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Chapter 43

Topologies and measures II

The first chapter of this volume was 'general' theory of topological measure spaces; I attempted to distinguish the most important properties a topological measure can have – inner regularity, τ-additivity – and describe their interactions at an abstract level. I now turn to rather more specialized investigations, looking for features which offer explanations of the behaviour of the most important spaces, radiating outwards from Lebesgue measure.

In effect, this chapter consists of three distinguishable parts and two appendices. The first three sections are based on ideas from descriptive set theory, in particular Souslin's operation (§431); the properties of this operation are the foundation for the theory of two classes of topological space of particular importance in measure theory, the K-analytic spaces (§432) and the analytic spaces (§433). The second part of the chapter, §§434-435, collects miscellaneous results on Borel and Baire measures, looking at the ways in which topological properties of a space determine properties of the measures it carries. In §436 I present the most important theorems on the representation of linear functionals by integrals; if you like, this is the inverse operation to the construction of integrals from measures in §122. The ideas continue into §437, where I discuss spaces of signed measures representing the duals of spaces of continuous functions, and topologies on spaces of measures. The first appendix, §438, looks at a special topic: the way in which the patterns in §§434-435 are affected if we assume that our spaces are not unreasonably complex in a rather special sense defined in terms of measures on discrete spaces. Finally, I end the chapter with a further collection of examples, mostly to exhibit boundaries to the theorems of the chapter, but also to show some of the variety of the structures we are dealing with.

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431 Souslin's operation

I begin the chapter with a short section on Souslin's operation (§421). The basic facts we need to know are that (in a complete locally determined measure space) the family of measurable sets is closed under Souslin's operation (431A), and that the kernel of a Souslin scheme can be approximated from within in measure (431D). As in §421, I write S for $\bigcup_{k\in\mathbb{N}}\mathbb{N}^k$ and S^* for $\bigcup_{k\ge 1}\mathbb{N}^k$.

431A Theorem Let (X,Σ,μ) be a complete locally determined measure space. Then Σ is closed under Souslin's operation.

proof Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A. If $F\in\Sigma$ and $\mu F<\infty$, then $A\cap F\in\Sigma$. **P** For each $\sigma\in S$, set

$$A_\sigma=\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}},\phi\supseteq\sigma}\bigcap_{n\ge 1}E_{\phi\restriction n},$$

and let G_σ be a measurable envelope of $A_\sigma\cap F$. Because $A_\sigma\subseteq E_\sigma$ (writing $E_\emptyset=X$), we may suppose that $G_\sigma\subseteq E_\sigma\cap F$. Now, for any $\sigma\in S$,

$$A_\sigma\cap F=\bigcup_{i\in\mathbb{N}}A_{\sigma^\frown<i>}\cap F\subseteq\bigcup_{i\in\mathbb{N}}G_{\sigma^\frown<i>},$$

so

$$H_\sigma=G_\sigma\setminus\bigcup_{i\in\mathbb{N}}G_{\sigma^\frown<i>}$$

is negligible.

Set $H=\bigcup_{\sigma\in S}H_\sigma$, so that H is negligible. Take any $x\in G_\emptyset\setminus H$. Choose $\langle\phi(i)\rangle_{i\in\mathbb{N}}$ inductively, as follows. Given that $\sigma=\langle\phi(i)\rangle_{i<k}$ has been chosen and $x\in G_\sigma$, then $x\notin H_\sigma$, so there must be some $j\in\mathbb{N}$ such that $x\in G_{\sigma^\frown<j>}$; set $\phi(k)=j$, and continue. Now

$$x\in\bigcap_{k\ge1}G_{\phi\restriction k}\subseteq\bigcap_{k\ge1}E_{\phi\restriction k}\subseteq A.$$

Thus we see that $G_\emptyset\setminus H\subseteq A$; as $G_\emptyset\subseteq F$, $G_\emptyset\setminus H\subseteq A\cap F$. On the other hand, $A\cap F\subseteq G_\emptyset$. Because H is negligible and μ is complete, $A\cap F\in\Sigma$. **Q**

Because μ is locally determined, it follows that $A\in\Sigma$. As $\langle E_\sigma\rangle_{\sigma\in S^*}$ is arbitrary, Σ is closed under Souslin's operation.

431B Corollary If $(X,\mathfrak{T},\Sigma,\mu)$ is a complete locally determined topological measure space, every Souslin-F set in X (definition: 421K) is measurable.

431C Corollary Let X be a set and θ an outer measure on X. Let μ be the measure defined by Carathéodory's method, and Σ its domain. Then Σ is closed under Souslin's operation.

proof Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A. Take any $C\subseteq X$ such that $\theta C<\infty$. Then $\theta_C=\theta\restriction\mathcal{P}C$ is an outer measure on C; let μ_C be the measure on C defined from θ_C by Carathéodory's method, and Σ_C its domain. If $\sigma\in S^*$ and $D\subseteq C$ then

$$\begin{aligned}\theta_C(D\cap C\cap E_\sigma)+\theta_C(D\setminus(C\cap E_\sigma))&=\theta(D\cap E_\sigma)+\theta(D\setminus E_\sigma)\\&=\theta D=\theta_C D;\end{aligned}$$

as D is arbitrary, $C\cap E_\sigma\in\Sigma_C$. μ_C is a complete totally finite measure, so 431A tells us that the kernel of the Souslin scheme $\langle C\cap E_\sigma\rangle_{\sigma\in S^*}$ belongs to Σ_C. But this is just $C\cap A$ (applying 421Cb to the identity map from C to X). So

$$\theta(C\cap A)+\theta(C\setminus A)=\theta_C(C\cap A)+\theta_C(C\setminus A)=\theta_C C=\theta C.$$

As C is arbitrary, $A\in\Sigma$ (113D). As $\langle E_\sigma\rangle_{\sigma\in S^*}$ is arbitrary, we have the result.

431D Theorem Let (X,Σ,μ) be a complete locally determined measure space, and $\langle E_\sigma\rangle_{\sigma\in S^*}$ a Souslin scheme in Σ with kernel A.

(a)

$$\begin{aligned}\mu A&=\sup\{\mu(\bigcup_{\phi\in K}\bigcap_{n\ge1}E_{\phi\restriction n}):K\subseteq\mathbb{N}^{\mathbb{N}}\text{ is compact}\}\\&=\sup\{\mu(\bigcup_{\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}):\psi\in\mathbb{N}^{\mathbb{N}}\},\end{aligned}$$

writing $\phi\le\psi$ if $\phi(i)\le\psi(i)$ for every $i\in\mathbb{N}$.

(b) If $\langle E_\sigma\rangle_{\sigma\in S^*}$ is fully regular, then $\mu A=\sup\{\mu(\bigcap_{n\ge1}E_{\psi\restriction n}):\psi\in\mathbb{N}^{\mathbb{N}}\}$, and if in addition μ is totally finite, $\mu A=\sup\{\inf_{n\ge1}\mu E_{\psi\restriction n}:\psi\in\mathbb{N}^{\mathbb{N}}\}$.

proof (a)(i) By 431A, A is measurable. For $K\subseteq\mathbb{N}^{\mathbb{N}}$, set $H_K=\bigcup_{\phi\in K}\bigcap_{n\ge1}E_{\phi\restriction n}$. Of course $H_K\subseteq A$, and we know from 421M (or otherwise) that $H_K\in\Sigma$ if K is compact. So surely $\mu A\ge\mu H_K$ for every compact $K\subseteq\mathbb{N}^{\mathbb{N}}$. If $\psi\in\mathbb{N}^{\mathbb{N}}$, then $\{\phi:\phi\le\psi\}=\prod_{i\in\mathbb{N}}(\psi(i)+1)$ is compact. We therefore have

$$\begin{aligned}\mu A&\ge\sup\{\mu(\bigcup_{\phi\in K}\bigcap_{n\ge1}E_{\phi\restriction n}):K\subseteq\mathbb{N}^{\mathbb{N}}\text{ is compact}\}\\&\ge\sup\{\mu(\bigcup_{\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}):\psi\in\mathbb{N}^{\mathbb{N}}\}.\end{aligned}$$

So what I need to prove is that

$$\mu A\le\sup\{\mu(\bigcup_{\phi\le\psi}\bigcap_{n\ge1}E_{\phi\restriction n}):\psi\in\mathbb{N}^{\mathbb{N}}\}.$$

(ii) Fix on a set $F \in \Sigma$ of finite measure. For $\sigma \in S$ set

$$A_\sigma = \bigcup_{\phi \in \mathbb{N}^\mathbb{N}, \phi \supseteq \sigma} \bigcap_{n \geq 1} E_{\phi \restriction n}.$$

We need to know that A_σ belongs to Σ; this follows from 431A, because writing $E'_\tau = E_\tau$ if $\tau \subseteq \sigma$ or $\sigma \subseteq \tau$, $\emptyset$ otherwise,

$$A_\sigma = \bigcup_{\phi \in \mathbb{N}^\mathbb{N}} \bigcap_{n \geq 1} E'_{\phi \restriction n} \in \mathcal{S}(\Sigma) = \Sigma,$$

writing $\mathcal{S}$ for Souslin's operation, as in §421.

Let $\epsilon > 0$, and take a family $\langle \epsilon_\sigma \rangle_{\sigma \in S}$ of strictly positive real numbers such that $\sum_{\sigma \in S} \epsilon_\sigma \leq \epsilon$. For each $\sigma \in S$ we have $A_\sigma = \bigcup_{i \in \mathbb{N}} A_{\sigma^\frown <i>}$, so there is an $m_\sigma \in \mathbb{N}$ such that

$$\mu(F \cap A_\sigma \setminus \bigcup_{i \leq m_\sigma} A_{\sigma^\frown <i>}) \leq \epsilon_\sigma.$$

Define $\psi \in \mathbb{N}^\mathbb{N}$ by saying that

$$\psi(k) = \max\{m_\sigma : \sigma \in \mathbb{N}^k,\ \sigma(i) \leq \psi(i) \text{ for every } i < k\}$$

for each $k \in \mathbb{N}$. Set

$$H = \bigcup_{\phi \in \mathbb{N}^\mathbb{N}, \phi \leq \psi} \bigcap_{n \geq 1} E_{\phi \restriction n}.$$

(iii) Set

$$G = \bigcup_{\sigma \in S} F \cap A_\sigma \setminus \bigcup_{i \leq m_\sigma} A_{\sigma^\frown <i>},$$

so that $\mu G \leq \epsilon$, by the choice of the ϵ_σ and the m_σ. Then $F \cap A \setminus G \subseteq H$. **P** If $x \in F \cap A \setminus G$, choose $\langle \phi(i) \rangle_{i \in \mathbb{N}}$ inductively, as follows. Given that $\phi(i) \leq \psi(i)$ for $i < k$ and $x \in A_\sigma$, where $\sigma = \langle \phi(i) \rangle_{i<k}$, then $x \notin A_\sigma \setminus \bigcup_{j \leq m_\sigma} A_{\sigma^\frown <j>}$, so there must be some $j \leq m_\sigma$ such that $x \in A_{\sigma^\frown <j>}$; set $\phi(k) = j$; because $\sigma \in \prod_{i<k}(\psi(i)+1)$, $j \leq m_\sigma \leq \psi(k)$, and the induction continues. At the end of the induction, $\phi \leq \psi$ and

$$x \in \bigcap_{n \geq 1} A_{\phi \restriction n} \subseteq \bigcap_{n \geq 1} E_{\phi \restriction n} \subseteq H. \text{ **Q**}$$

(iv) It follows that

$$\mu(A \cap F) \leq \mu G + \mu H \leq \epsilon + \mu H.$$

As F and ϵ are arbitrary, and μ is semi-finite,

$$\mu A \leq \sup\{\mu(\bigcup_{\phi \leq \psi} \bigcap_{n \geq 1} E_{\phi \restriction n}) : \psi \in \mathbb{N}^\mathbb{N}\},$$

and (a) is true.

(b) If $\langle E_\sigma \rangle_{\sigma \in S^*}$ is fully regular, and $\phi \leq \psi$, then $\phi \restriction n \leq \psi \restriction n$ and $E_{\phi \restriction n} \subseteq E_{\psi \restriction n}$ for every $n \geq 1$; consequently $\bigcup_{\phi \leq \psi} \bigcap_{n \geq 1} E_{\phi \restriction n} \subseteq \bigcap_{n \in \mathbb{N}} E_{\psi \restriction n}$ for every $\psi \in \mathbb{N}^\mathbb{N}$, and

$$\begin{aligned} \mu A &= \sup\{\mu(\bigcup_{\phi \leq \psi} \bigcap_{n \geq 1} E_{\phi \restriction n}) : \psi \in \mathbb{N}^\mathbb{N}\} \\ &= \sup\{\mu(\bigcap_{n \geq 1} E_{\psi \restriction n}) : \psi \in \mathbb{N}^\mathbb{N}\}. \end{aligned}$$

Moreover, $E_{\psi \restriction n} \subseteq E_{\psi \restriction m}$ whenever $1 \leq m \leq n$, so if μ is totally finite, $\mu(\bigcap_{n \geq 1} E_{\psi \restriction n}) = \inf_{n \geq 1} \mu E_{\psi \restriction n}$ for every ψ, and $\mu A = \sup_{\psi \in \mathbb{N}^\mathbb{N}} \inf_{n \geq 1} \mu E_{\psi \restriction n}$.

431E Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is a topological measure space and $E \subseteq X$ is a Souslin-F set with finite outer measure, then $\mu^* E = \sup\{\mu F : F \subseteq E \text{ is closed}\}$.

proof Let $\tilde{\mu}$ be the c.l.d. version of μ (213E). Let $\langle E_\sigma \rangle_{\sigma \in S^*}$ be a Souslin scheme of closed sets with kernel E. Then 213Fb and 431D tell us that

$$\mu^* E = \tilde{\mu} E = \sup_{K \subseteq \mathbb{N}^\mathbb{N} \text{ is compact}} \mu F_K,$$

where $F_K = \bigcup_{\phi \in K} \bigcap_{n \geq 1} E_{\phi \restriction n}$ for $K \subseteq \mathbb{N}^\mathbb{N}$. But every F_K is closed, by 421M. So $\mu^* E \leq \sup_{F \subseteq E \text{ is closed}} \mu F$; as the reverse inequality is trivial, we have the result.

***431F** Two further versions of the ideas in 431A will be useful. The first is topological.

Theorem Let X be any topological space, and $\widehat{\mathcal{B}}$ its Baire-property algebra.

(a) For any $A \subseteq X$, there is a Baire-property envelope of A, that is, a set $E \in \widehat{\mathcal{B}}$ such that $A \subseteq E$ and $E \setminus F$ is meager whenever $A \subseteq F \in \widehat{\mathcal{B}}$.

(b) $\widehat{\mathcal{B}}$ is closed under Souslin's operation.

proof (a) By 4A3Sa, there is an open set $H \subseteq X$ such that $A \setminus H$ is meager and $H \cap G$ is empty whenever $G \subseteq X$ is open and $A \cap G$ is meager. Set $E = A \cup H$; then $E \supseteq A$ and $E \triangle H = A \setminus H$ is meager, so $E \in \widehat{\mathcal{B}}$. If $A \subseteq F \in \widehat{\mathcal{B}}$, let G be an open set such that $G \triangle (X \setminus F)$ is meager. Then $G \cap A \subseteq G \cap F$ is meager, so $G \cap H$ is empty and $E \setminus F \subseteq (E \triangle H) \cup (G \triangle (X \setminus F))$ is meager. Thus E is a Baire-property envelope of A.

(b) Let $\langle E_\sigma \rangle_{\sigma \in S^*}$ be a Souslin scheme in $\widehat{\mathcal{B}}$ with kernel A. For each $\sigma \in S$, set

$$A_\sigma = \bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma} \bigcap_{n \geq 1} E_{\phi \restriction n},$$

and let G_σ be a Baire-property envelope of A_σ as described in (a). Because $A_\sigma \subseteq E_\sigma$ (writing $E_\emptyset = X$), we may suppose that $G_\sigma \subseteq E_\sigma$. Now, for any $\sigma \in S$,

$$A_\sigma = \bigcup_{i \in \mathbb{N}} A_{\sigma^\frown <i>} \subseteq \bigcup_{i \in \mathbb{N}} G_{\sigma^\frown <i>},$$

so

$$H_\sigma = G_\sigma \setminus \bigcup_{i \in \mathbb{N}} G_{\sigma^\frown <i>}$$

is meager.

Set $H = \bigcup_{\sigma \in S} H_\sigma$, so that H is meager. Take any $x \in G_\emptyset \setminus H$. Choose $\langle \phi(i) \rangle_{i \in \mathbb{N}}$ inductively, as follows. Given that $\sigma = \langle \phi(i) \rangle_{i<k}$ has been chosen and $x \in G_\sigma$, then $x \notin H_\sigma$, so there must be some $j \in \mathbb{N}$ such that $x \in G_{\sigma^\frown <j>}$; set $\phi(k) = j$, and continue. Now

$$x \in \bigcap_{k \geq 1} G_{\phi \restriction k} \subseteq \bigcap_{k \geq 1} E_{\phi \restriction k} \subseteq A.$$

Thus we see that $G_\emptyset \setminus H \subseteq A$. On the other hand, $A \subseteq G_\emptyset$, so $G_\emptyset \triangle A$ is meager and $A \in \widehat{\mathcal{B}}$. As $\langle E_\sigma \rangle_{\sigma \in S^*}$ is arbitrary, $\widehat{\mathcal{B}}$ is closed under Souslin's operation.

***431G** The second relies on a countable chain condition to give the same envelope property.

Theorem Let X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I} \subseteq \Sigma$ a σ-ideal of subsets of X. If $\Sigma/\mathcal{I}$ is ccc then Σ is closed under Souslin's operation.

proof (a) As before, the essential fact is that for every $A \subseteq X$ there is an $E \in \Sigma$ such that $A \subseteq E$ and $F \in \mathcal{I}$ whenever $F \in \Sigma$ and $F \subseteq E \setminus A$. **P** Let $\mathcal{E}$ be a maximal disjoint family of members of $\Sigma \setminus \mathcal{I}$ disjoint from A. Because $\Sigma/\mathcal{I}$ is ccc, $\mathcal{E}$ is countable (316C), so $E = X \setminus \bigcup \mathcal{E}$ belongs to Σ; now it is easy to see that this E serves. **Q**

In this case I will call E a 'measurable envelope' of A.

(b) Now we can argue as in 431A or 431F. Let $\langle E_\sigma \rangle_{\sigma \in S^*}$ be a Souslin scheme in Σ with kernel A; for $\sigma \in S$, set

$$A_\sigma = \bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \sigma \subseteq \phi} \bigcap_{n \geq 1} E_{\phi \restriction n},$$

and let $G_\sigma \subseteq E_\sigma$ be a measurable envelope of A_σ. Setting

$$H = \bigcup_{\sigma \in S} (G_\sigma \setminus \bigcup_{i \in \mathbb{N}} G_{\sigma^\frown <i>}),$$

$H \in \mathcal{I}$ and $G_\emptyset \triangle A \subseteq H$, so $A \in \Sigma$. As $\langle E_\sigma \rangle_{\sigma \in S^*}$ is arbitrary, Σ is closed under Souslin's operation.

431X Basic exercises (a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space, Y a topological space, and $f : X \to Y$ a measurable function. Let $\mathcal{B}(Y)$ be the Borel algebra of Y. Show that $f^{-1}[B] \in \Sigma$ for every $B \in \mathcal{S}(\mathcal{B}(Y))$.

(b) Let X be a topological space and μ a semi-finite topological measure on X which is inner regular with respect to the Souslin-F sets. Show that μ is inner regular with respect to the closed sets.

>(c) Let (X,Σ,μ) be a measure space with locally determined negligible sets (definition: 213I), and $\langle E_\sigma\rangle_{\sigma\in S^*}$ a Souslin scheme in Σ with kernel A. Show that

$$\mu^*A=\sup_{\psi\in\mathbb{N}^\mathbb{N}}\mu(\bigcup_{\phi\le\psi}\bigcap_{n\ge 1}E_{\phi\restriction n}).$$

>(d) Let (X,Σ,μ) be a semi-finite measure space, and $\langle E_\sigma\rangle_{\sigma\in S^*}$ a Souslin scheme in Σ with kernel A. Show that

$$\mu_*A=\sup_{\psi\in\mathbb{N}^\mathbb{N}}\mu(\bigcup_{\phi\le\psi}\bigcap_{n\ge 1}E_{\phi\restriction n}).$$

431Y Further exercises (a) Let (X,Σ,μ) be a complete measure space with the measurable envelope property. Show that Σ is closed under Souslin's operation.

(b) Let (X,Σ,μ) be a complete totally finite measure space, Y a set and T a σ-algebra of subsets of Y. Suppose that $A\in\mathcal{S}(\Sigma\widehat{\otimes}\mathrm{T})$. Show that $\{y:\mu A^{-1}[\{y\}]>\alpha\}$ belongs to $\mathcal{S}(\mathrm{T})$ for every $\alpha\in\mathbb{R}$.

(c) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X such that $\mathcal{I}\subseteq\Sigma$. Suppose that for every $A\subseteq X$ there is an $F\in\Sigma$ such that $A\subseteq F$ and $F\setminus E\in\mathcal{I}$ whenever $A\subseteq E\in\Sigma$. Show that Σ is closed under Souslin's operation.

(d) Let $r\ge 1$ be an integer, $D\subseteq\mathbb{R}^r$ a Borel set and $f:D\to\mathbb{R}$ a Borel measurable function. Show that the domain of its first partial derivative $\frac{\partial f}{\partial\xi_1}$ is coanalytic, therefore Lebesgue measurable, but may fail to be Borel.

431 Notes and comments From the point of view of measure theory, the most important property of Souslin's operation, after its idempotence, is the fact that (for many measure spaces) the family of measurable sets is closed under the operation (431A). The proof I give here is based on the concept of measurable envelope, which can be used in other cases of great interest (431F, 431G, 431Yc). But for some applications it is also very important to know that if A is the kernel of a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S^*}$, then A can be approximated from inside by sets of the form $H=\bigcup_{\phi\le\psi}\bigcap_{n\ge 1}E_{\phi\restriction n}$ (431D, 431Xc), which belong to the σ-algebra generated by the E_σ (421M). A typical application of this idea is when every E_σ is a Borel subset of $\mathbb{R}$; then we find not only that A is Lebesgue measurable (indeed, measured by every Radon measure on $\mathbb{R}$) but that (for any given Radon measure μ) the Souslin scheme itself provides Borel subsets H of A of measure approximating the measure of A. A similar result, based on rather different hypotheses, is in 432K.

Let me repeat that the essence of descriptive set theory is that we are not satisfied merely to know that a set of a certain type exists. We want also to know how to build it, because we expect that an explicit construction will be valuable later on. For instance, the construction given in 431D shows that if the Souslin scheme consists of closed compact sets, the sets H will be compact (421Xn).

I mention 431B as a typical application of 431A, even though it is both obvious and obviously less than what can be said. The algebras Σ of this section are algebras closed under Souslin's operation. In a complete locally determined topological measure space, the algebra Σ of measurable sets includes the open sets (by definition), therefore the Borel algebra $\mathcal{B}$, therefore $\mathcal{S}(\mathcal{B})$; but now we can take the algebra $\mathcal{A}_1$ generated by $\mathcal{S}(\mathcal{B})$, and $\mathcal{A}_1$ and $\mathcal{S}(\mathcal{A}_1)$ will also be included in Σ, so that Σ will included the algebra $\mathcal{A}_2$ generated by $\mathcal{S}(\mathcal{A}_1)$, and so on. (Note that $\mathcal{S}(\mathcal{A}_1)$ includes the σ-algebra generated by $\mathcal{A}_1$, by 421F, so I do not need to mention that separately.) We have to run through all the countable ordinals before we can be sure of getting to the smallest algebra $\mathcal{A}_{\omega_1}=\bigcup_{\xi<\omega_1}\mathcal{A}_\xi$ which contains every open set and is closed under Souslin's operation, and we shall then have $\mathcal{A}_{\omega_1}\subseteq\Sigma$.

Version of 2.10.13

432 K-analytic spaces

I describe the basic measure-theoretic properties of K-analytic spaces (§422). I start with 'elementary' results (432A-432C), assembling ideas from §§421, 422 and 431. The main theorem of the section is 432D, one of the leading cases of the general extension theorem 416P. An important corollary (432G) gives a sufficient condition for the existence of pull-back measures. I briefly mention 'capacities' (432J-432L).

432A Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined Hausdorff topological measure space. Then every K-analytic subset of X is measurable.

proof If $A\subseteq X$ is K-analytic, it is Souslin-F (422Ha), therefore measurable (431B).

432B Theorem Let X be a K-analytic Hausdorff space, and μ a semi-finite topological measure on X. Then

$$\mu X=\sup\{\mu K:K\subseteq X\text{ is compact}\}.$$

proof If $\gamma<\mu X$, there is an $E\in\operatorname{dom}\mu$ such that $\gamma<\mu E<\infty$; set $\nu F=\mu(E\cap F)$ for every Borel set $F\subseteq X$, so that ν is a totally finite Borel measure on X, and $\nu X>\gamma$. Let $\hat\nu$ be the completion of ν. Let $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}]=X$. Set $F_\sigma=\overline{R[I_\sigma]}$ for $\sigma\in S^*=\bigcup_{k\ge1}\mathbb{N}^k$, where $I_\sigma=\{\phi:\sigma\subseteq\phi\in\mathbb{N}^{\mathbb{N}}\}$. Because R is closed in $\mathbb{N}^{\mathbb{N}}\times X$ (422Da), X is the kernel of the Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$ (421I). By 431D, there is a compact $L\subseteq\mathbb{N}^{\mathbb{N}}$ such that $\hat\nu(\bigcup_{\phi\in L}\bigcap_{n\in\mathbb{N}}F_{\phi\restriction n})\ge\gamma$. But, by 421I, this is just $\hat\nu(R[L])$; and $R[L]$ is compact, by 422D(e-i). So $\mu R[L]$ is defined, with $\mu R[L]\ge\nu R[L]=\hat\nu R[L]$, and we have a compact subset of X of measure at least γ. As γ is arbitrary, the theorem is proved.

432C Proposition Let X be a Hausdorff space such that all its open sets are K-analytic, and μ a Borel measure on X.

(a) If μ is semi-finite, it is tight.

(b) If μ is locally finite, its completion is a Radon measure on X.

proof (a) By 422Hb, every open subset of X is Souslin-F. Applying 421F to the family $\mathcal{E}$ of closed subsets of X, we see that every Borel subset of X is Souslin-F, therefore K-analytic (422Ha). Now suppose that $E\subseteq X$ is a Borel set. Then the subspace measure μ_E is a semi-finite Borel measure on the K-analytic space E, so by 432B $\mu E=\sup_{K\subseteq E\text{ is compact}}\mu K$. As E is arbitrary, μ is inner regular with respect to the compact sets; but we are supposing that X is Hausdorff, so these are all closed, and μ is tight.

(b) Because X is Lindelöf (422Gg), μ is σ-finite (411Ge), therefore semi-finite. So (a) tells us that μ is tight. By 416F, its c.l.d. version is a Radon measure. But (because μ is σ-finite) this is just its completion (213Ha).

432D Theorem (ALDAZ & RENDER 00) Let X be a K-analytic Hausdorff space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Then μ has an extension to a Radon measure on X. In particular, μ is τ-additive.

proof The point is that if $\mu E>0$ then there is a compact $K\subseteq E$ such that $\mu^*K>0$. **P** Write Σ for the domain of μ. Take $\gamma<\mu E$. Because X is Lindelöf (422Gg again), μ is σ-finite (411Ge), therefore semi-finite. Let $E'\subseteq E$ be such that $\gamma<\mu E'<\infty$. Because μ is inner regular with respect to the closed sets, there is a closed set $F\subseteq E$ such that $\mu F>\gamma$. F is K-analytic (422Gf); let $R\subseteq\mathbb{N}^{\mathbb{N}}\times F$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}]=F$. For $\sigma\in S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ set

$$A_\sigma=\{x:(\phi,x)\in R\text{ for some }\phi\in\mathbb{N}^{\mathbb{N}}\text{ such that }\phi(i)\le\sigma(i)\text{ for every }i<\#(\sigma)\}.$$

Then $\langle A_{\sigma^\frown<i>}\rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence with union A_σ, so

$$\mu^*A_\sigma=\sup_{i\in\mathbb{N}}\mu^*A_{\sigma^\frown<i>}$$

for every $\sigma\in S$ (132Ae). We can therefore find a sequence $\psi\in\mathbb{N}^{\mathbb{N}}$ such that

$$\mu^*A_{\psi\restriction n}>\gamma$$

for every $n\in\mathbb{N}$. Set

$$K=\{\phi:\phi\in\mathbb{N}^{\mathbb{N}},\ \phi(i)\le\psi(i)\text{ for every }i\in\mathbb{N}\};$$

then $K=\prod_{n\in\mathbb{N}}(\psi(n)+1)$ is compact, so $R[K]$ is compact (422D(e-i) again).

? Suppose, if possible, that $\mu^*R[K]<\gamma$. Then there is an $H\in\Sigma$ such that $R[K]\subseteq H\subseteq F$ and $\mu(F\setminus H)>\mu F-\gamma$. Because μ is inner regular with respect to the closed sets, there is a closed set $F'\in\Sigma$

such that $F' \subseteq F \setminus H$ and $\mu F' > \mu F - \gamma$. Since $R[K] \cap F' = \emptyset$, $K \cap R^{-1}[F'] = \emptyset$. $R^{-1}[F']$ is closed, because R is usco-compact, so there is some n such that

$$L = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}},\ \phi{\restriction}n = \phi'{\restriction}n \text{ for some } \phi' \in K\}$$

does not meet $R^{-1}[F']$ (4A2F(h-vi)), and $R[L] \cap F' = \emptyset$. But L is just $\{\phi : \phi(i) \le \psi(i) \text{ for every } i < n\}$, so $R[L] = A_{\psi{\restriction}n}$, and

$$\gamma < \mu^* A_{\psi{\restriction}n} \le \mu(F \setminus F') < \gamma,$$

which is absurd. **X**

Thus $\mu^* R[K] \ge \gamma$. As $\gamma > 0$, we have the result. **Q**

Now the theorem follows at once from 416P(ii)⇒(i).

432E Corollary Let X be a K-analytic Hausdorff space, and μ a locally finite quasi-Radon measure on X. Then μ is a Radon measure.

proof By 432D, μ has an extension to a Radon measure μ'. But of course μ and μ' must coincide, by 415H or otherwise.

432F Corollary Let X be a K-analytic Hausdorff space, and ν a locally finite Baire measure on X. Then ν has an extension to a Radon measure on X; in particular, it is τ-additive. If the topology of X is regular, the extension is unique.

proof Because X is Lindelöf (422Gg once more), ν is σ-finite, therefore semi-finite; by 412D, it is inner regular with respect to the closed sets. So 432D tells us that it has an extension to a Radon measure on X. Since the extension is τ-additive, so is ν.

If X is regular, then it must be completely regular (4A2H(b-i)), and the family $\mathcal{G}$ of cozero sets is a base for the topology closed under finite unions. If μ, μ' are Radon measures extending ν, they agree on $\mathcal{G}$, and must be equal, by 415H(iv).

432G Corollary Let X be a K-analytic Hausdorff space, Y a Hausdorff space and ν a locally finite measure on Y which is inner regular with respect to the closed sets. Let $f : X \to Y$ be a continuous function such that $f[X]$ has full outer measure in Y. Then there is a Radon measure μ on X such that f is inverse-measure-preserving for μ and ν. If ν is Radon, it is precisely the image measure μf^{-1}.

proof (a) Write T for the domain of ν, and set $\Sigma_0 = \{f^{-1}[F] : f \in \mathrm{T}\}$, so that Σ_0 is a σ-algebra of subsets of X, and we have a measure μ_0 on X defined by setting $\mu_0 f^{-1}[F] = \nu F$ whenever $F \in \mathrm{T}$ (234F).

(b) If $E \in \Sigma_0$ and $\gamma < \mu_0 E$, there is an $F \in \mathrm{T}$ such that $E = f^{-1}[F]$. Now there is a closed set $F' \subseteq F$ such that $\nu F' \ge \gamma$. Because f is continuous, $f^{-1}[F']$ is closed, and we have $f^{-1}[F'] \subseteq E$ and $\mu_0 f^{-1}[F'] \ge \gamma$. As E and γ are arbitrary, μ_0 is inner regular with respect to the closed sets.

If $x \in X$, then (because ν is locally finite) there is an open set $H \subseteq Y$ such that $f(x) \in H$ and $\nu^* H < \infty$; as f is of course inverse-measure-preserving for μ_0 and ν, $\mu_0^* f^{-1}[H] \le \nu^* H$ (234B(f-i)) is finite, while $f^{-1}[H]$ is an open set containing x. Thus μ_0 is locally finite.

(c) By 432D, there is a Radon measure μ on X extending μ_0. Because f is inverse-measure-preserving for μ_0 and ν, it is surely inverse-measure-preserving for μ and ν.

The image measure μf^{-1} extends ν, so must be locally finite; it is therefore a Radon measure (418I). So if ν itself is a Radon measure, it must be identical with μf^{-1}, by 416Eb.

432H Corollary Suppose that X is a set and that $\mathfrak{S}$, $\mathfrak{T}$ are Hausdorff topologies on X such that $(X, \mathfrak{T})$ is K-analytic and $\mathfrak{S} \subseteq \mathfrak{T}$. Then the totally finite Radon measures on X are the same for $\mathfrak{S}$ and $\mathfrak{T}$.

proof Write f for the identity function on X regarded as a continuous function from $(X, \mathfrak{T})$ to $(X, \mathfrak{S})$. If μ is a totally finite $\mathfrak{T}$-Radon measure on X, then $\mu = \mu f^{-1}$ is $\mathfrak{S}$-Radon, by 418I again. If ν is a totally finite $\mathfrak{S}$-Radon measure on X, then 432G tells us that it is of the form $\mu = \mu f^{-1}$ for some $\mathfrak{T}$-Radon measure μ, that is, is itself $\mathfrak{T}$-Radon.

432I Corollary Let X be a K-analytic Hausdorff space, and $\mathcal{U}$ a subbase for the topology of X. Let (Y,T,ν) be a complete totally finite measure space and $\phi:Y\to X$ a function such that $\phi^{-1}[U]\in\mathrm{T}$ for every $U\in\mathcal{U}$. Then there is a Radon measure μ on X such that $\int f d\mu=\int f\phi\,d\nu$ for every bounded continuous $f:X\to\mathbb{R}$.

proof (a) Let $\nu\phi^{-1}$ be the image measure on X, and Σ_0 its domain. Then Σ_0 is a σ-algebra of subsets of X including $\mathcal{U}$. So if x, y are distinct points of X, there are disjoint open sets U, $V\in\Sigma_0$ containing x, y respectively. **P** Because X is Hausdorff, there are disjoint open sets U_0 and V_0 such that $x\in U_0$ and $y\in V_0$. Because Σ_0 is closed under finite intersections and includes the subbase $\mathcal{U}$, it includes a base for the topology of X (4A2B(a-i)), and there are open sets U, $V\in\Sigma_0$ such that $x\in U\subseteq U_0$ and $y\in V\subseteq V_0$. **Q**

Every cozero subset of X belongs to Σ_0. **P** If $G\subseteq X$ is a cozero set, there is a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of closed subsets of X with union G. For each n, F_n and $X\setminus G$ are disjoint K-analytic subsets of X (422Gf), so there is an $E_n\in\Sigma_0$ such that $F_n\subseteq E_n\subseteq G$ (422I). Now $G=\bigcup_{n\in\mathbb{N}}E_n$ belongs to Σ_0. **Q**

(b) It follows that the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X is included in Σ_0. So $\mu_0=\nu\phi^{-1}\restriction\mathcal{B}\mathfrak{a}(X)$ is a Baire measure on X. By 432F, μ_0 has an extension to a Radon measure μ on X.

If $f\in C_b(X)$, then f is μ_0-integrable; since ϕ is inverse-measure-preserving for ν and μ_0, $\int f\phi\,d\nu$ is defined and equal to $\int f d\mu_0$ (235G). Similarly $\int f d\mu=\int f d\mu_0$. So $\int f d\mu=\int f\phi\,d\nu$, as required.

432J Capacitability The next theorem is not exactly measure theory as studied in most of this treatise; but it is clearly very close to the other ideas of this section, and it has important applications to measure theory in the narrow sense.

Definitions Let $(X,\mathfrak{T})$ be a topological space.

(a) A **Choquet capacity** on X is a function $c:\mathcal{P}X\to[0,\infty]$ such that

(i) $c(A)\le c(B)$ whenever $A\subseteq B\subseteq X$;

(ii) $\lim_{n\to\infty}c(A_n)=c(A)$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A;

(iii) $c(K)=\inf\{c(G):G\supseteq K\text{ is open}\}$ for every compact set $K\subseteq X$.

(b) A Choquet capacity c on X is **outer regular** if $c(A)=\inf\{c(G):G\supseteq A\text{ is open}\}$ for every $A\subseteq X$.

432K Theorem (Choquet 55) Let X be a Hausdorff space and c a Choquet capacity on X. If $A\subseteq X$ is K-analytic, then $c(A)=\sup\{c(K):K\subseteq A\text{ is compact}\}$.

proof Take $\gamma<c(A)$. Let $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}]=A$; for $\sigma\in S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ set

$$A_\sigma=\{x:(\phi,x)\in R\text{ for some }\phi\in\mathbb{N}^{\mathbb{N}}\text{ such that }\phi(i)\le\sigma(i)\text{ for every }i<\#(\sigma)\}.$$

Then $\langle A_{\sigma^\frown<i>}\rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence with union A_σ, so

$$c(A_\sigma)=\sup_{i\in\mathbb{N}}c(A_{\sigma^\frown<i>})$$

for every $\sigma\in S$. We can therefore find a sequence $\psi\in\mathbb{N}^{\mathbb{N}}$ such that $c(A_{\psi\restriction n})>\gamma$ for every $n\in\mathbb{N}$. Set

$$K=\{\phi:\phi\in\mathbb{N}^{\mathbb{N}},\ \phi(i)\le\psi(i)\text{ for every }i\in\mathbb{N}\};$$

then $K=\prod_{n\in\mathbb{N}}(\psi(n)+1)$ is compact, so $R[K]$ is compact (422D(e-i) once more).

? Suppose, if possible, that $c(R[K])<\gamma$. Then, by (iii) of 432J, there is an open set $G\supseteq R[K]$ such that $c(G)<\gamma$. Set $F=X\setminus G$, so that F is closed and $K\cap R^{-1}[F]=\emptyset$. $R^{-1}[F]$ is closed, because R is usco-compact, so there is some n such that

$$L=\{\phi:\phi\in\mathbb{N}^{\mathbb{N}},\ \phi\restriction n=\phi'\restriction n\text{ for some }\phi'\in K\}$$

does not meet $R^{-1}[F]$ (4A2F(h-vi) again), and $R[L]\cap F=\emptyset$, that is, $R[L]\subseteq G$. But L is just $\{\phi:\phi(i)\le\psi(i)$ for every $i<n\}$, so $R[L]=A_{\psi\restriction n}$, and

$$\gamma<c(A_{\psi\restriction n})\le c(G)<\gamma,$$

which is absurd. **X**

Thus $c(R[K])\ge\gamma$. As γ is arbitrary and $R[K]$ is compact, we have the result.

432L Proposition Let $(X,\mathfrak{T})$ be a topological space.

(a) Let $c_0 : \mathfrak{T} \to [0,\infty]$ be a functional such that

$c_0(G) \le c_0(H)$ whenever G, $H \in \mathfrak{T}$ and $G \subseteq H$;

c_0 is submodular (definition: 413Qb)

$c_0(\bigcup_{n\in\mathbb{N}} G_n) = \lim_{n\to\infty} c_0(G_n)$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$.

Then c_0 has a unique extension to an outer regular Choquet capacity c on X, and c is submodular.

(b) Suppose that X is regular. Let $\mathcal{K}$ be the family of compact subsets of X, and $c_1 : \mathcal{K} \to [0,\infty]$ a functional such that

c_1 is submodular;

$c_1(K) = \inf_{G\in\mathfrak{T},G\supseteq K} \sup_{L\in\mathcal{K},L\subseteq G} c_1(L)$ for every $K \in \mathcal{K}$.

Then c_1 has a unique extension to an outer regular Choquet capacity c on X such that

$$c(G) = \sup\{c(K) : K \subseteq G \text{ is compact}\} \text{ for every open } G \subseteq X,$$

and c is submodular.

proof (a) For $A \subseteq X$, set $c(A) = \inf\{c_0(G) : A \subseteq G \in \mathfrak{T}\}$. Then $c : \mathcal{P}X \to [0,\infty]$ extends c_0 because c_0 is order-preserving. Conditions (i) and (iii) of 432J are obviously satisfied. As for (ii), let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of X with union A. Then $\lim_{n\to\infty} c(A_n)$ is defined and not greater than $c(A)$. If the limit is infinite, then certainly it is equal to $c(A)$. Otherwise, take $\epsilon > 0$. For each $n \in \mathbb{N}$, choose an open set $G_n \supseteq A_n$ such that $c_0(G_n) \le c(A_n) + 2^{-n}\epsilon$. Set $H_n = \bigcup_{i\le n} G_i$ for $n \in \mathbb{N}$. Then $c_0(H_n) \le c(A_n) + 2\epsilon - 2^{-n}\epsilon$ for every n. **P** Induce on n. If $n = 0$ then $H_0 = G_0$ and the result is immediate. For the inductive step to $n+1$, we have

$$c_0(H_{n+1}) + c_0(H_n \cap G_{n+1}) \le c_0(H_n) + c_0(G_{n+1})$$

(because c_0 is submodular)

$$\le c(A_n) + 2\epsilon - 2^{-n}\epsilon + c(A_{n+1}) + 2^{-n-1}\epsilon$$

(using the inductive hypothesis)

$$\le c_0(H_n \cap G_{n+1}) + 2\epsilon - 2^{-n-1}\epsilon + c(A_{n+1});$$

as $c_0(H_n \cap G_{n+1}) \le c_0(G_{n+1})$ is finite, $c_0(H_{n+1}) \le 2\epsilon - 2^{-n-1}\epsilon + c(A_{n+1})$, as required. **Q**

Set $H = \bigcup_{n\in\mathbb{N}} H_n$. As $\langle H_n\rangle_{n\in\mathbb{N}}$ is non-decreasing,

$$c(A) \le c_0(H) = \lim_{n\to\infty} c_0(H_n) \le \lim_{n\to\infty} c(A_n) + 2\epsilon.$$

As ϵ is arbitrary, $c(A) \le \lim_{n\to\infty} c(A_n)$ and the final condition of 432J is satisfied.

Of course c is the only Choquet capacity extending c_0 and outer regular with respect to the open sets.

As for the submodularity of c, if A, $B \subseteq X$ and $\epsilon > 0$, there are open sets $G \supseteq A$ and $H \supseteq B$ such that $c_0(G) + c_0(H) \le c(G) + c(H) + \epsilon$; so that

$$c(A \cup B) + c(A \cap B) \le c_0(G \cup H) + c_0(G \cap H)$$
$$\le c_0(G) + c_0(H) \le c(A) + c(B) + \epsilon.$$

As ϵ is arbitrary, $c(A \cup B) + c(A \cap B) \le c(A) + c(B)$, as required.

(b)(i) The key fact is this: if G, $H \in \mathfrak{T}$, K, $L \in \mathcal{K}$, $K \subseteq G \cup H$ and $L \subseteq G \cap H$, then there are K_1, $L_1 \in \mathcal{K}$ such that $K_1 \subseteq G$, $L_1 \subseteq H$, $K \subseteq K_1 \cup L_1$ and $L \subseteq K_1 \cap L_1$. **P** Because $(X,\mathfrak{T})$ is regular, there is an open set G_1 such that $K \setminus H \subseteq G_1$ and $\overline{G}_1 \subseteq G$ (4A2F(h-ii)). Set $K_1 = (K \cap \overline{G}_1) \cup L$, $L_1 = (K \setminus G_1) \cup L$; these work. **Q**

(ii) Define $c_0 : \mathfrak{T} \to [0,\infty]$ by setting $c_0(G) = \sup_{K\in\mathcal{K},K\subseteq G} c_1(K)$ for open $G \subseteq X$. Then c_0 satisfies the conditions of (a). **P** The first and third are elementary. As for the second, if G, $H \in \mathfrak{T}$ and $\gamma < c_0(G \cup H) + c_0(G \cap H)$, there are K, $L \in \mathcal{K}$ such that $K \subseteq G \cup H$, $L \subseteq G \cap H$ and $\gamma \le c_1(K) + c_2(L)$. Now (i) tells us that there are compact sets $K_1 \subseteq G$ and $L_1 \subseteq H$ such that $K \subseteq K_1 \cup L_1$ and $L \subseteq K_1 \cap L_1$, in which case

$$c_0(G)+c_0(H)\ge c_1(K_1)+c_1(L_1)\ge c_1(K_1\cup L_1)+c_1(K_1\cap L_1)$$
$$\ge c_1(K)+c_1(L)\ge\gamma.$$

As γ is arbitrary, $c_0(G)+c_0(H)\ge c_0(G\cup H)+c_0(G\cap H)$, as required. **Q**

(iii) We therefore have a submodular outer regular Choquet capacity $c:\mathcal{P}X\to[0,\infty]$ defined by setting $c(A)=\inf_{G\in\mathfrak{T},A\subseteq G}c_0(G)$ for every $A\subseteq X$. From the second condition on c_1, we see that c extends c_1. Clearly c satisfies the two regularity conditions, and is the only extension of c_1 which does so.

432X Basic exercises (a) Put 422Xe, 431Xb and 432D together to prove 432C.

(b) Let X be a K-analytic Hausdorff space, and μ a measure on X which is outer regular with respect to the open sets. Show that $\mu X=\sup_{K\subseteq X\text{ is compact}}\mu^*K$. (*Hint*: see the proof of 432D.)

>(c) Let X be a K-analytic Hausdorff space, and μ a semi-finite topological measure on X. Show that if *either* μ is inner regular with respect to the closed sets *or* X is regular and μ is a τ-additive Borel measure, then μ is tight.

(d) Use 422Gf, 432B and 416C to prove 432E.

>(e) Suppose that X is a set and that $\mathfrak{S}$, $\mathfrak{T}$ are Hausdorff topologies on X such that $(X,\mathfrak{T})$ is K-analytic and $\mathfrak{S}\subseteq\mathfrak{T}$. Let $(Z,\mathfrak{U},\mathrm{T},\nu)$ be a Radon measure space and $f:Z\to X$ a function which is almost continuous for $\mathfrak{U}$ and $\mathfrak{S}$. Show that f is almost continuous for $\mathfrak{U}$ and $\mathfrak{T}$. (*Hint*: it is enough to consider totally finite ν; show that νf^{-1} is $\mathfrak{T}$-Radon, so is inner regular for $\{K:\mathfrak{T}_K=\mathfrak{S}_K\}$, writing $\mathfrak{T}_K$ for the subspace topology induced by $\mathfrak{T}$ on K.)

(f) Let X be a topological space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Show that μ^* is a submodular Choquet capacity.

(g) Let X be a topological space and F a closed subset of X. Define $c:\mathcal{P}X\to\{0,1\}$ by setting $c(A)=1$ if A meets F, 0 otherwise. Show that c is a submodular Choquet capacity on X.

(h) Let X and Y be Hausdorff spaces, and $R\subseteq X\times Y$ an usco-compact relation. Show that if c is a Choquet capacity on Y, then $A\mapsto c(R[A])$ is a Choquet capacity on X, which is submodular if c is.

(i) Use 432K and 432Xf to shorten the proof of 432D.

(j) Let P be a lattice, and $c:P\to\mathbb{R}$ an order-preserving functional. Show that the following are equiveridical: (i) c is submodular; (ii) $(p,q)\mapsto 2c(p\vee q)-c(p)-c(q)$ is a pseudometric on P; (iii) setting $c_r(p)=c(p\vee r)-c(r)$, $c_r(p\vee q)\le c_r(p)+c_r(q)$ for all p, q, $r\in P$.

(k) Let X be a topological space, $c:X\to[0,\infty]$ a Choquet capacity, and $f:[0,\infty]\to[0,\infty]$ a non-decreasing function. (i) Show that if f is continuous then fc is a Choquet capacity. (ii) Show that if $f\restriction[0,\infty[$ is concave and c is submodular, then fc is submodular.

(l) Let X be a Hausdorff space, c a Choquet capacity on X, and $\mathcal{K}$ a non-empty downwards-directed family of compact subsets of X. Show that $c(\bigcap\mathcal{K})=\inf_{K\in\mathcal{K}}c(K)$.

432Y Further exercises (a) Show that there are a K-analytic Hausdorff space X and a probability measure μ on X such that (i) μ is inner regular with respect to the Borel sets (ii) the domain of μ includes a base for the topology of X (iii) every compact subset of X is negligible. Show that there is no extension of μ to a topological measure on X.

(b) Let X, Y be Hausdorff spaces, $R\subseteq X\times Y$ an usco-compact relation and μ a Radon probability measure on X such that $\mu_*R^{-1}[Y]=1$. Show that there is a Radon probability measure on Y such that $\nu_*R[A]\ge\mu_*A$ for every $A\subseteq X$.

432 Notes and comments The measure-theoretic properties of K-analytic spaces can largely be summarised in the slogan 'K-analytic spaces have lots of compact sets'. I said above that it is sometimes helpful to think of K-analytic spaces as an amalgam of compact Hausdorff spaces and Souslin-F subsets of $\mathbb{R}$. For the former, it is obvious that they have many compact subsets; for the latter, it is not obvious, but is of course one of their fundamental properties, deducible from 422De. 432B and the proof of 432D (repeated in 432K) are typical manifestations of the phenomenon. The real point of these theorems is that we can extend a Borel or Baire measure to a Radon measure with no prior assumption of τ-additivity (432F). A Radon measure must be τ-additive just because it is tight. A (locally finite) Borel or Baire measure must be τ-additive whenever the measurable open sets are K-analytic.

The condition 'every open set is K-analytic' in 432C is of course a very strong one in the context of compact Hausdorff spaces (422Xd). But for analytic spaces it is automatically satisfied (423Eb), and that is the side on which the principal applications of 432C appear.

The results which I call corollaries of 432D can mostly be proved by more direct methods (see 432Xd), but the line I choose here seems to be the most powerful technique. Indeed it can be used to deal with 432C as well (432Xa).

In §434 I will discuss 'universally measurable' sets in topological spaces. In fact K-analytic sets are universally measurable in a particularly strong sense (432A). The point here is that K-analyticity is intrinsic; a K-analytic space is measurable whenever embedded as a subspace of a (complete locally determined) Hausdorff topological measure space.

The theorems here touch on two phenomena of particular importance. First, in 432G we have an example of 'pulling back' a measure, that is, we have a measure ν on a set Y and a function $f : X \to Y$ and seek a Radon measure μ on X such that f is inverse-measure-preserving, or, even better, such that $\nu = \mu f^{-1}$. There was a similar result in 418L. In both cases we have to suppose that f is continuous and (in effect) that ν is a Radon measure. (This is not part of the hypotheses of 432G, but of course it is an easy consequence of them, using 432B.) In 418L, we need a special hypothesis to ensure that there are enough compact subsets of X to carry an appropriate Radon measure; in 432G, this is an automatic result of assuming that X is K-analytic. Both 418L and 432G can be regarded as consequences of Henry's theorem (416N). The difficulty arises from the requirement that μ should be a Radon measure; if we do not insist on this there is a much simpler solution, since we need suppose only that $f[X]$ has full outer measure (234F).

The next theme I wish to mention is a related one, the investigation of comparable topologies. If $\mathfrak{S}$ and $\mathfrak{T}$ are (Hausdorff) topologies on a set X, and $\mathfrak{S}$ is coarser than $\mathfrak{T}$ (so that $(X, \mathfrak{S})$ is a continuous image of $(X, \mathfrak{T})$), then 418I tells us that any totally finite $\mathfrak{T}$-Radon measure is $\mathfrak{S}$-Radon. We very much want to know when the reverse is true, so that the (totally finite) Radon measures for the two topologies are the same. 432H provides one of the important cases in which this occurs. The hypothesis '$(X, \mathfrak{T})$ is K-analytic' generalizes the alternative '$(X, \mathfrak{T})$ is compact'; in the latter case, $\mathfrak{S} = \mathfrak{T}$, so that the result is, from our point of view here, trivial. (But from the point of view of elementary general topology, of course, it is one of the pivots of the theory of compact Hausdorff spaces.) In a similar vein we have a variety of important topological consequences of the same hypotheses (422Yb, 423Fb).

The paragraphs 432J-432L may appear to be no more that a minor extension of ideas already set out. I ought therefore to say plainly that the topological and measure theory of K-analytic spaces have co-evolved with the notion of capacity, and that 432K ('K-analytic spaces are capacitable') is one of the cornerstones of a theory of which I am giving only a minuscule part. For a idea of the vitality and scope of this theory, see Dellacherie 80.

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433 Analytic spaces

We come now to the special properties of measures on 'analytic' spaces, that is, continuous images of $\mathbb{N}^{\mathbb{N}}$, as described in §423. I start with a couple of facts about spaces with countable networks.

433A Proposition Let $(X, \mathfrak{T})$ be a topological space with a countable network, and μ a localizable topological measure on X which is inner regular with respect to the Borel sets. Then μ has countable Maharam type.

proof Let $\tilde{\mu}$ be the c.l.d. version of μ (213E). Then the measure algebra $\mathfrak{A}$ of $\tilde{\mu}$ can be identified with the measure algebra of μ (322D(b-iii)). Also $\tilde{\mu}$ is complete, locally determined and localizable, so every subset of X has a measurable envelope with respect to $\tilde{\mu}$ (213J, 213L). Let $\tilde{\Sigma}$ be the domain of $\tilde{\mu}$, and $\mathcal{E}$ a countable network for $\mathfrak{T}$. For each $E \in \mathcal{E}$, let $F_E \in \tilde{\Sigma}$ be a measurable envelope of E.

Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{F_E^\bullet : E \in \mathcal{E}\}$, and set $\mathrm{T} = \{F : F \in \tilde{\Sigma},\ F^\bullet \in \mathfrak{B}\}$. Because $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$, T is a σ-subalgebra of $\tilde{\Sigma}$. Now $\mathfrak{T} \subseteq \mathrm{T}$. **P** If $G \in \mathfrak{T}$, set $\mathcal{E}_0 = \{E : E \in \mathcal{E},\ E \subseteq G\}$. Set $F = \bigcup_{E \in \mathcal{E}_0} F_E$, so that $F \in \mathrm{T}$ and $G \subseteq F$. For each $E \in \mathcal{E}_0$, $F_E \setminus G$ is negligible, so $F \setminus G$ is negligible, and $G^\bullet = F^\bullet \in \mathfrak{B}$, so $G \in \mathrm{T}$. **Q**

It follows that T includes the Borel σ-algebra of X. Because μ is inner regular with respect to the Borel sets, $\mathfrak{B}$ is order-dense in $\mathfrak{A}$, and $\mathfrak{B} = \mathfrak{A}$. Thus the countable set $\{F_E^\bullet : E \in \mathcal{E}\}$ τ-generates $\mathfrak{A}$, and the Maharam type of $\mathfrak{A}$, which is the Maharam type of μ, is countable.

433B Lemma If $(X, \mathfrak{T})$ is a Hausdorff space with a countable network, then any topological measure on X is countably separated in the sense of 343D.

proof By 4A2Nf, there is a countable family of open sets separating the points of X.

433C Theorem Let X be an analytic Hausdorff space, and μ a Borel measure on X.

(a) If μ is semi-finite, it is tight.

(b) If μ is locally finite, its completion is a Radon measure on X.

proof X is K-analytic (423C); moreover, every open subset of X is again analytic (423Eb). So 432C gives the result at once.

Remark Compare 256C.

433D Theorem Let X and Y be analytic Hausdorff spaces, ν a totally finite Radon measure on Y and $f : X \to Y$ a Borel measurable function such that $f[X]$ has full outer measure for ν. Then there is a Radon measure μ on X such that $\nu = \mu f^{-1}$.

proof By 423Ga, the graph R of f is an analytic set in $X \times Y$, therefore K-analytic. Set $\pi_1(x, y) = x$, $\pi_2(x, y) = y$ for $(x, y) \in R$, so that π_1 and π_2 are continuous. Now $\pi_2[R] = f[X]$ has full outer measure, so by 432G there is a Radon measure λ on R such that $\nu = \lambda\pi_2^{-1}$. Next, because π_1 is continuous, the image $\mu = \lambda\pi_1^{-1}$ is a Radon measure on X, by 418I. But $\pi_2 = f\pi_1$, so

$$\mu f^{-1} = (\lambda\pi_1^{-1})f^{-1} = \lambda(f\pi_1)^{-1} = \lambda\pi_2^{-1} = \nu,$$

as required.

433E Proposition Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Let $(Y, \mathfrak{S})$ be an analytic Hausdorff space and $f : X \to Y$ a measurable function. Then f is almost continuous.

proof Take $E \in \Sigma$ and $\gamma < \mu E$. Then there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. For Borel sets $H \subseteq Y$, set $\nu H = \mu(F \cap f^{-1}[H])$. Then ν is a totally finite Borel measure on Y, so is tight (433C); let $K \subseteq Y$ be a compact set such that $\nu K > \gamma$, so that $\mu(F \cap f^{-1}[K]) > \gamma$. The subspace measure on $L = F \cap f^{-1}[K]$ is still inner regular with respect to the (relatively) closed sets (412Pc), and $f{\restriction}L$ is still measurable; but $f{\restriction}L$ is a function from L to K, and K is metrizable, by 423Dc. So $f{\restriction}L$ is almost continuous, by 418J, and there is a set $F' \subseteq L$, of measure at least γ, such that $f{\restriction}F'$ is continuous.

As E and γ are arbitrary, f is almost continuous.

Remark Compare 418Yf.

433F I give some simple corollaries of the von Neumann-Jankow selection theorem (423N-423Q).

Proposition Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be analytic Hausdorff spaces, and $f : X \to Y$ a Borel measurable surjection. Let ν be a complete locally determined topological measure on Y, and T its domain. Then there is a T-measurable function $g : Y \to X$ such that gf is the identity on X.

proof By 423Q we know that there is a function $g : Y \to X$ such that fg is the identity and g is T_1-measurable, where T_1 is the σ-algebra generated by the Souslin-F subsets of Y. But T contains every Souslin-F subset of Y, by 431B, therefore includes T_1, and g is actually T-measurable.

433G Proposition Let $(X, \mathfrak{T})$ be an analytic Hausdorff space, (Y, T, ν) a complete locally determined measure space, and $f : X \to Y$ a surjection. Suppose that there is some countable family $\mathcal{F} \subseteq \mathrm{T}$ such that $\mathcal{F}$ separates the points of Y (that is, whenever y, y' are distinct points of Y there is a member of $\mathcal{F}$ containing one and not the other) and $f^{-1}[F]$ is a Borel subset of X for every $F \in \mathcal{F}$. Then there is a T-measurable function $g : Y \to X$ such that fg is the identity on Y.

proof Set $\mathcal{A} = \mathcal{F} \cup \{Y \setminus F : F \in \mathcal{F}\}$. The topology $\mathfrak{T}_1$ on X generated by $\mathfrak{T} \cup \{f^{-1}[A] : A \in \mathcal{A}\}$ is still analytic (423H). If we take $\mathfrak{S}$ to be the topology on Y generated by $\mathcal{A}$, then $\mathfrak{S}$ is Hausdorff and f is $(\mathfrak{T}_1, \mathfrak{S})$-continuous, so $\mathfrak{S}$ is analytic (423Bb).

Because $\mathfrak{S}$ is generated by a countable subset $\mathcal{A}$ of T, it is second-countable, and $\mathfrak{S} \subseteq \mathrm{T}$ (4A3Da/4A3E). So ν is a topological measure with respect to $\mathfrak{S}$. By 433F, there is a function $g : Y \to X$, measurable for T and the topology $\mathfrak{T}_1$, such that gf is the identity on X; and of course g is still measurable for T and the coarser original topology $\mathfrak{T}$ on X.

433H Proposition Let X be an analytic Hausdorff space, and (Y, T, ν) a complete locally determined measure space. Suppose that $W \subseteq X \times Y$ belongs to $\mathcal{S}(\mathcal{B}(X)\widehat{\otimes}\mathrm{T})$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Then $W[X] \in \mathrm{T}$ and there is a T-measurable function $g : W[X] \to X$ such that $(g(y), y) \in W$ for every $y \in W[X]$.

proof Set $\mathcal{V} = \mathcal{S}(\{E \times F : E \subseteq X \text{ is closed}, F \in \mathrm{T}\})$. Then $\mathcal{V}$ contains $H \times Y$ for every Souslin-F subset H of X (421Cb), and therefore for every $H \in \mathcal{B}(X)$ (423Eb); by 421F, it follows that $\mathcal{V}$ includes $\mathcal{B}(Y)\widehat{\otimes}\mathrm{T}$ and therefore $W \in \mathcal{S}(\mathcal{V}) = \mathcal{V}$ (421D). By 423N, $W[X] \in \mathcal{S}(\mathrm{T})$, which by 431A is just T, and there is a T-measurable function which is a selector for W^{-1}.

433I Because analytic spaces have countable networks (423C), and their compact subsets are therefore metrizable (423Dc), their measure theory is very close to that of $\mathbb{R}$ or $[0,1]$ or $\{0,1\}^{\mathbb{N}}$. I give some simple manifestations of this principle.

Proposition Let $\langle X_i \rangle_{i \in I}$ be a family of analytic Hausdorff spaces, and for each $i \in I$ let μ_i be a Radon probability measure on X_i. Let λ be the ordinary product measure on $X = \prod_{i \in I} X_i$, as defined in §254.

(a) If I is countable then λ is a Radon measure.

(b) If every μ_i is strictly positive, then λ is a quasi-Radon measure.

proof (a) In this case, X is analytic (423Bc), therefore hereditarily Lindelöf (423Da). Let Λ be the domain of λ and $\mathfrak{T}$ the topology of X. Then $\Lambda \cap \mathfrak{T}$ is a base for $\mathfrak{T}$; by 4A3Da, $\mathfrak{T} \subseteq \Lambda$. By 417Sb, λ is the τ-additive product measure on X; by 417Q, this is a Radon measure.

(b) By (a), the ordinary product measure on $\prod_{i \in J} X_i$ is a Radon measure for every finite set $J \subseteq I$. So 417Sc tells us that λ is the τ-additive product measure on X; by 417O, this is a quasi-Radon measure.

433J Proposition Let X be an analytic Hausdorff space, and T a countably generated σ-subalgebra of the Borel σ-algebra $\mathcal{B}(X)$ of X. Then any locally finite measure with domain T has an extension to a Radon measure on X.

proof Let μ_0 be a locally finite measure with domain T.

(a) Consider first the case in which μ_0 is totally finite. Let $\langle F_n \rangle_{n \in \mathbb{N}}$ be a sequence in T generating T as σ-algebra. Define $f : X \to \{0,1\}^{\mathbb{N}}$ by setting

$$f(x)(n) = \chi F_n(x) \text{ for } n \in \mathbb{N},\ x \in X.$$

Then f is T-measurable (use 418Bd), so we have a Borel measure ν_0 on $\{0,1\}^{\mathbb{N}}$ defined by setting $\nu_0 E = \mu_0 f^{-1}[E]$ for every Borel set $E \subseteq \{0,1\}^{\mathbb{N}}$. Now the completion ν of ν_0 is a Radon measure (433C). Also $f[X]$ must be analytic, by 423Gb, because f is $\mathcal{B}(X)$-measurable. So ν measures $f[X]$ (432A), and

$$\nu f[X] = \nu_0^* f[X] = \nu_0\{0,1\}^{\mathbb{N}},$$

that is, $f[X]$ is ν-conegligible. By 433D, there is a Radon measure μ on X such that $\nu = \mu f^{-1}$.

Because every F_n is expressible as $f^{-1}[E]$ for some $E \in \mathcal{B}(\{0,1\}^{\mathbb{N}})$, so is every member of T. If $F \in \mathrm{T}$, take $H \in \mathcal{B}(\{0,1\}^{\mathbb{N}})$ such that $F = f^{-1}[H]$; then

$$\mu F = \nu H = \nu_0 H = \mu_0 F.$$

Thus μ extends μ_0 and $\mu\restriction\Sigma$ will serve.

(b) In general, because X is Lindelöf and μ_0 is locally finite, μ_0 is σ-finite. Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a partition of X into members of T such that $\mu_0 X_n$ is finite for every n, and set $\mu_0^{(n)}F = \mu_0(F\cap X_n)$ for every n and every $F \in \mathrm{T}$; then every $\mu_0^{(n)}$ has an extension to a Radon measure $\mu^{(n)}$. Let μ be the sum $\sum_{n=0}^{\infty}\mu^{(n)}$ (234G[1]). Of course μ extends $\mu_0 = \sum_{n=0}^{\infty}\mu_0^{(n)}$. Because μ_0 is locally finite, so is μ, and μ is a Radon measure (416De).

433K I turn now to a brief mention of 'standard Borel spaces'. From the point of view of this chapter, it is natural to regard the following results as simple corollaries of theorems about Polish spaces. But, as remarked in §424, there are cases in which a standard Borel space is presented without any specific topology being attached; and in any case it is interesting to look at the ways in which we can express these ideas as theorems about σ-algebras rather than about topological spaces.

Proposition Let (X,Σ) be a standard Borel space and T a countably generated σ-subalgebra of Σ. Then any σ-finite measure with domain T has an extension to Σ.

proof Let μ_0 be a σ-finite measure with domain T.

(a) If μ_0 is totally finite, give X a Polish topology for which Σ is the Borel σ-algebra of X, and use 433J.

(b) In general, let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a partition of X into members of T such that $\mu_0 X_n < \infty$ for every n, and set $\mu_0^{(n)}F = \mu_0(F\cap X_n)$ for every n and every $F \in \mathrm{T}$; then every $\mu_0^{(n)}$ has an extension to a measure $\mu^{(n)}$ with domain Σ, and we can set $\mu = \sum_{n=0}^{\infty}\mu^{(n)}$.

433L Proposition Let $\langle(X_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces such that (X_n,Σ_n) is a standard Borel space for every n. Suppose that for each $n\in\mathbb{N}$ we are given an inverse-measure-preserving function $f_n : X_{n+1}\to X_n$. Then we can find a standard Borel space (X,Σ), a probability measure μ with domain Σ, and inverse-measure-preserving functions $g_n : X \to X_n$ such that $f_n g_{n+1} = g_n$ for every n.

proof For each n, choose a Polish topology $\mathfrak{T}_n$ on X_n such that Σ_n is the algebra of $\mathfrak{T}_n$-Borel sets. Let $\hat\mu_n$ be the completion of μ_n; then $\hat\mu_n$ is a Radon measure (433C). Every f_n is inverse-measure-preserving for $\hat\mu_{n+1}$ and $\hat\mu_n$, by 234Ba[2], and almost continuous, by 418J.

By 418Q, we have a Radon measure $\hat\mu$ on

$$X = \{x : x \in \prod_{n\in\mathbb{N}} X_n,\ f_n(x(n+1)) = x(n) \text{ for every } n\in\mathbb{N}\}$$

such that the continuous maps $x \mapsto x(n) = g_n(x) : X \to X_n$ are inverse-measure-preserving for every n. Now X is a Borel subset of $Z = \prod_{n\in\mathbb{N}} X_n$. **P** For each $n\in\mathbb{N}$, let $\mathcal{U}_n$ be a countable base for $\mathfrak{T}_n$. Then

$$Z\setminus X = \bigcup_{n\in\mathbb{N}}\bigcup_{U,V\in\mathcal{U}_n,U\cap V=\emptyset}\{z : z(n)\in U,\ f_n(z(n+1))\in V\}$$

is a countable union of Borel sets because $\{z : z(n)\in U\}$ is open and $\{z : z(n+1)\in f_n^{-1}[V]\}$ is Borel whenever $n\in\mathbb{N}$ and U, $V\in\mathcal{U}_n$. So $Z\setminus X$ and X are Borel sets. **Q**

Accordingly (X,Σ) is a standard Borel space, where Σ is the Borel σ-algebra of X (424G). So if we take $\mu = \hat\mu\restriction\Sigma$, we shall have a suitable measure on X.

433X Basic exercises (a) Let $(X,\mathfrak{T})$ be a topological space with a countable network, and μ a topological measure on X which is inner regular with respect to the Borel sets and has the measurable envelope property (213Xl). Show that μ has countable Maharam type.

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[1] Formerly 112Ya.

[2] Formerly 235Hc.

(b) Show that an effectively locally finite measure on a hereditarily Lindelöf space (in particular, on any analytic Hausdorff space) is σ-finite.

(c) Let $X \subseteq [0,1]$ be a set of Lebesgue outer measure 1 and inner measure 0. Show that the subspace measure on X is a totally finite Borel measure which is not tight.

(d) Let X be a Hausdorff space and μ a locally finite measure on X, inner regular with respect to the Borel sets, such that $\operatorname{dom}\mu$ includes a base for the topology of X. Suppose that $Y \subseteq X$ is an analytic set of full outer measure. Show that μ has a unique extension to a Radon measure $\tilde{\mu}$ on X, and that Y is $\tilde{\mu}$-conegligible.

(e) Let (X, Σ) be a standard Borel space. (i) Show that any semi-finite measure with domain Σ is a compact measure (definition: 342Ac, or 451Ab below), therefore perfect. (*Hint*: if X is given a suitable topology, the measure is tight.) (ii) Show that any measure with domain including Σ is countably separated.

>(f) (i) Let (X, Σ, μ) and (Y, T, ν) be atomless probability spaces such that (X, Σ) and (Y, T) are standard Borel spaces. Show that (X, Σ, μ) and (Y, T, ν) are isomorphic. (*Hint*: by 344I, their completions are isomorphic; by 344H, they have negligible sets with cardinal $\mathfrak{c}$; show that any isomorphism between the completions is (Σ, T)-measurable on a conegligible set; use 424Da to match residual negligible sets.) (ii) Let X be a Polish space and μ an atomless Radon measure on X. Show that there is a Borel isomorphism between X and $[0,1]$ which matches μ to Lebesgue measure on $[0,1]$.

(g) Let X be $[0,1] \times \{0,1\}$, with its usual topology, and $I^{\|}$ the split interval (419L); define $f : X \to I^{\|}$ by setting $f(t,0) = t^-$, $f(t,1) = t^+$ for $t \in [0,1]$. (i) Give $I^{\|}$ its usual Radon measure ν (343J, 419Lc). Show that there is no Radon measure λ on X such that $\nu = \lambda f^{-1}$. (ii) Let μ be the product Radon probability measure on X, starting from Lebesgue measure on $[0,1]$ and the usual fair-coin measure on $\{0,1\}$. Show that f is inverse-measure-preserving for μ and ν. Show that f is not almost continuous.

433Y Further exercises (a) Find a Hausdorff topological space X with a countable network and a semi-finite Borel measure on X which does not have countable Maharam type.

(b) Let X be an analytic Hausdorff space and μ an atomless Radon measure on X. Show that (X, μ) is isomorphic to Lebesgue measure on some measurable subset of $\mathbb{R}$. (*Hint*: 344I.)

(c) Let $(X, \mathfrak{T})$ be a Polish space without isolated points, and μ a σ-finite topological measure on X. Show that there is a conegligible meager set. (*Hint*: Show that every non-empty open set is uncountable. Find a countable dense negligible set and a negligible G_δ set including it.)

(d) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence of analytic Hausdorff spaces and for each $n \in \mathbb{N}$ let μ_n be a Borel probability measure on X_n. Suppose that for each $n \in \mathbb{N}$ we are given an inverse-measure-preserving function $f_n : X_{n+1} \to X_n$. Show that we can find a standard Borel space (X, Σ), a probability measure μ with domain Σ, and inverse-measure-preserving functions $g_n : X \to X_n$ such that $f_n g_{n+1} = g_n$ for every n.

433 Notes and comments The measure-theoretic results of 433C-433E are of much the same type as those in §432. A characteristic difference is that Borel measurable functions between analytic spaces behave in many ways like continuous functions. (Compare 433D and 432G.) You may feel that 423Yc offers some explanation of this. For any question which refers to the Borel algebra of an analytic space X, or to the class of its analytic subsets, we can expect to be able to suppose that X is separable and metrizable (see 423Xd), and that any single Borel measurable function appearing is continuous. (424H is a particularly remarkable instance of this principle.)

433I here amounts to spelling out a special case of ideas already treated in 417S. As this territory is relatively unfamiliar, I give detailed examples (423Xi, 433Xc, 433Xg, 439A, 439K) to show that the theorems of this section are not generally valid for compact Hausdorff spaces (the archetype of K-analytic spaces which need not be analytic), nor for separable metric spaces (the archetypical spaces with countable network). They really do depend on the particular combination of properties possessed by analytic spaces.

For large parts of probability theory, standard Borel spaces provide an adequate framework, and have a number of advantages; some of the technical problems concerning measurability which loom rather large in this treatise disappear in such contexts. Many authors accordingly give them great prominence. I myself believe that the simplifications are an entrapment rather than a liberation, that sooner or later everyone has to leave the comfortable environment of Borel algebras on Polish spaces, and that it is better to be properly equipped with a suitable general theory when one does. But it is surely important to know what the simplifications are, and the results in 433K-433L will I hope show at least that there are wonderful ideas here, even if my own presentation tends to leave them on one side.

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434 Borel measures

What one might call the fundamental question of topological measure theory is the following.

What kinds of measures can arise on what kinds of topological space?

Of course this question has inexhaustible ramifications, corresponding to all imaginable properties of measures and topologies and connexions between them. The challenge I face here is that of identifying particular ideas as being more important than others, and the chief difficulty lies in the bewildering variety of topological properties which have been studied, any of which may have implications for the measure theory of the spaces involved. In this section and the next I give a sample of what is known, necessarily biased and incomplete. I try however to include the results which are most often applied and enough others for the proofs to contain, between them, most of the non-trivial arguments which have been found effective in this area.

In 434A I set out a crude classification of Borel measures on topological spaces. For compact Hausdorff spaces, at least, the first question is whether they carry Borel measures which are not, in effect, Radon measures; this leads us to the definition of 'Radon' space (434C) which is also of interest in the context of general Hausdorff spaces. I give a brief account of the properties of Radon spaces (434F, 434Nd). I look also at two special topics: 'quasi-dyadic' spaces (434O-434Q) and a construction of Borel product measures by integration of sections (434R).

In the study of Radon spaces we find ourselves looking at 'universally measurable' subsets of topological spaces (434D-434E). These are interesting in themselves, and also interact with constructions from earlier parts of this treatise (434S-434T). Three further classes of topological space, defined in terms of the types of topological measure which they carry, are the 'Borel-measure-compact', 'Borel-measure-complete' and 'pre-Radon' spaces; I discuss them briefly in 434G-434J. They provide useful methods for deciding whether Hausdorff spaces are Radon (434K).

434A Types of Borel measures In §411 I introduced the following properties which a Borel measure may or may not have:

(i) inner regularity with respect to closed sets;

(ii) inner regularity with respect to zero sets;

(iii) tightness (that is, inner regularity with respect to closed compact sets);

(iv) τ-additivity.

These are of course interrelated. (ii)$\Rightarrow$(i) just because zero sets are closed, and (iii)$\Rightarrow$(iv) by 411E; in a Hausdorff space, (iii)$\Rightarrow$(i); and for an effectively locally finite measure on a regular topological space, (iv)$\Rightarrow$(i) (414Mb).

On a regular Hausdorff space, therefore, we can divide totally finite Borel measures into four classes:

(A) measures which are not inner regular with respect to the closed sets,

(B) measures which are inner regular with respect to the closed sets, but not τ-additive nor tight,

(C) measures which are τ-additive and inner regular with respect to the closed sets, but not inner regular with respect to the compact sets,

(D) tight measures;

and each of the classes (B)-(D) can be further subdivided into those which are completion regular (B_1, C_1, D_1) and those which are not (B_0, C_0, D_0). Examples may be found in 434Xg (type A), 411Q and 439Yf (type B_0), 439J (type B_1), 415Xc and 434Xa (type C_1) and 434Xb (type D_0), while Lebesgue measure itself is of type D_1, and any direct sum of spaces of types D_0 and C_1 will have type C_0. (The space in 439J depends for its construction on supposing that there is a cardinal which is not measure-free. It seems that no convincing example of a space of class B_1, that is, a completion regular, non-τ-additive Borel probability measure on a completely regular Hausdorff space, is known which does not depend on some special axiom beyond ordinary ZFC. For one of the obstacles to finding such a space, see 434Q.)

Note that a totally finite Borel measure μ on a regular Hausdorff space can be extended to a quasi-Radon measure iff μ is of class C or D (415M), and that in this case the quasi-Radon measure must be just the completion $\hat{\mu}$ of μ. $\hat{\mu}$ will be of the same type, on the classification here, as μ; in particular, $\hat{\mu}$ will be a Radon measure iff μ is of class D (416F).

434B Compact, analytic and K-analytic spaces For any class of topological spaces, we can enquire which of the seven types of measure described above can be realized by measures on spaces of that class. The enquiry is limited only by our enterprise and diligence in seeking out new classes of topological space. For the spaces studied in §§432-433, however, we have something worth repeating here. On a K-analytic Hausdorff space, a semi-finite Borel measure which is inner regular with respect to the closed sets is tight (432B, 432D); consequently classes B and C of 434A cannot appear, and we are left with only the types A, D_0 and D_1, all of which appear on compact Hausdorff spaces (434Xb, 434Xg). On an analytic Hausdorff space we have further simplifications: every semi-finite Borel measure is tight (433Ca), and (if X is regular) every closed set is a zero set (423Db). Thus on an analytic regular Hausdorff space only type D_1, of the seven types in 434A, can appear. (If the topology is not regular, we may also get measures of type D_0; see 434Ya.)

434C Radon spaces: Definition For K-analytic Hausdorff spaces, therefore, we have a large gap between the 'bad' measures of class A and the 'good' measures of class D; furthermore, we have an important class of spaces in which type A cannot appear. It is natural to enquire further into the spaces in which every (totally finite) Borel measure is of class D, and (given that no exact description can be found) we are led, as usual, to a definition. A Hausdorff space X is **Radon** if every totally finite Borel measure on X is tight.

434D Universally measurable sets Before going farther with the study of Radon spaces it will be useful to spend a couple of paragraphs on the following concept. Let X be a topological space.

(a) I will say that a subset E of X is **universally measurable** (in X) if it is measured by the completion of every Borel probability measure on X; that is, for every Borel probability measure μ on X there is a Borel set $F \subseteq X$ such that $E \triangle F$ is μ-negligible.

(b) A subset of X is universally measurable iff it is measured by every complete locally determined topological measure on X. **P** (i) Suppose that $A \subseteq X$ is universally measurable, and that μ is a complete locally determined topological measure on X. Take any $F \subseteq X$ be such that μF is defined and finite. If F is negligible then $F \cap A$ is negligible and $\mu(F \cap A)$ is defined. Otherwise, we have a Borel probability measure ν on X defined by setting $\nu E = \frac{1}{\mu F}\mu(F \cap E)$ for every Borel set $E \subseteq X$. Now there are Borel sets E, $B \subseteq X$ such that $A \triangle E \subseteq B$ and $\nu B = 0$. In this case, $(A \cap F) \triangle (E \cap F) \subseteq B \cap F$ and $\mu(B \cap F) = 0$, so that $A \cap F$ is measured by μ. Because F is arbitrary and μ is locally determined, A is measured by μ. (ii) Suppose that $A \subseteq X$ is measured by every complete locally determined topological measure on X. Then, in particular, it is measured by the completion of any Borel probability measure, so is universally measurable. **Q**

(c) The family Σ_{um} of universally measurable subsets of X is a σ-algebra closed under Souslin's operation and including the Borel σ-algebra. (For it is the intersection of the domains of a family of complete totally finite measures, and these are all σ-algebras including the Borel σ-algebra and closed under Souslin's operation, by 431A.) In particular, Souslin-F sets are universally measurable, so (if X is Hausdorff) K-analytic and analytic sets are (422Ha, 423C).

(d) Note that a function $f : X \to \mathbb{R}$ is Σ_{um}-measurable iff it is μ-virtually measurable (definition: 122Q) for every totally finite Borel measure μ on X (122Q, 212Fa). Generally, if Y is another topological space, I will say that $f : X \to Y$ is **universally measurable** if $f^{-1}[H] \in \Sigma_{\mathrm{um}}$ for every open set $H \subseteq Y$; that is, if f is $(\Sigma_{\mathrm{um}}, \mathcal{B}(Y))$-measurable, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. Continuous functions are universally measurable, of course.

(e) In fact, if $f : X \to Y$ is universally measurable, then it is $(\Sigma_{\mathrm{um}}, \Sigma_{\mathrm{um}}^{(Y)})$-measurable, where $\Sigma_{\mathrm{um}}^{(Y)}$ is the algebra of universally measurable subsets of Y. **P** Take $F \in \Sigma_{\mathrm{um}}^{(Y)}$ and a totally finite Borel measure μ on X. If $\hat{\mu}$ is the completion of μ, then the image measure $\nu = \hat{\mu}f^{-1}$ is a complete totally finite topological measure on Y, so measures F, and $f^{-1}[F] \in \operatorname{dom}\hat{\mu}$. As μ is arbitrary, $f^{-1}[F] \in \Sigma_{\mathrm{um}}$; as F is arbitrary, f is $(\Sigma_{\mathrm{um}}, \Sigma_{\mathrm{um}}^{(Y)})$-measurable. **Q**

(f) It follows that if Z is a third topological space and $f : X \to Y$, $g : Y \to Z$ are universally measurable, then $gf : X \to Z$ is universally measurable.

434E Universally Radon-measurable sets A companion idea is the following. Let X be a Hausdorff space.

(a) I will say that a subset E of X is **universally Radon-measurable** if it is measured by every Radon measure on X.

(b) The family Σ_{uRm} of universally Radon-measurable subsets of X is a σ-algebra closed under Souslin's operation and including the algebra Σ_{um} of universally measurable subsets of X (and, *a fortiori*, including the Borel σ-algebra). (Use 434Db and the idea of 434Dc.)

(c) If Y is another topological space, I will say that a function $f : X \to Y$ is **universally Radon-measurable** if $f^{-1}[H] \in \Sigma_{\mathrm{uRm}}$ for every open set $H \subseteq Y$. A function $f : X \to \mathbb{R}$ is universally Radon-measurable iff it is Σ_{uRm}-measurable iff it is μ-virtually measurable (that is, μ-almost continuous, see 418J) for every totally finite tight Borel measure μ on X. (Compare 434Dd.) A universally measurable function is universally Radon-measurable.

434F Elementary properties of Radon spaces: Proposition Let X be a Hausdorff space.

(a) The following are equiveridical:

(i) X is a Radon space;

(ii) every semi-finite Borel measure on X is tight;

(iii) if μ is a locally finite Borel measure on X, its c.l.d. version $\tilde{\mu}$ is a Radon measure;

(iv) whenever μ is a totally finite Borel measure on X, and $G \subseteq X$ is an open set with $\mu G > 0$, then there is a compact set $K \subseteq G$ such that $\mu K > 0$;

(v) whenever μ is a non-zero totally finite Borel measure on X, there is a Radon subspace Y of X such that $\mu^* Y > 0$.

(b) If $Y \subseteq X$ is a subspace which is a Radon space in its induced topology, then Y is universally measurable in X.

(c) If X is a Radon space and $Y \subseteq X$, then Y is Radon iff it is universally measurable in X iff it is universally Radon-measurable in X. In particular, all Borel subsets and all Souslin-F subsets of X are Radon spaces.

(d) The family of Radon subspaces of X is closed under Souslin's operation and set difference.

proof (a)(i)$\Rightarrow$(ii) Let μ be a semi-finite Borel measure on X, $E \subseteq X$ a Borel set and $\gamma < \mu E$. Because μ is semi-finite, there is a Borel set H of finite measure such that $\mu(E \cap H) > \gamma$. Set $\nu F = \mu(F \cap H)$ for every Borel set $F \subseteq X$; then ν is a totally finite Borel measure on X, and $\nu E > \gamma$. Because X is a Radon space, there is a compact set $K \subseteq E$ such that $\nu K \geq \gamma$, and now $\mu K \geq \gamma$. As μ, E and γ are arbitrary, (ii) is true.

(ii)$\Rightarrow$(i) and **(i)$\Rightarrow$(v)** are trivial.

(v)$\Rightarrow$(iv) Assume (v), and let μ be a totally finite Radon measure on X and G a non-negligible open set. Set $\nu E = \mu(E \cap G)$ for every Borel set $E \subseteq X$. Then ν is a non-zero totally finite Borel measure on X, so there is a Radon subspace Y of X such that $\nu^*Y>0$. The subspace measure ν_Y on Y is a Borel measure on Y, so is tight. Since $\nu_Y(Y\setminus G)=\nu(X\setminus G)=0$, $\nu_Y(Y\cap G)>0$ and there is a compact set $K\subseteq Y\cap G$ such that $\nu_YK>0$. Now $\mu K>0$. As μ and G are arbitrary, (iv) is true.

not-(i)⇒not-(iv) If X is not Radon, there is a totally finite Borel measure μ on X which is not tight. By 416F(iii), there is an open set $G\subseteq X$ such that

$$\mu G>\sup_{K\subseteq G\text{ is compact}}\mu K=\gamma$$

say. Let $\mathcal{K}$ be the family of compact subsets of G. By 215B(v), there is a non-decreasing sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $\mu(K\setminus F)=0$ for every $K\in\mathcal{K}$, where $F=\bigcup_{n\in\mathbb{N}}K_n$. Observe that

$$\mu F=\lim_{n\to\infty}\mu K_n\le\gamma<\mu G.$$

Now set $\nu E=\mu(E\cap G\setminus F)$ for every Borel set $E\subseteq X$. Then ν is a Borel measure on X, and $\nu G>0$. If $K\subseteq G$ is compact, then $\nu K=\mu(K\setminus F)=0$. So ν and G witness that (iv) is false.

(i)⇒(iii) The point is that $\tilde\mu$ is tight. **P** If $\tilde\mu E>\gamma$, then, because $\tilde\mu$ is semi-finite, there is a set $E'\subseteq E$ such that $\gamma<\tilde\mu E'<\infty$; now there is a Borel set $H\subseteq E'$ such that $\mu H=\tilde\mu E'$ (213Fc). Setting $\nu F=\mu(H\cap F)$ for every Borel set F, ν is a totally finite Borel measure on X and $\nu H>\gamma$, so there is a compact set $K\subseteq H$ such that $\nu K\ge\gamma$. Since $\mu K<\infty$, $\tilde\mu K=\mu K\ge\gamma$ (213Fa), while $K\subseteq E$. As E and γ are arbitrary, $\tilde\mu$ is tight. **Q**

On the other hand, every point of X belongs to an open set of finite measure for μ, which is still of finite measure for $\tilde\mu$ (213Fa again). So $\tilde\mu$ is locally finite; since it is surely complete and locally determined, it is a Radon measure.

(iii)⇒(i) Assume (iii), and let μ be a totally finite Borel measure on X. Then its c.l.d. version $\tilde\mu$ is tight. But $\tilde\mu$ extends μ (213Hc), so μ also is tight. As μ is arbitrary, X is a Radon space.

(b) Let μ be a totally finite Borel measure on X, and $\hat\mu$ its completion; let $\epsilon>0$. Let μ_Y be the subspace measure on Y, so that μ_Y is a totally finite Borel measure on Y, and is tight. There is a compact set $K\subseteq Y$ such that $\mu_YK\ge\mu_YY-\epsilon$. But this means that

$$\mu^*Y=\mu_YY\le\mu_YK+\epsilon=\mu^*(K\cap Y)+\epsilon=\mu K+\epsilon\le\mu_*Y+\epsilon.$$

As ϵ is arbitrary, $\mu_*Y=\mu^*Y$, and Y is measured by $\hat\mu$ (413Ef); as μ is arbitrary, Y is universally measurable.

(c)(i) If Y is Radon, it is universally measurable, by (b). (ii) If Y is universally measurable, it is universally Radon-measurable, by 434Eb. (iii) Suppose that Y is universally Radon-measurable, and that ν is a totally finite Borel measure on Y. For Borel sets $E\subseteq X$, set $\mu E=\nu(E\cap Y)$. Then μ is a totally finite Borel measure on X, so its c.l.d. version $\tilde\mu$ is a Radon measure on X, by (a-iii). We are supposing that Y is universally Radon-measurable, so, in particular, it must be measured by $\tilde\mu$. We have

$$\tilde\mu(X\setminus Y)=\sup_{K\subseteq X\setminus Y\text{ is compact}}\tilde\mu K=\sup_{K\subseteq X\setminus Y\text{ is compact}}\mu K$$

(213Ha, because μ is totally finite)

$$=\sup_{K\subseteq X\setminus Y\text{ is compact}}\nu(K\cap Y)=0,$$

and Y is $\tilde\mu$-conegligible.

Now suppose that $E\subseteq Y$ is a (relatively) Borel subset of Y. Then E is of the form $F\cap Y$ where F is a Borel subset of X, so that

$$\begin{aligned}\nu E&=\mu F=\tilde\mu F=\tilde\mu(Y\cap F)=\tilde\mu E\\&=\sup_{K\subseteq E\text{ is compact}}\mu K=\sup_{K\subseteq E\text{ is compact}}\nu K.\end{aligned}$$

As E is arbitrary, ν is tight; as ν is arbitrary, Y is a Radon space.

By 434Dc, it follows that all Borel subsets and all Souslin-F subsets of X are Radon spaces.

(d) The first step is to note that if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of Radon subspaces of X with union E, then E is Radon; this is immediate from (a-v) above.

Now let $\langle E_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme, consisting of Radon subsets of X, with kernel A. We know that $E=\bigcup_{\sigma\in S}E_\sigma$ is a Radon space. Every E_σ is universally measurable in E, by (b), so A also is (434Dc), and must be Radon, by (c). Thus the family of Radon subspaces of X is closed under Souslin's operation.

If E and F are Radon subsets of X, then $E\cup F$ is Radon, and, just as above, F is universally measurable in $E\cup F$. But this means that $E\setminus F=(E\cup F)\setminus F$ is universally measurable in $E\cup F$, so that $E\setminus F$ is Radon.

434G Just as we can address the question 'when can we be sure that every Borel measure is of class D?' in terms of the definition of 'Radon' space (434C), we can form other classes of topological space by declaring that the Borel measures they support must be of certain kinds. Three definitions which lead to interesting patterns of ideas are the following.

Definitions (a) A topological space X is **Borel-measure-compact** (Gardner & Pfeffer 84) if every totally finite Borel measure on X which is inner regular with respect to the closed sets is τ-additive, that is, X carries no measure of class B in the classification of 434A.

(b) A topological space X is **Borel-measure-complete** (Gardner & Pfeffer 84) if every totally finite Borel measure on X is τ-additive. (If X is regular and Hausdorff, this amounts to saying that X carries no measures of classes A or B in the classification of 434A.)

(c) A Hausdorff space X is **pre-Radon** (also called '**hypo-radonian**', '**semi-radonian**') if every τ-additive totally finite Borel measure on X is tight. (If X is regular, this amounts to saying that X carries no measure of class C in the classification of 434A.)

434H Proposition Let X be a topological space and $\mathcal{B}$ its Borel σ-algebra.

(a) The following are equiveridical:

(i) X is Borel-measure-compact;

(ii) every semi-finite Borel measure on X which is inner regular with respect to the closed sets is τ-additive;

(iii) every effectively locally finite Borel measure on X which is inner regular with respect to the closed sets has an extension to a quasi-Radon measure;

(iv) every totally finite Borel measure on X which is inner regular with respect to the closed sets has a support;

(v) if μ is a non-zero totally finite Borel measure on X, inner regular with respect to the closed sets, and $\mathcal{G}$ is an open cover of X, then there is some $G\in\mathcal{G}$ such that $\mu G>0$.

(b) If X is Lindelöf (in particular, if X is a K-analytic Hausdorff space), it is Borel-measure-compact.

(c) If X is Borel-measure-compact and $A\subseteq X$ is a Souslin-F set, then A is Borel-measure-compact in its subspace topology. In particular, any Baire subset of X is Borel-measure-compact.

proof (a)(i)⇒(ii) Assume (i), and let μ be a semi-finite Borel measure on X which is inner regular with respect to the closed sets. Let $\mathcal{G}$ be an upwards-directed family of open sets with union G^*, and $\gamma<\mu G^*$. Because μ is semi-finite, there is an $H\in\mathcal{B}$ such that $\mu H<\infty$ and $\mu(H\cap G^*)\ge\gamma$. Set $\nu E=\mu(E\cap H)$ for every $E\in\mathcal{B}$; then ν is a totally finite Borel measure on X. For any $E\in\mathcal{B}$,

$$\nu E=\mu(E\cap H)=\sup\{\mu F:F\subseteq E\cap H\text{ is closed}\}\le\sup\{\nu F:F\subseteq E\text{ is closed}\},$$

so ν is inner regular with respect to the closed sets, and must be τ-additive. Now

$$\gamma\le\nu G^*=\sup_{G\in\mathcal{G}}\nu G\le\sup_{G\in\mathcal{G}}\mu G.$$

As γ and $\mathcal{G}$ is arbitrary, μ is τ-additive.

(ii)⇒(iii) Assume (ii), and let μ be an effectively locally finite Borel measure on X which is inner regular with respect to the closed sets. Then it is semi-finite (411Gd), therefore τ-additive. By 415L, it has an extension to a quasi-Radon measure on X.

(iii)⇒(i) If (iii) is true and μ is a totally finite Borel measure on X which is inner regular with respect to the closed sets, then μ has an extension to a quasi-Radon measure, which is τ-additive, so μ also is τ-additive (411C).

(ii)⇒(iv) Use 411Nd.

(iv)⇒(v) Suppose that (iv) is true, that μ is a non-zero totally finite Borel measure on X which is inner regular with respect to the closed sets, and that $\mathcal{G}$ is an open cover of X. If F is the support of μ, then $\mu F > 0$ so $F \neq \emptyset$; there must be some $G \in \mathcal{G}$ meeting F, and now $\mu G > 0$.

not-(i)⇒not-(v) Suppose that there is a totally finite Borel measure μ on X, inner regular with respect to the closed sets, which is not τ-additive. Let $\mathcal{G}$ be an upwards-directed family of open sets such that $\mu G^* > \gamma$, where $G^* = \bigcup \mathcal{G}$ and $\gamma = \sup_{G \in \mathcal{G}} \mu G$. Let $\langle G_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\mu(G \setminus G_0^*)$ for every $G \in \mathcal{G}$, where $G_0^* = \bigcup_{n \in \mathbb{N}} G_n$ (215B(v) again). Then $\mu G_0^* \leq \gamma$, so there is a closed set $F \subseteq G^* \setminus G_0^*$ such that $\mu F > 0$.

Let ν be the Borel measure on X defined by setting $\nu E = \mu(E \cap F)$ for every $E \in \mathcal{B}$. As in the argument for (i)⇒(ii), ν is inner regular with respect to the closed sets. Consider $\mathcal{H} = \mathcal{G} \cup \{X \setminus F\}$; this is an open cover of X. If $G \in \mathcal{G}$ then $\nu G \leq \mu(G \setminus G_0^*) = 0$, so $\nu H = 0$ for every $H \in \mathcal{H}$; thus ν and $\mathcal{H}$ witness that (v) is false.

(b) Use (a-v) with 422Gg.

(c) Let μ be a Borel measure on A which is inner regular with respect to the closed sets, that is to say, the relatively closed sets in A. Let ν be the corresponding Borel measure on X, defined by setting $\nu E = \mu(A \cap E)$ for every $E \in \mathcal{B}$. Let $\hat{\nu}$ be the completion of ν. Putting 431D and 421M together, we see that $\hat{\nu} A = \sup\{\hat{\nu} F : F \subseteq A \text{ is closed in } X\}$, that is, $\nu X = \sup\{\mu F : F \subseteq A \text{ is closed in } X\}$. But this means that if $E \in \mathcal{B}$ and $\gamma < \nu E$, there is a closed set F in X such that $F \subseteq A$ and $\mu(E \cap F) > \gamma$; now there is a relatively closed set $F' \subseteq A$ such that $F' \subseteq E \cap F$ and $\mu F' \geq \gamma$, and as F' must be relatively closed in F it is closed in X, while $\nu F' \geq \gamma$. Since E and γ are arbitrary, ν is inner regular with respect to the closed sets, and will be τ-additive.

Now suppose that $\mathcal{G}$ is an upwards-directed family of relatively open subsets of A. Set $\mathcal{H} = \{H : H \subseteq X$ is open, $H \cap A \in \mathcal{G}\}$. Then $\mathcal{H}$ is upwards-directed, so

$$\mu(\bigcup \mathcal{G}) = \nu(\bigcup \mathcal{H}) = \sup_{H \in \mathcal{H}} \nu H = \sup_{G \in \mathcal{G}} \mu G.$$

As μ and $\mathcal{G}$ are arbitrary, A is Borel-measure-compact.

By 421L, it follows that any Baire subset of X is Borel-measure-compact.

434I Proposition Let X be a topological space.

(a) The following are equiveridical:

(i) X is Borel-measure-complete;

(ii) every semi-finite Borel measure on X is τ-additive;

(iii) every totally finite Borel measure on X has a support;

(iv) whenever μ is a totally finite Borel measure on X there is a base $\mathcal{U}$ for the topology of X such that $\mu(\bigcup\{U : U \in \mathcal{U}, \mu U = 0\}) = 0$.

(b) If X is regular, it is Borel-measure-complete iff every effectively locally finite Borel measure on X has an extension to a quasi-Radon measure.

(c) If X is Borel-measure-complete, it is Borel-measure-compact.

(d) If X is Borel-measure-complete, so is every subspace of X.

(e) If X is hereditarily Lindelöf (for instance, if X is separable and metrizable, see 4A2P(a-iii)), it is Borel-measure-complete, therefore Borel-measure-compact.

proof (a)(i)⇒(ii) Use the argument of (i)⇒(ii) of 434Ha; this case is simpler, because we do not need to check that the auxiliary measure ν is inner regular.

(ii)⇒(i) is trivial.

(i)⇒(iv) If X is Borel-measure-complete and μ is a totally finite Borel measure on X, take $\mathcal{U}$ to be the family of all open subsets of X. This is surely a base for the topology, and setting $\mathcal{U}_0 = \{U : U \in \mathcal{U}, \mu U = 0\}$, $\mathcal{U}_0$ is upwards-directed so $\mu(\bigcup \mathcal{U}_0) = \sup_{U \in \mathcal{U}_0} \mu U = 0$, as required.

(iv)⇒(iii) Assume (iv), and let μ be a totally finite Borel measure on X. Take a base $\mathcal{U}$ as in (iv), so that $\mu(\bigcup\mathcal{U}_0)=0$, where $\mathcal{U}_0$ is the family of negligible members of $\mathcal{U}$. Set $F=X\setminus\bigcup\mathcal{U}_0$, so that F is a conegligible closed set. If $G\subseteq X$ is an open set meeting F, there is a member U of $\mathcal{U}$ such that $U\subseteq G$ and $U\cap F\ne\emptyset$; now $U\notin\mathcal{U}$ so

$$\mu(G\cap F)=\mu G\ge\mu U>0.$$

As G is arbitrary, F is self-supporting and is the support of μ.

(iii)⇒(i) Assume (iii), and let μ be a totally finite Borel measure on X. Let $\mathcal{G}$ be an upwards-directed family of open sets with union G^*. Set $\gamma=\sup_{G\in\mathcal{G}}\mu G$. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\mu(G\setminus G_0^*)$ for every $G\in\mathcal{G}$, where $G_0^*=\bigcup_{n\in\mathbb{N}}G_n$ (215B(v) once more). Then $\mu G_0^*\le\gamma$. Let ν be the Borel measure on X defined by setting $\mu E=\mu(E\cap G^*\setminus G_0^*)$ for every Borel set $E\subseteq X$. Then ν has a support F say. Now $\nu G=0$ for every $G\in\mathcal{G}$, so $F\cap G=\emptyset$ for every $G\in\mathcal{G}$, and $F\cap G^*=\emptyset$; but this means that

$$\mu(G^*\setminus G_0^*)=\nu X=\nu F=\mu(F\cap G^*\setminus G_0^*)=0.$$

Accordingly $\mu G^*=\gamma$. As μ and $\mathcal{G}$ are arbitrary, X is Borel-measure-complete.

(b) If X is Borel-measure-complete and μ is an effectively locally finite Borel measure on X, then μ is τ-additive, by (a-ii), so extends to a quasi-Radon measure on X, by 415Cb. If effectively locally finite Borel measures on X extend to quasi-Radon measures, then any totally finite Borel measure is τ-additive, by 411C, and X is Borel-measure-complete.

(c) Immediate from the definitions.

(d) If $Y\subseteq X$ and μ is a totally finite Borel measure on Y, let ν be the Borel measure on X defined by setting $\nu E=\mu(E\cap Y)$ for every Borel set $E\subseteq X$. Then ν is τ-additive. So if $\mathcal{G}$ is an upwards-directed family of relatively open subsets of Y, and we set $\mathcal{H}=\{H:H\subseteq X\text{ is open, }H\cap Y\in\mathcal{G}\}$, we shall get

$$\mu(\bigcup\mathcal{G})=\nu(\bigcup\mathcal{H})=\sup_{H\in\mathcal{H}}\nu H=\sup_{G\in\mathcal{G}}\mu G.$$

As μ and $\mathcal{G}$ are arbitrary, Y is Borel-measure-complete.

(e) If μ is a totally finite Borel measure on X and $\mathcal{G}$ is a non-empty upwards-directed family of open subsets of X with union G^*, then there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ with union G^*, by 4A2H(c-i). Because $\mathcal{G}$ is upwards-directed, there is a non-decreasing sequence $\langle G'_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ such that $G'_n\supseteq G_n$ for every $n\in\mathbb{N}$, so that

$$\mu G^*=\lim_{n\to\infty}\mu G'_n\le\sup_{G\in\mathcal{G}}\mu G.$$

As μ and $\mathcal{G}$ are arbitrary, X is Borel-measure-complete.

434J Proposition Let X be a Hausdorff space.

(a) The following are equiveridical:

(i) X is pre-Radon;

(ii) every effectively locally finite τ-additive Borel measure on X is tight;

(iii) whenever μ is a non-zero totally finite τ-additive Borel measure on X, there is a compact set $K\subseteq X$ such that $\mu K>0$;

(iv) whenever μ is a totally finite τ-additive Borel measure on X, $\mu X=\sup_{K\subseteq X\text{ is compact}}\mu K$;

(v) whenever μ is a locally finite effectively locally finite τ-additive Borel measure on X, the c.l.d. version of μ is a Radon measure on X.

(b) If X is pre-Radon, then every locally finite quasi-Radon measure on X is a Radon measure.

(c) If X is regular and every totally finite quasi-Radon measure on X is a Radon measure, then X is pre-Radon.

(d) If X is pre-Radon, then any universally Radon-measurable subspace (in particular, any Borel subset or Souslin-F subset) of X is pre-Radon.

(e) If $A\subseteq X$ is pre-Radon in its subspace topology, it is universally Radon-measurable in X.

(f) If X is K-analytic (for instance, if it is compact), it is pre-Radon.

(g) If X is completely regular and Čech-complete (for instance, if it is locally compact (4A2Gk), or metrizable and complete under a metric inducing its topology (4A2Md)), it is pre-Radon.

(h) If $X = \prod_{i\in I} X_i$ where $\langle X_i\rangle_{i\in I}$ is a countable family of pre-Radon Hausdorff spaces, then X is pre-Radon.

(i) If every point of X belongs to a pre-Radon open subset of X, then X is pre-Radon.

proof (a)(i)$\Rightarrow$(ii) Suppose that X is pre-Radon, that μ is an effectively locally finite τ-additive Borel measure on X, that $E \subseteq X$ is Borel, and that $\gamma < \mu E$. Because μ is semi-finite, there is a Borel set $H \subseteq E$ of finite measure such that $\mu(H \cap E) > \gamma$. Set $\nu F = \mu(F \cap H)$ for every Borel set $F \subseteq X$; then ν is a totally finite Borel measure on X, and is τ-additive by 414Ea. Now $\nu E > \gamma$, so there is a compact set $K \subseteq E$ such that $\gamma \le \nu K \le \mu K$. As E is arbitrary, μ is tight.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(iv) Assume (iii), and let μ be a totally finite τ-additive Borel measure on X. Let $\mathcal{K}$ be the family of compact subsets of X and set $\alpha = \sup_{K\in\mathcal{K}} \mu K$. **?** Suppose, if possible, that $\mu X > \alpha$. Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}$ such that $\sup_{n\in\mathbb{N}} \mu K_n = \alpha$, and set $L = \bigcup_{n\in\mathbb{N}} K_n$; then

$$\mu L = \lim_{n\to\infty} \mu(\bigcup_{i\le n} K_i) = \alpha.$$

Set $\nu E = \mu(E \setminus L)$ for every Borel set $E \subseteq X$. Then ν is a non-zero totally finite Borel measure on X, and is τ-additive, by 414Ea again. So there is a $K \in \mathcal{K}$ such that $\nu K > 0$. But now there is an $n \in \mathbb{N}$ such that $\nu K + \mu K_n > \alpha$, and in this case $K \cup K_n \in \mathcal{K}$ and

$$\mu(K \cup K_n) = \mu(K \setminus K_n) + \mu K_n \ge \nu K + \mu K_n > \alpha,$$

which is impossible. **X** So $\mu X = \alpha$, as required.

(iv)$\Rightarrow$(i) Assume (iv), and let μ be a totally finite τ-additive Borel measure on X. Suppose that $E \subseteq X$ is Borel and that $\gamma < \mu E$. By (iv), there is a compact set $K \subseteq X$ such that $\mu K > \mu X - \mu E + \gamma$, so that $\mu(E \cap K) > \gamma$. Consider the subspace measure μ_K on the compact Hausdorff space K. By 414K, μ_K is τ-additive, so is inner regular with respect to the closed subsets of K (414Mb). There is therefore a relatively closed subset F of K such that $F \subseteq K \cap E$ and $\mu_K F \ge \gamma$; but now F is a compact subset of E and $\mu F \ge \gamma$. As E and γ are arbitrary, μ is tight. As μ is arbitrary, X is pre-Radon.

(ii)$\Rightarrow$(v) Assume (ii), and let μ be a locally finite effectively locally finite τ-additive Borel measure on X. Then μ is tight, so by 416F(ii) its c.l.d. version is a Radon measure.

(v)$\Rightarrow$(iv) Assume (v), and let μ be a totally finite τ-additive Borel measure on X. Then the c.l.d. version $\tilde\mu$ of μ is a Radon measure; but $\tilde\mu$ extends μ (213Hc), so

$$\sup_{K\subseteq X \text{ is compact}} \mu K = \sup_{K\subseteq X \text{ is compact}} \tilde\mu K = \tilde\mu X = \mu X.$$

(b) Let μ be a locally finite quasi-Radon measure on X. By (a-ii), applied to the restriction of μ to the Borel σ-algebra of X, μ is tight; by 416C, μ is a Radon measure.

(c) Let μ be a totally finite τ-additive Borel measure on X. Because X is regular, the c.l.d. version $\tilde\mu$ of μ is a quasi-Radon measure (415Cb again), therefore a Radon measure; but $\tilde\mu$ extends μ (213Hc once more), so μ, like $\tilde\mu$, must be tight. As μ is arbitrary, X is pre-Radon.

(d) Let A be a universally Radon-measurable subset of X, and μ a totally finite τ-additive Borel measure on A. Set $\nu E = \mu(E \cap A)$ for every Borel set $E \subseteq X$; then ν is a totally finite τ-additive Borel measure on X. So its c.l.d. version (that is, its completion $\hat\nu$, by 213Ha) is a Radon measure on X, by (a-v). Now $\hat\nu$ measures A, so

$$\mu A = \nu^* A = \hat\nu A = \sup\{\hat\nu K : K \subseteq A \text{ is compact}\} = \sup\{\mu K : K \subseteq A \text{ is compact}\}.$$

By (a-iv), A is pre-Radon.

(e) Let μ be a totally finite Radon measure on X. Then the subspace measure μ_A is τ-additive (414K), so its restriction ν to the Borel σ-algebra of A is still τ-additive. Because A is pre-Radon,

$$\mu^*A = \mu_A A = \nu A = \sup\{\nu K : K \subseteq A \text{ is compact}\}$$
$$= \sup\{\mu K : K \subseteq A \text{ is compact}\} = \mu_* A,$$

and μ measures A (413Ef again). As μ is arbitrary, A is universally Radon-measurable.

(f) Put 432B and (a-iv) together.

(g) If we identify X with a G_δ set in a compact Hausdorff space Z, then Z is pre-Radon, by (f), so X is pre-Radon, by (d).

(h) Let μ be a totally finite τ-additive Borel measure on X, and $\epsilon > 0$. Let $\langle \epsilon_i \rangle_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \epsilon_i \le \epsilon$ (4A1P). For each $i \in I$ and Borel set $F \subseteq X_i$, set $\mu_i F = \mu\pi_i^{-1}[F]$, where $\pi_i(x) = x(i)$ for $x \in X$; because $\pi_i : X \to X_i$ is continuous, μ_i is a totally finite τ-additive Borel measure on X_i. Because X_i is pre-Radon, we can find a compact set $K_i \subseteq X_i$ such that $\mu_i(X_i \setminus K_i) \le \epsilon_i$, by (a-iv). Now $K = \prod_{i\in I} K_i$ is compact (3A3J), and $X \setminus K \subseteq \bigcup_{i\in I} \pi_i^{-1}[X_i \setminus K_i]$, so

$$\mu(X \setminus K) \le \sum_{i\in I} \mu_i(X_i \setminus K_i) \le \sum_{i\in I} \epsilon_i \le \epsilon.$$

As ϵ and μ are arbitrary, X satisfies the condition of (a-iv), and is pre-Radon.

(i) Let $\mathcal{G}$ be a cover of X by pre-Radon open sets. Let μ be a non-zero totally finite τ-additive Borel measure on X. Then $\mu X = \sup\{\mu(\bigcup \mathcal{G}_0) : \mathcal{G}_0 \subseteq \mathcal{G} \text{ is finite}\}$, so there is some $G \in \mathcal{G}$ such that $\mu G > 0$. Now the subspace measure μ_G is a non-zero totally finite τ-additive Borel measure on G, so there is a compact set $K \subseteq G$ such that $\mu_G K > 0$, in which case $\mu K > 0$. As μ is arbitrary, X is pre-Radon, by (a-iii).

434K I return to criteria for deciding whether Hausdorff spaces are Radon.

Proposition (a) A Hausdorff space is Radon iff it is Borel-measure-complete and pre-Radon.

(b) An analytic Hausdorff space is Radon. In particular, any compact metrizable space is Radon and any Polish space is Radon.

(c) ω_1 and $\omega_1 + 1$, with their order topologies, are not Radon.

(d) For a set I, $[0,1]^I$ is Radon iff $\{0,1\}^I$ is Radon iff I is countable.

(e) A hereditarily Lindelöf K-analytic Hausdorff space is Radon; in particular, the split interval (343J, 419L) is Radon.

proof (a) Put the definitions 434C, 434Gb and 434Gc together, recalling that a tight measure is necessarily τ-additive (411E).

(b) 433Cb.

(c) Dieudonné's measure (411Q) is a Borel measure on ω_1 which is not tight, so ω_1 is certainly not a Radon space; as it is an open set in $\omega_1 + 1$, and the subspace topology inherited from $\omega_1 + 1$ is the order topology of ω_1 (4A2S(a-iii)), $\omega_1 + 1$ cannot be Radon (434Fc).

(d) If I is countable, then $\{0,1\}^I$ and $[0,1]^I$ are compact metrizable spaces, so are Radon. If I is uncountable, then $\omega_1 + 1$, with its order topology, is homeomorphic to a closed subset of $\{0,1\}^I$. **P** Set $\kappa = \#(I)$. For $\xi \le \omega_1$, $\eta < \kappa$ set $x_\xi(\eta) = 1$ if $\eta < \xi$, 0 if $\xi \le \eta$. The map $\xi \mapsto x_\xi : \omega_1 + 1 \to \{0,1\}^\kappa$ is injective because $\kappa \ge \omega_1$, and is continuous because all the sets $\{\xi : x_\xi(\eta) = 0\} = (\omega_1 + 1) \cap (\eta + 1)$ are open-and-closed in $\omega_1 + 1$. Since $\omega_1 + 1$ is compact in its order topology (4A2S(a-i)), it is homeomorphic to its image in $\{0,1\}^\kappa \cong \{0,1\}^I$. **Q**

By 434Fc, $\{0,1\}^I$ cannot be a Radon space. Since $\{0,1\}^I$ is a closed subset of $[0,1]^I$, $[0,1]^I$ also is not a Radon space.

(e) Suppose that X is a hereditarily Lindelöf K-analytic Hausdorff space. Then it is Borel-measure-complete by 434Ie and pre-Radon by 434Jf, so by (a) here it is Radon.

Since the split interval is compact and Hausdorff and hereditarily Lindelöf (419La), it is a Radon space.

434L It is worth noting an elementary special property of metric spaces.

Proposition If (X,ρ) is a metric space, then any quasi-Radon measure on X is inner regular with respect to the totally bounded subsets of X.

proof Let μ be a quasi-Radon measure on X and Σ its domain. Suppose that $E\in\Sigma$ and $\gamma<\mu E$. Then there is an open set G of finite measure such that $\mu(E\cap G)>\gamma$; set $\delta=\mu(E\cap G)-\gamma$. For $n\in\mathbb{N}$, $I\subseteq X$ set $H(n,I)=\bigcup_{x\in I}\{y:\rho(y,x)<2^{-n}\}$. Then $\{H(n,I):I\in[X]^{<\omega}\}$ is an upwards-directed family of open sets covering X. Because μ is τ-additive, there is a finite set $I_n\subseteq X$ such that $\mu(G\setminus H(n,I_n))\le 2^{-n-1}\delta$. Consider $F=\bigcap_{n\in\mathbb{N}}H(n,I_n)$. This is totally bounded and $\mu(G\setminus F)\le\delta$, so $E\cap F$ is totally bounded and $\mu(E\cap F)\ge\gamma$. As E and γ are arbitrary, μ is inner regular with respect to the totally bounded sets.

434M I turn next to a couple of ideas depending on countable compactness.

Lemma Let X be a countably compact topological space and $\mathcal{E}$ a non-empty family of closed subsets of X with the finite intersection property. Then there is a Borel probability measure μ on X, inner regular with respect to the closed sets, such that $\mu F=1$ for every $F\in\mathcal{E}$.

proof (a) By Zorn's lemma, $\mathcal{E}$ is included in a maximal family $\mathcal{E}^*$ of closed subsets of X with the finite intersection property.

(i) If $F\subseteq X$ is closed and $F\cap F_0\cap\ldots\cap F_n\ne\emptyset$ for every $F_0,\ldots,F_n\in\mathcal{E}^*$, then $\mathcal{E}^*\cup\{F\}$ has the finite intersection property, so $F\in\mathcal{E}^*$.

(ii) If F, $F'\in\mathcal{E}^*$, then $F\cap F'\cap F_0\cap\ldots\cap F_n\ne\emptyset$ for all $F_0,\ldots,F_n\in\mathcal{E}^*$, so $F\cap F'\in\mathcal{E}^*$.

(iii) If $F\subseteq X$ is closed and $F\cap F'\in\mathcal{E}^*$ for every $F'\in\mathcal{E}^*$, then $F\cap F_0\cap\ldots\cap F_n\in\mathcal{E}^*$ for every $F_0,\ldots,F_n\in\mathcal{E}^*$ (because $F_0\cap\ldots\cap F_n\in\mathcal{E}^*$, by (ii)), so $F\in\mathcal{E}^*$.

(iv) If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}^*$, with intersection F, and $F'\in\mathcal{E}^*$, then $F'\cap\bigcap_{i\le n}F_i\ne\emptyset$ for every $n\in\mathbb{N}$. Because X is countably compact, $F'\cap F\ne\emptyset$ (4A2G(f-ii)). As F' is arbitrary, $F\in\mathcal{E}^*$, by (iii). Thus $\mathcal{E}^*$ is closed under countable intersections.

(b) Set

$$\Sigma=\{E:E\subseteq X,\text{ there is an }F\in\mathcal{E}^*\text{ such that either }F\subseteq E\text{ or }F\cap E=\emptyset\},$$

and define $\hat\mu:\Sigma\to\{0,1\}$ by saying that $\hat\mu E=1$ if there is some $F\in\mathcal{E}^*$ such that $F\subseteq E$, 0 otherwise. Then $\hat\mu$ is a probability measure on X. **P**

(i) $\emptyset\in\Sigma$ because $\mathcal{E}^*\supseteq\mathcal{E}$ is not empty.

(ii) $X\setminus E\in\Sigma$ whenever $E\in\Sigma$ because the definition of Σ is symmetric between E and $X\setminus E$.

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is any sequence in Σ with union E, then either there are $n\in\mathbb{N}$ and $F\in\mathcal{E}^*$ such that $F\subseteq E_n\subseteq E$ and $E\in\Sigma$, or for every $n\in\mathbb{N}$ there is an $F_n\in\mathcal{E}^*$ such that $F_n\cap E_n=\emptyset$. In this case $F=\bigcap_{n\in\mathbb{N}}F_n\in\mathcal{E}^*$, by (a-iv), and $E\cap F=\emptyset$, so again $E\in\Sigma$. Thus Σ is a σ-algebra of subsets of X.

(iv) $\hat\mu\emptyset=0$ because $\emptyset$ cannot belong to $\mathcal{E}^*$.

(v) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is any disjoint sequence in Σ with union E, then either there is some n such that $\hat\mu E_n=1$, in which case $\hat\mu E_i=0$ for every $i\ne n$ (because any two members of $\mathcal{E}^*$ must meet) and $\hat\mu E=1=\sum_{i=0}^\infty\hat\mu E_i$, or $\hat\mu E_i=0$ for every i, in which case, just as in (iii), $\hat\mu E=0=\sum_{i=0}^\infty\hat\mu E_i$. Thus $\hat\mu$ is a measure.

(vi) Because $\mathcal{E}^*\ne\emptyset$, $\hat\mu X=1$. Thus $\hat\mu$ is a probability measure. **Q**

(c) If $F\subseteq X$ is a closed set, then either F itself belongs to $\mathcal{E}^*$, so $F\in\Sigma$, or there is some $F'\in\mathcal{E}^*$ such that $F\cap F'=\emptyset$, in which case again $F\in\Sigma$. So Σ contains every closed set, therefore every Borel set, and $\hat\mu$ is a topological measure. By construction, $\hat\mu$ is inner regular with respect to $\mathcal{E}^*$ and therefore with respect to the closed sets. Finally, if $F\in\mathcal{E}$ then $F\in\mathcal{E}^*$, so $\hat\mu F=1$. We may therefore take μ to be the restriction of $\hat\mu$ to the Borel σ-algebra of X, and μ will be a Borel measure on X, inner regular with respect to the closed sets, such that $\mu E=1$ for every $E\in\mathcal{E}$.

434N Proposition (a) Let X be a Borel-measure-compact topological space. Then closed countably compact subsets of X are compact.

(b) Let X be a Borel-measure-complete topological space. Then countably compact subsets of X are compact.

(c) Let X be a Hausdorff Borel-measure-complete topological space. Then compact subsets of X are countably tight.

(d) In particular, any Radon compact Hausdorff space is countably tight.

proof (a) Let C be a closed countably compact subset of X. Let $\mathcal{F}$ be an ultrafilter on C. Let $\mathcal{E}$ be the family of closed subsets of C belonging to $\mathcal{F}$. Then $\mathcal{E}$ has the finite intersection property, so by 434M there is a Borel probability measure μ on C, inner regular with respect to the closed sets, such that $\mu E = 1$ for every $E \in \mathcal{E}$. Let ν be the Borel measure on X defined by setting $\nu H = \mu(C \cap H)$ for every Borel set $H \subseteq X$. Then ν is also inner regular with respect to the closed sets (of either C or X); because X is Borel-measure-compact, ν has a support F (434H(a-iv)). Since $\nu F = \nu X = 1$, $F \cap C \neq \emptyset$; take $x \in F \cap C$. If G is any open set (in X) containing x, then $\mu(C \setminus G) = \nu(X \setminus G) < 1$, so $C \setminus G \notin \mathcal{F}$ and $C \cap G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to x$; as $\mathcal{F}$ is arbitrary, C is compact.

(b) Repeat the argument of (a). Let C be a countably compact subset of X and $\mathcal{F}$ an ultrafilter on C. Let $\mathcal{E}$ be the family of relatively closed subsets of C belonging to $\mathcal{F}$. Then there is a Borel probability measure μ on C such that $\mu E = 1$ for every $E \in \mathcal{E}$. Let ν be the Borel measure on X defined by setting $\nu H = \mu(C \cap H)$ for every Borel set $H \subseteq X$. Because X is Borel-measure-complete, ν has a support F (434I(a-iii)). Since $\nu F = \nu X = 1$, $F \cap C \neq \emptyset$; take $x \in F \cap C$. If G is any open set containing x, then $\nu(X \setminus G) < 1$, so $C \setminus G \notin \mathcal{F}$ and $C \cap G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to x$; as $\mathcal{F}$ is arbitrary, C is compact.

(c) Again let C be a (countably) compact subset of X. Take $A \subseteq C$, and set $C_0 = \bigcup\{\overline{B} : B \in [A]^{\le\omega}\}$. Then C_0 is countably compact. **P** If $\langle y_n \rangle_{n\in\mathbb{N}}$ is any sequence in C_0, it has a cluster point $y \in C$. For each $n \in \mathbb{N}$ there is a countable set $B_n \subseteq A$ such that $y_n \in \overline{B_n}$. Now $B = \bigcup_{n\in\mathbb{N}} B_n$ is a countable subset of A, and $y \in \overline{B} \subseteq C_0$, so y is a cluster point of $\langle y_n \rangle_{n\in\mathbb{N}}$ in C_0. As $\langle y_n \rangle_{n\in\mathbb{N}}$ is arbitrary, C_0 is countably compact. **Q**

By (b), C_0 is compact, therefore closed, and must include $\overline{A}$. Thus every point of $\overline{A}$ is in the closure of some countable subset of A. As A is arbitrary, C is countably tight.

(d) Finally, a compact Radon Hausdorff space is Borel-measure-complete (434Ka) and countably compact, therefore countably tight.

434O Quasi-dyadic spaces I wish now to present a result in an entirely different direction. Measures of type B_1 in the classification of 434A (completion regular, but not τ-additive) seem to be hard to come by. The next theorem shows that on a substantial class of spaces they cannot appear. First, we need a definition.

Definition A topological space X is **quasi-dyadic** if it is expressible as a continuous image of a product of separable metrizable spaces.

I give some elementary results to indicate what kind of spaces we have here.

434P Proposition (a) A continuous image of a quasi-dyadic space is quasi-dyadic.

(b) Any product of quasi-dyadic spaces is quasi-dyadic.

(c) A space with a countable network is quasi-dyadic.

(d) A Baire subset of a quasi-dyadic space is quasi-dyadic.

(e) If X is any topological space, a countable union of quasi-dyadic subspaces of X is quasi-dyadic.

(f) A quasi-dyadic space is ccc.

proof (a) Immediate from the definition.

(b) Again immediate; if X_i is a continuous image of $\prod_{j\in J_i} Y_{ij}$, where Y_{ij} is a separable metrizable space for every $i \in I$ and $j \in J_i$, then $\prod_{i\in I} X_i$ is a continuous image of $\prod_{i\in I, j\in J_i} Y_{ij}$.

(c) Let $\mathcal{E}$ be a countable network for the topology of X. On X let $\sim$ be the equivalence relation in which $x \sim y$ if they belong to just the same members of $\mathcal{E}$; let Y be the space $X/\sim$ of equivalence classes, and $\phi : X \to Y$ the canonical map. Y has a separable metrizable topology with base $\{\phi[E] : E \in \mathcal{E}\} \cup \{\phi[X \setminus E] : E \in \mathcal{E}\}$. Let I be any set such that $\#(\{0,1\}^I) \ge \#(X)$, and for each $y \in Y$ let $f_y : \{0,1\}^I \to y$ be a surjection. Then we have a continuous surjection $f : Y \times \{0,1\}^I \to X$ given by saying that $f(y,z) = f_y(z)$ for $y \in Y$ and $z \in \{0,1\}^I$.

(d) Let $\langle Y_i \rangle_{i\in I}$ be a family of separable metrizable spaces with product Y and $f : Y \to X$ a continuous surjection. If $W \subseteq Y$ is a Baire set, it is determined by coordinates in a countable subset of I (4A3Nb), so can be regarded as $W' \times \prod_{i\in I\setminus J} Y_i$, where $J \subseteq I$ is countable and $W' \subseteq \prod_{i\in J} Y_i$; as $\prod_{i\in J} Y_i$ and W' are separable metrizable spaces (4A2Pa), W can be thought of as a product of separable metrizable spaces. Now the set $\{E : E \subseteq X,\, f^{-1}[E]$ is a Baire set in $Y\}$ is a σ-algebra containing every zero set in X, so contains every Baire set. Thus every Baire subset of X is a continuous image of a Baire subset of Y, and is therefore quasi-dyadic.

(e) If $E_n \subseteq X$ is quasi-dyadic for each $n \in \mathbb{N}$, then $Z = \mathbb{N} \times \prod_{n\in\mathbb{N}} E_n$ is quasi-dyadic, and $f : Z \to \bigcup_{n\in\mathbb{N}} E_n$ is a continuous surjection, where $f(n, \langle x_i\rangle_{i\in\mathbb{N}}) = x_n$. So $\bigcup_{n\in\mathbb{N}} E_n$ is quasi-dyadic.

(f) Use 4A2E(a-iii) and (a-iv).

434Q Theorem (Fremlin & Grekas 95) A semi-finite completion regular topological measure on a quasi-dyadic space is τ-additive.

proof ? Suppose, if possible, otherwise.

(a) The first step is the standard reduction to the case in which $\mu X = 1$ and X is covered by open sets of zero measure. In detail: suppose that X is a quasi-dyadic space and μ_0 is a semi-finite completion regular topological measure on X which is not τ-additive. Let $\mathcal{G}$ be an upwards-directed family of open sets in X such that $\mu_0(\bigcup\mathcal{G})$ is strictly greater than $\sup_{G\in\mathcal{G}} \mu_0 G = \gamma$ say. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\lim_{n\to\infty} \mu_0 G_n = \gamma$, and set $H_0 = \bigcup_{n\in\mathbb{N}} G_n$, so that $\mu_0 H_0 = \gamma$; take a closed set $Z \subseteq \bigcup\mathcal{G}$ such that $\gamma < \mu_0 Z < \infty$. Set $\mu_1 E = \mu_0(E \cap Z \setminus H_0)$ for every Borel set $E \subseteq X$. Then μ_1 is a non-zero totally finite Borel measure on X, and is completion regular. **P** If $E \subseteq X$ is a Borel set and $\epsilon > 0$, there is a zero set $F \subseteq E \cap Z \setminus H_0$ such that $\mu_0 F \ge \mu_0(E \cap X \setminus H_0) - \epsilon$, and now $\mu_1 F \ge \mu_1 E - \epsilon$. **Q** Note that $\mu_1(X \setminus Z) = \mu_1 G = 0$ for every $G \in \mathcal{G}$.

For Borel sets $E \subseteq X$, set $\mu E = \mu_1 E/\mu_1 X$; then μ is a completion regular Borel probability measure on X, and $\mathcal{G} \cup \{X \setminus Z\}$ is a cover of X by open negligible sets.

(b) Now let $\langle Y_i\rangle_{i\in I}$ be a family of separable metrizable spaces such that there is a continuous surjection $f : Y \to X$, where $Y = \prod_{i\in I} Y_i$. For each $i \in I$ let $\mathcal{B}_i$ be a countable base for the topology of Y_i. For $J \subseteq I$ let $\mathcal{C}(J)$ be the family of all non-empty open cylinders in Y expressible in the form

$$\{s : s(i) \in B_i \ \forall\, i \in K\}$$

where K is a finite subset of J and $B_i \in \mathcal{B}_i$ for each $i \in K$; thus $\mathcal{C}(I)$ is a base for the topology of Y. Set $\mathcal{C}_0(J) = \{U : U \in \mathcal{C}(J),\, \mu^* f[U] = 0\}$ for each $J \subseteq I$. Note that (because every $\mathcal{B}_i$ is countable) $\mathcal{C}(J)$ and $\mathcal{C}_0(J)$ are countable for every countable subset J of I. It is easy to see that $\mathcal{C}(J) \cap \mathcal{C}(K) = \mathcal{C}(J \cap K)$ for all J, $K \subseteq I$, because if $U \in \mathcal{C}(I)$ is non-empty it belongs to $\mathcal{C}(J)$ iff its projection onto X_i is the whole of X_i for every $i \notin J$.

For each negligible set $E \subseteq X$, let $\langle F_n(E)\rangle_{n\in\mathbb{N}}$ be a family of zero sets, subsets of $X \setminus E$, such that $\sup_{n\in\mathbb{N}} \mu F_n(E) = 1$. Then each $f^{-1}[F_n(E)]$ is a zero set in Y, so there is a countable set $M(E) \subseteq I$ such that all the sets $f^{-1}[F_n(E)]$ are determined by coordinates in $M(E)$ (4A3Nc). Let $\mathcal{J}$ be the family of countable subsets J of I such that $M(f[U]) \subseteq J$ for every $U \in \mathcal{C}_0(J)$; then $\mathcal{J}$ is cofinal with $[I]^{\le\omega}$, that is, every countable subset of I is included in some member of $\mathcal{J}$. **P** If we start from any countable subset J_0 of I and set

$$J_{n+1} = J_n \cup \bigcup\{M(f[U]) : U \in \mathcal{C}_0(J_n)\}$$

for each $n \in \mathbb{N}$, then every J_n is countable, and $\bigcup_{n\in\mathbb{N}} J_n \in \mathcal{J}$, because $\langle J_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so $\mathcal{C}_0(\bigcup_{n\in\mathbb{N}} J_n) = \bigcup_{n\in\mathbb{N}} \mathcal{C}_0(J_n)$. **Q**

(c) For each $J \in \mathcal{J}$, set

$$Q_J = \bigcap\{\textstyle\bigcup_{n\in\mathbb{N}} F_n(f[U]) : U \in \mathcal{C}_0(J)\}.$$

Then $\mu Q_J = 1$ and $f^{-1}[Q_J]$ is determined by coordinates in J, while $f^{-1}[Q_J] \cap U = \emptyset$ whenever $U \in \mathcal{C}_0(J)$.

If $G \subseteq X$ is an open set, then $G \cap Q_J = \emptyset$ whenever $J \in \mathcal{J}$ and there is a negligible Baire set $Q \supseteq G$ such that $f^{-1}[Q]$ is determined by coordinates in J. **P** Set $H = \pi_J^{-1}[\pi_J[f^{-1}[G]]]$, where $\pi_J : Y \to \prod_{i\in J} Y_i$ is the canonical map; then H is a union of members of $\mathcal{C}(J)$, because $f^{-1}[G]$ is open in Y and $\pi_J[f^{-1}[G]]$ is open in $\prod_{i\in J} Y_i$. Also, because $f^{-1}[Q]$ is determined by coordinates in J, $H \subseteq f^{-1}[Q]$, so $f[H] \subseteq Q$ and

$\mu^* f[H] = 0$; thus all the members of $\mathcal{C}(J)$ included in H actually belong to $\mathcal{C}_0(J)$, and $H \cap f^{-1}[Q_J] = \emptyset$. But this means that $f^{-1}[G] \cap f^{-1}[Q_J] = \emptyset$ and (because f is a surjection) $G \cap Q_J = \emptyset$, as claimed. **Q** In particular, if G is a negligible open set in X, then $G \cap Q_J = \emptyset$ whenever $J \in \mathcal{J}$ and $J \supseteq M(G)$.

(d) If $J \in \mathcal{J}$, there are s, $s' \in f^{-1}[Q_J]$ such that $s{\restriction}J = s'{\restriction}J$ and $f(s)$, $f(s')$ can be separated by open sets in X. **P** Start from any $x \in Q_J$ and take a negligible open set G containing x (recall that our hypothesis is that X is covered by negligible open sets). For each $n \in \mathbb{N}$ let $h_n : X \to \mathbb{R}$ be a continuous function such that $F_n(G) = h_n^{-1}[\{0\}]$. We know that $G \cap Q_J \ne \emptyset$, while $G \subseteq X \setminus (\bigcup_{n\in\mathbb{N}} F_n(G) \cap Q_J)$, which is a negligible Baire set; by (c), $f^{-1}[X \setminus (\bigcup_{n\in\mathbb{N}} F_n(G) \cap Q_J)]$ is not determined by coordinates in J, and there must be some n such that $f^{-1}[F_n(G) \cap Q_J]$ is not determined by coordinates in J. Accordingly there must be s, $s' \in Y$ such that $s{\restriction}J = s'{\restriction}J$, $s \in f^{-1}[F_n(G) \cap Q_J]$ and $s' \notin f^{-1}[F_n(G) \cap Q_J]$. Now $s \in f^{-1}[Q_J]$, which is determined by coordinates in J; since $s{\restriction}J = s'{\restriction}J$, $s' \in f^{-1}[Q_J]$ and $s' \notin f^{-1}[F_n(G)]$. Accordingly $h_n(f(s)) = 0 \ne h_n(f(s'))$ and $f(s)$, $f(s')$ can be separated by open sets. **Q**

(e) We are now ready to embark on the central construction of the argument. We can choose inductively, for ordinals $\xi < \omega_1$, sets $J_\xi \in \mathcal{J}$, negligible open sets G_ξ, $G'_\xi \subseteq X$, points s_ξ, $s'_\xi \in Y$ and sets U_ξ, V_ξ, $V'_\xi \in \mathcal{C}(I)$ such that

$J_\eta \subseteq J_\xi$, U_η, V_η, V'_η all belong to $\mathcal{C}(J_\xi)$ and $G_\eta \cap Q_{J_\xi} = \emptyset$ whenever $\eta < \xi < \omega_1$ (using the results of (b) and (c) to choose J_ξ);

$s_\xi{\restriction}J_\xi = s'_\xi{\restriction}J_\xi$, $s_\xi \in f^{-1}[Q_{J_\xi}]$ and $f(s_\xi)$ and $f(s'_\xi)$ can be separated by open sets in X (using (d) to choose s_ξ, s'_ξ);

G_ξ, G'_ξ are disjoint negligible open sets containing $f(s_\xi)$, $f(s'_\xi)$ respectively (choosing G_ξ, G'_ξ);

$U_\xi \in \mathcal{C}(J_\xi)$, V_ξ, $V'_\xi \in \mathcal{C}(I \setminus J_\xi)$, $s_\xi \in U_\xi \cap V_\xi \subseteq f^{-1}[G_\xi]$, $s'_\xi \in U_\xi \cap V'_\xi \subseteq f^{-1}[G'_\xi]$ (choosing U_ξ, V_ξ, V'_ξ, using the fact that $s_\xi{\restriction}J_\xi = s'_\xi{\restriction}J_\xi$).

On completing this construction, take for each $\xi < \omega_1$ a finite set $K_\xi \subseteq J_{\xi+1}$ such that U_ξ, V_ξ and V'_ξ all belong to $\mathcal{C}(K_\xi)$. By the Δ-system Lemma (4A1Db), there is an uncountable $A \subseteq \omega_1$ such that $\langle K_\xi\rangle_{\xi\in A}$ is a Δ-system with root K say. For $\xi \in A$, express U_ξ as $\tilde{U}_\xi \cap U'_\xi$ where $\tilde{U}_\xi \in \mathcal{C}(K)$ and $U'_\xi \in \mathcal{C}(K_\xi \setminus K)$. Then there are only countably many possibilities for $\tilde{U}_\xi$, so there is an uncountable $B \subseteq A$ such that $\tilde{U}_\xi$ is constant for $\xi \in B$; write $\tilde{U}$ for the constant value. Let $C \subseteq B$ be an uncountable set, not containing $\min A$, such that $K_\xi \setminus K$ does not meet J_η whenever ξ, $\eta \in C$ and $\eta < \xi$ (4A1Eb). Let $D \subseteq C$ be such that D and $C \setminus D$ are both uncountable.

Note that $K \subseteq K_\eta \subseteq J_\xi$ whenever η, $\xi \in A$ and $\eta < \xi$, so that $K \subseteq J_\xi$ for every $\xi \in C$. Consequently U'_ξ, V_ξ and V'_ξ all belong to $\mathcal{C}(K_\xi \setminus K)$ for every $\xi \in C$.

(f) Consider the open set

$$G = \bigcup_{\xi\in D} G_\xi \subseteq X.$$

At this point the argument divides.

case 1 Suppose $\mu^*(G \cap f[\tilde{U}]) > 0$. Then there is a Baire set $Q \subseteq G$ such that $\mu^*(Q \cap f[\tilde{U}]) > 0$. Let $J \subseteq I$ be a countable set such that $f^{-1}[Q]$ is determined by coordinates in J. Let $\gamma \in C \setminus D$ be so large that $K_\xi \setminus K$ does not meet J for any $\xi \in A$ with $\xi \ge \gamma$. Then $Q \cap Q_{J_\gamma} \cap f[\tilde{U}]$ is not empty; take $s \in \tilde{U} \cap f^{-1}[Q \cap Q_{J_\gamma}]$. Because the $K_\xi \setminus K$ are disjoint from each other and from $J \cup J_\gamma$ for $\xi \in C \setminus (\gamma + 1)$, we can modify s to form s' such that $s'{\restriction}J \cup J_\gamma = s{\restriction}J \cup J_\gamma$ and $s' \in U'_\xi \cap V'_\xi$ whenever $\xi \in C$ and $\xi > \gamma$; now $s' \in \tilde{U}$ (because $K \subseteq J_\gamma$), so $s' \in \tilde{U} \cap U'_\xi \cap V'_\xi \subseteq f^{-1}[G'_\xi]$ and $f(s') \notin G_\xi$ whenever $\xi \in C$ and $\xi > \gamma$. On the other hand, if $\xi \in D$ and $\xi < \gamma$, $G_\xi \cap Q_{J_\gamma} = \emptyset$, while $s' \in f^{-1}[Q_{J_\gamma}]$ (because $f^{-1}[Q_{J_\gamma}]$ is determined by coordinates in J_γ), so again $f(s') \notin G_\xi$.

Thus $f(s') \notin G$. But $s'{\restriction}J = s{\restriction}J$ so $f(s) \in Q \subseteq G$; which is impossible.

This contradiction disposes of the possibility that $\mu^*(G \cap f[\tilde{U}]) > 0$.

case 2 Suppose that $\mu^*(G \cap f[\tilde{U}]) = 0$. In this case there is a negligible Baire set $Q \supseteq G \cap f[\tilde{U}]$. Let $J \subseteq I$ be a countable set such that $f^{-1}[Q]$ is determined by coordinates in J. Let $\gamma < \omega_1$ be such that $J \cap J_\gamma = J \cap \bigcup_{\xi<\omega_1} J_\xi$ and $J \cap K_\xi \setminus K = \emptyset$ for every $\xi \in A \setminus \gamma$. Take $\xi \in D$ such that $\xi \ge \gamma$. Then

$$\tilde{U} \cap U'_\xi \cap V_\xi \subseteq f^{-1}[G_\xi] \cap \tilde{U} \subseteq f^{-1}[G \cap f[\tilde{U}]] \subseteq f^{-1}[Q],$$

so $\tilde{U} \subseteq f^{-1}[Q]$, because $U'_\xi \cap V_\xi$ is a non-empty member of $\mathcal{C}(I \setminus J)$. But this means that $\mu^* f[\tilde{U}] = 0$ and $\mu^* f[U_\xi] = 0$. On the other hand, we have $s_\xi \in U_\xi \cap f^{-1}[Q_{J_\xi}]$, so $U_\xi \notin \mathcal{C}_0(J_\xi)$ and $\mu^* f[U_\xi] > 0$. **X**

Thus this route also is blocked and we must abandon the original hypothesis that there is a quasi-dyadic space with a semi-finite completion regular topological measure which is not τ-additive.

434R There is a useful construction of Borel product measures which can be fitted in here.

Proposition Let X and Y be topological spaces with Borel measures μ and ν; write $\mathcal{B}(X)$, $\mathcal{B}(Y)$ for the Borel σ-algebras of X and Y respectively. If *either* X is first-countable *or* ν is τ-additive and effectively locally finite, there is a Borel measure λ_B on $X \times Y$ defined by the formula

$$\lambda_B W = \sup_{F \in \mathcal{B}(Y), \nu F < \infty} \int \nu(W[\{x\}] \cap F)\mu(dx)$$

for every Borel set $W \subseteq X \times Y$. Moreover

(i) if μ is semi-finite, then λ_B agrees with the c.l.d. product measure λ on $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, and the c.l.d. version $\tilde{\lambda}_B$ of λ_B extends λ;

(ii) if ν is σ-finite, then $\lambda_B W = \int \nu W[\{x\}]\mu(dx)$ for every Borel set $W \subseteq X \times Y$;

(iii) if both μ and ν are τ-additive and effectively locally finite, then λ_B is just the restriction of the τ-additive product measure $\tilde{\lambda}$ (417D, 417F) to the Borel σ-algebra of $X \times Y$; in particular, λ_B is τ-additive.

proof (a) The point is that $x \mapsto \nu(W[\{x\}] \cap F)$ is lower semi-continuous whenever $W \subseteq X \times Y$ is open and $\nu F < \infty$. **P** Of course $W[\{x\}]$ is always open, so ν always measures $W[\{x\}] \cap F$. Take any $\alpha \in \mathbb{R}$ and set $G = \{x : x \in X, \nu(W[\{x\}] \cap F) > \alpha\}$; let $x_0 \in G$.

(α) Suppose that X is first-countable. Let $\langle U_n \rangle_{n \in \mathbb{N}}$ be a non-increasing sequence running over a base of open neighbourhoods of x_0. For each $n \in \mathbb{N}$, set

$$V_n = \bigcup\{V : V \subseteq Y \text{ is open}, U_n \times V \subseteq W\}.$$

Then $\langle V_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with union $W[\{x_0\}]$, so there is an $n \in \mathbb{N}$ such that $\nu(V_n \cap F) > \alpha$. Now $V_n \subseteq W[\{x\}]$ for every $x \in U_n$, so $U_n \subseteq G$.

(β) Suppose that ν is τ-additive and effectively locally finite. Set

$$\mathcal{V} = \{V : V \subseteq Y \text{ is open}, U \times V \subseteq W \text{ for some open set } U \text{ containing } x_0\}.$$

Then $\mathcal{V}$ is an upwards-directed family of open sets with union $W[\{x_0\}]$, so there is a $V \in \mathcal{V}$ such that $\nu(V \cap F) > \alpha$ (414Ea). Let U be an open set containing x_0 such that $U \times V \subseteq W$; then $V \subseteq W[\{x\}]$ for every $x \in U$, so $U \subseteq G$.

(γ) Thus in either case we have an open set containing x_0 and included in G. As x_0 is arbitrary, G is open; as α is arbitrary, $x \mapsto \nu(W[\{x\}] \cap F)$ is lower semi-continuous. **Q**

(b) It follows that $x \mapsto \nu(W[\{x\}] \cap F)$ is Borel measurable whenever $W \subseteq X \times Y$ is a Borel set and $\nu F < \infty$. **P** Let $\mathcal{W}$ be the family of sets $W \subseteq X \times Y$ such that $W[\{x\}]$ is a Borel set for every $x \in X$ and $x \mapsto \nu(W[\{x\}] \cap F)$ is Borel measurable. Then every open subset of $X \times Y$ belongs to $\mathcal{W}$ (by (a) above), $W \setminus W' \in \mathcal{W}$ whenever W, $W' \in \mathcal{W}$ and $W' \subseteq W$, and $\bigcup_{n \in \mathbb{N}} W_n \in \mathcal{W}$ whenever $\langle W_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$. By the Monotone Class Theorem (136B), $\mathcal{W}$ includes the σ-algebra generated by the open sets, that is, the Borel σ-algebra of $X \times Y$. **Q**

(c) It is now easy to check that $W \mapsto \int \nu(W[\{x\}] \cap F)\mu(dx)$ is a Borel measure on $X \times Y$ whenever $\nu F < \infty$, and therefore that λ_B, as defined here, is a Borel measure.

(d) Now suppose that μ is semi-finite, and that $W \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. Then

$$\lambda W = \sup_{\mu E<\infty,\nu F<\infty} \lambda(W \cap (E \times F))$$

(by the definition of 'c.l.d. product measure', 251F)

$$= \sup_{\mu E<\infty,\nu F<\infty} \int_E \nu(W[\{x\}] \cap F)\mu(dx)$$

(by Fubini's theorem, 252C, applied to the product of the subspace measures μ_E and ν_F)

$$= \sup_{\nu F<\infty} \int \nu(W[\{x\}] \cap F)\mu(dx)$$

(by 213B, because μ is semi-finite)

$$= \lambda_B W.$$

(e) If, on the other hand, ν is σ-finite, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering Y; then

$$\begin{aligned}\lambda_B W &= \sup_{\nu F<\infty} \int \nu(W[\{x\}] \cap F)\mu(dx) \ge \sup_{n\in\mathbb{N}} \int \nu(W[\{x\}] \cap F_n)\mu(dx)\\ &= \int \sup_{n\in\mathbb{N}} \nu(W[\{x\}] \cap F_n)\mu(dx) = \int \nu W[\{x\}]\mu(dx) \ge \lambda_B W\end{aligned}$$

for any Borel set $W \subseteq X$.

(f) If both μ and ν are τ-additive and effectively locally finite, so that we have a τ-additive product measure $\tilde\lambda$, then Fubini's theorem for such measures (417G) tells us that $\lambda_B W = \tilde\lambda W$ at least when $W \subseteq X \times Y$ is a Borel set and $\tilde\lambda W$ is finite. If W is any Borel subset of $X \times Y$, then, as in (d),

$$\begin{aligned}\lambda_B W &= \sup_{\mu E<\infty,\nu F<\infty} \int_E \nu(W[\{x\}] \cap F)\mu(dx)\\ &= \sup_{\mu E<\infty,\nu F<\infty} \tilde\lambda(W \cap (E \times F)) = \tilde\lambda W\end{aligned}$$

by 417C(b-iii).

Remark The case in which X is first-countable is due to Johnson 82.

***434S** The concept of 'universally measurable' set enables us to extend a number of ideas from earlier sections. First, recall a problem from the very beginning of measure theory on the real line: the composition of Lebesgue measurable functions need not be Lebesgue measurable (134Ib), while the composition of a Borel measurable function with a Lebesgue measurable function is measurable (121Eg). In fact we can replace 'Borel measurable' by 'universally measurable', as follows.

Proposition Let (X,Σ,μ) be a complete locally determined measure space, Y and Z topological spaces, $f : X \to Y$ a measurable function and $g : Y \to Z$ a universally measurable function. Then $gf : X \to Z$ is measurable. In particular, $f^{-1}[F] \in \Sigma$ for every universally measurable set $F \subseteq Y$.

proof Let $H \subseteq Z$ be an open set and $E \in \Sigma$ a set of finite measure. Let μ_E be the subspace measure on E. Then the image measure $\nu = \mu_E(f\restriction E)^{-1}$ is a complete totally finite topological measure on Y, so its domain contains $g^{-1}[H]$, and

$$E \cap (gf)^{-1}[H] = (f\restriction E)^{-1}[g^{-1}[H]] \in \operatorname{dom}\mu_E \subseteq \Sigma.$$

As E is arbitrary and μ is locally determined, $(gf)^{-1}[H] \in \Sigma$; as H is arbitrary, gf is measurable.

Applying this to $g = \chi F$, we see that $f^{-1}[F] \in \Sigma$ for every universally measurable $F \subseteq Y$.

***434T** The next remark concerns the concept $[\![u \in E]\!]$ of §364.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Write Σ_{um} for the algebra of universally measurable subsets of $\mathbb{R}$.

(a) For any $u \in L^0 = L^0(\mathfrak{A})$, we have a sequentially order-continuous Boolean homomorphism $E \mapsto [\![u \in E]\!] : \Sigma_{\mathrm{um}} \to \mathfrak{A}$ defined by saying that

$$\begin{aligned}[\![u \in E]\!] &= \sup\{[\![u \in F]\!] : F \subseteq E \text{ is Borel}\} = \sup\{[\![u \in K]\!] : K \subseteq E \text{ is compact}\}\\ &= \inf\{[\![u \in F]\!] : F \supseteq E \text{ is Borel}\} = \inf\{[\![u \in G]\!] : G \supseteq E \text{ is open}\}\end{aligned}$$

for every $E \in \Sigma_{\mathrm{um}}$.

(b) For any $u \in L^0$ and universally measurable function $h : \mathbb{R} \to \mathbb{R}$ we have a corresponding element $\bar{h}(u)$ of L^0 defined by the formula

$$[\![\bar{h}(u) \in E]\!] = [\![u \in h^{-1}[E]]\!] \text{ for every } E \in \Sigma_{\mathrm{um}},\ u \in L^0.$$

proof We can regard $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a complete strictly localizable measure space (X, Σ, μ) (322O), in which case L^0 can be identified with $L^0(\mu)$ (364Ic). Write $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$.

(a) Let $f : X \to \mathbb{R}$ be a Σ-measurable function representing u. Then $f^{-1}[E] \in \Sigma$ for every $E \in \Sigma_{\mathrm{um}}$, by 434S. Setting $\phi E = (f^{-1}[E])^\bullet$, $\phi : \Sigma_{\mathrm{um}} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism.

We find that

$$\phi E = \sup\{[\![u \in K]\!] : K \subseteq E \text{ is compact}\}$$

for every $E \in \Sigma_{\mathrm{um}}$. **P** If $H \in \Sigma$ and $\mu H < \infty$, then (writing μ_H for the subspace measure on H) the image measure $\mu_H(f{\restriction}H)^{-1}$ is a complete topological measure, and its restriction ν to the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ is a totally finite Borel measure. Now E is measured by the completion $\hat{\nu}$ of ν, which is a Radon measure (256C), so for any $\epsilon > 0$ there are a compact $K \subseteq E$ and a Borel $F \supseteq E$ such that $\nu F = \hat{\nu} E \le \nu K + \epsilon$. In this case,

$$[\![u \in K]\!] = (f^{-1}[K])^\bullet \subseteq \phi E \subseteq (f^{-1}[F])^\bullet = [\![u \in F]\!],$$

using the formula of 364Ib, while

$$\begin{aligned}\bar{\mu}(H^\bullet \cap \phi E) &\le \bar{\mu}(H^\bullet \cap [\![u \in F]\!]) = \mu(H \cap f^{-1}[F])\\ &= \nu F \le \nu K + \epsilon = \bar{\mu}(H^\bullet \cap [\![u \in K]\!]) + \epsilon.\end{aligned}$$

As ϵ is arbitrary,

$$H^\bullet \cap \phi E = \sup\{H^\bullet \cap [\![u \in K]\!] : K \subseteq E \text{ is compact}\};$$

as H is arbitrary, and $(\mathfrak{A}, \bar{\mu})$ is semi-finite, $\phi E = \sup\{[\![u \in K]\!] : K \subseteq E \text{ is compact}\}$. **Q**

Applying this to $\mathbb{R} \setminus E$, we see that $\phi E = \inf\{[\![u \in G]\!] : G \supseteq E \text{ is open}\}$. Of course it follows at once that

$$[\![u \in E]\!] = \sup\{[\![u \in F]\!] : F \subseteq E \text{ is Borel}\} = \inf\{[\![u \in F]\!] : F \supseteq E \text{ is Borel}\}.$$

We can therefore identify the sequentially order-continuous Boolean homomorphism ϕ with $E \mapsto [\![u \in E]\!]$, as described.

(b) Once again identifying u with $f^\bullet$ where f is Σ-measurable, we see that hf is Σ-measurable (by 434S), so we have a corresponding element $(hf)^\bullet$ of L^0. If $E \in \Sigma_{\mathrm{um}}$, then

$$[\![(hf)^\bullet \in E]\!] = ((hf)^{-1}[E])^\bullet = (f^{-1}[h^{-1}[E]])^\bullet = [\![u \in h^{-1}[E]]\!],$$

using 434De to check that $h^{-1}[E] \in \Sigma_{\mathrm{um}}$, so that we can identify $(hf)^\bullet$ with $\bar{h}(u)$, as described.

434U I give an elementary remark on images of completion regular measures.

Proposition Let X and Y be compact Hausdorff spaces and $f : X \to Y$ a continuous open map. If μ is a completion regular topological measure on X, then the image measure μf^{-1} on Y is completion regular.

proof If $F \subseteq Y$ is measured by $\nu = \mu f^{-1}$ and $\gamma < \nu F = \mu f^{-1}[F]$, there is a zero set $Z \subseteq f^{-1}[F]$ such that $\mu Z \ge \gamma$. Now $f[Z] \subseteq F$ is a zero set in Y (4A2G(c-ii)) and

$$\nu f[Z] = \mu f^{-1}[f[Z]] \ge \mu Z \ge \gamma.$$

As F and γ are arbitrary, ν is inner regular with respect to the zero sets, so is completion regular.

434X Basic exercises >(a) Let $A \subseteq [0,1]$ be any non-measurable set. Show that the subspace measure on A is completion regular and τ-additive but not tight.

>(b) Let X be any Hausdorff space with a point x such that $\{x\}$ is not a G_δ set; for instance, $X = \omega_1 + 1$ and $x = \omega_1$, or $X = \{0,1\}^I$ for any uncountable set I and x any point of X. Show that setting $\mu E = \chi E(x)$ we get a tight Borel measure on X which is not completion regular.

>(c) Let X be a topological space. (i) Show that if $A \subseteq X$ is universally measurable in X, then $A \cap Y$ is universally measurable in Y for any set $Y \subseteq X$. (ii) Show that if $Y \subseteq X$ is universally measurable in X, and $A \subseteq Y$ is universally measurable in Y, then A is universally measurable in X. (iii) Suppose that X is the product of a countable family $\langle X_i \rangle_{i \in I}$ of topological spaces, and $E_i \subseteq X_i$ is a universally measurable set for each $i \in I$. Show that $\prod_{i \in I} E_i$ is universally measurable in X.

(d) Let X be an analytic Hausdorff space. (i) Suppose that Y is a topological space and W is a Borel subset of $X \times Y$. Show that $W[X]$ is a universally measurable subset of Y. (*Hint*: 423P.) (ii) Let A be a subset of X. Show that the following are equiveridical: (α) A is universally measurable in X; (β) $f^{-1}[A]$ is Lebesgue measurable for every Borel measurable function $f : [0,1] \to X$; (γ) $f^{-1}[A]$ is measured by the usual measure on $\{0,1\}^{\mathbb{N}}$ for every continuous function $f : \{0,1\}^{\mathbb{N}} \to X$.

(e) Let Σ_{um} be the algebra of universally measurable subsets of $\mathbb{R}$, and μ the restriction of Lebesgue measure to Σ_{um}. Show that μ is translation-invariant, but has no translation-invariant lifting. (*Hint*: 345F.)

(f) Let X be a Hausdorff space. (i) Show that, for $A \subseteq X$, the following are equiveridical: (α) A is universally Radon-measurable in X; (β) A is measured by every atomless Radon probability measure on X; (γ) $A \cap K$ is universally Radon-measurable in K for every compact $K \subseteq X$. (ii) Show that if $A \subseteq X$ is universally Radon-measurable in X, then $A \cap Y$ is universally Radon-measurable in Y for any set $Y \subseteq X$. (iii) Show that if $Y \subseteq X$ is universally Radon-measurable in X, and $A \subseteq Y$ is universally Radon-measurable in Y, then A is universally Radon-measurable in X. (iv) Show that if $\mathcal{G}$ is an open cover of X, and $A \subseteq X$ is such that $A \cap G$ is universally Radon-measurable (in G or in X) for every $G \in \mathcal{G}$, then A is universally Radon-measurable in X. (v) Show that if Y is another Hausdorff space, and $\Sigma^{(X)}_{\mathrm{uRm}}$, $\Sigma^{(Y)}_{\mathrm{uRm}}$ are the algebras of universally Radon-measurable subsets of X, Y respectively, then every continuous function from X to Y is $(\Sigma^{(X)}_{\mathrm{uRm}}, \Sigma^{(Y)}_{\mathrm{uRm}})$-measurable. (vi) Suppose that X is the product of a countable family $\langle X_i \rangle_{i \in I}$ of topological spaces, and $E_i \subseteq X_i$ is a universally Radon-measurable set for each $i \in I$. Show that $\prod_{i \in I} E_i$ is universally Radon-measurable in X.

>(g)(i) Let μ_0 be Dieudonné's measure on ω_1. Give $\omega_1 + 1 = \omega_1 \cup \{\omega_1\}$ its compact Hausdorff order topology, and define a Borel measure μ on $\omega_1 + 1$ by setting $\mu E = \mu_0(E \cap \omega_1)$ for every Borel set $E \subseteq \omega_1 + 1$. Show that μ is a complete probability measure and is neither τ-additive nor inner regular with respect to the closed sets. (ii) Show that the universally measurable subsets of $\omega_1 + 1$ are just its Borel sets. (*Hint*: 4A3J, 411Q.) (iii) Show that every totally finite τ-additive topological measure on $\omega_1 + 1$ has a countable support. (iv) Show that every subset of $\omega_1 + 1$ is universally Radon-measurable.

(h)(i) Show that there is a set $X \subseteq [0,1]$ such that $K \cap X$ and $K \setminus X$ both have cardinal $\mathfrak{c}$ for every uncountable compact set $K \subseteq [0,1]$. (*Hint*: 4A3Fa, 423L.) (ii) Show that if we give X its subspace topology, then every subset of X is universally Radon-measurable, but not every subset is universally measurable. (*Hint*: every compact subset of X is countable, so every Radon measure on X is purely atomic, but X has full outer Lebesgue measure in $[0,1]$.)

(i) Show that a Hausdorff space X is Radon iff (α) every compact subset of X is Radon (β) for every non-zero totally finite Borel measure μ on X there is a compact subset K of X such that $\mu K > 0$. (*Hint*: 434F(a-v).)

>(j)(i) Let X and Y be K-analytic Hausdorff spaces and $f : X \to Y$ a continuous surjection. Suppose that $F \subseteq Y$ and that $f^{-1}[F]$ is universally Radon-measurable in X. Show that F is universally Radon-measurable in Y. (*Hint*: 432G.) (ii) Let X and Y be analytic Hausdorff spaces and $f : X \to Y$ a Borel measurable surjection. Suppose that $F \subseteq Y$ and that $f^{-1}[F]$ is universally Radon-measurable in X. Show that F is universally Radon-measurable in Y. (*Hint*: 433D.)

(k) Show that if X is a perfectly normal space then it is Borel-measure-compact iff it is Borel-measure-complete.

(l) Let X be a Radon Hausdorff space. (i) Show that $X \times Y$ is Borel-measure-compact whenever Y is Borel-measure-compact. (ii) Show that $X \times Y$ is Borel-measure-complete whenever Y is Borel-measure-complete.

(m) Show that if we give $\omega_1 + 1$ its order topology, it is Borel-measure-compact but not Borel-measure-complete or pre-Radon, and its open subset ω_1 is not Borel-measure-compact.

(n) Show that the Sorgenfrey line (415Xc, 439Q) is Borel-measure-complete and Borel-measure-compact, but not Radon or pre-Radon.

(o) Let X be a topological space. (i) Show that the family of Borel-measure-complete subsets of X is closed under Souslin's operation. (ii) Show that the union of a sequence of Borel-measure-compact subsets of X is Borel-measure-compact. (iii) Show that if X is Hausdorff then the family of pre-Radon subsets of X is closed under Souslin's operation. (*Hint*: in (i) and (iii), start by showing that the family under consideration is closed under countable unions.)

(p) Show that $]0,1[^{\omega_1}$ is not pre-Radon.

(q) Let X be a separable metrizable space. Show that the following are equiveridical: (i) X is a Radon space; (ii) X is a pre-Radon space; (iii) there is a metric on X, defining the topology of X, such that X is universally Radon-measurable in its completion; (iv) whenever Y is a separable metrizable space and X' is a subset of Y such that there is a Borel isomorphism between X and X', then X' is universally measurable in Y; (v) X is a Radon space under any separable metrizable topology giving rise to the same Borel sets as the original topology.

>(r) Show that a K-analytic Hausdorff space is Radon iff all its compact subsets are Radon. (*Hint*: 432B, 434Xi.)

(s) Suppose that X is a K-analytic Hausdorff space such that every Radon measure on X is completion regular. Show that X is a Radon space.

(t) Let X and Y be topological spaces, and suppose that Y has a countable network. (i) Show that if X is Borel-measure-complete, then $X \times Y$ is Borel-measure-complete. (ii) Show that if X and Y are Radon Hausdorff spaces, then $X \times Y$ is Radon.

(u) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of topological spaces; write $X = \prod_{n\in\mathbb{N}} X_n$ and $Z_n = \prod_{i\le n} X_i$ for each n. (i) Show that if every Z_n is Borel-measure-complete, so is X. (ii) Show that if every Z_n is Hausdorff and pre-Radon, so is X. (iii) Show that if every Z_n is Hausdorff and Radon, so is X.

(v)(i) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of Radon Hausdorff spaces such that $\prod_{i\le n} K_i$ is Radon whenever $n \in \mathbb{N}$ and $K_i \subseteq X_i$ is compact for every $i \le n$. Show that $X = \prod_{n\in\mathbb{N}} X_n$ is Radon. (ii) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of Radon Hausdorff spaces with countable networks. Show that $\prod_{n\in\mathbb{N}} X_n$ is Radon.

(w) Show that if, in 434R, ν is σ-finite, then $\int g\,d\lambda_B = \iint g(x,y)\nu(dy)\mu(dx)$ for every λ_B-integrable function $g : X \times Y \to \mathbb{R}$.

(x) Show that the product measure construction of 434R is 'associative' and 'distributive' in the sense that (under appropriate hypotheses) the product measures on $(X \times Y) \times Z$ and $X \times (Y \times Z)$ agree, and those on $\bigcup_{i,j\in\mathbb{N}}(X_i \times Y_j)$ and $(\bigcup_{i\in\mathbb{N}} X_i) \times (\bigcup_{j\in\mathbb{N}} Y_j)$ agree.

>(y) Show that the product measure construction of 434R is not 'commutative'; indeed, taking $\mu = \nu$ to be Dieudonné's measure on ω_1, show that the Borel measures λ_1, λ_2 on ω_1^2 defined by setting

$$\lambda_1 W = \int \nu W[\{\xi\}]\mu(d\xi), \quad \lambda_2 W = \int \mu W^{-1}[\{\eta\}]\nu(d\eta)$$

are different.

(z) Read through §271, looking for ways to apply the concept '$\Pr(\boldsymbol{X} \in E)$' for random variables $\boldsymbol{X}$ and universally measurable sets E.

434Y Further exercises (a) Set $X = \mathbb{N} \setminus \{0,1\}$. For m, $p \in X$ set $U_{mp} = m + p\mathbb{N}$; show that $\{U_{mp} : m, p \in X \text{ are coprime}\}$ is a base for a connected Hausdorff topology on X. (*Hint*: $pq \in \overline{U}_{mp}$ for every $q \ge 1$. See Steen & Seebach 78, ex. 60.) Show that X is a second-countable analytic Hausdorff space and carries a Radon measure which is not completion regular.

(b) Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a topological space with a countable network consisting of universally measurable sets, and that $f : X \to Y$ is measurable. Show that f is almost continuous.

(c) If X is a topological space, a set $A \subseteq X$ is **universally capacitable** if $c(A) = \sup\{c(K) : K \subseteq A$ is compact$\}$ for every Choquet capacity c on X. (i) Show that if X is a Hausdorff space and π_1, $\pi_2 : X \times X \to X$ are the coordinate maps, then we have a Choquet capacity c on $X \times X$ defined by saying that $c(A) = 0$ if $A \subseteq X \times X$ and there is a Borel set $E \subseteq X$ including $\pi_1[A]$ and disjoint from $\pi_2[A]$, and $c(A) = 1$ for other $A \subseteq X \times X$. (ii) Show that there is a universally measurable subset of $\mathbb{R}$ which is not universally capacitable. (*Hint*: 423M.)

(d) Let X be a Hausdorff space. Let Σ be the family of those subsets E of X such that $f^{-1}[E]$ has the Baire property in Z whenever Z is a compact Hausdorff space and $f : Z \to X$ is continuous. Show that Σ is a σ-algebra of subsets of X closed under Souslin's operation. Show that every member of Σ is universally Radon-measurable.

(e) Let X be a Hausdorff space such that there is a countable algebra $\mathcal{A}$ of universally Radon-measurable subsets of X which separates the points of X in the sense that whenever $I \in [X]^2$ there is an $A \in \mathcal{A}$ such that $\#(I \cap A) = 1$. Show that two Radon probability measures on X which agree on $\mathcal{A}$ are identical.

(f) Let X be a completely regular Hausdorff space. Show that the following are equiveridical: (α) X is pre-Radon; (β) X is a universally Radon-measurable subset of its Stone-Čech compactification; (γ) whenever Y is a Hausdorff space and X' is a subspace of Y which is homeomorphic to X, then X' is universally Radon-measurable in Y.

(g) Set $X = \omega_1 + 1$, with its order topology, and let Σ be the σ-algebra of subsets of X generated by the countable sets and the set Ω of limit ordinals in X. Show that there is a unique probability measure μ on X with domain Σ such that $\mu\xi = \mu\Omega = 0$ for every $\xi < \omega_1$. Show that μ is inner regular with respect to the Borel sets, is defined on a base for the topology of the compact Hausdorff space X, but has no extension to a topological measure on X.

(h) Let X be a metrizable space without isolated points, and μ a σ-finite Borel measure on X. Show that there is a conegligible meager set. (*Hint*: there is a dense set $D \subseteq X$ such that $\{\{d\} : d \in D\}$ is σ-metrically-discrete.)

(i) Give an example of a Hausdorff uniform space $(X, \mathcal{W})$ with a quasi-Radon probability measure which is not inner regular with respect to the totally bounded sets.

(j) Show that $\beta\mathbb{N}$ is not countably tight, therefore not Borel-measure-complete.

(k)(i) Show that the split interval is not quasi-dyadic. (ii) Show that the Sorgenfrey line is not quasi-dyadic. (iii) Show that ω_1 and $\omega_1 + 1$, with their order topologies, are not quasi-dyadic.

(l) Show that a perfectly normal quasi-dyadic space is Borel-measure-compact.

(m) Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be a sequence of first-countable spaces, and μ_n a Borel probability measure on X_n for each n. For $n \in \mathbb{N}$ set $Z_n = \prod_{i<n} X_i$, and let λ_n be the product Borel measure on Z_n constructed by repeatedly using the method of 434R (cf. 434Xx). (i) Show that there is a unique Borel measure λ on $Z = \prod_{n\in\mathbb{N}} X_n$ such that all the canonical maps from Z to Z_n are inverse-measure-preserving. (ii) Show that, for any n, λ can be identified with the product of λ_n and a suitable product measure on $\prod_{i\ge n} X_i$.

(n) (Aldaz 97) A topological space X is **countably metacompact** if whenever $\mathcal{G}$ is a countable open cover of X then there is a point-finite open cover $\mathcal{H}$ of X refining G. (i) Show that X is countably metacompact iff whenever $\langle F_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets with empty intersection in X then there is a sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ of open sets, with empty intersection, such that $F_n \subseteq G_n$ for every n. (ii) Let X be any topological space and $\nu : \mathcal{P}X \to [0,1]$ a finitely additive functional such that $\nu X = 1$. Show that there is a finitely additive $\nu' : \mathcal{P}X \to [0,1]$ such that $\nu F \le \nu' F = \inf\{\nu' G : G \supseteq F \text{ is open}\}$ for every closed $F \subseteq X$. (*Hint*: 413S.) (iii) Show that if X is countably metacompact and μ is any Borel probability measure on X, there is a Borel probability measure μ' on X, inner regular with respect to the closed sets, such that $\mu F \le \mu' F$ for every closed set $F \subseteq X$; so that μ and μ' agree on the Baire σ-algebra of X.

(o) Let X be a totally ordered set with its order topology. Show that any τ-additive Borel probability measure on X has countable Maharam type. (*Hint*: $\{]-\infty, x]^\bullet : x \in X\}$ generates the measure algebra.)

(p) If X is a topological space and ρ is a metric on X, X is **σ-fragmented** by ρ if for every $\epsilon > 0$ there is a countable cover $\mathcal{A}$ of X such that whenever $\emptyset \ne B \subseteq A \in \mathcal{A}$ there is a non-empty relatively open subset of B of ρ-diameter at most ϵ. Now suppose that X is a Hausdorff space which is σ-fragmented by a metric ρ such that (i) X is complete under ρ (ii) the topology generated by ρ is finer than the given topology on X. Show that X is a pre-Radon space.

(q) (Oxtoby 70) Let μ be an atomless strictly positive Radon probability measure on $\mathbb{N}^{\mathbb{N}}$. (i) Show that if $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ is any sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n = 1$, then there is a partition $\langle U_n \rangle_{n\in\mathbb{N}}$ of $\mathbb{N}^{\mathbb{N}}$ into open sets such that $\mu U_n = \alpha_n$ for every n. (ii) Show that if ν is any other atomless strictly positive Radon probability measure on $\mathbb{N}^{\mathbb{N}}$, there is a homeomorphism $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\nu = \mu f^{-1}$.

(r) Let X be a Hausdorff space and μ an atomless strictly localizable tight Borel measure on X. Show that μ is σ-finite. (*Hint*: Fremlin n05.)

434Z Problems (a) Must every Radon compact Hausdorff space be sequentially compact?

(b) Must a Hausdorff continuous image of a Radon compact Hausdorff space be Radon?

434 Notes and comments I said that the fundamental question of topological measure theory is 'which measures can appear on which topological spaces'? In this section I have concentrated on Borel measures, classified according to the scheme laid out in §411. (Of course there are other kinds of classification. One of the most interesting is the Maharam classification of Chapter 33: we can ask what measure algebras can appear from topological measures on a given topological space. I will return to this idea in §531 of Volume 5. For the moment I pass it by, with only 434Yo to give a taste.) We can ask this question from either of two directions. The obvious approach is to ask, for a given class of topological spaces, which types of measure can appear. But having discovered that (for instance) there are several types of topological space on which all (totally finite) Borel measures are tight, we can use this as a definition of a class of topological spaces, and ask the ordinary questions about this class. Thus we have 'Radon', 'Borel-measure-complete', 'Borel-measure-compact' and 'pre-Radon' spaces (434C, 434G). I have given precedence to the first partly to honour the influence of Schwartz 73 and partly because a compact Hausdorff space is always Borel-measure-compact and pre-Radon (434Hb, 434Jf) and is Borel-measure-complete iff it is Radon (434Ka). In effect, 'Borel-measure-complete' means 'Borel measures are quasi-Radon' (434Ib), 'pre-Radon' means 'quasi-Radon measures are Radon' (434Jb), and 'Radon' means 'Borel measures are Radon' (434F(a-iii)).

These slogans have to be interpreted with care; but it is true that a Hausdorff space is Radon iff it is both Borel-measure-complete and pre-Radon (434Ka).

The concept of 'Radon' space is in fact one of the important contributions of measure theory to general topology, offering a variety of challenging questions. One which has attracted some attention is the problem of determining when products of Radon spaces are Radon. Uncountable products hardly ever are (434Kd); for countable products it is enough to understand products of finitely many compact spaces (434Xv); but the product of two compact spaces already seems to lead us into undecidable questions (438Xq, Wage 80). Two more very natural questions are in 434Z. One of the obstacles to the investigation is the rather small number of Radon compact Hausdorff spaces which are known. I should remark that if the continuum hypothesis (for instance) is true, then every compact Hausdorff space in which countably compact sets are closed is sequentially compact (Ismail & Nyikos 80, or Fremlin 84, 24Nc), so that in this case we have a quick answer to 434Za, using 434Nb.

You will recognise the construction of 434M as a universal version of Dieudonné's measure (411Q). 'Tightness' is of great interest for other reasons (Engelking 89), and here is very helpful when showing that spaces are not Radon (434Yj).

A large proportion of the definitions in general topology can be regarded as different abstractions from the concept of metrizability. Countable tightness is an obvious example; so is 'first-countability' (434R). In quite a different direction we have 'metacompactness' (438J, 434Yn). The construction of the product measure in 434R is an obvious idea, as soon as you have seen Fubini's theorem, but it is not obvious just when it will work.

'Quasi-dyadic' spaces are a relatively recent invention; I introduce them here only as a vehicle for the argument of 434Q. Of course a dyadic space is quasi-dyadic; for basic facts on dyadic spaces, see 4A2D, 4A5T and Engelking 89, §3.12 and 4.5.9-4.5.11.

Version of 16.8.08

435 Baire measures

Imitating the programme of §434, I apply a similar analysis to Baire measures, starting with a simple-minded classification (435A). This time the central section (435D-435H) is devoted to 'measure-compact' spaces, those on which all (totally finite) Baire measures are τ-additive.

435A Types of Baire measures In 434A I looked at a list of four properties which a Borel measure may or may not possess: inner regularity with respect to closed sets, inner regularity with respect to zero sets, tightness (that is, inner regularity with respect to closed compact sets), and τ-additivity. Since every (semi-finite) Baire measure is inner regular with respect to the zero sets (412D), only two of the four are important considerations for Baire measures: tightness and τ-additivity. On the other hand, there is a new question we can ask. Given a Baire measure on a topological space, when can it be extended to a Borel measure? And in the case of a positive answer, we can ask whether the extension is unique, and whether we can find extensions to Borel measures satisfying the properties considered in 434A.

We already have some information on this. If X is a completely regular space, and μ is a τ-additive effectively locally finite Baire measure on X, then μ has a (unique) extension to a τ-additive Borel measure (415N). While if μ is tight, the extension will also be tight (cf. 416C). Perhaps I should remark immediately that while there can be only one τ-additive Borel measure extending μ, there might be another Borel measure, not τ-additive, also extending μ; see 435Xa. Of course if there is any completion regular Borel measure extending μ, there is only one; moreover, if μ is σ-finite, and there is a completion regular Borel measure extending μ, this is the only Borel measure extending μ. (For every Borel set will be measured by the completion of μ.)

A possible division of Baire measures is therefore into classes

(E) measures which are not τ-additive,

(F) measures which are τ-additive, but not tight,

(G) tight measures,

and within these classes we can distinguish measures with no extension to a Borel measure (type E_0), measures with more than one extension to a Borel measure (types E_1, F_1 and G_1), measures with exactly one extension to a Borel measure which is not completion regular (types E_2, F_2 and G_2) and measures with an extension to a completion regular Borel measure (types E_3, F_3 and G_3). For examples, see 439M and 439O (E_0), 439N (E_2), 439J (E_3), 435Xc (F_1), 435Xd (F_2), 415Xc and 434Xa (F_3), 435Xa (G_1), 435Xb (G_2) and the restriction of Lebesgue measure to the Baire subsets of $\mathbb{R}$ (G_3); other examples may be constructed as direct sums of these.

A separate question we can ask of a Baire measure is whether it can be extended to a Radon measure. For this there is a straightforward criterion (435B), which shows that (at least for totally finite measures on completely regular spaces) only the types F_1 and F_2 are divided by this question. (If a Baire measure μ can be extended to a Radon measure, it is surely τ-additive. If μ is tight, it satisfies the criteria of 435B, so has an extension to a Radon measure. If μ has an extension to a completion regular Borel measure μ_1 *and* has an extension to a Radon measure μ_2, then the completion $\hat{\mu}$ of μ extends μ_1, while μ_2 extends $\hat{\mu}$; so μ_1 is the restriction of μ_2 to the Borel sets and $\mu_2 = \hat{\mu}_1 = \hat{\mu}$ and μ, like μ_2, is tight, by 412Hb or otherwise. Thus no measure of type F_3 can be extended to a Radon measure.)

As with the classification of Borel measures that I offered in §434, any restriction on the topology of the underlying space may eliminate some of these possibilities. For instance, because a semi-finite Baire measure is inner regular with respect to the closed sets, we can have no (semi-finite) measure of classes E or F on a compact Hausdorff space. On a locally compact Hausdorff space we can have no effectively locally finite Baire measure of class F (435Xe), while on a K-analytic Hausdorff space we can have no locally finite Baire measure of class E (432F). In a metrizable space, or a regular space with a countable network (e.g., a regular analytic Hausdorff space), the Baire and Borel σ-algebras coincide (4A3Kb), so we can have no measures of type E_0, E_1, F_1 or G_1.

435B Theorem Let X be a Hausdorff space and μ a locally finite Baire measure on X. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) for every non-negligible Baire set $E \subseteq X$ there is a compact set $K \subseteq E$ such that $\mu^* K > 0$.

If μ is totally finite, we can add

(iii) $\sup\{\mu^* K : K \subseteq X \text{ is compact}\} = \mu X$.

proof Because μ is inner regular with respect to the closed sets (412D), this is just a special case of 416P.

435C Theorem (Mařík 57) Let X be a normal countably paracompact space. Then any semi-finite Baire measure on X has an extension to a semi-finite Borel measure which is inner regular with respect to the closed sets.

proof (a) Let ν be a semi-finite Baire measure on X. Let $\mathcal{K}$ be the family of those closed subsets of X which are included in zero sets of finite measure, and set $\phi_0 K = \nu^* K$ for $K \in \mathcal{K}$. Then $\mathcal{K}$ and ϕ_0 satisfy the conditions of 413J, that is,

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$,

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

P The first three are trivial.

(α) Take K, $L \in \mathcal{K}$ with $L \subseteq K$, and set $\gamma = \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$. (i) If $K' \subseteq K \setminus L$ is closed, then (because X is normal) there is a zero set F including K' and disjoint from L (4A2F(d-iv)), so

$$\phi_0 K' + \phi_0 L = \nu^*((K' \cup L) \cap F) + \nu^*((K' \cup L) \setminus F) = \nu^*(K' \cup L) \le \nu^* K.$$

As K' is arbitrary, $\gamma + \phi_0 L \le \phi_0 K$. (ii) Let $\epsilon > 0$. Let F_0 be a zero set of finite measure including K. Because ν is inner regular with respect to the zero sets (412D), there is a zero set $F \subseteq F_0 \setminus L$ such that $\nu F \ge \nu_*(F_0 \setminus L) - \epsilon$ (413Ee), so that $\nu(F_0 \setminus F) \le \nu^* L + \epsilon$ (413Ec). Set $K' = K \cap F$. Then

$$\nu^*K = \nu^*(K \setminus F) + \nu^*(K \cap F) \le \nu(F_0 \setminus F) + \nu^*K' \le \nu^*L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\nu^*K \le \nu^*L + \gamma$.

(β) If $\langle K_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then (because X is countably paracompact) there is a sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ of open sets such that $K_n \subseteq G_n$ for every n and $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$ (4A2Ff). Because X is normal, there are zero sets F_n such that $K_n \subseteq F_n \subseteq G_n$ for each n (4A2F(d-iv) again), so that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. We may suppose that F_0 has finite measure. In this case,

$$\lim_{n\to\infty} \nu^*K_n \le \lim_{n\to\infty} \nu(\bigcap_{i\le n} F_i) = 0.$$

Thus $\mathcal{K}$ and ϕ_0 satisfy the conditions (α) and (β) as well. **Q**

(b) By 413J, there is a complete locally determined measure μ on X, extending ϕ_0 and inner regular with respect to $\mathcal{K}$. If $F \subseteq X$ is closed, then $F \cap K \in \mathcal{K}$ for every $K \in \mathcal{K}$, so $F \in \operatorname{dom}\mu$ (413F(ii)); accordingly μ is a topological measure, and because ν also is inner regular with respect to $\mathcal{K}$, μ must extend ν. So the restriction of μ to the Borel sets is a Borel extension of ν which is inner regular with respect to the closed sets.

Remark If X is normal, but not countably paracompact, the result may fail; see 439O. I have stated the result in terms of 'countable paracompactness', but the formally distinct 'countable metacompactness' is also sufficient (435Ya). If we are told that the Baire measure is τ-additive and effectively locally finite, we have a much stronger result (415M).

435D Just as with the 'Radon' spaces of §434, we can look at classes of topological spaces defined by the behaviour of the Baire measures they carry. The class which has aroused most interest is the following.

Definition A completely regular topological space X is **measure-compact** (sometimes called **almost Lindelöf**) if every totally finite Baire measure on X is τ-additive, that is, has an extension to a quasi-Radon measure on X (415N).

435E The following lemma will make our path easier.

Lemma Let X be a completely regular topological space and ν a totally finite Baire measure on X. Suppose that $\sup_{G\in\mathcal{G}} \nu G = \nu X$ whenever $\mathcal{G}$ is an upwards-directed family of cozero sets with union X. Then ν is τ-additive.

proof Let $\mathcal{G}$ be an upwards-directed family of open Baire sets such that $G^* = \bigcup \mathcal{G}$ also is a Baire set, and $\epsilon > 0$. Because ν is inner regular with respect to the zero sets, there is a zero set $F \subseteq G^*$ such that $\nu F \ge \nu G^* - \epsilon$. Let $\mathcal{G}'$ be the family of cozero sets included in members of $\mathcal{G}$; because X is completely regular, so that the cozero sets are a base for its topology, $\bigcup \mathcal{G}' = G^*$, and of course $\mathcal{G}'$ is upwards-directed. Now

$$\mathcal{H} = \{G \cup (X \setminus F) : G \in \mathcal{G}'\}$$

is an upwards-directed family of cozero sets with union X, so there is a $G_0 \in \mathcal{G}'$ such that $\nu(G_0 \cup (X \setminus F)) \ge \nu X - \epsilon$. In this case

$$\sup_{G\in\mathcal{G}} \nu G \ge \nu G_0 \ge \nu X - \epsilon - \nu(X \setminus F) = \nu F - \epsilon \ge \nu G^* - 2\epsilon.$$

As $\mathcal{G}$ and ϵ are arbitrary, ν is τ-additive.

435F Elementary facts (a) If X is a completely regular space which is not measure-compact, there are a Baire probability measure μ on X and a cover of X by μ-negligible cozero sets. **P** There is a totally finite Baire measure ν on X which is not τ-additive. By 435E, there is an upwards-directed family $\mathcal{G}$ of cozero sets, covering X, such that $\sup_{G\in\mathcal{G}} \nu G < \nu X$. Let $\langle G_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ such that $\sup_{n\in\mathbb{N}} \nu G_n = \sup_{G\in\mathcal{G}} \nu G$. Then $\gamma = \nu(X \setminus \bigcup_{n\in\mathbb{N}} G_n) > 0$. Set

$$\mu H = \frac{1}{\gamma}\nu(H \setminus \bigcup_{n\in\mathbb{N}} G_n)$$

for Baire sets $H \subseteq X$; then μ is a Baire probability measure and $\mathcal{G}$ is a cover of X by μ-negligible cozero sets. **Q**

(b) Regular Lindelöf spaces are measure-compact. (For if a Lindelöf space can be covered by negligible open sets, it can be covered by countably many negligible open sets, so is itself negligible.) In particular, compact Hausdorff spaces, indeed all regular K-analytic Hausdorff spaces (422Gg), are measure-compact.

Note that regular Lindelöf spaces are normal and paracompact (4A2H(b-i)), so their measure-compactness is also a consequence of 435C and 434Hb.

(c) An open subset of a measure-compact space need not be measure-compact (435Xi(i)). A continuous image of a measure-compact space need not be measure-compact (435Xi(ii)). $\mathbb{N}^{\mathfrak{c}}$ is not measure-compact (439P). The product of two measure-compact spaces need not be measure-compact (439Q).

(d) If X is a measure-compact completely regular space it is Borel-measure-compact. **P** Let μ be a non-zero totally finite Borel measure on X and $\mathcal{G}$ an open cover of X. Let ν be the restriction of μ to the Baire σ-algebra of X, so that ν is τ-additive. Let $\mathcal{U}$ be the set of cozero sets $U \subseteq X$ included in members of $\mathcal{G}$; because the family of cozero sets is a base for the topology of X, $\bigcup\mathcal{U} = X$, and there is some $U \in \mathcal{U}$ such that $\nu U > 0$. This means that there is some $G \in \mathcal{G}$ such that $\mu G > 0$. By 434H(a-v), X is Borel-measure-compact. **Q**

435G Proposition A Souslin-F subset of a measure-compact completely regular space is measure-compact.

proof (a) Let X be a measure-compact completely regular space, $\langle F_\sigma\rangle_{\sigma\in S}$ a Souslin scheme consisting of closed subsets of X with kernel A, ν a totally finite Baire measure on A, and $\mathcal{G}$ an upwards-directed family of (relatively) cozero subsets of A covering A. Let ν_1 be the Baire measure on X defined by setting $\nu_1 H = \nu(A \cap H)$ for every Baire subset H of X. Because X is measure-compact, ν_1 has an extension to a quasi-Radon measure μ on X. Let μ_A be the subspace measure on A.

(b) By 431B, A is measured by μ. In fact $\mu A = \nu A$. **P** The construction of μ given in 415K-415N ensures that $\mu F = \nu_1^* F$ for every closed set F, and this is in any case a consequence of the facts that μ is τ-additive and $\operatorname{dom}\nu_1$ includes a base for the topology. For each $\sigma \in S$, in particular, $\mu F_\sigma = \nu_1^* F_\sigma$; let $F'_\sigma \supseteq F_\sigma$ be a Baire set such that $\nu_1 F'_\sigma = \nu_1^* F_\sigma$. Then

$$\mu F'_\sigma = \nu_1 F'_\sigma = \nu_1^* F_\sigma = \mu F_\sigma$$

and $\mu(F'_\sigma \setminus F_\sigma) = 0$ for every $\sigma \in S$. Let A' be the kernel of the Souslin scheme $\langle F'_\sigma\rangle_{\sigma\in S}$. Then $A \subseteq A'$ and

$$\mu(A' \setminus A) \le \textstyle\sum_{\sigma\in S} \mu(F'_\sigma \setminus F_\sigma) = 0,$$

so $\mu A = \mu A'$. On the other hand, writing $\hat\nu_1$ for the completion of ν_1, A' is measured by $\hat\nu_1$, by 431A, so that (because μ extends ν_1)

$$\mu A = \mu A' = \mu^* A' \le \nu_1^* A' = (\nu_1)_* A' \le \mu_* A' = \mu A'.$$

Thus $\mu A = \nu_1^* A'$. But of course

$$\nu A = \nu_1 X = \nu_1^* A = \nu_1^* A',$$

so that $\mu A = \nu A$. **Q**

Since we surely have

$$\mu X = \nu_1 X = \nu A,$$

we see that $\mu(X \setminus A) = 0$.

(c) It follows that $\mu F = \nu F$ for every (relatively) zero set $F \subseteq A$. **P** There is a closed set $F' \subseteq X$ such that $F = A \cap F'$. Now if $H \subseteq X$ is a Baire set including F', $H \cap A$ is a (relatively) Baire set including F, so $\nu F \le \nu(H \cap A) = \nu_1 H$; as H is arbitrary, $\nu F \le \nu_1^* F'$. But $\nu_1^* F' = \mu F'$, as remarked in (b) above, and $\mu(X \setminus A) = 0$, so

$$\mu F = \mu F' = \nu_1^* F' \ge \nu F.$$

On the other hand, $A \setminus F$ is (relatively) cozero, so there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of (relatively) zero subsets of A with union $A \setminus F$, and

$$\mu(A \setminus F) = \lim_{n\to\infty} \mu F_n \ge \lim_{n\to\infty} \nu F_n = \nu(A \setminus F).$$

Since we already know that $\mu A=\nu A$, it follows that

$$\mu F=\mu A-\mu(A\setminus F)\le\nu A-\nu(A\setminus F)=\nu F,$$

and $\mu F=\nu F$. **Q**

(d) The set

$$\{E:E\subseteq A\text{ is a (relative) Baire set, }\mu E=\nu E\}$$

therefore contains every (relatively) zero set, and by the Monotone Class Theorem (136C) contains every (relatively) Baire set. What this means is that μ actually extends ν; so the subspace measure $\mu_A=\mu\restriction\mathcal{P}A$ also extends ν. But μ_A is a quasi-Radon measure (415B), therefore τ-additive, and ν must also be τ-additive.

435H Corollary A Baire subset of a measure-compact completely regular space is measure-compact.

proof Put 435G and 421L together.

435X Basic exercises >(a) Give ω_1+1 its order topology. (i) Show that its Baire σ-algebra Σ is just the family of sets $E\subseteq\omega_1+1$ such that either E or its complement is a countable subset of ω_1. (ii) Show that there is a unique Baire probability measure ν on ω_1+1 such that $\nu\{\xi\}=0$ for every $\xi<\omega_1$. (iii) Show that ν is τ-additive. (iv) Show that there is exactly one Radon measure on ω_1+1 extending ν, but that the measure μ of 434Xg is another Borel measure also extending ν.

>(b) Let I be a set with cardinal ω_1, endowed with its discrete topology, and $X=I\cup\{\infty\}$ its one-point compactification (3A3O). Let μ be the Dirac measure on X concentrated at ∞. (i) Show that every subset of X is a Borel set. (ii) Show that $\{\infty\}$ is not a zero set. (iii) Let ν be the restriction of μ to the Baire σ-algebra of X. Show that ν is tight. Show that μ is the unique Borel measure extending ν (*hint*: you will need 419G), but is not completion regular. (iv) Show that the subspace measure ν_I on I is the countable-cocountable measure on I, and is not a Baire measure, nor has any extension to a Baire measure on I. (v) Show that X is measure-compact.

(c) On $\mathbb{R}^{\omega_1}$ let μ be the Baire measure defined by saying that $\mu E=1$ if $\chi\omega_1\in E$, 0 otherwise. (i) Show that μ is τ-additive, but not tight. (*Hint*: 4A3P.) (ii) Show that the map $\xi\mapsto\chi\xi:\omega_1+1\to\mathbb{R}^{\omega_1}$ is continuous, so that μ has more than one extension to a Borel measure. (iii) Show that μ has an extension to a Radon measure.

(d) Set $X=\omega_1+1$ with the topology $\mathcal{P}\omega_1\cup\{X\setminus A:A\subseteq\omega_1\text{ is countable}\}$. Let μ be the Baire measure on X defined by saying that, for Baire sets $E\subseteq X$, $\mu E=1$ if $\omega_1\in E$, 0 otherwise. (i) Show that a function $f:X\to\mathbb{R}$ is continuous iff $\{\xi:\xi\in X,\ f(\xi)\ne f(\omega_1)\}$ is countable; show that X is completely regular and Hausdorff. (ii) Show that μ is τ-additive. (iii) Show that every subset of X is Borel. (iv) Show that the only Borel measure extending μ is the Dirac measure concentrated at ω_1, and that this is a Radon measure. (v) Show that all compact subsets of X are finite, so that μ is not tight. *(vi) Show that X is Lindelöf.

(e) Let X be a locally compact Hausdorff space and μ an effectively locally finite τ-additive Baire measure on X. Show that μ is tight. (*Hint*: the relatively compact cozero sets cover X; use 414Ea and 412D.)

>(f) Let X be a completely regular space and μ a totally finite τ-additive Borel measure on X. Let μ_0 be the restriction of μ to the Baire σ-algebra of X. Show that $\mu F=\mu_0^*F$ for every closed set $F\subseteq X$.

(g) Show that if a semi-finite Baire measure ν on a normal countably paracompact space is extended to a Borel measure μ by the construction in 435C, then the measure algebra of ν becomes embedded as an order-dense subalgebra of the measure algebra of μ, so that $L^1(\mu)$ can be identified with $L^1(\nu)$.

(h) Show that a Borel-measure-compact normal countably paracompact space is measure-compact.

(i)(i) Show that ω_1+1 is measure-compact, in its order topology, but that its open subset ω_1 is not (cf. 434Xm). (ii) Show that a discrete space with cardinal ω_1 is measure-compact, but that it has a continuous image which is not measure-compact.

(j) Let X be a metacompact completely regular space and ν a totally finite strictly positive Baire measure on X. Show that X is Lindelöf, so that ν has an extension to a quasi-Radon measure on X. (*Hint*: if $\mathcal{H}$ is a point-finite open cover of X, not containing $\emptyset$, then for each $H \in \mathcal{H}$ choose a non-empty cozero set $G_H \subseteq H$; show that $\{H : \nu G_H \ge \delta\}$ is finite for every $\delta > 0$.)

(k) A completely regular space X is **strongly measure-compact** (Moran 69) if $\mu X = \sup\{\mu^* K : K \subseteq X \text{ is compact}\}$ for every totally finite Baire measure μ on X. (i) Show that a completely regular Hausdorff space X is strongly measure-compact iff every totally finite Baire measure on X has an extension to a Radon measure iff X is measure-compact and pre-Radon. (ii) Show that a Souslin-F subset of a strongly measure-compact completely regular space is strongly measure-compact. (iii) Show that a discrete space with cardinal ω_1 is strongly measure-compact. (iv) Show that a countable product of strongly measure-compact completely regular spaces is strongly measure-compact. (v) Show that $\mathbb{N}^{\omega_1}$ is not strongly measure-compact. (*Hint*: take a non-trivial probability measure on $\mathbb{N}$ and consider its power on $\mathbb{N}^{\omega_1}$.) (vi) Show that if X and Y are completely regular spaces, X is measure-compact and Y is strongly measure-compact then $X \times Y$ is measure-compact.

(l) (T.D.Austin) Let X be a topological space, μ an atomless Baire probability measure on X and $\hat{\mu}$ its completion. Show that there is a continuous function $f : X \to [0,1]$ which is inverse-measure-preserving for $\hat{\mu}$ and Lebesgue measure on $[0,1]$. (*Hint*: Check the case $X = [0,1]$ first. For the general case, let Z be the set of continuous functions from X to $[0,1]$ with the complete metric induced by $\| \|_\infty$, and set $\alpha(f) = \max\{\mu f^{-1}[\{t\}] : t \in [0,1]\}$ for $f \in Z$. Show that $\operatorname{int}\{f : \alpha(f) \le \epsilon\}$ is dense in Z for every $\epsilon > 0$, so that there is an $f \in Z$ such that μf^{-1} is atomless.)

(m) Let X be a normal space and μ a complete σ-finite topological probability measure on X which is inner regular with respect to the closed sets. (i) Let ν be the restriction of μ to the Baire σ-algebra of X. Show that μ and ν have isomorphic measure algebras. (ii) Show that if μ is an atomless probability measure there is a continuous $f : X \to [0,1]$ which is inverse-measure-preserving for μ and Lebesgue measure.

(n) Let X be a topological space and $\mathcal{G}$ the family of cozero sets in X. Show that a functional $\psi : \mathcal{G} \to [0,\infty[$ can be extended to a Baire measure on X iff ψ is modular (definition: 413Qc) and $\lim_{n\to\infty} \psi G_n = 0$ whenever $\langle G_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{G}$ with empty intersection. (*Hint*: if ψ satisfies the conditions, first check that $\psi\emptyset = 0$ and that $\psi G \le \psi H$ whenever $G \subseteq H$; now apply 413J with $\phi K = \inf\{\psi G : K \subseteq G \in \mathcal{G}\}$ for zero sets K.)

(o) Let X be a countably compact topological space and μ a totally finite Baire measure on X. Show that μ has an extension to a Borel measure which is inner regular with respect to the closed sets. (*Hint*: 413P.)

435Y Further exercises (a) Show that a normal countably metacompact space (434Yn) is countably paracompact.

(b) Let X be a completely regular Hausdorff space and βX its Stone-Čech compactification. Show that X is measure-compact iff whenever ν is a Radon measure on βX such that $\nu X = 0$, there is a ν-negligible Baire subset of βX including X.

435 Notes and comments The principal reason for studying Baire measures is actually outside the main line of this chapter. For a completely regular Hausdorff space X, write $C_b(X)$ for the M-space of bounded continuous real-valued functions on X. Then $C_b(X)^* = C_b(X)^\sim$ is an L-space (356N), and inside $C_b(X)^*$ we have the bands generated by the tight, smooth and sequentially smooth functionals (see 437A and 437F below), all identifiable, if we choose, with spaces of 'signed Baire measures'. Wheeler 83 argues convincingly that for the questions a functional analyst naturally asks, these Baire measures are often an effective aid.

From the point of view of the arguments in this section, the most fundamental difference between 'Baire' and 'Borel' measures lies in their action on subspaces. If X is a topological space and A is a subset of X, then any Borel or Baire measure μ on A provides us with a measure μ_1 of the same type on X, setting

$\mu_1 E = \mu(A \cap E)$ for the appropriate sets E. In the other direction, if μ is a Borel measure on X, then the subspace measure μ_A is a Borel measure on A, because the Borel σ-algebra of A is just the subspace σ-algebra derived from the Borel algebra of X (4A3Ca). But if μ is a Baire measure on X, it does not follow that μ_A is a Baire measure on A; this is because (in general) not every continuous function $f : A \to [0,1]$ has a continuous extension to X, so that not every zero set in A is the intersection of A with a zero set in X (see 435Xb). The analysis of those pairs (X, A) for which the Baire σ-algebra of A is just the subspace algebra derived from the Baire sets in X is a challenging problem in general topology which I pass by here. For the moment I note only that avoiding it is the principal technical problem in the proof of 435G.

I do not know if I ought to apologise for 'countably tight' spaces (434N), 'first-countable' spaces (434R), 'metacompact' spaces (438J), 'normal countably paracompact' spaces (435C), 'quasi-dyadic' spaces (434O) and 'sequential' spaces (436F). General topology is notorious for invoking arcane terminology to stretch arguments to their utmost limit of generality, and even specialists may find their patience tried by definitions which seem to have only one theorem each. In 438J, for instance, it is obvious that the original result concerned metrizable spaces (438H), and you may well feel at first that the extension is a baroque over-elaboration. On the other hand, there are (if you look for them) some very interesting metacompact spaces (Engelking 89, §5.3), and metacompactness has taken its place in the standard lists. In this book I try to follow a rule of introducing a class of topological spaces only when it is both genuinely interesting, from the point of view of general topology, and also a support for an idea which is interesting from the point of view of measure theory.

Version of 9.5.11

436 Representation of linear functionals

I began this treatise with the three steps which make measure theory, as we know it, possible: a construction of Lebesgue measure, a definition of an integral from a measure, and a proof of the convergence theorems. I used what I am sure is the best route: Lebesgue measure from Lebesgue outer measure, and integrable functions from simple functions. But of course there are many other paths to the same ends, and some of them show us slightly different aspects of the subject. In this section I come – rather later than many authors would – to an account of a procedure for constructing measures from integrals.

I start with three fundamental theorems, the first and third being the most important. A positive linear functional on a truncated Riesz space of functions is an integral iff it is sequentially smooth (436D); a smooth linear functional corresponds to a quasi-Radon measure (436H); and if X is a compact Hausdorff space, any positive linear functional on $C(X)$ corresponds to a Radon measure (436J-436K).

436A Definition Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a positive linear functional. I say that f is **sequentially smooth** if whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$, then $\lim_{n\to\infty} f(u_n) = 0$.

If (X, Σ, μ) is a measure space and U is a Riesz subspace of the space of real-valued μ-integrable functions defined everywhere on X, then $\int d\mu : U \to \mathbb{R}$ is sequentially smooth, by Fatou's Lemma or Lebesgue's Dominated Convergence Theorem.

Remark It is essential to distinguish between 'sequentially smooth', as defined here, and 'sequentially order-continuous', as in 313Hb or 355G. In the context here, a positive linear operator $f : U \to \mathbb{R}$ is sequentially order-continuous if $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that 0 is the greatest lower bound for $\{u_n : n \in \mathbb{N}\}$ in U; while f is sequentially smooth if $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that 0 is the greatest lower bound for $\{u_n : n \in \mathbb{N}\}$ in $\mathbb{R}^X$. So there can be sequentially smooth functionals which are not sequentially order-continuous, as in 436Xi. A sequentially order-continuous positive linear functional is of course sequentially smooth.

436B Definition Let X be a set. I will say that a Riesz subspace U of $\mathbb{R}^X$ is **truncated** (or satisfies **Stone's condition**) if $u \wedge \chi X \in U$ for every $u \in U$.

In this case, $u \wedge \gamma\chi X \in U$ for every $\gamma \geq 0$ and $u \in U$ (being $-u^-$ if $\gamma = 0$, $\gamma(\gamma^{-1}u \wedge \chi X)$ otherwise).

436C Lemma Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Write $\mathcal{K}$ for the family of sets of the form $\{x : x \in X,\, u(x) \ge 1\}$ as u runs over U. Let $f : U \to \mathbb{R}$ be a sequentially smooth positive linear functional, and μ a measure on X such that μK is defined and equal to $\inf\{f(u) : \chi K \le u \in U\}$ for every $K \in \mathcal{K}$. Then $\int u\,d\mu$ exists and is equal to $f(u)$ for every $u \in U$.

proof It is enough to deal with the case $u \ge 0$, since $U = U^+ - U^+$ and both f and $\int$ are linear. Note that if $v \in U$, $K \in \mathcal{K}$ and $v \le \chi K$, then $v \le w$ whenever $\chi K \le w \in U$, so $f(v) \le \mu K$. For k, $n \in \mathbb{N}$ set

$$K_{nk} = \{x : u(x) \ge 2^{-n}k\}, \quad u_{nk} = u \wedge 2^{-n}k\chi X.$$

Then, for $k \ge 1$,

$$K_{nk} = \{x : \frac{2^n}{k}u \ge 1\} \in \mathcal{K},$$

$$2^n(u_{n,k+1} - u_{nk}) \le \chi K_{nk} \le 2^n(u_{nk} - u_{n,k-1}).$$

So

$$2^n f(u_{n,k+1} - u_{nk}) \le \mu K_{nk} \le 2^n f(u_{nk} - u_{n,k-1}),$$

and

$$f(u_{n,4^n+1} - u_{n1}) \le \sum_{k=1}^{4^n} 2^{-n}\mu K_{nk} \le f(u_{n,4^n} \le f(u).$$

But setting $w_n = u_{n,4^n+1} - u_{n1}$, $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of functions in U and $\sup_{n\in\mathbb{N}} w_n(x) = u(x)$ for every x, so $\lim_{n\to\infty} f(u - w_n) = 0$ and $\lim_{n\to\infty} f(w_n) = f(u)$. Also, setting $v_n = \sum_{k=1}^{4^n} 2^{-n}\chi K_{nk}$, we have $w_n \le v_n \le u$ and $f(w_n) \le \int v_n \le f(u)$ for each n, so

$$\int u = \lim_{n\to\infty} \int v_n = f(u)$$

by B.Levi's theorem.

436D Theorem Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $f : U \to \mathbb{R}$ be a positive linear functional. Then the following are equiveridical:

(i) f is sequentially smooth;

(ii) there is a measure μ on X such that $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$.

proof I remarked in 436A that (ii)⇒(i) is a consequence of Fatou's Lemma. So the argument here is devoted to proving that (i)⇒(ii).

(a) Let $\mathcal{K}$ be the family of sets $K \subseteq X$ such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ for some sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U, taking the infimum in $\mathbb{R}^X$, so that $(\inf_{n\in\mathbb{N}} u_n)(x) = \inf_{n\in\mathbb{N}} u_n(x)$ for every $x \in X$. Then $\mathcal{K}$ is closed under finite unions and countable intersections. **P** (i) If K, $K' \in \mathcal{K}$ take sequences $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle u'_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ and $\chi K' = \inf_{n\in\mathbb{N}} u'_n$; then $\chi(K \cup K') = \inf_{m,n\in\mathbb{N}} u_m \vee u'_n$, so $K \cup K' \in \mathcal{K}$. (ii) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$, then for each $n \in \mathbb{N}$ we can choose a sequence $\langle u_{ni}\rangle_{i\in\mathbb{N}}$ in U such that $\chi K_n = \inf_{i\in\mathbb{N}} u_{ni}$; now $\chi(\bigcap_{n\in\mathbb{N}} K_n) = \inf_{n,i\in\mathbb{N}} u_{ni}$, so $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$. **Q**

Note that $\emptyset \in \mathcal{K}$ because $0 \in U$.

(b) We need to know that if $u \in U$ then $K = \{x : u(x) \ge 1\}$ belongs to $\mathcal{K}$. **P** Set

$$u_n = 2^n((u \wedge \chi X) - (u \wedge (1 - 2^{-n})\chi X)).$$

Because U is truncated, every u_n belongs to U, and it is easy to check that $\inf_{n\in\mathbb{N}} u_n = \chi K$. **Q** It follows that

$$\{x : u(x) \ge \alpha\} = \{x : \frac{1}{\alpha}u(x) \ge 1\} \in \mathcal{K}$$

whenever $u \in U$ and $\alpha > 0$.

(c) For $K \in \mathcal{K}$, set $\phi_0 K = \inf\{f(u) : u \in U,\, u \ge \chi K\}$. Then ϕ_0 satisfies the conditions of 413J. **P** I have already checked (†) and (‡) of 413J.

(α) Fix K, $L \in \mathcal{K}$ with $L \subseteq K$. Set $\gamma = \sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}$.

(i) Suppose that $K' \in \mathcal{K}$ is included in $K \setminus L$, and $\epsilon > 0$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle u'_n\rangle_{n\in\mathbb{N}}$ be sequences in U such that $\chi L = \inf_{n\in\mathbb{N}} u_n$ and $\chi K' = \inf_{n\in\mathbb{N}} u'_n$, and let $u \in U$ be such that $u \geq \chi K$ and $f(u) \leq \phi_0 K + \epsilon$. Set $v_n = u \wedge \inf_{i\leq n} u_i$, $v'_n = u \wedge \inf_{i\leq n} u'_i$ for each n. Then $\langle v_n \wedge v'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum $\chi L \wedge \chi K' = 0$, so there is an n such that $f(v_n \wedge v'_n) \leq \epsilon$. In this case

$$\begin{aligned}\phi_0 L + \phi_0 K' &\leq f(v_n) + f(v'_n) = f(v_n + v'_n)\\ &= f(v_n \vee v'_n) + f(v_n \wedge v'_n) \leq f(u) + \epsilon \leq \phi_0 K + 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\phi_0 L + \phi_0 K' \leq \phi_0 K$. As K' is arbitrary, $\phi_0 L + \gamma \leq \phi_0 K$.

(ii) Next, given $\epsilon \in \,]0,1[$, there are u, $v \in U$ such that $u \geq \chi K$, $v \geq \chi L$ and $f(v) \leq \phi_0 L + \epsilon$. Consider

$$K' = \{x : x \in K,\, \min(1, u(x)) - v(x) \geq \epsilon\} \subseteq K \setminus L.$$

By (b), $K' \in \mathcal{K}$. If $w \in U$ and $w \geq \chi K'$, then $v(x) + w(x) \geq 1 - \epsilon$ for every $x \in K$, so

$$\phi_0 K \leq \tfrac{1}{1-\epsilon} f(v + w) \leq \tfrac{1}{1-\epsilon}(\phi_0 L + \epsilon + f(w)).$$

As w is arbitrary,

$$(1 - \epsilon)\phi_0 K \leq \phi_0 L + \epsilon + \phi_0 K' \leq \phi_0 L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\phi_0 K \leq \phi_0 L + \gamma$ and we have equality, as required by (α) in 413J.

(β) Now suppose that $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection. For each $n \in \mathbb{N}$ let $\langle u_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence in U with infimum χK_n in $\mathbb{R}^X$. Set $v_n = \inf_{i,j\leq n} u_{ji}$ for each n; then $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum $\inf_{n\in\mathbb{N}} \chi K_n = 0$, so $\inf_{n\in\mathbb{N}} f(v_n) = 0$. But

$$v_n \geq \inf_{j\leq n} \chi K_j = \chi K_n, \quad \phi_0 K_n \leq f(v_n)$$

for every n, so $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$, as required by (β) of 413J. **Q**

(d) By 413J, there is a complete locally determined measure μ on X, inner regular with respect to $\mathcal{K}$, extending ϕ_0. By 436C, $f(u) = \int u\,d\mu$ for every $u \in U$, as required.

436E Proposition Let X be any topological space, and $C_b = C_b(X)$ the space of bounded continuous real-valued functions on X. Then there is a one-to-one correspondence between totally finite Baire measures μ on X and sequentially smooth positive linear functionals $f : C_b \to \mathbb{R}$, given by the formulae

$$f(u) = \int u\,d\mu \text{ for every } u \in C_b,$$

$$\mu Z = \inf\{f(u) : \chi Z \leq u \in C_b\} \text{ for every zero set } Z \subseteq X.$$

proof (a) If μ is a totally finite Baire measure on X, then every continuous bounded real-valued function is integrable, and $f = \int d\mu$ is a sequentially smooth positive linear operator on C_b, by Fatou's Lemma, as usual.

(b) If $f : C_b \to \mathbb{R}$ is a sequentially smooth positive linear operator, then 436D tells us that there is a measure μ_0 on X such that $\int u\,d\mu_0$ is defined and equal to $f(u)$ for every $u \in C_b$. By the construction in 436D, or otherwise, we may suppose that μ_0 is complete, so that every $u \in C_b$ is Σ-measurable, where Σ is the domain of μ_0. It follows by the definition of the Baire σ-algebra $\mathcal{B}\mathfrak{a}$ of X (4A3K) that $\mathcal{B}\mathfrak{a} \subseteq \Sigma$, so that $\mu = \mu_0\restriction\Sigma$ is a Baire measure; of course we still have $f(u) = \int u\,d\mu$ for every $u \in C_b$. Also, if $Z \subseteq X$ is a zero set, $\mu Z = \inf\{f(u) : \chi Z \leq u \in C_b\}$. **P** Express Z as $\{x : v(x) = 0\}$ where $v : X \to [0,1]$ is continuous. Set

$$u_n = (\chi X - 2^n v)^+$$

for $n \in \mathbb{N}$; then $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in C_b and $\langle u_n(x)\rangle_{n\in\mathbb{N}} \to (\chi Z)(x)$ for every $x \in X$, so

$$\begin{aligned}\mu Z &\leq \inf\{\int u\,d\mu : \chi Z \leq u \in C_b\} = \inf\{f(u) : \chi Z \leq u \in C_b\}\\ &\leq \inf_{n\in\mathbb{N}} f(u_n) = \lim_{n\to\infty}\int u_n d\mu = \mu Z. \ \mathbf{Q}\end{aligned}$$

(c) The argument of (b) shows that if two totally finite Baire measures give the same integrals to every member of C_b, then they must agree on all zero sets. By the Monotone Class Theorem (136C) they agree on the σ-algebra generated by the zero sets, that is, $\mathcal{B}\mathfrak{a}$, and are therefore equal. Thus the operator $\mu \mapsto \int d\mu$ from the set of totally finite Baire measures on X to the set of sequentially smooth positive linear operators on C_b is a bijection, and if $f = \int d\mu$ then $\mu Z = \inf\{f(u) : \chi Z \le u \in C_b\}$ for every zero set Z, as required.

436F Corresponding to 434R, we have the following construction for product Baire measures, applicable to a slightly larger class of spaces.

Proposition Let X be a sequential space, Y a topological space, and μ, ν totally finite Baire measures on X, Y respectively. Then there is a Baire measure λ on $X \times Y$ such that

$$\lambda W = \int \nu W[\{x\}]\mu(dx), \quad \int f d\lambda = \iint f(x,y)\nu(dy)\mu(dx)$$

for every Baire set $W \subseteq X \times Y$ and every bounded continuous function $f : X \times Y \to \mathbb{R}$.

proof (a) $\phi(f) = \iint f(x,y)dydx$ is defined in $\mathbb{R}$ for every bounded continuous function $f : X \times Y \to \mathbb{R}$. **P** For each $x \in X$, $g(x) = \int f(x,y)dy$ is defined because $y \mapsto f(x,y)$ is continuous. If $\langle x_n \rangle_{n\in\mathbb{N}}$ is any sequence in X converging to $x \in X$, then

$$g(x) = \int f(x,y)dy = \int \lim_{n\to\infty} f(x_n,y)dy = \lim_{n\to\infty} \int f(x_n,y)dy = \lim_{n\to\infty} g(x_n)$$

by Lebesgue's Dominated Convergence Theorem. So g is sequentially continuous; because X is sequential, g is continuous (4A2Kd). So $\iint f(x,y)dydx = \int g(x)dx$ is defined in $\mathbb{R}$. **Q**

(b) Of course ϕ is a positive linear functional on $C_b(X \times Y)$, and B.Levi's theorem shows that it is sequentially smooth. By 436E, there is a Baire measure λ on $X \times Y$ such that $\int f d\lambda = \phi(f)$ for every $f \in C_b(X \times Y)$.

(c) If $W \subseteq X \times Y$ is a zero set, there is a non-increasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $C_b(X \times Y)$ such that $\chi W = \inf_{n\in\mathbb{N}} f_n$. So B.Levi's theorem tells us that

$$\int \nu W[\{x\}]dx = \lim_{n\to\infty} \int f_n(x,y)dydx = \lim_{n\to\infty} \int f_n d\lambda = \lambda W.$$

Now the Monotone Class Theorem (136B) tells us that

$$\{W : W \subseteq X \times Y \text{ is Baire, } \int \nu W[\{x\}]dx \text{ exists} = \lambda W\}$$

includes the σ-algebra generated by the zero sets, that is, contains every Baire set in $X \times Y$. So λ has the required properties.

436G Definition Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a positive linear functional. I say that f is **smooth** if whenever A is a non-empty downwards-directed family in U such that $\inf_{u\in A} u(x) = 0$ for every $x \in X$, then $\inf_{u\in A} f(u) = 0$.

Of course a smooth functional is sequentially smooth. If $(X, \mathfrak{T}, \Sigma, \mu)$ is an effectively locally finite τ-additive topological measure space and U is a Riesz subspace of $\mathbb{R}^X$ consisting of integrable continuous functions, then $\int d\mu : U \to \mathbb{R}$ is smooth, by 414Bb. Corresponding to the remark in 436A, note that an order-continuous positive linear functional must be smooth, but that a smooth positive linear functional need not be order-continuous.

436H Theorem Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $f : U \to \mathbb{R}$ be a positive linear functional. Then the following are equiveridical:

(i) f is smooth;

(ii) there are a topology $\mathfrak{T}$ and a measure μ on X such that μ is a quasi-Radon measure with respect to $\mathfrak{T}$, $U \subseteq C(X)$ and $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$;

(iii) writing $\mathfrak{S}$ for the coarsest topology on X for which every member of U is continuous, there is a measure μ on X such that μ is a quasi-Radon measure with respect to $\mathfrak{S}$, and $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$.

proof As remarked in 436G, in a fractionally more general context, (ii)⇒(i) is a consequence of 414B. Of course (iii)⇒(ii). So the argument here is devoted to proving that (i)⇒(iii). Except for part (b) it is a simple adaptation of the method of 436D.

(a) Let $\mathcal{K}$ be the family of sets $K \subseteq X$ such that $\chi K = \inf A$ in $\mathbb{R}^X$ for some non-empty set $A \subseteq U$. Then $\mathcal{K}$ is closed under finite unions. **P** If K, $K' \in \mathcal{K}$ take A, $A' \subseteq U$ such that $\chi K = \inf A$, $\chi K' = A'$; then $\chi(K \cup K') = \inf\{u \vee u' : u \in A,\, u' \in A'\}$, so $K \cup K' \in \mathcal{K}$. **Q**

Note that $\emptyset \in \mathcal{K}$ because $0 \in U$.

As in part (b) of the proof of 436D, $\{x : u(x) \geq \alpha\} \in \mathcal{K}$ whenever $\alpha > 0$ and $u \in U$.

(b) Every member of $\mathcal{K}$ is closed for $\mathfrak{S}$, being of the form $\{x : u(x) \geq 1$ for every $u \in A\}$ for some $A \subseteq U$. We need to know that if $K \in \mathcal{K}$ and $G \in \mathfrak{S}$, then $K \setminus G \in \mathcal{K}$. **P** Take a non-empty set $B \subseteq U$ such that $\chi K = \inf B$. Because $\mathcal{K}$ is closed under finite unions and arbitrary intersections, $\mathfrak{S}_K = \{G : G \subseteq X,\, K \setminus G \in \mathcal{K}\}$ is a topology on X. (i) If $u \in U$ and $G = \{x : u(x) > 0\}$, then $\chi(K \setminus G) = \inf\{(v - ku)^+ : v \in B,\, k \in \mathbb{N}\}$ so $K \setminus G \in \mathcal{K}$ and $G \in \mathfrak{S}_K$. (ii) If $u \in U$ and $\alpha > 0$, then $\{x : u(x) > \alpha\} = \{x : (u - u \wedge \alpha\chi X)(x) > 0\}$ belongs to $\mathfrak{S}_K$, by (i). (iii) If $u \in U$ and $\alpha > 0$, set $G = \{x : u(x) < \alpha\}$. Then

$$K \setminus G = K \cap \{x : u(x) \geq \alpha\} \in \mathcal{K},$$

so $G \in \mathfrak{S}_K$. (iv) Thus every member of U^+ is $\mathfrak{S}_K$-continuous (2A3Bc), so every member of U is $\mathfrak{S}_K$-continuous (2A3Be), and $\mathfrak{S} \subseteq \mathfrak{S}_K$, that is, $K \setminus G \in \mathcal{K}$ for every $G \in \mathfrak{S}$. **Q**

(c) For $K \in \mathcal{K}$, set $\phi_0 K = \inf\{f(u) : u \in U,\, u \geq \chi K\}$. Then ϕ_0 satisfies the conditions of 415K. **P** I have already checked (†) and (‡) of 415K.

(α) Fix K, $L \in \mathcal{K}$ with $L \subseteq K$. Set $\gamma = \sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}$.

(i) Suppose that $K' \in \mathcal{K}$ is included in $K \setminus L$ and $\epsilon > 0$. Set $A = \{u : \chi L \leq u \in U\}$, $A' = \{u : \chi K' \leq u \in U\}$, so that $\chi L = \inf A$ and $\chi K' = \inf A'$, and let $v \in U$ be such that $v \geq \chi K$ and $f(v) \leq \phi_0 K + \epsilon$. Then $\{u \wedge u' : u \in A,\, u' \in A'\}$ is a downwards-directed family with infimum 0 in $\mathbb{R}^X$, so (because f is smooth) there are $u \in A$, $u' \in A'$ such that $f(u \wedge u') \leq \epsilon$. In this case

$$\phi_0 L + \phi_0 K' \leq f(v \wedge u) + f(v \wedge u') = f(v \wedge (u \vee u')) + f(v \wedge u \wedge u') \leq \phi_0 K + 2\epsilon.$$

As ϵ is arbitrary, $\phi_0 L + \phi_0 K' \leq \phi_0 K$. As K' is arbitrary, $\phi_0 L + \gamma \leq \phi_0 K$.

(ii) Next, given $\epsilon \in \,]0,1[$, there are u, $v \in U$ such that $u \geq \chi K$, $v \geq \chi L$ and $f(v) \leq \phi_0 L + \epsilon$. Consider

$$K' = \{x : x \in K,\, \min(1, u(x)) - v(x) \geq \epsilon\}.$$

By the last remark in (a), $K' \in \mathcal{K}$. If $w \in U$ and $w \geq \chi K'$, then $v(x) + w(x) \geq 1 - \epsilon$ for every $x \in K$, so

$$\phi_0 K \leq \tfrac{1}{1-\epsilon} f(v + w) \leq \tfrac{1}{1-\epsilon}(\phi_0 L + \epsilon + f(w)).$$

As w is arbitrary,

$$(1 - \epsilon)\phi_0 K \leq \phi_0 L + \epsilon + \phi_0 K' \leq \phi_0 L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\phi_0 K \leq \phi_0 L + \gamma$ and we have equality, as required by (α) in 415K.

(β) Now suppose that $\mathcal{K}'$ is a non-empty downwards-directed subset of $\mathcal{K}$ with empty intersection. Set

$$A = \bigcup_{K \in \mathcal{K}'} \{u : \chi K \leq u \in U\}.$$

Then A is a downwards-directed subset of U and $\inf A = 0$ in $\mathbb{R}^X$. Because f is smooth,

$$0 = \inf_{u \in A} f(u) = \inf_{K \in \mathcal{K}'} \phi_0 K.$$

Thus (β) of 415K is satisfied.

(γ) If $K \in \mathcal{K}$ and $\phi_0 K > 0$, take $u \in U$ such that $u \geq \chi K$, and consider $G = \{x : u(x) > \frac{1}{2}\}$. Then $K \subseteq G$, while $G \subseteq \{x : 2u(x) \geq 1\}$, so

$$\sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq G\} \leq 2f(u) < \infty.$$

Thus ϕ_0 satisfies (γ) of 415K. **Q**

(d) By 415K, there is a quasi-Radon measure μ on X extending ϕ_0. By 436C, $f(u) = \int u\, d\mu$ for every $u \in U$.

Remark It is worth noting explicitly that μ, as constructed here, is inner regular with respect to the family $\mathcal{K}$ of sets $K \subseteq X$ such that $\chi K = \inf A$ for some set $A \subseteq U$.

436I Lemma Let X be a topological space. Let $C_0 = C_0(X)$ be the space of continuous functions $u : X \to \mathbb{R}$ which 'vanish at infinity' in the sense that $\{x : |u(x)| \ge \epsilon\}$ is compact for every $\epsilon > 0$.

(a) C_0 is a norm-closed solid linear subspace of $C_b = C_b(X)$, so is a Banach lattice in its own right.

(b) $C_0^* = C_0^\sim$ is an L-space (definition: 354M).

(c) If $A \subseteq C_0$ is a non-empty downwards-directed set such that $\inf_{u \in A} u(x) = 0$ for every $x \in X$, then $\inf_{u \in A} \|u\|_\infty = 0$.

proof (a)(i) If $u \in C_0$, then $K = \{x : |u(x)| \ge 1\}$ is compact, so $\|u\|_\infty \le \sup(\{1\} \cup \{|u(x)| : x \in K\})$ is finite, and $u \in C_b$.

(ii) If u, $v \in C_0$ and $\alpha \in \mathbb{R}$ and $w \in C_b$ and $|w| \le |u|$, then for any $\epsilon > 0$

$$\{x : |u(x) + v(x)| \ge \epsilon\} \subseteq \{x : |u(x)| \ge \frac{1}{2}\epsilon\} \cup \{x : |v(x)| \ge \frac{1}{2}\epsilon\},$$

$$\{x : |\alpha u(x)| \ge \epsilon\} \subseteq \{x : |u(x)| \ge \tfrac{\epsilon}{1+|\alpha|}\},$$

$$\{x : |w(x)| \ge \epsilon\} \subseteq \{x : |u(x)| \ge \epsilon\}$$

are closed relatively compact sets, so are compact, and $u + v$, αu, w belong to C_0. Thus C_0 is a solid linear subspace of C_b.

(iii) If $\langle u_n \rangle_{n \in \mathbb{N}}$ is a sequence in C_0 which $\| \,\|_\infty$-converges to $u \in C_b$, then for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\|u - u_n\|_\infty \le \frac{1}{2}\epsilon$, so that

$$\{x : |u(x)| \ge \epsilon\} \subseteq \{x : |u_n(x)| \ge \frac{1}{2}\epsilon\}$$

is compact, and $u \in C_0$. Thus C_0 is norm-closed in C_b.

(iv) Being a norm-closed Riesz subspace of a Banach lattice, C_0 is itself a Banach lattice.

(b) By 356Dc, $C_0^* = C_0^\sim$ is a Banach lattice. Now $\|f + g\| = \|f\| + \|g\|$ for all non-negative f, $g \in C_0^*$. **P** Of course $\|f + g\| \le \|f\| + \|g\|$. On the other hand, for any $\epsilon > 0$ there are u, $v \in C_0$ such that $\|u\|_\infty \le 1$, $\|v\|_\infty \le 1$ and $|f(u)| \ge \|f\| - \epsilon$, $|g(v)| \ge \|g\| - \epsilon$. Set $w = |u| \vee |v|$; then $w \in C_0$ and

$$\|w\|_\infty = \max(\|u\|_\infty, \|v\|_\infty) \le 1.$$

So

$$\|f + g\| \ge (f + g)(w) \ge f(|u|) + g(|v|) \ge |f(u)| + |g(v)| \ge \|f\| + \|g\| - 2\epsilon.$$

As ϵ is arbitrary, $\|f + g\| \ge \|f\| + \|g\|$. **Q**

So C_0^* is an L-space.

(c) Let $\epsilon > 0$. For $u \in A$ set $K_u = \{x : u(x) \ge \epsilon\}$. Then $\{K_u : u \in A\}$ is a downwards-directed family of closed compact sets with empty intersection, so there must be some $u \in A$ such that $K_u = \emptyset$, and $\|u\|_\infty \le \epsilon$. As ϵ is arbitrary, we have the result.

Remark (c) is a version of **Dini's theorem**.

436J Riesz Representation Theorem (first form) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, and $C_k = C_k(X)$ the space of continuous real-valued functions on X with compact support. If $f : C_k \to \mathbb{R}$ is any positive linear functional, there is a unique Radon measure μ on X such that $f(u) = \int u \, d\mu$ for every $u \in C_k$.

proof (a) The point is that f is smooth. **P** Suppose that $A \subseteq C_k$ is non-empty and downwards-directed and that $\inf A = 0$ in $\mathbb{R}^X$. Fix $u_0 \in A$ and set $K = \overline{\{x : u_0(x) > 0\}}$, so that K is compact. Because X is locally compact, there is an open relatively compact set $G \supseteq K$. Now there is a continuous function

$u_1 : X \to [0,1]$ such that $u_1(x) = 1$ for $x \in K$ and $u_1(x) = 0$ for $x \in X \setminus G$ (4A2F(h-iii)). Because G is relatively compact, $u_1 \in C_k$.

Take any $\epsilon > 0$. By 436Ic, there is a $v \in A$ such that $\|v\|_\infty \le \epsilon$. Now there is a $v' \in A$ such that $v' \le v \wedge u_0$, so that $v'(x) \le \epsilon$ for every $x \in K$ and $v'(x) = 0$ for $x \notin K$. In this case $v' \le \epsilon u_1$, and

$$\inf_{u\in A} f(u) \le f(v') \le \epsilon f(u_1).$$

As ϵ is arbitrary, $\inf_{u\in A} f(u) = 0$; as A is arbitrary, f is smooth. **Q**

(b) Note that because $\mathfrak{T}$ is locally compact, it is the coarsest topology on X for which every function in C_k is continuous (4A2G(e-ii)). Also C_k is a truncated Riesz subspace of $\mathbb{R}^X$. So 436H tells us that there is a quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$. And μ is locally finite. **P** If $x_0 \in X$, then (as in (a) above) there is a $u_1 \in C_k^+$ such that $u_1(x_0) = 1$; now $G = \{x : u_1(x) > \frac{1}{2}\}$ is an open set containing x_0, and $\mu G \le 2f(u_1)$ is finite. **Q**

By 416G, or otherwise, μ is a Radon measure.

(c) By 416E(b-v), μ is unique.

436K Riesz Representation Theorem (second form) Let $(X,\mathfrak{T})$ be a locally compact Hausdorff space. If $f : C_0(X) \to \mathbb{R}$ is any positive linear functional, there is a unique totally finite Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_0 = C_0(X)$.

proof (a) As noted in 436Ib, $C_0^* = C_0^\sim$, so f is $\|\,\|_\infty$-continuous. $C_k(X)$ is a linear subspace of C_0, and $f\restriction C_k(X)$ is a positive linear functional; so by 436J there is a unique Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k(X)$. Now μ is totally finite. **P** By 414Ab,

$$\mu X = \sup\{f(u) : u \in C_k(X),\, 0 \le u \le \chi X\} \le \|f\| < \infty. \text{ **Q**}$$

(b) Accordingly $\int u\,d\mu$ is defined for every $u \in C_b(X)$, and in particular for every $u \in C_0$. Next, $C_k = C_k(X)$ is norm-dense in C_0. **P** If $u \in C_0^+$, then $u_n = (u - 2^{-n}\chi X)^+$ belongs to C_k and $\|u - u_n\|_\infty \le 2^{-n}$ for every $n \in \mathbb{N}$, so $u \in \overline{C}_k$; accordingly $C_0 = C_0^+ - C_0^+$ is included in $\overline{C}_k$. **Q** Since $\int d\mu$, regarded as a linear functional on C_0, is positive, therefore continuous, and agrees with f on C_k, it must be identical to f. Thus $f(u) = \int u\,d\mu$ for every $u \in C_0$.

(c) Because there is only one Radon measure giving the right integrals to members of C_k (436J), μ is unique.

***436L** The results here, by opening a path between measure theory and the study of linear functionals on spaces of continuous functions, provide an enormously powerful tool for the analysis of dual spaces $C(X)^*$ and their relatives. I will explore some of these ideas in the next section. Here I will give only a sample pair of facts to show how measure theory can tell us things about Banach lattices which seem difficult to reach by other methods.

Proposition Let X be a topological space; write C_b for $C_b(X)$. Suppose that U is a norm-closed linear subspace of C_b^* such that the functional $u \mapsto f(u \times v) : C_b \to \mathbb{R}$ belongs to U whenever $f \in U$ and $v \in C_b$. Then U is a band in the L-space C_b^*.

proof (a) Let $e = \chi X$ be the standard order unit of C_b, and if $f \in C_b^*$ and u, $v \in C_b$ write $f_v(u)$ for $f(u \times v)$. By 356Na, $C_b^* = C_b^\sim$ is an L-space.

(b) I show first that U is a Riesz subspace of $C_b^\sim$. **P** If $f \in U$ and $\epsilon > 0$, there is a $v \in C_b$ such that $|v| \le e$ and $f(v) \ge |f|(e) - \epsilon$ (356B). Now $f_v \le |f|$ and

$$\||f| - f_v\| = (|f| - f_v)(e) \le \epsilon$$

(356Nb), while $f_v \in U$. As ϵ is arbitrary, $|f| \in \overline{U} = U$; as f is arbitrary, U is a Riesz subspace of C_b^* (352Ic). **Q**

(c) Now suppose that X is a compact Hausdorff space. Then U is a solid linear subspace of $C_b^\sim = C(X)^\sim$. **P** Suppose that $f \in U$ and that $0 \le g \le f$. Let $\epsilon > 0$. By either 436J or 436K, there are Radon measures μ, ν on X such that $f(u) = \int u\,d\mu$ and $g(u) = \int u\,d\nu$ for every $u \in C(X)$. By 416Ea, $\nu \le \mu$ in the sense of 234P, so ν is an indefinite-integral measure over μ (415Oa, or otherwise); let $w : X \to [0,1]$ be such that $\int_E w\,d\mu = \nu E$ for every $E \in \operatorname{dom}\nu$. There is a continuous function $v : X \to \mathbb{R}$ such that $\int |w - v|d\mu \le \epsilon$ (416I), and now

$$|g(u) - f_v(u)| = |\int u\,d\nu - \int u \times v\,d\mu| = |\int u \times (w - v)d\mu|$$

(235K)

$$\le \|u\|_\infty \int |w - v|d\mu \le \epsilon\|u\|_\infty$$

for every $u \in C(X)$, so $\|g - f_v\| \le \epsilon$, while $f_v \in U$. As ϵ is arbitrary, $g \in U$; as f and g are arbitrary (and U is a Riesz subspace of $C(X)^*$), U is a solid linear subspace of $C(X)^*$. **Q**

Since every norm-closed solid linear subspace of an L-space is a band (354Eg), it follows that (provided X is a compact Hausdorff space) U is actually a band.

(d) For the general case, let Z be the set of all Riesz homomorphisms $z : C_b \to \mathbb{R}$ such that $z(e) = 1$. Then Z is a weak*-closed subset of the unit ball of C_b^* so is a compact Hausdorff space. We have a Banach lattice isomorphism $T : C_b \to C(Z)$ given by the formula $(Tu)(z) = z(u)$ for $u \in C_b$, $z \in Z$ (see the proofs of 353N[3] and 354K). But note also that T is multiplicative (353Qd[4]), and $T' : C(Z)^* \to C_b^*$ is a Banach lattice isomorphism. Let V be $(T')^{-1}[U] \subseteq C(Z)^*$; then V is a closed linear subspace of $C(Z)^*$. If $g \in V$ and v, $w \in C(Z)$, then

$$g(v \times w) = (T'g)(T^{-1}v \times T^{-1}w),$$

so g_w, defined in $C(Z)^*$ by the convention of (a) above, is just $(T')^{-1}((T'g)_{T^{-1}w})$, and belongs to V. By (b), V is a band in $C(Z)^*$ so U is a band in C_b^*, as required.

***436M Corollary** Let $\mathfrak{A}$ be a Boolean algebra, and $M(\mathfrak{A})$ the L-space of bounded finitely additive functionals on $\mathfrak{A}$ (362B). Let $U \subseteq M(\mathfrak{A})$ be a norm-closed linear subspace such that $a \mapsto \nu(a \cap b)$ belongs to U whenever $\nu \in U$ and $b \in \mathfrak{A}$. Then U is a band in $M(\mathfrak{A})$.

proof (a) Let Z be the Stone space of $\mathfrak{A}$, so that $C(Z)$ is the M-space $L^\infty(\mathfrak{A})$ (363A), and we have an L-space isomorphism $T : M(\mathfrak{A}) \to C(Z)^*$ defined by saying that $(T\nu)(\chi\widehat{a}) = \nu a$ whenever $\nu \in M(\mathfrak{A})$, $a \in \mathfrak{A}$ and $\widehat{a}$ is the open-and-closed subset of Z corresponding to a (363K). Now $V = T[U]$ is a norm-closed linear subspace of $C(Z)^*$.

(b) V satisfies the condition of 436L. **P** Suppose that $f \in V$ and $v \in C(Z)$; set $f_v(u) = f(u \times v)$ for $u \in C(Z)$, and $\nu = T^{-1}f \in U$. (i) If v is of the form $\chi\widehat{b}$, where $b \in \mathfrak{A}$, then $\nu_b \in U$, where $\nu a = \nu(a \cap b)$ for $a \in \mathfrak{A}$. Now $T\nu_b \in V$ and

$$(T\nu_b)(\chi\widehat{a}) = \nu_b a = \nu(a \cap b) = f(\chi(\widehat{a \cap b})) = f(\chi\widehat{a} \times \chi\widehat{b}) = f_v(\chi\widehat{a})$$

for every $a \in \mathfrak{A}$, so $f_v = T\nu_b$ belongs to U. (ii) If v is of the form $\sum_{i=0}^n \alpha_i\chi\widehat{b}_i$, where $b_0, \dots, b_n \in \mathfrak{A}$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$, then $f_v = \sum_{i=0}^n \alpha_i f_{v_i}$, where $v_i = \chi\widehat{b}_i$ for each i; as $f_{v_i} \in V$ for each i, $f_v \in V$, because V is a linear subspace. (iii) In general, given $v \in C(Z) = L^\infty(\mathfrak{A})$ and $\epsilon > 0$, there is a $w \in C(Z)$, expressible in the form of (ii), such that $\|v - w\|_\infty \le \epsilon$ (363C). In this case $f_w \in V$, by (ii), while

$$|f_v(u) - f_w(u)| = |f(u \times (v - w))| \le \|f\|\|u\|_\infty\epsilon$$

for every $u \in C(Z)$, and $\|f_v - f_w\| \le \epsilon\|f\|$. As ϵ is arbitrary and V is closed, $f_v \in V$. **Q**

(c) By 436L, V is a band in $C(Z)^*$; as T is a Riesz space isomorphism, U is a band in $M(\mathfrak{A})$.

436X Basic exercises >(a) Let (X, Σ, μ_0) be a measure space, and U the set of μ_0-integrable Σ-measurable real-valued functions defined everywhere on X. For $u \in U$ set $f(u) = \int u\,d\mu_0$. Show that U and f satisfy the conditions of 436D, and that the measure μ constructed from f by the procedure there is just the c.l.d. version of μ_0.

[3] Formerly 353M.

[4] Formerly 353Pd.

(b) Let μ and ν be two complete locally determined measures on a set X, and suppose that $\int f\,d\mu = \int f\,d\nu$ for every function $f : X \to \mathbb{R}$ for which either integral is defined in $\mathbb{R}$. Show that $\mu = \nu$.

>(c) Let X be a set, U a truncated Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a sequentially smooth positive linear functional. For $A \subseteq X$ set

$$\theta A = \inf\{\sup_{n\in\mathbb{N}} f(u_n) : \langle u_n\rangle_{n\in\mathbb{N}} \text{ is a non-decreasing sequence in } U^+,\\ \lim_{n\to\infty} u_n(x) = 1 \text{ for every } x \in A\},$$

taking $\inf\emptyset = \infty$ if need be. Show that θ is an outer measure on X. Let μ_0 be the measure defined from θ by Carathéodory's method. Show that $f(u) = \int u\,d\mu_0$ for every $u \in U$. Show that the measure μ constructed in 436D is the c.l.d. version of μ_0.

(d) Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $\tau : U \to [0,\infty[$ be a seminorm such that (i) $\tau(u) \le \tau(v)$ whenever $|u| \le |v|$ (ii) $\lim_{n\to\infty} \tau(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$. Show that for any $u_0 \in U^+$ there is a measure μ on x such that $\int u\,d\mu$ is defined, and less than or equal to $\tau(u)$, for every $u \in U$, and $\int u_0 d\mu = \tau(u_0)$. (*Hint*: put the Hahn-Banach theorem together with 436D.)

(e)(i) Let X be any topological space. Show that every positive linear functional on $C(X)$ is sequentially smooth (compare 375A), so corresponds to a totally finite Baire measure on X. (ii) Let X be a regular Lindelöf space. Show that every positive linear functional on $C(X)$ is smooth, so corresponds to a totally finite quasi-Radon measure on X. (iii) Let X be a K-analytic Hausdorff space. Show that every positive linear functional on $C(X)$ corresponds to at least one totally finite Radon measure on X.

(f) Let X be a completely regular space. Show that it is measure-compact iff every sequentially smooth positive linear functional on $C_b(X)$ is smooth. (*Hint*: 436Xj.)

(g) A completely regular topological space X is called **realcompact** if every Riesz homomorphism from $C(X)$ to $\mathbb{R}$ is of the form $u \mapsto \alpha u(x)$ for some $x \in X$ and $\alpha \ge 0$. (i) Show that, for any topological space X, any Riesz homomorphism from $C(X)$ to $\mathbb{R}$ is representable by a Baire measure on X which takes at most two values. (ii) Show that a completely regular space X is realcompact iff every $\{0,1\}$-valued Baire measure on X is of the form $E \mapsto \chi E(x)$. (iii) Show that a completely regular space X is realcompact iff every purely atomic totally finite Baire measure on X is τ-additive. (iii) Show that a measure-compact completely regular space is realcompact. (iv) Show that the discrete topology on $[0,1]$ is realcompact. (*Hint*: if ν is a Baire measure taking only the values 0 and 1, set $x_0 = \sup\{x : \nu[x,1] = 1\}$.) (v) Show that any product of realcompact completely regular spaces is realcompact. (vi) Show that a Souslin-F subset of a realcompact completely regular space is realcompact. (For realcompact spaces which are not measure-compact, see 439Xp.)

(h) In 436F, suppose that μ and ν are τ-additive. Let $\tilde\mu$ and $\tilde\nu$ be the corresponding quasi-Radon measures (415N), and $\tilde\lambda$ the quasi-Radon product of $\tilde\mu$ and $\tilde\nu$ (417N). Show that λ is the restriction of $\tilde\lambda$ to the Baire σ-algebra of $X \times Y$.

>(i) For $u \in C([0,1])$ let $f(u)$ be the Lebesgue integral of u. Show that f is smooth (therefore sequentially smooth) but not sequentially order-continuous (therefore not order-continuous). (*Hint*: enumerate $\mathbb{Q}\cap[0,1]$ as $\langle q_n\rangle_{n\in\mathbb{N}}$, and set $u_n(x) = \min_{i\le n} 2^{i+2}|x - q_i|$ for $n \in \mathbb{N}$, $x \in [0,1]$; show that $\inf_{n\in\mathbb{N}} u_n = 0$ in $C([0,1])$ but $\lim_{n\to\infty} f(u_n) > 0$.)

(j) In 436E, show that μ is τ-additive iff f is smooth.

(k) Suppose that X is a set, U is a truncated Riesz subspace of $\mathbb{R}^X$ and $f : U \to \mathbb{R}$ is a smooth positive linear functional. For $A \subseteq X$ set

$$\theta A = \inf\{\sup_{u\in B} f(u) : B \text{ is an upwards-directed family in } U^+$$
$$\text{such that } \sup_{u\in B} u(x) = 1 \text{ for every } x \in A\},$$

taking $\inf\emptyset = \infty$ if need be. Show that θ is an outer measure on X. Let μ_0 be the measure defined from θ by Carathéodory's method. Show that $f(u) = \int u\,d\mu_0$ for every $u \in U$. Show that the measure μ constructed in 436H is the c.l.d. version of μ_0.

(l) Let X be a completely regular topological space and f a smooth positive linear functional on $C_b(X)$. Show that there is a unique totally finite quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_b(X)$.

>(m) For $u \in C([0,1])$ let $f(u)$ be the Riemann integral of u (134K). Show that the Radon measure on $[0,1]$ constructed by the method of 436J is just Lebesgue measure on $[0,1]$. Explain how to construct Lebesgue measure on $\mathbb{R}$ from an appropriate version of the Riemann integral on $\mathbb{R}$.

(n) Let X be a topological space. Let $f : C_b(X) \to \mathbb{R}$ be a linear functional. (i) Show that the following are equiveridical: (α) f is **tight**, that is, for every $\epsilon > 0$ there is a closed compact $K \subseteq X$ such that $|f(u)| \le \epsilon$ whenever $0 \le u \le \chi(X\setminus K)$ (β) there is a tight totally finite Borel measure μ on X such that $|f(u)| \le \int |u|\,d\mu$ for every $u \in C_b(X)$. (*Hint*: show that a positive tight functional is smooth.) (ii) Show that the set of tight functionals on $C_b(X)$ is a band in $C_b(X)^\sim$.

>(o) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be locally compact Radon measure spaces. (i) Show that the function $x \mapsto \int w(x,y)\nu(dy)$ belongs to $C_k(X)$ for every $w \in C_k(X\times Y)$, so we have a positive linear functional $h : C_k(X\times Y) \to \mathbb{R}$ defined by setting

$$h(w) = \iint w(x,y)\nu(dy)\mu(dx)$$

for $w \in C_k(X\times Y)$. (ii) Show that the corresponding Radon measure on $X\times Y$ is just the product Radon measure as defined in 417P/417R.

>(p) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space; identify $L^\infty(\mathfrak{A})$ with $C(Z)$, as in 363A. Let ν be a non-negative finitely additive functional on $\mathfrak{A}$, f the corresponding positive linear functional on $L^\infty(\mathfrak{A})$ (363K), and μ the corresponding Radon measure on Z (416Qb). Show that $f(u) = \int u\,d\mu$ for every $u \in L^\infty(\mathfrak{A})$.

(q) Let X be a locally compact Hausdorff space. Show that a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $C_0(X)$ converges to 0 for the weak topology on $C_0(X)$ iff $\sup_{n\in\mathbb{N}}\|u_n\|_\infty$ is finite and $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$.

(r) Let X be a non-empty compact Hausdorff space, and $\phi : X \to X$ a continuous function. Show that there is a Radon probability measure μ on X such that ϕ is inverse-measure-preserving for μ. (*Hint*: let $\mathcal{F}$ be a non-principal ultrafilter on X, x_0 any point of X, and define μ by the formula $\int u d\mu = \lim_{n\to\mathcal{F}} \frac{1}{n+1}\sum_{k=0}^{n} u(\phi^k(x_0))$ for every $u \in C(X)$. Use 416E(b-v) to show that $\mu\phi^{-1} = \mu$.)

(s) Let X be a locally compact Hausdorff space. (i) Write $M_{\mathrm{R}}^{\infty+}$ for the set of all Radon measures on X. For $\mu \in M_{\mathrm{R}}^{\infty+}$, let $S\mu$ be the corresponding functional on $C_k(X)$, defined by setting $(S\mu)(u) = \int u\,d\mu$ for every $u \in C_k(X)$. Show that $S(\mu+\nu) = S\mu + S\nu$ and $S(\alpha\mu) = \alpha S\mu$ whenever μ, $\nu \in M_{\mathrm{R}}^{\infty+}$ and $\alpha \ge 0$, where addition and scalar multiplication of measures are defined as in 234G and 234Xf. (ii) Write M_{R}^{+} for the set of totally finite Radon measures on X. For $\mu \in M_{\mathrm{R}}^{+}$, let $T\mu$ be the corresponding functional on $C_b(X)$, defined by setting $(T\mu)(u) = \int u\,d\mu$ for every $u \in C_b(X)$. Show that $T(\mu+\nu) = T\mu + T\nu$ and $T(\alpha\mu) = \alpha T\mu$ whenever μ, $\nu \in M_{\mathrm{R}}^{+}$ and $\alpha \ge 0$.

436Y Further exercises (a) Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a sequentially smooth positive linear functional. (i) Write U_σ for the set of functions from X to $[0,\infty]$ expressible as the supremum of a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U^+ such that $\sup_{n\in\mathbb{N}} f(u_n) < \infty$. Show that there is a functional $f_\sigma : U_\sigma \to [0,\infty[$ such that $f_\sigma(u) = \sup_{n\in\mathbb{N}} f(u_n)$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U^+ with supremum $u \in U_\sigma$. (Compare 122I.) (ii) Show that $u + v \in U_\sigma$ and $f_\sigma(u + v) = f_\sigma(u) + f_\sigma(v)$ for all u, $v \in U_\sigma$. (iii) Suppose that u, $v \in U_\sigma$, $u \le v$ and $u(x) = v(x)$ whenever $v(x)$ is finite. Show that $f_\sigma(u) = f_\sigma(v)$. (*Hint*: take non-decreasing sequences $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ with suprema u, v. Consider $\langle f(v_k - u_n - \delta v_n)^+\rangle_{n\in\mathbb{N}}$ where $k \in \mathbb{N}$, $\delta > 0$.) (iv) Let V be the set of functions $v : X \to \mathbb{R}$ such that there are u_1, $u_2 \in U_\sigma$ such that $v(x) = u_1(x) - u_2(x)$ whenever $u_1(x)$, $u_2(x)$ are both finite. Show that V is a linear subspace of $\mathbb{R}^X$ and that there is a linear functional $g : V \to \mathbb{R}$ defined by setting $g(v) = f_\sigma(u_1) - f_\sigma(u_2)$ whenever $v = u_1 - u_2$ in the sense of the last sentence. (v) Show that V is a Riesz subspace of $\mathbb{R}^X$. (vi) Show that if $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in V and $\gamma = \sup_{n\in\mathbb{N}} g(v_n)$ is finite, then there is a $v \in V$ such that $g(v) = \sup_{n\in\mathbb{N}} g(v \wedge v_n) = \gamma$. (This is a version of the **Daniell integral**.)

(b) Develop further the theory of 436Ya, finding a version of Lebesgue's Dominated Convergence Theorem, a concept of 'negligible' subset of X, and an L-space of equivalence classes of 'integrable' functions.

(c) Let X be a countably compact topological space. Show that every positive linear functional on $C_b(X)$ is sequentially smooth, so corresponds to a totally finite Baire measure on X.

(d) Let X be a sequential space, Y a topological space, μ a semi-finite Baire measure on X and ν a σ-finite Baire measure on Y. Let $\tilde\mu$ be the c.l.d. version of μ. Show that there is a semi-finite Baire measure λ on $X \times Y$ such that

$$\lambda W = \int \nu W[\{x\}]\tilde\mu(dx), \qquad \int f d\lambda = \iint f(x,y)\nu(dy)\tilde\mu(dx)$$

for every Baire set $W \subseteq X \times Y$ and every non-negative continuous function $f : X \times Y \to \mathbb{R}$. Show that the c.l.d. version of λ extends the c.l.d. product measure of μ and ν.

(e) Let $X_0, \dots, X_n$ be sequential spaces and μ_i a totally finite Baire measure on X_i for each i. (i) Show that if $f : X_0 \times \ldots \times X_n \to \mathbb{R}$ is a bounded separately continuous function, then

$$\phi(f) = \int \ldots \int f(x_0, \dots, x_n)\mu_n(dx_n)\ldots\mu_0(dx_0)$$

is defined, so that we have a corresponding Baire product measure on $X_0 \times \ldots \times X_n$. (ii) Show that this product is associative.

(f) Let X and Y be compact Hausdorff spaces. (i) Show that there is a unique bilinear operator $\phi : C(X)^* \times C(Y)^* \to C(X \times Y)^*$ which is separately continuous for the weak* topologies and such that $\phi(\delta_x, \delta_y) = \delta_{(x,y)}$ for all $x \in X$ and $y \in Y$, setting $\delta_x(f) = f(x)$ for $f \in C(X)$ and $x \in X$. (ii) Show that if μ and ν are Radon measures on X, Y respectively with Radon measure product λ, then $\phi(\int d\mu, \int d\nu) = \int d\lambda$. (iii) Show that $\|\phi\| \le 1$ (definition: 253Ab).

(g) Let X be any topological space. (i) Let C_k be the set of continuous functions $u : X \to \mathbb{R}$ such that $\overline{\{x : u(x) \ne 0\}}$ is compact, and $f : C_k \to \mathbb{R}$ a positive linear functional. Show that there is a tight quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$. (ii) Let $\tilde C_k$ be the set of continuous functions $u : X \to \mathbb{R}$ such that $\{x : u(x) \ne 0\}$ is relatively compact, and $f : \tilde C_k \to \mathbb{R}$ a positive linear functional. Show that there is a tight quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$.

436 Notes and comments From the beginning, integration has been at the centre of measure theory. My own view – implicit in the arrangement of this treatise, from Chapter 11 onward – is that 'measure' and 'integration' are not quite the same thing. I freely acknowledge that my treatment of 'integration' is distorted by my presentation of it as part of measure theory; on the other side of the argument, I hold that regarding 'measure' as a concept subsidiary to 'integral', as many authors do, seriously interferes with the development of truly penetrating intuitions for the former. But it is undoubtedly true that every complete locally determined measure can be derived from its associated integral (436Xb). Moreover, it is clearly of the highest importance that we should be able to recognise integrals when we see them; I mean, given a linear functional on a linear space of functions, then if it can be expressed as an integral with respect to a measure this is something we need to know at once. And thirdly, investigation of linear functionals frequently leads us to measures of great importance and interest.

Concerning the conditions in 436D, an integral must surely be 'positive' (because measures in this treatise are always non-negative) and 'sequentially smooth' (because measures are supposed to be countably additive). But it is not clear that we are forced to restrict our attention to Riesz subspaces of $\mathbb{R}^X$, and even less clear that they have to be 'truncated'. In 439I below I give an example to show that this last condition is essential for 436D and 436H as stated. However it is not necessary for large parts of the theory. In many cases, if $U \subseteq \mathbb{R}^X$ is a Riesz subspace which is not truncated, we can take an element $e \in U^+$ and look at $Y = \{x : e(x) > 0\}$, $V_e = \{u : u(x) = 0$ for every $x \in X \setminus Y\} \cong W_e$, where $W_e = \{u/e : u \in V_e\}$ is a truncated Riesz subspace of $\mathbb{R}^Y$. But there are applications in which this approach is unsatisfactory and a more radical revision of the basic theory of integration, as in 436Ya, is useful.

I have based the arguments of this section on the inner measure constructions of §413. Of course it is also possible to approach them by means of outer measures (436Xc, 436Xk).

I emphasize the exercise 436Xo because it is prominent in 'Bourbakist' versions of the theory of Radon measures, in which (following BOURBAKI 65 rather than BOURBAKI 69) Radon measures are regarded as linear functionals on spaces of continuous functions. By 436J, this is a reasonably effective approach as long as we are interested only in locally compact spaces, and there are parts of the theory of topological groups (notably the duality theory of §445 below) in which it even has advantages. The construction of 436Xo shows that we can find a direct approach to the tensor product of linear functionals which does not require any attempt to measure sets rather than integrate functions. I trust that the prejudices I am expressing will not be taken as too sweeping a disparagement of such methods. Practically all correct arguments in mathematics (and not a few incorrect ones) are valuable in some way, suggesting new possibilities for investigation. In particular, one of the challenges of measure theory (not faced in this treatise) is that of devising effective theories of vector-valued measures. Typically this is much easier with Riemann-type integrals, and any techniques for working directly with these should be noted.

436L revisits ideas from Chapter 35, and the result would be easier to find if it were in §356. I include it here as an example of the way in which familiar material from measure theory (in particular, the Radon-Nikodým theorem) can be drafted to serve functional analysis. I should perhaps remark that there are alternative routes which do not use measure theory explicitly, and while longer are (in my view) more illuminating.

Version of 5.11.12

437 Spaces of measures

Once we have started to take the correspondence between measures and integrals as something which operates in both directions, we can go a very long way. While 'measures', as dealt with in this treatise, are essentially positive, an 'integral' can be thought of as a member of a linear space, dual in some sense to a space of functions. Since the principal spaces of functions are Riesz spaces, we find ourselves looking at dual Riesz spaces as discussed in §356; while the corresponding spaces of measures are close to those of §362. Here I try to draw these ideas together with an examination of spaces $U_\sigma^\sim$ and $U_\tau^\sim$ of sequentially smooth and smooth functionals, and the matching spaces M_σ and M_τ of countably additive and τ-additive measures (437A-437I). Because a (sequentially) smooth functional corresponds to a countably additive measure, which can be expected to integrate many more functions than those in the original Riesz space (typically, a space of continuous functions), we find that relatively large spaces of bounded measurable functions can be canonically embedded into the biduals $(U_\sigma^\sim)^*$ and $(U_\tau^\sim)^*$ (437C, 437H, 437I).

The guiding principles of functional analysis encourage us not only to form linear spaces, but also to examine linear space topologies, starting with norm and weak topologies. The theory of Banach lattices described in §354, particularly the theory of M- and L-spaces, is an important part of the structure here. In addition, our spaces $U_\sigma^\sim$ have natural weak* topologies which can be regarded as topologies on spaces of measures; these are the 'vague' topologies of 437J, which have already been considered, in a special case, in §285.

It turns out that on the positive cone of M_τ, at least, the vague topology may have an alternative description directly in terms of the behaviour of the measures on open sets (437L). This leads us to a parallel idea, the 'narrow' topology on non-negative additive functionals (437Jd). The second half of the section is devoted to the elementary properties of narrow topologies (437K-437N), with especial reference to compact sets in these topologies (437P, 437Rf, 437T). Seeking to identify narrowly compact sets, we come to the concept of 'uniform tightness' (437O). Bounded uniformly tight sets are narrowly relatively compact (437P); in 'Prokhorov spaces' (437U) the converse is true. I end the section with a list of the best-known Prokhorov spaces (437V).

437A Smooth and sequentially smooth duals Let X be a set, and U a Riesz subspace of $\mathbb{R}^X$. Recall that $U^\sim$ is the Dedekind complete Riesz space of order-bounded linear functionals on U, that $U^\sim_c$ is the band of differences of sequentially order-continuous positive linear functionals, and that $U^\times$ is the band of differences of order-continuous positive linear functionals (356A). A functional $f \in (U^\sim)^+$ is 'sequentially smooth' if $\inf_{n\in\mathbb{N}} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$, and 'smooth' if $\inf_{u\in A} f(u) = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$ (436A, 436G).

(a) Set $U^\sim_\sigma = \{f : f \in U^\sim, |f| \text{ is sequentially smooth}\}$, the **sequentially smooth dual** of U. Then $U^\sim_\sigma$ is a band in $U^\sim$. **P** (i) If $f \in U^\sim_\sigma$, $g \in U^\sim$ and $|g| \le |f|$, then

$$|g|(u_n) \le |f|(u_n) \to 0$$

whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every x, so $|g|$ is sequentially smooth and $g \in U^\sim_\sigma$. Thus $U^\sim_\sigma$ is a solid subset of $U^\sim$. (ii) If f, $g \in U^\sim_\sigma$ and $\alpha \in \mathbb{R}$, then

$$|f+g|(u_n) \le |f|(u_n) + |g|(u_n) \to 0, \quad |\alpha f|(u_n) = |\alpha||f|(u_n) \to 0$$

whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every x, so $|f+g|$ and $|\alpha f|$ are sequentially smooth. Thus $U^\sim_\sigma$ is a Riesz subspace of $U^\sim$. (iii) Now suppose that $B \subseteq (U^\sim_\sigma)^+$ is an upwards-directed set with supremum $g \in U^\sim$, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$. Then, given $\epsilon > 0$, there is an $f \in B$ such that $(g-f)(u_0) \le \epsilon$ (355Ed), so that $g(u_n) \le f(u_n) + \epsilon$ for every n, and

$$\limsup_{n\to\infty} g(u_n) \le \epsilon + \lim_{n\to\infty} f(u_n) \le \epsilon.$$

As ϵ and $\langle u_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $g \in U^\sim_\sigma$; as B is arbitrary, $U^\sim_\sigma$ is a band (352Ob). **Q**

As remarked in 436A, sequentially order-continuous positive linear functionals are sequentially smooth, so $U^\sim_c \subseteq U^\sim_\sigma$.

(b) Set $U^\sim_\tau = \{f : f \in U^\sim, |f| \text{ is smooth}\}$, the **smooth dual** of U. Then $U^\sim_\tau$ is a band in $U^\sim$. **P** (i) Suppose that f, $g \in U^\sim_\tau$, $\alpha \in \mathbb{R}$, $h \in U^\sim$ and $|h| \le |f|$. If $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$, and $\epsilon > 0$, then there are u_0, $u_1 \in A$ such that $|f|(u_0) \le \epsilon$ and $|g|(u_1) \le \epsilon$, and a $u \in A$ such that $u \le u_0 \wedge u_1$. In this case

$$|h|(u) \le |f|(u) \le \epsilon,$$

$$|f+g|(u) \le |f|(u) + |g|(u) \le 2\epsilon,$$

$$|\alpha f|(u) = |\alpha||f|(u) \le |\alpha|\epsilon.$$

As A and ϵ are arbitrary, h, $f+g$ and αf all belong to $U^\sim_\tau$; so that $U^\sim_\tau$ is a solid Riesz subspace of $U^\sim$. (ii) Now suppose that $B \subseteq (U^\sim_\tau)^+$ is an upwards-directed set with supremum $g \in U^\sim$, and that $A \subseteq U$ is a non-empty downwards-directed set such that $\inf_{u\in A} u(x) = 0$ for every $x \in X$. Fix any $u_0 \in A$. Then, given $\epsilon > 0$, there is an $f \in B$ such that $(g-f)(u_0) \le \epsilon$, so that $g(u) \le f(u) + \epsilon$ whenever $u \in A$ and $u \le u_0$. But $A_0 = \{u : u \in A, u \le u_0\}$ is also a downwards-directed set, and $\inf_{u\in A_0} u(x) = 0$ for every $x \in X$, so

$$\inf_{u\in A} g(u) \le \epsilon + \inf_{u\in A_0} f(u) \le \epsilon.$$

As ϵ and A are arbitrary, $g \in U^\sim_\tau$; as B is arbitrary, $U^\sim_\tau$ is a band. **Q**

Just as $U^\sim_c \subseteq U^\sim_\sigma$, $U^\times \subseteq U^\sim_\tau$.

437B Signed measures Collecting these ideas together with those of §§362-363, we are ready to approach 'signed measures'. Recall that if X is a set and Σ is a σ-algebra of subsets of X, we can identify $L^\infty = L^\infty(\Sigma)$, as defined in §363, with the space $\mathcal{L}^\infty = \mathcal{L}^\infty(\Sigma)$ of bounded Σ-measurable real-valued functions (363H). Now, because $\mathcal{L}^\infty$ is sequentially order-closed in $\mathbb{R}^X$, sequentially smooth functionals on $\mathcal{L}^\infty$ are actually sequentially order-continuous, so $(\mathcal{L}^\infty)^\sim_\sigma = (\mathcal{L}^\infty)^\sim_c$. Next, we can identify $(L^\infty)^\sim_c$ with the space M_σ of countably additive functionals, or 'signed measures', on Σ (363K); if $\nu \in M_\sigma$, the corresponding member of $(L^\infty)^\sim_c$ is the unique order-bounded (or norm-continuous) linear functional f on L^∞ such that $f(\chi E) = \nu E$ for every $E \in \Sigma$. If $\nu \ge 0$, so that ν is a totally finite measure with domain Σ, then of course f, when interpreted as a functional on $\mathcal{L}^\infty$, must be just integration with respect to ν.

The identification between $(L^\infty)^\sim_c$ and M_σ described in 363K is an L-space isomorphism. So it tells us, for instance, that if we are willing to use the symbol $⨍$ for the duality between L^∞ and the space of bounded finitely additive functionals on Σ, as in 363L, then we can write

$$⨍ u\,d(\mu+\nu) = ⨍ u\,d\mu + ⨍ u\,d\nu$$

for every $u \in \mathcal{L}^\infty$ and all μ, $\nu \in M_\sigma$.

437C Theorem Let X be a set and U a Riesz subspace of $\ell^\infty(X)$, the M-space of bounded real-valued functions on X, containing the constant functions.

(a) Let Σ be the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Let $M_\sigma = M_\sigma(\Sigma)$ be the L-space of countably additive functionals on Σ (326I[5], 362B). Then there is a Banach lattice isomorphism $T : M_\sigma \to U^\sim_\sigma$ defined by saying that $(T\mu)(u) = \int u\,d\mu$ whenever $\mu \in M_\sigma^+$ and $u \in U$.

(b) We now have a sequentially order-continuous norm-preserving Riesz homomorphism S, embedding the M-space $\mathcal{L}^\infty = \mathcal{L}^\infty(\Sigma)$ of bounded real-valued Σ-measurable functions on X (363Hb) into the M-space $(U^\sim_\sigma)^\sim = (U^\sim_\sigma)^* = (U^\sim_\sigma)^\times$, defined by saying that $(Sv)(T\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If $u \in U$, then $(Su)(f) = f(u)$ for every $f \in U^\sim_\sigma$.

proof (a)(i) The norm $\| \|_\infty$ is an order-unit norm on U (354Ga), so $U^* = U^\sim$ is an L-space (356N), and the band $U^\sim_\sigma$ (437Aa) is an L-space in its own right (354O).

(ii) As noted in 437B, we have a Banach lattice isomorphism $T_0 : M_\sigma \to (\mathcal{L}^\infty)^\sim_c$ defined by saying that $(T_0\mu)(u) = \int u\,d\mu$ whenever $u \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If we set $T\mu = T_0\mu\restriction U$, then T is a positive linear operator from M_σ to $U^\sim$, just because U is a linear subspace of $\mathcal{L}^\infty$; and since $T\mu \in U^\sim_\sigma$ for every $\mu \in M_\sigma^+$, T is an operator from M_σ to $U^\sim_\sigma$. Now every $f \in (U^\sim_\sigma)^+$ is of the form $T\mu$ for some $\mu \in M_\sigma^+$. **P** By 436D, there is some measure λ such that $\int u\,d\lambda = f(u)$ for every $u \in U$. Completing λ if necessary, we see that we may suppose that every member of U is $(\operatorname{dom}\lambda)$-measurable, that is, that $\Sigma \subseteq \operatorname{dom}\lambda$; take $\mu = \lambda\restriction\Sigma$. **Q** So T is surjective.

(iii) Write $\mathcal{K}$ for the family of sets $K \subseteq X$ such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ for some sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U. (See the proof of 436D.) We need to know the following. (α) $\mathcal{K} \subseteq \Sigma$. (β) If $K \in \mathcal{K}$, then there is a non-increasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$. (For if $\langle u'_n\rangle_{n\in\mathbb{N}}$ is any sequence in U such that $\chi K = \inf_{n\in\mathbb{N}} u'_n$, we can set $u_n = \inf_{i\le n} u'_i$ for each i.) (γ) The σ-algebra of subsets of X generated by $\mathcal{K}$ is Σ. **P** Let T be the σ-algebra of subsets of X generated by $\mathcal{K}$. $\mathrm{T} \subseteq \Sigma$ because $\mathcal{K} \subseteq \Sigma$. If $u \in U$ and $\alpha > 0$ then $\{x : u(x) \ge \alpha\} \in \mathcal{K}$ (see part (b) of the proof of 436D). So every member of U^+, therefore every member of U, is T-measurable, and $\Sigma \subseteq \mathrm{T}$. **Q**

(iv) T is injective. **P** If μ_1, $\mu_2 \in M_\sigma$ and $T\mu_1 = T\mu_2$, set $\nu_i = \mu_i + \mu_1^- + \mu_2^-$ for each i, so that ν_i is non-negative and $T\nu_1 = T\nu_2$. If $K \in \mathcal{K}$ then there is a non-increasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ in $\mathbb{R}^X$, so

$$\nu_1 K = \inf_{n\in\mathbb{N}} \int u_n d\nu_1 = \inf_{n\in\mathbb{N}} \int u_n d\nu_2 = \nu_2 K.$$

Now $\mathcal{K}$ contains X and is closed under finite intersections and ν_1 and ν_2 agree on $\mathcal{K}$. By the Monotone Class Theorem (136C), ν_1 and ν_2 agree on the σ-algebra generated by $\mathcal{K}$, which is Σ; so $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$. **Q**

Thus T is a linear space isomorphism between M_σ and $U^\sim_\sigma$.

(v) As noted in (ii), $T[M_\sigma^+] = (U^\sim_\sigma)^+$; so T is a Riesz space isomorphism.

[5]Formerly 326E.

(vi) Now if $\mu \in M_\sigma$,

$$\|T\mu\| = |T\mu|(\chi X)$$

(356Nb)

$$= (T|\mu|)(\chi X)$$

(because T is a Riesz homomorphism)

$$= |\mu|(X) = \|\mu\|$$

(362Ba). So T is norm-preserving and is an L-space isomorphism, as claimed.

(b)(i) By 356Pb, $(U_\sigma^\sim)^* = (U_\sigma^\sim)^\sim = (U_\sigma^\sim)^\times$ is an M-space.

(ii) We have a canonical map $S_0 : \mathcal{L}^\infty \to ((\mathcal{L}^\infty)_c^\sim)^\times$ defined by saying that $(S_0v)(h) = h(v)$ for every $v \in \mathcal{L}^\infty$ and $h \in (\mathcal{L}^\infty)_c^\sim$; and by 356F, S_0 is a Riesz homomorphism. If $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{L}^\infty$ with infimum 0, then $\inf_{n\in\mathbb{N}}(S_0v_n)(h) = \inf_{n\in\mathbb{N}} h(v_n) = 0$ for every $h \in ((\mathcal{L}^\infty)_c^\sim)^+$, so $\inf_{n\in\mathbb{N}} S_0v_n = 0$ (355Ee); as $\langle v_n\rangle_{n\in\mathbb{N}}$ is arbitrary, S_0 is sequentially order-continuous (351Gb).

Also S_0 is norm-preserving. **P** (α) If $h \in (\mathcal{L}^\infty)_c^\sim$ and $v \in \mathcal{L}^\infty$, then

$$|(S_0v)(h)| = |h(v)| \le \|h\|\|v\|_\infty,$$

so $\|S_0v\| \le \|v\|_\infty$. (β) If $v \in \mathcal{L}^\infty$ and $0 \le \gamma < \|v\|_\infty$, take $x \in X$ such that $|v(x)| > \gamma$, and define $h_x \in (\mathcal{L}^\infty)_c^\sim$ by setting $h_x(w) = w(x)$ for every $w \in \mathcal{L}^\infty$; then $\|h_x\| = 1$ and

$$|(S_0v)(h_x)| = |h_x(v)| = |v(x)| \ge \gamma,$$

so $\|S_0v\| \ge \gamma$. As γ is arbitrary, $\|S_0v\| \ge \|v\|_\infty$ and $\|S_0v\| = \|v\|_\infty$. **Q**

(iii) Now $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ and $T : M_\sigma \to U_\sigma^\sim$ are both norm-preserving Riesz space isomorphisms, so $T_0T^{-1} : U_\sigma^\sim \to (\mathcal{L}^\infty)_c^\sim$ is another, and its adjoint $S_1 : ((\mathcal{L}^\infty)_c^\sim)^* \to (U_\sigma^\sim)^*$ must also be a norm-preserving Riesz space isomorphism. So if we set $S = S_1S_0$, S will be a norm-preserving sequentially order-continuous Riesz homomorphism from $\mathcal{L}^\infty$ to $(U_\sigma^\sim)^\times = (U_\sigma^\sim)^*$.

(iv) Setting the construction out in this way tells us a lot about the properties of the operator S, but undeniably leaves it somewhat obscure. So let us start again from $v \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$, and seek to calculate $(Sv)(T\mu)$. We have

$$(Sv)(T\mu) = (S_1S_0v)(T\mu) = (S_0v)(T_0T^{-1}T\mu)$$

(because S_1 is the adjoint of T_0T^{-1})

$$= (S_0v)(T_0\mu) = (T_0\mu)(v) = \int v\,d\mu,$$

as claimed.

If $u \in U$, then $(T\mu)(u) = (T_0\mu)(u)$ for every $\mu \in M_\sigma$, so if $f \in U_\sigma^\sim$ then

$$(Su)(f) = (S_1S_0u)(f) = (S_0u)(T_0T^{-1}f) = (T_0T^{-1}f)(u) = (TT^{-1}f)(u) = f(u).$$

This completes the proof.

437D Remarks What is happening here is that the canonical Riesz homomorphism $u \mapsto \hat{u}$ from U to $(U_\sigma^\sim)^*$ (356F) has a natural extension to $\mathcal{L}^\infty(\Sigma)$. The original homomorphism $u \mapsto \hat{u}$ is not, as a rule, sequentially order-continuous, just because $U_\sigma^\sim$ is generally larger than $U_c^\sim$; but the extension to $\mathcal{L}^\infty$ is sequentially order-continuous. If you like, it is sequential smoothness which is carried over to the extension, and because the embedding of $\mathcal{L}^\infty$ in $\mathbb{R}^X$ is sequentially order-continuous, a sequentially smooth operator on $\mathcal{L}^\infty$ is sequentially order-continuous.

In the statement of 437C, I have used the formulae $(T\mu)(u) = \int u\,d\mu$ and $(Sv)(T\mu) = \int v\,d\mu$ on the assumption that $\mu \in M_\sigma^+$, so that μ is actually a measure on the definition used in this treatise, and $\int d\mu$ is the ordinary integral as constructed in §122. Since the functions u and v are bounded, measurable and defined everywhere, we can choose to extend the notion of integration to signed measures, as in 363L, in which case the formulae $(T\mu)(u) = \oint u\,d\mu$ and $(Sv)(T\mu) = \oint v\,d\mu$ become meaningful, and true, for all $\mu \in M_\sigma$, $u \in U$ and $v \in \mathcal{L}^\infty$.

In fact the ideas here can be pushed farther, as in 437Ib, 437Xf and 437Yd.

437E Corollary Let X be a completely regular Hausdorff space, and $\mathcal{B}\mathfrak{a} = \mathcal{B}\mathfrak{a}(X)$ its Baire σ-algebra. Then we can identify $C_b(X)^\sim_\sigma$ with the L-space $M_\sigma(\mathcal{B}\mathfrak{a})$ of countably additive functionals on $\mathcal{B}\mathfrak{a}$, and we have a norm-preserving sequentially order-continuous Riesz homomorphism S from $\mathcal{L}^\infty(\mathcal{B}\mathfrak{a})$ to $(C_b(X)^\sim_\sigma)^*$ defined by setting $(Sv)(f) = \int v\,d\mu_f$ for every $v \in \mathcal{L}^\infty$ and $f \in (C_b(X)^\sim_\sigma)^+$, where μ_f is the Baire measure associated with f.

proof Apply 437C with $U = C_b(X)$ (cf. 436E).

437F Proposition Let X be a topological space and $\mathcal{B} = \mathcal{B}(X)$ its Borel σ-algebra. Let M_σ be the L-space of countably additive functionals on $\mathcal{B}$.

(a) Write $M_\tau \subseteq M_\sigma$ for the set of differences of τ-additive totally finite Borel measures. Then M_τ is a band in M_σ, so is an L-space in its own right.

(b) Write $M_t \subseteq M_\tau$ for the set of differences of totally finite Borel measures which are tight (that is, inner regular with respect to the closed compact sets). Then M_t is a band in M_σ, so is an L-space in its own right.

proof (a)(i) Let μ_1, μ_2 be totally finite τ-additive Borel measures on X, $\alpha \ge 0$, and $\mu \in M_\sigma$ such that $0 \le \mu \le \mu_1$. Then $\mu_1 + \mu_2$, $\alpha\mu_1$ and μ are totally finite τ-additive Borel measures. **P** They all belong to M_σ, that is, are totally finite Borel measures. Now let $\mathcal{G}$ be a non-empty upwards-directed family of open sets in X with union H, and $\epsilon > 0$. Then there are G_1, $G_2 \in \mathcal{G}$ such that $\mu_1 G_1 \ge \mu_1 H - \epsilon$ and $\mu_2 G_2 \ge \mu_2 H - \epsilon$, and a $G \in \mathcal{G}$ such that $G \supseteq G_1 \cup G_2$. In this case,

$$(\mu_1 + \mu_2)(G) \ge (\mu_1 + \mu_2)(H) - 2\epsilon,$$

$$(\alpha\mu_1)(G) \ge (\alpha\mu_1)(H) - \alpha\epsilon$$

and

$$\mu G = \mu H - \mu(H \setminus G) \ge \mu H - \mu_1(H \setminus G) \ge \mu H - \epsilon.$$

As $\mathcal{G}$ and ϵ are arbitrary, $\mu_1 + \mu_2$, $\alpha\mu_1$ and μ are all τ-additive. **Q**

It follows that M_τ is a solid linear subspace of M_σ.

(ii) Now suppose that $B \subseteq M_\tau^+$ is non-empty and upwards-directed and has a supremum ν in M_σ. Then $\nu \in M_\tau$. **P** If $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H, then

$$\nu H = \sup_{\mu \in B} \mu H$$

(362Be)

$$= \sup_{\mu \in B, G \in \mathcal{G}} \mu G = \sup_{G \in \mathcal{G}} \nu G;$$

as $\mathcal{G}$ is arbitrary, ν is τ-additive and belongs to M_τ. **Q**

As B is arbitrary, M_τ is a band in M_σ. By 354O, it is itself an L-space.

(b) We can use the same arguments. Suppose that μ_1, $\mu_2 \in M_\sigma^+$ are tight, $\alpha \ge 0$, and $\mu \in M_\sigma$ is such that $0 \le \mu \le \mu_1$. If $E \in \mathcal{B}$ and $\epsilon > 0$, there are closed compact sets K_1, $K_2 \subseteq E$ such that $\mu_1 K_1 \ge \mu_1 E - \epsilon$ and $\mu_2 K_2 \ge \mu_2 E - \epsilon$. Set $K = K_1 \cup K_2$, so that K also is a closed compact subset of E. Then

$$(\mu_1 + \mu_2)(K) \ge (\mu_1 + \mu_2)(E) - 2\epsilon,$$

$$(\alpha\mu_1)(K) \ge (\alpha\mu_1)(E) - \alpha\epsilon$$

and

$$\mu K = \mu E - \mu(E \setminus K) \ge \mu E - \mu_1(E \setminus K) \ge \mu E - \epsilon.$$

As $\mathcal{G}$ and ϵ are arbitrary, $\mu_1 + \mu_2$, $\alpha\mu_1$ and μ are all tight; as μ_1, μ_2, μ and α are arbitrary, M_t is a solid linear subspace of M_σ.

Now suppose that $B \subseteq M_t^+$ is non-empty and upwards-directed and has a supremum ν in M_σ. Take any $E \in \mathcal{B}$ and $\epsilon > 0$. Then there is a $\mu \in B$ such that $\mu E \ge \nu E - \epsilon$; there is a closed compact set $K \subseteq E$ such that $\mu K \ge \mu E - \epsilon$; and now $\nu K \ge \nu E - 2\epsilon$. As E and ϵ are arbitrary, ν is tight; as B is arbitrary, M_t is a band in M_σ, and is in itself an L-space.

437G Definitions Let X be a topological space. A **signed Baire measure** on X will be a countably additive functional on the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$, which by the Jordan decomposition theorem (231F) is expressible as the difference of two totally finite Baire measures; a **signed Borel measure** will be a countably additive functional on the Borel σ-algebra $\mathcal{B}(X)$, that is, the difference of two totally finite Borel measures; a **signed τ-additive Borel measure** will be a member of the L-space M_τ as described in 437F, that is, the difference of two τ-additive totally finite Borel measures; and a **signed tight Borel measure** will be a member of the L-space M_t as described in 437F, that is, the difference of two tight totally finite Borel measures.

437H Theorem Let X be a set and U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions. Let $\mathfrak{T}$ be the coarsest topology on X rendering every member of U continuous, and $\mathcal{B}$ the corresponding Borel σ-algebra.

(a) Let M_τ be the L-space of signed τ-additive Borel measures on X. Then we have a Banach lattice isomorphism $T : M_\tau \to U_\tau^\sim$ defined by saying that $(T\mu)(u) = \int u\,d\mu$ whenever $\mu \in M_\tau^+$ and $u \in U$.

(b) We now have a sequentially order-continuous norm-preserving Riesz homomorphism S, embedding the M-space $\mathcal{L}^\infty = \mathcal{L}^\infty(\mathcal{B})$ of bounded Borel measurable functions on X into $(U_\tau^\sim)^\sim = (U_\tau^\sim)^* = (U_\tau^\sim)^\times$, defined by saying that $(Sv)(T\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty$ and $\mu \in M_\tau^+$. If $u \in U$, then $(Su)(f) = f(u)$ for every $f \in U_\tau^\sim$.

proof The proof follows the same lines as that of 437C.

(a)(i) As before, the norm $\| \|_\infty$ is an order-unit norm on U, so $U^* = U^\sim$ is an L-space, and the band $U_\tau^\sim$ (437Ab) is an L-space in its own right, like M_τ (437F).

(ii) Let M_σ be the L-space of all countably additive functionals on $\mathcal{B}$, so that M_τ is a band in M_σ. Let $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ be the canonical Banach lattice isomorphism defined by saying that $(T_0\mu)(u) = \int u\,d\mu$ whenever $u \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If we set $T\mu = T_0\mu{\restriction}U$ for $\mu \in M_\tau$, then T is a positive linear operator from M_τ to $U^\sim$, just because U is a Riesz subspace of $\mathcal{L}^\infty$; and since $T\mu \in U_\tau^\sim$ for every $\mu \in M_\tau^+$ (436H), T is an operator from M_τ to $U_\tau^\sim$. Now every $f \in (U_\tau^\sim)^+$ is of the form $T\mu$ for some $\mu \in M_\tau^+$. **P** By 436H, there is a quasi-Radon measure λ such that $\int u\,d\lambda = f(u)$ for every $u \in U$; set $\mu = \lambda{\restriction}\mathcal{B}$. **Q** So T is surjective.

(iii) Let $\mathcal{K}$ be the family of subsets K of X such that $\chi K = \inf A$ in $\mathbb{R}^X$ for some non-empty subset A of U. Then $\mathcal{K}$ is just the family of closed sets for $\mathfrak{T}$. **P** As noted in part (b) of the proof of 436H, every member of $\mathcal{K}$ is closed, and $K \setminus G \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $G \in \mathfrak{T}$; but as, in the present case, $X \in \mathcal{K}$, every closed set belongs to $\mathcal{K}$. **Q**

(iv) T is injective. **P** If μ_1, $\mu_2 \in M_\tau$ and $T\mu_1 = T\mu_2$, set $\nu_i = \mu_i + \mu_1^- + \mu_2^-$ for each i, so that ν_i is non-negative and $T\nu_1 = T\nu_2$. If $K \in \mathcal{K}$, set $A = \{u : u \in U, u \ge \chi K\}$, so that $\chi K = \inf A$ in $\mathbb{R}^X$, and A is downwards-directed. By 414Bb,

$$\nu_1 K = \inf_{u \in A} \int u\,d\nu_1 = \inf_{u \in A} \int u\,d\nu_2 = \nu_2 K.$$

Now $\mathcal{K}$ contains X and is closed under finite intersections and ν_1 and ν_2 agree on $\mathcal{K}$. By the Monotone Class Theorem, ν_1 and ν_2 agree on the σ-algebra generated by $\mathcal{K}$, which is $\mathcal{B}$; so $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$. **Q**

Thus T is a linear space isomorphism between M_τ and $U_\tau^\sim$.

(v) As noted in (ii), $T[M_\tau^+] = (U_\tau^\sim)^+$; so T is a Riesz space isomorphism.

(vi) Now if $\mu \in M_\tau$,

$$\|T\mu\| = |T\mu|(\chi X) = (T|\mu|)(\chi X) = |\mu|(X) = \|\mu\|.$$

So T is norm-preserving and is an L-space isomorphism, as claimed.

(b)(i) By 356Pb, $(U_\tau^\sim)^* = (U_\tau^\sim)^\sim = (U_\tau^\sim)^\times$ is an M-space.

(ii) Because $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ is a Banach lattice isomorphism, and M_τ is a band in M_σ, $W = T_0[M_\tau]$ is a band in $(\mathcal{L}^\infty)_c^\sim$. We therefore have a Riesz homomorphism $S_0 : \mathcal{L}^\infty \to W^\times$ defined by writing $(S_0 v)(h) = h(v)$ for $v \in \mathcal{L}^\infty$, $h \in W$ (356F). Just as in (b-ii) of the proof of 437C, S_0 is sequentially order-continuous and norm-preserving. (We need to observe that h_x in the second half of the argument there always belongs to W; this is because $h_x = T_0(\delta_x)$, where $\delta_x \in M_\tau$ is defined by setting $\delta_x(E) = \chi E(x)$ for every Borel set E.)

(iii) Now $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ and $T : M_\tau \to U_\tau^\sim$ are both norm-preserving Riesz space isomorphisms, so $T_0T^{-1} : U_\tau^\sim \to W$ is another, and its adjoint $S_1 : W^* \to (U_\tau^\sim)^*$ must also be a norm-preserving Riesz space isomorphism. So if we set $S = S_1S_0$, S will be a norm-preserving sequentially order-continuous Riesz homomorphism from $\mathcal{L}^\infty$ to $(U_\tau^\sim)^\times = (U_\tau^\sim)^*$.

(iv) If $v \in \mathcal{L}^\infty$ and $\mu \in M_\tau^+$,

$$\begin{aligned}(Sv)(T\mu) &= (S_1S_0v)(T\mu) = (S_0v)(T_0T^{-1}T\mu)\\ &= (S_0v)(T_0\mu) = (T_0\mu)(v) = \int v\,d\mu;\end{aligned}$$

if $u \in U$ and $f \in U_\tau^\sim$, then $(T\mu)(u) = (T_0\mu)(u)$ for every $\mu \in M_\tau$, so

$$(Su)(f) = (S_1S_0u)(f) = (S_0u)(T_0T^{-1}f) = (T_0T^{-1}f)(u) = (TT^{-1}f)(u) = f(u).$$

437I Proposition Let X be a locally compact Hausdorff space, $\mathcal{B} = \mathcal{B}(X)$ its Borel σ-algebra, and $\mathcal{L}^\infty(\mathcal{B})$ the M-space of bounded Borel measurable real-valued functions on X.

(a) Let M_t be the L-space of signed tight Borel measures on X. Then we have a Banach lattice isomorphism $T : M_t \to C_0(X)^*$ defined by saying that $(T\mu)(u) = \int u\,d\mu$ whenever $\mu \in M_t^+$ and $u \in C_0(X)$.

(b) Let Σ_{uRm} be the algebra of universally Radon-measurable subsets of X (definition: 434E), and $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ the M-space of bounded Σ_{uRm}-measurable real-valued functions on X. Then we have a norm-preserving sequentially order-continuous Riesz homomorphism $S : \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}}) \to C_0(X)^{**}$ defined by saying that $(Sv)(T\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ and $\mu \in M_t^+$; and $(Su)(f) = f(u)$ for every $u \in C_0(X)$, $f \in C_0(X)^*$.

proof (a) The point is just that in this context M_t is equal to M_τ, as defined in 437F-437H (416H), while $C_0(X)^* = C_0(X)_\tau^\sim$ (see part (a) of the proof of 436J), and the topology of X is completely regular, so we just have a special case of 437Ha.

(b)(i) As in 437Hb, we have a sequentially order-continuous Riesz homomorphism $S_0 : \mathcal{L}^\infty(\mathcal{B}) \to C_0(X)^{**}$ defined by saying that $(S_0v)(T\nu) = \int v\,d\nu$ whenever $v \in \mathcal{L}^\infty(\mathcal{B})$ and $\nu \in M_t^+$.

(ii) If $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$, then

$$\sup\{S_0w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v\} = \inf\{S_0w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \ge v\}$$

in $C_0(X)^{**}$. **P** Set

$$A = \{w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v\}, \quad B = \{w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \ge v\}.$$

Because the constant functions belong to $\mathcal{L}^\infty(\mathcal{B})$, A and B are both non-empty; of course $w \le w'$ and $S_0w \le S_0w'$ for every $w \in A$ and $w' \in B$; because $C_0(X)^{**}$ is Dedekind complete, $\phi = \sup S_0[A]$ and $\psi = \inf S_0[B]$ are both defined in $C_0(X)^{**}$, and $\phi \le \psi$. If $f \ge 0$ in $C_0(X)^*$, then there is a $\nu \in M_t^+$ such that $T\nu = f$. Since v is ν-virtually measurable (see 434Ec), there are (bounded) Borel measurable functions w, w' such that $w \le v \le w'$ and $w = w'$ ν-a.e., that is, $w \in A$, $w' \in B$ and

$$(S_0w)(f) = \int w\,d\nu = \int w'd\nu = (S_0w')(f).$$

But as

$$(S_0 w)(f) \le \phi(f) \le \psi(f) \le (S_0 w')(f),$$

$\phi(f) = \psi(f)$; as f is arbitrary, $\phi = \psi$. **Q**

(iii) We can therefore define $S : \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}}) \to C_0(X)^{**}$ by setting

$$Sv = \sup\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v\} = \inf\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \ge v\}$$

for every $v \in \mathcal{L}^\infty$. The argument in (ii) tells us also that $(Sv)(T\nu) = \int v\,d\nu$ for every $\nu \in M_t^+$; that is, that $(Sv)(T\mu) = \int v\,d\mu$ for every Radon measure μ on X.

(iv) Now S is a norm-preserving sequentially order-continuous Riesz homomorphism. **P** (Compare 355F.) (α) The non-trivial part of this is actually the check that S is additive. But the formula

$$Sv = \sup\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v\}$$

ensures that $Sv_1 + Sv_2 \le S(v_1 + v_2)$ for all v_1, $v_2 \in \mathcal{L}^\infty$, while the formula

$$Sv = \inf\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \ge v\}$$

ensures that $Sv_1 + Sv_2 \ge S(v_1 + v_2)$ for all v_1, v_2. (β) It is easy to check that $S(\alpha v) = \alpha Sv$ whenever $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ and $\alpha > 0$, so that S is linear. (γ) If $v_1 \wedge v_2 = 0$ in $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$,

$$\begin{aligned} Sv_1 \wedge Sv_2 &= \sup\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v_1\} \wedge \sup\{S_0 w : w \in \mathcal{L}^\infty(\mathcal{B}),\ w \le v_2\} \\ &= \sup\{S_0 w_1 \wedge S_0 w_2 : w_1,\ w_2 \in \mathcal{L}^\infty(\mathcal{B}),\ w_1 \le v_1,\ w_2 \le v_2\} \end{aligned}$$

(352Ea)

$$= \sup\{S_0(w_1 \wedge w_2) : w_1,\ w_2 \in \mathcal{L}^\infty(\mathcal{B}),\ w_1 \le v_1,\ w_2 \le v_2\}$$

(because S_0 is a Riesz homomorphism)

$$= 0.$$

So S is a Riesz homomorphism (352G(iv)). (δ) Now suppose that $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ with infimum 0 in $\mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$. Then $\inf_{n\in\mathbb{N}} v_n(x) = 0$ for every $x \in X$, so $\inf_{n\in\mathbb{N}} \int v_n d\nu = 0$ for every $\nu \in M_t^+$ and $\inf_{n\in\mathbb{N}} (Sv_n)(f) = 0$ for every $f \in (C_0(X)^*)^+$. So $\inf_{n\in\mathbb{N}} Sv_n = 0$; as $\langle v_n\rangle_{n\in\mathbb{N}}$ is arbitrary, S is sequentially order-continuous. (ϵ) If $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$, then $|v| \le \|v\|_\infty \chi X$, so

$$\|Sv\| \le \|v\|_\infty \|S(\chi X)\| = \|v\|_\infty.$$

On the other hand, for any $x \in X$ we have the Dirac measure δ_x on X concentrated at x, with the matching functional $h_x \in C_0(X)^*$, and

$$\|Sv\| = \||Sv|\| = \|S|v|\| \ge (S|v|)(h_x) = \int |v| d\delta_x = |v(x)|,$$

so $\|Sv\| \ge \|v\|_\infty$; thus S is norm-preserving. **Q**

Remark As in 437D, we can write $(Sv)(T\mu) = \fint v\,d\mu$ whenever $v \in \mathcal{L}^\infty(\mathcal{B})$ and $u \in U$ and $\mu \in M_\tau$ (in 437H) or $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}})$ and $u \in C_0(X)$ and $\mu \in M_t$ (in 437I).

437J Vague and narrow topologies We are ready for another look at 'vague' topologies on spaces of measures. Let X be a topological space.

(a) Let Σ be an algebra of subsets of X. I will say that Σ **separates zero sets** if whenever F, $F' \subseteq X$ are disjoint zero sets then there is an $E \in \Sigma$ such that $F \subseteq E$ and $E \cap F' = \emptyset$.

(b) If Σ is any algebra of subsets of X, we can identify the Banach algebra and Banach lattice $L^\infty(\Sigma)$, as defined in §363, with the $\| \|_\infty$-closed linear subspace of $\ell^\infty(X)$ generated by $\{\chi E : E \in \Sigma\}$ (363C, 363Ha). If we do this, then $C_b(X) \subseteq L^\infty(\Sigma)$ iff Σ separates zero sets. **P** (i) Suppose that $C_b(X) \subseteq L^\infty(\Sigma)$ and that F_1, $F_2 \subseteq X$ are disjoint zero sets. Let u_1, $u_2 : X \to \mathbb{R}$ be continuous functions such that $F_i = u_i^{-1}[\{0\}]$ for both i; then $|u_1(x)| + |u_2(x)| > 0$ for every x; set $v = \frac{|u_1|}{|u_1|+|u_2|}$, so that $v : X \to [0, 1]$ is continuous, $v(x) = 0$ for $x \in F_1$ and $v(x) = 1$ for $x \in F_2$. Now $v \in C_b(X) \subseteq L^\infty(\Sigma)$, so there is a $w \in S(\Sigma)$, the linear subspace of $L^\infty(\Sigma)$ generated by $\{\chi E : E \in \Sigma\}$, such that $\|v - w\|_\infty < \frac{1}{2}$ (363C). Set $E = \{x : w(x) \le \frac{1}{2}\}$; then $E \in \Sigma$ and $F_1 \subseteq E \subseteq X \setminus F_2$. As F_1 and F_2 are arbitrary, Σ separates zero sets.

(ii) Now suppose that Σ separates zero sets, that $u : X \to [0,1]$ is continuous, and that $n \ge 1$ is an integer. For $i \le n$, set $F_i = \{x : x \in X,\, u(x) \le \frac{i}{n}\}$, $F'_i = \{x : x \in X,\, u(x) \ge \frac{i+1}{n}\}$. Then F_i and F'_i are disjoint zero sets so there is an $E_i \in \Sigma$ such that $F'_i \subseteq E_i \subseteq X \setminus F_i$. Set $w = \frac{1}{n}\sum_{i=1}^n \chi E_i \in S(\Sigma)$. If $x \in X$, let $j \le n$ be such that $\frac{j}{n} \le u(x) < \frac{j+1}{n}$; then for $i \le n$

$$i < j \Longrightarrow x \in F'_i \Longrightarrow x \in E_i \Longrightarrow x \notin F_i \Longrightarrow i \le j,$$

and $w(x) = \frac{1}{n}\#(\{i : i \le n,\, x \in E_i\})$ is either $\frac{j}{n}$ or $\frac{j+1}{n}$. Thus $|w(x) - u(x)| \le \frac{1}{n}$. As x is arbitrary, $\|u - w\|_\infty \le \frac{1}{n}$; as n is arbitrary, $u \in L^\infty(\Sigma)$. As $L^\infty(\Sigma)$ is a linear subspace of $\ell^\infty(X)$, this is enough to show that $C_b(X) \subseteq L^\infty(\Sigma)$. **Q**

(c) It follows that if Σ is an algebra of subsets of X separating the zero sets, and $\nu : \Sigma \to \mathbb{R}$ is a bounded additive functional, we can speak of $⨍ u\,d\nu$ for any $u \in C_b(X)$. The map $\nu \mapsto ⨍ d\nu$ is a Banach lattice isomorphism from the L-space $M(\Sigma)$ of bounded additive functionals on Σ to $L^\infty(\Sigma)^* = L^\infty(\Sigma)^\sim$ (363K). We therefore have a positive linear operator $T : M(\Sigma) \to C_b(X)^*$ defined by setting $(T\nu)(u) = ⨍ u\,d\nu$ for every $\nu \in M(\Sigma)$ and $u \in C_b(X)$. Except in the trivial case $X = \emptyset$, $\|T\| = 1$ (if $x \in X$, we have $\delta_x \in M(\Sigma)$ defined by setting $\delta_x(E) = \chi E(x)$ for $E \in \Sigma$, and $\|T(\delta_x)\| = 1$).

The **vague topology** on $M(\Sigma)$ is now the topology generated by the functionals $\nu \mapsto ⨍ u\,d\nu$ as u runs over $C_b(X)$; that is, the coarsest topology on $M(\Sigma)$ such that the canonical map $T : M(\Sigma) \to C_b(X)^*$ is continuous for the weak* topology of $C_b(X)^*$. Because the functionals $\nu \mapsto |⨍ u\,d\nu|$ are seminorms on $M(\Sigma)$, the vague topology is a locally convex linear space topology.

(d) There is a variant of the vague topology which can be applied directly to spaces of (non-negative) totally finite measures. Let $\tilde M^+$ be the set of all non-negative real-valued additive functionals defined on algebras of subsets of X which contain every open set. The **narrow topology** on $\tilde M^+$ is that generated by sets of the form

$$\{\nu : \nu \in \tilde M^+,\, \nu G > \alpha\}, \quad \{\nu : \nu \in \tilde M^+,\, \nu X < \alpha\}$$

for open sets $G \subseteq X$ and real numbers α. (See Topsøe 70b, 8.1.)

Observe that $\nu \mapsto \nu X : \tilde M^+ \to [0, \infty[$ is continuous for the narrow topology, and if $G \subseteq X$ is open then $\nu \mapsto \nu G$ is lower semi-continuous for the narrow topology. Writing P_{top} for the set of topological probability measures on X, then the narrow topology on P_{top} is generated by sets of the form $\{\mu : \mu \in P_{\mathrm{top}},\, \mu G > \alpha\}$ for real numbers α and open sets $G \subseteq X$. Writing δ_x for the Dirac measure on X concentrated at x, $x \mapsto \delta_x : X \to P_{\mathrm{top}}$ is a homeomorphism between X and $\{\delta_x : x \in X\}$, since $\{x : \delta_x G > \alpha\}$ is X if $\alpha < 0$, G if $0 \le \alpha < 1$ and $\emptyset$ if $\alpha \ge 1$.

Writing $\tilde M^+_\sigma$ for the set of totally finite topological measures on X, then $\nu \mapsto \nu E : \tilde M^+_\sigma \to [0, \infty[$ is Borel measurable, for the narrow topology on $\tilde M^+_\sigma$, for every Borel set $E \subseteq X$ (because the family of sets E for which $\nu \mapsto \nu E$ is Borel measurable is a Dynkin class containing the open sets). Similarly, $\nu \mapsto \int u\,d\nu : \tilde M^+_\sigma \to \mathbb{R}$ is Borel measurable for every bounded Borel measurable function $u : X \to \mathbb{R}$, being the limit of a sequence of linear combinations of Borel measurable functions.

(e) Vague topologies, being linear space topologies, are necessarily associated with uniformities (3A4Ad), therefore completely regular (4A2Ja). In the very general context of (c) here, in which we have a space $M(\Sigma)$ of all finitely additive functionals on an algebra Σ, we do not expect the vague topology to be Hausdorff. But if we look at particular subspaces, such as the space $M_\sigma(\mathcal{B}\mathfrak{a}(X))$ of signed Baire measures, or the space M_τ of signed τ-additive Borel measures on a completely regular space X, we may well have a Hausdorff vague topology (437Xg).

Similarly, the narrow topology on $\tilde M^+$ is rarely Hausdorff. But on important subspaces we can get Hausdorff topologies. In particular, if X is Hausdorff, then the narrow topology on the space M^+_{R} of totally finite Radon measures on X is Hausdorff (437R(a-ii)).

(f) It will be useful to know that if $u : X \to \mathbb{R}$ is bounded and lower semi-continuous, then $\nu \mapsto ⨍ u\,d\nu : \tilde M^+ \to \mathbb{R}$ is lower semi-continuous for the narrow topology. **P** (i) Perhaps I should start by explaining why $⨍ u\,d\nu$ is always defined; this is because the algebra T generated by the open sets is always a subalgebra of

$\operatorname{dom}\nu$, and $\{x : u(x) > \alpha\} \in \mathfrak{T}$ for every α, so $u \in L^\infty(\mathfrak{T})$ (363Ha). (ii) Now suppose for a moment that $u \ge 0$. If $\nu_0 \in \tilde{M}^+$ and $\gamma < \oint u\,d\nu_0$, let $\epsilon > 0$ be such that $\gamma + \epsilon(1 + \nu_0 X) < \oint u\,d\nu_0$, let $n \ge 1$ be such that $\|u\|_\infty \le n\epsilon$, and for $i \le n$ set $G_i = \{x : u(x) > i\epsilon\}$. Then

$$\epsilon\textstyle\sum_{i=1}^n \chi G_i \le u \le \epsilon(\chi X + \sum_{i=1}^n \chi G_i),$$

$$\oint u\,d\nu_0 \le \epsilon(\nu_0 X + \textstyle\sum_{i=1}^n \nu_0 G_i),$$

$$V = \{\nu : \nu \in \tilde{M}^+, \textstyle\sum_{i=0}^n \nu G_i > \sum_{i=0}^n \nu_0 G_i - 1\}$$

is a neighbourhood of ν_0 in $\tilde{M}^+$, and

$$\oint u\,d\nu \ge \epsilon\textstyle\sum_{i=1}^n \nu G_i > \gamma$$

for every $\nu \in V$. As ν_0 and γ are arbitrary, $\nu \mapsto \oint u\,d\nu$ is lower semi-continuous. (iii) In general, u is expressible as the sum of a constant function and a non-negative lower semi-continuous function; as $\nu \mapsto \nu X$ is continuous, $\nu \mapsto \oint u\,d\nu$ is the sum of two lower semi-continuous functions and is lower semi-continuous. **Q**

Of course it follows at once that if $u : X \to \mathbb{R}$ is bounded and continuous, then $\nu \mapsto \oint u\,d\nu$ is continuous for the narrow topology; that is, the vague topology is coarser than the narrow topology in contexts in which both make sense.

(g) With the more liberal definitions I use when considering integrals with respect to σ-additive measures, we have another version of the same idea. If $u : X \to [0, \infty]$ is a lower semi-continuous function, then $\nu \mapsto \int u\,d\nu : \tilde{M}^+_\sigma \to [0, \infty]$ is lower semi-continuous for the narrow topology. **P** It is the supremum of the lower semi-continuous functions $\nu \mapsto \int (u \wedge n\chi X)d\nu$. **Q**

(h) Let X and Y be topological spaces, $\phi : X \to Y$ a continuous function, and $\tilde{M}^+(X)$, $\tilde{M}^+(Y)$ the spaces of functionals described in (d). For a functional ν defined on a subset of $\mathcal{P}X$, define $\nu\phi^{-1}$ by saying that $(\nu\phi^{-1})(F) = \nu(\phi^{-1}[F])$ whenever $F \subseteq Y$ and $\phi^{-1}[F] \in \operatorname{dom}\nu$. Then $\nu\phi^{-1} \in \tilde{M}^+(Y)$ whenever $\nu \in \tilde{M}^+(X)$, and the map $\nu \mapsto \nu\phi^{-1} : \tilde{M}^+(X) \to \tilde{M}^+(Y)$ is continuous for the narrow topologies (use 4A2B(a-ii)).

(i) I am trying to maintain the formal distinctions between 'quasi-Radon measure' and 'τ-additive effectively locally finite Borel measure inner regular with respect to the closed sets', and between 'Radon measure' and 'tight locally finite Borel measure'. There are obvious problems in interpreting the sum and difference of measures with different domains, which are readily soluble (see 234G and 416De) but in the context of this section are unilluminating. If, however, we take M^+_{qR} to be the set of totally finite quasi-Radon measures on X, and X is completely regular, we have a canonical embedding of M^+_{qR} into a cone in the L-space $C_b(X)^*$; more generally, even if our space X is not completely regular, the map $\mu \mapsto \mu\restriction\mathcal{B}(X) : M^+_{\mathrm{qR}} \to M_\sigma(\mathcal{B}(X))$ is still injective, and we can identify M^+_{qR} with a cone in the L-space M_τ of signed τ-additive Borel measures (often the whole positive cone of M_τ, as in 415M). Similarly, when X is Hausdorff, we can identify totally finite Radon measures with tight totally finite Borel measures (416F). The definition in 437Jd makes it plain that these identifications are homeomorphisms for the narrow topology,

It is even possible to extend these ideas to measures which are not totally finite (437Yi), though there may be new difficulties (415Ya).

(j) For a different kind of narrow topology, adapted to the space of all Radon measures on a Hausdorff space, see 495R below.

437K Proposition Let X be a topological space, and $\tilde{M}^+$ the set of all non-negative real-valued additive functionals defined on algebras of subsets of X containing every open set.

(a) We have a function $T : \tilde{M}^+ \to C_b(X)^*$ defined by the formula $(T\nu)(u) = \oint u\,d\nu$ whenever $\nu \in \tilde{M}^+$ and $u \in C_b(X)$.

(b) T is continuous for the narrow topology $\mathfrak{S}$ on $\tilde{M}^+$ and the weak* topology on $C_b(X)^*$.

(c) Suppose now that X is completely regular, and that $W \subseteq \tilde{M}^+$ is a family of τ-additive totally finite topological measures such that two members of W which agree on the Borel σ-algebra are equal. Then $T\restriction W$ is a homeomorphism between W, with the narrow topology, and $T[W]$, with the weak* topology.

proof (a) We have only to assemble the operators of 437Jc, noting that if an algebra of subsets of X contains every open set then it certainly separates the zero sets (indeed, it actually contains every zero set).

(b) As already noted in 437Jf, $\nu \mapsto (T\nu)(u) = \int u\,d\nu$ is $\mathfrak{S}$-continuous for every $u \in C_b(X)$. Since the weak* topology on $C_b(X)^*$ is the coarsest topology on $C_b(X)^*$ for which all the functionals $f \mapsto f(u)$ are continuous, T is continuous.

(c)(i) Write $\mathfrak{T}$ for the topology on W induced by T, that is, the family of sets of the form $W \cap T^{-1}[V]$ where $V \subseteq C_b(X)^*$ is weak*-open. If $G \subseteq X$ is open, then $A = \{u : u \in C_b(X), 0 \le u \le \chi G\}$ is upwards-directed and has supremum χG, so $\mu G = \sup_{u\in A} \int u\,d\mu$ for every $\mu \in W$ (414Ba). Accordingly $\{\mu : \mu \in W,\ \mu G > \alpha\} = \bigcup_{u\in A}\{\mu : (T\mu)(u) > \alpha\}$ belongs to $\mathfrak{T}$ for every $\alpha \in \mathbb{R}$. Also, of course, $\{\mu : \mu X < \alpha\} = \{\mu : (T\mu)(\chi X) < \alpha\} \in \mathfrak{T}$ for every α. So if $\mathfrak{S}'$ is the narrow topology on W, $\mathfrak{S}' \subseteq \mathfrak{T}$. Putting this together with (b), we see that $\mathfrak{S}' = \mathfrak{T}$.

(ii) Now the same formulae show that $T{\restriction}W$ is injective. **P** Suppose that μ_1, $\mu_2 \in W$ and that $T\mu_1 = T\mu_2$. Then $\mu_1 G = \mu_2 G$ for every open set $G \subseteq X$. By the Monotone Class Theorem, μ_1 and μ_2 agree on all Borel sets; but our hypothesis is that this is enough to ensure that $\mu_1 = \mu_2$. **Q**

Since $T : W \to T[W]$ is continuous and open, it is a homeomorphism.

437L Corollary Let X be a completely regular topological space, and M_τ the space of signed τ-additive Borel measures on X. Then the narrow and vague topologies on M_τ^+ coincide. In particular, the narrow topology on M_τ^+ is completely regular.

proof 437Kc, 3A4Ad, 4A2Ja.

437M Theorem (Ressel 77) For a topological space X, write $M_{\mathrm{qR}}^+(X)$ for the space of totally finite quasi-Radon measures on X, $P_{\mathrm{qR}}(X)$ for the space of quasi-Radon probability measures on X, and $M_\tau(X)$ for the L-space of signed τ-additive Borel measures on X.

(a) Let X and Y be topological spaces. If $\mu \in M_{\mathrm{qR}}^+(X)$ and $\nu \in M_{\mathrm{qR}}^+(Y)$, write $\mu\times\nu$ for their τ-additive product measure on $X \times Y$ (417C, 417F). Then $(\mu,\nu) \mapsto \mu\times\nu$ is continuous for the narrow topologies on $M_{\mathrm{qR}}^+(X)$, $M_{\mathrm{qR}}^+(Y)$ and $M_{\mathrm{qR}}^+(X\times Y)$.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, with product X. If $\langle \mu_i\rangle_{i\in I}$ is a family of probability measures such that $\mu_i \in P_{\mathrm{qR}}(X_i)$ for each i, write $\prod_{i\in I}\mu_i$ for its τ-additive product on X. Then $\langle\mu_i\rangle_{i\in I} \mapsto \prod_{i\in I}\mu_i$ is continuous for the narrow topology on $P_{\mathrm{qR}}(X)$ and the product of the narrow topologies on $\prod_{i\in I} P_{\mathrm{qR}}(X_i)$.

(c) Let X and Y be topological spaces.

(i) We have a unique bilinear operator $\psi : M_\tau(X) \times M_\tau(Y) \to M_\tau(X\times Y)$ such that $\psi(\mu,\nu)$ is the restriction of the τ-additive product of μ and ν to the Borel σ-algebra of $X\times Y$ whenever μ, ν are totally finite Borel measures on X, Y respectively.

(ii) $\|\psi\| \le 1$ (definition: 253Ab).

(iii) ψ is separately continuous for the vague topologies on $M_\tau(X)$, $M_\tau(Y)$ and $M_\tau(X\times Y)$.

(d) In (c), suppose that X and Y are compact and Hausdorff. If $B \subseteq M_\tau(X)$ and $B' \subseteq M_\tau(Y)$ are norm-bounded, then $\psi{\restriction}B\times B'$ is continuous for the vague topologies.

proof (a)(i) I ought to note that we need 417N to assure us that $\mu\times\nu \in M_{\mathrm{qR}}^+(X\times Y)$ whenever $\mu \in M_{\mathrm{qR}}^+(X)$ and $\nu \in M_{\mathrm{qR}}^+(Y)$.

(ii) If $W \subseteq X\times Y$ is open and $\alpha \in \mathbb{R}$, then $Q = \{(\mu,\nu) : (\mu\times\nu)(W) > \alpha\}$ is open in $M_{\mathrm{qR}}^+(X)\times M_{\mathrm{qR}}^+(Y)$. **P** Suppose that $(\mu_0,\nu_0) \in Q$. Because $\mu_0\times\nu_0$ is τ-additive, there is a subset $W' \subseteq W$, expressible in the form $\bigcup_{i\le n} G_i \times H_i$ where $G_i \subseteq X$ and $H_i \subseteq Y$ are open for every i, such that

$$\alpha < (\mu_0\times\nu_0)(W') = \int \nu_0 W'[\{x\}]\mu_0(dx)$$

(417C(b-v-β). Set $u(x) = \nu_0 W'[\{x\}]$ for $x \in X$, so that u is lower semi-continuous (417Ba). Let $\eta > 0$ be such that $\int u\,d\mu_0 > \alpha + (1+2\mu_0 X)\eta$, and set $E_i = \{x : u(x) > \eta i\}$ for $i \in \mathbb{N}$, so that $\eta\sum_{i=1}^\infty \mu_0 E_i > \int u\,d\mu_0 - \eta\mu_0 X$. Because every E_i is open, there is a neighbourhood U of μ_0 in $M_{\mathrm{qR}}^+(X)$ such that

$$\int u\,d\mu_0 - \eta\mu_0 X \le \eta\textstyle\sum_{i=1}^\infty \mu E_i \le \int u\,d\mu = (\mu\times\nu_0)(W')$$

for every $\mu \in U$; shrinking U if necessary, we can arrange at the same time that $\mu X < \mu_0 X + 1$ for every $\mu \in U$. Next, observe that $\mathcal{H} = \{W'[\{x\}] : x \in X\} \subseteq \{\bigcup_{i \in I} H_i : I \subseteq \{0, \dots, n\}\}$ is finite, so there is a neighbourhood V of ν_0 in $M^+_{\mathrm{qR}}(Y)$ such that $\nu H \geq \nu_0 H - \eta$ for every $H \in \mathcal{H}$ and $\nu \in V$. If $\mu \in U$ and $\nu \in V$, we have

$$(\mu \times \nu)(W) \geq (\mu \times \nu)(W') = \int \nu W'[\{x\}] \mu(dx) \geq \int u(x) - \eta \, \mu(dx)$$
$$= \int u \, d\mu - \eta \mu X \geq \int u \, d\mu_0 - \eta \mu_0 X - \eta(1 + \mu_0 X) > \alpha.$$

As μ_0 and ν_0 are arbitrary, Q is open. **Q**

(iii) Since $(\mu \times \nu)(X \times Y) = \mu X \cdot \nu Y$, the sets $\{(\mu, \nu) : (\mu \times \nu)(X \times Y) < \alpha\}$ are also open for every $\alpha \in \mathbb{R}$. So $(\mu, \nu) \mapsto \mu \times \nu$ is continuous (4A2B(a-ii) again).

(b) Similarly, we can refer to 417O to check that $\prod_{i \in I} \mu_i \in P_{\mathrm{qR}}(X)$ whenever $\mu_i \in P_{\mathrm{qR}}(X_i)$ for each i. For finite sets I, the result is a simple induction on $\#(I)$, using 417Db and part (a) just above. For infinite I, let $W \subseteq X$ be an open set and $\alpha \in \mathbb{R}$, and consider

$$Q = \{\langle \mu_i \rangle_{i \in I} : \mu_i \in P_{\mathrm{qR}}(X_i) \text{ for each } i, \ (\textstyle\prod_{i \in I} \mu_i)(W) > \alpha\}.$$

If $\langle \mu_i \rangle_{i \in I} \in Q$, then there is an open set $W' \subseteq W$, determined by coordinates in a finite set $J \subseteq I$, such that $(\prod_{i \in I} \mu_i)(W') > \alpha$. Setting $V = \{x \restriction J : x \in W'\}$, we have $(\prod_{i \in J} \mu_i)(V) > \alpha$. Now we can find open sets U_i in $P_{\mathrm{qR}}(X_i)$, for $i \in J$, such that $(\prod_{i \in J} \nu_i)(V) > \alpha$ whenever $\nu_i \in U_i$ for $i \in J$. If now $\langle \nu_i \rangle_{i \in I} \in \prod_{i \in I} P_{\mathrm{qR}}(X_i)$ is such that $\nu_i \in U_i$ for every $i \in J$,

$$(\textstyle\prod_{i \in I} \nu_i)(W) \geq (\prod_{i \in I} \nu_i)(W') = (\prod_{i \in J} \nu_i)(V) > \alpha,$$

so $\prod_{i \in I} \nu_i \in Q$. As $\langle \mu_i \rangle_{i \in I}$ is arbitrary, Q is open.

As W and α are arbitrary, $\langle \mu_i \rangle_{i \in I} \mapsto \prod_{i \in I} \mu_i$ is continuous.

(c)(i) Start by writing $\psi(\mu, \nu) = (\mu \times \nu) \restriction \mathcal{B}(X \times Y)$ for $\mu \in M^+_\tau(X)$ and $\nu \in M^+_\tau(Y)$, where $\mathcal{B}(X \times Y)$ is the Borel σ-algebra of $X \times Y$. If μ, μ_1, $\mu_2 \in M^+_\tau(X)$ and ν, ν_1, $\nu_2 \in M^+_\tau(Y)$ and $\alpha \geq 0$, then

$$\psi(\mu_1 + \mu_2, \nu) = \psi(\mu_1, \nu) + \psi(\mu_2, \nu).$$

P On each side of the equation we have a τ-additive Borel measure, and the two measures agree on the standard base $\mathcal{W}$ for the topology of $X \times Y$ consisting of products of open sets; since $\mathcal{W}$ is closed under finite intersections, they agree on the algebra generated by $\mathcal{W}$ and therefore on all open sets and therefore (using the Monotone Class Theorem yet again) on all Borel sets. **Q** Similarly,

$$\psi(\mu, \nu_1 + \nu_2) = \psi(\mu, \nu_1) + \psi(\mu, \nu_2), \quad \psi(\alpha\mu, \nu) = \psi(\mu, \alpha\nu) = \alpha\psi(\mu, \nu)$$

whenever $\mu \in M^+_\tau(X)$, ν, ν_1, $\nu_2 \in M^+_\tau(Y)$ and $\alpha \in \mathbb{R}$. Now if μ'_1, $\mu'_2 \in M^+_\tau(X)$ and ν'_1, $\nu'_2 \in M^+_\tau(Y)$ are such that $\mu_1 - \mu_2 = \mu'_1 - \mu'_2$ and $\nu_1 - \nu_2 = \nu'_1 - \nu'_2$, we shall have

$$\begin{aligned}
&\psi(\mu_1, \nu_1) - \psi(\mu_1, \nu_2) - \psi(\mu_2, \nu_1) + \psi(\mu_2, \nu_2) \\
&\qquad = \psi(\mu_1, \nu_1 + \nu'_2) - \psi(\mu_1 + \mu'_2, \nu'_2) + \psi(\mu'_2, \nu'_2) \\
&\qquad\qquad - \psi(\mu_1, \nu'_1 + \nu_2) + \psi(\mu_1 + \mu'_2, \nu'_1) - \psi(\mu'_2, \nu'_1) \\
&\qquad\qquad - \psi(\mu_2, \nu_1 + \nu'_2) + \psi(\mu_2 + \mu'_1, \nu'_2) - \psi(\mu'_1, \nu'_2) \\
&\qquad\qquad + \psi(\mu_2, \nu'_1 + \nu_2) - \psi(\mu_2 + \mu'_1, \nu'_1) + \psi(\mu'_1, \nu'_1) \\
&\qquad = \psi(\mu'_2, \nu'_2) - \psi(\mu'_2, \nu'_1) - \psi(\mu'_1, \nu'_2) + \psi(\mu'_1, \nu'_1).
\end{aligned}$$

We can therefore extend ψ to an operator on $M_\tau(X) \times M_\tau(Y)$ by setting

$$\psi(\mu_1 - \mu_2, \nu_1 - \nu_2) = \psi(\mu_1, \nu_1) - \psi(\mu_1, \nu_2) - \psi(\mu_2, \nu_1) + \psi(\mu_2, \nu_2)$$

whenever μ_1, $\mu_2 \in M^+_\tau(X)$ and ν_1, $\nu_2 \in M^+_\tau(Y)$, and it is straightforward to check that ψ is bilinear.

(ii) If $\mu \in M_\tau(X)$, then $\|\mu\| = \mu^+(X) + \mu^-(X)$, where μ^+ and μ^- are evaluated in the Riesz space $M_\tau(X)$. Now if $\nu \in M_\tau(Y)$,

$$\begin{aligned}|\psi(\mu,\nu)| &= |\psi(\mu^+,\nu^+)-\psi(\mu^+,\nu^-)-\psi(\mu^-,\nu^+)+\psi(\mu^-,\nu^-)|\\ &\le \psi(\mu^+,\nu^+)+\psi(\mu^+,\nu^-)+\psi(\mu^-,\nu^+)+\psi(\mu^-,\nu^-),\end{aligned}$$

so

$$\begin{aligned}\|\psi(\mu,\nu)\| &= |\psi(\mu,\nu)|(X\times Y)\\ &\le \psi(\mu^+,\nu^+)(X\times Y)+\psi(\mu^+,\nu^-)(X\times Y)\\ &\qquad +\psi(\mu^-,\nu^+)(X\times Y)+\psi(\mu^-,\nu^-)(X\times Y)\\ &= \mu^+(X)\cdot\nu^+(Y)+\mu^+(X)\cdot\nu^-(Y)+\mu^-(X)\cdot\nu^+(Y)+\mu^-(X)\cdot\nu^-(Y)\\ &= \|\mu\|\|\nu\|.\end{aligned}$$

As μ and ν are arbitrary, $\|\psi\|\le 1$.

(iii) Fix $\nu\in M_\tau^+(Y)$ and $w\in C_b(X\times Y)^+$, and consider the map $\mu\mapsto \int w\,d\psi(\mu,\nu):M_\tau(X)\to\mathbb{R}$. Note first that if $\mu\in M_\tau^+(X)$,

$$\int w\,d\psi(\mu,\nu)=\int w\,d(\mu\times\nu)=\iint w(x,y)\nu(dy)\mu(dx)=\iint w(x,y)\nu(dy)\mu(dx)$$

(417G). Since both sides of this equation are linear in μ, we have

$$\int w\,d\psi(\mu,\nu)=\iint w(x,y)\nu(dy)\mu(dx)$$

for every $\mu\in M_\tau(X)$. Now $x\mapsto\int w(x,y)\nu(dy)$ is continuous. **P** By 417Bc, it is lower semi-continuous; but if $\alpha\ge\|w\|_\infty$ and $w'=\alpha\chi(X\times Y)-w$, then $x\mapsto\int w'(x,y)\nu(dy)$ is lower semi-continuous, so

$$x\mapsto\alpha\nu Y-\int w'(x,y)\nu(dy)=\int w(x,y)\nu(dy)$$

is also upper semi-continuous, therefore continuous. **Q** It follows at once that $\mu\mapsto\iint w(x,y)\nu(dy)\mu(dx)$ is continuous for the vague topology on $M_\tau(X)$. The argument has supposed that w and ν are positive; but taking positive and negative parts as usual, we see that $\mu\mapsto\int w\,d\psi(\mu,\nu)$ is vaguely continuous for every $w\in C_b(X\times Y)$ and $\nu\in M_\tau(Y)$. As w is arbitrary, $\mu\mapsto\psi(\mu,\nu)$ is vaguely continuous, for every ν. Similarly, $\nu\mapsto\psi(\mu,\nu)$ is vaguely continuous for every μ, and ψ is separately continuous.

(d) Now suppose that X and Y are compact. Let W be the linear subspace of $C(X\times Y)$ generated by $\{u\otimes v:u\in C(X),\,v\in C(Y)\}$, writing $(u\otimes v)(x,y)=u(x)v(y)$ as in 253B. Then W is a subalgebra of $C(X\times Y)$ separating the points of $X\times Y$ and containing the constant functions, so is $\|\,\|_\infty$-dense in $C(X\times Y)$ (281E). Now

$$(\mu,\nu)\mapsto\int u\otimes v\,d\psi(\mu,\nu)=\int u\,d\mu\cdot\int v\,d\nu$$

is continuous whenever $u\in C(X)$ and $v\in C(Y)$, so

$$(\mu,\nu)\mapsto\int w\,d\psi(\mu,\nu)$$

is continuous whenever $w\in W$.

Next suppose that $B\subseteq M_\tau(X)$ and $B'\subseteq M_\tau(Y)$ are bounded. Let $\gamma\ge 0$ be such that $\|\mu\|\le\gamma$ for every $\mu\in B$ and $\|\nu\|\le\gamma$ for every $\nu\in B$. If $w\in C(X\times Y)$ and $\epsilon>0$, there is a $w'\in W$ such that $\|w-w'\|_\infty\le\epsilon$. In this case

$$\begin{aligned}|\int w\,d\psi(\mu,\nu)-\int w'\,d\psi(\mu,\nu)| &\le \|w-w'\|_\infty\|\psi(\mu,\nu)\|\\ &\le \epsilon\|\mu\|\|\nu\|\le\gamma^2\epsilon\end{aligned}$$

whenever $\mu\in B$ and $\nu\in B'$. As ϵ is arbitrary, the function $(\mu,\nu)\mapsto\int w\,d\psi(\mu,\nu)$ is uniformly approximated on $B\times B'$ by vaguely continuous functions, and is therefore itself vaguely continuous on $B\times B'$.

437N One of the standard constructions of Radon measures is as image measures. It leads naturally to maps between spaces of Radon measures, and of course we wish to know whether they are continuous.

Proposition (a) Let X and Y be Hausdorff spaces, and $\phi : X \to Y$ a continuous function. Let $M_{\mathrm{R}}^+(X)$, $M_{\mathrm{R}}^+(Y)$ be the spaces of totally finite Radon measures on X and Y respectively. Write $\tilde{\phi}(\mu)$ for the image measure $\mu\phi^{-1}$ for $\mu \in M_{\mathrm{R}}^+(X)$.

(i) $\tilde{\phi} : M_{\mathrm{R}}^+(X) \to M_{\mathrm{R}}^+(Y)$ is continuous for the narrow topologies on $M_{\mathrm{R}}^+(X)$ and $M_{\mathrm{R}}^+(Y)$.

(ii) $\tilde{\phi}(\mu+\nu) = \tilde{\phi}(\mu) + \tilde{\phi}(\nu)$ and $\tilde{\phi}(\alpha\mu) = \alpha\tilde{\phi}(\mu)$ for all μ, $\nu \in M_{\mathrm{R}}^+(X)$ and $\alpha \ge 0$.

(b) If Y is a Hausdorff space, X a subset of Y, and $\phi : X \to Y$ the identity map, then $\tilde{\phi}$ is a homeomorphism between $M_{\mathrm{R}}^+(X)$ and $\{\nu : \nu \in M_{\mathrm{R}}^+(Y),\ \nu(Y \setminus X) = 0\}$.

proof (a)(i) All we have to do is to recall from 418I that $\mu\phi^{-1} \in M_{\mathrm{R}}^+(Y)$ for every $\mu \in M_{\mathrm{R}}^+(X)$, and observe that

$$\{\mu : (\mu\phi^{-1})(H) > \alpha\} = \{\mu : \mu\phi^{-1}[H] > \alpha\}, \quad \{\mu : (\mu\phi^{-1})(Y) < \alpha\} = \{\mu : \mu X < \alpha\}$$

are narrowly open in $M_{\mathrm{R}}^+(X)$ for every open set $H \subseteq Y$ and $\alpha \in \mathbb{R}$.

(ii) As usual, since all the measures here are Radon measures, it is enough to check that $\tilde{\phi}(\mu+\nu)(E) = \tilde{\phi}(\mu)(E) + \tilde{\phi}(\nu)(E)$ and $\tilde{\phi}(\alpha\mu)(E) = \alpha\tilde{\phi}(\mu)(E)$ for every Borel set $E \subseteq X$, and this is easy.

(b) First note that if $\mu \in M_{\mathrm{R}}^+(X)$, then certainly $\tilde{\phi}(\mu)(Y \setminus X) = 0$; while if $\nu \in M_{\mathrm{R}}^+(Y)$ and $\nu(Y \setminus X) = 0$, then $\mu = \nu \restriction \mathcal{P}X$ is a Radon measure on X (416Rb) and $\nu = \tilde{\phi}(\mu)$. Thus $\tilde{\phi}$ is a continuous bijection from $M_{\mathrm{R}}^+(X)$ to $\{\nu : \nu \in M_{\mathrm{R}}^+(Y),\ \nu(Y \setminus X) = 0\}$. Now if $G \subseteq X$ is relatively open and $\alpha \in \mathbb{R}$, there is an open set $H \subseteq Y$ such that $G = H \cap X$, so that

$$\{\mu : \mu \in M_{\mathrm{R}}^+(X),\ \mu G > \alpha\} = \{\mu : \tilde{\phi}(\mu)(H) > \alpha\}$$

is the inverse image of a narrowly open set in $M_{\mathrm{R}}^+(Y)$; and of course

$$\{\mu : \mu \in M_{\mathrm{R}}^+(X),\ \mu X < \alpha\} = \{\mu : \tilde{\phi}(\mu)(Y) < \alpha\}$$

is also the inverse image of an open set. So $\tilde{\phi}$ is a homeomorphism between $M_{\mathrm{R}}^+(X)$ and $\{\nu : \nu \in M_{\mathrm{R}}^+(Y),\ \nu(Y \setminus X) = 0\}$.

437O Uniform tightness Let X be a topological space. If ν is a bounded additive functional on an algebra of subsets of X, I say that it is **tight** if

$$\nu E \in \overline{\{\nu K : K \subseteq E,\ K \in \operatorname{dom}\nu,\ K \text{ is closed and compact}\}}$$

for every $E \in \operatorname{dom}\nu$, and that a set A of tight functionals is **uniformly tight** if every member of A is tight and for every $\epsilon > 0$ there is a closed compact set $K \subseteq X$ such that νK is defined and $|\nu E| \le \epsilon$ whenever $\nu \in A$ and $E \in \operatorname{dom}\nu$ is disjoint from K.

437P Proposition Let X be a topological space.

(a) Let M_{qR}^+ be the set of totally finite quasi-Radon measures on X. Suppose that $A \subseteq M_{\mathrm{qR}}^+$ is uniformly totally finite (that is, $\{\mu X : \mu \in A\}$ has a finite upper bound) and for every $\epsilon > 0$ there is a closed compact $K \subseteq X$ such that $\mu(X \setminus K) \le \epsilon$ for every $\mu \in A$. Then A is relatively compact in M_{qR}^+ for the narrow topology.

(b) Suppose now that X is Hausdorff, and that M_{R}^+ is the set of Radon measures on X. If $A \subseteq M_{\mathrm{R}}^+$ is uniformly totally finite and uniformly tight, then it is relatively compact in M_{R}^+ for the narrow topology.

proof (a)(i) I show first that the closure $\overline{A}$ of A in M_{qR}^+ has the same two properties. **P** Because $\mu \mapsto \mu X$ is continuous for the narrow topology, $\{\mu X : \mu \in \overline{A}\} \subseteq \overline{\{\mu X : \mu \in A\}}$ is bounded. If $\epsilon > 0$, there is a closed compact set $K \subseteq X$ such that $\mu(X \setminus K) \le \epsilon$ for every $\mu \in A$. In this case $\{\mu : \mu \in M_{\mathrm{qR}}^+,\ \mu(X \setminus K) \le \epsilon\} = M_{\mathrm{qR}}^+ \setminus \{\mu : \mu(X \setminus K) > \epsilon\}$ is closed in M_{qR}^+, so includes $\overline{A}$. As ϵ is arbitrary, we have the result. **Q**

(ii) Now let $\mathcal{F}$ be an ultrafilter on M_{qR}^+ containing $\overline{A}$.

($\boldsymbol{\alpha}$) For Borel sets $E \subseteq X$, set $\theta E = \lim_{\nu \to \mathcal{F}} \nu E$; this is defined in $\mathbb{R}$ because $\sup_{\nu \in \overline{A}} \nu X$ is finite. θ is a non-negative additive functional on the Borel σ-algebra of X. The family $\mathcal{K}$ of closed compact subsets of X is a compact class containing $\emptyset$ and closed under finite unions and countable intersections, so 413Ub tells us that there is a complete measure μ on X such that $\mu X \le \theta X$, $\mathcal{K} \subseteq \operatorname{dom}\mu$, $\mu K \ge \theta K$ for every $K \in \mathcal{K}$, and μ is inner regular with respect to $\mathcal{K}$.

If $F \subseteq X$ is closed, then $F \cap K \in \mathcal{K}$ for every $K \in \mathcal{K}$, so μ measures F (412Ja). Thus μ is a topological measure. Because μ is tight, it is τ-additive (411E). So μ is a complete totally finite τ-additive measure which is inner regular with respect to the closed sets, and is a quasi-Radon measure.

(β) Given $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $\nu(X \setminus K) \le \epsilon$ for every $\nu \in \overline{A}$, so $\theta(X \setminus K) \le \epsilon$ and

$$\theta X - \epsilon \le \theta K \le \mu K \le \mu X \le \theta X.$$

As ϵ is arbitrary, $\mu X = \theta X = \lim_{\nu\to\mathcal{F}} \nu X$.

(γ) If $G \subseteq X$ is open and $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $\nu(X \setminus K) \le \epsilon$ for every $\nu \in \overline{A}$, so $\theta(X \setminus K) \le \epsilon$ and

$$\mu G = \mu X - \mu(X \setminus G) \le \theta X - \mu(K \setminus G) \le \theta X - \theta(K \setminus G)$$

(because $K \setminus G \in \mathcal{K}$)

$$\le \theta X - \theta(X \setminus G) + \theta(X \setminus K) \le \theta G + \epsilon.$$

As ϵ is arbitrary,

$$\mu G \le \theta G = \lim_{\nu\to\mathcal{F}} \nu G.$$

(δ) Putting (β) and (γ) together, we see that $\mathcal{F} \to \mu$ for the narrow topology on M^+_{qR}; it follows that $\mu \in \overline{A}$. Thus every ultrafilter on M^+_{qR} containing $\overline{A}$ has a limit in $\overline{A}$, and $\overline{A}$ is compact. Accordingly A is relatively compact in M^+_{qR}, as claimed.

(b) We know from the proof of (a) that the closure $\overline{A}$ of A in M^+_{qR} is compact, so it will be enough to show that $\overline{A} \subseteq M^+_{\mathrm{R}}$. If $\mu \in \overline{A}$ and $E \in \operatorname{dom}\mu$, then for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu(X \setminus K) \le \epsilon$, by (a-i) above; also there is a closed set $F \subseteq E$ such that $\mu(E \setminus F) \le \epsilon$. But now $F \cap K$ is compact and $\mu(E \setminus (F \cap K)) \le 2\epsilon$. As E and ϵ are arbitrary, μ is inner regular with respect to the compact sets, so is a Radon measure.

437Q Two metrics So far, as elsewhere in this volume, I have been writing about topologies with as few restrictions on their nature as possible. Of course the repeated invocation of L-spaces in the first part of the section indicates that there are norms and their associated metrics about, and when the underlying set X is metrizable we rather hope that the constructions of 437J will lead to metrizable topologies on the spaces of measures considered there. I offer two definitions which seem to give us interesting paths to explore.

(a)(i) If X is a set and μ, ν are bounded additive functionals defined on algebras of subsets of X, then $\mu - \nu : \operatorname{dom}\mu \cap \operatorname{dom}\nu \to \mathbb{R}$ is bounded and additive, and we can set

$$\rho_{\mathrm{tv}}(\mu, \nu) = |\mu - \nu|(X) = \sup_{E,F\in\operatorname{dom}\mu\cap\operatorname{dom}\nu} (\mu - \nu)(E) - (\mu - \nu)(F).$$

In this generality, ρ_{tv} is not even a pseudometric, but if we have a class M of totally finite measures on X all of which are inner regular with respect to a subset $\mathcal{K}$ of $\bigcap_{\mu\in M} \operatorname{dom}\mu$, then we have

$$\rho_{\mathrm{tv}}(\mu, \nu) = \sup_{K,L\in\mathcal{K}} (\mu K - \mu L) - (\nu K - \nu L)$$

for all μ, $\nu \in M$, and $\rho_{\mathrm{tv}} \restriction M \times M$ is a pseudometric on M. If moreover M is such that distinct members of M differ on $\mathcal{K}$ (as when $\mathcal{K}$ is the family of closed sets in a topological space X and $M = M^+_{\mathrm{qR}}(X)$, or when $\mathcal{K}$ the family of compact sets in a Hausdorff space X and $M = M^+_{\mathrm{R}}(X)$), then ρ_{tv} gives us a metric on M. In such a case I will call $\rho_{\mathrm{tv}} \restriction M \times M$ the **total variation metric** on M. (Compare the 'total variation norms' of 362B.)

(ii) Note that if $\Sigma \subseteq \operatorname{dom}\mu \cap \operatorname{dom}\nu$ is a σ-algebra then

$$|\int u\,d\mu - \int u\,d\nu| \le \|u\|_\infty \rho_{\mathrm{tv}}(\mu, \nu)$$

whenever $u \in \mathcal{L}^\infty(\Sigma)$. So if, for instance, X is a topological space and $M \subseteq M_{\mathrm{qR}}^+(X)$, then $u \mapsto \int u\,d\mu$ will be continuous for the total variation metric on M whenever $u : X \to \mathbb{R}$ is a bounded universally measurable function.

(iii) It is of course worth knowing when to expect a complete metric. When our set M can be identified with the positive cone of a band in some L-space M_σ of countably additive functions, as in 437F, then we naturally have a complete metric, because bands in L-spaces are closed subspaces (354Bd). In particular, for any Hausdorff space X, $M_{\mathrm{R}}^+(X)$ can be identified with the positive cone of the L-space of tight Borel measures on X, so is complete. See also 437Xo.

(iv) There is an obvious variation on ρ_{tv} as defined here: the function

$$(\mu, \nu) \mapsto \sup_{E \in \operatorname{dom}\mu \cap \operatorname{dom}\nu} |\mu E - \nu E|,$$

which will be a metric on nearly all occasions when ρ_{tv} is a metric, and will then be uniformly equivalent to ρ_{tv}. But the more complex formulation gives a better match to the Riesz norm metric of the leading examples.

(b) Suppose that (X, ρ) is a metric space. Write M_{qR}^+ for the set of totally finite quasi-Radon measures on X. For μ, $\nu \in M_{\mathrm{qR}}^+$ set

$$\rho_{\mathrm{KR}}(\mu, \nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to [-1, 1] \text{ is 1-Lipschitz}\}.$$

Then ρ_{KR} is a metric on M_{qR}^+. **P** It is immediate from the form of the definition that ρ_{KR} is a pseudometric. If μ, $\nu \in M_{\mathrm{qR}}^+$ are different, there is an open set G such that $\mu G \neq \nu G$ (415G(iii)); suppose that $\mu G < \nu G$. Set $u(x) = \rho(x, X \setminus G)$ for $x \in X$. There must be an $n \in \mathbb{N}$ such that $\mu G < \nu F_n$ where $F_n = \{x : u(x) \geq 2^{-n}\}$. In this case, setting $u' = u \wedge 2^{-n}\chi X$,

$$\int u' d\mu \leq 2^{-n}\mu G < 2^{-n}\nu F_n \leq \int u' d\nu,$$

so $\rho_{\mathrm{KR}}(\mu, \nu) \geq 2^{-n}(\nu F_n - \mu G) > 0$. As μ and ν are arbitrary, ρ_{KR} is a metric. **Q**

Remark ρ_{KR} here is taken from Bogachev 07, §8.3, where it is called the 'Kantorovich-Rubinshtein metric'. For its principal properties, see 437R(g)-(h) below. A variation of this construction will be used in 457L; see also 437Xs.

437R Theorem Let X be a topological space; write $M_{\mathrm{qR}}^+ = M_{\mathrm{qR}}^+(X)$ for the set of totally finite quasi-Radon measures on X, and if X is Hausdorff write $M_{\mathrm{R}}^+ = M_{\mathrm{R}}^+(X)$ for the set of totally finite Radon measures on X, both endowed with their narrow topologies.

(a)(i) If X is regular then M_{qR}^+ is Hausdorff.

(ii) If X is Hausdorff then M_{R}^+ is Hausdorff.

(b) If X has a countable network then M_{qR}^+ has a countable network.

(c) Suppose that X is separable.

(i) If X is a T_1 space, then M_{qR}^+ is separable.

(ii) If X is Hausdorff, M_{R}^+ is separable.

(d) If X is a K-analytic Hausdorff space, so is $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$.

(e) If X is an analytic Hausdorff space, so is $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$.

(f)(i) If X is compact, then for any real $\gamma \geq 0$ the sets $\{\mu : \mu \in M_{\mathrm{qR}}^+, \mu X \leq \gamma\}$ and $\{\mu : \mu \in M_{\mathrm{qR}}^+, \mu X = \gamma\}$ are compact.

(ii) If X is compact and Hausdorff, then for any real $\gamma \geq 0$ the sets $\{\mu : \mu \in M_{\mathrm{R}}^+, \mu X \leq \gamma\}$ and $\{\mu : \mu \in M_{\mathrm{R}}^+, \mu X = \gamma\}$ are compact. In particular, the set P_{R} of Radon probability measures on X is compact.

(g) Suppose that X is metrizable and ρ is a metric on X inducing its topology.

(i) The metric ρ_{KR} on M_{qR}^+ (437Qb) induces the narrow topology on M_{qR}^+.

(ii) If (X, ρ) is complete then $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$ is complete under ρ_{KR}.

(h) If X is Polish, so is $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$.

proof (a)(i) (Cf. 437Qb.) Take distinct μ_0, $\mu_1 \in M_{\mathrm{qR}}^+$. If $\mu_0 X \ne \mu_1 X$ then they can be separated by open sets of the form $\{\mu : \mu X < \alpha\}$, $\{\mu : \mu X > \alpha\}$. Otherwise, set $\gamma = \mu_0 X = \mu_1 X$. There is certainly an open set G such that $\mu_0 G \ne \mu_1 G$ (415H); suppose that $\mu_0 G < \mu_1 G$. Because μ_1 is inner regular with respect to the closed sets, there is a closed set $F \subseteq G$ such that $\mu_0 G < \mu_1 F$. Now consider

$$\mathcal{H} = \{H : H \text{ is open, } \overline{H} \subseteq G\}.$$

Then $\mathcal{H}$ is upwards-directed; because X is regular, $\bigcup \mathcal{H} = G$; because μ_1 is quasi-Radon,

$$\sup_{H\in\mathcal{H}} \mu_1 H \ge \sup_{H\in\mathcal{H}} \mu_1(H \cap F) = \mu_1 F > \mu_0 G$$

and there is an $H \in \mathcal{H}$ such that $\mu_1 H > \mu_0 G$. Now

$$\mu_1 H + \mu_0(X \setminus \overline{H}) \ge \mu_1 H + \gamma - \mu_0 G > \gamma.$$

Let α, β be such that $\mu_1 H > \alpha$, $\mu_0(X \setminus \overline{H}) > \beta$ and $\alpha + \beta > \gamma$. Then

$$\{\mu : \mu \in M_{\mathrm{qR}}^+,\, \mu(X \setminus \overline{H}) > \beta\}, \quad \{\mu : \mu \in M_{\mathrm{qR}}^+,\, \mu H > \alpha,\, \mu X < \alpha + \beta\}$$

are disjoint open sets containing μ_0, μ_1 respectively, so again we have separation.

(ii) We can use the same ideas. Take distinct μ_0, $\mu_1 \in M_{\mathrm{R}}^+$. If $\mu_0 X \ne \mu_1 X$ then μ_0 and μ_1 can be separated by open sets of the form $\{\mu : \mu X < \alpha\}$, $\{\mu : \mu X > \alpha\}$. Otherwise, set $\gamma = \mu_0 X = \mu_1 X$, and take an open set G such that $\mu_0 G \ne \mu_1 G$; suppose that $\mu_0 G < \mu_1 G$. Then $\mu_0(X \setminus G) + \mu_1 G > \gamma$. Because μ_0 and μ_1 are inner regular with respect to the compact sets, there are compact sets $K_0 \subseteq X \setminus G$, $K_1 \subseteq G$ such that $\mu_0 K_0 + \mu_1 K_1 > \gamma$. Now there are disjoint open sets H_0, H_1 such that $K_i \subseteq H_i$ for both i (4A2F(h-i)), in which case $\mu_0 H_0 + \mu_1 H_1 > \gamma$. Take $\alpha_0 < \mu_0 H_0$ and $\alpha_1 < \mu_1 H_1$ such that $\alpha_0 + \alpha_1 > \gamma$. In this case, $\{\mu : \mu H_0 > \alpha_0\}$ and $\{\mu : \mu H_1 > \alpha_1,\, \mu X < \alpha_0 + \alpha_1\}$ are disjoint open sets containing μ_0, μ_1 respectively.

(b) Let $\mathcal{A}$ be a countable network for the topology of X; replacing $\mathcal{A}$ by $\{\bigcup \mathcal{A}_0 : \mathcal{A}_0 \in [\mathcal{A}]^{<\omega}\}$ if necessary, we can suppose that $\mathcal{A}$ is closed under finite unions. Let $\mathcal{D}$ be the family of sets of the form

$$\{\mu : \mu \in M_{\mathrm{qR}}^+,\, \mu X < \gamma,\, \mu^* A_i > \gamma_i \text{ for } i \le n\}$$

where $n \in \mathbb{N}$, $A_0, \dots, A_n \in \mathcal{A}$ and $\gamma, \gamma_0, \dots, \gamma_n \in \mathbb{Q}$. Then $\mathcal{D}$ is countable. If $V \subseteq M_{\mathrm{qR}}^+$ is an open set and $\mu_0 \in V$, there must be open sets $G_0, \dots, G_n \subseteq X$ and $\gamma, \gamma_0, \dots, \gamma_n \in \mathbb{Q}$ such that

$$\mu_0 \in \{\mu : \mu X < \gamma,\, \mu G_i > \gamma_i \text{ for every } i \le n\} \subseteq V.$$

For each $i \le n$, $\{A : A \in \mathcal{A},\, A \subseteq G_i\}$ is a countable upwards-directed set with union G_i, so there is a non-decreasing sequence $\langle A_{ij}\rangle_{j\in\mathbb{N}}$ in $\mathcal{A}$ with union G_i, and there must be a $j_i \in \mathbb{N}$ such that $\mu_0^* A_{ij_i} > \gamma_i$ (132Ae). Now

$$\{\mu : \mu X < \gamma,\, \mu^* A_{ij_i} > \gamma_i \text{ for every } i \le n\}$$

belongs to $\mathcal{D}$, contains μ and is included in V. As μ and V are arbitrary, $\mathcal{D}$ is a countable network for the topology of M_{qR}^+.

(c)(i) If X is empty, then M_{qR}^+ is a singleton, and we can stop. Otherwise, let D be a countable dense subset of X. Set $D' = \{\sum_{i=0}^n \alpha_i \delta_{x_i} : x_0, \dots, x_n \in D,\, \alpha_0, \dots, \alpha_n \in \mathbb{Q} \cap [0, \infty[\}$, writing δ_x for the Dirac measure concentrated at x for each $x \in X$. Because X is T_1, $D' \subseteq M_{\mathrm{qR}}^+$. In fact D' is dense in M_{qR}^+. **P** Take any $\mu \in M_{\mathrm{qR}}^+$, a finite family $\mathcal{G}$ of open subsets of X, and $\epsilon > 0$. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{G}$, and $\mathcal{A}$ the set of atoms of $\mathcal{E}$. For each $E \in \mathcal{A}$ choose $x_E \in D \cap \bigcap\{G : E \subseteq G \in \mathcal{G}\}$ and $\alpha_E \in \mathbb{Q} \cap [0, \infty[$ such that $|\alpha_E - \mu E| \le \frac{\epsilon}{\#(\mathcal{A})}$. Try $\nu = \sum_{E\in\mathcal{A}} \alpha_E \delta_{x_E} \in D'$. If $G \in \mathcal{G}$, then

$$\begin{aligned} \mu G = \sum_{E\in\mathcal{A}, E\subseteq G} \mu E &\le \sum_{E\in\mathcal{A}, x_E\in G} \mu E \\ &\le \epsilon + \sum_{E\in\mathcal{A}, x_E\in G} \alpha_E = \epsilon + \nu G; \end{aligned}$$

while

$$\nu X = \sum_{E\in\mathcal{A}} \alpha_E \le \epsilon + \sum_{E\in\mathcal{A}} \mu E = \epsilon + \mu X.$$

As μ, $\mathcal{G}$ and ϵ are arbitrary, D' is dense in M_{qR}^+. **Q** So M_{qR}^+ is separable.

(ii) If X is Hausdorff, use the same construction; in this case $D' \subseteq M_{\mathrm{R}}^+$, so M_{R}^+ also is separable.

(d) Most of the argument will be devoted to proving that the set P_{R} of Radon probability measures on X is K-analytic in its narrow topology.

(i) We are supposing that there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = X$ (422F). Set $R_1 = \{(\alpha, x) : \text{there is a } \beta \le \alpha \text{ such that } (\beta, \alpha) \in R\}$; then R_1 also is usco-compact (422Dh).

Set

$$\begin{aligned}\tilde{R} &= \{(\boldsymbol{\alpha}, \mu) : \boldsymbol{\alpha} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}},\ \mu \in P_{\mathrm{R}},\ \mu R_1[\{\boldsymbol{\alpha}(n)\}] \ge 1 - 2^{-n} \text{ for every } n \in \mathbb{N}\} \\ &\subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times P_{\mathrm{R}}.\end{aligned}$$

(Of course $R_1[\{\boldsymbol{\alpha}(n)\}]$ is compact, therefore universally measurable, whenever $\boldsymbol{\alpha} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ and $n \in \mathbb{N}$.)

(ii) $\tilde{R}[(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}] = P_{\mathrm{R}}$. **P** $\mathbb{N}^{\mathbb{N}} \times X$ is K-analytic (422Ge), while R is a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$ (422Da), so is itself K-analytic (422Gf). Let $\pi_1 : R \to \mathbb{N}^{\mathbb{N}}$ and $\pi_2 : R \to X$ be the coordinate maps. If $\mu \in P_{\mathrm{R}}$, there is a Radon probability measure λ on R such that $\mu = \lambda\pi_2^{-1}$ (432G). For each $n \in \mathbb{N}$ let $L_n \subseteq R$ be a compact set such that $\lambda L_n > 1 - 2^{-n}$; then $\pi_1[L_n]$ is a non-empty compact subset of $\mathbb{N}^{\mathbb{N}}$. Define $\boldsymbol{\alpha}$ by setting

$$\boldsymbol{\alpha}(n)(m) = \sup\{\beta(m) : \beta \in \pi_1[L_n]\}$$

for m, $n \in \mathbb{N}$. Then

$$\mu R_1[\{\boldsymbol{\alpha}(n)\}] \ge \mu R[\pi_1[L_n]] \ge \mu\pi_2[L_n] \ge \lambda L_n \ge 1 - 2^{-n}$$

for every $n \in \mathbb{N}$, so $(\boldsymbol{\alpha}, \mu) \in \tilde{R}$ and $\mu \in \tilde{R}[(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}]$. **Q**

(iii) $\tilde{R}[\{\boldsymbol{\alpha}\}]$ is a compact subset of P_{R} for every $\boldsymbol{\alpha} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. **P** Since $R_1[\{\boldsymbol{\alpha}(n)\}]$ is compact for every n, $\tilde{R}[\{\boldsymbol{\alpha}\}]$ is uniformly tight, therefore relatively compact in M_{R}^+, by 437Pb. On the other hand, $\{\mu : \mu \in M_{\mathrm{R}}^+, \mu X = 1\}$ and $\{\mu : \mu \in M_{\mathrm{R}}^+,\ \mu(X \setminus R_1[\{\boldsymbol{\alpha}(n)\}]) \le 1 - 2^{-n}\}$ are closed for every n, so $\tilde{R}[\{\boldsymbol{\alpha}\}]$ is closed, therefore compact. **Q**

(iv) If $F \subseteq P_{\mathrm{R}}$ is closed, then $\tilde{R}^{-1}[F]$ is closed in $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. **P** Let $\langle\boldsymbol{\alpha}_k\rangle_{k\in\mathbb{N}}$ be a sequence in $\tilde{R}^{-1}[F]$ converging to $\boldsymbol{\alpha}$ in $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. For each $k \in \mathbb{N}$ choose $\mu_k \in F$ such that $(\boldsymbol{\alpha}_k, \mu_k) \in \tilde{R}$. For n, $k \in \mathbb{N}$ set $L_{nk} = \{\boldsymbol{\alpha}(n)\} \cup \{\boldsymbol{\alpha}_l(n) : l \ge k\}$. Then L_{nk} is a compact subset of $\mathbb{N}^{\mathbb{N}}$, so $R_1[L_{nk}]$ is a compact subset of X. **?** If $x \in \bigcap_{k\in\mathbb{N}} R_1[L_{nk}] \setminus R_1[\{\boldsymbol{\alpha}(n)\}]$, then for every $k \in \mathbb{N}$ there is an $l_k \ge k$ such that $(\boldsymbol{\alpha}_{l_k}(n), x) \in R_1$; but $R_1^{-1}[\{x\}]$ is closed in $\mathbb{N}^{\mathbb{N}}$, so contains $\lim_{k\to\infty} \boldsymbol{\alpha}_{l_k}(n) = \boldsymbol{\alpha}(n)$, and $x \in R_1[\{\boldsymbol{\alpha}(n)\}]$. **X** Thus $\bigcap_{k\in\mathbb{N}} R_1[L_{nk}] = R_1[\{\boldsymbol{\alpha}(n)\}]$.

For any n and k, $\mu_l R_1[L_{nk}] \ge \mu_l R_1[\{\boldsymbol{\alpha}_l(n)\}] \ge 1 - 2^{-n}$ for every $l \ge k$. In the first place, taking $k = 0$, $\{\mu_l : l \in \mathbb{N}\}$ is uniformly tight, therefore relatively compact and $\langle\mu_l\rangle_{l\in\mathbb{N}}$ has a cluster point μ say, which must belong to F. Now, for any n,

$$\mu R_1[\{\boldsymbol{\alpha}(n)\}] = \inf_{k\in\mathbb{N}} \mu R_1[L_{nk}] \ge \inf_{k\in\mathbb{N}, l\ge k} \mu_l R_1[L_{nk}]$$

(because $R_1[L_{nk}]$ is compact, therefore closed, and $\mu \in \overline{\{\mu_l : l \ge k\}}$, for each k)

$$\ge \inf_{k\in\mathbb{N}, l\ge k} \mu_l R_1[\{\boldsymbol{\alpha}_l(n)\}] \ge 1 - 2^{-n}.$$

So $(\boldsymbol{\alpha}, \mu) \in \tilde{R}$ and $\boldsymbol{\alpha} \in \tilde{R}^{-1}[F]$. As $\langle\boldsymbol{\alpha}_k\rangle_{k\in\mathbb{N}}$ is arbitrary, $\tilde{R}^{-1}[F]$ is closed. **Q**

(v) Thus $\tilde{R} \subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times P_{\mathrm{R}}$ is usco-compact. Since $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, like $\mathbb{N}^{\mathbb{N}}$, is Polish (4A2Ub, 4A2Qc), $\tilde{R}[(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}]$ is K-analytic (422Gd). But we saw in (ii) that $\tilde{R}[(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}] = P_{\mathrm{R}}$. So P_{R} is K-analytic.

(vi) Now observe that $(\alpha, \mu) \mapsto \alpha\mu : [0, \infty[\times P_{\mathrm{R}} \to M_{\mathrm{R}}^+$ is continuous. **P** We have only to note that

$$(\alpha, \mu) \mapsto (\alpha\mu)(X) = \alpha$$

is continuous, and that

$$\{(\alpha,\mu):(\alpha\mu)(G)>\gamma\}=\bigcup_{\beta>0}\{(\alpha,\mu):\alpha>\beta,\,\mu G>\tfrac{\gamma}{\beta}\}$$

is open for every open $G\subseteq X$ and $\gamma\ge 0$. **Q** Since $[0,\infty[$ and P_{R} are K-analytic, and (except in the trivial case $X=\emptyset$) every member of M_{R}^+ is expressible as a non-negative multiple of a probability measure, M_{R}^+ is K-analytic (using 422Ge and 422Gd again).

(vii) Finally, $M_{\mathrm{qR}}^+=M_{\mathrm{R}}^+$ by 432E.

(e) Put (d), (b) and 423C together.

(f)(i) Because X is compact, every quasi-Radon measure on X is tight, and M_{qR}^+ itself is uniformly tight; by 437Pa, $\{\mu:\mu\in M_{\mathrm{qR}}^+,\,\mu X\le\gamma\}$ is relatively compact in M_{qR}^+. But as it is also closed in M_{qR}^+, it is actually compact. The same argument applies to $\{\mu:\mu\in M_{\mathrm{qR}}^+,\,\mu X=\gamma\}$.

(ii) Use the same idea, but with 437Pb in place of 437Pa.

(g)(i) Write $\mathfrak{T}_{\mathrm{KR}}$ for the topology generated by ρ_{KR}.

(α) If $\mu\in M_{\mathrm{qR}}^+$ and $\mu X>\alpha$, then $\nu X>\alpha$ whenever $\nu\in M_{\mathrm{qR}}^+$ and $\rho_{\mathrm{KR}}(\mu,\nu)<\mu X-\alpha$, just because χX is a 1-Lipschitz function; so $\{\mu:\mu\in M_{\mathrm{qR}}^+,\,\mu X>\alpha\}\in\mathfrak{T}_{\mathrm{KR}}$ for every $\alpha\in\mathbb{R}$.

If $G\subseteq X$ is open, $\alpha\ge 0$ and $\mu\in M_{\mathrm{qR}}^+$ is such that $\mu G>\alpha$, there is a $\delta\in\,]0,1]$ such that $\mu F>\alpha+\delta$, where $F=\{x:\rho(x,X\setminus G)\ge\delta\}$. Let u be a 1-Lipschitz function such that $\delta\chi F\le u\le\delta\chi G$. If $\nu\in M_{\mathrm{qR}}^+$ and $\rho_{\mathrm{KR}}(\mu,\nu)\le\delta^2$, then

$$\delta\nu G\ge\int u\,d\nu\ge\int u\,d\mu-\delta^2\ge\delta\mu F-\delta^2>\delta\alpha$$

and $\nu G>\alpha$. This shows that $\{\mu:\mu\in M_{\mathrm{qR}}^+,\,\mu G>\alpha\}\in\mathfrak{T}_{\mathrm{KR}}$. As G and α are arbitrary, $\mathfrak{T}_{\mathrm{KR}}$ is finer than the narrow topology.

(β) Suppose that $\mu\in M_{\mathrm{qR}}^+$ and $\epsilon>0$; let $\delta>0$ be such that $\delta(3\delta+6\mu X+7)\le\epsilon$. Then there is a totally bounded closed set $F\subseteq X$ such that $\mu(X\setminus F)\le\delta$ (434L). Set $G=\{x:\rho(x,F)<\delta\}$. Let $x_0,\dots,x_n\in X$ be such that $F\subseteq\bigcup_{i\le n}B(x_i,\delta)$; then $G\subseteq\bigcup_{i\le n}B(x_i,2\delta)$ and there are $v_0,\dots,v_n\in C_b(X)^+$ such that $\chi G\le\sum_{i=0}^n v_i(x)\le\chi X$ and $\{x:v_i(x)>0\}\subseteq B(x_i,3\delta)$ for every $i\le n$. Let $w\in C_b(X)$ be such that $\chi(X\setminus G)\le w\le\chi(X\setminus F)$. By the choice of F, $\int w\,d\mu\le\delta$.

If $u:X\to[-1,1]$ is 1-Lipschitz then

$$|u-\textstyle\sum_{i=0}^n u(x_i)v_i|\le 3\delta\chi X+2w.$$

P If $x\in G$, then

$$|u(x)-\sum_{i=0}^n u(x_i)v_i(x)|=|\sum_{i=0}^n(u(x)-u(x_i))v_i(x)|$$

(because $\sum_{i=0}^n v_i(x)=1$)

$$\le\sum_{i=0}^n|u(x)-u(x_i)|v_i(x)\le\sum_{i=0}^n 3\delta v_i(x)$$

(because whenever $v_i(x)>0$, $|u(x)-u(x_i)|\le\rho(x,x_i)\le 3\delta$)

$$\le 3\delta.$$

If $x\in X\setminus G$, then

$$|u(x)-\textstyle\sum_{i=0}^n u(x_i)v_i(x)|\le|u(x)|+\sum_{i=0}^n|u(x_i)|v_i(x)\le 2=2w(x).\ \mathbf{Q}$$

So if $\nu\in M_{\mathrm{qR}}^+$,

$$|\textstyle\int u\,d\nu-\sum_{i=0}^n u(x_i)\int v_i d\nu|\le 3\delta\nu X+2\int w\,d\nu.$$

By 437Jf or 437L, there is a neighbourhood V of μ for the narrow topology in M_{qR}^+ such that if $\nu \in V$ then $\nu X \le \mu X + \delta$, $\int w\,d\nu \le 2\delta$ and $|\int v_i d\mu - \int v_i d\nu| \le \frac{\delta}{n+1}$ for every $i \le n$. So if $\nu \in V$ and $u : X \to [-1,1]$ is 1-Lipschitz, we shall have

$$\begin{aligned}
|\int u\,d\nu - \int u\,d\mu| &\le 3\delta(\nu X + \mu X) + 2(\int w\,d\nu + \int w\,d\mu) \\
&\qquad + \sum_{i=0}^{n} |\int v_i d\nu - \int v_i d\mu| \\
&\le 3\delta(2\mu X + \delta) + 6\delta + \sum_{i=0}^{n} \frac{\delta}{n+1} \le \delta(6\mu X + 3\delta + 7) \le \epsilon.
\end{aligned}$$

Thus $\{\nu : \rho_{\mathrm{KR}}(\nu,\mu) \le \epsilon\} \supseteq V$ is a neighbourhood of μ for the narrow topology; as μ and ϵ are arbitrary, the narrow topology is finer than $\mathfrak{T}_{\mathrm{KR}}$, and the two topologies are equal.

(ii) If X is ρ-complete then $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$ by 434Jg and 434Jb. Now suppose that $\langle \mu_n \rangle_{n\in\mathbb{N}}$ is a ρ_{KR}-Cauchy sequence in M_{R}^+.

(α) For every $\epsilon \in \,]0,1]$ there is a compact $K \subseteq X$ such that $\mu_n(X \setminus U(K,\epsilon)) \le \epsilon$ for every $n \in \mathbb{N}$, where $U(K,\epsilon) = \{x : \rho(x,y) < \epsilon$ for some $y \in K\}$. **P** Take $m \in \mathbb{N}$ such that $\rho_{\mathrm{KR}}(\mu_m,\mu_n) \le \frac{1}{2}\epsilon^2$ for every $n \ge m$. Let $K \subseteq X$ be a compact set such that $\mu_n(X \setminus K) \le \frac{1}{2}\epsilon$ for every $n \le m$. Set $G = U(K,\epsilon)$. There is a 1-Lipschitz function $u : X \to [0,\epsilon]$ such that $\epsilon\chi(X \setminus G) \le u \le \chi(X \setminus K)$. If $n \le m$, then of course $\mu_n(X \setminus G) \le \epsilon$. If $n \ge m$, then

$$\begin{aligned}
\mu_n(X \setminus G) &\le \frac{1}{\epsilon}\int u\,d\mu_n \le \frac{1}{\epsilon}\big(\rho_{\mathrm{KR}}(\mu_n,\mu_m) + \int u\,d\mu_m\big) \\
&\le \frac{1}{\epsilon}\big(\frac{\epsilon^2}{2} + \epsilon\mu_m(X \setminus K)\big) \le \frac{\epsilon}{2} + \mu_m(X \setminus K) \le \epsilon.
\end{aligned}$$

So we have an appropriate K. **Q**

(β) $\{\mu_n : n \in \mathbb{N}\}$ is uniformly totally finite and uniformly tight. **P** Since $|\mu_m X - \mu_n X| \le \rho_{\mathrm{KR}}(\mu_m,\mu_n)$ for all m, $n \in \mathbb{N}$, $\{\mu_n X : n \in \mathbb{N}\}$ is bounded. Of course all the μ_n are tight. Now take any $\epsilon \in \,]0,1]$. For each $m \in \mathbb{N}$, (i) tells us that there is a compact set $K_m \subseteq X$ such that $\mu_n(X \setminus U(K_m, 2^{-m}\epsilon)) \le 2^{-m}\epsilon$ for every $n \in \mathbb{N}$. Set $E = \bigcap_{m\in\mathbb{N}} U(K_m, 2^{-m}\epsilon)$, $K = \overline{E}$. Then E and K are totally bounded; because (X,ρ) is complete, K is compact. And

$$\mu_n(X \setminus K) \le \sum_{m=0}^{\infty} \mu_n(X \setminus U(K_m, 2^{-m}\epsilon)) \le 2\epsilon$$

for every $n \in \mathbb{N}$. As ϵ is arbitrary, $\{\mu_n : n \in \mathbb{N}\}$ is uniformly tight. **Q**

(γ) By 437Pb, $\langle \mu_n \rangle_{n\in\mathbb{N}}$ has a cluster point μ in M_{R}^+ for the narrow topology. Now, for $m \in \mathbb{N}$,

$$\begin{aligned}
\rho_{\mathrm{KR}}(\mu,\mu_m) &= \sup\{|\int u\,d\mu - \int u\,d\mu_m| : u : X \to [-1,1] \text{ is 1-Lipschitz}\} \\
&\le \sup\{|\int u\,d\mu_n - \int u\,d\mu_m| : u : X \to [-1,1] \text{ is 1-Lipschitz}, n \ge m\}
\end{aligned}$$

(437Jf again)

$$\le \sup_{n\ge m} \rho_{\mathrm{KR}}(\mu_n,\mu_m),$$

and $\lim_{n\to\infty} \rho_{\mathrm{KR}}(\mu,\mu_m) = 0$.

(δ) Thus every ρ_{KR}-Cauchy sequence in M_{R}^+ has a limit in M_{R}^+, and M_{R}^+ is complete.

(h) Put (g-ii) and (c-ii) together.

437S The sets of measures we have been considering have generally been convex, if addition and multiplication by non-negative scalars are defined as in 234G and 234Xf. We can therefore look for extreme points, in the hope that they will have straightforward characterizations, as in the following.

Proposition Let X be a Hausdorff space, and P_{R} the set of Radon probability measures on X. Then the extreme points of P_{R} are just the Dirac measures on X.

proof (a) Suppose that $x \in X$, and that δ_x is the Dirac measure on X concentrated at x. If μ_1, $\mu_2 \in P_{\mathrm{R}}$ are such that $\delta_x = \frac{1}{2}(\mu_1+\mu_2)$, then we must have $\mu_1 E \le 2\mu E$ for every Borel set E; in particular, $\mu_1(X\setminus\{x\}) = 0$ and $\mu_1\{x\} = 1$, that is, $\mu_1 = \delta_x$. Similarly, $\mu_2 = \delta_x$; as μ_1 and μ_2 are arbitrary, δ_x is an extreme point of P_{R}.

(b) Suppose that μ is an extreme point of P_{R}. Let K be the support of μ. **?** If K has more than one point, take distinct x, $y \in K$. As X is Hausdorff, there are disjoint open sets G, H such that $x \in G$ and $y \in H$. Set $E = G \cap K$, $\alpha = \mu E$. Because K is the support of μ, $\alpha > 0$. But similarly $\mu(H \cap K) > 0$ and $\alpha < 1$. Let μ_1, μ_2 be the indefinite-integral measures defined over μ by $\frac{1}{\alpha}\chi E$ and $\frac{1}{\beta}\chi(X \setminus E)$ respectively. Then both are Radon probability measures on X (416S), so belong to P_{R}. Now $\mu F = \alpha\mu_1 F + (1-\alpha)\mu_2 F$ for every Borel set F; as μ and $\alpha\mu_1 + (1-\alpha)\mu_2$ are both Radon measures, they coincide; as neither μ_1 nor μ_2 is equal to μ, μ is not extreme in P_{R}. **X**

Thus $K = \{x\}$ for some $x \in X$. But this means that $\mu\{x\} = 1$ and $\mu(X \setminus \{x\}) = 0$, so $\mu = \delta_x$ is of the declared form.

437T We now have a language in which to express a fundamental result in the theory of dynamical systems.

Theorem Let X be a non-empty compact Hausdorff space, and $\phi : X \to X$ a continuous function. Write Q_ϕ for the set of Radon probability measures on X for which ϕ is inverse-measure-preserving. Then Q_ϕ is convex and not empty, and is compact for the narrow topology.

proof (a) Write M_{R}^+ for the set of totally finite Radon measures on X, and let $\tilde{\phi} : M_{\mathrm{R}}^+ \to M_{\mathrm{R}}^+$ be the function corresponding to $\phi : X \to X$ as described in 437N. Now, for $\mu \in M_{\mathrm{R}}^+$, $\mu \in Q_\phi$ iff $\mu X = 1$ and $\mu(\phi^{-1}[E]) = \mu E$ whenever μ measures E, that is, iff the image measure $\mu\phi^{-1} = \tilde{\phi}(\mu)$ extends μ. But as $\tilde{\phi}(\mu)$ and μ are Radon measures, $\mu \in Q_\phi$ iff $\mu X = 1$ and $\tilde{\phi}(\mu) = \mu$.

Since $\tilde{\phi}$ is continuous (and M_{R}^+ is Hausdorff, see 437Ra), Q_ϕ is closed for the narrow topology. By 437Pb/437R(f-ii), it is compact. Because $\tilde{\phi}$ respects addition and scalar multiplication, Q_ϕ is convex.

(b) To see that Q_ϕ is not empty, take any $x_0 \in X$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Define $f : C(X) \to \mathbb{R}$ by setting $f(u) = \lim_{n\to\mathcal{F}} \frac{1}{n+1}\sum_{i=0}^{n} u(\phi^i(x_0))$ for every $u \in C(X)$. Then f is a positive linear functional and $f(\chi X) = 1$. So there is a $\mu \in M_{\mathrm{R}}^+$ such that $f(u) = \int u\,d\mu$ for every $u \in C(X)$.

If $u \in C(X)$, then $f(u) = f(u\phi)$. **P**

$$
\begin{aligned}
|f(u\phi) - f(u)| &= |\lim_{n\to\mathcal{F}} \frac{1}{n+1}\sum_{i=0}^{n} u(\phi^{i+1}(x_0)) - u(\phi^i(x_0))| \\
&= |\lim_{n\to\mathcal{F}} \frac{1}{n+1} u(\phi^{n+1}(x_0)) - u(x_0)| \\
&\le \lim_{n\to\mathcal{F}} \frac{1}{n+1}|u(\phi^{n+1}(x_0)) - u(x_0)| \le \lim_{n\to\mathcal{F}} \frac{2\|u\|_\infty}{n+1} = 0. \quad \mathbf{Q}
\end{aligned}
$$

On the other hand,

$$\int u\,d(\tilde{\phi}(\mu)) = \int u\phi\,d\mu$$

(235G)

$$= f(u\phi) = f(u) = \int u\,d\mu$$

for every $u \in C(X)$. By the uniqueness of the representation of f as an integral, $\mu = \tilde{\phi}(\mu)$. Of course $\mu X = f(\chi X) = 1$ so $\mu \in Q_\phi$, as required.

437U In important cases, the narrowly compact subsets of $M_{\mathrm{R}}^+(X)$ are exactly the bounded uniformly tight sets. Once again, it is worth introducing a word to describe when this happens.

Definition Let X be a Hausdorff space and $P_{\mathrm{R}}(X)$ the set of Radon probability measures on X. X is a **Prokhorov space** if every subset of $P_{\mathrm{R}}(X)$ which is compact for the narrow topology is uniformly tight.

437V Theorem (a) Compact Hausdorff spaces are Prokhorov spaces.
(b) A closed subspace of a Prokhorov Hausdorff space is a Prokhorov space.
(c) An open subspace of a Prokhorov Hausdorff space is a Prokhorov space.
(d) The product of a countable family of Prokhorov Hausdorff spaces is a Prokhorov space.
(e) Any G_δ subset of a Prokhorov Hausdorff space is a Prokhorov space.
(f) Čech-complete spaces are Prokhorov spaces.
(g) Polish spaces are Prokhorov spaces.

proof (a) This is trivial; on a compact Hausdorff space the set of all Radon probability measures is uniformly tight.

(b) Let X be a Prokhorov Hausdorff space, Y a closed subset of X, and $A \subseteq P_{\mathrm{R}}(Y)$ a narrowly compact set. Taking ϕ to be the identity map from Y to X, and defining $\tilde{\phi} : M_{\mathrm{R}}^+(Y) \to M_{\mathrm{R}}^+(X)$ as in 437N, $\tilde{\phi}[A]$ is narrowly compact in $P_{\mathrm{R}}(X)$, so is uniformly tight. For any $\epsilon > 0$, there is a compact set $K \subseteq X$ such that $\tilde{\phi}(\mu)(X \setminus K) \le \epsilon$ for every $\mu \in A$. Now $K \cap Y$ is a compact subset of Y and $\mu(Y \setminus (K \cap Y)) \le \epsilon$ for every $\mu \in A$. As ϵ is arbitrary, A is uniformly tight in $P_{\mathrm{R}}(Y)$.

(c) Let X be a Prokhorov Hausdorff space, Y an open subset of X, and $A \subseteq P_{\mathrm{R}}(Y)$ a narrowly compact set. Once again, take ϕ to be the identity map from Y to X, so that $\tilde{\phi}[A] \subseteq P_{\mathrm{R}}(X)$ is narrowly compact and uniformly tight in $P_{\mathrm{R}}(X)$.

? Suppose, if possible, that A is not uniformly tight in $P_{\mathrm{R}}(Y)$. Then there is an $\epsilon > 0$ such that $A_K = \{\mu : \mu \in A,\ \mu(Y \setminus K) \ge 5\epsilon\}$ is non-empty for every compact set $K \subseteq Y$. Note that $A_K \subseteq A_{K'}$ whenever $K \supseteq K'$, so $\{A_K : K \subseteq Y \text{ is compact}\}$ has the finite intersection property, and there is an ultrafilter $\mathcal{F}$ on $P_{\mathrm{R}}(Y)$ containing every A_K. Because A is narrowly compact, there is a $\lambda \in P_{\mathrm{R}}(Y)$ such that $\mathcal{F} \to \lambda$. Let $K^* \subseteq Y$ be a compact set such that $\lambda(Y \setminus K^*) \le \epsilon$.

As $\tilde{\phi}[A]$ is uniformly tight, there is a compact set $L \subseteq X$ such that $\mu(Y \setminus L) = \tilde{\phi}(\mu)(X \setminus L) \le \epsilon$ for every $\mu \in A$. Now K^* and $L \setminus Y$ are disjoint compact sets in the Hausdorff space X, so there are disjoint open sets G, $H \subseteq X$ such that $K^* \subseteq G$ and $L \setminus Y \subseteq H$ (4A2F(h-i) again). Set $K = L \setminus H \supseteq L \cap G$; then K is a compact subset of Y. As $A_K \in \mathcal{F}$, there must be a $\mu \in A_K$ such that $\mu Y \le \lambda Y + \epsilon$ and $\mu(G \cap Y) \ge \lambda(G \cap Y) - \epsilon$. Accordingly

$$
\begin{aligned}
\mu(Y \setminus L) &\le \epsilon, \\
\mu(Y \setminus G) &= \mu Y - \mu(G \cap Y) \le \lambda Y + \epsilon - \lambda(G \cap Y) + \epsilon \\
&= \lambda(Y \setminus G) + 2\epsilon \le \lambda(Y \setminus K^*) + 2\epsilon \le 3\epsilon, \\
\mu((Y \setminus L) \cup (Y \setminus G)) &= \mu(Y \setminus (L \cap G)) \ge \mu(Y \setminus K) \ge 5\epsilon,
\end{aligned}
$$

which is impossible. **X**

Thus A is uniformly tight. As A is arbitrary, Y is a Prokhorov space.

(d) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence of Prokhorov Hausdorff spaces with product X. Let $A \subseteq P_{\mathrm{R}}(X)$ be a narrowly compact set. Let $\epsilon > 0$. For each $n \in \mathbb{N}$ let $\pi_n : X \to X_n$ be the canonical map and $\tilde{\pi}_n : M_{\mathrm{R}}^+(X) \to M_{\mathrm{R}}^+(X_n)$ the associated function. Then $\tilde{\pi}_n[A]$ is narrowly compact in $P_{\mathrm{R}}(X_n)$, therefore uniformly tight, and there is a compact set $K_n \subseteq X_n$ such that $(\tilde{\pi}_n \mu)(X_n \setminus K_n) \le 2^{-n-1}\epsilon$ for every $\mu \in A$. Set $K = \prod_{n \in \mathbb{N}} K_n$, so that K is a compact subset of X and $X \setminus K = \bigcup_{n \in \mathbb{N}} \pi_n^{-1}[X_n \setminus K_n]$. If $\mu \in A$, then

$$\mu(X \setminus K) \le \sum_{n=0}^{\infty} \mu \pi_n^{-1}[X_n \setminus K_n] \le \sum_{n=0}^{\infty} 2^{-n-1}\epsilon = \epsilon.$$

As ϵ is arbitrary, A is uniformly tight; as A is arbitrary, X is a Prokhorov space.

(e) Let X be a Prokhorov Hausdorff space and Y a G_δ subset of X. Express Y as $\bigcap_{n\in\mathbb{N}} Y_n$ where every $Y_n \subseteq X$ is open. Set $Z = \{z : z \in \prod_{n\in\mathbb{N}} Y_n,\ z(m) = z(n) \text{ for all } m, n \in \mathbb{N}\}$. Because X is Hausdorff, Z is a closed subspace of $\prod_{n\in\mathbb{N}} Y_n$ homeomorphic to Y. Putting (c), (d) and (b) together, Z and Y are Prokhorov spaces.

(f) Put (a), (e) and the definition of 'Čech-complete' together.

(g) This is a special case of (f) (4A2Md).

437X Basic exercises (a) Let X be a set, U a Riesz subspace of $\mathbb{R}^X$ and $f \in U^\sim$. (i) Show that $f \in U^\sim_\sigma$ iff $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$. (*Hint*: show that in this case, if $0 \le v_n \le u_n$, we can find $k(n)$ such that $|f(v_n \vee u_{k(n)}) - f(v_n)| \le 2^{-n}$.) (ii) Show that $f \in U^\sim_\tau$ iff $\inf_{u\in A} |f(u)| = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$. (*Hint*: given $\epsilon > 0$, set $B = \{v : f(v) \ge \inf_{u\in A} f^+(u) - \epsilon,\ \exists\, w \in A,\ v \ge w\}$ and show that B is a downwards-directed set with infimum 0 in $\mathbb{R}^X$.)

(b) Let X be a set, U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions, and Σ the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Let μ and ν be two totally finite measures on X with domain Σ, and f, g the corresponding linear functionals on U. Show that $f \wedge g = 0$ in $U^\sim$ iff there is an $E \in \Sigma$ such that $\mu E = \nu(X \setminus E) = 0$. (*Hint*: 326M[6].)

>(c) Let $I \subseteq \mathbb{R}$ be a interval with at least two points. (i) Show that if $g : I \to \mathbb{R}$ is of bounded variation on every compact subinterval of I, there is a unique signed tight Borel measure μ_g on I such that $\mu_g[a,b] = \lim_{x\downarrow b} g(x) - \lim_{x\uparrow a} g(x)$ whenever $a \le b$ in I, counting $\lim_{x\uparrow a} g(x)$ as $g(a)$ if $a = \min I$, and $\lim_{x\downarrow b} g(x)$ as $g(b)$ if $b = \max I$. (ii) Show that if $h : I \to \mathbb{R}$ is another function of bounded variation on every compact subinterval, then $\mu_h = \mu_g$ iff $\{x : h(x) \ne g(x)\}$ is countable iff $\{x : h(x) = g(x)\}$ is dense in I. (iii) Show that if ν is any signed Baire measure on I there is a g of bounded variation on every compact subinterval such that $\nu = \mu_g$.

(d)(i) Show that S, in 437C, is the unique sequentially order-continuous positive linear operator from $\mathcal{L}^\infty$ to $(U^\sim_\sigma)^*$ which extends the canonical embedding of U in $(U^\sim_\sigma)^*$. (ii) Show that S, in 437H, is the unique sequentially order-continuous positive linear operator from $\mathcal{L}^\infty$ to $(U^\sim_\tau)^*$ which extends the canonical embedding of U in $(U^\sim_\tau)^*$ and is 'τ-additive' in the sense that whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H then $S(\chi H) = \sup_{G\in\mathcal{G}} S(\chi G)$ in $(U^\sim_\tau)^*$.

(e) Let X and Y be completely regular topological spaces and $\phi : X \to Y$ a continuous function. Define $T : C_b(Y) \to C_b(X)$ by setting $T(v) = v\phi$ for every $v \in C_b(Y)$, and let $T' : C_b(X)^* \to C_b(Y)^*$ be its adjoint. (i) Show that T' is a norm-preserving Riesz homomorphism. (ii) Show that $T'[C_b(X)^\sim_\sigma] \subseteq C_b(Y)^\sim_\sigma$, and that if $f \in C_b(X)^\sim_\sigma$ corresponds to a Baire measure μ on X, then $T'f$ corresponds to the Baire measure $\mu\phi^{-1}\restriction\mathcal{B}\mathfrak{a}(Y)$. (iii) Show that $T'[C_b(X)^\sim_\tau] \subseteq C_b(Y)^\sim_\tau$, and that if $f \in C_b(X)^\sim_\tau$ corresponds to a Borel measure μ on X, then $T'f$ corresponds to the Borel measure $\mu\phi^{-1}\restriction\mathcal{B}(Y)$. (iv) Write $\mathcal{L}^\infty_X$ and $\mathcal{L}^\infty_Y$ for the M-spaces of bounded real-valued Borel measurable functions on X, Y respectively, and $S_X : \mathcal{L}^\infty_X \to (C_b(X)^\sim_\tau)^*$, $S_Y : \mathcal{L}^\infty_Y \to (C_b(Y)^\sim_\tau)^*$ for the canonical Riesz homomorphisms as constructed in 437Hb. Show that if $T'' : (C_b(Y)^\sim_\tau)^* \to (C_b(X)^\sim_\tau)^*$ is the adjoint of $T'\restriction C_b(X)^\sim_\tau$, then $T''S_Y(v) = S_X(v\phi)$ for every $v \in \mathcal{L}^\infty_Y$.

(f) Let X be a topological space, $\mathcal{L}^\infty(\Sigma_{\mathrm{um}})$ the space of bounded universally measurable real-valued functions on X, and M_σ the space of countably additive functionals on the Borel σ-algebra of X. Show that we have a sequentially order-continuous Riesz homomorphism $S : \mathcal{L}^\infty(\Sigma_{\mathrm{um}}) \to M^*_\sigma$ defined by the formula $(Sv)(\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty(\Sigma_{\mathrm{um}})$ and $\mu \in M^+_\sigma$.

(g) Let X be a completely regular topological space. Show that the vague topology on the space M_τ of differences of τ-additive totally finite Borel measures on X is Hausdorff.

>(h) Let X and Y be topological spaces, and $\phi : X \to Y$ a continuous function. Write $M_\#(X)$ for any of $M(\mathcal{B}\mathfrak{a}(X))$, $M_\sigma(\mathcal{B}\mathfrak{a}(X))$, $M(\mathcal{B}(X))$, $M_\sigma(\mathcal{B}(X))$, $M_\tau(X)$ or $M_t(X)$, where $M_\tau(X) \subseteq M_\sigma(\mathcal{B}(X))$ is

[6] Formerly 326I.

the space of signed τ-additive Borel measures and $M_t(X) \subseteq M_\tau(X)$ is the space of signed tight Borel measures; and $M_\#(Y)$ for the corresponding space based on Y. Show that there is a positive linear operator $\tilde{\phi} : M_\#(X) \to M_\#(Y)$ defined by saying that $\tilde{\phi}(\mu)(E) = \mu\phi^{-1}[E]$ whenever $\mu \in M_\#(X)$ and E belongs to $\mathcal{B}\mathfrak{a}(Y)$ or $\mathcal{B}(Y)$, as appropriate, and that $\tilde{\phi}$ is continuous for the vague topologies on $M_\#(X)$ and $M_\#(Y)$.

>(i) Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. (i) Show that $\mathcal{E}$ separates zero sets. (ii) Show that the vague topology on $M(\mathcal{E})$ is just the topology of pointwise convergence induced by the usual topology of $\mathbb{R}^{\mathcal{E}}$. (iii) Writing M_t for the space of signed tight Borel measures on X, show that $\mu \mapsto \mu\restriction\mathcal{E} : M_t \to M(\mathcal{E})$ is a Banach lattice isomorphism between the L-spaces M_t and $M(\mathcal{E})$, and is also a homeomorphism when M_t and $M(\mathcal{E})$ are given their vague topologies.

(j) (i) Let X be a topological space, and Σ an algebra of subsets of X containing every open set; let $M(\Sigma)^+$ be the set of non-negative real-valued additive functionals on Σ, endowed with its narrow topology, E a member of Σ, and ∂E its boundary. Show that $\nu \mapsto \nu E : M(\Sigma)^+ \to [0, \infty[$ is continuous at $\nu_0 \in M(\Sigma)^+$ iff $\nu_0(\partial E) = 0$. (ii) Let X be a completely regular topological space, and Σ a σ-algebra of subsets of X including the Baire σ-algebra. Write M_σ for the L-space of countably additive functionals on Σ. Let $\mathcal{F}$ be a filter on the positive cone M_σ^+ and μ a member of M_σ^+. Show that $\mathcal{F} \to \mu$ for the vague topology on M_σ iff $\mu E = \lim_{\nu\to\mathcal{F}} \nu E$ whenever $E \in \Sigma$ and $\mu(\partial E) = 0$.

(k) Let X be a compact Hausdorff space, M_{R}^+ the set of Radon measures on X and P_{R} the set of Radon probability measures on X. (i) Show that M_{R}^+, with its narrow topology and its natural convex structure, can be identified with the positive cone of $C(X)^*$ with its weak* topology. (ii) Show that P_{R}, with its narrow topology and its natural convex structure, can be identified with $\{f : f \in C(X)^*,\ f \ge 0,\ f(\chi X) = 1\}$ with its weak* topology.

(l) In 437Mc, show that $|\psi(\mu, \nu)| = \psi(|\mu|, |\nu|)$ for every $\mu \in M_\tau(X)$ and $\nu \in M_\tau(Y)$.

(m) Let X be a topological space, Y a regular topological space and $M_{\mathrm{qR}}^+(X)$, $M_{\mathrm{qR}}^+(Y)$ the spaces of totally finite quasi-Radon measures on X, Y respectively. For a continuous $\phi : X \to Y$ define $\tilde{\phi} : M_{\mathrm{qR}}^+(X) \to M_{\mathrm{qR}}^+(Y)$ by saying that ϕ is inverse-measure-preserving for μ and $\tilde{\phi}(\mu)$ for every $\mu \in M_{\mathrm{qR}}^+(X)$ (418Hb). Show that $\tilde{\phi}$ is continuous for the narrow topologies.

>(n) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of Radon probability spaces, and Q the set of Radon probability measures μ on $X = \prod_{i\in I} X_i$ such that the image of μ under the map $x \mapsto x(i)$ is μ_i for every $i \in I$. Show that Q is uniformly tight and is compact for the narrow topology on the set of totally finite topological measures on X.

(o) Let X be any topological space, and M_{qR}^+ the space of totally finite quasi-Radon measures on X. Show that M_{qR}^+ is complete in the total variation metric.

(p) Let X and Y be topological spaces, and $\rho_{\mathrm{tv}}^{(X)}$, $\rho_{\mathrm{tv}}^{(Y)}$, $\rho_{\mathrm{tv}}^{(X\times Y)}$ the total variation metrics on the spaces $M_{\mathrm{qR}}^+(X)$, $M_{\mathrm{qR}}^+(Y)$ and $M_{\mathrm{qR}}^+(X \times Y)$ of quasi-Radon measures. Let μ_1, μ_2 be totally finite quasi-Radon measures on X, ν_1, ν_2 totally finite quasi-Radon measures on Y, and $\mu_1 \times \nu_1$, $\mu_2 \times \nu_2$ the quasi-Radon product measures. Show that

$$\rho_{\mathrm{tv}}^{(X\times Y)}(\mu_1 \times \nu_1, \mu_2 \times \nu_2) \le \rho_{\mathrm{tv}}^{(X)}(\mu_1, \mu_2) \cdot \nu_2 Y + \mu_1 X \cdot \rho_{\mathrm{tv}}^{(Y)}(\nu_1, \nu_2).$$

(q)(i) Show that the set $M_\sigma^+(\mathcal{B}(X))$ of totally finite Borel probability measures on X is T_0 in its narrow topology for any topological space X. (ii) Give $X = \omega_1 + 1$ its order topology. Show that the narrow topology on $M_\sigma^+(\mathcal{B}(X))$ is not T_1. (*Hint*: consider interpretations of Dieudonné's measure on ω_1 and the Dirac measure concentrated at ω_1 as Borel measures on X.)

(r) Let X be any topological space and $\tilde{M}^+$ the set of non-negative additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. For μ, $\nu \in \tilde{M}^+$ define $\mu + \nu \in \tilde{M}^+$ by setting $(\mu + \nu)(E) = \mu E + \nu E$ for $E \in \operatorname{dom}\mu \cap \operatorname{dom}\nu$. (i) Show that addition on $\tilde{M}^+$ is continuous for the narrow topology. (ii)

Show that $(\alpha,\mu)\mapsto\alpha\mu:[0,\infty[\,\times\tilde M^+\to\tilde M^+$ is continuous for the narrow topology on $\tilde M^+$. (iii) Writing $\tilde P$ for $\{\mu:\mu\in\tilde M^+,\,\mu X=1\}$, and δ_x for the Dirac measure concentrated at x for each $x\in X$, show that the convex hull of $\{\delta_x:x\in X\}$ is dense in $\tilde P$ for the narrow topology. (iv) Suppose that A and B are uniformly tight subsets of $\tilde M^+$ and $\gamma\ge 0$. Show that $A\cup B$, $A+B=\{\mu+\nu:\mu\in A,\,\nu\in B\}$ and $\{\alpha\mu:\mu\in A,\,0\le\alpha\le\gamma\}$ are uniformly tight.

(s) Let (X,ρ) be a metric space, Σ a subalgebra of $\mathcal{P}X$ containing all the open sets, and $M=M(\Sigma)$ the set of bounded finitely additive functionals on Σ. For μ, $\nu\in M$ set

$$\rho_{\mathrm{KR}}(\mu,\nu)=\sup\{|⨍u\,d\mu-⨍u\,d\nu|:u:X\to[-1,1]\text{ is 1-Lipschitz}\},$$

$$\rho_{\mathrm{LP}}(\mu,\nu)=\inf\{\epsilon:\epsilon>0,\,\nu F-\mu U(F;\epsilon)\le\epsilon\text{ and }\mu F-\nu U(F;\epsilon)\le\epsilon$$
$$\text{for every non-empty closed }F\subseteq X\},$$

where $U(F;\epsilon)=\{x:\rho(x,F)<\epsilon\}$ for non-empty subsets F of X. (i) Show that ρ_{KR} and ρ_{LP} are pseudometrics on M. (ii) Show that if μ, $\nu\in M$ and $\rho_{\mathrm{LP}}(\mu,\nu)=\delta$ then $\min(1,\delta^2)\le\rho_{\mathrm{KR}}(\mu,\nu)\le 2\delta(1+\delta+|\nu|X)$. (iii) Show that ρ_{KR} and ρ_{LP} induce the same topology on M and the same uniformity on $\{\mu:\mu\in M,\,|\mu|X\le\gamma\}$, for any $\gamma\ge 0$. (*Hint*: Bogachev 07, 8.10.43. ρ_{LP} is the **Lévy-Prokhorov** pseudometric.)

(t) Let X be a Hausdorff space and P'_{R} the set of Radon measures μ on X such that $\mu X\le 1$, with its narrow topology. (i) Show that the extreme points of P'_{R} are the Dirac measures on X, as in 437S, together with the zero measure. (ii) For $x\in X$ let δ_x be the Dirac measure on X concentrated at x. Show that $x\mapsto\delta_x$ is a homeomorphism between X and its image in P'_{R}.

(u) Let X be a non-empty compact metrizable space, and $\phi:X\to X$ a continuous function. Show that there is an $x\in X$ such that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n u(\phi^i(x))$ is defined for every $u\in C(X)$. (*Hint*: 372H[7], 4A2Pe.)

(v) Let X be a T_0 topological space, and P_{qR} the set of quasi-Radon probability measures on X. Show that the extreme points of P_{qR} are just the Dirac measures on X concentrated on points x such that $\{x\}$ is closed.

>(w) Let X be a Prokhorov Hausdorff space, and A a set of totally finite Radon measures on X which is compact for the narrow topology. Show that A is uniformly tight. (*Hint*: (i) $\gamma=\sup_{\mu\in A}\mu X$ is finite; (ii) for any $\epsilon>0$ the set $\{\frac{1}{\mu X}\mu:\mu\in A,\,\mu X\ge\epsilon\}$ is narrowly compact, therefore uniformly tight; for any $\epsilon>0$ the set $\{\mu:\mu\in A,\,\mu X\ge\epsilon\}$ is uniformly tight.)

(x) Give ω_1 its order topology, and let M_t be the L-space of signed tight Borel measures on ω_1. (i) Show that ω_1 is a Prokhorov space. (ii) For $\xi<\omega_1$, define $\mu_\xi\in M_t$ by setting $\mu_\xi(E)=\chi E(\xi)-\chi E(\xi+1)$ for every Borel set $E\subseteq\omega_1$. Show that $\{\mu_\xi:\xi<\omega_1\}$ is relatively compact in M_t for the vague topology, but is not uniformly tight. (Compare 437Yy below.)

(y) Let X and Y be analytic spaces, and $P=P_{\mathrm{R}}(X)$ the space of Radon probability measures on X with its narrow topology. (i) Let V be an analytic subset of $(P\times Y)\times X$. Show that $\{(\mu,y):\mu\in P,\,y\in Y,\,\mu V[\{(\mu,y)\}]>\alpha\}$ is analytic for every $\alpha\in\mathbb{R}$. (*Hint*: start with $V=C\times F\times E$ where C, F and E are closed, recalling 437Re and 437Jd, and apply 431Db; compare 431Yb.) (ii) Let W be a coanalytic subset of $(P\times Y)\times X$. Show that $\{(\mu,y):\mu\in P,\,y\in Y,\,W[\{(\mu,y)\}]\text{ is not }\mu\text{-negligible}\}$ is coanalytic.

(z) Let X be a second-countable topological space, $P=P_\sigma(X)$ the space of topological probability measures on X with its narrow topology and $\mathcal{C}$ the space of closed subsets of X with the Fell topology. For $\mu\in P$ write $\operatorname{supp}\mu$ for the support of μ. (i) Show that $\{(x,C):C\in\mathcal{C},\,x\in C\}$ is a Borel subset of $X\times\mathcal{C}$. (ii) Show that $\{(\mu,x):\mu\in P,\,x\in\operatorname{supp}\mu\}$ is a Borel subset of $P\times X$. (iii) Show that $\mu\mapsto\operatorname{supp}\mu:P\to\mathcal{C}$ is Borel measurable. (Cf. 424Ya.)

[7] Formerly 372I.

437Y Further exercises (a) Let X be a set and U a Riesz subspace of $\mathbb{R}^X$. Give formulae for the components of a given element of $U^\sim$ in the bands $U^\sim_\sigma$, $(U^\sim_\sigma)^\perp$, $U^\sim_\tau$ and $(U^\sim_\tau)^\perp$. (*Hint*: 356Yb.)

(b) Let X be a compact Hausdorff space. Show that the dual $C(X;\mathbb{C})^*$ of the complex linear space of continuous functions from X to $\mathbb{C}$ can be identified with the space of 'complex tight Borel measures' on X, that is, the space of functionals $\mu : \mathcal{B}(X) \to \mathbb{C}$ expressible as a complex linear combination of tight totally finite Borel measures; explain how this may be identified, as Banach space, with the complexification of the L-space M_t of signed tight Borel measures as described in 354Yl. Show that the complex Banach space $\mathcal{L}^\infty_{\mathbb{C}}(\mathcal{B}(X))$ is canonically embedded in $C(X;\mathbb{C})^{**}$.

(c) Write μ_c for counting measure on $[0,1]$, and μ_L for Lebesgue measure; write $\mu_c \times \mu_L$ for the product measure on $[0,1]^2$, and μ for the direct sum of μ_c and $\mu_c \times \mu_L$. Show that the L-space $C([0,1])^\sim$ is isomorphic, as L-space, to $L^1(\mu)$. (*Hint*: every Radon measure on $[0,1]$ has countable Maharam type.)

(d) Let X be a set, U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions, and Σ the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Write $\tilde\Sigma$ for the intersection of the domains of the completions of the totally finite measures with domain Σ. Show that there is a unique sequentially order-continuous norm-preserving Riesz homomorphism from $\mathcal{L}^\infty(\tilde\Sigma)$ to $(U^\sim_\sigma)^* \cong M^*_\sigma$ such that $(Su)(f) = f(u)$ whenever $u \in U$ and $f \in U^\sim_\sigma$.

(e) Show that in 437Ib the operator $S : \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}}) \to C_0(X)^{**}$ is multiplicative if $C_0(X)^{**}$ is given the Arens multiplication described in 4A6O based on the ordinary multiplication $(u,v) \mapsto u \times v$ on $C_0(X)$.

(f) Explain how to express the proof of 285L(iii)$\Rightarrow$(ii) as (α) a proof that if the characteristic functions of a sequence $\langle \nu_n \rangle_{n\in\mathbb{N}}$ of Radon probability measures on $\mathbb{R}^r$ converge pointwise to a characteristic function, then $\{\nu_n : n \in \mathbb{N}\}$ is uniformly tight (β) the observation that any subalgebra of $C_b(\mathbb{R}^r)$ which separates the points of $\mathbb{R}^r$ and contains the constant functions will define the vague topology on any vaguely compact set of measures.

(g) Let X be a topological space. (i) Let $\mathcal{B}a$ be the Baire σ-algebra of X, $M_\sigma(\mathcal{B}a)$ the space of signed Baire measures on X, and $u : X \to \mathbb{R}$ a bounded Baire measurable function. Show that we have a linear functional $\mu \mapsto \fint u\,d\mu : M_\sigma(\mathcal{B}\mathfrak{a}) \to \mathbb{R}$ agreeing with ordinary integration with respect to non-negative measures. Show that this functional is Baire measurable with respect to the vague topology on $M_\sigma(\mathcal{B}\mathfrak{a})$. (ii) Let M_τ be the space of signed τ-additive Borel measures on X, and $u : X \to \mathbb{R}$ a bounded Borel measurable function. Show that we have a linear functional $\mu \mapsto \fint u\,d\mu : M_\tau \to \mathbb{R}$ agreeing with ordinary integration with respect to non-negative measures. Show that this functional is Borel measurable with respect to the vague topology on M_τ.

(h) Let X be a topological space, μ_0 a totally finite τ-additive topological measure on X, and $u : X \to \mathbb{R}$ a bounded function which is continuous μ_0-a.e. Let $\tilde M^+_\sigma$ be the set of totally finite topological measures on X, with its narrow topology. Show that $\nu \mapsto \overline{\int} u\,d\nu : \tilde M^+_\sigma \to \mathbb{R}$ is continuous at μ_0.

(i) Let X be a Hausdorff space, and $M^{\infty+}_{\mathrm{R}}$ the set of all Radon measures on X. Define addition and scalar multiplication (by positive scalars) on $M^{\infty+}_{\mathrm{R}}$ as in 234G, 234Xf and 416De and $\le$ by the formulae of 234P or 416Ea. (i) Show that $M^{\infty+}_{\mathrm{R}}$ is a Dedekind complete lattice. (ii) Show that if $A \subseteq M^{\infty+}_{\mathrm{R}}$ is upwards-directed and non-empty, it is bounded above iff $\{G : G \subseteq X \text{ is open}, \sup_{\nu\in A} \nu G < \infty\}$ covers X, and in this case $\mathrm{dom}(\sup A) = \bigcap_{\nu\in A} \mathrm{dom}\,\nu$ and $(\sup A)(E) = \sup_{\nu\in A} \nu E$ for every $E \in \mathrm{dom}(\sup A)$. (iii) Show that if μ, $\nu \in M^{\infty+}_{\mathrm{R}}$ then $\nu = \sup_{n\in\mathbb{N}} \nu \wedge n\mu$ iff every μ-negligible set is ν-negligible. (iv) Show that if μ, $\nu \in M^{\infty+}_{\mathrm{R}}$ then ν is uniquely expressible as $\nu_s + \nu_{ac}$ where ν_s, $\nu_{ac} \in\in M^{\infty+}_{\mathrm{R}}$, $\mu \wedge \nu_s = 0$ and $\nu_{ac} = \sup_{n\in\mathbb{N}} \nu_{ac} \wedge n\mu$. (v) Show that if μ, $\nu \in M^{\infty+}_{\mathrm{R}}$ then $\mathrm{dom}(\mu\vee\nu) = \mathrm{dom}\,\mu \cap \mathrm{dom}\,\nu$ and $(\mu\vee\nu)(E) = \sup\{\mu F + \nu(E\setminus F) : F \in \mathrm{dom}\,\mu\cap\mathrm{dom}\,\nu, F \subseteq E\}$ for every $E \in \mathrm{dom}(\mu \vee \nu)$. (vi) Show that if μ, $\nu \in M^{\infty+}_{\mathrm{R}}$ then $\mathrm{dom}(\mu \wedge \nu) = \{E \cup F : E \in \mathrm{dom}\,\mu, F \in \mathrm{dom}\,\nu\}$. (vii) Show that if μ, $\nu \in M^{\infty+}_{\mathrm{R}}$ then $\mu \wedge \nu = 0$ iff there is a set $E \subseteq X$ which is μ-negligible and ν-conegligible. (viii) Show that there is a Dedekind complete Riesz space V such that the positive cone of V is isomorphic to $M^{\infty+}_{\mathrm{R}}$.

(j) For a topological space X let $M_\tau(X)$ be the L-space of signed τ-additive Borel measures on X, and $\psi : M_\tau(X) \times M_\tau(X) \to M_\tau(X \times X)$ the canonical bilinear operator (437M); give $M_\tau(X)$ and $M_\tau(X \times X)$ their vague topologies. (i) Show that if $X = [0,1]$ then ψ is not continuous. (ii) Show that $X = \mathbb{N}$ and B is the unit ball of $M_\tau(X)$ then $\psi{\restriction}B \times B$ is not continuous.

(k) Let X and Y be topological spaces, and $\psi : M_\tau(X)\times M_\tau(Y) \to M_\tau(X\times Y)$ the bilinear map of 437Mc. Write $M_t(X)$, etc., for the spaces of signed tight Borel measures. (i) Show that $\psi(\mu,\nu) \in M_t(X \times Y)$ for every $\mu \in M_t(X)$, $\nu \in M_t(Y)$. (ii) Show that if $B \subseteq M_t(X)$, $B' \subseteq M_t(Y)$ are norm-bounded and uniformly tight, then $\psi{\restriction}B \times B'$ is continuous for the vague topologies.

(l) Let X be a topological space, and $\tilde M$ the space of bounded additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. For $\nu \in \tilde M$, say that $|\nu|(E) = \sup\{\nu F - \nu(E \setminus F) : F \in \operatorname{dom}\nu,\ F \subseteq E\}$ for $E \in \operatorname{dom}\nu$. Show that a set $A \subseteq \tilde M$ is uniformly tight in the sense of 437O iff every member of A is tight and $\{|\nu| : \nu \in A\}$ is uniformly tight.

(m) Let X be a completely regular space and P_{qR} the space of quasi-Radon probability measures on X. Let $B \subseteq P_{\mathrm{qR}}$ be a non-empty set. Show that the following are equiveridical: (i) B is relatively compact in P_{qR} for the narrow topology; (ii) whenever $A \subseteq C_b(X)$ is non-empty and downwards-directed and $\inf_{u\in A} u(x) = 0$ for every $x \in A$, then $\inf_{u\in A}\sup_{\mu\in B}\int u\,d\mu = 0$; (iii) whenever $\mathcal{G}$ is an upwards-directed family of open sets with union X, then $\sup_{G\in\mathcal{G}}\inf_{\mu\in B}\mu G = 1$.

(n) (i) Let X be a regular topological space, and M^+_{qR} the space of totally finite quasi-Radon measures on X, with its narrow topology. Show that M^+_{qR} is regular. (ii) Find a second-countable Hausdorff space X such that the space P_{qR} of quasi-Radon probability measures on X is not Hausdorff in its narrow topology.

(o) Let (X,ρ) and (Y,σ) be metric spaces, and give $M^+_{\mathrm{qR}}(X)$ and $M^+_{\mathrm{qR}}(Y)$ the corresponding metrics ρ_{KR}, σ_{KR} as in 437Rg. For a continuous function $\phi : X \to Y$, let $\tilde\phi : M^+_{\mathrm{qR}}(X) \to M^+_{\mathrm{qR}}(Y)$ be the map described in 437Xm. (i) Show that if ϕ is γ-Lipschitz, where $\gamma \ge 1$, then $\tilde\phi$ is γ-Lipschitz. (ii) (J.Pachl) Show that if ϕ is uniformly continuous, then $\tilde\phi$ is uniformly continuous on any uniformly totally finite subset of $M^+_{\mathrm{qR}}(X)$. (iii) Show that if (X,ρ) is $\mathbb{R}$ with its usual metric, then ρ_{KR} is not uniformly equivalent to Lévy's metric as described in 274Yc[8]. (For a discussion of various metrics related to ρ_{KR}, see Bogachev 07, 8.10.43-8.10.48.)

(p) Let (X,ρ) be a metric space. For $f \in C_b(X)^\sim_\sigma$, set $\|f\|_{\mathrm{KR}} = \sup\{|f(u)| : u \in C_b(X),\ \|u\|_\infty \le 1,\ u$ is 1-Lipschitz$\}$. (i) Show that $\|\ \|_{\mathrm{KR}}$ is a norm on $C_b(X)^\sim_\sigma$. (ii) Let (X',ρ') and (X'',ρ'') be metric spaces, and ρ the ℓ^1-product metric on $X = X' \times X''$ defined by saying that $\rho((x',x''),(y',y'')) = \rho'(x',y') + \rho''(x'',y'')$. Identifying the spaces $M_\tau(X')$, $M_\tau(X'')$ and $M_\tau(X)$ of signed τ-additive Borel measures with subspaces of $C_b(X')^\sim_\sigma$, $C_b(X'')^\sim_\sigma$ and $C_b(X')^\sim_\sigma$, as in 437E-437H, show that the bilinear map $\psi : M_\tau(X') \times M_\tau(X'') \to M_\tau(X)$ described in 437Mc has norm 1 when $M_\tau(X')$, $M_\tau(X'')$ and $M_\tau(X)$ are given the appropriate norms $\|\ \|_{\mathrm{KR}}$.

(q) Let X be a topological space and $\tilde M^+$ the set of non-negative real-valued additive functionals defined on algebras of subsets of X containing every open set, endowed with its narrow topology. Show that the weight $w(\tilde M^+)$ of $\tilde M^+$ is at most $\max(\omega, w(X))$.

(r) Let X be a Čech-complete completely regular Hausdorff space and P_{R} the set of Radon probability measures on X, with its narrow topology. Show that P_{R} is Čech-complete.

(s) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $\mathcal{F}$ an ultrafilter on I. Let $(\mathfrak{A},\bar\mu) = \prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$ be the reduced product as defined in 328C. For each $i \in I$, let (Z_i,ν_i) be the Stone space of $(\mathfrak{A}_i,\bar\mu_i)$; give $W = \{(z,i) : i \in I,\ z \in Z_i\}$ its disjoint union topology, and let βW be the Stone-Čech compactification of W. For each $i \in I$, define $\phi_i : Z_i \to W \subseteq \beta W$ by setting $\phi_i(z) = (z,i)$ for $z \in Z_i$, and let $\nu_i\phi_i^{-1}$ be the image measure on βW. Let ν be the limit $\lim_{i\to\mathcal{F}}\nu_i\phi_i^{-1}$ for the narrow topology on the space of Radon probability measures on βW, and Z its support. Show that (Z,ν) can be identified with the Stone space of $(\mathfrak{A},\bar\mu)$.

[8] Formerly 274Ya.

(t) (i) Show that there are a continuous $\phi : \{0,1\}^{\mathbb{N}} \to [0,1]$ and a positive linear operator $T : C(\{0,1\}^{\mathbb{N}}) \to C([0,1])$ such that $T(u\phi) = u$ for every $u \in C([0,1])$. (*Hint*: if $I_\sigma = \{x : \sigma \subseteq x \in \{0,1\}^{\mathbb{N}}$ for $\sigma \in \bigcup_{n\in\mathbb{N}}\{0,1\}^n$, arrange that $\{t : T(\chi I_\sigma)(t) > 0\}$ is always an interval of length $(\frac{2}{3})^{\#(\sigma)}$.) (ii) Show that there are a continuous $\tilde{\phi} : (\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ and a positive linear operator $\tilde{T} : C((\{0,1\}^{\mathbb{N}})^{\mathbb{N}}) \to C([0,1]^{\mathbb{N}})$ such that $\tilde{T}(h\tilde{\phi}) = h$ for every $h \in C([0,1]^{\mathbb{N}})$. (*Hint*: if, in (i), $(Tg)(t) = \int g\,d\nu_t$ for $t \in [0,1]$ and $g \in C(\{0,1\}^{\mathbb{N}})$, take $\nu_{\boldsymbol{t}}$ to be the product measure $\prod_{n\in\mathbb{N}} \nu_{t_n}$ for $\boldsymbol{t} = \langle t_n\rangle_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$.)

(u) Let X be a separable metrizable space and $P = P_{\mathrm{R}}(X)$ the set of Radon probability measures on X, with its narrow topology. Show that there is a family $\langle f_\mu\rangle_{\mu\in P}$ of functions from $[0,1]$ to X such that (i) $(\mu,t) \mapsto f_\mu(t)$ is Borel measurable (ii) writing μ_L for Lebesgue measure on $[0,1]$, $\mu = \mu_L f_\mu^{-1}$ for every $\mu \in P$ (iii) whenever $\langle \mu_n\rangle_{n\in\mathbb{N}}$ is a sequence in P converging to $\mu \in P$, there is a countable set $A \subseteq [0,1]$ such that $f_\mu(t) = \lim_{n\to\infty} f_{\mu_n}(t)$ for every $t \in [0,1] \setminus A$. (*Hint*: first consider the cases $X = [0,1]$ and $X = \{0,1\}^{\mathbb{N}}$, then use 437Yt to deal with $[0,1]^{\mathbb{N}}$ and its subspaces. See Bogachev 07, §8.5.)

(v) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\mathfrak{A}^f$ the ideal of elements of $\mathfrak{A}$ of finite measure. For $a \in \mathfrak{A}^f$ and $u \in L^0 = L^0(\mathfrak{A})$, let ν_{au} be the totally finite Radon measure on $\mathbb{R}$ defined by saying that $\nu_{au}(E) = \bar{\mu}(a \cap [\![u \in E]\!])$ (definition: 364G, 434T) for Borel sets $E \subseteq \mathbb{R}$. For $a \in \mathfrak{A}^f$ and u, $v \in L^0$ set $\bar{\rho}_a(u,v) = \rho_{KR}(\nu_{au},\nu_{av})$, where ρ_{KR} is the metric on $M_{\mathrm{R}}^+ = M_{\mathrm{R}}^+(\mathbb{R})$ defined from the usual metric on $\mathbb{R}$. (i) Show that the family $\mathrm{P} = \{\bar{\rho}_a : a \in \mathfrak{A}^f\}$ of pseudometrics defines the topology of convergence in measure on L^0 (definition: 367L). (ii) Show that if $(\mathfrak{A},\bar{\mu})$ is semi-finite then the uniformity $\mathcal{U}$ defined from P is metrizable iff $(\mathfrak{A},\bar{\mu})$ is σ-finite and $\mathfrak{A}$ has countable Maharam type. (iii) Show that if $(\mathfrak{A},\bar{\mu})$ is semi-finite then L^0 is complete under $\mathcal{U}$ (definition: 3A4F) iff $\mathfrak{A}$ is purely atomic. (*Hint*: if $(\mathfrak{A},\bar{\mu})$ is an atomless probability algebra and $\langle c_n\rangle_{n\in\mathbb{N}}$ is an independent sequence of elements of $\mathfrak{A}$ of measure $\frac{1}{2}$, show that $\langle \nu_{a,\chi c_n}\rangle_{n\in\mathbb{N}}$ is convergent in M_{R}^+ for every $a \in \mathfrak{A}$, so $\langle \chi c_n\rangle_{n\in\mathbb{N}}$ is $\mathcal{U}$-Cauchy.)

(x)(i) Let X be a metrizable space, and A a narrowly compact subset of the set of Radon probability measures on X. Show that there is a separable subset Y of X which is conegligible for every measure in A. (ii) Show that a metrizable space is Prokhorov iff all its closed separable subspaces are Prokhorov.

(y) I say that a completely regular Hausdorff space X is **strongly Prokhorov** if every vaguely compact subset of the space $M_t(X)$ of signed tight Borel measures on X is uniformly tight. (i) Check that a strongly Prokhorov completely regular Hausdorff space is Prokhorov. (ii) Show that a closed subspace of a strongly Prokhorov completely regular Hausdorff space is strongly Prokhorov. (iii) Show that the product of a countable family of strongly Prokhorov completely regular Hausdorff spaces is strongly Prokhorov. (iv) Show that a G_δ subset of a strongly Prokhorov metrizable space is strongly Prokhorov. (v) Show that if (X,ρ) is a complete metric space then X is strongly Prokhorov.

(z) Let X be a regular Hausdorff topological space and C a non-empty narrowly compact set of totally finite topological measures on X, all inner regular with respect to the closed sets. Set $c(A) = \sup_{\mu\in C} \mu^*A$ for $A \subseteq X$. Show that $c : \mathcal{P}X \to [0,\infty[$ is a Choquet capacity.

437 Notes and comments The ramifications of the results here are enormous. For completely regular topological spaces X, the theorems of §436 give effective descriptions of the totally finite Baire, quasi-Radon and Radon measures on X as linear functionals on $C_b(X)$ (436E, 436Xl, 436Xn). This makes it possible, and natural, to integrate the topological measure theory of X into functional analysis, through the theory of $C_b(X)^*$. (See Wheeler 83 for an extensive discussion of this approach.) For the rest of this volume we shall never be far away from such considerations. In 437C-437I I give only a sample of the results, heavily slanted towards the abstract theory of Riesz spaces in Chapter 35 and the first part of Chapter 36.

Note that while the constructions of the dual spaces $U^\sim$, $U_c^\sim$ and $U^\times$ are 'intrinsic' to a Riesz space U, in that we can identify these functions as soon as we know the linear and order structure of U, the spaces $U_\sigma^\sim$ and $U_\tau^\sim$ are definable only when U is presented as a Riesz subspace of $\mathbb{R}^X$. In the same way, while the space $M_\sigma(\Sigma)$ of countably additive functionals on a σ-algebra Σ depends only on the Boolean algebra structure, the spaces M_τ here (not to be confused with the space of completely additive functionals considered in 362B) depend on the topology as well as the Borel algebra. (For an example in which radically different topologies give rise to the same Borel algebra, see Juhász Kunen & Rudin 76.)

You may have been puzzled by the shift from 'quasi-Radon' measures in 436H to 'τ-additive' measures in 437H; somewhere the requirement of inner regularity has got lost. The point is that the topologies being considered here, being defined by declaring certain families of functions continuous, are (completely) regular; so that τ-additive measures are necessarily inner regular with respect to the closed sets (414Mb).

The theory of 'vague' and 'narrow' topologies in 437J-437V here hardly impinges on the questions considered in §§274 and 285, where vague topologies first appeared. This is because the earlier investigation was dominated by the very special position of the functions $x \mapsto e^{iy\cdot x}$ (what we shall in §445 come to call the 'characters' of the additive groups of $\mathbb{R}$ or $\mathbb{R}^r$). One idea which does appear essentially in the proof of 285L, and has a natural interpretation in the general theory, is that of a 'uniformly tight' family of Radon measures (437O). In 445Yh below I set out a generalization of 285L to abelian locally compact groups.

In §461 I will return to the general theory of extreme points in compact convex sets. Here I remark only that it is never surprising that extreme points should be special in some way, as in 437S and 461Q-461R; but the precise ways in which they are special are often unexpected. A good deal of work has been done on relationships between the topological properties of a topological space X and the space P_{R} of Radon probability measures on X with the narrow topology. Here I give only a sample of basic facts in 437R and 437Yq-437Yr. Having observed that $M^+_{\mathrm{qR}}(X)$ is metrizable whenever X is (437Rg), it is natural to seek ways of defining a metric on $M^+_{\mathrm{qR}}(X)$ from a metric on X. The Kantorovich-Rubinstein metric ρ_{KR} I have chosen here is only one of many possibilities; compare the Lévy metric of 274Yc and the metric ρ_{W} of 457K below. Note that it can make a difference whether we look at quasi-Radon (or τ-additive) probability measures, or at general Borel measures (438Yl).

The terms 'vague' and 'narrow' both appear in the literature on this topic, and I take the opportunity to use them both, meaning slightly different things. Vague topologies, in my usage, are linear space topologies on linear spaces of functionals; narrow topologies are topologies on spaces of (finitely additive) measures, which are not linear spaces, though we can in some cases define addition and multiplication by non-negative scalars (437N, 437Xr, 437Yi). I must warn you that this distinction is not standard. I see that the word 'narrow' appears above a good deal oftener than the word 'vague', which is in part a reflection of a simple prejudice against signed measures; but from the point of view of this treatise as a whole, it is more natural to work with a concept well adapted to measures with variable domains, even if we are considering questions (like compactness of sets of measures) which originate in linear analysis. I should mention also that the definition in 437Jc includes a choice. The duality considered there uses the space $C_b(X)$; for locally compact X, we have the rival spaces $C_0(X)$ and $C_k(X)$ (see 436J and 436K), and there are occasions when one of these gives a more suitable topology on a space of measures (as in 495Xl below).

The elementary theory of uniform tightness and Prokhorov spaces (437O-437V) is both pretty and useful. The emphasis I give it here, however, is partly because it provides the background to a remarkable construction by D.Preiss (439S below), showing that $\mathbb{Q}$ is not a Prokhorov space.

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438 Measure-free cardinals

At several points in §418, and again in §434, we had theorems about separable metrizable spaces in which the proofs undoubtedly needed some special property of these spaces (e.g., the fact that they are Lindelöf), but left it unclear whether something more general could be said. When we come to investigate further, asking (for instance) whether complete metric spaces in general are Radon (438H), we find ourselves once again approaching the Banach-Ulam problem, already mentioned at several points in previous volumes, and in particular in 363S. It seems to be undecidable, in ordinary set theory with the axiom of choice, whether or not every discrete space is Radon in the sense of 434C. On the other hand it is known that discrete spaces with cardinal at most ω_{ω_1} (for instance) are indeed always Radon. While as a rule I am deferring questions of this type to Volume 5, this particular phenomenon is so pervasive that I think it is worth taking a section now to clarify it.

The central definition is that of 'measure-free cardinal' (438A), and the basic results are 438B-438D. In particular, 'small' infinite cardinals are measure-free (438C). From the point of view of measure theory, a metrizable space whose weight is measure-free is almost separable, and most of the results in §418 concerning separable metrizable spaces can be extended (438E-438G). In fact 'measure-free weight' exactly determines whether a metrizable space is measure-compact (438J, 438Xm) and whether a complete metric space is Radon (438H). If $\mathfrak{c}$ is measure-free, some interesting spaces of functions are Radon (438T). I approach these last spaces through the concept of 'hereditary weak θ-refinability' (438K), which enables us to do most of the work without invoking any special axiom.

438A Measure-free cardinals: Definition A cardinal κ is **measure-free** or **of measure zero** if whenever μ is a probability measure with domain $\mathcal{P}\kappa$ then there is a $\xi < \kappa$ such that $\mu\{\xi\} > 0$. In 363S I discussed some statements equiveridical with the assertion 'every cardinal is measure-free'.

438B It is worth getting some basic facts out into the open immediately.

Lemma Let (X, Σ, μ) be a semi-finite measure space and $\langle E_i\rangle_{i\in I}$ a point-finite family of subsets of X such that $\#(I)$ is measure-free and $\bigcup_{i\in J} E_i \in \Sigma$ for every $J \subseteq I$. Set $E = \bigcup_{i\in I} E_i$.

(a) $\mu E = \sup_{J\subseteq I \text{ is finite}} \mu(\bigcup_{i\in J} E_i)$.

(b) If $\langle E_i\rangle_{i\in I}$ is disjoint, then $\mu E = \sum_{i\in I} \mu E_i$. In particular, if $\Sigma = \mathcal{P}X$ and $A \subseteq X$ has measure-free cardinal, then $\mu A = \sum_{x\in A} \mu\{x\}$.

(c) If μ is σ-finite, then $L = \{i : i \in I, \mu E_i > 0\}$ is countable and $\bigcup_{i\in I\setminus L} E_i$ is negligible.

proof (a)(i) The first step is to show, by induction on n, that the result is true if $\mu X < \infty$ and every E_i is negligible and $\#(\{i : i \in I, x \in E_i\}) \le n$ for every $x \in X$. If $n = 0$ this is trivial, since every E_i must be empty. For the inductive step to $n \ge 1$, define $\nu : \mathcal{P}I \to [0, \infty[$ by setting $\nu J = \mu(\bigcup_{i\in J} E_i)$ for every $J \subseteq I$. Then ν is a measure on I. **P** Write $F_J = \bigcup_{i\in J} E_j$ for $J \subseteq I$. (α) If J, $K \subseteq I$ are disjoint, then for $i \in I$ set $E'_i = E_i \cap F_K$ for $i \in J$, $\emptyset$ for $i \in I \setminus J$. In this case, $\langle E'_i\rangle_{i\in I}$ is a family of negligible subsets of X, $\bigcup_{i\in J'} E'_i = F_{J'\cap J} \cap F_K$ is measurable for every $J' \subseteq I$, and $\#(\{i : x \in E'_i\}) \le n - 1$ for every $x \in X$; so the inductive hypothesis tells us that

$$\mu(\textstyle\bigcup_{i\in I} E'_i) = \sup_{J'\subseteq I \text{ is finite}} \mu(\bigcup_{i\in J'} \mu E'_i) = 0,$$

that is, $F_J \cap F_K$ is negligible. But this means that

$$\nu(J \cup K) = \mu F_{J\cup K} = \mu(F_J \cup F_K) = \mu F_J + \mu F_K = \nu J + \nu K.$$

As J and K are arbitrary, ν is additive. (β) If $\langle J_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{P}I$, then

$$\nu(\bigcup_{n\in\mathbb{N}} J_n) = \mu(\bigcup_{n\in\mathbb{N}} F_{J_n}) = \lim_{n\to\infty} \mu(\bigcup_{m\le n} F_{J_m})$$
$$= \lim_{n\to\infty} \sum_{m=0}^{n} \nu J_m = \sum_{n=0}^{\infty} \nu J_n,$$

so ν is countably additive and is a measure. **Q**

At the same time, $\nu\{i\} = \mu E_i = 0$ for every i. Because $\#(I)$ is measure-free, $\nu I = 0$. **P?** Otherwise, let $f : I \to \kappa = \#(I)$ be any bijection and set $\lambda A = \frac{1}{\nu I}\nu f^{-1}[A]$ for every $A \subseteq \kappa$; then λ is a probability measure with domain $\mathcal{P}\kappa$ which is zero on singletons, and κ is not measure-free. **XQ** But this means just that $\mu(\bigcup_{i\in I} E_i) = 0$. Thus the induction proceeds.

(ii) ? Now suppose, if possible, that the general result is false. For finite sets $J \subseteq I$ set $F_J = \bigcup_{i\in J} E_i$, as before, and consider $\mathcal{E} = \{F_J : J \in [\kappa]^{<\omega}\}$ (see 3A1J for this notation). Then $\mathcal{E}$ is closed under finite unions and $\gamma = \sup_{H\in\mathcal{E}} \mu H$ is finite, because it is less than μE; let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{E}$ such that $\mu(H \setminus H^*) = 0$ for every $H \in \mathcal{E}$, where $H^* = \bigcup_{n\in\mathbb{N}} H_n$ and $\mu H^* = \gamma$ (215Ab).

Because μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. For each $n \in \mathbb{N}$, set

$$Y_n = \{x : x \in F \setminus H^*, \#(\{i : x \in E_i\}) \le n\}.$$

Then there is some $n \in \mathbb{N}$ such that $\mu^* Y_n > 0$. Let ν be the subspace measure on Y_n, so that ν is non-zero and totally finite. Now $\langle E_i \cap Y_n\rangle_{i\in I}$ is a family of negligible subsets of Y_n, $\bigcup_{i\in J} E_i \cap Y_n = Y_n \cap \bigcup_{i\in J} E_i$ is measured by ν for every $J \subseteq I$, and $\#(\{i : x \in E_i \cap Y_n\}) \le n$ for every $x \in Y_n$. But this contradicts (i) above. **X**

This proves (a).

(b) If $\langle E_i\rangle_{i\in I}$ is disjoint, then

$$\sup_{J\in[I]^{<\omega}}\mu(\bigcup_{i\in J}E_i)=\sup_{J\in[I]^{<\omega}}\sum_{i\in J}\mu E_i=\sum_{i\in I}\mu E_i.$$

Setting $I=A$, $E_x=\{x\}$ for $x\in A$, we get the special case.

(c) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of measurable sets of finite measure covering X. For each n, set $L_n=\{i:i\in I,\,\mu(E_i\cap X_n)\ge 2^{-n}\}$. **?** If L_n is infinite, take a sequence $\langle i_k\rangle_{k\in\mathbb{N}}$ of distinct elements in L_n, and consider $G_m=\bigcup_{k\ge m}E_{i_k}$ for $m\in\mathbb{N}$; then every G_m has measure at least 2^{-n}, G_0 has finite measure, $\langle G_m\rangle_{m\in\mathbb{N}}$ is non-increasing, and $\bigcap_{m\in\mathbb{N}}G_m$ is empty, because $\langle E_{i_k}\rangle_{k\in\mathbb{N}}$ is point-finite. But this is impossible. **X**

Thus every L_n is finite and $L=\bigcup_{n\in\mathbb{N}}L_n$ is countable.

Now (a), applied to $\langle E_i'\rangle_{i\in I}$ where $E_i'=E_i$ if $i\in I\setminus L$, $\emptyset$ if $i\in L$, tells us that $\bigcup_{i\in I\setminus L}E_i$ is negligible.

438C I do not think we are ready for the most interesting set-theoretic results concerning measure-free cardinals. But the following facts may help to make sense of the general pattern.

Theorem (Ulam 1930) (a) ω is measure-free.

(b) If κ is a measure-free cardinal and $\kappa'\le\kappa$ is a smaller cardinal, then κ' is measure-free.

(c) If $\langle\kappa_\xi\rangle_{\xi<\lambda}$ is a family of measure-free cardinals, and λ also is measure-free, then $\kappa=\sup_{\xi<\lambda}\kappa_\xi$ is measure-free.

(d) If κ is a measure-free cardinal so is κ^+.

(e) The following are equiveridical:

(i) $\mathfrak{c}$ is not measure-free;

(ii) there is a semi-finite measure space $(X,\mathcal{P}X,\mu)$ which is not purely atomic;

(iii) there is a measure μ on $[0,1]$ extending Lebesgue measure and measuring every subset of $[0,1]$.

(f) If $\kappa\ge\mathfrak{c}$ is a measure-free cardinal then 2^κ is measure-free.

proof (a) This is trivial.

(b) If μ is a probability measure with domain $\mathcal{P}\kappa'$, set $\nu A=\mu(\kappa'\cap A)$ for every $A\subseteq\kappa$. Then ν is a probability measure with domain $\mathcal{P}\kappa$, so there is a $\xi<\kappa$ such that $\nu\{\xi\}>0$; evidently $\xi<\kappa'$ and $\mu\{\xi\}>0$.

(c) Let μ be a probability measure on κ with domain $\mathcal{P}\kappa$. Define $f:\kappa\to\lambda$ by setting $f(\alpha)=\min\{\xi:\alpha<\kappa_\xi\}$ for $\alpha<\kappa$. Then the image measure μf^{-1} is a probability measure on λ with domain $\mathcal{P}\lambda$, so there is a $\xi<\lambda$ such that $\mu f^{-1}[\{\xi\}]>0$. Now $\mu\kappa_\xi>0$. Applying 438Bb to $A=\kappa_\xi$, we see that there is an $\alpha<\kappa_\xi$ such that $\mu\{\alpha\}>0$. As μ is arbitrary, κ is measure-free.

(d) By (a) and (b), we need consider only the case $\kappa\ge\omega$. **?** Suppose, if possible, that μ is a probability measure with domain $\mathcal{P}\kappa^+$ such that $\mu\{\alpha\}=0$ for every $\alpha<\kappa^+$. For each $\alpha<\kappa^+$, choose an injection $f_\alpha:\alpha\to\kappa$. For $\beta<\kappa^+$, $\xi<\kappa$ set $A(\beta,\xi)=\{\alpha:\beta<\alpha<\kappa^+,\,f_\alpha(\beta)=\xi\}$. Then $\kappa^+\setminus\bigcup_{\xi<\kappa}A(\beta,\xi)=\beta+1$ has cardinal at most κ, which is measure-free, so $\mu(\beta+1)=0$ and $\mu(\bigcup_{\xi<\kappa}A(\beta,\xi))>0$. Also $\langle A(\beta,\xi)\rangle_{\xi<\kappa}$ is disjoint. There is therefore a $\xi_\beta<\kappa$ such that $\mu A(\beta,\xi_\beta)>0$, by 438Bb. Now $\kappa^+>\max(\omega,\kappa)$, so there must be an $\eta<\kappa$ such that $B=\{\beta:\xi_\beta=\eta\}$ is uncountable. In this case, however, $\langle A(\beta,\eta)\rangle_{\beta\in B}$ is an uncountable family of sets of measure greater than zero, and cannot be disjoint, because μ is totally finite (215B(iii)); but if $\alpha\in A(\beta,\eta)\cap A(\beta',\eta)$, where $\beta\ne\beta'$, then $f_\alpha(\beta)=f_\alpha(\beta')=\eta$, which is impossible, because f_α is supposed to be injective. **X**

So there is no such measure μ, and κ^+ is measure-free.

(e)(i)⇒(ii) Suppose that $\mathfrak{c}$ is not measure-free; let μ be a probability measure with domain $\mathcal{P}\mathfrak{c}$ such that $\mu\{\xi\}=0$ for every $\xi<\mathfrak{c}$. Then μ is atomless. **P?** Suppose, if possible, that $A\subseteq\mathfrak{c}$ is an atom for μ. Let $f:\mathfrak{c}\to\mathcal{P}\mathbb{N}$ be a bijection. For each $n\in\mathbb{N}$, set $E_n=\{\xi:n\in f(\xi)\}$. Set $D=\{n:\mu(A\cap E_n)=\mu A\}$. Because A is an atom, $\mu(A\cap E_n)=0$ for every $n\in\mathbb{N}\setminus D$. This means that $B=\bigcap_{n\in D}E_n\setminus\bigcup_{n\in\mathbb{N}\setminus D}E_n$ has measure $\mu A>0$; but $f(\xi)=D$ for every $\xi\in B$, so $\#(B)\le 1$, and $\mu\{\xi\}>0$ for some ξ, contrary to hypothesis. **XQ**

So (ii) is true.

(ii)⇒(iii) Suppose that there is a semi-finite measure space $(X,\mathcal{P}X,\mu)$ which is not purely atomic. Then there is a non-negligible set $E\subseteq X$ which does not include any atom; let $F\subseteq E$ be a set of non-zero finite measure. If we take ν to be $\frac{1}{\mu F}\mu_F$, where μ_F is the subspace measure on F, then ν is an atomless probability measure with domain $\mathcal{P}F$. Consequently there is a function $g:F\to[0,1]$ which is inverse-measure-preserving for ν and Lebesgue measure (343Cb). But this means that the image measure νg^{-1} is a measure defined on every subset of $[0,1]$ which extends Lebesgue measure.

not-(i)⇒not-(iii) Conversely, if $\mathfrak{c}$ is measure-free, then any probability measure on $[0,1]$ measuring every subset must give positive measure to some singleton, and cannot extend Lebesgue measure.

(f) We are supposing that $\kappa\ge\mathfrak{c}$ is measure-free, so, in particular, $\mathfrak{c}$ is measure-free. Let μ be a probability measure with domain $\mathcal{P}(2^\kappa)$. By (e), it cannot be atomless; let $E\subseteq 2^\kappa$ be an atom. Let $f:2^\kappa\to\mathcal{P}\kappa$ be a bijection, and for $\xi<\kappa$ set $E_\xi=\{\alpha:\alpha<2^\kappa,\,\xi\in f(\alpha)\}$; set $D=\{\xi:\xi<\kappa,\,\mu(E\cap E_\xi)=\mu E\}$. Note that $\mu(E\cap E_\xi)$ must be zero for every $\xi\in\kappa\setminus D$, so that $E\cap\{\alpha:\xi\in D\triangle f(\alpha)\}$ is always negligible. Consider

$$A_\xi=\{\alpha:\alpha\in E,\,\xi=\min(D\triangle f(\alpha))\}$$

for $\xi<\kappa$. Then $\langle A_\xi\rangle_{\xi<\kappa}$ is a disjoint family of negligible sets, so its union A is negligible, by 438Bb, because κ is measure-free. But $E\setminus A\subseteq f^{-1}[\{D\}]$ has at most one element, and is not negligible; so $\mu\{\alpha\}>0$ for some α. As μ is arbitrary, 2^κ is measure-free.

Remark This extends the result of 419G, which used a different approach to show that ω_1 is measure-free.

We see from (d) above that $\omega_2,\,\omega_3,\dots$ are all measure-free; so, by (c), ω_ω also is; generally, if κ is any measure-free cardinal, so is ω_κ (438Xa). I ought to point out that there are more powerful arguments showing that any cardinal which is not measure-free must be enormous (see 541L in Volume 5). In this context, however, $\mathfrak{c}=2^\omega$ can be 'large', at least in the absence of an axiom like the continuum hypothesis to locate it in the hierarchy $\langle\omega_\xi\rangle_{\xi\in\mathrm{On}}$; it is generally believed that it is consistent to suppose that $\mathfrak{c}$ is not measure-free.

438D I turn now to the contexts in which measure-free cardinals behave as if they were 'small'.

Proposition Let (X,Σ,μ) be a σ-finite measure space, Y a metrizable space with measure-free weight, and $f:X\to Y$ a measurable function. Then there is a closed separable set $Y_0\subseteq Y$ such that $f^{-1}[Y_0]$ is conegligible; that is, there is a conegligible measurable set $X_0\subseteq X$ such that $f[X_0]$ is separable.

proof Let $\mathcal{U}$ be a σ-disjoint base for the topology of Y (4A2L(g-ii)); express it as $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ where each $\mathcal{U}_n$ is a disjoint family of open sets. If $n\in\mathbb{N}$, $\#(\mathcal{U}_n)\le w(Y)$ (4A2Db) is a measure-free cardinal (438Cb), and $\langle f^{-1}[U]\rangle_{U\in U_n}$ is a disjoint family in Σ such that $\bigcup_{u\in\mathcal{V}}f^{-1}[U]=f^{-1}[\bigcup\mathcal{V}]$ is measurable for every $\mathcal{V}\subseteq\mathcal{U}_n$; so 438Bc tells us that there is a countable set $\mathcal{V}_n\subseteq\mathcal{U}_n$ such that

$$f^{-1}[\bigcup(\mathcal{U}_n\setminus\mathcal{V}_n)]=\bigcup_{U\in\mathcal{U}_n\setminus\mathcal{V}_n}f^{-1}[U]$$

is negligible. Set

$$Y_0=Y\setminus\bigcup_{n\in\mathbb{N}}\bigcup(\mathcal{U}_n\setminus\mathcal{V}_n).$$

Then $f^{-1}[Y\setminus Y_0]=\bigcup_{n\in\mathbb{N}}f^{-1}[\bigcup(\mathcal{U}_n\setminus\mathcal{V}_n)]$ is negligible. On the other hand,

$$\{U\cap Y_0:U\in\mathcal{U}\}\subseteq\{\emptyset\}\cup\{V\cap Y_0:V\in\bigcup_{n\in\mathbb{N}}\mathcal{V}_n\}$$

is countable, and is a base for the subspace topology of Y_0 (4A2B(a-vi)); so Y_0 is second-countable and must be separable (4A2Oc).

Thus we have an appropriate Y_0. Now $X_0=f^{-1}[Y_0]$ is conegligible and measurable and $f[X_0]\subseteq Y_0$ is separable (4A2P(a-iv)).

438E Proposition (cf. 418B) Let (X,Σ,μ) be a complete locally determined measure space.

(a) If Y is a topological space, Z is a metrizable space, $w(Z)$ is measure-free, and $f:X\to Y$, $g:X\to Z$ are measurable functions, then $x\mapsto(f(x),g(x)):X\to Y\times Z$ is measurable.

(b) If $\langle Y_n\rangle_{n\in\mathbb{N}}$ is a sequence of metrizable spaces, with product Y, $w(Y_n)$ is measure-free for every $n\in\mathbb{N}$, and $f_n:X\to Y_n$ is measurable for every $n\in\mathbb{N}$, then $x\mapsto f(x)=\langle f_n(x)\rangle_{n\in\mathbb{N}}:X\to\prod_{n\in\mathbb{N}}Y_n$ is measurable.

proof (a)(i) Consider first the case in which μ is totally finite. Then there is a conegligible set $X_0 \subseteq X$ such that $g[X_0]$ is separable (438D). Applying 418Bb to $f\restriction X_0$ and $g\restriction X_0$, we see that $x \mapsto (f(x), g(x)) : X_0 \to Y \times g[Z_0]$ is measurable. As μ is complete, it follows that $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable.

(ii) In the general case, take any open set $W \subseteq Y \times Z$ and any measurable set $F \subseteq X$ of finite measure. Set $Q = \{x : (f(x), g(x)) \in W\}$. By (i), applied to $f\restriction F$ and $g\restriction F$, $F \cap Q \in \Sigma$; as F is arbitrary and μ is locally determined, $Q \in \Sigma$; as W is arbitrary, $x \mapsto (f(x), g(x))$ is measurable.

(b) As in (a), it is enough to consider the case in which μ is totally finite. In this case, we have for each $n \in \mathbb{N}$ a conegligible set X_n such that $f_n[X_n]$ is separable. Set $X' = \bigcap_{n\in\mathbb{N}} X_n$; then 418Bd tells us that $f\restriction X'$ is measurable, so that f is measurable.

438F Proposition (cf. 418J) Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a metrizable space, $w(Y)$ is measure-free and $f : X \to Y$ is measurable. Then f is almost continuous.

proof Take $E \in \Sigma$ and $\gamma < \mu E$. Then there is a measurable set $F \subseteq E$ such that $\gamma < \mu F < \infty$. Let $F_0 \subseteq F$ be a measurable set such that $F \setminus F_0$ is negligible and $f[F_0]$ is separable (438D). By 412Pc, the subspace measure on F_0 is still inner regular with respect to the closed sets, so $f\restriction F_0$ is almost continuous (418J), and there is a measurable set $H \subseteq F_0$, of measure at least γ, such that $f\restriction H$ is continuous. As E and γ are arbitrary, f is almost continuous.

438G Corollary (cf. 418K) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and Y a metrizable space such that $w(Y)$ is measure-free. Then a function $f : X \to Y$ is measurable iff it is almost continuous.

438H Now let us turn to questions which arose in §434.

Proposition A complete metric space is Radon iff its weight is measure-free.

proof Let (X, ρ) be a complete metric space, and $\kappa = w(X)$ its weight.

(a) If κ is measure-free, let μ be any totally finite Borel measure on X. Applying 438D to the identity map from X to itself, we see that there is a closed separable conegligible subspace X_0. Now X_0 is complete, so is a Polish space, and by 434Kb is a Radon space. The subspace measure μ_{X_0} is therefore tight (that is, inner regular with respect to the compact sets); as X_0 is conegligible, it follows at once that μ also is. As μ is arbitrary, X is a Radon space.

(b) If κ is not measure-free, take any σ-disjoint base $\mathcal{U}$ for the topology of X. Express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ where every $\mathcal{U}_n$ is disjoint. Then $\kappa \le \#(\mathcal{U})$ and there is a probability measure ν on $\mathcal{U}$, with domain $\mathcal{P}\mathcal{U}$, such that $\nu\{U\} = 0$ for every $U \in \mathcal{U}$. Let $n \in \mathbb{N}$ be such that $\nu\mathcal{U}_n > 0$. For each $U \in \mathcal{U}_n$ choose $x_U \in U$. For Borel sets $E \subseteq X$ set $\mu E = \nu\{U : U \in \mathcal{U}_n,\ x_U \in E\}$; then μ is a Borel measure on X and $\mu(\bigcup \mathcal{U}_n) = \nu\mathcal{U}_n > 0$, while $\mu(\bigcup \mathcal{V}) = \nu\mathcal{V} = 0$ for every finite $\mathcal{V} \subseteq \mathcal{U}_n$. Thus μ is not τ-additive and cannot be tight, and X is not a Radon space.

438I Proposition Let X be a metrizable space and $\langle F_\xi\rangle_{\xi<\kappa}$ a non-decreasing family of closed subsets of X, where κ is a measure-free cardinal. Then

$$\mu(\bigcup_{\xi<\kappa} F_\xi) = \sum_{\xi<\kappa} \mu(F_\xi \setminus \bigcup_{\eta<\xi} F_\eta)$$

for every semi-finite Borel measure μ on X.

proof (a) I had better begin by remarking that $H_\xi = \bigcup_{\eta<\xi} F_\eta$ is an F_σ set for every ordinal $\xi \le \kappa$, by 4A2Ld and 4A2Ka. So, setting $E_\xi = F_\xi \setminus H_\xi$, $\sum_{\xi<\kappa} \mu E_\xi$ is defined.

(b) I show by induction on ζ that $\mu H_\zeta = \sum_{\xi<\zeta} \mu E_\xi$ for every $\zeta \le \kappa$. The induction starts trivially with $\mu H_0 = 0$. The inductive step to a successor ordinal $\zeta + 1$ is also immediate, as $H_{\zeta+1} = H_\zeta \cup E_\zeta$. For the inductive step to a limit ordinal ζ of countable cofinality, let $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in ζ with supremum ζ; then

$$\mu H_\zeta = \sup_{n\in\mathbb{N}} \mu H_{\zeta_n} = \sup_{n\in\mathbb{N}} \sum_{\xi<\zeta_n} \mu E_\xi = \sum_{\xi<\zeta} \mu E_\xi,$$

as required.

(c) So we are left with the case in which ζ is a limit ordinal of uncountable cofinality. In this case, $\mu(E\cap H_\zeta)\le\sum_{\xi<\zeta}\mu E_\xi$ whenever μE is finite. **P** Let $\mathcal{U}$ be a σ-disjoint base for the topology of X (4A2L(g-ii)), and express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ where each $\mathcal{U}_n$ is disjoint. For $n\in\mathbb{N}$, $\xi\le\zeta$ set

$$\mathcal{V}_{n\xi}=\{U:U\in\mathcal{U}_n,\,U\cap H_\xi\ne\emptyset\},\quad V_{n\xi}=\bigcup\mathcal{V}_{n\xi}.$$

Define $\phi_n:V_{n\zeta}\to\zeta$ by saying that $\phi_n(x)=\min(\{\xi:\xi<\zeta,\,x\in V_{n\xi}\})$. Then, for any $D\subseteq\zeta$,

$$\phi_n^{-1}[D]=\bigcup_{\xi\in D}\bigcup(\mathcal{V}_{n\xi}\setminus\bigcup_{\eta<\xi}\mathcal{V}_{n\eta})$$

is a union of members of $\mathcal{U}_n$, so is open. We therefore have a measure ν_n on $\mathcal{P}\zeta$ defined by saying that $\nu_nD=\mu(E\cap\phi_n^{-1}[D])$ for every $D\subseteq\zeta$. At this point, recall that we are supposing that κ is measure-free, so $\#(\zeta)$ also is measure-free (438Cb) and $\nu_n\zeta=\sum_{\xi<\zeta}\nu_n\{\xi\}=\sup_{\xi<\zeta}\nu_n\xi$ (438Bb). Interpreting this in X, we have $\mu(E\cap V_{n\zeta})=\sup_{\xi<\zeta}\mu(E\cap V_{n\xi})$.

This is true for every $n\in\mathbb{N}$. So there is a countable set $C\subseteq\zeta$ such that $\mu(E\cap V_{n\zeta})=\sup_{\xi\in C}\mu(E\cap V_{n\xi})$ for every $n\in\mathbb{N}$. Because $\operatorname{cf}\zeta>\omega$, there is an $\alpha<\zeta$ such that $C\subseteq\alpha$, and $\mu(E\cap V_{n\zeta})=\mu(E\cap V_{n\alpha})$, that is, $E\cap V_{n\zeta}\setminus V_{n\alpha}$ is negligible, for every $n\in\mathbb{N}$.

Now note that F_α is closed. So

$$\begin{aligned}H_\zeta\setminus F_\alpha&\subseteq\bigcup\{U:U\in\mathcal{U},\,H_\zeta\cap U\ne\emptyset,\,U\cap F_\alpha=\emptyset\}\\&=\bigcup_{n\in\mathbb{N}}V_{n\zeta}\setminus V_{n,\alpha+1},\end{aligned}$$

and $E\cap H_\zeta\setminus F_\alpha$ is negligible. Accordingly, using the inductive hypothesis,

$$\mu(E\cap H_\zeta)\le\mu F_\alpha=\mu H_{\alpha+1}\le\sum_{\xi\le\alpha}\mu E_\xi\le\sum_{\xi<\zeta}\mu E_\xi,$$

as claimed. **Q**

Because μ is semi-finite, and E is arbitrary, $\mu H_\zeta\le\sum_{\xi<\zeta}E_\xi$; but the reverse inequality is trivial, so we have equality, and the induction proceeds in this case also.

(d) At the end of the induction we have $\mu H_\kappa=\sum_{\xi<\kappa}\mu E_\xi$, as stated.

438J So far we have been looking at metrizable spaces, the obvious first step. But it turns out that the concept of 'metacompactness' leads to generalizations of some of the results above.

Proposition (Moran 70, Haydon 74) Let X be a metacompact space with measure-free weight.

(a) X is Borel-measure-compact.

(b) If X is normal, it is measure-compact.

(c) If X is perfectly normal (for instance, if it is metrizable), it is Borel-measure-complete.

proof (a) ? If X is not Borel-measure-compact, there are a non-zero totally finite Borel measure μ on X and a cover $\mathcal{G}$ of X by negligible open sets (434H(a-v)). Let $\mathcal{H}$ be a point-finite open cover of X refining $\mathcal{G}$. By 4A2Dc, $\#(\mathcal{H})$ is at most $\max(\omega,w(X))$, so is measure-free, by 438C. Because μ is a Borel measure, $\bigcup\mathcal{H}'$ is measurable for every $\mathcal{H}'\subseteq\mathcal{H}$; $\mu H=0$ for every $H\in\mathcal{H}$; while $\mu(\bigcup\mathcal{H})=\mu X>0$. But this contradicts 438Ba. **X**

(b) Now suppose that X is normal, and that μ is a totally finite Baire measure on X. Because a normal metacompact space is countably paracompact (4A2F(g-iii)), μ has an extension to a Borel measure μ_1 which is inner regular with respect to the closed sets, by Mařík's theorem (435C). Now μ_1 is τ-additive, by (a) above, so μ also is (411C). As μ is arbitrary, X is measure-compact.

(c) Since on a perfectly normal space the Baire and Borel measures are the same, X is Borel-measure-complete iff it is measure-compact, and we can use (b).

Remark The arguments here can be adapted in various ways, and in particular the hypotheses can be weakened; see 438Yd-438Yf.

438K Hereditarily weakly θ-refinable spaces A topological space X is **hereditarily weakly θ-refinable** (also called **hereditarily σ-relatively metacompact**, **hereditarily weakly submetacompact**) if for every family $\mathcal{G}$ of open subsets of X there is a σ-isolated family $\mathcal{A}$ of subsets of X, refining $\mathcal{G}$, such that $\bigcup\mathcal{A} = \bigcup\mathcal{G}$.

438L Lemma (a) Any subspace of a hereditarily weakly θ-refinable topological space is hereditarily weakly θ-refinable.

(b) A hereditarily metacompact space (e.g., any metrizable space, see 4A2Lb) is hereditarily weakly θ-refinable.

(c) A hereditarily Lindelöf space is hereditarily weakly θ-refinable.

(d) A topological space with a σ-isolated network is hereditarily weakly θ-refinable.

proof (a) If X is hereditarily weakly θ-refinable, Y is a subspace of X, and $\mathcal{H}$ is a family of open subsets of Y, set $\mathcal{G} = \{G : G \subseteq X \text{ is open}, G \cap Y \in \mathcal{H}\}$. Then there is a σ-isolated family $\mathcal{A}$, refining $\mathcal{G}$, with union $\bigcup\mathcal{G}$; and $\{A \cap Y : A \in \mathcal{A}\}$ is σ-isolated (4A2B(a-viii)), refines $\mathcal{H}$, and has union $\bigcup\mathcal{H}$. As $\mathcal{H}$ is arbitrary, Y is hereditarily weakly θ-refinable.

(b) If X is hereditarily metacompact, and $\mathcal{G}$ is any family of open sets in X with union W, then $\mathcal{G}$ is an open cover of the metacompact space W, so has a point-finite open refinement $\mathcal{H}$ with the same union. For each $x \in W$, set $\mathcal{H}_x = \{H : x \in H \in \mathcal{H}\}$, $V_x = \bigcap\mathcal{H}_x$, so that $\mathcal{H}_x$ is a non-empty finite set and V_x is an open set containing x. For $n \ge 1$, set $A_n = \{x : x \in W, \#(\mathcal{H}_x) = n\}$; then for any distinct x, $y \in A_n$, either $\mathcal{H}_x = \mathcal{H}_y$ and $V_x = V_y$, or $\#(\mathcal{H}_x \cup \mathcal{H}_y) > n$ and $V_x \cap V_y \cap A_n = \emptyset$. This means that $\mathcal{A}_n = \{V_x \cap A_n : x \in A_n\}$ is a partition of A_n into relatively open sets, and is an isolated family. Also, $\mathcal{A}_n$ is a refinement of $\mathcal{H}$ and therefore of $\mathcal{G}$; so $\bigcup_{n\ge1}\mathcal{A}_n$ is a σ-isolated refinement of $\mathcal{G}$, and its union is $\bigcup_{n\ge1} A_n = W$. As $\mathcal{G}$ is arbitrary, X is hereditarily weakly θ-refinable.

(c) If X is hereditarily Lindelöf and $\mathcal{G}$ is a family of open subsets of X, there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ with the same union; now $\mathcal{G}_0$, being countable, is σ-isolated. As $\mathcal{G}$ is arbitrary, X is hereditarily weakly θ-refinable.

(d) If X has a σ-isolated network $\mathcal{A}$, and $\mathcal{G}$ is any family of open subsets of X, then

$$\mathcal{E} = \{A : A \in \mathcal{A}, A \text{ is included in some member of } \mathcal{G}\}$$

is a σ-isolated family (4A2B(a-viii) again), refining $\mathcal{G}$, with union $\bigcup\mathcal{G}$.

438M Proposition (Gardner 75) If X is a hereditarily weakly θ-refinable topological space with measure-free weight, it is Borel-measure-complete.

proof Let μ be a Borel probability measure on X, and $\mathcal{G}$ the family of μ-negligible open sets. Let $\mathcal{A}$ be a σ-isolated family refining $\mathcal{G}$ with union $\bigcup\mathcal{G}$. Express $\mathcal{A}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$ where each $\mathcal{A}_n$ is an isolated family; for each $n \in \mathbb{N}$, set $X_n = \bigcup\mathcal{A}_n$ and let μ_n be the subspace measure on X_n. Then $\mathcal{A}_n$ is a disjoint family of relatively open μ_n-negligible sets; as $\#(\mathcal{A}_n) \le w(X_n) \le w(X)$ (4A2D) is measure-free, and μ_n is a totally finite Borel measure on X_n,

$$\mu^*X_n = \mu_n X_n = \mu_n(\bigcup\mathcal{A}_n) = 0,$$

by 438Bb. Now $\mu(\bigcup\mathcal{G}) = \mu^*(\bigcup_{n\in\mathbb{N}} X_n) = 0$. As μ is arbitrary, X is Borel-measure-complete (434I(a-iv)).

438N For the next few paragraphs, I will use the following notation. Let X be a topological space and $\mathcal{G}$ a family of subsets of X. Then $\mathcal{J}(\mathcal{G})$ will be the family of subsets of X expressible as $\bigcup\mathcal{A}$ for some σ-isolated family $\mathcal{A}$ refining $\mathcal{G}$. Observe that X is hereditarily weakly θ-refinable iff $\bigcup\mathcal{G}$ belongs to $\mathcal{J}(\mathcal{G})$ for every family $\mathcal{G}$ of open subsets of X.

(a) $\mathcal{J}(\mathcal{G})$ is always a σ-ideal of subsets of X. **P** If $\mathcal{A}$ is a σ-isolated family of subsets of X, refining $\mathcal{G}$, and B is any set, then $\{B \cap A : A \in \mathcal{A}\}$ is still σ-isolated and still refines $\mathcal{G}$; so any subset of a member of $\mathcal{J}(\mathcal{G})$ belongs to $\mathcal{J}(\mathcal{G})$. If $\langle\mathcal{A}_n\rangle_{n\in\mathbb{N}}$ is a sequence of σ-isolated families refining $\mathcal{G}$, then $\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$ is σ-isolated and refines $\mathcal{G}$; so the union of any sequence in $\mathcal{J}(\mathcal{G})$ belongs to $\mathcal{J}(\mathcal{G})$. **Q**

(b) If $\mathcal{H}$ refines $\mathcal{G}$, then $\mathcal{J}(\mathcal{H}) \subseteq \mathcal{J}(\mathcal{G})$. **P** All we need to remember is that any family refining $\mathcal{H}$ also refines $\mathcal{G}$. **Q**

(c) If X and Y are topological spaces, $A \subseteq X$, $f : A \to Y$ is continuous, and $\mathcal{H}$ is a family of subsets of Y, set $\mathcal{G} = \{f^{-1}[H] : H \in \mathcal{H}\}$. Then $\mathcal{J}(\mathcal{G}) \supseteq \{f^{-1}[B] : B \in \mathcal{J}(\mathcal{H})\}$. **P** If $B \in \mathcal{J}(\mathcal{H})$, there is a σ-isolated family $\mathcal{D}$ of subsets of Y, refining $\mathcal{H}$, and with union B. Now $\mathcal{A} = \{f^{-1}[D] : D \in \mathcal{D}\}$ refines $\mathcal{G}$ and has union $f^{-1}[B]$. We can express $\mathcal{D}$ as $\bigcup_{n\in\mathbb{N}} \mathcal{D}_n$, where each $\mathcal{D}_n$ is an isolated family; set $\mathcal{A}_n = \{f^{-1}[D] : D \in \mathcal{D}_n\}$, so that $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. For each n, $\mathcal{A}_n$ is disjoint, because $\mathcal{D}_n$ is. Moreover, if $D \in \mathcal{D}_n$, then $D = H \cap \bigcup \mathcal{D}_n$ for some open set $H \subseteq Y$, so that $f^{-1}[D] = f^{-1}[H] \cap \bigcup \mathcal{A}_n$ is relatively open in $\bigcup \mathcal{A}_n$; this shows that $\mathcal{A}_n$ is an isolated family. Accordingly $\mathcal{A}$ is σ-isolated and witnesses that $f^{-1}[B] \in \mathcal{J}(\mathcal{G})$. As B is arbitrary, we have the result. **Q**

(d) If X is a topological space, $\mathcal{G}$ is a family of subsets of X, and $\langle D_i\rangle_{i\in I}$ is an isolated family in $\mathcal{J}(\mathcal{G})$, then $\bigcup_{i\in I} D_i \in \mathcal{J}(\mathcal{G})$. **P** For each $i \in I$, let $\langle \mathcal{A}_{ni}\rangle_{n\in\mathbb{N}}$ be a sequence of isolated families, all refining $\mathcal{G}$, such that $D_i = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{A}_{in}$. Set $\mathcal{A}_n = \bigcup_{i\in I} \mathcal{A}_{in}$ for each n. Then $\mathcal{A}_n$ refines $\mathcal{G}$, and $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{A}_n = \bigcup_{i\in I} D_i$. It is easy to check that every $\mathcal{A}_n$ is isolated, so that $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ witnesses that $\bigcup_{i\in I} D_i$ belongs to $\mathcal{J}(\mathcal{G})$. **Q**

438O Lemma Give $\mathbb{R}$ the topology $\mathfrak{S}$ generated by the closed intervals $]-\infty, t]$ for $t \in \mathbb{R}$, and let $r \geq 1$. Then $\mathbb{R}^r$, with the product topology corresponding to $\mathfrak{S}$, is hereditarily weakly θ-refinable.

proof Induce on r. Write $\leq$ for the usual partial order of $\mathbb{R}^r$, and $]-\infty, x]$ for $\{y : y \leq x\}$; set $V_A = \bigcup_{x\in A}]-\infty, x]$ for $A \subseteq \mathbb{R}^r$. The sets $]-\infty, x]$, as x runs over $\mathbb{R}^r$, form a base for the topology of $\mathbb{R}^r$.

The induction starts easily because $\mathfrak{S}$ itself is hereditarily Lindelöf. **P** If $\mathcal{G} \subseteq \mathfrak{S}$, set

$$A = \{x : x \in \mathbb{R}, \text{ there is some } G \in \mathcal{G} \text{ such that }]-\infty, x] \subseteq G\}.$$

Then A has a countable cofinal set D, so that there is a corresponding countable subset of $\mathcal{G}$ with the same union as $\mathcal{G}$. **Q** By 438Lc, $\mathfrak{S}$ is hereditarily weakly θ-refinable.

For the inductive step to $r + 1$, where $r \geq 1$, let $\mathcal{G}$ be a family of open subsets of $\mathbb{R}^{r+1}$, and set

$$A = \{x : x \in \mathbb{R}^r, \text{ there is some } G \in \mathcal{G} \text{ such that }]-\infty, x] \subseteq G\}.$$

For each $k \leq r$, $q \in \mathbb{Q}$ set $K_k = (r + 1) \setminus \{k\}$ and let $B_{kq} \subseteq \mathbb{R}^{K_k}$ be the set $\{z : z^\frown\!\!<q> \in V_A\}$, writing $z^\frown\!\!<q>$ for that member x of $\mathbb{R}^{r+1}$ such that $x\restriction K_k = z$ and $x(k) = q$. (I am thinking of members of $\mathbb{R}^{r+1}$ as functions from $r + 1 = \{0, \ldots, r\}$ to $\mathbb{R}$.) Set $A_{kq} = \{x : x \in V_A,\ x(k) = q\}$, $\mathcal{G}_{kq} = \{]-\infty, x] : x \in A_{kq}\}$. Then, in the notation of 438N, $V_{A_{kq}} \in \mathcal{J}(\mathcal{G})$. **P** Set $f(x) = x\restriction K_k$ for each $x \in V_{A_{kq}}$, so that $f : V_{A_{kq}} \to \mathbb{R}^{K_k}$ is continuous. For $x \in A_{kq}$, $]-\infty, x] = f^{-1}[\,]-\infty, f(x)]\,]$. Now $\mathbb{R}^{K_k}$ is hereditarily weakly θ-refinable, by the inductive hypothesis, so if we set $\mathcal{H}_{kq} = \{]-\infty, f(x)] : x \in A_{kq}\}$, $\bigcup \mathcal{H}_{kq} \in \mathcal{J}(\mathcal{H}_{kq})$ and (by 438Nc)

$$V_{A_{kq}} = f^{-1}[\bigcup \mathcal{H}_{kq}] \in \mathcal{J}(\mathcal{G}_{kq}) \subseteq \mathcal{J}(\mathcal{G}). \text{ **Q**}$$

Accordingly $W \in \mathcal{J}(\mathcal{G})$, where $W = \bigcup_{k\leq r, q\in\mathbb{Q}} V_{A_{kq}}$, by 438Na.

Now consider $V_A \setminus W$. If x, $x' \in V_A \setminus W$ and $x \leq x'$ then $x = x'$. **P?** Otherwise, there are a $k \leq n$ and a $q \in \mathbb{Q}$ such that $x(k) \leq q \leq x'(k)$. In this case, setting $y\restriction K_k = x\restriction K_k$ and $y(k) = q$, we have $y \in A_{kq}$ and $x \in V_{A_{kq}}$. **XQ**

But this means that the subspace topology of $V_A \setminus W$ is discrete, so that $\{\{x\} : x \in V_A \setminus W\}$ is an isolated family covering $V_A \setminus W$ and refining $\mathcal{G}$; thus $V_A \setminus W \in \mathcal{J}(\mathcal{G})$ and $\bigcup \mathcal{G} = V_A$ belongs to $\mathcal{J}(\mathcal{G})$. As $\mathcal{G}$ is arbitrary, $\mathbb{R}^{r+1}$ is hereditarily weakly θ-refinable and the induction proceeds.

438P Lemma Let X be a Polish space, and $\tilde{C}^{\Vert} = \tilde{C}^{\Vert}(X)$ the family of functions $\omega : \mathbb{R} \to X$ such that $\lim_{s\uparrow t} \omega(s)$ and $\lim_{s\downarrow t} \omega(s)$ are defined in X for every $t \in \mathbb{R}$.

(a) For $A \subseteq B \subseteq \mathbb{R}$ and $f \in X^B$, set

$$\mathrm{jump}_A(f, \epsilon) = \sup\{n : \text{ there is an } I \in [A]^n \text{ such that } \rho(f(s), f(t)) > \epsilon$$
$$\text{whenever } s < t \text{ are successive elements of } I\}.$$

Now a function $\omega \in X^{\mathbb{R}}$ belongs to $\tilde{C}^{\Vert}$ iff $\mathrm{jump}_{[-n,n]}(\omega, \epsilon)$ is finite for every $n \in \mathbb{N}$ and $\epsilon > 0$.

(b) If $\omega \in \tilde{C}^{\Vert}$ then ω is continuous at all but countably many points of $\mathbb{R}$.

(c) If $\omega \in \tilde{C}^{\Downarrow}$ then $\omega[[-n,n]]$ is relatively compact in X for every $n \in \mathbb{N}$.

proof Fix on a complete metric ρ inducing the topology of X.

(a)(i) Suppose that $\omega \in \tilde{C}^{\Downarrow}$, $n \in \mathbb{N}$ and $\epsilon > 0$. For every $t \in [-n,n]$, there is a $\delta_t > 0$ such that $\rho(\omega(s),\omega(s')) \le \epsilon$ whenever either $t < s \le s' \le t+\delta_t$ or $t-\delta_t \le s \le s' < t$. Now there is an $m \ge 1$ such that whenever s, $s' \in [-n,n]$ and $|s-s'| \le \frac{2n}{m}$ there is a $t \in [-n,n]$ such that both s and s' belong to $[t-\delta_t, t+\delta_t]$. Suppose now that $-n \le t_0 < t_1 < \ldots < t_{3m} \le n$. Then there must be an $i < m$ such that $t_{3i+3} - t_{3i} \le \frac{2n}{m}$. Let t be such that both t_{3i} and t_{3i+3} belong to $[t-\delta_t, t+\delta_t]$. There is at least one j such that $3i \le j \le 3i+2$ and $t \notin [t_j, t_{j+1}]$; in which case $\rho(\omega(t_j),\omega(t_{j+1})) \le \epsilon$. So $\mathrm{jump}_{[-n,n]}(\omega,\epsilon) \le 3m$. As n and ϵ are arbitrary, the condition is satisfied.

(ii) Suppose that ω satisfies the condition. If $t \in \mathbb{R}$ and $\epsilon > 0$, take $n \ge |t|+1$ and $m = \mathrm{jump}_{[-n,n]}(\omega,\epsilon)$; then there must be a $\delta > 0$ such that $\mathrm{diam}\{\omega(s) : t < s \le t+\delta\} \le 2\epsilon$, since otherwise we should be able to find $t_0 > t_1 > \ldots > t_m > t$ such that $t_0 = t+1$ and $\rho(\omega(t_{i+1}),\omega(t_i)) > \epsilon$ for $i < m$. Because X is ρ-complete, $\lim_{s\downarrow t}\omega(s)$ is defined. Similarly, $\lim_{s\uparrow t}\omega(s)$ is defined; as t is arbitrary, $\omega \in \tilde{C}^{\Downarrow}$.

(b) For $k \in \mathbb{N}$, set set $A_k = \{t : t \in \mathbb{R}, \limsup_{s\to t}\rho(\omega(s),\omega(t)) > 2^{-k+1}\}$. Then $\#(A_k \cap [-n,n[) \le \mathrm{jump}_{[-n,n]}(\omega,2^{-k})$ for every $n \in \mathbb{N}$. **P** If $t_0,\ldots,t_m \in A_k$ and $-n \le t_0 < \ldots < t_m < n$, then we can choose $s_0,\ldots,s_m$ such that $s_0 = t_0$,

$$s_{i-1} < s_i < t_{i+1}, \quad \rho(\omega(s_i),\omega(s_{i-1})) > 2^{-k}$$

whenever $1 \le i \le m$, interpreting t_{m+1} as n. Now $\{s_i : i \le m\}$ witnesses that $\mathrm{jump}_{[-n,n]}(\omega,2^{-k}) > m$. **Q**

By (a), $A_k \cap [-n,n[$ is finite for every n, and

$$\{t : \omega \text{ is discontinuous at } t\} = \bigcup_{k\in\mathbb{N}} A_k$$

is countable.

(c) If $\langle t_k\rangle_{k\in\mathbb{N}}$ is any monotonic sequence in $\mathbb{R}$ with limit t, $\langle\omega(t_k)\rangle_{k\in\mathbb{N}}$ is convergent to one of $\lim_{s\uparrow t}\omega(s)$, $\lim_{s\downarrow t}\omega(s)$. But this means that if $\langle t_k\rangle_{k\in\mathbb{N}}$ is any sequence in $[-n,n]$, $\langle\omega(t_k)\rangle_{k\in\mathbb{N}}$ has a subsequence which is convergent in X; by 4A2Le, $\omega[[-n,n]]$ is relatively compact in X.

438Q Theorem Let X be a Polish space, and $\tilde{C}^{\Downarrow} = \tilde{C}^{\Downarrow}(X)$ the family of functions $\omega : \mathbb{R} \to X$ such that $\lim_{s\uparrow t}\omega(s)$ and $\lim_{s\downarrow t}\omega(s)$ are defined in X for every $t \in \mathbb{R}$.

(a) $\tilde{C}^{\Downarrow}$, with its topology of pointwise convergence inherited from the product topology of $X^{\mathbb{R}}$, is K-analytic.

(b) $\tilde{C}^{\Downarrow}$ is hereditarily weakly θ-refinable.

proof Fix on a complete metric ρ inducing the topology of X.

(a) By 4A2Qg, X can be regarded as a G_δ set in a compact metrizable space Z.

(i) Give the space $\mathcal{C} = \mathcal{C}(Z)$ of closed subsets of Z its Fell topology; then $\mathcal{C}$ is compact and metrizable (4A2T(b-iii), 4A2Tf). Let $\mathcal{K}$ be the family of compact subsets of X, that is, the set of those $K \in \mathcal{C}$ which are included in X. Then $\mathcal{K}$ is a G_δ set in $\mathcal{C}$. **P** $Z \setminus X$ is an F_σ set in Z, so is expressible as the union of a sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ of compact sets; now $\mathcal{K} = \bigcap_{n\in\mathbb{N}}\{K : K \in \mathcal{C},\, K \cap L_n = \emptyset\}$ is G_δ, by the definition of the Fell topology (4A2T(a-ii)). **Q**

(ii) For $n \in \mathbb{N}$, set

$$Q_n = \{\omega : \omega \in Z^{\mathbb{R}},\, \omega[[-n,n]] \text{ is a relatively compact subset of } X\}.$$

Then Q_n is K-analytic. **P** Set

$$R_n = \{(K,\omega) : K \in \mathcal{K},\, \omega \in Z^{\mathbb{R}},\, \omega[[-n,n]] \subseteq K\}.$$

Then

$$R_n = \bigcap_{t\in[-n,n]}\{(K,\omega) : K \in \mathcal{K},\, \omega \in Z^{\mathbb{R}},\, \omega(t) \in K\}$$

is a closed set in $\mathcal{K}\times Z^{\mathbb{R}}$, by 4A2T(e-i). Since $\mathcal{K}$ is Polish and $Z^{\mathbb{R}}$ is compact, R_n is K-analytic (423Ba, 423C, 422Ge, 422Gf). But Q_n is the projection of R_n onto the second coordinate, so it too is K-analytic (422Gd). **Q**

(iii) Next, for m, $k\in\mathbb{N}$, defining the function $\mathrm{jump}_{[-n,n]}$ from ρ as in 438P,

$$V_{mk}=\{\omega:\omega\in Q_n,\,\mathrm{jump}_{[-n,n]}(\omega,2^{-k})\le m\}$$

is relatively closed in Q_n, therefore K-analytic, and

$$Q'_n=Q_n\cap\bigcap_{k\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}V_{mk}$$

is K-analytic (422Hc).

(iv) Consequently $Q=\bigcap_{n\in\mathbb{N}}Q'_n$ is K-analytic. But $Q=\tilde C^{\Bbbk}$. **P** For n, $k\in\mathbb{N}$, $\tilde C^{\Bbbk}\subseteq Q_n$ by 438Pc, and $\tilde C^{\Bbbk}\subseteq\bigcup_{m\in\mathbb{N}}V_{mk}$ by 438Pa. So $\tilde C^{\Bbbk}\subseteq Q$. Conversely, if $\omega\in Q$, then surely $\omega(t)\in X$ for every $t\in\mathbb{R}$, and $\mathrm{jump}_{[-n,n]}(\omega,2^{-k})$ is finite for all n, $k\in\mathbb{N}$; so $\omega\in\tilde C^{\Bbbk}$ by 438Pa in the other direction. **Q**

Accordingly $\tilde C^{\Bbbk}$ is K-analytic, as claimed.

(b) Let $\mathcal{G}$ be a family of open sets in $\tilde C^{\Bbbk}$.

(i) In the notation of 438N, I seek to show that $\bigcup\mathcal{G}$ belongs to $\mathcal{J}(\mathcal{G})$. Of course it will be enough to consider the case in which $\bigcup\mathcal{G}$ is non-empty. The following elementary remarks will be useful.

($\boldsymbol{\alpha}$) If $D\subseteq\bigcup\mathcal{G}$, and $\mathcal{E}$ is a partition of D into relatively open sets such that $\mathcal{E}$ refines $\mathcal{G}$, then $D\in\mathcal{J}(\mathcal{G})$.

($\boldsymbol{\beta}$) If $\langle D_i\rangle_{i\in I}$ is any family in $\mathcal{J}(\mathcal{G})$, and $\langle H_i\rangle_{i\in I}$ is a family of open sets, and $D\subseteq\{\omega:\#(\{i:\omega\in H_i\})=1\}$, then $D\cap\bigcup_{i\in I}H_i\cap D_i$ belongs to $\mathcal{J}(\mathcal{G})$. **P** $\langle D\cap H_i\cap D_i\rangle_{i\in I}$ is an isolated family in $\mathcal{J}(\mathcal{G})$; use 438Nd. **Q**

(ii) Let $\mathcal{I}$ be the family of non-empty open intervals in $\mathbb{R}$ with rational endpoints, and $\mathcal{U}$ a countable base for the topology of X. Write Q for the family of all finite sequences

$$\boldsymbol{q}=(I_0,U_0,V_0,W_0,I_1,U_1,V_1,W_1,\dots,I_n,U_n,V_n,W_n)$$

where $I_0,I_1,\dots,I_n$ are disjoint members of $\mathcal{I}$, all the U_i, V_i, W_i belong to $\mathcal{U}$, and, for each $i\le n$, any pair of U_i, V_i, W_i are either equal or disjoint. Fix $\boldsymbol{q}=(I_0,\dots,W_n)\in Q$ for the moment.

(iii) Set $T_{\boldsymbol{q}}=\prod_{i\le n}I_i$. For $\tau\in T_{\boldsymbol{q}}$, write $F_{\boldsymbol{q}\tau}$ for the set of those $\omega\in\tilde C^{\Bbbk}$ such that, for every $i\le n$, $\omega(s)\in U_i$ for $s\in I_i\cap]-\infty,\tau(i)[$, $\omega(\tau(i))\in V_i$ and $\omega(s)\in W_i$ for $s\in I_i\cap]\tau(i),\infty[$. Set $\Omega_{\boldsymbol{q}}=\bigcup\{F_{\boldsymbol{q}\tau}:\tau\in T_{\boldsymbol{q}}\}$, and for $\tau\in T_{\boldsymbol{q}}$ set $H_{\boldsymbol{q}\tau}=\{\omega:\omega\in\Omega_{\boldsymbol{q}},\,\omega(\tau(i))\in V_i$ for every $i\le n\}$. Finally, set

$$S_{\boldsymbol{q}}=\{\tau:\tau\in T_{\boldsymbol{q}}\text{ and }H_{\boldsymbol{q}\tau}\text{ is included in some member of }\mathcal{G}\}.$$

(iv) If $T\subseteq S_{\boldsymbol{q}}$ then $H=\bigcup_{\tau\in T}H_{\boldsymbol{q}\tau}$ belongs to $\mathcal{J}(\mathcal{G})$. **P** Induce on $\#(L(T))$, where

$$L(T)=\{i:i\le n,\text{ there are }\tau,\,\tau'\in T\text{ such that }\tau(i)\ne\tau'(i)\}.$$

If $L(T)=\emptyset$, then $\#(T)\le 1$, so H is either empty or included in some member of $\mathcal{G}$, and the induction starts. For the inductive step to $\#(L(T))=k\ge 1$, consider three cases.

case 1 Suppose there is a $j\in L(T)$ such that $U_j=V_j=W_j$. Then $\omega(t)\in V_j$ whenever $\omega\in\Omega_{\boldsymbol{q}}$ and $t\in I_j$. Fix any $t^*\in I_j$ and for $\tau\in T_{\boldsymbol{q}}$ define $\tau^*\in T_{\boldsymbol{q}}$ by setting $\tau^*(j)=t^*$, $\tau^*(i)=\tau(i)$ for $i\ne j$; then $H_{\boldsymbol{q}\tau^*}=H_{\boldsymbol{q}\tau}$. Set $T^*=\{\tau^*:\tau\in T\}$; then $L(T^*)=L(T)\setminus\{j\}$, so $\#(L(T^*))<\#(L(T))$, while $T^*\subseteq S_{\boldsymbol{q}}$. By the inductive hypothesis, $H=\bigcup_{\tau\in T^*}H_{\boldsymbol{q}\tau}$ belongs to $\mathcal{J}(\mathcal{G})$.

case 2 Suppose there is a $j\in L(T)$ such that $U_j\ne V_j$ and $V_j\ne W_j$. Then $V_j\cap(U_j\cup W_j)=\emptyset$. For $s\in I_j$ set $T^*_s=\{\tau:\tau\in T,\,\tau(j)=s\}$. Then $\#(L(T^*_s))<\#(L(T))$ so, by the inductive hypothesis, $H^*_s\in\mathcal{J}(\mathcal{G})$, where $H^*_s=\bigcup_{\tau\in T^*_s}H_{\boldsymbol{q}\tau}$. But, for $\tau\in T$ and $\omega\in H_{\boldsymbol{q}\tau}$, $\omega(s)\in V_j$ iff $s=\tau(j)$; so $H^*_s=\{\omega:\omega\in H,\,\omega(s)\in V_j\}$ and $\langle H^*_s\rangle_{s\in I_j}$ is a partition of H into relatively open sets. By (i-β), $H\in\mathcal{J}(\mathcal{G})$.

case 3 Otherwise, $L(T)=J\cup J'$ where $J=\{i:i\in L(T),\,U_i=V_i\}$ and $J'=\{i:i\in L(T),\,V_i=W_i\}$ are disjoint. For $\omega\in H$ and $i\in J$ we see that there is a largest $t\in I_i$ such that $\omega(t)\in V_i$; set $\phi_i(\omega)=-t$.

Similarly, if $\omega\in H$, $i\in J'$ there is a smallest $t\in I_i$ such that $\omega(t)\in V_i$; in this case, set $\phi_i(\omega)=t$. Observe that, for $i\in J$ and $s\in\mathbb{R}$,

$$\begin{aligned}\{\omega:\omega\in H,\,\phi_i(\omega)\le s\}&=\emptyset\text{ if }s<-t\text{ for every }t\in I_i,\\&=H\text{ if }-t<s\text{ for every }t\in I_i,\\&=\{\omega:\omega\in H,\,\omega(-s)\in V_i\}\text{ if }-s\in I_i,\end{aligned}$$

so is always relatively open in H, and $\phi_i:H\to\mathbb{R}$ is continuous if $\mathbb{R}$ is given the left-facing topology $\mathfrak{S}$ of Lemma 438O. Similarly, for $i\in J'$, $s\in\mathbb{R}$,

$$\begin{aligned}\{\omega:\omega\in H,\,\phi_i(\omega)\le s\}&=\emptyset\text{ if }s<t\text{ for every }t\in I_i,\\&=H\text{ if }t<s\text{ for every }t\in I_i,\\&=\{\omega:\omega\in H,\,\omega(s)\in V_i\}\text{ if }s\in I_i.\end{aligned}$$

So in this case also ϕ_i is continuous.

Accordingly, giving $\mathbb{R}^{L(T)}$ the product topology corresponding to $\mathfrak{S}$, we have a continuous map $\phi:H\to\mathbb{R}^{L(T)}$ defined by setting $\phi(\omega)=\langle\phi_i(\omega)\rangle_{i\in L(T)}$ for $\omega\in H$. For $\tau\in T$, set $\tilde\tau(i)=-\tau(i)$ if $i\in J$, $\tau(i)$ if $i\in J'$, and $\tilde H_\tau=\,]-\infty,\tilde\tau]\subseteq\mathbb{R}^{L(T)}$. Then

$$\begin{aligned}H_{\boldsymbol{q}\tau}&=\{\omega:\omega\in H,\,\omega(\tau(i))\in V_i\text{ for every }i\le n\}\\&=\{\omega:\omega\in H,\,\omega(\tau(i))\in V_i\text{ for every }i\in L(T)\}\end{aligned}$$

(because if $\omega\in H$, $i\le n$ and $i\notin L(T)$ then there is some $\tau'\in T$ such that $\omega\in H_{\boldsymbol{q}\tau'}$, so that $\omega(\tau'(i))\in V_i$ and therefore $\omega(\tau(i))\in V_i$)

$$=\{\omega:\omega\in H,\,\phi_i(\omega)\le\tilde\tau(i)\text{ for every }i\in L(T)\}=\phi^{-1}[\tilde H_\tau].$$

Set $\tilde{\mathcal{G}}=\{\tilde H_\tau:\tau\in T\}$. Because $\mathbb{R}^{L(T)}$ is hereditarily weakly θ-refinable (438O), and $\tilde{\mathcal{G}}$ is a family of open subsets of $\mathbb{R}^{L(T)}$, $\bigcup\tilde{\mathcal{G}}\in\mathcal{J}(\tilde{\mathcal{G}})$. By 438Nc, $H=\bigcup_{\tau\in T}H_{\boldsymbol{q}\tau}=\phi^{-1}[\bigcup\tilde{\mathcal{G}}]$ belongs to $\mathcal{J}(\{H_{\boldsymbol{q}\tau}:\tau\in T\}$ and therefore to $\mathcal{J}(\mathcal{G})$, by 438Nb.

Thus in all three cases the induction proceeds. **Q**

(v) This means that, for any $\boldsymbol{q}\in Q$, $Y_{\boldsymbol{q}}=\bigcup\{H_{\boldsymbol{q}\tau}:\tau\in S_{\boldsymbol{q}}\}$ belongs to $\mathcal{J}(\mathcal{G})$. Since Q is countable, $Y\in\mathcal{J}(\mathcal{G})$, where $Y=\bigcup_{\boldsymbol{q}\in Q}Y_{\boldsymbol{q}}$. But $\bigcup\mathcal{G}\subseteq Y$. **P** If $\omega\in G\in\mathcal{G}$, there are $t_0<\ldots<t_n$ and $V_i'\in\mathcal{I}$, for $i\le n$, such that

$$\omega\in\{\omega':\omega'\in\tilde C^{\Vert},\,\omega'(t_i)\in V_i'\text{ for every }i\le n\}\subseteq G.$$

Set $x_i=\omega(t_i)$, $x_i^-=\lim_{s\uparrow t_i}\omega(s)$, $x_i^+=\lim_{s\downarrow t_i}\omega(s)$ for $i\le n$; let U_i, V_i, $W_i\in\mathcal{U}$ be such that $x_i^-\in U_i$, $x_i\in V_i\subseteq V_i'$, $x_i^+\in W_i$ and any pair of U_i, V_i, W_i are either equal or disjoint; and let $I_0,\ldots,I_n\in\mathcal{I}$ be disjoint and such that $t_i\in I_i$, $\omega(s)\in U_i$ for $s\in I_i\cap\,]-\infty,t_i[$ and $\omega(s)\in W_i$ for $s\in I_i\cap\,]t_i,\infty[$ for each $i\le n$. Then, setting $\boldsymbol{q}=(I_0,\ldots,W_n)$ and $\tau(i)=t_i$ for $i\le n$,

$$\omega\in F_{\boldsymbol{q}\tau}\subseteq H_{\boldsymbol{q}\tau}\subseteq G,$$

so that $\tau\in S_{\boldsymbol{q}}$ and $\omega\in Y_{\boldsymbol{q}}\subseteq Y$. **Q**

As $\mathcal{G}$ is arbitrary, $\tilde C^{\Vert}$ is hereditarily weakly θ-refinable.

438R Corollary (a) Let $I^{\Vert}$ be the split interval (419L). Then any countable power of $I^{\Vert}$ is a hereditarily weakly θ-refinable compact Hausdorff space.

(b) Let Y be the 'Helly space', the space of non-decreasing functions from $[0,1]$ to itself with the topology of pointwise convergence inherited from the product topology on $[0,1]^{[0,1]}$ (Kelley 55, Ex. 5M). Then Y is a hereditarily weakly θ-refinable compact Hausdorff space.

proof These are both (homeomorphic to) subspaces of the space $\tilde C^{\Vert}$ of Proposition 438Q, if we take X there to be $\mathbb{R}$. To see this, argue as follows. For (a), observe that we have a function $f:I^{\Vert}\to\tilde C^{\Vert}$

defined by setting $f(t^-)(s) = f(t^+)(s) = 1$ if $s < t$, $f(t^-)(s) = f(t^+)(s) = 0$ if $s > t$, and $f(t^-)(t) = 0$, $f(t^+)(t) = 1$, and that f is a homeomorphism between $I^{\|}$ and its image. Next, for any $L \subseteq \mathbb{N}$, we can define $g : (I^{\|})^L \to \tilde{C}^{\mathbb{1}}$ by setting

$$\begin{aligned} g(\mathbf{t})(s) &= f(t_n)(s - 2n) \text{ if } n \in L \text{ and } 2n \le s \le 2n+1, \\ &= 0 \text{ if } s \in \mathbb{R} \setminus \bigcup_{n \in L} [2n, 2n+1] \end{aligned}$$

for $\mathbf{t} = \langle t_n \rangle_{n \in L} \in (I^{\|})^L$; it is easy to check that g is a homeomorphism between $(I^{\|})^L$ and its image in $\tilde{C}^{\mathbb{1}}$. As for (b), if we take $g(y)$ to be the extension of the function $y : [0,1] \to [0,1]$ to the function which is constant on each of the intervals $]-\infty, 0]$ and $[1, \infty[$, then $g : Y \to \tilde{C}^{\mathbb{1}}$ is a homeomorphism between Y and its image $g[Y]$.

Since both $(I^{\|})^L$ and Y are compact, they are homeomorphic to closed subsets of $\tilde{C}^{\mathbb{1}}$, and are hereditarily weakly θ-refinable (438La).

***438S Càllàl functions** To support some of the theory of Lévy processes which I will present in §455, I give a further consequence of 438Q.

Proposition Let X be a Polish space. Let $C^{\mathbb{1}} = C^{\mathbb{1}}(X)$ be the set of càllàl functions (definition: 4A2A) from $[0, \infty[$ to X, with its topology of pointwise convergence inherited from the product topology of $X^{[0,\infty[}$.

(a)(i) If $\omega \in C^{\mathbb{1}}$, then ω is continuous at all but countably many points of $[0, \infty[$.

(ii) If ω, $\omega' \in C^{\mathbb{1}}$, D is a dense subset of $[0, \infty[$ containing every point at which ω is discontinuous, and $\omega' \restriction D = \omega \restriction D$, then $\omega' = \omega$.

(b) $C^{\mathbb{1}}$ is hereditarily weakly θ-refinable.

(c) $C^{\mathbb{1}}$ is K-analytic.

proof Fix a complete metric ρ on X defining its topology. Let $\tilde{C}^{\mathbb{1}} \subseteq X^{\mathbb{R}}$ be the space of 438P-438Q.

(a) If $X = \emptyset$ the results are trivial. Otherwise, fix $x_0 \in X$, and for $\omega \in X^{[0,\infty[}$ define $\tilde{\omega} \in X^{\mathbb{R}}$ to be that extension of ω which takes the value x_0 everywhere on $]-\infty, 0[$.

(i) If $\omega \in C^{\mathbb{1}}$, then $\tilde{\omega} \in \tilde{C}^{\mathbb{1}}$; so the result follows from 438Pb.

(ii) If $t \in D$, $\omega'(t)$ is certainly equal to $\omega(t)$. Next,

$$\omega'(0) = \lim_{s \downarrow 0} \omega'(s) = \lim_{s \in D, s \downarrow 0} \omega'(s) = \lim_{s \in D, s \downarrow 0} \omega(s) = \omega(0).$$

If $t \in]0, \infty[\setminus D$, then ω is continuous at t, so

$$\lim_{s \uparrow t} \omega'(s) = \lim_{s \in D, s \uparrow t} \omega'(s) = \lim_{s \in D, s \uparrow 0} \omega(s) = \omega(t),$$

$$\lim_{s \downarrow t} \omega'(s) = \lim_{s \in D, s \downarrow t} \omega'(s) = \lim_{s \in D, s \downarrow 0} \omega(s) = \omega(t).$$

Since $\omega'(t)$ must be either $\lim_{s \uparrow t} \omega'(s)$ or $\lim_{s \downarrow t} \omega'(s)$, it is again equal to $\omega(t)$. So $\omega' = \omega$.

(b) Since $C^{\mathbb{1}}$ is homeomorphic to a subspace of $\tilde{C}^{\mathbb{1}}$, it is hereditarily weakly θ-refinable (438Qb, 438La).

(c) Set $\tilde{Q} = \{\tilde{\omega} : \omega \in C^{\mathbb{1}}\}$; then $\tilde{Q} \subseteq \tilde{C}^{\mathbb{1}}$ is homeomorphic to $C^{\mathbb{1}}$. But $\tilde{Q}$ is a Souslin-F set in $\tilde{C}^{\mathbb{1}}$. **P** If $\omega \in C^{\mathbb{1}}$ then it belongs to $\tilde{Q}$ iff $\omega(t) = x_0$ for every $t < 0$. $\lim_{t \downarrow 0} \omega(t) = \omega(0)$, and $\omega(t) \in \{\lim_{s \uparrow t} \omega(s), \lim_{s \downarrow t} \omega(s)\}$ for every $t > 0$. Now

$$\{\omega : \omega(t) = x_0 \text{ for every } t < 0\}$$

is closed, while

$$\{\omega : \omega \in \tilde{C}^{\mathbb{1}}, \omega(0) = \lim_{t \downarrow 0} \omega(t)\} = \{\omega : \omega \in \tilde{C}^{\mathbb{1}}, \omega(0) = \lim_{i \to \infty} \omega(2^{-i})\}$$

(because $\lim_{t \downarrow 0} \omega(t)$ is defined for every $\omega \in \tilde{C}^{\mathbb{1}}$)

$$= \bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{i\ge m} \{\omega : \omega \in \tilde{C}^{\Vdash}, \, \rho(\omega(2^{-i}), \omega(0)) \le 2^{-k}\}$$

is Souslin-F. As for the other condition, note that for $t > 0$ and $\omega \in \tilde{C}^{\Vdash}$, $\omega(t)$ belongs to $\{\lim_{s\uparrow t}\omega(s), \lim_{s\downarrow t}\omega(s)\}$ iff for every $\epsilon > 0$ there are distinct rational numbers q, $q' \in [t-\epsilon, t+\epsilon]$ such that $\omega(q)$ and $\omega(q')$ belong to $B(\omega(t), \epsilon)$. Let $\mathcal{U}$ be a countable base for the topology of X and $\mathcal{I}$ a countable base for the topology of $]0,\infty[$ not containing $\emptyset$; then for $\omega \in \tilde{C}^{\Vdash}$, $\omega(t) \in \{\lim_{s\uparrow t}\omega(s), \lim_{s\downarrow t}\omega(s)\}$ for every $t > 0$ if and only if

for every $U \in \mathcal{U}$ and $I \in \mathcal{I}$ either $I \cap \omega^{-1}[U] = \emptyset$ or $I \cap \mathbb{Q} \cap \omega^{-1}[\overline{U}]$ has at least two members.

Since, for $U \in \mathcal{U}$ and $I \in \mathcal{I}$,

$$\{\omega : \omega \in \tilde{C}^{\Vdash}, \, I \cap \omega^{-1}[U] = \emptyset\} = \bigcap_{t\in I}\{\omega : \omega \in \tilde{C}^{\Vdash}, \, \omega(t) \notin U\}$$

is closed in $\tilde{C}^{\Vdash}$, and

$$\{\omega : \omega \in \tilde{C}^{\Vdash}, \, I \cap \mathbb{Q} \cap \omega^{-1}[\overline{U}] \text{ has at least two members}\}$$
$$= \bigcup_{\substack{q,q'\in I\cap\mathbb{Q} \\ q<q'}} \{\omega : \omega(q), \, \omega(q') \in \overline{U}\}$$

is F_σ in $\tilde{C}^{\Vdash}$, while $\mathcal{I}$ and $\mathcal{U}$ are countable,

$$\{\omega : \omega \in \tilde{C}^{\Vdash}, \, \omega(t) \in \{\lim_{s\uparrow t}\omega(s), \lim_{s\downarrow t}\omega(s)\} \text{ for every } t > 0\}$$

is Souslin-F in $\tilde{C}^{\Vdash}$. Taking the intersection, we see that $\tilde{Q}$ is Souslin-F. **Q**

Accordingly $\tilde{Q}$ and $C^{\Vdash}$ are K-analytic (422Hb).

438T Proposition Assume that $\mathfrak{c}$ is measure-free. Then $(I^{\|})^{\mathbb{N}}$, the Helly space (438Rb) and the spaces $\tilde{C}^{\Vdash}(X)$, $C^{\Vdash}(X)$ of 438Q and 438S, for any Polish space X, are all Radon spaces.

proof By 438Q-438S, they are K-analytic and hereditarily weakly θ-refinable, also they have weight at most $w(X^{\mathbb{R}}) \le \mathfrak{c}$. They are therefore pre-Radon (434Jf), Borel-measure-complete (438M) and Radon (434Ka).

438U In 434R I described a construction of product measures. In accordance with my general practice of examining the measure algebra of any new measure, I give the following result.

Proposition Let X and Y be topological spaces with σ-finite Borel measures μ, ν respectively. Suppose that *either* X is first-countable *or* ν is τ-additive and effectively locally finite. Write λ for the Borel measure on $X \times Y$ defined by the formula

$$\lambda W = \int \nu W[\{x\}]\mu(dx) \text{ for every Borel set } W \subseteq X \times Y$$

as in 434R(ii). If *either* the weight of X *or* the Maharam type of ν is a measure-free cardinal, then for every Borel set $W \subseteq X \times Y$ there is a set $W' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $\lambda(W \triangle W') = 0$; consequently, the measure algebra of λ can be identified with the localizable measure algebra free product of the measure algebras of μ and ν.

proof (a) Write $(\mathfrak{B}, \bar{\nu})$ for the measure algebra of ν. With its measure-algebra topology, this is metrizable (323Gb). Let $\langle Y_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of Borel sets of finite measure in Y with union Y.

(b) For the moment (down to the end of (e) below) fix on an open set $W \subseteq X \times Y$. For $x \in X$, set $f(x) = W[\{x\}]^\bullet$ in $\mathfrak{B}$. Then $f : X \to \mathfrak{B}$ is Borel measurable.

P (i) Let $H \subseteq \mathfrak{B}$ be an open set. For k, $n \in \mathbb{N}$ set

$$E_{nk} = \{x : x \in X, \, 2^{-n}k \le \nu(Y_n \cap W[\{x\}]) < 2^{-n}(k+1)\}.$$

Just as in part (a) of the proof of 434R, the function $x \mapsto \nu(Y_n \cap W[\{x\}])$ is lower semi-continuous, so E_{nk} is a Borel set. Set

$$G_{nk} = \bigcup\{G : G \subseteq X \text{ is open}, \, G \cap E_{nk} \subseteq f^{-1}[H]\};$$

then $E=\bigcup_{n,k\in\mathbb{N}}(G_{nk}\cap E_{nk})$ is a Borel set included in $f^{-1}[H]$.

(ii) The point is that $E=f^{-1}[H]$. To see this, take any x such that $f(x)\in H$. Then there are $b\in\mathfrak{B}$, $\epsilon>0$ such that $\bar\nu b<\infty$ and $c\in H$ whenever $c\in\mathfrak{B}$ and $\bar\nu(b\cap(c\triangle f(x)))\le 5\epsilon$. Since $b=\sup_{n\in\mathbb{N}}b\cap Y_n^\bullet$, there is an $n\in\mathbb{N}$ such that $\bar\nu(b\setminus Y_n^\bullet)\le\epsilon$ and $2^{-n}\le\epsilon$. In this case $c\in H$ whenever $c\in\mathfrak{B}$ and $\bar\nu(Y_n^\bullet\cap(c\triangle f(x)))\le 4\epsilon$; thus

$$\{x':\nu(Y_n\cap(W[\{x'\}]\triangle W[\{x\}]))\le 4\epsilon\}\subseteq f^{-1}[H].$$

Let $k\in\mathbb{N}$ be such that $2^{-n}k\le\nu(Y_n\cap W[\{x\}])<2^{-n}(k+1)$, that is, $x\in E_{nk}$. Again using the ideas of part (a) of the proof of 434R, there are an open set G containing x and an open set $V\subseteq Y$ such that $G\times V\subseteq W$ and $\nu(Y_n\cap V)\ge 2^{-n}(k-1)$. Now if $x'\in G\cap E_{nk}$, $V\subseteq W[\{x'\}]\cap W[\{x\}]$, so

$$\begin{aligned}\nu(Y_n\cap(W[\{x'\}]\triangle W[\{x\}]))&\le\nu(Y_n\cap W[\{x'\}])+\nu(Y_n\cap W[\{x\}])-2\nu(Y_n\cap V)\\&\le 2^{-n}((k+1)+(k+1)-2(k-1))=4\cdot 2^{-n}\le 4\epsilon.\end{aligned}$$

But this means that $G\cap E_{nk}\subseteq f^{-1}[H]$, so $G\subseteq G_{nk}$ and $x\in G\cap E_{nk}\subseteq E$. As x is arbitrary, $f^{-1}[H]\subseteq E$ and $E=f^{-1}[H]$.

(iii) Thus $f^{-1}[H]$ is a Borel set. As H is arbitrary, f is Borel measurable. **Q**

(c) We need to know also that if $\mathcal{H}$ is a disjoint family of open subsets of $\mathfrak{B}$ all meeting $f[X]$, then

$$\#(\mathcal{H})\le\max(\omega,\min(w(X),\tau(\mathfrak{B})))$$

P Repeat the ideas of (b) above, setting

$$G_{nk}^{(H)}=\bigcup\{G:G\subseteq X\text{ is open, }G\cap E_{nk}\subseteq f^{-1}[H]\}$$

for $H\in\mathcal{H}$ and k, $n\in\mathbb{N}$, so that $f^{-1}[H]=\bigcup_{n,k\in\mathbb{N}}G_{nk}^{(H)}\cap E_{nk}$. For fixed n and k the family $\langle G_{nk}^{(H)}\cap E_{nk}\rangle_{H\in\mathcal{H}}$ is disjoint, so can have at most $w(E_{nk})\le w(X)$ non-empty members (4A2D again). But this means that

$$\mathcal{H}=\bigcup_{n,k\in\mathbb{N}}\{H:G_{nk}^{(H)}\cap E_{nk}\ne\emptyset\}$$

has cardinal at most $\max(\omega,w(X))$.

On the other hand, there is a set $B\subseteq\mathfrak{B}$, with cardinal $\tau(\mathfrak{B})$, which τ-generates $\mathfrak{B}$. The algebra $\mathfrak{B}_0$ generated by B has cardinal at most $\max(\omega,\#(B))$ (331Gc), and $\mathfrak{B}_0$ is topologically dense in $\mathfrak{B}$ (323J), so every member of $\mathcal{H}$ meets $\mathfrak{B}_0$, and

$$\#(\mathcal{H})\le\#(\mathfrak{B}_0)\le\max(\omega,\tau(\mathfrak{B})).$$

Putting these together, we have the result. **Q**

In particular, under the hypotheses above, $\#(\mathcal{H})$ is measure-free whenever $\mathcal{H}$ is a disjoint family of open subsets of $\mathfrak{B}$ all meeting $f[X]$.

(d) The next step is to observe that there is a conegligible Borel set $Z\subseteq X$ such that $f[Z]$ is separable. **P** Let $\mathcal{H}$ be a σ-disjoint base for the topology of $\mathfrak{B}$; express it as $\bigcup_{n\in\mathbb{N}}\mathcal{H}_n$ where each $\mathcal{H}_n$ is disjoint. Let $\langle X_m\rangle_{m\in\mathbb{N}}$ be a cover of X by Borel sets of finite measure. For $n\in\mathbb{N}$ consider $\mathcal{H}'_n=\{H:H\in\mathcal{H}_n,\ H\cap f[X]\ne\emptyset\}$. For $m\in\mathbb{N}$, we have a totally finite measure ν_{nm} with domain $\mathcal{P}\mathcal{H}'_n$ defined by saying

$$\nu_{nm}\mathcal{E}=\mu(X_m\cap f^{-1}(\bigcup\mathcal{E}))$$

for every $\mathcal{E}\subseteq\mathcal{H}'_n$. Since $\mathcal{H}'_n$ has measure-free cardinal, by (c), there must be a countable set $\mathcal{E}_{nm}\subseteq\mathcal{H}'_n$ such that $\nu_{nm}(\mathcal{H}'_n\setminus\mathcal{E}_{nm})=0$. Set

$$Z=X\setminus\bigcup_{m,n\in\mathbb{N}}(X_m\cap f^{-1}[\bigcup(\mathcal{H}'_n\setminus\mathcal{E}_{nm})]);$$

then Z is conegligible. If $x\in Z$ and $f(x)\in H\in\mathcal{H}_n$, then there is some $m\in\mathbb{N}$ such that $x\in X_m$, while $H\in\mathcal{H}'_n$, so H must belong to $\mathcal{E}_{nm}$. But this means that $\{f[Z]\cap H:H\in\mathcal{H}\}$, which is a base for the topology of $f[Z]$, is just $\{f[Z]\cap H:H\in\bigcup_{m,n\in\mathbb{N}}\mathcal{E}_{mn}\}$, and is countable. So $f[Z]$ is separable (4A2Oc), as required. **Q**

(e) 418T(a-ii) now tells us that there is a set $W'\in\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $f(x)=W'[\{x\}]^\bullet$ for every $x\in Z$, so that $\nu(W[\{x\}]\triangle W'[\{x\}])=0$ for almost every x, that is, $\lambda(W\triangle W')=0$. And this is true for every open set $W\subseteq X\times Y$.

(f) Now let $\mathcal{W}$ be the family of those Borel sets $W \subseteq X \times Y$ for which there is a $W' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $\lambda(W \triangle W') = 0$. This is a σ-algebra containing every open set, so is the whole Borel σ-algebra, as required.

Since the c.l.d. product measure λ_0 on $X \times Y$ is just the completion of its restriction to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ (251K), and λ_0 and λ agree on $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ (by Fubini's theorem), the embedding $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y) \subsetneqq \mathcal{B}(X \times Y)$ induces an isomorphism between the measure algebras of λ and λ_0. As remarked in 325Eb, because μ and ν are strictly localizable, the latter may be identified with 'the' localizable measure algebra free product of the measure algebras of μ and ν.

Remark The hypothesis on the weight of X can be slightly weakened; see 438Yg. 439L below shows that some restriction on (X, μ) and (Y, ν) is necessary.

438X Basic exercises (a) Show that a cardinal κ is measure-free iff $M_\sigma = M_\tau$, where M_σ, M_τ are the spaces of countably additive and completely additive functionals on the algebra $\mathcal{P}\kappa$ (362B).

(b) Let (X, Σ, μ) be a localizable measure space. Show that the following are equiveridical: (α) the magnitude of μ (definition: 332Ga) is either finite or a measure-free cardinal (β) every absolutely continuous countably additive functional $\nu : \Sigma \to \mathbb{R}$ is truly continuous. (*Hint*: 363S.)

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with measure-free cellularity. Show that any countably additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is completely additive.

(d) Let U be a Dedekind complete Riesz space such that any disjoint order-bounded family in U^+ has measure-free cardinal. Show that $U_c^\sim = U^\times$.

(e) Let (X, Σ, μ) be a complete locally determined measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Suppose that the magnitude of ν is either finite or a measure-free cardinal. Show that μ is strictly localizable.

(f) Let (X_1, Σ_1, μ_1), (X_2, Σ_2, μ_2), $(Y_1, \mathrm{T}_1, \nu_1)$ and $(Y_2, \mathrm{T}_2, \nu_2)$ be measure spaces, and λ_1, λ_2 the c.l.d. product measures on $X_1 \times Y_1$, $X_2 \times Y_2$ respectively; suppose that $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ are inverse-measure-preserving functions, and that $h(x, y) = (f(x), g(y))$ for $x \in X_1$, $y \in Y_1$. Show that if μ_2 and ν_2 are both strictly localizable, with magnitudes which are either finite or measure-free cardinals, then h is inverse-measure-preserving. (Compare 251L.)

>(g) Show that if κ is a measure-free cardinal, so is ω_κ. (*Hint*: show by induction on ordinals ξ that if $\#(\xi)$ is measure-free, then so is ω_ξ.)

>(h) Let (X, Σ, μ) be a complete locally determined measure space, (Y, ρ) a complete metric space with measure-free weight, and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of measurable functions from X to Y. Show that $\{x : \lim_{n\to\infty} f_n(x)$ is defined in $Y\}$ is measurable. (Cf. 418C.)

(i) Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Give $L^0(\nu)$ the topology of convergence in measure. Suppose that $f : X \to L^0(\nu)$ is measurable and there is a conegligible set $X_0 \subseteq X$ such that $w(f[X_0])$ is measure-free. Show that there is an $h \in \mathcal{L}^0(\lambda)$ such that $f(x) = h_x^\bullet$ for every $x \in X$, where $h_x(y) = h(x, y)$ for $(x, y) \in \operatorname{dom} h$. (Cf. 418S.)

(j) Let (Y, T, ν) be a σ-finite measure space, and $(\mathfrak{B}, \bar{\nu})$ its measure algebra, with its usual topology; assume that the Maharam type of $\mathfrak{B}$ is measure-free. Let (X, Σ, μ) be a σ-finite measure space and Λ the domain of the c.l.d. product measure λ on $X \times Y$. Show that if $f : X \to \mathfrak{B}$ is measurable, then there is a $W \in \Lambda$ such that $f(x) = W[\{x\}]^\bullet$ for every $x \in X$. (Cf. 418T(b-ii).)

(k) Let (X, Σ, μ) be a complete locally determined measure space and V a normed space such that $w(V)$ is measure-free. (i) Show that the space $\mathcal{L}$ of measurable functions from X to V is a linear space, setting $(f + g)(x) = f(x) + g(x)$, etc. (ii) Show that if V is a Riesz space with a Riesz norm then $\mathcal{L}$ is a Riesz space under the natural operations.

>(l) Let X be a topological space and $\mathcal{G}$ a point-finite open cover of X such that $\#(\mathcal{G})$ is measure-free. Suppose that $E \subseteq X$ is such that $E \cap G$ is universally measurable for every $G \in \mathcal{G}$. Show that E is universally measurable. (Compare 434Xf(iv).)

>(m) Show that for a metrizable space X, the following are equiveridical: (i) X is Borel-measure-compact; (ii) X is Borel-measure-complete; (iii) X is measure-compact; (iv) $w(X)$ is measure-free.

(n) Let X be a topological space and $\mathcal{G}$ a family of open subsets of X. Show that the following are equiveridical: (i) there is a σ-isolated family $\mathcal{A}$ of sets, refining $\mathcal{G}$, such that $\bigcup \mathcal{A} = \bigcup \mathcal{G}$; (ii) there is a sequence $\mathcal{H}_n$ of families of open sets, all refining $\mathcal{G}$, such that for every $x \in \bigcup \mathcal{G}$ there is an $n \in \mathbb{N}$ such that $\{H : x \in H \in \mathcal{H}_n\}$ is finite and not empty.

(o) Let Y be the Helly space. (i) Show that Y is a compact convex subset of $\mathbb{R}^{[0,1]}$ with its usual topology. (ii) Show that there is a natural one-to-one correspondence between the split interval $I^{\Vert}$ and the set of extreme points of Y, matching $t^- \in I^{\Vert}$ with the function $\chi\,[0,t[$ and t^+ with $\chi[0,t]$. (iii) Let P_{R} be the set of Radon probability measures on $I^{\Vert}$ with its narrow topology (437J). Show that there is a natural homeomorphism $\phi : P_{\mathrm{R}} \to Y$ defined by setting $\phi(\mu)(t) = \mu[0^-, t^-]$ for $\mu \in P_{\mathrm{R}}$, $t \in [0,1]$.

(p) Show that any countable power of the Sorgenfrey line (415Xc, 439Q) is hereditarily weakly θ-refinable.

>(q) Let $I^{\Vert}$ be the split interval. Show that $I^{\Vert} \times I^{\Vert}$ is a Radon space iff $\mathfrak{c}$ is measure-free. (*Hint*: $\{(\alpha^+, (1-\alpha)^+) : \alpha \in [0,1]\}$ is a discrete Borel subset with cardinal $\mathfrak{c}$.)

(r) Give $\mathbb{R}$ the right-facing Sorgenfrey topology (415Xc). Show that the following are equiveridical: (i) $\mathfrak{c}$ is measure-free; (ii) $\mathbb{R}^{\mathbb{N}}$, with the corresponding product topology, is Borel-measure-complete; (iii) $\mathbb{R}^2$, with the product topology, is Borel-measure-compact. (Compare 439Q.)

(s) Suppose that $\mathfrak{c}$ is measure-free. Let $X \subseteq \mathbb{R}^{\mathbb{R}}$ be the set of functions of bounded variation on $\mathbb{R}$, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^{\mathbb{R}}$. Show that X is a Radon space. (*Hint*: X is an F_σ subset of the space $\tilde{C}^{\Downarrow\!\Uparrow}$ of 438Q.)

438Y Further exercises (a) Let (X, Σ, μ) be a probability space and $\langle E_i \rangle_{i \in I}$ a point-finite family of measurable sets such that $\nu J = \mu(\bigcup_{i \in J} E_i)$ is defined for every $J \subseteq I$. Show directly that ν is a uniformly exhaustive Maharam submeasure on $\mathcal{P}I$, and use the Kalton-Roberts theorem to prove 438Ba.

(b) Suppose that $\mathfrak{c}$ is measure-free, but that $\kappa > \mathfrak{c}$ is not measure-free. Show that there is a non-principal ω_1-complete ultrafilter on κ. (*Hint*: part (b) of the proof of 451Q.)

(c) Show that if X is a metrizable space and $\min(\mathfrak{c}, w(X))$ is measure-free, then every σ-finite Borel measure on X has countable Maharam type.

(d) Let X be a metacompact T_1 space. Show that X is Borel-measure-compact iff every closed discrete subspace has measure-free cardinal.

(e) Let X be a topological space such that every subspace of X is metacompact and has measure-free cellularity. Show that X is Borel-measure-complete.

(f) Let X be a normal metacompact Hausdorff space. Show that it is measure-compact iff every closed discrete subspace has measure-free cardinal.

(g) In 438U, show that it would be enough to suppose that every discrete subset of X has measure-free cardinal.

(h) Suppose that $\mathfrak{c}$ is measure-free. Let D be any subset of $\mathbb{R}$ and $X \subseteq \mathbb{R}^D$ the set of functions of bounded variation on D, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^D$. Show that X is a Radon space.

(i) Let X be a totally ordered set with its order topology. Show that if $c(X)$ is measure-free then every σ-finite Borel measure on X has countable Maharam type. (Cf. 434Yo.)

(j) Suppose that X is a normal metacompact Hausdorff space which is not realcompact. Show that there are a closed discrete subset D of X and a non-principal ω_1-complete ultrafilter on D. (*Hint*: in 435C, if we start with a $\{0,1\}$-valued Baire measure we obtain a $\{0,1\}$-valued Borel measure; in the proof of 438Ba, if μ is $\{0,1\}$-valued then ν is $\{0,1\}$-valued.)

(k) Let Z be a regular Hausdorff space, T a Dedekind complete totally ordered space with least and greatest elements a, b, and $x : T \to Z$ a function such that $\lim_{s\uparrow t} x(s)$ and $\lim_{s\downarrow t} x(s)$ are defined in Z for every $t \in T$ (except $t = a$ in the first case and $t = b$ in the second). Show that $x[T]$ is relatively compact in Z.

(l) Let (X, ρ) be a metric space, and P_{Bor} the set of Borel probability measures on X. For μ, $\nu \in P_{\mathrm{Bor}}$ set $\bar\rho_{\mathrm{KR}}(\mu, \nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to [-1, 1]$ is 1-Lipschitz$\}$. (i) Show that $\bar\rho_{\mathrm{KR}}$ is a metric on P_{Bor}. (ii) Let $\mathfrak{T}_{\mathrm{KR}}$ be the topology it induces on P_{Bor}. Show that $\mathfrak{T}_{\mathrm{KR}}$ is finer than the narrow topology on P_{Bor}. (iii) Show that the following are equiveridical: (α) the narrow topology on P_{Bor} is metrizable; (β) $\mathfrak{T}_{\mathrm{KR}}$ is the narrow topology on P_{Bor}; (γ) $w(X)$ is measure-free. (Cf. 437Rg, 437Yp.)

(m) Let (X, Σ, μ) be a complete locally determined measure space, and $\theta = \frac{1}{2}(\mu^* + \mu_*)$ the outer measure described in 413Xd. Show that if the measure μ_θ defined by Carathéodory's method is not equal to μ, then there is a set $A \subseteq X$ such that $0 < \mu^* A < \infty$ and the subspace measure on A induced by μ measures every subset of A.

438 Notes and comments Since the axiom 'every cardinal is measure-free' is admissible – that is, will not lead to a paradox unless one is already latent in the Zermelo-Fraenkel axioms for set theory – it is tempting, in the context of this section, to assume it; so that 'every complete metric space is Radon' becomes a theorem, along with 'every measurable function from a quasi-Radon measure space to a metrizable space is almost continuous' (438G), '$U_c^\sim = U^\times$ for every Dedekind complete Riesz space U' (438Xd), 'metacompact spaces are Borel-measure-compact' (438J), 'the sum of two measurable functions from a complete probability space to a normed space is measurable' (438Xk) and 'the Helly space is Radon' (438T). Undoubtedly the consequent mathematical universe is tidier. In my view, the tidiness is the tidiness of poverty. Apart from anything else, it leads us to neglect such questions as 'is every measurable function from a Radon measure space to a metrizable space almost continuous?', which have answers in ZFC (451T).

From the point of view of measure theory, the really interesting question is whether $\mathfrak{c}$ is measure-free. It is not quite clear from the results above why this should be so; 438T is a very small part of the story. There is a larger hint in 438Ce-438Cf: if $\mathfrak{c}$ is measure-free, but $\kappa > \mathfrak{c}$ is not measure-free, then the witnessing measures will be purely atomic. I will return to this point in §543 of Volume 5. For a general exploration of universes in which $\mathfrak{c}$ is *not* measure-free, see §544 and Fremlin 93. For fragments of what happens if we suppose that we have an atom for a measure which witnesses that κ is not measure-free, see 438Yb and the notes on normal filters in 4A1I-4A1L.

There are many further applications of 438Q besides those in 438R and 438Xp-438Xs. But the most obvious candidate, the space $C(\mathbb{R})$ of continuous real-valued functions on $\mathbb{R}$, although indeed it is a Borel subset of the potentially Radon space of 438Q, is in fact Radon whether or not $\mathfrak{c}$ is measure-free (454Sa). As soon as we start using any such special axiom as '$\mathfrak{c} = \omega_1$' or '$\mathfrak{c}$ is measure-free', we must make a determined effort to check, through such examples as 438Xq, that our new theorems do indeed depend on something more than ZFC.

Version of 7.7.14

439 Examples

As in Chapter 41, I end this chapter with a number of examples, exhibiting some of the boundaries around the results in the rest of the chapter, and filling in a gap with basic facts about Lebesgue measure

(439E). The first three examples (439A) are measures defined on σ-subalgebras of the Borel σ-algebra of $[0,1]$ which have no extensions to the whole Borel algebra. The next part of the section (439B-439G) deals with 'universally negligible' sets; I use properties of these to show that Hausdorff measures are generally not semi-finite (439H), closing some unfinished business from §264, and that smooth linear functionals may fail to be representable by integrals in the absence of Stone's condition (439I). In 439J-439R I set out some examples relevant to §§434-435, filling out the classification schemes of 434A and 435A, with spaces which just miss being Radon (439K) or measure-compact (439N, 439P, 439Q). In 439S I present the canonical example of a non-Prokhorov topological space, answering an obvious question from §437.

439A Example Let $\mathcal{B}$ be the Borel σ-algebra of $[0,1]$. There is a probability measure ν defined on a σ-subalgebra T of $\mathcal{B}$ which has no extension to a measure on $\mathcal{B}$.

first construction Let $A \subseteq [0,1]$ be an analytic set which is not Borel (423M). Let $\mathcal{I}$ be the family of sets of the form $E \cup F$ where E, F are Borel sets, $E \subseteq A$ and $F \subseteq [0,1] \setminus A$. Then $\mathcal{I}$ is a σ-ideal of $\mathcal{B}$ not containing $[0,1]$. Set $\mathrm{T} = \mathcal{I} \cup \{[0,1] \setminus H : H \in \mathcal{I}\}$, and define $\nu : \mathrm{T} \to \{0,1\}$ by setting $\nu H = 0$, $\nu([0,1] \setminus H) = 1$ for every $H \in \mathcal{I}$; then ν is a probability measure (cf. countable-cocountable measures (211R) or Dieudonné's measure (411Q)).

? If $\mu : \mathcal{B} \to [0,1]$ is a measure extending ν, then its completion $\hat{\mu}$ measures A (432A). Also $\hat{\mu}$ is a Radon measure (433Cb). Now every compact subset of A belongs to $\mathcal{I}$, so

$$\hat{\mu}A = \sup\{\hat{\mu}K : K \subseteq A \text{ is compact}\} = \sup\{\nu K : K \subseteq A \text{ is compact}\} = 0.$$

Similarly $\hat{\mu}([0,1] \setminus A) = 0$, which is absurd. **X**

second construction This time, let $\mathcal{I}$ be the family of meager Borel sets in $[0,1]$. As before, let T be $\mathcal{I} \cup \{[0,1] \setminus E : E \in \mathcal{I}\}$, and set $\nu E = 0$, $\nu([0,1] \setminus E) = 1$ for $E \in \mathcal{I}$. **?** If μ is a Borel measure extending ν, then $\mu([0,1] \setminus \mathbb{Q}) = 1$, and μ is tight (that is, inner regular with respect to the compact sets), so there is a closed subset F of $[0,1] \setminus \mathbb{Q}$ such that $\mu F > 0$. But F is nowhere dense, so $\nu F = 0$. **X**

third construction[9] There is a function $f : [0,1] \to \{0,1\}^{\mathfrak{c}}$ which is $(\mathcal{B}, \mathcal{B}\mathfrak{a})$-measurable, where $\mathcal{B}\mathfrak{a}$ is the Baire σ-algebra of $\{0,1\}^{\mathfrak{c}}$, and such that $f[\,[0,1]\,]$ meets every non-empty member of $\mathcal{B}\mathfrak{a}$. **P** Set $X = C([0,1])^{\mathbb{N}}$ with the product of the norm topologies, so that X is an uncountable Polish space (4A2Pe, 4A2Qc), and $([0,1], \mathcal{B})$ is isomorphic to $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X (424Da). Define $g : X \to \{0,1\}^{[0,1]}$ by saying that $g(\langle u_i \rangle_{i\in\mathbb{N}})(t) = 1$ iff $\lim_{i\to\infty} u_i(t) = 1$. For each $t \in [0,1]$,

$$\{\langle u_i \rangle_{i\in\mathbb{N}} : \lim_{i\to\infty} u_i(t) = 1\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \{\langle u_i \rangle_{i\in\mathbb{N}} : |u_i(t) - 1| \le 2^{-m} \text{ for every } i \ge n\}$$

is a Borel subset of X, so g is $(\mathcal{B}(X), \mathcal{B}\mathfrak{a}(\{0,1\}^{[0,1]}))$-measurable, where $\mathcal{B}\mathfrak{a}(\{0,1\}^{[0,1]})$ is the Baire σ-algebra of $\{0,1\}^{[0,1]}$ (4A3Ne). If $E \in \mathcal{B}\mathfrak{a}(\{0,1\}^{[0,1]})$ is non-empty, there is a countable set $I \subseteq [0,1]$ such that E is determined by coordinates in I (4A3Nb), so that $E \supseteq \{w : w{\restriction}I = z\}$ for some $z \in \{0,1\}^I$. Now we can find a sequence $\langle u_i \rangle_{i\in\mathbb{N}}$ in $C([0,1])$ such that $\lim_{i\to\infty} u_i(t) = z(t)$ for every $t \in I$ (if $I \subseteq \{t_j : j \in \mathbb{N}\}$, take u_i such that $|u_i(t_j) - z(t_j)| \le 2^{-i}$ whenever $j \le i$), and in this case $g(\langle u_i \rangle_{i\in\mathbb{N}}) \in E$.

Because $(X, \mathcal{B}(X)) \cong ([0,1], \mathcal{B})$ and $(\{0,1\}^{[0,1]}, \mathcal{B}\mathfrak{a}(\{0,1\}^{[0,1]})) \cong (\{0,1\}^{\mathfrak{c}}, \mathcal{B}\mathfrak{a})$, we can copy g to a function f with the required properties. **Q**

In particular, $f[\,[0,1]\,]$ has full outer measure for the usual measure $\nu_{\mathfrak{c}}$ on $\{0,1\}^{\mathfrak{c}}$, because $\nu_{\mathfrak{c}}$ is completion regular (415E). Setting $\mathrm{T} = \{f^{-1}[H] : H \in \mathcal{B}\mathfrak{a}\}$, we have a measure ν with domain T such that f is inverse-measure-preserving for ν and $\nu_{\mathfrak{c}}$ (234F). The map $H^\bullet \mapsto f^{-1}[H]^\bullet$ from the measure algebra of $\nu_{\mathfrak{c}}$ to the measure algebra of ν is measure-preserving; since it is surely surjective, the measure algebras are isomorphic, and ν has Maharam type $\mathfrak{c}$.

However, any probability measure on the whole algebra $\mathcal{B}$ has countable Maharam type (433A), so cannot extend ν.

Remark Compare 433J-433K.

[9] I am grateful to M.Laczkovich and D.Preiss for showing this to me.

439B Definition Let X be a Hausdorff space. I will call X **universally negligible** if there is no Borel probability measure μ defined on X such that $\mu\{x\} = 0$ for every $x \in X$. A subset of X will be 'universally negligible' if it is universally negligible in its subspace topology.

439C Proposition Let X be a Hausdorff space.

(a) If A is a subset of X, the following are equiveridical:

(i) A is universally negligible;

(ii) $\mu^*A = 0$ whenever μ is a Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$;

(iii) $\mu^*A = 0$ whenever μ is a σ-finite topological measure on X such that $\mu\{x\} = 0$ for every $x \in A$;

(iv) for every σ-finite topological measure μ on X there is a countable set $B \subseteq A$ such that $\mu^*A = \mu B$;

(v) A is a Radon space and every compact subset of A is scattered.

In particular, countable subsets of X are universally negligible.

(b) The family of universally negligible subsets of X is a σ-ideal.

(c) Suppose that Y is a universally negligible Hausdorff space and that $f : X \to Y$ is a Borel measurable function such that $f^{-1}[\{y\}]$ is universally negligible for every $y \in Y$. Then X is universally negligible.

(d) If the topology on X is discrete, X is universally negligible iff $\#(X)$ is measure-free.

proof (a)(i)$\Rightarrow$(iii) If A is universally negligible and μ is a σ-finite topological measure on X such that $\mu\{x\} = 0$ for every $x \in A$, let μ_A be the subspace measure on A. **?** If $\mu^*A = \alpha > 0$, then (because μ is σ-finite) there is a measurable set $E \subseteq X$ such that $\gamma = \mu^*(E \cap A)$ is finite and non-zero. The subspace measure $\mu_{E\cap A}$ is a topological measure on $E\cap A$; set $\nu F = \gamma^{-1}\mu_{E\cap A}(E\cap F)$ for relatively Borel sets $F \subseteq A$; then ν is a Borel probability measure on A which is zero on singletons. **X** So $\mu^*A = 0$.

(iii)$\Rightarrow$(iv) If (iii) is true and μ is a σ-finite topological measure on X, set $B = \{x : x \in X, \mu\{x\} > 0\}$. Because μ is σ-finite, B must be countable, therefore measurable, and if we set $\nu E = \mu(E \setminus B)$ for every Borel set $E \subseteq X$, ν is a σ-finite Borel measure on X and $\nu\{x\} = 0$ for every $x \in X$. By (iii), $\nu^*A = 0$, that is, there is a Borel set $E \supseteq A$ such that $\mu(E \setminus B) = 0$; in which case

$$\mu^*A \le \mu(E \setminus (B \setminus A)) = \mu(E \setminus B) + \mu(A \cap B) = \mu(A \cap B) \le \mu^*A,$$

so $\mu^*A = \mu(A \cap B)$. As μ is arbitrary, (iv) is true.

(iv)$\Rightarrow$(ii) is trivial.

not-(i)$\Rightarrow$not-(ii) If A is not universally negligible, let μ be a Borel probability measure on A which is zero on singletons. Set $\nu E = \mu(E \cap A)$ for any Borel set $E \subseteq X$; then ν is a Borel probability measure on X which is zero on singletons, and $\nu^*A = 1$.

(i)$\Rightarrow$(v) Suppose that A is universally negligible. Let μ be a totally finite Borel measure on A. Applying (i)$\Rightarrow$(iv) with $X = A$, we see that there is a countable set $B \subseteq A$ such that $\mu B = \mu A$; but this means that μ is inner regular with respect to the finite subsets of B, which of course are compact. As μ is arbitrary, A is a Radon space.

? Suppose, if possible, that A has a compact set K which is not scattered. In this case there is a continuous surjection $f : K \to [0, 1]$ (4A2G(j-iv)). Now there is a Radon probability measure ν on K such that f is inverse-measure-preserving for ν and Lebesgue measure on $[0, 1]$ and induces an isomorphism of the measure algebras, so that ν is atomless (418L). Accordingly we have a Borel probability measure μ on A defined by setting $\mu E = \nu(K \cap E)$ for every relatively Borel set $E \subseteq A$, and $\mu\{x\} = 0$ for every $x \in A$, so A is not universally negligible. **X** Thus all compact subsets of A are scattered, and (v) is true.

(v)$\Rightarrow$(i) Now suppose that (v) is true and that that μ is a Borel probability measure on A. Then μ has an extension to a Radon measure $\tilde\mu$ (434F(a-iii)). Let $K \subseteq A$ be a non-empty compact set which is self-supporting for $\tilde\mu$ (416Dc). K is scattered, so has an isolated point $\{x\}$; because K is self-supporting, $\mu\{x\} = \tilde\mu\{x\} > 0$. As μ is arbitrary, A is universally negligible.

(b) This is immediate from (a-ii).

(c) Let μ be a Borel probability measure on X. Then $F \mapsto \nu f^{-1}[F]$ is a Borel probability measure on Y. Because Y is universally negligible, there must be a $y \in Y$ such that $\mu f^{-1}[\{y\}] > 0$. Set $E = f^{-1}[\{y\}]$ and let μ_E be the subspace measure on E. Then μ_E is a non-zero totally finite Borel measure on E. Since E is supposed to be universally negligible, there must be some $x \in E$ such that $0 < \mu_E\{x\} = \mu\{x\}$.

(d) This is just a re-phrasing of the definition in 438A.

439D Remarks (a) The following will be useful when interpreting the definition in 439B. Let X be a hereditarily Lindelöf Hausdorff space and μ a topological probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. Then μ is atomless.

P Suppose that $\mu H > 0$. Write

$$\mathcal{G} = \{G : G \subseteq X \text{ is open}, \mu(G \cap H) = 0\}.$$

Then there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}$ (4A2H(c-i)), so

$$\mu(H \cap \bigcup \mathcal{G}) = \mu(H \cap \bigcup \mathcal{G}_0) = 0,$$

and $\mu(H \setminus \bigcup \mathcal{G}) > 0$. Because μ is zero on singletons, $H \setminus \bigcup \mathcal{G}$ has at least two points x, y say. Now there are disjoint open sets G_0, G_1 containing x, y respectively, and neither belongs to $\mathcal{G}$, so $H \cap G_0$, $H \cap G_1$ are disjoint subsets of H of positive measure. Thus H is not an atom. As H is arbitrary, μ is atomless. **Q**

(b) The obvious applications of (a) are when X is separable and metrizable; but, more generally, we can use it on any Hausdorff space with a countable network, e.g., on any analytic space.

439E Lemma (a) Let E, $B \subseteq \mathbb{R}$ be such that E is measurable and $\mu_L E$, $\mu_L^* B$ are both greater than 0, where μ_L is Lebesgue measure. Then $E - B = \{x - y : x \in E, y \in B\}$ includes a non-trivial interval.

(b) If $A \subseteq \mathbb{R}$ and $\mu_L^* A > 0$, then $A + \mathbb{Q}$ is of full outer measure in $\mathbb{R}$.

proof (a) By 223B or 261Da, there are $a \in E$, $b \in B$ such that

$$\lim_{\delta \downarrow 0} \frac{1}{2\delta} \mu(E \cap [a - \delta, a + \delta]) = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \mu^*(B \cap [b - \delta, b + \delta]) = 1.$$

Let $\gamma > 0$ be such that

$$\mu_L(E \cap [a - \delta, a + \delta]) > \frac{3}{2}\delta, \quad \mu_L^*(B \cap [b - \delta, b + \delta]) > \frac{3}{2}\delta$$

whenever $0 < \delta \le \gamma$. Now suppose that $0 < \delta \le \gamma$. Then

$$\begin{aligned} \mu_L((E + b) \cap [a + b, a + b + \delta]) &= \mu_L(E \cap [a, a + \delta]) \\ &\ge \mu_L(E \cap [a - \delta, a + \delta]) - \delta > \frac{1}{2}\delta, \end{aligned}$$

and similarly

$$\begin{aligned} \mu_L^*((B + a + \delta) \cap [a + b, a + b + \delta]) &= \mu_L^*(B \cap [b - \delta, b]) \\ &\ge \mu_L^*(B \cap [b - \delta, b + \delta]) - \delta > \frac{1}{2}\delta. \end{aligned}$$

But this means that $(E + b) \cap (B + a + \delta)$ cannot be empty. If $u \in (E + b) \cap (B + a + \delta)$, then $u - b \in E$ and $u - a - \delta \in B$ so

$$a - b + \delta = (u - b) - (u - a - \delta) \in E - B.$$

As δ is arbitrary, $E - B$ includes the interval $]a - b, a - b + \gamma]$.

(b) ? Suppose, if possible, otherwise; that there is a measurable set $E \subseteq \mathbb{R}$ such that $\mu_L E > 0$ and $E \cap (A + \mathbb{Q}) = \emptyset$. Then $E - A$ does not meet $\mathbb{Q}$ and cannot include any non-trivial interval. **X**

Remark There will be a dramatic generalization of (a) in 443Db.

439F Proposition Let κ be the least cardinal of any set of non-zero Lebesgue outer measure in $\mathbb{R}$.

(a) There is a set $X \subseteq [0,1]$ with cardinal κ and full outer Lebesgue measure.

(b) If (Z, T, ν) is any atomless complete locally determined measure space and $A \subseteq Z$ has cardinal less than κ, then $\nu^*A = 0$.

(c) (Grzegorek 81) There is a universally negligible set $Y \subseteq [0,1]$ with cardinal κ.

proof (a) Take any set $A \subseteq \mathbb{R}$ such that $\#(A) = \kappa$ and $\mu_L^* A > 0$, where μ_L is Lebesgue measure. Set $B = A + \mathbb{Q}$. Then $(\mu_L)_*(\mathbb{R} \setminus B) = 0$, by 439Eb. Set $X = [0,1] \cap B$; then $\mu_L^* X = 1$ while $\#(X) \le \#(B) = \kappa$. By the definition of κ, $\#(X)$ must be exactly κ.

(b) ? Otherwise, by 412Jc, there is a set $F \subseteq Z$ such that $\nu F < \infty$ and $\nu^*(F \cap A) > 0$. By 343Cc, there is a function $f : F \to [0, \nu F]$ which is inverse-measure-preserving for the subspace measure ν_F and Lebesgue measure on $[0, \nu F]$. But $f[A \cap F]$ has cardinal less than κ, so $\mu_L f[A \cap F] = 0$ and

$$0 < \nu^*(A \cap F) \le \nu f^{-1}[f[A \cap F]] = 0,$$

which is absurd. **X**

(c)(i) Enumerate X as $\langle x_\xi \rangle_{\xi<\kappa}$. For each $\xi < \kappa$, $A_\xi = \{x_\eta : \eta \le \xi\}$ has cardinal less than κ, so is Lebesgue negligible; let $\langle I_{\xi n} \rangle_{n\in\mathbb{N}}$ be a sequence of intervals covering A_ξ with $\sum_{n=0}^\infty \mu_L I_{\xi n} < \frac{1}{2}$. Enlarging the intervals slightly if necessary, we may suppose that every $I_{\xi n}$ has rational endpoints; let $\langle J_m \rangle_{m\in\mathbb{N}}$ enumerate the family of intervals in $\mathbb{R}$ with rational endpoints.

Set

$$C_{mn} = \{\xi : \xi < \kappa,\ I_{\xi n} = J_m\}$$

for each $m, n \in \mathbb{N}$.

(ii) If ν is an atomless totally finite measure on κ which measures every C_{mn}, then $\nu\kappa = 0$. **P** Note first that (by (b), applied to the completion of ν) $\nu^*\xi = 0$ for every $\xi < \kappa$. Let λ be the (c.l.d.) product of μ_X, the subspace measure on X, with ν. Set

$$B = \bigcup_{m,n\in\mathbb{N}} ((X \cap J_m) \times C_{mn}) \subseteq X \times \kappa.$$

Then B is measured by λ, so, by Fubini's theorem,

$$\int \nu B[\{x\}] \mu_X(dx) = \int \mu_X B^{-1}[\{\xi\}] \nu(d\xi)$$

(252D).

Now look at the sectional measures $\nu B[\{x\}]$, $\mu_X B^{-1}[\{\xi\}]$. (Because B is actually a countable union of measurable rectangles, these are always defined.) For any $x \in X$, there is an $\eta < \kappa$ such that $x = x_\eta$, and now

$$\begin{aligned} B[\{x\}] &= \{\xi : \text{ there are } m, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } \xi \in C_{mn}\} \\ &= \{\xi : \text{ there are } m, n \in \mathbb{N} \text{ such that } x_\eta \in J_m \text{ and } I_{\xi n} = J_m\} \\ &= \{\xi : \text{ there is an } n \in \mathbb{N} \text{ such that } x_\eta \in I_{\xi n}\} \supseteq \kappa \setminus \eta \end{aligned}$$

by the choice of the $I_{\xi n}$. But as $\nu^*\eta = 0$, this means that $\nu B[\{x\}] = \nu\kappa$.

On the other hand, if $\xi < \kappa$, then

$$\begin{aligned} B^{-1}[\{\xi\}] &= \{x : \text{ there are } m, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } \xi \in C_{mn}\} \\ &= \{x : \text{ there are } m, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } I_{\xi n} = J_m\} \\ &= \{x : \text{ there is an } n \in \mathbb{N} \text{ such that } x \in I_{\xi n}\} = X \cap \bigcup_{n\in\mathbb{N}} I_{\xi n}, \end{aligned}$$

so that

$$\mu_X B^{-1}[\{\xi\}] \le \sum_{n=0}^\infty \mu_L I_{\xi n} \le \frac{1}{2}.$$

Returning to the integrals, we have

$$\nu\kappa = \int \nu B[\{x\}] \mu_X(dx) = \int \mu_X B^{-1}[\{\xi\}] \nu(d\xi) \le \frac{1}{2}\nu\kappa,$$

so that $\nu\kappa$ must be 0, as claimed. **Q**

(iii) Now there is an injective function $g:\kappa\to[0,1]$ such that $g[C_{mn}]$ is relatively Borel in $g[\kappa]$ for every m, $n\in\mathbb{N}$. **P** Define $h:\kappa\to\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ by setting

$$\begin{aligned} h(\xi)(m,n,k) &= 1 \text{ if } \xi\in C_{mn} \text{ and } x_\xi\in J_k, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then h is injective (because if $\xi\ne\eta$ then $x_\xi\ne x_\eta$, so there is some k such that $x_\xi\in J_k$ and $x_\eta\notin J_k$), and

$$h[C_{mn}]=h[\kappa]\cap\{w: \text{ there is some } k \text{ such that } w(m,n,k)=1\}$$

is relatively Borel in $h[\kappa]$ for every m, $n\in\mathbb{N}$. But now recall that $\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}\cong\{0,1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set $C\subseteq[0,1]$ (4A2Uc). If $\phi:\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}\to C$ is any homeomorphism, then ϕh has the required properties. **Q**

(iv) Set $Y=g[\kappa]$. Because g is injective, $\#(Y)=\kappa$. Also Y is universally negligible. **P** Suppose that $\tilde\nu$ is a Borel measure on Y which is zero on singletons. Then it is atomless, because Y is separable and metrizable (439D). So its copy $\nu=\tilde\nu(g^{-1})^{-1}$ on κ is atomless. Because $g[C_{mn}]$ is a Borel subset of Y, ν measures C_{mn} for all m, $n\in\mathbb{N}$, so $\tilde\nu Y=\nu\kappa=0$, by (ii) above. **Q**

439G Corollary A metrizable continuous image of a universally negligible metrizable space need not be universally negligible.

proof Take X and Y from 439Fa and 439Fc above, and let $f:X\to Y$ be any bijection. Let Γ be the graph of f. The projection map $(x,y)\mapsto y:\Gamma\to Y$ is continuous and injective, so Γ is universally negligible, by 439Cc. On the other hand, the projection map $(x,y)\mapsto x:\Gamma\to X$ is continuous and surjective, and X is surely not universally negligible, since it is not Lebesgue negligible.

439H Corollary One-dimensional Hausdorff measure on $\mathbb{R}^2$ is not semi-finite.

proof Let μ_{H1} be one-dimensional Hausdorff measure on $\mathbb{R}^2$. Let X, Γ be the sets described in 439F and the proof of 439G.

(a) $\mu^*_{H1}\Gamma>0$. **P** The first-coordinate map $\pi_1:\mathbb{R}^2\to\mathbb{R}$ is 1-Lipschitz, so, writing μ_L for Lebesgue measure on $\mathbb{R}$,

$$1=\mu^*_L X=\mu^*_L\pi_1[\Gamma]\le\mu^*_{H1}\Gamma$$

by 264G/471J and 264I. **Q**

(b) If $E\subseteq\mathbb{R}^2$ and $\mu_{H1}E<\infty$, then $\mu_{H1}(E\cap\Gamma)=0$, because Γ is universally negligible, so $E\cap\Gamma$ is universally negligible (439Cb), and μ_{H1} is a topological measure (264E/471Da) which is zero on singletons.

(c) ? Suppose, if possible, that Γ is not measured by μ_{H1}. Then there is a set $A\subseteq\mathbb{R}^2$ such that $\mu^*_{H1}A<\mu^*_{H1}(A\cap\Gamma)+\mu^*_{H1}(A\setminus\Gamma)$ (264Fb/471Dc). Let E be a Borel set including A such that $\mu_{H1}E=\mu^*_{H1}A$ (264Fa/471Db); then $\mu_{H1}(E\cap\Gamma)=0$, so

$$\mu^*_{H1}(A\cap\Gamma)+\mu^*_{H1}(A\setminus\Gamma)\le\mu_{H1}(E\cap\Gamma)+\mu^*_{H1}A=\mu^*_{H1}A. \ \mathbf{X}$$

(d) Since Γ is measurable, not negligible, and meets every measurable set of finite measure in a negligible set, it is purely infinite, and μ_{H1} is not semi-finite.

Remark Compare 471S below.

439I Example There are a set X, a Riesz subspace U of $\mathbb{R}^X$ and a smooth positive linear functional $h:U\to\mathbb{R}$ which is not expressible as an integral.

proof By 439F, we have a non-negligible X and a universally negligible set Y, both subsets of $[0,1]$, of the same cardinality. Replacing Y by $Y\setminus\{0\}$ if need be, we may suppose that $0\notin Y$. Let $f:X\to Y$ be any bijection.

Let U be the Riesz subspace $\{u \times f : u \in C_b\} \subseteq \mathbb{R}^X$, where C_b is the space of bounded continuous functions from X to $\mathbb{R}$. Because f is strictly positive, $u \mapsto u \times f : C_b \to U$ is a bijection, therefore a Riesz space isomorphism; moreover, for a non-empty set $A \subseteq C_b$, $\inf_{u \in A} u(x) = 0$ for every $x \in X$ iff $\inf_{u \in A} u(x)f(x) = 0$ for every $x \in X$. We therefore have a smooth linear functional $h : U \to \mathbb{R}$ defined by setting $h(u \times f) = \int u\,d\mu_X$ for every $u \in C_b$, where μ_X is the subspace measure on X induced by Lebesgue measure. (By 415B, μ_X is quasi-Radon, so the integral it defines on C_b is smooth, as noted in 436H.)

? But suppose, if possible, that h is the integral with respect to some measure ν on X. Since $f \in U$, it must be T-measurable, where T is the domain of the completion $\hat\nu$ of ν. Note that $\hat\nu\{x\} = 0$ for every $x \in X$. **P** Set $u_n(y) = \max(0, 1 - 2^n|y - x|)$ for $y \in X$. Then

$$f(x)\hat\nu\{x\} = \lim_{n\to\infty} \int u_n \times f\,d\nu = \lim_{n\to\infty} h(u_n \times f)$$
$$= \lim_{n\to\infty} \int u_n\,d\mu_X = \mu_X\{x\} = 0,$$

so $\hat\nu\{x\} = 0$. **Q**

For Borel sets $E \subseteq [0,1]$ set $\lambda E = \hat\nu f^{-1}[E]$. Then the completion $\hat\lambda$ of λ is a Radon measure on $[0,1]$ (433Cb or 256C). If $t \in [0,1]$ then $f^{-1}[\{t\}]$ contains at most one point, so $\hat\lambda\{t\} = \lambda\{t\} = 0$. But Y is supposed to be universally negligible, so $\lambda^*Y = \hat\lambda^*Y = 0$ (439Ca), that is, there is a Borel set $E \supseteq Y$ with $\lambda E = 0$; in which case $\nu X = \hat\nu f^{-1}[E] = 0$, which is impossible. **X**

Thus h is not an integral, despite being a smooth linear functional on a Riesz subspace of $\mathbb{R}^X$.

Remark Compare 436H. This example is adapted from FREMLIN & TALAGRAND 78.

439J Example Assume that there is some cardinal κ which is not measure-free. Give κ its discrete topology, and let μ be a probability measure with domain $\mathcal{P}\kappa$ such that $\mu\{\xi\} = 0$ for every $\xi < \kappa$. Now every subset of κ is open-and-closed, so μ is simultaneously a Baire probability measure and a completion regular Borel probability measure. Of course it is not τ-additive. In the classification schemes of 434A and 435A, we have a measure which is of type B_1 as a Borel measure and type E_3 as a Baire measure.

439K Example There is a first-countable compact Hausdorff space which is not Radon.

proof The construction starts from a compact metrizable space $(Z, \mathfrak{S})$ with an atomless Radon probability measure μ. The obvious candidate is $[0,1]$ with Lebesgue measure; but for technical convenience in a later application I will instead use $Z = \{0,1\}^{\mathbb{N}}$ with its usual product topology and measure (254J).

(a) There is a topology $\mathfrak{T}_{\mathfrak{c}}$ on Z such that

(α) $\mathfrak{S} \subseteq \mathfrak{T}_{\mathfrak{c}}$;

(β) every point of Z belongs to a countable set which is compact and open for $\mathfrak{T}_{\mathfrak{c}}$;

(γ) if $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of $\mathfrak{T}_{\mathfrak{c}}$-closed sets with empty intersection, then $\bigcap_{n\in\mathbb{N}} \overline{F}_n^{\mathfrak{S}}$ is countable, where I write $\overline{F}^{\mathfrak{S}}$ for the $\mathfrak{S}$-closure of F.

P(i) $\mathfrak{T}_{\mathfrak{c}}$ will be the last in a family $\langle \mathfrak{T}_\xi\rangle_{\xi\le\mathfrak{c}}$ of topologies. We must begin by enumerating Z as $\langle z_\xi\rangle_{\xi<\mathfrak{c}}$ and taking a family $\langle\langle I_{\xi n}\rangle_{n\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$ running over $([Z]^{\le\omega})^{\mathbb{N}}$ with cofinal repetitions. (This can be done because $\#([Z]^{\le\omega}) = \mathfrak{c}$, by 2A1Hb.) Together with $\langle\mathfrak{T}_\xi\rangle_{\xi\le\mathfrak{c}}$ we choose simultaneously families $\langle x_\xi\rangle_{\xi<\mathfrak{c}}$, $\langle y_\xi\rangle_{\xi<\mathfrak{c}}$ of points in Z, and the inductive hypothesis will be

$\mathfrak{T}_\xi$ is a topology on $X_\xi = \{x_\eta : \eta < \xi\} \cup \{y_\eta : \eta < \xi\}$ finer than the topology on X_ξ induced by $\mathfrak{S}$;

if $\eta < \xi \le \mathfrak{c}$, then $X_\eta \in \mathfrak{T}_\xi$ and $\mathfrak{T}_\eta$ is the subspace topology on X_η induced by $\mathfrak{T}_\xi$;

every point of X_ξ belongs to a countable set which is compact and open for $\mathfrak{T}_\xi$.

The induction starts with $X_0 = \emptyset$, $\mathfrak{T}_0 = \{\emptyset\}$.

(ii) *Inductive step to a successor ordinal* Suppose that we have found X_ξ and $\mathfrak{T}_\xi$ where $\xi < \mathfrak{c}$.

(α) Start by picking $y_\xi \in Z \setminus X_\xi$ such that $y_\xi = z_\xi$ if $z_\xi \notin X_\xi$. Examine the sequence $\langle I_{\xi n}\rangle_{n\in\mathbb{N}}$. If either $\bigcup_{n\in\mathbb{N}} I_{\xi n} \not\subseteq X_\xi$ or $\bigcap_{n\in\mathbb{N}} \overline{I}_{\xi n}^{\mathfrak{S}}$ is countable, take x_ξ to be any point of $Z \setminus (X_\xi \cup \{y_\xi\})$ and set $K_m = \emptyset$ for every m before proceeding to (γ) below.

(β) If $I_{\xi n} \subseteq X_\xi$ for every n and $\bigcap_{n\in\mathbb{N}} \overline{I}^{\mathfrak{S}}_{\xi n}$ is uncountable, it must have cardinal $\mathfrak{c}$, by 423L, so cannot be included in $X_\xi \cup \{y_\xi\}$. Take any $x_\xi \in \bigcap_{n\in\mathbb{N}} \overline{I}^{\mathfrak{S}}_{\xi n} \setminus (X_\xi \cup \{y_\xi\})$. Let $\langle t_m\rangle_{m\in\mathbb{N}}$ be a sequence in Z such that $t_m{\restriction}m = x_\xi{\restriction}m$ for every $m \in \mathbb{N}$ and $t_m \in I_{\xi n}$ whenever $r \in \mathbb{N}$, $n \le 2r$ and $m = r^2 + n$. (Thus, for each n, $t_m \in I_{\xi n}$ for infinitely many m, while $\langle t_m\rangle_{m\in\mathbb{N}} \to x_\xi$ in the ordinary sense.) By the inductive hypothesis, we can find countable sets $K_m \subseteq X_\xi$, compact and open for $\mathfrak{T}_\xi$, such that $t_m \in K_m$ for each m. Because $\{t : t \in X_\xi,\ t{\restriction}m = x_\xi{\restriction}m\}$ is open-and-closed for $\mathfrak{T}_\xi$ and contains t_m, we may suppose that $t{\restriction}m = x_\xi{\restriction}m$ for every $t \in K_m$.

(γ) Let $\mathfrak{T}_{\xi+1}$ be the topology on $X_{\xi+1} = X_\xi \cup \{x_\xi, y_\xi\}$ generated by

$$\mathfrak{T}_\xi \cup \{\{y_\xi\}\} \cup \{L_n : n \in \mathbb{N}\},$$

where $L_n = \{x_\xi\} \cup \bigcup_{m\ge n} K_m$ for each n.

(δ) Because $\mathfrak{T}_\xi \subseteq \mathfrak{T}_{\xi+1}$, X_ξ will be open in $X_{\xi+1}$. Because the K_m are always $\mathfrak{T}_\xi$-open, and x_ξ, y_ξ are distinct points of $Z \setminus X_\xi$, the topology on X_ξ induced by $\mathfrak{T}_{\xi+1}$ is just $\mathfrak{T}_\xi$. Consequently (by the inductive hypothesis) the topology on X_η induced by $\mathfrak{T}_{\xi+1}$ is $\mathfrak{T}_\eta$ for every $\eta \le \xi$. We have $t{\restriction}n = x_\xi{\restriction}n$ for every $t \in L_n$, so $\mathfrak{T}_{\xi+1}$ is finer than the usual topology on $X_{\xi+1}$.

If $x \in X_\xi$, then there is a countable $\mathfrak{T}_\xi$-open $\mathfrak{T}_\xi$-compact set containing x, which is still $\mathfrak{T}_{\xi+1}$-open and $\mathfrak{T}_{\xi+1}$-compact. Of course $\{y_\xi\}$ is a countable $\mathfrak{T}_{\xi+1}$-open $\mathfrak{T}_{\xi+1}$-compact set containing y_ξ. As for x_ξ, L_0 is surely countable and $\mathfrak{T}_{\xi+1}$-open. To see that it is $\mathfrak{T}_{\xi+1}$-compact, observe that any ultrafilter containing L_0 either contains every L_n, and converges to x_ξ, or contains some K_m and converges to a point of K_m.

Thus the induction proceeds at successor stages.

(iii) *Inductive step to a limit ordinal* If $\xi \le \mathfrak{c}$ is a non-zero limit ordinal, then we have $X_\xi = \bigcup_{\eta<\xi} X_\eta$, and can take $\mathfrak{T}_\xi$ to be the topology generated by $\bigcup_{\eta<\xi} \mathfrak{T}_\eta$. It is easy to check that this works (because the topologies $\mathfrak{T}_\eta$ are consistent with each other).

(iv) At the end of the induction, we have $X_\mathfrak{c} = Z$ because $z_\xi \in X_{\xi+1} \subseteq X_\mathfrak{c}$ for every ξ. The final topology $\mathfrak{T}_\mathfrak{c}$ on Z will have the properties (α) and (β) required. **?** Now suppose, if possible, that $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of $\mathfrak{T}_\mathfrak{c}$-closed sets with empty intersection, and that $\bigcap_{n\in\mathbb{N}} \overline{F}^{\mathfrak{S}}_n$ is uncountable. For each $n \in \mathbb{N}$, let $J_n \subseteq F_n$ be a countable $\mathfrak{S}$-dense set. Then there is some $\zeta < \mathfrak{c}$ such that $\bigcup_{n\in\mathbb{N}} J_n \subseteq X_\zeta$ (because $\operatorname{cf}\mathfrak{c} > \omega$, see 4A1A(c-iii)). Let $\xi \ge \zeta$ be such that $J_n = I_{\xi n}$ for every $n \in \mathbb{N}$. Then in the construction of $\mathfrak{T}_{\xi+1}$ we must be in case (β) of (ii) above. Taking $\langle t_m\rangle_{m\in\mathbb{N}}$ as described there, we have $\langle t_m\rangle_{m\in\mathbb{N}} \to x_\xi$ for $\mathfrak{T}_{\xi+1}$, and therefore for $\mathfrak{T}_\mathfrak{c}$. But for any $n \in \mathbb{N}$, $t_m \in J_n \subseteq F_n$ for infinitely many m, so $x_\xi \in F_n$. Thus $x_\xi \in \bigcap_{n\in\mathbb{N}} F_n$; but this is impossible. **X**

So we have a topology of the type required. **Q**

(b) There is a probability measure ν on Z, extending the usual measure μ, such that with respect to $\mathfrak{T}_\mathfrak{c}$ ν is a topological measure inner regular with respect to the closed sets, but is not τ-additive.

P Let $\mathcal{K}$ be the family of $\mathfrak{T}_\mathfrak{c}$-closed subsets of Z. For $F \in \mathcal{K}$, set $\phi F = \mu\overline{F}^{\mathfrak{S}}$.

(i) If E, F are disjoint $\mathfrak{T}_\mathfrak{c}$-closed sets, then $\overline{E}^{\mathfrak{S}} \cap \overline{F}^{\mathfrak{S}}$ must be countable (take $F_{2n} = E$, $F_{2n+1} = F$ in (a-γ)). So

$$\begin{aligned}\phi(E \cup F) = \mu(\overline{E \cup F}^{\mathfrak{S}}) &= \mu\overline{E}^{\mathfrak{S}} + \mu\overline{F}^{\mathfrak{S}} - \mu(\overline{E}^{\mathfrak{S}} \cap \overline{F}^{\mathfrak{S}})\\ &= \mu\overline{E}^{\mathfrak{S}} + \mu\overline{F}^{\mathfrak{S}} = \phi E + \phi F.\end{aligned}$$

(ii) If E, $F \in \mathcal{K}$, $E \subseteq F$ and $\epsilon > 0$, there is an $\mathfrak{S}$-open set $G \supseteq \overline{E}^{\mathfrak{S}}$ such that $\mu G \le \mu\overline{E}^{\mathfrak{S}} + \epsilon$. Now $F \setminus G \in \mathcal{K}$ and

$$\phi F = \mu(\overline{F}^{\mathfrak{S}} \cap G) + \mu(\overline{F}^{\mathfrak{S}} \setminus G) \le \mu G + \phi(F \setminus G) \le \phi E + \phi(F \setminus G) + \epsilon.$$

Putting this together with (i), we see that

$$\phi F = \phi E + \sup\{\phi E' : E' \in \mathcal{K},\ E' \subseteq F \setminus E\}.$$

(iii) If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then

$$\lim_{n\to\infty}\phi F_n=\lim_{n\to\infty}\mu\overline{F}_n^{\mathfrak{S}}=\mu(\bigcap_{n\in\mathbb{N}}\overline{F}_n^{\mathfrak{S}})=0.$$

(iv) Thus $\mathcal{K}$ and ϕ satisfy all the conditions of 413J, and there is a measure ν, extending ϕ, which is defined on every member of $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, and therefore is (for $\mathfrak{T}_\mathfrak{c}$) a topological measure inner regular with respect to the closed sets.

If we write $\mathcal{V}$ for the family of $\mathfrak{T}_\mathfrak{c}$-compact $\mathfrak{T}_\mathfrak{c}$-open countable subsets of Z, then for any $K\in\mathcal{V}$

$$\nu K=\phi K=\mu K=0,$$

while $\mathcal{V}$ is upwards-directed and has union Z; so that ν is not τ-additive. **Q**

(c) So far we seem to have very little more than is provided by ω_1 with the order topology and Dieudonné's measure. The point of doing all this work is the next step. Set $X=Z\times\{0,1\}$ and give X the topology $\mathfrak{T}$ generated by

$$\{G\times\{0,1\}:G\in\mathfrak{S}\}\cup\{H\times\{1\}:H\in\mathfrak{T}_\mathfrak{c}\}\cup\{X\setminus(K\times\{1\}):K\text{ is }\mathfrak{T}_\mathfrak{c}\text{-compact}\}.$$

(i) $\mathfrak{T}$ is Hausdorff. **P** If w, z are distinct points of X, then either their first coordinates differ and they are separated by sets of the form $G_0\times\{0,1\}$, $G_1\times\{0,1\}$ where G_0, G_1 belong to $\mathfrak{S}$, or they are of the form $(x,1)$, $(x,0)$ and are separated by open sets of the form $K\times\{1\}$, $X\setminus(K\times\{1\})$ for some set K which is compact and open for $\mathfrak{T}_\mathfrak{c}$. **Q**

(ii) $\mathfrak{T}$ is compact. **P** Let $\mathcal{F}$ be an ultrafilter on X. Writing $\pi_1(x,0)=\pi_1(x,1)=x$ for $x\in Z$, $\pi_1[[\mathcal{F}]]$ is $\mathfrak{S}$-convergent, to x_0 say. If $K\times\{1\}\in\mathcal{F}$ for some $\mathfrak{T}_\mathfrak{c}$-compact set K, then $\mathcal{F}$ is $\mathfrak{T}$-convergent to $(x,1)$; otherwise, it is $\mathfrak{T}$-convergent to $(x,0)$ (using 4A2B(a-iv)). **Q**

(iii) $\mathfrak{T}$ is first-countable. **P** If $x\in Z$, then $\{(x,0),(x,1)\}=\pi_1^{-1}[\{x\}]$ is a G_δ set in X because $\{x\}$ is a G_δ set in Z and π_1 is continuous (4A2C(a-iii)). Now $\{(x,0)\}$ and $\{(x,1)\}$ are relatively open in $\{(x,0),(x,1)\}$, so are G_δ sets in X (4A2C(a-iv)). Thus singletons are G_δ sets. Because $\mathfrak{T}$ is compact and Hausdorff, it is first-countable (4A2Kf). **Q**

(iv) $(X,\mathfrak{T})$ is not a Radon space. **P** $Z\times\{1\}$ is an open subset of X, homeomorphic to Z with the topology $\mathfrak{T}_\mathfrak{c}$. But the measure ν of (b) above (or, if you prefer, its restriction to the $\mathfrak{T}_\mathfrak{c}$-Borel algebra) witnesses that $\mathfrak{T}_\mathfrak{c}$ is not a Radon topology, so $\mathfrak{T}$ also cannot be a Radon topology, by 434Fc. **Q**

Remark Aficionados will recognise $\mathfrak{T}_\mathfrak{c}$ as a kind of 'JKR-space', derived from the construction in Juhász Kunen & Rudin 76.

439L Example Suppose that κ is a cardinal which is not measure-free; let μ be a probability measure with domain $\mathcal{P}\kappa$ which is zero on singletons. Give κ its discrete topology, so that μ is a Borel measure and κ is first-countable. Let ν be the restriction of the usual measure on $Y=\{0,1\}^\kappa$ to the algebra $\mathcal{B}$ of Borel subsets of Y, so that ν is a τ-additive probability measure, and λ the product measure on $\kappa\times Y$ constructed by the method of 434R. Then

$$W=\{(\xi,y):\xi<\kappa,\,y(\xi)=1\}=\bigcup_{\xi<\kappa}\{\xi\}\times\{y:y(\xi)=1\}$$

is open in $\kappa\times Y$.

If $W'\in\mathcal{P}\kappa\widehat{\otimes}\mathcal{B}$ then $\lambda(W\triangle W')=\frac{1}{2}$. **P** There is a countable set $\mathcal{E}\subseteq\mathcal{B}$ such that W' belongs to the σ-algebra generated by $\{A\times E:A\subseteq\kappa,\,E\in\mathcal{E}\}$ (331Gd). For $J\subseteq\kappa$, write $\pi_J(y)=y{\restriction}J$ for $y\in Y$, let ν_J be the usual measure on $\{0,1\}^J$ and T_J its domain, and let T'_J be the family of sets $E\subseteq Y$ such that there are $H,\,H'\in\mathrm{T}_J$ such that $\pi_J^{-1}[H]\subseteq E\subseteq\pi_J^{-1}[H']$ and $\nu_J(H'\setminus H)=0$. Then $\mathrm{T}'_J\subseteq\mathrm{T}'_K$ whenever $J\subseteq K\subseteq\kappa$, and every set measured by ν belongs to T'_J for some countable J (254Oc). There is therefore a countable set $J\subseteq\kappa$ such that $\mathcal{E}\subseteq\mathrm{T}'_J$. Also, of course, T'_J is a σ-algebra of subsets of Y.

The set

$$\{V:V\subseteq\kappa\times Y,\,V[\{\xi\}]\in\mathrm{T}'_J\text{ for every }\xi<\kappa\}$$

is a σ-algebra of subsets of $\kappa\times Y$ containing $A\times E$ whenever $A\subseteq\kappa$ and $E\in\mathcal{E}$, so contains W'. But this means that if $\xi\in\kappa\setminus J$, $W[\{\xi\}]$ and $W'[\{\xi\}]$ are stochastically independent, and $\nu(W[\{\xi\}]\triangle W'[\{\xi\}])=\frac{1}{2}$. Since $\mu(\kappa\setminus J)=1$,

$$\lambda(W\triangle W') = \int \nu(W[\{\xi\}]\triangle W'[\{\xi\}])\mu(d\xi) = \frac{1}{2},$$

as claimed. **Q**

In particular, $W^\bullet$ in the the measure algebra of λ cannot be represented by a member of $\mathcal{P}\kappa\widehat{\otimes}\mathcal{B}$.

439M Example There is a first-countable locally compact Hausdorff space X with a Baire probability measure μ which is not τ-additive and has no extension to a Borel measure. In the classification of 435A, μ is of type E_0.

proof Let Ω be the set of non-zero countable limit ordinals, and for each $\xi \in \Omega$ let $\langle\theta_\xi(i)\rangle_{i\in\mathbb{N}}$ be a strictly increasing sequence of ordinals with supremum ξ. Set $X = \omega_1 \times (\omega + 1)$, and define a topology $\mathfrak{T}$ on X by saying that $G \subseteq X$ is open iff

$\{\xi : (\xi, n) \in G\}$ is open in the order topology of ω_1 for every $n < \omega$,

whenever $\xi \in \Omega$ and $(\xi, \omega) \in G$ then there is some $n < \omega$ such that $(\eta, i) \in G$ whenever $n \le i < \omega$ and $\theta_\xi(i) < \eta \le \xi$.

This is finer than the product of the order topologies, so is Hausdorff. For every $\xi < \omega_1$ and $n < \omega$, $(\xi + 1) \times \{n\}$ is a countable compact open set containing (ξ, n); for every $\xi \in \omega_1 \setminus \Omega$, $\{(\xi, \omega)\}$ is a countable compact open set containg (ξ, ω); and for every $\xi \in \Omega$,

$$\{(\xi, \omega)\} \cup \{(\eta, i) : i < \omega,\, \theta_\xi(i) < \eta \le \xi\}$$

is a countable compact open subset of X containing (ξ, ω). Thus $\mathfrak{T}$ is locally compact, and every singleton subset of X is G_δ, so $\mathfrak{T}$ is first-countable (4A2Kf again).

If $f : X \to \mathbb{R}$ is continuous, then for every $n < \omega$ there is a $\zeta_n < \omega_1$ such that f is constant on $\{(\xi, n) : \zeta_n \le \xi < \omega_1\}$ (4A2S(b-iii)). Setting $\zeta = \sup_{n<\omega} \zeta_n$, f must be constant on $\{(\xi, \omega) : \xi \in \Omega,\, \xi > \zeta\}$. **P** If $\xi, \eta \in \Omega\setminus(\zeta+1)$, then $f(\xi, \omega) = \lim_{i\to\infty} f(\theta_\xi(i)+1, i)$ and $f(\eta, \omega) = \lim_{i\to\infty} f(\theta_\eta(i)+1, i)$. But there is some n such that both $\theta_\xi(i)$ and $\theta_\eta(i)$ are greater than ζ for every $i \ge n$, so that $f(\theta_\xi(i)+1, i) = f(\theta_\eta(i)+1, i)$ for every $i \ge n$ and $f(\xi, \omega) = f(\eta, \omega)$. **Q**

Writing Σ for the family of subsets E of X such that $\{\xi : \xi \in \Omega,\, (\xi, \omega) \in E\}$ is either countable or cocountable in Ω, Σ is a σ-algebra of subsets of X such that every continuous function is Σ-measurable, so every Baire set belongs to Σ. We therefore have a Baire measure μ_0 on X defined by saying that $\mu_0 E = 0$ if $E \cap (\Omega \times \{\omega\})$ is countable, 1 otherwise. $\{(\xi + 1) \times (\omega + 1) : \xi < \omega_1\}$ is a cover of X by negligible open-and-closed sets, so μ_0 is not τ-additive.

? Suppose, if possible, that μ were a Borel measure on X extending μ_0. Then we must have $\mu(\omega_1\times\{n\}) = \mu_0(\omega_1 \times \{n\}) = 0$ for every $n < \omega$, so $\mu(\omega_1 \times \{\omega\}) = 1$. Let λ be the subspace measure on $\omega_1 \times \{\omega\}$ induced by μ. If $A \subseteq \omega_1$, $(A \times \{\omega\}) \cup (\omega_1 \times \omega)$ is an open set, so λ is defined on every subset of $\omega_1 \times \{\omega\}$; and if $\xi < \omega_1$, then $\mu_0((\xi + 1) \times (\omega + 1)) = 0$, so λ is zero on singletons. And this contradicts Ulam's theorem (419G, 438Cd). **X**

439N Example Give ω_1 its order topology.

(i) ω_1 is a normal Hausdorff space which is not measure-compact.

(ii) There is a Baire probability measure μ_0 on ω_1 which is not τ-additive and has a unique extension to a Borel measure, which is not completion regular; that is, μ_0 is of type E_2 in the classification of 435A.

proof (a) As noted in 4A2Rc, order topologies are always normal and Hausdorff.

(b) Let μ be Dieudonné's measure on ω_1, and μ_0 its restriction to the Baire σ-algebra, which is also the countable-cocountable algebra (4A3P), so that μ_0 is the countable-cocountable measure. Then μ is the only Borel measure extending μ_0. **P** Let ν be any Borel measure extending μ_0. Every set $[0, \xi] = [0, \xi + 1[$, where $\xi < \omega_1$, is open-and-closed, so

$$\nu[0, \xi] = \mu_0[0, \xi] = \mu[0, \xi] = 0;$$

also, of course, $\nu\omega_1 = 1$. Let $F \subseteq \omega_1$ be any closed set. If F is countable, then it is included in some initial segment $[0, \xi]$, so $\nu F = \mu F = 0$. Now suppose that F is uncountable. Set $G = \omega_1 \setminus F$. For each $\xi \in F$, set $\zeta_\xi = \min\{\eta : \xi < \eta \in F\}$ and $G_\xi =]\xi, \zeta_\xi[$. Then $\langle G_\xi\rangle_{\xi\in F}$ is a disjoint family of open sets. By 438Bb and 419G/438Cd,

$$\nu(\bigcup_{\xi\in F} G_\xi) = \sum_{\xi\in F} \nu G_\xi = 0.$$

But now

$$1 = \nu\omega_1 = \nu F + \nu\,[0, \min F[\, + \nu(\bigcup_{\xi\in F} G_\xi) = \nu F = \mu F.$$

Thus μ and ν agree on the family $\mathcal{E}$ of closed sets. By the Monotone Class Theorem (136C), they agree on the σ-algebra generated by $\mathcal{E}$, which is their common domain; so they are equal. **Q**

(c) I have already remarked in 411Q-411R that μ and μ_0 are not τ-additive and μ is not completion regular. So of course ω_1 is not measure-compact.

439O In 439M I described a Baire measure with no extension to a Borel measure. In view of Mařík's theorem (435C), it is natural to ask whether this can be done with a normal space. This leads us into relatively deep water, and the only examples known need special assumptions.

Example Assume Ostaszewski's $\clubsuit$. Then there is a normal Hausdorff space with a Baire probability measure μ which is not τ-additive and not extendable to a Borel measure. (In the classification of 435A, μ is of type E_0.)

proof (a) $\clubsuit$ implies that there is a family $\langle C_\xi\rangle_{\xi<\omega_1}$ of sets such that (i) $C_\xi \subseteq \xi$ for every $\xi < \omega_1$ (ii) $C_\xi \cap \eta$ is finite whenever $\eta < \xi < \omega_1$ (iii) for any uncountable sets A, $B \subseteq \omega_1$ there is a $\xi < \omega_1$ such that $A \cap C_\xi$ and $B \cap C_\xi$ are both infinite (4A1N). For $A \subseteq \omega_1$, set $A' = \{\xi : \xi < \omega_1,\ A \cap C_\xi \text{ is infinite}\}$; then $A' \cap B'$ is non-empty whenever A, $B \subseteq \omega_1$ are uncountable. But this means that $A' \cap B'$ is actually uncountable for uncountable A, B, since $A' \cap B' \setminus \gamma \supseteq (A \setminus \gamma)' \cap (B \setminus \gamma)'$ is non-empty for every $\gamma < \omega_1$.

Set $X = \omega_1 \times \mathbb{N}$. For $x = (\xi, n) \in X$, say that

$$I_x = C_\xi \times \{n-1\} \text{ if } n \ge 1,$$
$$= \emptyset \text{ otherwise .}$$

(b) Define a topology $\mathfrak{T}$ on X by saying that a set $G \subseteq X$ is open iff $I_x \setminus G$ is finite for every $x \in G$.

The form of the construction ensures that $\mathfrak{T}$ is T_1. In fact, $I_x \cap I_y$ is finite whenever $x \ne y$ in X. **P** Express x as (ξ, m) and y as (η, n) where $\eta \le \xi$. If either $m = 0$ or $n = 0$ or $m \ne n$, $I_x \cap I_y = \emptyset$. If $n \ge 1$ and $\eta < \xi$, then

$$I_x \cap I_y \subseteq (C_\xi \cap \eta) \times \{n-1\}$$

is finite. Similarly, $I_x \cap I_y$ is finite if $m \ge 1$ and $\xi < \eta$. **Q** Consequently $\{x\} \cup J$ is closed whenever $x \in X$ and $J \subseteq I_x$.

Observe that $(\xi+1) \times \mathbb{N}$ is open and closed for every $\xi < \omega_1$, again because $C_\eta \cap (\xi+1)$ is finite whenever $\eta \in \Omega$ and $\eta > \xi$, while $C_\xi \subseteq \xi$ for every ξ.

(c) The next step is to understand the uncountable closed subsets of X. First, if $F \subseteq X$ is closed and $n \in \mathbb{N}$, then $F^{-1}[\{n\}]'$, as defined in (a), is a subset of $F^{-1}[\{n+1\}]$, since if $\xi \in F^{-1}[\{n\}]'$ then $I_{(\xi,n+1)} \cap F$ is infinite. If F is uncountable, there is some $n \in \mathbb{N}$ such that $F^{-1}[\{n\}]$ is uncountable, so that (inducing on m) $F^{-1}[\{m\}]$ is uncountable for every $m \ge n$. Finally, this means that if E, $F \subseteq X$ are uncountable closed sets, there is an $m \in \mathbb{N}$ such that $E^{-1}[\{m\}]$ and $F^{-1}[\{m\}]$ are both uncountable, so that $E^{-1}[\{m\}]' \cap F^{-1}[\{m\}]'$ is non-empty and $E \cap F$ is non-empty.

(d) It follows that X is normal. **P** Let E and F be disjoint closed sets in X. By (c), at least one of them is countable; let us take it that $E \subseteq \zeta \times \mathbb{N}$ where $\zeta < \omega_1$. Enumerate the open-and-closed set $W = (\zeta+1) \times \mathbb{N}$ as $\langle x_n\rangle_{n\in\mathbb{N}}$. Choose $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $U_0 = E$, $V_0 = F \cap W$. If $x_n \in U_n$, then $U_{n+1} = U_n \cup (I_{x_n} \setminus V_n)$ and $V_{n+1} = V_n$; if $x_n \notin U_n$, then $U_{n+1} = U_n$ and $V_{n+1} = V_n \cup \{x_n\} \cup (I_{x_n} \setminus U_n)$. An easy induction shows that, for every n, (α) $U_n \cap V_n = \emptyset$ (β) $U_n \cup V_n \subseteq W$ (γ) $I_x \cap (U_n \cup V_n)$ is finite for every $x \in X \setminus (U_n \cup V_n)$ (δ) $I_x \cap V_n$ is finite for every $x \in U_n$ (ϵ) $I_x \cap U_n$ is finite for every $x \in V_n$.

At the end of the induction, set $G = \bigcup_{n\in\mathbb{N}} U_n$, $H = \bigcup_{n\in\mathbb{N}} V_n \cup (X \setminus W)$. Then $E \subseteq G$, $F \subseteq H$ and $G \cap H = \emptyset$. If $x \in G$, it is of the form x_n for some n, in which case $x_n \in U_n$ (because $x_n \notin V_{n+1}$) and $I_x \setminus U_{n+1} = I_x \cap V_n$ is finite; thus G is open. If $x \in H \cap W$, again it is of the form x_n where this time $x_n \notin U_n$, so that $I_x \setminus V_{n+1} = I_x \cap U_n$ is finite; so H is open.

Thus E and F are separated by open sets; since E and F are arbitrary, X is normal. **Q**

Being T_1 (see (b)), X is also Hausdorff.

(e) Because disjoint closed sets in X cannot both be uncountable ((c) above), any bounded continuous function on X must be constant on a cocountable set. (Compare 4A2S(b-iii).) The countable-cocountable measure μ_0 is therefore a Baire measure on X (cf. 411R). But it has no extension to a Borel measure. **P** The point is that if A is any subset of ω_1, and $n \in \mathbb{N}$, then

$$(A \times \{n\}) \cup (\omega_1 \times n), \quad \omega_1 \times n$$

are both open, so $A \times \{n\}$ is Borel; accordingly every subset of X is a Borel set. But ω_1 is measure-free (419G, 438Cd), so there can be no Borel probability measure on X which is zero on singletons. **Q**

Of course μ_0 is not τ-additive, because $\{(\xi+1)\times\mathbb{N} : \xi < \omega_1\}$ is a cover of X by open-and-closed negligible sets.

Remark Thus in Mařík's theorem we really do need 'countably paracompact' as well as 'normal', at least if we want a theorem valid in ZFC.

Observe that any example of this phenomenon must involve a **Dowker space**, that is, a normal Hausdorff space which is not countably paracompact. The one here is based on de Caux 76. Such spaces are hard to come by in ZFC if we do not allow ourselves to use special principles like ♣. 'Real' Dowker spaces have been described by Rudin 71 and Balogh 96; for a survey, see Rudin 84. I do not know if either of these can be adapted to provide a ZFC example to replace the one above.

439P Example (cf. Moran 68) $\mathbb{N}^{\mathfrak{c}}$ is not Borel-measure-compact, therefore not Borel-measure-complete, measure-compact or Radon.

proof Consider the topology $\mathfrak{T}_{\mathfrak{c}}$ on $Z = \{0,1\}^{\mathbb{N}}$, as constructed in 439K. Then $(Z, \mathfrak{T}_{\mathfrak{c}})$ is homeomorphic to a closed subset of $\mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$, where in this product the second factor $\{0,1\}^{\mathbb{N}}$ is given its usual topology $\mathfrak{S}$. **P** For each $x \in Z$, let L_x be a $\mathfrak{T}_{\mathfrak{c}}$-open $\mathfrak{T}_{\mathfrak{c}}$-compact subset of Z. The first thing to observe is that if $x \in Z$, and we write $V_{xm} = \{y : y \in Z, y{\restriction}m = x{\restriction}m\}$ for each $m \in \mathbb{N}$, then $\mathcal{U}_x = \{L_x \cap V_{xm} : m \in \mathbb{N}\}$ is a downwards-directed family of compact open neighbourhoods of x with intersection $\{x\}$, so is a base of neighbourhoods of x (4A2Gd); thus $\mathcal{U} = \{L_x : x \in Z\} \cup \mathfrak{S}$ generates $\mathfrak{T}_{\mathfrak{c}}$. Now, for $x \in Z$, define $\phi_x : Z \to \mathbb{N}$ by setting

$$\begin{aligned}\phi_x(y) &= 0 \text{ if } y \in L_x,\\ &= m+1 \text{ if } y \in V_{xm} \setminus (L_x \cup V_{x,m+1}).\end{aligned}$$

Then every ϕ_x is $\mathfrak{T}_{\mathfrak{c}}$-continuous, so we have a $\mathfrak{T}_{\mathfrak{c}}$-continuous function $\phi : Z \to \mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$ defined by setting $\phi(y) = (\langle\phi_z(y)\rangle_{z\in Z}, y)$ for $y \in Z$. Because every element of $\mathcal{U}$ is of the form $\phi^{-1}[H]$ for some open set $H \subseteq \mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$, Z is homeomorphic to its image $\phi[Z]$.

Now suppose that $(w, z) \in \overline{\phi[Z]}$. In this case, there is a filter $\mathcal{G}$ containing $\phi[Z]$ which converges to (w, z) (4A2Bc). Let $\mathcal{F}$ be an ultrafilter on Z including $\{\phi^{-1}[A] : A \in \mathcal{G}\}$; then $\phi[[\mathcal{F}]]$ includes $\mathcal{G}$ so converges to (w, z), and $\mathcal{F} \to z$ for $\mathfrak{S}$. **?** If z is not the $\mathfrak{T}_{\mathfrak{c}}$-limit of $\mathcal{F}$, then $\mathcal{F}$ can have no $\mathfrak{T}_{\mathfrak{c}}$-limit, and can contain no $\mathfrak{T}_{\mathfrak{c}}$-compact set (2A3R). In particular, $L_z \notin \mathcal{F}$; but in this case $V_{zm} \setminus L_z \in \mathcal{F}$ for every m, so that $\{(v, y) : v(x) > m\} \in \phi[[\mathcal{F}]]$ for every m, and $w(x) > m$ for every m, which is impossible. **X** Thus $\mathcal{F} \to z$, and (as ϕ is continuous) $(w, z) = \phi(z)$.

This shows that $\phi[Z]$ is closed, so we have the required homeomorphism between Z and a closed subset of $\mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$. **Q**

Of course $\mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$ is a closed subset of $\mathbb{N}^Z \times \mathbb{N}^{\mathbb{N}} \cong \mathbb{N}^{\mathfrak{c}}$. So Z is homeomorphic to a closed subset of $\mathbb{N}^{\mathfrak{c}}$. But Z, with $\mathfrak{T}_c$, carries a Borel probability measure ν which is inner regular with respect to the closed sets and is not τ-additive (439Kb). So $(Z, \mathfrak{T}_{\mathfrak{c}})$ is not Borel-measure-compact. By 434Hc, $\mathbb{N}^{\mathfrak{c}}$ is not Borel-measure-compact. By 434Ic, $\mathbb{N}^{\mathfrak{c}}$ is not Borel-measure-complete; by 434Ka, it is not Radon; by 435Fd, it is not measure-compact.

439Q Example Let X be the Sorgenfrey line (415Xc). Then X is measure-compact but X^2 is not.

proof (a) Note that every set $[a, b[$ is open-and-closed in X, so that the topology is zero-dimensional, therefore completely regular; and it is finer than the usual topology of $\mathbb{R}$, so is Hausdorff.

X is Lindelöf. **P** Let $\mathcal{G}$ be an open cover of X. For each $q \in \mathbb{Q}$, set

$$A_q = \{a : a \in]-\infty, q[\text{ and there is some } G \in \mathcal{G} \text{ such that } [a, q[\subseteq G\}.$$

Then $\bigcup_{q\in\mathbb{Q}} A_q = \mathbb{R}$. For each $q \in \mathbb{Q}$, there is a countable set $A'_q \subseteq A_q$ such that $\inf A'_q = \inf A_q$ in $[-\infty, \infty]$ and A'_q contains $\min A_q$ if A_q has a least element. Now, for each pair (a, q) where $q \in \mathbb{Q}$ and $a \in A'_q$, choose $G_{aq} \in \mathcal{G}$ such that $[a, q[\subseteq G_{aq}$. It is easy to see that $\bigcup\{G_{aq} : a \in A'_q\} \supseteq A_q$, so that the countable family $\{G_{aq} : q \in \mathbb{Q}, a \in A'_q\}$ covers X. As $\mathcal{G}$ is arbitrary, X is Lindelöf. **Q**

It follows that X is measure-compact (435Fb).

(b) Let $\mathfrak{S}$ be the usual topology on $\mathbb{R}^2$, and $\mathfrak{T}$ the product topology on X^2.

(i) Whenever G, H are disjoint $\mathfrak{T}$-open sets, there is an $\mathfrak{S}$-Borel set E such that $G \subseteq E \subseteq X^2 \setminus H$. **P** For $n \in \mathbb{N}$, set

$$A_n = \{(a, b) : [a, a + 2^{-n}[\times [b, b + 2^{-n}[\subseteq G\}.$$

? Suppose, if possible, that there is a point $(x, y) \in \overline{A_n}^{\mathfrak{S}} \cap H$, where I write $\overline{}^{\mathfrak{S}}$ to denote closure for the topology $\mathfrak{S}$. Let $\delta > 0$ be such that $[x, x + 2\delta[\times [y, y + 2\delta[\subseteq H$ and $2\delta < 2^{-n}$. Then there must be $(a, b) \in A_n$ such that $|a - x| \le \delta$ and $|b - y| \le \delta$. In this case, $a \le x + \delta < a + 2^{-n}$ and $b \le y + \delta < b + 2^{-n}$, so $(x + \delta, y + \delta) \in G$; while δ was chosen so that $(x + \delta, y + \delta)$ would belong to H. **X**

Accordingly $E = \bigcup_{n\in\mathbb{N}} \overline{A_n}^{\mathfrak{S}}$ is an $\mathfrak{S}$-Borel set disjoint from H. But $G = \bigcup_{n\in\mathbb{N}} A_n$, so $G \subseteq E$. **Q**

(ii) Consequently every $\mathfrak{T}$-continuous real-valued function is $\mathfrak{S}$-Borel measurable. **P** If $f : X^2 \to \mathbb{R}$ is $\mathfrak{T}$-continuous and $\alpha \in \mathbb{R}$, then there is an $\mathfrak{S}$-Borel set E_α such that

$$\{(x, y) : f(x, y) < \alpha\} \subseteq E_\alpha \subseteq \{(x, y) : f(x, y) \le \alpha\}.$$

But this means that $\{(x, y) : f(x, y) < \alpha\} = \bigcup_{n\in\mathbb{N}} E_{\alpha-2^{-n}}$ is $\mathfrak{S}$-Borel. **Q**

(iii) It follows that every $\mathfrak{T}$-Baire set is $\mathfrak{S}$-Borel. We therefore have a $\mathfrak{T}$-Baire probability measure ν on X^2 defined by setting

$$\nu E = \mu_L\{t : t \in [0, 1], (t, 1 - t) \in E\}$$

for every $\mathfrak{T}$-Baire subset of X^2, where μ_L is Lebesgue measure on $\mathbb{R}$. In this case every point (x, y) of X^2 belongs to a $\mathfrak{T}$-open set of zero measure for ν. **P** Set $K = \{(t, 1-t) : t \in [0, 1]\}$. Then K is $\mathfrak{S}$-closed, therefore $\mathfrak{T}$-closed, and $\nu(X^2 \setminus K) = 0$, so if $(x, y) \notin K$ then we can stop. If $(x, y) \in K$, then $[x, x + 1[\times [y, y + 1[$ is a $\mathfrak{T}$-open $\mathfrak{T}$-closed set meeting K in the single point (x, y), so is a negligible $\mathfrak{T}$-neighbourhood of (x, y). **Q**

Thus ν is not τ-additive and X^2 is not measure-compact.

Remark Contrast this with 438Xr.

439R Example There are first-countable completely regular Hausdorff spaces X, Y with Baire probability measures μ, ν such that the Baire measures λ, λ' on $X \times Y$ defined by the formulae

$$\int f d\lambda = \iint f(x, y)\nu(dy)\mu(dx), \quad \int f d\lambda' = \iint f(x, y)\mu(dx)\nu(dy)$$

(436F) are different.

proof Let X, Y be disjoint stationary subsets of ω_1 (4A1Cd). Give each the topology induced by the order topology of ω_1. Let $\tilde{\mu}$ be Dieudonné's measure on ω_1, and $\tilde{\mu}_X$, $\tilde{\mu}_Y$ the subspace measures induced on X and Y by $\tilde{\mu}$; let μ and ν be the restrictions of $\tilde{\mu}_X$, $\tilde{\mu}_Y$ to the Baire σ-algebras of X, Y respectively. Then

$$\mu X = \tilde{\mu}_X X = \tilde{\mu}^* X = 1$$

because X meets every cofinal closed set in ω_1; similarly, $\nu Y = 1$.

Set

$$W = \{(x, y) : x \in X, y \in Y, x < y\} = \{(x, y) : x \in X, y \in Y, x \le y\}.$$

Then W is open-and-closed in $X \times Y$ (use 4A2Rl), so that $f = \chi W$ is continuous. But

$$\iint f(x, y)\nu(dy)\mu(dx) = \int \nu\{y : y \in Y, x < y\}\mu(dx) = 1,$$

$$\iint f(x,y)\mu(dx)\nu(dy) = \int \mu\{x : x \in X,\ x < y\}\nu(dy) = 0.$$

Remark Contrast this with 434Xy and 439Yi.

439S The results of 437V leave open the question of which familiar spaces, beyond Čech-complete spaces, can be Prokhorov. In fact rather few are. The basis of any further investigation must be the following result.

Theorem (Preiss 73) $\mathbb{Q}$ is not a Prokhorov space.

proof (a) There is a non-decreasing sequence $\langle X_k\rangle_{k\in\mathbb{N}}$ of non-empty compact subsets of $X = \mathbb{Q} \cap [0,1]$, with union X, such that whenever $k \in \mathbb{N}$, $x \in X_k$ and $\delta > 0$, then $X_{k+1} \cap [x-\delta, x+\delta]$ is infinite. **P** Start by enumerating X as $\langle q_k\rangle_{k\in\mathbb{N}}$. Set $X_0 = \{q_0\}$. Given that $X_k \subseteq X$ is compact, then for each $m \in \mathbb{N}$ let $\mathcal{E}_m$ be a finite cover of X_k by open intervals of length at most 2^{-m} all meeting X_k, and let I_{km} be a finite subset of $X \setminus X_k$ meeting every member of $\mathcal{E}_m$; set $X_{k+1} = X_k \cup \{q_{k+1}\} \cup \bigcup_{m\in\mathbb{N}} I_{km}$. If $\mathcal{H}$ is any cover of X_{k+1} by open sets in $\mathbb{R}$, then there is a finite $\mathcal{H}_0 \subseteq \mathcal{H}$ covering X_k. There must be an $m \in \mathbb{N}$ such that $[x-2^{-m}, x+2^{-m}] \subseteq \bigcup \mathcal{H}_0$ for every $x \in X_k$ (2A2Ed), so that $I_{kl} \subseteq \bigcup \mathcal{H}_0$ for every $l \geq m$, and $X_{k+1} \setminus \bigcup \mathcal{H}_0$ is finite; accordingly there is a finite $\mathcal{H}_1 \subseteq \mathcal{H}$ covering X_{k+1}. As $\mathcal{H}$ is arbitrary, X_{k+1} is compact, and the induction can proceed. If $x \in X_k$ and $\delta > 0$, then for every $m \in \mathbb{N}$ there is an $x' \in X_{k+1} \setminus X_k$ such that $|x' - x| \leq 2^{-m}$, so that $[x-\delta, x+\delta] \cap X_{k+1}$ must be infinite. **Q**

(b) If $\langle \epsilon_k\rangle_{k\in\mathbb{N}}$ is any sequence in $]0,\infty[$, and $F \subseteq [0,1]$ is a countable closed set, then there is an $x^* \in X \setminus F$ such that $\rho(x^*, X_k) < \epsilon_k$ for every $k \in \mathbb{N}$. **P** We can suppose that $\lim_{k\to\infty} \epsilon_k = 0$. Define $\langle H_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $H_0 = \mathbb{R}$. Given H_k, set $H_{k+1} = H_k \cap \{x : \rho(x, X_k \cap H_k) < \epsilon_k\}$, where $\rho(x, A) = \inf_{y\in A} |x-y|$ when $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ is non-empty. Observe that every H_k is an open subset of $\mathbb{R}$ and that $X_k \cap H_k \subseteq H_{k+1} \subseteq H_k$ for every k; consequently, setting $E = \bigcap_{k\in\mathbb{N}} H_k$, E is a G_δ subset of $\mathbb{R}$ and $X_k \cap H_k \subseteq E$ for every k. In particular, $E \cap X$ contains q_0 and is not empty. Next, for each k, $\rho(x, E \cap X_k) < \epsilon_k$ for every $x \in H_{k+1}$ and therefore for every $x \in E$; accordingly $E \cap X$ is dense in E. Moreover, if $x \in E \cap X$, there is a $k \in \mathbb{N}$ such that $x \in X_k$; we must have $x \in H_k$, and in this case H_{k+1} is a neighbourhood of x. So every neighbourhood of x contains infinitely many points of $H_{k+1} \cap X_{k+1} \subseteq E \cap X$. Thus $E \cap X$ has no isolated points; it follows that E has no isolated points. By 4A2Mc and 4A2Me, E is uncountable.

There is therefore a point $z \in E \setminus F$. Let $m \in \mathbb{N}$ be such that $\rho(z, F) \geq \epsilon_m$. As $z \in H_{m+1}$, there is an $x^* \in H_m \cap X_m$ such that $|z - x^*| < \epsilon_m$ and $x^* \notin F$. Let $k \in \mathbb{N}$. If $k \geq m$ then certainly $\rho(x^*, X_k) = 0 < \epsilon_k$. If $k < m$ then $x^* \in H_{k+1}$ so $\rho(x^*, X_k) \leq \rho(x^*, H_k \cap X_k) < \epsilon_k$. So we have a suitable x^*. **Q**

(c) For n, $k \in \mathbb{N}$ set

$$G_{kn} = \{x : x \in \mathbb{R} \setminus X_k,\ \rho(x, X_n) > 2^{-k}\}.$$

Then G_{kn} is an open subset of $\mathbb{R}$. Let A be the set of Radon probability measures μ on X such that $\mu(G_{kn} \cap X) \leq 2^{-n}$ for all n, $k \in \mathbb{N}$.

(d) Write $\tilde{A}$ for the set of Radon probability measures μ on $[0,1]$ such that $\mu(G_{kn} \cap [0,1]) \leq 2^{-n}$ for all k, $n \in \mathbb{N}$. Then $\tilde{A}$ is a narrowly closed subset of the set of Radon probability measures on $[0,1]$, which is itself narrowly compact (437R(f-ii)). Also $\mu([0,1] \setminus X) = 0$ for every $\mu \in \tilde{A}$. **P** Let $K \subseteq [0,1] \setminus X$ be compact, and $n \in \mathbb{N}$. Then K and X_n are disjoint compact sets, so there is some $k \in \mathbb{N}$ such that $|x-y| > 2^{-k}$ for every $x \in X_n$ and $y \in K$. In this case $K \subseteq G_{kn}$ so $\mu K \leq 2^{-n}$. As n is arbitrary, $\mu K = 0$; as K is arbitrary, $\mu([0,1] \setminus X) = 0$. **Q**

A is compact in the narrow topology. **P** The identity map $\phi : X \to [0,1]$ induces a map $\tilde{\phi} : M_{\mathrm{R}}^+(X) \to M_{\mathrm{R}}^+([0,1])$ which is a homeomorphism between $M_{\mathrm{R}}^+(X)$ and $\{\mu : \mu \in M_{\mathrm{R}}^+([0,1]),\ \mu([0,1] \setminus X) = 0\}$ (437Nb). The definition of A makes it plain that it is $\tilde{\phi}^{-1}[\tilde{A}]$; since $\tilde{A} \subseteq \{\mu : \mu \in M_{\mathrm{R}}^+([0,1]),\ \mu([0,1] \setminus X) = 0\}$, $\tilde{\phi}\restriction A$ is a homeomorphism between A and $\tilde{A}$, and A is compact. **Q**

(e) A, regarded as a subset of $M_{\mathrm{R}}^+(X)$, is not uniformly tight. **P** Let $K \subseteq X$ be compact. Consider the set C of those $w \in [0,1]^X$ such that $w(x) = 0$ for every $x \in K$, $\sum_{x\in X} w(x) \leq 1$ and $\sum_{x\in G_{kn}\cap X} w(x) \leq 2^{-n}$ for all k, $n \in \mathbb{N}$. Then C is a compact subset of $[0,1]^X$. If $D \subseteq C$ is any non-empty upwards-directed set, then $\sup D$, taken in $[0,1]^X$, belongs to C. By Zorn's Lemma, C has a maximal member w say. **?** Suppose, if possible, that $\sum_{x\in X} w(x) = \gamma < 1$. For each $n \in \mathbb{N}$, let $L_n \subseteq X$ be a finite set such that $\sum_{x\in L_n} w(x) \ge \gamma - 2^{-n-1}$, and $m_n \in \mathbb{N}$ such that $L_n \subseteq X_{m_n}$. By (b), there is an $x^* \in X \setminus K$ such that $\rho(x^*, X_n) < 2^{-m_n}$ for every $n \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $x^* \in X_r$ and $\gamma + 2^{-r} \le 1$, and set $w'(x^*) = w(x^*)+2^{-r}$, $w'(x) = w(x)$ for every $x \in X\setminus\{x^*\}$. Then certainly $w' \in [0,1]^X$ and $\sum_{x\in X} w'(x) \le 1$. If k, $n \in \mathbb{N}$ and $x^* \notin G_{kn}$, then $\sum_{x\in G_{kn}\cap X} w'(x) = \sum_{x\in G_{kn}\cap X} w(x) \le 2^{-n}$. If $x^* \in G_{kn}$, then $n < r$ and $2^{-k} < \rho(x^*, X_n) < 2^{-m_n}$, so $m_n < k$ and $L_n \subseteq X_k$ and

$$\sum_{x\in G_{kn}\cap X} w(x) \le \sum_{x\in X\setminus X_k} w(x) \le \sum_{x\in X\setminus L_n} w(x) \le 2^{-n-1},$$

$$\sum_{x\in G_{kn}\cap X} w'(x) \le 2^{-n-1} + 2^{-r} \le 2^{-n}.$$

Thus $w' \in C$ and w was not maximal. **X**

Accordingly $\sum_{x\in X} w(x) = 1$ and the point-supported measure μ defined by w is a probability measure on X. By the definition of C, $\mu \in A$ and $\mu(X \setminus K) = 1$. As K is arbitrary, A cannot be uniformly tight. **Q**

(f) Thus A witnesses that $X = \mathbb{Q} \cap [0,1]$ is not a Prokhorov space. Since X is a closed subset of $\mathbb{Q}$, 437Vb tells us that $\mathbb{Q}$ is not a Prokhorov space.

439X Basic exercises (a)(i) Show that there is a set $A \subseteq [0,1]$ such that $\mu_L^* A = 1$, where μ_L is Lebesgue measure, and every member of $[0,1]$ is uniquely expressible as $a + q$ where $a \in A$ and $q \in \mathbb{Q}$. (*Hint*: 134B.) (ii) Define $f : [0,1] \to A$ by setting $f(x) = a$ when $x \in a + \mathbb{Q}$. Show that the image measure $\mu_L f^{-1}$ takes only the values 0 and 1. (Aldaz 95. Compare 342Xg.)

(b) Let X be a Radon Hausdorff space and A a subset of X. Show that A is universally negligible iff $\mu A = 0$ for every atomless Radon measure on X.

>(c) Let X be a Hausdorff space. Show that a set $A \subseteq X$ is universally negligible iff $\mu A = 0$ whenever μ is a topological measure on X with locally determined negligible sets such that $\mu\{x\} = 0$ for every $x \in X$.

(d) Let X be a Hausdorff space. Show that any universally negligible subset of X is universally measurable in the sense of 434D.

(e)(i) Show that there is an analytic set $A \subseteq \mathbb{R}$ such that for any Borel subset E of $\mathbb{R} \setminus A$ there is an uncountable Borel subset of $\mathbb{R} \setminus (A \cup E)$. (*Hint*: 423Sb.) (ii) Show that A is universally measurable, but there is no Borel set E such that $A \triangle E$ is universally negligible.

(f) Show that a first-countable compact Hausdorff space is universally negligible iff it is scattered iff it is countable.

(g) Show that the product of two universally negligible Hausdorff spaces is universally negligible.

(h) Let us say that a Hausdorff space X is **universally τ-negligible** if there is no τ-additive Borel probability measure on X which is zero on singletons. (i) Show that if X is a Hausdorff space and $A \subseteq X$, then A is universally τ-negligible iff $\mu^* A = 0$ for every τ-additive Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. (ii) Show that if X is a regular Hausdorff space, then a subset A of X is universally τ-negligible iff $\mu A = 0$ for every atomless quasi-Radon measure on X. (iii) Show that if X is a completely regular Hausdorff space, it is universally τ-negligible iff whenever μ is an atomless Radon measure on a space Z, and $X' \subseteq Z$ is homeomorphic to X, then $\mu X' = 0$. (iv) Show that a Hausdorff space X is universally negligible iff it is Borel-measure-complete and universally τ-negligible. (v) Show that if X is a Hausdorff space, Y is a universally τ-negligible Hausdorff space, and $f : X \to Y$ is a continuous function such that $f^{-1}[\{y\}]$ is universally τ-negligible for every $y \in Y$, then X is universally τ-negligible. (vi) Show that the product of two universally τ-negligible Hausdorff spaces is universally τ-negligible. (vi) Show that a scattered Hausdorff space (in particular, any discrete space) is universally τ-negligible. (vii) Show that a compact Hausdorff space is universally τ-negligible iff it is scattered.

(i) Let X be a Polish space, $A \subseteq X$ an analytic set which is not Borel (423Sb, 423Ye), and $\langle E_\xi \rangle_{\xi<\omega_1}$ a family of Borel constituents of $X \setminus A$ (423R). Suppose that $x_\xi \in E_\xi \setminus \bigcup_{\eta<\xi} E_\eta$ for every $\xi < \omega_1$. Show that $\{x_\xi : \xi < \omega_1\}$ is universally negligible. Hence show that any probability measure with domain $\mathcal{P}\omega_1$ is point-supported.

(j) Let $(X,\leq)$ be any well-ordered set and μ a non-zero σ-finite measure on X such that every singleton is negligible. Show that $\{(x,y):x\leq y\}$ is not measured by the (c.l.d.) product measure on $X\times X$. (*Hint*: Reduce to the case in which μ is complete and totally finite, $X=\zeta$ is an ordinal and $\mu\xi=0$ for every $\xi<\zeta$. You will probably need 251Q.)

>(k) Show that 439Fc, or any of the examples of 439A, can be regarded as an example of a probability space (X,μ) and a function $f:X\to[0,1]$ such that there is no extension of μ to a measure ν such that f is $\operatorname{dom}\nu$-measurable; and accordingly can provide an example of a probability space (X,μ) with a countable totally ordered family $\mathcal{A}$ of subsets of X such that there is no extension of μ to a measure measuring every member of $\mathcal{A}$. Contrast with 214P, 214Xm-214Xn and 214Yb.

(l) Show that 1-dimensional Hausdorff measure on $\mathbb{R}^2$ is not inner regular with respect to the closed sets. (*Hint*: 439H, 439C(a-v); see also 471S.)

(m) Show that the one-point compactification of the space $(Z,\mathfrak{T}_{\mathfrak{c}})$ described in 439K is a scattered compact Hausdorff space with an atomless Borel probability measure.

(n) Show that a semi-finite Borel measure on ω_1, with its order topology, must be purely atomic.

(o) Show that $\mathbb{N}^I$ is not pre-Radon for any uncountable set I. (*Hint*: 417Xq.)

(p)(i) Suppose that X is a completely regular space and there is a continuous function f from X to a realcompact completely regular space Z such that $f^{-1}[\{z\}]$ is realcompact for every $z\in Z$. Show that X is realcompact (definition: 436Xg). (ii) Show that the spaces X of 439K and X^2 of 439Q are realcompact.

439Y Further exercises (a) Show that a subset A of $\mathbb{R}$ is universally negligible iff $f[A]$ is Lebesgue negligible for every continuous injective function $f:\mathbb{R}\to\mathbb{R}$. (*Hint*: if ν is an atomless Borel probability measure on $\mathbb{R}$, set $f(x)=x+\nu[0,x]$ for $x\geq 0$, and show that $\mu_L f[E]=\mu_L E+\nu E$ for every Borel set $E\subseteq[0,\infty[$.)

(b) For this exercise only, let us say that a 'universally negligible measurable space' is a pair (X,Σ) where X is a set and Σ a σ-algebra of subsets of X containing every countable subset of X such that there is no probability measure μ with domain Σ such that $\mu\{x\}=0$ for every $x\in X$. (i) Let X be a set, Σ a σ-algebra of subsets of X containing all countable subsets of X, $A\subseteq X$ and Σ_A the subspace σ-algebra. Show that (A,Σ_A) is universally negligible iff $\mu^*A=0$ whenever μ is a probability measure with domain Σ which is zero on singletons. Show that if (X,Σ) is universally negligible so is (A,Σ_A). (ii) Let X and Y be sets, Σ and T σ-algebras of subsets of X and Y containing all appropriate countable sets, and $f:X\to Y$ a (Σ,T)-measurable function. Suppose that (Y,T) and $(f^{-1}[\{y\}],\Sigma_{f^{-1}[\{y\}]})$ are universally negligible for every $y\in Y$. Show that (X,Σ) is universally negligible. (iii) Let X be a set and Σ a σ-algebra of subsets of X containing all countable subsets of X. Show that the set of those $A\subseteq X$ such that (A,Σ_A) is universally negligible is a σ-ideal of subsets of X.

(c) Let X be an analytic Hausdorff space and A an analytic subset of X. Show that $X\setminus A$ is universally negligible iff all the constituents of $X\setminus A$ (for any Souslin scheme defining A) are countable.

(d)(i) A pair $(\langle a_\xi\rangle_{\xi<\omega_1},\langle b_\xi\rangle_{\xi<\omega_1})$ of families of subsets of $\mathbb{N}$ is a **Hausdorff gap** if $a_\xi\setminus a_\eta$, $a_\xi\setminus b_\xi$ and $b_\eta\setminus b_\xi$ are finite whenever $\xi\leq\eta<\omega_1$, $a_\eta\setminus a_\xi$ and $b_\xi\setminus b_\eta$ are infinite whenever $\xi<\eta<\omega_1$, and moreover $\{\xi:\xi<\eta,\ a_\xi\subseteq b_\eta\cup n\}$ is finite for every $\eta<\omega_1$. (For a construction of a Hausdorff gap, see Fremlin 84, 21L.) Show that in this case there is no $c\subseteq\mathbb{N}$ such that $a_\xi\setminus c$ and $c\setminus b_\xi$ are finite for every $\xi<\omega_1$, and that $\{a_\xi:\xi<\omega_1\}\cup\{b_\xi:\xi<\omega_1\}$ is universally negligible in $\mathcal{P}\mathbb{N}$. (ii) Let $\phi:(\mathcal{P}\mathbb{N})^{\mathbb{N}}\to\mathcal{P}\mathbb{N}$ be a homeomorphism. For $0<\xi<\omega_1$ let $\langle\theta(\xi,n)\rangle_{n\in\mathbb{N}}$ be a sequence running over ξ. Set $a_0=\emptyset$ and for $0<\xi<\omega_1$ set $a_\xi=\phi(\langle a_{\theta(\xi,n)}\rangle_{n\in\mathbb{N}})$. Show that $\{a_\xi:\xi<\omega_1\}$ is universally negligible.

(e)(i) Let X be a metrizable space such that $f[X]$ is Lebesgue negligible for every continuous function $f:X\to\mathbb{R}$. Show that X is universally negligible. (ii) Let X be a completely regular Hausdorff space such that $f[X]$ is Lebesgue negligible for every continuous function $f:X\to\mathbb{R}$. Show that X is universally τ-negligible.

(f) Let $\mathfrak{T}_{\mathfrak{c}}$ be the topology on $\{0,1\}^{\mathbb{N}}$ constructed in the proof of 439K. (i) Show that it is normal and countably paracompact. (ii) Show that any $\mathfrak{T}_{\mathfrak{c}}$-zero set is an $\mathfrak{S}$-Borel set. (iii) Show that the measure ν of part (b) of the proof of 439K is not completion regular, so that its restriction to the $\mathfrak{T}_{\mathfrak{c}}$-Borel σ-algebra is of type B_0 in the classification of 434A.

(g) Show that the space of 439O is locally compact and locally countable, therefore first-countable.

(h) Show that the Sorgenfrey line is hereditarily Lindelöf, but that its square is not Lindelöf.

(i) Show that if ω_1 is given its order topology, and $f:\omega_1^2\to\mathbb{R}$ is continuous, then there is a $\zeta<\omega_1$ such that f is constant on $(\omega_1\setminus\zeta)^2$. (*Hint*: 4A2S(b-iii).) Show that if μ and ν are Baire probability measures on ω_1, then the Baire probability measures $\mu\times\nu$, $\nu\times\mu$ on ω_1^2 defined by the formulae of 436F coincide.

(j)(i) Let $X\subseteq[0,1]$ be a dense set with no uncountable compact subset. Show that X is not a Prokhorov space. (ii) Show that $\mathbb{R}^{\mathfrak{c}}$ is not a Prokhorov space.

(k) Show that the Sorgenfrey line is not a Prokhorov space. (*Hint*: Fremlin n15.)

439 Notes and comments I give three separate constructions in 439A because the phenomenon here is particularly important. For two chapters I have, piecemeal, been offering theorems on the extension of measures. The principal ones so far seem to be 413P, 415L, 416N, 417C, 417E and 435C, and I have used methods reflecting my belief that the essential feature on which each such theorem depends is inner regularity of an appropriate kind. I think we should simultaneously seek to develop an intuition for measures which do *not* extend, and those in 439A are especially significant because they refer to the Borel algebra of the unit interval, which in so many other contexts is comfortably clear of the obstacles which beset more exotic structures.

Note that because the Borel σ-algebra of $\mathbb{R}$ is countably generated, the examples here are examples of measures which cannot be extended to measure every member of a countable family of sets. Recall that in 214P I showed that measures can be extended to measure the sets in arbitrary *well-ordered* families.

Outside the context of Polish spaces, the terms 'universally measurable' and 'universally negligible' are not properly settled. I have tried to select definitions which lead to a reasonable pattern. At least a universally negligible subset of a Hausdorff space is universally measurable (439Xd), and both concepts can be expressed in terms of sets with σ-algebras, as in 439Yb. It is important to notice, in 439B, that I write '$\mu\{x\}=0$ for every $x\in X$', not 'μ is atomless'. For instance, Dieudonné's measure shows that ω_1, with its order topology, is not universally negligible on the definition here; but it is easy to show that there is no atomless Borel probability measure on ω_1 (439Xn). In many cases, of course, we do not need to make this distinction (439D).

The cardinal κ of 439F (the 'uniformity' of the Lebesgue null ideal) is one of a large family of cardinals which will be examined in Chapter 52 in the next volume.

In some of the arguments above (439J, 439L, 439O) I appeal to (different) principles ('there is a cardinal which is not measure-free', $\clubsuit$) which are not theorems according to the rules I follow in this book. Such examples would in some ways fit better into Volume 5, where I mean to investigate such principles properly. I include the examples here because they do at least exhibit bounds on what can be proved in ZFC. I should not want anyone to waste her time trying to show, for instance, that all completion regular Borel measures are τ-additive. Nevertheless, the absence of a 'real' counter-example (obviously we want a probability measure on a completely regular Hausdorff space) remains in my view a significant gap. It remains conceivable that there is a mathematical world in which no such space exists. Clearly the discovery of such a world is likely to require familiarity with the many worlds already known, and I am not going to embark on any such exploration in this volume. On the other hand, it is also very possible that all we need is a bit of extra ingenuity to construct a counter-example in ZFC. In this section we have two examples of successes of this kind. In 439F-439H, for instance, we have results which were long known as consequences of the continuum hypothesis; the particular insight of Grzegorek 81 was the observation that they depended on determinate properties of the cardinal κ of 439F, and that its indeterminate position between ω_1 and $\mathfrak{c}$ was unimportant. In 439K I show how a re-working of ideas in Juhász Kunen & Rudin 76, where a similar example was constructed (for an entirely different purpose) assuming the continuum hypothesis, provides us with an interesting space (a first-countable non-Radon compact Hausdorff space) in ZFC. Let me emphasize that these ideas were originally set out in a framework supported by an extra axiom, where some technical details were easier and the prize aimed at (a non-Lindelöf hereditarily separable space) more important.

The examples in 439K-439R are mostly based on constructions more or less familiar from general topology. I have already mentioned the origins of 439K. 439M is related to the Tychonoff and Dieudonné planks (Steen & Seebach 78, §§86-89). 439N and 439Q revisit yet again ω_1 and the Sorgenfrey line. 439O is adapted from one of the standard constructions of Dowker spaces. Products of disjoint stationary sets (439R) have also been used elsewhere.

Version of 21.11.10

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

432I Capacitability 432I, referred to in the 2008 edition of Volume 5, is now 432J.

434S-434T Vague topologies The material on vague topologies, referred to in the 2001 edition of Volume 2, has been moved to §437.

439H τ-smooth functionals The example of a τ-smooth functional which is not representable as an integral, referred to in BOGACHEV 07, is now 439I.

439J A non-Radon space The example of a first-countable compact Hausdorff space which is not Radon, referred to in BOGACHEV 07, is now 439K.

439N Baire measure The example of a Baire probability measure with no extension to a Borel measure, referred to in BOGACHEV 07, is now 439M.

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