\mathfrak{T}_s(L^2, L^2)$, so that $B$ is compact (4A4Ka). Let $\bullet_l$ be the left action of $X$ on $L^2$ as in 443G and 444S.

If $f$, $g \in \mathcal{L}^2$ and $a \in X$, then

$$(f * g)(a) = \int f(x)g(x^{-1}a)dx = \int f(x)\overleftrightarrow{g}(a^{-1}x)dx = \int f \times a\bullet_l \overleftrightarrow{g}.$$

(Note that because $X$ is unimodular, $\overleftrightarrow{g}$ and $a\bullet_l\overleftrightarrow{g}$ are square-integrable whenever $g$ is.) So if $u$, $w \in L^2$ and $a \in X$, $(u * w)(a) = (u|a\bullet_l\overleftrightarrow{w})$. It follows that, for any $w \in L^2$, the function $(a, u) \mapsto (u * w)(a) : X \times B \to \mathbb{R}$ is continuous. **P** We know that $\bullet_l : X \times L^2 \to L^2$ is continuous when $L^2$ is given its norm topology (443Gf). Now $(u, v) \mapsto (u|v)$ is continuous, so $(a, u) \mapsto (u * w)(a) = (u|a\bullet_l\overleftrightarrow{w})$ must be continuous. **Q**

Because $X$ is compact, this means that $u \mapsto u * w : B \to C(X)$ is continuous when $C(X)$ is given its norm topology and $B$ is given the weak topology (4A2G(g-ii)). Because $B$ is compact in the weak topology, $\{u*w : u \in B\}$ is compact in $C(X)$. But this implies that $u \mapsto u*w$ is a compact linear operator (definition: 3A5La).

**(iii)** Again because $X$ is compact, $\mu$ is totally finite, so, for $f \in C(X)$, $\|f\|_2 \le \|f\|_\infty\sqrt{\mu X}$, and the natural map $f \mapsto f^\bullet : C(X) \to L^2$ is a bounded linear operator. Consequently the map $u \mapsto (u * w)^\bullet : L^2 \to L^2$ is a compact operator, by 4A4La.

**(c)** Now suppose that $w = \overleftrightarrow{w}$. In this case $(u * w|v) = (u|v * w)$ for all $u$, $v \in L^2$. **P** Express $u$, $v$ and $w$ as $f^\bullet$, $g^\bullet$ and $h^\bullet$ where $f$, $g$ and $h$ are square-integrable Borel measurable functions defined everywhere on $X$. We have

$$\begin{aligned}(u * w|v) &= \int (f * h)(x)g(x)dx = \iint f(y)h(y^{-1}x)g(x)dydx \\ &= \iint f(y)h(y^{-1}x)g(x)dxdy\end{aligned}$$

(because $(x, y) \mapsto f(y)h(y^{-1}x)g(x)$ is Borel measurable, $\mu$ is totally finite and $\iint |f(y)h(y^{-1}x)g(x)|dydx = (|u| * |w|||v|)$ is finite)

$$\begin{aligned}&= \iint f(y)\overleftrightarrow{h}(x^{-1}y)g(x)dxdy = \int f(y)(g * \overleftrightarrow{h})(y)dy \\ &= (u|v * \overleftrightarrow{w}) = (u|v * w). \textbf{ Q}\end{aligned}$$

As $u$ and $v$ are arbitrary, this shows that $u \mapsto u * w : L^2 \to L^2$ is self-adjoint.

**444X Basic exercises >(a)** Let $X$ be a Hausdorff topological group. Show that if $\lambda$ and $\nu$ are totally finite Radon measures on $X$ then $\lambda * \nu$ is the image measure $(\lambda \times \nu)\phi^{-1}$, where $\phi(x, y) = xy$ for $x$, $y \in X$, and in particular is a Radon measure.

**>(b)** Let $X$ be a topological group and $\lambda$, $\nu$ two totally finite quasi-Radon measures on $X$. Writing $\operatorname{supp}\lambda$ for the support of $\lambda$, show that $\operatorname{supp}(\lambda * \nu) = \overline{(\operatorname{supp}\lambda)(\operatorname{supp}\nu)}$.

**(c)** Let $X$ be a topological group and $M^+_{\mathrm{qR}}$ the family of totally finite quasi-Radon measures on $X$. Show that $(\lambda, \nu) \mapsto \lambda * \nu : M^+_{\mathrm{qR}} \times M^+_{\mathrm{qR}} \to M^+_{\mathrm{qR}}$ is continuous for the narrow topology on $M^+_{\mathrm{qR}}$. (*Hint*: 437Ma, 437N.)

**(d)** Let $X$ be a Hausdorff topological group. Show that $X$ is abelian iff its Banach algebra of signed $\tau$-additive Borel measures is commutative.

**(e)** Let $X$ be a topological group, and $M_\tau$ its Banach algebra of signed $\tau$-additive Borel measures. (i) Show that we have actions $\bullet_l$, $\bullet_r$ of $X$ on $M_\tau$ defined by writing $(a\bullet_l\nu)(E) = \nu(aE)$, $(a\bullet_r\nu)(E) = \nu(Ea^{-1})$. (ii) Show that $(a\bullet_l\lambda) * \nu = a\bullet_l(\lambda * \nu)$, $\lambda * (a\bullet_r\nu) = a\bullet_r(\lambda * \nu)$ for all $a \in X$ and $\lambda$, $\nu \in M_\tau$.

**(f)** Let $X$ be a compact Hausdorff topological group, and $B$ a norm-bounded subset of the Banach algebra $M_\tau$ of signed $\tau$-additive Borel measures on $X$. Show that $(\lambda, \nu) \mapsto \lambda * \nu : B \times B \to M_\tau$ is continuous for the vague topology on $M_\tau$. (*Hint*: 437Md.)

**(g)** Let $X$ be a topological group, and $\nu$ a totally finite quasi-Radon measure on $X$. Show that for any Borel sets $E$, $F \subseteq X$, the function $(g, h) \mapsto \nu(gE \cap Fh)$ is Borel measurable. (*Hint*: for Borel sets $W \subseteq X \times X$, set $\nu'W = \nu\{x : (x, x) \in W\}$. Consider the action of $X \times X$ on itself defined by writing $(g, h)\bullet(x, y) = (gx, yh^{-1})$.)

**(h)** Let $X$ be a topological group and $f$ a real-valued function defined on a subset of $X$. (i) Show that $a\bullet_r(\nu * f) = \nu * (a\bullet_r f)$ (definition: 4A5Cc) whenever $a \in X$ and $\nu$ is a measure on $X$. (ii) Show that if $X$ carries Haar measures, then $a\bullet_l(f * \nu) = (a\bullet_l f) * \nu$ whenever $a \in X$ and $\nu$ is a measure on $X$.

**(i)** Let $X$ be a topological group carrying Haar measures, $f : X \to \mathbb{R}$ a bounded continuous function and $\nu$ a totally finite quasi-Radon measure on $X$. Show that $f * \nu$ is continuous.

**(j)** Let $X$ be a topological group carrying Haar measures, $f$ a real-valued function defined on a subset of $X$, and $\lambda$, $\nu$ totally finite quasi-Radon measures on $X$. Show that $((f * \nu) * \lambda)(x) = (f * (\nu * \lambda))(x)$ whenever the right-hand side is defined. (See also 444Yj.)

**(k)** Let $X$ be an abelian topological group carrying Haar measures. Show that $f * \nu = \nu * f$ for every measure $\nu$ on $X$ and every real-valued function $f$ defined on a subset of $X$.

**>(l)** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. (i) Let $\nu$ be a totally finite quasi-Radon measure on $X$ such that $x \mapsto \nu(xF)$ is continuous for every closed set $F \subseteq X$. Show that $\nu$ is truly continuous with respect to $\mu$. (*Hint*: if $\mu F = 0$, apply 444K to $\nu * \chi F^{-1}$ to see that $\nu(xF) = 0$ for $\mu$-almost every $x$.) (ii) Let $\nu$ be a totally finite Radon measure on $X$ such that $x \mapsto \nu(xK)$ is continuous for every compact set $K \subseteq X$. Show that $\nu$ is truly continuous with respect to $\mu$.

**(m)** Let $X$ be a topological group carrying Haar measures, $E \subseteq X$ a Haar negligible set and $\nu$ a $\sigma$-finite quasi-Radon measure on $X$. Show that $\nu(xE) = \nu(Ex) = 0$ for Haar-a.e. $x \in X$.

**(n)** Let $X$ be a topological group carrying Haar measures, and $\nu$ a non-zero totally finite quasi-Radon measure on $X$ such that $\nu(xE) = 0$ whenever $x \in X$ and $\nu E = 0$. (i) Show that $\nu$ is strictly positive, so that $X$ is ccc. (ii) Show that a subset of $X$ is $\nu$-negligible iff it is Haar negligible.

**(o)** Use the method of part (b) of the proof of 444M to prove part (a) there.

**>(p)** Let $X$ be the group $S^1 \times S^1$, with the topology defined by giving the first coordinate the usual topology of $S^1$ and the second coordinate its discrete topology, so that $X$ is a locally compact abelian group. Let $\mu$ be a Haar measure on $X$. (i) Find a Borel measurable function $f : X \times X \to \{0, 1\}$ such that $\iint f(x, y)\mu(dx)\mu(dy) \ne \iint f(x, y)\mu(dy)\mu(dx)$. (ii) Let $\nu$ be the Radon measure on $X$ defined by setting $\nu E = \#(\{s : (s, s^{-1}) \in E\})$ if this is finite, $\infty$ otherwise. Define $g : X \to \{0, 1\}$ by setting $g(s, t) = 1$ if $s = t$, 0 otherwise. Show that $\iint g(xy)\nu(dy)\mu(dx) = \infty$, $\iint g(xy)\mu(dx)\nu(dy) = 0$. (iii) Find a closed set $F \subseteq X$ such that $x \mapsto \nu(xF)$ is not Haar measurable.

**>(q)** Let $X$ be a Hausdorff topological group and for $a \in X$ write $\delta_a$ for the Dirac measure on $X$ concentrated at $a$. (i) Show that $\delta_a * \delta_b = \delta_{ab}$ for all $a$, $b \in X$. (ii) Show that, in the notation of 444Xe, $\delta_a * \hat{\nu}$ is the completion of $a^{-1}\bullet_l\nu$ and $\hat{\nu} * \delta_a$ is the completion of $a\bullet_r\nu$ for every $a \in X$ and every totally finite $\tau$-additive Borel measure $\nu$ on $X$ with completion $\hat{\nu}$. (iii) Show that $\delta_a * f = a\bullet_l f$ for every $a \in X$ and every real-valued function $f$ defined on a subset of $X$. (iv) Show that if $X$ carries Haar measures, and has left modular function $\Delta$, $f * \delta_a = \Delta(a^{-1})a^{-1}\bullet_r f$ for every $a \in X$ and every real-valued function $f$ defined on a subset of $X$. (v) Use these formulae to relate 444Of to 444B.

**(r)** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Show that if $f$, $g \in \mathcal{L}^0(X)$ then $(f*g)^{\leftrightarrow} = \overset{\leftrightarrow}{g} * \overset{\leftrightarrow}{f}$.

**(s)** Let $X$ be a locally compact Hausdorff topological group and $\mu$ a left Haar measure on $X$. Show that if $f$, $g : X \to \mathbb{R}$ are continuous functions with compact support, then $f*g$ is a continuous function with compact support.

**(t)** In 444Rc, show that $f * \overset{\leftrightarrow}{g}$ is uniformly continuous for the bilateral uniformity. (*Hint*: in 443Gf, $x \mapsto x{\bullet_l}u$ is uniformly continuous.)

**(u)** Let $X$ be a topological group with a totally finite Haar measure $\mu$. Show that (i) $(u{*}w|v) = (u|v{*}\overset{\leftrightarrow}{w})$ for any $u$, $v$, $w \in L^2 = L^2(\mu)$, where $\overset{\leftrightarrow}{w}$ and $u{*}v$ are defined as in 443Af and 444V (ii) the map $u \mapsto u{*}w : L^2 \to L^2$ is a compact linear operator for any $w \in L^2$. (*Hint*: for (ii), use 443L.)

**(v)** Let $X$ be a topological group with a Haar probability measure $\mu$. Show that $L^2(\mu)$ with convolution is a Banach algebra.

**>(w)**(i) Let $X_1$, $X_2$ be topological groups with totally finite quasi-Radon measures $\lambda_i$, $\nu_i$ on $X_i$ for each $i$. Let $\lambda = \lambda_1 \times \lambda_2$, $\nu = \nu_1 \times \nu_2$ be the quasi-Radon product measure on the topological group $X = X_1 \times X_2$. Show that $\lambda * \nu = (\lambda_1 * \nu_1) \times (\lambda_2 * \nu_2)$. (ii) Let $\langle X_i \rangle_{i\in I}$ be a family of topological groups, and $\lambda_i$, $\nu_i$ quasi-Radon probability measures on $X_i$ for each $i$. Let $\lambda = \prod_{i\in I} \lambda_i$, $\nu = \prod_{i\in I} \nu_i$ be the quasi-Radon product measures on the topological group $\prod_{i\in I} X_i$. Show that $\lambda * \nu = \prod_{i\in I} \lambda_i * \nu_i$.

**>(x)** Show that 444C, 444O, 444P, 444Qb and 444R-444U remain valid if we work with complex-valued, rather than real-valued, functions, and with $\mathcal{L}^p_{\mathbb{C}}$ and $L^p_{\mathbb{C}}$ rather than $\mathcal{L}^p$ and $L^p$.

**(y)** Let $X$ be a topological group with a left Haar measure $\mu$ and left modular function $\Delta$. Write $\boldsymbol{\Delta} \in L^0 = L^0(\mu)$ for the equivalence class of the function $\Delta$. For $u \in L^0$ write $u^*$ for $\overset{\leftrightarrow}{u} \times \overset{\leftrightarrow}{\boldsymbol{\Delta}}$. Show that (i) $(u^*)^* = u$ for every $u \in L^0$ (ii) $u \mapsto u^* : L^0 \to L^0$ is a Riesz space automorphism (iii) $u^* \in L^1$ for every $u \in L^1 = L^1(\mu)$ (iv) $u \mapsto u^* : L^1 \to L^1$ is an $L$-space automorphism (v) $u^* * v^* = (v * u)^*$ for all $u$, $v \in L^1$ (v) defining $T : L^1 \to M_\tau$ as in 444Sc, show that $Tu^* = \overset{\leftrightarrow}{T}u$ (that is, $(Tu^*)(E) = (Tu)(E^{-1})$ for Borel sets $E$) for every $u \in L^1$.

**444Y Further exercises (a)** Find a subgroup $X$ of $\{0,1\}^{\mathbb{N}}$ and quasi-Radon probability measures $\lambda$, $\nu$ on $X$ and a set $A \subseteq X$ such that $(\lambda * \nu)^*(A) = 1$ but $(\lambda \times \nu)\{(x,y) : x, y \in X, x + y \in A\} = 0$.

**(b)** Let $X$ be a **topological semigroup**, that is, a semigroup with a topology such that multiplication is continuous. (i) For totally finite $\tau$-additive Borel measures $\lambda$, $\nu$ on $X$, show that there is a $\tau$-additive Borel measure $\lambda * \nu$ defined by saying that $(\lambda * \nu)(E) = (\lambda \times \nu)\{(x,y) : xy \in E\}$ for every Borel set $E \subseteq X$. (ii) Show that in this context $(\lambda_1 * \lambda_2) * \lambda_3 = \lambda_1 * (\lambda_2 * \lambda_3)$. (iii) Show that $\int f d(\lambda * \nu) = \int f(xy)\lambda(dx)\nu(dy)$ whenever $f$ is $(\lambda * \nu)$-integrable. (iv) Show that if the topology is Hausdorff and $\lambda$ and $\nu$ are tight (that is, inner regular with respect to the compact sets) so is $\lambda * \nu$. (v) Show that we have a Banach algebra of signed $\tau$-additive Borel measures on $X$, as in 444E.

**(c)** Let $X$ be a topological group, and write $M_\tau^{(\mathbb{C})}$ for the complexification of the $L$-space $M_\tau$ of 444E, as described in 354Yl. Show that $M_\tau^{(\mathbb{C})}$, with the natural extension of the convolution operator of 444E, is a complex Banach algebra, and that we still have $|\lambda * \nu| \le |\lambda| * |\nu|$ for $\lambda$, $\nu \in M_\tau^{(\mathbb{C})}$.

**(d)** Find a locally compact Hausdorff topological group $X$, a Radon probability measure $\nu$ on $X$ and an open set $G \subseteq X$ such that $\{(xGx^{-1})^\bullet : x \in X\}$ is not a separable subset of the measure algebra of $\nu$.

**(e)** Let $X$ be a metrizable group. We say that a subset $A$ of $X$ is **Haar null** if there are a universally Radon-measurable set $E \supseteq A$ and a non-zero Radon measure $\nu$ on $X$ such that $\nu(xEy) = 0$ for every $x$, $y \in X$. (i) Show that the family of Haar null sets is a translation-invariant $\sigma$-ideal of subsets of $X$. (*Hint*: if $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence of universally Radon-measurable Haar null sets, we can find Radon probability measures $\nu_n$ concentrated on compact sets near the identity such that $\nu_n(xE_ny)=0$ for every $x$, $y$ and $n$; now construct an infinite convolution product $\nu=\nu_0*\nu_1*\ldots$ from the probability product of the $\nu_n$ and show that $\nu(xE_ny)=0$ for every $x$, $y$ and $n$.) (ii) Show that if $X$ and $Y$ are Polish groups, $\phi:X\to Y$ is a surjective continuous homomorphism and $B\subseteq Y$ is Haar null, then $\phi^{-1}[B]$ is Haar null in $X$. (iii) Show that if $X$ is a locally compact Polish group then a subset of $X$ is Haar null iff it is Haar negligible in the sense of 442H. (See Solecki 01.)

**(f)** Suppose that the continuum hypothesis is true. Let $\nu$ be Cantor measure on $\mathbb{R}$ (256Hc). Show that there is a set $A\subseteq\mathbb{R}$ such that $\nu(x+A)=0$ for every $x\in\mathbb{R}$, but $A$ is not Haar negligible.

**(g)** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Let $\tau$ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mu)$ (§374). Show that if $\nu$ is any totally finite quasi-Radon measure on $X$, then we have a linear operator $f^\bullet\mapsto(\nu*f)^\bullet$ from $L^\tau$ to itself, of norm at most $\nu X$.

**(h)** Let $X$ be a topological group with a left Haar measure $\mu$, $M_\tau$ the Banach algebra of signed $\tau$-additive Borel measures on $X$, and $p\in[1,\infty]$. (i) Show that we have a multiplicative linear operator $T$ from $M_\tau$ to the Banach algebra $\mathsf{B}(L^p(\mu);L^p(\mu))$ defined by writing $(T\nu)(f^\bullet)=(\hat\nu*f)^\bullet$ whenever $\nu$ is a totally finite $\tau$-additive Borel measure on $X$ with completion $\hat\nu$ and $f\in\mathcal{L}^p(\mu)$. (*Hint*: Use 444K and 444B to show that $(\lambda*\nu)*f=_{\text{a.e.}}\lambda*(\nu*f)$ for enough $\lambda$, $\nu$ and $f$. See also 444Yj.) (ii) Show that $\|T\nu\|\le\|\nu\|$ for every $\nu\in M_\tau^+$. (iii) Give an example in which $\|T\nu\|<\|\nu\|$.

**(i)** Let $X$ be a unimodular topological group with left Haar measure $\mu$. Suppose that $p$, $q$, $r\in[1,\infty]$ are such that $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$, interpreting $\frac{1}{\infty}$ as 0. Show that if $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$ then $f*g\in\mathcal{L}^r(\mu)$ and $\|f*g\|_r\le\|f\|_p\|g\|_q$. (*Hint*: 255Yl. Take care to justify any changes in order of integration.)

**(j)** Let $X$ be a topological group carrying Haar measures. Investigate conditions under which the associative laws

$$\lambda*(\nu*f)=(\lambda*\nu)*f,\quad \lambda*(f*\nu)=(\lambda*f)*\nu,\quad f*(\lambda*\nu)=(f*\lambda)*\nu,$$

$$f*(g*\nu)=(f*g)*\nu,\quad f*(\nu*g)=(f*\nu)*g,\quad \nu*(f*g)=(\nu*f)*g$$

will be valid, where $\lambda$ and $\nu$ are quasi-Radon measures on $X$ and $f$, $g$ are real-valued functions. Relate your results to 444Xq.

**(k)** Let $X$ be a topological group with a left Haar measure $\mu$ and left modular function $\Delta$. (i) Suppose that $f\in\mathcal{L}^0(\mu)$. Show that the following are equiveridical: ($\alpha$) $f(yx)=\Delta(y)f(xy)$ for $(\mu\times\mu)$-almost every $x$, $y\in X$; ($\beta$) $(a\bullet_c f)^\bullet=\Delta(a^{-1})f^\bullet$ for every $a\in X$. (ii) Show that in this case $f(x)=0$ for almost every $x$ such that $\Delta(x)\ne1$. (iii) Suppose that $f\in\mathcal{L}^1(\mu)$. Show that the following are equiveridical: ($\alpha$) $f(yx)=\Delta(y)f(xy)$ for $(\mu\times\mu)$-almost every $x$, $y\in X$; ($\beta$) $(f*g)^\bullet=(g*f)^\bullet$ for every $g\in\mathcal{L}^1(\mu)$.

**(l)** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Let $\tau$ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mu)$ such that $\tau\restriction L^\tau$ is an order-continuous norm. For a totally finite quasi-Radon measure $\nu$ on $X$, let $T_\nu:L^\tau\to L^\tau$ be the corresponding linear operator (444Yg). Show that for any $u\in L^\tau$ and $\epsilon>0$ there is a neighbourhood $U$ of the identity in $X$ such that $\tau(T_\nu u-u)\le\epsilon$ whenever $\nu U=\nu X=1$.

**(m)** Let $X$ be a topological group with a left Haar measure $\mu$. For $u\in L^2=L^2(\mu)$, set $A_u=\{a\bullet_l u:a\in X\}$ (443G) in $L^2$, and $D=\{v*u:v\in L^1(\mu),\,v\ge0,\,\int v=1\}$. (i) Show that the closed convex hull of $A_u$ in $L^2$ is the closure of $D$. (*Hint*: ($\alpha$) use 444Od to show that if $w\in L^2$ and $(w'|w)\ge\gamma$ for every $w'\in A_u$, then $(w'|w)\ge\gamma$ for every $w'\in D$ ($\beta$) use 444U to show that $A_u\subseteq\overline{D}$.) (ii) Show that the closed linear subspace $W_u$ generated by $A_u$ is the closure of $\{v*u:v\in L^1\}$. (iii) Show that if $w\in L^2$ and $w\in A_u^\perp$, that is, $(u'|w)=0$ for every $u'\in A$, then $W_w\subseteq W_u^\perp$. (iv) Show that if $X$ is $\sigma$-compact, then $W_u$ is separable. (*Hint*: $A_u$ is $\sigma$-compact, by 443Gf.) (v) Set $C=\{f^\bullet:f\in\mathcal{L}^2\cap C(X)\}$. Show that $C\cap W_u$ is dense in $W_u$. (*Hint*: $v*u\in C$ for many $v$, by 444Rc.) (vi) Show that if $X$ is $\sigma$-compact, then $W_u$ has an orthonormal basis in $C$. (vii) Show that $L^2$ has an orthonormal basis in $C$. (*Hint*: if $X$ is $\sigma$-compact, take a maximal orthogonal family of subspaces $W_u$, find a suitable orthonormal basis of each, and use (iii) to see that these assemble to form a basis of $L^2$. For a general locally compact Hausdorff group, start with a $\sigma$-compact open subgroup, and then deal with its cosets. For a general topological group with a Haar measure, use 443L.) (Compare 416Yh.)

**(n)** Let $X$ be a topological group with a left Haar measure $\mu$. Let $\lambda$ be the quasi-Radon product measure on $X \times X$. Let $\mathcal{U}$ be the set of those $h \in \mathcal{L}^1(\lambda)$ such that $(X \times X) \setminus \{(x, y) : (x, y) \in \operatorname{dom} h,\ h(x, y) = 0\}$ can be covered by a sequence of open sets of finite measure. (i) Show that if $h \in \mathcal{U}$, then $(x, y) \mapsto h(y, y^{-1}x)$ belongs to $\mathcal{U}$. (*Hint*: 443Xa.) (ii) Show that if $h \in \mathcal{U}$, then $(Th)(x) = \int h(y, y^{-1}x)\mu(dy)$ is defined for almost every $x \in X$ and $Th$ is $\mu$-integrable, with $\|Th\|_1 \le \|h\|_1$. (*Hint*: 255Xj.) (iii) Show that if $h_1$, $h_2 \in \mathcal{U}$ are equal $\lambda$-a.e. then $Th_1 = Th_2$ $\mu$-a.e. (iv) Show that every member of $L^1(\lambda)$ can be represented by a member of $\mathcal{U}$. (*Hint*: 443Xk.) (v) Show that if $f$, $g \in \mathcal{L}^1(\mu)$ and both are zero outside some countable union of open sets of finite measure, then $T(f \otimes g) = f * g$, where $(f \otimes g)(x, y) = f(x)g(y)$. (vi) Show that if we set $\tilde{T}(h^\bullet) = (Th)^\bullet$ for $h \in \mathcal{U}$, then $\tilde{T} : L^1(\lambda) \to L^1(\mu)$ is the unique continuous linear operator such that $\tilde{T}(u \otimes v) = u * v$ for all $u$, $v \in L^1(\mu)$, where $u * v$ is defined in 444S and $\otimes : L^1(\mu) \times L^1(\mu) \to L^1(\lambda)$ is the canonical bilinear operator (253E).

**(o)** In 444Yn, suppose that $\mu X = 1$. (i) Show that the map $\tilde{T}$ belongs to the class $\mathcal{T}_{\bar{\lambda}, \bar{\mu}}$ of §373. (ii) Show that if $p \in [1, \infty]$ then $\|Th\|_p \le \|h\|_p$ whenever $h \in \mathcal{U} \cap \mathcal{L}^p(\lambda)$.

**(p)** Rewrite this section in terms of right Haar measures instead of left Haar measures.

**(q)** Let $X$ be a topological group and $M_{\mathrm{qR}}^+$ the set of totally finite quasi-Radon measures on $X$. For $\nu \in M_{\mathrm{qR}}^+$, define $\overleftrightarrow{\nu} \in M_{\mathrm{qR}}^+$ by saying that $\overleftrightarrow{\nu}(E) = \nu E^{-1}$ whenever $E \subseteq X$ and $\nu$ measures $E^{-1}$. (i) Show that if $\lambda$, $\nu \in M_{\mathrm{qR}}^+$ then $\overleftrightarrow{\lambda} * \overleftrightarrow{\nu} = (\nu * \lambda)^{\leftrightarrow}$. (ii) Taking $\bullet_l$, $\bullet_r$ to be the left and right actions of $X$ on itself, and defining corresponding actions of $X$ on $M_{\mathrm{qR}}^+$ as in 441Yp, show that $a \bullet_l (\lambda * \nu) = (a \bullet_l \lambda) * \nu$ and $a \bullet_r (\lambda * \nu) = \lambda * (a \bullet_r \nu)$ for $\lambda$, $\nu \in M_{\mathrm{qR}}^+$ and $a \in X$.

**444 Notes and comments** The aim of this section and the next is to work through ideas from the second half of Chapter 25, and Chapter 28, in forms natural in the context of general topological groups. (It is of course possible to go farther; see 444Yb. It is the glory and confusion of twentieth-century mathematics that it has no firm stopping points.) The move from $\mathbb{R}$ to an arbitrary topological group is a large one, and I think it is worth examining the various aspects of this leap as they affect the theorems here. The most conspicuous change, and the one which most greatly affects the forms of the results, is the loss of commutativity. We are forced to re-examine every formula to determine exactly which manipulations can still be justified. Multiplications must be written the right way round, and inversions especially must be watched. But while there are undoubtedly some surprises, we find that in fact (provided we take care over the definitions) the most important results survive. Of course I wrote the earlier results out with a view to what I expected to do here, but no dramatic manoeuvers are needed to turn the fundamental results 255G, 255H, 255J, 257B, 257E, 257F into the new versions 444Od, 444Qb, 444Oe, 444C, 444B, 444Qa. (The changed order of presentation is an indication of the high connectivity of the web here, not of any new pattern.) In fact what makes the biggest difference is not commutativity, as such, but unimodularity. In groups which are not unimodular we do have new phenomena, as in 444Mb and 444Of, and these lead to complications in the proofs of such results as 444U, even though the result there is exactly what one would expect.

In this section I ignore right Haar measures entirely. I do not even put them in the exercises. If you wish to take this theory farther, you may some day have to work out the formulae appropriate to right Haar measures. (You can check your results in Hewitt & Ross 63, 20.32.) But for the moment, I think that they are likely to be just a source of confusion. There is one point which you may have noticed. The theory of groups is essentially symmetric. In the definition of 'group' there is no distinction between left and right. In the formulae defining group actions, we do have such a distinction, because they must reflect the fact that we write $g \bullet x$ rather than $x \bullet g$. With $\bullet_l$ and $\bullet_r$, for instance (444Of), if we want them to be actions in the standard sense we have to put an $^{-1}$ into the definition of $\bullet_l$ but not into the definition of $\bullet_r$. But we still expect that, for instance, $\lambda * \nu$ and $\nu * \lambda$ will be related in some transparent way. However there is an exception to this rule in the definitions of $\nu * f$ and $f * \nu$ (444H, 444J). The modular function appears in the latter, so in fact the definition applies only in a more restricted class of groups. In abelian groups we assume that $f * \nu$ and $\nu * f$ will be equal, and they are (444Xk), but strictly speaking, on the definitions here, we can write $f * \nu = \nu * f$ only for abelian topological groups carrying Haar measures.

From the point of view of the proofs in this section, the principal change is that the Haar measures here are no longer assumed to be $\sigma$-finite. I am well aware that non-$\sigma$-finite measures are a minority interest, especially in harmonic analysis, but I do think it interesting that $\sigma$-finiteness is not relevant to the main results, and the techniques required to demonstrate this are very much in the spirit of this treatise (see, in particular, the proof of 444N, and the repeated applications of 443Jb). The basic difficulty is that we can no longer exchange repeated integrals, even of non-negative Borel measurable functions, quite automatically. Let me emphasize that the result in 444N is really rather special. If we try to generalize it to other measures or other types of function we encounter the usual obstacles (444Xp).

A difficulty of a different kind arises in the proof of 444Fc. Here I wish to show that the function $g \mapsto (g\bullet E)^\bullet : G \to \mathfrak{A}$ is Borel measurable for every Borel measurable set $E$. The first step is to deal with open sets $E$, and it would be nice if we could then apply the Monotone Class Theorem. But the difficulty is that even though the map $(a, b) \mapsto a \setminus b : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is continuous, it does not quite follow that the map

$$g \mapsto (g\bullet(E \setminus F))^\bullet = (g\bullet E)^\bullet \setminus (g\bullet F)^\bullet$$

is Borel measurable whenever $g \mapsto (g\bullet E)^\bullet$ and $g \mapsto (g\bullet F)^\bullet$ are, because the map $g \mapsto ((g\bullet E)^\bullet, (g\bullet F)^\bullet) : G \to \mathfrak{A} \times \mathfrak{A}$ might conceivably fail to be Borel measurable, if the metric space $\mathfrak{A}$ is not separable, that is, if the Maharam type of the measure $\nu$ is uncountable. Of course the difficulty is easily resolved by an extra twist in the argument.

I use different techniques for the two parts of 444M as an excuse to recall the ideas of §371; in fact part (a) is slightly easier than part (b) when proved by the method of the latter (444Xo).

444U is a kind of density theorem. Compared with the density theorems in §§223 and 261, it is a 'mean' rather than 'pointwise' density theorem; if $E$ is concentrated near the identity, then $f_E^\bullet$ approximates $f^\bullet$ in $L^p$, but there is no suggestion that we can be sure that $f_E(x) \doteq f(x)$ for any particular $x$s unless we know much more about the set $E$. In fact this is to be expected from the form of the results concerning Lebesgue measure. The sets $E$ considered in Volume 2 are generally intervals or balls, and even in such a general form as 223Ya we need a notion of scalar multiplication separate from the group operation.

Version of 20.3.08

## 445 The duality theorem

In this section I present a proof of the Pontryagin-van Kampen duality theorem (445U). As in Chapter 28, and for the same reasons, we need to use complex-valued functions; the relevant formulae in §§443 and 444 apply unchanged, and I shall not repeat them here, but you may wish to re-read parts of those sections taking functions to be complex- rather than real-valued. (It *is* possible to avoid complex-valued measures, which I relegate to the exercises.) The duality theorem itself applies only to abelian locally compact Hausdorff groups, and it would be reasonable, on first reading, to take it for granted that all groups here are of this type, which simplifies some of the proofs a little.

My exposition is based on that of Rudin 67. I start with the definition of 'dual group', including a description of a topology on the dual (445A), and the simplest examples (445B), with a mention of Fourier-Stieltjes transforms of measures (445C-445D). The elementary special properties of dual groups of groups carrying Haar measures are in 445E-445G; in particular, in these cases, the bidual of a group begins to make sense, and we can start talking about Fourier transforms of functions.

Serious harmonic analysis begins with the identification of the dual group with the maximal ideal space of $L^1$ (445H-445K). The next idea is that of 'positive definite' function (445L-445M). Putting these together, we get the first result here which asserts that the dual group of an abelian group $X$ carrying Haar measures is sufficiently large to effectively describe functions on $X$ (Bochner's theorem, 445N). It is now easy to establish that $X$ can be faithfully embedded in its bidual (445O). We also have most of the machinery necessary to describe the correctly normalized Haar measure of the dual group, with a first step towards identifying functions whose Fourier transforms will have inverse Fourier transforms (the Inversion Theorem, 445P). This leads directly to the Plancherel Theorem, identifying the $L^2$ spaces of $X$ and its dual (445R). At this point it is clear that the bidual $\mathfrak{X}$ cannot be substantially larger than $X$, since they must have essentially the same $L^2$ spaces. A little manipulation of shifts and convolutions in $L^2$ (445S-445T) shows that $X$ must be dense in $\mathfrak{X}$, and a final appeal to local compactness shows that $X$ is closed in $\mathfrak{X}$.

**445A Dual groups** Let $X$ be any topological group.

**(a)** A **character** on $X$ is a continuous group homomorphism from $X$ to $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$. It is easy to see that the set $\mathcal{X}$ of all characters on $X$ is a subgroup of the group $(S^1)^X$, just because $S^1$ is an abelian topological group. (If $\chi$, $\theta \in \mathcal{X}$, then $x \mapsto \chi(x)\theta(x)$ is continuous, and

$$(\chi\theta)(xy) = \chi(xy)\theta(xy) = \chi(x)\chi(y)\theta(x)\theta(y) = \chi(x)\theta(x)\chi(y)\theta(y) = (\chi\theta)(x)(\chi\theta)(y).)$$

So $\mathcal{X}$ itself is an abelian group.

**(b)** Give $\mathcal{X}$ the topology of uniform convergence on subsets of $X$ which are totally bounded for the bilateral uniformity on $X$ (4A5Hb, 4A5O). (If $\mathcal{E}$ is the set of totally bounded subsets of $X$, then the topology of $\mathcal{X}$ is generated by the pseudometrics $\rho_E$, where $\rho_E(\chi, \theta) = \sup_{x \in E} |\chi(x) - \theta(x)|$ for $E \in \mathcal{E}$ and $\chi$, $\theta \in \mathcal{X}$. It will be useful, in this formula, to interpret $\sup \emptyset = 0$, so that $\rho_\emptyset$ is the zero pseudometric. Note that $\mathcal{E}$ is closed under finite unions, so $\{\rho_E : E \in \mathcal{E}\}$ is upwards-directed, as in 2A3Fe.) Then $\mathcal{X}$ is a Hausdorff topological group. (If $x \in E \in \mathcal{E}$ and $\chi$, $\chi_0$, $\theta$, $\theta_0 \in \mathcal{X}$,

$$\begin{aligned}|(\chi\theta)(x) - (\chi_0\theta_0)(x)| &= |\chi(x)(\theta(x) - \theta_0(x)) + \theta_0(x)(\chi(x) - \chi_0(x))| \\ &\le |\theta(x) - \theta_0(x)| + |\chi(x) - \chi_0(x)|,\end{aligned}$$

$$|\chi^{-1}(x) - \chi_0^{-1}(x)| = |\overline{\chi(x)} - \overline{\chi_0(x)}| = |\chi(x) - \chi_0(x)|,$$

so

$$\rho_E(\chi\theta, \chi_0\theta_0) \le \rho_E(\chi, \chi_0) + \rho_E(\theta, \theta_0), \quad \rho_E(\chi^{-1}, \chi_0^{-1}) = \rho_E(\chi, \chi_0).$$

If $\chi \ne \theta$ then there is an $x \in X$ such that $\chi(x) \ne \theta(x)$, and now $\{x\} \in \mathcal{E}$ and $\rho_{\{x\}}(\chi, \theta) > 0$.)

**(c)** Note that if $X$ is locally compact, then its totally bounded sets are just its relatively compact sets (4A5Oe), so the topology of $\mathcal{X}$ is the topology of uniform convergence on compact subsets of $X$.

**(d)** If $X$ is compact, then $\mathcal{X}$ is discrete. **P** $X$ itself is totally bounded, so $U = \{\chi : |\chi(x) - 1| \le 1$ for every $x \in X\}$ is a neighbourhood of the identity $\iota$ in $\mathcal{X}$. But if $\chi \in U$ and $x \in X$ then $|\chi(x)^n - 1| \le 1$ for every $n \in \mathbb{N}$, so $\chi(x) = 1$. Thus $U = \{\iota\}$ and $\iota$ is an isolated point of $\mathcal{X}$; it follows that every point of $\mathcal{X}$ is isolated. **Q**

**(e)** If $X$ is discrete then $\mathcal{X}$ is compact. **P** The only totally bounded sets in $X$ are the finite sets, so the topology of $\mathcal{X}$ is just that induced by its embedding in $(S^1)^X$. On the other hand, every homomorphism from $X$ to $S^1$ is continuous, so $\mathcal{X}$ is a closed set in $(S^1)^X$, which is compact by Tychonoff's theorem. **Q**

**(f)** I ought to remark that to most group theorists the word 'character' means something rather different. For a finite abelian group $X$ with its discrete topology, the 'characters' on $X$, as defined in (a) above, are just the group homomorphisms from $X$ to $\mathbb{C} \setminus \{0\}$, which in this context can be identified with the characters of the irreducible complex representations of $X$.

**445B Examples (a)** If $X = \mathbb{R}$ with addition, then $\mathcal{X}$ can also be identified with the additive group $\mathbb{R}$, if we write $\chi_y(x) = e^{iyx}$ for $x$, $y \in \mathbb{R}$.

**P** It is easy to check that every $\chi_y$, so defined, is a character on $\mathbb{R}$, and that $y \mapsto \chi_y : \mathbb{R} \to \mathcal{X}$ is a homomorphism. On the other hand, if $\chi$ is a character, then (because it is continuous) there is a $\delta \ge 0$ such that $|\chi(x) - 1| \le 1$ whenever $|x| \le \delta$. $\chi(\delta)$ is uniquely expressible as $e^{i\alpha}$ where $|\alpha| \le \frac{\pi}{2}$. Set $y = \alpha/\delta$, so that $\chi(\delta) = \chi_y(\delta)$. Now $\chi(\frac{1}{2}\delta)$ must be one of the square roots of $\chi(\delta)$, so is $\pm\chi_y(\frac{1}{2}\delta)$; but as $|\chi(\frac{1}{2}\delta) - 1| \le 1$, it must be $+\chi_y(\frac{1}{2}\delta)$. Inducing on $n$, we see that $\chi(2^{-n}\delta) = \chi_y(2^{-n}\delta)$ for every $n \in \mathbb{N}$, so that $\chi(2^{-n}k\delta) = \chi_y(2^{-n}k\delta)$ for every $k \in \mathbb{Z}$, $n \in \mathbb{N}$; as $\chi$ and $\chi_y$ are continuous, $\chi = \chi_y$. Thus the map $y \mapsto \chi_y$ is surjective and is a group isomorphism between $\mathbb{R}$ and $\mathcal{X}$.

As for the topology of $\mathcal{X}$, $\mathbb{R}$ is a locally compact topological group, so the totally bounded sets are just the relatively compact sets (4A5Oe again), that is, the bounded sets in the usual sense (2A2F). Now a straightforward calculation shows that for any $\alpha \ge 0$ in $\mathbb{R}$ and $\epsilon \in \,]0, 2[$,

$$\rho_{[-\alpha,\alpha]}(\chi_y,\chi_z) \le \epsilon \iff \alpha|y-z| \le 2\arcsin\frac{\epsilon}{2},$$

so that the topology of $\mathcal{X}$ agrees with that of $\mathbb{R}$. **Q**

**(b)** Let $X$ be the group $\mathbb{Z}$ with its discrete topology. Then we may identify its dual group $\mathcal{X}$ with $S^1$ itself, writing $\chi_\zeta(n) = \zeta^n$ for $\zeta \in S^1$, $n \in \mathbb{Z}$. **P** Once again, it is elementary to check that every $\chi_\zeta$ is a character, and that $\zeta \mapsto \chi_\zeta$ is an injective group homomorphism from $S^1$ to $\mathcal{X}$. If $\chi \in \mathcal{X}$, set $\zeta = \chi(1)$; then $\chi = \chi_\zeta$. So $\mathcal{X} \cong S^1$. And because the only totally bounded sets in $X$ are finite, $\chi \mapsto \chi_\zeta$ is continuous, therefore a homeomorphism. **Q**

**(c)** On the other hand, if $X = S^1$ with its usual topology, then we may identify its dual group $\mathcal{X}$ with $\mathbb{Z}$, writing $\chi_n(\zeta) = \zeta^n$ for $n \in \mathbb{Z}$, $\zeta \in S^1$. **P** The verification follows the same lines as in (a) and (b). As usual, the key step is to show that the map $n \mapsto \chi_n : \mathbb{Z} \to \mathcal{X}$ is surjective. We can do this by applying (a). If $\chi \in \mathcal{X}$, then $x \mapsto \chi(e^{ix})$ is a character of $\mathbb{R}$, so there is a $y \in \mathbb{R}$ such that $\chi(e^{ix}) = e^{iyx}$ for every $x \in \mathbb{R}$. In particular, $e^{2iy\pi} = \chi(1) = 1$, so $y \in \mathbb{Z}$, and $\chi = \chi_y$. Concerning the topology of $\mathcal{X}$, we know from 445Ad that it must be discrete, so that also matches the usual topology of $\mathbb{Z}$. **Q**

**(d)** Let $\langle X_j\rangle_{j\in J}$ be any family of topological groups, and $X$ their product (4A5G). For each $j \in J$ let $\mathcal{X}_j$ be the dual group of $X_j$. Then the dual group of $X$ can be identified with the subgroup $\mathcal{X}$ of $\prod_{j\in I}\mathcal{X}_j$ consisting of those $\chi \in \prod_{j\in J}\mathcal{X}_j$ such that $\{j : \chi(j)$ is not the identity$\}$ is finite; the action of $\mathcal{X}$ on $X$ is defined by the formula

$$\chi\bullet x = \prod_{j\in J}\chi(j)(x(j)).$$

(This is well-defined because only finitely many terms in the product are not equal to 1.) If $I$ is finite, so that $\mathcal{X} = \prod_{j\in I}\mathcal{X}_j$, the topology of $\mathcal{X}$ is the product topology.

**P** As usual, it is easy to check that $\bullet$, as defined above, defines an injective homomorphism from $\mathcal{X}$ to the dual group of $X$. If $\theta$ is any character on $X$, then for each $j \in I$ we have a continuous group homomorphism $\varepsilon_j : X_j \to X$ defined by setting $\varepsilon_j(\xi)(j) = \xi$, $\varepsilon_j(\xi)(k) = e_k$, the identity of $X_k$, for every $k \ne j$. Setting $\chi(j) = \theta\varepsilon_j$ for each $j$, we obtain $\chi \in \prod_{j\in I}\mathcal{X}_j$. Now there is a neighbourhood $U$ of the identity of $X$ such that $|\theta(x) - 1| \le 1$ for every $x \in U$, and we may suppose that $U$ is of the form $\{x : x(j) \in G_j$ for every $j \in J\}$, where $J \subseteq I$ is finite and $G_j$ is a neighbourhood of $e_j$ for every $j \in J$. If $k \in I \setminus J$, $\varepsilon_k(\xi) \in U$ for every $\xi \in X_k$, so that $|\chi(k)(\xi) - 1| \le 1$ for every $\xi$, and $\chi(k)$ must be the identity character on $X_k$; this shows that $\chi \in \mathcal{X}$. If $x \in X$ and $x(j) = e_j$ for $j \in J$, then again $|\theta(x^n) - 1| \le 1$ for every $n \in \mathbb{N}$, so $\theta(x) = \chi\bullet x = 1$. For any $x \in X$, we can express it as a finite product $y\prod_{j\in J}\varepsilon_j(x(j))$ where $y(j) = e_j$ for every $j \in J$, so that

$$\theta(x) = \theta(y)\prod_{j\in J}\theta\varepsilon_j(x(j)) = \prod_{j\in J}\chi(j)(x(j)) = \chi\bullet x.$$

Thus $\bullet$ defines an isomorphism between $\mathcal{X}$ and the dual group of $X$.

As for the topology of $\mathcal{X}$, a subset of $X$ is totally bounded iff it is included in a product of totally bounded sets (4A5Od). If $E = \prod_{j\in I}E_j$ is such a product (and not empty), then for $\chi$, $\theta \in \mathcal{X}$

$$\sup_{j\in I}\rho_{E_j}(\chi(j),\theta(j)) \le \rho_E(\chi,\theta) \le \sum_{j\in I}\rho_{E_j}(\chi(j),\theta(j)),$$

so (if $I$ is finite) the topology on $\mathcal{X}$ is just the product topology. **Q**

**445C Fourier-Stieltjes transforms** Let $X$ be a topological group, and $\mathcal{X}$ its dual group. For any totally finite topological measure $\nu$ on $X$, we can form its 'characteristic function' or **Fourier-Stieltjes transform** $\hat\nu : \mathcal{X} \to \mathbb{C}$ by writing $\hat\nu(\chi) = \int\chi(x)\nu(dx)$. (This generalizes the 'characteristic functions' of §285.)

**445D Theorem** Let $X$ be a topological group, and $\mathcal{X}$ its dual group. If $\lambda$ and $\nu$ are totally finite quasi-Radon measures on $X$, then $(\lambda * \nu)^\wedge = \hat\lambda \times \hat\nu$.

**proof** If $\chi \in \mathcal{X}$, then, by 444C,

$$(\lambda*\nu)^\wedge(\chi)=\int\chi\,d(\lambda*\nu)=\iint\chi(xy)\lambda(dx)\nu(dy)$$
$$=\iint\chi(x)\chi(y)\lambda(dx)\nu(dy)=\int\chi(x)\lambda(dx)\cdot\int\chi(y)\nu(dy)=\hat\lambda(\chi)\hat\nu(\chi).$$

**445E** Let us turn now to groups carrying Haar measures. I start with three welcome properties.

**Proposition** Let $X$ be a topological group with a neighbourhood of the identity which is totally bounded for the bilateral uniformity on $X$, and $\mathcal{X}$ its dual group, with its dual group topology.

(a) The map $(\chi,x)\mapsto\chi(x):\mathcal{X}\times X\to S^1$ is continuous.

(b) Let $\mathfrak{X}$ be the dual group of $\mathcal{X}$, again with its dual group topology, the topology of uniform convergence on totally bounded subsets of $\mathcal{X}$. Then we have a continuous homomorphism $x\mapsto\hat x:X\to\mathfrak{X}$ defined by setting $\hat x(\chi)=\chi(x)$ for $x\in X$ and $\chi\in\mathcal{X}$.

(c) For any totally finite quasi-Radon measure $\nu$ on $X$, its Fourier-Stieltjes transform $\hat\nu:\mathcal{X}\to\mathbb{C}$ is uniformly continuous.

**Remark** Note that the condition here is satisfied by any topological group $X$ carrying Haar measures (443H).

**proof** Fix an open totally bounded set $U_0$ containing the identity.

**(a)** Let $\chi_0\in\mathcal{X}$, $x_0\in X$ and $\epsilon>0$. Then $x_0U_0$ is totally bounded, so

$$V=\{\chi:|\chi(y)-\chi_0(y)|\le\frac{1}{2}\epsilon\text{ for every }y\in x_0U_0\}$$

is a neighbourhood of $\chi_0$. Also

$$U=\{x:x\in x_0U_0,\,|\chi_0(x)-\chi_0(x_0)|\le\frac{1}{2}\epsilon\}$$

is a neighbourhood of $x_0$. And if $\chi\in V$, $x\in U$ we have

$$|\chi(x)-\chi_0(x_0)|\le|\chi(x)-\chi_0(x)|+|\chi_0(x)-\chi_0(x_0)|\le\epsilon.$$

As $\chi_0$, $x_0$ and $\epsilon$ are arbitrary, $(\chi,x)\mapsto\chi(x)$ is continuous.

**(b)(i)** It is easy to check that $\hat x$, as defined above, is always a homomorphism from $\mathcal{X}$ to $S^1$, and that $x\mapsto\hat x:X\to(S^1)^{\mathcal{X}}$ is a homomorphism. Because $\rho_{\{x\}}$ is always one of the defining pseudometrics for the topology of $\mathcal{X}$ (445Ab), $\hat x$ is always continuous, so belongs to $\mathfrak{X}$.

**(ii)** To see that $\hat{}$ is continuous, I argue as follows. Take an open set $H\subseteq\mathfrak{X}$ and $x_0\in X$ such that $\hat x_0\in H$. Then there are a totally bounded set $F\subseteq\mathcal{X}$ and an $\epsilon>0$ such that $\mathfrak{x}\in H$ whenever $\mathfrak{x}\in\mathfrak{X}$ and $\rho_F(\mathfrak{x},\hat x_0)\le\epsilon$. Now $x_0U_0$ is a totally bounded neighbourhood of $x_0$, so

$$V=\{\theta:\theta\in\mathcal{X},\,|\theta(y)-1|\le\frac{1}{2}\epsilon\text{ for every }y\in x_0U_0\}$$

is a neighbourhood of the identity in $\mathcal{X}$. There are therefore $\chi_0,\dots,\chi_n\in\mathcal{X}$ such that $F\subseteq\bigcup_{k\le n}\chi_kV$. Set

$$U=\{x:x\in x_0U_0,\,|\chi_k(x)-\chi_k(x_0)|<\frac{1}{2}\epsilon\text{ for every }k\le n\}.$$

Then $U$ is an open neighbourhood of $x_0$ in $X$.

If $x\in U$ and $\chi\in F$ then there is a $k\le n$ such that $\theta=\chi_k^{-1}\chi\in V$, so that

$$|\chi(x)-\chi(x_0)|=|\chi_k(x)\theta(x)-\chi_k(x_0)\theta(x_0)|$$
$$\le|\chi_k(x)-\chi_k(x_0)|+|\theta(x)-\theta(x_0)|\le\frac{1}{2}\epsilon+\frac{1}{2}\epsilon=\epsilon.$$

But this shows that $\rho_F(\hat x,\hat x_0)\le\epsilon$, so $\hat x\in H$.

So we have $x_0\in U\subseteq\{x:\hat x\in H\}$. As $x_0$ is arbitrary, $\{x:\hat x\in H\}$ is open; as $H$ is arbitrary, $x\mapsto\hat x$ is continuous.

**(c)** Let $\epsilon > 0$. Because $\nu$ is $\tau$-additive, there are $x_0, \dots, x_n \in X$ such that $\nu(X \setminus \bigcup_{k\le n} x_k U_0) \le \frac{1}{3}\epsilon$. Set $E = \bigcup_{k\le n} x_k U_0$; then $E$ is totally bounded. So

$$V = \{\theta : |\theta(x) - 1| \le \tfrac{\epsilon}{1+3\nu X} \text{ for every } x \in E\}$$

is a neighbourhood of the identity in $\mathcal{X}$. If $\chi$, $\chi' \in \mathcal{X}$ are such that $\theta = \chi^{-1}\chi'$ belongs to $V$, then

$$\begin{aligned}|\hat{\nu}(\chi) - \hat{\nu}(\chi')| &\le \int |\chi(x) - \chi'(x)|\nu(dx) = \int |1 - \theta(x)|\nu(dx) \\ &\le 2\nu(X \setminus E) + \tfrac{\epsilon\nu E}{1+3\nu X} \le \epsilon.\end{aligned}$$

Since $\epsilon$ is arbitrary (and $\mathcal{X}$ is abelian), this is enough to show that $\hat{\nu}$ is uniformly continuous.

**445F Fourier transforms of functions** Let $X$ be a topological group with a left Haar measure $\mu$. For any $\mu$-integrable complex-valued function $f$, define its **Fourier transform** $\hat{f} : \mathcal{X} \to \mathbb{C}$ by setting $\hat{f}(\chi) = \int f(x)\chi(x)\mu(dx)$ for every character $\chi$ of $X$. (Compare 283A. If $f$ is real and non-negative, then $\hat{f} = (f\mu)^\wedge$ as defined in 445C, where $f\mu$ is the indefinite-integral measure, as in 444K.) Note that $\hat{f} = \hat{g}$ whenever $f =_{\text{a.e.}} g$, so we can equally well write $\hat{u}(\chi) = \hat{f}(\chi)$ whenever $u = f^\bullet$ in $L^1_{\mathbb{C}}(\mu)$.

**445G Proposition** Let $X$ be a topological group with a left Haar measure $\mu$. Then for any $\mu$-integrable complex-valued functions $f$ and $g$, $(f * g)^\wedge = \hat{f} \times \hat{g}$; that is, $(u * v)^\wedge = \hat{u} \times \hat{v}$ for all $u$, $v \in L^1_{\mathbb{C}}(\mu)$.

**proof** For any character $\chi$ on $X$,

$$\begin{aligned}\int \chi(x)(f * g)(x)dx &= \iint \chi(xy)f(x)g(y)dxdy \\ &= \iint \chi(x)\chi(y)f(x)g(y)dxdy = \hat{f}(\chi)\hat{g}(\chi)\end{aligned}$$

(using 444Od).

**445H Theorem** Let $X$ be a topological group with a left Haar measure $\mu$; let $\mathcal{X}$ be its dual group and let $\Phi$ be the set of non-zero multiplicative linear functionals on the complex Banach algebra $L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$ (444Sb). Then there is a one-to-one correspondence between $\mathcal{X}$ and $\Phi$, defined by the formulae

$$\phi(f^\bullet) = \int f \times \chi\, d\mu = \hat{f}(\chi) \text{ for every } f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu),$$

$$\phi(a\bullet_l u) = \chi(a)\phi(u) \text{ for every } u \in L^1_{\mathbb{C}},\ a \in X,$$

for $\chi \in \mathcal{X}$ and $\phi \in \Phi$.

**Remark** I follow 443G in writing $a\bullet_l f^\bullet = (a\bullet_l f)^\bullet$, where $(a\bullet_l f)(x) = f(a^{-1}x)$ for $f \in \mathcal{L}^1_{\mathbb{C}}$ and $a$, $x \in X$, as in 4A5Cc.

**proof (a)** If $\chi \in \mathcal{X}$ then we can think of its equivalence class $\chi^\bullet$ as a member of $L^\infty_{\mathbb{C}} = L^\infty_{\mathbb{C}}(\mu)$, so that we can define $\phi_\chi \in (L^1_{\mathbb{C}})^*$ by writing $\phi_\chi(u) = \int \chi^\bullet \times u$ for every $u \in L^1_{\mathbb{C}}$; that is, $\phi_\chi(f^\bullet) = \int f \times \chi = \hat{f}(\chi)$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. 445G tells us that $\phi_\chi$ is multiplicative. To see that it is non-zero, recall that $\mu$ is strictly positive (442Aa) and that $\chi$ is continuous. Let $G$ be an open set containing the identity $e$ of $X$ such that $|\chi(x) - 1| \le \frac{1}{2}$ for every $x \in G$; then $\mathcal{R}\mathrm{e}(\chi(x)) \ge \frac{1}{2}$ for every $x \in G$, so

$$|\int_G \chi(x)dx| \ge \mathcal{R}\mathrm{e} \int_G \chi(x)dx = \int_G \mathcal{R}\mathrm{e}(\chi(x))dx \ge \tfrac{1}{2}\mu G > 0.$$

Accordingly $\phi_\chi(\chi G)^\bullet \ne 0$ and $\phi_\chi \ne 0$. (I hope that no confusion will arise if I continue occasionally to write $\chi E$ for the indicator function of a set $E$, even if the symbol $\chi$ is already active in the sentence.)

**(b)** Now suppose that $\phi$ is a non-zero multiplicative linear functional on $L^1_{\mathbb{C}}$. Fix on some $g_0 \in \mathcal{L}^1_{\mathbb{C}}$ such that $\phi(g_0^\bullet) = 1$. Let $\Delta$ be the left modular function of $X$. (If you are reading this proof on the assumption that $X$ is abelian, then $a\bullet_r f = a^{-1}\bullet_l f$ and $\Delta \equiv 1$, so the argument below simplifies usefully.)

**(i)** For any $u \in L^1_{\mathbb{C}}$ and $a \in X$, $\phi(a^{-1}\bullet_l u) = \Delta(a)\phi(a\bullet_r u)$. **P** Let $f \in \mathcal{L}^1_{\mathbb{C}}$ be such that $f^\bullet = u$. Take any $\epsilon > 0$. Then for any sufficiently small open neighbourhood $U$ of the identity, if we set $h = \frac{1}{\mu U}\chi U$, we shall have

$$\|(a\bullet_r f) * h - a\bullet_r f\|_1 \le \epsilon, \quad \|h * f - f\|_1 \le \epsilon$$

(444T, with 444P; see 444U). Setting $w = h^\bullet$, we have

$$\|(a\bullet_r u) * w - a\bullet_r u\|_1 \le \epsilon, \quad \|w * u - u\|_1 \le \epsilon,$$

$$|\phi(a^{-1}\bullet_l u) - \Delta(a)\phi(a\bullet_r u)| \le |\phi(a^{-1}\bullet_l u) - \phi(u * (a^{-1}\bullet_l w))|$$

(444Sa)

$$+ |\Delta(a)\phi((a\bullet_r u) * w) - \Delta(a)\phi(a\bullet_r u)|$$
$$= |\phi(a^{-1}\bullet_l u) - \phi(u)\phi(a^{-1}\bullet_l w)|$$
$$+ \Delta(a)|\phi((a\bullet_r u) * w) - \phi(a\bullet_r u)|$$
$$\le |\phi(a^{-1}\bullet_l u) - \phi(a^{-1}\bullet_l w)\phi(u)|$$
$$+ \Delta(a)\|(a\bullet_r u) * w - a\bullet_r u\|_1$$

(because $\|\phi\| \le 1$ in $(L^1_{\mathbb{C}})^*$, by 4A6F)

$$\le |\phi(a^{-1}\bullet_l u) - \phi((a^{-1}\bullet_l w) * u)| + \epsilon\Delta(a)$$
$$\le \|a^{-1}\bullet_l u - (a^{-1}\bullet_l w) * u\|_1 + \epsilon\Delta(a)$$
$$= \|a^{-1}\bullet_l (u - w * u)\|_1 + \epsilon\Delta(a)$$

(by another of the formulae in 444Sa)

$$= \|u - w * u\|_1 + \epsilon\Delta(a)$$

(443Ge)

$$\le (1 + \Delta(a))\epsilon.$$

As $\epsilon$ is arbitrary, we have the result. **Q**

**(ii)** For any $u$, $v \in L^1_{\mathbb{C}}$ and $a \in X$, $\phi(a\bullet_l u)\phi(v) = \phi(u)\phi(a\bullet_l v)$. **P**

$$\phi(a\bullet_l u)\phi(v) = \phi((a\bullet_l u) * v) = \phi(a\bullet_l (u * v)) = \Delta(a^{-1})\phi(a^{-1}\bullet_r (u * v))$$
$$= \Delta(a^{-1})\phi(u * a^{-1}\bullet_r v) = \phi(u)\Delta(a^{-1})\phi(a^{-1}\bullet_r v) = \phi(u)\phi(a\bullet_l v),$$

using (i) for the third and sixth equalities, and 444Sa for the second and fourth. **Q**

**(iii)** Let $v_0$ be $g_0^\bullet$, so that $\phi(v_0) = 1$, and set $\chi(a) = \phi(a\bullet_l v_0)$ for every $a \in X$. Then if $a$, $b \in X$,

$$\chi(ab) = \phi(ab\bullet_l v_0) = \phi(a\bullet_l (b\bullet_l v_0))$$

(because $\bullet_l$ is an action of $X$ on $L^1_{\mathbb{C}}$, as noted in 443Ge for $L^1$)

$$= \phi(a\bullet_l (b\bullet_l v_0))\phi(v_0) = \phi(b\bullet_l v_0)\phi(a\bullet_l v_0)$$

(by (ii) above)

$$= \chi(a)\chi(b).$$

So $\chi : X \to \mathbb{C}$ is a group homomorphism. Moreover, because $\bullet_l$ is continuous (443Gf), and $\phi$ also is continuous (indeed, of norm at most 1), $\chi$ is continuous. Finally,

$$|\chi(a)| \le \|a\bullet_l v_0\|_1 = \|v_0\|_1$$

for every $a\in X$, by 443Gb; it follows at once that $\{\chi(a)^n:n\in\mathbb{Z}\}$ is bounded, so that $|\chi(a)|=1$, for every $a\in X$. Thus $\chi\in\mathcal{X}$. Moreover, for any $u\in L^1_{\mathbb{C}}$,

$$\phi(a\bullet_l u)=\phi(a\bullet_l u)\phi(v_0)=\phi(u)\phi(a\bullet_l v_0)=\chi(a)\phi(u).$$

**(iv)** Now $\phi=\phi_\chi$. **P** Because $\phi\in(L^1_{\mathbb{C}})^*$, there is some $h\in\mathcal{L}^\infty_{\mathbb{C}}(\mu)$ such that $\phi(f^\bullet)=\int h(x)f(x)dx$ for every $f\in\mathcal{L}^1_{\mathbb{C}}$ (243Gb/243K; recall that by the rules of 441D, $\mu$ is suppose to be a quasi-Radon measure, therefore strictly localizable, by 415A). In this case, for any $f\in\mathcal{L}^1_{\mathbb{C}}$,

$$\phi(f^\bullet)=\phi(f*g_0)^\bullet=\int h(x)(f*g_0)(x)dx=\iint h(xy)f(x)g_0(y)dydx$$

(444Od)

$$=\iint h(y)f(x)g_0(x^{-1}y)dydx=\int\phi(x\bullet_l g_0)^\bullet f(x)dx$$

$$=\int\chi(x)f(x)dx=\phi_\chi(f^\bullet).\ \mathbf{Q}$$

**(c)** Thus we see that the formulae announced do define a surjection from $\mathcal{X}$ onto $\Phi$. We have still to confirm that it is injective. But if $\chi$, $\theta$ are distinct members of $\mathcal{X}$, then $\{x:\chi(x)\ne\theta(x)\}$ is a non-empty open set, so has positive measure, because $\mu$ is strictly positive; because $\mu$ is semi-finite, they represent different linear functionals on $L^1_{\mathbb{C}}$, and $\phi_\chi\ne\phi_\theta$.

This completes the proof.

**445I The topology of the dual group: Proposition** Let $X$ be a topological group with a left Haar measure $\mu$, and $\mathcal{X}$ its dual group. For $\chi\in\mathcal{X}$, let $\chi^\bullet$ be its equivalence class in $L^0_{\mathbb{C}}=L^0_{\mathbb{C}}(\mu)$, and $\phi_\chi\in(L^1_{\mathbb{C}})^*=(L^1_{\mathbb{C}}(\mu))^*$ the multiplicative linear functional corresponding to $\chi$, as in 445H. Then the maps $\chi\mapsto\chi^\bullet$ and $\chi\mapsto\phi_\chi$ are homeomorphisms between $\mathcal{X}$ and its images in $L^0_{\mathbb{C}}$ and $(L^1_{\mathbb{C}})^*$, if we give $L^0_{\mathbb{C}}$ the topology of convergence in measure (245A/245M) and $(L^1_{\mathbb{C}})^*$ the weak* topology (2A5Ig).

**proof (a)** Note that $\chi\mapsto\chi^\bullet$ is injective because $\mu$ is strictly positive, so that if $\chi$, $\theta$ are distinct members of $\mathcal{X}$ then the non-empty open set $\{x:\chi(x)\ne\theta(x)\}$ has non-zero measure; and that $\chi\mapsto\phi_\chi$ is injective by 445H. So we have one-to-one correspondences between $\mathcal{X}$ and its images in $L^0_{\mathbb{C}}$ and $(L^1_{\mathbb{C}})^*$.

Write $\mathfrak{T}$ for the topology of $\mathcal{X}$ as defined in 445Ab, $\mathfrak{T}_m$ for the topology induced by its identification with its image in $L^0_{\mathbb{C}}$, and $\mathfrak{T}_w$ for the topology induced by its identification with its image in $(L^1_{\mathbb{C}})^*$. Let $\mathcal{E}$ be the family of non-empty totally bounded subsets of $X$ and $\Sigma^f$ the set of measurable sets of finite measure; for $E\in\mathcal{E}$, $F\in\Sigma^f$ and $f\in\mathcal{L}^1_{\mathbb{C}}=\mathcal{L}^1_{\mathbb{C}}(\mu)$ set

$$\rho_E(\chi,\theta)=\sup_{x\in E}|\chi(x)-\theta(x)|,$$

$$\rho'_F(\chi,\theta)=\int_F\min(1,|\chi(x)-\theta(x)|)\mu(dx),$$

$$\rho''_f(\chi,\theta)=|\int f(x)\chi(x)\mu(dx)-\int f(x)\theta(x)\mu(dx)|$$

for $\chi$, $\theta\in\mathcal{X}$. Then $\mathfrak{T}$ is generated by the pseudometrics $\{\rho_E:E\in\mathcal{E}\}$, $\mathfrak{T}_m$ is generated by $\{\rho'_F:F\in\Sigma^f\}$ and $\mathfrak{T}_w$ is generated by $\{\rho''_f:f\in\mathcal{L}^1_{\mathbb{C}}\}$.

**(b)** $\mathfrak{T}_m\subseteq\mathfrak{T}$. **P** Suppose that $F\subseteq X$ is a measurable set of finite measure, and $\epsilon>0$. There is a non-empty totally bounded open set $U\subseteq X$ (443H). Since $\{xU:x\in X\}$ is an open cover of $X$ and $\mu$ is $\tau$-additive, there are $y_0,\dots,y_n\in X$ such that $\mu(F\setminus\bigcup_{j\le n}y_jU)\le\frac{1}{3}\epsilon$; set $E=\bigcup_{j\le n}y_jU$. Then $E$ is totally bounded, and $\rho'_F(\chi,\theta)\le\epsilon$ whenever $\rho_E(\chi,\theta)\le\frac{\epsilon}{1+3\mu E}$. As $F$ and $\epsilon$ are arbitrary, the identity map $(\mathcal{X},\mathfrak{T})\to(\mathcal{X},\mathfrak{T}_m)$ is continuous (2A3H), that is, $\mathfrak{T}_m\subseteq\mathfrak{T}$. **Q**

**(c)** $\mathfrak{T}_w\subseteq\mathfrak{T}_m$. **P** If $f\in\mathcal{L}^1_{\mathbb{C}}$ and $\epsilon>0$ let $F\in\Sigma^f$, $M>0$ be such that $\int(|f|-M\chi F)^+d\mu\le\frac{1}{4}\epsilon$. If $\chi$, $\theta\in\mathcal{X}$ and $\rho'_F(\chi,\theta)\le\frac{\epsilon}{4M}$, then

$$\begin{aligned}\rho''_f(\chi,\theta) &= \Bigl|\int(\chi-\theta)\times f\Bigr| \le \int|\chi-\theta|\times|f| \\ &\le 2\int(|f|-M\chi F)^+ + M\int_F|\chi-\theta| \\ &\le \tfrac{1}{2}\epsilon + 2M\int_F\min(1,|\chi-\theta|) = \tfrac{1}{2}\epsilon + 2M\rho'_F(\chi,\theta) \le \epsilon.\end{aligned}$$

As $f$ and $\epsilon$ are arbitrary, this shows that $\mathfrak{T}_w \subseteq \mathfrak{T}_m$. **Q**

**(d)** Finally, $\mathfrak{T} \subseteq \mathfrak{T}_w$. **P** Fix $\chi \in \mathcal{X}$, $E \in \mathcal{E}$ and $\epsilon > 0$. Let $u \in L^1_{\mathbb{C}}$ be such that $\phi_\chi(u) = 1$, and represent $u$ as $f^\bullet$ where $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$. Set $U = \{a : a \in X, \|a\bullet_l u - u\|_1 < \frac{1}{4}\epsilon\}$; then $U$ is an open neighbourhood of the identity $e$ of $X$, because $a \mapsto a\bullet_l u$ is continuous (443Gf). Because $E$ is totally bounded, there are $y_0,\dots,y_n \in X$ such that $E \subseteq \bigcup_{k\le n} y_k U$. Set $f_k = y_k\bullet_l f$, so that $f_k^\bullet = y_k\bullet_l u$ for each $k \le n$.

Now suppose that $\theta \in \mathcal{X}$ is such that

$$\rho''_f(\theta,\chi) \le \frac{\epsilon}{4}, \quad \rho''_{f_k}(\theta,\chi) \le \frac{\epsilon}{4} \text{ for every } k \le n.$$

Take any $x \in E$. Then there is a $k \le n$ such that $x \in y_k U$, so that $y_k^{-1}x \in U$ and

$$\|x\bullet_l u - y_k\bullet_l u\|_1 = \|y_k\bullet_l(y_k^{-1}x\bullet_l u - u)\|_1 = \|y_k^{-1}x\bullet_l u - u\|_1 \le \frac{\epsilon}{4}$$

(using 443Ge for the second equality). Now $\phi_\chi(x\bullet_l u) = \chi(x)$ (445H), so

$$\begin{aligned}|\phi_\theta(x\bullet_l u) - \chi(x)| &\le |\phi_\theta(x\bullet_l u - y_k\bullet_l u)| + |\phi_\theta(y_k\bullet_l u) - \phi_\chi(y_k\bullet_l u)| + |\phi_\chi(y_k\bullet_l u - x\bullet_l u)| \\ &\le 2\|x\bullet_l u - y_k\bullet_l u\|_1 + \rho''_{f_k}(\theta,\chi) \le \frac{3}{4}\epsilon.\end{aligned}$$

On the other hand,

$$|\theta(x) - \phi_\theta(x\bullet_l u)| = |\theta(x)||1 - \phi_\theta(u)| = \rho''_f(\theta,\chi) \le \frac{\epsilon}{4}.$$

So $|\theta(x) - \chi(x)| \le \epsilon$. As $x$ is arbitrary, $\rho_E(\theta,\chi) \le \epsilon$.

As $\chi$, $E$ and $\epsilon$ are arbitrary, this shows that $\mathfrak{T} \subseteq \mathfrak{T}_w$. **Q**

**445J Corollary** For any topological group $X$ carrying Haar measures, its dual group $\mathcal{X}$ is locally compact and Hausdorff.

**proof** Let $\Phi$ be the set of non-zero multiplicative linear functionals on $L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$, for some left Haar measure $\mu$ on $X$, and give $\Phi$ its weak* topology. Then $\Phi \cup \{0\} \subseteq (L^1_{\mathbb{C}})^*$ is the set of all multiplicative linear functionals on $L^1_{\mathbb{C}}$, and is closed for the weak* topology, because

$$\{\phi : \phi \in (L^1_{\mathbb{C}})^*, \ \phi(u*v) = \phi(u)\phi(v)\}$$

is closed for all $u$, $v \in L^1_{\mathbb{C}}$. Because the unit ball of $(L^1_{\mathbb{C}})^*$ includes $\Phi$ (4A6F again), and is a compact Hausdorff space for the weak* topology (3A5F), so is $\Phi \cup \{0\}$. So $\Phi$ itself is an open subset of a compact Hausdorff space and is a locally compact Hausdorff space (3A3Bg). Since the topology on $\mathcal{X}$ can be identified with the weak* topology on $\Phi$ (445I), $\mathcal{X}$ also is locally compact and Hausdorff.

**445K Proposition** Let $X$ be a topological group and $\mu$ a left Haar measure on $X$. Let $\mathcal{X}$ be the dual group of $X$, and write $C_0 = C_0(\mathcal{X};\mathbb{C})$ for the Banach algebra of continuous functions $h : \mathcal{X} \to \mathbb{C}$ such that $\{\chi : |h(\chi)| \ge \epsilon\}$ is compact for every $\epsilon > 0$.

(a) For any $u \in L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$, its Fourier transform $\hat{u}$ belongs to $C_0$.

(b) The map $u \mapsto \hat{u} : L^1_{\mathbb{C}} \to C_0$ is a multiplicative linear operator, of norm at most 1.

(c) Suppose that $X$ is abelian. For $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$, set $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined. Then $\tilde{f} \in \mathcal{L}^1_{\mathbb{C}}$ and $\|\tilde{f}\|_1 = \|f\|_1$. For $u \in L^1_{\mathbb{C}}$, we may define $\tilde{u} \in L^1_{\mathbb{C}}$ by setting $\tilde{u} = \tilde{f}^\bullet$ whenever $u = f^\bullet$. Now $\hat{\tilde{u}}$ is the complex conjugate of $\hat{u}$, so $(u*\tilde{u})^\wedge = |\hat{u}|^2$.

(d) Still supposing that $X$ is abelian, $\{\hat{u} : u \in L^1_{\mathbb{C}}\}$ is a norm-dense subalgebra of $C_0$, and $\|\hat{u}\|_\infty = r(u)$, the spectral radius of $u$ (4A6G), for every $u \in L^1_{\mathbb{C}}$.

**proof (a)** As in 445H and 445J, let $\Phi$ be the set of non-zero multiplicative linear functionals on $L^1_{\mathbb{C}}$, so that $\Phi \cup \{0\}$ is compact for the weak* topology of $(L^1_{\mathbb{C}})^*$, and $\hat{u}(\chi) = \phi_\chi(u)$ for every $\chi \in \mathcal{X}$. By the definition of the weak* topology, $\phi \mapsto \phi(u)$ is continuous; since we can identify the weak* topology on $\Phi$ with the dual group topology of $\mathcal{X}$ (445I), $\hat{u}$ is continuous. Also, for any $\epsilon > 0$,

$$\{\chi : \chi \in \mathcal{X},\ |\hat{u}(\chi)| \ge \epsilon\} \cong \{\phi : \phi \in \Phi,\ |\phi(u)| \ge \epsilon\},$$

which is a closed subset of $\Phi \cup \{0\}$, therefore compact.

**(b)** It is immediate from the definition of ^ that it is a linear operator from $L^1_{\mathbb{C}}$ to $\mathbb{C}^{\mathcal{X}}$, and therefore from $L^1_{\mathbb{C}}$ to $C_0$; it is multiplicative by 445G, and of norm at most 1 because all the multiplicative linear functionals $u \mapsto \hat{u}(\chi)$ must be of norm at most 1.

**(c)** Now suppose that $X$ is abelian. If $f \in \mathcal{L}^1_{\mathbb{C}}$, then

$$\int \tilde{f}(x)dx = \overline{\int f(x^{-1})dx} = \overline{\int f(x)dx}$$

by 442Kb, so $\tilde{f} \in \mathcal{L}^1_{\mathbb{C}}$; the same formulae tell us that $\|\tilde{f}\|_1 = \|f\|_1$. If $f =_{\text{a.e.}} g$ then $\tilde{f} =_{\text{a.e.}} \tilde{g}$ (442G, or otherwise), so $\tilde{u}$ is well-defined. If $\chi \in \mathcal{X}$, and $u = f^\bullet$, then

$$\begin{aligned}\hat{\tilde{u}}(\chi) &= \int \tilde{f}(x)\chi(x)dx = \int \overline{f(x^{-1})}\chi(x)dx = \int \overline{f(x)}\chi(x^{-1})dx \\ &= \int \overline{f(x)\chi(x)}dx = \overline{\int f(x)\chi(x)dx} = \overline{\hat{u}(\chi)},\end{aligned}$$

so $\hat{\tilde{u}}$ is the complex conjugate of $\hat{u}$, and

$$(u * \tilde{u})^\wedge = \hat{u} \times \hat{\tilde{u}} = |\hat{u}|^2.$$

**(d)** To see that $A = \{\hat{u} : u \in L^1_{\mathbb{C}}\}$ is dense in $C_0$, we can use the Stone-Weierstrass theorem in the form 4A6B. $A$ is a subalgebra of $C_0$; it separates the points (because the canonical map from $\mathcal{X}$ to $(L^1_{\mathbb{C}})^*$ is injective); if $\chi \in \mathcal{X}$, there is an $h \in A$ such that $h(\chi) \ne 0$ (because elements of $\mathcal{X}$ act on $L^1_{\mathbb{C}}$ as non-zero functionals); and the complex conjugate of any function in $A$ belongs to $A$, by (c) above.

Accordingly $A$ is dense in $C_0$, by 4A6B.

The calculation of $\|\hat{u}\|_\infty$ is an immediate consequence of the characterization of $r(u)$ as $\max\{|\phi(u)| : \phi \in \Phi\}$ (4A6K) and the identification of $\Phi$ with $\mathcal{X}$.

**Remark** This is the first point in this section where we really need to know whether or not our group is abelian.

**445L Positive definite functions** Let $X$ be a group.

**(a)** A function $h : X \to \mathbb{C}$ is called **positive definite** if

$$\sum_{j,k=0}^n \zeta_j \bar{\zeta}_k h(x_k^{-1}x_j) \ge 0$$

for all $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $x_0, \dots, x_n \in X$.

**(b)** Suppose that $h : X \to \mathbb{C}$ is positive definite. Then, writing $e$ for the identity of $X$,

(i) $|h(x)| \le h(e)$ for every $x \in X$;

(ii) $h(x^{-1}) = \overline{h(x)}$ for every $x \in X$.

**P** If $\zeta \in \mathbb{C}$ and $x \in X$, take $n = 1$, $x_0 = e$, $x_1 = x$, $\zeta_0 = 1$ and $\zeta_1 = \zeta$ in the definition in (a) above, and observe that

$$(1 + |\zeta|^2)h(e) + \zeta h(x) + \bar{\zeta}h(x^{-1}) = h(e^{-1}e) + \zeta h(e^{-1}x) + \bar{\zeta}h(x^{-1}e) + \zeta\bar{\zeta}h(x^{-1}x) \ge 0.$$

Taking $\zeta = 0$, $x = e$ we get $h(e) \ge 0$. Taking $\zeta = 1$ we see that $h(x) + h(x^{-1})$ is real, and taking $\zeta = i$, we see that $h(x) - h(x^{-1})$ is purely imaginary; that is, $h(x^{-1}) = \overline{h(x)}$, for any $x$. Taking $\zeta$ such that $|\zeta| = 1$, $\zeta h(x) = -|h(x)|$ we get $2h(e) - 2|h(x)| \ge 0$, that is, $|h(x)| \le h(e)$ for every $x \in X$. **Q**

**(c)** If $h : X \to \mathbb{C}$ is positive definite and $\chi : X \to S^1$ is a homomorphism, then $h \times \chi$ is positive definite. **P** If $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $x_0, \dots, x_n \in X$ then

$$\sum_{j,k=0}^n \zeta_j \bar{\zeta}_k (h \times \chi)(x_k^{-1}x_j) = \sum_{j,k=0}^n \zeta_j \chi(x_j)\overline{\zeta_k \chi(x_k)} h(x_k^{-1}x_j) \geq 0. \text{ **Q**}$$

**(d)** If $X$ is an abelian topological group and $\mu$ a Haar measure on $X$, then for any $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ the convolution $f * \tilde{f} : X \to \mathbb{C}$ is continuous and positive definite, where $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined. **P** As in 444Rc, $f * \tilde{f}$ is defined everywhere on $X$ and is continuous. (The definition of $\tilde{}$ has shifted since §444, but the argument there applies unchanged to the present situation.) Now, if $x_0, \dots, x_n \in X$ and $\zeta_0, \dots, \zeta_n \in \mathbb{C}$,

$$\begin{aligned}
\sum_{j,k=0}^n \zeta_j \bar{\zeta}_k (f * \tilde{f})(x_k^{-1}x_j) &= \sum_{j,k=0}^n \zeta_j \bar{\zeta}_k \int f(y)\tilde{f}(y^{-1}x_k^{-1}x_j)dy \\
&= \int \sum_{j,k=0}^n \zeta_j \bar{\zeta}_k f(y)\overline{f(x_j^{-1}x_k y)}dy \\
&= \int \sum_{j,k=0}^n \zeta_j \bar{\zeta}_k f(x_j y)\overline{f(x_j^{-1}x_k x_j y)}dy \\
&= \int \sum_{j,k=0}^n \zeta_j \bar{\zeta}_k f(x_j y)\overline{f(x_k y)}dy \\
&= \int |\sum_{j=0}^n \zeta_j f(x_j y)|^2 dy \geq 0.
\end{aligned}$$

So $f * \tilde{f}$ is positive definite. **Q**

**445M Proposition** Let $X$ be a topological group and $\nu$ a quasi-Radon measure on $X$. If $h : X \to \mathbb{C}$ is a continuous positive definite function, then $\iint h(y^{-1}x)f(x)\overline{f(y)}\nu(dx)\nu(dy) \geq 0$ for every $\nu$-integrable function $f$.

**proof (a)** Extend $f$, if necessary, to the whole of $X$; since the hypothesis implies that $\operatorname{dom} f$ is conegligible, this does not affect the integrals. Let $\lambda$ be the product quasi-Radon measure on $X \times X$; because $h$ is continuous (by hypothesis) and bounded (by 445L(b-i)), the function $(x, y) \mapsto h(y^{-1}x)f(x)\overline{f(y)}$ is $\lambda$-integrable, and (because $\{x : f(x) \neq 0\}$ can be covered by a sequence of sets of finite measure)

$$I = \iint h(y^{-1}x)f(x)\overline{f(y)}\nu(dx)\nu(dy) = \int h(y^{-1}x)f(x)\overline{f(y)}\lambda(d(x, y))$$

(417G).

**(b)** Let $\epsilon > 0$. Set $\gamma = \sup_{x \in X} |h(x)| = h(e)$ (445L(b-i)). Let $F \subseteq X$ be a non-empty measurable set of finite measure for $\nu$ such that $\gamma \int_{(X \times X) \setminus (F \times F)} |f(x)\overline{f(y)}|\lambda(d(x, y)) \leq \frac{1}{2}\epsilon$ and $f$ is bounded on $F$; say $|f(x)| \leq M$ for every $x \in F$. Let $\delta > 0$ be such that

$$\delta(M^2 + 2M\gamma)(\nu F)^2 + 2M^2\gamma\delta \leq \tfrac{1}{2}\epsilon.$$

Let $\mathcal{G}$ be the set

$$\{G \times H : G, H \subseteq X \text{ are open}, |h(y^{-1}x) - h(y_1^{-1}x_1)| \leq \delta \\ \text{whenever } x, x_1 \in G, y, y_1 \in H\}.$$

Because $h$ is continuous, $\mathcal{G}$ is a cover of $X \times X$. Because $\lambda$ is $\tau$-additive, there is a finite set $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\lambda((F \times F) \setminus \bigcup \mathcal{G}_0) \leq \delta$; we may suppose that $\mathcal{G}_0$ is non-empty. Set $W = (F \times F) \cap \bigcup \mathcal{G}_0$. Enumerate $\mathcal{G}_0$ as $\langle G_i \times H_i \rangle_{i \leq n}$.

Let $\mathcal{F}$ be a finite partition of $F$ into measurable sets such that $|f(x) - f(x')| \leq \delta$ whenever $x, x'$ belong to the same member of $\mathcal{F}$. Let $\mathcal{E}$ be the partition of $F$ generated by $\mathcal{F} \cup \{F \cap G_j : j \leq n\} \cup \{F \cap H_j : j \leq n\}$.

Enumerate $\mathcal{E}$ as $\langle E_k\rangle_{k\le m}$; for each $k\le m$ choose $x_k\in E_k$. Set $J=\{(j,k): j\le m,\ k\le m,\ E_j\times E_k\subseteq W\}$; then $W=\bigcup_{(j,k)\in J}E_j\times E_k$.

**(c)** If $(j,k)\in J$, $x\in E_j$ and $y\in E_k$ then

$$|h(y^{-1}x)f(x)\overline{f(y)}-h(x_k^{-1}x_j)f(x_j)\overline{f(x_k)}|\le\delta(M^2+2M\gamma),$$

because there must be some $r\le n$ such that $E_j\times E_k\subseteq G_r\times H_r$, so that $|h(y^{-1}x)-h(x_k^{-1}x_j)|\le\delta$, while there are members of $\mathcal{F}$ including $E_j$ and $E_k$, so that $|f(x)-f(x_j)|\le\delta$ and $|\overline{f(y)}-\overline{f(x_k)}|\le\delta$; at the same time,

$$|f(x)\overline{f(y)}|\le M^2,\quad |h(x_k^{-1}x_j)||\overline{f(y)}|\le M\gamma,\quad |h(x_k^{-1}x_j)||f(x_j)|\le M\gamma$$

because $x$, $y$, $x_j$ and $x_k$ all belong to $F$.

**(d)** Set $\zeta_j=f(x_j)\nu E_j$ for $j\le m$, so that $\bar\zeta_j=\overline{f(x_j)}\nu E_j$. Now consider

$$\begin{aligned}
&\Bigl|\int_W h(y^{-1}x)f(x)\overline{f(y)}\lambda(d(x,y))-\sum_{(j,k)\in J}\zeta_j\bar\zeta_k h(x_k^{-1}x_j)\Bigr|\\
&=\Bigl|\sum_{(j,k)\in J}\int_{E_j\times E_k}h(y^{-1}x)f(x)\overline{f(y)}-h(x_k^{-1}x_j)f(x_j)\overline{f(x_k)}\lambda(d(x,y))\Bigr|\\
&\le\sum_{(j,k)\in J}\int_{E_j\times E_k}|h(y^{-1}x)f(x)\overline{f(y)}-h(x_k^{-1}x_j)f(x_j)\overline{f(x_k)}|\lambda(d(x,y))\\
&\le\sum_{(j,k)\in J}\delta(M^2+2M\gamma)\nu E_j\nu E_k\\
&\le\delta(M^2+2M\gamma)\lambda W\le\delta(M^2+2M\gamma)(\nu F)^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Bigl|\int_{(X\times X)\setminus W}h(y^{-1}x)f(x)\overline{f(y)}\lambda(d(x,y))\Bigr|&\le\gamma\int_{(X\times X)\setminus(F\times F)}|f(x)\overline{f(y)}|\lambda(d(x,y))\\
&\quad+\gamma\int_{(F\times F)\setminus W}|f(x)\overline{f(y)}|\lambda(d(x,y))\\
&\le\frac{1}{2}\epsilon+\gamma\delta M^2,
\end{aligned}$$

and

$$\begin{aligned}
\Bigl|\sum_{j\le m,k\le m,(j,k)\notin J}h(x_k^{-1}x_j)f(x_j)\tilde f(x_k)\Bigr|&\le\gamma M^2\sum_{j\le m,k\le m,(j,k)\notin J}\nu E_j\nu E_k\\
&=\gamma M^2\lambda((F\times F)\setminus W)\le\gamma M^2\delta.
\end{aligned}$$

Putting these together,

$$|I-\sum_{j,k=0}^m\zeta_j\bar\zeta_k h(x_k^{-1}x_j)|\le\delta(M^2+2M\gamma)(\nu F)^2+\tfrac{1}{2}\epsilon+\gamma\delta M^2+\gamma M^2\delta\le\epsilon.$$

But $\sum_{j,k=0}^m\zeta_j\bar\zeta_k h(x_k^{-1}x_j)\ge0$, because $h$ is positive definite. As $\epsilon$ is arbitrary, $I\ge0$, as required.

**445N Bochner's theorem** (Herglotz 1911, Bochner 1933, Weil 1940) Let $X$ be an abelian topological group with a Haar measure $\mu$, and $\mathcal{X}$ its dual group. Then for any continuous positive definite function $h:X\to\mathbb{C}$ there is a unique totally finite Radon measure $\nu$ on $\mathcal{X}$ such that

$$\int h\times f\,d\mu=\int\hat f\,d\nu\text{ for every }f\in\mathcal{L}^1_{\mathbb{C}}=\mathcal{L}^1_{\mathbb{C}}(\mu),$$

$$h(x)=\int\chi(x)\nu(d\chi)\text{ for every }x\in X.$$

**proof (a)** If $h(e)=0$, where $e$ is the identity in $X$, then $h=0$, by 445L(b-i), and we can take $\nu$ to be the zero measure. Otherwise, since multiplying $h$ by a positive scalar leaves $h$ positive definite and does not affect the result, we may suppose that $h(e)=1$. For $f$, $g\in\mathcal{L}^1_{\mathbb{C}}=\mathcal{L}^1_{\mathbb{C}}(\mu)$ set

$$(f|g)=\iint f(x)\overline{g(y)}h(y^{-1}x)dxdy=\iint f(x)\overline{g(y^{-1})}h(yx)dxdy$$

(by 442Kb, since $X$ is unimodular). Then, by 445M, $(f|f)\ge 0$ for every $f\in\mathcal{L}^1_{\mathbb{C}}$. Also $(f_1+f_2|g)=(f_1|g)+(f_2|g)$, $(\zeta f|g)=\zeta(f|g)$ and $(g|f)=\overline{(f|g)}$ for all $f$, $g$, $f_1$, $f_2\in\mathcal{L}^1_{\mathbb{C}}$ and $\zeta\in\mathbb{C}$. **P** Only the last is anything but trivial, and for this we have

$$\begin{aligned}(g|f)&=\iint g(x)\overline{f(y)}h(y^{-1}x)dxdy\\&=\iint g(x)\overline{f(y)}h(y^{-1}x)dydx\end{aligned}$$

(by 417Ga, because $(x,y)\mapsto g(x)\overline{f(y)}h(y^{-1}x)$ is integrable for the product measure and zero off the square of a countable union of sets of finite measure)

$$=\iint g(x)\overline{f(y)h(x^{-1}y)}dydx$$

(using 445L(b-ii))

$$\begin{aligned}&=\overline{\iint f(y)\overline{g(x)}h(x^{-1}y)dydx}\\&=\overline{(f|g)}.\ \mathbf{Q}\end{aligned}$$

**(b)** If $f$, $g\in\mathcal{L}^1_{\mathbb{C}}$, $|(f|g)|^2\le(f|f)(g|g)$. **P** (Really this is just Cauchy's inequality.) For any $\alpha$, $\beta\in\mathbb{C}$,

$$|\alpha|^2(f|f)+2\,\mathcal{R}\mathrm{e}(\alpha\bar\beta(f|g))+|\beta|^2(g|g)=(\alpha f+\beta g|\alpha f+\beta g)\ge 0.$$

If $(f|f)=0$ we have $2\,\mathcal{R}\mathrm{e}(\alpha(f|g))+(g|g)\ge 0$ for every $\alpha\in\mathbb{C}$ so in this case $(f|g)=0$; similarly $(f|g)=0$ if $(g|g)=0$; otherwise we can find non-zero $\alpha$, $\beta$ such that $|\alpha|^2=(g|g)$, $|\beta|^2=(f|f)$ and $\alpha\bar\beta(f|g)=-|\alpha\beta(f|g)|$, in which case the inequality simplifies to $|(f|g)|\le|\alpha\beta|$ and $|(f|g)|^2\le(f|f)(g|g)$, as required. **Q**

**(c)** Now consider the functional $\psi\in(L^1_{\mathbb{C}})^*=(L^1_{\mathbb{C}}(\mu))^*$ corresponding to $h$, so that $\psi(f^\bullet)=\int h\times f\,d\mu$ for every $f\in\mathcal{L}^1_{\mathbb{C}}$. Then $|\psi(f^\bullet)|^2\le(f|f)$ for every $f\in\mathcal{L}^1_{\mathbb{C}}$. **P** Let $\epsilon>0$. Then there is an open neighbourhood $U$ of $e$ such that $U=U^{-1}$ and

$$|h(y^{-1}x)-h(e)|\le\epsilon\text{ whenever }x,\ y\in U,$$

$$\|a\bullet_l f-f\|_1\le\epsilon\text{ for every }a\in U$$

where $(a\bullet_l f)(x)=f(a^{-1}x)$ whenever this is defined, as usual (443Gf). Shrinking $U$ if need be, we may suppose that $\mu U<\infty$, and of course $\mu U>0$. Set $g=\frac{1}{\mu U}\chi U\in\mathcal{L}^1_{\mathbb{C}}$. Then

$$\begin{aligned}|(f|g)-\psi(f^\bullet)|&=\frac{1}{\mu U}\Bigl|\int_U\int_X f(x)h(y^{-1}x)dxdy-\int_U\int_X f(x)h(x)dxdy\Bigr|\\&=\frac{1}{\mu U}\Bigl|\int_U\int_X f(yx)h(x)dxdy-\int_U\int_X f(x)h(x)dxdy\Bigr|\\&=\frac{1}{\mu U}\Bigl|\int_U\int_X((y^{-1}\bullet_l f)(x)-f(x))h(x)dxdy\Bigr|\\&\le\frac{1}{\mu U}\int_U\int_X|(y^{-1}\bullet_l f)(x)-f(x)||h(x)|dxdy\\&\le\frac{1}{\mu U}\int_U\|y^{-1}\bullet_l f-f\|_1\mu(dy)\le\epsilon.\end{aligned}$$

Also

$$\begin{aligned}|(g|g)-1| &= \Bigl|\frac{1}{\mu U^2}\int_U\int_U(h(y^{-1}x)-1)dxdy\Bigr| \\ &\le \frac{1}{\mu U^2}\int_U\int_U|h(y^{-1}x)-1|dxdy \le \epsilon.\end{aligned}$$

So

$$\max(0,|\psi(f^\bullet)|-\epsilon)^2 \le |(f|g)|^2 \le (f|f)(g|g) \le (1+\epsilon)(f|f).$$

Letting $\epsilon\downarrow 0$ we have the result. **Q**

**(d)** If we look at $(f|f)$, however, and apply 444Od, we see that

$$\begin{aligned}(f|f) &= \iint f(x)\overline{f(y)}h(y^{-1}x)dxdy \\ &= \iint f(x)\overline{f(y^{-1})}h(yx)dxdy = \int h(x)(f*\tilde f)(x)dx,\end{aligned}$$

where $\tilde f(x)=\overline{f(x^{-1})}$ whenever this is defined; that is,

$$(f|f)=\psi(f*\tilde f)^\bullet.$$

(Note that $\tilde f\in\mathcal{L}^1_{\mathbb{C}}$ because $X$ is unimodular, as in part (c) of the proof of 445K.) So (c) tells us that

$$|\psi(f^\bullet)|^2 \le \psi(f*\tilde f)^\bullet$$

for every $f\in\mathcal{L}^1_{\mathbb{C}}$, that is, $|\psi(u)|^2\le\psi(u*\tilde u)$ for every $u\in L^1_{\mathbb{C}}$, defining $\tilde u$ as in 445Kc.

**(e)** In fact

$$|\psi(u)|\le\|\hat u\|_\infty$$

for every $u\in L^1_{\mathbb{C}}$. **P** Set $u_0=u$ and $u_{k+1}=u_k*\tilde u_k$ for every $k\in\mathbb{N}$. We need to know that $u_k=\tilde u_k$ for $k\ge 1$. To see this, represent $u_{k-1}$ as $f^\bullet$ where $f\in\mathcal{L}^1_{\mathbb{C}}$, so that $u_k=(f*\tilde f)^\bullet$. Now

$$\begin{aligned}(f*\tilde f)^\sim(x) &= \overline{(f*\tilde f)(x^{-1})} = \overline{\int f(x^{-1}y)\tilde f(y^{-1})dy} \\ &= \int\overline{f(x^{-1}y)}f(y)dy = \int\tilde f(y^{-1}x)f(y)dy = (f*\tilde f)(x)\end{aligned}$$

for every $x$, so $(f*\tilde f)^\sim=f*\tilde f$ and $\tilde u_k=u_k$. Accordingly $u_{k+1}=u_k*u_k$ for $k\ge1$ and we have $u_k=(u_1)^{2^{k-1}}$ for every $k\ge1$.

At the same time, we have $|\psi(u_k)|^2\le\psi(u_{k+1})$ for every $k$, by (d), so that, for $k\ge1$,

$$|\psi(u)|^{2^k}\le\psi(u_k)\le\|u_k\|_1=\|u_1^{2^{k-1}}\|_1,$$

$$|\psi(u)|\le\|u_1^{2^{k-1}}\|_1^{1/2^k}.$$

Letting $k\to\infty$, $|\psi(u)|\le\sqrt{r(u_1)}$, where $r(u_1)$ is the spectral radius of $u_1$.

At this point, recall that $r(u_1)=\|\hat u_1\|_\infty$ (445Kd), while $|\hat u|^2=\hat u_1$ (445Kc), so $r(u_1)=\|\hat u\|_\infty^2$ and $|\psi(u)|\le\|\hat u\|_\infty$. **Q**

**(f)** Now consider $\hat{}$ as a linear operator from $L^1_{\mathbb{C}}$ to $C_0=C_0(\mathcal{X};\mathbb{C})$, as in 445K. If $\hat u=0$ then $\psi(u)=0$, by (e), so setting $A=\{\hat u:u\in L^1_{\mathbb{C}}\}$ we have a linear functional $\psi_0:A\to\mathbb{C}$ defined by saying that $\psi_0(\hat u)=\psi(u)$ for every $u\in L^1_{\mathbb{C}}$. By (e), $\|\psi_0\|\le1$. Now $\psi_0$ has an extension to a bounded linear operator $\psi_1$, still of norm at most 1, from $C_0$ to $\mathbb{C}$ (3A5Ab).

**(g)** Suppose that $q\in C_0$ and $0\le q\le\mathbf{1}$, writing $\mathbf{1}$ for the constant function with value 1; set $\alpha=\psi_1(q)$. Then for any $\zeta\in\mathbb{C}$ and $\gamma\ge0$ we have $|\zeta-\gamma\alpha|\le\max(|\zeta|,|\gamma|,|\zeta-\gamma|)$. **P** Let $\epsilon>0$. Set $V=\{x:|1-h(x)|<\epsilon\}$; then $V$ is an open neighbourhood of $e$; set $f=\frac{1}{\mu V}\chi V$ and $u=f^\bullet$, so that

$$\|\hat u\|_\infty=r(u)\le\|u\|_1=1,$$

$$|1-\psi(u)| = |\frac{1}{\mu V}\int_V (1-h(x))dx| \le \epsilon.$$

Set $v = u * \tilde{u}$; then

$$\psi_1(\hat{v}) = \psi(v) \ge |\psi(u)|^2 \ge (1-\epsilon)^2,$$

using part (d) for the central inequality. But $\hat{v} = |\hat{u}|^2$, so that $0 \le \hat{v} \le \mathbf{1}$ and $\psi_1(\hat{v}) \le 1$.

Now consider $\|\zeta\hat{v} - \gamma q\|_\infty$. If $\chi \in \mathcal{X}$, then $\zeta\hat{v}(\chi)$ and $\gamma q(\chi)$ both lie in the triangle with vertices 0, $\zeta$ and $\gamma$, because $0 \le \hat{v} \le \mathbf{1}$ and $0 \le q \le \mathbf{1}$. So

$$|\zeta\hat{v}(\chi) - \gamma q(\chi)| \le \max(|\gamma|, |\zeta|, |\gamma - \zeta|).$$

As $\chi$ is arbitrary,

$$\|\zeta\hat{v} - \gamma q\|_\infty \le \max(|\gamma|, |\zeta|, |\gamma - \zeta|).$$

Accordingly

$$\begin{aligned}|\zeta - \gamma\alpha| &\le |\zeta - \zeta\psi_1(\hat{v})| + |\psi_1(\zeta\hat{v} - \gamma q)| \\ &\le |\zeta|(1-(1-\epsilon)^2) + \|\zeta\hat{v} - \gamma q\|_\infty \le 2\epsilon|\zeta| + \max(|\zeta|, |\gamma|, |\zeta - \gamma|).\end{aligned}$$

As $\epsilon$ is arbitrary, we have the result. **Q**

Taking $\zeta = \gamma = 1$ we see that $|1-\alpha| \le 1$, so that $\mathcal{R}\mathrm{e}\,\alpha \ge 0$. Taking $\zeta = \pm i$, we see that $|i \pm \gamma\alpha| \le \sqrt{1+\gamma^2}$ for every $\gamma \ge 0$, so that $\mathcal{I}\mathrm{m}\,\alpha = 0$. Thus $\psi_1(q) \ge 0$; and this is true whenever $0 \le q \le \mathbf{1}$ in $C_0$.

**(h)** It follows at once that $\psi_1(q) \ge 0$ whenever $q \ge 0$ in $C_0$. Applying the Riesz Representation Theorem, in the form 436K, to the restriction of $\psi_1$ to $C_0(\mathcal{X};\mathbb{R})$, we see that there is a totally finite Radon measure $\nu$ on $\mathcal{X}$ such that $\psi_1(q) = \int q\,d\nu$ for every real-valued $q \in C_0$; of course it follows that $\psi_1(q) = \int q\,d\nu$ for every $q \in C_0$. Unwrapping the definition of $\psi_1$, we see that

$$\int h(x)f(x)dx = \psi(f^\bullet) = \psi_1(\hat{f}) = \int \hat{f}\,d\nu$$

for every $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$.

**(i)** For the second formula, argue as follows. Given $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, consider the function $(x,\chi) \mapsto f(x)\chi(x) : X \times \mathcal{X} \to \mathbb{C}$. Because $(\chi, x) \mapsto \chi(x)$ is continuous (445Ea), this is $\Lambda$-measurable, where $\Lambda$ is the domain of the product quasi-Radon measure $\mu \times \nu$ on $X \times \mathcal{X}$. It is integrable because $\nu\mathcal{X} < \infty$ and $|\chi(x)| = 1$ for every $\chi$, $x$; moreover, it is zero off the set $\{x : f(x) \ne 0\} \times \mathcal{X}$, which is a countable union of products of sets of finite measure. Note also that because $\chi \mapsto \chi(x)$ is continuous and bounded for every $x \in X$, $h_1(x) = \int \chi(x)\nu(d\chi)$ is defined, and $|h_1(x)| \le \nu X$, for every $x \in X$. What is more, $h_1$ is continuous. **P** Let $\mathfrak{X}$ be the dual group of $\mathcal{X}$, and for $x \in X$ let $\hat{x}$ be the corresponding member of $\mathfrak{X}$. Then, in the language of 445C, applied to the topological group $\mathcal{X}$,

$$h_1(x) = \int \hat{x}\,d\nu = \hat{\nu}(\hat{x})$$

for every $x \in X$. But $\hat{\nu} : \mathfrak{X} \to \mathbb{C}$ is continuous, by 445Ec, and $x \mapsto \hat{x} : X \to \mathfrak{X}$ is continuous, by 445Eb; so $h_1$ also is continuous. **Q**

We may therefore apply Fubini's theorem (417G) to see that

$$\begin{aligned}\int f(x)h_1(x)\mu(dx) &= \iint f(x)\chi(x)\nu(d\chi)\mu(dx) = \iint f(x)\chi(x)\mu(dx)\nu(d\chi) \\ &= \int \hat{f}(\chi)\nu(d\chi) = \int f(x)h(x)\mu(dx).\end{aligned}$$

Since this is true for every $f \in \mathcal{L}^1_{\mathbb{C}}$, $h_1 =_{\text{a.e.}} h$; since both are continuous, $h_1 = h$, as required.

**(j)** Finally, to see that $\nu$ is uniquely defined, note that $\{\hat{f} : f \in \mathcal{L}^1_{\mathbb{C}}\}$ is $\|\,\|_\infty$-dense in $C_0$ (445Kd), so 436K tells us that there can be at most one totally finite Radon measure $\nu$ on $\mathcal{X}$ such that $\int h \times f\,d\mu = \int \hat{f}\,d\nu$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$.

**445O Proposition** Let $X$ be a Hausdorff abelian topological group carrying Haar measures. Then the map $x \mapsto \hat{x}$ from $X$ to its bidual group $\mathfrak{X}$ is a homeomorphism between $X$ and its image in $\mathfrak{X}$. In particular, the dual group $\mathcal{X}$ of $X$ separates the points of $X$.

**proof** We already know that $\widehat{\ }$ is continuous (445Eb) and that $\mathcal{X}$ is locally compact and Hausdorff (445J). Now let $U$ be any neighbourhood of the identity $e$ of $X$. Let $V \subseteq U$ be an open neighbourhood of $e$ such that $VV^{-1} \subseteq U$ and $\mu V < \infty$. Then $f = \chi V \in \mathcal{L}^2_{\mathbb{C}}(\mu)$, so $f * \tilde{f}$ is positive definite and continuous (445Ld) and there is a totally finite Radon measure $\nu$ on $\mathcal{X}$ such that $(f * \tilde{f})(x) = \int \chi(x)\nu(d\chi)$ for every $x \in X$ (445N). Note that, writing $\mathfrak{e}$ for the identity of $\mathfrak{X}$ and $\hat{\nu} : \mathfrak{X} \to \mathbb{C}$ for the Fourier-Stieltjes transform of $\nu$,

$$\hat{\nu}(\mathfrak{e}) = \int \mathfrak{e}(\chi)\nu(d\chi) = \int \chi(e)\nu(d\chi) = (f * \tilde{f})(e)$$
$$= \int f(y)\tilde{f}(y^{-1})\mu(dy) = \int |f(y)|^2\mu(dy) \neq 0.$$

Now $\hat{\nu}$ is continuous (445Ec), so $W = \{\mathfrak{x} : \hat{\nu}(\mathfrak{x}) \neq 0\}$ is a neighbourhood of $\mathfrak{e}$. If $x \in X$ and $\hat{x} \in W$, then

$$(f * \tilde{f})(x) = \int \chi(x)\nu(d\chi) = \int \hat{x}(\chi)\nu(d\chi) = \hat{\nu}(\hat{x}) \neq 0,$$

so there is some $y \in X$ such that $f(y)\tilde{f}(y^{-1}x) \neq 0$, that is, $f(y) \neq 0$ and $f(x^{-1}y) \neq 0$, that is, $y$ and $x^{-1}y$ both belong to $V$; in which case $x \in VV^{-1} \subseteq U$.

Thus $U \supseteq \{x : \hat{x} \in W\}$. This means that, writing $\mathfrak{S}$ for $\{\{x : \hat{x} \in H\} : H \subseteq \mathfrak{X}$ is open$\}$, every neighbourhood of $e$ for the original topology $\mathfrak{T}$ of $X$ is a neighbourhood of $e$ for $\mathfrak{S}$. But (it is easy to check) $(X, \mathfrak{S})$ is a topological group because $\mathfrak{X}$ is a topological group and $\widehat{\ }$ is a homomorphism. So $\mathfrak{T} \subseteq \mathfrak{S}$ (4A5Fb). As we know already that $\mathfrak{S} \subseteq \mathfrak{T}$, the two topologies are equal.

It follows at once that if $\mathfrak{T}$ is Hausdorff, then (because $\mathfrak{S}$ is Hausdorff) the map $\widehat{\ }$ is an injection and is a homeomorphism between $X$ and its image in $\mathfrak{X}$.

**445P The Inversion Theorem** Let $X$ be an abelian topological group and $\mu$ a Haar measure on $X$. Then there is a unique Haar measure $\lambda$ on the dual group $\mathcal{X}$ of $X$ such that whenever $f : X \to \mathbb{C}$ is continuous, $\mu$-integrable and positive definite, then $\hat{f} : \mathcal{X} \to \mathbb{C}$ is $\lambda$-integrable and

$$f(x) = \int \hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi)$$

for every $x \in X$.

**proof (a)** Write $P$ for the set of $\mu$-integrable positive definite continuous functions $h : X \to \mathbb{C}$. For $h \in P$, let $\nu_h$ be the corresponding totally finite Radon measure on $\mathcal{X}$ defined in 445N, so that

$$\int f \times h \, d\mu = \int \hat{f} \, d\nu_h$$

for every $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$.

**(b)** The basis of the argument is the following fact. If $f \in \mathcal{L}^1_{\mathbb{C}}$ and $h_1$, $h_2 \in P$, then

$$\int \hat{f} \times \hat{\bar{h}}_2 d\nu_{h_1} = \int \hat{f} \times \hat{\bar{h}}_1 d\nu_{h_2}.$$

**P** $$\int \hat{f} \times \hat{\bar{h}}_1 d\nu_{h_2} = \int (f * \bar{h}_1)^\wedge d\nu_{h_2}$$

(445G)

$$= \int h_2(x)(f * \bar{h}_1)(x)\mu(dx) = \int \bar{h}_2(x^{-1})(f * \bar{h}_1)(x)\mu(dx)$$

(by 445Lb)

$$= ((f * \bar{h}_1) * \bar{h}_2)(e) = ((f * \bar{h}_2) * \bar{h}_1)(e)$$

(because $*$ is associative and commutative, by 444Oe and 444Og)

$$= \int \hat{f} \times \hat{\hat{h}}_2 d\nu_{h_1}. \mathbf{Q}$$

Now because $\hat{\hat{h}}_1$ and $\hat{\hat{h}}_2$ are both bounded (by $\int |h_1| d\mu$ and $\int |h_2| d\mu$ respectively), and $\nu_{h_1}$ and $\nu_{h_2}$ are both totally finite measures, and $\{\hat{f} : f \in \mathcal{L}^1_{\mathbb{C}}(\mu)\}$ is $\| \|_\infty$-dense in $C_0 = C_0(\mathcal{X};\mathbb{C})$ (445Kd), we must have

$$\int p \times \hat{\hat{h}}_2 d\nu_{h_1} = \int p \times \hat{\hat{h}}_1 d\nu_{h_2}$$

for every $p \in C_0$.

**(c)** Let $\mathcal{K}$ be the family of compact subsets of $\mathcal{X}$. For $K \in \mathcal{K}$ set

$$P_K = \{h : h \in P, \hat{\hat{h}}(\chi) > 0 \text{ for every } \chi \in K\}.$$

Then $P_K$ is non-empty. **P** Set

$$U = \{x : x \in X, |1 - \chi(x)| \le \tfrac{1}{2} \text{ for every } \chi \in K\}.$$

Then $U$ is a neighbourhood of the identity $e$ of $X$, by 445Eb. Let $V$ be an open neighbourhood of $e$, of finite measure, such that $VV^{-1} \subseteq U$, set $g = \frac{1}{\mu V}\chi V$, and try $h = g * \tilde{g}$. Then $h$ is continuous and positive definite (445Ld), real-valued and non-negative (because $g$ and $\tilde{g}$ are), zero outside $U$ (because $VV^{-1} \subseteq U$, as in the proof of 445O), and

$$\int h \, d\mu = \int g \, d\mu \cdot \int \tilde{g} \, d\mu = 1$$

(444Qb). Next,

$$\hat{\hat{h}} = \hat{h} = |\hat{g}|^2$$

(445Kc) is non-negative, and if $\chi \in K$ then

$$|1 - \hat{h}(\chi)| = |\int h(x) - h(x)\chi(x)\mu(dx)| \le \int h(x)|1 - \chi(x)|\mu(dx) \le \tfrac{1}{2}$$

because $|1 - \chi(x)| \le \frac{1}{2}$ if $x \in U$ and $h(x) = 0$ if $x \in X \setminus U$. So

$$\hat{\hat{h}}(\chi) = \hat{h}(\chi) \ge \tfrac{1}{2}$$

for every $\chi \in K$, and $h \in P_K$. **Q**

**(d)** Because $\mathcal{K}$ is upwards-directed, $\{P_K : K \in \mathcal{K}\}$ is downwards-directed and generates a filter $\mathcal{F}$ on $P$. Let $C_k = C_k(\mathcal{X};\mathbb{C})$ be the space of continuous complex-valued functions on $\mathcal{X}$ with compact support. If $q \in C_k$, then

$$\phi(q) = \lim_{h \to \mathcal{F}} \int \frac{q}{\hat{\hat{h}}} d\nu_h$$

is defined in $\mathbb{C}$, where in the division $q/\hat{\hat{h}}$ we interpret $0/0$ as $0$ if necessary. **P** Setting $K = \overline{\{\chi : q(\chi) \ne 0\}}$, we see in fact that for any $h_1$, $h_2 \in P_K$ we may define a function $p \in C_k$ by setting

$$\begin{aligned} p(\chi) &= \frac{q(\chi)}{\hat{\hat{h}}_1(\chi)\hat{\hat{h}}_2(\chi)} \text{ if } \chi \in K, \\ &= 0 \text{ if } q(\chi) = 0, \end{aligned}$$

so that

$$\int \frac{q}{\hat{\hat{h}}_1} d\nu_{h_1} = \int p \times \hat{\hat{h}}_2 \, d\nu_{h_1} = \int p \times \hat{\hat{h}}_1 \, d\nu_{h_2}$$

(by (b) above)

$$= \int \frac{q}{\hat{\hat{h}}_2} d\nu_{h_2}.$$

So this common value must be $\phi(q)$. **Q**

If $q$, $q'\in C_k$ and $\alpha\in\mathbb{C}$, then

$$\int\frac{q+q'}{\hat{\hat h}}d\nu_h=\int\frac{q}{\hat{\hat h}}d\nu_h+\int\frac{q'}{\hat{\hat h}}d\nu_h,$$

$$\int\frac{\alpha q}{\hat{\hat h}}d\nu_h=\alpha\int\frac{q}{\hat{\hat h}}d\nu_h$$

whenever $h\in P_K$, where $K=\overline{\{\chi:|q(\chi)|+|q'(\chi)|>0\}}$; so $\phi(q+q')=\phi(q)+\phi(q')$ and $\phi(\alpha q)=\alpha\phi(q)$. Moreover, if $q\ge 0$, then $q/\hat{\hat h}\ge 0$ for every $h\in P_K$, so $\phi(q)\ge 0$.

**(e)** By the Riesz Representation Theorem (in the form 436J) there is a Radon measure $\lambda$ on $\mathcal{X}$ such that $\int q\,d\lambda=\phi(q)$ for any continuous function $q$ of compact support. (As in part (h) of the proof of 445N, the shift from real-valued $q$ to complex-valued $q$ is elementary.)

**(f)** Now $\lambda$ is translation-invariant. **P** Take $\theta\in\mathcal{X}$ and $q\in C_k$. Set $K=\overline{\{\chi:q(\chi)\ne 0\}}$ and $L=\theta^{-1}K$, and take any $h\in P_K$. Set $h_1(x)=h\times\theta^{-1}$. Then $h_1$ is positive definite (445Lc); of course it is continuous and $\mu$-integrable; and for any $\chi\in L$,

$$\hat{\hat h}_1(\chi)=\int\overline{h(x)}\theta(x)\chi(x)\mu(dx)=\hat{\hat h}(\theta\chi)>0.$$

So $h_1\in P_L$.

To relate $\nu_{h_1}$ to $\nu_h$, observe that if $f\in\mathcal{L}^1_{\mathbb{C}}$ then

$$\hat{f}(\theta\chi)=\int f(x)\theta(x)\chi(x)\mu(dx)=(f\times\theta)^\wedge(\chi),$$

so

$$\int\hat{f}(\theta\chi)\nu_{h_1}(d\chi)=\int(f\times\theta)(x)h_1(x)\mu(dx)=\int f(x)h(x)\mu(dx)=\int\hat{f}(\chi)\nu_h(d\chi).$$

So we see that the equation

$$\int p(\theta\chi)\nu_{h_1}(d\chi)=\int p(\chi)\nu_h(d\chi)$$

is valid whenever $p$ is of the form $\hat f$, for some $f\in\mathcal{L}^1_{\mathbb{C}}$, and therefore for every $p\in C_0$.

Set $q_1(\chi)=q(\theta\chi)$ for every $\chi\in\mathcal{X}$, so that $q_1\in C_k$ and $L=\overline{\{\chi:q_1(\chi)\ne 0\}}$. Accordingly

$$\begin{aligned}\phi(q_1)&=\int\frac{q_1(\chi)}{\hat{\hat h}_1(\chi)}\nu_{h_1}(d\chi)=\int\frac{q(\theta\chi)}{\hat{\hat h}(\theta\chi)}\nu_{h_1}(d\chi)\\&=\int\frac{q(\chi)}{\hat{\hat h}(\chi)}\nu_h(d\chi)=\phi(q).\end{aligned}$$

So

$$\int q(\theta\chi)\lambda(d\chi)=\int q(\chi)\lambda(d\chi).$$

As $q$ and $\theta$ are arbitrary, $\lambda$ is translation-invariant (441L). **Q**

**(g)** Thus $\lambda$ is either zero or a Haar measure on $\mathcal{X}$. I have still to confirm that

$$f(x)=\int\hat f(\chi)\overline{\chi(x)}\lambda(d\chi)$$

whenever $f$ is continuous, positive definite and $\mu$-integrable, and $x\in X$. But recall the formula from (b) above. If $q\in C_k$, $K=\overline{\{\chi:q(\chi)\ne 0\}}$ and $h\in P_K$, then we must have

$$\int q\times\hat{\hat f}d\lambda=\int\frac{q\times\hat{\hat f}}{\hat{\hat h}}d\nu_h=\int\frac{q\times\hat{\hat h}}{\hat{\hat h}}d\nu_f=\int q\,d\nu_f.$$

In particular, $\int q\times\hat{\hat{f}}d\lambda\ge 0$ whenever $q\ge 0$; since $\hat{\hat{f}}$ is continuous (445Ka), and $\lambda$, being a Radon measure, is strictly positive, this shows that $\hat{\hat{f}}\ge 0$. Also

$$\int\hat{\hat{f}}d\lambda=\sup\{\int q\times\hat{\hat{f}}d\lambda:q\in C_k,\,0\le q\le\mathbf{1}\}=\nu_f(\mathcal{X})<\infty,$$

so we have an indefinite-integral measure $\hat{\hat{f}}\lambda$; since this is a Radon measure (416Sa), and acts on $C_k$ in the same way as $\nu_f$, it is actually equal to $\nu_f$ (by the uniqueness guaranteed in 436J). In particular, for any $x\in X$,

$$f(x)=\int\chi(x)\nu_f(d\chi)=\int\chi(x)\hat{\hat{f}}(\chi)\lambda(d\chi)$$

by the second formula in 445N, and 235K. But

$$\hat{\hat{f}}(\chi)=\int\overline{f(x)}\chi(x)\mu(dx)=\int f(x^{-1})\chi(x)\mu(dx)$$

(by 445Lb)

$$=\int f(x)\chi(x^{-1})\mu(dx)$$

(because $X$ is abelian, therefore unimodular)

$$=\hat{f}(\chi^{-1})$$

for every $\chi\in\mathcal{X}$. So

$$f(x)=\int\chi(x)\hat{f}(\chi^{-1})\lambda(d\chi)=\int\chi^{-1}(x)\hat{f}(\chi)\lambda(d\chi)$$

(because $\mathcal{X}$ is abelian)

$$=\int\hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi).$$

**(h)** We should check that $\lambda$ is non-zero and unique. But the construction in part (c) of the proof shows that there are many $f\in P$ such that $f(e)\ne 0$, and for any such $f$ we have $f(e)=\int\hat{\hat{f}}\,d\lambda$. This shows simultaneously that $\lambda$ is non-zero, therefore a Haar measure; and as all the Haar measures on $\mathcal{X}$ are scalar multiples of each other, there is at most one suitable $\lambda$.

**445Q Remark** We can extract the following useful fact from part (g) of the proof above. If $h:X\to\mathbb{C}$ is $\mu$-integrable, continuous and positive definite, then $\hat{\hat{h}}$ is non-negative and $\lambda$-integrable, and the Radon measure $\nu_h$ of 445N is just the indefinite-integral measure $\hat{\hat{h}}\lambda$.

Note also that $\lambda$ is actually a Radon measure; of course it has to be, because $\mathcal{X}$ is locally compact and Hausdorff (445J).

**445R The Plancherel Theorem** Let $X$ be an abelian topological group with a Haar measure $\mu$, and $\mathcal{X}$ its dual group. Let $\lambda$ be the Haar measure on $\mathcal{X}$ corresponding to $\mu$ (445P). Then there is a normed space isomorphism $T:L^2_{\mathbb{C}}(\mu)\to L^2_{\mathbb{C}}(\lambda)$ defined by setting $T(f^\bullet)=\hat{f}^\bullet$ whenever $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)\cap\mathcal{L}^2_{\mathbb{C}}(\mu)$.

**proof (a)** Since $\hat{f}=\hat{g}$ whenever $f=_{\text{a.e.}}g$, the formula certainly defines an operator $T$ from $L^1_{\mathbb{C}}(\mu)\cap L^2_{\mathbb{C}}(\mu)$ to $L^0_{\mathbb{C}}(\lambda)$, and of course it is linear.

If $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)\cap\mathcal{L}^2_{\mathbb{C}}(\mu)$, $h=f*\tilde{f}$ is continuous and positive definite (445Ld) and integrable, and $\hat{h}=|\hat{f}|^2$ (445Kc). Now

$$\int |\hat{f}|^2 \, d\lambda = \int \hat{h} \, d\lambda = h(e)$$

(445P)

$$= \int f(x)\tilde{f}(x^{-1})\mu(dx) = \int |f|^2 d\mu.$$

Thus $\|Tu\|_2 = \|u\|_2$ whenever $u \in L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$; since $L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$ is $\| \;\|_2$-dense in $L^2_{\mathbb{C}}(\mu)$ (244Ha/244Pb, or otherwise), we have a unique isometry $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ agreeing with the given formula on $L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$ (2A4I).

**(b)** The meat of the theorem is of course the proof that $T$ is surjective. **P?** Suppose, if possible, that $W = T[L^2_{\mathbb{C}}(\mu)]$ is not equal to $L^2_{\mathbb{C}}(\lambda)$. Because $T$ is linear, $W$ is a linear subspace of $L^2_{\mathbb{C}}(\lambda)$; because $T$ is an isometry, $W$ is isometric to $L^2_{\mathbb{C}}(\mu)$, and in particular is complete, therefore closed in $L^2_{\mathbb{C}}(\lambda)$ (3A4Fd). There is therefore a non-zero continuous linear functional on $L^2_{\mathbb{C}}(\lambda)$ which is zero on $W$ (3A5Ad), and this is of the form $u \mapsto \int u \times v$ for some $v \in L^2_{\mathbb{C}}(\lambda)$ (244J/244Pc). What this means is that there is a $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$ such that $g^\bullet \neq 0$ in $L^2_{\mathbb{C}}(\lambda)$ but $\int \hat{f} \times g \, d\lambda = 0$ for every $f \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$.

Suppose that $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$ and that $h$ is any $\mu$-integrable continuous positive definite function on $X$. Then $h$ is bounded (445Lb), so $|h|^2$ also is $\mu$-integrable, and $f * \bar{h} \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$ (444Ra). Accordingly

$$\int g \times \hat{f} \times \hat{\bar{h}} \, d\lambda = \int g \times (f * \bar{h})^\wedge d\lambda = 0.$$

Thus $\int g \times \hat{f} \, d\nu_h = 0$, where $\nu_h = \hat{\bar{h}}\lambda$ is the Radon measure on $\mathcal{X}$ corresponding to $h$ constructed in 445N (see 445Q). And this is true for every $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$. But as $\{\hat{f} : f \in \mathcal{L}^1_{\mathbb{C}}(\mu)\}$ is $\| \;\|_\infty$-dense in $C_0(\mathcal{X};\mathbb{C})$ (445Kd), and $g$ is $\nu_h$-integrable (because $\hat{\bar{h}} \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, by (a), so $\int |g \times \hat{\bar{h}}| d\lambda < \infty$), $g$ must be zero $\nu_h$-a.e., that is, $g \times \hat{\bar{h}} = 0$ $\lambda$-a.e. Now recall (from part (c) of the proof of 445P, for instance) that for every compact set $K \subseteq \mathcal{X}$ there is a $\mu$-integrable continuous positive definite $h$ such that $\hat{\bar{h}}(\chi) \neq 0$ for every $\chi \in K$, so $g = 0$ a.e. on $K$. Since $\lambda$ is tight, $g = 0$ a.e. (412Jc), which is impossible. **XQ**

Thus $T$ is surjective and we have the result.

**445S** While we do not have a direct definition of $\hat{f}$ for $f \in \mathcal{L}^2_{\mathbb{C}} \setminus \mathcal{L}^1_{\mathbb{C}}$, the map $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ does correspond to the map $f \mapsto \hat{f}$ in many ways. In particular, we have the following useful properties.

**Proposition** Let $X$ be an abelian topological group with a Haar measure $\mu$, $\mathcal{X}$ its dual group, $\lambda$ the associated Haar measure on $\mathcal{X}$ and $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ the standard isometry described in 445R. Suppose that $f_0$, $f_1 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $g_0$, $g_1 \in \mathcal{L}^2_{\mathbb{C}}(\nu)$ are such that $Tf_0^\bullet = g_0^\bullet$ and $Tf_1^\bullet = g_1^\bullet$, and take any $\theta \in \mathcal{X}$. Then

(a) setting $f_2 = \bar{f}_0$, $g_2(\chi) = \overline{g_0(\chi^{-1})}$ whenever this is defined, $Tf_2^\bullet = g_2^\bullet$;

(b) setting $f_3 = f_1 \times \theta$, $g_3(\chi) = g_1(\theta\chi)$ whenever this is defined, $Tf_3^\bullet = g_3^\bullet$;

(c) setting $f_4 = f_0 \times f_1 \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, $\hat{f}_4(\theta) = (g_0 * g_1)(\theta)$.

**proof (a)** We have isometries $R_1 : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\mu)$ and $R_2 : L^2_{\mathbb{C}}(\lambda) \to L^2_{\mathbb{C}}(\lambda)$ defined by setting $R_1 f^\bullet = (\bar{f})^\bullet$ for every $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $R_2 g^\bullet = (\tilde{g})^\bullet$ for every $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, where $\tilde{g}(\chi) = \overline{g(\chi^{-1})}$ whenever this is defined. Now if $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then

$$\hat{\bar{f}}(\chi) = \int \overline{f(x)}\chi(x)\mu(dx) = \overline{\int f(x)\chi^{-1}(x)\mu(dx)} = \tilde{\hat{f}}(\chi)$$

for every $\chi \in \mathcal{X}$. So $TR_1 u = R_2 T u$ for every $u \in L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$; as $L^1_{\mathbb{C}} \cap L^2_{\mathbb{C}}$ is dense in $L^2_{\mathbb{C}}$, $TR_1 = R_2 T$, which is what we need to know.

**(b)** This time, set $R_1 f^\bullet = (f \times \theta)^\bullet$ for every $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $R_2 g^\bullet = (\theta^{-1} \bullet_l g)^\bullet$ for $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, where $(\theta \bullet_l g)(\chi) = g(\theta^{-1}\chi)$ whenever this is defined. Once again, $R_1 : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\mu)$ and $R_2 : L^2_{\mathbb{C}}(\lambda) \to L^2_{\mathbb{C}}(\lambda)$ are isometries. If $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then

$$(f\times\theta)^\wedge(\chi)=\int f(x)\theta(x)\chi(x)\mu(dx)=\hat{f}(\theta\chi)$$

for every $\chi$, so $TR_1f^\bullet=R_2Tf^\bullet$; once again, this is enough to prove that $TR_1=R_2T$, as required.

**(c)** We have

$$(g_0*g_1)(\theta)=\int g_0(\chi^{-1})g_1(\theta\chi)\lambda(d\chi)=\int\overline{g_2(\chi)}g_3(\chi)\lambda(d\chi)=(g_3^\bullet|g_2^\bullet)$$

(where $(\ |\ )$ is the standard inner product of $L^2_{\mathbb{C}}(\lambda)$)

$$=(Tf_3^\bullet|Tf_2^\bullet)$$

(using (a) and (b))

$$=(f_3^\bullet|f_2^\bullet)$$

(because linear isometries of Hilbert space preserve inner products, see 4A4Jc)

$$=\int f_3(x)\overline{f_2(x)}\mu(dx)=\int f_1(x)\theta(x)f_0(x)\mu(dx)=(f_0\times f_1)^\wedge(\theta).$$

**445T Corollary** Let $X$ be an abelian topological group with a Haar measure $\mu$, and $\lambda$ the corresponding Haar measure on the dual group $\mathcal{X}$ of $X$ (445P). Then for any non-empty open set $H\subseteq\mathcal{X}$, there is an $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)$ such that $\hat{f}\ne0$ and $\hat{f}(\chi)=0$ for $\chi\in\mathcal{X}\setminus H$.

**proof** Let $V_1$ and $V_2$ be non-empty open sets of finite measure such that $V_1V_2\subseteq H$, and let $g_1$, $g_2$ be their indicator functions. Then there are $f_1$, $f_2\in\mathcal{L}^2_{\mathbb{C}}(\mu)$ such that $Tf_j^\bullet=g_j^\bullet$ for both $j$, where $T:L^2_{\mathbb{C}}(\mu)\to L^2_{\mathbb{C}}(\lambda)$ is the isometry of 445R. In this case, by 445Sc, $(f_1\times f_2)^\wedge=g_1*g_2$. But it is easy to check that $g_1*g_2$ is non-zero and zero outside $H$.

**445U The Duality Theorem** (Pontryagin 1934, Kampen 1935) Let $X$ be a locally compact Hausdorff abelian topological group. Then the canonical map $x\mapsto\hat{x}$ from $X$ to its bidual $\mathfrak{X}$ (445E) is an isomorphism between $X$ and $\mathfrak{X}$ as topological groups.

**proof** By 445O, $\hat{}$ is a homeomorphism between $X$ and its image $\hat{X}\subseteq\mathfrak{X}$. Accordingly $\hat{X}$ is itself, with its subspace topology, a locally compact topological group, and is closed in $\mathfrak{X}$ (4A5Mc).

**?** Suppose, if possible, that $\hat{X}\ne\mathfrak{X}$. Let $\mu$ be a Haar measure on $X$ (441E) and $\lambda$ the associated Haar measure on the dual $\mathcal{X}$ of $X$ (445P). By 445T, there is a $g\in\mathcal{L}^1_{\mathbb{C}}(\lambda)$ such that $\hat{g}$ is zero on $\hat{X}$ but not zero everywhere, so that $g$ is not zero a.e. We may suppose that $g$ is defined everywhere on $\mathcal{X}$. In this case we have

$$0=\hat{g}(\hat{x})=\int g(\chi)\hat{x}(\chi)\lambda(d\chi)=\int g(\chi)\chi(x)\lambda(d\chi)$$

for every $x\in X$.

By 418J, $g:\mathcal{X}\to\mathbb{C}$ is almost continuous. Let $K\subseteq\{\chi:\chi\in\mathcal{X},\ g(\chi)\ne0\}$ be a compact set such that $\int_K|g|d\lambda\ge\frac{3}{4}\int_{\mathcal{X}}|g|d\lambda$ and $g\restriction K$ is continuous. Set $q(\chi)=\overline{g(\chi)}/|g(\chi)|$ for $\chi\in K$. Now consider the linear span $A$ of $\{\hat{x}:x\in X\}$ as a linear space of complex-valued functions on $\mathcal{X}$. Since $\widehat{xy}=\hat{x}\times\hat{y}$ for all $x$, $y\in X$, $A$ is a subalgebra of $C_b=C_b(\mathcal{X};\mathbb{C})$; since $\widehat{x^{-1}}=\bar{\hat{x}}$ for every $x\in X$, $\bar{h}\in A$ for every $h\in A$; the constant function $\hat{e}$ belongs to $A$; and $A$ separates the points of $\mathcal{X}$. By the Stone-Weierstrass theorem, in the form 281G, there is an $h\in A$ such that $|h(\chi)-q(\chi)|\le\frac{1}{2}$ for every $\chi\in K$ and $|h(\chi)|\le1$ for every $\chi\in\mathcal{X}$. Of course $\int g\times h\,d\lambda=0$ for every $h\in A$ because $\int g\times\hat{x}\,d\lambda=0$ for every $x\in X$.

Now, however,

$$\begin{aligned}\int_K|g|d\lambda&=\int_K g\times q\,d\lambda\le\Bigl|\int_K g\times h\,d\lambda\Bigr|+\int_K|g|\times|h-q|d\lambda\\&\le\Bigl|\int_{\mathcal{X}\setminus K}g\times h\,d\lambda\Bigr|+\frac{1}{2}\int_K|g|d\lambda\le\int_{\mathcal{X}\setminus K}|g|d\lambda+\frac{1}{2}\int_K|g|d\lambda<\int_K|g|d\lambda,\end{aligned}$$

which is impossible. **X**

Thus $\hat{X}=\mathfrak{X}$ and the proof is complete.

**445X Basic exercises (a)** Consider the additive group $\mathbb{Q}$ with its usual topology. Show that its dual group can be identified with the additive group $\mathbb{R}$ with its usual topology.

**(b)** Let $X$ be any topological group, and $\mathcal{X}$ its dual group. Show that if $\nu$ is a totally finite Radon measure on $X$, then its Fourier-Stieltjes transform $\hat{\nu}:\mathcal{X}\to\mathbb{C}$ is continuous.

**(c)** Let $X$ be a topological group carrying Haar measures and $\mathcal{X}$ its dual group. For a totally finite quasi-Radon measure $\nu$ on $\mathcal{X}$ set $\hat{\nu}(x)=\int\chi(x)\nu(d\chi)$ for every $x\in X$. (i) Show that $\hat{\nu}:X\to\mathbb{C}$ is continuous. (ii) Show that for any totally finite quasi-Radon measure $\mu$ on $X$ with Fourier-Stieltjes transform $\hat{\mu}$, $\int\hat{\mu}\,d\nu=\int\hat{\nu}\,d\mu$.

**>(d)** Let $X$ be a group and $h_1$, $h_2:X\to\mathbb{C}$ positive definite functions. Show that $h_1+h_2$, $\alpha h_1$ and $\bar{h}_1$ are also positive definite for any $\alpha\ge 0$.

**(e)**(i) Let $X$ be a group, $Y$ a subgroup of $X$ and $h:Y\to\mathbb{C}$ a positive definite function. Set $h_1(x)=h(x)$ if $x\in Y$, $h_1(x)=0$ if $x\in X\setminus Y$. Show that $h_1$ is positive definite. (ii) Let $X$ and $Y$ be groups, $\phi:X\to Y$ a group homomorphism and $h:Y\to\mathbb{C}$ a positive definite function. Show that $h\phi:X\to\mathbb{C}$ is positive definite.

**(f)** Let $X$ be a topological group and $\mathcal{X}$ its dual group. (i) Let $\nu$ be any totally finite topological measure on $\mathcal{X}$ and set $h(x)=\int\chi(x)\nu(d\chi)$ for $x\in X$. Show that $h:X\to\mathbb{C}$ is positive definite. (ii) Let $\nu$ be any totally finite topological measure on $X$. Show that its Fourier transform $\hat{\nu}:\mathcal{X}\to\mathbb{C}$ is positive definite.

**>(g)** Let $X$ be a topological group with a left Haar measure, and $h:X\to\mathbb{C}$ a bounded continuous function. Show that $h$ is positive definite iff $\int h(x^{-1}y)f(y)\overline{f(x)}dxdy\ge 0$ for every integrable function $f$.

**>(h)** Let $\phi:\mathbb{R}^r\to\mathbb{C}$ be a function. Show that it is the characteristic function of a probability distribution on $\mathbb{R}^r$ iff it is continuous and positive definite and $\phi(0)=1$.

**>(i)** Let $X$ be a compact Hausdorff abelian topological group, and $\mu$ the Haar probability measure on $X$. Show that the corresponding Haar measure on the dual group $\mathcal{X}$ of $X$ is just counting measure on $\mathcal{X}$.

**>(j)** Let $X$ be an abelian group with its discrete topology, and $\mu$ counting measure on $X$. Let $\mathcal{X}$ be the dual group and $\lambda$ the corresponding Haar measure on $\mathcal{X}$. Show that $\lambda\mathcal{X}=1$.

**(k)** Let $X$ be the topological group $\mathbb{R}$, and $\mu=\frac{1}{\sqrt{2\pi}}\mu_L$, where $\mu_L$ is Lebesgue measure. (i) Show that if we identify the dual group $\mathcal{X}$ of $X$ with $\mathbb{R}$, writing $\chi(x)=e^{-i\chi x}$ for $x$, $\chi\in\mathbb{R}$, then the Haar measure on $\mathcal{X}$ corresponding to the Haar measure $\mu$ on $X$ is $\mu$ itself. (ii) Show that if we change the action of $\mathbb{R}$ on itself by setting $\chi(x)=e^{-2\pi i\chi x}$, then the Haar measure on $\mathcal{X}$ corresponding to $\mu_L$ is $\mu_L$.

**(l)** Let $X_0,\dots,X_n$ be abelian topological groups with Haar measures $\mu_0,\dots,\mu_n$, and let $X=X_0\times\ldots\times X_n$ be the product group with its Haar measure $\mu=\mu_0\times\ldots\times\mu_n$. For each $k\le n$ let $\mathcal{X}_k$ be the dual group of $X_k$ and $\lambda_k$ the Haar measure on $\mathcal{X}_k$ corresponding to $\mu_k$. Show that if we identify $\mathcal{X}=\mathcal{X}_0\times\ldots\times\mathcal{X}_n$ with the dual group of $X$, then the Haar measure on $\mathcal{X}$ corresponding to $\mu$ is just the product measure $\lambda_0\times\ldots\times\lambda_n$.

**>(m)** Let $X$ be a compact Hausdorff abelian topological group, with dual group $\mathcal{X}$, and $\mu$ the Haar probability measure on $X$. (i) Show that $\int\chi\,d\mu=0$ for every $\chi\in\mathcal{X}$ except the identity. (ii) Show that $\{\chi^\bullet:\chi\in\mathcal{X}\}$ is an orthonormal basis of the Hilbert space $L^2_{\mathbb{C}}(\mu)$. (*Hint*: $(u|\chi^\bullet)=(Tu|\hat{\chi}^\bullet)$ where $T$ is the operator of 445R.)

**(n)** Let $X$ be an abelian topological group with a Haar measure $\mu$, $\lambda$ the associated Haar measure on the dual group $\mathcal{X}$ and $T:L^2_{\mathbb{C}}(\mu)\to L^2_{\mathbb{C}}(\lambda)$ the standard isometry. Suppose that $u$, $v\in L^2_{\mathbb{C}}(\mu)$. Suppose that $f_0$, $f_1\in\mathcal{L}^2_{\mathbb{C}}(\mu)$, $g_0$, $g_1\in\mathcal{L}^2_{\mathbb{C}}(\nu)$ are such that $Tf_0^\bullet=g_0^\bullet$ and $Tf_1^\bullet=g_1^\bullet$. Show that $(f_0*f_1)(x)=\int g_0(\chi)g_1(\chi)\overline{\chi(x)}\lambda(d\chi)$ for any $x\in X$.

**(o)** Let $X$ be a locally compact Hausdorff abelian topological group and $\mathcal{X}$ its dual group. Show that a function $h : \mathcal{X} \to \mathbb{C}$ is the Fourier-Stieltjes transform of a totally finite Radon measure on $X$ iff it is continuous and positive definite.

**(p)**(i) Show that we can define a binary operation $+_{\text{2adic}}$ on $X = \{0,1\}^{\mathbb{N}}$ by setting $x +_{\text{2adic}} y = z$ whenever $x$, $y$, $z \in X$ and $\sum_{i=0}^{k} 2^i(x(i)+y(i)-z(i))$ is divisible by $2^{k+1}$ for every $k$. (ii) Show that if we give $X$ its usual topology then $(X, +_{\text{2adic}})$ is an abelian topological group. (iii) Show that the usual measure on $X$ is the Haar probability measure for this group operation. (iv) Show that $G = \{\zeta : \zeta \in \mathbb{C}, \exists\, n \in \mathbb{N}, \zeta^{2^n} = 1\}$ is a subgroup of $\mathbb{C}$. (v) Show that the dual of $(X, +_{\text{2adic}})$ is $\{\chi_\zeta : \zeta \in G\}$ where $\chi_\zeta(x) = \prod_{i=0}^{\infty} \zeta^{2^i x(i)}$ for $\zeta \in G$ and $x \in X$. (vi) Show that the functions $f$, $g : X \to X$ described in 388E are of the form $x \mapsto x \pm_{\text{2adic}} x_0$ for a certain $x_0 \in X$.

**(q)** Let $X$ be a locally compact Hausdorff abelian topological group. Show that if two totally finite Radon measures on $X$ have the same Fourier-Stieltjes transform, they are equal. (*Hint*: 281G.)

**445Y Further exercises (a)** Let $X$ be any Hausdorff topological group. Let $\widehat{X}$ be its completion under its bilateral uniformity. Show that the dual groups of $X$ and $\widehat{X}$ can be identified as groups. Show that they can be identified as topological groups if *either* $X$ is metrizable *or* $X$ has a totally bounded neighbourhood of the identity.

**(b)** Let $\langle X_j \rangle_{j\in I}$ be a countable family of topological groups, with product $X$; let $\mathcal{X}_j$ be the dual group of each $X_j$, and $\mathcal{X}$ the dual group of $X$. Show that the topology of $\mathcal{X}$ is generated by sets of the form $\mathcal{X} \cap \prod_{j\in I} H_j$ where $H_j \subseteq \mathcal{X}_j$ is open for each $j$.

**(c)** Let $X$ be a real linear topological space, with addition as its group operation. Show that its dual group is just the set of functionals $x \mapsto e^{if(x)}$ where $f : X \to \mathbb{R}$ is a continuous linear functional. Hence show that there are abelian groups with trivial duals.

**(d)** Let $X$ be the group of rotations of $\mathbb{R}^3$, that is, the group of orthogonal real $3 \times 3$ matrices with determinant 1, and give $X$ its usual topology, corresponding to its embedding as a subspace of $\mathbb{R}^9$. Show that the only character on $X$ is the constant function 1. (*Hint*: (i) show that two rotations through the same angle are conjugate in $X$; (ii) show that if $0 < \theta \le \frac{\pi}{2}$ then the product of two rotations through the angle $\theta$ about orthogonal axes is not a rotation through an angle $2\theta$.)

**(e)** Let $X$ be a finite abelian group, endowed with its discrete topology. Show that its dual is isomorphic to $X$. (*Hint*: $X$ is a product of cyclic groups.)

**(f)** Show that if $I$ is any uncountable set, then there is a quasi-Radon probability measure $\nu$ on the topological group $\mathbb{R}^I$ such that its Fourier-Stieltjes transform $\hat{\nu}$ is not continuous. (*Hint*: take $\nu$ to be a power of a suitably widely spread probability distribution on $\mathbb{R}$.)

**(g)** Let $X$ be an abelian topological group with a Haar measure $\mu$, and $\lambda$ the associated Haar measure on the dual group $\mathcal{X}$ of $X$. Let $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ be the standard isomorphism. Suppose that $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $g \in \mathcal{L}^1_{\mathbb{C}}(\lambda) \cap \mathcal{L}^2_{\mathbb{C}}(\lambda)$ are such that $Tf^\bullet = g^\bullet$. Show that $f(x) = \int g(\chi)\overline{\chi(x)}\lambda(d\chi)$ for almost every $x \in X$. (*Hint*: look first at locally compact Hausdorff $X$.)

**(h)** Let $X$ be a locally compact Hausdorff abelian topological group with dual group $\mathcal{X}$, $P_{\text{R}}$ the set of Radon probability measures on $X$, $\langle \nu_n \rangle_{n\in\mathbb{N}}$ a sequence in $P_{\text{R}}$ and $\nu$ a member of $P_{\text{R}}$. Show that the following are equiveridical: (i) $\langle \nu_n \rangle_{n\in\mathbb{N}} \to \nu$ for the narrow topology on $P_{\text{R}}$; (ii) $\lim_{n\to\infty} \hat{\nu}_n(\chi) = \hat{\nu}(\chi)$ for every $\chi \in \mathcal{X}$. (*Hint*: compare 285L. For the critical step, showing that $\{\nu_n : n \in \mathbb{N}\}$ is uniformly tight, use the formulae in 445N to show that there is an integrable $f : \mathcal{X} \to \mathbb{C}$ such that $0 \le \check{f} \le \mathbf{1}$ and $\int \check{f}(x)\nu(dx) \ge 1 - \frac{1}{2}\epsilon$.)

**(i)** Let $X$ be a topological group carrying Haar measures and $\mathcal{X}$ its dual group. Let $M_\tau$ be the complex Banach space of signed totally finite $\tau$-additive Borel measures on $\mathcal{X}$ (put the ideas of 437F and 437Yb together). Show that $X$ separates the points of $M_\tau$ in the sense that if $\nu \in M_\tau$ is non-zero, there is an $x \in X$ such that $\int \chi(x)\nu(d\chi) \ne 0$, if the integral is appropriately interpreted. (*Hint*: use the method in the proof of 445U.)

**(j)** Let $X$ be an abelian topological group carrying Haar measures. Let $M_\tau$ be the complex Banach space of signed totally finite $\tau$-additive Borel measures on $X$. Show that the dual $\mathcal{X}$ of $X$ separates the points of $M_\tau$ in the sense that if $\nu \in M_\tau$ is non-zero, there is a $\chi \in \mathcal{X}$ such that $\int \chi(x)\nu(dx) \neq 0$. (*Hint*: use 443L to reduce to the case in which $X$ is locally compact and Hausdorff; now use 445U and 445Yi.)

**(k)** Let $X$ be an abelian topological group and $\mu$ a Haar measure on $X$. Show that the spectral radius of any non-zero element of $L^1_{\mathbb{C}}(\mu)$ is non-zero. (*Hint*: 445Yj, 445Kd.)

**(l)** Show that for any integer $p \geq 2$ there is an operation $+_{\mathrm{padic}}$ on $\{0,\ldots,p-1\}^{\mathbb{N}}$ with properties similar to those of the operation $+_{\mathrm{2adic}}$ of 445Xp.

**(m)** Let $\mu$ be Lebesgue measure on $[0,\infty[$. (i) For $f$, $g \in \mathcal{L}^1(\mu)$ define $(f*g)(x) = \int_0^x f(y)g(x-y)\mu(dy)$ whenever the integral is defined. Show that $f*g \in \mathcal{L}^1(\mu)$. (ii) Show that we can define a bilinear operator $*$ on $L^1(\mu)$ by setting $f^\bullet * g^\bullet = (f*g)^\bullet$ for $f$, $g \in \mathcal{L}^1(\mu)$, and that under this multiplication $L^1(\mu)$ is a Banach algebra. (iii) Show that if $\phi : L^1(\mu) \to \mathbb{R}$ is a multiplicative linear operator then there is some $s \geq 0$ such that $\phi(f^\bullet) = \int_0^\infty f(x)e^{-sx}\mu(dx)$ for every $f \in \mathcal{L}^1(\mu)$.

**445 Notes and comments** I repeat that this section is intended to be a more or less direct attack on the duality theorem. At every point the clause 'let $X$ be a locally compact Hausdorff abelian topological group' is present in spirit. The actual statement of each theorem involves some subset of these properties, purely in accordance with the principle of omission of irrelevant hypotheses, not because I expect to employ the results in any more general setting.

In 445Ab I describe a topology on the dual group in a context so abstract that we have rather a lot of choice. For groups carrying Haar measures, the alternative descriptions of the topology on the dual (445I) make it plain that this must be the first topology to study. By 445E it is already becoming fairly convincing. But it is not clear that there is any such pre-eminent topology in the general case.

Fourier-Stieltjes transforms hardly enter into the arguments of this section; I mention them mostly because they form the obvious generalization of the ideas in §285. But I note that the principal theorem of §285 (that sequential convergence of characteristic functions corresponds to sequential convergence of distributions, 285L) generalizes directly to the context here (445Yh).

I have tried to lay this treatise out in such a way that we periodically return to themes from past chapters at a higher level of sophistication. There seem to be four really important differences between this section and the previous treatment in Chapter 28. (i) The first is the obvious one; we are dealing with general locally compact Hausdorff abelian groups, rather than with $\mathbb{R}$ and $S^1$ and $\mathbb{Z}$. Of course this puts much heavier demands on our technique, and, to begin with, leaves our imaginations unfocused. (ii) The second concerns differentiation, or rather its absence; since we no longer have any differential structure on our groups, a substantial part of the theory evaporates, and we are forced to employ new tactics in the rest. (iii) The third concerns the normalization of the measure on the dual group. As soon as we know that $\mathcal{X}$ is a locally compact group (445J) we know that it carries Haar measures. The problem is to describe the particular one we need in appropriate terms. In the case of the dual pairs $(\mathbb{R},\mathbb{R})$ or $(S^1,\mathbb{Z})$, we have measures already presented (counting measure on $\mathbb{Z}$, Lebesgue measure on $]-\pi,\pi]$ and $\mathbb{R}$). (They are not in fact dual in the sense we need here, at least not if we use the simplest formulae for the duality, and have to be corrected in each case by a factor of $2\pi$. See 445Xk.) But since we do have dual pairs already to hand, we can simultaneously develop theories of Fourier transforms and inverse Fourier transforms (for the pair $(S^1,\mathbb{Z})$ the inverse Fourier transform is just summation of trigonometric series), and the problem is to successfully match operations which have independent existences. (iv) The final change concerns an interesting feature of $\mathbb{Z}$ and $\mathbb{R}$. Repeatedly, in §§282-283, the formulae invoked symmetric limits $\lim_{n\to\infty}\sum_{k=-n}^{n}$ or $\lim_{a\to\infty}\int_{-a}^{a}$ to approach some conditionally convergent sum or integral. Elsewhere one sometimes deals with singularities by examining 'Cauchy principal values'; if $\int_{-1}^{1} f$ is undefined, try $\lim_{a\downarrow 0}(\int_{-1}^{-a} f + \int_a^1 f)$. This particular method seems to disappear in the general context. But the general challenge of the subject remains the same: to develop a theory of the transform $u \mapsto \hat{u}$ which will apply to the largest possible family of objects $u$ and will enable us to justify, in the widest possible contexts, the manipulations listed in the notes to §284. The calculations in 445S and 445Xn, treating 'shift' and 'convolution' in $L^2$, are typical.

In terms of the actual proofs of the results here, 'test functions' (284A) have gone, and in their place we take a lot more trouble over the Banach algebra $L^1$. This algebra is the key to one of the magic bits, which turns up in rather undignified corners in 445Kd and part (e) of the proof of 445N. Down to 445O, the dominating problem is that we do not know that the dual group $\mathcal{X}$ of a group $X$ is large enough to tell us anything interesting about $X$. (After that, the problem reverses; we have to show that $\mathcal{X}$ is not too big.) We find that (under rather specially arranged circumstances) we are able to say something useful about the spectral radius of a member of $L^1$, and we use this to guarantee that it has a non-trivial Fourier transform. If we identify $\mathcal{X}$ with the maximal ideal space of $L^1$ (445H), then the Fourier transform on $L^1$ becomes the 'Gelfand map', a general construction of great power in the theory of commutative Banach algebras.

There is one similarity between the methods of this section and those of §284. In both cases we have isomorphisms between $L^2_{\mathbb{C}}(\mu)$ and $L^2_{\mathbb{C}}(\lambda)$ (the Plancherel Theorem), but cannot define the Fourier transform of a function in $\mathcal{L}^2_{\mathbb{C}}$ in any direct way; indeed, while the Fourier transform of a function in $\mathcal{L}^1_{\mathbb{C}}$, or even of a (totally finite) measure, can really be thought of as a (continuous) function, the transform of a function in $L^2$ is at best a member of $L^2$, not a function at all. We manoeuvre around this difficulty by establishing that our (genuine) Fourier transforms match dense subspaces isometrically. In §284 I used test functions, and in the present section I use $\mathcal{L}^1 \cap \mathcal{L}^2$. Test functions are easier partly because the Fourier transform of a test function is again a test function, and all the formulae we need are easy to establish for such functions.

Searching for classes of functions which will be readily manageable in general locally compact abelian groups, we come to the 'positive definite' functions. The phrase is unsettling, since the functions themselves are in no obvious sense positive (nor even, as a rule, real-valued). Also their natural analogues in the theory of bilinear forms are commonly called 'positive semi-definite'. However, their Fourier transforms, whether regarded as measures (445N) or as functions (445Q), are positive, and, as a bonus, we get a characterization of the Fourier transforms of measures (445Xf, 445Xh), answering a question left hanging in 285Xu.

Version of 8.10.13

## 446 The structure of locally compact groups

I develop those fragments of the structure theory of locally compact Hausdorff topological groups which are needed for the main theorem of the next section. Theorem 446B here is of independent interest, being both itself important and with a proof which uses the measure theory of this chapter in an interesting way; but the rest of the section, from 446D on, is starred. Note that in this section, unlike the last, groups are not expected to be abelian.

**446A Finite-dimensional representations (a) Definitions (i)** For any $r \in \mathbb{N}$, write $M_r = M_r(\mathbb{R})$ for the space of $r \times r$ real matrices. If we identify it with the space $\mathsf{B}(\mathbb{R}^r;\mathbb{R}^r)$, where $\mathbb{R}^r$ is given its Euclidean norm, then $M_r$ becomes a unital Banach algebra (4A6C), with identity $I$, the $r \times r$ identity matrix. Write $GL(r,\mathbb{R})$ for the group of invertible elements of $M_r$.

**(ii)** Let $X$ be a topological group. A **finite-dimensional representation** of $X$ is a continuous homomorphism from $X$ to a group of the form $GL(r,\mathbb{R})$ for some $r \in \mathbb{N}$. If the homomorphism is injective the representation is called **faithful** (cf. 4A5Be).

**(b)** Observe that if $X$ is any topological group and $\phi$ is a finite-dimensional representation with kernel $Y$, then $X/Y$ has a faithful finite-dimensional representation $\psi$ defined by writing $\psi(x^\bullet) = \phi(x)$ for every $x \in X$ (4A5La).

**446B Theorem** Let $X$ be a compact Hausdorff topological group. Then for any $a \in X$, other than the identity, there is a finite-dimensional representation $\phi$ of $X$ such that $\phi(a) \neq I$; and we can arrange that $\phi(x)$ is an orthogonal matrix for every $x \in X$.

**proof (a)** Let $U$ be a symmetric neighbourhood of the identity $e$ in $X$ such that $a \notin UU$. Because $X$ is completely regular (3A3Bb), there is a non-zero continuous function $h : X \to [0,\infty[$ such that $h(x) = 0$ for every $x \in X \setminus U$; replacing $h$ by $x \mapsto h(x) + h(x^{-1})$ if necessary, we may suppose that $h(x) = h(x^{-1})$ for every $x$. Let $\mu$ be a (left) Haar measure on $X$ (441E), and set $w = h^\bullet$ in $L^0(\mu)$.

**(b)** Define an operator $T$ from $L^2 = L^2(\mu)$ to itself by setting

$$Tu = u * w \text{ for every } u \in L^2,$$

where $*$ is convolution; that is, $Tf^\bullet = (f*h)^\bullet$ for every $f \in \mathcal{L}^2 = \mathcal{L}^2(\mu)$. Then $T$ is a compact self-adjoint operator on the real Hilbert space $L^2$ (444V).

**(c)** For any $z \in X$, define $S_z : L^2 \to L^2$ by setting $S_z u = z\bullet_l u$ for $u \in L^2$, where $\bullet_l$ is the left shift action, so that $S_z$ is a norm-preserving linear operator (443Ge). Also $S_z$ commutes with $T$. **P** If $f \in \mathcal{L}^2$, then

$$S_z T f^\bullet = (z\bullet_l (f*h))^\bullet = ((z\bullet_l f)*h)^\bullet$$

(444Of)

$$= TS_z f^\bullet. \mathbf{Q}$$

**(d)** Now $S_a Tw \neq Tw$. **P** Set $g = h*h$, so that $Tw = g^\bullet$ and $g$, $a\bullet_l g$ are both continuous functions (444Rc). Then

$$(a\bullet_l g)(e) = g(a^{-1}) = \int h(y)h(y^{-1}a^{-1})dy = 0$$

because if $y \in U$ then $y^{-1}a^{-1} \notin U$, while

$$g(e) = \int h(y)h(y^{-1})dy = \int h(y)^2 dy > 0.$$

So the open set $\{x : g(x) \neq (a\bullet_l g)(x)\}$ is non-empty; because $\mu$ is strictly positive (442Aa), it is non-negligible, and

$$S_a Tw = (a\bullet_l g)^\bullet \neq g^\bullet = Tw. \mathbf{Q}$$

**(e)** The closed linear subspace $\{u : S_a u = u\}$ therefore does not include $T[L^2]$. But the linear span of $\{Tv : v \text{ is an eigenvector of } T\}$ is dense in $T[L^2]$ (4A4M), so there is an eigenvector $v^*$ of $T$ such that $S_a Tv^* \neq Tv^*$. Let $\gamma \in \mathbb{R}$ be such that $Tv^* = \gamma v^*$; since $Tv^* \neq 0$, $\gamma \neq 0$, and $V = \{u : u \in L^2, Tu = \gamma u\}$ is finite-dimensional (4A4Lb).

**(f)** $S_z[V] \subseteq V$ for every $z \in X$. **P**

$$TS_z u = S_z Tu = S_z(\gamma u) = \gamma S_z u$$

for every $u \in V$. **Q**

We therefore have a map $z \mapsto S_z \restriction V : X \to \mathsf{B}(V;V)$. As observed in 443Gc, this is actually a semigroup homomorphism, and of course $S_e \restriction V$ is the identity of $\mathsf{B}(V;V)$, so $S_z \restriction V$ is always invertible, and we have a group homomorphism from $X$ to the group of invertible elements of $\mathsf{B}(V;V)$. Taking an orthonormal basis $(v_1, \dots, v_r)$ of $V$, we have a homomorphism $\phi$ from $X$ to $GL(r,\mathbb{R})$, defined by setting $\phi(z) = \langle (S_z v_i | v_j)\rangle_{1\le i,j\le r}$ for every $z \in X$. Moreover, $\phi$ is continuous. **P** For any $u \in L^2$, $z \mapsto S_z u : X \to L^2$ is continuous, by 443Gf. But this means that all the maps $z \mapsto (S_z v_i | v_j)$ are continuous; since the topology of $GL(r,\mathbb{R})$ can be defined in terms of these functionals (see the formulae in 262H), $\phi$ is continuous. **Q**

Thus $\phi$ is a finite-dimensional representation of $X$. But $V$ was chosen to contain $v^*$; of course $Tv^* \in V$, while $S_a Tv^* \neq Tv^*$; so that $\phi(a)$ is not the identity.

**(g)** Finally, $\phi(z)$ is an orthogonal matrix for every $z \in X$. **P** As observed in (c), $S_z$ is norm-preserving, so $S_z \restriction V$ is again norm-preserving. By 4A4Jb, $(S_z v_i | S_z v_j) = (v_i | v_j)$ for $1 \le i,j \le r$, that is, $\phi(z)$ is orthogonal. **Q**

**446C Corollary** Let $X$ be a compact Hausdorff topological group. Then for any neighbourhood $U$ of the identity of $X$ there is a finite-dimensional representation of $X$ with kernel included in $U$.

**proof** Let $\Phi$ be the set of finite-dimensional representations of $X$. By 446B, $\bigcap_{\phi\in\Phi} \ker(\phi) = \{e\}$, where $e$ is the identity of $X$. Because $X \setminus \operatorname{int} U$ is compact and disjoint from $\bigcap_{\phi\in\Phi}\ker(\phi)$ (and $\ker(\phi)$ is closed for every $\phi$), there must be $\phi_0, \dots, \phi_n \in \Phi$ such that $\bigcap_{i\le n} \ker(\phi_i) \subseteq U$. For each $i \le n$, let $r_i \in \mathbb{N}$ be the integer such that $\phi_i$ is a continuous homomorphism from $X$ to $GL(r_i,\mathbb{R})$. Set $r = \sum_{i=0}^n r_i$. Then we have a map $\phi : X \to GL(r,\mathbb{R})$ given by the formula

$$\phi(x)=\begin{pmatrix}\phi_0(x) & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{0} & \phi_1(x) & \dots & \mathbf{0}\\ \dots & \dots & \dots & \dots\\ \mathbf{0} & \mathbf{0} & \dots & \phi_n(x)\end{pmatrix}$$

for every $x\in X$. It is easy to check that $\phi$ is a continuous homomorphism, and $\ker(\phi)=\bigcap_{i\le n}\ker(\phi_i)\subseteq U$. So we have an appropriate representation of $X$.

***446D Notation (a)** It will help to be clear on an elementary point of notation. If $X$ is a group and $A$ is a subset of $X$ I will write $A^0=\{e\}$, where $e$ is the identity of $X$, and $A^{n+1}=AA^n$ for $n\in\mathbb{N}$, so that $A^3=\{x_1x_2x_3:x_1,\,x_2,\,x_3\in A\}$, etc. Now we find that $A^{m+n}=A^mA^n$ and $A^{mn}=(A^m)^n$ for all $m,\,n\in\mathbb{N}$. Writing $A^{-1}=\{x^{-1}:x\in A\}$ as usual, we have $(A^r)^{-1}=(A^{-1})^r$. But note that if we also continue to write $A^{-1}=\{x^{-1}:x\in A\}$, then $AA^{-1}$ is not in general equal to $A^0$; and that there is no simple relation between $A^r$, $B^r$ and $(AB)^r$, unless $X$ is abelian.

**(b)** In the rest of this section, I shall make extensive use of the following device. If $X$ is a group with identity $e$, $e\in A\subseteq X$ and $n\in\mathbb{N}$, write $D_n(A)=\{x:x\in X,\,x^i\in A\text{ for every }i\le n\}$.

(i) $D_0(A)=X$.

(ii) $D_1(A)=A$.

(iii) $D_n(A)\subseteq D_m(A)$ whenever $m\le n$.

(iv) $D_{mn}(A)\subseteq D_m(D_n(A))$ for all $m,\,n\in\mathbb{N}$.

**P** If $x\in D_{mn}(A)$ then $(x^i)^j\in A$ whenever $j\le n$ and $i\le m$. **Q**

(v) If $r\in\mathbb{N}$ and $A^r\subseteq B$, then $D_n(A)\subseteq D_{nr}(B)$ for every $n\in\mathbb{N}$; in particular, $A\subseteq D_r(B)$.

**P** For $r=0$ this is trivial. Otherwise, take $x\in D_n(A)$ and $i\le nr$. Then we can express $i$ as $i_1+\ldots+i_r$ where $i_j\le n$ for each $j$, so that

$$x^i=\prod_{j=1}^r x^{i_j}\in A^r\subseteq B.\ \mathbf{Q}$$

(vi) If $A=A^{-1}$ then $D_n(A)=D_n(A)^{-1}$ for every $n\in\mathbb{N}$.

(vii) If $D_m(A)\subseteq B$ where $m\in\mathbb{N}$, then $D_{mn}(A)\subseteq D_n(B)$ for every $n\in\mathbb{N}$.

**P** If $x\in D_{mn}(A)$ and $i\le n$, then $x^{ij}\in A$ for every $j\le m$, so $x^i\in D_m(A)\subseteq B$. **Q**

**(c)** In (b), if $X$ is a topological group and $A$ is closed, then every $D_n(A)$ is closed; if moreover $A$ is compact, then $D_n(A)$ is compact for every $n\ge 1$. If $A$ is a neighbourhood of $e$, then so is every $D_n(A)$, because the map $x\mapsto x^i$ is continuous for every $i\le n$.

***446E Lemma** Let $X$ be a group, and $U\subseteq X$. Let $f:X\to[0,\infty[$ be a bounded function such that $f(x)=0$ for $x\in X\setminus U$; set $\alpha=\sup_{x\in X}f(x)$. Let $A\subseteq X$ be a symmetric set containing $e$, and $K$ a set including $A^k$, where $k\ge 1$. Define $g:X\to[0,\infty[$ by setting

$$g(x)=\frac{1}{k}\sum_{i=0}^{k-1}\sup\{f(yx):y\in A^i\}$$

for $x\in X$. Then

(a) $f(x)\le g(x)\le\alpha$ for every $x\in X$, and $g(x)=0$ if $x\notin K^{-1}U$.

(b) $|g(ax)-g(x)|\le\frac{j\alpha}{k}$ if $j\in\mathbb{N}$, $a\in A^j$ and $x\in X$.

(c) For any $x,\,z\in X$, $|g(x)-g(z)|\le\sup_{y\in K}|f(yx)-f(yz)|$.

**proof (a)** Of course

$$f(x)=\frac{1}{k}\sum_{i=0}^{k-1}f(ex)\le g(x)\le\frac{1}{k}\sum_{i=0}^{k-1}\alpha=\alpha$$

for every $x$. Suppose that $g(x)\ne 0$. Then there must be an $i<k$ and a $y\in A^i$ such that $f(yx)\ne 0$, so that $yx\in U$. But also, because $e\in A$, $y\in A^k\subseteq K$, so $x\in y^{-1}U\subseteq K^{-1}U$.

**(b)** Suppose first that $j=1$, so that $a\in A$. If $\epsilon>0$ there are $y_i\in A^i$, for $i<k$, such that $g(ax)\le\frac{1}{k}\sum_{i=0}^{k-1}f(y_iax)+\epsilon$. Now $y_ia\in A^{i+1}$ for each $i$, so

$$g(x)\ge\frac{1}{k}\sum_{i=0}^{k-2}f(y_iax)\ge g(ax)-\epsilon-\frac{\alpha}{k}.$$

As $\epsilon$ is arbitrary, $g(x)\ge g(ax)-\frac{\alpha}{k}$. Similarly, as $a^{-1}\in A$ (because $A$ is symmetric), $g(ax)\ge g(x)-\frac{\alpha}{k}$ and $|g(ax)-g(x)|\le\frac{\alpha}{k}$.

For the general case, induce on $j$. (If $j=0$, then $a=e$ and the result is trivial.)

**(c)** Set $\gamma=\sup_{y\in K}|f(yx)-f(yz)|$. If $\epsilon>0$, there are $y_i\in A^i$, for $i<k$, such that $g(x)\le\frac{1}{k}\sum_{i=0}^{k-1}f(y_ix)+\epsilon$. Now every $y_i$ belongs to $K$, so

$$g(z)\ge\frac{1}{k}\sum_{i=0}^{k-1}f(y_iz)\ge\frac{1}{k}\sum_{i=0}^{k-1}(f(y_ix)-\gamma)\ge g(x)-\epsilon-\gamma.$$

As $\epsilon$ is arbitrary, $g(z)\ge g(x)-\gamma$; similarly, $g(x)\ge g(z)-\gamma$.

***446F Lemma** Let $X$ be a locally compact Hausdorff topological group and $\langle A_n\rangle_{n\in\mathbb{N}}$ a sequence of closed symmetric subsets of $X$ all containing the identity $e$ of $X$. Suppose that for every neighbourhood $W$ of $e$ there is an $n_0\in\mathbb{N}$ such that $A_n\subseteq W$ for every $n\ge n_0$. Let $U$ be a compact neighbourhood of $e$ and suppose that for each $n\in\mathbb{N}$ we have $k(n)\in\mathbb{N}$ such that $A_n^{k(n)}\subseteq U$ and $A_n^{k(n)+1}\not\subseteq U$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and write $Q$ for the limit $\lim_{n\to\mathcal{F}}A_n^{k(n)}$ in the space $\mathcal{C}$ of closed subsets of $X$ with the Fell topology.

(i) If $Q^2=Q$ then $Q$ is a compact subgroup of $X$ included in $U$ and meeting the boundary of $U$.

(ii) If $Q^2\ne Q$ then there are a neighbourhood $W$ of $e$ and an infinite set $I\subseteq\mathbb{N}$ such that for every $n\in I$ there are an $x\in A_n$ and an $i\le k(n)$ such that $x^i\notin W$.

**proof (a)** I ought to begin by explaining why the limit $\lim_{n\to\mathcal{F}}A_n^{k(n)}$ is defined; this is just because the Fell topology is always compact (4A2T(b-iii)) and when based on a locally compact Hausdorff space is Hausdorff (4A2T(e-ii)).

Because $U$ is closed, $\{F:F\in\mathcal{C},\ F\subseteq U\}$ is closed, by the definition of the Fell topology (4A2T(a-ii)); because every $A_n^{k(n)}$ is included in $U$, so is $Q$, and $Q$ is compact. Because $x\mapsto x^{-1}$ is a homeomorphism of $X$, $F\mapsto F^{-1}$ is a homeomorphism of $\mathcal{C}$, and

$$Q^{-1}=\lim_{n\to\mathcal{F}}(A_n^{k(n)})^{-1}=\lim_{n\to\mathcal{F}}(A_n^{-1})^{k(n)}=\lim_{n\to\mathcal{F}}A_n^{k(n)}=Q.$$

And of course $e\in Q$ because $e\in A_n^{k(n)}$ for every $n$ and $\{(x,F):x\in F\in\mathcal{C}\}$ is closed in $X\times\mathcal{C}$ (4A2T(e-i)).

For each $n\in\mathbb{N}$, we have an $a_n\in A_n^{k(n)}$ and an $x_n\in A_n$ such that $a_nx_n\notin U$. Now $a=\lim_{n\to\mathcal{F}}a_n$ is defined (because $U$ is compact), and belongs to $Q$. Also $\lim_{n\to\infty}x_n=e$, because every neighbourhood of $e$ includes all but finitely many of the $A_n$, so $a=\lim_{n\to\mathcal{F}}a_nx_n\notin\operatorname{int}U$, and $a$ belongs to the boundary of $U$. Thus $Q$ meets the boundary of $U$.

**(b)** From (a) we see that if $Q^2=Q$ then $Q$ is a compact subgroup of $X$, included in $U$ and meeting the boundary of $U$. So henceforth let us suppose that $Q^2\ne Q$ and seek to prove (ii).

Let $w\in Q^2\setminus Q$. Let $W_0\subseteq U$ be an open neighbourhood of $e$ such that $W_0wW_0^2\cap QW_0^2=\emptyset$ (4A5Ee).

**(c)** Fix a left Haar measure $\mu$ on $X$. Let $f:X\to[0,\infty[$ be a continuous function such that $\{x:f(x)>0\}\subseteq W_0$ and $\int f(x)dx=1$. Set $\alpha=\sup_{x\in X}f(x)$ and $\beta=\int f(x)^2dx$, so that $\alpha$ is finite and $\beta>0$. $W_0U^2W_0\subseteq U^4$ is open and relatively compact, so has finite measure, and there is an $\eta>0$ such that

$$2\eta(1+\alpha\mu(W_0U^2W_0))<\beta.$$

Let $W$ be a neighbourhood of $e$ such that $W\subseteq W_0$ and $|f(yax)-f(ybx)|\le\eta$ whenever $y\in(U^{-1})^2\cup U$, $x\in X$ and $ab^{-1}\in W$ (4A5Pa).

**(d)** Express $w$ as $w'w''$ where $w'$, $w''\in Q$. Then

$$\{n:A_n^{k(n)}\cap W_0w'\ne\emptyset\},\quad\{n:A_n^{k(n)}\cap w''W_0\ne\emptyset\},$$

$$\{n : A_n^{k(n)} \subseteq QW_0\} = \{n : A_n^{k(n)} \cap (U \setminus QW_0) = \emptyset\}$$

all belong to $\mathcal{F}$, by the definition of the Fell topology. Also

$$\{n : k(n) \geq 1\}$$

is cofinite in $\mathbb{N}$, because $A_n \subseteq U$ for all $n$ large enough. Let $I$ be the intersection of these four sets, so that $I$ belongs to $\mathcal{F}$ and must be infinite.

**(e)** Let $n \in I$. **?** Suppose, if possible, that $x^i \in W$ for every $x \in A_n$ and $i \leq k(n)$. (The rest of the proof will be a search for a contradiction.) Note that $k(n) \geq 1$.

Choose $x_j \in A_n$, for $j < 2k(n)$, such that the products $x_{2k(n)-1}x_{2k(n)-2}\ldots x_{k(n)}$, $x_{k(n)-1}\ldots x_0$ belong to $W_0w'$, $w''W_0$ respectively; set $\tilde{w} = x_{2k(n)-1}\ldots x_0$, so that

$$\tilde{w} \in A_n^{2k(n)} \cap W_0w'w''W_0 \subseteq W_0wW_0.$$

Since $A_n^{k(n)} \subseteq QW_0$ and $W_0wW_0^2 \cap QW_0^2$ is empty, $\tilde{w}W_0$ does not meet $A_n^{k(n)}W_0$.

**(f)** Define $g : X \to [0,\infty[$ by setting

$$g(x) = \frac{1}{k(n)}\sum_{i=0}^{k(n)-1} \sup\{f(yx) : y \in A_n^i\}.$$

Then $g(\tilde{w}x)f(x) = 0$ for every $x \in X$. **P** If $f(x) \neq 0$, then $x \in W_0$, so $\tilde{w}x \notin A_n^{k(n)}W_0$, and $g(\tilde{w}x) = 0$, by 446Ea. **Q** Accordingly

$$\int (g(x) - g(\tilde{w}x))f(x)dx = \int g(x)f(x)dx \geq \beta$$

since $g \geq f$ (446Ea).

Set $y_0 = e$ and $y_{i+1} = x_iy_i$ for $i < 2k(n)$, so that $y_{2k(n)} = \tilde{w}$ and

$$y_i \in A_n^i \subseteq A_n^{2k(n)} = (A_n^{k(n)})^2 \subseteq U^2$$

for every $i \leq 2k(n)$. Then

$$g(x) - g(\tilde{w}x) = \sum_{i=0}^{2k(n)-1} g(y_ix) - g(y_{i+1}x)$$

for every $x \in X$. Let $i < 2k(n)$ be such that

$$\int (g(y_ix) - g(y_{i+1}x))f(x)dx \geq \frac{1}{2k(n)}\int (g(x) - g(\tilde{w}x))f(x)dx \geq \frac{\beta}{2k(n)}.$$

Set $u = x_i$ and $v = y_i$, so that $u \in A_n$, $v \in U^2$ and

$$\begin{aligned}\int (g(x) - g(ux))f(v^{-1}x)dx &= \int (g(vx) - g(uvx))f(x)dx \\ &= \int (g(y_ix) - g(y_{i+1}x))f(x)dx \geq \frac{\beta}{2k(n)}.\end{aligned}$$

**(g)** We have

$$\begin{aligned}&\int (g(x) - g(u^{k(n)}x))f(v^{-1}x)dx \\ &\quad= \sum_{j=0}^{k(n)-1}\int (g(u^jx) - g(u^{j+1}x))f(v^{-1}x)dx \\ &\quad= \sum_{j=0}^{k(n)-1}\int (g(x) - g(ux))f(v^{-1}u^{-j}x)dx \\ &\quad= k(n)\int (g(x) - g(ux))f(v^{-1}x)dx \\ &\qquad\quad+ \sum_{j=0}^{k(n)-1}\int (g(x) - g(ux))(f(v^{-1}u^{-j}x) - f(v^{-1}x))dx,\end{aligned}$$

that is,

$$k(n)\int(g(x)-g(ux))f(v^{-1}x)dx$$
$$=\int(g(x)-g(u^{k(n)}x))f(v^{-1}x)dx$$
$$-\sum_{j=0}^{k(n)-1}\int(g(x)-g(ux))(f(v^{-1}u^{-j}x)-f(v^{-1}x))dx.$$

Set

$$\beta_1=\sum_{j=0}^{k(n)-1}\int(g(x)-g(ux))(f(v^{-1}u^{-j}x)-f(v^{-1}x))dx,$$

$$\beta_2=\int(g(x)-g(u^{k(n)}x))f(v^{-1}x)dx;$$

then

$$\beta_2-\beta_1=k(n)\int(g(x)-g(ux))f(v^{-1}x)dx\ge\tfrac{1}{2}\beta.$$

**(h)($\boldsymbol{\alpha}$)** Examine $\beta_1$. We know that, because $u\in A_n$, $|g(x)-g(ux)|\le\frac{\alpha}{k(n)}$ for every $x$ (see 446Eb). On the other hand, we are supposing that $x^j\in W$ for every $j\le k(n)$ and every $x\in A_n$, so, in particular, $u^j\in W\subseteq W_0$ for every $j\le k(n)$. Also, as noted in (f), $v\in U^2$. So for any $j<k(n)$ we must have $|f(v^{-1}u^{-j}x)-f(v^{-1}x)|\le\eta$ for every $x\in X$, by the choice of $W$, while $f(v^{-1}u^{-j}x)-f(v^{-1}x)=0$ unless $x\in W_0U^2W_0$. So

$$|\beta_1|\le\sum_{j=0}^{k(n)-1}\int|g(x)-g(ux)||f(v^{-1}u^{-j}x)-f(v^{-1}x)|dx$$
$$\le\sum_{j=0}^{k(n)-1}\frac{\alpha}{k(n)}\eta\mu(W_0U^2W_0)=\alpha\eta\mu(W_0U^2W_0).$$

**($\boldsymbol{\beta}$)** Now consider $\beta_2$. As $u^{k(n)}\in W$, $|f(zu^{k(n)}x)-f(zx)|\le\eta$ for every $z\in U$ and $x\in X$, by the choice of $W$, so (because $A_n^{k(n)}\subseteq U$) $|g(u^{k(n)}x)-g(x)|\le\eta$ for every $x$ (446Ec). Accordingly

$$|\beta_2|\le\eta\int f(v^{-1}x)dx=\eta\int f(x)dx=\eta.$$

**(i)** But this means that

$$\beta\le2(|\beta_1|+|\beta_2|)\le2\eta(1+\alpha\mu(W_0U^2W_0))<\beta,$$

which is absurd. **X**

**(j)** Thus for every $n\in I$ there are an $x\in A_n$ and an $i\le k(n)$ such that $x^i\notin W$, and (ii) is true. This completes the proof.

***446G 'Groups with no small subgroups' (a) Definition** Let $X$ be a topological group. We say that $X$ **has no small subgroups** if there is a neighbourhood $U$ of the identity $e$ of $X$ such that the only subgroup of $X$ included in $U$ is $\{e\}$.

**(b)** If $X$ is a Hausdorff topological group and $U$ is a compact symmetric neighbourhood of the identity $e$ such that the only subgroup of $X$ included in $U$ is $\{e\}$, then $\{D_n(U):n\in\mathbb{N}\}$ is a base of neighbourhoods of $e$, where $D_n(U)=\{x:x\in X,\ x^i\in U$ for every $i\le n\}$. **P** By 446Dc, $\langle D_n(U)\rangle_{n\ge1}$ is a non-increasing sequence of compact neighbourhoods of $e$, and if $x\in\bigcap_{n\in\mathbb{N}}D_n(U)$ then $x^i\in U$ for every $i\in\mathbb{N}$; as $U^{-1}=U$, $U$ includes the subgroup $\{x^i:i\in\mathbb{Z}\}$, so $x=e$. Thus $\bigcap_{n\in\mathbb{N}}D_n(U)=\{e\}$ and $\{D_n(U):n\in\mathbb{N}\}$ is a base of neighbourhoods of $e$ (4A2Gd). **Q**

**(c)** In particular, a locally compact Hausdorff topological group with no small subgroups is metrizable (4A5Q).

***446H Lemma** Let $X$ be a locally compact Hausdorff topological group. For $A \subseteq X$, $n \in \mathbb{N}$ set $D_n(A) = \{x : x^i \in A \text{ for every } i \leq n\}$. If $U \subseteq X$ is a compact symmetric neighbourhood of the identity which does not include any subgroup of $X$ other than $\{e\}$, then there is an $r \geq 1$ such that $D_{rn}(U)^n \subseteq U$ for every $n \in \mathbb{N}$.

**proof ?** Suppose, if possible, that for every $r \geq 1$ there is an $n_r \in \mathbb{N}$ such that $D_{rn_r}(U)^{n_r} \not\subseteq U$. Of course $n_r \geq 1$. Set $A_0 = U$ and $A_r = D_{rn_r}(U)$ for $r \geq 1$. Note that $D_{n_1}(U) \subseteq A_0$ but $D_{n_1}(U)^{n_1} \not\subseteq U$, so $A_0^{n_1} \not\subseteq U$. We therefore have, for every $r \in \mathbb{N}$, a $k_r$ such that $A_r^{k(r)} \subseteq U$ but $A_r^{k(r)+1} \not\subseteq U$. Also, by 446Gb, every neighbourhood of $e$ includes all but finitely many of the $A_r$. We can therefore apply 446F to the sequence $\langle A_r \rangle_{r\in\mathbb{N}}$. Of course $k(r) \geq 1$ for every $r$, while $k(r) < n_r$ for $r \geq 1$. Since $U$ includes no non-trivial subgroup, (i) of 446F is impossible, and we are left with (ii). Let $W$, $I$ be as declared there, so that $A_r \not\subseteq D_{k(r)}(W)$ for every $r \in I$. There must be some $m \geq 1$ such that $D_m(U) \subseteq W$ (446Gb). Take $r \in I$ such that $r \geq m$; then $rn_r \geq mk(r)$, so

$$A_r = D_{rn_r}(U) \subseteq D_{mk(r)}(U) \subseteq D_{k(r)}(D_m(U))$$

(446D(b-iv))

$$\subseteq D_{k(r)}(W),$$

which is impossible. **X**

***446I Lemma** Let $X$ be a locally compact Hausdorff topological group and $U$ a compact symmetric neighbourhood of the identity in $X$ such that $U$ does not include any subgroup of $X$ other than $\{e\}$. For $n \in \mathbb{N}$, set $D_n(U) = \{x : x^i \in U \text{ for every } i \leq n\}$, and let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Suppose that $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in $X$ such that $x_n \in D_n(U)$ for every $n \in \mathbb{N}$. Then we have a continuous homomorphism $q : \mathbb{R} \to X$ defined by setting $q(t) = \lim_{n\to\mathcal{F}} x_n^{i(n)}$ whenever $\langle i(n) \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n} = t$ in $\mathbb{R}$.

**proof (a)** If $\langle i(n) \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathbb{N}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n}$ is defined in $\mathbb{R}$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in $X$. **P** There is some $m \in \mathbb{N}$ such that $m > \lim_{n\to\mathcal{F}} \frac{i(n)}{n}$, so that $J = \{n : i(n) \leq mn\} \in \mathcal{F}$; but if $n \in J$, then

$$x_n \in D_n(U) \subseteq D_{mn}(U^m) \subseteq D_{i(n)}(U^m),$$

by 446D(b-v), and $x_n^{i(n)} \in U^m$. But this means that $\mathcal{F}$ contains $\{n : x_n^{i(n)} \in U^m\}$; as $U^m$ is compact, $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in $X$. **Q**

More generally, if $\langle i(n) \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n}$ is defined in $\mathbb{R}$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in $X$. **P** At least one of $\{n : i(n) \geq 0\}$, $\{n : i(n) \leq 0\}$ belongs to $\mathcal{F}$. In the former case, $\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{\max(0,i(n))}$ is defined; in the latter case,

$$\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} (x_n^{\max(0,-i(n))})^{-1} = (\lim_{n\to\mathcal{F}} x_n^{\max(0,-i(n))})^{-1}$$

is defined. **Q**

**(b)** If $V$ is any neighbourhood of $e$, there is a $\delta > 0$ such that $\lim_{n\to\mathcal{F}} x_n^{i(n)} \in V$ whenever $\langle i(n) \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} |\frac{i(n)}{n}| \leq \delta$. **P** By 446Gb, there is an $m \geq 1$ such that $D_m(U) \subseteq V$. Take $\delta < \frac{1}{m}$. If $\lim_{n\to\mathcal{F}} |\frac{i(n)}{n}| \leq \delta$, then $J = \{n : m|i(n)| \leq n\} \in \mathcal{F}$. But if $n \in J$, then

$$x_n \in D_n(U) \subseteq D_{m|i(n)|}(U) \subseteq D_{|i(n)|}(D_m(U))$$

and $x_n^{|i(n)|} \in D_m(U)$; since $D_m(U)$, like $U$, is symmetric (446D(b-vi)), $x_n^{i(n)} \in D_m(U)$. This is true for every $n \in J$, so $\lim_{n\to\mathcal{F}} x_n^{i(n)} \in D_m(U) \subseteq V$, as required. **Q**

**(c)** It follows at once that if $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n} = 0$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)} = e$. Consequently, if $\langle i(n)\rangle_{n\in\mathbb{N}}$, $\langle j(n)\rangle_{n\in\mathbb{N}}$ are sequences in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n}$ and $\lim_{n\to\mathcal{F}} \frac{j(n)}{n}$ both exist in $\mathbb{R}$ and are equal, then $\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)}$. **P** Set $k(n) = i(n) - j(n)$. Then $\lim_{n\to\mathcal{F}} x_n^{k(n)} = e$ because $\lim_{n\to\mathcal{F}} \frac{k(n)}{n} = 0$. But now

$$\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)} x_n^{k(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)}. \text{ **Q**}$$

**(d)** We do therefore have a function $q : \mathbb{R} \to X$ defined by the given formula. Now $q(s+t) = q(s)q(t)$ for all $s$, $t \in \mathbb{R}$. **P** Take sequences $\langle i(n)\rangle_{n\in\mathbb{N}}$, $\langle j(n)\rangle_{n\in\mathbb{N}}$ such that $s = \lim_{n\to\infty} \frac{i(n)}{n}$, $t = \lim_{n\to\infty} \frac{j(n)}{n}$; then $s + t = \lim_{n\to\infty} \frac{i(n)+j(n)}{n}$, so

$$q(s+t) = \lim_{n\to\mathcal{F}} x_n^{i(n)+j(n)} = \lim_{n\to\mathcal{F}} x_n^{i(n)} x_n^{j(n)} = q(s)q(t). \text{ **Q**}$$

**(e)** Thus $q$ is a homomorphism. Finally, (b) shows that it is continuous at 0, so it must be continuous (4A5Fa).

***446J Lemma** Let $X$ be a locally compact Hausdorff topological group with no small subgroups. Then there is a neighbourhood $V$ of the identity $e$ such that $x = y$ whenever $x$, $y \in V$ and $x^2 = y^2$.

**proof** Let $U$ be a symmetric compact neighbourhood of $e$ not including any subgroup of $X$ except $\{e\}$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of neighbourhoods of $e$ comprising a base of neighbourhoods of $e$ (446Gc), and with $U_0 = U$.

**?** Suppose, if possible, that for each $n \in \mathbb{N}$ there are distinct $x_n$, $y_n \in U_n$ such that $x_n^2 = y_n^2$. Set $a_n = x_n^{-1}y_n$; then

$$x_n^{-1}a_nx_n = x_n^{-2}y_nx_n = y_n^{-1}x_n = a_n^{-1}.$$

Accordingly

$$x_n^{-1}a_n^m x_n = (x_n^{-1}a_nx_n)^m = a_n^{-m}$$

for every $m \in \mathbb{N}$.

Since $U$ includes no non-trivial subgroup, and $a_n \ne e$, there is a $k(n) \in \mathbb{N}$ such that $a_n^i \in U$ for $i \le k(n)$ and $a_n^{k(n)+1} \notin U$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$; then $a = \lim_{n\to\mathcal{F}} a_n^{k(n)}$ is defined and belongs to $U$. Also, because $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ both converge to $e$, so does $\langle a_n\rangle_{n\in\mathbb{N}}$, and

$$a = \lim_{n\to\mathcal{F}} a_n^{k(n)+1} \in X \setminus \operatorname{int} U;$$

thus $a$ cannot be $e$.

However, $x_n^{-1}a_n^{k(n)}x_n = a_n^{-k(n)}$ for each $n$, so

$$e^{-1}ae = \lim_{n\to\mathcal{F}} x_n^{-1}a_n^{k(n)}x_n = \lim_{n\to\mathcal{F}}(a_n^{k(n)})^{-1} = a^{-1}.$$

So $a = a^{-1}$ and $\{e, a\}$ is a non-trivial subgroup of $X$ included in $U$, which is supposed to be impossible. **X**

Thus some $U_n$ serves for $V$.

***446K Lemma** Let $X$ be a locally compact Hausdorff topological group with no small subgroups. For $A \subseteq X$ set $D_n(A) = \{x : x^i \in A \text{ for every } i \le n\}$. Then there is a compact symmetric neighbourhood $U$ of the identity $e$ such that whenever $V$ is a neighbourhood of $e$ there are an $n_0 \in \mathbb{N}$ and a neighbourhood $W$ of $e$ such that whenever $n \ge n_0$, $x \in D_n(U)$, $y \in D_n(U)$ and $x^ny^n \in W$, then $xy \in D_n(V)$.

**proof (a)** Let $U_0$ be a compact symmetric neighbourhood of $e$ such that $(\alpha)$ $U_0^3$ includes no subgroup of $X$ other than $\{e\}$ $(\beta)$ whenever $x$, $y \in U_0$ and $x^2 = y^2$, then $x = y$; such a neighbourhood exists by 446J. Let

$r \ge 1$ be such that $D_{rn}(U_0)^n \subseteq U_0$ for every $n \in \mathbb{N}$ (446H). Let $U$ be a compact symmetric neighbourhood of $e$ such that $U^r \subseteq U_0$. In this case $D_n(U) \subseteq D_{rn}(U_0)$ for every $n$, by 446D(b-v). So $D_n(U)^n \subseteq U_0$ for every $n$.

**(b)** Fix a left Haar measure $\mu$ on $X$. Let $f : X \to [0,\infty[$ be a continuous function such that $\int f(x)dx = 1$ and $f(x) = 0$ for $x \in X \setminus U_0$. Set $\alpha = \sup_{x\in X}|f(x)|$, $\beta = \int f(x)^2 dx$, so that $\alpha$ is finite and $\beta > 0$. For $n \ge 1$, set

$$f_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \sup\{f(yx) : y \in D_n(U)^i\}$$

for $x \in X$. Because $D_n(U)^n \subseteq U_0$, we can apply 446E to see that, for each $n$,

(i) $f_n \ge f$,

(ii) $f_n(x) = 0$ if $x \notin U_0^2$,

(iii) $|f_n(ax) - f_n(x)| \le \frac{j\alpha}{n}$ if $j \in \mathbb{N}$, $a \in D_n(U)^j$ and $x \in X$,

(iv) for any $x$, $z \in X$, $|f_n(x) - f_n(z)| \le \sup_{y\in U_0}|f(yx) - f(yz)|$.

It follows that

(v) for any $\epsilon > 0$ there is a neighbourhood $W$ of $e$ such that $|f_n(ax) - f_n(bx)| \le \epsilon$ whenever $a$, $b$, $x \in X$, $n \in \mathbb{N}$ and $ab^{-1} \in W$

(4A5Pa again).

**(c)** It will help to have the following fact available. Suppose we are given sequences $\langle x_n\rangle_{n\in\mathbb{N}}$, $\langle y_n\rangle_{n\in\mathbb{N}}$ such that $x_n$ and $y_n$ belong to $D_n(U)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n^n y_n^n = e$. Write

$$\gamma_n = \sup\{|f_n(y_n^j x) - f_n(x_n^{-j}x)| : j \le n,\ x \in X\}$$

for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty}\gamma_n = 0$. **P?** Otherwise, there is an $\eta > 0$ such that $J = \{n : \gamma_n > \eta\}$ is infinite. Let $W$ be a neighbourhood of $e$ such that $|f_n(ax) - f_n(bx)| \le \eta$ whenever $n \in \mathbb{N}$, $x$, $a$, $b \in X$ and $ab^{-1} \in W$ ((b-v) above); let $W'$ be a neighbourhood of $e$ such that $ab \in W$ whenever $a$, $b \in U$ and $ba \in W'$ (4A5Ej). Then for each $n \in J$ there must be a $j(n) \le n$ such that $y_n^{j(n)} x_n^{j(n)} \notin W$, while $x_n^{j(n)}$ and $y_n^{j(n)}$ both belong to $U$, so that $x^{j(n)}y^{j(n)} \notin W'$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ containing $J$. By 446I, there are continuous homomorphisms $q$, $\tilde{q}$ from $\mathbb{R}$ to $X$ such that $q(t) = \lim_{n\to\mathcal{F}} x_n^{-i(n)}$, $\tilde{q}(t) = \lim_{n\to\mathcal{F}} y_n^{i(n)}$ whenever $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} \frac{i(n)}{n} = t$ in $\mathbb{R}$. (Of course $x_n^{-1} \in D_n(U)$ for every $n$, by 446D(b-vi).) Setting $t_0 = \lim_{n\to\mathcal{F}} \frac{j(n)}{n} \in [0,1]$,

$$q(-t_0)\tilde{q}(t_0) = \lim_{n\to\mathcal{F}} x_n^{j(n)} y_n^{j(n)} \notin \operatorname{int} W',$$

so $q(t_0) \ne \tilde{q}(t_0)$ and $q \ne \tilde{q}$. But

$$q(-1)\tilde{q}(1) = \lim_{n\to\mathcal{F}} x_n^n y_n^n = e,$$

so $q(1) = \tilde{q}(1)$. Now if $0 \le i(n) \le n$, then $x_n^{-i(n)} \in D_n(U)^n \subseteq U_0$; so if $0 \le t \le 1$, $q(t) \in U_0$. Similarly, $\tilde{q}(t) \in U_0$ whenever $t \in [0,1]$. But recall that $U_0$ was chosen so that if $x$, $y \in U_0$ and $x^2 = y^2$ then $x = y$. In particular, since $q(\frac{1}{2})$ and $\tilde{q}(\frac{1}{2})$ both belong to $U_0$, and their squares $q(1)$, $\tilde{q}(1)$ are equal, $q(\frac{1}{2}) = \tilde{q}(\frac{1}{2})$. Repeating this argument, we see that $q(2^{-k}) = \tilde{q}(2^{-k})$ for every $k \in \mathbb{N}$, so that $q(2^{-k}i) = \tilde{q}(2^{-k}i)$ for every $k \in \mathbb{N}$, $i \in \mathbb{Z}$; since $q$ and $\tilde{q}$ are supposed to be continuous, they must be equal; but $q(t_0) \ne \tilde{q}(t_0)$. **XQ**

**(d)** Now let $V$ be any neighbourhood of $e$.

**?** Suppose, if possible, that for every neighbourhood $W$ of $e$ and $n_0 \in \mathbb{N}$ there are $n \ge n_0$ and $x$, $y \in D_n(U)$ such that $x^n y^n \in W$ but $xy \notin D_n(V)$. For $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ and $\tilde{x}_k$, $\tilde{y}_k \in D_{n_k}(U)$ such that $\tilde{x}_k^{n_k}\tilde{y}_k^{n_k} \in D_k(U)$ but $\tilde{x}_k\tilde{y}_k \notin D_{n_k}(V)$, and $n_k > n_{k-1}$ if $k \ge 1$. Now we know that $\langle D_k(U)\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence constituting a base of neighbourhoods of $e$ (446Gb), so $\lim_{k\to\infty}\tilde{x}_k^{n_k}\tilde{y}_k^{n_k} = e$. Set $J = \{n_k : k \in \mathbb{N}\}$. For $n = n_k$, set $x_n = \tilde{x}_k$, $y_n = \tilde{y}_k$; for $n \in \mathbb{N} \setminus J$, set $x_n = y_n = e$. Then $x_n$, $y_n \in D_n(U)$ for every $n \in \mathbb{N}$ and $\langle x_n^n y_n^n\rangle_{n\in\mathbb{N}} \to e$ as $n \to \infty$, while $x_n y_n \notin D_n(V)$ for $n \in J$.

We know from (c) that

$$\gamma_n = \sup\{|f_n(y_n^j x) - f_n(x_n^{-j}x)| : j \le n,\ x \in X\} \to 0$$

as $n \to \infty$; for future reference, take a sequence $\langle j(n) \rangle_{n \ge 1}$ such that $1 \le j(n) \le n$ for every $n \ge 1$, $\lim_{n\to\infty} \frac{j(n)}{n} = 0$ and $\lim_{n\to\infty} \frac{n\gamma_n}{j(n)} = 0$.

**(e)** Let $l \ge 1$ be such that $D_l(U_0^3) \subseteq V$ (446Gb again), and set $K = U_0^{2l+1}$. Then $D_{nl}(U_0^3) \subseteq D_n(V)$ for every $n$ (446D(b-vii)). So $x_n y_n \notin D_{nl}(U_0^3)$ for $n \in J$. For each $n \in J$ choose $m(n) \le nl$ such that $(x_n y_n)^{m(n)} \notin U_0^3$; for $n \in \mathbb{N} \setminus J$, set $m(n) = 1$. Then $(x_n y_n)^{-m(n)} x \notin U_0^2$ for any $x \in U_0$ and $n \in J$, and $f_n((x_n y_n)^{-m(n)} x) f(x) = 0$ for any $x \in X$ and $n \in J$. So

$$|\int (f_n((x_n y_n)^{-m(n)} x) - f_n(x)) f(x) dx| = \int f_n(x) f(x) dx \ge \beta$$

for $n \in J$, because $f_n \ge f$ ((b-i) above).

**(f)** Of course $m(n) > 0$ for every $n \in J$, therefore for every $n$. So we can set

$$g_n(x) = \frac{1}{m(n)} \sum_{i=0}^{m(n)-1} f((x_n y_n)^i x)$$

for $x \in X$ and $n \in \mathbb{N}$. Note that $g_n$, like $f$, is non-negative, and also that $\int g_n(x) dx = \int f(x) dx = 1$. We need to know that $g_n(x) = 0$ if $x \notin K$; this is because $(x_n y_n)^i \in D_n(U)^{2nl} \subseteq U_0^{2l}$ for every $i \le m(n)$, while $f(x) = 0$ if $x \notin U_0$, and $U_0$ is symmetric.

We also have

$$\begin{aligned} |g_n(ax) - g_n(x)| &\le \sup_{i < m(n)} |f((x_n y_n)^i ax) - f((x_n y_n)^i x)| \\ &\le \sup\{|f(wax) - f(wx)| : w \in U_0^{2l}\} \end{aligned}$$

for every $n \in \mathbb{N}$ and $a, x \in X$ (cf. 446Ec), so for every $\eta > 0$ there must be a neighbourhood $W$ of $e$ such that $|g_n(ax) - g(x)| \le \eta$ whenever $n \in \mathbb{N}$, $a \in W$ and $x \in X$, by 4A5Pa once more. Since also $g_n(ax) = g_n(x) = 0$ if $a \in U_0$ and $x \notin U_0^{-1} K$, and $U_0^{-1} K$ has finite measure, we see that for every $\eta > 0$ there is a neighbourhood $W$ of $e$ such that $\int |g_n(ax) - g(x)| dx \le \eta$ for every $a \in W$, $n \in \mathbb{N}$.

**(g)** Returning to the formula in (e), we see that, for any $n \in J$,

$$\begin{aligned} \beta &\le \Bigl|\int \bigl(f_n((x_n y_n)^{-m(n)} x) - f_n(x)\bigr) f(x) dx\Bigr| \\ &= \Bigl|\sum_{i=0}^{m(n)-1} \int \bigl(f_n((x_n y_n)^{-i-1} x) - f_n((x_n y_n)^{-i} x)\bigr) f(x) dx\Bigr| \\ &= \Bigl|\sum_{i=0}^{m(n)-1} \int \bigl(f_n((x_n y_n)^{-1} x) - f_n(x)\bigr) f((x_n y_n)^i x) dx\Bigr| \\ &= m(n)\Bigl|\int \bigl(f_n((x_n y_n)^{-1} x) - f_n(x)\bigr) g_n(x) dx\Bigr| \\ &\le ln\Bigl|\int \bigl(f_n(y_n^{-1} x_n^{-1} x) - f_n(x)\bigr) g_n(x) dx\Bigr| \\ &\le ln\Bigl|\int \bigl(f_n(y_n^{-1} x) - f_n(x) - f_n(y_n^{-1} x_n^{-1} x) + f_n(x_n^{-1} x)\bigr) g_n(x) dx\Bigr| \\ &\qquad + ln\Bigl|\int \bigl(2f_n(x) - f_n(x_n^{-1} x) - f_n(y_n^{-1} x)\bigr) g_n(x) dx\Bigr|. \end{aligned}$$

Next,

$$\begin{aligned} &j(n)\bigl(2f_n(x) - f_n(x_n^{-1} x) - f_n(y_n^{-1} x)\bigr) \\ &\quad = j(n)\bigl(f_n(x) - f_n(x_n^{-1} x)\bigr) - f_n(x) + f_n(x_n^{-j(n)} x) \\ &\qquad + j(n)\bigl(f_n(x) - f_n(y_n^{-1} x)\bigr) - f_n(x) + f_n(y_n^{-j(n)} x) \\ &\qquad + 2f_n(x) - f_n(x_n^{-j(n)} x) - f_n(y_n^{-j(n)} x) \end{aligned}$$

for every $x$, and finally

$$\begin{aligned}
&\int\bigl(2f_n(x)-f_n(x_n^{-j(n)}x)-f_n(y_n^{-j(n)}x)\bigr)g_n(x)dx\\
&\qquad=\int\bigl(f_n(x)-f_n(y_n^{-j(n)}x)\bigr)g_n(x)dx-\int\bigl(f_n(y_n^{j(n)}x)-f_n(x)\bigr)g_n(x)dx\\
&\qquad\qquad+\int\bigl(f_n(y_n^{j(n)}x)-f_n(x_n^{-j(n)}x)\bigr)g_n(x)dx\\
&\qquad=\int\bigl(f_n(y_n^{j(n)}x)-f_n(x)\bigr)g_n(y_n^{j(n)}x)dx-\int\bigl(f_n(y_n^{j(n)}x)-f_n(x)\bigr)g_n(x)dx\\
&\qquad\qquad+\int\bigl(f_n(y_n^{j(n)}x)-f_n(x_n^{-j(n)}x)\bigr)g_n(x)dx\\
&\qquad=\int\bigl(f_n(y_n^{j(n)}x)-f_n(x)\bigr)\bigl(g_n(y_n^{j(n)}x)-g_n(x)\bigr)dx\\
&\qquad\qquad+\int\bigl(f_n(y_n^{j(n)}x)-f_n(x_n^{-j(n)}x)\bigr)g_n(x)dx.
\end{aligned}$$

So if we write

$$\beta_{1n}=ln\int\Bigl(f_n(y_n^{-1}x)-f_n(x)-f_n(y_n^{-1}x_n^{-1}x)+f_n(x_n^{-1}x)\Bigr)g_n(x)dx,$$

$$\beta_{2n}=\frac{ln}{j(n)}\int\Bigl(j(n)(f_n(x)-f_n(x_n^{-1}x))-f_n(x)+f_n(x_n^{-j(n)}x)\Bigr)g_n(x)dx,$$

$$\beta_{3n}=\frac{ln}{j(n)}\int\Bigl(j(n)(f_n(x)-f_n(y_n^{-1}x))-f_n(x)+f_n(y_n^{-j(n)}x)\Bigr)g_n(x)dx,$$

$$\beta_{4n}=\frac{ln}{j(n)}\int\Bigl(f_n(y_n^{j(n)}x)-f_n(x)\Bigr)\Bigl(g_n(y_n^{j(n)}x)-g_n(x)\Bigr)dx,$$

$$\beta_{5n}=\frac{ln}{j(n)}\int\Bigl(f_n(y_n^{j(n)}x)-f_n(x_n^{-j(n)}x)\Bigr)g_n(x)dx$$

for $n\ge 1$, we have

$$|\beta_{1n}|+|\beta_{2n}|+|\beta_{3n}|+|\beta_{4n}|+|\beta_{5n}|\ge\beta$$

for every $n\in J$.

**(h)** Now $\beta_{1n}\to 0$ as $n\to\infty$. **P**

$$\begin{aligned}
&|\int\bigl(f_n(y_n^{-1}x)-f_n(x)-f_n(y_n^{-1}x_n^{-1}x)+f_n(x_n^{-1}x)\bigr)g_n(x)dx|\\
&\qquad=|\int\bigl(f_n(y_n^{-1}x)-f_n(x)\bigr)g_n(x)dx-\int\bigl(f_n(y_n^{-1}x_n^{-1}x)-f_n(x_n^{-1}x)\bigr)g_n(x)dx|\\
&\qquad=|\int\bigl(f_n(y_n^{-1}x)-f_n(x)\bigr)g_n(x)dx-\int\bigl(f_n(y_n^{-1}x)-f_n(x)\bigr)g_n(x_nx)dx|\\
&\qquad=|\int\bigl(f_n(y_n^{-1}x)-f_n(x)\bigr)\bigl(g_n(x)-g_n(x_nx)\bigr)dx|\\
&\qquad\le\frac{\alpha}{n}\int|g_n(x)-g_n(x_nx)|dx
\end{aligned}$$

by (b-iii). So

$$|\beta_{1n}|\le\alpha l\int|g_n(x)-g_n(x_nx)|dx\to 0$$

as $n\to\infty$, by (f), since surely $\langle x_n\rangle_{n\in\mathbb{N}}\to e$. **Q**

**(i)** Now look at $\beta_{2n}$. If we set

$$\gamma'_n = \sup\{\int |g_n(ax) - g(x)|dx : a \in D_n(U)^{j(n)}\},$$

then $\gamma'_n \to 0$ as $n \to \infty$. **P** Given $\epsilon > 0$, there is a neighbourhood $W$ of $e$ such that $\int |g_n(ax) - g_n(x)|dx \le \epsilon$ whenever $n \ge 1$ and $a \in W$, as noted at the end of (f). Let $p$ be such that $D_p(U_0) \subseteq W$. Then, for all $n$ large enough, $pj(n) \le n$, so that $D_n(U)^{pj(n)} \subseteq U_0$ and $D_n(U)^{j(n)} \subseteq D_p(U_0) \subseteq W$ (446D(b-v)) and $\int |g_n(ax) - g(x)|dx \le \epsilon$ for every $a \in D_n(U)^{j(n)}$. **Q**

We have

$$\begin{aligned}
&|\int \big(j(n)(f_n(x) - f_n(x_n^{-1}x)) - f_n(x) + f_n(x_n^{-j(n)}x)\big)g_n(x)dx| \\
&= |\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x) - f_n(x_n^{-i}x) + f_n(x_n^{-i-1}x)\big)g_n(x)dx| \\
&= |\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x)dx - \int \big(f_n(x_n^{-i}x) - f_n(x_n^{-i-1}x)\big)g_n(x)dx| \\
&= |\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x)dx - \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x_n^i x)dx| \\
&= |\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)\big(g_n(x) - g_n(x_n^i x)\big)dx| \\
&\le \sum_{i=0}^{j(n)-1} \int |f_n(x) - f_n(x_n^{-1}x)||g_n(x) - g_n(x_n^i x)|dx \le j(n)\frac{\alpha}{n}\gamma'_n.
\end{aligned}$$

So

$$|\beta_{2n}| \le l\alpha\gamma'_n \to 0$$

as $n \to \infty$. Similarly, $\langle\beta_{3n}\rangle_{n\ge1} \to 0$.

**(j)** As for $\beta_{4n}$, we have

$$\int |f_n(y_n^{j(n)}x) - f_n(x)||g_n(y_n^{j(n)}x) - g_n(x)|dx \le \frac{\alpha j(n)}{n}\gamma'_n,$$

putting (b-iii) and the definition of $\gamma'_n$ together. So

$$|\beta_{4n}| \le l\alpha\gamma'_n \to 0$$

as $n \to \infty$.

**(k)** We come at last to $\beta_{5n}$. Here, for every $n \ge 1$,

$$\begin{aligned}
|\int \big(f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big)g_n(x)dx| &\le \int \big|f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big|g_n(x)dx \\
&\le \gamma_n \int g_n(x)dx = \gamma_n
\end{aligned}$$

by the definition of $\gamma_n$ in (c) above. So

$$|\beta_{5n}| \le l\frac{n}{j(n)}\gamma_n \to 0$$

by the choice of the $j(n)$.

**(l)** Thus $\beta_{in} \to 0$ as $n \to \infty$ for every $i$. But this is impossible, because $0 < \beta \le \sum_{i=1}^5 |\beta_{in}|$ for every $n \in J$. **X**

This contradiction shows that we must be able to find a neighbourhood $W$ of $e$ and an $n_0 \in \mathbb{N}$ such that $xy \in D_n(V)$ whenever $n \ge n_0$, $x$, $y \in D_n(U)$ and $x^n y^n \in W$; as $V$ is arbitrary, $U$ has the property required.

***446L Definition** Let $X$ be a topological group. A **$B$-sequence** in $X$ is a non-increasing sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of closed neighbourhoods of the identity, constituting a base of neighbourhoods of the identity, such that there is some $M$ such that for every $n\in\mathbb{N}$ the set $V_nV_n^{-1}$ can be covered by at most $M$ left translates of $V_n$.

***446M Proposition** Let $X$ be a locally compact Hausdorff topological group with no small subgroups. Then it has a $B$-sequence.

**proof (a)** For $A\subseteq X$ set $D_n(A)=\{x:x^i\in A$ for every $i\le n\}$. We know from 446K that there is a compact symmetric neighbourhood $U$ of the identity $e$ such that whenever $V$ is a neighbourhood of $e$ there are an $n_0\in\mathbb{N}$ and a neighbourhood $W$ of $e$ such that whenever $n\ge n_0$, $x\in D_n(U)$, $y\in D_n(U)$ and $x^ny^n\in W$, then $xy\in D_n(V)$. Shrinking $U$ if necessary, we may suppose that $U$ includes no subgroup of $X$ other than $\{e\}$, so that there is an $r\ge 1$ such that $D_{rn}(U)^n\subseteq U$ for every $n\in\mathbb{N}$ (446H).

Let $V$ be a closed symmetric neighbourhood of $e$ such that $V^{2r}\subseteq U$. Then $D_n(V)^2\subseteq D_n(U)$ for every $n\in\mathbb{N}$. **P** $D_n(V)\subseteq D_{2rn}(U)$, by 446D(b-v), so

$$(D_n(V)^2)^n\subseteq D_{2rn}(U)^{2n}\subseteq U,$$

and $D_n(V)^2\subseteq D_n(U)$ (446D(b-v) again). **Q** Take $n_0\in\mathbb{N}$ and a neighbourhood $W$ of $e$ such that whenever $n\ge n_0$, $x$, $y\in D_n(U)$ and $x^ny^n\in W^{-1}W$, then $xy\in D_n(V)$.

**(b)** Let $M$ be so large that $U$ can be covered by $M$ left translates of $W$. Then for any $n\ge n_0$, $D_n(V)D_n(V)^{-1}=D_n(V)^2$ can be covered by $M$ left translates of $D_n(V)$.

**P** Let $z_0,\dots,z_{M-1}$ be such that $U\subseteq\bigcup_{i\le M}z_iW$. For each $i<M$, set $A_i=\{x:x\in D_n(U),\,x^n\in z_iW\}$; if $A_i\ne\emptyset$ choose $x_i\in A_i$; otherwise, set $x_i=e$.

For any $y\in D_n(V)^2$, $y\in D_n(U)$, so $y^n\in U$ and there is some $i<M$ such that $y\in A_i$. In this case $x_i$ also belongs to $A_i$. Now $z_i^{-1}y^n$ and $z_i^{-1}x_i^n$ both belong to $W$, so $x_i^{-n}y^n$ belongs to $W^{-1}W$, and $x_i^{-1}y\in D_n(V)$, by the choice of $W$ and $n_0$. But this means that $y\in x_iD_n(V)$. As $y$ is arbitrary, $D_n(V)^2\subseteq\bigcup_{i<M}x_iD_n(V)$ is covered by $M$ left translates of $D_n(V)$. **Q**

**(c)** But this means that $\langle D_{n+n_0}(V)\rangle_{n\in\mathbb{N}}$ is a $B$-sequence in $X$. (It constitutes a base of neighbourhoods of $e$ by 446Gb, as usual.)

***446N Proposition** Let $X$ be a locally compact Hausdorff topological group with a faithful finite-dimensional representation. Then it has a $B$-sequence.

**proof (a)** Let $\phi:X\to GL(r,\mathbb{R})$ be a faithful finite-dimensional representation. Identifying $M_r$ with the Banach algebra $\mathsf{B}=\mathsf{B}(\mathbb{R}^r;\mathbb{R}^r)$, where $\mathbb{R}^r$ is given the Euclidean norm, we see that $GL(r,\mathbb{R})$ is an open subset of $\mathsf{B}$ (4A6H). Note also that the operator norm $\|\;\|$ of $\mathsf{B}$ is equivalent to its 'Euclidean' norm corresponding to an identification with $\mathbb{R}^{r^2}$, that is, writing $\|T\|_{HS}=\sqrt{\sum_{i=1}^r\sum_{j=1}^r\tau_{ij}^2}$ if $T=\langle\tau_{ij}\rangle_{1\le i,j\le r}$, $\|\;\|_{HS}$ is equivalent to $\|\;\|$. (See the inequalities in 262H.) In particular, all the balls $B(T,\delta)=\{S:\|S-T\|\le\delta\}$ are closed for the Euclidean norm (4A2Lj). If we write $\mu_L$ for Lebesgue measure on $\mathsf{B}$, identified with $\mathbb{R}^{r^2}$, and set $\gamma=\mu_LB(\mathbf{0},1)$, then $0<\gamma<\infty$ (because $B(\mathbf{0},1)$ includes, and is included in, non-trivial Euclidean balls) and $\mu_LB(T,\delta)=\delta^{r^2}\gamma$ for every $T\in\mathsf{B}$ and $\delta\ge 0$ (using 263A, or otherwise).

**(b)** We need to recall a basic inequality concerning inversion in Banach algebras. If $T\in\mathsf{B}$ and $\|T-I\|\le\frac{1}{2}$, then $T$ is invertible and

$$\|T^{-1}-I\|\le\frac{\|T-I\|}{1-\|T-I\|}\le 1$$

(4A6H), so $\|T^{-1}\|\le 2$.

**(c)** Now let $V$ be a compact neighbourhood of the identity $e$ of $X$. Let $V_1$ be a neighbourhood of $e$ such that $(V_1V_1^{-1})^{-1}V_1V_1^{-1}\subseteq V$. For $\delta>0$, set $U_\delta=\{x:x\in V,\,\|\phi(x)-I\|\le\delta\}$. Then each $U_\delta$ is a compact neighbourhood of $e$, because $\phi$ is continuous. Also

$$\bigcap_{\delta>0}U_\delta=\{x:x\in V,\,\phi(x)=I\}=\{e\}.$$

So $\{U_\delta : \delta > 0\}$ is a base of neighbourhoods of $e$ (4A2Gd), and there is a $\delta_1 > 0$ such that $U_{\delta_1} \subseteq V_1$; of course we may suppose that $\delta_1 \le \frac{1}{8}$.

**(d)** If $\delta \le \frac{1}{2}$ and $x \in U_\delta U_\delta^{-1}$, then $\|\phi(x) - I\| \le 4\delta$. **P** Express $x$ as $yz^{-1}$ where $y$, $z \in U_\delta$. Then $\|\phi(z) - I\| \le \frac{1}{2}$, so $\|\phi(z^{-1})\| \le 2$ and

$$\|\phi(x) - I\| = \|(\phi(y) - \phi(z))\phi(z^{-1})\| \le 2\|\phi(y) - \phi(z)\| \le 4\delta. \text{ } \mathbf{Q}$$

**(e)** Now if $\delta \le \delta_1$, $U_\delta U_\delta^{-1}$ can be covered by at most $m = 17^{r^2}$ left translates of $U_\delta$. **P?** Suppose, if possible, otherwise. Then we can choose $x_0, \dots, x_m \in U_\delta U_\delta^{-1}$ such that $x_j \notin x_i U_\delta$ whenever $i < j \le m$. If $i < j \le m$, then

$$x_i^{-1}x_j \in (U_\delta U_\delta^{-1})^{-1}U_\delta U_\delta^{-1} \subseteq (V_1V_1^{-1})^{-1}V_1V_1^{-1} \subseteq V,$$

and $x_i^{-1}x_j \notin U_\delta$, so $\|\phi(x_i^{-1}x_j) - I\| > \delta$. Set $T_i = \phi(x_i)$ for each $i \le m$; then

$$\|T_i - I\| \le 4\delta \le \tfrac{1}{2}$$

for each $i$, by (d), while

$$\delta < \|\phi(x_i^{-1}x_j) - I\| = \|T_i^{-1}T_j - I\| \le \|T_i^{-1}\|\|T_j - T_i\| \le 2\|T_j - T_i\|$$

whenever $i < j \le m$. Write $B_i = B(T_i, \frac{\delta}{4})$ for each $i$; then all the $B_i$ are disjoint. But also they are all included in $B(I, \frac{17\delta}{4})$, so we have

$$(17^{r^2} + 1)\bigl(\tfrac{\delta}{4}\bigr)^{r^2}\gamma \le \bigl(\tfrac{17\delta}{4}\bigr)^{r^2}\gamma,$$

which is impossible. **XQ**

**(f)** Accordingly, setting $W_n = U_{2^{-n}\delta_1}$, $\langle W_n\rangle_{n\in\mathbb{N}}$ is a $B$-sequence in $X$.

***446O Theorem** Let $X$ be a locally compact Hausdorff topological group. Then it has an open subgroup $Y$ which has a compact normal subgroup $Z$ such that $Y/Z$ has no small subgroups.

**proof (a)** Let $U$ be a compact neighbourhood of the identity $e$ of $X$. Then there are a subgroup $Y_0$ of $X$, included in $U$, and a neighbourhood $W_0$ of $e$ such that every subgroup of $X$ included in $W_0$ is also included in $Y_0$.

**P(i)** To begin with (down to the end of (iii)) let us suppose that $X$ is metrizable. Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of closed symmetric neighbourhoods of $e$ running over a base of neighbourhoods of $e$, and such that $V_1^2 \subseteq V_0 \subseteq U$. For each $n \in \mathbb{N}$, set $A_n = \{x : x^i \in V_n \text{ for every } i \in \mathbb{N}\}$.

**(ii)?** Suppose, if possible, that $\bigcup_{k\in\mathbb{N}} A_n^k \not\subseteq U$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $k(n) \in \mathbb{N}$ be such that $A_n^{k(n)} \subseteq U$, $A_n^{k(n)+1} \not\subseteq U$. Then $\langle A_n\rangle_{n\in\mathbb{N}}$ and $U$ satisfy the conditions of 446F (because $A_m \subseteq V_n$ whenever $m \ge n$, and $\{V_n : n \in \mathbb{N}\}$ is a base of neighbourhoods of $e$). Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and set $Q = \lim_{n\to\mathcal{F}} A_n^{k(n)}$ in the space $\mathcal{C}$ of closed subsets of $X$ with the Fell topology.

If $W$ is any neighbourhood of $e$, there is an $n \in \mathbb{N}$ such that $V_n \subseteq W$, so that $x^i \in W$ whenever $x \in \bigcup_{m\ge n} A_m$ and $i \in \mathbb{N}$. Thus (ii) of 446F is not true, and $Q$ must be a closed subgroup of $X$ included in $U$ and meeting the boundary of $U$.

By 446C, there are an $r \in \mathbb{N}$ and a continuous homomorphism $\phi : Q \to GL(r, \mathbb{R})$ such that the kernel $Z$ of $\phi$ is included in $\operatorname{int} V_1$. Let $G \subseteq X$ be an open set including $Z$ and with closure disjoint from $(X \setminus \operatorname{int} V_1) \cup \{x : x \in Q, \|\phi(x) - I\| \ge \frac{1}{6}\}$; such can be found because $Z$ is compact and $X$ is regular (4A2F(h-ii)). Then $Z \subseteq G$, and any subgroup $Z'$ of $Q$ included in $\overline{G}$ has $\|\phi(x)^i - I\| \le \frac{1}{6}$ for every $x \in Z'$ and $i \in \mathbb{N}$, so that $Z' \subseteq Z$, by 4A6N. Set $V = \overline{G}$.

Since $V \subseteq U$ and $A_n^{k(n)+1} \not\subseteq U$ for every $n$, we can find $j(n) \le k(n)$ such that $A_n^{j(n)} \subseteq V$ and $A_n^{j(n)+1} \not\subseteq V$ for every $n$. Set $Q' = \lim_{n\to\mathcal{F}} A_n^{j(n)}$. As before, (ii) of 446F cannot be true of $Q'$, and $Q'$ must be a closed subgroup of $X$ meeting the boundary of $V$. Because $e \in A_n$, $A_n^{j(n)} \subseteq A_n^{k(n)}$ for every $n$, and $Q' \subseteq Q$, because $\{(E, F) : E \subseteq F\}$ is closed in $\mathcal{C}$ (4A2T(e-i)); also $Q' \subseteq V$, so $Q' \subseteq Z$. But $Z$ does not meet the boundary of $V$. **X**

**(iii)** So there is some $n \in \mathbb{N}$ such that $A_n^k \subseteq U$ for every $k \in \mathbb{N}$. Because $A_n^{-1} = A_n$, $Y_0 = \bigcup_{k\in\mathbb{N}} A_n^k$ is a subgroup of $X$. Any subgroup of $X$ included in $V_n$ is a subset of $A_n$ so is included in $Y_0$. Thus we have a pair $Y_0$, $W_0 = V_n$ of the kind required, at least when $X$ is metrizable.

**(iv)** Now suppose that $X$ is $\sigma$-compact. Let $U_1$ be a neighbourhood of $e$ such that $U_1^2 \subseteq U$. Then there is a closed normal subgroup $X_0$ of $X$ such that $X_0 \subseteq U_1$ and $X' = X/X_0$ is metrizable (4A5S). By (i)-(iii), there are a subgroup $Y_0'$ of $X'$, included in the image of $U_1$ in $X'$, and a neighbourhood $W_0'$ of the identity in $X'$ such that any subgroup of $X'$ included in $W_0'$ must also be included in $Y_0'$. Write $\pi : X \to X'$ for the canonical homomorphism and consider $Y_0 = \pi^{-1}[Y_0']$, $W_0 = \pi^{-1}[W_0']$. Then $W_0$ is a neighbourhood of $e$ and $Y_0$ is a subgroup of $X$ included in

$$\pi^{-1}[\pi[U_1]] = U_1 X_0 \subseteq U_1^2 \subseteq U.$$

And if $Z$ is any subgroup of $X$ included in $W_0$, then $\pi[Z] \subseteq W_0'$ so $\pi[Z] \subseteq Y_0'$ and $Z \subseteq Y_0$. Thus in this case also we have the result.

**(v)** Finally, for the general case, observe that $X$ has a $\sigma$-compact open subgroup $X_1$ (4A5El). So we can find a subgroup $Y_0$ of $X_1$, included in $U \cap X_1$, and a neighbourhood $W_0$ of the identity in $X_1$ such that any subgroup of $X_1$ included in $W_0$ is also included in $Y_0$. But of course $Y_0$ and $W_0$ also serve for $X$ and $U$.

This completes the proof of (a). **Q**

**(b)** Of course $\overline{Y_0}$ is a subgroup of $X$ (4A5Em); being included in $U$, it is compact. By 446C, there is a finite-dimensional representation $\phi : \overline{Y_0} \to GL(r, \mathbb{R})$, for some $r \in \mathbb{N}$, such that the kernel $Z$ of $\phi$ is included in $\operatorname{int} W_0$. Let $W_1$ be a neighbourhood of $e$ in $X$ such that $\|\phi(x) - I\| \le \frac{1}{6}$ for every $x \in W_1 \cap \overline{Y_0}$, and set $W = W_1 Z \cap W_0$. Note that if $x \in W \cap \overline{Y_0}$, there is a $z \in Z$ such that $xz \in W_1 \cap \overline{Y_0}$, so that $\|\phi(x) - I\| = \|\phi(xz) - I\| \le \frac{1}{6}$. Of course $Z \subseteq \operatorname{int} W_1 Z$, so $Z \subseteq \operatorname{int} W$.

If $Y'$ is a subgroup of $\overline{Y_0}$ included in $W$, then $\|\phi(x)^i - I\| \le \frac{1}{6}$ for every $i \in \mathbb{N}$ and $x \in Y'$, so $Y' \subseteq Z$. Consequently any subgroup of $X$ included in $W$ is a subgroup of $Z$, since by the choice of $\overline{Y_0}$ and $W_0$ it is a subgroup of $\overline{Y_0}$.

Now let $Y$ be the normalizer of $Z$ in $X$. $Z$ is compact, so $G = \{x : xZx^{-1} \subseteq \operatorname{int} W\}$ is open (4A5Ei), and contains $e$; but also $G \subseteq Y$, because if $x \in G$ then $xZx^{-1}$ is a subgroup of $X$ included in $W$, and must be included in $Z$. Accordingly $Y = GY$ is open.

Since any subgroup of $Y$ included in $W$ is a subgroup of $Z$, we see that any subgroup of $Y/Z$ included in the image of $W$ is the trivial subgroup, and $Y/Z$ has no small subgroups, as required.

***446P Corollary** Let $X$ be a locally compact Hausdorff topological group. Then it has a chain $\langle X_\xi \rangle_{\xi \le \kappa}$ of closed subgroups, where $\kappa$ is an infinite cardinal, such that

(i) $X_0$ is open,

(ii) $X_{\xi+1}$ is a normal subgroup of $X_\xi$ for every $\xi < \kappa$,

(iii) $X_\xi$ is compact for $\xi \ge 1$,

(iv) $X_\xi = \bigcap_{\eta<\xi} X_\eta$ for non-zero limit ordinals $\xi \le \kappa$,

(v) $X_\xi / X_{\xi+1}$ has a $B$-sequence for every $\xi < \kappa$,

(vi) $X_\kappa = \{e\}$, where $e$ is the identity of $X$.

**proof (a)** By 446O, $X$ has an open subgroup $X_0$ with a compact normal subgroup $X_1$ such that $X_0/X_1$ has no small subgroups. By 446M, $X_0/X_1$ has a $B$-sequence.

**(b)** Let $\Phi$ be the set of finite-dimensional representations of $X_1$; if we distinguish the trivial homomorphisms from $X_1$ to each $GL(r, \mathbb{R})$, $\Phi$ is infinite. Set $\kappa = \#(\Phi)$ and let $\langle \phi_\xi \rangle_{1 \le \xi < \kappa}$ run over $\Phi$. For $1 \le \xi \le \kappa$, set

$$X_\xi = \{x : x \in X_1,\ \phi_\eta(x) = I \text{ for } 1 \le \eta < \xi\}.$$

Then $\langle X_\xi \rangle_{\xi \le \kappa}$ satisfies conditions (i)-(iv). As for (v), I have already checked the case $\xi = 0$, and if $1 \le \xi < \kappa$, then $\phi_\xi \restriction X_\xi$ is a finite-dimensional representation of $X_\xi$ with kernel $X_{\xi+1}$, so $X_\xi/X_{\xi+1}$ has a faithful finite-dimensional representation (446Ab), and therefore has a $B$-sequence (446N).

Finally, $X_\kappa = \{e\}$ by 446B; if $x \in X_1$ and $x \ne e$, there is a $\phi \in \Phi$ such that $\phi(x) \ne \phi(e)$, so that $x \notin X_\kappa$.

**446X Basic exercises (a)** Let $X$ be a locally compact Hausdorff abelian topological group. Show that for every element $a$ of $X$, other than the identity, there is a two-dimensional representation $\phi$ of $X$ such that $\phi(a) \neq I$. (*Hint*: 445O.)

**(b)** Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be any sequence of groups with their discrete topologies, and $X$ the product topological group $\prod_{n\in\mathbb{N}} X_n$. Show that $X$ has a $B$-sequence. (*Hint*: set $V_n = \{x : x(i) = e(i) \text{ for } i < n\}$.)

**446Y Further exercises (a)** Let $X$ be the countable group of all permutations of $\mathbb{N}$ which are products of an even number of transpositions. Give $X$ its discrete topology, so that it is a locally compact topological group. Show that any finite-dimensional representation of $X$ is trivial. (*Hint*: $X$ is simple and has many commuting involutions.)

**(b)** Let $X$ be a compact Hausdorff topological group, $f \in C(X)$ and $\epsilon > 0$. Show that there are a finite-dimensional representation $\phi : X \to GL(r, \mathbb{R})$ and $a$, $b \in \mathbb{R}^r$ such that $|f(x) - (\phi(x)(a)|b)| \leq \epsilon$ for every $x \in X$.

**(c)** Let $\kappa$ be an infinite cardinal, and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ the measure algebra of the usual measure on $\{0,1\}^\kappa$. Give the group $\mathrm{Aut}_{\bar{\nu}_\kappa}(\mathfrak{B}_\kappa)$ of measure-preserving automorphisms of $\mathfrak{B}_\kappa$ its topology of pointwise convergence. Let $X$ be a compact Hausdorff topological group of weight at most $\kappa$. Show that there is a continuous injective homomorphism from $X$ to $\mathrm{Aut}_{\bar{\nu}_\kappa}(\mathfrak{B}_\kappa)$.

**446 Notes and comments** The ideas above are extracted from the structure theory for locally compact groups, as described in Montgomery & Zippin 55. (A brisker and sometimes neater, but less complete, exposition can be found in Kaplansky 71.) The full theory goes very much deeper into the analysis of groups with no small subgroups. One of the most important ideas, hidden away in 446I and part (c) of the proof of 446K, is that of 'one-parameter subgroup'; if $X$ is a group with no small subgroups, there are enough continuous homomorphisms from $\mathbb{R}$ to $X$ not only to provide a great deal of information on the topological group structure of $X$, but even to set up a differential structure (Kaplansky 71, §II.3). For our purposes here, however, all we need to know is that groups with no small subgroups have '$B$-sequences' (446L-446M), which can form the basis of a theory corresponding to Vitali's theorem and Lebesgue's Density Theorem in $\mathbb{R}^r$ (447C-447D below).

There are four essential elements in the argument here. Working from the outside, the first step is 446O: starting from a locally compact Hausdorff group $X$, we can find an open subgroup $X_0$ of $X$ and a compact normal subgroup $X_1$ of $X_0$ such that $X_0/X_1$ has no small subgroups. This depends on a subtle argument based on the first key lemma, the dichotomy in 446F, which in turn uses the 'smoothing' construction in 446E and a careful analysis of inequalities involving integrals. (Naturally enough, the translation-invariance of the Haar integral is a leitmotiv of this investigation.) Note the remarkable transition in 446H. The sets $D_n(U)$ are defined solely in terms of powers, while the sets $D_n(U)^n$ involve products. We are able to obtain information about products $x_1 \ldots x_n$ from information about the powers $x_j^i$ for $i$, $j \leq n$.

Next, we need to find a chain of closed subnormal subgroups of $X_1$, decreasing to $\{e\}$, such that the quotients all have faithful finite-dimensional representations (in this context, this means that they are isomorphic to compact subgroups of $GL(r, \mathbb{R})$). This step depends on the older ideas in 446B-446C, where we use the theory of compact operators on Hilbert spaces to show that a compact group has many representations as actions on finite-dimensional subspaces of its $L^2$ space. (Observe that in this section I revert to real-valued, rather than complex-valued, functions.) This can be thought of as a development of the result of 445O. If $X$ is a locally compact abelian group, its characters separate its points (cf. 446Xa); if $X$ is compact but not necessarily abelian, its finite-dimensional representations separate its points. (But if $X$ is neither compact nor abelian, there are further difficulties; see 446Ya.)

The other two necessary facts are that both groups with no small subgroups, and groups with faithful finite-dimensional representations, have $B$-sequences. The latter is reasonably straightforward (446N); any complications are due entirely to the fact that the natural measure on $GL(r, \mathbb{R})$, inherited from $\mathbb{R}^{r^2}$, is not quite invariant under multiplication, so we have to manipulate some inequalities. For groups with no small subgroups (446M) we have much more to do. The proof I give here depends on a second key lemma, 446K, refining the methods of 446F; a slightly stronger version of this result is the basis of the analysis of one-parameter subgroups in the general theory (compare Montgomery & Zippin 55, §3.8).

Version of 7.1.10

## 447 Translation-invariant liftings

I devote a section to the main theorem of Ionescu Tulcea & Ionescu Tulcea 67: a group carrying Haar measures has a translation-invariant lifting (447J). The argument uses an inductive construction of the same type as that used in §341 for the ordinary Lifting Theorem. It depends on the structure theory for locally compact groups described in §446. On the way I describe a Vitali theorem for certain metrizable groups (447C), with a corresponding density theorem (447D).

**447A Liftings and lower densities** Let $X$ be a group carrying Haar measures, $\Sigma$ its algebra of Haar measurable sets and $\mathfrak{A}$ its Haar measure algebra (442H, 443A).

**(a)** Recall that a **lifting** of $\mathfrak{A}$ is either a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$, or a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that $E \triangle \phi E$ is Haar negligible for every $E \in \Sigma$ and $\phi E = \emptyset$ whenever $E$ is Haar negligible (341A). Such a lifting $\theta$ or $\phi$ is **left-translation-invariant** if $\theta((xE)^\bullet) = x(\theta E^\bullet)$ or $\phi(xE) = x(\phi E)$ for every $E \in \Sigma$ and $x \in X$. (In the notation of 443C, a lifting $\theta : \mathfrak{A} \to \Sigma$ is left-translation-invariant if $\theta(x \bullet_l a) = x\theta a$ for every $x \in X$, $a \in \mathfrak{A}$.)

The language of 341A demanded a named measure; I spoke there of a lifting for a measure space $(X, \Sigma, \mu)$ or a measure $\mu$. But (as noted in 341Lh) what the concept really depends on is a triple $(X, \Sigma, \mathcal{I})$, where $\Sigma$ is an algebra of subsets of $X$ and $\mathcal{I}$ is an ideal of $\Sigma$. Variations in the measure which do not affect the algebra of measurable sets or the null ideal are irrelevant. So, in the present context, we can speak of a 'lifting for Haar measure' without declaring which Haar measure we are using, nor even whether it is a left or right Haar measure.

**(b)** Now suppose that $\Sigma_0$ is a $\sigma$-subalgebra of $\Sigma$. In this case, a **partial lower density** on $\Sigma_0$ is a function $\underline{\phi} : \Sigma_0 \to \Sigma$ such that $\underline{\phi}E = \underline{\phi}F$ whenever $E$, $F \in \Sigma_0$ and $E \triangle F$ is negligible, $E \triangle \underline{\phi}E$ is negligible for every $E \in \Sigma_0$, $\underline{\phi}\emptyset = \emptyset$ and $\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all $E$, $F \in \Sigma_0$. (See 341C-341D.) As in (a), such a function is **left-translation-invariant** if $xE \in \Sigma_0$ and $\underline{\phi}(xE) = x(\underline{\phi}E)$ for every $x \in X$ and $E \in \Sigma_0$.

**447B Lemma** Let $X$ be a group carrying Haar measures and $Y$ a subgroup of $X$. Write $\Sigma_Y$ for the algebra of Haar measurable subsets $E$ of $X$ such that $EY = E$, and suppose that $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ is a left-translation-invariant partial lower density. Then $G \subseteq \underline{\phi}(GY)$ for every open set $G \subseteq X$.

**proof** Of course $GY$ is open (4A5Ed), so belongs to $\Sigma_Y$. Let $a \in G$ and let $U$ be an open neighbourhood of the identity in $X$ such that $U^{-1}Ua \subseteq G$. Then $UaY$ is a non-empty open set, therefore not negligible (442Aa), and there is an $x \in UaY \cap \underline{\phi}(UaY)$. Express $x$ as $uay$ where $u \in U$ and $y \in Y$; then $ua = xy^{-1} \in Ua \cap \underline{\phi}(UaY)$, because $\underline{\phi}(UaY) \in \Sigma_Y$. So

$$a = u^{-1}ua \in u^{-1}\underline{\phi}(UaY) = \underline{\phi}(u^{-1}UaY) \subseteq \underline{\phi}(GY).$$

As $a$ is arbitrary, $G \subseteq \underline{\phi}(GY)$.

**447C Vitali's theorem** Let $X$ be a topological group with a left Haar measure $\mu$, and $\langle V_n \rangle_{n\in\mathbb{N}}$ a $B$-sequence in $X$ (definition: 446L). If $A \subseteq X$ is any set and $K_x$ is an infinite subset of $\mathbb{N}$ for every $x \in A$, then there is a disjoint family $\mathcal{V}$ of sets such that $A \setminus \bigcup \mathcal{V}$ is negligible and every member of $\mathcal{V}$ is of the form $xV_n$ for some $x \in A$ and $n \in K_x$.

**proof (a)** There is surely some $r$ such that $V_r$ is totally bounded for the bilateral uniformity on $X$ (443H); replacing $V_i$ by $V_r$ for $i < r$ and $K_x$ by $K_x \setminus r$ for each $x$, we may suppose that $V_0$ is totally bounded.

**(b)** Choose $\langle I_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. Given $I_j \subseteq X$ for $j < n$, choose a set $I_n \subseteq A$ which is maximal subject to the conditions

$$n \in K_x \text{ for every } x \in I_n,$$

$$xV_n \cap yV_j = \emptyset \text{ whenever } x \in I_n,\ j < n \text{ and } y \in I_j,$$

$$xV_n \cap yV_n = \emptyset \text{ whenever } x,\ y \in I_n \text{ are distinct.}$$

On completing the induction, set $\mathcal{V}=\{xV_n:n\in\mathbb{N},\,x\in I_n\}$; this is a disjoint family.

**(c) ?** Suppose, if possible, that $A\setminus\bigcup\mathcal{V}$ is not negligible. By 415B, the subspace measure on $A\setminus\bigcup\mathcal{V}$ is $\tau$-additive and has a non-empty support. Take any $a$ belonging to this support and set $G=\operatorname{int}(aV_0)$; then $G$ is totally bounded and $\delta=\mu^*(G\cap A\setminus\bigcup\mathcal{V})$ is non-zero. Let $M$ be such that every $V_nV_n^{-1}$ can be covered by $M$ left translates of $V_n$, so that $\mu(V_nV_n^{-1})\le M\mu V_n$ for every $n$. Set

$$J_n=I_n\cap GV_0^{-1},\quad E_n=J_nV_n,\quad \tilde E_n=E_nV_n^{-1}$$

for each $n$.

If $n\in\mathbb{N}$ and $x\in J_n$, then $xV_n\subseteq GV_0^{-1}V_0$. Accordingly (because $\langle xV_n\rangle_{x\in J_n}$ is disjoint)

$$\#(J_n)\mu V_n=\sum_{x\in J_n}\mu(xV_n)\le\mu(GV_0^{-1}V_0)<\infty$$

because $GV_0^{-1}V_0$ is totally bounded (4A5Ob). So $J_n$ is finite and $E_n$ is closed. Note that if $x\in I_n$ and $G\cap xV_n\ne\emptyset$, then $x\in GV_n^{-1}\subseteq GV_0^{-1}$ and $x\in J_n$; so $G\cap\bigcup\mathcal{V}=G\cap\bigcup_{n\in\mathbb{N}}E_n$. Also, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of subsets of $GV_0^{-1}V_0$; accordingly $\sum_{n=0}^\infty\mu E_n$ is finite, and there is an $m\in\mathbb{N}$ such that $M\sum_{n=m}^\infty\mu E_n<\delta$.

Observe next that, for any $x\in X$ and $n\in\mathbb{N}$,

$$\mu(xV_nV_n^{-1})=\mu(V_nV_n^{-1})\le M\mu V_n=M\mu(xV_n).$$

So

$$\mu\tilde E_n\le\sum_{x\in J_n}\mu(xV_nV_n^{-1})\le M\sum_{x\in J_n}\mu(xV_n)=M\mu E_n$$

for each $n$, and $\mu(\bigcup_{n\ge m})\tilde E_n<\delta$.

This means that $\bigcup_{n\ge m}\tilde E_n$ cannot include $A\cap G\setminus\bigcup\mathcal{V}$, and there is a $z$ belonging to

$$\begin{aligned}A\cap G\setminus(\bigcup\mathcal{V}\cup\bigcup_{n\ge m}\tilde E_n)&=A\cap G\setminus(\bigcup_{n\in\mathbb{N}}E_n\cup\bigcup_{n\ge m}\tilde E_n)\\&=A\cap G\setminus(\bigcup_{n<m}E_n\cup\bigcup_{n\ge m}\tilde E_n).\end{aligned}$$

Now there must be a first $k\ge m$ such that $k\in K_z$ and $zV_k\subseteq G\setminus\bigcup_{n<m}E_n$. (This is where we use the hypothesis that $\{V_n:n\in\mathbb{N}\}$ is a base of neighbourhoods of the identity.) Since $z\in G\setminus J_k$, $z\notin I_k$, and there are $j\le k$, $x\in I_j$ such that $zV_k\cap xV_j\ne\emptyset$. In this case, $x\in GV_0^{-1}$, so $x\in J_j$. Accordingly

$$z\in xV_jV_k^{-1}\subseteq xV_jV_j^{-1}\subseteq\tilde E_j$$

and $j<m$; but this means that $zV_k\cap xV_j\subseteq zV_k\cap E_j$ must be empty, which is impossible. **X**

**(d)** Thus $\mu(A\setminus\bigcup\mathcal{V})=0$, and $\mathcal{V}$ is an appropriate family.

**447D Theorem** Let $X$ be a topological group with a left Haar measure $\mu$, and $\langle V_n\rangle_{n\in\mathbb{N}}$ a $B$-sequence in $X$. Then for any Haar measurable set $E\subseteq X$,

$$\lim_{n\to\infty}\frac{\mu(E\cap xV_n)}{\mu V_n}=\chi E(x)$$

for almost every $x\in X$.

**proof (a)** Let $\alpha<1$, and set

$$A=\{x:x\in E,\ \liminf_{n\to\infty}\frac{\mu(E\cap xV_n)}{\mu V_n}<\alpha\}.$$

**?** Suppose, if possible, that $A$ is not negligible. Then there is an open set $G$ of finite measure such that $\mu^*(G\cap A)=\gamma>0$. Let $\delta>0$ be such that $\gamma>\alpha(\gamma+\delta)+\delta$. Take a Borel set $F$ which is a measurable envelope of $G\cap A$ and a closed set $F_1\subseteq G\setminus F$ such that $\mu F_1\ge\mu(G\setminus F)-\delta$. Writing $H=G\setminus F_1$, we see that $H\cap A=G\cap A$ and

$$\mu H\le\mu^*(H\cap A)+\delta=\gamma+\delta.$$

For each $x \in H \cap A$, set

$$K_x = \{n : xV_n \subseteq H,\ \mu(E \cap xV_n) \le \alpha\mu V_n\}.$$

Then $K_x$ is infinite. By Vitali's theorem in the form 447C, there is a disjoint family $\mathcal{V} \subseteq \{xV_n : x \in H \cap A,\ n \in K_x\}$ such that $(H \cap A) \setminus \bigcup \mathcal{V}$ is negligible. Since every member of $\mathcal{V}$ has non-zero measure, while $\mu H$ is finite, $\mathcal{V}$ is countable. Now $\mu(\bigcup \mathcal{V}) \ge \mu^*(H \cap A)$, so $\mu(H \setminus \bigcup \mathcal{V}) \le \delta$; also, because $\mu(E \cap V) \le \alpha\mu V$ for every $V \in \mathcal{V}$, and $\mathcal{V}$ is disjoint,

$$\mu(E \cap \bigcup \mathcal{V}) \le \alpha\mu(\bigcup \mathcal{V}) \le \alpha\mu H$$

and

$$\gamma = \mu^*(A \cap H) \le \mu(E \cap H) \le \mu(E \cap \bigcup \mathcal{V}) + \delta \le \alpha\mu H + \delta \le \alpha(\gamma + \delta) + \delta,$$

which is impossible, by the choice of $\delta$. **X**

**(b)** As $\alpha$ is arbitrary,

$$\liminf_{n\to\infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = 1$$

for almost every $x \in E$. Similarly,

$$\liminf_{n\to\infty} \frac{\mu(xV_n \setminus E)}{\mu V_n} = 1, \quad \limsup_{n\to\infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = 0$$

for almost every $x \in X \setminus E$, so

$$\lim_{n\to\infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = \chi E(x)$$

for almost every $x \in X$.

**447E** We need to recall some results from 443P-443R. Let $X$ be a locally compact Hausdorff topological group, and $Y$ a closed subgroup of $X$ such that the modular function of $Y$ is the restriction to $Y$ of the modular function of $X$. Let $\mu$ be a left Haar measure on $X$ and $\mu_Y$ a left Haar measure on $Y$.

**(a)** Writing $C_k(X)$ for the space of continuous real-valued functions on $X$ with compact support, and $X/Y$ for the set of left cosets of $Y$ in $X$ with the quotient topology, we have a linear operator $T : C_k(X) \to C_k(X/Y)$ defined by writing $(Tf)(x^\bullet) = \int_Y f(xy)\mu_Y(dy)$ whenever $x \in X$ and $f \in C_k(X)$ (443P); moreover, $T[C_k(X)^+] = C_k(X/Y)^+$ (443Pa), and we have an invariant Radon measure $\lambda$ on $X/Y$ such that $\int Tf d\lambda = \int f d\mu$ for every $f \in C_k(X)$ (see part (b) of the proof of 443R). Turning this structure round, we see from 443Qb that $\mu$, $\mu_Y$ and $\lambda$ here are related in exactly the same way as $\mu$, $\nu$ and $\lambda$ in 443Q. If $Y$ is a normal subgroup of $X$, so that $X/Y$ is the quotient group, $\lambda$ is a left Haar measure. If $Y$ is compact and $\mu_Y$ is the Haar probability measure on $Y$, then $\lambda$ is the image measure $\mu\pi^{-1}$, where $\pi(x) = x^\bullet = xY$ for every $x \in X$ (443Qd).

**(b)** If $E \subseteq X$ and $EY = Y$, then $E$ is Haar measurable iff $\tilde{E} = \{x^\bullet : x \in E\}$ belongs to the domain of $\lambda$, and $E$ is Haar negligible iff $\tilde{E}$ is $\lambda$-negligible (443Qc).

**(c)** Now suppose that $X$ is $\sigma$-compact. Then for any Haar measurable $E \subseteq X$, $\mu E = \int g\,d\lambda$ in $[0,\infty]$, where $g(x^\bullet) = \mu_Y(Y \cap x^{-1}E)$ is defined for almost every $x \in X$ (443Qe). In particular, $E$ is Haar negligible iff $\mu_Y(Y \cap x^{-1}E) = 0$ for almost every $x \in X$.

**(d)** Again suppose that $X$ is $\sigma$-compact. Then we can extend the operator $T$ of part (a) to an operator from $\mathcal{L}^1(\mu)$ to $\mathcal{L}^1(\lambda)$ by writing $(Tf)(x^\bullet) = \int f(xy)\mu_Y(dy)$ whenever $f \in \mathcal{L}^1(\mu)$, $x \in X$ and the integral is defined, and $\int Tf d\lambda = \int f d\mu$ for every $f \in \mathcal{L}^1(\mu)$ (443Qe). If $f \in \mathcal{L}^1(\mu)$, and we set $f_x(y) = f(xy)$ for all those $x \in X$, $y \in Y$ for which $xy \in \operatorname{dom} f$, then $Q = \{x : f_x \in \mathcal{L}^1(\mu_Y)\}$ is $\mu$-conegligible, and $x \mapsto f_x^\bullet : Q \to L^1(\mu_Y)$ is almost continuous (443Qf).

**(e)** If $X$ is $\sigma$-compact, $Y$ is compact and $\mu_Y$ is the Haar probability measure on $Y$, so that $\lambda$ is the image measure $\mu\pi^{-1}$, then we can apply 235G to the formula in (d) to see that

$$\iint f(xy)\mu_Y(dy)\mu(dx) = \int (Tf)(x^\bullet)\mu(dx) = \int Tf\,d\lambda = \int f d\mu$$

for every $\mu$-integrable function $f$, and therefore (because $\mu$ is $\sigma$-finite) for every function $f$ such that $\int f d\mu$ is defined in $[-\infty, \infty]$. In particular, $\mu E = \int \nu(Y \cap x^{-1}E)\mu(dx)$ for every Haar measurable set $E \subseteq X$.

**447F Lemma** Let $X$ be a $\sigma$-compact locally compact Hausdorff topological group and $Y$ a closed subgroup of $X$ such that the modular function of $Y$ is the restriction to $Y$ of the modular function of $X$. Let $Z$ be a compact normal subgroup of $Y$ such that the quotient group $Y/Z$ has a $B$-sequence. Let $\Sigma_Y$ be the $\sigma$-algebra of those Haar measurable subsets $E$ of $X$ such that $EY = E$, and $\Sigma_Z$ the algebra of Haar measurable sets $E \subseteq X$ such that $EZ = E$. Let $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ be a left-translation-invariant partial lower density. Then there is a left-translation-invariant partial lower density $\underline{\psi} : \Sigma_Z \to \Sigma_Z$ extending $\underline{\phi}$.

**proof (a)** Let $\mu$ be a left Haar measure on $X$, $\mu_Y$ a left Haar measure on $Y$ and $\mu_Z$ the Haar probability measure on $Z$; then there is a left Haar measure $\nu$ on $Y/Z$ such that $\int g(y)\mu_Y(dy) = \int (Tg)(u)\nu(du)$ for every $g \in C_k(Y)$, where $(Tg)(y^\bullet) = \int g(yz)\mu_Z(dz)$ for every $y \in Y$ (447Ea). We are supposing that $Y/Z$ has a $B$-sequence $\langle V_n\rangle_{n\in\mathbb{N}}$. It follows that there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in $C_k(Y)^+$ such that (i) $\int h_n(y)\mu_Y(dy) = 1$ for every $n$ (ii) whenever $F \subseteq Y$ is Haar measurable (regarded as a subset of $Y$, that is), and $FZ = Z$, then

$$\lim_{n\to\infty} \int \chi F(by)h_n(y)\mu_Y(dy) = \chi F(b)$$

for $\mu_Y$-almost every $b \in Y$.

**P** Since any subsequence of $\langle V_n\rangle_{n\in\mathbb{N}}$ is a $B$-sequence, and $Y/Z$ is locally compact, we may suppose that every $V_n$ is compact. For each $n \in \mathbb{N}$, choose a non-negative $h'_n \in C_k(Y/Z)$ such that

$$\int h'_n d\nu = 1, \quad \int |h'_n - \frac{1}{\nu V_n}\chi V_n| d\nu \le 2^{-n}.$$

(This is possible by 416I, or otherwise.) Let $h_n \in C_k(Y)^+$ be such that $Th_n = h'_n$ (447Ea again); then $\int h_n d\mu_Y = \int h'_n d\nu = 1$. Now if $F \subseteq Y$ is Haar measurable and $FZ = Z$, there is a Haar measurable $\tilde{F} \subseteq Y/Z$ such that $F = \{y : y^\bullet \in \tilde{F}\}$ (447Eb). Take $b \in Y$ and $n \in \mathbb{N}$. Because

$$\int \chi F(byz)h_n(yz)\mu_Z(dz) = \int \chi\tilde{F}(b^\bullet y^\bullet)h_n(yz)\mu_Z(dz) = \chi\tilde{F}(b^\bullet y^\bullet)h'_n(y^\bullet)$$

for every $y \in Y$,

$$\int \chi F(by)h_n(y)\mu_Y(dy) = \iint \chi F(byz)h_n(yz)\mu_Z(dz)\mu_Y(dy)$$

(447Ee)

$$= \int \chi\tilde{F}(b^\bullet y^\bullet)h'_n(y^\bullet)\mu_Y(dy) = \int \chi\tilde{F}(b^\bullet u)h'_n(u)\nu(du)$$

because $y \mapsto y^\bullet$ is inverse-measure-preserving for $\mu_Y$ and $\nu$ (447Ee). So

$$\begin{aligned}|\chi F(b) - \int \chi F(by)h_n(y)\mu_Y(dy)| &= |\chi\tilde{F}(b^\bullet) - \int \chi\tilde{F}(b^\bullet u)h'_n(u)\nu(du)| \\ &\le |\chi\tilde{F}(b^\bullet) - \frac{1}{\nu V_n}\int \chi\tilde{F}(b^\bullet u)\chi V_n(u)\nu(du)| \\ &\qquad + \int |h'_n(u) - \frac{1}{\nu V_n}\chi V_n(u)|\nu(du) \\ &\le |\chi\tilde{F}(b^\bullet) - \frac{\nu(\tilde{F}\cap b^\bullet V_n)}{\nu V_n}| + 2^{-n}.\end{aligned}$$

Since

$$\{v : v \in Y/Z, \lim_{n\to\infty} \frac{\nu(\tilde{F}\cap vV_n)}{\nu V_n} \ne \chi\tilde{F}(v)\}$$

is $\nu$-negligible (447D), its inverse image in $Y$ is $\mu_Y$-negligible, so $\chi F(b) = \lim_{n\to\infty} \int \chi F(by)h_n(y)\mu_Y(dy)$ for almost every $b$, as claimed. **Q**

**(b)** We find now that if $E \in \Sigma_Z$, then $\lim_{n\to\infty} \int \chi E(xy)h_n(y)\mu_Y(dy) = \chi E(x)$ for $\mu$-almost every $x \in X$. **P** Set $E_x = Y \cap x^{-1}E$ for $x \in X$. Because $X$ is $\sigma$-compact, we can express $E$ as the union of a non-decreasing sequence $\langle F^{(k)}\rangle_{k\in\mathbb{N}}$ where each $F^{(k)}$ is Haar measurable and relatively compact; set $F^{(k)}_x = Y \cap x^{-1}F^{(k)}$ for each $x$. In this case, for any $k \in \mathbb{N}$, $Q_k = \{x : F^{(k)}_x \in \operatorname{dom}(\mu_Y)\}$ is conegligible, and $x \mapsto (\chi F^{(k)}_x)^\bullet : Q_k \to L^1(\mu_Y)$ is almost continuous (447Ed), so that $x \mapsto \int \chi F^{(k)}_x(y)h_n(y)\mu_Y(dy) : Q_k \to [0,1]$ is measurable, for each $n$. But this means that, setting $Q = \bigcap_{k\in\mathbb{N}} Q_k$,

$$x \mapsto \int \chi E_x(y)h_n(y)\mu_Y(dy) = \lim_{k\to\infty} \int \chi F^{(k)}_x(y)h_n(y)\mu_Y(dy) : Q \mapsto [0,1]$$

is measurable, for every $n \in \mathbb{N}$. Note that if $x \in X$ and $y \in Y$ then $F^{(k)}_{xy} = y^{-1}F^{(k)}_x$, so that $Q_kY = Q_k$ for every $k \in \mathbb{N}$, and $QY = Q$.

Now consider $\tilde{Q} = \{x : x \in Q, \lim_{n\to\infty} \int \chi E(xy)h_n(y)\mu_Y(dy) = \chi E(x)\}$. This is a Haar measurable subset of $X$. If $a \in Q$, then

$$Y \cap a^{-1}\tilde{Q} = \{y : y \in Y, \lim_{n\to\infty} \chi E(ays)h_n(s)\mu_Y(ds) = \chi E(ay)\}$$

is $\mu_Y$-conegligible, by the choice of the $h_n$ in (a) above. Because $Q$ is $\mu$-conegligible, $Q \setminus \tilde{Q}$ is $\mu$-negligible (447Ec) and $\tilde{Q}$ is conegligible, as required. **Q**

**(c)** We are now ready for the formulae at the centre of this proof. For any Haar measurable set $E \subseteq X$, $n \in \mathbb{N}$ and $\gamma < 1$, set

$$\psi_{n\gamma}(E) = \bigcup\{G \cap \underline{\phi}F : G \subseteq X \text{ is open, } F \in \Sigma_Y,$$
$$\int \chi E(xy)h_n(y)\mu_Y(dy) \text{ is defined and at least } \gamma \text{ for every } x \in G \cap F\},$$

$$\underline{\psi}E = \bigcap_{\gamma<1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \psi_{m\gamma}(E).$$

The rest of the proof is devoted to checking that $\underline{\psi}\restriction\Sigma_Z$ is a left-translation-invariant partial lower density extending $\underline{\phi}$.

**(d)** I had better make one remark straight away. If $G \subseteq X$ is open, then $G \subseteq \underline{\phi}(GY)$ (447B). It follows that if $G \subseteq X$ is open, $F \in \Sigma_Y$ and $G \cap \underline{\phi}F \ne \emptyset$, then $G \cap F \ne \emptyset$. **P** If $a \in G \cap \underline{\phi}F$, then $a \in \underline{\phi}(GY) \cap \underline{\phi}F = \underline{\phi}(GY \cap F)$, so $GY \cap F \ne \emptyset$, that is, $G \cap F = G \cap FY^{-1}$ is non-empty. **Q** I mention this now because we need to know that the condition

$$\int \chi E(xy)h_n(y)\mu_Y(dy) \text{ is defined and at least } \gamma \text{ for every } x \in G \cap F$$

in the definition of $\psi_{n\gamma}(E)$ is never vacuously satisfied if $G \cap \underline{\phi}F \ne \emptyset$. In particular, $\psi_{n\gamma}(\emptyset) = \emptyset$ whenever $n \in \mathbb{N}$ and $0 < \gamma < 1$, so that $\underline{\psi}\emptyset = \emptyset$.

**(e)** If $E \subseteq X$ is Haar measurable, $n \in \mathbb{N}$ and $\epsilon > 0$, then for almost every $a \in X$ there are an open set $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int |\chi E(xy) - \chi E(ay)|h_n(y)\mu_Y(dy) \le \epsilon$ whenever $x \in G \cap F$. **P** Let $\tilde{\lambda}$ be the invariant Radon measure on $X/Y$ derived from $\mu$ and $\mu_Y$ as in 447Ea. Take $\delta > 0$ such that $\delta(1 + 2\|h_n\|_\infty) \le \epsilon$. Because $X$ is $\sigma$-compact, locally compact and Hausdorff, therefore Lindelöf and regular (4A2Hd, 3A3Bb), there is a sequence $\langle f_r\rangle_{r\in\mathbb{N}}$ of continuous functions from $X$ to $[0,1]$ such that $\int |\chi E(x) - f_r(x)|\mu(dx) \le 2^{-r}$ for every $r$ (415Pb). Set $g_r = |\chi E - f_r|$ for each $r$. For $f \in \mathcal{L}^1(\mu)$, define $\tilde{T}f \in \mathcal{L}^1(\tilde{\lambda})$ by writing $(\tilde{T}f)(x^\bullet) = \int f(xy)\mu_Y(dy)$ whenever this is defined (447Ed); we have $\int \tilde{T}f d\tilde{\lambda} = \int f d\mu$.

Set $Q = \{x : Y \cap x^{-1}E \in \operatorname{dom}\mu_Y\}$, so that $Q \in \Sigma_Y$ is conegligible (447Ec). For each $r$, set $\tilde{F}_r = \{u : u \in X/Y, \tilde{T}g_r(u) \text{ is defined and at least } \delta\}$; then

$$\tilde{\lambda}\tilde{F}_r \le \frac{1}{\delta}\int Tg_r d\tilde{\lambda} = \frac{1}{\delta}\int g_r d\mu \le 2^{-r}/\delta.$$

So $\bigcap_{r\in\mathbb{N}} \tilde{F}_r$ is $\tilde{\lambda}$-negligible. Set

$$\begin{aligned} F_r &= \{x : x \in X,\, x^\bullet \in \tilde{F}_r\} = \{x : \tilde{T}g_r(x^\bullet) \ge \delta\} \\ &= \{x : \int g_r(xy)\mu_Y(dy) \ge \delta\} = \{x : \int |\chi E(xy) - f_r(xy)|\mu_Y(dy) \ge \delta\} \end{aligned}$$

for each $r$; then $F_rY = F_r$, $F_r$ is Haar measurable (447Eb) and $\bigcap_{r\in\mathbb{N}} F_r$ is $\mu$-negligible (also by 447Eb). Since $(X \setminus F_r)\triangle\underline{\phi}(X \setminus F_r)$ is negligible for each $r$, $Q_1 = Q \cap \bigcup_{r\in\mathbb{N}} \underline{\phi}(X \setminus F_r) \setminus F_r$ is conegligible. Note that $Q_1Y = Q_1$.

Suppose that $a \in Q_1$. Then there is an $r \in \mathbb{N}$ such that

$$a \in Q_1 \cap \underline{\phi}(X \setminus F_r) \setminus F_r = (Q_1 \setminus F_r) \cap \underline{\phi}(Q_1 \setminus F_r).$$

Set $F = Q_1 \setminus F_r \in \Sigma_Y$. Consider the function $x \mapsto \int f_r(xy)h_n(y)\mu_Y(dy)$. We chose $h_n$ with compact support $L \subseteq Y$ say. If $V$ is a compact neighbourhood of $a$ in $X$, then $f_r$ is uniformly continuous on $VL$ for the right uniformity on $X$ (4A5Ha, 4A2Jf). There is therefore an open neighbourhood $U$ of the identity of $X$ such that $|f_r(x') - f_r(x)| \le \delta$ whenever $x$, $x' \in VL$ and $x'x^{-1} \in U$; of course we may suppose that $G = Ua$ is a subset of $V$.

Take any $x \in G \cap F$. Then if $y \in L$, we have $ay$, $xy$ both in $VL$, while $xy(ay)^{-1} \in U$, so that $|f_r(xy) - f_r(ay)| \le \delta$. Accordingly $|f_r(xy) - f_r(ay)|h_n(y) \le \delta h_n(y)$ for every $y \in Y$, and

$$\int |f_r(ay) - f_r(xy)|h_n(y)\mu_Y(dy) \le \delta.$$

At the same time, because both $x$ and $a$ belong to $F = Q_1 \setminus F_r$,

$$\int |\chi E(ay) - f_r(ay)|h_n(y)\mu_Y(dy) \le \delta\|h_n\|_\infty,$$

$$\int |\chi E(xy) - f_r(xy)|h_n(y)\mu_Y(dy) \le \delta\|h_n\|_\infty.$$

Putting these together,

$$\int |\chi E(ay) - \chi E(xy)|h_n(y)\mu_Y(dy) \le \delta(1 + 2\|h_n\|_\infty) \le \epsilon.$$

Thus $G$ and $F$ witness that $a$ has the property required; as $a$ is any member of the conegligible set $Q_1$, we have the result. **Q**

**(f)** If $E \in \Sigma_Z$ then $E\triangle\underline{\psi}E$ is negligible. **P** By (e), applied in turn to every $n$ and every $\epsilon$ of the form $2^{-i}$, there is a conegligible set $Q_1 \subseteq X$ such that whenever $a \in Q_1$, $n \in \mathbb{N}$ and $\epsilon > 0$ there are an open set $G$ containing $a$ and an $F \in \Sigma_Y$ such that $a \in \underline{\phi}F$ and $\int |\chi E(xy) - \chi E(ay)|h_n(y)\mu_Y(dy) \le \epsilon$ for every $x \in G \cap F$. By (b), there is a conegligible set $Q_2 \subseteq X$ such that $\lim_{n\to\infty} \int \chi E(ay)h_n(y)\mu_Y(dy) = \chi E(a)$ for every $a \in Q_2$.

Suppose that $a \in Q_1 \cap Q_2 \cap E$. Let $\gamma < 1$; set $\epsilon = \frac{1}{2}(1 - \gamma)$. Because $a \in Q_2$, there is an $n \in \mathbb{N}$ such that $\int \chi E(ay)h_m(y)\mu_Y(dy) \ge 1 - \epsilon$ for every $m \ge n$. Take any $m \ge n$. Because $a \in Q_1$, there are an open set $G$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int |\chi E(ay) - \chi E(xy)|h_m(y)\mu_Y(dy) \le \epsilon$ whenever $x \in G \cap F$. But now

$$\int \chi E(xy)h_m(y)\mu_Y(dy) \ge 1 - 2\epsilon = \gamma$$

for every $x \in G \cap F$, so $a \in \psi_{m\gamma}(E)$. This is true for every $m \ge n$; as $\gamma$ is arbitrary, $a \in \underline{\psi}E$. As $a$ is arbitrary, $Q_1 \cap Q_2 \cap E \subseteq \underline{\psi}E$.

Now suppose that $a \in Q_1 \cap Q_2 \cap \underline{\psi}E$. Then there is an $n \in \mathbb{N}$ such that $a \in \psi_{m,3/4}(E)$ for every $m \ge n$. There is an $m \ge n$ such that $|\int \chi E(ay)h_m(y)\mu_Y(dy) - \chi E(a)| \le \frac{1}{4}$. There are an open set $G_1$ and an $F_1 \in \Sigma_Y$ such that $a \in G_1 \cap \underline{\phi}F_1$ and $\int \chi E(xy)h_m(y)\mu_Y(dy) \ge \frac{3}{4}$ for every $x \in G_1 \cap F_1$. There are also an open set $G_2$ and an $F_2 \in \Sigma_Y$ such that $a \in G_2 \cap \underline{\phi}F_2$ and $\int |\chi E(ay) - \chi E(xy)|h_m(y)\mu_Y(dy) \le \frac{1}{4}$ for every $x \in G_2 \cap F_2$. Set $G = G_1 \cap G_2$, $F = F_1 \cap F_2$; then $a \in G \cap \underline{\phi}F$, so $G \cap F$ is not empty ((d) above). Take $x \in G \cap F$. Then

$$\int \chi E(xy)h_m(y)\mu_Y(dy) \ge \frac{3}{4},$$

$$\int |\chi E(ay) - \chi E(xy)|h_m(y)\mu_Y(dy) \le \frac{1}{4},$$

$$|\int \chi E(ay)h_m(y)\mu_Y(dy) - \chi E(a)| \le \frac{1}{4},$$

so $\chi E(a) \ge \frac{1}{4}$ and $a \in E$. This shows that $Q_1 \cap Q_2 \cap \underline{\psi}E \subseteq E$.

Accordingly $E \triangle \underline{\psi}E \subseteq X \setminus (Q_1 \cap Q_2)$ is negligible, as required. **Q**

In particular, $\underline{\psi}E$ is Haar measurable for every $E \in \Sigma_Z$.

**(g)** If $E \in \Sigma_Z$ then $\underline{\psi}E \in \Sigma_Z$. **P** We have just seen that $\underline{\psi}E$ is Haar measurable. Take $z \in Z$, $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E)$. Then there are an open $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E(xy)h_n(y)\mu_Y(dy) \ge \gamma$ for every $x \in G \cap F$. Because $F$ and $\underline{\phi}F$ belong to $\Sigma_Y$, $az \in \underline{\phi}F$. Of course $Gz$ is an open set containing $az$. If $x \in Gz \cap F$, then $xz^{-1} \in G \cap F$ and $\int \chi E(xz^{-1}y)h_n(y)\mu_Y(dy) \ge \gamma$. But

$$\chi E(xz^{-1}y) = \chi E(xy \cdot y^{-1}z^{-1}y) = \chi E(xy)$$

for every $y \in Y$, because $Z \vartriangleleft Y$ (so $z' = y^{-1}z^{-1}y \in Z$) and we are supposing that $E \in \Sigma_Z$ (so $xyz' \in E$ iff $xy \in E$). So

$$\int \chi E(xy)h_n(y)\mu_Y(dy) = \int \chi E(xz^{-1}y)h_n(y)\mu_Y(dy) \ge \gamma.$$

As $x$ is arbitrary, $Gz$ and $F$ witness that $az \in \psi_{n\gamma}E$.

This shows that, for any $n$ and $\gamma$, $az \in \psi_{n\gamma}(E)$ whenever $a \in \psi_{n\gamma}(E)$ and $z \in Z$. It follows at once that $az \in \underline{\psi}E$ whenever $a \in \underline{\psi}E$ and $z \in Z$, as claimed. **Q**

**(h)** For any Haar measurable $E \subseteq X$ and $c \in X$, $\underline{\psi}(cE) = c\underline{\psi}E$. **P** Suppose that $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E)$. Then there are an open set $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E(xy)h_n(y)\mu_Y(dy) \ge \gamma$ for every $x \in G \cap F$. Now $cG$ is an open set containing $ca$, $cF \in \Sigma_Y$ and $\underline{\phi}(cF) = c\underline{\phi}F$ contains $ca$, and if $x \in cG \cap cF$ we have

$$\int \chi(cE)(xy)h_n(xy)\mu_Y(dy) = \int \chi E(c^{-1}xy)h_n(xy)\mu_Y(dy) \ge \gamma$$

because $c^{-1}x \in G \cap F$. But this means that $cG$, $cF$ witness that $ca \in \psi_{n\gamma}(cE)$. Since $a$ is arbitrary, $c\psi_{n\gamma}(E) \subseteq \psi_{n\gamma}(cE)$; as $n$ and $\gamma$ are arbitrary, $c\underline{\psi}E \subseteq \underline{\psi}(cE)$. Similarly, of course, $c^{-1}\underline{\psi}(cE) \subseteq \underline{\psi}E$, so in fact $\underline{\psi}(cE) = c\underline{\psi}E$, as claimed. **Q**

**(i)** If $E_1$, $E_2 \subseteq X$ are Haar measurable and $E_1 \setminus E_2$ is $\mu$-negligible, $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$. **P** Take $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E_1)$. Then there are an open set $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E_1(xy)h_n(y)\mu_Y(dy) \ge \gamma$ for every $x \in G \cap F$. Let $Q$ be the set of those $x \in X$ such that $\mu_Y$ measures $Y \cap x^{-1}(E_1 \setminus E_2)$ and $\int \chi(E_1 \setminus E_2)(xy)\mu_Y(dy) = 0$; then $QY = Q$ is conegligible (447Ec). Now $Q \cap F \in \Sigma_Y$, and $\underline{\phi}(Q \cap F) = \underline{\phi}F$ contains $a$. But if $x \in G \cap Q \cap F$, $\chi E_1(xy) \le \chi E_2(xy)$ for $\mu_Y$-almost every $y$. So

$$\int \chi E_2(xy)h_n(y)\mu_Y(dy) \ge \int \chi E_1(xy)h_n(y)\mu_Y(dy) \ge \gamma.$$

Thus $G$ and $Q \cap F$ witness that $a \in \psi_{n\gamma}(E_2)$. As $a$ is arbitrary, $\psi_{n\gamma}(E_1) \subseteq \psi_{n\gamma}(E_2)$; as $n$ and $\gamma$ are arbitrary, $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$. **Q**

In particular, (i) $\underline{\psi}E_1 = \underline{\psi}E_2$ whenever $E_1 \triangle E_2$ is negligible (ii) $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$ whenever $E_1 \subseteq E_2$.

**(j)** If $E_1$, $E_2 \subseteq X$ are Haar measurable, $\underline{\psi}(E_1 \cap E_2) = \underline{\psi}E_1 \cap \underline{\psi}E_2$. **P** By (i), $\underline{\psi}(E_1 \cap E_2) \subseteq \underline{\psi}E_1 \cap \underline{\psi}E_2$. So take $a \in \underline{\psi}E_1 \cap \underline{\psi}E_2$. Let $\gamma < 1$. Set $\delta = \frac{1}{2}(1+\gamma) < 1$. Then there are $n_1$, $n_2 \in \mathbb{N}$ such that $a \in \psi_{m\delta}(E_1)$ for every $m \ge n_1$ and $a \in \psi_{m\delta}(E_2)$ for every $m \ge n_2$. Set $n = \max(n_1, n_2)$ and take any $m \ge n$. Then there are open sets $G_1$, $G_2 \subseteq X$ and $F_1$, $F_2 \in \Sigma_Y$ such that $a \in G_1 \cap G_2 \cap \underline{\phi}F_1 \cap \underline{\phi}F_2$, $\int \chi E_1(xy)h_m(y)\mu_Y(dy) \ge \delta$ for every $x \in G_1 \cap F_1$ and $\int \chi E_2(xy)h_m(y)\mu_Y(dy) \ge \delta$ for every $x \in G_2 \cap F_2$. Let $Q \in \Sigma_Y$ be the conegligible set of those $x \in X$ such that $\int \chi(E_1 \cap E_2)(xy)\mu_Y(dy)$ is defined. Set $G = G_1 \cap G_2$, $F = F_1 \cap F_2 \cap Q$; then $G$ is open, $F \in \Sigma_Y$ and

$$\underline{\phi}F = \underline{\phi}(F_1 \cap F_2) = \underline{\phi}F_1 \cap \underline{\phi}F_2,$$

so that $a \in G \cap \underline{\phi}F$. Now take any $x \in G \cap F$. We have

$$\int(1-\chi(E_1\cap E_2))(xy)h_m(y)\mu_Y(dy)$$
$$\le\int(1-\chi E_1(xy))h_m(y)\mu(dy)+\int(1-\chi E_2(xy))h_m(y)\mu(dy)$$
$$\le 2(1-\delta)=1-\gamma,$$

and $\int\chi(E_1\cap E_2)(xy)h_m(y)\mu_Y(dy)\ge\gamma$, because $\int h_m(y)\mu_Y(dy)=1$.

As $x$ is arbitrary, $G$ and $F$ witness that $a\in\psi_{m\gamma}(E_1\cap E_2)$. And this is true for every $m\ge n$. As $\gamma$ is arbitrary, $a\in\underline{\psi}(E_1\cap E_2)$. As $a$ is arbitrary, $\underline{\psi}E_1\cap\underline{\psi}E_2\subseteq\underline{\psi}(E_1\cap E_2)$ and the two are equal. **Q**

**(k)** If $E\in\Sigma_Y$, $\underline{\psi}E=\underline{\phi}E$. **P** (i) Suppose $a\in\underline{\phi}E$, $n\in\mathbb{N}$ and $\gamma<1$. Set $F=E\cap\underline{\phi}E\in\Sigma_Y$, $G=X$. Then $G\cap\underline{\phi}F=\underline{\phi}(E\cap\underline{\phi}E)=\underline{\phi}E$ contains $a$. Take any $x\in F$. Then

$$\int\chi E(xy)h_n(y)\mu_Y(dy)=\int h_n(y)\mu_Y(dy)=1;$$

as $x$ is arbitrary, $a\in\psi_{n\gamma}(E)$. As $n$ and $\gamma$ are arbitrary, $a\in\underline{\psi}E$; as $a$ is arbitrary, $\underline{\phi}E\subseteq\underline{\psi}E$. (ii) Suppose $a\in\underline{\psi}E$. Then there must be some open $G\subseteq X$ and $F\in\Sigma_Y$ and $n\in\mathbb{N}$ such that $a\in G\cap\underline{\phi}F$ and $\int\chi E(xy)h_n(y)\mu_Y(dy)>0$ for every $x\in G\cap F$. This surely implies that $G\cap F\subseteq EY=E$, so that $GY\cap F\subseteq E$. But $a\in G\subseteq\underline{\phi}(GY)$, by 447B, so

$$a\in\underline{\phi}(GY)\cap\underline{\phi}F=\underline{\phi}(GY\cap F)\subseteq\underline{\phi}E.$$

This shows that $\underline{\psi}E\subseteq\underline{\phi}E$. **Q**

**(l)** Thus we have assembled all the facts required to establish that $\underline{\psi}\restriction\Sigma_Z$ is a left-translation-invariant partial lower density extending $\underline{\phi}$.

**447G Lemma** Let $X$ be a $\sigma$-compact locally compact Hausdorff topological group, and $\langle Y_n\rangle_{n\in\mathbb{N}}$ a non-increasing sequence of compact subgroups of $X$ with intersection $Y$. Let $\Sigma$ be the algebra of Haar measurable subsets of $X$; set $\Sigma_{Y_n}=\{E:E\in\Sigma,\ EY_n=E\}$ for each $n$, and $\Sigma_Y=\{E:E\in\Sigma,\ EY=E\}$. Suppose that for each $n\in\mathbb{N}$ we are given a left-translation-invariant partial lower density $\underline{\phi}_n:\Sigma_{Y_n}\to\Sigma_{Y_n}$, and that $\underline{\phi}_{n+1}$ extends $\underline{\phi}_n$ for every $n$. Then there is a left-translation-invariant partial lower density $\underline{\phi}:\Sigma_Y\to\Sigma_Y$ extending every $\underline{\phi}_n$.

**proof (a)** Fix a left Haar measure $\mu$ on $X$, and for each $n\in\mathbb{N}$ let $\nu_n$ be the Haar probability measure on $Y_n$ (442Ie). As noted in 443Sb, the modular function of $X$ must be equal to 1, and equal to the modular function of $Y_n$, everywhere in every $Y_n$.

**(b)** We need to know that for any $E\in\Sigma_Y$ there is an $F$ in the $\sigma$-algebra $\Lambda$ generated by $\bigcup_{n\in\mathbb{N}}\Sigma_{Y_n}$ such that $E\triangle F$ is negligible. **P** Because $X$ is $\sigma$-compact and $\mu$ is a Radon measure (442Ac), there is a sequence $\langle K_i\rangle_{i\in\mathbb{N}}$ of compact sets such that $K_i\subseteq E$ for every $i$ and $E\setminus\bigcup_{i\in\mathbb{N}}K_i$ is negligible. For each $i\in\mathbb{N}$, $\bigcap_{n\in\mathbb{N}}K_iY_n=K_iY$ (4A5Eh), so is included in $E$. Set $F=\bigcup_{i\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}K_iY_n$; then $F$ belongs to the $\sigma$-algebra generated by $\bigcup_{n\in\mathbb{N}}\Sigma_{Y_n}$, and $K_i\subseteq F\subseteq E$ for every $i$, so $E\triangle F$ is negligible. **Q**

**(c)** For each $E\in\Sigma_Y$, $n\in\mathbb{N}$ set

$$g_{En}(x)=\nu_n(Y_n\cap x^{-1}E)\text{ whenever this is defined.}$$

By 447Ee, $g_{En}$ is defined $\mu$-almost everywhere and is $\Sigma$-measurable. In fact $g_{En}$ is $\Sigma_{Y_n}$-measurable, because $g_{En}(xy)=g_{En}(x)$ whenever $x\in X$, $y\in Y_n$ and either is defined. If $F\in\Sigma_{Y_n}$ then $g_{En}(x)\chi F(x)=\nu_n(Y_n\cap x^{-1}(E\cap F))$ whenever this is defined, which is almost everywhere; so $\int_F g_{En}d\mu=\mu(E\cap F)$, by 447Ee. If $E$, $E'\in\Sigma_Y$ and $E\triangle E'$ is negligible, then $g_{En}=_{\text{a.e.}}g_{E'n}$, because $g_{E\triangle E',n}=0$ a.e.

It follows that $\langle g_{En}\rangle_{n\in\mathbb{N}}\to\chi E$ $\mu$-a.e. for every $E\in\Sigma_Y$. **P** Let $G\subseteq X$ be any non-empty relatively compact open set, and set $U=GY_0$, so that $U$ also is a non-empty relatively compact open set, $UY_n=U$ for every $n$ and $UY=U$. Set $\mu_U(F)=\dfrac{\mu F}{\mu U}$ whenever $F\in\Sigma$ and $F\subseteq U$, so that $\mu_U$ is a probability measure on $U$.

Writing $\Sigma^{(U)}_{Y_n}$, $\Sigma^{(U)}_Y$ and $\Lambda^{(U)}$ for the subspace $\sigma$-algebras on $U$ generated by $\Sigma_{Y_n}$, $\Sigma_Y$ and $\Lambda$, we see that if $F\in\Sigma^{(U)}_{Y_n}$ then

$$\int_F g_{En} d\mu_U = \frac{1}{\mu U}\int_F g_{En} d\mu = \mu_U(E \cap F).$$

So $g_{En}\restriction U$ is a conditional expectation of $\chi(E\cap U)$ on $\Sigma^{(U)}_{Y_n}$. By Lévy's martingale theorem (275I), $\langle g_{En}\rangle_{n\in\mathbb{N}}$ converges almost everywhere in $U$ to a conditional expectation $g$ of $\chi(E \cap U)$ on $\Lambda^{(U)}$, because of course $\Lambda^{(U)}$ is the $\sigma$-algebra of subsets of $U$ generated by $\bigcup_{n\in\mathbb{N}} \Sigma^{(U)}_{Y_n}$. But as there is an $F \in \Lambda$ such that $E\triangle F$ is negligible, by (b) above, $g$ must be equal to $\chi E$ almost everywhere in $U$.

Thus $g_{En} \to \chi E$ almost everywhere in $U$ and therefore almost everywhere in $G$. As $G$ is arbitrary, $g_{En} \to \chi E$ almost everywhere in $X$, by 412Jb (applied to the family $\mathcal{K}$ of subsets of relatively compact open sets). **Q**

**(d)** Now we can use the method of 341G, as follows. For $E \in \Sigma_Y$, $k \ge 1$ and $n \in \mathbb{N}$ set

$$H_{kn}(E) = \{x : x \in \operatorname{dom}(g_{En}),\ g_{En}(x) \ge 1 - 2^{-k}\} \in \Sigma_{Y_n},$$

$$\tilde{H}_{kn}(E) = \underline{\phi}_n(H_{kn}(E)), \quad \underline{\phi}E = \bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde{H}_{km}(E).$$

By the arguments of parts (e)-(i) of the proof of 341G, $\underline{\phi}$ is a lower density on $\Sigma_Y$ extending every $\underline{\phi}_n$.

**(e)** To see that $\underline{\phi}$ is left-translation-invariant, we may argue as follows. Let $E \in \Sigma_Y$ and $a \in X$. Then, for any $n$,

$$g_{aE,n}(x) = \nu_n(Y_n \cap x^{-1}aE) = g_{En}(a^{-1}x)$$

for almost every $x$, so $aH_{kn}(E)\triangle H_{kn}(aE)$ is negligible, and

$$\tilde{H}_{kn}(aE) = \underline{\phi}_n(H_{kn}(aE)) = \underline{\phi}_n(aH_{kn}(E)) = a\underline{\phi}_n(H_{kn}(E)) = a\tilde{H}_{kn}(E),$$

for every $k$. Accordingly

$$\underline{\phi}(aE) = \bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde{H}_{km}(aE) = a(\bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde{H}_{km}(E)) = a\underline{\phi}(E),$$

as required.

**447H Lemma** Let $X$ be a locally compact Hausdorff topological group, and $\Sigma$ the algebra of Haar measurable sets in $X$. Then there is a left-translation-invariant lower density $\underline{\phi} : \Sigma \to \Sigma$.

**proof (a)** To begin with (down to the end of (c) below) let us suppose that $X$ is $\sigma$-compact. By 446P, there is a family $\langle X_\xi\rangle_{\xi\le\kappa}$ of closed subgroups of $X$, where $\kappa$ is an infinite cardinal, such that

$X_0$ is an open subgroup of $X$,
for every $\xi < \kappa$, $X_{\xi+1}$ is a normal subgroup of $X_\xi$ and $X_\xi/X_{\xi+1}$ has a $B$-sequence,
for every non-zero limit ordinal $\xi \le \kappa$, $X_\xi = \bigcap_{\eta<\xi} X_\eta$,
$X_1$ is compact,
$X_\kappa = \{e\}$, where $e$ is the identity of $X$.

Note that for every $\xi \le \kappa$, the modular function $\Delta_\xi$ of $X_\xi$ is just the restriction to $X_\xi$ of the modular function $\Delta$ of $X$. **P** For $\xi = 0$ this is because $X_0$ is an open subgroup of $X$ (443Sd). For $\xi \ge 1$, $X_\xi$ is compact, so $\Delta_\xi$ and $\Delta\restriction X_\xi$ are both constant with value 1, as noted in 443Sb. **Q**

**(b)** For each $\xi \le \kappa$, write $\Sigma_\xi$ for the $\sigma$-algebra $\{E : E \in \Sigma,\ EX_\xi = E\}$. I seek to choose inductively a family $\langle\underline{\phi}_\xi\rangle_{\xi\le\kappa}$ such that each $\underline{\phi}_\xi : \Sigma_\xi \to \Sigma_\xi$ is a left-translation-invariant partial lower density, and $\underline{\phi}_\xi$ extends $\underline{\phi}_\eta$ whenever $\eta < \xi$.

**(i)** *Start* Since $X_0$ is an open subgroup of $X$, every member of $\Sigma_0$ is open, and we can start the induction by setting $\underline{\phi}_0 E = E$ for every $E \in \Sigma_0$.

**(ii)** *Inductive step to a successor ordinal* If we have defined $\langle\underline{\phi}_\eta\rangle_{\eta\le\xi}$, where $\xi < \kappa$, then $X_{\xi+1} \lhd X_\xi$ is compact and $X_\xi/X_{\xi+1}$ has a $B$-sequence. So the conditions of 447F are satisfied and $\underline{\phi}_\xi$ has an extension to a left-translation-invariant partial lower density $\underline{\phi}_{\xi+1} : \Sigma_{\xi+1} \to \Sigma_{\xi+1}$. Of course $\underline{\phi}_{\xi+1}$ extends $\underline{\phi}_\eta$ for every $\eta \le \xi$ because $\underline{\phi}_\xi$ does.

**(iii)** *Inductive step to a limit ordinal of countable cofinality* If we have defined $\langle\underline{\phi}_\eta\rangle_{\eta<\xi}$, where $\xi \le \kappa$ is a non-zero limit ordinal of countable cofinality, let $\langle\xi_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence with supremum $\xi$; we may suppose that $\xi_0 \ge 1$, so that every $X_{\xi_n}$ is compact. Then $X_\xi = \bigcap_{n\in\mathbb{N}} X_{\xi_n}$, so by 447G there is a left-translation-invariant partial lower density $\underline{\phi} : \Sigma_\xi \to \Sigma_\xi$ extending every $\underline{\phi}_{\xi_n}$, and therefore extending $\underline{\phi}_\eta$ whenever $\eta < \xi$.

**(iv)** *Inductive step to a limit ordinal of uncountable cofinality* Suppose we have defined $\langle \underline{\phi}_\eta \rangle_{\eta<\xi}$, where $\xi \le \kappa$ is a limit ordinal of uncountable cofinality. Then for every $E \in \Sigma_\xi$ there are an $\eta < \xi$ and an $F \in \Sigma_\eta$ such that $E \triangle F$ is negligible. **P** (Cf. part (b) of the proof of 447G.) Because $X$ is $\sigma$-compact, there are non-decreasing sequences $\langle K_i \rangle_{i\in\mathbb{N}}$, $\langle L_i \rangle_{i\in\mathbb{N}}$ of compact subsets of $E$, $X \setminus E$ respectively such that $E \setminus \bigcup_{i\in I} K_i$ and $(X \setminus E) \setminus \bigcup_{i\in\mathbb{N}} L_i$ are negligible. For each $i \in \mathbb{N}$, $K_i X_\xi \cap L_i \subseteq EX_\xi \setminus E$ is empty; by 4A5Eh again, there is an $\eta_i < \xi$ such that $K_i X_{\eta_i} \cap L_i$ is empty. Set $\eta = \sup_{i\in\mathbb{N}} \eta_i$, $F = \bigcup_{i\in\mathbb{N}} K_i X_\eta$; this works. **Q**

Accordingly we have a function $\underline{\phi}_\xi : \Sigma_\xi \to \Sigma_\xi$ defined by writing $\underline{\phi}_\xi(E) = \underline{\phi}_\eta(F)$ whenever $E \in \Sigma_\xi$, $\eta < \xi$, $F \in \Sigma_\eta$ and $E \triangle F$ is negligible. **P** If $\eta \le \eta' < \xi$ and $F \in \Sigma_\eta$, $F' \in \Sigma_{\eta'}$ are such that $E \triangle F$ and $E \triangle F'$ are both negligible, then $F \triangle F'$ is negligible so $\underline{\phi}_\eta(F) = \underline{\phi}_{\eta'}(F) = \underline{\phi}_{\eta'}(F')$. **Q** It is easy to check that $\underline{\phi}_\xi$ is a left-translation-invariant partial lower density (cf. part (A-d) of the proof of 341H), and of course it extends $\underline{\phi}_\eta$ for every $\eta < \xi$.

**(c)** On completing the induction, we see that $\Sigma_\kappa = \Sigma$, so that $\underline{\phi}_\kappa : \Sigma \to \Sigma$ is a left-translation-invariant lower density.

**(d)** For the general case, recall that $X$ certainly has an open $\sigma$-compact subgroup $Y$ say (4A5El). If $\Sigma$ is the algebra of Haar measurable subsets of $X$, and $\mathrm{T}$ is the algebra of Haar measurable subsets of $Y$, then $\mathrm{T}$ is just $\Sigma \cap \mathcal{P}Y = \{E \cap Y : E \in \Sigma\}$, and the Haar negligible subsets of $Y$ are just sets of the form $E \cap Y$ where $E$ is a Haar negligible subset of $X$ (443F).

Let $\underline{\psi} : \mathrm{T} \to \mathrm{T}$ be a left-translation-invariant lower density. For $E \in \Sigma$ set

$$\underline{\phi}E = \{x : x \in X,\ e \in \underline{\psi}(Y \cap x^{-1}E)\},$$

where $e$ is the identity of $X$. It is easy to check that

$$\underline{\phi}\emptyset = \emptyset,$$

$$\underline{\phi}E = \underline{\phi}F \text{ if } E \triangle F \text{ is negligible},$$

$$\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F \text{ for all } E,\ F \in \Sigma,$$

directly from the corresponding properties of $\underline{\psi}$. If $E \in \Sigma$ and $a \in X$, then

$$x \in \underline{\phi}E \iff e \in \underline{\psi}(Y \cap x^{-1}E) \iff e \in \underline{\psi}(Y \cap (ax)^{-1}aE) \iff ax \in \underline{\phi}(aE),$$

so $\underline{\phi}(aE) = a\underline{\phi}E$.

I have not yet checked that $E \triangle \underline{\phi}E$ is always negligible. But if $E \in \Sigma$, then

$$E \triangle \underline{\phi}E = \{x : e \in \underline{\psi}(Y \cap x^{-1}E) \triangle (Y \cap x^{-1}E)\},$$

so

$$\begin{aligned}(E \triangle \underline{\phi}E) \cap Y &= \{x : x \in Y,\ e \in \underline{\psi}(x^{-1}(E \cap Y))\} \triangle (E \cap Y) \\ &= \{x : x \in Y,\ e \in x^{-1}\underline{\psi}(E \cap Y)\} \triangle (E \cap Y) = \underline{\psi}(E \cap Y) \triangle (E \cap Y)\end{aligned}$$

is negligible. Moreover, for any $a \in X$,

$$(E \triangle \underline{\phi}E) \cap aY = a((a^{-1}E \triangle \underline{\phi}(a^{-1}E)) \cap Y)$$

because $\underline{\phi}$ is translation-invariant, so $(E \triangle \underline{\phi}E) \cap aY$ is negligible. Since $\{aY : a \in X\}$ is an open cover of $X$, $E \triangle \underline{\phi}E$ is negligible (412Jb again). In particular, $\underline{\phi}E \in \Sigma$. So $\underline{\phi} : \Sigma \to \Sigma$ is a left-translation-invariant lower density, as required.

**447I Theorem** (Ionescu Tulcea & Ionescu Tulcea 67) Let $X$ be a locally compact Hausdorff topological group. Then it has a left-translation-invariant lifting for its Haar measures.

**proof** (Cf. 345B-345C.) Write $\Sigma$ for the algebra of Haar measurable subsets of $X$, and let $\underline{\phi} : \Sigma \to \Sigma$ be a left-translation-invariant lower density (447H). Let $\phi_0 : \Sigma \to \Sigma$ be any lifting such that $\phi_0 E \supseteq \underline{\phi}E$ for every $E \in \Sigma$ (341Jb). For $E \in \Sigma$, set

$$\phi E = \{x : e \in \phi_0(x^{-1}E)\},$$

where $e$ is the identity of $X$. It is easy to check that $\phi : \Sigma \to \mathcal{P}X$ is a Boolean homomorphism. Also

$$x \in \underline{\phi}E \Longrightarrow e \in \underline{\phi}(x^{-1}E) \Longrightarrow e \in \phi_0(x^{-1}E) \Longrightarrow x \in \phi E.$$

So $\phi$ is a lifting (341Ib). Finally, $\phi$ is left-translation-invariant by the argument used in (d) of the proof of 447H (and also in (e) of the proof of 345B).

**447J Corollary** Let $X$ be any topological group carrying Haar measures. Then it has a left-translation-invariant lifting for its left Haar measures.

**proof** Let $\mu$ be a left Haar measure on $X$. By 443L, we have a locally compact Hausdorff topological group $Z$ and a continuous homomorphism $f : X \to Z$, inverse-measure-preserving for $\mu$ and an appropriate left Haar measure $\nu$ on $Z$, such that for every $E$ in the domain $\Sigma$ of $\mu$ there is an $F$ in the domain $\mathrm{T}$ of $\nu$ such that $f^{-1}[F] \subseteq E$ and $E \setminus f^{-1}[F]$ is negligible. Let $\psi$ be a left-translation-invariant lifting for $\nu$. Since $F^\bullet \mapsto f^{-1}[F]^\bullet$ is an isomorphism between the measure algebras of $\mu$ and $\nu$, we have a lifting $\phi : \Sigma \to \Sigma$ given by saying that $\phi E = f^{-1}[\psi F]$ whenever $F \in \mathrm{T}$ and $E \triangle f^{-1}[F]$ is negligible (346D). Now $\phi$ is left-translation-invariant because $f$ is a group homomorphism and $\psi$ is left-translation-invariant.

**447X Basic exercises >(a)** Let $X = \mathbb{R} \times \{-1, 1\}$, with its usual topology, and define a multiplication on $X$ by setting $(x, \delta)(y, \epsilon) = (x + \delta y, \delta\epsilon)$. Show that $X$ is a locally compact topological group. Show that there is no lifting for the Haar measure algebra of $X$ which is both left- and right-translation-invariant. (*Hint*: 345Xc.)

**(b)** Let $X$ be a topological group carrying Haar measures which has a $B$-sequence. Show that it has a $B$-sequence $\langle V_n \rangle_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \mu(V_n V_n^{-1})/\mu V_n$ is finite for any Haar measure $\mu$ on $X$, whether left or right.

**(c)** Let $X$ be a topological group with a left Haar measure $\mu$, and $\langle V_n \rangle_{n\in\mathbb{N}}$ a $B$-sequence for $X$. Show that if $f \in \mathcal{L}^0(\mu)$ is locally integrable, then $f(x) = \lim_{n\to\infty} \frac{1}{\mu V_n} \int_{xV_n} f d\mu$ for almost every $x$.

**447Y Further exercises (a)** Describe a compact Hausdorff topological group such that its Haar measure has no lifting which is both left- and right-translation-invariant.

**(b)** Let $(X, \Sigma, \mu)$ be a measure space, with measure algebra $\mathfrak{A}$, and $\underline{\theta} : \mathfrak{A} \to \Sigma$ a lower density. Show that we have a function $q : L^\infty(\mathfrak{A})^+ \to L^\infty(\Sigma)^+$ such that $\{x : q(u)(x) > \alpha\} = \bigcup_{\beta>\alpha} \underline{\theta}[\![u > \beta]\!]$ for every $\alpha \ge 0$ and $u \in L^\infty(\mathfrak{A})^+$. Show that $q(u)^\bullet = u$, $q(\alpha u) = \alpha q(u)$, $q(u \wedge v) = q(u) \wedge q(v)$ and $q(\chi a) = \chi(\underline{\theta} a)$ for every $u$, $v \in L^\infty(\mathfrak{A})^+$, $\alpha \ge 0$ and $a \in \mathfrak{A}$.

**447 Notes and comments** The structure of the proof of 447I is exactly that of the proof of the ordinary Lifting Theorem in §341; the lifting is built from a lower density which is constructed inductively on a family of sub-$\sigma$-algebras. To get a translation-invariant lifting it is natural to look for a translation-invariant lower density, and a simple trick (already used in §345) ensures that this is indeed enough. The refinements we need here are dramatic but natural. To make the final lower density $\underline{\phi}$ (in 447H) translation-invariant, it is clearly sensible (if we can do it) to keep all the partial lower densities $\underline{\phi}_\xi$ translation-invariant. This means that their domains $\Sigma_\xi$ should be translation-invariant. It does not quite follow that they have to be of the form $\Sigma_\xi = \{E : EX_\xi = X_\xi\}$ for closed subgroups $X_\xi$, but if we look at the leading example of $\{0, 1\}^I$ (345C) this also is a reasonable thing to try first. So now we have to consider what extra hypotheses will be needed to make the induction work. The inductive step to limit ordinals of uncountable cardinality remains elementary, at least if the $X_\xi$ are compact (part (b-iv) of the proof of 447H). The inductive step to limit ordinals of countable cofinality (447G) is harder, but can be managed with ideas already presented. Indeed, compared with the version in 341G, we have the advantage of a formula for the auxiliary functions $g_{En}$, which is very helpful when we come to translation-invariance. We have to do something about the fact that we are no longer working with a probability space – that is the point of the $\mu_U$ in part (c) of the proof of 447G. (Another expression of the manoeuvre here is in 369Xq.)

Where we do need a new idea is in the inductive step to a successor ordinal. If $\Sigma_{\xi+1}$ is to be translation-invariant, it must be much bigger than the $\sigma$-algebra generated by $\Sigma_\xi \cup \{E\}$, as discussed in 341F. To make the step a small one (and therefore presumably easier), we want $X_{\xi+1}$ to be a large subgroup of $X_\xi$ in some sense; as it turns out, a helpful approach is to ask for $X_{\xi+1}$ to be a normal subgroup of $X_\xi$ and for $X_\xi/X_{\xi+1}$ to be small. At this point we have to know something of the structure theory of locally compact topological groups. The right place to start is surely the theory of compact Hausdorff groups. Such a group $X$ actually has a continuous decreasing chain $\langle X_\xi\rangle_{\xi\le\kappa}$ of closed normal subgroups, from $X_0 = X$ to $X_\kappa = \{e\}$, such that all the quotients $X_\xi/X_{\xi+1}$ are Lie groups. I do not define 'Lie group' here, because for our purposes it is enough to know that the quotients have faithful finite-dimensional representations, and therefore have '$B$-sequences' in the sense of 446L. Having identified this as a relevant property, it is not hard to repeat arguments from §221 and §261 to prove versions of Vitali's theorem and Lebesgue's Density Theorem in such groups (447C-447D). This will evidently provide translation-invariant lower densities for groups of this special type, just as Lebesgue lower density is a translation-invariant lower density on $\mathbb{R}^r$ (345B).

Of course we still have to find a way of combining this construction with a translation-invariant lower density on $\Sigma_\xi$ to produce a translation-invariant lower density on $\Sigma_{\xi+1}$, and this is what I do in 447F. The argument I offer is essentially that of Ionescu Tulcea & Ionescu Tulcea 67, §7, and is the deepest part of this section.

For compact groups, these ideas are all we need, and indeed the step to a limit ordinal of countable cofinality is a little easier, since we have a Haar probability measure on the whole group. The next step, to general $\sigma$-compact locally compact groups, demands much deeper ideas from the structure theory, but from the point of view of the present section the modifications are minor. The subgroups $X_\xi$ are now not always normal subgroups of $X$, which means that we have to be more careful in the description of the quotient spaces $X/X_\xi$ (they must consist of *left* cosets), and we have to watch the modular functions of the $X_\xi$ in order to be sure that there are invariant measures on the quotients. An extra obstacle at the beginning is that we may have to start the chain with a proper subgroup $X_0$ of $X$, but since $X_0$ can be taken to be open, it is pretty clear that this will not be serious, and in fact it gives no trouble (part (b-i) of the proof of 447H). For $\xi \ge 1$, the $X_\xi$ are compact, so the inductive steps to limit ordinals are nearly the same.

The step to a general locally compact Hausdorff topological group (part (d) of the proof of 447H) is essentially elementary. And finally I note that the whole thing applies to general topological groups with Haar measures (447J), for the usual reasons. There is an implicit challenge here: find expressions of the arguments used in this section which will be valid in the more general context. The measure-theoretic part of such a programme might be achievable, but I do not see any hope of a workable structure theory to match that of §446 which does not use 443L or something like it.

Version of 12.4.13

## 448 Polish group actions

I devote this section to two quite separate theorems. The first is an interesting result about measures on Polish spaces which are invariant under actions of Polish groups. In contrast to §441, we no longer have a strong general existence theorem for such measures, but instead have a natural necessary and sufficient condition in terms of countable dissections: there is an invariant probability measure on $X$ if and only if there is no countable dissection of $X$ into Borel sets which can be rearranged, by the action of the group, into two copies of $X$ (448P).

The principal ideas needed here have already been set out in §395, and in many of the proofs I allow myself to direct you to the corresponding arguments there rather than write the formulae out again. I do not think you need read through §395 before embarking on this section; I will try to give sufficiently detailed references so that you can take them one paragraph at a time, and many of the arguments referred to are in any case elementary. But unless you are already familiar with this topic, you will need a copy of §395 to hand to fully follow the proofs below.

The second theorem concerns the representation of group actions on measure algebras in terms of group actions on measure spaces. If we have a locally compact Polish group $G$ (so that we do have Haar measures), and a Borel measurable action of $G$ on the measure algebra of a Radon measure $\mu$ on a Polish space $X$, then it can be represented by a Borel measurable action of $G$ on $X$ (448S). The proof is mostly descriptive set theory based on §§423-424, but it also uses some interesting facts about $L^0$ spaces (448Q-448R).

**448A Definitions** (Compare 395A.) Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra, and $G$ a subgroup of $\operatorname{Aut}\mathfrak{A}$. For $a$, $b \in \mathfrak{A}$ I will say that an isomorphism $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ between the corresponding principal ideals belongs to the **countably full local semigroup generated by** $G$ if there are a countable partition of unity $\langle a_i \rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \pi_i \rangle_{i\in I}$ in $G$ such that $\phi c = \pi_i c$ whenever $i \in I$ and $c \subseteq a_i$. If such an isomorphism exists I will say that $a$ and $b$ are **$G$-$\sigma$-equidecomposable**.

I write $a \preccurlyeq^\sigma_G b$ to mean that there is a $b' \subseteq b$ such that $a$ and $b'$ are $G$-$\sigma$-equidecomposable.

As in §395, I will say that a function $f$ with domain $\mathfrak{A}$ is **$G$-invariant** if $f(\pi a) = f(a)$ whenever $a \in \mathfrak{A}$ and $\pi \in G$.

I have expressed these definitions, and most of the work below, in terms of abstract Dedekind $\sigma$-complete Boolean algebras. The applications I have in mind for this section are to $\sigma$-algebras of sets. If you have already worked through §395, the version here should come very easily; but even if you have not, I think that the extra abstraction clarifies some of the ideas.

**448B** I begin with results corresponding to 395B-395D; there is hardly any difference, except that we must now occasionally pause to check that a partition of unity is countable.

**Lemma** Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra and $G$ a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $G^*_\sigma$ for the countably full local semigroup generated by $G$.

(a) If $a$, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to $G^*_\sigma$, then $\phi^{-1} : \mathfrak{A}_b \to \mathfrak{A}_a$ also belongs to $G^*_\sigma$.

(b) Suppose that $a$, $b$, $a'$, $b' \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_{a'}$, $\psi : \mathfrak{A}_b \to \mathfrak{A}_{b'}$ belong to $G^*_\sigma$. Then $\psi\phi \in G^*_\sigma$; its domain is $\mathfrak{A}_c$ where $c = \phi^{-1}(b \cap a')$, and its set of values is $\mathfrak{A}_{c'}$ where $c' = \psi(b \cap a')$.

(c) If $a$, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to $G^*_\sigma$, then $\phi{\restriction}\mathfrak{A}_c \in G^*_\sigma$ for any $c \subseteq a$.

(d) Suppose that $a$, $b \in \mathfrak{A}$ and that $\psi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism such that there are a countable partition of unity $\langle a_i \rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \phi_i \rangle_{i\in I}$ in $G^*_\sigma$ such that $\psi c = \phi_i c$ whenever $i \in I$ and $c \subseteq a_i$. Then $\psi \in G^*_\sigma$.

**proof (a)** As 395Bb.

**(b)** As 395Bc.

**(c)** As 395Bd.

**(d)** For each $i \in I$, let $\langle a_{ij} \rangle_{j\in J(i)}$, $\langle \pi_{ij} \rangle_{j\in J(i)}$ witness that $\phi_i \in G^*_\sigma$; then $\langle a_i \cap a_{ij} \rangle_{i\in I, j\in J(i)}$ and $\langle \pi_{ij} \rangle_{i\in I, j\in J(i)}$ witness that $\psi \in G^*_\sigma$.

**448C Lemma** Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra and $G$ a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $G^*_\sigma$ for the countably full local semigroup generated by $G$.

(a) For $a$, $b \in \mathfrak{A}$, $a \preccurlyeq^\sigma_G b$ iff there is a $\phi \in G^*_\sigma$ such that $a \in \operatorname{dom}\phi$ and $\phi a \subseteq b$.

(b)(i) $\preccurlyeq^\sigma_G$ is transitive and reflexive;

(ii) if $a \preccurlyeq^\sigma_G b$ and $b \preccurlyeq^\sigma_G a$ then $a$ and $b$ are $G$-$\sigma$-equidecomposable.

(c) $G$-$\sigma$-equidecomposability is an equivalence relation on $\mathfrak{A}$.

(d) If $\langle a_i \rangle_{i\in I}$ and $\langle b_i \rangle_{i\in I}$ are countable families in $\mathfrak{A}$, of which $\langle b_i \rangle_{i\in I}$ is disjoint, and $a_i \preccurlyeq^\sigma_G b_i$ for every $i \in I$, then $\sup_{i\in I} a_i \preccurlyeq^\sigma_G \sup_{i\in I} b_i$.

**proof** The arguments of 395C apply unchanged, calling on 448B in place of 395B.

**448D Theorem** Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra and $G$ a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is an $a \neq 1$ such that $a$ is $G$-$\sigma$-equidecomposable with 1;

(ii) there is a disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all $G$-$\sigma$-equidecomposable;

(iii) there are non-zero $G$-$\sigma$-equidecomposable $a$, $b$, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) there are $G$-$\sigma$-equidecomposable $a$, $b \in \mathfrak{A}$ such that $a \subset b$.

**proof** As 395D.

**448E Definition** (Compare 395E.) Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra and $G$ a subgroup of $\operatorname{Aut}\mathfrak{A}$. I will say that $G$ is **countably non-paradoxical** if the statements of 448D are false; that is, if one of the following equiveridical statements is true:

(i) if $a$ is $G$-$\sigma$-equidecomposable with 1 then $a = 1$;

(ii) there is no disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all $G$-$\sigma$-equidecomposable;

(iii) there are no non-zero $G$-$\sigma$-equidecomposable $a$, $b$, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) if $a$, $b \in \mathfrak{A}$ are $G$-$\sigma$-equidecomposable and $a \subseteq b$ then $a = b$.

**448F** We now come to one of the points where we need to find a new path because we are looking at algebras which need not be Dedekind complete. Provided the original group $G$ is *countable*, we can still follow the general line of §395, as follows.

**Lemma** (Compare 395G.) Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra and $G$ a countable subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of $G$.

(a) For any $a \in \mathfrak{A}$, $\operatorname{upr}(a,\mathfrak{C})$ (313S) is defined, and is given by the formula

$$\operatorname{upr}(a,\mathfrak{C}) = \sup\{\pi a : \pi \in G\}.$$

(b) If $G^*_\sigma$ is the countably full local semigroup generated by $G$, then $\phi(c\cap a) = c \cap \phi a$ whenever $\phi \in G^*_\sigma$, $a \in \operatorname{dom}\phi$ and $c \in \mathfrak{C}$.

(c) $\operatorname{upr}(\phi a,\mathfrak{C}) = \operatorname{upr}(a,\mathfrak{C})$ whenever $\phi \in G^*_\sigma$ and $a \in \operatorname{dom}\phi$; consequently, $\operatorname{upr}(a,\mathfrak{C}) \subseteq \operatorname{upr}(b,\mathfrak{C})$ whenever $a \preccurlyeq^\sigma_G b$.

(d) If $a \preccurlyeq^\sigma_G b$ and $c \in \mathfrak{C}$ then $a \cap c \preccurlyeq^\sigma_G b \cap c$. So $a \cap c$ and $b \cap c$ are $G$-$\sigma$-equidecomposable whenever $a$ and $b$ are $G$-$\sigma$-equidecomposable and $c \in \mathfrak{C}$.

**proof (a)** As remarked in 395Ga, $\mathfrak{C}$ is order-closed. Because $G$ is countable and $\mathfrak{A}$ is Dedekind $\sigma$-complete, $c^* = \sup\{\pi a : \pi \in G\}$ is defined in $\mathfrak{A}$ If $\phi \in G$, then

$$\phi c^* = \sup\{\phi\pi a : \pi \in G\} \subseteq c^*$$

because $\phi$ is order-continuous and $\phi\pi \in G$ for every $\pi \in G$. Similarly $\phi^{-1}c^* \subseteq c^*$ and $c^* \subseteq \phi c^*$. Thus $\phi c^* = c^*$; as $\phi$ is arbitrary, $c^* \in \mathfrak{C}$.

If $c \in \mathfrak{C}$, then

$$\begin{aligned} a \subseteq c &\iff \pi a \subseteq \pi c \text{ for every } \pi \in G \\ &\iff \pi a \subseteq c \text{ for every } \pi \in G \\ &\iff c^* \subseteq c, \end{aligned}$$

so $c^* = \inf\{c : a \subseteq c \in \mathfrak{C}\}$, taking the infimum in $\mathfrak{C}$, as required in the definition of $\operatorname{upr}(a,\mathfrak{C})$.

**(b)** Suppose that $\langle a_i\rangle_{i\in I}$, $\langle \pi_i\rangle_{i\in I}$ witness that $\phi \in G^*_\sigma$. Then

$$\phi(a\cap c) = \sup_{i\in I}\pi_i(a_i \cap a \cap c) = \sup_{i\in I}\pi_i(a_i\cap a)\cap c = c \cap \phi a.$$

**(c)** For $c \in \mathfrak{C}$,

$$a \subseteq c \iff a \cap c = a \iff \phi(a\cap c) = \phi a \iff c \cap \phi a = \phi a \iff \phi a \subseteq c.$$

**(d)** There is a $\phi \in G^*_\sigma$ such that $\phi a \subseteq b$; now

$$a \cap c \preccurlyeq^\sigma_G \phi(a\cap c) = c \cap \phi a \subseteq b \cap c.$$

**448G** With this support, we can now continue with the ideas of 395H-395L, adding at each step the hypothesis '$G$ is countable' to compensate for the weakening of the hypotheses '$\mathfrak{A}$ is Dedekind complete, $G$ is fully non-paradoxical' to '$\mathfrak{A}$ is Dedekind $\sigma$-complete, $G$ is countably non-paradoxical'.

**Lemma** (Compare 395H.) Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra and $G$ a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $\mathfrak{C}$ for the fixed-point subalgebra of $G$. Take any $a$, $b \in \mathfrak{A}$. Then $c_0 = \sup\{c : c \in \mathfrak{C},\ a \cap c \preccurlyeq^\sigma_G b\}$ is defined in $\mathfrak{A}$ and belongs to $\mathfrak{C}$; $a \cap c_0 \preccurlyeq^\sigma_G b$ and $b \setminus c_0 \preccurlyeq^\sigma_G a$.

**proof** Let $\langle\pi_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $G$. Define $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, setting

$$a_n = (a \setminus \sup_{i<n} a_i) \cap \pi_n^{-1}(b \setminus \sup_{i<n} b_i), \quad b_n = \pi_n a_n.$$

Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_a$ and $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_b$, and $\sup_{n\in\mathbb{N}} a_n$ is $G$-$\sigma$-equidecomposable with $\sup_{n\in\mathbb{N}} b_n$. Set

$$a' = a \setminus \sup_{n\in\mathbb{N}} a_n, \quad b' = b \setminus \sup_{n\in\mathbb{N}} b_n, \quad c_0 = 1 \setminus \operatorname{upr}(a', \mathfrak{C}) \subseteq \sup_{n\in\mathbb{N}} a_n.$$

Then

$$a \cap c_0 \subseteq \sup_{n\in\mathbb{N}} a_n \preccurlyeq_G^\sigma b.$$

Now $b' \subseteq c_0$. **P?** Otherwise, because $c_0 = 1 \setminus \sup_{n\in\mathbb{N}} \pi_n a'$, there must be an $n \in \mathbb{N}$ such that $b' \cap \pi_n a' \neq 0$. But in this case $d = a' \cap \pi_n^{-1} b' \neq 0$, and we have

$$d \subseteq (a \setminus \sup_{i<n} a_i) \cap \pi_n^{-1}(b \setminus \sup_{i<n} b_i),$$

so that $d \subseteq a_n$, which is absurd. **XQ** Consequently

$$b \setminus c_0 \subseteq b \setminus b' = \sup_{n\in\mathbb{N}} b_n \preccurlyeq_G^\sigma a.$$

Now take any $c \in \mathfrak{C}$ such that $a \cap c \preccurlyeq_G^\sigma b$, and consider $c' = c \setminus c_0$. Then $b' \cap c' = 0$, that is, $b \cap c' = \sup_{n\in\mathbb{N}} b_n \cap c'$, which is $G$-$\sigma$-equidecomposable with $\sup_{n\in\mathbb{N}} a_n \cap c' = (a \setminus a') \cap c'$. But now

$$a \cap c' = a \cap c_0 \cap c' \preccurlyeq_G^\sigma b \cap c' \preccurlyeq_G^\sigma (a \cap c') \setminus (a' \cap c');$$

because $G$ is countably non-paradoxical, $a' \cap c'$ must be 0, that is, $c' \subseteq c_0$ and $c \subseteq c_0$. So $c_0$ has the required properties.

**448H Lemma** (Compare 395I.) Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra, not $\{0\}$, and $G$ a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of $G$. Suppose that $a$, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a, \mathfrak{C}) = 1$. Then there are non-negative $u$, $v \in L^0(\mathfrak{C})$ such that

$$[\![u \geq n]\!] = \max\{c : c \in \mathfrak{C}, \text{ there is a disjoint family } \langle d_i\rangle_{i<n} \text{ such that } a \cap c \preccurlyeq_G^\sigma d_i \subseteq b \text{ for every } i < n\},$$

$$[\![v \leq n]\!] = \max\{c : c \in \mathfrak{C}, \text{ there is a family } \langle d_i\rangle_{i<n} \text{ such that } d_i \preccurlyeq_G^\sigma a \text{ for every } i < n \text{ and } b \cap c \subseteq \sup_{i<n} d_i\}$$

for every $n \in \mathbb{N}$. Moreover, we have

(i) $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$,

(ii) $[\![v > 0]\!] = \operatorname{upr}(b, \mathfrak{C})$,

(iii) $u \leq v \leq u + \chi 1$.

**proof** The argument of 395I applies unchanged, except that every $\preccurlyeq_G^\tau$ must be replaced with a $\preccurlyeq_G^\sigma$, and we use 448F and 448G in place of 395G and 395H. $\mathfrak{C}$ is Dedekind $\sigma$-complete because it is order-closed in the Dedekind $\sigma$-complete algebra $\mathfrak{A}$ (314Eb).

**448I Notation** (Compare 395J.) In the context of 448H, I will write $\lfloor b : a\rfloor$ for $u$, $\lceil b : a\rceil$ for $v$.

**448J Lemma** (Compare 395K-395L.) Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra, not $\{0\}$, and $G$ a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that $a$, $a_1$, $a_2$, $b$, $b_1$, $b_2 \in \mathfrak{A}$ and that

$$\operatorname{upr}(a, \mathfrak{C}) = \operatorname{upr}(a_1, \mathfrak{C}) = \operatorname{upr}(a_2, \mathfrak{C}) = 1.$$

Then

(a) $\lfloor 0 : a\rfloor = \lceil 0 : a\rceil = 0$ and $\lfloor 1 : a\rfloor \geq \chi 1$.

(b) If $b_1 \preccurlyeq_G^\sigma b_2$ then $\lfloor b_1 : a\rfloor \leq \lfloor b_2 : a\rfloor$ and $\lceil b_1 : a\rceil \leq \lceil b_2 : a\rceil$.

(c) $\lceil b_1 \cup b_2 : a\rceil \leq \lceil b_1 : a\rceil + \lceil b_2 : a\rceil$.

(d) If $b_1 \cap b_2 = 0$, then $\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor \leq \lfloor b_1 \cup b_2 : a\rfloor$.

(e) If $c \in \mathfrak{C}$ is such that $a \cap c$ is a relative atom over $\mathfrak{C}$, then $c \subseteq [\![\lceil b:a\rceil - \lfloor b:a\rfloor = 0]\!]$.

(f) $\lfloor b:a_2\rfloor \ge \lfloor b:a_1\rfloor \times \lfloor a_1:a_2\rfloor$, $\lceil b:a_2\rceil \le \lceil b:a_1\rceil \times \lceil a_1:a_2\rceil$.

**proof** As in 395K-395L.

**448K** For the result corresponding to 395Mb, we again need to find a new approach; I deal with it by adding a further hypothesis to the list which has already accreted.

**Definition** Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra and $G$ a countable subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. I will say that $G$ has the **$\sigma$-refinement property** if for every $a \in \mathfrak{A}$ there is a $d \subseteq a$ such that $d \preccurlyeq^\sigma_G a \setminus d$ and $a' = a \setminus \operatorname{upr}(d, \mathfrak{C})$ is a relative atom over $\mathfrak{C}$, that is, every $b \subseteq a'$ is expressible as $a' \cap c$ for some $c \in \mathfrak{C}$.

(If we replace $\preccurlyeq^\sigma_G$ with $\preccurlyeq^\tau_G$, as used in §395, we see that 395Ma could be read as 'if $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then any subgroup of $\operatorname{Aut}\mathfrak{A}$ has the $\tau$-refinement property'.)

**448L** I give the principal case in which the '$\sigma$-refinement property' just defined arises.

**Proposition** Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra with countable Maharam type (definition: 331F). Then any countable subgroup of $\operatorname{Aut}\mathfrak{A}$ has the $\sigma$-refinement property.

**proof (a)** Let $E$ be a countable subset of $\mathfrak{A}$ which $\tau$-generates $\mathfrak{A}$, and $\mathfrak{C}$ the subalgebra of $\mathfrak{A}$ generated by $E$; then $\mathfrak{C}$ is countable (331Gc), and the smallest order-closed subset of $\mathfrak{A}$ including $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$ (313Fc), so must be $\mathfrak{A}$ itself.

**(b)** Suppose that $b \in \mathfrak{A} \setminus \{0\}$ and $\pi \in \operatorname{Aut}\mathfrak{A}$ are such that $b \cap \pi b = 0$. Then there is an $e \in \mathfrak{C}$ such that $b \cap e \setminus \pi e \ne 0$. **P?** Otherwise, set

$$D = \{d : d \in \mathfrak{A},\, b \cap d \setminus \pi d = 0\}.$$

Then $\mathfrak{C} \subseteq D$, but $b \notin D$. So $D$ cannot be order-closed. **case 1** If $D_0 \subseteq D$ is a non-empty upwards-directed set with supremum $d_0 \notin D$, then $b \cap d_0 \setminus \pi d_0 \ne 0$, so there is a $d \in D_0$ such that $b \cap d \setminus \pi d_0 \ne 0$; but now $d \notin D$, which is impossible. **case 2** If $D_0 \subseteq D$ is a non-empty downwards-directed subset of $D$ with infimum $d_0 \notin D$, then $b \cap d_0 \setminus \pi d_0 \ne 0$. But $\pi$ is order-continuous, so there is a $d \in D_0$ such that $b \cap d_0 \setminus \pi d \ne 0$; and now $d \notin D$, which is impossible. Thus in either case we have a contradiction. **XQ**

**(c)** Now let $G$ be a countable subgroup of $\operatorname{Aut}\mathfrak{A}$, with fixed-point subalgebra $\mathfrak{C}$, and let $\langle(\pi_n, e_n)\rangle_{n\in\mathbb{N}}$ be a sequence running over $G \times \mathfrak{C}$. Take any $a \in \mathfrak{A}$. For $k \in \mathbb{N}$ set

$$a_k = a \cap e_k \cap \pi_k^{-1}(a \setminus e_k),$$

$$a'_k = a_k \setminus \sup_{j<k} \operatorname{upr}(a'_j, \mathfrak{C}).$$

Then

$$a'_k \cap \pi_k a'_k \subseteq a_k \cap \pi_k a_k = 0$$

for every $k \in \mathbb{N}$, and whenever $j < k$ in $\mathbb{N}$ we have

$$a'_j \cap a'_k = 0, \quad \pi_j a'_j \cap a'_k = 0,$$

$$a'_j \cap \pi_k a'_k = \pi_k(\pi_k^{-1} a'_j \cap a'_k) = 0, \quad \pi_j a'_j \cap \pi_k a'_k = \pi_k(a'_k \cap \pi_k^{-1}\pi_j a'_j) = 0.$$

So, setting $d = \sup_{k\in\mathbb{N}} a'_k$ and $d' = \sup_{k\in\mathbb{N}} \pi_k a'_k$, $d$ and $d'$ are disjoint and $G$-$\sigma$-equidecomposable and included in $a$, and $d \preccurlyeq^\sigma_G a \setminus d$.

Consider $a' = a \setminus \operatorname{upr}(d, \mathfrak{C})$. Since $a'_k = a_k \setminus \sup_{j<k} \operatorname{upr}(a'_j, \mathfrak{C})$ for each $k$,

$$\operatorname{upr}(d, \mathfrak{C}) = \sup_{k\in\mathbb{N}} \operatorname{upr}(a'_k, \mathfrak{C}) = \sup_{k\in\mathbb{N}} \operatorname{upr}(a_k, \mathfrak{C}).$$

**?** Suppose, if possible, that $a'$ is not a relative atom over $\mathfrak{C}$; that is, that there is a $b \subseteq a'$ such that $b \ne a' \cap c$ for any $c \in \mathfrak{C}$. Then, in particular, $b \ne a' \cap \operatorname{upr}(b, \mathfrak{C})$, and there is a $\pi \in G$ such that $b' = a' \cap \pi b \setminus b \ne 0$. Then $b' \cup \pi^{-1} b' \subseteq a$, while $b' \cap \pi^{-1} b' = 0$, so $\pi b' \cap b' = 0$. By (b), there is an $e \in \mathfrak{C}$ such that $b'' = b' \cap e \setminus \pi e \ne 0$. Let $k$ be such that $\pi^{-1} = \pi_k$ and $e = e_k$, so that

$$b'' = b' \cap e_k \setminus \pi_k^{-1} e_k \subseteq a \cap e_k \cap \pi_k^{-1}(a \setminus e_k) = a_k.$$

(Because $\pi^{-1}b' \subseteq a$, $b' \subseteq \pi_k^{-1}a$.) Since also

$$b'' \cap \operatorname{upr}(a'_j, \mathfrak{C}) \subseteq a' \cap \operatorname{upr}(d, \mathfrak{C}) = 0$$

for every $j$, $b'' \subseteq a'_k \subseteq d$, which is impossible, because $b'' \subseteq a'$. **X**

Thus $a'$ is a relative atom over $\mathfrak{C}$, as required.

**448M Lemma** Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra, not $\{0\}$, and $G$ a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. If $G$ has the $\sigma$-refinement property, then for any $\epsilon > 0$ there is an $a^* \in \mathfrak{A}$ such that $\operatorname{upr}(a^*, \mathfrak{C}) = 1$ and $\lceil b : a^* \rceil \le \lfloor b : a^* \rfloor + \epsilon \lfloor 1 : a^* \rfloor$ for every $b \in \mathfrak{A}$.

**proof** As part (b) of the proof of 395M.

**448N Theorem** (Compare 395N.) Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra and $G$ a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that $G$ has the $\sigma$-refinement property of 448K. Then there is a function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ such that

(i) $\theta$ is additive, non-negative and sequentially order-continuous;

(ii) $\theta a = 0$ iff $a = 0$, $\theta 1 = \chi 1$;

(iii) $\theta(a \cap c) = \theta a \times \chi c$ for every $a \in \mathfrak{A}$, $c \in \mathfrak{C}$; in particular, $\theta c = \chi c$ for every $c \in \mathfrak{C}$;

(iv) if $a$, $b \in \mathfrak{A}$ are $G$-$\sigma$-equidecomposable, then $\theta a = \theta b$; in particular, $\theta$ is $G$-invariant.

**proof** The arguments of the proof of 395N apply here also, though we have to take things in a slightly different order. As in 395N, set

$$\theta_a(b) = \frac{\lceil b : a \rceil}{\lfloor 1 : a \rfloor} \in L^0(\mathfrak{C})$$

whenever $\operatorname{upr}(a, \mathfrak{C}) = 1$ and $b \in \mathfrak{A}$. This time, turn immediately to part (c) of the proof to see that if $e_n$ is chosen (using 448M) such that $\operatorname{upr}(e_n, \mathfrak{C}) = 1$ and $\lceil b : e_n \rceil \le \lfloor b : e_n \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor$ for every $b \in \mathfrak{A}$, then $\theta_{e_n} b \le \theta_a b + 2^{-n} \lceil 1 : a \rceil$ whenever $\operatorname{upr}(a, \mathfrak{C}) = 1$ and $b \in \mathfrak{A}$. So we can write

$$\theta b = \inf_{n \in \mathbb{N}} \theta_{e_n} b = \inf_{\operatorname{upr}(a,\mathfrak{C})=1} \theta_a b$$

for every $b \in \mathfrak{A}$, and we have a function $\theta : \mathfrak{A} \to L^0$ as before. The rest of the proof is unchanged, except that we have a simplification in (h), since we need consider only the case $\kappa = \omega$.

**448O** This concludes the adaptations we need from §395. I now return to the specific problem addressed in the present section. The first step is a variation on 448N.

**Theorem** Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra, not $\{0\}$, and $G$ a countable subgroup of $\operatorname{Aut}\mathfrak{A}$ with the $\sigma$-refinement property. Let $\mathfrak{C}$ be the fixed-point subalgebra of $G$. Then the following are equiveridical:

(i) there are a Dedekind $\sigma$-complete Boolean algebra $\mathfrak{D}$, not $\{0\}$, and a $G$-invariant sequentially order-continuous non-negative additive function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ such that $\theta 1 = \chi 1$;

(ii) if $a \in \mathfrak{A}$ and $1 \preccurlyeq_G^\sigma a$, then $\operatorname{upr}(1 \setminus a, \mathfrak{C}) \ne 1$;

(iii) if $a \in \mathfrak{A}$ and $1 \preccurlyeq_G^\sigma a$, then $1 \not\preccurlyeq_G^\sigma 1 \setminus a$.

**proof (a)(i)$\Rightarrow$(iii)** Take $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ as in (i). If $1 \preccurlyeq_G^\sigma a$, then there is a $b \subseteq a$ which is $G$-$\sigma$-equidecomposable with 1, so that $\theta b = \chi 1$, just as in 395N(v)/448N(iv). But this means that $\theta a = \chi 1$; so that $\theta(1 \setminus a) \ne \chi 1$ and $1 \not\preccurlyeq_G^\sigma 1 \setminus a$.

**(b)not-(ii)$\Rightarrow$not-(iii)** Suppose that (ii) is false; that there is an $a \in \mathfrak{A}$ such that $1 \preccurlyeq_G^\sigma a$ and $\operatorname{upr}(1 \setminus a, \mathfrak{C}) = 1$. Let $G_\sigma^*$ be the countably full local semigroup generated by $G$; then there is a $\psi \in G_\sigma^*$ such that $\psi 1 \subseteq a$. Set $b_0 = 1 \setminus \psi 1$ and $b_n = \psi^n b_0$ for every $n \ge 1$; then

$$b_0 \cap b_n \subseteq b_0 \cap \psi 1 = 0$$

for every $n \ge 1$, so

$$b_m \cap b_n = \psi^m (b_0 \cap b_{n-m}) = 0$$

whenever $m < n$.

Let $\langle\pi_i\rangle_{i\in\mathbb{N}}$ be a sequence running over $G$; then

$$\sup_{i\in\mathbb{N}}\pi_i b_0=\operatorname{upr}(b_0,\mathfrak{C})\supseteq\operatorname{upr}(1\setminus a,\mathfrak{C})=1.$$

Set

$$a_j=\pi_j b_0\setminus\sup_{i<j}\pi_i b_0$$

for every $j\in\mathbb{N}$, so that $\langle a_j\rangle_{j\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. Define $\psi_1$, $\psi_2:\mathfrak{A}\to\mathfrak{A}$ by setting

$$\psi_1 d=\sup_{i\in\mathbb{N}}\psi^{2i}\pi_i^{-1}(d\cap a_i),\quad\psi_2 d=\sup_{i\in\mathbb{N}}\psi^{2i+1}\pi_i^{-1}(d\cap a_i)$$

for every $d\in\mathfrak{A}$. Because $\psi^{2i}\pi_i^{-1}a_i\subseteq b_{2i}$ for every $i$, $\langle\psi^{2i}\pi_i^{-1}a_i\rangle_{i\in\mathbb{N}}$ is disjoint, so $\psi_1\in G^*_\sigma$ (448Bd); similarly, $\psi_2\in G^*_\sigma$. Thus

$$1\preccurlyeq^\sigma_G\psi_1 1\subseteq\sup_{i\in\mathbb{N}}b_{2i},$$

$$1\preccurlyeq^\sigma_G\psi_2 1\subseteq\sup_{i\in\mathbb{N}}b_{2i+1}\subseteq 1\setminus\psi_1 1$$

and (iii) is false.

**(c)** For the rest of this proof I will suppose that (ii) is true and seek to prove (i).

Let $\mathcal{I}$ be the $\sigma$-ideal of $\mathfrak{A}$ generated by $\{1\setminus a:a\in\mathfrak{A},\ 1\preccurlyeq^\sigma_G a\}$. Then $1\notin\mathcal{I}$. **P?** Otherwise, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $1\preccurlyeq^\sigma_G a_n$ for every $n$ and $\sup_{n\in\mathbb{N}}1\setminus a_n=1$. Choose $\psi_n\in G^*_\sigma$ such that $\psi_n 1\subseteq a_n$, and set $c_n=\operatorname{upr}(1\setminus\psi_n 1,\mathfrak{C})$ for each $n$, so that $\sup_{n\in\mathbb{N}}c_n=1$. Set $c'_n=c_n\setminus\sup_{i<n}c_i$ for each $n$, and write

$$\psi d=\sup_{n\in\mathbb{N}}\psi_n(d\cap c'_n)$$

for each $d\in\mathfrak{A}$. Because every $c'_n$ belongs to $\mathfrak{C}$, $\langle\psi_n c'_n\rangle_{n\in\mathbb{N}}=\langle c'_n\rangle_{n\in\mathbb{N}}$ is disjoint, and $\psi\in G^*_\sigma$.

By (ii), $c=\operatorname{upr}(1\setminus\psi 1,\mathfrak{C})$ is not 1; let $n$ be such that $c'=c'_n\setminus c\ne 0$. Because $c'\subseteq c'_n$, $c'\setminus\psi_n 1=c'\setminus\psi 1$; because $c'\in\mathfrak{C}$,

$$\begin{aligned}0\ne c'\cap c'_n\subseteq c'\cap\operatorname{upr}(1\setminus\psi_n 1,\mathfrak{C})&=\operatorname{upr}(c'\setminus\psi_n 1,\mathfrak{C})\\&=\operatorname{upr}(c'\setminus\psi 1,\mathfrak{C})=c'\cap\operatorname{upr}(1\setminus\psi 1,\mathfrak{C})=0\end{aligned}$$

which is absurd. **XQ**

Let $\mathfrak{B}$ be the quotient Boolean algebra $\mathfrak{A}/\mathcal{I}$; then $\mathfrak{B}$ is Dedekind $\sigma$-complete and the canonical homomorphism $a\mapsto a^\bullet:\mathfrak{A}\to\mathfrak{B}$ is sequentially order-continuous (314C, 313Qb).

**(d)** Next, $\pi b\in\mathcal{I}$ whenever $b\in\mathcal{I}$ and $\pi\in G$. **P** The sets $\{a:1\preccurlyeq^\sigma_G a\}$ and $\{1\setminus a:1\preccurlyeq^\sigma_G a\}$ are both invariant under the action of $G$, so $\mathcal{I}$ also must be invariant. **Q** We can therefore define, for each $\pi\in G$, a Boolean automorphism $\tilde\pi:\mathfrak{B}\to\mathfrak{B}$, setting $\tilde\pi a^\bullet=(\pi a)^\bullet$ for every $a\in\mathfrak{A}$. Because $(\pi\phi)^\sim=\tilde\pi\tilde\phi$ for all $\pi$, $\phi\in G$, $\tilde G=\{\tilde\pi:\pi\in G\}$ is a subgroup of $\operatorname{Aut}\mathfrak{B}$; of course it is countable. Let $\mathfrak{D}$ be the fixed-point subalgebra of $\tilde G$ in $\mathfrak{B}$. Because $\mathfrak{B}$ is not $\{0\}$, nor is $\mathfrak{D}$.

**(e)** $\tilde G$ is countably non-paradoxical. **P** Suppose that $b$ is $\tilde G$-$\sigma$-equidecomposable with 1 in $\mathfrak{B}$. Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a partition of unity in $\mathfrak{B}$ and $\langle\pi_n\rangle_{n\in\mathbb{N}}$ a sequence in $G$ such that $\langle\tilde\pi_n b_n\rangle_{n\in\mathbb{N}}$ is disjoint and has supremum $b$. For each $n\in\mathbb{N}$, let $a_n\in\mathfrak{A}$ be such that $a_n^\bullet=b_n$. We have

$$(a_m\cap a_n)^\bullet=(\pi_m a_m\cap\pi_n a_n)^\bullet=0$$

whenever $m\ne n$, so

$$d=\sup_{m\ne n}(a_m\cap a_n)\cup\sup_{m\ne n}(a_m\cap\pi_m^{-1}\pi_n a_n)$$

belongs to $\mathcal{I}$, while $\langle a_n\setminus d\rangle_{n\in\mathbb{N}}$, $\langle\pi_n(a_n\setminus d)\rangle_{n\in\mathbb{N}}$ are disjoint.

Because $\sup_{n\in\mathbb{N}}b_n=1$ in $\mathfrak{B}$, $d'=1\setminus\sup_{n\in\mathbb{N}}a_n\in\mathcal{I}$. Because $\mathcal{I}$ is a $\sigma$-ideal and $G$ is countable,

$$c^*=\operatorname{upr}(d\cup d',\mathfrak{C})=\sup_{\pi\in G}\pi(d\cup d')$$

belongs to $\mathcal{I}$, while $\{c^*\}\cup\{a_n\setminus c^*:n\in\mathbb{N}\}$ is a partition of unity in $\mathfrak{A}$.

Define $\psi\in G^*_\sigma$ by setting

$$\psi a=\sup_{n\in\mathbb{N}}\pi_n(a\cap a_n\setminus c^*)\cup(a\cap c^*)$$

for every $a \in \mathfrak{A}$. Then $1 \setminus \psi 1 \in \mathcal{I}$, by the definition of $\mathcal{I}$, so $(\psi 1)^\bullet = 1$ in $\mathfrak{B}$. But

$$\begin{aligned}(\psi 1)^\bullet &= \sup_{n\in\mathbb{N}}(\pi_n(a_n \setminus c^*))^\bullet = \sup_{n\in\mathbb{N}}(\pi_n a_n)^\bullet \\ &= \sup_{n\in\mathbb{N}} \tilde\pi_n b_n = b.\end{aligned}$$

So $b = 1$. As $b$ is arbitrary, $\tilde G$ is countably non-paradoxical. **Q**

**(f)** $\tilde G$ has the $\sigma$-refinement property. **P** Let $b \in \mathfrak{B}$. Then there is an $a \in \mathfrak{A}$ such that $a^\bullet = b$. Because $G$ is supposed to have the $\sigma$-refinement property, there is a $d \subseteq a$ such that $d \preccurlyeq^\sigma_G a \setminus d$ and $a \setminus \operatorname{upr}(d, \mathfrak{C})$ is a relative atom over $\mathfrak{C}$. Set $e = d^\bullet \subseteq b$.

We know that there are a partition of unity $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_d$ and a sequence $\langle \pi_n\rangle_{n\in\mathbb{N}}$ in $G$ such that $\pi_n d_n \subseteq a \setminus d$ for every $n$ and $\langle \pi_n d_n\rangle_{n\in\mathbb{N}}$ is disjoint. Now $\langle d_n^\bullet\rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{B}_e$, $\tilde\pi_n d_n^\bullet \subseteq b \setminus e$ for every $n$, and $\langle \tilde\pi_n d_n^\bullet\rangle_{n\in\mathbb{N}}$ is disjoint; so $e \preccurlyeq^\sigma_{\tilde G} b \setminus e$.

Suppose that $b_0 \subseteq b \setminus \operatorname{upr}(e, \mathfrak{D})$. Then it is expressible as $a_0^\bullet$ where $a_0 \subseteq a$ and

$$(a_0 \cap \pi d)^\bullet = b_0 \cap \tilde\pi e = 0$$

for every $\pi \in G$. So if we set $a_1 = a_0 \setminus \sup_{\pi\in G} \pi d$, we shall have $a_1 \subseteq a \setminus \operatorname{upr}(d, \mathfrak{C})$ and $a_1^\bullet = b_0$. Now $a \setminus \operatorname{upr}(d, \mathfrak{C})$ is supposed to be a relative atom over $\mathfrak{C}$, so $a_1 = a \cap c$ for some $c \in \mathfrak{C}$. In this case,

$$\tilde\pi c^\bullet = (\pi c)^\bullet = c^\bullet$$

for every $\pi \in G$, so $c^\bullet \in \mathfrak{D}$, while $b_0 = b \cap c^\bullet$. As $b_0$ is arbitrary, $b \setminus \operatorname{upr}(e, \mathfrak{D})$ is a relative atom over $\mathfrak{D}$.

Thus $e$ has both the properties required by the definition 448K. As $b$ is arbitrary, $\tilde G$ has the $\sigma$-refinement property. **Q**

**(g)** 448N now tells us that there is a sequentially order-continuous non-negative additive functional $\theta_0 : \mathfrak{B} \to L^\infty(\mathfrak{D})$ such that $\theta_0 1 = \chi 1$ and $\theta_0(\tilde\pi b) = \theta_0 b$ whenever $b \in \mathfrak{B}$ and $\pi \in G$. If we set $\theta a = \theta_0 a^\bullet$ for $a \in \mathfrak{A}$, it is easy to see that $\theta$ has all the properties required by (i) of this theorem. Thus (ii)$\Rightarrow$(i), and the proof is complete.

**448P** At last we come to Polish spaces.

**Theorem** (Nadkarni 90, Becker & Kechris 96) Let $G$ be a Polish group acting on a non-empty Polish space $(X, \mathfrak{T})$ with a Borel measurable action $\bullet$. For Borel sets $E$, $F \subseteq X$ say that $E \preccurlyeq^\sigma_G F$ if there are a countable partition $\langle E_i\rangle_{i\in I}$ of $E$ into Borel sets, and a family $\langle g_i\rangle_{i\in I}$ in $G$, such that $g_i \bullet E_i \subseteq F$ for every $i$ and $\langle g_i \bullet E_i\rangle_{i\in I}$ is disjoint. Then the following are equiveridical:

(i) there is a $G$-invariant Radon probability measure $\mu$ on $X$;

(ii) if $F \subseteq X$ is a Borel set such that $X \preccurlyeq^\sigma_G F$, then $\bigcap_{n\in\mathbb{N}} g_n \bullet F \neq \emptyset$ for any sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in $G$;

(iii) there are no disjoint Borel sets $E$, $F \subseteq X$ such that $X \preccurlyeq^\sigma_G E$ and $X \preccurlyeq^\sigma_G F$.

**proof (a)** Let us start with the easy parts.

**(i)$\Rightarrow$(ii)** Let $\mu$ be a $G$-invariant Radon probability measure on $X$, and suppose that $X \preccurlyeq^\sigma_G F$. Let $\langle E_i\rangle_{i\in I}$ be a countable partition of $X$ into Borel sets and $\langle h_i\rangle_{i\in I}$ a family in $G$ such that $\langle h_i \bullet E_i\rangle_{i\in I}$ is disjoint and $h_i \bullet E_i \subseteq F$ for every $i$. Then

$$\mu F \ge \sum_{i\in I} \mu(h_i \bullet E_i) = \sum_{i\in I} \mu E_i = \mu X,$$

so $F$ is conegligible. Consequently $\bigcap_{n\in\mathbb{N}} g_n \bullet F$ must be conegligible and cannot be empty, for any sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in $G$. As $F$ and $\langle g_n\rangle_{n\in\mathbb{N}}$ are arbitrary, (ii) is true.

**(ii)$\Rightarrow$(iii)** Assume (ii). **?** If there are disjoint $E$, $F$ such that $X \preccurlyeq^\sigma_G E$ and $X \preccurlyeq^\sigma_G F$, then we have a countable partition $\langle E_i\rangle_{i\in I}$ of $X$ into Borel sets and a family $\langle g_i\rangle_{i\in I}$ in $G$ such that $g_i \bullet E_i \subseteq E$ for every $i \in I$. But there is an $x \in \bigcap_{i\in I} g_i^{-1} \bullet F$, by (ii). In this case there is a $j \in I$ such that $x \in E_j$ and $g_j \bullet x \in E \cap F$, which is impossible. **X** So (iii) must be true.

**(b)** For the rest of the proof, therefore, I shall assume (iii) and seek to prove (i).

Let $\mathfrak{T}$ be the topology of $X$, and $\mathcal{B} = \mathcal{B}(X)$ its Borel $\sigma$-algebra. For $g \in G$ define $\pi_g : \mathcal{B} \to \mathcal{B}$ by writing $\pi_g E = g \bullet E$ for every $E \in \mathcal{B}$. Then $\pi_{gh} = \pi_g \pi_h$ for all $g$, $h \in G$, so $\tilde G = \{\pi_g : g \in G\}$ is a subgroup of $\operatorname{Aut}\mathcal{B}$. Observe that for $E$, $F \in \mathcal{B}$, $E \preccurlyeq^\sigma_G F$, in the sense here, iff $E \preccurlyeq^\sigma_{\tilde G} F$ in the sense of 448A.

By the Becker-Kechris theorem (424H), there is a Polish topology $\mathfrak{T}_1$ on $X$, giving rise to the same Borel $\sigma$-algebra $\mathcal{B}$ as the original topology, for which the action of $G$ is continuous. Let $\mathcal{U}$ be a countable base for $\mathfrak{T}_1$ closed under finite unions. (We are going to have three Polish topologies on $X$ in this proof, so watch carefully.)

**(c)** For the time being (down to the end of (f) below) let us suppose that $G$, and therefore $\tilde{G}$, are countable. In this case, because $\mathcal{B}$ is countably generated, $\tilde{G}$ has the $\sigma$-refinement property, by 448L. We can therefore apply 448O to see that (iii) implies that

(i)$'$ there are a Dedekind $\sigma$-complete Boolean algebra $\mathfrak{D}$, not $\{0\}$, and a $\tilde{G}$-invariant sequentially order-continuous non-negative additive functional $\theta : \mathcal{B} \to L^\infty(\mathfrak{D})$ such that $\theta X = \chi 1$.

Express $\mathfrak{D}$ as $\Sigma/\mathcal{J}$ where $\Sigma$ is a $\sigma$-algebra of subsets of a set $Z$ and $\mathcal{J}$ is a $\sigma$-ideal of $\Sigma$ (314N). Then we can identify $L^\infty(\mathfrak{D})$ with the quotient $\mathcal{L}^\infty/W$, where $\mathcal{L}^\infty$ is the space of bounded $\Sigma$-measurable real-valued functions on $Z$ and $W$ is the set $\{f : f \in \mathcal{L}^\infty, \{z : f(z) \neq 0\} \in \mathcal{J}\}$ (363Hb). For each $E \in \mathcal{B}$, let $f_E \in \mathcal{L}^\infty$ be a representative of $\theta E \in L^\infty(\mathfrak{D})$; because $\theta(\pi E) = \theta E$ whenever $E \in \mathcal{B}$ and $\pi \in \tilde{G}$, we may suppose that $f_{\pi E} = f_E$ whenever $E \in \mathcal{B}$ and $\pi \in \tilde{G}$.

Let $\mathfrak{B}$ be the subalgebra of $\mathcal{B}$ generated by $\{\pi U : U \in \mathcal{U}, \pi \in \tilde{G}\}$. Then $\mathfrak{B}$ is countable and $\pi E \in \mathfrak{B}$ for every $E \in \mathfrak{B}$, $\pi \in \tilde{G}$. By 4A3I, there is yet another Polish topology $\mathfrak{S}$ on $X$ which is zero-dimensional and such that every member of $\mathfrak{B}$ is open-and-closed for $\mathfrak{S}$. Of course $\mathfrak{S} \supseteq \mathcal{U}$, so $\mathcal{B}$ is still the algebra of $\mathfrak{S}$-Borel sets (423Fb). Let $\mathcal{W}$ be a countable base for $\mathfrak{S}$ consisting of sets which are open-and-closed for $\mathfrak{S}$, and let $\mathfrak{B}_1$ be the subalgebra of $\mathcal{B}$ generated by $\mathcal{W} \cup \mathfrak{B}$; then $\mathfrak{B}_1$ is countable and consists of open-and-closed sets for $\mathfrak{S}$. Let $\langle W_n \rangle_{n \in \mathbb{N}}$ be a sequence running over $\mathcal{W}$. Let $\rho$ be a complete metric on $X$ defining the topology $\mathfrak{S}$, and for $m$, $n \in \mathbb{N}$ set

$$W_{mn} = \bigcup \{W_i : i \leq n, \operatorname{diam}_\rho(W_i) \leq 2^{-m}\};$$

then for each $m \in \mathbb{N}$, $\langle W_{mn} \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{B}_1$ with union $X$.

**(d)** Consider the subsets of $Z$ of the following types:

$$P_E = \{z : f_E(z) < 0\}, \text{ where } E \in \mathfrak{B}_1,$$

$$Q_{EF} = \{z : f_{E \cup F}(z) \neq f_E(z) + f_F(z)\}, \text{ where } E,\, F \in \mathfrak{B}_1 \text{ and } E \cap F = \emptyset,$$

$$R = \{z : f_X(z) \neq 1\},$$

$$S_m = \{z : \sup_{n \in \mathbb{N}} f_{W_{mn}}(z) \neq 1\}, \text{ where } m \in \mathbb{N}.$$

Because

$$f_E^\bullet = \theta E \geq 0 \text{ for every } E \in \mathcal{B},$$

$$f_{E \cup F}^\bullet = \theta(E \cup F) = \theta E + \theta F = f_E^\bullet + f_F^\bullet \text{ whenever } E \cap F = \emptyset,$$

$$f_X^\bullet = \theta X = \chi 1,$$

$$\sup_{n \in \mathbb{N}} f_{W_{mn}}^\bullet = \sup_{n \in \mathbb{N}} \theta W_{mn} = \theta(\bigcup_{n \in \mathbb{N}} W_{mn}) = \theta X = \chi 1$$

for every $m \in \mathbb{N}$, all the sets $P_E$, $Q_{EF}$, $R$ and $S_m$ belong to $\mathcal{J}$. Since $\mathfrak{D} \neq \{0\}$, $Z \notin \mathcal{J}$; so there is a $z_0 \in Z$ not belonging to $R$ or $P_E$ or $Q_{EF}$ or $S_m$ whenever $m \in \mathbb{N}$ and $E$, $F \in \mathfrak{B}_1$ are disjoint.

Set $\nu E = f_E(z_0)$ for every $E \in \mathfrak{B}_1$. If $E$, $F \in \mathfrak{B}_1$ are disjoint, then $\nu(E \cup F) = \nu E + \nu F$ because $z_0 \notin Q_{EF}$; thus $\nu : \mathfrak{B}_1 \to \mathbb{R}$ is additive. If $E \in \mathfrak{B}_1$ then $\nu E \geq 0$ because $z_0 \notin P_E$, so $\nu$ is non-negative. $\nu X = 1$ because $z_0 \notin R$. For each $m \in \mathbb{N}$, $\sup_{n \in \mathbb{N}} \nu W_{mn} = 1$ because $z_0 \notin S_m$.

**(e)** For any $\epsilon > 0$ there is an $\mathfrak{S}$-compact set $K \subseteq X$ such that $\nu E \geq 1 - \epsilon$ whenever $E \in \mathfrak{B}_1$ and $E \supseteq K$. **P** For each $m \in \mathbb{N}$ we have a $k(m) \in \mathbb{N}$ such that $\nu W_{m,k(m)} \geq 1 - 2^{-m-1}\epsilon$. Set $K = \bigcap_{m \in \mathbb{N}} W_{m,k(m)}$. Because every $W_{m,k(m)}$ is $\mathfrak{S}$-closed, $K$ is $\mathfrak{S}$-closed, therefore $\rho$-complete; because every $W_{m,k(m)}$ is a finite union of sets of diameter at most $2^{-m}$, $K$ is $\rho$-totally bounded, therefore $\mathfrak{S}$-compact (4A2Je). **?** Suppose, if possible, that $E \in \mathfrak{B}_1$ is such that $K \subseteq E$ and $\nu E < 1 - \epsilon$. For every $m \in \mathbb{N}$,

$$\nu(\bigcap_{i \leq m} W_{i,k(i)}) \geq 1 - \sum_{i=0}^m \nu(X \setminus W_{i,k(i)}) \geq 1 - \sum_{i=0}^m 2^{-i-1}\epsilon > 1 - \epsilon > \nu E$$

because $\nu$ is non-negative and finitely additive. So $\bigcap_{i\le m} W_{i,k(i)} \setminus E$ must be non-empty. There is therefore an ultrafilter $\mathcal{F}$ on $X$ containing $W_{i,k(i)} \setminus E$ for every $i \in \mathbb{N}$. Now for each $i$ there must be a $j \le k(i)$ such that $\operatorname{diam} W_j \le 2^{-i}$ and $W_j \in \mathcal{F}$, so $\mathcal{F}$ is a $\rho$-Cauchy filter, and $\mathfrak{S}$-converges to $x$ say. Because every $W_{i,k(i)}$ is $\mathfrak{S}$-closed, $x \in \bigcap_{i\in\mathbb{N}} W_{i,k(i)} = K$; because $E \in \mathfrak{S}$, $x \notin E$; but $K$ is supposed to be included in $E$. **X**

Thus $\inf\{\nu E : K \subseteq E \in \mathfrak{B}_1\} \ge 1 - \epsilon$. As $\epsilon$ is arbitrary, we have the result. **Q**

**(f)** By 416O, there is an $\mathfrak{S}$-Radon measure $\mu$ on $X$ extending $\nu$. Because $\mu$ is just the completion of its restriction to $\mathcal{B}$, it is also $\mathfrak{T}$-Radon and $\mathfrak{T}_1$-Radon (433Cb).

Now $\mu$ is $G$-invariant. **P** Take any $g \in G$. Set $\mu_g E = \mu(g{\bullet}E)$ whenever $E \subseteq X$ and $g{\bullet}E \in \operatorname{dom}\mu$. The map $x \mapsto g{\bullet}x$ is a homeomorphism for $\mathfrak{T}_1$, so $\mu_g$ also is a $\mathfrak{T}_1$-Radon measure. (Setting $\phi(x) = g^{-1}{\bullet}x$, $\mu_g$ is the image measure $\mu\phi^{-1}$.) Again because $\mathfrak{T}$ and $\mathfrak{T}_1$ have the same Borel $\sigma$-algebras, $\mu_g$ is $\mathfrak{T}$-Radon. If $E \in \mathfrak{B}$, then $E$ and $g{\bullet}E$ belong to $\mathfrak{B} \subseteq \mathfrak{B}_1$, so

$$\mu_g E = \mu(g{\bullet}E) = \nu(g{\bullet}E) = \nu(\pi_g E) = f_{\pi_g E}(z_0) = f_E(z_0)$$

(because $f_{\pi_g E} = f_E$, as declared in (c) above)

$$= \nu E = \mu E.$$

In particular, $\mu_g E = \mu E$ for every $E$ in the algebra generated by $\mathcal{U}$. But $\mu_g$ and $\mu$ are both $\mathfrak{T}$-Radon measures, and $\mathcal{U}$ is a base for $\mathfrak{T}_1$ closed under finite unions, so $\mu_g = \mu$ (415H(iv)). As $g$ is arbitrary, $\mu$ is $G$-invariant. **Q**

Thus we have found a $G$-invariant Radon probability measure, and (i) is true.

**(g)** Thus (iii)$\Rightarrow$(i) if $G$ is countable. Now let us consider the general case. Because $G$ is a Polish group, it has a countable dense subgroup $H$. (Take $H$ to be the subgroup generated by any countable dense subset of $G$.) Of course there can be no disjoint $E$, $F \in \mathcal{B}$ such that $X \preccurlyeq^\sigma_H E$ and $X \preccurlyeq^\sigma_H F$, so there must be an $H$-invariant Radon probability measure $\mu$ on $X$, by the arguments of (b)-(f). ($H$ need not be a Polish group in its subspace topology. But if we give it its discrete topology, then $x \mapsto h{\bullet}x$ is still a $\mathfrak{T}_1$-homeomorphism for every $h \in H$, so the action of $H$ on $X$ is still continuous if $H$ is given its discrete topology and $X$ is given $\mathfrak{T}_1$.)

Now $\mu$ is $G$-invariant. **P** For any $g \in G$, let $\mu_g$ be the Radon probability measure defined by setting $\mu_g E = \mu(g{\bullet}E)$ whenever this is defined. (As in (f) above, this formula does define a probability measure which is Radon for either $\mathfrak{T}$ or $\mathfrak{T}_1$.) Let $f : X \to \mathbb{R}$ be any bounded $\mathfrak{T}_1$-continuous function. Then

$$\int f d\mu_g = \int f(g^{-1}{\bullet}x)\mu(dx)$$

(applying 235G with $\phi(x) = g^{-1}{\bullet}x$). Now there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in $H$ converging to $g$. In this case, because $G$ is a topological group, $g^{-1} = \lim_{n\to\infty} h_n^{-1}$. Because the action of $G$ on $X$ is $\mathfrak{T}_1$-continuous, $g^{-1}{\bullet}x = \lim_{n\to\infty} h_n^{-1}{\bullet}x$, for $\mathfrak{T}_1$, for every $x \in X$. Because $f$ is $\mathfrak{T}_1$-continuous, $f(g^{-1}{\bullet}x) = \lim_{n\to\infty} f(h_n^{-1}{\bullet}x)$ in $\mathbb{R}$ for every $x \in X$. By Lebesgue's Dominated Convergence Theorem,

$$\int f d\mu_g = \int f(g^{-1}{\bullet}x)\mu(dx) = \lim_{n\to\infty} f(h_n^{-1}{\bullet}x)\mu(dx) = \lim_{n\to\infty} \int f d\mu_{h_n} = \int f d\mu$$

because $\mu$ is $H$-invariant, so $\mu_{h_n} = \mu$ for every $n$. As $f$ is arbitrary, $\mu_g = \mu$, by 415I. As $g$ is arbitrary, $\mu$ is $G$-invariant. **Q**

Thus (iii)$\Rightarrow$(i) in all cases, and the proof is complete.

**448Q** I turn now to Mackey's theorem. I pave the way with a couple of lemmas which are of independent interest.

**Lemma** Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space with countable Maharam type. Write $L^0(\Sigma)$ for the set of $\Sigma$-measurable functions from $X$ to $\mathbb{R}$. Then there is a function $T : L^0(\mu) \to L^0(\Sigma)$ such that

($\alpha$) $u = (Tu)^\bullet$ for every $u \in L^0$,

($\beta$) $(u, x) \mapsto (Tu)(x) : L^0 \times X \to \mathbb{R}$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable,

where $\mathcal{B} = \mathcal{B}(L^0)$ is the Borel $\sigma$-algebra of $L^0$ with its topology of convergence in measure.

**proof (a)** Consider the first the case in which $\mu$ is a probability measure.

**(i)** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\Sigma$ such that the measure algebra $\mathfrak{A}$ of $\mu$ is $\tau$-generated by $\{E_n^\bullet : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$ let $\Sigma_n$ be the finite subalgebra of $\Sigma$ generated by $\{E_i : i \le n\}$, and for $n \in \mathbb{N}$, $u \in L^\infty = L^\infty(\mu)$ and $x \in X$ set

$$\begin{aligned}(S_n u)(x) &= \frac{1}{\mu E}\int_E u \text{ if } E \text{ is the atom of } \Sigma_n \text{ containing } x \text{ and } \mu E > 0,\\ &= 0 \text{ if the atom of } \Sigma_n \text{ containing } x \text{ is negligible.}\end{aligned}$$

Then $(u,x) \mapsto (S_n u)(x)$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable, because $u \mapsto \int_E u : L^\infty \to \mathbb{R}$ is continuous (for the topology of convergence in measure) for every $E \in \Sigma$. So if we set $Su = \limsup_{n\to\infty} S_n u$ for $u \in L^\infty$, $(u,x) \mapsto (Su)(x)$ will be $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable.

On the other hand, if $f \in \mathcal{L}^\infty$, $S_n f^\bullet$ is a conditional expectation of $f$ on $\Sigma_n$ for each $n$. So Lévy's martingale theorem (275I) tells us that if $f \in \mathcal{L}^\infty$ then $\langle S_n f^\bullet\rangle_{n\in\mathbb{N}}$ converges a.e. to a conditional expectation $g$ of $f$ on the $\sigma$-algebra $\Sigma_\infty$ generated by $\bigcup_{n\in\mathbb{N}}\Sigma_n$. But we chose $\langle E_n\rangle_{n\in\mathbb{N}}$ to generate $\mathfrak{A}$, so $\mathfrak{A} = \{E^\bullet : E \in \Sigma_\infty\}$. If now $E \in \Sigma$, there is an $F \in \Sigma_\infty$ such that $E\triangle F$ is negligible, so

$$\int_E g = \int_F g = \int_F f = \int_E f.$$

As $E$ is arbitrary,

$$f =_{\text{a.e.}} g =_{\text{a.e.}} \limsup_{n\to\infty} S_n f^\bullet = S f^\bullet.$$

Turning this round, $(Su)^\bullet = u$ for every $u \in L^\infty$.

**(ii)** Now define $R : L^0 \to L^\infty$ by setting

$$Rf^\bullet = (\arctan f)^\bullet$$

for $f \in \mathcal{L}^0$ (see 241I). Then $R$ is continuous for the topology of convergence in measure (245Dd), so $(u,x) \mapsto (SRu)(x) : L^0 \times X \to \mathbb{R}$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable. Note that if $u \in L^0$, then $-\frac{\pi}{2} < (S_n Ru)(x) < \frac{\pi}{2}$ for every $x$ and $n$, so $-\frac{\pi}{2} \le (SRu)(x) \le \frac{\pi}{2}$ for every $x$; also, if $u = f^\bullet$, then $-\frac{\pi}{2} < \arctan f(x) < \frac{\pi}{2}$ whenever $f(x)$ is defined, so $-\frac{\pi}{2} < (SRu)(x) < \frac{\pi}{2}$ for almost every $x$. If now we set

$$\begin{aligned}\tan_0 t &= \tan t \text{ for } -\frac{\pi}{2} < t < \frac{\pi}{2},\\ &= 0 \text{ for } t = \pm\frac{\pi}{2},\end{aligned}$$

and $Tu = \tan_0 SRu$, we shall have $Tu \in L^0(\Sigma)$ and $(Tu)^\bullet = u$ for every $u \in L^0$, while $(u,x) \mapsto (Tu)(x)$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable.

**(b)** For the general case, if $\mu X = 0$ the result is trivial, as we can just set $(Tu)(x) = 0$ for all $u$ and $x$. So suppose otherwise. Let $\nu$ be a probability measure with the same domain and the same negligible sets as $\mu$ (215B(vii)). Then the measure algebra of $\nu$, regarded as a Boolean algebra, is the same as that of $\mu$, so $\nu$ also has countable Maharam type; similarly, $L^0 = L^0(\nu)$. Moreover, the topology of convergence in measure on $L^0$ is the same, whichever measure we take to define it (245Xm, 367T). So we can apply (a) to $(X,\Sigma,\nu)$.

**448R Lemma** Let $(X,\Sigma,\mu)$ be a $\sigma$-finite measure space with countable Maharam type.

(a) $L^0 = L^0(\mu)$, with its topology of convergence in measure, is a Polish space.

(b) Let $\mathfrak{A}$ be the measure algebra of $\mu$, and $\mathfrak{A}^f$ the set $\{a : a \in \mathfrak{A}, \bar\mu a < \infty\}$. Then the Borel $\sigma$-algebra $\mathcal{B} = \mathcal{B}(L^0)$ is the $\sigma$-algebra of subsets of $L^0$ generated by sets of the form $\{u : \bar\mu(a \cap [\![u \in F]\!]) > \alpha\}$, where $a \in \mathfrak{A}^f$, $F \subseteq \mathbb{R}$ is Borel, and $\alpha \in \mathbb{R}$.

**proof (a)** By 245Eb, $L^0$ is metrizable, and complete when regarded as a linear topological space; so by 4A4Bj there is a metric on $L^0$, defining its topology, under which $L^0$ is complete. By 367Rb, $L^0$ is separable, so it is a Polish space.

**(b)** Write $\Upsilon$ for the $\sigma$-algebra of subsets of $L^0$ generated by sets of the form $\{u : \bar\mu(a \cap [\![u \in F]\!]) > \alpha\}$, where $a \in \mathfrak{A}^f$, $F \subseteq \mathbb{R}$ is Borel, and $\alpha \in \mathbb{R}$.

**(i)** If $a \in \mathfrak{A}^f$, $\alpha \in \mathbb{R}$ and $H \subseteq \mathbb{R}$ is open, then $U = \{u : \bar{\mu}(a \cap [\![u \in H]\!]) > \alpha\}$ is open in $L^0$. **P** If $u \in U$, there are a compact set $K \subseteq H$ and a $\delta > 0$ such that $\bar{\mu}(a \cap [\![u \in K]\!]) > \alpha + \delta$. Now there is an $\eta \in ]0,1]$ such that $|\alpha - \beta| > \eta$ whenever $\alpha \in K$ and $\beta \in \mathbb{R} \setminus H$. In this case, $V = \{v : v \in L^0, \bar{\mu}(a \cap [\![|u-v| > \eta]\!]) \le \delta\}$ is a neighbourhood of $u$ in $L^0$ (367L). If $v \in V$, then

$$[\![v \in H]\!] \supseteq [\![u \in K]\!] \cap [\![|u-v| \le \eta]\!],$$

$$\begin{aligned}\bar{\mu}(a \cap [\![v \in H]\!]) &\ge \bar{\mu}(a \cap [\![u \in K]\!]) - \bar{\mu}(a \cap [\![|u-v| > \eta]\!]) \\ &\ge \bar{\mu}(a \cap [\![u \in K]\!]) - \delta > \alpha.\end{aligned}$$

Thus $V \subseteq U$ and $U$ is a neighbourhood of $u$; as $u$ is arbitrary, $U$ is open. **Q**

**(ii)** Thus $u \mapsto \bar{\mu}(a \cap [\![u \in H]\!])$ is $\mathcal{B}$-measurable for every $a \in \mathfrak{A}^f$ and open $H \subseteq \mathbb{R}$. Now the set

$$\{F : F \subseteq \mathbb{R} \text{ is Borel, } u \mapsto \bar{\mu}(a \cap [\![u \in F]\!]) \text{ is } \mathcal{B}\text{-measurable for every } a \in \mathfrak{A}^f\}$$

is a Dynkin class containing all open sets, so is the Borel $\sigma$-algebra of $\mathbb{R}$ (136B), and $u \mapsto \bar{\mu}(a \cap [\![u \in F]\!])$ is $\mathcal{B}$-measurable for every $a \in \mathfrak{A}^f$ and Borel $F \subseteq \mathbb{R}$. Thus $\Upsilon \subseteq \mathcal{B}$.

**(iii)** In the other direction, we know that $\mathfrak{A}$ is separable, by 331O; let $\langle c_k \rangle_{k \in \mathbb{N}}$ run over a dense subset of $\mathfrak{A}$. We also know that there is a sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}^f$ with supremum 1. Set

$$E_{nkqq'} = \{u : u \in L^0, \bar{\mu}(a_n \cap c_k \cap [\![u > q]\!]) > q'\} \in \Upsilon$$

for $n$, $k \in \mathbb{N}$ and $q$, $q' \in \mathbb{Q}$. If $u$, $v \in L^0$ are different, there are $n$, $k$, $q$ and $q'$ such that $E_{nkqq'}$ contains one of $u$, $v$ and not the other. **P** Choose $q \in \mathbb{Q}$ such that $[\![u > q]\!] \ne [\![v > q]\!]$. Suppose for the moment that $c = [\![u > q]\!] \setminus [\![v > q]\!] \ne 0$. Let $n \in \mathbb{N}$ be such that $\bar{\mu}(a_n \cap c) > 0$. Let $k \in \mathbb{N}$ be such that $\bar{\mu}(a_n \cap (c \triangle c_k)) < \bar{\mu}(a_n \cap c)$. Then

$$\begin{aligned}\bar{\mu}(a_n \cap c_k \cap [\![v > q]\!]) &\le \bar{\mu}(a_n \cap c_k \setminus c) \\ &< \bar{\mu}(a_n \cap c) - \bar{\mu}(a_n \cap c \setminus c_k) \le \bar{\mu}(a_n \cap c_k \cap [\![u > q]\!]),\end{aligned}$$

so there is a $q' \in \mathbb{Q}$ such that $u \in E_{nkqq'}$ and $v \notin E_{nkqq'}$. Similarly, if $[\![v > q]\!] \not\subseteq [\![u > q]\!]$ there are $n$, $k \in \mathbb{N}$ and $q' \in \mathbb{Q}$ such that $v \in E_{nkqq'}$ and $u \notin E_{nkqq'}$. **Q**

By 423J, the $\sigma$-algebra generated by $\{E_{nkqq'} : n, k \in \mathbb{N}, q, q' \in \mathbb{Q}\}$ is the whole of $\mathcal{B}$, and $\Upsilon$ must be equal to $\mathcal{B}$, as claimed.

**448S Mackey's theorem** (Mackey 62) Let $G$ be a locally compact Polish group, $(X, \Sigma)$ a standard Borel space and $\mu$ a $\sigma$-finite measure with domain $\Sigma$. Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of $\mu$ with its measure-algebra topology. Let $\circ$ be a Borel measurable action of $G$ on $\mathfrak{A}$ such that $a \mapsto g \circ a$ is a Boolean automorphism for every $g \in G$. Then we have a $(\mathcal{B}(G)\widehat{\otimes}\Sigma, \Sigma)$-measurable action $\bullet$ of $G$ on $X$ such that

$$g \circ E^\bullet = (g \bullet E)^\bullet$$

for every $g \in G$ and $E \in \Sigma$, writing $g \bullet E$ for $\{g \bullet x : x \in E\}$ as usual.

**proof (a)** To begin with (down to the end of (j) below) suppose that $X = \mathbb{R}$, with $\Sigma = \mathcal{B}(\mathbb{R})$ its Borel $\sigma$-algebra, and that $\mu$ is totally finite. The first thing to note is that for every $g \in G$ the automorphism $a \mapsto g \circ a$ can be represented by a Borel automorphism $f_g : \mathbb{R} \to \mathbb{R}$ such that $g \circ E^\bullet = f_g^{-1}[E]^\bullet$ for every $E \in \mathcal{B}(\mathbb{R})$ (425Ac). Of course $f_g$ belongs to the space $L^0(\Sigma)$ of $\Sigma$-measurable functions from $\mathbb{R}$ to itself, so we can speak of its equivalence class $f_g^\bullet \in L^0(\mu)$. If we give $L^0(\mu)$ its topology of convergence in measure, it is a Polish space (448Ra).

The function $g \mapsto f_g^\bullet : G \to L^0(\mu)$ is Borel measurable. **P** If $E \in \mathcal{B}(\mathbb{R})$, $a \in \mathfrak{A}$ and $\alpha \in \mathbb{R}$, then, setting $b = E^\bullet$,

$$[\![f_g^\bullet \in E]\!] = f_g^{-1}[E]^\bullet = g \circ b$$

for every $g \in G$, so

$$\{g : \bar{\mu}(a \cap [\![f_g^\bullet \in E]\!]) > \alpha\} = \{g : \bar{\mu}(a \cap (g \circ b)) > \alpha\}$$

is a Borel set in $G$, because $\circ$ is Borel measurable and $\{c : \bar\mu(a\cap c) > \alpha\}$ is open in $\mathfrak{A}$. Thus $\mathcal{B}(G)$ contains the inverse images of the sets generating $\mathcal{B}(L^0(\mu))$ described in 448Rb, and therefore the inverse image of every set in $\mathcal{B}(L^0(\mu))$, as required. **Q**

**(b)** By 448Q, there is a function $T : L^0(\mu) \to L^0(\Sigma)$ such that $(Tu)^\bullet = u$ for every $u \in L^0(\mu)$ and $(u,x) \mapsto (Tu)(x)$ is $(\mathcal{B}(L^0(\mu))\widehat{\otimes}\mathcal{B}(\mathbb{R}))$-measurable. Define $\phi : G \times \mathbb{R} \to \mathbb{R}$ by setting

$$\phi(g,x) = (Tf_g^\bullet)(x)$$

for $g \in G$ and $x \in \mathbb{R}$. Then $\phi$ is a composition of the Borel measurable functions $(g,x) \mapsto (f_g^\bullet, x)$ and $(u,x) \mapsto (Tu)(x)$, so is Borel measurable; and if $g \in G$ then $\phi(g,x) = f_g(x)$ for $\mu$-almost every $x$, because $f_g =_{\text{a.e.}} Tf_g^\bullet$. Now

$$g{\circ}E^\bullet = (f_g^{-1}[E])^\bullet = \{x : \phi(g,x) \in E\}^\bullet$$

for every $g \in G$ and $E \in \mathcal{B}(\mathbb{R})$.

**(c)** Let $\lambda$ be a Haar measure on $G$. Because $G$ is a Polish space and $\lambda$ is a Radon measure on $G$, $\lambda$ is $\sigma$-finite (411Ge) and $L^0(\lambda)$, with its topology of convergence in measure, is a Polish space (448Ra again). For $x \in \mathbb{R}$, set $\phi_x(g) = \phi(g,x)$ for $g \in G$; then $\phi_x : G \to \mathbb{R}$ is Borel measurable. Set $\theta(x) = \phi_x^\bullet$ in $L^0(\lambda)$. Then $\theta : \mathbb{R} \to L^0(\lambda)$ is Borel measurable. **P** Again I use the characterization of the Borel $\sigma$-algebra of $L^0(\lambda)$ in 448Rb. Let $(\mathfrak{C},\bar\lambda)$ be the measure algebra of $\lambda$. If $c \in \mathfrak{C}$, $\bar\lambda c < \infty$, $E \subseteq \mathbb{R}$ is Borel, and $\alpha \in \mathbb{R}$, take a Borel set $F \subseteq G$ such that $c = F^\bullet$; then

$$\begin{aligned}\{x : \bar\lambda(c \cap [\![\theta(x) \in E]\!]) > \alpha\} &= \{x : \lambda(F \cap \phi_x^{-1}[E]) > \alpha\}\\ &= \{x : \lambda\{g : g \in F,\, \phi_x(g) \in E\} > \alpha\}\\ &= \{x : \lambda\{g : g \in F,\, \phi(g,x) \in E\} > \alpha\}\\ &= \{x : \lambda W^{-1}[\{x\}] > \alpha\}\end{aligned}$$

where $W = \{(g,x) : g \in F,\ x \in \mathbb{R},\ \phi(g,x) \in E\}$ is a Borel subset of $G \times \mathbb{R}$. But this means that $W \in \mathcal{B}(G)\widehat{\otimes}\mathcal{B}(\mathbb{R})$ (4A3Ga) and $x \mapsto \lambda W^{-1}[\{x\}]$ is Borel measurable (252P), so $\{x : \bar\lambda(c \cap [\![\theta(x) \in E]\!]) > \alpha\}$ is a Borel subset of $\mathbb{R}$. Thus the inverse image of every set in the generating family for the Borel $\sigma$-algebra of $L^0(\lambda)$ is a Borel set, and we have a Borel measurable function. **Q**

Let $\nu$ be the totally finite Borel measure on $L^0(\lambda)$ defined by setting $\nu F = \mu\theta^{-1}[F]$ for every Borel set $F \subseteq L^0(\lambda)$.

**(d)** If $E \subseteq \mathbb{R}$ is Borel, there is a set $A \subseteq L^0(\lambda)$ such that $E \triangle \theta^{-1}[A]$ is $\mu$-negligible. **P** Let $\hat\mu$, $\hat\nu$ be the completions of $\mu$ and $\nu$, so that $\theta$ is inverse-measure-preserving for $\hat\mu$ and $\hat\nu$ (234Ba). Because $E$ is a Borel subset of $\mathbb{R}$, $\theta[E]$ is an analytic subset of $L^0(\lambda)$ (423Gb), therefore Souslin-F (423Eb); accordingly $\hat\nu$ measures $\theta[E]$ (431B). Let $\mathrm{T}_0$ be the $\sigma$-algebra of subsets of $L^0(\lambda)$ generated by the Souslin-F subsets of $L^0(\lambda)$. By 423Q, there is a $\mathrm{T}_0$-measurable function $\theta' : \theta[E] \to E$ such that $\theta\theta'$ is the identity on $\theta[E]$. Now $\mathrm{T}_0$ is included in the domain $\hat{\mathrm{T}}$ of $\hat\nu$, so $\theta'$ is $\hat{\mathrm{T}}$-measurable, and there is a Borel set $F_0 \subseteq \theta[E]$ such that $\theta'\restriction F_0$ is Borel measurable and $\theta[E] \setminus F_0$ is $\hat\nu$-negligible (212Fa). Since $\theta'$ is surely injective, $E_0 = \theta'[F_0]$ is a Borel subset of $E$ (423Ib) and $\theta\restriction E_0$ is a bijection from $E_0$ to $F_0$ with inverse $\theta'$. Note that

$$\mu(E \setminus \theta^{-1}[F_0]) \le \hat\mu(\theta^{-1}[\theta[E]] \setminus \theta^{-1}[F_0]) = \hat\nu(\theta[E] \setminus F_0) = 0.$$

Define $\psi : \mathbb{R} \to \mathbb{R}$ by setting

$$\begin{aligned}\psi(x) &= \theta'\theta(x) \text{ if } x \in \theta^{-1}[F_0],\\ &= x \text{ otherwise.}\end{aligned}$$

Then $\psi$ is a Borel measurable function and $\theta\psi = \theta$, that is, $\phi_x^\bullet = \phi_{\psi(x)}^\bullet$ for every $x \in \mathbb{R}$. Consequently

$$\{(g,x) : g \in G,\, x \in \mathbb{R},\, \phi(g,x) \ne \phi(g,\psi(x))\}$$

has $\lambda$-negligible horizontal sections. Since it is a Borel set, it must have many $\mu$-negligible vertical sections; let $g_0 \in G$ be such that $\{x : \phi(g_0,x) \ne \phi(g_0,\psi(x))\}$ is $\mu$-negligible. By (b), we also have $\phi(g_0,x) = f_{g_0}(x)$ for $\mu$-almost every $x$. So the Borel set $H = \{x : f_{g_0}(x) = \phi(g_0,x) = \phi(g_0,\psi(x))\}$ is $\mu$-conegligible.

Set $A=\theta[E\cap\theta^{-1}[F_0]\cap H]$. Of course $A\subseteq F_0$. If $x\in\theta^{-1}[A]\setminus E$, then there is a $y\in E\cap\theta^{-1}[F_0]\cap H$ such that $\theta(y)=\theta(x)$; now $\psi(y)=\psi(x)$ and $x\ne y$, so

$$\phi(g_0,\psi(x))=\phi(g_0,\psi(y))=f_{g_0}(y)\ne f_{g_0}(x)$$

and $x\notin H$. Thus $\theta^{-1}[A]\setminus E\subseteq\mathbb{R}\setminus H$ is $\mu$-negligible. On the other hand, $E\setminus\theta^{-1}[A]\subseteq E\setminus(\theta^{-1}[F_0]\cap H)$ is also $\mu$-negligible. So $E\triangle\theta^{-1}[A]$ is $\mu$-negligible, as required. **Q**

**(e)** There is a $\mu$-conegligible Borel set $H\subseteq\mathbb{R}$ such that $\theta\restriction H$ is injective. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of Borel sets in $\mathbb{R}$ such that whenever $x$, $y\in\mathbb{R}$ are distinct there is an $n$ such that $x\in E_n$ and $y\notin E_n$. For each $n\in\mathbb{N}$ let $A_n\subseteq L^0(\lambda)$ be such that $E_n\triangle\theta^{-1}[A_n]$ is $\mu$-negligible; let $H$ be a $\mu$-conegligible Borel set disjoint from $\bigcup_{n\in\mathbb{N}}(E_n\triangle\theta^{-1}[A_n])$. If $x$, $y\in H$ are distinct, there is an $n\in\mathbb{N}$ such that $x\in E_n$ and $y\notin E_n$; now $x\in\theta^{-1}[A_n]$ and $y\notin\theta^{-1}[A_n]$, so $\theta(x)\ne\theta(y)$. **Q**

**(f)** Of course $\theta[H]$ is now a Borel subset of $L^0(\lambda)$, and must be $\hat\nu$-conegligible. Let $\mathfrak{B}$ be the measure algebra of $\nu$, and $\pi:\mathfrak{B}\to\mathfrak{A}$ the measure-preserving homomorphism defined by setting $\pi F^\bullet=\theta^{-1}[F]^\bullet$ for every Borel set $F$. If $E\subseteq\mathbb{R}$ is Borel, then $E^\bullet=\pi(\theta[E\cap H])^\bullet$ belongs to $\pi[\mathfrak{B}]$, so $\pi$ is surjective and is an isomorphism.

**(g)** Recall that we have a continuous action $\bullet_l$ of $G$ on $L^0(\lambda)$ defined as in 443G. If $g\in G$, then

$$g\bullet_l\theta(x)=\theta(\phi(g^{-1},x))=\theta(f_{g^{-1}}(x))$$

for $\mu$-almost every $x\in\mathbb{R}$. **P** Consider the set $\{(h,x):\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))\}\subseteq G\times\mathbb{R}$. Because $h\mapsto g^{-1}h$ is continuous, it is Borel measurable, so $(h,x)\mapsto\phi(g^{-1}h,x)$ is Borel measurable; the same is true of $(h,x)\mapsto\phi(h,\phi(g^{-1},x))$, so $\{(h,x):\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))\}$ is a Borel set. For given $h\in G$ and $E\in\mathcal{B}(\mathbb{R})$, set $F=\{x:\phi(h,x)\in E\}$; then

$$\begin{aligned}\{x:\phi(g^{-1}h,x)\in E\}^\bullet&=(g^{-1}h)\circ E^\bullet=g^{-1}\circ(h\circ E^\bullet)=g^{-1}\circ\{x:\phi(h,x)\in E\}^\bullet\\&=g^{-1}\circ F^\bullet=\{x:\phi(g^{-1},x)\in F\}^\bullet=\{x:\phi(h,\phi(g^{-1},x))\in E\}^\bullet\end{aligned}$$

so $\{x:\phi(g^{-1}h,x)\in E\}\triangle\{x:\phi(h,\phi(g^{-1},x))\in E\}$ is $\mu$-negligible. As $E$ is arbitrary, $\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))$ for $\mu$-almost every $x$.

This is true for every $h\in G$. So there is a $\mu$-conegligible Borel set $H'\subseteq\mathbb{R}$ such that if $x\in H'$ then $\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))$ for $\lambda$-almost every $h$. But this means that if $x\in H'$ then

$$(g\bullet_l\phi_x)(h)=\phi_x(g^{-1}h)=\phi(g^{-1}h,x)=\phi(h,\phi(g^{-1},x))=\phi_{\phi(g^{-1},x)}(h)$$

for $\lambda$-almost every $h$, and

$$g\bullet_l\theta(x)=g\bullet_l\phi_x^\bullet=(g\bullet_l\phi_x)^\bullet=\phi^\bullet_{\phi(g^{-1},x)}=\theta(\phi(g^{-1},x)).$$

Thus $g\bullet_l\theta(x)=\theta(\phi(g^{-1},x))$ for almost every $x$. And of course we already know from (b) that $\phi(g^{-1},x)=f_{g^{-1}}(x)$ for almost every $x$. **Q**

**(h)** We have a function $\circ_l:G\times\mathfrak{B}\to\mathfrak{B}$ defined by setting

$$g\circ_lF^\bullet=(g\bullet_lF)^\bullet$$

for every Borel set $F\subseteq L^0(\lambda)$ and $g\in G$, writing $g\bullet_lF=\{g\bullet_lu:u\in F\}$ as in 4A5Bc. **P** Take any $g\in G$. By (g) just above, applied to $g^{-1}$, $g^{-1}\bullet_l\theta(x)=\theta(f_g(x))$ for $\mu$-almost every $x$. Because the shift operator $u\mapsto g\bullet_lu:L^0(\lambda)\to L^0(\lambda)$ is a homeomorphism, it is a Borel automorphism, and $g\bullet_lF$ is a Borel set for every Borel set $F\subseteq L^0(\lambda)$. If $\nu F=0$, then

$$\begin{aligned}\nu(g\bullet_lF)&=\mu\{x:\theta(x)\in g\bullet_lF\}=\mu\{x:g^{-1}\bullet_l\theta(x)\in F\}\\&=\mu\{x:\theta(f_g(x))\in F\}=\mu(f_g^{-1}[\theta^{-1}[F]])=0\end{aligned}$$

because $\theta^{-1}[F]$ is $\mu$-negligible and $f_g$ represents an automorphism of the measure algebra $\mathfrak{A}$ of $\mu$. It follows that $(g\bullet_lF_0)\triangle(g\bullet_lF_1)=g\bullet_l(F_0\triangle F_1)$ is $\nu$-negligible and $(g\bullet_lF_0)^\bullet=(g\bullet_lF_1)^\bullet$ whenever $F_0^\bullet=F_1^\bullet$, which is what we need to know. **Q**

**(i)** For any $b\in\mathfrak{B}$ and $g\in G$, $g\circ\pi b=\pi(g\circ_lb)$. **P** Let $F\subseteq L^0(\lambda)$ be a Borel set such that $b=F^\bullet$, and set $E=\theta^{-1}[F]$, $a=E^\bullet=\pi b$. Then $g\circ_lb=(g\bullet_lF)^\bullet$, so

$$\pi(g\circ_l b) = \{x : \theta(x) \in g\bullet_l F\}^\bullet = \{x : g^{-1}\bullet_l \theta(x) \in F\}^\bullet = \{x : \theta(f_g(x)) \in F\}^\bullet$$
$$= \{x : f_g(x) \in E\}^\bullet = (f_g^{-1}[E])^\bullet = g\circ a = g\circ\pi b,$$

as required. **Q**

**(j)** Now observe that because $\lambda$ is a Haar measure, $\lambda G > 0$, so $L^0(\lambda) \neq \{0\}$, $L^0(\lambda)$ is uncountable and $\#(L^0(\lambda)) = \mathfrak{c} = \#(\mathbb{R})$ (423L). By 425Ad, there is a Borel isomorphism $\tilde\theta : \mathbb{R} \to L^0(\lambda)$ which represents $\pi$. Set

$$g\bullet x = \tilde\theta^{-1}(g\bullet_l \tilde\theta(x))$$

for $g \in G$ and $x \in \mathbb{R}$. Then $\bullet : G \times \mathbb{R} \to \mathbb{R}$ is a composition of the Borel measurable functions $(g, x) \mapsto (g, \tilde\theta(x))$, $(g, u) \mapsto g\bullet_l u$ and $u \mapsto \tilde\theta^{-1}(u)$, so is Borel measurable. Because $\Sigma = \mathcal{B}(\mathbb{R})$ and $\mathcal{B}(G \times \mathbb{R}) = \mathcal{B}(G)\widehat\otimes\mathcal{B}(\mathbb{R})$ (4A3Ga again), $\bullet$ is $(\mathcal{B}(G)\widehat\otimes\Sigma, \Sigma)$-measurable. If $g$, $h \in G$ and $x \in \mathbb{R}$,

$$gh\bullet x = \tilde\theta^{-1}(gh\bullet_l \tilde\theta(x)) = \tilde\theta^{-1}(g\bullet_l(h\bullet_l \tilde\theta(x))) = \tilde\theta^{-1}(g\bullet_l \tilde\theta(h\bullet x)) = g\bullet(h\bullet x),$$

and if $e$ is the identity of $G$ and $x \in \mathbb{R}$,

$$e\bullet x = \tilde\theta^{-1}(e\bullet_l \tilde\theta(x)) = \tilde\theta^{-1}\tilde\theta(x) = x.$$

Thus $\bullet$ is an action of $G$ on $\mathbb{R}$. If $g \in G$ and $E \in \mathcal{B}(\mathbb{R})$, set $F = \tilde\theta[E]$. Then $F^\bullet = \pi^{-1}[E^\bullet]$, so

$$(\tilde\theta^{-1}[g\bullet_l F])^\bullet = \pi((g\bullet_l F)^\bullet) = \pi(g\circ_l F^\bullet) = g\circ\pi F^\bullet = g\circ E^\bullet.$$

As

$$g\bullet E = \{g\bullet x : x \in E\} = \{\tilde\theta^{-1}(g\bullet_l \tilde\theta(x)) : x \in E\} = \{\tilde\theta^{-1}(g\bullet_l u) : u \in F\} = \tilde\theta^{-1}[g\bullet_l F],$$

we see that

$$g\circ E^\bullet = (g\bullet E)^\bullet$$

as required in the statement of this theorem.

**(k)** Thus the result is true if $(X, \Sigma) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\mu$ is totally finite. As for non-totally-finite $\mu$, there will always be a totally finite measure $\mu_1$ with the same domain and the same null ideal (215B again), in which case the measure algebra of $\mu_1$ will have the same Boolean algebra $\mathfrak{A}$, though with a different measure $\bar\mu_1$. However the measure-algebra topology of $\mathfrak{A}$ is unchanged (324H), so $\circ$ is still Borel measurable, and we can use the Borel measurable action of $G$ on $\mathbb{R}$ found by the method of (a)-(j) above. Since we are assuming that $(X, \Sigma)$ is a standard Borel space, this covers all the cases in which $X$ is uncountable, by 424C-424D.

**(l)** We are left with the case of countable $X$. This is of course essentially trivial. $\Sigma = \mathcal{P}X$ and $\mu$ is a point-supported measure. Let $Y$ be the set of atoms of $\mu$, that is, the set $\{x : \mu\{x\} > 0\}$. Then we can identify the measure algebra $\mathfrak{A} = \mathcal{P}X/\mathcal{P}(X \setminus Y)$ with $\mathcal{P}Y$, in which case the equivalence class $E^\bullet$ of any $E \subseteq X$ becomes identified with $E \cap Y$. As in part (a) of the proof above, we can represent each automorphism $a \mapsto g\circ a : \mathfrak{A} \to \mathfrak{A}$ by a permutation $f_g : X \to X$, and we must have $f_g^{-1}[Y] = Y$. Try

$$g\bullet x = f_g^{-1}(x) \text{ if } x \in Y,$$
$$= x \text{ if } x \in X \setminus Y$$

for every $g \in G$. If $g$, $h \in G$ and $x \in Y$,

$$\{gh\bullet x\} = \{f_{gh}^{-1}(x)\} = gh\circ\{x\} = g\circ(h\circ\{x\}) = g\circ f_h^{-1}[\{x\}]$$
$$= f_g^{-1}[f_h^{-1}[\{x\}]] = f_g^{-1}[\{f_h^{-1}(x)\}] = f_g^{-1}\{h\bullet x\} = \{g\bullet(h\bullet x)\}$$

and $gh\bullet x = g\bullet(h\bullet x)$; if $x \in X \setminus Y$, then $gh\bullet x = x = g\bullet(h\bullet x)$. Of course $e\bullet x = x$ for every $x \in X$. So $\bullet$ is an action of $G$ on $X$. To see that it is $(\mathcal{B}(G)\widehat\otimes\Sigma, \Sigma)$-measurable, note that the measure-algebra topology of $\mathfrak{A} \cong \mathcal{P}Y$ is the discrete topology. If $y \in Y$, then $\{g : g\bullet y = z\} = \{g : g\circ\{y\} = \{z\}\}$ is a Borel set for every $z \in X$; if $x \in X \setminus Y$, then $\{g : g\bullet x = z\}$ is either $G$ or $\emptyset$ for every $z \in X$. So

$$\{(g, x) : g\bullet x \in W\} = \bigcup_{z\in W, x\in X}\{g : g\bullet x = z\} \times \{x\}$$

belongs to $\mathcal{B}(G)\widehat{\otimes}\Sigma$ for every subset $W$ of $X$, and $\bullet$ is $(\mathcal{B}(G)\widehat{\otimes}\Sigma,\Sigma)$-measurable. Finally, if $g\in G$,

$$(g\bullet E)^\bullet=(g\bullet E)\cap Y=Y\cap f_g^{-1}[E]=(f_g^{-1}[E])^\bullet=g\circ E^\bullet$$

for every $E\subseteq X$. So again we have a suitable action of $G$ on $X$.

This completes the proof.

**448T Corollary** Let $G$ be a $\sigma$-compact locally compact Hausdorff group, $X$ a Polish space, $\mu$ a $\sigma$-finite Borel measure on $X$, and $(\mathfrak{A},\bar\mu)$ the measure algebra of $\mu$, with its measure-algebra topology. Let $\circ$ be a continuous action of $G$ on $\mathfrak{A}$ such that $a\mapsto g\circ a$ is a Boolean automorphism for every $g\in G$. Then we have a Borel measurable action $\bullet$ of $G$ on $X$ such that

$$g\circ E^\bullet=(g\bullet E)^\bullet$$

for every $g\in G$ and $E\in\mathcal{B}(X)$.

**proof** We know that $\mathfrak{A}$ is separable (331O again) and metrizable (323Gb); let $\langle a_n\rangle_{n\in\mathbb{N}}$ run over a topologically dense subset of $\mathfrak{A}$ and $\langle U_n\rangle_{n\in\mathbb{N}}$ over a base for its topology. For each $(m,n)$ such that $a_m\in U_n$, $V_{mn}=\{g:g\circ a_m\in U_n\}$ is a neighbourhood of the identity $e$ of $G$. By 4A5S, there is a compact normal subgroup $H$ of $G$ such that $H\subseteq\bigcap\{V_{mn}:m,\,n\in\mathbb{N},\,a_m\in U_n\}$ and $G/H$ is Polish. Now we have a continuous action $\bar\circ$ of $G/H$ on $\mathfrak{A}$ such that $g^\bullet\bar\circ a=g\circ a$ for every $g\in G$ and $a\in\mathfrak{A}$. **P** If $g$, $h\in G$ are such that $g^\bullet=h^\bullet$, $m\in\mathbb{N}$, and $a_m\in U_n$, then $g^{-1}h\in V_{mn}$ so $g^{-1}h\circ a_m\in U_n$. As $n$ is arbitrary, $g^{-1}h\circ a_m=a_m$; as $a\mapsto g^{-1}h\circ a$ is continuous, and $m$ is arbitrary, $g^{-1}h\circ a=a$ and $h\circ a=g\circ a$ for every $a\in\mathfrak{A}$. This shows that the given formula defines a function $\bar\circ$ from $(G/H)\times\mathfrak{A}$ to $\mathfrak{A}$. It is easy to check that $\bar\circ$ is an action of $G/H$ on $\mathfrak{A}$.

Now suppose that $v\in G/H$, $a\in\mathfrak{A}$ and $U$ is a neighbourhood of $v\bar\circ a$ in $\mathfrak{A}$. Let $g\in G$ be such that $g^\bullet=v$; then $g\circ a=v\bar\circ a$, so there are open sets $V\subseteq G$ and $U'\subseteq\mathfrak{A}$ such that $g\in V$, $a\in U'$ and $h\circ b\in U$ whenever $h\in V$ and $b\in U'$. By 4A5Ja, $W=\{h^\bullet:h\in V\}$ is open in $G/H$; now $v\in W$ and $w\bar\circ b\in U$ whenever $w\in W$ and $b\in U'$. As $v$, $a$ and $U$ are arbitrary, $\bar\circ$ is continuous. **Q**

There is therefore a Borel measurable action $\bar\bullet:(G/H)\times X\to X$ such that $v\bar\circ E^\bullet=(v\bar\bullet E)^\bullet$ whenever $v\in G/H$ and $E\in\mathcal{B}(X)$ (448S). Set $g\bullet x=g^\bullet\bar\bullet x$ for $g\in G$ and $x\in X$. It is elementary to check that $\bullet$ is an action of $G$ on $X$. Also it is Borel measurable, because $(g,x)\mapsto(g^\bullet,x)$ is continuous, therefore Borel measurable, and $(g^\bullet,x)\mapsto g^\bullet\bar\bullet x$ is Borel measurable. If $g\in G$ and $E\in\mathcal{B}(X)$, then

$$g\circ E^\bullet=g^\bullet\bar\circ E^\bullet=(g^\bullet\bar\bullet E)^\bullet=(g\bullet E)^\bullet,$$

so $\bullet$ is an action of the kind we seek.

**448X Basic exercises (a)** Show that the results in 448Fb and 448Fd remain true if $G$ is not assumed to be countable.

**(b)** In part (c) of the proof of 448O, show that $\mathcal{I}$ is just the set of those $d\in\mathfrak{A}$ such that $d\subseteq\operatorname{upr}(1\setminus a,\mathfrak{C})$ for some $a$ such that $1\preccurlyeq_G^\sigma a$.

**(c)** Show that, in part (c) of the proof of 448P, we can if we wish take $Z=X$ and $\Sigma=\mathcal{B}$.

**>(d)** Let $(X,\Sigma)$ be a standard Borel space and $\Sigma_0$ a countable subalgebra of $\Sigma$. Show that there is a sequence $\langle\langle E_{ni}\rangle_{i\in\mathbb{N}}\rangle_{n\in\mathbb{N}}$ of partitions of unity in $\Sigma$ such that whenever $\nu:\Sigma\to\mathbb{R}$ is a finitely additive functional and $\nu X=\sum_{i=0}^\infty\nu E_{ni}$ for every $n\in\mathbb{N}$, then $\nu\restriction\Sigma_0$ is countably additive.

**>(e)** Set $X=[0,1]\setminus\mathbb{Q}$, $G=\mathbb{Q}$ and define $\bullet:G\times X\to X$ by requiring that $g\bullet x-g-x\in\mathbb{Z}$ for $g\in G$ and $x\in X$. Show that this is a Borel measurable action and that Lebesgue measure on $X$ is $G$-invariant. Find a metric on $X$, inducing its topology, for which all the maps $x\mapsto g\bullet x$ are isometries.

**>(f)** Show that a Polish group carries Haar measures iff it is locally compact. (*Hint*: 443E.)

**(g)** Give $\mathbb{Z}^{\mathbb{N}}$ its usual (product) topology and abelian group structure. Show that it is a Polish group, and has no Haar measure.

**>(h)** Let $(X,\rho)$ be a metric space, $G$ a group and $\bullet$ an action of $G$ on $X$ such that $x \mapsto g\bullet x$ is an isometry for every $g \in G$. (i) Show that if $\mu$ is a $G$-invariant quasi-Radon probability measure on $X$ then $\{g\bullet x : g \in G\}$ is totally bounded for every $x$ in the support of $\mu$. (ii) Show that if the action is transitive and there is a non-zero $G$-invariant quasi-Radon measure on $X$, then $X$ is covered by totally bounded open sets. (iii) Suppose that $X$ has measure-free weight (see §438; for instance, $X$ could be separable). Show that if the action is transitive and there is a $G$-invariant topological probability measure on $X$ then $X$ is totally bounded.

**(i)** Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, with the operation $\triangle$ and the measure-algebra topology. (i) Show that $\mathfrak{A}$ is a topological group. (ii) Show that if $\bar\mu$ is $\sigma$-finite and $\mathfrak{A}$ has countable Maharam type, it is a Polish group. (iii) Show that if $(\mathfrak{A},\bar\mu)$ is semi-finite and not purely atomic, then $\mathfrak{A}$ has no Haar measure.

**(j)** Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$, with its measure metric, and $G = \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ the group of measure-preserving automorphisms on $\mathfrak{A}$. (i) Show that if we give $G$ the topology induced by the topology of pointwise convergence on the isometry group of $\mathfrak{A}$, then it is a Polish group. (*Hint*: 441Xq.) (ii) Show that if $\nu$ is a $G$-invariant topological probability measure on $\mathfrak{A}$, then $\nu\{0,1\} = 1$.

**448Y Further exercises (a)** Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra, and $G$ a subgroup of $\operatorname{Aut}\mathfrak{A}$; let $G^*_\sigma$ be the countably full local semigroup generated by $G$, and write $H$ for the union of all the full local semigroups generated by countable subgroups of $G$ (following the definition in 395A as written, without troubling about whether $\mathfrak{A}$ is Dedekind complete). (i) Show that $G^*_\sigma \subseteq H$. (ii) Find an example in which $H \ne G^*_\sigma$. (iii) Show that if $\mathfrak{A}$ is Dedekind complete then $G^*_\sigma = H$. (iv) Show that if $\mathfrak{A}$ is ccc then $G^*_\sigma = H$ is the full local semigroup generated by $G$.

**(b)** In 448N, show that $\theta$ is uniquely defined.

**(c)** Let $(X,\Sigma)$ be a standard Borel space, $Y$ any set, $\mathrm{T}$ a $\sigma$-algebra of subsets of $Y$ and $\mathcal{J}$ a $\sigma$-ideal of subsets of $\mathrm{T}$. Let $\theta : \Sigma \to L^\infty(\mathrm{T}/\mathcal{J})$ be a non-negative, sequentially order-continuous additive function. Show that there is a non-negative, sequentially order-continuous additive function $\phi : \Sigma \to L^\infty(\mathrm{T})$ such that (identifying $L^\infty(\mathrm{T}/\mathcal{J})$ with a quotient space of $L^\infty(\mathrm{T})$) $\theta E = (\phi E)^\bullet$ for every $E \in \Sigma$.

**448 Notes and comments** The keys to the first part of the section are in 448F, 448G and 448L. Even though we no longer have a Dedekind complete algebra, the fact that we are working with countable groups means that the suprema we actually need are defined. The final step, however, uses yet another idea. In a standard Borel space, given a finitely additive functional on the $\sigma$-algebra, we can sometimes confirm an adequate approximation to countable additivity by looking at only countably many sequences (448Xd). This enables us to pass from a $G$-invariant map $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ to a $G$-invariant Radon measure (parts (d)-(f) of the proof of 448P), without needing to know anything about the algebra $\mathfrak{D}$ except that it is Dedekind $\sigma$-complete. In particular (and in contrast to the corresponding step in 395P) we do not need to suppose that $\mathfrak{D}$ is a measurable algebra. I do not know whether there is a useful ergodicity condition which could be added to the hypotheses of 448O to ensure that $\mathfrak{D}$ there becomes $\{0,1\}$.

448P was proved in the case $G = \mathbb{Z}$ by Nadkarni 90; the extension to general Borel actions by Polish groups is due to Becker & Kechris 96. (See Nadkarni 90 for notes on the history of the problem, and Kechris 95 for the basic general theory of Polish groups and Borel actions.) It is a remarkable result, but its application is limited by the difficulty of determining whether either condition (ii) or condition (iii) is satisfied. Much commoner situations are those like 448Xe-448Xj, where either there is no invariant measure or we can find one easily.

The second main theorem makes no reference to the first. But it has something in common. It is an example of the power of descriptive set theory to dramatically extend a result on group actions, which is comparatively straightforward when the group in question is $\mathbb{Z}$, to arbitrary Polish groups. Nadkarni's theorem is not obvious, but it is a lot easier than the general result here. Mackey's theorem for countable groups also requires a little care, but is essentially covered (in usefully greater generality) by 344C. The descriptive set theory the theorem here relies on does not go as deep as the Becker-Kechris theorem, but in exchange it calls on a kind of lifting theorem quite different from those in Chapter 34. Looked at from the standpoint of Chapter 34, 448Q is a rank impossibility (see 341Xg); but the point is that we have abandoned the ordinary algebraic requirements on a lifting and replaced them by a strong measurability property.

Of course a lifting was used in 344C as well, but in a quite different way. There the hypotheses were adjusted to give a slightly more general context in which we could be sure that individual homomorphisms from the measure algebra to itself were representable by functions from the measure space to itself; and I relied indirectly on the lifting theorem 341K to set up the functions. For the context of the present section, this step was done in 425A with no mention of liftings, but using the classification of standard Borel spaces in 424D. In view of 424Yf, it is plain that we do not get much extra generality by using the argument through 341K. The real difference in 344B-344C is that we can deal with semigroups of homomorphisms as well as groups of automorphisms.

The proof of Mackey's theorem is based on there being a Haar measure on $G$, so that we can use Fubini's theorem (three times, in parts (c), (d) and (g) of the proof). There are non-locally-compact groups $G$ for which a corresponding result is true (KWIATOWSKA & SOLECKI 11); it remains quite unclear when to expect this.

Version of 13.6.13

## 449 Amenable groups

I end this chapter with a brief introduction to 'amenable' topological groups. I start with the definition (449A) and straightforward results assuring us that there are many amenable groups (449C). At a slightly deeper level we have a condition for a group to be amenable in terms of a universal object constructible from the group, not invoking 'all compact Hausdorff spaces' (449E). I give some notes on amenable locally compact groups, concentrating on a long list of properties equivalent to amenability (449J), and a version of Tarski's theorem characterizing amenable discrete groups (449M). I end with Banach's theorem on extending Lebesgue measure in one and two dimensions.

**449A Definition** A topological group $G$ is **amenable** if whenever $X$ is a non-empty compact Hausdorff space and $\bullet$ is a continuous action of $G$ on $X$, then there is a $G$-invariant Radon probability measure on $X$.

**Warning**: other definitions have been used, commonly based on conditions equivalent to amenability for locally compact Hausdorff groups, such as those listed in 449J(ii)-449J(xiv). In addition, many authors use the phrase 'amenable group' to mean a group which is amenable in its discrete topology. The danger of this to the non-specialist is that many theorems concerning amenable discrete groups do not generalize in the ways one might expect.

**449B Lemma** Let $G$ be a topological group, $X$ a locally compact Hausdorff space, and $\bullet$ a continuous action of $G$ on $X$.

(a) Writing $C_0$ for the Banach space of continuous real-valued functions on $X$ vanishing at $\infty$ (436I), the map $a \mapsto a^{-1}\bullet f : C_0 \to C_0$ (definition: 4A5Cc) is uniformly continuous for the right uniformity on $G$ and the norm uniformity of $C_0$, for any $f \in C_0$.

(b) If $\mu$ is a $G$-invariant Radon measure on $X$ and $1 \le p < \infty$, then $a \mapsto a^{-1}\bullet u : G \to L^p$ (definition: 441Kc) is uniformly continuous for the right uniformity on $G$ and the norm uniformity of $L^p = L^p(\mu)$, for any $u \in L^p$.

**proof (a)(i)** Note first that if $a \in G$ and $f \in C_0$, then $x \mapsto a\bullet x : X \to X$ is a homeomorphism (4A5Bd), so $x \mapsto f(a\bullet x)$ belongs to $C_0$; but this is just the function $a^{-1}\bullet f$.

**(ii)** For any $\epsilon > 0$ and $f \in C_0$ there is a neighbourhood $V$ of the identity $e$ of $G$ such that $\|f - a^{-1}\bullet_l f\|_\infty \le \epsilon$ for every $a \in V$. **P?** Suppose, if possible, otherwise. For each symmetric neighbourhood $V$ of $e$ set

$$Q_V = \{(a,x) : a \in V,\, x \in X,\, |f(x)| \ge \tfrac{\epsilon}{2},\, |f(x) - f(a\bullet x)| \ge \epsilon\}.$$

We are supposing that there are $a \in V$ and $x \in X$ such that $|f(x) - f(a\bullet x)| \ge \epsilon$. If $|f(x)| \ge \frac{1}{2}\epsilon$ then $(a, x) \in Q_V$. Otherwise, $|f(a\bullet x)| \ge \frac{1}{2}\epsilon$, $a^{-1} \in V$ and $|f(a\bullet x) - f(a^{-1}a\bullet x)| \ge \epsilon$, so $(a^{-1}, a\bullet x) \in Q_V$. Thus $Q_V$ is never empty. Since $Q_V \subseteq Q_{V'}$ whenever $V \subseteq V'$, there is an ultrafilter $\mathcal{F}$ on $G \times X$ such that $Q_V \in \mathcal{F}$ for every neighbourhood $V$ of $e$. Setting $\pi_1(a, x) = a$ and $\pi_2(a, x) = x$ for $(a, x) \in G \times X$, we see that the image filter $\pi_1[[\mathcal{F}]]$ contains every neighbourhood of $e$, so converges to $e$, while $\pi_2[[\mathcal{F}]]$ contains the compact set $\{x : |f(x)| \ge \frac{1}{2}\epsilon\}$, so must have a limit $x_0$ in $X$. So $\mathcal{F} \to (e, x_0)$ in $G \times X$. Next, because the action is continuous, $\bullet[[\mathcal{F}]] \to e\bullet x_0 = x_0$, and there must be an $F \in \mathcal{F}$ such that $|f(x_0) - f(a\bullet x)| \le \frac{1}{3}\epsilon$ for every $(a, x) \in F$. Also, of course, there is an $F' \in \mathcal{F}$ such that $|f(x_0) - f(x)| \le \frac{1}{3}\epsilon$ whenever $(a, x) \in F'$. But now there is an $(a, x) \in Q_G \cap F \cap F'$, and we have

$$|f(x) - f(a\bullet x)| \ge \epsilon, \quad |f(x_0) - f(a\bullet x)| \le \tfrac{1}{3}\epsilon, \quad |f(x_0) - f(x)| \le \tfrac{1}{3}\epsilon$$

simultaneously, which is impossible. **XQ**

Now we find that if $a$, $b \in G$, $ab^{-1} \in V$ and $x \in X$, then

$$|(a^{-1}\bullet f)(x) - (b^{-1}\bullet f)(x)| = |f(a\bullet x) - f(b\bullet x)| = |f(ab^{-1}\bullet(b\bullet x)) - f(b\bullet x)| \le \epsilon.$$

As $x$ is arbitrary, $\|a^{-1}\bullet f - b^{-1}\bullet f\|_\infty \le \epsilon$; as $\epsilon$ is arbitrary, $a \mapsto a^{-1}\bullet f$ is uniformly continuous for the right uniformity.

**(b)(i)** Suppose that $f : X \to \mathbb{R}$ is continuous and has compact support $K = \overline{\{x : f(x) \ne 0\}}$. Let $H \supseteq K$ be an open set of finite measure. Then $V_0 = \{a : a \in G,\ a\bullet x \in H \text{ for every } x \in K\}$ is a neighbourhood of $e$. **P** If we take a continuous function $f_0$ with compact support such that $\chi K \le f_0 \le \chi H$ (4A2G(e-i)), then $V_0 \supseteq \{a : \|f_0 - a^{-1}\bullet f_0\|_\infty < 1\}$, which is a neighbourhood of $e$ by (a). **Q** Let $\epsilon > 0$. By (a) again, there is a symmetric neighbourhood $V_1$ of $e$ such that $(\|f - a^{-1}\bullet f\|_\infty)^p \mu H \le \epsilon^p$ for every $a \in V_1$; we may suppose that $V_1 \subseteq V_0$. If $a \in V_1$, $f(x) = f(a\bullet x) = 0$ for every $x \in X \setminus H$, so that

$$\|f - a^{-1}\bullet f\|_p^p = \int_H |f - a^{-1}\bullet f|^p d\mu \le (\|f - a^{-1}\bullet f\|_\infty)^p \mu H \le \epsilon^p.$$

Now suppose that $a$, $b \in G$ and that $ab^{-1} \in V_1$. Then

$$\begin{aligned}\|a^{-1}\bullet f - b^{-1}\bullet f\|_p^p &= \int |f(a\bullet x) - f(b\bullet x)|^p \mu(dx) \\ &= \int |f(a\bullet(b^{-1}\bullet x)) - f(b\bullet(b^{-1}\bullet x))|^p \mu(dx)\end{aligned}$$

(because $\mu$ is $G$-invariant, see 441L)

$$= \int |ba^{-1}\bullet f - f)|^p d\mu \le \epsilon^p,$$

and $\|a^{-1}\bullet f - b^{-1}\bullet f\|_p \le \epsilon$. As $\epsilon$ is arbitrary, $a \mapsto (a^{-1}\bullet f)^\bullet$ is uniformly continuous for the right uniformity.

**(ii)** In general, given $u \in L^p(\mu)$ and $\epsilon > 0$, there is an $f \in C_k(X)$ such that $\|u - f^\bullet\|_p \le \epsilon$ (416I). Let $V$ be a neighbourhood of $e$ such that $\|a^{-1}\bullet f - a^{-1}\bullet f\|_\infty \le \epsilon$ whenever $ab^{-1} \in V$; then $\|a^{-1}\bullet u - (a^{-1}\bullet f)^\bullet\|_p = \|u - f^\bullet\|_p$ (because $\mu$ is $G$-invariant), so

$$\|a^{-1}\bullet u - b^{-1}\bullet u\|_p \le \|a^{-1}\bullet u - a^{-1}\bullet f^\bullet\|_p + \|a^{-1}\bullet f - b^{-1}\bullet f\|_p + \|b^{-1}\bullet f^\bullet - b^{-1}\bullet u\|_p \le 3\epsilon$$

whenever $ab^{-1} \in V$ (using 441Kc). As $\epsilon$ is arbitrary, $a \mapsto a^{-1}\bullet u$ is uniformly continuous for the right uniformity. This completes the proof.

**449C Theorem** (a) Let $G$ and $H$ be topological groups such that there is a continuous surjective homomorphism from $G$ onto $H$. If $G$ is amenable, so is $H$.

(b) Let $G$ be a topological group and suppose that there is a dense subset $A$ of $G$ such that every finite subset of $A$ is included in an amenable subgroup of $G$. Then $G$ is amenable.

(c) Let $G$ be a topological group and $H$ a normal subgroup of $G$. If $H$ and $G/H$ are both amenable, so is $G$.

(d) Let $G$ be a topological group with two amenable subgroups $H_0$ and $H_1$ such that $H_0$ is normal and $H_0 H_1 = G$. Then $G$ is amenable.

(e) The product of any family of amenable topological groups is amenable.
(f) Any abelian topological group is amenable.
(g) Any compact Hausdorff topological group is amenable.

**proof (a)** Let $\phi : G \to H$ be a continuous surjective homomorphism. Let $X$ be a non-empty compact Hausdorff space and $\bullet : H \times X \to X$ a continuous action. For $a \in G$ and $x \in X$, set $a\bullet_1 x = \phi(a)\bullet x$. Then $\bullet_1$ is a continuous action of $G$ on $X$, so there is a $G$-invariant Radon probability measure $\mu$ on $X$. Because $\phi[G] = H$, $\mu$ is also $H$-invariant; as $X$ and $\bullet$ are arbitrary, $H$ is amenable.

**(b)(i)** Let $X$ be a non-empty compact Hausdorff space and $\bullet$ a continuous action of $G$ on $X$. Let $P = P_{\mathrm{R}}(X)$ be the set of Radon probability measures on $X$ with the narrow topology (437Jd), so that $P$ is a compact Hausdorff space (437R(f-ii)); recall that in this context the vague and narrow topologies coincide (437Kc). For $a \in G$ and $x \in X$, set $T_a(x) = a\bullet x$, so that $T_a : X \to X$ is a homeomorphism. For $a \in G$ and $\mu \in P$ write $a\bullet\mu$ for the image measure $\mu T_a^{-1}$, so that $a\bullet\mu \in P$ (418I); it is easy to check that $(a, \mu) \mapsto a\bullet\mu : G \times P \to P$ is an action. The point is that it is continuous. **P** Let $f \in C(X)$, $a_0 \in G$, $\mu_0 \in P$ and $\epsilon > 0$. By 449Ba, there is a neighbourhood $V$ of $a_0$ in $G$ such that $\|a^{-1}\bullet f - a_0^{-1}\bullet f\|_\infty \le \frac{1}{2}\epsilon$ for every $a \in V$. Next, there is a neighbourhood $W$ of $\mu_0$ in $P$ such that $|\int a_0^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0| \le \frac{1}{2}\epsilon$ for every $\mu \in W$. But now, if $a \in V$ and $\mu \in W$,

$$|\int f d(a\bullet\mu) - \int f d(a_0\bullet\mu_0)| = |\int f T_a d\mu - \int f T_{a_0} d\mu_0|$$

(235J)

$$= |\int a^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0|$$

$$\le |\int a^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu| + |\int a_0^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0|$$

$$\le \|a^{-1}\bullet f - a_0^{-1}\bullet f\|_\infty + \frac{1}{2}\epsilon \le \epsilon.$$

As $\epsilon$, $a_0$ and $\mu_0$ are arbitrary, $(a, \mu) \mapsto \int f d(a\bullet\mu)$ is continuous; as $f$ is arbitrary, $\bullet$ is continuous. **Q**

**(ii)** Because the topology of $P$ is Hausdorff, it follows that $Q_a = \{\mu : \mu \in P, a\bullet\mu = \mu\}$ is closed in $P$ for any $a \in G$, and that $G_\mu = \{a : a \in G, a\bullet\mu = \mu\}$ is closed in $G$ for any $\mu \in P$. Now for any finite subset $I$ of $A$ there is an amenable subgroup $H_I$ of $G$ including $I$. The restriction of the action to $H_I \times X$ is a continuous action of $H_I$ on $X$, so has an $H_I$-invariant Radon probability measure, and $\bigcap_{a\in I} Q_a \supseteq \bigcap_{a\in H_I} Q_a$ is non-empty. Because $P$ is compact, there is a $\mu \in \bigcap_{a\in A} Q_a$. Since $G_\mu$ includes the dense set $A$, it is the whole of $G$, and $\mu$ is $G$-invariant. As $X$ and $\bullet$ are arbitrary, $G$ is amenable.

**(c)** Let $X$ be a compact Hausdorff space and $\bullet$ a continuous action of $G$ on $X$. Let $P$ be the space of Radon probability measures on $X$ with its narrow topology. Define $a\bullet\mu$, for $a \in G$ and $\mu \in P$, as in (b-i) above, so that this is a continuous action of $G$ on $P$. Set $Q = \{\mu : \mu \in P, a\bullet\mu = \mu \text{ for every } a \in H\}$; then $Q$ is a closed subset of $P$ and, because $H$ is amenable, is non-empty, since it is the set of $H$-invariant Radon probability measures on $X$. Next, $b\bullet\mu \in Q$ for every $\mu \in Q$ and $b \in G$. **P** If $a \in H$, then

$$a\bullet(b\bullet\mu) = (ab)\bullet\mu = (bb^{-1}ab)\bullet\mu = b\bullet((b^{-1}ab)\bullet\mu) = b\bullet\mu,$$

because $H$ is normal, so $b^{-1}ab \in H$. As $a$ is arbitrary, $b\bullet\mu \in Q$. **Q** Accordingly we have a continuous action of $G$ on the compact Hausdorff space $Q$.

If $a \in H$ and $b \in G$, then $b\bullet\mu = (ba)\bullet\mu$ for every $\mu \in Q$. We therefore have a map $\circ : G/H \times Q \to Q$ defined by setting $b^\bullet\circ\mu = b\bullet\mu$ whenever $b \in G$ and $\mu \in Q$. It is easy to check that this is an action. Moreover, it is continuous, because if $W \subseteq Q$ is relatively open then $\{(b, \mu) : b\bullet\mu \in W\}$ is open in $G \times Q$, so its image $\{(b^\bullet, \mu) : b^\bullet\circ\mu \in W\}$ is open in $(G/H) \times Q$ (using 4A2B(f-iv)). Because $G/H$ is amenable, there is a $(G/H)$-invariant Radon probability measure $\lambda$ on $Q$.

Now consider the formula $p(f) = \int_Q(\int_X f(x)\mu(dx))\lambda(d\mu)$. If $f \in C(X)$, then $\mu \mapsto \int_X f(x)\mu(dx)$ is continuous for the vague topology on $Q$, so $p(f)$ is well-defined. Clearly $p$ is a linear functional, $p(f) \ge 0$ if

$f \geq 0$, and $p(\chi X) = 1$; so there is a Radon probability measure $\nu$ on $X$ such that $p(f) = \int f d\nu$ for every $f \in C(X)$ (436J/436K). If $b \in G$, then, in the language of (b) above,

$$\int f d(b\bullet\nu) = \int fT_b d\nu = p(fT_b) = \int_Q (\int_X fT_b d\mu)\lambda(d\mu)$$
$$= \int_Q \int_X f d(b\bullet\mu)\lambda(d\mu) = \int_Q \int_X f d(b^\bullet\circ\mu)\lambda(d\mu) = \int_Q \int_X f d\mu\,\lambda(d\mu)$$

(because $\lambda$ is $G/H$-invariant)

$$= \int f d\nu$$

for every $f \in C(X)$, so that $b\bullet\nu = \nu$. Thus $\nu$ is $G$-invariant. As $X$ and $\bullet$ are arbitrary, $G$ is amenable.

**(d)** The canonical map from $H_1$ to $G/H_0$ is a continuous surjective homomorphism. By (a), $G/H_0$ is amenable; by (c), $G$ is amenable.

**(e)** By (c) or (d), the product of two amenable topological groups is amenable, since each can be regarded as a normal subgroup of the product. It follows that the product of finitely many amenable topological groups is amenable. Now let $\langle G_i\rangle_{i\in I}$ be any family of amenable topological groups with product $G$. For finite $J \subseteq I$ let $H_J$ be the set of those $a \in G$ such that $a(i)$ is the identity in $G_i$ for every $i \in I\setminus J$. Then $H_J$ is isomorphic (as topological group) to $\prod_{i\in J} G_i$, so is amenable. Since $\{H_J : J \in [I]^{<\omega}\}$ is an upwards-directed family of subgroups of $G$ with dense union, (b) tells us that $G$ is amenable.

**(f)(i)** The first step is to observe that the group $\mathbb{Z}$, with its discrete topology, is amenable. **P** Let $X$ be a compact Hausdorff space and $\bullet$ a continuous action of $\mathbb{Z}$ on $X$. Set $\phi(x) = 1\bullet x$ for $x \in X$. Then $\phi : X \to X$ is continuous, so by 437T there is a Radon probability measure $\mu$ on $X$ such that $\mu$ is equal to the image measure $\mu\phi^{-1}$. Because $\phi$ is bijective, we see that, for $E \subseteq X$, $E \in \operatorname{dom}\mu$ iff $\phi[E] \in \operatorname{dom}(\mu\phi^{-1}) = \operatorname{dom}\mu$, and in this case $\mu\phi[E] = \mu E$; that is, $\phi^{-1}$, like $\phi$, is inverse-measure-preserving. Now we can induce on $n$ to see that $\mu(\phi^n)^{-1}$ and $\mu(\phi^{-n})^{-1}$ are equal to $\mu$ for every $n$. Since $n\bullet x = \phi^n(x)$ for every $x \in X$ and $n \in \mathbb{Z}$, $\mu$ is $\mathbb{Z}$-invariant. As $X$ and $\bullet$ are arbitrary, $\mathbb{Z}$ is amenable. **Q**

**(ii)** Now let $G$ be any abelian topological group. For each finite set $I \subseteq G$ let $\phi_I : \mathbb{Z}^I \to G$ be the continuous homomorphism defined by setting $\phi_I(z) = \prod_{a\in I} a^{z(a)}$ for $z \in \mathbb{Z}^I$. By (i) just above and (e), we know that $\mathbb{Z}^I$ (with its discrete topology) is amenable, so (a) tells us that the subgroup $G_I = \phi_I[\mathbb{Z}^I]$ is amenable. But now $\{G_I : I \in [G]^{<\omega}\}$ is an upwards-directed family of amenable subgroups of $G$ with union $G$, so from (b) we see that $G$ is amenable.

**(g)** This is immediate from 443Ub. (See also 449Xe.)

**449D Theorem** Let $G$ be a topological group.

(a) Write $U$ for the set of bounded real-valued functions on $G$ which are uniformly continuous for the right uniformity of $G$. Then $U$ is an $M$-space, and we have an action $\bullet_l$ of $G$ on $U$ defined by the formula $(a\bullet_l f)(y) = f(a^{-1}y)$ for $a$, $y \in G$ and $f \in U$.

(b) Let $Z \subseteq \mathbb{R}^U$ be the set of Riesz homomorphisms $z : U \to \mathbb{R}$ such that $z(\chi G) = 1$. Then $Z$ is a compact Hausdorff space, and we have a continuous action of $G$ on $Z$ defined by the formula $(a\bullet z)(f) = z(a^{-1}\bullet_l f)$ for $a \in G$, $z \in Z$ and $f \in U$.

(c) Setting $\hat{a}(f) = f(a)$ for $a \in G$ and $f \in U$, the map $a \mapsto \hat{a} : G \to Z$ is a continuous function from $G$ onto a dense subset of $Z$. If $a$, $b \in G$ then $a\bullet\hat{b} = \widehat{ab}$.

(d) Now suppose that $X$ is a compact Hausdorff space, $(a, x) \mapsto a\bullet x$ is a continuous action of $G$ on $X$, and $x_0 \in X$. Then there is a unique continuous function $\phi : Z \to X$ such that $\phi(\hat{e}) = x_0$ and $\phi(a\bullet z) = a\bullet\phi(z)$ for every $a \in G$ and $z \in Z$.

(e) If $G$ is Hausdorff then the action of $G$ on $Z$ is faithful and the map $a \mapsto \hat{a}$ is a homeomorphism between $G$ and its image in $Z$.

**proof (a)** Because $U$ is a norm-closed Riesz subspace of $C_b(G)$ containing the constant functions (4A2Jh), it is an $M$-space. To see that the given formula defines an action, we need to check that $a\bullet_l f$ belongs to $U$

whenever $a \in G$ and $f \in U$. Of course $a \bullet_l f$ is continuous and $\|a \bullet_l f\|_\infty = \|f\|_\infty$ is finite. If $\epsilon > 0$ there is a neighbourhood $V$ of the identity $e$ in $G$ such that $|f(b) - f(c)| \le \epsilon$ whenever $b$, $c \in G$ and $bc^{-1} \in V$; now $a^{-1}Va$ is a neighbourhood of $e$, and if $bc^{-1} \in aVa^{-1}$ then $(a^{-1}b)(a^{-1}c)^{-1} \in V$, so $|(a \bullet_l f)(b) - (a \bullet_l f)(c)| = |f(a^{-1}b) - f(a^{-1}c)| \le \epsilon$. As $\epsilon$ is arbitrary, $a \bullet_l f$ is uniformly continuous with respect to the right uniformity. Now $\bullet_l$ is an action, just as in 4A5Cc.

**(b)(i)** Because $U$ is an $M$-space with standard order unit $\chi G$, $Z$ is a compact Hausdorff space and $U$ can be identified, as normed Riesz space, with $C(Z)$ (354L). For any $a \in G$, the map $f \mapsto a \bullet_l f : U \to U$ is a Riesz homomorphism leaving the constant functions fixed. So we can define $a \bullet z$, for $z \in Z$, by saying that $(a \bullet z)(f) = z(a^{-1} \bullet_l f)$ for any $f \in U$, and $a \bullet z$ will belong to $Z$ for any $a \in G$ and $z \in Z$. As usual, it is easy to check that this formula defines an action of $G$ on $Z$.

**(ii)** $(a, z) \mapsto a \bullet z$ is continuous. **P** Take $a_0 \in G$, $z_0 \in Z$ and any neighbourhood $W$ of $a_0 \bullet z_0$ in $Z$. Because $U$ corresponds to the whole of $C(Z)$, and $Z$ is completely regular, there is an $f \in U^+$ such that $(a_0 \bullet z_0)(f) = 1$ and $z(f) = 0$ for every $z \in Z \setminus W$. Set $W_0 = \{z : z(a_0^{-1} \bullet_l f) > \frac{1}{2}\}$. Observe that $z_0(a_0^{-1} \bullet_l f) = 1$, so $W_0$ is an open subset of $Z$ containing $z_0$. Next, set $V_0 = \{a : a \in G, \|a^{-1} \bullet_l f - a_0^{-1} \bullet_l f\|_\infty \le \frac{1}{2}\}$. There is a neighbourhood $V$ of $e$ such that $|f(b) - f(c)| \le \frac{1}{2}$ whenever $b$, $c \in G$ and $bc^{-1} \in V$. If $a \in Va_0$ then $ab(a_0 b)^{-1} = aa_0^{-1} \in V$ so

$$|(a^{-1} \bullet_l f)(b) - (a_0^{-1} \bullet_l f)(b)| = |f(ab) - f(a_0 b)| \le \frac{1}{2}$$

for every $b \in G$, and $a \in V_0$. Thus $V_0 \supseteq Va_0$ is a neighbourhood of $a_0$.

Now if $a \in V_0$ and $z \in W_0$ we shall have

$$(a \bullet z)(f) = z(a^{-1} \bullet_l f) \ge z(a_0^{-1} \bullet_l f) - \frac{1}{2} > 0$$

and $a \bullet z \in W$. As $a_0$, $z_0$ and $W$ are arbitrary, the action of $G$ on $Z$ is continuous. **Q**

**(c)** Of course $\hat{a}$, as defined, is a Riesz homomorphism taking the correct value at $\chi G$, so belongs to $Z$. Because $U \subseteq C(X)$, the map $a \mapsto \hat{a}$ is continuous. **?** If $\{\hat{a} : a \in G\}$ is not dense in $Z$, there is a non-zero $h \in C(Z)$ such that $h(\hat{a}) = 0$ for every $a \in G$; but as $U$ is identified with $C(Z)$, there is an $f \in U$ such that $z(f) = h(z)$ for every $z \in Z$. In this case, $f$ cannot be the zero function, but $f(a) = \hat{a}(f) = h(\hat{a}) = 0$ for every $a \in G$. **X** Thus the image of $G$ is dense, as claimed.

If $a$, $b \in G$ and $f \in U$ then

$$(a \bullet \hat{b})(f) = \hat{b}(a^{-1} \bullet_l f) = (a^{-1} \bullet_l f)(b) = f(ab) = \widehat{ab}(f),$$

so $a \bullet \hat{b} = \widehat{ab}$.

**(d)** We have a Riesz homomorphism $T : C(X) \to \mathbb{R}^G$ defined by setting $(Tg)(a) = g(a \bullet x_0)$ for every $g \in C(X)$ and $a \in G$. Now $Tg \in U$ for every $g \in C(X)$. **P** $Tg(a) = (a^{-1} \bullet g)(x_0)$; since the map $a \mapsto a^{-1} \bullet g$ is uniformly continuous (449Ba), so is $Tg$. $\|Tg\|_\infty \le \|g\|_\infty$ is finite, so $Tg \in U$. **Q**

Of course $T(\chi X) = \chi G$. So if $z \in Z$, $zT : C(X) \to \mathbb{R}$ is a Riesz homomorphism such that $(zT)(\chi X) = 1$. There is therefore a unique $\phi(z) \in X$ such that $(zT)(g) = g(\phi(z))$ for every $g \in C(X)$ (354L again). Since the function $z \mapsto g(\phi(z)) = z(Tg)$ is continuous for every $g \in C(X)$, $\phi$ is continuous.

Now suppose that $a \in G$. Then $\phi(\hat{a}) = a \bullet x_0$. **P** If $g \in C(X)$, then

$$g(\phi(\hat{a})) = \hat{a}(Tg) = (Tg)(a) = g(a \bullet x_0). \textbf{ Q}$$

So if $a$, $b \in G$, then

$$\phi(a \bullet \hat{b}) = \phi(\widehat{ab}) = (ab) \bullet x_0 = a \bullet (b \bullet x_0) = a \bullet \phi(b).$$

Since $\{\hat{b} : b \in G\}$ is dense in $Z$, and all the functions here are continuous, $\phi(a \bullet z) = a \bullet \phi(z)$ for all $a \in G$ and $z \in Z$.

To see that $\phi$ is unique, observe that if $a \in G$ then $\phi(\hat{a}) = \phi(\widehat{ae}) = \phi(a \bullet \hat{e})$ must be $a \bullet \phi(\hat{e}) = a \bullet x_0$; since $\{\hat{a} : a \in G\}$ is dense in $Z$, $X$ is Hausdorff and $\phi$ is declared to be continuous, $\phi$ is uniquely defined.

**(e)** Now suppose that the topology of $G$ is Hausdorff. Then it is defined by the bounded uniformly continuous functions (4A2Ja); the map $a \mapsto \hat{a}$ is therefore injective and is a homeomorphism between $G$ and its image in $Z$. If $a$, $b \in G$ are distinct, then $a \bullet \hat{e} = \hat{a} \ne \hat{b} = b \bullet \hat{e}$, so the action is faithful.

**Definition** Following Brook 70, the space $Z$, together with the canonical action of $G$ on it and the map $a \mapsto \hat{a} : G \to Z$, is called the **greatest ambit** of the topological group $G$.

**449E Corollary** Let $G$ be a topological group. Then the following are equiveridical:
(i) $G$ is amenable;
(ii) there is a $G$-invariant Radon probability measure on the greatest ambit of $G$;
(iii) writing $U$ for the space of bounded real-valued functions on $G$ which are uniformly continuous for the right uniformity, there is a positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ for every $f \in U$ and $a \in G$.

**proof** Let $Z$ be the greatest ambit of $G$.

**(i)⇒(ii)** As soon as we know that $Z$ is a compact Hausdorff space and the action of $G$ on $Z$ is continuous (449Db), this becomes a special case of the definition of 'amenable topological group'.

**(ii)⇒(i)** Let $\mu$ be a $G$-invariant Radon probability measure on $Z$. Given any continuous action of $G$ on a non-empty compact Hausdorff space $X$, fix $x_0 \in X$ and let $\phi : Z \to X$ be a continuous function such that $\phi(a\bullet z) = a\bullet\phi(z)$ for every $a \in G$ and $z \in Z$, as in 449Dd. Let $\nu$ be the image measure $\mu\phi^{-1}$. Then $\nu$ is a Radon probability measure on $X$ (418I again). If $F \in \operatorname{dom}\nu$ and $a \in G$, then

$$\nu(a\bullet F) = \mu\phi^{-1}[a\bullet F] = \mu(a\bullet\phi^{-1}[F]) = \mu\phi^{-1}[F] = \nu F.$$

As $a$ and $F$ are arbitrary, $\nu$ is $G$-invariant; as $X$ and $\bullet$ are arbitrary, $G$ is amenable.

**(ii)⇔(iii)** The identification of $U$ with $C(Z)$ (see (b-i) of the proof of 449D) means that we have a one-to-one correspondence between Radon probability measures $\mu$ on $Z$ and positive linear functionals $p$ on $U$ such that $p(\chi G) = 1$, given by the formula $p(f) = \int z(f)\mu(dz)$ for $f \in U$ (436J/436K again). Now

$$\begin{aligned}
&\mu \text{ is } G\text{-invariant}\\
&\qquad\iff \int (a\bullet z)(f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G\\
&\text{(441L)}\\
&\qquad\iff \int z(a^{-1}\bullet_l f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G\\
&\qquad\iff \int z(a\bullet_l f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G\\
&\qquad\iff p(a\bullet_l f) = p(f) \text{ for every } f \in U,\ a \in G.
\end{aligned}$$

So there is a $G$-invariant $\mu$, as required by (ii), iff there is a $G$-invariant $p$ as required by (iii).

**449F Corollary** Let $G$ be a topological group.
(a) If $G$ is amenable, then
 (i) every open subgroup of $G$ is amenable;
 (ii) every dense subgroup of $G$ is amenable.
(b) Suppose that for every sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of the identity $e$ of $G$ there is a normal subgroup $H$ of $G$ such that $H \subseteq \bigcap_{n\in\mathbb{N}} V_n$ and $G/H$ is amenable. Then $G$ is amenable.

**proof** Write $U_G$ for the set of bounded real-valued functions on $G$ which are uniformly continuous for the right uniformity of $G$; if $H$ is a subgroup of $G$, let $U_H$ the set of bounded real-valued functions on $H$ which are uniformly continuous for the right uniformity of $H$; and if $H \lhd G$, let $U_{G/H}$ the set of bounded real-valued functions on the quotient $G/H$ which are uniformly continuous for the right uniformity of $G/H$.

**(a)(i)($\boldsymbol{\alpha}$)** Let $H$ be an open subgroup of $G$. Take a set $A \subseteq G$ meeting each right coset of $H$ in just one point, so that each member of $G$ is uniquely expressible as $ya$ where $y \in H$ and $a \in A$. Define $T : U_H \to \mathbb{R}^G$ by setting $(Tf)(ya) = f(y)$ whenever $f \in U_H$, $y \in H$ and $a \in A$. Then $T$ is a positive linear operator. Also $T[U_H] \subseteq U_G$. **P** Let $f \in U_H$. Of course $Tf$ is bounded. If $\epsilon > 0$, there is a neighbourhood $W$ of the identity in $H$ such that $|f(x)-f(y)|\le\epsilon$ whenever $x$, $y\in H$ and $xy^{-1}\in W$. Because $H$ is open, $W$ is also a neighbourhood of the identity in $G$. Now suppose that $x$, $y\in G$ and $xy^{-1}\in W$. Express $x$ as $x_0a$ and $y$ as $y_0b$ where $x_0$, $y_0\in H$ and $a$, $b\in A$. Then

$$x_0ab^{-1}y_0^{-1}\in W\subseteq H,$$

so $ab^{-1}\in H$ and $a\in Hb$ and $a=b$ and $x_0y_0^{-1}\in W$ and

$$|(Tf)(x)-(Tf)(y)|=|f(x_0)-f(y_0)|\le\epsilon.$$

As $\epsilon$ is arbitrary, $Tf$ is uniformly continuous and belongs to $U_G$. **Q**

**($\beta$)** Next, $b\bullet_l(Tf)=T(b\bullet_l f)$ whenever $f\in U_H$ and $b\in H$. **P** If $x\in G$, express it as $ya$ where $y\in H$ and $a\in A$. Then

$$(b\bullet_l Tf)(x)=(Tf)(b^{-1}x)=(Tf)(b^{-1}ya)=f(b^{-1}y)=(b\bullet_l f)(y)=T(b\bullet_l f)(x).\ \mathbf{Q}$$

**($\gamma$)** By 449E, there is a positive linear functional $p:U_G\to\mathbb{R}$ such that $p(\chi G)=1$ and $p(a\bullet_l f)=p(f)$ whenever $f\in U_G$ and $a\in G$. Set $q(f)=p(Tf)$ for $f\in U_H$; then $q$ is a positive linear operator, $q(\chi H)=1$ and $q$ is $H$-invariant, by (ii). So by 449E in the other direction, $H$ is amenable.

**(ii)** Now suppose that $H$ is a dense subgroup of $G$. It is easy to see that the right uniformity of $H$ is the subspace uniformity induced by the right uniformity of $G$ (3A4D), so that $f\restriction H\in U_H$ for every $f\in U_G$. In the other direction, if $g\in U_H$, then $g$ extends uniquely to a member of $U_G$, by 3A4G; write $Tg$ for the extension. In this case, $b\bullet_l Tg=T(b\bullet_l g)$ for every $g\in U_H$ and $b\in H$. **P** $b\bullet_l Tg$ and $T(b\bullet_l g)$ are continuous, and for $a\in H$,

$$(b\bullet_l Tg)(a)=Tg(b^{-1}a)=g(b^{-1}a)=(b\bullet_l g)(a)=T(b\bullet_l g)(a);$$

as $H$ is dense in $G$, $b\bullet_l Tg=T(b\bullet_l g)$. **Q**

Now we can use the same argument as in (i-$\gamma$) above to see that $H$ is amenable.

**(b)(i)** Let $\mathcal{H}$ be the family of normal subgroups $H$ of $G$ such that $G/H$ is amenable.

**($\alpha$)** For $H\in\mathcal{H}$, let $\pi_H:G\to G/H$ be the canonical homomorphism and $p_H:U_{G/H}\to\mathbb{R}$ a positive linear functional such that $p_H(\chi(G/H))=1$ and $p_H(c\bullet_l g)=p_H(g)$ whenever $g\in U_{G/H}$ and $c\in G/H$. Let $U'_H$ be $\{f:f\in U_G,\ f(x)=f(y)$ whenever $x$, $y\in G$ and $xy^{-1}\in H\}$. Then $U'_H$ is a linear subspace of $U_G$ containing $\chi G$.

If $f\in U'_H$ then there is a unique $g\in U_{G/H}$ such that $f=g\pi_H$. **P** Because $f(x)=f(y)$ whenever $\pi_Hx=\pi_Hy$, and $\pi_H$ is surjective, there is a unique function $g:G/H\to\mathbb{R}$ such that $f=g\pi_H$; because $f$ is bounded, so is $g$. Given $\epsilon>0$, there is an open neighbourhood $W$ of $e$ such that $|f(x)-f(y)|\le\epsilon$ whenever $xy^{-1}\in W$. In this case, $\pi_H[W]$ is a neighbourhood of the identity in $G/H$ (4A5J(a-i)). Suppose that $c_0$, $c_1\in G/H$ are such that $c_0c_1^{-1}\in\pi_H[W]$. Then there are $x_0$, $x_1\in G$ and $x\in W$ such that $\pi_Hx_0=c_0$, $\pi_Hx_1=c_1$ and $\pi_Hx=c_0c_1^{-1}$. As $\pi_H(x_0x_1^{-1})=\pi_Hx$, there is a $y\in H$ such that $yx_0x_1^{-1}=x$ belongs to $W$; so that $\pi_H(yx_0)=c_0$ and

$$|g_H(c_0)-g_H(c_1)|=|f(yx_0)-f(x_1)|\le\epsilon.$$

As $\epsilon$ is arbitrary, $g\in U_{G/H}$. **Q**

We therefore have a functional $p'_H:U'_H\to\mathbb{R}$ defined by setting $p'_H(g\pi_H)=p_H(g)$ whenever $g\in U_{G/H}$. Of course $g\ge0$ whenever $g\pi_H\ge0$, so $p'_H$ is a positive linear functional, and $p'_H(\chi G)=1$.

**($\beta$)** If $f\in U'_H$ and $a\in G$ then $a\bullet_l f\in U'_H$ and $p'_H(a\bullet_l f)=p'_H(f)$. **P** Let $g\in U_{G/H}$ be such that $f=g\pi_H$. Then

$$(a\bullet_l f)(x)=f(a^{-1}x)=g\pi_H(a^{-1}x)=g(\pi_H(a)^{-1}\pi_H(x))=(\pi_H(a)\bullet_l g)(\pi_H(x))$$

for every $x\in G$; so $a\bullet_l f=(\pi_H(a)\bullet_l g)\pi_H$ belongs to $U'_H$, and

$$p'_H(a\bullet_l f)=p_H(\pi_H(a)\bullet_l g)=p_H(g)=p'_H(f).\ \mathbf{Q}$$

**(ii)** For any family $\mathcal{V}$ of neighbourhoods of $e$, set $\mathcal{H}_{\mathcal{V}}=\{H:H\in\mathcal{H},\ H\subseteq\bigcap\mathcal{V}\}$. Now for any $f\in U_G$ there is a countable family $\mathcal{V}$ of neighbourhoods of $e$ such that $f\in U'_H$ for every $H\in\mathcal{H}_{\mathcal{V}}$. **P** For each $n \in \mathbb{N}$ choose a neighbourhood $V_n$ of $e$ such that $|f(x) - f(y)| \le 2^{-n}$ whenever $xy^{-1} \in V_n$, and set $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$. **Q**

**(iii)** We are supposing that $\mathcal{H}_{\mathcal{V}}$ is non-empty for every countable family $\mathcal{V}$ of neighbourhoods of $e$. There is therefore an ultrafilter $\mathcal{F}$ on $\mathcal{H}$ containing $\mathcal{H}_{\mathcal{V}}$ for every countable $\mathcal{V}$. Now we see from (ii) that for any $f \in U_G$ the set $\{H : H \in \mathcal{H},\ f \in U'_H\}$ belongs to $\mathcal{F}$, while $|p'_H(f)| \le \|f\|_\infty$ for any $H$ such that $f \in U'_H$; so we can set $p(f) = \lim_{H\to\mathcal{F}} p'_H(f)$ for every $f \in U_G$. In this case, of course, $p$ is a positive linear functional and $p(\chi G) = 1$. Also, given $f \in U_G$ and $a \in G$, then $p'_H(f) = p'_H(a\bullet_l f)$ whenever $f \in U'_H$, by (i-$\beta$), so $p(f) = p(a\bullet_l f)$. Thus $p$ satisfies (iii) of 449E and $G$ is amenable.

**449G Example** Let $F_2$ be the free group on two generators, with its discrete topology. Then $F_2$ is a $\sigma$-compact unimodular locally compact Polish group. But it is not amenable. **P** Let $a$ and $b$ be the generators of $F_2$. Then every element of $F_2$ is uniquely expressible as a word (possibly empty) in the letters $a$, $b$, $a^{-1}$, $b^{-1}$ in which the letters $a$, $a^{-1}$ are never adjacent and the letters $b$, $b^{-1}$ are never adjacent. Write $A$ for the set of elements of $F_2$ for which the canonical word does not begin with either $b$ or $b^{-1}$, and $B$ for the set of elements of $F_2$ for which the canonical word does not begin with either $a$ or $a^{-1}$. Then $A \cup B = F_2$ and $A \cap B = \{e\}$. **?** Suppose, if possible, that $F_2$ is amenable. Every member of $\ell^\infty(F_2)$ is uniformly continuous with respect to the right uniformity. So there is an $F_2$-invariant positive linear functional $p : \ell^\infty(F_2) \to \mathbb{R}$ such that $p(\chi F_2) = 1$. Let $\nu$ be the corresponding non-negative additive functional on $\mathcal{P}F_2$, so that $\nu C = p(\chi C)$ for every $C \subseteq F_2$. For $c \in F_2$ and $C \subseteq F_2$, $c\bullet_l \chi C = \chi(cC)$, so $\nu(cC) = \nu C$ for every $C \subseteq F_2$ and $c \in F_2$. In particular, $\nu(b^n A) = \nu A$ for every $n \in \mathbb{Z}$; but as all the $b^n A$, for $n \in \mathbb{Z}$, are disjoint, $\nu A = 0$. Similarly $\nu B = 0$ and

$$0 = \nu(A \cup B) = \nu F_2 = p(\chi F_2) = 1,$$

which is absurd. **X** Thus $F_2$ is not amenable, as claimed. **Q**

**449H** In this section so far, I have taken care to avoid assuming that groups are locally compact. Some of the most interesting amenable groups are very far from being locally compact (e.g., 449Xh). But of course a great deal of work has been done on amenable locally compact groups. In particular, there is a remarkable list of equivalent properties, some of which I will present in the next theorem. It will be useful to have the following facts to hand.

**Lemma** Let $G$ be a locally compact Hausdorff topological group, and $U$ the space of bounded real-valued functions on $G$ which are uniformly continuous for the right uniformity, as in 449D-449E. Let $\mu$ be a left Haar measure on $G$, and $*$ the corresponding convolution on $\mathcal{L}^0(\mu)$ (444O).

(a) If $h \in \mathcal{L}^1(\mu)$ and $f \in \mathcal{L}^\infty(\mu)$ then $h * f \in U$.

(b) Let $p : U \to \mathbb{R}$ be a positive linear functional such that $p(a\bullet_l f) = p(f)$ whenever $f \in U$ and $a \in G$. Then $p(h * f) = p(f)\int h\,d\mu$ for every $h \in \mathcal{L}^1(\mu)$ and $f \in U$.

**proof (a)** Recall that we know from 444Rc that $h * f$ is defined everywhere in $G$ and is continuous. For any $x \in G$,

$$(h * f)(x) = \int h(xy)f(y^{-1})\mu(dy) = \int (x^{-1}\bullet_l h) \times \overset{\leftrightarrow}{f},$$

where $\overset{\leftrightarrow}{f}(y) = f(y^{-1})$ whenever $y^{-1} \in \operatorname{dom} f$ (4A5C(c-ii)). By 449Bb, applied to the left action of $G$ on itself, $x \mapsto (x^{-1}\bullet_l h)^\bullet : G \to L^1(\mu)$ is uniformly continuous for the right uniformity of $G$ and the norm uniformity of $L^1(\mu)$. Since $u \mapsto \int u \times v : L^1(\mu) \to \mathbb{R}$ is uniformly continuous for every $v \in L^\infty(\mu)$, $x \mapsto (h * f)(x) = \int (x^{-1}\bullet_l h)^\bullet \times \overset{\leftrightarrow}{f}{}^\bullet$ is uniformly continuous for the right uniformity (3A4Cb). Of course $\sup_{x\in G} |(h * f)(x)| \le \|h\|_1\|f\|_\infty$ is finite, so $h * f \in U$.

**(b)** Let $\epsilon > 0$. Then there are a compact set $K \subseteq G$ such that $\int_{G\setminus K} |h|d\mu \le \epsilon$ (412Je) and a symmetric open neighbourhood $V_0$ of $e$ such that $|f(x) - f(y)| \le \epsilon$ whenever $xy^{-1} \in V_0$. Let $a_0, \dots, a_n \in G$ be such that $K \subseteq \bigcup_{i\le n} a_i V_0$, and set $E_i = a_i V_0 \setminus \bigcup_{j<i} a_j V_0$, $\alpha_i = \int_{E_i} h\,d\mu$ for each $i \le n$ and $F = G \setminus \bigcup_{j\le n} a_j V_0$. If $x \in G$ and $y \in E_i$, then $y^{-1}x(a_i^{-1}x)^{-1} = (a_i^{-1}y)^{-1}$ belongs to $V_0$, so $|f(y^{-1}x) - f(a_i^{-1}x)| \le \epsilon$. So, for any $x \in G$,

$$
\begin{aligned}
&\bigl|(h*f)(x)-\sum_{i=0}^{n}\alpha_i f(a_i^{-1}x)\bigr|\\
&\qquad=\bigl|\int h(y)f(y^{-1}x)\mu(dy)-\sum_{i=0}^{n}\alpha_i f(a_i^{-1}x)\bigr|\\
&\qquad=\bigl|\int_F h(y)f(y^{-1}x)\mu(dy)+\sum_{i=0}^{n}\bigl(\int_{E_i}h(y)f(y^{-1}x)\mu(dy)-\alpha_i f(a_i^{-1}x)\bigr)\bigr|\\
&\qquad\le\|f\|_\infty\int_F|h|d\mu+\sum_{i=0}^{n}\bigl|\int_{E_i}h(y)(f(y^{-1}x)-f(a_i^{-1}x))\mu(dy)\bigr|\\
&\qquad\le\|f\|_\infty\int_{X\setminus K}|h|d\mu+\epsilon\sum_{i=0}^{n}\int_{E_i}|h|d\mu\le\epsilon(\|f\|_\infty+\|h\|_1).
\end{aligned}
$$

Thus

$$\|h*f-\textstyle\sum_{i=0}^{n}\alpha_i a_i\bullet_l f\|_\infty\le\epsilon(\|f\|_\infty+\|h\|_1).$$

Since $p(a_i\bullet_l f)=p(f)$ for every $i$, it follows that

$$
\begin{aligned}
|p(h*f)-p(f)\int h\,d\mu|&\le\epsilon(\|f\|_\infty+\|h\|_1)p(\chi G)+|\sum_{i=0}^{n}\alpha_i-\int h\,d\mu||p(f)|\\
&\le\epsilon(\|f\|_\infty+\|h\|_1)p(\chi G)+|\int_F h\,d\mu|\|f\|_\infty p(\chi G)\\
&\le\epsilon(2\|f\|_\infty+\|h\|_1)p(\chi G).
\end{aligned}
$$

As $\epsilon$ is arbitrary, $p(h*f)=p(f)\int h\,d\mu$, as claimed.

**449I Notation** It will save repeated explanations if I say now that for the next two results, given a locally compact Hausdorff group $G$, $\Sigma_G$ will be the algebra of Haar measurable subsets of $G$ and $\mathcal{N}_G$ the ideal of Haar negligible subsets of $G$ (443A), while $\mathcal{B}_G$ will be the Borel $\sigma$-algebra of $G$. Recall that all three are left- and right-translation-invariant and inversion-invariant, and indeed autohomeomorphism-invariant, in that if $\gamma:G\to G$ is a function of any of the types

$$x\mapsto ax,\quad x\mapsto xa,\quad x\mapsto x^{-1}$$

or is a group automorphism which is also a homeomorphism, and $E\subseteq G$, then $\gamma[E]$ belongs to $\Sigma_G$, $\mathcal{N}_G$ or $\mathcal{B}_G$ iff $E$ does (443Aa).

**449J Theorem** Let $G$ be a locally compact Hausdorff group; fix a left Haar measure $\mu$ on $G$. Write $\mathcal{L}^1$ for $\mathcal{L}^1(\mu)$ and $L^\infty$ for $L^\infty(\mu)$, etc. Let $C_{k1}^+$ be the set of continuous functions $h:G\to[0,\infty[$ with compact supports such that $\int h\,d\mu=1$, and suppose that $q\in[1,\infty[$. Then the following are equiveridical:

(i) $G$ is amenable;

(ii) there is a positive linear functional $p:C_b(G)\to\mathbb{R}$ such that $p(\chi G)=1$ and $p(a\bullet_l f)=p(f)$ for every $f\in C_b(G)$ and every $a\in G$;

(iii) there is a finitely additive functional $\phi:\mathcal{B}_G\to[0,1]$ such that $\phi G=1$, $\phi(aE)=\phi E$ for every $E\in\mathcal{B}_G$ and $a\in G$, and $\phi E=0$ for every Haar negligible $E\in\mathcal{B}_G$;

(iv) there is a finitely additive functional $\phi:\Sigma_G\to[0,1]$ such that $\phi G=1$, $\phi(aE)=\phi(Ea)=\phi(E^{-1})=\phi E$ for every $E\in\Sigma_G$ and $a\in G$, and $\phi E=0$ for every $E\in\mathcal{N}_G$;

(v) there is a positive linear functional $\tilde p:L^\infty\to\mathbb{R}$ such that $\tilde p(\chi G^\bullet)=1$ and $\tilde p(a\bullet_l u)=\tilde p(a\bullet_r u)=\tilde p(a\bullet_c u)=\tilde p(\overset{\leftrightarrow}{u})=\tilde p(u)$ for every $u\in L^\infty$ and every $a\in G$, where $\bullet_l$, $\bullet_r$, $\bullet_c$ and $\leftrightarrow$ are defined as in 443Af and 443Gc;

(vi) there is a positive linear functional $\tilde p:L^\infty\to\mathbb{R}$ such that $\tilde p(\chi G^\bullet)=1$ and $\tilde p(a\bullet_l u)=\tilde p(u)$ for every $u\in L^\infty$ and every $a\in G$;

(vii) there is a positive linear functional $\tilde{p} : L^\infty \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(\nu * u) = \nu G \cdot \tilde{p}(u)$ for every $u \in L^\infty$ and every totally finite Radon measure $\nu$ on $G$, where $\nu * u$ is defined as in 444Ma;

(viii) there is a positive linear functional $\tilde{p} : L^\infty \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(v * u) = \tilde{p}(u) \int v$ for every $v \in L^1$ and $u \in L^\infty$;

(ix) for every finite set $J \subseteq \mathcal{L}^1$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $\|g * h - (\int g\,d\mu)h\|_1 \le \epsilon$ for every $g \in J$;

(x) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $\|a \bullet_l h - h\|_1 \le \epsilon$ for every $a \in K$;

(xi) for any finite set $I \subseteq G$ and $\epsilon > 0$, there is a $u \in L^q$ such that $\|u\|_q = 1$ and $\|u - a \bullet_l u\|_q \le \epsilon$ for every $a \in I$;

(xii) for any finite set $I \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(L \triangle aL) \le \epsilon \mu L$ for every $a \in I$;

(xiii) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a symmetric compact neighbourhood $L$ of the identity $e$ in $G$ such that $\mu(L \triangle aL) \le \epsilon \mu L$ for every $a \in K$;

(xiv) (Emerson & Greenleaf 67) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(KL) \le (1 + \epsilon)\mu L$.

**proof (a)(i)⇒(vii)** Write $U$ for the space of bounded real-valued functions on $G$ which are uniformly continuous for the right uniformity. Then we have a positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a \bullet_l f) = p(f)$ for every $f \in U$ and $a \in G$ (449E). Now if $f \in \mathcal{L}^\infty$, $h_1$, $h_2 \in \mathcal{L}^1$ and $\int h_1 d\mu = \int h_2 d\mu$, then $p(h_1 * f) = p(h_2 * f)$. **P** By 449Ha, both $h_1 * f$ and $h_2 * f$ belong to $U$. Set $h = h_1 - h_2$. By 444T, there is a neighbourhood $V$ of $e$ such that $\|h * \nu - h\|_1 \le \epsilon$ whenever $\nu$ is a quasi-Radon measure on $G$ such that $\nu V = \nu G = 1$, defining $h * \nu$ as in 444J. In particular, taking $\nu$ to be the indefinite-integral measure over $\mu$ defined from $g = \frac{1}{\mu V}\chi V$, $\|h * g - h\|_1 \le \epsilon$ (using 444Pb). Now

$$|p(h_1 * f) - p(h_2 * f)| = |p(h * f)| \le |p((h * g) * f)| + |p((h * g - h) * f)|$$
$$\le |p(h * (g * f))| + \|(h * g - h) * f\|_\infty$$

(because $*$ is associative, 444Oe)

$$\le |p(g * f) \int h\,d\mu| + \|h * g - h\|_1 \|f\|_\infty$$

(449Hb)

$$\le \epsilon \|f\|_\infty.$$

As $\epsilon$ is arbitrary, $p(h_1 * f) = p(h_2 * f)$, as claimed. **Q**

Of course $p(h * f) = 0$ whenever $h \in \mathcal{L}^1$, $f \in \mathcal{L}^\infty$ and $f = 0$ a.e. (444Ob). We can therefore define a functional $\tilde{p} : L^\infty \to \mathbb{R}$ by saying that $\tilde{p}(f^\bullet) = p(h * f)$ whenever $f \in \mathcal{L}^\infty$, $h \in \mathcal{L}^1$ and $\int h\,d\mu = 1$. $\tilde{p}$ is positive and linear because $p$ is. It follows that $p(h * f) = \tilde{p}(f^\bullet) \int h\,d\mu$ whenever $h \in \mathcal{L}^1$ and $f \in \mathcal{L}^\infty$. Also $\tilde{p}(\chi G^\bullet) = p(\chi G) = 1$ because $h * \chi G = (\int h\,d\mu)\chi G$ for every $h \in \mathcal{L}^1$.

If $u \in L^\infty$ and $\nu$ is a totally finite Radon measure on $G$, express $u$ as $f^\bullet$ where $f \in \mathcal{L}^\infty$, so that $\nu * u = (\nu * f)^\bullet$ (444Ma). Taking any non-negative $h \in \mathcal{L}^1$ such that $\int h\,d\mu = 1$, we have

$$h * (\nu * f) = h\mu * (\nu * f)$$

(444Pa; here $h\mu$ is the indefinite-integral measure, as in 444J)

$$= (h\mu * \nu) * f$$

(444Ic)

$$= (h * \nu)\mu * f$$

(444K)

$$= (h * \nu) * f$$

(444Pa again). So

$$\begin{aligned}\tilde{p}(\nu * u) &= \tilde{p}((\nu * f)^\bullet) = p(h * (\nu * f)) \\ &= p((h * \nu) * f) = \int h * \nu \, d\mu \cdot \tilde{p}(u) = \nu G \cdot \tilde{p}(u)\end{aligned}$$

(444K). As $\nu$ and $u$ are arbitrary, $\tilde{p}$ has the required properties.

**(b)(vii)⇒(vi)** Take $\tilde{p}$ from (vii). If $a \in G$ and $u \in L^\infty$, consider the Dirac measure $\delta_a$ on $G$ concentrated at $a$. Then $\delta_a * u = a \bullet_l u$. **P** Take $f \in \mathcal{L}^\infty$ such that $f^\bullet = u$. Then

$$(\delta_a * f)(x) = \int f(y^{-1}x)\delta_a(dy) = f(a^{-1}x) = (a \bullet_l f)(x)$$

whenever $a^{-1}x \in \operatorname{dom} f$, so $\delta_a * f = a \bullet_l f$ and

$$\delta_a * u = \delta_a * f^\bullet = (\delta_a * f)^\bullet = (a \bullet_l f)^\bullet = a \bullet_l u. \text{ } \mathbf{Q}$$

Accordingly, using (vii),

$$\tilde{p}(a \bullet_l u) = \tilde{p}(\delta_a * u) = \delta_a(G)\tilde{p}(u) = \tilde{p}(u),$$

as required by (vi).

**(c)(vi)⇒(v)($\alpha$)** The first step is to note that since there is a left-invariant mean there must also be a right-invariant mean, that is, a positive linear functional $\tilde{q} : L^\infty \to \mathbb{R}$ such that $\tilde{q}(\chi G^\bullet) = 1$ and $\tilde{q}(a \bullet_r u) = \tilde{q}(u)$ for every $u \in L^\infty$ and every $a \in G$. **P** Set $\tilde{q}(u) = \tilde{p}(\overleftrightarrow{u})$ for $u \in L^\infty$. Evidently $\tilde{q}$ is a positive linear functional and $\tilde{q}(\chi G^\bullet) = 1$. By 443Gc,

$$\tilde{q}(a \bullet_r u) = \tilde{p}((a \bullet_r u)^\leftrightarrow) = \tilde{p}(a \bullet_l \overleftrightarrow{u}) = \tilde{p}(\overleftrightarrow{u}) = \tilde{q}(u)$$

whenever $u \in L^\infty$ and $a \in G$. **Q**

**($\beta$)** At this point, recall that $L^1$ is a Banach algebra under convolution (444Sb), and that $L^\infty$ can be identified with its normed space dual, because $\mu$ is a quasi-Radon measure, therefore localizable (415A), and we can use 243Gb. We therefore have an Arens multiplication on $(L^\infty)^* \cong (L^1)^{**}$ defined by the formulae of 4A6O. Of course $\tilde{p}$ and $\tilde{q}$ both belong to $(L^\infty)^*$; write $\tilde{r}_0 = \tilde{p} \circ \tilde{q}$ for their Arens product. To see that $\tilde{r}_0(\chi G^\bullet) = 1$, note that if $u$, $v \in L^1$ then, defining $\chi G^\bullet \circ u$ and $\tilde{q} \circ \chi G^\bullet$ as in 4A6O, we have

$$\int (\chi G^\bullet \circ u) \times v = \int \chi G^\bullet \times (u * v) = \int u \int v$$

as noted in 444Sb; consequently $\chi G^\bullet \circ u = (\int u)\chi G^\bullet$,

$$\int (\tilde{q} \circ \chi G^\bullet) \times u = \tilde{q}(\chi G^\bullet \circ u) = \tilde{q}((\int u)\chi G^\bullet) = \int u$$

and $\tilde{q} \circ \chi G^\bullet = \chi G^\bullet$. Now, of course,

$$\tilde{r}_0(\chi G^\bullet) = \tilde{p}(\tilde{q} \circ \chi G^\bullet) = \tilde{p}(\chi G^\bullet) = 1.$$

As noted in 4A6O, $\|\tilde{r}_0\| \le \|\tilde{p}\|\|\tilde{q}\| = 1$, so $\tilde{r}_0$ must be a positive linear functional.

**($\gamma$)** We find next that $\tilde{r}_0(a \bullet_l u) = \tilde{r}_0(u)$ whenever $u \in L^\infty$ and $a \in G$. **P** By 443Ge, we have a bounded linear operator $S : L^1 \to L^1$ defined by setting $Sv = a^{-1} \bullet_l v$ for every $v \in L^1$. By 444Sa, $S(u * v) = (Su) * v$ for all $u$, $v \in L^1$. Identifying $L^\infty$ with $(L^1)^*$, we have the adjoint operator $S' : L^\infty \to L^\infty$ given by saying that

$$\begin{aligned}\int S'u \times v &= \int u \times Sv = \int u \times (a^{-1} \bullet_l v) \\ &= \int a \bullet_l (u \times a^{-1} \bullet_l v) = \int (a \bullet_l u) \times v\end{aligned}$$

whenever $u \in L^\infty$ and $v \in L^1$, so that $S'u = a \bullet_l u$ for every $u \in L^\infty$. But this means that

$$(S''\tilde{p})(u) = \tilde{p}(a \bullet_l u) = \tilde{p}(u)$$

for every $u$, so that $S''\tilde{p} = \tilde{p}$. By 4A6O(b-i),

$$S''\tilde{r}_0 = S''(\tilde{p} \circ \tilde{q}) = (S''\tilde{p}) \circ \tilde{q} = \tilde{p} \circ \tilde{q} = \tilde{r}_0,$$

that is, $\tilde{r}_0(a\bullet_l u) = \tilde{r}_0(u)$ for every $u \in L^\infty$. **Q**

**($\delta$)** In the same way, $\tilde{r}_0(a\bullet_r u) = \tilde{r}_0(u)$ whenever $u \in L^\infty$ and $a \in G$. **P** This time, define $T : L^1 \to L^1$ by setting $Tv = \Delta(a^{-1})a^{-1}\bullet_r v$ for every $v \in L^1$, where $\Delta$ is the left modular function of $G$; 444Sa tells us that $T(u * v) = u * Tv$ for all $u$, $v \in L^1$. Since $\int f d\mu = \Delta(a) \int a\bullet_r f d\mu$ for every $f \in \mathcal{L}^1$ (442Kc), $\int v = \Delta(a) \int a\bullet_r v$ for every $v \in L^1$, and

$$\begin{aligned}\int T'u \times v = \int u \times Tv = \Delta(a^{-1}) \int u \times (a^{-1}\bullet_r v)\\ = \int a\bullet_r (u \times a^{-1}\bullet_r v) = \int (a\bullet_r u) \times v\end{aligned}$$

whenever $u \in L^\infty$ and $v \in L^1$. Thus $T'u = a\bullet_r u$ for every $u \in L^\infty$. But now we have

$$(T''\tilde{q})(u) = \tilde{q}(a\bullet_r u) = \tilde{q}(u)$$

for every $u$, so that $T''\tilde{q} = \tilde{q}$. By 4A6O(b-ii), $T''\tilde{r}_0 = \tilde{r}_0$, that is, $\tilde{r}_0(a\bullet_r u) = \tilde{r}_0(u)$ for every $u \in L^\infty$. **Q**

**($\epsilon$)** Thus $\tilde{r}_0$ is both left- and right-invariant. To get reversal-invariance, set

$$\tilde{r}(u) = \frac{1}{2}(\tilde{r}_0(u) + \tilde{r}_0(\overset{\leftrightarrow}{u}))$$

for $u \in L^\infty$. Then $\tilde{r}_0$ is a positive linear functional and $\tilde{r}_0(\chi G^\bullet) = 1$. Because

$$a\bullet_l \overset{\leftrightarrow}{u} = (a\bullet_r u)^\leftrightarrow, \quad a\bullet_r \overset{\leftrightarrow}{u} = (a\bullet_l u)^\leftrightarrow,$$

$u \mapsto \tilde{r}_0(\overset{\leftrightarrow}{u})$ and $\tilde{r}$ are also both left- and right-invariant, and of course $\tilde{r}(\overset{\leftrightarrow}{u}) = \tilde{r}(u)$ for every $u$. Finally,

$$\tilde{r}(a\bullet_c u) = \tilde{r}(a\bullet_l(a\bullet_r u)) = \tilde{r}(u)$$

for every $u \in L^\infty$ and $a \in G$, so $\tilde{r}$ has all the properties required by (v).

**(d)(v)$\Rightarrow$(iv)** Take $\tilde{p}$ from (v), and set $\phi E = \tilde{p}(\chi E^\bullet)$ for every $E \in \Sigma_G$. Then $\phi : \Sigma_G \to [0, 1]$ is additive and $\phi G = 1$; also, if $E \in \mathcal{N}_G$, $\chi E^\bullet = 0$ in $L^\infty$ and $\phi E = 0$. If $E \in \Sigma_G$ and $a \in G$, then $\chi(aE) = a\bullet_l(\chi E)$ (4A5C(c-ii)) and

$$\phi(aE) = \tilde{p}(\chi(aE)^\bullet) = \tilde{p}((a\bullet_l \chi E)^\bullet) = \tilde{p}(a\bullet_l(\chi E^\bullet)) = \tilde{p}(\chi E^\bullet) = \phi E.$$

Next, $\chi E^{-1} = (\chi E)^\leftrightarrow$ and $(\chi E^{-1})^\bullet = (\chi E^\bullet)^\leftrightarrow$, so

$$\phi(E^{-1}) = \tilde{p}((\chi E^\bullet)^\leftrightarrow) = \tilde{p}(\chi E^\bullet) = \phi E.$$

Consequently, for $E \in \Sigma_G$ and $a \in G$,

$$\phi(Ea) = \phi((Ea)^{-1}) = \phi(a^{-1}E^{-1}) = \phi(E^{-1}) = \phi E.$$

Thus $\phi$ satisfies the requirements of (iv).

**(e)(iv)$\Rightarrow$(iii)** This is trivial; we have only to take $\phi : \Sigma_G \to [0, 1]$ as in (iv) and consider $\phi\restriction\mathcal{B}_G$.

**(f)(iii)$\Rightarrow$(ii)** Given $\phi : \mathcal{B}_G \to [0, 1]$ as in (iii), set $p(f) = ⨍ f d\phi$ for $f \in C_b(G)$, where $⨍ d\phi$ is as defined in 363L, that is, the unique $\| \|_\infty$-continuous linear functional on the space $L^\infty(\mathcal{B}_G)$ of bounded Borel measurable functions from $G$ to $\mathbb{R}$ such that $⨍ \chi E\, d\phi = \phi E$ for every $E \in \mathcal{B}_G$. $p$ is positive because $\phi$ is non-negative (363Lc), and $p(\chi G) = \phi G = 1$. If $a \in G$, then

$$⨍ a\bullet_l \chi E\, d\phi = ⨍ \chi(aE) d\phi = \phi(aE) = \phi E = ⨍ \chi E\, d\phi$$

for every $E \in \mathcal{B}_G$; because $f \mapsto ⨍ f d\phi$ and $f \mapsto ⨍ a\bullet_l f d\phi$ are both linear and $\| \|_\infty$-continuous, they agree on $L^\infty(\mathcal{B}_G) \supseteq C_b(G)$, and

$$p(a\bullet_l f) = ⨍ a\bullet_l f d\phi = ⨍ f d\phi = p(f)$$

for every $f \in C_b(G)$, as required.

**(g)(ii)$\Rightarrow$(i)** Given $p$ as in (ii), its restriction to the space of bounded right-uniformly-continuous functions is positive, linear and $G$-invariant, so $G$ is amenable, by 449E.

**(h)(vii)$\Rightarrow$(viii)** Take $\tilde{p}$ as in (vii). If $g \in \mathcal{L}^1$, $f \in \mathcal{L}^\infty$ and $g \geq 0$, then

$$\begin{aligned}\tilde{p}(g*f)^\bullet &= \tilde{p}(g\mu*f)^\bullet = \tilde{p}(g\mu*f^\bullet)\\ &= (g\mu)(G)\tilde{p}(f^\bullet) = \int g\,d\mu\cdot\tilde{p}(f^\bullet);\end{aligned}$$

translating into terms of $L^1$ and $L^\infty$ as in 444Sa, we get $\tilde{p}(v*u)=\int v\cdot\tilde{p}(u)$ for all $u\in L^\infty$ and $v\in(L^1)^+$. By linearity, the same is true for all $v\in L^1$, as required by (viii).

**(i)(viii)⇒(ix)** Suppose that (viii) is true.

**($\alpha$)** Note first that if $J\subseteq\mathcal{L}^\infty$ is finite and $\epsilon>0$, then

$$A(J,\epsilon)=\{h:h\in C_{k1}^+,\ |\int f\times h\,d\mu-\tilde{p}(f^\bullet)|\le\epsilon\text{ for every }f\in J\}$$

is non-empty. **P** It is enough to consider the case in which $\chi G\in J$. Let $\eta\in\left]0,\frac{1}{2}\right[$ be such that $\eta+5\eta\sup_{f\in J}\|f^\bullet\|_\infty\le\epsilon$. Because $\tilde{p}\in(L^\infty)^*\cong(L^1)^{**}$, there is a $u_0\in L^1$ such that $\|u_0\|_1\le1$ and $|\tilde{p}(f^\bullet)-\int f^\bullet\times u_0|\le\eta$ for every $f\in J$ (4A4If). In particular, $\int u_0\ge\tilde{p}(\chi G^\bullet)-\eta=1-\eta$. By 416I again, there is a continuous $h_0:G\to\mathbb{R}$ with compact support such that $\|u_0-h_0^\bullet\|_1\le\eta$. Now $\int h_0d\mu\ge1-2\eta$ and $\int|h_0|d\mu\le1+\eta$. So if we set $h_0^+=h_0\vee0$, $\gamma=\int h_0^+d\mu$ and $h=\frac{1}{\gamma}h_0^+$, we shall have

$$\gamma\le1+\eta,\quad \|h_0^+-h_0\|_1=\frac{1}{2}\int|h_0|-h_0\le2\eta,\quad \|h-h_0^+\|=|\gamma-1|\le2\eta,$$

so $\|u_0-h^\bullet\|_1\le5\eta$, while $h\in C_{k1}^+$. This will mean that

$$|\tilde{p}(f^\bullet)-\int f\times h\,d\mu|\le\eta+5\eta\|f^\bullet\|_\infty\le\epsilon$$

for every $f\in J$. **Q**

We therefore have a filter $\mathcal{F}$ on $C_{k1}^+$ containing every $A(J,\epsilon)$, and $\tilde{p}(f^\bullet)=\lim_{h\to\mathcal{F}}\int f\times h\,d\mu$ for every $f\in\mathcal{L}^\infty$.

**($\beta$)** Now $0=\lim_{h\to\mathcal{F}}(g*h)^\bullet-(\int g\,d\mu)h^\bullet$ for the weak topology of $L^1$, for every $g\in\mathcal{L}^1$. **P** Set $\gamma=\int g\,d\mu$. Let $f\in\mathcal{L}^\infty$. Define $g'$ by setting $g'(x)=\Delta(x^{-1})g(x^{-1})$ whenever this is defined, where $\Delta$ is the left modular function of $G$, as before; then $g'\in\mathcal{L}^1$ and $\int g'd\mu=\gamma$ (442K(b-ii)). Set $v=(g')^\bullet\in L^1$. If $h\in C_{k1}^+$, then

$$\int f\times(g*h)d\mu=\iint f(xy)g(x)h(y)\mu(dx)\mu(dy)$$

(444Od)

$$\begin{aligned}&=\int\Bigl(\int\Delta(x^{-1})g'(x^{-1})f(xy)\mu(dx)\Bigr)h(y)\mu(dy)\\ &=\int(g'*f)(y)h(y)\mu(dy)\end{aligned}$$

(444Oa)

$$=\int(g'*f)\times h\,d\mu.$$

By (viii), we have

$$\tilde{p}(g'*f)^\bullet=\tilde{p}(v*f^\bullet)=\tilde{p}(f^\bullet)\int v=\gamma\tilde{p}(f^\bullet),$$

so we get

$$\begin{aligned}\lim_{h\to\mathcal{F}}\int f\times(g*h-\gamma h)d\mu&=\lim_{h\to\mathcal{F}}\int(g'*f-\gamma f)\times h\,d\mu\\ &=\tilde{p}(g'*f)^\bullet-\gamma\tilde{p}(f^\bullet)=0.\end{aligned}$$

As $f$ is arbitrary, and $(L^1)^*$ can be identified with $L^\infty$, this is all we need. **Q**

**($\gamma$)** Now take any finite set $J \subseteq \mathcal{L}^1$ and $\epsilon > 0$. On $(L^1)^J$ let $\mathfrak{T}$ be the locally convex linear space topology which is the product topology when each copy of $L^1$ is given the norm topology, and $\mathfrak{S}$ the corresponding weak topology. Define $T : C_k(G) \to (L^1)^J$ by setting

$$Th = \langle (g * h)^\bullet - (\int g\,d\mu)h^\bullet \rangle_{g \in J}$$

for $h \in C_k(G)$, where $C_k(G)$ is the linear space of continuous real-valued functions on $G$ with compact support. Then $T$ is linear. Moreover, by ($\beta$), $\lim_{h \to \mathcal{F}} Th = 0$ in $(L^1)^J$ for the product topology, if each copy of $L^1$ is given its weak topology. By 4A4Be, this is just $\mathfrak{S}$. In particular, 0 belongs to the $\mathfrak{S}$-closure of $T[C_{k1}^+]$. But $C_{k1}^+$ is convex and $T$ is linear, so $T[C_{k1}^+]$ is convex; by 4A4Ed, 0 belongs to the $\mathfrak{T}$-closure of $T[C_{k1}^+]$. There is therefore an $h \in C_{k1}^+$ such that $\|(g * h)^\bullet - (\int g\,d\mu)h^\bullet\|_1 \le \epsilon$ for every $g \in J$. As $J$ and $\epsilon$ are arbitrary, (ix) is true.

**(j)(ix)$\Rightarrow$(x)** Suppose that (ix) is true and that we are given a compact set $K \subseteq G$ and $\epsilon > 0$. Set $\eta = \frac{1}{3}\epsilon$. Fix any $h_0 \in C_{k1}^+$. Let $V$ be a neighbourhood of $e$ such that $\|c \bullet_l h_0 - h_0\|_1 \le \eta$ whenever $c \in V$ (443Gf). Let $I \subseteq G$ be a finite set such that $K \subseteq IV$. By (ix), there is an $h_1 \in C_{k1}^+$ such that

$$\|b \bullet_l h_0 * h_1 - h_1\|_1 \le \eta \text{ for every } b \in I, \quad \|h_0 * h_1 - h_1\|_1 \le \eta.$$

(I omit brackets because $(b \bullet_l h_0) * h_1 = b \bullet_l (h_0 * h_1)$, see 444Of.) Set $h = h_0 * h_1$. Then $h \in C_{k1}^+$. **P** $h$ is continuous, by 444Rc (or otherwise). If we write $M_i$ for the support $\operatorname{supp}(h_i)$ of $h_i$ for both $i$, then $h(x) = 0$ for every $x \in G \setminus M_0 M_1$, so $h$ has compact support. Of course $h \ge 0$, and $\int h\,d\mu = \int h_0 d\mu \int h_1 d\mu = 1$ (444Qb), so $h \in C_{k1}^+$. **Q**

If $a \in K$, there are $b \in I$, $c \in V$ such that $a = bc$, so that

$$\begin{aligned}\|a \bullet_l h - h\|_1 &= \|b \bullet_l (c \bullet_l h_0 * h_1) - h_0 * h_1\|_1 \\ &\le \|b \bullet_l (c \bullet_l h_0 - h_0) * h_1\|_1 + \|b \bullet_l h_0 * h_1 - h_1\|_1 + \|h_1 - h_0 * h_1\|_1 \\ &\le \|c \bullet_l h_0 - h_0\|_1 + \eta + \eta \le 3\eta = \epsilon.\end{aligned}$$

So this $h$ will serve.

**(k)(x)$\Rightarrow$(xiii)** Suppose that (x) is true.

**($\alpha$)** I show first that for any compact set $K \subseteq G$ and $\epsilon > 0$ there is an $h \in C_{k1}^+$ such that

$$\overleftrightarrow{h} = h, \quad h(e) = \|h\|_\infty,$$

$$\|a \bullet_l h - h\|_1 \le \epsilon \text{ for every } a \in K.$$

**P** By (x), there is an $h_0 \in C_{k1}^+$ such that $\|a \bullet_l h_0 - h_0\|_1 \le \epsilon$ for every $a \in K$. Set $\gamma = \int \overleftrightarrow{h}_0 d\mu$; because $\overleftrightarrow{h}_0 \in C_k(X)^+ \setminus \{0\}$, $\gamma$ is finite and not 0. Try $h = h_0 * \frac{1}{\gamma}\overleftrightarrow{h}_0$, so that $h(x) = \frac{1}{\gamma}\int h_0(y) h_0(x^{-1}y)\mu(dy)$ for every $x \in G$. By 444Rc, $h \in C_b(G)$ and

$$\|h\|_\infty \le \|h_0\|_2 \|\frac{1}{\gamma} h_0\|_2 = \frac{1}{\gamma}\int h_0^2 d\mu = h(e).$$

Because $h_0 \ge 0$, $h \ge 0$; by 444Qb, $\int h\,d\mu = 1$; and (as in (j) above) $\operatorname{supp}(h)$ is included in the compact set $\operatorname{supp}(h_0)\operatorname{supp}(\overleftrightarrow{h}_0)$, so $h \in C_{k1}^+$. By 444Rb, or otherwise, $h = \overleftrightarrow{h}$.

Finally, if $a \in K$, then

$$\|a \bullet_l h - h\|_1 = \frac{1}{\gamma}\|a \bullet_l (h_0 * \overleftrightarrow{h}_0) - h_0 * \overleftrightarrow{h}_0\|_1 = \frac{1}{\gamma}\|(a \bullet_l h_0 - h_0) * \overleftrightarrow{h}_0\|_1$$

(444Of once more)

$$\le \frac{1}{\gamma}\|a \bullet_l h_0 - h_0\|_1 \|\overleftrightarrow{h}_0\|_1$$

(444Qb again)

$$= \|a \bullet_l h_0 - h_0\|_1 \le \epsilon,$$

as required. **Q**

**(β)** Next, for any $\epsilon$, $\delta > 0$ and any compact set $K \subseteq G$ there are an open symmetric neighbourhood $V$ of $e$ and a closed set $F$ such that $\mu V < \infty$, $\mu F \le \delta$ and $\mu(aV \triangle V) \le \epsilon\mu V$ whenever $a \in K \setminus F$. **P** Of course it is enough to deal with the case in which $\mu K > 0$. Set $\eta = \epsilon\delta/\mu K$. By ($\alpha$), there is an $h \in C_{k1}^{+}$ such that $h(e) = \|h\|_\infty$, $h = \overset{\leftrightarrow}{h}$ and $\|a\bullet_l h - h\|_1 < \eta$ for every $a \in K$.

Set $K_0 = \operatorname{supp}(h)$ and $K^* = K_0 \cup KK_0$, so that $K^* \subseteq G$ is compact. Set

$$Q = \{(a, x, t) : a \in K,\, x \in G,\, t \in \mathbb{R},\ \text{either } h(x) \le t < h(a^{-1}x) \text{ or } h(a^{-1}x) \le t < h(x)\}.$$

Then $Q$ is a Borel subset of $G \times G \times \mathbb{R}$ included in the compact set $K \times K^* \times [0, h(e)]$. Let $\mu_L$ be Lebesgue measure on $\mathbb{R}$, and let $\mu \times \mu \times \mu_L$ be the $\tau$-additive product measure on $G \times G \times \mathbb{R}$ (417D). (Of course this is actually a Radon measure.) For $t \in \mathbb{R}$ let $V_t$ be the open set $\{x : h(x) > t\}$. Now 417G tells us that

$$(\mu \times \mu \times \mu_L)(Q) = \int_{G\times G} \mu_L\{t : (a, x, t) \in Q\}(\mu \times \mu)(d(a, x))$$

(where $\mu\times\mu$ is the $\tau$-additive product measure on $G\times G$, so that we can identify $\mu\times\mu\times\mu_L$ with $(\mu\times\mu)\times\mu_L$), as in 417Db)

$$= \int_K \int_G |h(a^{-1}x) - h(x)|\mu(dx)\mu(da)$$

(we can use 417G again because $\{x : h(a^{-1}x) \ne h(x)\} \subseteq K^*$ if $a \in K$, and $\mu K^*$ is finite)

$$= \int_K \|a\bullet_l h - h\|_1 \mu(da) < \eta\mu K$$

(by the choice of $h$)

$$= \eta\mu K \int h\, d\mu = \eta\mu K(\mu \times \mu_L)\{(x, t) : 0 \le t < h(x)\}$$

$$= \eta\mu K \int_0^{h(e)} \mu V_t\, \mu_L(dt)$$

as in 252N. (The c.l.d. and $\tau$-additive product measures on $G \times \mathbb{R}$ coincide, by 417T.) On the other hand,

$$(\mu \times \mu \times \mu_L)(Q) = \int_{K\times\mathbb{R}} \mu\{x : (a, x, t) \in Q\}(\mu \times \mu_L)(d(a, t))$$

(again, we can use 417G because $x \in K^*$ whenever $(a, x, t) \in Q$)

$$= \int_{K\times\mathbb{R}} \mu(V_t \triangle aV_t)(\mu \times \mu_L)(d(a, t))$$

$$= \int_0^\infty \int_K \mu(V_t \triangle aV_t)\mu(da)\mu_L(dt)$$

because $V_t = aV_t = G$ whenever $t < 0$ and $a \in G$. So there must be some $t \in \,]0, h(e)[$ such that

$$\textstyle\int_K \mu(V_t \triangle aV_t)\mu(da) < \eta\mu K\mu V_t = \epsilon\delta\mu V_t.$$

Set $V = V_t$ and $F = \{a : a \in K,\, \mu(V_t \triangle aV_t) \ge \epsilon\mu V_t\}$; then $V$ is open, $F$ is closed (443C) and $\mu(V \triangle aV) \le \epsilon\mu V$ for every $a \in K \setminus F$; also $0 < \mu V < \infty$, $V$ is symmetric (because $h = \overset{\leftrightarrow}{h}$) and $e \in V$ (because $t < h(e)$). **Q**

**(γ)** Now let $K \subseteq G$ be a compact set and $\epsilon > 0$, as in the statement of (xiii); enlarging $K$ and lowering $\epsilon$ if necessary, we may suppose that $\mu K > 0$ and $\epsilon \le 1$. Set $K_1 = K \cup KK$, so that $K_1$ is still compact. By ($\beta$), we have a symmetric open neighbourhood $V$ of $e$, of finite measure, such that $W = \{a : a \in K_1,\, \mu(aV \triangle V) > \frac{1}{3}\epsilon\mu V\}$ has measure less than $\frac{1}{2}\mu K$. If $a \in K$, then $W \cup a^{-1}W$ cannot cover $K$, so there is a $b \in K \setminus W$ such that $ab \notin W$; thus $b$ and $ab$ both belong to $K_1 \setminus W$, and

$$\mu(aV\triangle V)\le\mu(aV\triangle abV)+\mu(abV\triangle V)\le\mu(V\triangle bV)+\frac{1}{3}\epsilon\mu V\le\frac{2}{3}\epsilon\mu V.$$

We can now find a compact symmetric neighbourhood $L$ of $e$, included in $V$, with $8\mu(V\setminus L)\le\epsilon\mu L$. In this case, we shall have $\mu L>0$ and

$$\begin{aligned}\mu(aL\triangle L)&\le\mu(aV\setminus aL)+\mu(aV\triangle V)+\mu(V\setminus L)\\&\le 2\mu(V\setminus L)+\frac{2}{3}\epsilon(\mu L+\mu(V\setminus L))\le\epsilon\mu L\end{aligned}$$

for every $a\in K$, as required.

**(l)(xiii)⇒(xiv)** Suppose that (xiii) is true, and that $K\subseteq G$ is compact and $\epsilon>0$. Enlarging $K$ if necessary, we may suppose that it includes a neighbourhood of $e$. Of course we may also suppose that $\epsilon\le 1$.

**($\alpha$)** The first thing to note is that there is a set $I\subseteq G$ such that $KI=G$ and $m=\sup_{y\in G}\#(\{x:x\in I,\,y\in Kx\})$ is finite. **P** Let $V$ be an open neighbourhood of $e$ such that $VV^{-1}\subseteq K$. Let $I\subseteq G$ be maximal subject to $x^{-1}V\cap y^{-1}V=\emptyset$ for all distinct $x$, $y\in I$. If $x\in G$, there must be a $y\in I$ such that $x^{-1}V\cap y^{-1}V\ne\emptyset$, so that $x^{-1}\in y^{-1}VV^{-1}$ and $x\in VV^{-1}y\subseteq Ky\subseteq KI$; as $x$ is arbitrary, $G\subseteq KI$. If $y\in G$ and $I_y=\{x:x\in I,\,y\in Kx\}$, then $I_y\subseteq K^{-1}y$, so that $\{x^{-1}V:x\in I_y\}$ is a disjoint family of subsets of $y^{-1}KV$. But this means that $\#(I_y)\mu V\le\mu(y^{-1}KV)=\mu(KV)$. Accordingly $\sup_{y\in G}\#(I_y)\le\frac{\mu(KV)}{\mu V}$ is finite. **Q**

**($\beta$)** Set $\gamma=\sup_{a\in K}\Delta(a)$, $K^*=KKK^{-1}$. Let $\delta>0$ be such that

$$\delta\gamma m<\epsilon(\mu K-\delta\gamma),$$

and let $\eta>0$ be such that

$$1+\frac{\delta\gamma m}{\mu K-\delta\gamma}\le(1+\epsilon)(1-\frac{\eta}{\delta}\mu K^*).$$

By (xiii), there is a non-negligible compact set $L^*\subseteq G$ such that $\mu(aL^*\triangle L^*)\le\eta\mu L^*$ for every $a\in K^*$. Set $L=\{x:x\in L^*,\,\mu(K^*\setminus L^*x^{-1})\le\delta\}$; note that $L$ is closed (because $x\mapsto\mu(K^*\cap L^*x^{-1})$ is continuous, see 443C), therefore compact.

**($\gamma$)** $\mu L\ge(1-\frac{\eta}{\delta}\mu K^*)\mu L^*$. **P** Set $W=\{(x,y):x\in K^*,\,y\in L^*\setminus L,\,xy\notin L^*\}$. Then $W$ is a relatively compact Borel subset of $G\times G$, so we may apply Fubini's theorem (in the form 417G, as usual) to see that

$$\begin{aligned}\delta\mu(L^*\setminus L)&\le\int_{L^*\setminus L}\mu(K^*\setminus L^*y^{-1})\mu(dy)=\int\mu W^{-1}[\{y\}]\mu(dy)\\&=\int\mu W[\{x\}]\mu(dx)=\int_{K^*}\mu((L^*\setminus L)\setminus x^{-1}L^*)\mu(dx)\\&\le\int_{K^*}\mu(L^*\setminus x^{-1}L^*)\mu(dx)=\int_{K^*}\mu(xL^*\setminus L^*)\mu(dx)\le\eta\mu K^*\mu L^*\end{aligned}$$

by the choice of $L^*$. Accordingly

$$\mu L=\mu L^*-\mu(L^*\setminus L)\ge(1-\frac{\eta}{\delta}\mu K^*)\mu L^*.\ \mathbf{Q}$$

In particular, $\mu L>0$.

**($\delta$)** Set $J=\{x:x\in I,\,L\cap Kx\ne\emptyset\}$. For each $x\in J$, choose $z_x\in Kx\cap L$. Then $\Delta(z_x)\le\gamma\Delta(x)$ and $\Delta(x)(\mu K-\gamma\delta)\le\mu(Kx\cap L^*)$ for every $x\in J$. **P** $\Delta(z_x)=\Delta(z_xx^{-1})\Delta(x)\le\gamma\Delta(x)$ because $z_xx^{-1}\in K$. Next, because $z_x\in L$,

$$\mu(K^*z_x\setminus L^*)=\Delta(z_x)\mu(K^*\setminus L^*z_x^{-1})\le\delta\Delta(z_x).$$

Since $x\in K^{-1}z_x$, $Kx\subseteq KK^{-1}z_x\subseteq K^*z_x$,

$$\mu(Kx\setminus L^*)\le\mu(K^*z_x\setminus L^*)\le\delta\Delta(z_x)\le\delta\gamma\Delta(x)$$

and

$$\mu(Kx\cap L^*)=\mu(Kx)-\mu(Kx\setminus L^*)\ge\Delta(x)\mu K-\delta\gamma\Delta(x)=\Delta(x)(\mu K-\delta\gamma).\ \mathbf{Q}$$

**($\epsilon$)** Now recall that $\sum_{x\in I}\chi(Kx)\le m\chi G$, by the definition of $m$, so that

$$(\mu K-\delta\gamma)\sum_{x\in J}\Delta(x)\le\sum_{x\in J}\mu(Kx\cap L^*)\le m\mu L^*.$$

Since $KI=G$, $L\subseteq KJ$ and

$$KL\subseteq\bigcup_{x\in J}KKx\subseteq\bigcup_{x\in J}KKK^{-1}z_x=\bigcup_{x\in J}K^*z_x,$$

$$\begin{aligned}\mu(KL\setminus L^*)&\le\sum_{x\in J}\mu(K^*z_x\setminus L^*)=\sum_{x\in J}\Delta(z_x)\mu(K^*\setminus L^*z_x^{-1})\\&\le\delta\sum_{x\in J}\Delta(z_x)\le\delta\gamma\sum_{x\in J}\Delta(x)\le\tfrac{\delta\gamma m}{\mu K-\delta\gamma}\mu L^*.\end{aligned}$$

Accordingly

$$\mu(KL)\le\mu L^*(1+\tfrac{\delta\gamma m}{\mu K-\delta\gamma})\le\mu L\cdot\frac{1+\frac{\delta\gamma m}{\mu K-\delta\gamma}}{1-\frac{\eta}{\delta}\mu K^*}$$

(by ($\gamma$) above)

$$\le(1+\epsilon)\mu L,$$

by the choice of $\delta$ and $\eta$. Thus we have found an appropriate set $L$.

**(m)(xiv)⇒(xii)** If $I\subseteq G$ is finite and $\epsilon>0$, $I\cup\{e\}$ is compact, so there is a compact set $L\subseteq G$, of non-zero measure, such that $\mu(IL\cup L)\le(1+\frac{1}{2}\epsilon)\mu L$. Consequently

$$\mu(L\triangle aL)=2\mu(aL\setminus L)\le2\mu(IL\setminus L)\le\epsilon\mu L$$

for every $a\in I$, as required by (xii).

**(n)(xii)⇒(ii)** Write $\mathcal{L}$ for the family of all compact subsets of $G$ with non-zero measure. For $L\in\mathcal{L}$, define $p_L:C_b(G)\to\mathbb{R}$ by setting $p_L(f)=\frac{1}{\mu L}\int_L fd\mu$ for $f\in C_b(G)$. Of course $|p_L(f)|\le\|f\|_\infty$. For finite $I\subseteq G$, $\epsilon>0$ set

$$\mathcal{A}(I,\epsilon)=\{L:L\in\mathcal{L},\ \mu(aL\triangle L)\le\epsilon\mu L\text{ for every }a\in I\}.$$

By (xii), no $\mathcal{A}(I,\epsilon)$ is empty. So we have an ultrafilter $\mathcal{F}$ on $\mathcal{L}$ containing every $\mathcal{A}(I,\epsilon)$. Set $p(f)=\lim_{L\to\mathcal{F}}p_L(f)$ for $f\in C_b(G)$; then $p:C_b(G)\to\mathbb{R}$ is a positive linear functional and $p(\chi G)=1$.

If $a\in G$, $f\in C_b(G)$ and $L\in\mathcal{A}(\{a\},\epsilon)$, then

$$\begin{aligned}|p_L(a\bullet_l f)-p_L(f)|&=\frac{1}{\mu L}\Big|\int_L f(a^{-1}x)\mu(dx)-\int_L f(x)\mu(dx)\Big|\\&=\frac{1}{\mu L}\Big|\int_{aL}fd\mu-\int_L fd\mu\Big|\le\frac{1}{\mu L}\|f\|_\infty\mu(aL\triangle L)\le\epsilon\|f\|_\infty.\end{aligned}$$

Since $\mathcal{F}$ contains $\mathcal{A}(\{a\},\epsilon)$ for every $\epsilon>0$,

$$|p(a\bullet_l f)-p(f)|=\lim_{L\to\mathcal{F}}|p_L(a\bullet_l f)-p_L(f)|=0.$$

As $f$ and $a$ are arbitrary, $p$ witnesses that (ii) is true.

**(o)(xii)⇒(xi)** Given a finite set $I\subseteq G$ and $\epsilon>0$, (xii) tells us that there is a compact set $L\subseteq G$ of non-zero measure such that $\mu(aL\triangle L)\le\epsilon^q\mu L$ for every $a\in I$. Try $u=\frac{1}{(\mu L)^{1/q}}(\chi L)^\bullet$. Then $\|u\|_q=1$. If $a\in I$, then $a\bullet_l u=\frac{1}{(\mu L)^{1/q}}\chi(aL)^\bullet$, so

$$\int |u - a\bullet_l u|^q = \int (\frac{1}{(\mu L)^{1/q}}\chi(aL\triangle L)^\bullet)^q = \frac{\mu(aL\triangle L)}{\mu L} \le \epsilon^q$$

and $\|u - a\bullet_l u\|_q \le \epsilon$, as required by (xi).

**(p)(xi)⇒(xii)** Now assume that (xi) is true. Let $I \subseteq G$ be a finite set, and $\epsilon > 0$. Set $\delta = \frac{\epsilon}{4+\epsilon}$, and let $\eta > 0$ be such that that $(1+\eta\#(I))^q \le 1+\delta$.

Take $u \in L^q$ such that $\|u\|_q = 1$ and $\|u - a\bullet_l u\|_q \le \eta$ for every $a \in I$. Setting $v = |u|$, we see that $a\bullet_l v = |a\bullet_l u|$, so $|v - a\bullet_l v| \le |u - a\bullet_l u|$ and $\|v - a\bullet_l v\|_q \le \eta$ for every $a \in I$, while $\|v\|_q = 1$. Let $f : G \to [0,\infty[$ be a function such that $f^\bullet = v$ in $L^q$; then $\|f\|_q = 1$ while $\|f - a\bullet_l f\|_q \le \eta$ for every $a \in I$. Set $g = \sup_{a\in I\cup\{e\}} a\bullet_l f$; then

$$f \le g \le f + \textstyle\sum_{a\in I}(a\bullet_l f - f)^+,$$

so

$$\|g\|_q \le 1 + \textstyle\sum_{a\in I}\|a\bullet_l f - f\|_q \le 1 + \eta\#(I), \qquad \int g^q d\mu \le (1+\eta\#(I))^q \le 1+\delta.$$

For $t > 0$, set $E_t = \{x : f(x)^q \ge t\}$, $F_t = \{t : g(x)^q \ge t\}$. Then

$$\int_0^\infty \mu E_t dt = \int f^q d\mu = 1, \qquad \int_0^\infty \mu F_t dt = \int g^q d\mu \le 1+\delta,$$

where the integrals here are with respect to Lebesgue measure (252O). There must therefore be a $t > 0$ such that $\mu E_t > 0$ and $\mu F_t \le (1+\delta)\mu E_t$.

If $a \in I$, then $a\bullet_l f \le g$, so $aE_t = \{x : (a\bullet_l f)(x) \ge t\}$ is included in $F_t$; also, of course, $E_t \subseteq F_t$. We therefore have

$$\mu(E_t\triangle aE_t) = 2\mu(aE_t \setminus E_t) \le 2\mu(F_t \setminus E_t) \le 2\delta\mu E_t.$$

There is no reason why $E_t$ should be compact, so it may not itself be the $L$ we seek. However, $\mu E_t$ is certainly finite, so there must be a compact $L \subseteq E$ such that $\mu L \ge (1-\delta)\mu E_t$. In this case, $\mu L > 0$ and

$$\mu(aL\triangle L) \le \mu(aL\triangle aE_t) + \mu(aE_t\triangle E_t) + \mu(E_t\triangle L)$$
$$\le 2\mu(E_t \setminus L) + 2\delta\mu E_t \le 4\delta\mu E_t \le \frac{4\delta}{1-\delta}\mu L = \epsilon\mu L$$

for every $a \in I$. So this $L$ will serve.

**Remark** Of course there are many variations possible in the conditions listed above, some of which are in 449Xk-449Xm.

**449K Proposition** Let $G$ be an amenable locally compact Hausdorff group, and $H$ a subgroup of $G$. Then $H$ is amenable.

**proof** (Paterson 88, 1.12) **(a)** For most of the proof (down to the end of (g) below), suppose that $H$ is closed. Let $V$ be a compact neighbourhood of the identity in $G$. Let $I \subseteq G$ be a maximal set such that $VzH \cap Vz'H = \emptyset$ for all distinct $z$, $z' \in I$. Then $V^{-1}VIH = G$. **P** If $x \in G$, there is a $z \in I$ such that $VxH \cap VzH \ne \emptyset$, that is,

$$x \in V^{-1}VzHH^{-1} = V^{-1}VzH \subseteq V^{-1}VIH. \textbf{ Q}$$

**(b)** If $x \in G$ then

$$I \cap V^{-1}xH = \{z : z \in I, z \in V^{-1}xH\} = \{z : z \in I, x \in VzH^{-1} = VzH\}$$

has at most one element. If $K \subseteq G$ is compact, then there is a finite set $J \subseteq G$ such that $K \subseteq V^{-1}J$, and now

$$I \cap KH \subseteq \textstyle\bigcup_{x\in J} I \cap V^{-1}xH$$

is finite.

**(c)** Let $h \in C_k(X)^+$ be such that $h \ge \chi(V^{-1}V)$; write $W$ for the support of $h$. Set $g(x) = \sum_{z\in I} h(xz^{-1})$ for $x \in G$.

If $K \subseteq G$ is compact, then

$$\begin{aligned}\{z : z \in I,\, h(xyz^{-1}) \neq 0 \text{ for some } x \in K \text{ and } y \in H\}\\ \subseteq I \cap \{z : KHz^{-1} \cap W \neq \emptyset\}\\ = I \cap \{z : zH^{-1}K^{-1} \cap W^{-1} \neq \emptyset\} = I \cap W^{-1}KH\end{aligned}$$

is finite. In particular, $\{z : z \in I,\, h(xz^{-1}) \neq 0\}$ and $g(x)$ are finite for every $x \in G$. Next, if $x_0 \in G$, then $J = \{z : h(xz^{-1}) \neq 0 \text{ for some } x \in x_0V\}$ is finite, and $g(x) = \sum_{z \in J} h(xz^{-1})$ for $x \in x_0V$, so $g$ is continuous at $x_0$; as $x_0$ is arbitrary, $g \in C(G)$. Of course $g \geq 0$ because $h \geq 0$.

**(d)(i)** For $x \in G$, set $g_x(y) = g(x^{-1}y)$ for $y \in H$. Then $g_x \in C_k(H)$. **P** $g_x$ is continuous because $g$ is. Now $J = \{z : z \in I,\, h(x^{-1}yz^{-1}) \neq 0 \text{ for some } y \in H\}$ is finite, and

$$\begin{aligned}\{y : g_x(y) \neq 0\} &\subseteq \{y : \text{ there is a } z \in I \text{ such that } h(x^{-1}yz^{-1}) \neq 0\}\\ &\subseteq \{y : \text{ there is a } z \in J \text{ such that } x^{-1}yz^{-1} \in W\} = xWJ\end{aligned}$$

which is compact. **Q**

**(ii)** Moreover, for any $x_0 \in G$ there is a compact set $L \subseteq H$ such that $\{x : |g_x - g_{x_0}| \leq \epsilon\chi L\}$ is a neighbourhood of $x_0$ for every $\epsilon > 0$. **P**

$$J = \{z : z \in I,\, h(x^{-1}yz^{-1}) \neq 0 \text{ for some } x \in x_0V \text{ and } y \in H\}$$

is finite, by (c) in its full strength. Let $L$ be the compact set $H \cap x_0VWJ$.

Take any $\epsilon > 0$. If $x \in x_0V$, then $g_x(y) = \sum_{z \in J} h(x^{-1}yz^{-1})$ for every $y \in H$, and $g_x(y) = 0$ for $y \in H \setminus L$. Moreover, setting $g'_x(y) = \sum_{z \in J} h(x^{-1}yz^{-1})$ for $x \in G$ and $y \in H$, $(x, y) \mapsto g'_x(y)$ is continuous, so $x \mapsto g'_x : G \to C(H)$ is continuous if we give $C(H)$ the topology of uniform convergence on compact subsets of $H$ (4A2G(g-i)). In particular, $x \mapsto g'_x\restriction L$ is continuous for the norm topology of $C(L)$, and $U = \{x : x \in x_0V,\, \|g'_x\restriction L - g'_{x_0}\restriction L\|_\infty \leq \epsilon\}$ is a neighbourhood of $x_0$. But if $x \in U$, then

$$g_x(y) = g_{x_0}(y) = 0 \text{ for } y \in H \setminus L,$$

$$|g_x(y) - g_{x_0}(y)| = |g'_x(y) - g'_{x_0}(y)| \leq \epsilon \text{ for } y \in L,$$

so $|g_x - g_{x_0}| \leq \epsilon\chi L$. **Q**

**(e)** Now take a left Haar measure $\nu$ on $H$. (This is where it really matters whether $H$ is closed.) Define $T : C_b(H) \to \mathbb{R}^G$ by setting

$$(Tf)(x) = \int g_x \times f d\nu = \int_H g(x^{-1}y)f(y)\nu(dy)$$

for $f \in C_b(H)$ and $x \in G$. Then $Tf \in C(G)$ for every $f \in C_b(H)$. **P** Given $x_0 \in G$ and $\epsilon > 0$, let $L \subseteq H$ be a compact set as in (d-ii). Let $\delta > 0$ be such that $\delta \int_L |f| d\nu \leq \epsilon$. Then

$$\{x : |(Tf)(x) - (Tf)(x_0)| \leq \epsilon\} \supseteq \{x : |g_x - g_{x_0}| \leq \delta\chi L\}$$

is a neighbourhood of $x_0$. As $x_0$ and $\epsilon$ are arbitrary, $Tf$ is continuous. **Q**

Clearly, $T : C_b(H) \to C(G)$ is a positive linear operator. Next, if $f \in C_b(H)$ and $b \in H$, $T(b\bullet_l f) = b\bullet_l(Tf)$. **P**

$$\begin{aligned}T(b\bullet_l f)(x) &= \int_H g(x^{-1}y)(b\bullet_l f)(y)\nu(dy) = \int_H g(x^{-1}y)f(b^{-1}y)\nu(dy)\\ &= \int_H g(x^{-1}by)f(y)\nu(dy) = (Tf)(b^{-1}x) = (b\bullet_l Tf)(x)\end{aligned}$$

for every $x \in G$. **Q**

We need to know that $T(\chi H)(x) > 0$ for every $x \in G$. **P** There is a $z \in I$ such that $x^{-1} \in V^{-1}VzH$, as remarked in (a). Now $x^{-1}Hz^{-1}$ meets $V^{-1}V$, so there is a $y \in H$ such that

$$1 \leq h(x^{-1}yz^{-1}) \leq g(x^{-1}y) = g_x(y)$$

and $T(\chi H)(x) = \int_H g_x d\nu > 0$ because $g_x$ is continuous and non-negative and $\nu$ is strictly positive. **Q**

**(f)** We therefore have a positive linear operator $S : C_b(H) \to C(G)$ defined by setting $Sf = \dfrac{Tf}{T(\chi H)}$ for $f \in C_b(H)$. Since $S(\chi H) = \chi G$, $S[C_b(H)] \subseteq C_b(G)$; moreover, for $f \in C_b(H)$ and $b \in H$,

$$\begin{aligned} S(b\bullet_l f) &= \frac{T(b\bullet_l f)}{T(\chi H)} = \frac{T(b\bullet_l f)}{T(b\bullet_l \chi H)} \\ &= \frac{b\bullet_l (Tf)}{b\bullet_l (T\chi H)} = b\bullet_l \frac{Tf}{T\chi H} = b\bullet_l Sf. \end{aligned}$$

**(g)** At this point, recall that by 449J(ii) there is a positive linear functional $p : C_b(G) \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ whenever $f \in C_b(G)$ and $a \in G$. Set $q = pS : C_b(H) \to \mathbb{R}$; then $q$ is a positive linear functional, $q(\chi H) = 1$ and $q(b\bullet_l f) = q(f)$ whenever $f \in C_b(H)$ and $b \in H$. So $q$ witnesses that 449J(ii) is true for $H$, and $H$ is amenable.

**(h)** All this has been on the assumption that $H$ is closed. But in general $\overline{H}$ is a closed subgroup of $G$, therefore amenable by (a)-(g) here, and $H$ is dense in $\overline{H}$, therefore amenable by 449F(a-ii).

**449L** If we make a further step back towards the origin of this topic, and suppose that our group is discrete, then we have a striking further condition to add to the lists above. I give this as a corollary of a general result on group actions recalling the main theorems of §§395 and 448.

**Tarski's theorem** Let $G$ be a group acting on a non-empty set $X$. Then the following are equiveridical:

(i) there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu X = 1$ and $\nu(a\bullet A) = \nu A$ whenever $A \subseteq X$ and $a \in G$;

(ii) there are no $A_0, \dots, A_n$, $a_0, \dots, a_n$, $b_0, \dots, b_n$ such that $A_0, \dots, A_n$ are subsets of $X$ covering $X$, $a_0, \dots, a_n, b_0, \dots, b_n$ belong to $G$, and $a_0\bullet A_0$, $b_0\bullet A_0$, $a_1\bullet A_1$, $b_1\bullet A_1, \dots, b_n\bullet A_n$ are all disjoint.

**proof (a)(i)⇒(ii)** This is elementary, for if $\nu : \mathcal{P}X \to [0,1]$ is a non-zero additive functional and $A_0, \dots, A_n$ cover $X$ and $a_0, \dots, b_n \in G$, then

$$\textstyle\sum_{i=0}^n \nu(a_i\bullet A_i) + \sum_{i=0}^n \nu(b_i\bullet A_i) = 2\sum_{i=0}^n \nu A_i \ge 2\nu X > \nu X,$$

and $a_0\bullet A_0, \dots, b_n\bullet A_n$ cannot all be disjoint.

**(b)(ii)⇒(i)** Now suppose that (ii) is true.

**(α)** Suppose that $c_0, \dots, c_n \in G$. Then there is a finite set $I \subseteq X$ such that $\#(\{c_i\bullet x : i \le n,\, x \in I\}) < 2\#(I)$. **P?** Otherwise, by the Marriage Lemma in the form 4A1H, applied to the set

$$R = \{((x,j), c_i\bullet x) : x \in X,\, j \in \{0,1\},\, i \le n\} \subseteq (X \times \{0,1\}) \times X,$$

there is an injective function $\phi : X \times \{0,1\} \to X$ such that $\phi(x,j) \in \{c_i\bullet x : i \le n\}$ for every $x \in X$ and $j \in \{0,1\}$. Now set $B_{ij} = \{x : \phi(x,0) = c_i\bullet x,\, \phi(x,1) = c_j\bullet x\}$ for $i$, $j \le n$, so that $X = \bigcup_{i,j\le n} B_{ij}$. Let $A_{ij} \subseteq B_{ij}$ be such that $\langle A_{ij}\rangle_{i,j\le n}$ is a partition of $X$, and set $a_{ij} = c_i$, $b_{ij} = c_j$ for $i$, $j \le n$; then $a_{ij}\bullet A_{ij} \subseteq \phi[A_{ij} \times \{0\}]$, $b_{ij}\bullet A_{ij} \subseteq \phi[A_{ij} \times \{1\}]$ are all disjoint, which is supposed to be impossible. **XQ**

**(β)** Suppose that $J \subseteq G$ is finite and $\epsilon > 0$. Then there is a non-empty finite set $I \subseteq X$ such that $\#(I \triangle c\bullet I) \le \epsilon\#(I)$ for every $c \in J$. **P** It is enough to consider the case in which the identity $e$ of $G$ belongs to $J$. **?** Suppose, if possible, that there is no such set $I$. Let $m \ge 1$ be such that $(1 + \frac{1}{2}\epsilon)^m \ge 2$. Set $K = J^m = \{c_1c_2\dots c_m : c_1, \dots, c_m \in J\}$. By (α), there is a finite set $I_0 \subseteq X$ such that $\#(I_0^*) < 2\#(I_0)$, where $I_0^* = \{a\bullet x : a \in K,\, x \in I_0\}$. Choose $c_1, \dots, c_m$ and $I_1, \dots, I_m$ inductively such that

> given that $I_k$ is a non-empty finite subset of $X$, where $0 \le k < m$, take $c_{k+1} \in J$ such that $\#(I_k \triangle c_{k+1}\bullet I_k) > \epsilon\#(I_k)$ and set $I_{k+1} = I_k \cup c_{k+1}\bullet I_k$.

Then $I_k \subseteq \{a\bullet x : a \in J^k,\, x \in I_0\}$ for each $k \le m$, and in particular $I_m \subseteq I_0^*$. But also

$$\begin{aligned} \#(I_{k+1}) &= \#(I_k) + \#((c_{k+1}\bullet I_k) \setminus I_k) \\ &= \#(I_k) + \frac{1}{2}\#((c_{k+1}\bullet I_k) \triangle I_k) \ge (1 + \frac{1}{2}\epsilon)\#(I_k) \end{aligned}$$

for every $k < m$, so

$$\#(I_0^*) \ge \#(I_m) \ge (1+\frac{1}{2}\epsilon)^m\#(I_0) \ge 2\#(I_0),$$

contrary to the choice of $I_0$. **XQ**

**($\gamma$)** There is therefore an ultrafilter $\mathcal{F}$ on $[X]^{<\omega}\setminus\{\emptyset\}$ such that

$$\mathcal{A}_{c\epsilon} = \{I : I \in [X]^{<\omega}\setminus\{\emptyset\},\ \#(I\triangle c{\bullet}I) \le \epsilon\#(I)\}$$

belongs to $\mathcal{F}$ for every $c \in G$ and $\epsilon > 0$. For $I \in [X]^{<\omega}\setminus\{\emptyset\}$ and $A \subseteq X$ set $\nu_I(A) = \#(A\cap I)/\#(I)$, and set $\nu A = \lim_{I\to\mathcal{F}}\nu_I A$ for every $A \subseteq X$, so that $\nu : \mathcal{P}X \to [0,1]$ is an additive functional and $\nu X = 1$.

Now $\nu$ is $G$-invariant. **P** If $A \subseteq X$ and $c \in G$ and $\epsilon > 0$, then $\mathcal{A}_{c^{-1},\epsilon} \in \mathcal{F}$. If $I \in \mathcal{A}_{c^{-1},\epsilon}$, then

$$\begin{aligned}|\nu_I(c{\bullet}A) - \nu_I(A)| &= \frac{1}{\#(I)}|\#(I\cap(c{\bullet}A)) - \#(I\cap A)|\\ &= \frac{1}{\#(I)}|\#((c^{-1}{\bullet}I)\cap A) - \#(I\cap A)|\\ &\le \frac{1}{\#(I)}\#((c^{-1}{\bullet}I)\triangle I) \le \epsilon.\end{aligned}$$

As $\epsilon$ is arbitrary, $\lim_{I\to\mathcal{F}}\nu_I(c{\bullet}A) - \nu_I(A) = 0$, and $\nu(c{\bullet}A) = \nu A$. As $A$ and $c$ are arbitrary, $\nu$ is $G$-invariant. **Q**

So $\nu$ witnesses that (i) is true, and the proof is complete.

**449M Corollary** Let $G$ be a group with its discrete topology. Then the following are equiveridical:

(i) $G$ is amenable;

(ii) there are no $A_0,\dots,A_n$, $a_0,\dots,a_n$, $b_0,\dots,b_n$ such that $G = \bigcup_{i\le n} A_i$ $a_0,\dots,a_n,b_0,\dots,b_n$ belong to $G$, and $a_0A_0$, $b_0A_0$, $a_1A_1$, $b_1A_1,\dots,b_nA_n$ are disjoint.

**proof** All we have to observe is that ($\alpha$) every function from $G$ to $\mathbb{R}$ is uniformly continuous for the right uniformity of $G$, so that $G$ is amenable iff there is an invariant positive linear functional $p : \ell^\infty(G) \to \mathbb{R}$ such that $p(\chi G) = 1$ ($\beta$) that a positive linear functional on $\ell^\infty(G)$ is $G$-invariant iff the corresponding additive functional on $\mathcal{P}G$ is $G$-invariant. So (i) of 449L is equivalent to amenability of $G$ as defined in 449A.

**449N Theorem** Let $G$ be a group which is amenable in its discrete topology, $X$ a set, and $\bullet$ an action of $G$ on $X$. Let $\mathcal{E}$ be a subring of $\mathcal{P}X$ and $\nu : \mathcal{E} \to [0,\infty[$ a finitely additive functional which is $G$-invariant in the sense that $g{\bullet}E \in \mathcal{E}$ and $\nu(g{\bullet}E) = \nu E$ whenever $E \in \mathcal{E}$ and $g \in G$. Then there is an extension of $\nu$ to a $G$-invariant non-negative finitely additive functional $\tilde\nu$ defined on the ideal $\mathcal{I}$ of subsets of $X$ generated by $\mathcal{E}$.

**proof (a)** There is a non-negative finitely additive functional $\theta : \mathcal{I} \to \mathbb{R}$ extending $\nu$. **P** Let $V$ be the linear subspace of $\ell^\infty(X)$ generated by $\{\chi E : E \in \mathcal{E}\}$, so that $V$ can be identified with the Riesz space $S(\mathcal{E})$ (361L). Let $U$ be the solid linear subspace of $\ell^\infty(X)$ generated by $V$. For $u \in U$ set $q(u) = \inf\{⨍ v\,d\nu : v \in V, |u| \le v\}$, where $⨍\, d\nu : S(\mathcal{E}) \to \mathbb{R}$ is the positive linear functional corresponding to $\nu : \mathcal{E} \to [0,\infty[$ as in 361F-361G. Then $q$ is a seminorm, and $|⨍ v\,d\nu| \le q(v)$ for every $v \in V$. So there is a linear functional $f : U \to \mathbb{R}$ such that $f(v) = ⨍ v\,d\nu$ for every $v \in V$ and $|f(u)| \le q(u)$ for every $u \in U$ (4A4D(a-i)). Set $\theta A = f(\chi A)$ for $A \in \mathcal{I}$. Then $\theta : \mathcal{I} \to \mathbb{R}$ is additive and extends $\nu$. If $A \in \mathcal{I}$, there is an $E \in \mathcal{E}$ including $A$. Now we have

$$\theta(E\setminus A) = f(\chi E - \chi A) \le q(\chi E - \chi A) \le ⨍ \chi E\,d\nu = \nu E = \theta E,$$

so $\theta A \ge 0$. So $\theta$ is non-negative. **Q**

**(b)** As in the proof of 449M, we have a positive $G$-invariant linear functional $p : \ell^\infty(G) \to \mathbb{R}$ such that $p(\chi G) = 1$. For $A \in \mathcal{I}$, set $f_A(a) = \theta(a^{-1}{\bullet}A)$ for $a \in G$, and $\tilde\nu A = p(f_A)$. Then $\tilde\nu : \mathcal{I} \to [0,\infty[$ is additive. If $E \in \mathcal{E}$ then $f_A(a) = \nu E$ for every $a$, so $\tilde\nu$ extends $\nu$. If $A \in \mathcal{I}$ and $a$, $b \in G$, then $f_{aA}(b) = \theta(b^{-1}aA) = f_A(a^{-1}b)$, so $f_{aA} = a{\bullet_l}f_A$ and

$$\tilde\nu(aA) = p(f_{aA}) = p(f_A) = \tilde\nu A.$$

Thus $\tilde\nu$ is $G$-invariant, as required.

**449O Corollary** (Banach 1923) If $r = 1$ or $r = 2$, there is a functional $\theta : \mathcal{P}\mathbb{R}^r \to [0,\infty]$ such that (i) $\theta(A \cup B) = \theta A + \theta B$ whenever $A$, $B \subseteq \mathbb{R}^r$ are disjoint (ii) $\theta E$ is the Lebesgue measure of $E$ whenever $E \subseteq \mathbb{R}^r$ is Lebesgue measurable (iii) $\theta(g[A]) = \theta A$ whenever $A \subseteq \mathbb{R}^r$ and $g : \mathbb{R}^r \to \mathbb{R}^r$ is an isometry.

**proof (a)** The point is that the group $G$ of all isometries of $\mathbb{R}^r$, with its discrete topology, is amenable. **P** Let $G_0 \subseteq G$ be the subgroup consisting of rotations about $\mathbf{0}$; because $r \le 2$, this is abelian, therefore amenable (449Cf). Let $G_1 \subseteq G$ be the subgroup consisting of isometries keeping $\mathbf{0}$ fixed; then $G_0$ is a normal subgroup of $G_1$, and $G_1/G_0$ is abelian, so $G_1$ is amenable (449Cc). Let $G_2 \subseteq G$ be the normal subgroup consisting of translations; then $G_2$ is abelian, therefore amenable. Now $G = G_1G_2$, so $G$ is amenable (449Cd). **Q**

**(b)** Let $\mathcal{E}$ be the ring of subsets of $\mathbb{R}^r$ with finite Lebesgue measure, and let $\nu$ be the restriction of Lebesgue measure to $\mathcal{E}$. Then $\nu$ is $G$-invariant. By 449N, there is a $G$-invariant extension $\tilde\nu$ of $\nu$ to the ideal $\mathcal{I}$ generated by $\mathcal{E}$. Setting $\theta A = \tilde\nu A$ for $A \in \mathcal{I}$, $\infty$ for $A \in \mathcal{P}\mathbb{R}^r \setminus \mathcal{I}$, we have a suitable functional $\theta$.

**449X Basic exercises >(a)** Let $G$ be a topological group. On $G$ define a binary operation $\diamond$ by saying that $x \diamond y = yx$ for all $x$, $y \in G$. Show that $(G, \diamond)$ is a topological group isomorphic to $G$, so is amenable iff $G$ is.

**(b)** Show that any finite topological group is amenable.

**>(c)** Show that, for any $r \in \mathbb{N}$, the isometry group of $\mathbb{R}^r$, with the topology of pointwise convergence, is amenable. (*Hint*: 443Xw, 449Cd.)

**(d)** Find a locally compact Polish group which is amenable but not unimodular. (*Hint*: 442Xf, 449Cd.)

**(e)** Prove 449Cg directly from 441C, without mentioning Haar measure.

**(f)** Let $G$ be a topological group and $U$ the space of bounded real-valued functions on $G$ which are uniformly continuous for the right uniformity of $G$. Show that $\bullet_r$, as defined in 4A5Cc, gives an action of $G$ on $U$.

**(g)** Let $\bullet$ be an action of a group $G$ on a set $X$, and $U$ a Riesz subspace of $\ell^\infty(X)$, containing the constant functions, such that $a{\bullet}f \in U$ whenever $f \in U$ and $a \in G$. Show that the following are equiveridical: (i) there is a $G$-invariant positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi X) = 1$; (ii) $\sup_{x \in X} \sum_{i=0}^n f_i(x) - f_i(a_i{\bullet}x) \ge 0$ whenever $f_0, \dots, f_n \in U$ and $a_0, \dots, a_n \in G$. (*Hint*: if (ii) is true, let $V$ be the linear subspace generated by $\{f - a{\bullet}f : f \in U,\ a \in G\}$ and show that $\inf_{g \in V} \|g - \chi X\|_\infty = 1$.)

**>(h)** Let $X$ be a set and $G$ the group of all permutations of $X$. (i) Give $X$ the zero-one metric, so that $G$ is the isometry group of $X$. Show that $G$, with the topology of pointwise convergence (441G), is amenable. (*Hint*: for any $I \in [X]^{<\omega}$, $\{a : a \in G,\ a(x) = x$ for every $x \in X \setminus I\}$ is amenable.) (ii) Show that if $X$ is infinite then $G$, with its discrete topology, is not amenable. (*Hint*: the left action of $F_2$ on itself can be regarded as an embedding of $F_2$ in $G$.)

**(i)** Let $G$ be a Hausdorff topological group, and $\hat G$ its completion with respect to its bilateral uniformity (definition: 4A5Hb). Show that $G$ is amenable iff $\hat G$ is.

**(j)**(i) Let $G$ be the group with generators $a$, $b$ and relations $a^2 = b^3 = e$ (that is, the quotient of the free group on two generators $a$ and $b$ by the normal subgroup generated by $\{a^2, b^3\}$). Show that $G$, with its discrete topology, is not amenable. (ii) Let $G$ be the group with generators $a$, $b$ and relations $a^2 = b^2 = e$. Show that $G$, with its discrete topology, is amenable. (*Hint*: we have a function length : $G \to \mathbb{N}$ such that $\mathrm{length}(ab) \le \mathrm{length}(a) + \mathrm{length}(b)$ for all $a$, $b \in G$ and $\limsup_{n\to\infty} \frac{1}{n}\#(\{a : \mathrm{length}(a) \le n\})$ is finite. See also 449Yf.)

**(k)** Let $G$ be a locally compact Hausdorff group, and $\mu$ a left Haar measure on $G$. Show that $G$ is amenable iff for every finite set $I \subseteq G$, finite set $J \subseteq \mathcal{L}^\infty(\mu)$ and $\epsilon > 0$, there is an $h \in C_{k1}(G)^+$ (definition: 449J) such that $|\int f(ax)h(x)\mu(dx) - \int f(x)h(x)\mu(dx)| \le \epsilon$ for every $a \in I$ and $f \in J$. (*Hint*: the image of the unit ball in $L^1$ is weak* dense in the unit ball of $(L^\infty)^*$.)

**(l)** Let $G$ be a locally compact Hausdorff group, and $\mu$ a left Haar measure on $G$. Show that the following are equiveridical: (i) $G$ is amenable; (ii) there is a positive linear functional $p^\# : L^\infty(\mu) \to \mathbb{R}$ such that $p^\#(\chi G^\bullet) = 1$ and $p^\#(a\bullet_r u) = p^\#(u)$ for every $u \in L^\infty(\mu)$ and every $a \in G$; (iii) for every finite set $I \subseteq G$, finite set $J \subseteq \mathcal{L}^\infty(\mu)$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $|\int f(xa)h(x)\mu(dx) - \int f(x)h(x)\mu(dx)| \le \epsilon$ for every $a \in I$ and $f \in J$.

**(m)** Let $G$ be a locally compact Hausdorff group and $\mathcal{B}\mathfrak{a}_G$ its Baire $\sigma$-algebra. Show that $G$ is amenable iff there is a non-zero finitely additive $\phi : \mathcal{B}\mathfrak{a}_G \to [0,1]$ such that $\phi(aE) = \nu E$ for every $a \in G$ and $E \in \mathcal{B}\mathfrak{a}_G$.

**(n)** A **symmetric Følner sequence** in a group $G$ is a sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ of non-empty finite symmetric subsets of $G$ such that $\lim_{n\to\infty} \dfrac{\#(L_n \triangle aL_n)}{\#(L_n)} = 0$ for every $a \in G$. Show that a group $G$ has a symmetric Følner sequence iff it is countable and amenable when given its discrete topology.

**>(o)** Let $G$ be a group which is amenable when given its discrete topology. Let $\phi : \mathcal{P}G \to [0,1]$ be an additive functional such that $\phi G = 1$ and $\phi(aE) = \phi E$ whenever $E \subseteq G$ and $a \in G$. For $E \subseteq G$ set $\psi E = ⨍\phi(Ex)\phi(dx)$. Show that $\psi : \mathcal{P}G \to [0,1]$ is an additive functional, that $\psi G = 1$ and that $\psi(aE) = \psi(Ea) = \psi E$ for every $E \subseteq G$ and $a \in G$.

**(p)** Let $X$ be a non-empty set, $G$ a group and $\bullet$ an action of $G$ on $X$. Suppose that $G$ is an amenable group when given its discrete topology. Show that there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu X = 1$ and $\nu(a\bullet A) = \nu A$ for every $A \subseteq X$ and every $a \in G$.

**(q)** Let $G$ be a locally compact Hausdorff group and $\mu$ a left Haar measure on $G$. Show that if $G$, with its discrete topology, is amenable, then there is a functional $\phi : \mathcal{P}G \to [0,\infty]$, extending $\mu$, such that $\phi(A \cup B) = \phi A + \phi B$ whenever $A$, $B \subseteq G$ are disjoint and $\phi(xA) = \phi A$ for every $x \in G$ and $A \subseteq G$.

**(r)** Let $X$ be a compact metrizable space, $\phi : X \to X$ a continuous function and $\mu$ a Radon probability measure on $X$ such that $\mu\phi^{-1} = \mu$. Show that for $\mu$-almost every $x \in X$, $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$ is defined for every $f \in C(X)$. (*Hint*: 4A2Pe, 372J.)

**449Y Further exercises (a)** If $S$ is a semigroup with identity $e$ and $X$ is a set, an **action** of $S$ on $X$ is a map $(s,x) \mapsto s\bullet x : S \times X \to X$ such that $s\bullet(t\bullet x) = (st)\bullet x$ and $e\bullet x = x$ for every $s$, $t \in S$ and $x \in X$. A topological semigroup $S$ with identity is **amenable** if for every non-empty compact Hausdorff space $X$ and every continuous action of $S$ on $X$ there is a Radon probability measure $\mu$ on $X$ such that $\int f(s\bullet x)\mu(dx) = \int f(x)\mu(dx)$ for every $s \in S$ and $f \in C(X)$. Show that

(i) $(\mathbb{N}, +)$, with its discrete topology, is amenable;

(ii) if $S$ is a topological semigroup and $\mathcal{S}$ is an upwards-directed family of amenable sub-semigroups of $S$ with dense union in $S$, then $S$ is amenable;

(iii) if $\langle S_i\rangle_{i\in I}$ is a family of amenable topological semigroups with product $S$ then $S$ is amenable;

(iv) if $S$ is an amenable topological semigroup, $S'$ is a topological semigroup, and there is a continuous multiplicative surjection from $S$ onto $S'$, then $S'$ is amenable;

(v) if $S$ is an abelian topological semigroup, then it is amenable.

**(b)** Give an example of a topological semigroup $S$ with identity such that $S$ is amenable in the sense of 449Ya but $(S, \diamond)$ is not, where $a \diamond b = ba$ for $a$, $b \in S$.

**(c)** Let $G$ be a topological group and $U$ the space of bounded real-valued functions on $G$ which are uniformly continuous for the right uniformity. Let $M_{\mathrm{qR}}^+$ be the space of totally finite quasi-Radon measures on $G$. (i) Show that if $\nu \in M_{\mathrm{qR}}^+$ then $\nu * f$ (definition: 444H) belongs to $U$ for every $f \in U$. (ii) Show that $(\nu, f) \mapsto \nu * f : M_{\mathrm{qR}}^+ \times U \to U$ is continuous if $M_{\mathrm{qR}}^+$ is given its narrow topology and $U$ is given its norm topology. (iii) Show that if $p : U \to \mathbb{R}$ is a continuous linear functional such that $p(a\bullet_l f) = p(f)$ for every $f \in U$ and $a \in G$, then $p(\nu * f) = \nu G \cdot p(f)$ for every $f \in U$ and every totally finite quasi-Radon measure $\nu$ on $G$.

**(d)** Re-work 449J for general groups carrying Haar measures.

**(e)** Let $G$ be a group with a symmetric Følner sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ (449Xn), and $\bullet$ an action of $G$ on a reflexive Banach space $U$ such that $u\mapsto a\bullet u$ is a linear operator of norm at most 1 for every $a\in G$. For $n\in\mathbb{N}$ set $T_nu=\frac{1}{\#(L_n)}\sum_{a\in L_n}a\bullet u$ for $u\in U$. Show that for every $u\in U$ the sequence $\langle T_nu\rangle_{n\in\mathbb{N}}$ is norm-convergent to a $v\in U$ such that $a\bullet v=v$ for every $a\in G$. (*Hint*: 372A. See also 461Yg below.)

**(f)** Let $G$ be a locally compact Hausdorff group and $\mu$ a left Haar measure on $G$. Suppose that $G$ is **exponentially bounded**, that is, $\limsup_{n\to\infty}(\mu(K^n))^{1/n}\le 1$ for every compact set $K\subseteq G$. Show that $G$ is amenable.

**(g)** Let $G$ be a group and $\bullet$ an action of $G$ on a set $X$. Let $\mathrm{T}$ be an algebra of subsets of $X$ such that $g\bullet E\in\mathrm{T}$ for every $E\in\mathrm{T}$ and $g\in G$, and $H$ a member of $\mathrm{T}$; write $\mathrm{T}_H$ for $\{E:E\in\mathrm{T},\ E\subseteq H\}$. Let $\nu:\mathrm{T}_H\to[0,\infty]$ be a functional which is additive in the sense that $\nu\emptyset=0$ and $\nu(E\cup F)=\nu E+\nu F$ whenever $E$, $F\in\mathrm{T}_H$ are disjoint, and locally $G$-invariant in the sense that $g\bullet E\in\mathrm{T}$ and $\nu(g\bullet E)=\nu E$ whenever $E\in\mathrm{T}_H$, $g\in G$ and $g\bullet E\subseteq H$. Show that there is an extension of $\nu$ to a $G$-invariant additive functional $\tilde\nu:\mathrm{T}\to[0,\infty]$.

**(h)** Let $X$ be a set, $A$ a subset of $X$, and $\bullet$ an action of a group $G$ on $X$. Show that the following are equiveridical: (i) there is a functional $\theta:\mathcal{P}X\to[0,\infty]$ such that $\theta A=1$, $\theta(B\cup C)=\theta B+\theta C$ and $\theta(a\bullet B)=\theta B$ for all disjoint $B$, $C\subseteq X$ and $a\in G$; (ii) there are no $A_0,\dots,A_n$, $a_0,\dots,a_n$, $b_0,\dots,b_n$ such that $A_0,\dots,A_n$ are subsets of $G$ covering $A$, $a_0,\dots,b_n$ belong to $G$, and $a_0\bullet A_0$, $b_0\bullet A_0$, $a_1\bullet A_1$, $b_1\bullet A_1,\dots,b_n\bullet A_n$ are disjoint subsets of $A$.

**(i)** (Świerczkowski 58) Let $G$ be the group of orthogonal $3\times 3$ matrices. Set $S=\begin{pmatrix}\frac35&\frac45&0\\-\frac45&\frac35&0\\0&0&1\end{pmatrix}$ and $T=\begin{pmatrix}1&0&0\\0&\frac35&\frac45\\0&-\frac45&\frac35\end{pmatrix}$. Show that $S$ and $T$ are free in $G$ (that is, no non-trivial product of the form $S^{n_0}T^{n_1}S^{n_2}T^{n_3}\dots S^{n_{2k}}$ can be the identity), so that $G$ is not amenable in its discrete topology. (*Hint*: let $R$ be the ring of $3\times 3$ matrices over the field $\mathbb{Z}_5$. In $R$ set $\sigma=\begin{pmatrix}3&4&0\\1&3&0\\0&0&0\end{pmatrix}$, $\tau=\begin{pmatrix}0&0&0\\0&3&4\\0&1&3\end{pmatrix}$. Show that $\sigma^2=\sigma$. Now suppose $\rho\in R$ is defined as a non-trivial product of the elements $\sigma$, $\tau$ and their transposes $\sigma^\top$, $\tau^\top$ in which $\sigma$ and $\sigma^\top$ are never adjacent, $\tau$ and $\tau^\top$ are never adjacent, and the last term is $\sigma$. Set $\begin{pmatrix}a\\b\\c\end{pmatrix}=\rho\begin{pmatrix}1\\0\\0\end{pmatrix}$. Show that if the first term in the product is $\sigma$ or $\sigma^\top$, then $c=0$ and $b\ne0$, and otherwise $a=0$ and $b\ne0$.)

**(j)** Let $F_2$ be the free group on two generators $a$, $b$. (i) Show that there is a partition $(A,B,C,D)$ of $F_2$ such that $aA=A\cup B\cup C$ and $bB=A\cup B\cup D$. (ii) Let $S_2$ be the unit sphere in $\mathbb{R}^3$. Show that if $S$, $T$ are the matrices of 449Yi, there is a partition $(A,B,C,D,E)$ of $S^2$ such that $E$ is countable, $S[A]=A\cup B\cup C$ and $T[B]=A\cup B\cup D$. (This is a version of the **Hausdorff paradox**.) (iii) Show that there is no non-zero rotation-invariant additive functional from $\mathcal{P}S_2$ to $[0,1]$. (iv) Show that there is no rotation-invariant additive extension of Lebesgue measure to all subsets of the unit ball $B(\mathbf{0},1)$. (See Wagon 85.)

**449 Notes and comments** The general theory of amenable groups is outside the scope of this book. Here I have tried only to indicate some of the specifically measure-theoretic arguments which are used in the theory. Primarily we have the Riesz representation theorem, enabling us to move between linear functionals and measures. Since the invariant measures considered in the definition of 'amenable group' are all Radon measures on compact Hausdorff spaces, they can equally well be thought of as linear functionals on spaces of continuous functions. What is striking is that the definition in terms of continuous actions on arbitrary compact Hausdorff spaces can be reduced to a question concerning an invariant mean on a single space easily constructed from the group (449E).

The first part of this section deals with general topological groups. It is a remarkable fact that some of the most important non-locally-compact topological groups are amenable. For most of these we shall have to wait until we have done 'concentration of measure' (§§476, 492) and can approach 'extremely amenable' groups (§493). But there is an easy example in 449Xh which already indicates one of the basic methods.

For a much fuller account of the theory of amenable locally compact groups, see Paterson 88. Theorem 449J here is mostly taken from Greenleaf 69, where you will find many references to its development. Historically the subject was dominated by the case of discrete groups, in which combinatorial rather than measure-theoretic formulations seem more appropriate. In 449J, conditions (ii)-(viii) relate to invariant means of one kind or another, strengthening that of 449E. Note that the means of 449E(iii) and 449J(ii)-(viii) are normalized by conditions $p(\chi G) = \tilde{p}(\chi G^\bullet) = \phi G = 1$, while the left Haar measure $\mu$ of 449J has a degree of freedom; so that when they come together in 449J(viii) the two sides of the equation $\tilde{p}(g * f)^\bullet = \tilde{p}(f^\bullet) \int g\,d\mu$ must move together if we change $\mu$ by a scalar factor. Of course this happens through the hidden dependence of the convolution operation on $\mu$. (The convolutions in 449J(vii) do *not* involve $\mu$.) Between 449J(i) and 449J(viii) there is a double step. First we note that a convolution $g * f$ is a kind of weighted average of left translates of $f$, so that if we have a mean which is invariant under translations we can hope that it will be invariant under convolutions (449H, 449Yc). What is more remarkable is that an invariant mean on the space $U$ of bounded uniformly continuous functions should in the first place give rise to a left-invariant mean on the space $L^\infty$ of (equivalence classes of) bounded Haar measurable functions (449J(vi)), and then even a two-sided-invariant mean (449J(v)).

Condition (xii) in 449J looks at a different aspect of the phenomenon. In effect, it amounts to saying that not only is there an invariant mean, but that there is an invariant mean defined by the formula $p(f) = \lim_{L\to\mathcal{F}} \frac{1}{\mu L}\int_L f$ for a suitable filter $\mathcal{F}$ on the family of sets of non-zero finite measure. This may be called a 'Følner condition', following Følner 55. (449Xk looks for an invariant mean of the form $p(f) = \lim_{h\to\mathcal{F}} \int f \times h\,d\mu$, where $\mathcal{F}$ is a suitable filter on $\mathcal{L}^1(\mu)$.) In 449J(xi), the case $q = 1$ is just a weaker version of condition (x), but the case $q = 2$ tells us something new.

The techniques developed in §444 to handle Haar measures on groups which are not locally compact can also be used in 449H-449J, using 'totally bounded for the bilateral uniformity' in place of 'compact' when appropriate (449Yd). 443L provides another route to the same generalization.

In 449K, it is natural to ask whether the hypothesis 'locally compact' is necessary. It certainly cannot be dropped completely; there is an important amenable Polish group with a closed subgroup which is not amenable (493Xf).

The words 'right' and 'left' appear repeatedly in this section, and it is not perhaps immediately clear which of the ordinary symmetries we can expect to find. The fact that the operation $x \mapsto x^{-1} : G \to G$ always gives us an isomorphism between a group and the same set with the multiplication reversed (449Xa) means that we do not have to distinguish between 'left amenable' and 'right amenable' groups, at least if we start from the definition in 449A. In 449C also there is nothing to break the symmetry between left and right. In 449B and 449D-449E, however, we must commit ourselves to the *left* action of the group on the space of functions which are uniformly continuous with respect to the *right* uniformity. If we wish to change one, we must also expect to change the other. In the list of conditions in 449J, some can be reflected straightforwardly (see 449Xl), but in groups which are not unimodular there seem to be difficulties. For semigroups we do have a difference between 'left' and 'right' amenability (449Yb).

It is not surprising that in the search for invariant means we should repeatedly use averaging and limiting processes. The 'finitely additive integrals' $\oint d\phi$, $\oint d\nu$ in part (f) the proof of 449J and part (a) of the proof of 449N are an effective way of using one invariant additive functional $\phi$ or $\nu$ to build another. Similarly, because we are looking only for finite additivity, we can be optimistic about taking cluster points of families of almost-invariant functionals, as in the proofs of 449F, 449J and 449L.

In the case of discrete groups, in which all considerations of measurability and continuity evaporate, we have a completely different technique available, as in 449L. Here we can go directly from a non-paradoxicality condition, a weaker version of conditions already introduced in 395E and 448E, to a Følner condition (($\beta$) in part (b) of the proof of 449L) which easily implies amenability. I remind you that I still do not know how far these ideas can be applied to other algebras than $\mathcal{P}X$ (395Z). The difficulty is that the unscrupulous use of the axiom of choice in the infinitary Marriage Lemma seems to give us no control over the nature of the sets $A_{ij}$ described in (b-$\alpha$) of the proof of 449L; moreover, the structure of the proof depends on having a suitable invariant measure (counting measure on $X$) to begin with. For more on amenable discrete groups and their connexions with measure theory see Laczkovich 02.

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## Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

**441A Shift actions** The definition of shift actions in 441Ac of the 2003 and 2006 editions, called on in the 2008 edition of Volume 5, has been moved to 4A5Cc.

**444Xn Orthonormal bases** The sketch of a construction of an orthonormal basis in $L^2$ consisting of equivalence classes of continuous functions, referred to in Bogachev 07, is now 444Ym.

§**445 Convolutions** The material mentioned in the notes to §257 in the May 2001 edition of Volume 2 has been moved to §444. The particular result referred to as '445K' is now 444R.

**445Xq** The exercise 445Xq, referred to in the May 2002 edition of Volume 3, has been moved to 445Xp.

**449I, 449J Tarski's theorem** The proof of Tarski's theorem, referred to in the 2002 and 2004 editions of Volume 3, is now in 449L.

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