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Chapter 42

Descriptive set theory

At this point, I interpolate an auxiliary chapter, in the same spirit as Chapters 31 and 35 in the last volume. As with Boolean algebras and Riesz spaces, it is not just that descriptive set theory provides essential tools for modern measure theory; it also offers deep intuitions, and for this reason demands study well beyond an occasional glance at an appendix. Several excellent accounts have been published; the closest to what we need here is probably ROGERS 80; at a deeper level we have MOSCHOVAKIS 80, and an admirable recent treatment is KECHRIS 95. Once again, however, I indulge myself by extracting those parts of the theory which I shall use directly, giving proofs and exercises adapted to the ideas I am trying to emphasize in this volume and the next.

The first section describes Souslin's operation and its basic set-theoretic properties up to first steps in the theory of 'constituents' (421N-421Q), mostly steering away from topological ideas, but with some remarks on σ-algebras and Souslin-F sets. §422 deals with usco-compact relations and K-analytic spaces, working through the topological properties which will be useful later, and giving a version of the First Separation Theorem (422I-422J). §423 looks at 'analytic' or 'Souslin' spaces, treating them primarily as a special kind of K-analytic space, with the von Neumann-Jankow selection theorem (423P). §424 is devoted to 'standard Borel spaces'; it is largely a series of easy applications of results in §423, but there is a substantial theorem on Borel measurable actions of Polish groups (424H). Finally, I add a note on A.Törnquist's theorem on representation of groups of automorphisms of quotient algebras (425D).

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421 Souslin's operation

I introduce Souslin's operation $\mathcal{S}$ (421B) and show that it is idempotent (421D). I describe alternative characterizations of members of $\mathcal{S}(\mathcal{E})$, where $\mathcal{E} \subseteq \mathcal{P}X$, as projections of sets in $\mathbb{N}^{\mathbb{N}} \times X$ (421G-421J). I briefly mention Souslin-F sets (421J-421L) and a special property of 'inner Souslin kernels' (421M). At the end of the section I set up an abstract theory of 'constituents' for kernels of Souslin schemes and their complements (421N-421Q).

421A Notation S will always be the set $\bigcup_{k\in\mathbb{N}} \mathbb{N}^k$, and $S^* = S \setminus \{\emptyset\}$ the set $\bigcup_{k\ge 1} \mathbb{N}^k$; for $\sigma \in S$, I_σ will be $\{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma\}$. If $\sigma \in \mathbb{N}^k$ and $i \in \mathbb{N}$ I write $\sigma^\frown{<}i{>}$ for the member τ of $\mathbb{N}^{k+1}$ such that $\tau(k) = i$ and $\tau(j) = \sigma(j)$ for $j < k$.

421B Definition If $\mathcal{E}$ is a family of sets, I write $\mathcal{S}(\mathcal{E})$ for the family of sets expressible in the form

$$\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge 1} E_{\phi\restriction k}$$

for some family $\langle E_\sigma\rangle_{\sigma\in S^*}$ in $\mathcal{E}$.

A family $\langle E_\sigma\rangle_{\sigma\in S^*}$ is called a **Souslin scheme**; the corresponding set $\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge 1} E_{\phi\restriction k}$ is its **kernel**; the operation

$$\langle E_\sigma\rangle_{\sigma\in S^*} \mapsto \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge 1} E_{\phi\restriction k}$$

is **Souslin's operation** or **operation $\mathcal{A}$**. Thus $\mathcal{S}(\mathcal{E})$ is the family of sets obtainable from sets in $\mathcal{E}$ by Souslin's operation. If $\mathcal{E} = \mathcal{S}(\mathcal{E})$, we say that $\mathcal{E}$ is **closed under Souslin's operation**.

421C Elementary facts (a) It is worth noting straight away that if $\mathcal{E}$ is any family of sets, then $\bigcup_{n\in\mathbb{N}} E_n$ and $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{S}(\mathcal{E})$ for any sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$.
In particular, $\mathcal{E}\subseteq\mathcal{S}(\mathcal{E})$.

(b) Let X and Y be sets, and $f:X\to Y$ a function. Let $\langle F_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{P}Y$, with kernel B. Then $f^{-1}[B]$ is the kernel of the Souslin scheme $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S^*}$.

(c) Let X and Y be sets, and $f:X\to Y$ a function. Let $\mathcal{F}$ be a family of subsets of Y. Then

$$\{f^{-1}[B]:B\in\mathcal{S}(\mathcal{F})\}=\mathcal{S}(\{f^{-1}[F]:F\in\mathcal{F}\}).$$

(d) Let X and Y be sets, and $f:X\to Y$ a surjective function. Let $\mathcal{F}$ be a family of subsets of Y. Then

$$\mathcal{S}(\mathcal{F})=\{B:B\subseteq Y,\,f^{-1}[B]\in\mathcal{S}(\{f^{-1}[F]:F\in\mathcal{F}\})\}.$$

(e) Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme with kernel A. Set

$$R=\bigcap_{n\ge1}\bigcup_{\sigma\in\mathbb{N}^n} I_\sigma\times E_\sigma.$$

Then $R[\mathbb{N}^\mathbb{N}]=A$.

(f)(i) I will say that a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S^*}$ is **fully regular** if $E_\sigma\subseteq E_\tau$ whenever σ, $\tau\in S^*$, $\#(\tau)\le\#(\sigma)$ and $\sigma(i)\le\tau(i)$ for every $i<\#(\sigma)$.

(ii) Let $\mathcal{E}$ be a family of sets such that $E\cup F$ and $E\cap F$ belong to $\mathcal{E}$ for all E, $F\in\mathcal{E}$. Then every member of $\mathcal{S}(\mathcal{E})$ can be expressed as the kernel of a regular Souslin scheme in $\mathcal{E}$.

421D Theorem For any family $\mathcal{E}$ of sets, $\mathcal{S}(\mathcal{E})$ is closed under Souslin's operation.

421E Corollary For any family $\mathcal{E}$ of sets, $\mathcal{S}(\mathcal{E})$ is closed under countable unions and intersections.

421F Corollary Let X be a set and $\mathcal{E}$ a family of subsets of X. Suppose that X and $\emptyset$ belong to $\mathcal{S}(\mathcal{E})$ and that $X\setminus E\in\mathcal{S}(\mathcal{E})$ for every $E\in\mathcal{E}$. Then $\mathcal{S}(\mathcal{E})$ includes the σ-algebra of subsets of X generated by $\mathcal{E}$.

421G Proposition Let $\mathcal{E}$ be a family of sets such that $\emptyset\in\mathcal{E}$. Then

$$\begin{aligned}\mathcal{S}(\mathcal{E})&=\{R[\mathbb{N}^\mathbb{N}]:R\in\mathcal{S}(\{I_\sigma\times E:\sigma\in S^*,\,E\in\mathcal{E}\})\}\\&=\{R[\mathbb{N}^\mathbb{N}]:R\in\mathcal{S}(\{I_\sigma\times E:\sigma\in S^*,\,E\in\mathcal{E}\}),\,R^{-1}[\{x\}]\text{ is closed for every }x\}.\end{aligned}$$

421H Proposition Let X be a set, and Σ a σ-algebra of subsets of X. Let $\mathcal{B}$ be the algebra of Borel subsets of $\mathbb{N}^\mathbb{N}$. Then

$$\begin{aligned}\mathcal{S}(\Sigma)&=\{R[\mathbb{N}^\mathbb{N}]:R\in\mathcal{B}\widehat{\otimes}\Sigma\}\\&=\{R[\mathbb{N}^\mathbb{N}]:R\in\mathcal{S}(\{I_\sigma\times E:\sigma\in S^*,\,E\in\Sigma\})\}\\&=\{R[\mathbb{N}^\mathbb{N}]:R\in\mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma)\}.\end{aligned}$$

421I Lemma Let X be a topological space and $R\subseteq\mathbb{N}^\mathbb{N}\times X$ a closed set. Then

$$R[A]=\bigcup_{\phi\in A}\bigcap_{n\ge1}\overline{R[I_{\phi\restriction n}]}.$$

for any $A\subseteq\mathbb{N}^\mathbb{N}$. In particular, $R[\mathbb{N}^\mathbb{N}]$ is the kernel of the Souslin scheme $\langle\overline{R[I_\sigma]}\rangle_{\sigma\in S^*}$.

421J Proposition Let X be a topological space, and $\mathcal{F}$ the family of closed subsets of X. Then a set $A\subseteq X$ belongs to $\mathcal{S}(\mathcal{F})$ iff there is a closed set $R\subseteq\mathbb{N}^\mathbb{N}\times X$ such that A is the projection of R on X.

421K Definition Let X be a topological space. A subset of X is a **Souslin-F** set in X if it is obtainable from closed subsets of X by Souslin's operation.

421L Proposition Let X be any topological space. Then every Baire subset of X is Souslin-F.

421M Proposition Let $\mathcal{E}$ be any family of sets such that $\emptyset \in \mathcal{E}$ and $E \cup E'$, $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{E}$ for every E, $E' \in \mathcal{E}$ and all sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{E}$, and $K \subseteq \mathbb{N}^{\mathbb{N}}$ a set which is compact for the usual topology on $\mathbb{N}^{\mathbb{N}}$. Then $\bigcup_{\phi\in K}\bigcap_{n\ge 1} E_{\phi\restriction n} \in \mathcal{E}$.

***421N Trees and derived trees (a)** Let $\mathcal{T}$ be the family of subsets T of S^* such that $\sigma\restriction k \in T$ whenever $\sigma \in T$ and $1 \le k \le \#(\sigma)$. Note that the intersection and union of any non-empty family of members of $\mathcal{T}$ again belong to $\mathcal{T}$.

(b) For $T \in \mathcal{T}$, set

$$\partial T = \{\sigma : \sigma \in S^*, \exists\, i \in \mathbb{N}, \sigma^\frown\!<\!i\!> \in T\},$$

so that $\partial T \in \mathcal{T}$ and $\partial T \subseteq T$. $\partial T_0 \subseteq \partial T_1$ whenever T_0, $T_1 \in \mathcal{T}$ and $T_0 \subseteq T_1$.

(c) For $T \in \mathcal{T}$, define $\langle \partial^\xi T\rangle_{\xi<\omega_1}$ inductively by setting $\partial^0 T = T$ and, for $\xi > 0$, $\partial^\xi T = \bigcap_{\eta<\xi}\partial(\partial^\eta T)$. $\partial^\xi T \in \mathcal{T}$, $\partial^\xi T \subseteq \partial^\eta T$ and $\partial^{\xi+1}T = \partial(\partial^\xi T)$ whenever $\eta \le \xi < \omega_1$.

(d) For any $T \in \mathcal{T}$, there is a $\xi < \omega_1$ such that $\partial^\xi T = \partial^\eta T$ whenever $\xi \le \eta < \omega_1$.

(e) For $T \in \mathcal{T}$, its **rank** is the first ordinal $r(T) < \omega_1$ such that $\partial^{r(T)}T = \partial^{r(T)+1}T$; $\partial^{r(T)}T = \partial^\eta T$ whenever $r(T) \le \eta < \omega_1$, and $\partial(\partial^{r(T)}T) = \partial^{r(T)}T$.

(f) For $T \in \mathcal{T}$, the following are equiveridical: (α) $\partial^{r(T)}T \ne \emptyset$; (β) there is a $\phi \in \mathbb{N}^{\mathbb{N}}$ such that $\phi\restriction n \in T$ for every $n \ge 1$.

(g) Now suppose that $\langle A_\sigma\rangle_{\sigma\in S^*}$ is a Souslin scheme. For any x we have a tree $T_x \in \mathcal{T}$ defined by saying that

$$T_x = \{\sigma : \sigma \in S^*, x \in \textstyle\bigcap_{1\le i\le\#(\sigma)} A_{\sigma\restriction i}\}.$$

Now the kernel of $\langle A_\sigma\rangle_{\sigma\in S^*}$ is just

$$\begin{aligned} A &= \{x : \exists\, \phi \in \mathbb{N}^{\mathbb{N}}, x \in \bigcap_{n\ge 1} A_{\phi\restriction n}\} \\ &= \{x : \exists\, \phi \in \mathbb{N}^{\mathbb{N}}, \phi\restriction n \in T_x \;\forall\, n \ge 1\} = \{x : \partial^{r(T)}T \ne \emptyset\}. \end{aligned}$$

The sets

$$\{x : x \in X \setminus A, r(T_x) = \xi\} = \{x : x \in X, r(T_x) = \xi, \partial^\xi T_x = \emptyset\},$$

for $\xi < \omega_1$, are called **constituents** of $X \setminus A$.

***421O Theorem** Let X be a set and Σ a σ-algebra of subsets of X. Let $\langle A_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A, and for $x \in X$ set

$$T_x = \{\sigma : \sigma \in S^*, x \in \textstyle\bigcap_{1\le i\le\#(\sigma)} A_{\sigma\restriction i}\}.$$

(a) For every $\xi < \omega_1$ and $\sigma \in S^*$, $\{x : x \in X, \sigma \in \partial^\xi T_x\} \in \Sigma$.
(b) For every $\xi < \omega_1$, $\{x : x \in A, r(T_x) \le \xi\}$ and $\{x : x \in X \setminus A, r(T_x) \le \xi\}$ belong to Σ. In particular, all the constituents of $X \setminus A$ belong to Σ.

***421P Corollary** Let X be a set and Σ a σ-algebra of subsets of X. If $A \in \mathcal{S}(\Sigma)$ then both A and $X \setminus A$ can be expressed as the union of at most ω_1 members of Σ.

***421Q Lemma** Let X be a set and $\langle A_\sigma\rangle_{\sigma\in S^*}$ and $\langle B_\sigma\rangle_{\sigma\in S^*}$ two Souslin schemes of subsets of X. Suppose that whenever ϕ, $\psi\in\mathbb{N}^{\mathbb{N}}$ there is an $n\ge 1$ such that $\bigcap_{1\le i\le n}A_{\phi\restriction i}\cap B_{\psi\restriction i}=\emptyset$. For $x\in X$ set

$$T_x=\bigcup_{n\ge 1}\{\sigma:\sigma\in\mathbb{N}^n,\,x\in\bigcap_{1\le i\le n}A_{\sigma\restriction i}\}$$

and let B be the kernel of $\langle B_\sigma\rangle_{\sigma\in S^*}$. Then $\sup_{x\in B}r(T_x)<\omega_1$.

Version of 12.4.16

422 K-analytic spaces

I introduce K-analytic spaces, defined in terms of usco-compact relations. The first step is to define the latter (422A) and give their fundamental properties (422B-422E). I reach K-analytic spaces themselves in 422F, with an outline of the most important facts about them in 422G-422K.

422A Definition Let X and Y be Hausdorff spaces. A relation $R\subseteq X\times Y$ is **usco-compact** if

(α) $R[\{x\}]$ is a compact subset of Y for every $x\in X$,

(β) $R^{-1}[F]$ is a closed subset of X for every closed set $F\subseteq Y$.

422B Lemma Let X and Y be Hausdorff spaces and $R\subseteq X\times Y$ an usco-compact relation. If $x\in X$ and H is an open subset of Y including $R[\{x\}]$, there is an open set $G\subseteq X$, containing x, such that $R[G]\subseteq H$.

422C Proposition Let X and Y be Hausdorff spaces. Then a subset R of $X\times Y$ is an usco-compact relation iff whenever $\mathcal{F}$ is an ultrafilter on $X\times Y$, containing R, such that the first-coordinate image $\pi_1[[\mathcal{F}]]$ of $\mathcal{F}$ has a limit in X, then $\mathcal{F}$ has a limit in R.

422D Lemma (a) Let X and Y be Hausdorff spaces. If $R\subseteq X\times Y$ is an usco-compact relation, then R is closed in $X\times Y$.

(b) Let X and Y be Hausdorff spaces. If $R\subseteq X\times Y$ is an usco-compact relation and $R'\subseteq R$ is a closed set, then R' is usco-compact.

(c) Let X and Y be Hausdorff spaces. If $f:X\to Y$ is a continuous function, then its graph is an usco-compact relation.

(d) Let $\langle X_i\rangle_{i\in I}$ and $\langle Y_i\rangle_{i\in I}$ be families of Hausdorff spaces, and $R_i\subseteq X_i\times Y_i$ an usco-compact relation for each i. Set $X=\prod_{i\in I}X_i$, $Y=\prod_{i\in I}Y_i$ and

$$R=\{(x,y):x\in X,\,y\in Y,\,(x(i),y(i))\in R_i\text{ for every }i\in I\}.$$

Then R is usco-compact in $X\times Y$.

(e) Let X and Y be Hausdorff spaces, and $R\subseteq X\times Y$ an usco-compact relation. Then (i) $R[K]$ is a compact subset of Y for any compact subset K of X (ii) $R[L]$ is a Lindelöf subset of Y for any Lindelöf subset L of X.

(f) Let X, Y and Z be Hausdorff spaces, and $R\subseteq X\times Y$, $S\subseteq Y\times Z$ usco-compact relations. Then the composition

$$S\circ R=\{(x,z):\text{there is some }y\in Y\text{ such that }(x,y)\in R\text{ and }(y,z)\in S\}$$

is usco-compact in $X\times Z$.

(g) Let X and Y be Hausdorff spaces and Y_0 any subset of Y. Then a relation $R\subseteq X\times Y_0$ is usco-compact when regarded as a relation between X and Y_0 iff it is usco-compact when regarded as a relation between X and Y.

(h) Let Y be a Hausdorff space and $R\subseteq\mathbb{N}^{\mathbb{N}}\times Y$ an usco-compact relation. Set

$$R'=\{(\alpha,y):\alpha\in\mathbb{N}^{\mathbb{N}},\,y\in Y\text{ and there is a }\beta\le\alpha\text{ such that }(\beta,y)\in R\}.$$

Then R' is usco-compact.

422E Lemma Let X and Y be Hausdorff spaces, and $R\subseteq X\times Y$ an usco-compact relation. If X is regular, so is R.

422F Definition Let X be a Hausdorff space. Then X is **K-analytic** if there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = X$.

If X is a Hausdorff space, we call a subset of X K-analytic if it is a K-analytic space in its subspace topology.

422G Theorem (a) Let X be a Hausdorff space. Then a subset A of X is K-analytic iff there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$.

(b) $\mathbb{N}^{\mathbb{N}}$ is K-analytic.

(c) Compact Hausdorff spaces are K-analytic.

(d) If X and Y are Hausdorff spaces and $R \subseteq X \times Y$ is an usco-compact relation, then $R[A]$ is K-analytic whenever $A \subseteq X$ is K-analytic. In particular, a Hausdorff continuous image of a K-analytic Hausdorff space is K-analytic.

(e) A product of countably many K-analytic Hausdorff spaces is K-analytic.

(f) A closed subset of a K-analytic Hausdorff space is K-analytic.

(g) A K-analytic Hausdorff space is Lindelöf, so a regular K-analytic Hausdorff space is completely regular.

422H Theorem (a) If X is a Hausdorff space, then any K-analytic subset of X is Souslin-F in X.

(b) If X is a K-analytic Hausdorff space, then a subset of X is K-analytic iff it is Souslin-F in X.

(c) For any Hausdorff space X, the family of K-analytic subsets of X is closed under Souslin's operation.

422I Lemma Let X be a Hausdorff space. Let $\mathcal{E}$ be a family of subsets of X such that (i) $\bigcup_{n\in\mathbb{N}} E_n$ and $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{E}$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}$ (ii) whenever x, y are distinct points of X, there are disjoint E, $F \in \mathcal{E}$ such that $x \in \operatorname{int} E$ and $y \in \operatorname{int} F$. Then whenever A, B are disjoint non-empty K-analytic subsets of X, there are disjoint E, $F \in \mathcal{E}$ such that $A \subseteq E$ and $B \subseteq F$.

422J Corollary Let X be a Hausdorff space and A, B disjoint K-analytic subsets of X. Then there is a Borel set which includes A and is disjoint from B.

***422K Theorem** Let X be a Hausdorff space.

(i) Suppose that X is regular. Let $A \subseteq X$ be a K-analytic set. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every Souslin-F subset of X disjoint from A is included in some E_ξ.

(ii) Suppose that X is regular. Let $A \subseteq X$ be a Souslin-F set. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every K-analytic subset of $X \setminus A$ is included in some E_ξ.

(iii) Let $A \subseteq X$ be a K-analytic set. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every K-analytic subset of $X \setminus A$ is included in some E_ξ.

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423 Analytic spaces

We come now to the original class of K-analytic spaces, the 'analytic' spaces. I define these as continuous images of $\mathbb{N}^{\mathbb{N}}$ (423A), but move as quickly as possible to their characterization as K-analytic spaces with countable networks (423C), so that many other fundamental facts (423E-423G) can be regarded as simple corollaries of results in §422. I give two versions of Lusin's theorem on injective images of Borel sets (423I), and a form of the von Neumann-Jankow measurable selection theorem (423P). I end with notes on constituents of coanalytic sets (423R-423S).

423A Definition A Hausdorff space is **analytic** or **Souslin** if it is either empty or a continuous image of $\mathbb{N}^{\mathbb{N}}$.

423B Proposition (a) A Polish space is analytic.

(b) A Hausdorff continuous image of an analytic Hausdorff space is analytic.

(c) A product of countably many analytic Hausdorff spaces is analytic.

(d) A closed subset of an analytic Hausdorff space is analytic.

(e) An analytic Hausdorff space has a countable network consisting of analytic sets.

423C Theorem A Hausdorff space is analytic iff it is K-analytic and has a countable network.

423D Corollary (a) An analytic Hausdorff space is hereditarily Lindelöf.

(b) In a regular analytic Hausdorff space, closed sets are zero sets and the Baire and Borel σ-algebras coincide.

(c) A compact subset of an analytic Hausdorff space is metrizable.

(d) A metrizable space is analytic iff it is K-analytic.

423E Theorem (a) For any Hausdorff space X, the family of subsets of X which are analytic in their subspace topologies is closed under Souslin's operation.

(b) Let $(X,\mathfrak{T})$ be an analytic Hausdorff space. For a subset A of X, the following are equiveridical:

(i) A is analytic;

(ii) A is K-analytic;

(iii) A is Souslin-F;

(iv) A can be obtained by Souslin's operation from the family of Borel subsets of X.

In particular, all Borel sets in X are analytic.

423F Proposition Let $(X,\mathfrak{T})$ be an analytic Hausdorff space.

(a) A set $E \subseteq X$ is Borel iff both E and $X \setminus E$ are analytic.

(b) If $\mathfrak{S}$ is a coarser Hausdorff topology on X, then $\mathfrak{S}$ and $\mathfrak{T}$ have the same Borel sets.

423G Lemma Let X and Y be analytic Hausdorff spaces and $f : X \to Y$ a Borel measurable function.

(a) f is an analytic set.

(b) $f[A]$ is an analytic set in Y for any analytic set $A \subseteq X$.

(c) $f^{-1}[B]$ is an analytic set in X for any analytic set $B \subseteq Y$.

423H Lemma Let $(X,\mathfrak{T})$ be an analytic Hausdorff space, and $\langle E_n\rangle_{n\in\mathbb{N}}$ any sequence of Borel sets in X. Then the topology $\mathfrak{T}'$ generated by $\mathfrak{T} \cup \{E_n : n \in \mathbb{N}\}$ is analytic.

423I Theorem Let X be a Polish space, $E \subseteq X$ a Borel set, Y a Hausdorff space and $f : E \to Y$ an injective function.

(a) If f is continuous, then $f[E]$ is Borel.

(b) If Y has a countable network and f is Borel measurable, then $f[E]$ is Borel.

423J Proposition Let $(X,\mathfrak{T})$ be an analytic Hausdorff space, and Σ a countably generated σ-subalgebra of the Borel σ-algebra $\mathcal{B}(X,\mathfrak{T})$ which separates the points of X. Then $\Sigma = \mathcal{B}(X,\mathfrak{T})$.

423K Lemma If X is an uncountable analytic Hausdorff space, it has subsets homeomorphic to $\{0,1\}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

423L Corollary Any uncountable Borel set in any analytic Hausdorff space has cardinal $\mathfrak{c}$.

423M Proposition Let X be an uncountable analytic Hausdorff space. Then it has a non-Borel analytic subset.

423N Theorem Let X be an analytic Hausdorff space, Y a set, and $\mathcal{C} \subseteq \mathcal{P}Y$. Write T for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathcal{C})$, where $\mathcal{S}$ is Souslin's operation, and $\mathcal{V}$ for $\mathcal{S}(\{F \times C : F \subseteq X \text{ is closed}, C \in \mathcal{C}\})$. If $W \in \mathcal{V}$, then $W[X] \in \mathcal{S}(\mathcal{C})$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

423O Corollary Let X be an analytic Hausdorff space, Y a set and T a σ-algebra of subsets of Y which is closed under Souslin's operation. Suppose that $W \in \mathcal{S}(\mathcal{B}(X)\widehat{\otimes}\mathrm{T})$ where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

423P Corollary Let X be an analytic Hausdorff space and Y any topological space. Let T be the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. If $W \in \mathcal{S}(\mathcal{B}(X\times Y))$, then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

423Q Corollary Let X and Y be analytic Hausdorff spaces, A an analytic subset of X and $f : A \to Y$ a Borel measurable function. Let T be the σ-algebra of subsets of Y generated by the Souslin-F subsets of Y. Then $f[A] \in \mathrm{T}$ and there is a T-measurable function $g : f[A] \to A$ such that fg is the identity on $f[A]$.

***423R Constituents of coanalytic sets: Theorem** Let X be a Hausdorff space, and $A \subseteq X$ an analytic subset of X. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel subsets of X, with union $X \setminus A$, such that every analytic subset of $X \setminus A$ is included in some E_ξ.

***423S Remarks (a)** Let A be an analytic set in an analytic space X and $\langle E_\xi\rangle_{\xi<\omega_1}$ a family of Borel sets as in 423R. There is nothing unique about the E_ξ. But if $\langle E'_\xi\rangle_{\xi<\omega_1}$ is another such family, then there is a cofinal closed set C in ω_1 such that $\bigcup_{\eta<\xi} E_\eta = \bigcup_{\eta<\xi} E'_\eta$ for every $\xi \in C$.

Another way of expressing the result in 423R is to say that if we write $\mathcal{I} = \{B : B \subseteq X \setminus A \text{ is analytic}\}$, then $\{E : E \in \mathcal{I},\ E \text{ is Borel}\}$ is cofinal with $\mathcal{I}$ and $\operatorname{cf}\mathcal{I} \le \omega_1$.

(b) It is a remarkable fact that, in some models of set theory, we can have non-Borel coanalytic sets in Polish spaces such that all their constituents are countable. But in 'ordinary' cases we shall have, for any family $\langle G_\xi\rangle_{\xi<\omega_1}$ of Borel constituents of $X \setminus A$, uncountable G_ξ.

***423T Coanalytic and PCA sets** Let X be a Polish space.

(a) A subset A of X is **coanalytic** if $X\setminus A$ is analytic, and **PCA** if there is a coanalytic set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$.

(b) Every PCA set $A \subseteq X$ can be expressed as the union of at most ω_1 Borel sets.

(c) A subset of X is Borel iff it is both analytic and coanalytic. The union and intersection of a sequence of coanalytic subsets of X are coanalytic. If Y is another Polish space and $h : X \to Y$ is Borel measurable, then $h^{-1}[B]$ is coanalytic in X for every coanalytic $B \subseteq Y$. If Y is a G_δ subset of X and $B \subseteq Y$ is coanalytic in Y then B is coanalytic in X.

(d) If X and Y are Polish spaces, $A \subseteq Y$ is PCA and $f : X \to Y$ is Borel measurable, then $f^{-1}[A]$ is a PCA set in X.

Version of 21.3.08

424 Standard Borel spaces

This volume is concerned with topological measure spaces, and it will come as no surprise that the topological properties of Polish spaces are central to the theory. But even from the point of view of unadorned measure theory, not looking for topological structures on the underlying spaces, it turns out that the Borel algebras of Polish spaces have a very special position. It will be useful later on to be able to refer to some fundamental facts concerning them.

424A Definition Let X be a set and Σ a σ-algebra of subsets of X. We say that (X,Σ) is a **standard Borel space** if there is a Polish topology on X for which Σ is the algebra of Borel sets.

424B Proposition (a) If (X,Σ) is a standard Borel space, then Σ is countably generated as σ-algebra of sets.

(b) If $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ is a countable family of standard Borel spaces, then $(\prod_{i\in I}X_i,\widehat{\bigotimes}_{i\in I}\Sigma_i)$ is a standard Borel space.

(c) Let (X,Σ) and (Y,T) be standard Borel spaces and $f:X\to Y$ a (Σ,T)-measurable surjection. Then

(i) if $E\in\Sigma$ is such that $f[E]\cap f[X\setminus E]=\emptyset$, then $f[E]\in\mathrm{T}$;

(ii) $\mathrm{T}=\{F:F\subseteq Y,\,f^{-1}[F]\in\Sigma\}$;

(iii) if f is a bijection it is an isomorphism.

(d) Let (X,Σ) and (Y,T) be standard Borel spaces and $f:X\to Y$ a (Σ,T)-measurable injection. Then $Z=f[X]\in\mathrm{T}$ and f is an isomorphism between (X,Σ) and (Z,T_Z), where T_Z is the subspace σ-algebra.

424C Theorem Let (X,Σ) be a standard Borel space.

(a) If X is countable then $\Sigma=\mathcal{P}X$.

(b) If X is uncountable then (X,Σ) is isomorphic to $(\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$, where $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$ is the algebra of Borel subsets of $\mathbb{N}^{\mathbb{N}}$.

424D Corollary (a) If (X,Σ) and (Y,T) are standard Borel spaces and $\#(X)=\#(Y)$, then (X,Σ) and (Y,T) are isomorphic.

(b) If (X,Σ) is an uncountable standard Borel space then $\#(X)=\#(\Sigma)=\mathfrak{c}$.

424E Proposition Let X be a set and Σ a σ-algebra of subsets of X; suppose that (X,Σ) is countably separated in the sense that there is a countable set $\mathcal{E}\subseteq\Sigma$ separating the points of X. If $A\subseteq X$ is such that (A,Σ_A) is a standard Borel space, where Σ_A is the subspace σ-algebra, then $A\in\Sigma$.

424F Corollary Let X be a Polish space and $A\subseteq X$ any set which is not Borel. Let $\mathcal{B}(A)$ be the Borel σ-algebra of A. Then $(A,\mathcal{B}(A))$ is not a standard Borel space.

424G Proposition Let (X,Σ) be a standard Borel space. Then (E,Σ_E) is a standard Borel space for every $E\in\Sigma$.

***424H Theorem** Let G be a Polish group, $(X,\mathfrak{T})$ a Polish space and $\bullet$ a Borel measurable action of G on X. Then there is a Polish topology $\mathfrak{T}'$ on X, yielding the same Borel sets as $\mathfrak{T}$, such that the action is continuous for $\mathfrak{T}'$ and the given topology of G.

Version of 9.8.13

*425 Realization of automorphisms

In §344 I presented some results on the representation of a countable semigroup of Boolean homomorphisms in a measure algebra by a semigroup of functions on the measure space underlying the algebra. §424 provides us with the tools needed for a remarkable extension, in the case of the Lebesgue measure algebra, to groups with cardinal ω_1 (Theorem 425D). The expression of the ideas is made smoother by using the language of group actions.

425A I begin with what amounts to a special case of the main theorem, with some refinements which will be useful elsewhere.

Proposition (a) Let (X,Σ) and (Y,T) be non-empty standard Borel spaces, and $\mathcal{I}$, $\mathcal{J}$ σ-subalgebras of Σ, T respectively; write $\mathfrak{A}=\Sigma/\mathcal{I}$ and $\mathfrak{B}=\mathrm{T}/\mathcal{J}$ for the quotient algebras. For $E\in\Sigma$, $F\in\mathrm{T}$ write Σ_E, T_F for the subspace σ-algebras on E, F respectively.

(a) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, there is a (T,Σ)-measurable $f:Y\to X$ which represents π in the sense that $\pi E^\bullet=f^{-1}[E]^\bullet$ for every $E\in\Sigma$.

(b) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a Boolean isomorphism, there are $G\in\mathcal{I}$, $H\in\mathcal{J}$ and a bijection $h:Y\setminus H\to X\setminus G$ which is an isomorphism between $(Y\setminus H,\mathrm{T}_{Y\setminus H})$ and $(X\setminus G,\Sigma_{X\setminus G})$, and represents π in the sense that $\pi E^\bullet=h^{-1}[E\setminus G]^\bullet$ for every $E\in\Sigma$.

(c) If $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean automorphism, there is a bijection $h:X\to X$ which is an automorphism of (X,Σ) and represents π in the sense of (a).

(d) If $\#(X)=\#(Y)=\mathfrak{c}$, $\mathfrak{A}$ and $\mathfrak{B}$ are ccc, and $\pi:\mathfrak{A}\to\mathfrak{B}$ is a Boolean isomorphism, there is a bijection $h:Y\to X$ which is an isomorphism between (Y,T) and (X,Σ), and represents π in the sense of (a).

425B Lemma Let G be a group, G_0 a subgroup of G, H another group, and X, Z sets; let $\bullet_r$ be the right shift action of H on Z^H. Suppose we are given a group homomorphism $\theta:G\to H$, an injective function $f:\mathbb{N}\times Z^H\to X$ and an action $\bullet_0$ of G_0 on X such that $\pi\bullet_0 f(n,z)=f(n,\theta(\pi)\bullet_r z)$ whenever $n\in\mathbb{N}$ and $z\in Z^H$.

(a) If $\#(X\setminus f[\mathbb{N}\times Z^H])\le\#(Z)$, there is an action $\bullet$ of G on X extending $\bullet_0$.

(b) Suppose moreover that H is countable, X and Z are Polish spaces, and f is Borel measurable when $\mathbb{N}\times Z^H$ is given the product topology. If $x\mapsto\pi\bullet_0 x$ is Borel measurable for every $\pi\in G_0$, then $\bullet$ can be chosen in such a way that $x\mapsto\psi\bullet x$ is Borel measurable for every $\psi\in G$.

425C Master actions (a) For each $R\subseteq\mathbb{N}^2$, consider the family F_R of injective functions f from countable ordinals to $\mathbb{N}$ such that

$$\text{for every } \beta\in\operatorname{dom}f,\ f(\beta) \text{ is the unique member of } \mathbb{N} \text{ such that } R[\{f(\beta)\}]=f[\beta].$$

$f_R=\bigcup F_R$ is the unique maximal element of F_R.

For a countable ordinal α, let $\mathcal{R}_\alpha$ be the set of those $R\subseteq\mathbb{N}^2$ such that $\alpha\le\operatorname{dom}f_R$. Note that if $f:\alpha\to\mathbb{N}$ is injective, then there is an $R\in\mathcal{R}_\alpha$ such that $f=f_R\restriction\alpha$.

(b) Let ✠ be the family of group operations $\star$ on $\mathbb{N}$. We are going to need the natural Borel structure on ✠ corresponding to the identification of each $\star\in$ ✠, which is a function from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$, with the set

$$\{(i,j,k):i\star j=k\}\subseteq\mathbb{N}^3.$$

For $\star\in$ ✠, let $\bullet_r^\star$ be the corresponding right shift action of $\mathbb{N}$ on $\mathbb{R}^{\mathbb{N}}$, so that $(m\bullet_r^\star z)(i)=z(i\star m)$ whenever $z\in\mathbb{R}^{\mathbb{N}}$ and i, $m\in\mathbb{N}$.

(c) Let G be a group, with cardinal ω_1, with identity ι; let $\langle\pi_\alpha\rangle_{\alpha<\omega_1}$ enumerate G, with $\pi_0=\iota$. Let $F\subseteq\omega_1$ be the set of those α such that $G_\alpha=\{\pi_\beta:\beta<\alpha\}$ is a subgroup of G. For $\alpha\in F$, set

$$\mathcal{S}_\alpha=\{(R,\star):R\in\mathcal{R}_\alpha,\ \star\in\text{✠ and } f_R(\beta)\star f_R(\gamma)=f_R(\delta)$$
$$\text{whenever } \beta,\ \gamma,\ \delta<\alpha \text{ and } \pi_\beta\pi_\gamma=\pi_\delta\}.$$

(d) For $\alpha \in F$, set

$$\mathcal{M}_\alpha = \{(R, \star, z) : (R, \star) \in \mathcal{S}_\alpha,\ z \in \mathbb{R}^{\mathbb{N}}\}.$$

Then we have an action $\bullet'_\alpha$ of G_α on $\mathcal{M}_\alpha$ defined by saying that

$$\pi_\beta \bullet'_\alpha (R, \star, z) = (R, \star, f_R(\beta) \bullet^\star_r z)$$

whenever $(R, \star) \in \mathcal{S}_\alpha$, $z \in \mathbb{R}^{\mathbb{N}}$ and $\beta < \alpha$.

(e) If α, $\beta \in F$ and $\alpha \le \beta$, then $\mathcal{R}_\beta \subseteq \mathcal{R}_\alpha$, $\mathcal{S}_\beta \subseteq \mathcal{S}_\alpha$, $\mathcal{M}_\beta \subseteq \mathcal{M}_\alpha$ and $\pi \bullet'_\beta (R, \star, z) = \pi \bullet'_\alpha (R, \star, z)$ whenever $(R, \star, z) \in \mathcal{M}_\beta$ and $\pi \in G_\alpha$. If $\beta \in F$ and $\beta = \sup(\beta \cap F)$, then $\mathcal{R}_\beta = \bigcap_{\alpha \in \beta \cap F} \mathcal{R}_\alpha$ and $\mathcal{M}_\beta = \bigcap_{\alpha \in \beta \cap F} \mathcal{M}_\alpha$.

(f)(i) For $\alpha < \omega_1$, $\mathcal{R}_\alpha$ belongs to the Borel σ-algebra $\mathcal{B}(\mathcal{P}\mathbb{N}^2)$, and $\{(R, m) : R \subseteq \mathbb{N}^2,\ (\beta, m) \in f_R\} \in \mathcal{B}(\mathcal{P}\mathbb{N}^2 \times \mathbb{N})$ whenever $m \in \mathbb{N}$ and $\beta < \alpha$.

(ii) ✠ is a Borel subset of $\mathcal{P}(\mathbb{N}^3)$, $\mathcal{S}_\alpha$ is a Borel subset of $\mathcal{P}(\mathbb{N}^2) \times \mathcal{P}(\mathbb{N}^3)$, and $\mathcal{M}_\alpha$ is a Borel subset of $\mathcal{P}(\mathbb{N}^2) \times \mathcal{P}(\mathbb{N}^3) \times \mathbb{R}^{\mathbb{N}}$, for every $\alpha \in F$. $(R, \star, z) \mapsto \pi \bullet'_\alpha (R, \star, z) : \mathcal{M}_\alpha \to \mathcal{M}_\alpha$ is Borel measurable whenever $\alpha \in F$ and $\pi \in G_\alpha$.

425D Törnquist's theorem Let (X, Σ) be a standard Borel space and $\mathcal{I}$ a σ-ideal of Σ containing an uncountable set. Let $\mathfrak{A}$ be the quotient algebra $\Sigma/\mathcal{I}$, and $G \subseteq \operatorname{Aut}\mathfrak{A}$ a subgroup with cardinal at most ω_1. Then there is an action $\bullet$ of G on X which represents G in the sense that $\pi \bullet E$ belongs to Σ, and $(\pi \bullet E)^\bullet = \pi(E^\bullet)$, for every $E \in \Sigma$ and $\pi \in G$.

425E Scholium The theorem here applies to groups with cardinal at most ω_1. So it is worth noting that in the context of 425D the whole group $\operatorname{Aut}\mathfrak{A}$ has cardinal at most $\mathfrak{c}$.

We therefore have

> Suppose the continuum hypothesis is true. Let (X, Σ) be a standard Borel space and $\mathcal{I}$ a σ-ideal of subsets of Σ containing an uncountable set. Then there is an action $\bullet$ of $\operatorname{Aut}(\Sigma/\mathcal{I})$ on X such that $\pi E^\bullet = (\pi \bullet E)^\bullet$ whenever $E \in \Sigma$ and $\pi \in \operatorname{Aut}(\Sigma/\mathcal{I})$.

425Z Problems (a) Suppose that (X, Σ) is an uncountable standard Borel space and $\mathcal{I}$ the ideal $[X]^{\le\omega}$. Which subgroups G of $\operatorname{Aut}(\Sigma/\mathcal{I})$ can be represented by actions of G on X?

(b) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of Lebesgue measure μ on $[0,1]$, and G a semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself with $\#(G) = \omega_1$. Must there be a family $\langle f_\pi \rangle_{\pi \in G}$ of inverse-measure-preserving functions from $[0,1]$ to itself such that $f_{\pi\phi} = f_\phi f_\pi$ for all π, $\phi \in G$ and f_π represents π, in the sense of 425A, for every $\pi \in G$?

(c) Let $(\mathfrak{B}_{\omega_1}, \bar\nu_{\omega_1})$ be the measure algebra of the usual measure on $\{0,1\}^{\omega_1}$, and G a group of measure-preserving automorphisms of $\mathfrak{B}_{\omega_1}$ with $\#(G) = \omega_1$. Must there be a family $\langle f_\pi \rangle_{\pi \in G}$ of inverse-measure-preserving functions from $\{0,1\}^{\omega_1}$ to itself such that $f_{\pi\phi} = f_\phi f_\pi$ for all π, $\phi \in G$ and f_π represents π for every $\pi \in G$?

Version of 30.11.16

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

423J Extra results have been interpolated into §423, so the second half of that section (423J-423R), referred to in the 2008 and 2015 editions of Volume 5, is now 423K-423T.