Version of 21.8.15

Chapter 43

Topologies and measures II

The first chapter of this volume was 'general' theory of topological measure spaces; I attempted to distinguish the most important properties a topological measure can have – inner regularity, τ-additivity – and describe their interactions at an abstract level. I now turn to rather more specialized investigations, looking for features which offer explanations of the behaviour of the most important spaces, radiating outwards from Lebesgue measure.

In effect, this chapter consists of three distinguishable parts and two appendices. The first three sections are based on ideas from descriptive set theory, in particular Souslin's operation (§431); the properties of this operation are the foundation for the theory of two classes of topological space of particular importance in measure theory, the K-analytic spaces (§432) and the analytic spaces (§433). The second part of the chapter, §§434-435, collects miscellaneous results on Borel and Baire measures, looking at the ways in which topological properties of a space determine properties of the measures it carries. In §436 I present the most important theorems on the representation of linear functionals by integrals; if you like, this is the inverse operation to the construction of integrals from measures in §122. The ideas continue into §437, where I discuss spaces of signed measures representing the duals of spaces of continuous functions, and topologies on spaces of measures. The first appendix, §438, looks at a special topic: the way in which the patterns in §§434-435 are affected if we assume that our spaces are not unreasonably complex in a rather special sense defined in terms of measures on discrete spaces. Finally, I end the chapter with a further collection of examples, mostly to exhibit boundaries to the theorems of the chapter, but also to show some of the variety of the structures we are dealing with.

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431 Souslin's operation

I begin the chapter with a short section on Souslin's operation. The basic facts we need to know are that (in a complete locally determined measure space) the family of measurable sets is closed under Souslin's operation (431A), and that the kernel of a Souslin scheme can be approximated from within in measure (431D). I write S^* for $\bigcup_{k\ge 1}\mathbb{N}^k$.

431A Theorem Let (X,Σ,μ) be a complete locally determined measure space. Then Σ is closed under Souslin's operation.

431B Corollary If $(X,\mathfrak{T},\Sigma,\mu)$ is a complete locally determined topological measure space, every Souslin-F set in X is measurable.

431C Corollary Let X be a set and θ an outer measure on X. Let μ be the measure defined by Carathéodory's method, and Σ its domain. Then Σ is closed under Souslin's operation.

431D Theorem Let (X,Σ,μ) be a complete locally determined measure space, and $\langle E_\sigma\rangle_{\sigma\in S^*}$ a Souslin scheme in Σ with kernel A.

(a)

$$\mu A = \sup\{\mu(\bigcup_{\phi\in K}\bigcap_{n\geq 1} E_{\phi\restriction n}) : K \subseteq \mathbb{N}^{\mathbb{N}} \text{ is compact}\}$$
$$= \sup\{\mu(\bigcup_{\phi\leq\psi}\bigcap_{n\geq 1} E_{\phi\restriction n}) : \psi \in \mathbb{N}^{\mathbb{N}}\},$$

writing $\phi \leq \psi$ if $\phi(i) \leq \psi(i)$ for every $i \in \mathbb{N}$.

(b) If $\langle E_\sigma\rangle_{\sigma\in S^*}$ is fully regular, then $\mu A = \sup\{\mu(\bigcap_{n\geq 1} E_{\psi\restriction n}) : \psi \in \mathbb{N}^{\mathbb{N}}\}$, and if in addition μ is totally finite, $\mu A = \sup\{\inf_{n\geq 1} \mu E_{\psi\restriction n} : \psi \in \mathbb{N}^{\mathbb{N}}\}$.

431E Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is a topological measure space and $E \subseteq X$ is a Souslin-F set with finite outer measure, then $\mu^* E = \sup\{\mu F : F \subseteq E \text{ is closed}\}$.

***431F Theorem** Let X be any topological space, and $\widehat{\mathcal{B}}$ its Baire-property algebra.

(a) For any $A \subseteq X$, there is a Baire-property envelope of A, that is, a set $E \in \widehat{\mathcal{B}}$ such that $A \subseteq E$ and $E \setminus F$ is meager whenever $A \subseteq F \in \widehat{\mathcal{B}}$.

(b) $\widehat{\mathcal{B}}$ is closed under Souslin's operation.

***431G Theorem** Let X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I} \subseteq \Sigma$ a σ-ideal of subsets of X. If $\Sigma/\mathcal{I}$ is ccc then Σ is closed under Souslin's operation.

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432 K-analytic spaces

I describe the basic measure-theoretic properties of K-analytic spaces. I start with 'elementary' results (432A-432C), assembling ideas from §§421, 422 and 431. The main theorem of the section is 432D, one of the leading cases of the general extension theorem 416P. An important corollary (432G) gives a sufficient condition for the existence of pull-back measures. I briefly mention 'capacities' (432J-432L).

432A Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined Hausdorff topological measure space. Then every K-analytic subset of X is measurable.

432B Theorem Let X be a K-analytic Hausdorff space, and μ a semi-finite topological measure on X. Then

$$\mu X = \sup\{\mu K : K \subseteq X \text{ is compact}\}.$$

432C Proposition Let X be a Hausdorff space such that all its open sets are K-analytic, and μ a Borel measure on X.

(a) If μ is semi-finite, it is tight.

(b) If μ is locally finite, its completion is a Radon measure on X.

432D Theorem Let X be a K-analytic Hausdorff space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Then μ has an extension to a Radon measure on X. In particular, μ is τ-additive.

432E Corollary Let X be a K-analytic Hausdorff space, and μ a locally finite quasi-Radon measure on X. Then μ is a Radon measure.

432F Corollary Let X be a K-analytic Hausdorff space, and ν a locally finite Baire measure on X. Then ν has an extension to a Radon measure on X; in particular, it is τ-additive. If the topology of X is regular, the extension is unique.

432G Corollary Let X be a K-analytic Hausdorff space, Y a Hausdorff space and ν a locally finite measure on Y which is inner regular with respect to the closed sets. Let $f : X \to Y$ be a continuous function such that $f[X]$ has full outer measure in Y. Then there is a Radon measure μ on X such that f is inverse-measure-preserving for μ and ν. If ν is Radon, it is precisely the image measure μf^{-1}.

432H Corollary Suppose that X is a set and that $\mathfrak{S}$, $\mathfrak{T}$ are Hausdorff topologies on X such that $(X, \mathfrak{T})$ is K-analytic and $\mathfrak{S} \subseteq \mathfrak{T}$. Then the totally finite Radon measures on X are the same for $\mathfrak{S}$ and $\mathfrak{T}$.

432I Corollary Let X be a K-analytic Hausdorff space, and $\mathcal{U}$ a subbase for the topology of X. Let (Y, T, ν) be a complete totally finite measure space and $\phi : Y \to X$ a function such that $\phi^{-1}[U] \in \mathrm{T}$ for every $U \in \mathcal{U}$. Then there is a Radon measure μ on X such that $\int f d\mu = \int f\phi\, d\nu$ for every bounded continuous $f : X \to \mathbb{R}$.

432J Capacitability: Definitions Let $(X, \mathfrak{T})$ be a topological space.

(a) A **Choquet capacity** on X is a function $c : \mathcal{P}X \to [0, \infty]$ such that
(i) $c(A) \le c(B)$ whenever $A \subseteq B \subseteq X$;
(ii) $\lim_{n\to\infty} c(A_n) = c(A)$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A;
(iii) $c(K) = \inf\{c(G) : G \supseteq K \text{ is open}\}$ for every compact set $K \subseteq X$.

(b) A Choquet capacity c on X is **outer regular** if $c(A) = \inf\{c(G) : G \supseteq A \text{ is open}\}$ for every $A \subseteq X$.

432K Theorem Let X be a Hausdorff space and c a Choquet capacity on X. If $A \subseteq X$ is K-analytic, then $c(A) = \sup\{c(K) : K \subseteq A \text{ is compact}\}$.

432L Proposition Let $(X, \mathfrak{T})$ be a topological space.
(a) Let $c_0 : \mathfrak{T} \to [0, \infty]$ be a functional such that
$c_0(G) \le c_0(H)$ whenever G, $H \in \mathfrak{T}$ and $G \subseteq H$;
c_0 is submodular
$c_0(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} c_0(G_n)$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$.
Then c_0 has a unique extension to an outer regular Choquet capacity c on X, and c is submodular.
(b) Suppose that X is regular. Let $\mathcal{K}$ be the family of compact subsets of X, and $c_1 : \mathcal{K} \to [0, \infty]$ a functional such that
c_1 is submodular;
$c_1(K) = \inf_{G\in\mathfrak{T}, G\supseteq K} \sup_{L\in\mathcal{K}, L\subseteq G} c_1(L)$ for every $K \in \mathcal{K}$.
Then c_1 has a unique extension to an outer regular Choquet capacity c on X such that

$$c(G) = \sup\{c(K) : K \subseteq G \text{ is compact}\} \text{ for every open } G \subseteq X,$$

and c is submodular.

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433 Analytic spaces

We come now to the special properties of measures on 'analytic' spaces. I start with a couple of facts about spaces with countable networks.

433A Proposition Let $(X, \mathfrak{T})$ be a topological space with a countable network, and μ a localizable topological measure on X which is inner regular with respect to the Borel sets. Then μ has countable Maharam type.

433B Lemma If $(X,\mathfrak{T})$ is a Hausdorff space with a countable network, then any topological measure on X is countably separated.

433C Theorem Let X be an analytic Hausdorff space, and μ a Borel measure on X.
(a) If μ is semi-finite, it is tight.
(b) If μ is locally finite, its completion is a Radon measure on X.

433D Theorem Let X and Y be analytic Hausdorff spaces, ν a totally finite Radon measure on Y and $f : X \to Y$ a Borel measurable function such that $f[X]$ has full outer measure for ν. Then there is a Radon measure μ on X such that $\nu = \mu f^{-1}$.

433E Proposition Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Let $(Y,\mathfrak{S})$ be an analytic Hausdorff space and $f : X \to Y$ a measurable function. Then f is almost continuous.

433F Proposition Let $(X,\mathfrak{T})$ and $(Y,\mathfrak{S})$ be analytic Hausdorff spaces, and $f : X \to Y$ a Borel measurable surjection. Let ν be a complete locally determined topological measure on Y, and T its domain. Then there is a T-measurable function $g : Y \to X$ such that gf is the identity on X.

433G Proposition Let $(X,\mathfrak{T})$ be an analytic Hausdorff space, (Y,T,ν) a complete locally determined measure space, and $f : X \to Y$ a surjection. Suppose that there is some countable family $\mathcal{F} \subseteq \mathrm{T}$ such that $\mathcal{F}$ separates the points of Y and $f^{-1}[F]$ is a Borel subset of X for every $F \in \mathcal{F}$. Then there is a T-measurable function $g : Y \to X$ such that fg is the identity on Y.

433H Proposition Let X be an analytic Hausdorff space, and (Y,T,ν) a complete locally determined measure space. Suppose that $W \subseteq X \times Y$ belongs to $\mathcal{S}(\mathcal{B}(X)\widehat{\otimes}\mathrm{T})$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Then $W[X] \in \mathrm{T}$ and there is a T-measurable function $g : W[X] \to X$ such that $(g(y),y) \in W$ for every $y \in W[X]$.

433I Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of analytic Hausdorff spaces, and for each $i \in I$ let μ_i be a Radon probability measure on X_i. Let λ be the ordinary product measure on $X = \prod_{i\in I} X_i$.
(a) If I is countable then λ is a Radon measure.
(b) If every μ_i is strictly positive, then λ is a quasi-Radon measure.

433J Proposition Let X be an analytic Hausdorff space, and T a countably generated σ-subalgebra of the Borel σ-algebra $\mathcal{B}(X)$ of X. Then any locally finite measure with domain T has an extension to a Radon measure on X.

433K Proposition Let (X,Σ) be a standard Borel space and T a countably generated σ-subalgebra of Σ. Then any σ-finite measure with domain T has an extension to Σ.

433L Proposition Let $\langle(X_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces such that (X_n,Σ_n) is a standard Borel space for every n. Suppose that for each $n \in \mathbb{N}$ we are given an inverse-measure-preserving function $f_n : X_{n+1} \to X_n$. Then we can find a standard Borel space (X,Σ), a probability measure μ with domain Σ, and inverse-measure-preserving functions $g_n : X \to X_n$ such that $f_n g_{n+1} = g_n$ for every n.

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434 Borel measures

What one might call the fundamental question of topological measure theory is the following.

What kinds of measures can arise on what kinds of topological space?

Of course this question has inexhaustible ramifications, corresponding to all imaginable properties of measures and topologies and connexions between them. The challenge I face here is that of identifying particular ideas as being more important than others, and the chief difficulty lies in the bewildering variety of topological properties which have been studied, any of which may have implications for the measure theory of the spaces involved. In this section and the next I give a sample of what is known, necessarily biased and incomplete. I try however to include the results which are most often applied and enough others for the proofs to contain, between them, most of the non-trivial arguments which have been found effective in this area.

In 434A I set out a crude classification of Borel measures on topological spaces. For compact Hausdorff spaces, at least, the first question is whether they carry Borel measures which are not, in effect, Radon measures; this leads us to the definition of 'Radon' space (434C) which is also of interest in the context of general Hausdorff spaces. I give a brief account of the properties of Radon spaces (434F, 434Nd). I look also at two special topics: 'quasi-dyadic' spaces (434O-434Q) and a construction of Borel product measures by integration of sections (434R).

In the study of Radon spaces we find ourselves looking at 'universally measurable' subsets of topological spaces (434D-434E). These are interesting in themselves, and also interact with constructions from earlier parts of this treatise (434S-434T). Three further classes of topological space, defined in terms of the types of topological measure which they carry, are the 'Borel-measure-compact', 'Borel-measure-complete' and 'pre-Radon' spaces; I discuss them briefly in 434G-434J. They provide useful methods for deciding whether Hausdorff spaces are Radon (434K).

434C Radon spaces: Definition A Hausdorff space X is **Radon** if every totally finite Borel measure on X is tight.

434D Universally measurable sets Let X be a topological space.

(a) I will say that a subset E of X is **universally measurable** (in X) if it is measured by the completion of every Borel probability measure on X.

(b) A subset of X is universally measurable iff it is measured by every complete locally determined topological measure on X.

(c) The family Σ_{um} of universally measurable subsets of X is a σ-algebra closed under Souslin's operation and including the Borel σ-algebra. In particular, Souslin-F sets are universally measurable, so (if X is Hausdorff) K-analytic and analytic sets are.

(d) Note that a function $f : X \to \mathbb{R}$ is Σ_{um}-measurable iff it is μ-virtually measurable for every totally finite Borel measure μ on X. Generally, if Y is another topological space, I will say that $f : X \to Y$ is **universally measurable** if $f^{-1}[H] \in \Sigma_{\mathrm{um}}$ for every open set $H \subseteq Y$. Continuous functions are universally measurable.

(e) In fact, if $f : X \to Y$ is universally measurable, then it is $(\Sigma_{\mathrm{um}}, \Sigma_{\mathrm{um}}^{(Y)})$-measurable, where $\Sigma_{\mathrm{um}}^{(Y)}$ is the algebra of universally measurable subsets of Y.

(f) It follows that if Z is a third topological space and $f : X \to Y$, $g : Y \to Z$ are universally measurable, then $gf : X \to Z$ is universally measurable.

434E Universally Radon-measurable sets Let X be a Hausdorff space.

(a) I will say that a subset E of X is **universally Radon-measurable** if it is measured by every Radon measure on X.

(b) The family Σ_{uRm} of universally Radon-measurable subsets of X is a σ-algebra closed under Souslin's operation and including the algebra of universally measurable subsets of X.

(c) If Y is another topological space, I will say that a function $f : X \to Y$ is **universally Radon-measurable** if $f^{-1}[H] \in \Sigma_{\mathrm{uRm}}$ for every open set $H \subseteq Y$. A function $f : X \to \mathbb{R}$ is universally Radon-measurable iff it is Σ_{uRm}-measurable iff it is μ-virtually measurable for every totally finite tight Borel measure μ on X. A universally measurable function is universally Radon-measurable.

434F Elementary properties of Radon spaces: Proposition Let X be a Hausdorff space.

(a) The following are equiveridical:

(i) X is a Radon space;

(ii) every semi-finite Borel measure on X is tight;

(iii) if μ is a locally finite Borel measure on X, its c.l.d. version $\tilde{\mu}$ is a Radon measure;

(iv) whenever μ is a totally finite Borel measure on X, and $G \subseteq X$ is an open set with $\mu G > 0$, then there is a compact set $K \subseteq G$ such that $\mu K > 0$;

(v) whenever μ is a non-zero totally finite Borel measure on X, there is a Radon subspace Y of X such that $\mu^* Y > 0$.

(b) If $Y \subseteq X$ is a subspace which is a Radon space in its induced topology, then Y is universally measurable in X.

(c) If X is a Radon space and $Y \subseteq X$, then Y is Radon iff it is universally measurable in X iff it is universally Radon-measurable in X. In particular, all Borel subsets and all Souslin-F subsets of X are Radon spaces.

(d) The family of Radon subspaces of X is closed under Souslin's operation and set difference.

434G Definitions (a) A topological space X is **Borel-measure-compact** if every totally finite Borel measure on X which is inner regular with respect to the closed sets is τ-additive.

(b) A topological space X is **Borel-measure-complete** if every totally finite Borel measure on X is τ-additive.

(c) A Hausdorff space X is **pre-Radon** if every τ-additive totally finite Borel measure on X is tight.

434H Proposition Let X be a topological space and $\mathcal{B}$ its Borel σ-algebra.

(a) The following are equiveridical:

(i) X is Borel-measure-compact;

(ii) every semi-finite Borel measure on X which is inner regular with respect to the closed sets is τ-additive;

(iii) every effectively locally finite Borel measure on X which is inner regular with respect to the closed sets has an extension to a quasi-Radon measure;

(iv) every totally finite Borel measure on X which is inner regular with respect to the closed sets has a support;

(v) if μ is a non-zero totally finite Borel measure on X, inner regular with respect to the closed sets, and $\mathcal{G}$ is an open cover of X, then there is some $G \in \mathcal{G}$ such that $\mu G > 0$.

(b) If X is Lindelöf, it is Borel-measure-compact.

(c) If X is Borel-measure-compact and $A \subseteq X$ is a Souslin-F set, then A is Borel-measure-compact in its subspace topology. In particular, any Baire subset of X is Borel-measure-compact.

434I Proposition Let X be a topological space.

(a) The following are equiveridical:

(i) X is Borel-measure-complete;

(ii) every semi-finite Borel measure on X is τ-additive;

(iii) every totally finite Borel measure on X has a support;

(iv) whenever μ is a totally finite Borel measure on X there is a base $\mathcal{U}$ for the topology of X such that $\mu(\bigcup\{U : U \in \mathcal{U},\, \mu U = 0\}) = 0$.

(b) If X is regular, it is Borel-measure-complete iff every effectively locally finite Borel measure on X has an extension to a quasi-Radon measure.

(c) If X is Borel-measure-complete, it is Borel-measure-compact.

(d) If X is Borel-measure-complete, so is every subspace of X.

(e) If X is hereditarily Lindelöf, it is Borel-measure-complete.

434J Proposition Let X be a Hausdorff space.

(a) The following are equiveridical:

(i) X is pre-Radon;

(ii) every effectively locally finite τ-additive Borel measure on X is tight;

(iii) whenever μ is a non-zero totally finite τ-additive Borel measure on X, there is a compact set $K \subseteq X$ such that $\mu K > 0$;

(iv) whenever μ is a totally finite τ-additive Borel measure on X, $\mu X = \sup_{K \subseteq X \text{ is compact}} \mu K$;

(v) whenever μ is a locally finite effectively locally finite τ-additive Borel measure on X, the c.l.d. version of μ is a Radon measure on X.

(b) If X is pre-Radon, then every locally finite quasi-Radon measure on X is a Radon measure.

(c) If X is regular and every totally finite quasi-Radon measure on X is a Radon measure, then X is pre-Radon.

(d) If X is pre-Radon, then any universally Radon-measurable subspace of X is pre-Radon.

(e) If $A \subseteq X$ is pre-Radon in its subspace topology, it is universally Radon-measurable in X.

(f) If X is K-analytic, it is pre-Radon.

(g) If X is completely regular and Čech-complete, it is pre-Radon.

(h) If $X = \prod_{i \in I} X_i$ where $\langle X_i \rangle_{i \in I}$ is a countable family of pre-Radon Hausdorff spaces, then X is pre-Radon.

(i) If every point of X belongs to a pre-Radon open subset of X, then X is pre-Radon.

434K Proposition (a) A Hausdorff space is Radon iff it is Borel-measure-complete and pre-Radon.

(b) An analytic Hausdorff space is Radon.

(c) ω_1 and $\omega_1 + 1$, with their order topologies, are not Radon.

(d) For a set I, $[0,1]^I$ is Radon iff $\{0,1\}^I$ is Radon iff I is countable.

(e) A hereditarily Lindelöf K-analytic Hausdorff space is Radon; in particular, the split interval is Radon.

434L Proposition If (X, ρ) is a metric space, then any quasi-Radon measure on X is inner regular with respect to the totally bounded subsets of X.

434M Lemma Let X be a countably compact topological space and $\mathcal{E}$ a non-empty family of closed subsets of X with the finite intersection property. Then there is a Borel probability measure μ on X, inner regular with respect to the closed sets, such that $\mu F = 1$ for every $F \in \mathcal{E}$.

434N Proposition (a) Let X be a Borel-measure-compact topological space. Then closed countably compact subsets of X are compact.

(b) Let X be a Borel-measure-complete topological space. Then countably compact subsets of X are compact.

(c) Let X be a Hausdorff Borel-measure-complete topological space. Then compact subsets of X are countably tight.

(d) In particular, any Radon compact Hausdorff space is countably tight.

434O Quasi-dyadic spaces: Definition A topological space X is **quasi-dyadic** if it is expressible as a continuous image of a product of separable metrizable spaces.

434P Proposition (a) A continuous image of a quasi-dyadic space is quasi-dyadic.

(b) Any product of quasi-dyadic spaces is quasi-dyadic.

(c) A space with a countable network is quasi-dyadic.

(d) A Baire subset of a quasi-dyadic space is quasi-dyadic.

(e) If X is any topological space, a countable union of quasi-dyadic subspaces of X is quasi-dyadic.

(f) A quasi-dyadic space is ccc.

434Q Theorem A semi-finite completion regular topological measure on a quasi-dyadic space is τ-additive.

434R Proposition Let X and Y be topological spaces with Borel measures μ and ν; write $\mathcal{B}(X)$, $\mathcal{B}(Y)$ for the Borel σ-algebras of X and Y respectively. If *either* X is first-countable *or* ν is τ-additive and effectively locally finite, there is a Borel measure λ_B on $X \times Y$ defined by the formula

$$\lambda_B W = \sup_{F \in \mathcal{B}(Y), \nu F < \infty} \int \nu(W[\{x\}] \cap F)\mu(dx)$$

for every Borel set $W \subseteq X \times Y$. Moreover

(i) if μ is semi-finite, then λ_B agrees with the c.l.d. product measure λ on $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, and the c.l.d. version $\tilde{\lambda}_B$ of λ_B extends λ;

(ii) if ν is σ-finite, then $\lambda_B W = \int \nu W[\{x\}]\mu(dx)$ for every Borel set $W \subseteq X \times Y$;

(iii) if both μ and ν are τ-additive and effectively locally finite, then λ_B is just the restriction of the τ-additive product measure $\tilde{\lambda}$ to the Borel σ-algebra of $X \times Y$; in particular, λ_B is τ-additive.

***434S Proposition** Let (X, Σ, μ) be a complete locally determined measure space, Y and Z topological spaces, $f : X \to Y$ a measurable function and $g : Y \to Z$ a universally measurable function. Then $gf : X \to Z$ is measurable. In particular, $f^{-1}[F] \in \Sigma$ for every universally measurable set $F \subseteq Y$.

***434T Proposition** Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Write Σ_{um} for the algebra of universally measurable subsets of $\mathbb{R}$.

(a) For any $u \in L^0 = L^0(\mathfrak{A})$, we have a sequentially order-continuous Boolean homomorphism $E \mapsto [\![u \in E]\!] : \Sigma_{\mathrm{um}} \to \mathfrak{A}$ defined by saying that

$$\begin{aligned} [\![u \in E]\!] &= \sup\{[\![u \in F]\!] : F \subseteq E \text{ is Borel}\} = \sup\{[\![u \in K]\!] : K \subseteq E \text{ is compact}\} \\ &= \inf\{[\![u \in F]\!] : F \supseteq E \text{ is Borel}\} = \inf\{[\![u \in G]\!] : G \supseteq E \text{ is open}\} \end{aligned}$$

for every $E \in \Sigma_{\mathrm{um}}$.

(b) For any $u \in L^0$ and universally measurable function $h : \mathbb{R} \to \mathbb{R}$ we have a corresponding element $\bar{h}(u)$ of L^0 defined by the formula

$$[\![\bar{h}(u) \in E]\!] = [\![u \in h^{-1}[E]]\!] \text{ for every } E \in \Sigma_{\mathrm{um}},\ u \in L^0.$$

434U Proposition Let X and Y be compact Hausdorff spaces and $f : X \to Y$ a continuous open map. If μ is a completion regular topological measure on X, then the image measure μf^{-1} on Y is completion regular.

434Z Problems (a) Must every Radon compact Hausdorff space be sequentially compact?

(b) Must a Hausdorff continuous image of a Radon compact Hausdorff space be Radon?

Version of 16.8.08

435 Baire measures

Imitating the programme of §434, I apply a similar analysis to Baire measures, starting with a simple-minded classification. This time the central section (435D-435H) is devoted to 'measure-compact' spaces, those on which all (totally finite) Baire measures are τ-additive.

435B Theorem Let X be a Hausdorff space and μ a locally finite Baire measure on X. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) for every non-negligible Baire set $E \subseteq X$ there is a compact set $K \subseteq E$ such that $\mu^* K > 0$.

If μ is totally finite, we can add

(iii) $\sup\{\mu^* K : K \subseteq X \text{ is compact}\} = \mu X$.

435C Theorem Let X be a normal countably paracompact space. Then any semi-finite Baire measure on X has an extension to a semi-finite Borel measure which is inner regular with respect to the closed sets.

435D Definition A completely regular topological space X is **measure-compact** if every totally finite Baire measure on X is τ-additive.

435E Lemma Let X be a completely regular topological space and ν a totally finite Baire measure on X. Suppose that $\sup_{G\in\mathcal{G}} \nu G = \nu X$ whenever $\mathcal{G}$ is an upwards-directed family of cozero sets with union X. Then ν is τ-additive.

435F Elementary facts (a) If X is a completely regular space which is not measure-compact, there are a Baire probability measure μ on X and a cover of X by μ-negligible cozero sets.

(b) Regular Lindelöf spaces are measure-compact.

(c) An open subset of a measure-compact space need not be measure-compact. A continuous image of a measure-compact space need not be measure-compact. $\mathbb{N}^{\mathfrak{c}}$ is not measure-compact. The product of two measure-compact spaces need not be measure-compact.

(d) If X is a measure-compact completely regular space it is Borel-measure-compact.

435G Proposition A Souslin-F subset of a measure-compact completely regular space is measure-compact.

435H Corollary A Baire subset of a measure-compact completely regular space is measure-compact.

Version of 9.5.11

436 Representation of linear functionals

I began this treatise with the three steps which make measure theory, as we know it, possible: a construction of Lebesgue measure, a definition of an integral from a measure, and a proof of the convergence theorems. I used what I am sure is the best route: Lebesgue measure from Lebesgue outer measure, and integrable functions from simple functions. But of course there are many other paths to the same ends, and some of them show us slightly different aspects of the subject. In this section I come – rather later than many authors would – to an account of a procedure for constructing measures from integrals.

I start with three fundamental theorems, the first and third being the most important. A positive linear functional on a truncated Riesz space of functions is an integral iff it is sequentially smooth (436D); a smooth linear functional corresponds to a quasi-Radon measure (436H); and if X is a compact Hausdorff space, any positive linear functional on $C(X)$ corresponds to a Radon measure (436J-436K).

436A Definition Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a positive linear functional. I say that f is **sequentially smooth** if whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$, then $\lim_{n\to\infty} f(u_n) = 0$.

If (X, Σ, μ) is a measure space and U is a Riesz subspace of the space of real-valued μ-integrable functions defined everywhere on X, then $\int d\mu : U \to \mathbb{R}$ is sequentially smooth.

436B Definition Let X be a set. I will say that a Riesz subspace U of $\mathbb{R}^X$ is **truncated** (or satisfies **Stone's condition**) if $u \wedge \chi X \in U$ for every $u \in U$.

In this case, $u \wedge \gamma\chi X \in U$ for every $\gamma \ge 0$ and $u \in U$.

436C Lemma Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Write $\mathcal{K}$ for the family of sets of the form $\{x : x \in X,\, u(x) \ge 1\}$ as u runs over U. Let $f : U \to \mathbb{R}$ be a sequentially smooth positive linear functional, and μ a measure on X such that μK is defined and equal to $\inf\{f(u) : \chi K \le u \in U\}$ for every $K \in \mathcal{K}$. Then $\int u\, d\mu$ exists and is equal to $f(u)$ for every $u \in U$.

436D Theorem Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $f : U \to \mathbb{R}$ be a positive linear functional. Then the following are equiveridical:

(i) f is sequentially smooth;

(ii) there is a measure μ on X such that $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$.

436E Proposition Let X be any topological space, and $C_b = C_b(X)$ the space of bounded continuous real-valued functions on X. Then there is a one-to-one correspondence between totally finite Baire measures μ on X and sequentially smooth positive linear functionals $f : C_b \to \mathbb{R}$, given by the formulae

$$f(u) = \int u\,d\mu \text{ for every } u \in C_b,$$

$$\mu Z = \inf\{f(u) : \chi Z \le u \in C_b\} \text{ for every zero set } Z \subseteq X.$$

436F Proposition Let X be a sequential space, Y a topological space, and μ, ν totally finite Baire measures on X, Y respectively. Then there is a Baire measure λ on $X \times Y$ such that

$$\lambda W = \int \nu W[\{x\}]\mu(dx), \quad \int f d\lambda = \iint f(x,y)\nu(dy)\mu(dx)$$

for every Baire set $W \subseteq X \times Y$ and every bounded continuous function $f : X \times Y \to \mathbb{R}$.

436G Definition Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a positive linear functional. I say that f is **smooth** if whenever A is a non-empty downwards-directed family in U such that $\inf_{u\in A} u(x) = 0$ for every $x \in X$, then $\inf_{u\in A} f(u) = 0$.

Of course a smooth functional is sequentially smooth. If $(X, \mathfrak{T}, \Sigma, \mu)$ is an effectively locally finite τ-additive topological measure space and U is a Riesz subspace of $\mathbb{R}^X$ consisting of integrable continuous functions, then $\int d\mu : U \to \mathbb{R}$ is smooth.

436H Theorem Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $f : U \to \mathbb{R}$ be a positive linear functional. Then the following are equiveridical:

(i) f is smooth;

(ii) there are a topology $\mathfrak{T}$ and a measure μ on X such that μ is a quasi-Radon measure with respect to $\mathfrak{T}$, $U \subseteq C(X)$ and $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$;

(iii) writing $\mathfrak{S}$ for the coarsest topology on X for which every member of U is continuous, there is a measure μ on X such that μ is a quasi-Radon measure with respect to $\mathfrak{S}$, and $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$.

Remark μ, as constructed here, is inner regular with respect to the family $\mathcal{K}$ of sets $K \subseteq X$ such that $\chi K = \inf A$ for some set $A \subseteq U$.

436I Lemma Let X be a topological space. Let $C_0 = C_0(X)$ be the space of continuous functions $u : X \to \mathbb{R}$ which 'vanish at infinity' in the sense that $\{x : |u(x)| \ge \epsilon\}$ is compact for every $\epsilon > 0$.

(a) C_0 is a norm-closed solid linear subspace of $C_b = C_b(X)$, so is a Banach lattice in its own right.

(b) $C_0^* = C_0^\sim$ is an L-space.

(c) If $A \subseteq C_0$ is a non-empty downwards-directed set such that $\inf_{u\in A} u(x) = 0$ for every $x \in X$, then $\inf_{u\in A} \|u\|_\infty = 0$.

436J Riesz Representation Theorem (first form) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, and $C_k = C_k(X)$ the space of continuous real-valued functions on X with compact support. If $f : C_k \to \mathbb{R}$ is any positive linear functional, there is a unique Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$.

436K Riesz Representation Theorem (second form) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space. If $f : C_0(X) \to \mathbb{R}$ is any positive linear functional, there is a unique totally finite Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_0 = C_0(X)$.

***436L Proposition** Let X be a topological space; write C_b for $C_b(X)$. Suppose that U is a norm-closed linear subspace of C_b^* such that the functional $u \mapsto f(u \times v) : C_b \to \mathbb{R}$ belongs to U whenever $f \in U$ and $v \in C_b$. Then U is a band in the L-space C_b^*.

***436M Corollary** Let $\mathfrak{A}$ be a Boolean algebra, and $M(\mathfrak{A})$ the L-space of bounded finitely additive functionals on $\mathfrak{A}$. Let $U \subseteq M(\mathfrak{A})$ be a norm-closed linear subspace such that $a \mapsto \nu(a \cap b)$ belongs to U whenever $\nu \in U$ and $b \in \mathfrak{A}$. Then U is a band in $M(\mathfrak{A})$.

Version of 5.11.12

437 Spaces of measures

Once we have started to take the correspondence between measures and integrals as something which operates in both directions, we can go a very long way. While 'measures', as dealt with in this treatise, are essentially positive, an 'integral' can be thought of as a member of a linear space, dual in some sense to a space of functions. Since the principal spaces of functions are Riesz spaces, we find ourselves looking at dual Riesz spaces as discussed in §356; while the corresponding spaces of measures are close to those of §362. Here I try to draw these ideas together with an examination of spaces $U_\sigma^\sim$ and $U_\tau^\sim$ of sequentially smooth and smooth functionals, and the matching spaces M_σ and M_τ of countably additive and τ-additive measures (437A-437I). Because a (sequentially) smooth functional corresponds to a countably additive measure, which can be expected to integrate many more functions than those in the original Riesz space (typically, a space of continuous functions), we find that relatively large spaces of bounded measurable functions can be canonically embedded into the biduals $(U_\sigma^\sim)^*$ and $(U_\tau^\sim)^*$ (437C, 437H, 437I).

The guiding principles of functional analysis encourage us not only to form linear spaces, but also to examine linear space topologies, starting with norm and weak topologies. The theory of Banach lattices described in §354, particularly the theory of M- and L-spaces, is an important part of the structure here. In addition, our spaces $U_\sigma^\sim$ have natural weak* topologies which can be regarded as topologies on spaces of measures; these are the 'vague' topologies of 437J, which have already been considered, in a special case, in §285.

It turns out that on the positive cone of M_τ, at least, the vague topology may have an alternative description directly in terms of the behaviour of the measures on open sets (437L). This leads us to a parallel idea, the 'narrow' topology on non-negative additive functionals (437Jd). The second half of the section is devoted to the elementary properties of narrow topologies (437K-437N), with especial reference to compact sets in these topologies (437P, 437Rf, 437T). Seeking to identify narrowly compact sets, we come to the concept of 'uniform tightness' (437O). Bounded uniformly tight sets are narrowly relatively compact (437P); in 'Prokhorov spaces' (437U) the converse is true. I end the section with a list of the best-known Prokhorov spaces (437V).

437A Smooth and sequentially smooth duals Let X be a set, and U a Riesz subspace of $\mathbb{R}^X$.

(a) Set $U_\sigma^\sim = \{f : f \in U^\sim, |f| \text{ is sequentially smooth}\}$, the **sequentially smooth dual** of U. Then $U_\sigma^\sim$ is a band in $U^\sim$.
$U_c^\sim \subseteq U_\sigma^\sim$.

(b) Set $U_\tau^\sim = \{f : f \in U^\sim, |f| \text{ is smooth}\}$, the **smooth dual** of U. Then $U_\tau^\sim$ is a band in $U^\sim$.
$U^\times \subseteq U_\tau^\sim$.

437B Signed measures Recall that if X is a set and Σ is a σ-algebra of subsets of X, we can identify $L^\infty = L^\infty(\Sigma)$ with the space $\mathcal{L}^\infty = \mathcal{L}^\infty(\Sigma)$ of bounded Σ-measurable real-valued functions. $(\mathcal{L}^\infty)_\sigma^\sim = (\mathcal{L}^\infty)_c^\sim$. Next, we can identify $(L^\infty)_c^\sim$ with the space M_σ of countably additive functionals on Σ; if $\nu \in M_\sigma$, the corresponding member of $(L^\infty)_c^\sim$ is the unique order-bounded linear functional f on L^∞ such that $f(\chi E) = \nu E$ for every $E \in \Sigma$.

The identification between $(L^\infty)_c^\sim$ and M_σ is an L-space isomorphism. So

$$\int u\,d(\mu+\nu)=\int u\,d\mu+\int u\,d\nu$$

for every $u\in\mathcal{L}^\infty$ and all μ, $\nu\in M_\sigma$.

437C Theorem Let X be a set and U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions.

(a) Let Σ be the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Let $M_\sigma=M_\sigma(\Sigma)$ be the L-space of countably additive functionals on Σ. Then there is a Banach lattice isomorphism $T:M_\sigma\to U_\sigma^\sim$ defined by saying that $(T\mu)(u)=\int u\,d\mu$ whenever $\mu\in M_\sigma^+$ and $u\in U$.

(b) We have a sequentially order-continuous norm-preserving Riesz homomorphism S, embedding the M-space $\mathcal{L}^\infty=\mathcal{L}^\infty(\Sigma)$ of bounded real-valued Σ-measurable functions on X into the M-space $(U_\sigma^\sim)^\sim=(U_\sigma^\sim)^*=(U_\sigma^\sim)^\times$, defined by saying that $(Sv)(T\mu)=\int v\,d\mu$ whenever $v\in\mathcal{L}^\infty$ and $\mu\in M_\sigma^+$. If $u\in U$, then $(Su)(f)=f(u)$ for every $f\in U_\sigma^\sim$.

437E Corollary Let X be a completely regular Hausdorff space, and $\mathcal{B}\mathfrak{a}$ its Baire σ-algebra. Then we can identify $C_b(X)_\sigma^\sim$ with the L-space $M_\sigma(\mathcal{B}\mathfrak{a})$ of countably additive functionals on $\mathcal{B}\mathfrak{a}$, and we have a norm-preserving sequentially order-continuous Riesz homomorphism S from $\mathcal{L}^\infty(\mathcal{B}\mathfrak{a})$ to $(C_b(X)_\sigma^\sim)^*$ defined by setting $(Sv)(f)=\int v\,d\mu_f$ for every $v\in\mathcal{L}^\infty$ and $f\in(C_b(X)_\sigma^\sim)^+$, where μ_f is the Baire measure associated with f.

437F Proposition Let X be a topological space and $\mathcal{B}$ its Borel σ-algebra. Let M_σ be the L-space of countably additive functionals on $\mathcal{B}$.

(a) Write $M_\tau\subseteq M_\sigma$ for the set of differences of τ-additive totally finite Borel measures. Then M_τ is a band in M_σ, so is an L-space in its own right.

(b) Write $M_t\subseteq M_\tau$ for the set of differences of totally finite Borel measures which are tight. Then M_t is a band in M_σ, so is an L-space in its own right.

437G Definitions Let X be a topological space. A **signed Baire measure** on X will be a countably additive functional on the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$; a **signed Borel measure** will be a countably additive functional on the Borel σ-algebra $\mathcal{B}(X)$; a **signed τ-additive Borel measure** will be the difference of two τ-additive totally finite Borel measures; and a **signed tight Borel measure** will be the difference of two tight totally finite Borel measures.

437H Theorem Let X be a set and U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions. Let $\mathfrak{T}$ be the coarsest topology on X rendering every member of U continuous, and $\mathcal{B}$ the corresponding Borel σ-algebra.

(a) Let M_τ be the L-space of signed τ-additive Borel measures on X. Then we have a Banach lattice isomorphism $T:M_\tau\to U_\tau^\sim$ defined by saying that $(T\mu)(u)=\int u\,d\mu$ whenever $\mu\in M_\tau^+$ and $u\in U$.

(b) We have a sequentially order-continuous norm-preserving Riesz homomorphism S, embedding $\mathcal{L}^\infty(\mathcal{B})$ into $(U_\tau^\sim)^\sim=(U_\tau^\sim)^*=(U_\tau^\sim)^\times$, defined by saying that $(Sv)(T\mu)=\int v\,d\mu$ whenever $v\in\mathcal{L}^\infty$ and $\mu\in M_\tau^+$. If $u\in U$, then $(Su)(f)=f(u)$ for every $f\in U_\tau^\sim$.

437I Proposition Let X be a locally compact Hausdorff space, $\mathcal{B}$ its Borel σ-algebra, and $\mathcal{L}^\infty(\mathcal{B})$ the M-space of bounded Borel measurable real-valued functions on X.

(a) Let M_t be the L-space of signed tight Borel measures on X. Then we have a Banach lattice isomorphism $T:M_t\to C_0(X)^*$ defined by saying that $(T\mu)(u)=\int u\,d\mu$ whenever $\mu\in M_t^+$ and $u\in C_0(X)$.

(b) Let Σ_{uRm} be the algebra of universally Radon-measurable subsets of X, and $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ the M-space of bounded Σ_{uRm}-measurable real-valued functions on X. Then we have a norm-preserving sequentially order-continuous Riesz homomorphism $S:\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})\to C_0(X)^{**}$ defined by saying that $(Sv)(T\mu)=\int v\,d\mu$ whenever $v\in\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ and $\mu\in M_t^+$; and $(Su)(f)=f(u)$ for every $u\in C_0(X)$, $f\in C_0(X)^*$.

437J Vague and narrow topologies Let X be a topological space.

(a) Let Σ be an algebra of subsets of X. I will say that Σ **separates zero sets** if whenever F, $F'\subseteq X$ are disjoint zero sets then there is an $E\in\Sigma$ such that $F\subseteq E$ and $E\cap F'=\emptyset$.

(b) If Σ is any algebra of subsets of X, we can identify the Banach algebra and Banach lattice $L^\infty(\Sigma)$ with the $\| \|_\infty$-closed linear subspace of $\ell^\infty(X)$ generated by $\{\chi E : E \in \Sigma\}$. If we do this, then $C_b(X) \subseteq L^\infty(\Sigma)$ iff Σ separates zero sets.

(c) It follows that if Σ is an algebra of subsets of X separating the zero sets, and $\nu : \Sigma \to \mathbb{R}$ is a bounded additive functional, we can speak of $⨍ u\,d\nu$ for any $u \in C_b(X)$. The map $\nu \mapsto ⨍ d\nu$ is a Banach lattice isomorphism from the L-space $M(\Sigma)$ of bounded additive functionals on Σ to $L^\infty(\Sigma)^* = L^\infty(\Sigma)^\sim$. We therefore have a positive linear operator $T : M(\Sigma) \to C_b(X)^*$ defined by setting $(T\nu)(u) = ⨍ u\,d\nu$ for every $\nu \in M(\Sigma)$ and $u \in C_b(X)$. Except in the trivial case $X = \emptyset$, $\|T\| = 1$.

The **vague topology** on $M(\Sigma)$ is the topology generated by the functionals $\nu \mapsto ⨍ u\,d\nu$ as u runs over $C_b(X)$; that is, the coarsest topology on $M(\Sigma)$ such that the canonical map $T : M(\Sigma) \to C_b(X)^*$ is continuous for the weak* topology of $C_b(X)^*$. Because the functionals $\nu \mapsto |⨍ u\,d\nu|$ are seminorms on $M(\Sigma)$, the vague topology is a locally convex linear space topology.

(d) Let $\tilde M^+$ be the set of all non-negative real-valued additive functionals defined on algebras of subsets of X which contain every open set. The **narrow topology** on $\tilde M^+$ is that generated by sets of the form

$$\{\nu : \nu \in \tilde M^+,\, \nu G > \alpha\}, \quad \{\nu : \nu \in \tilde M^+,\, \nu X < \alpha\}$$

for open sets $G \subseteq X$ and real numbers α.

Observe that $\nu \mapsto \nu X : \tilde M^+ \to [0,\infty[$ is continuous for the narrow topology, and if $G \subseteq X$ is open then $\nu \mapsto \nu G$ is lower semi-continuous for the narrow topology. Writing P_{top} for the set of topological probability measures on X, then the narrow topology on P_{top} is generated by sets of the form $\{\mu : \mu \in P_{\mathrm{top}},\, \mu G > \alpha\}$ for real numbers α and open sets $G \subseteq X$. Writing δ_x for the Dirac measure on X concentrated at x, $x \mapsto \delta_x : X \to P_{\mathrm{top}}$ is a homeomorphism between X and $\{\delta_x : x \in X\}$.

Writing $\tilde M^+_\sigma$ for the set of totally finite topological measures on X, then $\nu \mapsto \nu E : \tilde M^+_\sigma \to [0,\infty[$ is Borel measurable, for the narrow topology on $\tilde M^+_\sigma$, for every Borel set $E \subseteq X$. Similarly, $\nu \mapsto \int u\,d\nu : \tilde M^+_\sigma \to \mathbb{R}$ is Borel measurable for every bounded Borel measurable function $u : X \to \mathbb{R}$.

(e) Vague topologies, being linear space topologies, are necessarily completely regular. In the very general context of (c) here, we do not expect the vague topology to be Hausdorff.

Similarly, the narrow topology on $\tilde M^+$ is rarely Hausdorff. But on important subspaces we can get Hausdorff topologies. In particular, if X is Hausdorff, then the narrow topology on the space M^+_{R} of totally finite Radon measures on X is Hausdorff.

(f) It will be useful to know that if $u : X \to \mathbb{R}$ is bounded and lower semi-continuous, then $\nu \mapsto ⨍ u\,d\nu : \tilde M^+ \to \mathbb{R}$ is lower semi-continuous for the narrow topology.

Of course it follows at once that if $u : X \to \mathbb{R}$ is bounded and continuous, then $\nu \mapsto ⨍ u\,d\nu$ is continuous for the narrow topology; that is, the vague topology is coarser than the narrow topology in contexts in which both make sense.

(g) If $u : X \to [0,\infty]$ is a lower semi-continuous function, then $\nu \mapsto \int u\,d\nu : \tilde M^+_\sigma \to [0,\infty]$ is lower semi-continuous for the narrow topology.

(h) Let X and Y be topological spaces, $\phi : X \to Y$ a continuous function, and $\tilde M^+(X)$, $\tilde M^+(Y)$ the spaces of functionals described in (d). For a functional ν defined on a subset of $\mathcal{P}X$, define $\nu\phi^{-1}$ by saying that $(\nu\phi^{-1})(F) = \nu(\phi^{-1}[F])$ whenever $F \subseteq Y$ and $\phi^{-1}[F] \in \operatorname{dom}\nu$. Then $\nu\phi^{-1} \in \tilde M^+(Y)$ whenever $\nu \in \tilde M^+(X)$, and the map $\nu \mapsto \nu\phi^{-1} : \tilde M^+(X) \to \tilde M^+(Y)$ is continuous for the narrow topologies.

(i) I am trying to maintain the formal distinctions between 'quasi-Radon measure' and 'τ-additive effectively locally finite Borel measure inner regular with respect to the closed sets', and between 'Radon measure' and 'tight locally finite Borel measure'. If we take M^+_{qR} to be the set of totally finite quasi-Radon measures on X, and X is completely regular, we have a canonical embedding of M^+_{qR} into a cone in the L-space $C_b(X)^*$; even if X is not completely regular, the map $\mu \mapsto \mu\restriction\mathcal{B}(X) : M^+_{\mathrm{qR}} \to M_\sigma(\mathcal{B}(X))$ is still injective, and we can identify M^+_{qR} with a cone in the L-space M_τ of signed τ-additive Borel measures. Similarly, when X is Hausdorff, we can identify totally finite Radon measures with tight totally finite Borel measures. The definition in 437Jd makes it plain that these identifications are homeomorphisms for the narrow topology,

437K Proposition Let X be a topological space, and $\tilde{M}^+$ the set of all non-negative real-valued additive functionals defined on algebras of subsets of X containing every open set.

(a) We have a function $T : \tilde{M}^+ \to C_b(X)^*$ defined by the formula $(T\nu)(u) = \oint u\,d\nu$ whenever $\nu \in \tilde{M}^+$ and $u \in C_b(X)$.

(b) T is continuous for the narrow topology $\mathfrak{S}$ on $\tilde{M}^+$ and the weak* topology on $C_b(X)^*$.

(c) Suppose now that X is completely regular, and that $W \subseteq \tilde{M}^+$ is a family of τ-additive totally finite topological measures such that two members of W which agree on the Borel σ-algebra are equal. Then $T\restriction W$ is a homeomorphism between W, with the narrow topology, and $T[W]$, with the weak* topology.

437L Corollary Let X be a completely regular topological space, and M_τ the space of signed τ-additive Borel measures on X. Then the narrow and vague topologies on M_τ^+ coincide. In particular, the narrow topology on M_τ^+ is completely regular.

437M Theorem For a topological space X, write $M_{\mathrm{qR}}^+(X)$ for the space of totally finite quasi-Radon measures on X, $P_{\mathrm{qR}}(X)$ for the space of quasi-Radon probability measures on X, and $M_\tau(X)$ for the L-space of signed τ-additive Borel measures on X.

(a) Let X and Y be topological spaces. If $\mu \in M_{\mathrm{qR}}^+(X)$ and $\nu \in M_{\mathrm{qR}}^+(Y)$, write $\mu \times \nu$ for their τ-additive product measure on $X \times Y$. Then $(\mu, \nu) \mapsto \mu \times \nu$ is continuous for the narrow topologies on $M_{\mathrm{qR}}^+(X)$, $M_{\mathrm{qR}}^+(Y)$ and $M_{\mathrm{qR}}^+(X \times Y)$.

(b) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces, with product X. If $\langle \mu_i \rangle_{i \in I}$ is a family of probability measures such that $\mu_i \in P_{\mathrm{qR}}(X_i)$ for each i, write $\prod_{i \in I} \mu_i$ for its τ-additive product on X. Then $\langle \mu_i \rangle_{i \in I} \mapsto \prod_{i \in I} \mu_i$ is continuous for the narrow topology on $P_{\mathrm{qR}}(X)$ and the product of the narrow topologies on $\prod_{i \in I} P_{\mathrm{qR}}(X_i)$.

(c) Let X and Y be topological spaces.

(i) We have a unique bilinear operator $\psi : M_\tau(X) \times M_\tau(Y) \to M_\tau(X \times Y)$ such that $\psi(\mu, \nu)$ is the restriction of the τ-additive product of μ and ν to the Borel σ-algebra of $X \times Y$ whenever μ, ν are totally finite Borel measures on X, Y respectively.

(ii) $\|\psi\| \le 1$.

(iii) ψ is separately continuous for the vague topologies on $M_\tau(X)$, $M_\tau(Y)$ and $M_\tau(X \times Y)$.

(d) In (c), suppose that X and Y are compact and Hausdorff. If $B \subseteq M_\tau(X)$ and $B' \subseteq M_\tau(Y)$ are norm-bounded, then $\psi\restriction B \times B'$ is continuous for the vague topologies.

437N Proposition (a) Let X and Y be Hausdorff spaces, and $\phi : X \to Y$ a continuous function. Let $M_{\mathrm{R}}^+(X)$, $M_{\mathrm{R}}^+(Y)$ be the spaces of totally finite Radon measures on X and Y respectively. Write $\tilde{\phi}(\mu)$ for the image measure $\mu\phi^{-1}$ for $\mu \in M_{\mathrm{R}}^+(X)$.

(i) $\tilde{\phi} : M_{\mathrm{R}}^+(X) \to M_{\mathrm{R}}^+(Y)$ is continuous for the narrow topologies on $M_{\mathrm{R}}^+(X)$ and $M_{\mathrm{R}}^+(Y)$.

(ii) $\tilde{\phi}(\mu + \nu) = \tilde{\phi}(\mu) + \tilde{\phi}(\nu)$ and $\tilde{\phi}(\alpha\mu) = \alpha\tilde{\phi}(\mu)$ for all μ, $\nu \in M_{\mathrm{R}}^+(X)$ and $\alpha \ge 0$.

(b) If Y is a Hausdorff space, X a subset of Y, and $\phi : X \to Y$ the identity map, then $\tilde{\phi}$ is a homeomorphism between $M_{\mathrm{R}}^+(X)$ and $\{\nu : \nu \in M_{\mathrm{R}}^+(Y), \nu(Y \setminus X) = 0\}$.

437O Uniform tightness Let X be a topological space. If ν is a bounded additive functional on an algebra of subsets of X, it is **tight** if

$$\nu E \in \overline{\{\nu K : K \subseteq E,\ K \in \operatorname{dom}\nu,\ K \text{ is closed and compact}\}}$$

for every $E \in \operatorname{dom}\nu$, and that a set A of tight functionals is **uniformly tight** if every member of A is tight and for every $\epsilon > 0$ there is a closed compact set $K \subseteq X$ such that νK is defined and $|\nu E| \le \epsilon$ whenever $\nu \in A$ and $E \in \operatorname{dom}\nu$ is disjoint from K.

437P Proposition Let X be a topological space.

(a) Let M_{qR}^+ be the set of totally finite quasi-Radon measures on X. Suppose that $A \subseteq M_{\mathrm{qR}}^+$ is uniformly totally finite and for every $\epsilon > 0$ there is a closed compact $K \subseteq X$ such that $\mu(X \setminus K) \le \epsilon$ for every $\mu \in A$. Then A is relatively compact in M_{qR}^+ for the narrow topology.

(b) Suppose now that X is Hausdorff, and that M_{R}^+ is the set of Radon measures on X. If $A \subseteq M_{\mathrm{R}}^+$ is uniformly totally finite and uniformly tight, then it is relatively compact in M_{R}^+ for the narrow topology.

437Q Two metrics (a)(i) If X is a set and μ, ν are bounded additive functionals defined on algebras of subsets of X, then $\mu - \nu : \operatorname{dom}\mu \cap \operatorname{dom}\nu \to \mathbb{R}$ is bounded and additive, and we can set

$$\rho_{\mathrm{tv}}(\mu,\nu) = |\mu-\nu|(X) = \sup_{E,F\in\operatorname{dom}\mu\cap\operatorname{dom}\nu}(\mu-\nu)(E) - (\mu-\nu)(F).$$

In this generality, ρ_{tv} is not even a pseudometric, but if we have a class M of totally finite measures on X all of which are inner regular with respect to a subset $\mathcal{K}$ of $\bigcap_{\mu\in M}\operatorname{dom}\mu$, then we have

$$\rho_{\mathrm{tv}}(\mu,\nu) = \sup_{K,L\in\mathcal{K}}(\mu K - \mu L) - (\nu K - \nu L)$$

for all μ, $\nu \in M$, and $\rho_{\mathrm{tv}}\restriction M \times M$ is a pseudometric on M. If moreover M is such that distinct members of M differ on $\mathcal{K}$ (as when $\mathcal{K}$ is the family of closed sets in a topological space X and $M = M_{\mathrm{qR}}^+(X)$, or when $\mathcal{K}$ the family of compact sets in a Hausdorff space X and $M = M_{\mathrm{R}}^+(X)$), then ρ_{tv} gives us a metric on M. In such a case I will call $\rho_{\mathrm{tv}}\restriction M \times M$ the **total variation metric** on M.

(ii) Note that if $\Sigma \subseteq \operatorname{dom}\mu \cap \operatorname{dom}\nu$ is a σ-algebra then

$$|\int u\,d\mu - \int u\,d\nu| \le \|u\|_\infty \rho_{\mathrm{tv}}(\mu,\nu)$$

whenever $u \in \mathcal{L}^\infty(\Sigma)$. So if, for instance, X is a topological space and $M \subseteq M_{\mathrm{qR}}^+(X)$, then $u \mapsto \int u\,d\mu$ will be continuous for the total variation metric on M whenever $u : X \to \mathbb{R}$ is a bounded universally measurable function.

(iii) When our set M can be identified with the positive cone of a band in some L-space M_σ of countably additive functions, then we have a complete metric. In particular, for any Hausdorff space X, $M_{\mathrm{R}}^+(X)$ can be identified with the positive cone of the L-space of tight Borel measures on X, so is complete.

(b) Suppose that (X,ρ) is a metric space. Write M_{qR}^+ for the set of totally finite quasi-Radon measures on X. For μ, $\nu \in M_{\mathrm{qR}}^+$ set

$$\rho_{\mathrm{KR}}(\mu,\nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to [-1,1] \text{ is 1-Lipschitz}\}.$$

Then ρ_{KR} is a metric on M_{qR}^+.

437R Theorem Let X be a topological space; write $M_{\mathrm{qR}}^+ = M_{\mathrm{qR}}^+(X)$ for the set of totally finite quasi-Radon measures on X, and if X is Hausdorff write $M_{\mathrm{R}}^+ = M_{\mathrm{R}}^+(X)$ for the set of totally finite Radon measures on X, both endowed with their narrow topologies.

(a)(i) If X is regular then M_{qR}^+ is Hausdorff.

(ii) If X is Hausdorff then M_{R}^+ is Hausdorff.

(b) If X has a countable network then M_{qR}^+ has a countable network.

(c) Suppose that X is separable.

(i) If X is a T_1 space, then M_{qR}^+ is separable.

(ii) If X is Hausdorff, M_{R}^+ is separable.

(d) If X is a K-analytic Hausdorff space, so is $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$.

(e) If X is an analytic Hausdorff space, so is $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$.

(f)(i) If X is compact, then for any real $\gamma \ge 0$ the sets $\{\mu : \mu \in M_{\mathrm{qR}}^+, \mu X \le \gamma\}$ and $\{\mu : \mu \in M_{\mathrm{qR}}^+, \mu X = \gamma\}$ are compact.

(ii) If X is compact and Hausdorff, then for any real $\gamma \ge 0$ the sets $\{\mu : \mu \in M_{\mathrm{R}}^+, \mu X \le \gamma\}$ and $\{\mu : \mu \in M_{\mathrm{R}}^+, \mu X = \gamma\}$ are compact. In particular, the set P_{R} of Radon probability measures on X is compact.

(g) Suppose that X is metrizable and ρ is a metric on X inducing its topology.

(i) The metric ρ_{KR} on M^+_{qR} induces the narrow topology on M^+_{qR}.

(ii) If (X,ρ) is complete then $M^+_{\mathrm{qR}}=M^+_{\mathrm{R}}$ is complete under ρ_{KR}.

(h) If X is Polish, so is $M^+_{\mathrm{qR}}=M^+_{\mathrm{R}}$.

437S Proposition Let X be a Hausdorff space, and P_{R} the set of Radon probability measures on X. Then the extreme points of P_{R} are just the Dirac measures on X.

437T Theorem Let X be a non-empty compact Hausdorff space, and $\phi:X\to X$ a continuous function. Write Q_ϕ for the set of Radon probability measures on X for which ϕ is inverse-measure-preserving. Then Q_ϕ is convex and not empty, and is compact for the narrow topology.

437U Definition Let X be a Hausdorff space and $P_{\mathrm{R}}(X)$ the set of Radon probability measures on X. X is a **Prokhorov space** if every subset of $P_{\mathrm{R}}(X)$ which is compact for the narrow topology is uniformly tight.

437V Theorem (a) Compact Hausdorff spaces are Prokhorov spaces.

(b) A closed subspace of a Prokhorov Hausdorff space is a Prokhorov space.

(c) An open subspace of a Prokhorov Hausdorff space is a Prokhorov space.

(d) The product of a countable family of Prokhorov Hausdorff spaces is a Prokhorov space.

(e) Any G_δ subset of a Prokhorov Hausdorff space is a Prokhorov space.

(f) Čech-complete spaces are Prokhorov spaces.

(g) Polish spaces are Prokhorov spaces.

Version of 13.12.06/10.10.07

438 Measure-free cardinals

At several points in §418, and again in §434, we had theorems about separable metrizable spaces in which the proofs undoubtedly needed some special property of these spaces (e.g., the fact that they are Lindelöf), but left it unclear whether something more general could be said. When we come to investigate further, asking (for instance) whether complete metric spaces in general are Radon (438H), we find ourselves once again approaching the Banach-Ulam problem, already mentioned at several points in previous volumes. It seems to be undecidable, in ordinary set theory with the axiom of choice, whether or not every discrete space is Radon in the sense of 434C. On the other hand it is known that discrete spaces with cardinal at most ω_{ω_1} (for instance) are indeed always Radon. While as a rule I am deferring questions of this type to Volume 5, this particular phenomenon is so pervasive that I think it is worth taking a section now to clarify it.

The central definition is that of 'measure-free cardinal' (438A), and the basic results are 438B-438D. In particular, 'small' infinite cardinals are measure-free (438C). From the point of view of measure theory, a metrizable space whose weight is measure-free is almost separable, and most of the results in §418 concerning separable metrizable spaces can be extended (438E-438G). In fact 'measure-free weight' exactly determines whether a metrizable space is measure-compact (438J) and whether a complete metric space is Radon (438H). If $\mathfrak{c}$ is measure-free, some interesting spaces of functions are Radon (438T). I approach these last spaces through the concept of 'hereditary weak θ-refinability' (438K), which enables us to do most of the work without invoking any special axiom.

438A Measure-free cardinals: Definition A cardinal κ is **measure-free** or **of measure zero** if whenever μ is a probability measure with domain $\mathcal{P}\kappa$ then there is a $\xi<\kappa$ such that $\mu\{\xi\}>0$.

438B Lemma Let (X,Σ,μ) be a semi-finite measure space and $\langle E_i\rangle_{i\in I}$ a point-finite family of subsets of X such that $\#(I)$ is measure-free and $\bigcup_{i\in J}E_i\in\Sigma$ for every $J\subseteq I$. Set $E=\bigcup_{i\in I}E_i$.

(a) $\mu E=\sup_{J\subseteq I\text{ is finite}}\mu(\bigcup_{i\in J}E_i)$.

(b) If $\langle E_i\rangle_{i\in I}$ is disjoint, then $\mu E = \sum_{i\in I}\mu E_i$. In particular, if $\Sigma = \mathcal{P}X$ and $A \subseteq X$ has measure-free cardinal, then $\mu A = \sum_{x\in A}\mu\{x\}$.

(c) If μ is σ-finite, then $L = \{i : i \in I,\ \mu E_i > 0\}$ is countable and $\bigcup_{i\in I\setminus L} E_i$ is negligible.

438C Theorem (a) ω is measure-free.

(b) If κ is a measure-free cardinal and $\kappa' \le \kappa$ is a smaller cardinal, then κ' is measure-free.

(c) If $\langle\kappa_\xi\rangle_{\xi<\lambda}$ is a family of measure-free cardinals, and λ also is measure-free, then $\kappa = \sup_{\xi<\lambda}\kappa_\xi$ is measure-free.

(d) If κ is a measure-free cardinal so is κ^+.

(e) The following are equiveridical:

(i) $\mathfrak{c}$ is not measure-free;

(ii) there is a semi-finite measure space $(X, \mathcal{P}X, \mu)$ which is not purely atomic;

(iii) there is a measure μ on $[0,1]$ extending Lebesgue measure and measuring every subset of $[0,1]$.

(f) If $\kappa \ge \mathfrak{c}$ is a measure-free cardinal then 2^κ is measure-free.

438D Proposition Let (X,Σ,μ) be a σ-finite measure space, Y a metrizable space with measure-free weight, and $f : X \to Y$ a measurable function. Then there is a closed separable set $Y_0 \subseteq Y$ such that $f^{-1}[Y_0]$ is conegligible; that is, there is a conegligible measurable set $X_0 \subseteq X$ such that $f[X_0]$ is separable.

438E Proposition Let (X,Σ,μ) be a complete locally determined measure space.

(a) If Y is a topological space, Z is a metrizable space, $w(Z)$ is measure-free, and $f : X \to Y$, $g : X \to Z$ are measurable functions, then $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable.

(b) If $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a sequence of metrizable spaces, with product Y, $w(Y_n)$ is measure-free for every $n \in \mathbb{N}$, and $f_n : X \to Y_n$ is measurable for every $n \in \mathbb{N}$, then $x \mapsto f(x) = \langle f_n(x)\rangle_{n\in\mathbb{N}} : X \to \prod_{n\in\mathbb{N}} Y_n$ is measurable.

438F Proposition Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a metrizable space, $w(Y)$ is measure-free and $f : X \to Y$ is measurable. Then f is almost continuous.

438G Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space and Y a metrizable space such that $w(Y)$ is measure-free. Then a function $f : X \to Y$ is measurable iff it is almost continuous.

438H Proposition A complete metric space is Radon iff its weight is measure-free.

438I Proposition Let X be a metrizable space and $\langle F_\xi\rangle_{\xi<\kappa}$ a non-decreasing family of closed subsets of X, where κ is a measure-free cardinal. Then

$$\mu(\bigcup_{\xi<\kappa} F_\xi) = \sum_{\xi<\kappa}\mu(F_\xi \setminus \bigcup_{\eta<\xi} F_\eta)$$

for every semi-finite Borel measure μ on X.

438J Proposition Let X be a metacompact space with measure-free weight.

(a) X is Borel-measure-compact.

(b) If X is normal, it is measure-compact.

(c) If X is perfectly normal, it is Borel-measure-complete.

438K Hereditarily weakly θ-refinable spaces A topological space X is **hereditarily weakly θ-refinable** if for every family $\mathcal{G}$ of open subsets of X there is a σ-isolated family $\mathcal{A}$ of subsets of X, refining $\mathcal{G}$, such that $\bigcup\mathcal{A} = \bigcup\mathcal{G}$.

438L Lemma (a) Any subspace of a hereditarily weakly θ-refinable topological space is hereditarily weakly θ-refinable.

(b) A hereditarily metacompact space is hereditarily weakly θ-refinable.

(c) A hereditarily Lindelöf space is hereditarily weakly θ-refinable.

(d) A topological space with a σ-isolated network is hereditarily weakly θ-refinable.

438M Proposition If X is a hereditarily weakly θ-refinable topological space with measure-free weight, it is Borel-measure-complete.

438N Let X be a topological space and $\mathcal{G}$ a family of subsets of X. Then $\mathcal{J}(\mathcal{G})$ will be the family of subsets of X expressible as $\bigcup\mathcal{A}$ for some σ-isolated family $\mathcal{A}$ refining $\mathcal{G}$. X is hereditarily weakly θ-refinable iff $\bigcup\mathcal{G}$ belongs to $\mathcal{J}(\mathcal{G})$ for every family $\mathcal{G}$ of open subsets of X.

(a) $\mathcal{J}(\mathcal{G})$ is always a σ-ideal of subsets of X.

(b) If $\mathcal{H}$ refines $\mathcal{G}$, then $\mathcal{J}(\mathcal{H}) \subseteq \mathcal{J}(\mathcal{G})$.

(c) If X and Y are topological spaces, $A \subseteq X$, $f : A \to Y$ is continuous, and $\mathcal{H}$ is a family of subsets of Y, set $\mathcal{G} = \{f^{-1}[H] : H \in \mathcal{H}\}$. Then $\mathcal{J}(\mathcal{G}) \supseteq \{f^{-1}[B] : B \in \mathcal{J}(\mathcal{H})\}$.

(d) If X is a topological space, $\mathcal{G}$ is a family of subsets of X, and $\langle D_i\rangle_{i\in I}$ is an isolated family in $\mathcal{J}(\mathcal{G})$, then $\bigcup_{i\in I} D_i \in \mathcal{J}(\mathcal{G})$.

438O Lemma Give $\mathbb{R}$ the topology $\mathfrak{S}$ generated by the closed intervals $]-\infty, t]$ for $t \in \mathbb{R}$, and let $r \ge 1$. Then $\mathbb{R}^r$, with the product topology corresponding to $\mathfrak{S}$, is hereditarily weakly θ-refinable.

438P Lemma Let X be a Polish space, and $\tilde{C}^{\Downarrow} = \tilde{C}^{\Downarrow}(X)$ the family of functions $\omega : \mathbb{R} \to X$ such that $\lim_{s\uparrow t} \omega(s)$ and $\lim_{s\downarrow t} \omega(s)$ are defined in X for every $t \in \mathbb{R}$.

(a) For $A \subseteq B \subseteq \mathbb{R}$ and $f \in X^B$, set

$$\begin{aligned}\operatorname{jump}_A(f, \epsilon) = \sup\{n :\ &\text{there is an } I \in [A]^n \text{ such that } \rho(f(s), f(t)) > \epsilon\\ &\text{whenever } s < t \text{ are successive elements of } I\}.\end{aligned}$$

Now a function $\omega \in X^{\mathbb{R}}$ belongs to $\tilde{C}^{\Downarrow}$ iff $\operatorname{jump}_{[-n,n]}(\omega, \epsilon)$ is finite for every $n \in \mathbb{N}$ and $\epsilon > 0$.

(b) If $\omega \in \tilde{C}^{\Downarrow}$ then ω is continuous at all but countably many points of $\mathbb{R}$.

(c) If $\omega \in \tilde{C}^{\Downarrow}$ then $\omega[[-n, n]]$ is relatively compact in X for every $n \in \mathbb{N}$.

438Q Theorem Let X be a Polish space, and $\tilde{C}^{\Downarrow} = \tilde{C}^{\Downarrow}(X)$ the family of functions $\omega : \mathbb{R} \to X$ such that $\lim_{s\uparrow t} \omega(s)$ and $\lim_{s\downarrow t} \omega(s)$ are defined in X for every $t \in \mathbb{R}$.

(a) $\tilde{C}^{\Downarrow}$, with its topology of pointwise convergence inherited from the product topology of $X^{\mathbb{R}}$, is K-analytic.

(b) $\tilde{C}^{\Downarrow}$ is hereditarily weakly θ-refinable.

438R Corollary (a) Let $I^{\|}$ be the split interval. Then any countable power of $I^{\|}$ is a hereditarily weakly θ-refinable compact Hausdorff space.

(b) Let Y be the 'Helly space', the space of non-decreasing functions from $[0, 1]$ to itself with the topology of pointwise convergence inherited from the product topology on $[0, 1]^{[0,1]}$. Then Y is a hereditarily weakly θ-refinable compact Hausdorff space.

***438S Càllàl functions: Proposition** Let X be a Polish space. Let $C^{\Downarrow} = C^{\Downarrow}(X)$ be the set of càllàl functions from $[0, \infty[$ to X, with its topology of pointwise convergence inherited from the product topology of $X^{[0,\infty[}$.

(a)(i) If $\omega \in C^{\Downarrow}$, then ω is continuous at all but countably many points of $[0, \infty[$.

(ii) If ω, $\omega' \in C^{\Downarrow}$, D is a dense subset of $[0, \infty[$ containing every point at which ω is discontinuous, and $\omega'\restriction D = \omega\restriction D$, then $\omega' = \omega$.

(b) $C^{\Downarrow}$ is hereditarily weakly θ-refinable.

(c) $C^{\Downarrow}$ is K-analytic.

438T Proposition Assume that $\mathfrak{c}$ is measure-free. Then $(I^{\|})^{\mathbb{N}}$, the Helly space and the spaces $\tilde{C}^{\mathbb{1}}(X)$, $C^{\mathbb{1}}(X)$ of 438Q and 438S, for any Polish space X, are all Radon spaces.

438U Proposition Let X and Y be topological spaces with σ-finite Borel measures μ, ν respectively. Suppose that *either* X is first-countable *or* ν is τ-additive and effectively locally finite. Write λ for the Borel measure on $X \times Y$ defined by the formula

$$\lambda W = \int \nu W[\{x\}]\mu(dx) \text{ for every Borel set } W \subseteq X \times Y$$

as in 434R(ii). If *either* the weight of X *or* the Maharam type of ν is a measure-free cardinal, then for every Borel set $W \subseteq X \times Y$ there is a set $W' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $\lambda(W \triangle W') = 0$; consequently, the measure algebra of λ can be identified with the localizable measure algebra free product of the measure algebras of μ and ν.

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439 Examples

As in Chapter 41, I end this chapter with a number of examples, exhibiting some of the boundaries around the results in the rest of the chapter, and filling in a gap with basic facts about Lebesgue measure (439E). The first three examples (439A) are measures defined on σ-subalgebras of the Borel σ-algebra of $[0, 1]$ which have no extensions to the whole Borel algebra. The next part of the section (439B-439G) deals with 'universally negligible' sets; I use properties of these to show that Hausdorff measures are generally not semi-finite (439H), closing some unfinished business from §264, and that smooth linear functionals may fail to be representable by integrals in the absence of Stone's condition (439I). In 439J-439R I set out some examples relevant to §§434-435, filling out the classification schemes of 434A and 435A, with spaces which just miss being Radon (439K) or measure-compact (439N, 439P, 439Q). In 439S I present the canonical example of a non-Prokhorov topological space, answering an obvious question from §437.

439A Example Let $\mathcal{B}$ be the Borel σ-algebra of $[0, 1]$. There is a probability measure ν defined on a σ-subalgebra T of $\mathcal{B}$ which has no extension to a measure on $\mathcal{B}$.

439B Definition Let X be a Hausdorff space. I will call X **universally negligible** if there is no Borel probability measure μ defined on X such that $\mu\{x\} = 0$ for every $x \in X$. A subset of X will be 'universally negligible' if it is universally negligible in its subspace topology.

439C Proposition Let X be a Hausdorff space.

(a) If A is a subset of X, the following are equiveridical:

(i) A is universally negligible;

(ii) $\mu^* A = 0$ whenever μ is a Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$;

(iii) $\mu^* A = 0$ whenever μ is a σ-finite topological measure on X such that $\mu\{x\} = 0$ for every $x \in A$;

(iv) for every σ-finite topological measure μ on X there is a countable set $B \subseteq A$ such that $\mu^* A = \mu B$;

(v) A is a Radon space and every compact subset of A is scattered.

In particular, countable subsets of X are universally negligible.

(b) The family of universally negligible subsets of X is a σ-ideal.

(c) Suppose that Y is a universally negligible Hausdorff space and that $f : X \to Y$ is a Borel measurable function such that $f^{-1}[\{y\}]$ is universally negligible for every $y \in Y$. Then X is universally negligible.

(d) If the topology on X is discrete, X is universally negligible iff $\#(X)$ is measure-free.

439D Remark Let X be a hereditarily Lindelöf Hausdorff space and μ a topological probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. Then μ is atomless.

439E Lemma (a) Let E, $B \subseteq \mathbb{R}$ be such that E is measurable and $\mu_L E$, $\mu_L^* B$ are both greater than 0, where μ_L is Lebesgue measure. Then $E - B = \{x - y : x \in E,\, y \in B\}$ includes a non-trivial interval.

(b) If $A \subseteq \mathbb{R}$ and $\mu_L^* A > 0$, then $A + \mathbb{Q}$ is of full outer measure in $\mathbb{R}$.

439F Proposition Let κ be the least cardinal of any set of non-zero Lebesgue outer measure in $\mathbb{R}$.

(a) There is a set $X \subseteq [0,1]$ with cardinal κ and full outer Lebesgue measure.

(b) If (Z, T, ν) is any atomless complete locally determined measure space and $A \subseteq Z$ has cardinal less than κ, then $\nu^* A = 0$.

(c) There is a universally negligible set $Y \subseteq [0,1]$ with cardinal κ.

439G Corollary A metrizable continuous image of a universally negligible metrizable space need not be universally negligible.

439H Corollary One-dimensional Hausdorff measure on $\mathbb{R}^2$ is not semi-finite.

439I Example There are a set X, a Riesz subspace U of $\mathbb{R}^X$ and a smooth positive linear functional $h : U \to \mathbb{R}$ which is not expressible as an integral.

439J Example Assume that there is some cardinal κ which is not measure-free. Give κ its discrete topology, and let μ be a probability measure with domain $\mathcal{P}\kappa$ such that $\mu\{\xi\} = 0$ for every $\xi < \kappa$. Now every subset of κ is open-and-closed, so μ is simultaneously a Baire probability measure and a completion regular Borel probability measure. Of course it is not τ-additive.

439K Example There is a first-countable compact Hausdorff space which is not Radon.

439L Example Suppose that κ is a cardinal which is not measure-free; let μ be a probability measure with domain $\mathcal{P}\kappa$ which is zero on singletons. Give κ its discrete topology. Let ν be the restriction of the usual measure on $Y = \{0,1\}^\kappa$ to the algebra $\mathcal{B}$ of Borel subsets of Y, so that ν is a τ-additive probability measure, and λ the product measure on $\kappa \times Y$ constructed by the method of 434R. Then

$$W = \{(\xi, y) : \xi < \kappa,\, y(\xi) = 1\} = \bigcup_{\xi<\kappa} \{\xi\} \times \{y : y(\xi) = 1\}$$

is open in $\kappa \times Y$.

If $W' \in \mathcal{P}\kappa \widehat{\otimes} \mathcal{B}$ then $\lambda(W \triangle W') = \frac{1}{2}$.

In particular, $W^\bullet$ in the the measure algebra of λ cannot be represented by a member of $\mathcal{P}\kappa \widehat{\otimes} \mathcal{B}$.

439M Example There is a first-countable locally compact Hausdorff space X with a Baire probability measure μ which is not τ-additive and has no extension to a Borel measure.

439N Example Give ω_1 its order topology.

(i) ω_1 is a normal Hausdorff space which is not measure-compact.

(ii) There is a Baire probability measure μ_0 on ω_1 which is not τ-additive and has a unique extension to a Borel measure, which is not completion regular.

439O Example Assume Ostaszewski's $\clubsuit$. Then there is a normal Hausdorff space with a Baire probability measure μ which is not τ-additive and not extendable to a Borel measure.

439P Example $\mathbb{N}^{\mathfrak{c}}$ is not Borel-measure-compact, therefore not Borel-measure-complete, measure-compact or Radon.

439Q Example Let X be the Sorgenfrey line. Then X is measure-compact but X^2 is not.

439R Example There are first-countable completely regular Hausdorff spaces X, Y with Baire probability measures μ, ν such that the Baire measures λ, λ' on $X \times Y$ defined by the formulae

$$\int f d\lambda = \iint f(x,y)\nu(dy)\mu(dx), \qquad \int f d\lambda' = \iint f(x,y)\mu(dx)\nu(dy)$$

are different.

439S Theorem $\mathbb{Q}$ is not a Prokhorov space.

Version of 21.11.10

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

432I Capacitability 432I, referred to in the 2008 edition of Volume 5, is now 432J.

434S-434T Vague topologies The material on vague topologies, referred to in the 2001 edition of Volume 2, has been moved to §437.

439H τ-smooth functionals The example of a τ-smooth functional which is not representable as an integral, referred to in Bogachev 07, is now 439I.

439J A non-Radon space The example of a first-countable compact Hausdorff space which is not Radon, referred to in Bogachev 07, is now 439K.

439N Baire measure The example of a Baire probability measure with no extension to a Borel measure, referred to in Bogachev 07, is now 439M.

References

Bogachev V.I. [07] *Measure theory.* Springer, 2007.