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Chapter 42

Descriptive set theory

At this point, I interpolate an auxiliary chapter, in the same spirit as Chapters 31 and 35 in the last volume. As with Boolean algebras and Riesz spaces, it is not just that descriptive set theory provides essential tools for modern measure theory; it also offers deep intuitions, and for this reason demands study well beyond an occasional glance at an appendix. Several excellent accounts have been published; the closest to what we need here is probably Rogers 80; at a deeper level we have Moschovakis 80, and an admirable recent treatment is Kechris 95. Once again, however, I indulge myself by extracting those parts of the theory which I shall use directly, giving proofs and exercises adapted to the ideas I am trying to emphasize in this volume and the next.

The first section describes Souslin's operation and its basic set-theoretic properties up to first steps in the theory of 'constituents' (421N-421Q), mostly steering away from topological ideas, but with some remarks on σ-algebras and Souslin-F sets. §422 deals with usco-compact relations and K-analytic spaces, working through the topological properties which will be useful later, and giving a version of the First Separation Theorem (422I-422J). §423 looks at 'analytic' or 'Souslin' spaces, treating them primarily as a special kind of K-analytic space, with the von Neumann-Jankow selection theorem (423P). §424 is devoted to 'standard Borel spaces'; it is largely a series of easy applications of results in §423, but there is a substantial theorem on Borel measurable actions of Polish groups (424H). Finally, I add a note on A.Törnquist's theorem on representation of groups of automorphisms of quotient algebras (425D).

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421 Souslin's operation

I introduce Souslin's operation $\mathcal{S}$ (421B) and show that it is idempotent (421D). I describe alternative characterizations of members of $\mathcal{S}(\mathcal{E})$, where $\mathcal{E} \subseteq \mathcal{P}X$, as projections of sets in $\mathbb{N}^{\mathbb{N}} \times X$ (421G-421J). I briefly mention Souslin-F sets (421J-421L) and a special property of 'inner Souslin kernels' (421M). At the end of the section I set up an abstract theory of 'constituents' for kernels of Souslin schemes and their complements (421N-421Q).

421A Notation Throughout this chapter, and frequently in the next, I shall regard a member of $\mathbb{N}$ as the set of its predecessors, so that a finite power X^k can be identified with the set of functions from k to X, and if $\phi \in X^{\mathbb{N}}$ and $k \in \mathbb{N}$, we can speak of the restriction $\phi\restriction k \in X^k$. In the same spirit, identifying functions with their graphs, I can write '$\sigma \subseteq \phi$' when $\sigma \in X^k$, $\phi \in X^{\mathbb{N}}$ and ϕ extends σ. On occasion I may write $\#(\sigma)$ for the 'length' of a finite function σ − again identifying σ with its graph − so that $\#(\sigma) = k$ if $\sigma \in X^k$. And if $k = 0$, identified with $\emptyset$, then the only function from k to X is the empty function, so X^0 becomes $\{\emptyset\}$.

I shall sometimes refer to the 'usual topology of $\mathbb{N}^{\mathbb{N}}$'; this is the product topology if each copy of $\mathbb{N}$ is given its discrete topology. S will always be the set $\bigcup_{k\in\mathbb{N}} \mathbb{N}^k$, and $S^* = S \setminus \{\emptyset\}$ the set $\bigcup_{k\ge1} \mathbb{N}^k$; for $\sigma \in S$, I_σ will be $\{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma\}$. Then $I_\emptyset = \mathbb{N}^{\mathbb{N}}$ and $\{I_\sigma : \sigma \in S^*\}$ is a base for the topology of $\mathbb{N}^{\mathbb{N}}$ consisting of open-and-closed sets. If $\sigma \in \mathbb{N}^k$ and $i \in \mathbb{N}$ I write $\sigma^\frown{<}i{>}$ for the member τ of $\mathbb{N}^{k+1}$ such that $\tau(k) = i$ and $\tau(j) = \sigma(j)$ for $j < k$.

421B Definition If $\mathcal{E}$ is a family of sets, I write $\mathcal{S}(\mathcal{E})$ for the family of sets expressible in the form

$$\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{k\ge 1}E_{\phi\restriction k}$$

for some family $\langle E_\sigma\rangle_{\sigma\in S^*}$ in $\mathcal{E}$.

A family $\langle E_\sigma\rangle_{\sigma\in S^*}$ is called a **Souslin scheme**; the corresponding set $\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{k\ge 1}E_{\phi\restriction k}$ is its **kernel**; the operation

$$\langle E_\sigma\rangle_{\sigma\in S^*}\mapsto\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{k\ge 1}E_{\phi\restriction k}$$

is **Souslin's operation** or **operation** $\mathcal{A}$. Thus $\mathcal{S}(\mathcal{E})$ is the family of sets obtainable from sets in $\mathcal{E}$ by Souslin's operation. If $\mathcal{E}=\mathcal{S}(\mathcal{E})$, we say that $\mathcal{E}$ is **closed under Souslin's operation**.

Remark I should perhaps warn you that some authors use $\bigcup_{k\in\mathbb{N}}\mathbb{N}^k$ here in place of S^*; so that their Souslin kernels are of the form $\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{k\ge 0}E_{\phi\restriction k}\subseteq E_\emptyset$. Consequently, for such authors, any member of $\mathcal{S}(\mathcal{E})$ is included in some member of $\mathcal{E}$. If $\mathcal{E}$ has a greatest member (or, fractionally more generally, if any sequence in $\mathcal{E}$ is bounded above in $\mathcal{E}$) this makes no difference; but if, for instance, $\mathcal{E}$ is the family of compact subsets of a topological space, the two definitions of $\mathcal{S}$ may not quite coincide. I believe that on this point, for once, I am following the majority.

421C Elementary facts (a) It is worth noting straight away that if $\mathcal{E}$ is any family of sets, then $\bigcup_{n\in\mathbb{N}}E_n$ and $\bigcap_{n\in\mathbb{N}}E_n$ belong to $\mathcal{S}(\mathcal{E})$ for any sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. **P** Set

$$F_\sigma=E_{\sigma(0)}\text{ for every }\sigma\in S^*,$$

$$G_\sigma=E_k\text{ whenever }k\in\mathbb{N},\ \sigma\in\mathbb{N}^{k+1};$$

then

$$\bigcup_{n\in\mathbb{N}}E_n=\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{k\ge 1}F_{\phi\restriction k}\in\mathcal{S}(\mathcal{E}),$$

$$\bigcap_{n\in\mathbb{N}}E_n=\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{k\ge 1}G_{\phi\restriction k}\in\mathcal{S}(\mathcal{E}).\ \mathbf{Q}$$

In particular, $\mathcal{E}\subseteq\mathcal{S}(\mathcal{E})$. But note that there is no reason why $E\setminus F$ should belong to $\mathcal{S}(\mathcal{E})$ for E, $F\in\mathcal{E}$.

(b) Let X and Y be sets, and $f:X\to Y$ a function. Let $\langle F_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{P}Y$, with kernel B. Then $f^{-1}[B]$ is the kernel of the Souslin scheme $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S^*}$. **P**

$$f^{-1}[B]=f^{-1}[\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n\ge 1}F_{\phi\restriction n}]=\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}}\bigcap_{n\ge 1}f^{-1}[F_{\phi\restriction n}].\ \mathbf{Q}$$

(c) Let X and Y be sets, and $f:X\to Y$ a function. Let $\mathcal{F}$ be a family of subsets of Y. Then

$$\{f^{-1}[B]:B\in\mathcal{S}(\mathcal{F})\}=\mathcal{S}(\{f^{-1}[F]:F\in\mathcal{F}\}).$$

P For a set $A\subseteq X$, $A\in\mathcal{S}(\{f^{-1}[F]:F\in\mathcal{F}\})$ iff there is some Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S^*}$ in $\{f^{-1}[F]:F\in\mathcal{F}\}$ such that A is the kernel of $\langle E_\sigma\rangle_{\sigma\in S^*}$, that is, iff there is some Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$ in $\mathcal{F}$ such that A is the kernel of $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S^*}$, that is, iff $A=f^{-1}[B]$ where B is the kernel of some Souslin scheme in $\mathcal{F}$. **Q**

(d) Let X and Y be sets, and $f:X\to Y$ a surjective function. Let $\mathcal{F}$ be a family of subsets of Y. Then

$$\mathcal{S}(\mathcal{F})=\{B:B\subseteq Y,\ f^{-1}[B]\in\mathcal{S}(\{f^{-1}[F]:F\in\mathcal{F}\})\}.$$

P If $B\in\mathcal{S}(\mathcal{F})$, then $f^{-1}[B]\in\mathcal{S}(\{f^{-1}[F]:F\in\mathcal{F}\})$, by (c) above. If $B\subseteq Y$ and $f^{-1}[B]\in\mathcal{S}(\{f^{-1}[F]:F\in\mathcal{F}\})$, then there is a Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$ in $\mathcal{F}$ such that $f^{-1}[B]$ is the kernel of $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S^*}$, that is, $f^{-1}[B]=f^{-1}[C]$ where C is the kernel of $\langle F_\sigma\rangle_{\sigma\in S^*}$. Because f is surjective, $B=C\in\mathcal{S}(\mathcal{F})$. **Q**

(e) Souslin's operation can be thought of as a projection operator, as follows. Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme with kernel A. Set

$$R=\bigcap_{n\ge 1}\bigcup_{\sigma\in\mathbb{N}^n}I_\sigma\times E_\sigma.$$

Then $R[\mathbb{N}^{\mathbb{N}}] = A$. **P** For any x, and any $\phi \in \mathbb{N}^{\mathbb{N}}$,

$$\begin{aligned}(\phi, x) \in R &\iff \text{for every } n \ge 1 \text{ there is a } \sigma \in \mathbb{N}^n \text{ such that } x \in E_\sigma,\ \phi \in I_\sigma \\ &\iff x \in E_{\phi\restriction n} \text{ for every } n \ge 1.\end{aligned}$$

But this means that

$$\begin{aligned}x \in R[\mathbb{N}^{\mathbb{N}}] &\iff \text{there is a } \phi \in \mathbb{N}^{\mathbb{N}} \text{ such that } (\phi, x) \in R \\ &\iff \text{there is a } \phi \in \mathbb{N}^{\mathbb{N}} \text{ such that } x \in \bigcap_{n\ge1} E_{\phi\restriction n} \iff x \in A. \textbf{ Q}\end{aligned}$$

(f)(i) I will say that a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S^*}$ is **fully regular** if $E_\sigma \subseteq E_\tau$ whenever σ, $\tau \in S^*$, $\#(\tau) \le \#(\sigma)$ and $\sigma(i) \le \tau(i)$ for every $i < \#(\sigma)$.

(ii) Let $\mathcal{E}$ be a family of sets such that $E \cup F$ and $E \cap F$ belong to $\mathcal{E}$ for all E, $F \in \mathcal{E}$. Then every member of $\mathcal{S}(\mathcal{E})$ can be expressed as the kernel of a regular Souslin scheme in $\mathcal{E}$. **P** Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{E}$ with kernel A. Set $F_\sigma = \bigcap_{1\le k\le n} E_{\sigma\restriction k}$, $G_\sigma = \bigcup_{\tau\in\mathbb{N}^n,\tau\le\sigma} F_\tau$ for $n \ge 1$ and $\sigma \in \mathbb{N}^n$; then F_σ, $G_\sigma \in \mathcal{E}$ for every $\sigma \in S^*$. Write B, C for the kernels of $\langle F_\sigma\rangle_{\sigma\in S^*}$, $\langle G_\sigma\rangle_{\sigma\in S^*}$ respectively. If $\phi \in \mathbb{N}^{\mathbb{N}}$, then $\bigcap_{n\ge1} F_{\phi\restriction n} = \bigcap_{n\ge1} E_{\phi\restriction n}$, so $A' = A$. Because $F_\sigma \subseteq G_\sigma$ for every σ, $A' \subseteq A''$. If $x \in A''$, let $\phi \in \mathbb{N}^{\mathbb{N}}$ be such that $x \in \bigcap_{n\ge1} G_{\phi\restriction n}$. Then for each $k \ge 1$ there is a $\tau_k \in \mathbb{N}^k$ such that $\tau_k \le \phi\restriction k$ and $x \in F_{\tau_k}$. For $k \ge 1$, set

$$\begin{aligned}\psi_k(i) &= \tau_k(i) \text{ for } i < k, \\ &= 0 \text{ otherwise;}\end{aligned}$$

then ψ_k belongs to the compact set $\prod_{i\in\mathbb{N}} \phi(i) + 1$ for every n, so $\langle\psi_k\rangle_{k\ge1}$ has a cluster point ψ in $\mathbb{N}^{\mathbb{N}}$. For any $n \ge 1$, there are infinitely many k such that $\psi\restriction n = \psi_k\restriction n$, so there is such a k with $k \ge n$, in which case $\psi\restriction n = \tau_k\restriction n$ and

$$x \in F_{\tau_k} \subseteq E_{\tau_k\restriction n} = E_{\psi\restriction n}.$$

Thus $x \in \bigcap_{n\ge1} E_{\psi\restriction n} \subseteq A$. As x is arbitrary, $A'' \subseteq A$ and $A'' = A$.

On the other hand, if σ, $\tau \in S^*$, $m = \#(\tau) \le \#(\sigma) = n$ and $\sigma\restriction m \le \tau$, then take any $\sigma' \in \mathbb{N}^n$ such that $\sigma' \le \sigma$; in this case $\sigma'\restriction m \le \tau$ so

$$F_{\sigma'} \subseteq F_{\sigma'\restriction m} \subseteq G_\tau.$$

As σ' is arbitrary, $G_\sigma \subseteq G_\tau$; as σ and τ are arbitrary, $\langle G_\sigma\rangle_{\sigma\in S^*}$ is fully regular.

Thus $A = A''$ is the kernel of a fully regular Souslin scheme in $\mathcal{E}$. **Q**

421D The first fundamental theorem is that the operation $\mathcal{S}$ is idempotent.

Theorem (Souslin 1917) For any family $\mathcal{E}$ of sets, $\mathcal{S}(\mathcal{E})$ is closed under Souslin's operation.

proof (a) Let $\langle A_\sigma\rangle_{\sigma\in S^*}$ be a family in $\mathcal{S}(\mathcal{E})$, and set $A = \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge1} A_{\phi\restriction k}$; I have to show that $A \in \mathcal{S}(\mathcal{E})$. For each $\sigma \in S$, let $\langle E_{\sigma\tau}\rangle_{\tau\in S^*}$ be a family in $\mathcal{E}$ such that $A_\sigma = \bigcup_{\psi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{m\ge1} E_{\sigma,\psi\restriction m}$. Then

$$A = \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\ge1} \bigcup_{\psi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{m\ge1} E_{\phi\restriction k,\psi\restriction m} = \bigcup_{\substack{\phi\in\mathbb{N}^{\mathbb{N}} \\ \boldsymbol{\psi}\in(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}}} \bigcap_{k,m\ge1} E_{\phi\restriction k,\psi_k\restriction m},$$

writing $\boldsymbol{\psi} = \langle\psi_k\rangle_{k\ge1}$ for $\boldsymbol{\psi} \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}$. The idea of the proof is simply that $\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}$ is essentially identical to $\mathbb{N}^{\mathbb{N}}$, so that all we have to do is to organize new names for the $E_{\sigma\tau}$. But as it is by no means a trivial matter to devise a coding scheme which really works, I give the details at length.

(b) The first step is to note that S^* and $(S^*)^2$ are countable, so there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ running over $\{E_{\sigma\tau} : \sigma, \tau \in S^*\}$. Next, choose any injective function $q : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\setminus\{0\}$ such that $q(0,0) = 1$ and $q(0,1) = 2$. For k, $m \ge 1$ set $J_{km} = \{(i,0) : i < k\} \cup \{(i,k) : i < m\}$, so that $J_{11} = \{(0,0),(0,1)\}$, and choose a family $\langle(k_n, m_n)\rangle_{n\ge 3}$ running over $(\mathbb{N}\setminus\{0\})^2$ such that $q[J_{k_n,m_n}]\subseteq n$ for every $n\ge 3$. (The pairs (k_n, m_n) need not all be distinct, so this is easy to achieve.)

Now, for $\upsilon\in\mathbb{N}^n$, where $n\ge 3$, set $F_\upsilon = E_{\sigma\tau}$ where

$$\sigma\in\mathbb{N}^{k_n},\ \sigma(i)=\upsilon(q(i,0)) \text{ for } i<k_n,$$

$$\tau\in\mathbb{N}^{m_n},\ \tau(i)=\upsilon(q(i,k_n)) \text{ for } i<m_n;$$

these are well-defined because $q[J_{k_n,m_n}]\subseteq n$. For $\upsilon\in\mathbb{N}^1\cup\mathbb{N}^2$, set $F_\upsilon = H_{\upsilon(0)}$.

(c) This defines a Souslin scheme $\langle F_\upsilon\rangle_{\upsilon\in S^*}$ in $\mathcal{E}$. Let A' be its kernel, so that $A'\in\mathcal{S}(\mathcal{E})$. The point is that $A'=A$.

P (i) If $x\in A$, there must be $\phi\in\mathbb{N}^{\mathbb{N}}$, $\boldsymbol{\psi}\in(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}$ such that $x\in\bigcap_{k,m\ge 1}E_{\phi\restriction k,\psi_k\restriction m}$. Choose $\theta\in\mathbb{N}^{\mathbb{N}}$ such that

$$H_{\theta(0)} = E_{\phi\restriction 1,\psi_1\restriction 1},$$
$$\theta(q(i,0)) = \phi(i) \text{ for every } i\in\mathbb{N},$$
$$\theta(q(i,k)) = \psi_k(i) \text{ for every } k\ge 1,\ i\in\mathbb{N}.$$

(This is possible because $q:\mathbb{N}^2\to\mathbb{N}\setminus\{0\}$ is injective.) Now

$$F_{\theta\restriction 1} = F_{\theta\restriction 2} = H_{\theta(0)} = E_{\phi\restriction 1,\psi_1\restriction 1}$$

certainly contains x. And for $n\ge 3$, $F_{\theta\restriction n} = E_{\sigma\tau}$ where $\sigma(i)=\theta(q(i,0))$ for $i<k_n$, $\tau(i)=\theta(q(i,k_n))$ for $i<m_n$, that is, $\sigma=\phi\restriction k_n$ and $\tau=\psi_{k_n}\restriction m_n$, so again $x\in F_{\theta\restriction n}$. Thus

$$x\in\bigcap_{n\ge 1}F_{\theta\restriction n}\subseteq A'.$$

As x is arbitrary, $A\subseteq A'$.

(ii) Now take any $x\in A'$. Let $\theta\in\mathbb{N}^{\mathbb{N}}$ be such that $x\in\bigcap_{n\ge 1}F_{\theta\restriction n}$. Define $\phi\in\mathbb{N}^{\mathbb{N}}$, $\boldsymbol{\psi}\in(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}\setminus\{0\}}$ by setting

$$\phi(i)=\theta(q(i,0)) \text{ for } i\in\mathbb{N},$$
$$\psi_k(i)=\theta(q(i,k)) \text{ for } k\ge 1,\ i\in\mathbb{N}.$$

If k, $m\ge 1$, let $n\ge 3$ be such that $k=k_n$, $m=m_n$. Then $x\in F_{\theta\restriction n}=E_{\sigma\tau}$, where

$$\sigma(i)=\theta(q(i,0)) \text{ for } i<k_n,\quad \tau(i)=\theta(q(i,k_n)) \text{ for } i<m_n,$$

that is, $\sigma=\phi\restriction k_n=\phi\restriction k$ and $\tau=\psi_{k_n}\restriction m_n=\psi_k\restriction m$. As m and n are arbitrary,

$$x\in\bigcap_{m,n\ge 1}E_{\phi\restriction k,\psi_k\restriction m}\subseteq A.$$

As x is arbitrary, $A'\subseteq A$. **Q**

Accordingly we must have $A\in\mathcal{S}(\mathcal{E})$, and the proof is complete.

421E Corollary For any family $\mathcal{E}$ of sets, $\mathcal{S}(\mathcal{E})$ is closed under countable unions and intersections.

proof For 421Ca tells us that the union and intersection of any sequence in $\mathcal{S}(\mathcal{E})$ will belong to $\mathcal{S}\mathcal{S}(\mathcal{E})=\mathcal{S}(\mathcal{E})$.

421F Corollary Let X be a set and $\mathcal{E}$ a family of subsets of X. Suppose that X and $\emptyset$ belong to $\mathcal{S}(\mathcal{E})$ and that $X\setminus E\in\mathcal{S}(\mathcal{E})$ for every $E\in\mathcal{E}$. Then $\mathcal{S}(\mathcal{E})$ includes the σ-algebra of subsets of X generated by $\mathcal{E}$.

proof The set

$$\Sigma=\{F: F\in\mathcal{S}(\mathcal{E}),\ X\setminus F\in\mathcal{S}(\mathcal{E})\}$$

is closed under complements (necessarily), contains $\emptyset$ (because $\emptyset$ and X belong to $\mathcal{S}(\mathcal{E})$), and is also closed under countable unions, by 421E. So it is a σ-algebra; but the hypotheses also ensure that $\mathcal{E}\subseteq\Sigma$, so that the σ-algebra generated by $\mathcal{E}$ is included in Σ and in $\mathcal{S}(\mathcal{E})$.

421G Proposition Let $\mathcal{E}$ be a family of sets such that $\emptyset \in \mathcal{E}$. Then

$$\begin{aligned}\mathcal{S}(\mathcal{E}) &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S^*,\, E \in \mathcal{E}\})\} \\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S^*,\, E \in \mathcal{E}\}),\, R^{-1}[\{x\}] \text{ is closed for every } x\}.\end{aligned}$$

proof Set $\mathcal{F} = \{I_\sigma \times E : \sigma \in S^*,\, E \in \mathcal{E}\}$.

(a) Suppose first that $A \in \mathcal{S}(\mathcal{E})$. Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{E}$ with kernel A. Set

$$R = \bigcap_{k\ge 1}\bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times E_\sigma.$$

Then $R \in \mathcal{S}(\mathcal{F})$, by 421E, and $R[\mathbb{N}^{\mathbb{N}}] = A$, by 421Ce. Also

$$R^{-1}[\{x\}] = \bigcap_{k\ge 1}\bigcup\{I_\sigma : \sigma \in \mathbb{N}^k,\, x \in E_\sigma\}$$

is closed, for every x.

(b) Now suppose that $A = R[\mathbb{N}^{\mathbb{N}}]$ for some $R \in \mathcal{S}(\mathcal{F})$. Let $\langle I_{\tau(\sigma)} \times E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{F}$ with kernel R. For $k \ge 1$, $\sigma \in \mathbb{N}^k$ set

$$\begin{aligned}F_\sigma &= E_\sigma \text{ if } \bigcap_{1\le n\le k} I_{\tau(\sigma\restriction n)} \ne \emptyset, \\ &= \emptyset \text{ otherwise.}\end{aligned}$$

Then $\langle F_\sigma\rangle_{\sigma\in S^*}$ is a Souslin scheme in $\mathcal{E}$, so its kernel A' belongs to $\mathcal{S}(\mathcal{E})$.

The point is that $A' = A$. **P** (i) If $x \in A$, there are a $\phi \in \mathbb{N}^{\mathbb{N}}$ such that $(\phi, x) \in R$ and a $\psi \in \mathbb{N}^{\mathbb{N}}$ such that $(\phi, x) \in \bigcap_{n\ge 1} I_{\tau(\psi\restriction n)} \times E_{\psi\restriction n}$. Now, for any $k \ge 1$, we have

$$\phi \in \bigcap_{1\le n\le k} I_{\tau(\psi\restriction n)} = \bigcap_{1\le n\le k} I_{\tau((\psi\restriction k)\restriction n)},$$

so that $F_{\psi\restriction k} = E_{\psi\restriction k}$ contains x; thus $x \in \bigcap_{k\ge 1} F_{\psi\restriction k} \subseteq A'$. As x is arbitrary, $A \subseteq A'$. (ii) If $x \in A'$, take $\psi \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \bigcap_{n\ge 1} F_{\psi\restriction n}$. In this case we must have $F_{\psi\restriction k} \ne \emptyset$, so $\bigcap_{1\le n\le k} I_{\tau(\psi\restriction n)} \ne \emptyset$, for every $k \ge 1$. But what this means is that, setting $\tau_n = \tau(\psi\restriction n)$ for each $n \ge 1$, $\tau_n(i) = \tau_m(i)$ whenever $i \in \mathbb{N}$ is such that both are defined. So $\{\tau_n : n \ge 1\}$ must have a common extension $\phi \in \mathbb{N}^{\mathbb{N}}$, and $\phi \in \bigcap_{n\ge 1} I_{\tau(\psi\restriction n)}$. Now

$$(\phi, x) \in \bigcap_{n\ge 1} I_{\tau(\psi\restriction n)} \times E_{\psi\restriction n} \subseteq R,$$

so $x \in A$. Thus $A' \subseteq A$ and the two are equal. **Q**

This shows that

$$\{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{F})\} \subseteq \mathcal{S}(\mathcal{E}),$$

and the proof is complete.

421H When the class $\mathcal{E}$ is a σ-algebra, the last proposition can be extended.

Proposition Let X be a set, and Σ a σ-algebra of subsets of X. Let $\mathcal{B}$ be the algebra of Borel subsets of $\mathbb{N}^{\mathbb{N}}$. Then

$$\begin{aligned}\mathcal{S}(\Sigma) &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\} \\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S^*,\, E \in \Sigma\})\} \\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma)\}.\end{aligned}$$

Notation Recall that $\mathcal{B}\widehat{\otimes}\Sigma$ is the σ-algebra of subsets of $\mathbb{N}^{\mathbb{N}} \times X$ generated by $\{H \times E : H \in \mathcal{B},\, E \in \Sigma\}$.

proof (a) Suppose first that $A \in \mathcal{S}(\Sigma)$. As in 421G, let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A, and set

$$R = \bigcap_{k\ge 1}\bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times E_\sigma,$$

so that $A = R[\mathbb{N}^{\mathbb{N}}]$ (421Ce again). Because every I_σ is an open-and-closed set in $\mathbb{N}^{\mathbb{N}}$, $R \in \mathcal{B}\widehat{\otimes}\Sigma$. Thus

$$\mathcal{S}(\Sigma) \subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\}.$$

(b) Set $\mathcal{F} = \{I_\sigma \times E : \sigma \in S^*, \, E \in \Sigma\}$. Then $\mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma) = \mathcal{S}(\mathcal{F})$. **P** If $E \in \Sigma$ and $\sigma \in \mathbb{N}^k$ then

$$(\mathbb{N}^{\mathbb{N}} \times X) \setminus (I_\sigma \times E) = (I_\sigma \times (X \setminus E)) \cup \bigcup_{\tau \in \mathbb{N}^k, \tau \neq \sigma} I_\tau \times X \in \mathcal{S}(\mathcal{F}).$$

Also

$$\mathbb{N}^{\mathbb{N}} \times X = \bigcup_{\sigma \in \mathbb{N}^1} I_\sigma \times X, \quad \emptyset = I_\tau \times \emptyset$$

(where τ is any member of S^*) belong to $\mathcal{S}(\mathcal{F})$. By 421F, $\mathcal{S}(\mathcal{F})$ includes the σ-algebra Λ of sets generated by $\mathcal{F}$. Now if $E \in \Sigma$ and $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open, $H = \bigcup_{\sigma \in T} I_\sigma$ for some $T \subseteq S^*$; as T is necessarily countable,

$$H \times E = \bigcup_{\sigma \in T} I_\sigma \times E \in \Lambda.$$

Since $\{F : F \subseteq \mathbb{N}^{\mathbb{N}}, \, F \times E \in \Lambda\}$ is a σ-algebra of subsets of $\mathbb{N}^{\mathbb{N}}$, and we have just seen that it contains all the open sets, it must include $\mathcal{B}$; thus $F \times E \in \Lambda$ for every $F \in \mathcal{B}$, $E \in \Sigma$. So $\mathcal{B}\widehat{\otimes}\Sigma \subseteq \Lambda \subseteq \mathcal{S}(\mathcal{F})$, and

$$\mathcal{S}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma) \subseteq \mathcal{S}\mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$$

(421D). **Q**

(c) Now we have

$$\mathcal{S}(\Sigma) \subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\}$$

(by (a))

$$\subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma)\} = \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{F})\}$$

(by (b))

$$= \mathcal{S}(\Sigma)$$

by 421G.

Remark A more general form of this result is in 423O below.

421I There is a particularly simple description of sets obtainable by Souslin's operation from closed sets in a topological space.

Lemma Let X be a topological space and $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ a closed set. Then

$$R[A] = \bigcup_{\phi \in A} \bigcap_{n \geq 1} \overline{R[I_{\phi \restriction n}]}.$$

for any $A \subseteq \mathbb{N}^{\mathbb{N}}$. In particular, $R[\mathbb{N}^{\mathbb{N}}]$ is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma \in S^*}$.

proof Set

$$B = \bigcup_{\phi \in A} \bigcap_{n \geq 1} \overline{R[I_{\phi \restriction n}]}.$$

(i) If $x \in R[A]$, there is a $\phi \in A$ such that $(\phi, x) \in R$. In this case, $\phi \in I_{\phi \restriction n}$ so

$$x \in R[I_{\phi \restriction n}] \subseteq \overline{R[I_{\phi \restriction n}]}$$

for every n, and $x \in B$. Thus $R[A] \subseteq B$. (ii) If $x \in B$, let $\phi \in A$ be such that $x \in \overline{R[I_{\phi \restriction n}]}$ for every $n \in \mathbb{N}$. **?** If $(\phi, x) \notin R$, then (because R is closed) there are a $\sigma \in S^*$ and an open $G \subseteq X$ such that $\phi \in I_\sigma$, $x \in G$ and $(I_\sigma \times G) \cap R = \emptyset$. But this means that $G \cap R[I_\sigma] = \emptyset$ so $G \cap \overline{R[I_\sigma]} = \emptyset$ and $x \notin \overline{R[I_\sigma]}$; which is absurd, because $\sigma = \phi \restriction n$ for some $n \geq 1$. **X** Thus $(\phi, x) \in R$ and $x \in R[A]$. As x is arbitrary, $B \subseteq R[A]$ and $B = R[A]$, as required.

421J Proposition Let X be a topological space, and $\mathcal{F}$ the family of closed subsets of X. Then a set $A \subseteq X$ belongs to $\mathcal{S}(\mathcal{F})$ iff there is a closed set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that A is the projection of R on X.

proof (a) Suppose that $A \in \mathcal{S}(\mathcal{F})$. Let $\langle F_\sigma \rangle_{\sigma \in S^*}$ be a Souslin scheme in $\mathcal{F}$ with kernel A. Set

$$R = \bigcap_{n \geq 1} \bigcup_{\sigma \in \mathbb{N}^n} I_\sigma \times F_\sigma.$$

For each $n \ge 1$,

$$\bigcup_{\sigma\in\mathbb{N}^n} I_\sigma \times F_\sigma = (\mathbb{N}^\mathbb{N} \times X) \setminus \bigcup_{\sigma\in\mathbb{N}^n} I_\sigma \times (X \setminus F_\sigma)$$

is closed in $\mathbb{N}^\mathbb{N} \times X$, so R is closed; and the projection $R[\mathbb{N}^\mathbb{N}]$ is A, by 421Ce.

(b) Suppose that $R \subseteq \mathbb{N}^\mathbb{N}$ is a closed set with projection A. Then A is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]}\rangle_{\sigma\in S^*}$, by 421I, so belongs to $\mathcal{S}(\mathcal{F})$.

421K Definition Let X be a topological space. A subset of X is a **Souslin-F** set in X if it is obtainable from closed subsets of X by Souslin's operation; that is, is the projection of a closed subset of $\mathbb{N}^\mathbb{N} \times X$.

For a subset of $\mathbb{R}^r$, or, more generally, of any Polish space, it is common to say 'Souslin set' for 'Souslin-F set'; see 421Xl.

421L Proposition Let X be any topological space. Then every Baire subset of X is Souslin-F.

proof Let $\mathcal{Z}$ be the family of zero sets in X. If $F \in \mathcal{Z}$ then $X \setminus F$ is a countable union of zero sets (4A2C(b-vi)), so belongs to $\mathcal{S}(\mathcal{Z})$. By 421F, the σ-algebra generated by $\mathcal{Z}$ is included in $\mathcal{S}(\mathcal{Z}) \subseteq \mathcal{S}(\mathcal{F})$, where $\mathcal{F}$ is the family of closed subsets of X; that is, every Baire set is Souslin-F.

421M Proposition Let $\mathcal{E}$ be any family of sets such that $\emptyset \in \mathcal{E}$ and $E \cup E'$, $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{E}$ for every E, $E' \in \mathcal{E}$ and all sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. (For instance, $\mathcal{E}$ could be the family of closed subsets of a topological space, or a σ-algebra of sets.) Let $\langle E_\sigma\rangle_{\sigma\in S^*}$ be a Souslin scheme in $\mathcal{E}$, and $K \subseteq \mathbb{N}^\mathbb{N}$ a set which is compact for the usual topology on $\mathbb{N}^\mathbb{N}$. Then $\bigcup_{\phi\in K}\bigcap_{n\ge1} E_{\phi\restriction n} \in \mathcal{E}$.

proof Set $A = \bigcup_{\phi\in K}\bigcap_{n\ge1} E_{\phi\restriction n}$. For $k \in \mathbb{N}$, set $K_k = \{\phi\restriction k : \phi \in K\}$; note that $K_k \subseteq \mathbb{N}^k$ is compact, because $\phi \mapsto \phi\restriction k$ is continuous, therefore finite, because the topology of $\mathbb{N}^k$ is discrete. Set

$$H = \bigcap_{k\ge1}\bigcup_{\phi\in K_k}\bigcap_{1\le n\le k} E_{\phi\restriction n}.$$

Because $\mathcal{E}$ is closed under finite unions and countable intersections, $H \in \mathcal{E}$. Now $A = H$. **P** (i) If $x \in A$, take $\phi \in K$ such that $x \in E_{\phi\restriction n}$ for every $n \ge 1$; then $\phi\restriction k \in K_k$ and $x \in \bigcap_{1\le n\le k} E_{(\phi\restriction k)\restriction n}$ for every $k \ge 1$, so $x \in H$. Thus $A \subseteq H$. (ii) If $x \in H$, then for each $k \in \mathbb{N}$ we have a $\sigma_k \in K_k$ such that $x \in \bigcap_{1\le n\le k} E_{\sigma_k\restriction n}$. Choose $\phi_k \in K$ such that $\phi_k\restriction k = \sigma_k$ for each k. Now K is supposed to be compact, so the sequence $\langle\phi_k\rangle_{k\in\mathbb{N}}$ has a cluster point ϕ in K.

If $n \ge 1$, then $I_{\phi\restriction n}$ is a neighbourhood of ϕ in $\mathbb{N}^\mathbb{N}$, so must contain ϕ_k for infinitely many k; let $k \ge n$ be such that $\phi_k\restriction n = \phi\restriction n$. In this case

$$x \in E_{\sigma_k\restriction n} = E_{\phi_k\restriction n} = E_{\phi\restriction n}.$$

As n is arbitrary,

$$x \in \bigcap_{n\ge1} E_{\phi\restriction n} \subseteq A.$$

As x is arbitrary, $H \subseteq A$ and $H = A$, as claimed. **Q**

So $A \in \mathcal{E}$.

***421N** I now embark on preparations for the theory of 'constituents' of analytic and coanalytic sets. It turns out that much of the work can be done in the abstract context of this section.

Trees and derived trees (a) Let $\mathcal{T}$ be the family of subsets T of $S^* = \bigcup_{n\ge1}\mathbb{N}^n$ such that $\sigma\restriction k \in T$ whenever $\sigma \in T$ and $1 \le k \le \#(\sigma)$. Note that the intersection and union of any non-empty family of members of $\mathcal{T}$ again belong to $\mathcal{T}$. Members of $\mathcal{T}$ are often called **trees**.

(b) For $T \in \mathcal{T}$, set

$$\partial T = \{\sigma : \sigma \in S^*,\ \exists\, i \in \mathbb{N},\ \sigma^\frown{<}i{>} \in T\},$$

so that $\partial T \in \mathcal{T}$ and $\partial T \subseteq T$. Of course $\partial T_0 \subseteq \partial T_1$ whenever T_0, $T_1 \in \mathcal{T}$ and $T_0 \subseteq T_1$.

(c) For $T \in \mathcal{T}$, define $\langle \partial^\xi T \rangle_{\xi<\omega_1}$ inductively by setting $\partial^0 T = T$ and, for $\xi > 0$, $\partial^\xi T = \bigcap_{\eta<\xi} \partial(\partial^\eta T)$. An easy induction shows that $\partial^\xi T \in \mathcal{T}$, $\partial^\xi T \subseteq \partial^\eta T$ and $\partial^{\xi+1} T = \partial(\partial^\xi T)$ whenever $\eta \le \xi < \omega_1$.

(d) For any $T \in \mathcal{T}$, there is a $\xi < \omega_1$ such that $\partial^\xi T = \partial^\eta T$ whenever $\xi \le \eta < \omega_1$. **P** Set $T_1 = \bigcap_{\xi<\omega_1} \partial^\xi T$. For each $\sigma \in S^* \setminus T_1$, there is a $\xi_\sigma < \omega_1$ such that $\sigma \notin \partial^{\xi_\sigma} T$. Set $\xi = \sup\{\xi_\sigma : \sigma \in S^* \setminus T_1\}$; because S^* is countable, $\xi < \omega_1$, and we must now have $\partial^\xi T = T_1$, so that $\partial^\xi T = \partial^\eta T$ whenever $\xi \le \eta < \omega_1$. **Q**

(e) For $T \in \mathcal{T}$, its **rank** is the first ordinal $r(T) < \omega_1$ such that $\partial^{r(T)} T = \partial^{r(T)+1} T$; of course $\partial^{r(T)} T = \partial^\eta T$ whenever $r(T) \le \eta < \omega_1$, and $\partial(\partial^{r(T)} T) = \partial^{r(T)} T$.

(f) For $T \in \mathcal{T}$, the following are equiveridical: (α) $\partial^{r(T)} T \ne \emptyset$; (β) there is a $\phi \in \mathbb{N}^\mathbb{N}$ such that $\phi{\restriction}n \in T$ for every $n \ge 1$. **P** (i) If $\sigma \in \partial^{r(T)} T$ then $\sigma \in \partial(\partial^{r(T)} T)$ so there is an $i \in \mathbb{N}$ such that $\sigma^\frown{<}i{>} \in \partial^{r(T)} T$. We can therefore choose $\langle \sigma_n \rangle_{n\in\mathbb{N}}$ inductively so that $\sigma_n \in \partial^{r(T)} T$ and σ_{n+1} properly extends σ_n for every n. At the end of the induction, $\phi = \bigcup_{n\in\mathbb{N}} \sigma_n$ belongs to $\mathbb{N}^\mathbb{N}$ and

$$\phi{\restriction}n = \sigma_n{\restriction}n \in \partial^{r(T)} T \subseteq T$$

for every $n \ge 1$. (ii) If $\phi \in \mathbb{N}^\mathbb{N}$ is such that $\phi{\restriction}n \in T$ for every $n \ge 1$, then an easy induction shows that $\phi{\restriction}n \in \partial^\xi T$ for every $\xi < \omega_1$ and every $n \ge 1$, so that $\partial^{r(T)} T$ is non-empty. **Q**

(g) Now suppose that $\langle A_\sigma \rangle_{\sigma\in S^*}$ is a Souslin scheme. For any x we have a tree $T_x \in \mathcal{T}$ defined by saying that

$$T_x = \{\sigma : \sigma \in S^*,\ x \in \textstyle\bigcap_{1\le i\le\#(\sigma)} A_{\sigma{\restriction}i}\}.$$

Now the kernel of $\langle A_\sigma \rangle_{\sigma\in S^*}$ is just

$$\begin{aligned} A &= \{x : \exists\, \phi \in \mathbb{N}^\mathbb{N},\ x \in \bigcap_{n\ge1} A_{\phi{\restriction}n}\} \\ &= \{x : \exists\, \phi \in \mathbb{N}^\mathbb{N},\ \phi{\restriction}n \in T_x\ \forall\, n \ge 1\} = \{x : \partial^{r(T)} T \ne \emptyset\} \end{aligned}$$

by (f).

The sets

$$\{x : x \in X \setminus A,\ r(T_x) = \xi\} = \{x : x \in X,\ r(T_x) = \xi,\ \partial^\xi T_x = \emptyset\},$$

for $\xi < \omega_1$, are called **constituents** of $X \setminus A$. (Of course they should properly be called 'the constituents of the Souslin scheme $\langle A_\sigma \rangle_{\sigma\in S^*}$'.)

***421O Theorem** Let X be a set and Σ a σ-algebra of subsets of X. Let $\langle A_\sigma \rangle_{\sigma\in S^*}$ be a Souslin scheme in Σ with kernel A, and for $x \in X$ set

$$T_x = \{\sigma : \sigma \in S^*,\ x \in \textstyle\bigcap_{1\le i\le\#(\sigma)} A_{\sigma{\restriction}i}\} \in \mathcal{T}$$

as in 421Ng.

(a) For every $\xi < \omega_1$ and $\sigma \in S^*$, $\{x : x \in X,\ \sigma \in \partial^\xi T_x\} \in \Sigma$.

(b) For every $\xi < \omega_1$, $\{x : x \in A,\ r(T_x) \le \xi\}$ and $\{x : x \in X \setminus A,\ r(T_x) \le \xi\}$ belong to Σ. In particular, all the constituents of $X \setminus A$ belong to Σ.

proof (a) Induce on ξ. For $\xi = 0$, we have

$$\{x : x \in X,\ \sigma \in \partial^0 T_x\} = \{x : x \in X,\ \sigma \in T_x\} = \textstyle\bigcap_{1\le i\le\#(\sigma)} A_{\sigma{\restriction}i} \in \Sigma.$$

For the inductive step to $\xi > 0$, we have

$$\begin{aligned} \{x : \sigma \in \partial^\xi T_x\} &= \{x : \sigma \in \bigcap_{\eta<\xi} \partial(\partial^\eta T_x)\} \\ &= \bigcap_{\eta<\xi} \bigcup_{i\in\mathbb{N}} \{x : \sigma^\frown{<}i{>} \in \partial^\eta T_x\} \in \Sigma \end{aligned}$$

because ξ is countable and all the sets $\{x : \sigma^\frown{<}i{>} \in \partial^\eta T_x\}$ belong to Σ by the inductive hypothesis.

(b) Now, given $\xi<\omega_1$, we see that $r(T_x)\le\xi$ iff $\partial^{\xi+1}T_x\supseteq\partial^\xi T_x$, so that if we set $E_\xi=\{x:x\in X,\,r(T_x)\le\xi\}$ then

$$E_\xi=\bigcap_{\sigma\in S^*}\{x:x\in X,\,\sigma\in\partial^{\xi+1}T_x\text{ or }\sigma\notin\partial^\xi T_x\}$$

belongs to Σ. If $x\in E_\xi$, so that $\partial^{r(T_x)}T_x=\partial^\xi T_x$, 421Ng tells us that $x\in A$ iff $\partial^\xi T_x\ne\emptyset$; so that

$$E_\xi\cap A=E_\xi\cap\bigcup_{\sigma\in S^*}\{x:\sigma\in\partial^\xi T_x\}$$

and $E_\xi\setminus A$ both belong to Σ.

Now the constituents of $X\setminus A$ are the sets $(E_\xi\setminus A)\setminus\bigcup_{\eta<\xi}E_\eta$ for $\xi<\omega_1$, which all belong to Σ.

***421P Corollary** Let X be a set and Σ a σ-algebra of subsets of X. If $A\in\mathcal{S}(\Sigma)$ then both A and $X\setminus A$ can be expressed as the union of at most ω_1 members of Σ.

proof In the language of 421O, we have

$$A=\bigcup_{\xi<\omega_1}E_\xi\cap A,\quad X\setminus A=\bigcup_{\xi<\omega_1}E_\xi\setminus A.$$

***421Q Lemma** Let X be a set and $\langle A_\sigma\rangle_{\sigma\in S^*}$ and $\langle B_\sigma\rangle_{\sigma\in S^*}$ two Souslin schemes of subsets of X. Suppose that whenever ϕ, $\psi\in\mathbb{N}^\mathbb{N}$ there is an $n\ge1$ such that $\bigcap_{1\le i\le n}A_{\phi\restriction i}\cap B_{\psi\restriction i}=\emptyset$. For $x\in X$ set

$$T_x=\bigcup_{n\ge1}\{\sigma:\sigma\in\mathbb{N}^n,\,x\in\bigcap_{1\le i\le n}A_{\sigma\restriction i}\}$$

as in 421Ng, and let B be the kernel of $\langle B_\sigma\rangle_{\sigma\in S^*}$. Then $\sup_{x\in B}r(T_x)<\omega_1$.

proof For $\sigma\in S^*$ set $A'_\sigma=\bigcap_{1\le i\le\#(\sigma)}A_{\sigma\restriction i}$, $B'_\sigma=\bigcap_{1\le i\le\#(\sigma)}B_{\sigma\restriction i}$. Then $T_x=\{\sigma:\sigma\in S^*,\,x\in A'_\sigma\}$ for each $x\in X$, B is the kernel of $\langle B'_\sigma\rangle_{\sigma\in S^*}$, and for every ϕ, $\psi\in\mathbb{N}^\mathbb{N}$ there is an $n\in\mathbb{N}$ such that $A'_{\phi\restriction n}\cap B'_{\psi\restriction n}=\emptyset$.

Define $\langle Q_\xi\rangle_{\xi<\omega_1}$ inductively by setting

$$Q_0=\{(\sigma,\tau):\sigma,\,\tau\in S^*,\,A'_\sigma\cap B'_\tau\ne\emptyset\},$$

and, for $0<\xi<\omega_1$,

$$Q_\xi=\bigcap_{\eta<\xi}\{(\sigma,\tau):\sigma,\,\tau\in S^*,\,\exists\,i,\,j\in\mathbb{N},\,(\sigma^\frown{<}i{>},\tau^\frown{<}j{>})\in Q_\eta\}.$$

Then the same arguments as in 421Na-421Nd show that there is a $\zeta<\omega_1$ such that $Q_{\zeta+1}=Q_\zeta$. **?** If $Q_\zeta\ne\emptyset$, then, just as in 421Nf, there must be ϕ, $\psi\in\mathbb{N}^\mathbb{N}$ such that $(\phi\restriction m,\psi\restriction n)\in Q_\zeta\subseteq Q_0$ for every m, $n\ge1$; but this means that $A'_{\phi\restriction n}\cap B'_{\psi\restriction n}\ne\emptyset$ for every $n\ge1$, which is supposed to be impossible. **X**

Now suppose that $x\in B$. Then there is a $\psi\in\mathbb{N}^\mathbb{N}$ such that $x\in B'_{\psi\restriction n}$ for every $n\ge1$. But this means that $(\sigma,\psi\restriction n)\in Q_0$ for every $\sigma\in T_x$ and every $n\ge1$. An easy induction shows that $(\sigma,\psi\restriction n)\in Q_\xi$ whenever $\xi<\omega_1$, $\sigma\in\partial^\xi T_x$ and $n\ge1$. But as $Q_\zeta=\emptyset$ we must have $\partial^\zeta T_x=\emptyset$ and $r(T_x)\le\zeta$. Thus $\sup_{x\in B}r(T_x)\le\zeta<\omega_1$, and the proof is complete.

421X Basic exercises (a) Let X be a set and $\mathcal{E}$ a family of subsets of X. (i) Show that $\emptyset\in\mathcal{S}(\mathcal{E})$ iff there is a sequence in $\mathcal{E}$ with empty intersection. (ii) Show that $X\in\mathcal{S}(\mathcal{E})$ iff there is a sequence in $\mathcal{E}$ with union X.

(b) Let $\mathcal{E}$ be a family of sets and F any set. Show that

$$\mathcal{S}(\{E\cap F:E\in\mathcal{E}\})=\{A\cap F:A\in\mathcal{S}(\mathcal{E})\},$$

$$\mathcal{S}(\{E\cup F:E\in\mathcal{E}\})=\{A\cup F:A\in\mathcal{S}(\mathcal{E})\}.$$

(c) Suppose that $\mathcal{E}$ is a family of sets with $\#(\mathcal{E})\le\mathfrak{c}$. Show that $\#(\mathcal{S}(\mathcal{E}))\le\mathfrak{c}$. (*Hint*: $\#(\mathcal{E}^{S^*})\le\#((\mathcal{P}\mathbb{N})^{S^*})=\#(\mathcal{P}(\mathbb{N}\times S^*))$.)

(d) Let $\mathcal{E}$ be the family of half-open intervals $[2^{-n}k,2^{-n}(k+1)[$, where $n\in\mathbb{N}$, $k\in\mathbb{Z}$; let $\mathcal{G}$ be the set of open subsets of $\mathbb{R}$; let $\mathcal{F}$ be the set of closed subsets of $\mathbb{R}$; let $\mathcal{K}$ be the set of compact subsets of $\mathbb{R}$; let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. Show that $\mathcal{S}(\mathcal{E})=\mathcal{S}(\mathcal{F})=\mathcal{S}(\mathcal{G})=\mathcal{S}(\mathcal{K})=\mathcal{S}(\mathcal{B})$. (*Hint*: 421F.)

(e) Let $\mathcal{I}$ be the family $\{I_\sigma : \sigma \in \bigcup_{k\in\mathbb{N}} \mathbb{N}^k\}$ (421A); let $\mathcal{G}$ be the set of open subsets of $\mathbb{N}^\mathbb{N}$; let $\mathcal{F}$ be the set of closed subsets of $\mathbb{N}^\mathbb{N}$; let $\mathcal{K}$ be the set of compact subsets of $\mathbb{N}^\mathbb{N}$; let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{N}^\mathbb{N}$. Show that $\mathcal{S}(\mathcal{I}) = \mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{B})$, but that $\mathcal{S}(\mathcal{K})$ is strictly smaller than these. (*Hint*: if $\langle K_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{K}$, set $\phi(i) = 1 + \sup_{\psi\in K_i} \psi(i)$ for each $i \in \mathbb{N}$, so that $\phi \notin \bigcup_{n\in\mathbb{N}} K_n$; hence show that $\mathbb{N}^\mathbb{N} \notin \mathcal{S}(\mathcal{K})$.)

(f) Let X be a separable metrizable space with at least two points; let $\mathcal{U}$ be any base for its topology, and $\mathcal{B}$ its Borel σ-algebra. Show that $\mathcal{S}(\mathcal{U}) = \mathcal{S}(\mathcal{B})$. What can happen if $\#(X) \le 1$? What about hereditarily Lindelöf spaces?

(g) Let X be a topological space; let $\mathcal{Z}$ be the set of zero sets in X, $\mathcal{G}$ the set of cozero sets, and $\mathcal{B}\mathfrak{a}$ the Baire σ-algebra. Show that $\mathcal{S}(\mathcal{Z}) = \mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{B}\mathfrak{a})$.

(h) Let X be a set, $\mathcal{E}$ a family of subsets of X, and Σ the σ-algebra of subsets of X generated by $\mathcal{E}$. Show that if $\#(\mathcal{E}) \le \mathfrak{c}$ then $\#(\Sigma) \le \mathfrak{c}$. (*Hint*: $\#(\mathfrak{c}^{S^*}) = \#(\mathcal{P}(\mathbb{N}\times\mathbb{N})) = \mathfrak{c}$ and $\Sigma \subseteq \mathcal{S}(\mathcal{E} \cup \{X \setminus E : E \in \mathcal{E}\})$.)

(i) Let X be a topological space such that every open set is Souslin-F. Show that every Borel set is Souslin-F.

(j) Let X be a topological space and $\mathcal{B}(X)$ its Borel σ-algebra. Show that $\mathcal{S}(\mathcal{B}(X))$ is just the set of projections on X of Borel subsets of $\mathbb{N}^\mathbb{N} \times X$. (*Hint*: 4A3G.)

(k) Let X and Y be topological spaces, $f : X \to Y$ a continuous function and $F \subseteq Y$ a Souslin-F set. Show that $f^{-1}[F]$ is a Souslin-F set in X.

(l) Let X be any perfectly normal topological space (e.g., any metrizable space); let $\mathcal{G}$ be the set of open subsets of X, $\mathcal{F}$ the set of closed subsets, and $\mathcal{B}$ the Borel σ-algebra. Show that $\mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{B})$.

>(n) Let X be a Hausdorff topological space and $\langle K_\sigma\rangle_{\sigma\in S^*}$ a Souslin scheme in which every K_σ is a compact subset in X. Show that $\bigcup_{\phi\in K}\bigcap_{n\ge 1} K_{\phi\restriction n}$ is compact for any compact $K \subseteq \mathbb{N}^\mathbb{N}$.

421Y Further exercises (a) Let X be a topological space, Y a Hausdorff space and $f : X \to Y$ a continuous function. Let $\mathcal{K}$ be the family of closed countably compact subsets of X. Show that for any $\mathcal{E} \subseteq \mathcal{K}$ such that $E \cap F \in \mathcal{E}$ for all $E, F \in \mathcal{E}$,

$$\{f[A] : A \in \mathcal{S}(\mathcal{E})\} = \mathcal{S}(\{f[E] : E \in \mathcal{E}\}).$$

(b) Let $\mathcal{E}$ be a family of sets and F any set. Show that

$$\begin{aligned}\mathcal{S}(\mathcal{E} \cup \{F\}) = \{F\} \cup \{A \cap F : A \in \mathcal{S}(\mathcal{E})\} \cup \{B \cup F : B \in \mathcal{S}(\mathcal{E})\}\\ \cup \{(A \cap F) \cup B : A, B \in \mathcal{S}(\mathcal{E})\}.\end{aligned}$$

(c) Let X be a topological space, and $\mathcal{B}\mathfrak{a}$ its Baire σ-algebra. Show that $\mathcal{S}(\mathcal{B}\mathfrak{a})$ is just the family of sets expressible as $f^{-1}[B]$ where f is a continuous function from X to some metrizable space Y and $B \subseteq Y$ is Souslin-F.

(d) Let X be a set, $\mathcal{E}$ a family of subsets of X, and Σ the smallest σ-algebra of subsets of X including $\mathcal{E}$ and closed under Souslin's operation. Show that if $\#(\mathcal{E}) \le \mathfrak{c}$ then $\#(\Sigma) \le \mathfrak{c}$. (*Hint*: define $\langle\mathcal{E}_\xi\rangle_{\xi<\omega_1}$ by setting $\mathcal{E}_\xi = \mathcal{S}(\{X \setminus E : E \in \mathcal{E} \cup \bigcup_{\eta<\xi} \mathcal{E}_\xi\})$ for each ξ. Show that $\#(\mathcal{E}_\xi) \le \mathfrak{c}$ for every ξ and that $\Sigma = \bigcup_{\xi<\omega_1} \mathcal{E}_\xi$.)

(e) Let X be a compact space and A a Souslin-F set in X. Show that there is a family $\langle F_\xi\rangle_{\xi<\omega_1}$ of Borel sets such that $X \setminus A = \bigcup_{\xi<\omega} F_\xi$ and whenever $B \subseteq X \setminus A$ is a Souslin-F set there is a $\xi < \omega_1$ such that $B \subseteq F_\xi$. (*Hint*: take $F_\xi = \{x : r(T_x) \le \xi\} \setminus A$ as in 421Ob, and apply 421Q.)

421 Notes and comments In 111G, I defined the Borel sets of $\mathbb{R}$ to be the members of the smallest σ-algebra containing every open set. In 114E, I defined a set to be Lebesgue measurable if it behaves in the right way with respect to Lebesgue outer measure. The latter formulation, at least, provides some sort of testing principle to determine whether a set is Lebesgue measurable. But the definition of 'Borel set' does not. The only tool so far available for proving that a set $E \subseteq \mathbb{R}$ is *not* Borel is to find a σ-algebra containing all open sets and not containing E; conversely, the only method we have for proving properties of Borel sets is to show that a property is possessed by every member of some σ-algebra containing every open set. The revolutionary insight of Souslin 1917 was a construction which could build every Borel set from rational intervals. (See 421Xd.) For fundamental reasons, no construction of this kind can provide all Borel sets without also producing other sets, and to actually characterize the Borel σ-algebra a further idea is needed (423Fa); but the class of analytic sets, being those constructible by Souslin's operation from rational intervals (or open sets, or closed sets, or Borel sets – the operation is robust under such variations), turns out to have remarkable properties which make it as important in modern real analysis as the Borel algebra itself.

The guiding principle of 'descriptive set theory' is that the properties of a set may be analysed in the light of a construction for that set. Thus we can think of a closed set $F \subseteq \mathbb{R}$ as

$$\mathbb{R} \setminus \bigcup_{(q,q') \in I}]q, q'[$$

where $I \subseteq \mathbb{Q} \times \mathbb{Q}$. The principle can be effective because we often have such descriptions in terms of objects fundamentally simpler than the set being described. In the formula above, for instance, $\mathbb{Q} \times \mathbb{Q}$ is simpler than the set F, being a countable set with a straightforward description from $\mathbb{N}$. The set $\mathcal{P}(\mathbb{Q} \times \mathbb{Q})$ is relatively complex; but a single subset I of $\mathbb{Q} \times \mathbb{Q}$ can easily be coded as a single subset of $\mathbb{N}$ (taking some more or less natural enumeration of $\mathbb{Q}^2$ as a sequence $\langle (q_n, q'_n) \rangle_{n \in \mathbb{N}}$, and matching I with $\{n : (q_n, q'_n) \in I\}$). So, subject to an appropriate coding, we have a description of closed subsets of $\mathbb{R}$ in terms of subsets of $\mathbb{N}$. At the most elementary level, this shows that there are at most $\mathfrak{c}$ closed subsets of $\mathbb{R}$. But we can also set out to analyse such operations as intersection, union, closure in terms of these descriptions. The details are complex, and I shall go no farther along this path until Chapter 56 in Volume 5; but investigations of this kind are at the heart of some of the most exciting developments of twentieth-century real analysis.

The particular descriptive method which concerns us in the present section is Souslin's operation. Starting from a relatively simple class $\mathcal{E}$, we proceed to the larger class $\mathcal{S}(\mathcal{E})$. The most fundamental property of $\mathcal{S}$ is 421D: $\mathcal{S}\mathcal{S}(\mathcal{E}) = \mathcal{S}(\mathcal{E})$. This means, for instance, that if $\mathcal{E} \subseteq \mathcal{S}(\mathcal{F})$ and $\mathcal{F} \subseteq \mathcal{S}(\mathcal{E})$, then $\mathcal{S}(\mathcal{E})$ will be equal to $\mathcal{S}(\mathcal{F})$; consequently, different classes of sets will often have the same Souslin closures, as in 421Xd-421Xg. After a little practice you will find that it is often easy to see when two classes $\mathcal{E}$ and $\mathcal{F}$ are at the same level in this sense; but watch out for traps like the class of compact subsets of $\mathbb{N}^{\mathbb{N}}$ (421Xe) and odd technical questions (421Xf).

Souslin's operation, and variations on it, will be the basis of much of the next chapter; it has dramatic applications in general topology and functional analysis as well as in real analysis and measure theory. An important way of looking at the kernel of a Souslin scheme $\langle E_\sigma \rangle_{\sigma \in S^*}$ is to regard it as the projection on the second coordinate of the corresponding set $R = \bigcap_{k \geq 1} \bigcup_{\sigma \in \mathbb{N}^k} I_\sigma \times E_\sigma$ (421Ce). We find that many other sets $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ will also have projections in $\mathcal{S}(\mathcal{E})$ (421G, 421H). Let me remark that it is essential here that the first coordinate should be of the right type. In one sense, indeed, $\mathbb{N}^{\mathbb{N}}$ is the only thing that will do; but its virtue transfers to analytic spaces, as we shall see in 423N-423Q below. We shall often want to deal with members of $\mathcal{S}(\mathcal{E})$ which are most naturally defined in terms of some such auxiliary space.

I have moved into slightly higher gear for 421N-421Q because these are not essential for most of the work of the next chapter. From the point of view of this section 421P is very striking but the significance of 421Q is unlikely to be apparent. It becomes important in contexts in which the condition

$$\forall\, \phi,\, \psi \in \mathbb{N}^{\mathbb{N}}\ \exists\, n \geq 1,\ \bigcap_{1 \leq i \leq n} A_{\phi \restriction i} \cap B_{\psi \restriction i} = \emptyset$$

is satisfied for natural reasons. I will expand on these in the next two sections. In the meantime, I offer 421Ye as an example of what 421O and 421Q together can tell us.

Version of 12.4.16

422 K-analytic spaces

I introduce K-analytic spaces, defined in terms of usco-compact relations. The first step is to define the latter (422A) and give their fundamental properties (422B-422E). I reach K-analytic spaces themselves in 422F, with an outline of the most important facts about them in 422G-422K.

422A Definition Let X and Y be Hausdorff spaces. A relation $R \subseteq X \times Y$ is **usco-compact** if

(α) $R[\{x\}]$ is a compact subset of Y for every $x \in X$,

(β) $R^{-1}[F]$ is a closed subset of X for every closed set $F \subseteq Y$.

(Relations satisfying condition (β) are sometimes called 'upper semi-continuous'.)

422B The following elementary remark will be useful.

Lemma Let X and Y be Hausdorff spaces and $R \subseteq X \times Y$ an usco-compact relation. If $x \in X$ and H is an open subset of Y including $R[\{x\}]$, there is an open set $G \subseteq X$, containing x, such that $R[G] \subseteq H$.

proof Set $G = X \setminus R^{-1}[Y \setminus H]$. Because $Y \setminus H$ is closed, so is $R^{-1}[Y \setminus H]$, and G is open. Of course $R[G] \subseteq H$, and $x \in G$ because $R[\{x\}] \subseteq H$.

422C Proposition Let X and Y be Hausdorff spaces. Then a subset R of $X \times Y$ is an usco-compact relation iff whenever $\mathcal{F}$ is an ultrafilter on $X \times Y$, containing R, such that the first-coordinate image $\pi_1[[\mathcal{F}]]$ of $\mathcal{F}$ has a limit in X, then $\mathcal{F}$ has a limit in R.

proof Recall that, writing $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for $(x, y) \in X \times Y$,

$$\pi_1[[\mathcal{F}]] = \{A : A \subseteq X,\ \pi_1^{-1}[A] \in \mathcal{F}\} = \{A : A \subseteq X,\ A \times Y \in \mathcal{F}\}$$

(2A1Ib), and that $\mathcal{F} \to (x, y)$ iff $\pi_1[[\mathcal{F}]] \to x$ and $\pi_2[[\mathcal{F}]] \to y$ (3A3Ic).

(a) Suppose that R is usco-compact and that $\mathcal{F}$ is an ultrafilter on $X \times Y$, containing R, such that $\pi_1[[\mathcal{F}]]$ has a limit $x \in X$. **?** If $\mathcal{F}$ has no limit in R, then, in particular, it does not converge to (x, y) for any $y \in R[\{x\}]$; that is, $\pi_2[[\mathcal{F}]]$ does not converge to any point of $R[\{x\}]$, that is, every point of $R[\{x\}]$ belongs to an open set not belonging to $\pi_2[[\mathcal{F}]]$. Because $R[\{x\}]$ is compact, it is covered by a finite union of open sets not belonging to $\pi_2[[\mathcal{F}]]$; but as $\pi_2[[\mathcal{F}]]$ is an ultrafilter (2A1N), there is an open set $H \supseteq R[\{x\}]$ such that $Y \setminus H \in \pi_2[[\mathcal{F}]]$.

Now 422B tells us that there is an open set G containing x such that $R[G] \subseteq H$. In this case, $G \in \pi_1[[\mathcal{F}]]$ so $G \times Y \in \mathcal{F}$; at the same time, $X \times (Y \setminus H) \in \mathcal{F}$. So

$$R \cap (G \times Y) \cap (X \times (Y \setminus H)) \in \mathcal{F}.$$

But this is an empty set, by the choice of G; which is intolerable. **X**

Thus $\mathcal{F}$ has a limit in R, as required.

(b) Now suppose that R has the property described.

(i) Let $x \in X$, and suppose that $\mathcal{G}$ is an ultrafilter on Y containing $R[\{x\}]$. Set $h(y) = (x, y)$ for $y \in Y$; then $\mathcal{F} = h[[\mathcal{G}]]$ is an ultrafilter on $X \times Y$ containing R. The image $\pi_1[[\mathcal{F}]]$ is just the principal filter generated by $\{x\}$, so certainly converges to x; accordingly $\mathcal{F}$ must converge to some point $(x, y) \in R$, and

$$\pi_2[[\mathcal{F}]] = \pi_2 h[[\mathcal{G}]] = \mathcal{G}$$

converges to $y \in R[\{x\}]$. As $\mathcal{G}$ is arbitrary, $R[\{x\}]$ is compact (2A3R).

(ii) Let $F \subseteq Y$ be closed, and take $x \in \overline{R^{-1}[F]} \subseteq X$. Consider

$$\mathcal{E} = \{R, X \times F\} \cup \{G \times Y : G \subseteq X \text{ is open, } x \in G\}.$$

Then $\mathcal{E}$ has the finite intersection property. **P** If $G_0, \dots, G_n$ are open sets containing x, then $R^{-1}[F]$ meets $G_0 \cap \dots \cap G_n$ in z say, and now $(z, y) \in R \cap (X \times F) \cap \bigcap_{i \le n}(G_i \times Y)$ for some $y \in F$. **Q** Let $\mathcal{F}$ be an ultrafilter on $X \times Y$ including $\mathcal{E}$ (4A1Ia). Because $G \times Y \in \mathcal{E} \subseteq \mathcal{F}$ for every open set G containing x, $\pi_1[[\mathcal{F}]] \to x$, so $\mathcal{F}$ converges to some point (x, y) of R. Because $X \times F$ is a closed set belonging to $\mathcal{E} \subseteq \mathcal{F}$, $y \in F$ and $x \in R^{-1}[F]$. As x is arbitrary, $R^{-1}[F]$ is closed; as F is arbitrary, R satisfies condition (β) of 422A, and is usco-compact.

422D Lemma (a) Let X and Y be Hausdorff spaces. If $R \subseteq X \times Y$ is an usco-compact relation, then R is closed in $X \times Y$.

(b) Let X and Y be Hausdorff spaces. If $R \subseteq X \times Y$ is an usco-compact relation and $R' \subseteq R$ is a closed set, then R' is usco-compact.

(c) Let X and Y be Hausdorff spaces. If $f : X \to Y$ is a continuous function, then its graph is an usco-compact relation.

(d) Let $\langle X_i \rangle_{i \in I}$ and $\langle Y_i \rangle_{i \in I}$ be families of Hausdorff spaces, and $R_i \subseteq X_i \times Y_i$ an usco-compact relation for each i. Set $X = \prod_{i \in I} X_i$, $Y = \prod_{i \in I} Y_i$ and

$$R = \{(x, y) : x \in X,\ y \in Y,\ (x(i), y(i)) \in R_i \text{ for every } i \in I\}.$$

Then R is usco-compact in $X \times Y$.

(e) Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an usco-compact relation. Then (i) $R[K]$ is a compact subset of Y for any compact subset K of X (ii) $R[L]$ is a Lindelöf subset of Y for any Lindelöf subset L of X.

(f) Let X, Y and Z be Hausdorff spaces, and $R \subseteq X \times Y$, $S \subseteq Y \times Z$ usco-compact relations. Then the composition

$$S \circ R = \{(x, z) : \text{there is some } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

is usco-compact in $X \times Z$.

(g) Let X and Y be Hausdorff spaces and Y_0 any subset of Y. Then a relation $R \subseteq X \times Y_0$ is usco-compact when regarded as a relation between X and Y_0 iff it is usco-compact when regarded as a relation between X and Y.

(h) Let Y be a Hausdorff space and $R \subseteq \mathbb{N}^{\mathbb{N}} \times Y$ an usco-compact relation. Set

$$R' = \{(\alpha, y) : \alpha \in \mathbb{N}^{\mathbb{N}},\ y \in Y \text{ and there is a } \beta \leq \alpha \text{ such that } (\beta, y) \in R\}.$$

Then R' is usco-compact.

proof (a) If $(x, y) \in \overline{R}$, there is an ultrafilter $\mathcal{F}$ containing R and converging to (x, y) (4A2Bc). By 422C, $\mathcal{F}$ must have a limit in R; but as $X \times Y$ is Hausdorff, this limit must be (x, y), and $(x, y) \in R$. As (x, y) is arbitrary, R is closed.

(b) It is obvious that R' will satisfy the condition of 422C if R does.

(c) $f[\{x\}] = \{f(x)\}$ is surely compact for every $x \in X$, and $f^{-1}[F]$ is closed for every closed set $F \subseteq Y$ because f is continuous.

(d) For $i \in I$, $x \in X$, $y \in Y$ set $\phi_i(x, y) = (x(i), y(i))$. If $\mathcal{F}$ is an ultrafilter on $X \times Y$ containing R such that $\pi_1[[\mathcal{F}]]$ has a limit in X, then

$$\pi_1 \phi_i[[\mathcal{F}]] = \psi_i \pi_1[[\mathcal{F}]]$$

has a limit in X_i for every $i \in I$, writing $\psi_i(x) = x(i)$ for $i \in I$ and $x \in X$. But $\phi_i[[\mathcal{F}]]$ is an ultrafilter containing R_i, so has a limit $(x_0(i), y_0(i))$ in R_i, for each i. Accordingly (x_0, y_0) is a limit of $\mathcal{F}$ in $X \times Y$ (3A3Ic), while $(x_0, y_0) \in R$. As $\mathcal{F}$ is arbitrary, R is usco-compact.

(e) For the moment, let L be any subset of X. Let $\mathcal{H}$ be a family of open sets in Y covering $R[L]$. Let $\mathcal{G}$ be the family of those open sets $G \subseteq X$ such that $R[G]$ can be covered by finitely many members of $\mathcal{H}$. Then $\mathcal{G}$ covers L. **P** If $x \in L$, then $R[\{x\}]$ is a compact subset of $R[L] \subseteq \bigcup \mathcal{H}$, so there is a finite set $\mathcal{H}' \subseteq \mathcal{H}$ covering $R[\{x\}]$. Now there is an open set G containing x such that $R[G] \subseteq \bigcup \mathcal{H}'$, by 422B. **Q**

(i) If L is compact, then there must be a finite subfamily $\mathcal{G}'$ of $\mathcal{G}$ covering L; now $R[L] \subseteq R[\bigcup \mathcal{G}']$ is covered by finitely many members of $\mathcal{H}$. As $\mathcal{H}$ is arbitrary, $R[L]$ is compact.

(ii) If L is Lindelöf, then there must be a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ covering L; now $R[L] \subseteq R[\bigcup \mathcal{G}']$ is covered by countably many members of $\mathcal{H}$. As $\mathcal{H}$ is arbitrary, $R[L]$ is Lindelöf.

(f) If $x \in X$ then $R[\{x\}] \subseteq Y$ is compact, so $(SR)[\{x\}] = S[R[\{x\}]]$ is compact, by (e-i). If $F \subseteq Z$ is closed then $S^{-1}[F] \subseteq Y$ is closed so $(SR)^{-1}[F] = R^{-1}[S^{-1}[F]]$ is closed.

(g)(i) Suppose that R is usco-compact when regarded as a subset of $X \times Y_0$. Set $S = \{(y, y) : y \in Y_0\}$; by (c), S is usco-compact when regarded as a subset of $Y_0 \times Y$, so by (f) $R = SR$ is usco-compact when regarded as a subset of $X \times Y$.

(ii) If R is usco-compact when regarded as a subset of $X \times Y$, and $x \in X$, then $R[\{x\}]$ is a subset of Y_0 which is compact for the topology of Y, therefore for the subspace topology of Y_0. If $F \subseteq Y_0$ is closed for the subspace topology, it is of the form $F' \cap Y_0$ for some closed $F' \subseteq Y$, so $R^{-1}[F] = R^{-1}[F']$ is closed in X. As x and F are arbitrary, R is usco-compact in $X \times Y_0$.

(h) Set $S = \{(\alpha, \beta) : \beta \le \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Then S is usco-compact in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. **P** $S[\{\alpha\}] = \{\beta : \beta \le \alpha\}$ is a product of finite sets, so is compact, for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. If $F \subseteq \mathbb{N}^{\mathbb{N}}$ is closed and $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is a convergent sequence in $S^{-1}[F]$ with limit $\alpha \in \mathbb{N}^{\mathbb{N}}$, then for every $n \in \mathbb{N}$ there is a $\beta_n \in F$ such that $\beta_n \le \alpha_n$. For any $i \in \mathbb{N}$, $\sup_{n \in \mathbb{N}} \beta_n(i) \le \sup_{n \in \mathbb{N}} \alpha_n(i)$ is finite, so $\{\beta_n : n \in \mathbb{N}\}$ is relatively compact and $\langle \beta_n \rangle_{n \in \mathbb{N}}$ has a cluster point β say; now

$$\beta(i) \le \limsup_{n \to \infty} \beta_n(i) \le \limsup_{n \to \infty} \alpha_n(i) = \alpha(i)$$

for every $i \in \mathbb{N}$, so $\beta \le \alpha$. Also, of course, $\beta \in F$, so $\alpha \in S^{-1}[F]$. As $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $S^{-1}[F]$ is closed; as F is arbitrary, S is usco-compact. **Q**

Now $R' = R \circ S$ is usco-compact, by (f) above.

422E The following lemma is actually very important in the structure theory of K-analytic spaces (see 422Yc). It will be useful to us in 423C below.

Lemma Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an usco-compact relation. If X is regular, so is R (in its subspace topology).

proof ? Suppose, if possible, otherwise; that there are a closed set $F \subseteq R$ and an $(x, y) \in R \setminus F$ which cannot be separated from F by open sets (in R). If G, H are open sets containing x, y respectively, then $R \cap (G \times H)$, $R \setminus (\overline{G} \times \overline{H})$ are disjoint relatively open sets in R, so the latter cannot include F; that is, $F \cap (\overline{G} \times \overline{H}) \ne \emptyset$ whenever G, H are open, $x \in G$ and $y \in H$. Accordingly there is an ultrafilter $\mathcal{F}$ on $X \times Y$ such that $F \cap (\overline{G} \times \overline{H}) \in \mathcal{F}$ whenever $G \subseteq X$ and $H \subseteq Y$ are open sets containing x, y respectively. In this case $R \in \mathcal{F}$, and $\overline{G} \in \pi_1[[\mathcal{F}]]$ for every open set G containing x. Because the topology of X is regular, every open set containing x includes $\overline{G}$ for some smaller open set G containing x, and belongs to $\pi_1[[\mathcal{F}]]$; thus $\pi_1[[\mathcal{F}]] \to x$ in X. Because R is usco-compact, $\mathcal{F}$ has a limit in R, which must be of the form (x, y') (422C). Because $F \in \mathcal{F}$ is closed (in R), $(x, y') \in F$. But also $y' \in \overline{H}$ for every open set H containing y, since $X \times \overline{H}$ is a closed set belonging to $\mathcal{F}$; because the topology of Y is Hausdorff, y' must be equal to y, and $(x, y) \in F$, which is absurd. **X**

422F Definition (Frolík 61) Let X be a Hausdorff space. Then X is **K-analytic** if there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = X$.

If X is a Hausdorff space, we call a subset of X K-analytic if it is a K-analytic space in its subspace topology.

422G Theorem (a) Let X be a Hausdorff space. Then a subset A of X is K-analytic iff there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$.

(b) $\mathbb{N}^{\mathbb{N}}$ is K-analytic.

(c) Compact Hausdorff spaces are K-analytic.

(d) If X and Y are Hausdorff spaces and $R \subseteq X \times Y$ is an usco-compact relation, then $R[A]$ is K-analytic whenever $A \subseteq X$ is K-analytic. In particular, a Hausdorff continuous image of a K-analytic Hausdorff space is K-analytic.

(e) A product of countably many K-analytic Hausdorff spaces is K-analytic.

(f) A closed subset of a K-analytic Hausdorff space is K-analytic.

(g) A K-analytic Hausdorff space is Lindelöf, so a regular K-analytic Hausdorff space is completely regular.

proof (a) A is K-analytic iff there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times A$ with projection A. But a subset of $\mathbb{N}^{\mathbb{N}} \times X$ with projection A is usco-compact in $\mathbb{N}^{\mathbb{N}} \times A$ iff it is usco-compact in $\mathbb{N}^{\mathbb{N}} \times X$, by 422Dg.

(b) The identity function from $\mathbb{N}^{\mathbb{N}}$ to itself is an usco-compact relation, by 422Dc.

(c) If X is compact, then $R = \mathbb{N}^{\mathbb{N}} \times X$ is an usco-compact relation (because $R[\{\phi\}] = X$ is compact for every $\phi \in \mathbb{N}^{\mathbb{N}}$, while $R^{-1}[F]$ is either $\mathbb{N}^{\mathbb{N}}$ or $\emptyset$ for every closed $F \subseteq X$), so $X = R[\mathbb{N}^{\mathbb{N}}]$ is K-analytic.

(d) By (a), there is an usco-compact relation $S \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $S[\mathbb{N}^{\mathbb{N}}] = A$. Now $R \circ S \subseteq \mathbb{N}^{\mathbb{N}} \times Y$ is usco-compact, by 422Df, and $RS[\mathbb{N}^{\mathbb{N}}] = R[A]$.

In particular, if X itself is K-analytic and $f : X \to Y$ is a continuous surjection, f is an usco-compact relation (422Dc), so $Y = f[X]$ is K-analytic.

(e) Let $\langle X_i \rangle_{i \in I}$ be a countable family of K-analytic Hausdorff spaces with product X. If $I = \emptyset$ then $X = \{\emptyset\}$ is compact, therefore K-analytic. Otherwise, choose for each $i \in I$ an usco-compact relation $R_i \subseteq \mathbb{N}^{\mathbb{N}} \times X_i$ such that $R_i[\mathbb{N}^{\mathbb{N}}] = X_i$. Set

$$R = \{(\phi, x) : \phi \in (\mathbb{N}^{\mathbb{N}})^I,\ x \in X,\ (\phi(i), x(i)) \in R_i \text{ for every } i \in I\}.$$

By 422Dd, R is an usco-compact relation in $(\mathbb{N}^{\mathbb{N}})^I \times X$, and it is easy to see that $R[(\mathbb{N}^{\mathbb{N}})^I] = X$. But $(\mathbb{N}^{\mathbb{N}})^I \cong \mathbb{N}^{\mathbb{N} \times I}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, because I is countable, so we can identify R with a relation in $\mathbb{N}^{\mathbb{N}} \times X$ which is still usco-compact, and X is K-analytic.

(f) Let X be a K-analytic Hausdorff space and F a closed subset. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. Set $R' = R \cap (\mathbb{N}^{\mathbb{N}} \times F)$. Then R' is a closed subset of R, so is usco-compact (422Db). By (a) here, $F = R'[\mathbb{N}^{\mathbb{N}}]$ is K-analytic.

(g) Let X be a K-analytic Hausdorff space. $\mathbb{N}^{\mathbb{N}}$ is Lindelöf (4A2Ub), and there is an usco-compact relation R such that $R[\mathbb{N}^{\mathbb{N}}] = X$, so that X is Lindelöf, by 422D(e-ii). 4A2H(b-i) now tells us that if X is regular it is completely regular.

422H Theorem (a) If X is a Hausdorff space, then any K-analytic subset of X is Souslin-F in X.

(b) If X is a K-analytic Hausdorff space, then a subset of X is K-analytic iff it is Souslin-F in X.

(c) For any Hausdorff space X, the family of K-analytic subsets of X is closed under Souslin's operation.

proof (a) If $A \subseteq X$ is K-analytic, there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$, by 422Ga. By 422Da, R is a closed set; so A is Souslin-F by 421J.

(b) Now suppose that X itself is K-analytic, and that $A \subseteq X$ is Souslin-F in X. Then there is a closed set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$ (421J, in the other direction). $\mathbb{N}^{\mathbb{N}} \times X$ is K-analytic (422Gb, 422Ge), and R is closed, therefore itself K-analytic (422Gf); so its continuous image A is K-analytic, by 422Gd.

(c)(i) The first step is to show that the union of a sequence of K-analytic subsets of X is K-analytic. **P** Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of K-analytic sets, with union A. For each $n \in \mathbb{N}$, let $R_n \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R_n[\mathbb{N}^{\mathbb{N}}] = A_n$. In $(\mathbb{N} \times \mathbb{N}^{\mathbb{N}}) \times X$ let R be the set

$$\{((n, \phi), x) : n \in \mathbb{N},\ (\phi, x) \in R_n\}.$$

If $(n, \phi) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, then $R[\{(n, \phi)\}] = R_n[\{\phi\}]$ is compact; if $F \subseteq X$ is closed, then

$$R^{-1}[F] = \{(n, \phi) : n \in \mathbb{N},\ \phi \in R_n^{-1}[F]\}$$

is closed in $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$. So R is usco-compact, and of course

$$R[\mathbb{N} \times \mathbb{N}^{\mathbb{N}}] = \bigcup_{n \in \mathbb{N}} R_n[\mathbb{N}^{\mathbb{N}}] = A.$$

As $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ is K-analytic (in fact, homeomorphic to $\mathbb{N}^{\mathbb{N}}$), A is K-analytic. **Q**

(ii) Now suppose that $\langle A_\sigma \rangle_{\sigma \in S^*}$ is a Souslin scheme consisting of K-analytic sets with kernel A. Then $X' = \bigcup_{\sigma \in S^*} A_\sigma$ is K-analytic, by (i). By (a), every A_σ is Souslin-F when regarded as a subset of X'. But since the family of Souslin-F subsets of X' is closed under Souslin's operation, by 421D, A also is Souslin-F in X'. By (b) of this theorem, A is K-analytic. As $\langle A_\sigma \rangle_{\sigma \in S^*}$ is arbitrary, we have the result.

422I It seems that for the measure-theoretic results of §432, at least, the following result (the 'First Separation Theorem') is not essential. However I do not think it possible to get a firm grasp on K-analytic and analytic spaces without knowing some version of it, so I present it here. It is most often used through the forms in 422J and 422Xf below.

Lemma Let X be a Hausdorff space. Let $\mathcal{E}$ be a family of subsets of X such that (i) $\bigcup_{n\in\mathbb{N}} E_n$ and $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{E}$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}$ (ii) whenever x, y are distinct points of X, there are disjoint E, $F \in \mathcal{E}$ such that $x \in \operatorname{int} E$ and $y \in \operatorname{int} F$. Then whenever A, B are disjoint non-empty K-analytic subsets of X, there are disjoint E, $F \in \mathcal{E}$ such that $A \subseteq E$ and $B \subseteq F$.

proof (a) We need to know that if K, L are disjoint non-empty compact subsets of X, there are disjoint E, $F \in \mathcal{E}$ such that $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$. **P** For any point $(x,y) \in K \times L$, we can find disjoint E_{xy}, $F_{xy} \in \mathcal{E}$ such that $x \in \operatorname{int} E_{xy}$ and $y \in \operatorname{int} F_{xy}$. Because L is compact and non-empty, there is for each $x \in K$ a non-empty finite set $I_x \subseteq L$ such that $L \subseteq \bigcup_{y\in I_x} \operatorname{int} F_{xy}$. Set $E_x = \bigcap_{y\in I_x} E_{xy}$, $F_x = \bigcup_{y\in I_x} F_{xy}$; then E_x, F_x are disjoint members of $\mathcal{E}$, $x \in \operatorname{int} E_x$ and $L \subseteq \operatorname{int} F_x$. Because K is compact and not empty, there is a non-empty finite set $J \subseteq K$ such that $K \subseteq \bigcup_{x\in J} \operatorname{int} E_x$. Set $E = \bigcup_{x\in J} E_x$, $F = \bigcap_{x\in J} F_x$; then E, $F \in \mathcal{E}$, $E \cap F = \emptyset$, $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$, as required. **Q**

(b) Let Q, $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be usco-compact relations such that $Q[\mathbb{N}^{\mathbb{N}}] = A$ and $R[\mathbb{N}^{\mathbb{N}}] = B$. For each $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, set

$$I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}, \quad A_\sigma = Q[I_\sigma], \quad B_\sigma = R[I_\sigma],$$

so that $A = A_\emptyset$ and $A_\sigma = \bigcup_{i\in\mathbb{N}} A_{\sigma^\frown<i>}$ for every σ.

(c) Write T for the set of pairs

$$\{(\sigma,\tau) : \sigma,\, \tau \in S \text{ and there are disjoint } E,\, F \in \mathcal{E} \text{ such that } A_\sigma \subseteq E \text{ and } B_\tau \subseteq F\}.$$

If σ, $\tau \in S$ are such that $(\sigma^\frown<i>, \tau^\frown<j>) \in T$ for every i, $j \in \mathbb{N}$, then $(\sigma,\tau) \in T$. **P** For each i, $j \in \mathbb{N}$ take disjoint E_{ij}, $F_{ij} \in \mathcal{E}$ such that

$$A_{\sigma^\frown<i>} \subseteq E_{ij}, \quad B_{\tau^\frown<j>} \subseteq F_{ij}.$$

Then $E = \bigcup_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} E_{ij}$, $F = \bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} F_{ij}$ are disjoint and belong to $\mathcal{E}$, and $A_\sigma \subseteq E$, $B_\tau \subseteq F$. So $(\sigma,\tau) \in T$. **Q**

(d) ? Now suppose, if possible, that there are no disjoint E, $F \in \mathcal{E}$ such that $A \subseteq E$ and $B \subseteq F$; that is, that $(\emptyset,\emptyset) \notin T$. By (c), used repeatedly, we can find sequences $\langle\phi(i)\rangle_{i\in\mathbb{N}}$, $\langle\psi(i)\rangle_{i\in\mathbb{N}}$ such that $(\phi{\restriction}n, \psi{\restriction}n) \notin T$ for every $n \in \mathbb{N}$. Set $K = Q[\{\phi\}]$, $L = R[\{\psi\}]$. These are compact (because R is usco-compact) and disjoint (because $K \subseteq A$ and $L \subseteq B$). By (a), there are disjoint E, $F \in \mathcal{E}$ such that $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$.

By 422B, there are open sets U, $V \subseteq \mathbb{N}^{\mathbb{N}}$ such that

$$\phi \in U, \quad Q[U] \subseteq \operatorname{int} E, \quad \psi \in V, \quad R[V] \subseteq \operatorname{int} F.$$

But now there is some $n \in \mathbb{N}$ such that $I_{\phi\restriction n} \subseteq U$ and $I_{\psi\restriction n} \subseteq V$, in which case

$$A_{\phi\restriction n} \subseteq E, \quad B_{\psi\restriction n} \subseteq F,$$

and $(\phi{\restriction}n, \psi{\restriction}n) \in T$, contrary to the choice of ϕ and ψ. **X**

This contradiction shows that the lemma is true.

422J Corollary Let X be a Hausdorff space and A, B disjoint K-analytic subsets of X. Then there is a Borel set which includes A and is disjoint from B.

proof Apply 422I with $\mathcal{E}$ the Borel σ-algebra of X.

***422K** I give the next step in the theory of 'constituents' begun in 421N-421Q.

Theorem Let X be a Hausdorff space.

(i) Suppose that X is regular. Let $A \subseteq X$ be a K-analytic set. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every Souslin-F subset of X disjoint from A is included in some E_ξ.

(ii) Suppose that X is regular. Let $A \subseteq X$ be a Souslin-F set. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every K-analytic subset of $X \setminus A$ is included in some E_ξ.

(iii) Let $A \subseteq X$ be a K-analytic set. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every K-analytic subset of $X \setminus A$ is included in some E_ξ.

proof (a) The first two parts depend on the following fact: if X is regular, $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ is usco-compact, $\langle F_\sigma \rangle_{\sigma \in S^*}$ is a Souslin scheme consisting of closed sets with kernel B, and $R[\mathbb{N}^{\mathbb{N}}] \cap B = \emptyset$, then for any ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ there is an $n \geq 1$ such that $\overline{R[I_{\phi \restriction n}]} \cap \bigcap_{1 \leq i \leq n} F_{\psi \restriction i}$ is empty, where I write $I_\sigma = \{\theta : \sigma \subseteq \theta \in \mathbb{N}^{\mathbb{N}}\}$ for $\sigma \in S^* = \bigcup_{n \geq 1} \mathbb{N}^n$. **P** We know that $K = R[\{\phi\}]$ is a compact set disjoint from the closed set $\bigcap_{n \geq 1} F_{\psi \restriction n}$. So there is some $m \geq 1$ such that $K \cap F = \emptyset$ where $F = \bigcap_{1 \leq i \leq m} F_{\psi \restriction i}$. Because X is regular, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $F \subseteq H$ (4A2F(h-ii)). By 422B, there is some n such that $R[I_{\phi \restriction n}] \subseteq G$. Of course we can take $n \geq m$, and in this case

$$\overline{R[I_{\phi \restriction n}]} \cap \bigcap\nolimits_{1 \leq i \leq n} F_{\psi \restriction i} \subseteq \overline{G} \cap F = \emptyset,$$

as required. **Q**

(b)(i) Suppose that $A \subseteq X$ is K-analytic. Then there is an usco-compact set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$, and R is closed (422Da), so that A is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma \in S^*}$ (421I). For $x \in X$ set $T_x = \{\sigma : \sigma \in S^*, x \in \overline{R[I_\sigma]}\}$, as in 421Ng, and let $r(T_x) < \omega_1$ be the rank of the tree T_x (421Ne). Then $E_\xi = \{x : x \in X \setminus A, r(T_x) \leq \xi\}$ is a Borel set for every $\xi < \omega_1$, by 421Ob. Now suppose that $B \subseteq X \setminus A$ is a Souslin-F set. Then it is the kernel of a Souslin scheme $\langle F_\sigma \rangle_{\sigma \in S^*}$ consisting of closed sets. If ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ then by (a) above there is an $n \geq 1$ such that $\overline{R[I_{\phi \restriction n}]} \cap \bigcap_{1 \leq i \leq n} F_{\psi \restriction i}$ is empty. By 421Q, there must be some $\xi < \omega_1$ such that $r(T_x) \leq \xi$ for every $x \in B$, that is, $B \subseteq E_\xi$. So $\langle E_\xi \rangle_{\xi < \omega_1}$ is a suitable family.

(ii) The other part is almost the same. Suppose that $A \subseteq X$ is Souslin-F. Then it is the kernel of a Souslin scheme $\langle F_\sigma \rangle_{\sigma \in S^*}$ consisting of closed sets. For $x \in X$ set

$$T_x = \bigcup\nolimits_{n \geq 1} \{\sigma : \sigma \in \mathbb{N}^n, x \in \bigcap\nolimits_{1 \leq i \leq n} F_{\sigma \restriction i}\},$$

and let $r(T_x) < \omega_1$ be the rank of the tree T_x. Then $E_\xi = \{x : x \in X \setminus A, r(T_x) \leq \xi\}$ is a Borel set for every $\xi < \omega_1$. Now let $B \subseteq X \setminus A$ be a K-analytic set. There is an usco-compact set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = B$, and B is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma \in S^*}$. If ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ then by (a) above there is an $n \geq 1$ such that $\bigcap_{1 \leq i \leq n} F_{\psi \restriction i} \cap \overline{R[I_{\phi \restriction n}]}$ is empty. So there must be some $\xi < \omega_1$ such that $B \subseteq E_\xi$. Thus here again $\langle E_\xi \rangle_{\xi < \omega_1}$ is a suitable family.

(c) If X is not regular, we still have a version of the result in (a), as follows: if R, $S \subseteq \mathbb{N}^{\mathbb{N}} \times X$ are usco-compact and $R[\mathbb{N}^{\mathbb{N}}] \cap S[\mathbb{N}^{\mathbb{N}}] = \emptyset$, then for any ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ there is an $n \geq 1$ such that $\overline{R[I_{\phi \restriction n}]} \cap S[I_{\psi \restriction n}]$ is empty. **P** This time, $R[\{\phi\}]$ and $S[\{\psi\}]$ are disjoint compact sets, so there are disjoint open sets G, H with $R[\{\phi\}] \subseteq G$ and $S[\{\psi\}] \subseteq H$ (4A2F(h-i)). Now $S[I_{\psi \restriction n}] \subseteq H$, $R[I_{\phi \restriction n}] \subseteq G$ and $\overline{R[I_{\phi \restriction n}]} \cap H = \emptyset$ for all n large enough. **Q**

Now the argument of (b-i), with $F_\sigma = S[I_\sigma]$, gives part (iii).

422X Basic exercises (a) Let X and Y be Hausdorff spaces and X_0 a closed subset of X. Show that a relation $R \subseteq X_0 \times Y$ is usco-compact when regarded as a relation between X_0 and Y iff it is usco-compact when regarded as a relation between X and Y.

(b) Show that a locally compact Hausdorff space is K-analytic iff it is Lindelöf iff it is σ-compact.

>(c) Prove 422Hc from first principles, without using 421D. (*Hint*: if $\langle R_\sigma \rangle_{\sigma \in S^*}$ is a Souslin scheme of usco-compact relations in $\mathbb{N}^{\mathbb{N}} \times X$,

$$\{((\phi, \langle \psi_\sigma \rangle_{\sigma \in S^*}), x) : (\psi_{\phi \restriction n}, x) \in R_{\phi \restriction n} \text{ for every } n \geq 1\}$$

is usco-compact in $(\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{S^*}) \times X$.)

(d) Let X be a K-analytic Hausdorff space. (i) Show that every Baire subset of X is K-analytic. (*Hint*: apply 136Xi to the family of K-analytic subsets of X.) (ii) Show that if X is regular, it is perfectly normal iff it is hereditarily Lindelöf iff every open subset of X is K-analytic.

(e) Let X be a Hausdorff space in which every open set is K-analytic. Show that every Borel set is K-analytic.

(f) Let X be a completely regular Hausdorff space and A, B disjoint K-analytic subsets of X. Show that there is a Baire set including A and disjoint from B.

422Y Further exercises (a) Let X be a completely regular Hausdorff space, and βX its Stone-Čech compactification. Show that X is K-analytic iff it is a Souslin-F set in βX.

(b) Let X be a set and $\mathfrak{S}$, $\mathfrak{T}$ two Hausdorff topologies on X such that $\mathfrak{S} \subseteq \mathfrak{T}$ and $(X, \mathfrak{T})$ is K-analytic. Show that $\mathfrak{S}$ and $\mathfrak{T}$ yield the same K-analytic subspaces of X.

(c) Show that a Hausdorff space is K-analytic iff it is a continuous image of a $\mathrm{K}_{\sigma\delta}$ set in a compact Hausdorff space, that is, of a set expressible as $\bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} K_{mn}$ where every K_{mn} is compact. (*Hint*: Write $\mathcal{K}^*$ for the class of Hausdorff continuous images of $\mathrm{K}_{\sigma\delta}$ subsets of compact Hausdorff spaces. (i) Show that $\mathbb{N}^{\mathbb{N}}$ is a $\mathrm{K}_{\sigma\delta}$ set in $Y^{\mathbb{N}}$, where Y is the one-point compactification of $\mathbb{N}$. (ii) Show that if X is a compact Hausdorff space and $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ is closed, then $R \in \mathcal{K}^*$. (iii) Show that if X is a compact Hausdorff space, then every Souslin-F subset of X belongs to $\mathcal{K}^*$. (iv) Show that if X is a regular K-analytic Hausdorff space, then $X \in \mathcal{K}^*$. (v) Show that if X is any Hausdorff space and $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ is an usco-compact relation, then $R \in \mathcal{K}^*$. See Jayne 76.)

(d) Let X be a normal space and $\mathcal{C}$ the family of countably compact closed subsets of X. Let A, B be disjoint sets obtainable from $\mathcal{C}$ by Souslin's operation. (For instance, if X itself is countably compact, A and B could be disjoint Souslin-F sets.) Show that there is a Borel set including A and disjoint from B.

(e) Let X be a Hausdorff space and $\langle A_n \rangle_{n\in\mathbb{N}}$ a sequence of K-analytic subsets of X such that $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Show that there is a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of Borel sets such that $A_n \subseteq E_n$ for every $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. (*Hint*: for each $n \in \mathbb{N}$ choose an usco-compact $R_n \subseteq \mathbb{N}^{\mathbb{N}} \times X$ with projection A_n. Consider the set $T = \{\langle \sigma_n \rangle_{n\in\mathbb{N}} : \exists \text{ Borel } E_n,\ R_n[I_{\sigma_n}] \subseteq E_n\ \forall\ n,\ \bigcap_{n\in\mathbb{N}} E_n = \emptyset\}$.)

(f) Explain how to prove 422J from 421Q, without using 422I.

(g) Let X be a Hausdorff space, $\mathcal{K}$ the family of K-analytic subsets of X, Y a set and $\mathcal{H}$ a family of subsets of Y containing $\emptyset$. Show that $R[X] \in \mathcal{S}(\mathcal{H})$ for every $R \in \mathcal{S}(\{K \times H : K \in \mathcal{K},\ H \in \mathcal{H}\})$.

422 Notes and comments In a sense, this section starts at the deep end of its topic. 'Descriptive set theory' originally developed in the context of the real line and associated spaces, and this remains the centre of the subject. But it turns out that some of the same arguments can be used in much more general contexts, and in particular greatly illuminate the theory of Radon measures on Hausdorff spaces. I find that a helpful way to look at K-analytic spaces is to regard them as a common generalization of compact Hausdorff spaces and Souslin-F subsets of $\mathbb{R}$; if you like, any theorem which is true of both these classes has a fair chance of being true of all K-analytic spaces. In the next section we shall come to the special properties of the more restricted class of 'analytic' spaces, which are much closer to the separable metric spaces of the original theory.

The phrase 'usco-compact' is neither elegant nor transparent, but is adequately established and (in view of the frequency with which it is needed) seems preferable to less concise alternatives. If we think of a relation $R \subseteq X \times Y$ as a function $x \mapsto R[\{x\}]$ from x to $\mathcal{P}Y$, then an usco-compact relation is one which takes compact values and is 'upper semi-continuous' in the sense that $\{x : R[\{x\}] \subseteq H\}$ is open for every open set $H \subseteq Y$; just as a real-valued function is upper semi-continuous if $\{x : f(x) < \alpha\}$ is open for every α.

This is not supposed to be a book on general topology, and in my account of the topological properties of K-analytic spaces I have concentrated on facts which are useful when proving that spaces are K-analytic, on the assumption that these will be valuable when we seek to apply the results of §432 below. Other properties are mentioned only when they are relevant to the measure-theoretic results which are my real concern, and readers already acquainted with this area may be startled by some of my omissions. For a proper treatment of the subject, I refer you to Rogers 80. As usual, however, I take technical details seriously in the material I do cover. I hope you will not find that such results as 422Dg and 422Ga try your patience too far. I think a moment's thought will persuade you that it is of the highest importance that K-analyticity (like compactness) is an intrinsic property. In contrast, the property of being 'Souslin-F', like the property of being closed, depends on the surrounding space. A completely regular Hausdorff space is compact iff it must be a closed set in any surrounding Hausdorff space iff it is closed in its Stone-Čech compactification; and it is K-analytic iff it must be a Souslin-F set in any surrounding Hausdorff space iff it is a Souslin-F set in its Stone-Čech compactification (422Ya).

For regular spaces, 422K gives us another version of the First Separation Theorem. But this one is simultaneously more restricted in its scope (it does not seem to have applications to Baire σ-algebras, for instance) and much more powerful in its application. When all Borel sets are Souslin-F, as in the next section, it tells us something very important about the cofinal structure of the Souslin-F subsets of the complement of a K-analytic set.

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423 Analytic spaces

We come now to the original class of K-analytic spaces, the 'analytic' spaces. I define these as continuous images of $\mathbb{N}^{\mathbb{N}}$ (423A), but move as quickly as possible to their characterization as K-analytic spaces with countable networks (423C), so that many other fundamental facts (423E-423G) can be regarded as simple corollaries of results in §422. I give two versions of Lusin's theorem on injective images of Borel sets (423I), and a form of the von Neumann-Jankow measurable selection theorem (423P). I end with notes on constituents of coanalytic sets (423R-423S).

423A Definition A Hausdorff space is **analytic** or **Souslin** if it is either empty or a continuous image of $\mathbb{N}^{\mathbb{N}}$.

423B Proposition (a) A Polish space is analytic.

(b) A Hausdorff continuous image of an analytic Hausdorff space is analytic.

(c) A product of countably many analytic Hausdorff spaces is analytic.

(d) A closed subset of an analytic Hausdorff space is analytic.

(e) An analytic Hausdorff space has a countable network consisting of analytic sets.

proof (a) Let X be a Polish space. If $X=\emptyset$, we can stop. Otherwise, let ρ be a metric on X, inducing its topology, under which X is complete. For $\sigma\in S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ choose $X_\sigma\subseteq X$ as follows. $X_\emptyset=X$. Given that X_σ is a closed non-empty subset of X, where $\sigma\in S$, then X_σ is separable, because X is separable and metrizable (4A2P(a-iv)), and we can choose a sequence $\langle x_{\sigma i}\rangle_{i\in\mathbb{N}}$ in X_σ such that $\{x_{\sigma i}:i\in\mathbb{N}\}$ is dense in X_σ. Set $X_{\sigma^\frown<i>}=X_\sigma\cap B(x_{\sigma i},2^{-n})$ for each $i\in\mathbb{N}$, where $B(x,\delta)=\{y:\rho(y,x)\le\delta\}$, and continue. Note that because $\{x_{\sigma i}:i\in\mathbb{N}\}$ is dense in X_σ, $X_\sigma=\bigcup_{i\in\mathbb{N}}X_{\sigma^\frown<i>}$, for every $\sigma\in S$.

For each $\phi\in\mathbb{N}^{\mathbb{N}}$, $\langle X_{\phi\restriction n}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty closed sets, and $\operatorname{diam}(X_{\phi\restriction n+1})\le 2^{-n+1}$ for every n. Because X is complete under ρ, $\bigcap_{n\in\mathbb{N}}X_{\phi\restriction n}$ is a singleton $\{f(\phi)\}$ say. ($f(\phi)$ is the limit of the Cauchy sequence $\langle x_{\phi\restriction n,\phi(n)}\rangle_{n\in\mathbb{N}}$.) Thus we have a function $f:\mathbb{N}^{\mathbb{N}}\to X$. f is continuous because $\rho(f(\psi),f(\phi))\le 2^{-n+1}$ whenever $\phi\restriction n+1=\psi\restriction n+1$ (since in this case both $f(\psi)$ and $f(\phi)$ belong to $X_{\phi\restriction n+1}$). f is surjective because, given $x\in X$, we can choose $\langle\phi(i)\rangle_{i\in\mathbb{N}}$ inductively so that $x\in X_{\phi\restriction n}$ for every n; at the inductive step, we have $x\in X_{\phi\restriction n}=\bigcup_{i\in\mathbb{N}}X_{(\phi\restriction n)^\frown<i>}$, so we can take $\phi(n)$ such that $x\in X_{(\phi\restriction n)^\frown<\phi(n)>}=X_{\phi\restriction n+1}$.

Thus X is a continuous image of $\mathbb{N}^{\mathbb{N}}$, as claimed.

(b) If X is an analytic Hausdorff space and Y is a Hausdorff continuous image of X, then either X is a continuous image of $\mathbb{N}^{\mathbb{N}}$ and Y is a continuous image of $\mathbb{N}^{\mathbb{N}}$, or $X=\emptyset$ and $Y=\emptyset$.

(c) Let $\langle X_i\rangle_{i\in I}$ be a countable family of analytic Hausdorff spaces, with product X. Then X is Hausdorff (3A3Id). If $I=\emptyset$ then $X=\{\emptyset\}$ is a continuous image of $\mathbb{N}^{\mathbb{N}}$, therefore analytic. If there is some $i\in I$ such that $X_i=\emptyset$, then $X=\emptyset$ is analytic. Otherwise, we have for each $i\in I$ a continuous surjection $f_i:\mathbb{N}^{\mathbb{N}}\to X_i$. Setting $f(\phi)=\langle f_i(\phi(i))\rangle_{i\in I}$ for $\phi\in(\mathbb{N}^{\mathbb{N}})^I$, $f:(\mathbb{N}^{\mathbb{N}})^I\to X$ is a continuous surjection. But $(\mathbb{N}^{\mathbb{N}})^I\cong\mathbb{N}^{\mathbb{N}\times I}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, so X is analytic.

(d) Let X be an analytic Hausdorff space and F a closed subset of X. Then F is Hausdorff in its subspace topology (4A2F(a-i)). If $X=\emptyset$ then $F=\emptyset$ is analytic. Otherwise, there is a continuous surjection $f:\mathbb{N}^{\mathbb{N}}\to X$. Now $H=f^{-1}[F]$ is a closed subset of the Polish space $\mathbb{N}^{\mathbb{N}}$, therefore Polish in its induced topology (4A2Qd). By (a), H is analytic, so its continuous image $F=f[H]$ also is analytic, by (b).

(e) Let X be an analytic Hausdorff space. If it is empty then of course it has a countable network consisting of analytic sets. Otherwise, there is a continuous surjection $f:\mathbb{N}^{\mathbb{N}}\to X$. For $\sigma\in S$ set $I_\sigma=\{\phi:\sigma\subseteq\phi\in\mathbb{N}^{\mathbb{N}}\}$; then $\{I_\sigma:\sigma\in S\}$ is a base for the topology of $\mathbb{N}^{\mathbb{N}}$, so $\{f[I_\sigma]:\sigma\in S\}$ is a network for the topology of X (see the proof of 4A2Nd). But I_σ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, so $f[I_\sigma]$ is analytic, for every $\sigma\in S$, and $\{f[I_\sigma]:\sigma\in S\}$ is a countable network consisting of analytic sets.

423C Theorem A Hausdorff space is analytic iff it is K-analytic and has a countable network.

proof (a) Let X be an analytic Hausdorff space. By 423Be, it has a countable network. If $X=\emptyset$ then surely it is K-analytic. Otherwise, X is a continuous image of $\mathbb{N}^{\mathbb{N}}$. But $\mathbb{N}^{\mathbb{N}}$ is K-analytic (422Gb), so X also is K-analytic, by 422Gd.

(b) Now suppose that X is a K-analytic Hausdorff space and has a countable network.

(i) If $X\subseteq\mathbb{N}^{\mathbb{N}}$ then X is analytic. **P** Let $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}]=X$. Then R is still usco-compact when regarded as a subset of $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ (422Dg), so is closed in $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ (422Da). But $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}\cong\mathbb{N}^{\mathbb{N}}$ is analytic, so R is in itself an analytic space (423Bd), and its continuous image X is analytic, by 423Bb. **Q**

(ii) Now suppose that X is regular. By 4A2Ng, X has a countable network $\mathcal{E}$ consisting of closed sets. Adding $\emptyset$ to $\mathcal{E}$ if need be, we may suppose that $\mathcal{E}\ne\emptyset$. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{E}$. For each $n\in\mathbb{N}$, let $\langle F_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence running over $\{E_n\}\cup\{E:E\in\mathcal{E},\,E\cap E_n=\emptyset\}$. Now consider the relation

$$R=\{(\phi,x):\phi\in\mathbb{N}^{\mathbb{N}},\,x\in\bigcap_{n\in\mathbb{N}}F_{n,\phi(n)}\}\subseteq\mathbb{N}^{\mathbb{N}}\times X.$$

(α) R is closed in $\mathbb{N}^{\mathbb{N}}\times X$. **P** Because every F_{ni} is closed,

$$(\mathbb{N}^{\mathbb{N}}\times X)\setminus R=\bigcup_{i,n\in\mathbb{N}}\{(\phi,x)\colon\ \phi(n)=i\text{ and }x\notin F_{ni}\}$$

is open. **Q**

(β) $R[\mathbb{N}^{\mathbb{N}}]=X$. **P** For every $n\in\mathbb{N}$,

$$X\setminus E_n=\bigcup\{E:E\in\mathcal{E},\,E\subseteq X\setminus E_n\}$$

because $\mathcal{E}$ is a network and E_n is closed, so $\bigcup_{i\in\mathbb{N}}F_{ni}=X$. So, given $x\in X$, we can find for each n a $\phi(n)$ such that $x\in F_{n,\phi(n)}$, and $(\phi,x)\in R$. **Q**

(γ) R is the graph of a function. **P?** Suppose that we have (ϕ,x) and (ϕ,y) in R where $x\ne y$. Because the topology of X is Hausdorff, there is an $n\in\mathbb{N}$ such that $x\in E_n$ and $y\notin E_n$. But in this case $x\in E_n\cap F_{n,\phi(n)}$, so $F_{n,\phi(n)}=E_n$, while $y\in F_{n,\phi(n)}\setminus E_n$, so $F_{n,\phi(n)}\ne E_n$; which is absurd. **XQ**

(δ) Set $A=R^{-1}[X]$, so that R is the graph of a function from A to X; in recognition of its new status, give it a new name f. Then f is continuous. **P** Suppose that $\phi\in A$ and that $x=f(\phi)\in G$, where $G\subseteq X$ is open. Then there is an $n\in\mathbb{N}$ such that $x\in E_n\subseteq G$. In this case $x\in F_{n,\phi(n)}$, because $(\phi,x)\in R$, so $F_{n,\phi(n)}=E_n$. Now if $\psi\in A$ and $\psi(n)=\phi(n)$, we must have

$$f(\psi)\in F_{n,\psi(n)}=E_n\subseteq G.$$

Thus $f^{-1}[G]$ includes a neighbourhood of ϕ in A. As ϕ and G are arbitrary, f is continuous. **Q**

(ϵ) At this point recall that X is K-analytic. It follows that $\mathbb{N}^{\mathbb{N}}\times X$ is K-analytic (422Ge), so that its closed subset R is K-analytic (422Gf) and A, which is a continuous image of R, is K-analytic (422Gd). But now A is a K-analytic subset of $\mathbb{N}^{\mathbb{N}}$, so is analytic, by (i) just above. And, finally, X is a continuous image of A, so is analytic.

(iii) Thus any regular K-analytic space with a countable network is analytic. Now suppose that X is an arbitrary K-analytic Hausdorff space with a countable network. Let $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. Then R is a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$, so is itself a K-analytic space with a countable network. But it is also regular, by 422E. So R is analytic and its continuous image X is analytic.

This completes the proof.

423D Corollary (a) An analytic Hausdorff space is hereditarily Lindelöf.

(b) In a regular analytic Hausdorff space, closed sets are zero sets and the Baire and Borel σ-algebras coincide.

(c) A compact subset of an analytic Hausdorff space is metrizable.

(d) A metrizable space is analytic iff it is K-analytic.

proof (a)-(c) These are true just because there is a countable network (4A2Nb, 4A3Kb, 4A2Na, 4A2Qh).

(d) Let X be a metrizable space. If X is analytic, of course it is K-analytic. If X is K-analytic, it is Lindelöf (422Gg) therefore separable (4A2Pd) and has a countable network (4A2P(a-iii)), so is analytic.

423E Theorem (a) For any Hausdorff space X, the family of subsets of X which are analytic in their subspace topologies is closed under Souslin's operation.

(b) Let $(X, \mathfrak{T})$ be an analytic Hausdorff space. For a subset A of X, the following are equiveridical:

(i) A is analytic;

(ii) A is K-analytic;

(iii) A is Souslin-F;

(iv) A can be obtained by Souslin's operation from the family of Borel subsets of X.

In particular, all Borel sets in X are analytic.

proof (a) Let X be a Hausdorff space and $\mathcal{A}$ the family of analytic subsets of X. Let $\langle A_\sigma \rangle_{\sigma \in S^*}$ be a Souslin scheme in $\mathcal{A}$ with kernel A. Then every A_σ is K-analytic, so A is K-analytic, by 422Hc. Also every A_σ has a countable network, so $A' = \bigcup_{\sigma \in S^*} A_\sigma$ has a countable network (4A2Nc); as $A \subseteq A'$, A also has a countable network (4A2Na) and is analytic.

(b) Because X has a countable network, so does A. So 423C tells us at once that (i)$\Leftrightarrow$(ii). In particular, X is K-analytic, so 422Hb tells us that (ii)$\Leftrightarrow$(iii). Of course (iii)$\Rightarrow$(iv).

Now suppose that $G \subseteq X$ is open. Then $G \in \mathcal{A}$. **P** If $X = \emptyset$ then $G = \emptyset$ is analytic. Otherwise, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to X$. Set $H = f^{-1}[G]$, so that $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open and $G = f[H]$. Being an open set in a metric space, H is F_σ (4A2Lc), so, in particular, is Souslin-F; but $\mathbb{N}^{\mathbb{N}}$ is analytic, so H is analytic and its continuous image G is analytic. **Q**

We have already seen that closed subsets of X belong to $\mathcal{A}$ (423Bd). Because $\mathcal{A}$ is closed under Souslin's operation, it contains every Borel set, by 421F. It therefore contains every set obtainable by Souslin's operation from Borel sets, and (iv)$\Rightarrow$(i).

Remark See also 423Yb below.

423F Proposition Let $(X, \mathfrak{T})$ be an analytic Hausdorff space.

(a) A set $E \subseteq X$ is Borel iff both E and $X \setminus E$ are analytic.

(b) If $\mathfrak{S}$ is a coarser (= smaller) Hausdorff topology on X, then $\mathfrak{S}$ and $\mathfrak{T}$ have the same Borel sets.

proof (a) If E is Borel, then E and $X \setminus E$ are analytic, by 423Eb. If E and $X \setminus E$ are analytic, they are K-analytic (423Eb) and disjoint, so there is a Borel set $F \supseteq E$ which is disjoint from $X \setminus E$ (422J); but now of course $F = E$, so E must be Borel.

(b) Because the identity map from $(X, \mathfrak{T})$ to $(X, \mathfrak{S})$ is continuous, $\mathfrak{S}$ is an analytic topology (423Bb) and every $\mathfrak{S}$-Borel set is $\mathfrak{T}$-Borel. If $E \subseteq X$ is $\mathfrak{T}$-Borel, then it and its complement are $\mathfrak{T}$-analytic, therefore $\mathfrak{S}$-analytic (423Bb), and E is $\mathfrak{S}$-Borel by (a).

423G Lemma Let X and Y be analytic Hausdorff spaces and $f : X \to Y$ a Borel measurable function.

(a) (The graph of) f is an analytic set.

(b) $f[A]$ is an analytic set in Y for any analytic set (in particular, any Borel set) $A \subseteq X$.

(c) $f^{-1}[B]$ is an analytic set in X for any analytic set (in particular, any Borel set) $B \subseteq Y$.

proof (a) Let $\mathcal{E}$ be a countable network for the topology of Y. Set

$$R = \bigcap_{E\in\mathcal{E}}(X \times \overline{E}) \cup ((X \setminus f^{-1}[\overline{E}]) \times Y).$$

Then R is a Borel set in $X \times Y$. But also R is the graph of f. **P** If $f(x) = y$, then surely $y \in \overline{E}$ whenever $x \in f^{-1}[\overline{E}]$, so $(x, y) \in R$. On the other hand, if $x \in X$, $y \in Y$ and $f(x) \neq y$, there are disjoint open sets G, $H \subseteq Y$ such that $f(x) \in G$ and $y \in H$; now there is an $E \in \mathcal{E}$ such that $f(x) \in E \subseteq G$, so that $f(x) \in \overline{E}$ but $y \notin \overline{E}$, and $(x, y) \notin R$. **Q**

Because $X \times Y$ is analytic (423Bc), R is analytic (423Eb).

(b) If $A \subseteq X$ is analytic, then $A \times Y$ and $R \cap (A \times Y)$ are analytic (423Ea), so $f[A] = R[A]$, which is a continuous image of $R \cap (A \times Y)$, is analytic.

(c) Similarly, if $B \subseteq Y$ is analytic, then $f^{-1}[B]$ is a continuous image of $R \cap (X \times B)$, so is analytic.

423H Lemma Let $(X, \mathfrak{T})$ be an analytic Hausdorff space, and $\langle E_n\rangle_{n\in\mathbb{N}}$ any sequence of Borel sets in X. Then the topology $\mathfrak{T}'$ generated by $\mathfrak{T} \cup \{E_n : n \in \mathbb{N}\}$ is analytic.

proof If $X = \emptyset$ this is trivial. Otherwise, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to X$. Set $F_n = f^{-1}[E_n]$ for each n; then F_n is a Borel subset of $\mathbb{N}^{\mathbb{N}}$, so there is a Polish topology $\mathfrak{S}'$ on $\mathbb{N}^{\mathbb{N}}$, finer than the usual topology, for which every F_n is open, by 4A3I. But now f is continuous for $\mathfrak{S}'$ and $\mathfrak{T}'$, so $\mathfrak{T}'$ is analytic, by 423Ba and 423Bb. (Of course $\mathfrak{T}'$ is Hausdorff, because it is finer than $\mathfrak{T}$.)

423I Theorem Let X be a Polish space, $E \subseteq X$ a Borel set, Y a Hausdorff space and $f : E \to Y$ an injective function.

(a) If f is continuous, then $f[E]$ is Borel.

(b) If Y has a countable network (e.g., is an analytic space or a separable metrizable space), and f is Borel measurable, then $f[E]$ is Borel.

proof (a)(i) Since there is a finer Polish topology on X for which E is closed (4A3I again), therefore Polish in the subspace topology (4A2Qd), and f will still be continuous for this topology, we may suppose that $E = X$.

(ii) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of X (4A2P(a-i)). For each pair m, $n \in \mathbb{N}$ such that $U_m \cap U_n$ is empty, $f[U_m]$ and $f[U_n]$ are analytic sets in Y (423Eb, 423Bb) and are disjoint (because f is injective), so there is a Borel set H_{mn} including $f[U_m]$ and disjoint from $f[U_n]$ (422J). Set

$$E_n = \overline{f[U_n]} \cap \bigcap\{H_{nm} \setminus H_{mn} : m \in \mathbb{N},\, U_m \cap U_n = \emptyset\}$$

for each $n \in \mathbb{N}$; then E_n is a Borel set in Y including $f[U_n]$. Note that if $U_m \cap U_n$ is empty, then $E_m \cap E_n \subseteq (H_{mn} \setminus H_{nm}) \cap (H_{nm} \setminus H_{mn})$ is also empty.

Fix a metric ρ on X, inducing its topology, for which X is complete, and for $k \in \mathbb{N}$ set

$$F_k = \bigcup\{E_n : n \in \mathbb{N},\, \operatorname{diam} U_n \leq 2^{-k}\},$$

so that F_k is Borel. Let $F = \bigcap_{k\in\mathbb{N}} F_k$; then F also is a Borel subset of Y.

The point is that $F = f[X]$. **P** (i) If $x \in X$, then for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that

$$x \in U_n \subseteq \{y : \rho(y, x) \leq 2^{-k-1}\};$$

now $\operatorname{diam} U_n \leq 2^{-k}$, so

$$f(x) \in f[U_n] \subseteq E_n \subseteq F_k.$$

As k is arbitrary, $f(x) \in F$; as x is arbitrary, $f[X] \subseteq F$. (ii) If $y \in F$, then for each $k \in \mathbb{N}$ we can find an $n(k)$ such that $y \in E_{n(k)}$ and $\operatorname{diam} U_{n(k)} \leq 2^{-k}$. Since $\overline{f[U_{n(k)}]} \supseteq E_{n(k)}$ is not empty, nor is $U_{n(k)}$, and we can choose $x_k \in U_{n(k)}$. Indeed, for any j, $k \in \mathbb{N}$, $E_{n(j)} \cap E_{n(k)}$ contains y, so is not empty, and $U_{n(j)} \cap U_{n(k)}$ cannot be empty; but this means that there is some x in the intersection, and

$$\rho(x_j, x_k) \leq \rho(x_j, x) + \rho(x, x_k) \leq \operatorname{diam} U_{n(j)} + \operatorname{diam} U_{n(k)} \leq 2^{-j} + 2^{-k}.$$

This means that $\langle x_k\rangle_{k\in\mathbb{N}}$ is a Cauchy sequence. But X is supposed to be complete, so $\langle x_k\rangle_{k\in\mathbb{N}}$ has a limit x say.

? If $f(x) \neq y$, then (because Y is Hausdorff) there is an open set H containing $f(x)$ such that $y \notin \overline{H}$. Now f is continuous, so there is a $\delta > 0$ such that $f(x') \in H$ whenever $\rho(x', x) \leq \delta$. There is a $k \in \mathbb{N}$ such that $2^{-k} + \rho(x_k, x) \leq \delta$. If $x' \in U_{n(k)}$, then

$$\rho(x', x) \leq \rho(x', x_k) + \rho(x_k, x) \leq \delta;$$

thus $f[U_{n(k)}] \subseteq H$, and

$$E_{n(k)} \subseteq \overline{f[U_{n(k)}]} \subseteq \overline{H}.$$

But $y \in E_{n(k)} \setminus \overline{H}$. **X**

Thus $y = f(x)$ belongs to $f[X]$; as y is arbitrary, $F \subseteq f[X]$. **Q** Accordingly $f[X] = F$ is a Borel subset of Y, as claimed.

(b) By 4A2Nf, there is a countable family $\mathcal{V}$ of open sets in Y such that whenever y, y' are distinct points of Y there are disjoint V, $V' \in \mathcal{V}$ such that $y \in V$ and $y' \in V'$. Let $\mathfrak{S}'$ be the topology generated by $\mathcal{V}$; then $\mathfrak{S}'$ is Hausdorff. For each $V \in \mathcal{V}$, $f^{-1}[V]$ is a Borel set in X, so there is a Polish topology $\mathfrak{T}'$ on X, finer than the original topology, for which every $f^{-1}[V]$ is open (4A3I once more). Now f is continuous for $\mathfrak{T}'$ and $\mathfrak{S}'$ (4A2B(a-ii)), and E is $\mathfrak{T}'$-Borel, so $f[E]$ is a $\mathfrak{S}'$-Borel set in Y, by (a). Since $\mathfrak{S}'$ is coarser than the original topology $\mathfrak{S}$ on Y, $f[E]$ is also $\mathfrak{S}$-Borel.

423J Proposition Let $(X, \mathfrak{T})$ be an analytic Hausdorff space, and Σ a countably generated σ-subalgebra of the Borel σ-algebra $\mathcal{B}(X, \mathfrak{T})$ which separates the points of X. Then $\Sigma = \mathcal{B}(X, \mathfrak{T})$.

proof Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence in Σ which σ-generates Σ. If x, $y \in X$ are distinct, the set $\{E : E \subseteq X, x \in E \iff y \in E\}$ is a σ-algebra of subsets of X not including Σ, so there is an $n \in \mathbb{N}$ such that just one of x, y belongs to E_n, and the topology $\mathfrak{S}$ generated by $\{E_n : n \in \mathbb{N}\} \cup \{X \setminus E_n : n \in \mathbb{N}\}$ is Hausdorff. Write $\mathfrak{T} \vee \mathfrak{S}$ for the topology generated by $\mathfrak{T} \cup \mathfrak{S}$, that is, the topology generated by $\mathfrak{T} \cup \{E_n : n \in \mathbb{N}\} \cup \{X \setminus E_n : n \in \mathbb{N}\}$. By 423H, $\mathfrak{T} \vee \mathfrak{S}$ is analytic; because both $\mathfrak{T}$ and $\mathfrak{S}$ are Hausdorff topologies coarser than $\mathfrak{T} \vee \mathfrak{S}$, the Borel σ-algebras $\mathcal{B}(X, \mathfrak{T})$, $\mathcal{B}(X, \mathfrak{T} \vee \mathfrak{S})$ and $\mathcal{B}(X, \mathfrak{S})$ are all the same (423Fb). Next, $\mathfrak{S}$ is second-countable, therefore hereditarily Lindelöf (4A2Oc), with a subbase included in Σ, so $\mathcal{B}(X, \mathfrak{S}) \subseteq \Sigma$ (4A3Da) and $\mathcal{B}(X, \mathfrak{T})$ must be equal to Σ.

423K Lemma If X is an uncountable analytic Hausdorff space, it has subsets homeomorphic to $\{0,1\}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

proof (a) Let $f : \mathbb{N}^{\mathbb{N}} \to X$ be a continuous surjection. Write $S = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$, $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$ for $\sigma \in S$,

$$T = \{\sigma : \sigma \in S,\ f[I_\sigma] \text{ is uncountable}\}.$$

Then if $\sigma \in T$ there are τ, $\tau' \in T$, both extending σ, such that $f[I_\tau] \cap f[I_{\tau'}] = \emptyset$. **P** Set

$$A = \bigcup \{f[I_\tau] : \tau \in S \setminus T\}.$$

Then A is a countable union of countable sets, so is countable. There must therefore be distinct points x, y of $f[I_\sigma] \setminus A$; express x as $f(\phi)$ and y as $f(\psi)$ where ϕ and ψ belong to I_σ. Because X is Hausdorff, there are disjoint open sets G, H such that $x \in G$ and $y \in H$. Because f is continuous, there are m, $n \in \mathbb{N}$ such that $I_{\phi \restriction m} \subseteq f^{-1}[G]$ and $I_{\psi \restriction n} \subseteq f^{-1}[H]$. Of course both $\tau = \phi \restriction m$ and $\tau' = \psi \restriction n$ must extend σ, and they belong to T because $x \in f[I_\tau] \setminus A$ and $y \in f[I_{\tau'}] \setminus A$. **Q**

(b) We can therefore choose inductively a family $\langle \tau(v) \rangle_{v \in S_2}$ in T, where $S_2 = \bigcup_{n \in \mathbb{N}} \{0,1\}^n$, such that

$$\tau(\emptyset) = \emptyset,$$

$$\tau(v^\frown\!<\!i\!>) \supseteq \tau(v) \text{ whenever } v \in S_2,\ i \in \{0,1\},$$

$$f[I_{\tau(v^\frown<0>)}] \cap f[I_{\tau(v^\frown<1>)}] = \emptyset \text{ for every } v \in S_2.$$

Note that $\#(\tau(v)) \geq \#(v)$ for every $v \in S_2$. For each $z \in \{0,1\}^{\mathbb{N}}$, $\langle \tau(z \restriction n) \rangle_{n \in \mathbb{N}}$ is a sequence in S in which each term strictly extends its predecessor, so there is a unique $g(z) \in \mathbb{N}^{\mathbb{N}}$ such that $\tau(z \restriction n) \subseteq g(z)$ for every n. Now $g(z') \restriction n = g(z) \restriction n$ whenever $z \restriction n = z' \restriction n$, so $g : \{0,1\}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $fg : \{0,1\}^{\mathbb{N}} \to X$ are continuous. If w, z are distinct points of $\{0,1\}^{\mathbb{N}}$, there is a first n such that $w(n) \neq z(n)$, in which case $fg(w) \in f[I_{\tau(w\restriction n)^\frown<w(n)>}]$ and $fg(z) \in f[I_{\tau(w\restriction n)^\frown<z(n)>}]$ are distinct. So $fg : \{0,1\}^{\mathbb{N}} \to X$ is a continuous injection, therefore a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and its image, because $\{0,1\}^{\mathbb{N}}$ is compact (3A3Dd).

(c) Thus X has a subspace homeomorphic to $\{0,1\}^{\mathbb{N}}$. Now $\{0,1\}^{\mathbb{N}}$ has a subspace homeomorphic to $\mathbb{N}$. **P** For instance, setting $d_n(n) = 1$, $d_n(i) = 0$ for $i \neq n$, $D = \{d_n : n \in \mathbb{N}\}$ is homeomorphic to $\mathbb{N}$. **Q** Now $D^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and is a subspace of $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}\times\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$, so $\{0,1\}^{\mathbb{N}}$ has a subspace homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Accordingly X also has a subspace homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

423L Corollary Any uncountable Borel set in any analytic Hausdorff space has cardinal $\mathfrak{c}$.

proof If X is an analytic space and $E \subseteq X$ is an uncountable Borel set, then E is analytic (423Eb), so includes a copy of $\{0,1\}^{\mathbb{N}}$ and must have cardinal at least $\#(\{0,1\}^{\mathbb{N}}) = \mathfrak{c}$. On the other hand, E is also a continuous image of $\mathbb{N}^{\mathbb{N}}$, so has cardinal at most $\#(\mathbb{N}^{\mathbb{N}}) = \mathfrak{c}$.

423M Proposition Let X be an uncountable analytic Hausdorff space. Then it has a non-Borel analytic subset.

proof (a) I show first that there is an analytic set $A \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that every analytic subset of $\mathbb{N}^{\mathbb{N}}$ is a vertical section of A. **P** Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, containing $\emptyset$, and $\langle U_n\rangle_{n\in\mathbb{N}}$ an enumeration of $\mathcal{U}$. Write

$$M = (\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \setminus \bigcup_{m,n\in\mathbb{N}}(\{x : x(m) = n\} \times U_n).$$

Then M is a closed subset of $(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}}$, therefore analytic (423Ba, 423Bd), so its continuous image

$$A = \{(x,z) : \text{there is some } y \text{ such that } (x,y,z) \in M\}$$

is analytic (423Bb).

Now let E be any analytic subset of $\mathbb{N}^{\mathbb{N}}$. By 423Eb, E is Souslin-F; by 421J, there is a closed set $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $E = \{z : \exists\, y, (y,z) \in F\}$. Let $\langle x(m)\rangle_{m\in\mathbb{N}}$ be a sequence running over $\{n : n \in \mathbb{N}, U_n \cap F = \emptyset\}$, so that

$$F = (\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \setminus \bigcup_{m\in\mathbb{N}} U_{x(m)} = \{(y,z) : (x,y,z) \in M\}.$$

Now

$$\begin{aligned}\{z : (x,z) \in A\} &= \{z : \text{ there is some } y \text{ such that } (x,y,z) \in M\}\\ &= \{z : \text{ there is some } y \text{ such that } (y,z) \in F\} = E,\end{aligned}$$

and E is a vertical section of A, as required. **Q**

(b) It follows that there is a non-Borel analytic set $B \subseteq \mathbb{N}^{\mathbb{N}}$. **P** Take A from (a) above, and try

$$B = \{x : (x,x) \in A\}.$$

Because B is the inverse image of A under the continuous map $x \mapsto (x,x)$, it is analytic (423Gc). **?** If B were a Borel set, then $B' = \mathbb{N}^{\mathbb{N}} \setminus B$ would also be Borel, therefore analytic (423Eb), and there would be an $x \in \mathbb{N}^{\mathbb{N}}$ such that $B' = \{y : (x,y) \in A\}$. But in this case

$$x \in B \iff (x,x) \in A \iff x \in B',$$

which is a difficulty you may have met before. **XQ**

(c) Now return to our arbitrary uncountable analytic Hausdorff space X. By 423K, X has a subset Z homeomorphic to $\mathbb{N}^{\mathbb{N}}$. By (b), Z has an analytic subset A which is not Borel in Z, therefore cannot be a Borel subset of X.

423N I devote a few paragraphs to an important method of constructing selectors.

Theorem Let X be an analytic Hausdorff space, Y a set, and $\mathcal{C} \subseteq \mathcal{P}Y$. Write T for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathcal{C})$, where $\mathcal{S}$ is Souslin's operation, and $\mathcal{V}$ for $\mathcal{S}(\{F \times C : F \subseteq X \text{ is closed}, C \in \mathcal{C}\})$. If

$W \in \mathcal{V}$, then $W[X] \in \mathcal{S}(\mathcal{C})$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

proof Write $\mathcal{F}$ for $\{F \times C : F \subseteq X$ is closed, $C \in \mathcal{C}\}$.

(a) Consider first the case in which $X = \mathbb{N}^{\mathbb{N}}$ and all the horizontal sections $W^{-1}[\{y\}]$ of W are closed. Let $\mathcal{E}$ be the family of closed subsets of Y. For $\sigma \in S = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$. Then $W \cap (I_\sigma \times Y) \in \mathcal{V}$. **P** Because Souslin's operation is idempotent (421D), $\mathcal{S}(\mathcal{V}) = \mathcal{V}$. The set $\{V : V \cap (I_\sigma \times Y) \in \mathcal{V}\}$ is therefore closed under Souslin's operation (apply 421Cc to the identity map from $I_\sigma \times Y$ to $X \times Y$, or otherwise); since it includes $\mathcal{F}$, it is the whole of $\mathcal{V}$, and contains W. **Q**

By 421G, $W[I_\sigma] = (W \cap (I_\sigma \times Y))[\mathbb{N}^{\mathbb{N}}]$ belongs to $\mathcal{S}(\mathcal{C}) \subseteq \mathrm{T}$ for every σ. In particular, $W[\mathbb{N}^{\mathbb{N}}] = W[I_\emptyset] \in \mathcal{S}(\mathcal{C})$. Define $\langle Y_\sigma\rangle_{\sigma\in S}$ in T inductively, as follows. $Y_\emptyset = W[\mathbb{N}^{\mathbb{N}}]$. Given that $Y_\sigma \in \mathrm{T}$ and that $Y_\sigma \subseteq W[I_\sigma]$, set

$$Y_{\sigma^\frown<j>} = Y_\sigma \cap W[I_{\sigma^\frown<j>}] \setminus \bigcup_{i<j} W[I_{\sigma^\frown<i>}] \in \mathrm{T}$$

for every $j \in \mathbb{N}$. Continue.

At the end of the induction, we have

$$\bigcup_{j\in\mathbb{N}} Y_{\sigma^\frown<j>} = Y_\sigma \cap \bigcup_{j\in\mathbb{N}} W[I_{\sigma^\frown<j>}] = Y_\sigma \cap W[I_\sigma] = Y_\sigma$$

for every $\sigma \in S$, while $\langle Y_{\sigma^\frown<j>}\rangle_{j\in\mathbb{N}}$ is always disjoint. So for each $y \in Y_\emptyset = W[\mathbb{N}^{\mathbb{N}}]$ we have a unique $f(y) \in \mathbb{N}^{\mathbb{N}}$ such that $y \in Y_{f(y)\restriction n}$ for every n. Since $f^{-1}[I_\sigma] = Y_\sigma \in \mathrm{T}$ for every $\sigma \in S$, f is T-measurable (4A3Db). Also $(f(y), y) \in W$ for every $y \in W[\mathbb{N}^{\mathbb{N}}]$. **P** For each $n \in \mathbb{N}$, $y \in Y_{f(y)\restriction n} = W[I_{f(y)\restriction n}]$, so there is an $x_n \in \mathbb{N}^{\mathbb{N}}$ such that $x_n\restriction n = f(y)\restriction n$ and $(x_n, y) \in W$. But this means that $f(y) = \lim_{n\to\infty} x_n$; since we are supposing that the horizontal sections of W are closed, $(f(y), y) \in W$. **Q**

Thus the theorem is true if $X = \mathbb{N}^{\mathbb{N}}$ and W has closed horizontal sections.

(b) Now suppose that $X = \mathbb{N}^{\mathbb{N}}$ and that $W \subseteq \mathbb{N}^{\mathbb{N}} \times Y$ is any set in $\mathcal{V}$. Then there is a Souslin scheme $\langle F_\sigma \times C_\sigma\rangle_{\sigma\in S^*}$ in $\mathcal{F}$ with kernel W; of course I mean you to suppose that $F_\sigma \subseteq \mathbb{N}^{\mathbb{N}}$ is closed and $C_\sigma \in \mathcal{C}$ for every σ. Set

$$\tilde W = \bigcap_{k\ge1} \bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times F_\sigma \times C_\sigma \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times Y.$$

Then W is the projection of $\tilde W$ onto the last two coordinates, by 421Ce. If $y \in Y$, then

$$\{(\phi, \psi) : (\phi, \psi, y) \in \tilde W\} = \bigcap_{k\ge1} \bigcup\{I_\sigma \times F_\sigma : \sigma \in \mathbb{N}^k,\, y \in C_\sigma\}$$

is closed in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. (If J is any subset of $\mathbb{N}^k$, then

$$(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \setminus \bigcup_{\sigma\in J} I_\sigma \times F_\sigma = \bigcup_{\sigma\in J} I_\sigma \times (\mathbb{N}^{\mathbb{N}} \setminus F_\sigma) \cup \bigcup_{\sigma\in\mathbb{N}^k\setminus J} I_\sigma \times \mathbb{N}^{\mathbb{N}}$$

is open.) Also $I_\sigma \times F_\sigma$ is a closed subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ for every σ, and $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. We can therefore apply (a) to $\tilde W$, regarded as a subset of $(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \times Y$, to see that $W[\mathbb{N}^{\mathbb{N}}] = \tilde W[\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}] \in \mathcal{S}(\mathcal{C})$ and that there is a T-measurable function $h = (g, f) : W[\mathbb{N}^{\mathbb{N}}] \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $(g(y), f(y), y) \in \tilde W$ for every $y \in W[\mathbb{N}^{\mathbb{N}}]$. Now, of course, $f : W[\mathbb{N}^{\mathbb{N}}] \to \mathbb{N}^{\mathbb{N}}$ is T-measurable and $(f(y), y) \in W$ for every $y \in W[\mathbb{N}^{\mathbb{N}}]$.

(c) Finally, suppose only that X is an analytic Hausdorff space and that $W \in \mathcal{V}$. If X is empty, so is Y, and the result is trivial. Otherwise, there is a continuous surjection $h : \mathbb{N}^{\mathbb{N}} \to X$. Set $\tilde h(\phi, y) = (h(\phi), y)$ for $\phi \in \mathbb{N}^{\mathbb{N}}$ and $y \in Y$; then $\tilde h : \mathbb{N}^{\mathbb{N}} \times Y \to X \times Y$ is a continuous surjection, and $\tilde W = \tilde h^{-1}[W]$ is the kernel of a Souslin scheme in

$$\{\tilde h^{-1}[F \times C] : F \subseteq \mathbb{N}^{\mathbb{N}} \text{ is closed}, C \in \mathcal{C}\} = \{h^{-1}[F] \times C : F \subseteq \mathbb{N}^{\mathbb{N}} \text{ is closed}, C \in \mathcal{C}\}$$

by 421Cb. So we can apply (b) to see that $W[X] = \tilde W[\mathbb{N}^{\mathbb{N}}] \in \mathcal{S}(\mathcal{C})$ and there is a T-measurable $g : W[X] \to \mathbb{N}^{\mathbb{N}}$ such that $(g(y), y) \in \tilde W$ for every $y \in Y$. Finally $f = hg : W[X] \to X$ is T-measurable and $(f(y), y) \in W$ for every $y \in Y$. This completes the proof.

423O The expression

$$\mathcal{V} = \mathcal{S}(\{F \times C : F \subseteq X \text{ is closed, } C \in \mathcal{C}\})$$

in 423N is a new formulation, and I had better describe a couple of the basic cases in which we can use the result. The first really is elementary.

Corollary Let X be an analytic Hausdorff space, Y a set and T a σ-algebra of subsets of Y which is closed under Souslin's operation. Suppose that $W \in \mathcal{S}(\mathcal{B}(X)\widehat{\otimes}\mathrm{T})$ where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

proof Set $\mathcal{V} = \mathcal{S}(\{F \times C : F \subseteq X \text{ is closed}, C \in \mathrm{T}\}$ and $\mathcal{E} = \{F \times Y : F \subseteq X \text{ is closed}\} \cup \{X \times C : C \in \mathrm{T}\}$. Then $E \times Y \in \mathcal{S}(\mathcal{E})$ for every Souslin-F subset E of X, by 421Cc, and in particular for every Borel subset E of X (423Eb). Accordingly $\mathcal{E}' = \{E \times Y : E \in \mathcal{B}(X)\} \cup \{X \times C : C \in \mathrm{T}\}$ is included in $\mathcal{S}(\mathcal{E})$. Since $\mathcal{E}'$ contains $\emptyset$ and the complement of any member of $\mathcal{E}'$ belongs to $\mathcal{E}'$, $\mathcal{S}(\mathcal{E}')$ includes the σ-algebra of subsets of $X \times Y$ generated by $\mathcal{E}'$ (421F), and we have

$$W \in \mathcal{S}(\mathcal{B}(X)\widehat{\otimes}\mathrm{T}) \subseteq \mathcal{S}(\mathcal{E}') \subseteq \mathcal{S}(\mathcal{S}(\mathcal{E})) = \mathcal{S}(\mathcal{E}) \subseteq \mathcal{V}.$$

So $W \in \mathcal{V}$ and we can read the conclusion off from 423N.

423P Corollary Let X be an analytic Hausdorff space and Y any topological space. Let T be the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. If $W \in \mathcal{S}(\mathcal{B}(X \times Y))$, then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

proof (a) Suppose to begin with that $X = \mathbb{N}^{\mathbb{N}}$. In 423N, set $\mathcal{C} = \mathcal{B}(Y)$. Then every open subset and every closed subset of $X \times Y$ belongs to $\mathcal{V}$ as defined in 423N. **P** For $\sigma \in S = \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$, set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$. If $V \subseteq X \times Y$ is open, set

$$H_\sigma = \bigcup\{H : H \subseteq Y \text{ is open}, I_\sigma \times H_\sigma \subseteq V\}$$

for each $\sigma \in S$. Because $\{I_\sigma : \sigma \in S\}$ is a base for the topology of $\mathbb{N}^{\mathbb{N}}$, $V = \bigcup_{\sigma\in S} I_\sigma \times H_\sigma \in \mathcal{V}$.

As for the complement of V, we have

$$\begin{aligned}(\mathbb{N}^{\mathbb{N}} \times Y) \setminus V &= \bigcap_{\substack{k\in\mathbb{N}\\ \sigma\in\mathbb{N}^k}} (\mathbb{N}^{\mathbb{N}} \times Y) \setminus (I_\sigma \times H_\sigma)\\ &= \bigcap_{\substack{k\in\mathbb{N}\\ \sigma\in\mathbb{N}^k}} ((\mathbb{N}^{\mathbb{N}} \setminus I_\sigma) \times Y) \cup (\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma))\\ &= \bigcap_{\substack{k\in\mathbb{N}\\ \sigma\in\mathbb{N}^k}} \bigcup_{\substack{\tau\in\mathbb{N}^k\\ \tau\neq\sigma}} (I_\tau \times Y) \cup (\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma))\end{aligned}$$

which again belongs to $\mathcal{V}$, because $\mathcal{V}$ is closed under countable unions and intersections and contains $I_\tau \times Y$ and $\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma)$ for all σ, $\tau \in S$. **Q**

By 421F, $\mathcal{V}$ contains every Borel subset of $\mathbb{N}^{\mathbb{N}} \times Y$, so includes $\mathcal{S}(\mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y))$. So in this case we can apply 423N directly to get the result.

(b) Now suppose that X is any analytic space. If X is empty, the result is trivial. Otherwise, let $h : \mathbb{N}^{\mathbb{N}} \to X$ be a continuous surjection. Set $\tilde{h}(\phi, y) = (h(\phi), y)$ for $\phi \in \mathbb{N}^{\mathbb{N}}$ and $y \in Y$, so that $\tilde{h} : \mathbb{N}^{\mathbb{N}} \times Y \to X \times Y$ is continuous. Set $\tilde{W} = \tilde{h}^{-1}[W]$. If $V \in \mathcal{B}(X \times Y)$ then $\tilde{h}^{-1}[V] \in \mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y)$ (4A3Cd), so $\tilde{W} \in \mathcal{S}(\mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y))$ (421Cc). By (a), $\tilde{W}[\mathbb{N}^{\mathbb{N}}] \in \mathrm{T}$ and there is a T-measurable function $g : \tilde{W}[\mathbb{N}^{\mathbb{N}}] \to \mathbb{N}^{\mathbb{N}}$ such that $(g(y), y) \in \tilde{W}$ for every $y \in \tilde{W}[\mathbb{N}^{\mathbb{N}}]$. It is now easy to check that $W[X] = \tilde{W}[\mathbb{N}^{\mathbb{N}}] \in \mathrm{T}$ (this is where we need to know that h is surjective), that $f = hg : W[X] \to X$ is T-measurable, and that $(f(y), y) \in W$ for every $y \in W[X]$, as required.

Remark This is a version of the **von Neumann-Jankow selection theorem**.

423Q Corollary Let X and Y be analytic Hausdorff spaces, A an analytic subset of X and $f : A \to Y$ a Borel measurable function. Let T be the σ-algebra of subsets of Y generated by the Souslin-F subsets of Y. Then $f[A] \in \mathrm{T}$ and there is a T-measurable function $g : f[A] \to A$ such that fg is the identity on $f[A]$.

proof In this context, every Borel subset of Y is Souslin-F (423Eb), so every member of $\mathcal{S}(\mathcal{B}(Y))$ is Souslin-F (421D) and $\mathrm{T} \supseteq \mathcal{S}(\mathcal{B}(Y))$. If we think of f as a subset of $X \times Y$, it is analytic (423Ga), therefore Souslin-F in $X \times Y$; now we can use 423P to find a T-measurable function $g : f[A] \to A$ such that $(g(y), y) \in f$, that is, $f(g(y)) = y$, for every $y \in f[A]$.

***423R Constituents of coanalytic sets: Theorem** Let X be a Hausdorff space, and $A \subseteq X$ an analytic subset of X. Then there is a non-decreasing family $\langle E_\xi \rangle_{\xi<\omega_1}$ of Borel subsets of X, with union $X \setminus A$, such that every analytic subset of $X \setminus A$ is included in some E_ξ.

proof Put 422K(iii) and 423C together.

***423S Remarks (a)** Let A be an analytic set in an analytic space X and $\langle E_\xi \rangle_{\xi<\omega_1}$ a family of Borel sets as in 423R. There is nothing unique about the E_ξ. But if $\langle E'_\xi \rangle_{\xi<\omega_1}$ is another such family, then every E'_ξ is an analytic subset of $X \setminus A$, by 423Eb, so is included in some E_η; and, similarly, every E_ξ is included in some E'_η. We therefore have a function $f : \omega_1 \to \omega_1$ such that $E'_\xi \subseteq E_{f(\xi)}$ and $E_\xi \subseteq E'_{f(\xi)}$ for every $\xi < \omega$. If we set $C = \{\xi : \xi < \omega_1,\ f(\eta) < \xi \text{ for every } \eta < \xi\}$, then C is a closed cofinal set in ω_1 (4A1Bc), and $\bigcup_{\eta<\xi} E_\eta = \bigcup_{\eta<\xi} E'_\eta$ for every $\xi \in C$. If $X \setminus A$ is itself analytic, that is, if A is a Borel set, then we shall have to have $X \setminus A = E_\xi = E'_\xi$ for some $\xi < \omega_1$.

Another way of expressing the result in 423R is to say that if we write $\mathcal{I} = \{B : B \subseteq X \setminus A \text{ is analytic}\}$, then $\{E : E \in \mathcal{I},\ E \text{ is Borel}\}$ is cofinal with $\mathcal{I}$ (this is the First Separation Theorem) and $\operatorname{cf} \mathcal{I} \le \omega_1$.

(b) It is a remarkable fact that, in some models of set theory, we can have non-Borel coanalytic sets in Polish spaces such that all their constituents are countable (Jech 78, p. 529, Cor. 2). (Note that, by (a), this is the same thing as saying that $X \setminus A$ is uncountable but all its Borel subsets are countable.) But in 'ordinary' cases we shall have, for every Borel subset E of $X \setminus A$, an uncountable Borel subset of $(X \setminus A) \setminus E$; so that for any family $\langle G_\xi \rangle_{\xi<\omega_1}$ of Borel constituents of $X \setminus A$, there must be uncountably many uncountable G_ξ. To see that this happens at least sometimes, take any non-Borel analytic subset A_0 of $\mathbb{N}^\mathbb{N}$ (423M), and consider $A = A_0 \times \mathbb{N}^\mathbb{N} \subseteq (\mathbb{N}^\mathbb{N})^2$. Then A is analytic (423B). If $E \subseteq (\mathbb{N}^\mathbb{N})^2 \setminus A$ is Borel, then $\pi_1[E] = \{x : (x, y) \in E\}$ is an analytic subset of $\mathbb{N}^\mathbb{N} \setminus A_0$, so is not the whole of $\mathbb{N}^\mathbb{N} \setminus A_0$ (by 423Fa). Taking any $x \in (\mathbb{N}^\mathbb{N} \setminus A_0) \setminus \pi_1[E]$, $\{x\} \times \mathbb{N}^\mathbb{N}$ is an uncountable Borel subset of $((\mathbb{N}^\mathbb{N})^2 \setminus A) \setminus E$. For an alternative construction, see 423Ye.

***423T Coanalytic and PCA sets** In the case of Polish spaces, we can go a great deal farther. I mention only some fragments which will be used in Volume 5. Let X be a Polish space.

(a) A subset A of X is **coanalytic** or $\mathbf{\Pi}^1_1$ if $X \setminus A$ is analytic, and **PCA** or $\mathbf{\Sigma}^1_2$ if there is a coanalytic set $R \subseteq \mathbb{N}^\mathbb{N} \times X$ such that $R[\mathbb{N}^\mathbb{N}] = A$. Generally, for $n \ge 1$, $A \subseteq X$ is $\mathbf{\Pi}^1_n$ if $X \setminus A$ is $\mathbf{\Sigma}^1_n$ (A is $\mathbf{\Sigma}^1_1$ if it is analytic), and $\mathbf{\Sigma}^1_{n+1}$ if there is a $\mathbf{\Pi}^1_n$ set $R \subseteq \mathbb{N}^\mathbb{N} \times X$ such that $R[\mathbb{N}^\mathbb{N}] = A$. If A is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, we say that A is $\mathbf{\Delta}^1_n$.

(b) Analytic subsets of X are Souslin-F (423Eb). Applying 421P to the Borel σ-algebra of X, we see that a subset of X which is either analytic or coanalytic can be expressed as the union of at most ω_1 Borel sets. It follows that every PCA set $A \subseteq X$ can be expressed as the union of at most ω_1 Borel sets. **P** Let $R \subseteq \mathbb{N}^\mathbb{N} \times X$ be a coanalytic set such that $A = R[\mathbb{N}^\mathbb{N}]$. Express R as $\bigcup_{\xi<\omega_1} R_\xi$ where every R_ξ is a Borel subset of the Polish space $\mathbb{N}^\mathbb{N} \times X$. For each $\xi < \omega_1$, R_ξ is analytic (423Eb) so $A_\xi = R_\xi[\mathbb{N}^\mathbb{N}]$ is analytic (423Bb) and can be expressed as $\bigcup_{\eta<\omega_1} E_{\xi\eta}$ where every $E_{\xi\eta}$ is a Borel subset of X. Now $A = \bigcup_{\xi,\eta<\omega_1} E_{\xi\eta}$ is the union of at most ω_1 Borel sets. **Q**

(c) A subset of X is Borel iff it is $\mathbf{\Delta}^1_1$, that is, is both analytic and coanalytic (423Eb, 423Fa). Since the intersection and union of a sequence of analytic subsets of X are analytic (423Ea), the union and intersection of a sequence of coanalytic subsets of X are coanalytic. If Y is another Polish space and $h : X \to Y$ is Borel measurable, then $h^{-1}[A]$ is analytic for every analytic $A \subseteq Y$ (423Gc), so $h^{-1}[B]$ is coanalytic in X for every coanalytic $B \subseteq Y$. If Y is a G_δ subset of X and $B \subseteq Y$ is coanalytic in Y (remember that Y, with its subspace topology, is Polish, by 4A2Qd), then B is coanalytic in X, because $X \setminus B = (X \setminus Y) \cup (Y \setminus B)$ is the union of two analytic sets.

(d) If X and Y are Polish spaces, $A \subseteq Y$ is PCA and $f : X \to Y$ is Borel measurable, then $f^{-1}[A]$ is a PCA set in X. **P** Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be a coanalytic set such that $R[\mathbb{N}^{\mathbb{N}}] = A$. Set $g(\phi, x) = (\phi, f(x))$ for $\phi \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$; writing $\mathcal{B}(X)$ for the Borel σ-algebra of X, etc., g is $(\mathcal{B}(\mathbb{N}^{\mathbb{N}})\widehat{\otimes}\mathcal{B}(X), \mathcal{B}(\mathbb{N}^{\mathbb{N}})\widehat{\otimes}\mathcal{B}(Y))$-measurable (use 4A3Bc). But $\mathcal{B}(\mathbb{N}^{\mathbb{N}})\widehat{\otimes}\mathcal{B}(X) = \mathcal{B}(\mathbb{N}^{\mathbb{N}} \times X)$ and $\mathcal{B}(\mathbb{N}^{\mathbb{N}})\widehat{\otimes}\mathcal{B}(Y) = \mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y)$ (4A3Ga), so g is Borel measurable. By (c), $R' = g^{-1}[R]$ is a coanalytic set in $\mathbb{N}^{\mathbb{N}} \times X$. Now

$$R'[\mathbb{N}^{\mathbb{N}}] = \{x : \exists\, \phi,\ (\phi, x) \in g^{-1}[R]\} = \{x : \exists\, \phi,\ (\phi, f(x)) \in R\} = f^{-1}[A],$$

so $f^{-1}[A]$ is PCA. **Q**

(e) For a fuller account of this material, see Kechris 95 or Kuratowski 66.

423X Basic exercises >(a) For a Hausdorff space X, show that the following are equiveridical: (i) X is analytic; (ii) X is a continuous image of a Polish space; (iii) X is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$.

(b) Write out a direct proof of 423Ea, not quoting 423C or 421D.

>(c) Let X be a set, $\mathfrak{S}$ a Hausdorff topology on X and $\mathfrak{T}$ an analytic topology on X such that $\mathfrak{S} \subseteq \mathfrak{T}$. Show that $\mathfrak{S}$ and $\mathfrak{T}$ have the same analytic sets. (*Hint*: 423F.)

(d) Let X be an analytic Hausdorff space. (i) Show that its Borel σ-algebra $\mathcal{B}(X)$ is countably generated as σ-algebra. (*Hint*: use 4A2Nf and 423Fb.) (ii) Show that there is an analytic subset Y of $\mathbb{R}$ such that $(X, \mathcal{A}_X)$ is isomorphic to $(Y, \mathcal{A}_Y)$, where $\mathcal{A}_X$, $\mathcal{A}_Y$ are the families of Souslin-F subsets of X, Y respectively. (*Hint*: show that there is an injective Borel measurable function from X to $\mathbb{R}$ (cf. 343E), and use 423G.) (iii) Show that $(X, \mathcal{B}(X))$ is isomorphic to $(Y, \mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. (*Hint*: 423Fa.) (iv) Let T_X, T_Y be the σ-algebras generated by $\mathcal{A}_X$, $\mathcal{A}_Y$ respectively. Show that (X, T_X) and (Y, T_Y) are isomorphic.

(e) Let $\mathfrak{S}$ be the right-facing Sorgenfrey topology on $\mathbb{R}$ (415Xc). Show that $\mathfrak{S}$ has the same Borel sets as the usual topology $\mathfrak{T}$ on $\mathbb{R}$. (*Hint*: $\mathfrak{S}$ is hereditarily Lindelöf (419Xf) and has a base consisting of $\mathfrak{T}$-Borel sets.) Show that $\mathfrak{S}$ is not analytic. (*Hint*: in the product topology on $\mathbb{R}^2$ the counter-diagonal $\{(x, -x)\} : x \in \mathbb{R}\}$ is closed and discrete.)

(f) Let X be an analytic Hausdorff space and Y any topological space. Let T be the σ-algebra of subsets of Y generated by the Souslin-F sets. Show that if $W \subseteq X \times Y$ is Souslin-F, then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$. (*Hint*: start with $X = \mathbb{N}^{\mathbb{N}}$, as in 423P.)

(g) Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an analytic set such that $R^{-1}[Y] = X$. Show that there is a function $g : X \to Y$, measurable with respect to the σ-algebra generated by the Souslin-F subsets of X, such that $(x, g(x)) \in R$ for every $x \in X$.

>(h)(i) Show that the family of analytic subsets of $[0,1]$ has cardinal $\mathfrak{c}$. (*Hint*: 421Xc.) (ii) Show that the σ-algebra T of subsets of $[0,1]$ generated by the analytic sets has cardinal $\mathfrak{c}$. (*Hint*: 421Xh.) (iii) Show that there is a set $A \subseteq [0,1]$ which does not belong to T.

(i) Let $X = Y = [0,1]$. Give Y the usual topology, and give X the topology corresponding to the one-point compactification of the discrete topology on $[0,1[$, that is, a set $G \subseteq X$ is open if either $1 \notin G$ or G is cofinite. Show that the identity map $f : X \to Y$ is a Borel measurable bijection, but that f^{-1} is not measurable for the σ-algebra of subsets of Y generated by the Souslin-F sets.

423Y Further exercises (a) Show that a space with a countable network is hereditarily separable, therefore countably tight.

(b) Show that if X is a Hausdorff space with a countable network, then every analytic subset of X is obtainable by Souslin's operation from the open subsets of X.

(c) Let $(X,\mathfrak{T})$ and $(Y,\mathfrak{S})$ be analytic Hausdorff spaces and $f:X\to Y$ a Borel measurable function. (i) Show that there is a zero-dimensional separable metrizable topology $\mathfrak{S}'$ on Y with the same Borel sets and the same analytic sets as $\mathfrak{S}$. (*Hint*: 423Xd.) (ii) Show that there is a zero-dimensional separable metrizable topology $\mathfrak{T}'$ on X, with the same Borel sets and the same analytic sets as $\mathfrak{T}$, such that f is continuous for the topologies $\mathfrak{T}'$ and $\mathfrak{S}'$. (iii) Explain how to elaborate these ideas to deal with any countable family of analytic spaces and Borel measurable functions between them.

(d) Let X be an analytic Hausdorff space, Y a Hausdorff space with a countable network, and $f:X\to Y$ a Borel measurable surjection. Let T be the σ-algebra of subsets of Y generated by the Souslin-F sets in Y. Show that there is a T-measurable function $g:Y\to X$ such that fg is the identity on Y.

(e) Set $S^*=\bigcup_{n\ge 1}\mathbb{N}^n$ and consider $\mathcal{P}(S^*)$ with its usual topology. Let $\mathcal{T}\subseteq\mathcal{P}S^*$ be the set of trees (421N); show that $\mathcal{T}$ is closed, therefore a compact metrizable space. Set $F_\sigma=\{T:\sigma\in T\in\mathcal{T}\}$ for $\sigma\in S^*$, and let A be the kernel of the Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S^*}$. Show that the constituents of $\mathcal{T}\setminus A$ for this scheme are just the sets $G_\xi=\{T:T\in\mathcal{T}\setminus A,\ r(T)=\xi\}$, where r is the rank function of 421Ne. Show by induction on ξ that all the G_ξ are non-empty, so that A is not a Borel set. Show that $\#(G_\xi)=\mathfrak{c}$ for $1\le\xi<\omega_1$. Show that if X is any topological space and $B\subseteq X$ is a Souslin-F set, there is a Borel measurable function $f:X\to\mathcal{T}$ such that $B=f^{-1}[A]$.

423 Notes and comments We have been dealing, in this section and the last, with three classes of topological space: the class of analytic spaces, the class of K-analytic spaces and the class of spaces with countable networks. The first is more important than the other two put together, and I am sure many people would find it more comfortable, if more time-consuming, to learn the theory of analytic spaces thoroughly first, before proceeding to the others. This was indeed my own route into the subject. But I think that the theory of K-analytic spaces has now matured to the point that it can stand on its own, without constant reference to its origin as an extension of descriptive set theory on the real line; and that our understanding of analytic spaces is usefully advanced by seeing how easily their properties can be deduced from the fact that they are K-analytic spaces with countable networks.

As in §422, I have made no attempt to cover the general theory of analytic spaces, nor even to give a balanced introduction. I have tried instead to give a condensed account of the principal methods for showing that spaces are analytic, with some of the ideas which can be applied to make them more accessible to the imagination (423K, 423Xc-423Xd, 423Yb-423Yc). Lusin's theorem 423I does not mention 'analytic' sets in its statement, but it depends essentially on the separation theorem 422J, so cannot really be put with the other results on Polish spaces in 4A2Q. You must of course know that not all analytic sets are Borel (423M) and that not all sets are analytic (423Xh). For further information about this fascinating subject, see Rogers 80, Kechris 95 and Moschovakis 80.

'Selection theorems' appear everywhere in mathematics. The axiom of choice is a selection theorem; it says that whenever $R\subseteq X\times Y$ is a relation and $R[X]=Y$, there is a function $f:Y\to X$ such that $(f(y),y)\in R$ for every $y\in Y$. The Lifting Theorem (§341) asks for a selector which is a Boolean homomorphism. In general topology we look for continuous selectors. In measure theory, naturally, we are interested in measurable selectors, as in 423N-423Q. Any selection theorem will be expressible either as a theorem on right inverses of functions, as in 423Q, or as a theorem on selectors for relations, as in 423N-423P. In the language here, however, we get better theorems by examining relations, because the essence of the method is that we can find measurable functions into analytic spaces, and the relations of 423N can be very far from being analytic, even when there is a natural topology on the space Y. The value of these results will become clearer in §433, when we shall see that the σ-algebras T of 423N-423Q are often included in familiar σ-algebras. Typical applications are in 433F-433G below.

Version of 21.3.08

424 Standard Borel spaces

This volume is concerned with topological measure spaces, and it will come as no surprise that the topological properties of Polish spaces are central to the theory. But even from the point of view of unadorned measure theory, not looking for topological structures on the underlying spaces, it turns out that the Borel algebras of Polish spaces have a very special position. It will be useful later on to be able to refer to some fundamental facts concerning them.

424A Definition Let X be a set and Σ a σ-algebra of subsets of X. We say that (X,Σ) is a **standard Borel space** if there is a Polish topology on X for which Σ is the algebra of Borel sets.

Warning! Many authors reserve the phrase 'standard Borel space' for the case in which X is uncountable. I have seen the phrase 'Borel space' used for what I call a 'standard Borel space'.

424B Proposition (a) If (X,Σ) is a standard Borel space, then Σ is countably generated as σ-algebra of sets.

(b) If $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ is a countable family of standard Borel spaces, then $(\prod_{i\in I}X_i,\widehat{\bigotimes}_{i\in I}\Sigma_i)$ (definition: 254E) is a standard Borel space.

(c) Let (X,Σ) and (Y,T) be standard Borel spaces and $f:X\to Y$ a (Σ,T)-measurable surjection. Then

(i) if $E\in\Sigma$ is such that $f[E]\cap f[X\setminus E]=\emptyset$, then $f[E]\in\mathrm{T}$;

(ii) $\mathrm{T}=\{F:F\subseteq Y,\ f^{-1}[F]\in\Sigma\}$;

(iii) if f is a bijection it is an isomorphism.

(d) Let (X,Σ) and (Y,T) be standard Borel spaces and $f:X\to Y$ a (Σ,T)-measurable injection. Then $Z=f[X]\in\mathrm{T}$ and f is an isomorphism between (X,Σ) and (Z,T_Z), where T_Z is the subspace σ-algebra.

proof (a) Let $\mathfrak{T}$ be a Polish topology on X such that Σ is the algebra of Borel sets. Then $\mathfrak{T}$ has a countable base $\mathcal{U}$, which generates Σ (4A3Da/4A3E).

(b) For each $i\in I$ let $\mathfrak{T}_i$ be a Polish topology on X_i such that Σ_i is the algebra of $\mathfrak{T}_i$-Borel sets. Then $X=\prod_{i\in I}\mathfrak{T}_i$, with the product topology $\mathfrak{T}$, is Polish (4A2Qc). By 4A3Dc/4A3E, $\Sigma=\widehat{\bigotimes}_{i\in I}\Sigma_i$ is just the Borel σ-algebra of X, so (X,Σ) is a standard Borel space.

(c) Let $\mathfrak{T}$, $\mathfrak{S}$ be Polish topologies on X, Y respectively for which Σ and T are the Borel σ-algebras. Then f is Borel measurable.

(i) By 423Eb and 423Gb, $f[E]$ and $f[X\setminus E]$ are analytic subsets of Y. But they are complementary, so they are Borel sets, by 423Fa.

(ii) $f^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$, just because f is measurable. On the other hand, if $F\subseteq Y$ and $E=f^{-1}[F]\in\Sigma$, then $F=f[E]\in\mathrm{T}$ by (i).

(iii) follows at once.

(d) Give X and Y Polish topologies for which Σ, T are the Borel σ-algebras. By 423Ib, $f[E]\in\mathrm{T}$ for every $E\in\Sigma$; in particular, $Z=f[X]$ belongs to T. Also $f^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}_Z$, so f is an isomorphism between (X,Σ) and (Z,T_Z).

424C Theorem Let (X,Σ) be a standard Borel space.

(a) If X is countable then $\Sigma=\mathcal{P}X$.

(b) If X is uncountable then (X,Σ) is isomorphic to $(\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$, where $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$ is the algebra of Borel subsets of $\mathbb{N}^{\mathbb{N}}$.

proof Let $\mathfrak{T}$ be a Polish topology on X such that Σ is its Borel σ-algebra.

(a) Every singleton subset of X is closed, so must belong to Σ. If X is countable, every subset of X is a countable union of singletons, so belongs to Σ.

(b) (Rao & Srivastava 94) The strategy of the proof is to find Borel sets $Z\subseteq X$, $W\subseteq\mathbb{N}^{\mathbb{N}}$ such that $(Z,\Sigma_Z)\cong(\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and $(W,\mathcal{B}(W))\cong(X,\Sigma)$ (writing Σ_Z, $\mathcal{B}(W)$ for the subspace σ-algebras), and use a form of the Schröder-Bernstein theorem.

(i) By 423K, X has a subset Z homeomorphic to $\mathbb{N}^{\mathbb{N}}$; let $h:\mathbb{N}^{\mathbb{N}}\to Z$ be a homeomorphism. By 424Bd, h is an isomorphism between $(\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and (Z,Σ_Z).

(ii) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of X. Define $g : X \to \{0,1\}^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ by setting $g(x) = \langle \chi U_n(x)\rangle_{n\in\mathbb{N}}$ for every $x \in \mathbb{N}$. Then g is injective, because X is Hausdorff. Also g is Borel measurable, by 4A3D(c-ii). By 424Bd, g is an isomorphism between (X,Σ) and $(W,\mathcal{B}(W))$, where $W = g[X]$ belongs to $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$.

(iii) We have $Z \in \Sigma$, $W \in \mathcal{B}(\mathbb{N}^{\mathbb{N}})$ such that $(Z,\Sigma_Z) \cong (\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and $(W,\mathcal{B}(W)) \cong (X,\Sigma)$. By 344D, $(X,\Sigma) \cong (\mathbb{N}^{\mathbb{N}},\mathcal{B}(\mathbb{N}^{\mathbb{N}}))$, as claimed.

424D Corollary (a) If (X,Σ) and (Y,T) are standard Borel spaces and $\#(X) = \#(Y)$, then (X,Σ) and (Y,T) are isomorphic.

(b) If (X,Σ) is an uncountable standard Borel space then $\#(X) = \#(\Sigma) = \mathfrak{c}$.

proof These follow immediately from 424C, if we recall that $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) = \mathfrak{c}$ (4A3Fb).

424E Proposition Let X be a set and Σ a σ-algebra of subsets of X; suppose that (X,Σ) is countably separated in the sense that there is a countable set $\mathcal{E} \subseteq \Sigma$ separating the points of X. If $A \subseteq X$ is such that (A,Σ_A) is a standard Borel space, where Σ_A is the subspace σ-algebra, then $A \in \Sigma$.

proof Give A a Polish topology $\mathfrak{T}$ such that Σ_A is the Borel σ-algebra of A, and let $\mathfrak{S}$ be the topology on X generated by $\mathcal{E} \cup \{X \setminus E : E \in \mathcal{E}\}$. Then $\mathfrak{S}$ is second-countable (4A2Oa), so has a countable network (4A2Oc), and is Hausdorff because $\mathcal{E}$ separates the points of X. The identity map from A to X is Borel measurable for $\mathfrak{T}$ and $\mathfrak{S}$, so 423Ib tells us that A is $\mathfrak{S}$-Borel; but of course the $\mathfrak{S}$-Borel σ-algebra is just the σ-algebra generated by $\mathcal{E}$ (4A3Da), so is included in Σ.

424F Corollary Let X be a Polish space and $A \subseteq X$ any set which is not Borel. Let $\mathcal{B}(A)$ be the Borel σ-algebra of A. Then $(A,\mathcal{B}(A))$ is not a standard Borel space.

424G Proposition Let (X,Σ) be a standard Borel space. Then (E,Σ_E) is a standard Borel space for every $E \in \Sigma$, writing Σ_E for the subspace σ-algebra.

proof Let $\mathfrak{T}$ be a Polish topology on X for which Σ is the Borel σ-algebra. Then there is a Polish topology $\mathfrak{T}' \supseteq \mathfrak{T}$ for which E is closed (4A3I), therefore itself Polish in the subspace topology $\mathfrak{T}'_E$ (4A2Qd). But $\mathfrak{T}'$ and $\mathfrak{T}$ have the same Borel sets (423Fb), so Σ_E is just the Borel σ-algebra of E for $\mathfrak{T}'_E$, and (E,Σ_E) is a standard Borel space.

***424H** For the full strength of a theorem in §448 we need a remarkable result concerning group actions on Polish spaces.

Theorem (Becker & Kechris 96) Let G be a Polish group, $(X,\mathfrak{T})$ a Polish space and $\bullet$ a Borel measurable action of G on X. Then there is a Polish topology $\mathfrak{T}'$ on X, yielding the same Borel sets as $\mathfrak{T}$, such that the action is continuous for $\mathfrak{T}'$ and the given topology of G.

proof (a) Fix on a right-translation-invariant metric ρ on G defining the topology of G (4A5Q), and let D be a countable dense subset of G; write e for the identity of G. Let Z be the set of 1-Lipschitz functions from G to $[0,1]$, that is, functions $f : G \to [0,1]$ such that $|f(g) - f(h)| \le \rho(g,h)$ for all g, $h \in G$. Then Z, with the topology of pointwise convergence inherited from the product topology of $[0,1]^G$, is a compact metrizable space. **P** It is a closed subset of $[0,1]^G$, so is a compact Hausdorff space. Writing $q(f) = f\restriction D$ for $f \in Z$, $q : Z \to [0,1]^D$ is injective, because D is dense and every member of Z is continuous; but this means that Z is homeomorphic to $q[Z]$, which is metrizable, by 4A2Pc. **Q**

We see also that the Borel σ-algebra of Z is the σ-algebra generated by sets of the form $W_{g\alpha} = \{f : f(g) < \alpha\}$ where $g \in G$ and $\alpha \in [0,1]$. **P** This σ-algebra contains every set of the form $\{f : f \in Z,\ \alpha < f(g) < \beta\}$, where $g \in G$ and α, $\beta \in \mathbb{R}$; since these sets generate the topology of Z, the σ-algebra they generate is the Borel σ-algebra of Z, by 4A3Da. **Q**

(b) There is a continuous action of G on Z defined by setting

$$(g \bullet_r f)(h) = f(hg)$$

for $f \in Z$ and $g, h \in G$. **P** (i) If $f \in Z$ and $g, h_1, h_2 \in G$, then

$$|(g\bullet_r f)(h_1) - (g\bullet_r f)(h_2)| = |f(h_1 g) - f(h_2 g)| \le \rho(h_1 g, h_2 g) = \rho(h_1, h_2)$$

because ρ is right-translation-invariant. So $g\bullet_r f \in Z$ for every $f \in Z$, $g \in G$. (ii) As in 4A5Cc-4A5Cd, $\bullet_r$ is an action of G on Z. (iii) Suppose that $g_0 \in G$, $f_0 \in Z$, $h \in G$ and $\epsilon > 0$. Set

$$V = \{g : g \in G,\, \rho(hg, hg_0) < \tfrac{1}{2}\epsilon\},$$

$$W = \{f : f \in Z,\, |f(hg_0) - f_0(hg_0)| < \tfrac{1}{2}\epsilon\}.$$

Then V is an open set in G containing g_0 (because $g \mapsto \rho(hg, hg_0)$ is continuous) and W is an open set in Z containing f_0. If $g \in V$ and $f \in W$,

$$\begin{aligned}|(g\bullet_r f)(h) - (g_0\bullet_r f_0)(h)| &= |f(hg) - f_0(hg_0)| \\ &\le |f(hg) - f(hg_0)| + |f(hg_0) - f_0(hg_0)| \\ &\le \rho(hg, hg_0) + \frac{1}{2}\epsilon \le \epsilon.\end{aligned}$$

As f_0, g_0, ϵ are arbitrary, the map $(g, f) \mapsto (g\bullet_r f)(h)$ is continuous; as h is arbitrary, the map $(g, f) \mapsto g\bullet_r f$ is continuous. **Q**

(c) Let $\mathcal{B}(X)$ be the Borel σ-algebra of X. For $x \in X$ and $B \in \mathcal{B}(X)$, set

$$P_B(x) = \{g : g \in G,\, g\bullet x \in B\},$$

$$Q_B(x) = \bigcup\{V : V \subseteq G \text{ is open, } V \setminus P_B(x) \text{ is meager}\},$$

$$f_B(x)(g) = \inf(\{1\} \cup \{\rho(g, h) : h \in G \setminus Q_B(x)\})$$

for $g \in G$. It is easy to check that

$$f_B(x)(g') \le \rho(g, g') + f_B(x)(g')$$

for all $g, g' \in G$, so that every $f_B(x)$ belongs to Z.

Every $P_B(x)$ is a Borel set, because $\bullet$ is Borel measurable, so has the Baire property in X (4A3S(b-i)). Because X is a Baire space (4A2Ma), $Q_B(x) \subseteq \overline{P_B(x)}$ (4A3Sa).

Let $\mathcal{V}$ be a countable base for the topology of G containing G.

(d) For each $B \in \mathcal{B}(X)$, the map $f_B : X \to Z$ is Borel measurable. **P** Because the Borel σ-algebra of Z is generated by the sets $W_{g\alpha}$ of (a) above, it is enough to show that

$$\{x : f_B(x) \in W_{g\alpha}\} = \{x : f_B(x)(g) < \alpha\}$$

always belongs to $\mathcal{B}(X)$, because $\{W : f_B^{-1}[W] \in \mathcal{B}\}$ is surely a σ-algebra of subsets of Z. But if $\alpha > 1$ this set is X, while if $\alpha \le 1$ it is

$$\begin{aligned}\{x :\ &\text{there is some } h \in X \setminus Q_B(x) \text{ such that } \rho(g, h) < \alpha\} \\ &= \{x : \text{ there is some } V \in \mathcal{V} \text{ such that } V \setminus Q_B(x) \ne \emptyset \\ &\qquad\qquad \text{and } \rho(g, h) < \alpha \text{ for every } h \in V\} \\ &= \bigcup_{V \in \mathcal{V}'} \{x : V \not\subseteq Q_B(x)\}\end{aligned}$$

(where $\mathcal{V}' = \{V : V \in \mathcal{V},\, \rho(g, h) < \alpha \text{ for every } h \in V\}$)

$$= \bigcup_{V \in \mathcal{V}'} \{x : V \setminus P_B(x) \text{ is not meager}\}.$$

But for any fixed $V \in \mathcal{V}$,

$$\{x : V \setminus P_B(x) \text{ is not meager}\} = X \setminus \{x : W[\{x\}] \text{ is meager}\}$$

where

$$W = \{(y,g) : y \in X,\, g \in V,\, g\bullet y \in X \setminus B\}$$

is a Borel subset of $X \times G$, because $\bullet$ is supposed to be Borel measurable; and therefore $W \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(G)$, writing $\mathcal{B}(G)$ for the Borel σ-algebra of G (4A3Ga). Now $\mathcal{B}(G) \subseteq \widehat{\mathcal{B}}(G)$, the Baire-property algebra of G (4A3S(b-i) again), so $W \in \mathcal{B}(X)\widehat{\otimes}\widehat{\mathcal{B}}(G)$. By 4A3Sc, the quotient algebra $\widehat{\mathcal{B}}/\mathcal{M}$ has a countable order-dense set, where $\mathcal{M}$ is the σ-ideal of meager sets, so 4A3Ta tells us that $\{x : W[\{x\}]$ is meager$\}$ is a Borel subset of X. Accordingly $\{x : V \setminus P_B(x)$ is not meager$\}$ is Borel for every V, and $\{x : f_B(x)(g) < \alpha\}$ is Borel. **Q**

(e) If $g \in G$, $B \in \mathcal{B}(X)$ and $x \in X$, then $g\bullet_r f_B(x) = f_B(g\bullet x)$. **P**

$$P_B(g\bullet x) = \{h : h\bullet(g\bullet x) \in B\} = \{h : hg \in P_B(x)\} = P_B(x)g^{-1}.$$

Because the map $h \mapsto hg^{-1} : G \to G$ is a homeomorphism, $Q_B(g\bullet x) = Q_B(x)g^{-1}$; because it is an isometry,

$$\begin{aligned} f_B(g\bullet x)(h) &= \min(1, \rho(h, X \setminus Q_B(g\bullet x))) = \min(1, \rho(h, X \setminus Q_B(x)g^{-1})) \\ &= \min(1, \rho(hg, X \setminus Q_B(x))) = f_B(x)(hg) = (g\bullet_r f_B(x))(h) \end{aligned}$$

for every $h \in G$. **Q**

(f) Let $\langle B_{0m}\rangle_{m\in\mathbb{N}}$ run over a base for $\mathfrak{T}$ containing X. We can now find a countable set $\mathcal{E} \subseteq \mathcal{B}(X)$ such that (i) the topology $\mathfrak{T}^*$ generated by $\mathcal{E}$ is a Polish topology finer than $\mathfrak{T}$ (ii) f_E is $\mathfrak{T}^*$-continuous for every $E \in \mathcal{E}$ (iii) $X \in \mathcal{E}$. **P** Let $\mathcal{W}$ be a countable base for the topology of Z. Enumerate $\mathbb{N} \times \mathbb{N} \times \mathcal{W}$ as $\langle(k_n, m_n, W_n)\rangle_{n\in\mathbb{N}}$ in such a way that $k_n \le n$ for every n. Having chosen Borel sets $B_{ij} \subseteq X$ for $i \le n$, $j \in \mathbb{N}$ in such a way that the topology $\mathfrak{S}_n$ generated by $\{B_{ij} : i \le n,\, j \in \mathbb{N}\}$ is a Polish topology finer than $\mathfrak{T}$, consider the set

$$C_n = \{x : f_{B_{k_n,m_n}}(x) \in W_n\}.$$

This is $\mathfrak{T}$-Borel, therefore $\mathfrak{S}_n$-Borel, so by 4A3H there is a Polish topology $\mathfrak{S}_{n+1} \supseteq \mathfrak{S}_n$ such that $C_n \in \mathfrak{S}_{n+1}$. Let $\langle B_{n+1,m}\rangle_{m\in\mathbb{N}}$ run over a base for $\mathfrak{S}_{n+1}$; by 423Fb, every $B_{n+1,m}$ belongs to $\mathcal{B}(X)$. Continue.

Let $\mathfrak{T}^*$ be the topology generated by $\mathcal{E} = \{B_{ij} : i,\, j \in \mathbb{N}\}$. By 4A2Qf, $\mathfrak{T}^*$ is Polish, because it is the topology generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{S}_n$. If $W \in \mathcal{W}$ and $E \in \mathcal{E}$ there are i, $j \in \mathbb{N}$ such that $B = B_{ij}$ and an $n \in \mathbb{N}$ such that $(i,j,W) = (k_n, m_n, W_n)$; now

$$f_E^{-1}[W] = C_n \in \mathfrak{S}_{n+1} \subseteq \mathfrak{T}^*.$$

As W is arbitrary, f_E is $\mathfrak{T}^*$-continuous. Also

$$X \in \{B_{0m} : m \in \mathbb{N}\} \subseteq \mathcal{E},$$

as required. **Q**

(g) Define $\theta : X \to Z^{\mathcal{E}}$ by setting $\theta(x)(E) = f_E(x)$ for $x \in X$ and $E \in \mathcal{E}$. Then θ is injective. **P** Suppose that x, $y \in X$ and that $x \ne y$. For every $g \in G$,

$$g^{-1}\bullet(g\bullet x) = x \ne y = g^{-1}\bullet(g\bullet y),$$

so $g\bullet x \ne g\bullet y$ and there is some $m \in \mathbb{N}$ such that $g\bullet x \in B_{0m}$ while $g\bullet y \notin B_{0m}$, that is, $g \in P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$. Thus $G = \bigcup_{m\in\mathbb{N}} P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$; by Baire's theorem, there is some $m \in \mathbb{N}$ such that $P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$ is non-meager. Because $Q_{B_{0m}}(x)\triangle P_{B_{0m}}(x)$ and $Q_{B_{0m}}(y)\triangle P_{B_{0m}}(y)$ are both meager,

$$\{g : f_{B_{0m}}(x)(g) > 0\} = Q_{B_{0m}}(x) \ne Q_{B_{0m}}(y) = \{g : f_{B_{0m}}(y)(g) > 0\},$$

and

$$\theta(x)(B_{0m}) = f_{B_{0m}}(x) \ne f_{B_{0m}}(y) = \theta(y)(B_{0m}).$$

So $\theta(x) \ne \theta(y)$. **Q** Because f_E is $\mathfrak{T}^*$-continuous for every $E \in \mathcal{E}$, θ is $\mathfrak{T}^*$-continuous.

(h) Let $\mathfrak{T}'$ be the topology on X induced by θ; that is, the topology which renders θ a homeomorphism between X and $\theta[X]$. Because $\theta[X] \subseteq Z^{\mathcal{E}}$ is separable and metrizable, $\mathfrak{T}'$ is separable and metrizable. Because θ is $\mathfrak{T}^*$-continuous, $\mathfrak{T}' \subseteq \mathfrak{T}^*$.

(i) The action of G on X is continuous for the given topology $\mathfrak{S}$ on G and $\mathfrak{T}'$ on X. **P** For any $E \in \mathcal{E}$,

$$(g,x) \mapsto \theta(g\bullet x)(E) = f_E(g\bullet x) = g\bullet_r f_E(x)$$

((e) above) is $\mathfrak{S}\times\mathfrak{T}'$-continuous because the action of G on Z is continuous ((b) above) and $f_E:X\to Z$ is $\mathfrak{T}'$-continuous (by the definition of $\mathfrak{T}'$). But this means that $(g,x)\mapsto\theta(g\bullet x)$ is $\mathfrak{S}\times\mathfrak{T}'$-continuous, so that $(g,x)\mapsto g\bullet x$ is $(\mathfrak{S}\times\mathfrak{T}',\mathfrak{T}')$-continuous. **Q**

(j) Let σ be a complete metric on G defining the topology $\mathfrak{S}$, and τ a complete metric on X defining the topology $\mathfrak{T}^*$. (We do not need to relate σ to ρ in any way beyond the fact that they both give rise to the same topology $\mathfrak{S}$.) For $E\in\mathcal{E}$, $V\in\mathcal{V}$ and $n\in\mathbb{N}$ let S_{EVn} be the set of those $\phi\in Z^{\mathcal{E}}$ such that *either* $\phi(E)(g)=0$ for every $g\in V$ *or* there is an $F\in\mathcal{E}$ such that $F\subseteq E$, $\operatorname{diam}_\tau(F)\le 2^{-n}$ and $\phi(F)(g)>0$ for some $g\in V$. Then S_{EVn} is the union of a closed set and an open set, so is a G_δ set in $Z^{\mathcal{E}}$ (4A2C(a-i)). Consequently

$$Y=\{\phi:\phi\in Z^{\mathcal{E}},\,\phi(X)=\chi G\}\cap\overline{\theta[X]}\cap\bigcap_{E\in\mathcal{E},V\in\mathcal{V},n\in\mathbb{N}}S_{EVn}$$

is a G_δ subset of $Z^{\mathcal{E}}$, being the intersection of countably many G_δ sets.

(k) $\theta[X]\subseteq Y$. **P** Let $x\in X$. (i) $P_X(x)=G$ so $Q_X(x)=G$ and $\theta(x)(X)=f_X(x)=\chi G$. (ii) Of course $\theta(x)\in\overline{\theta[X]}$. (iii) Suppose that $E\in\mathcal{E}$, $V\in\mathcal{V}$ and $n\in\mathbb{N}$. If $Q_E(x)\cap V=\emptyset$ then $\theta(x)(E)(g)=f_E(x)(g)=0$ for every $g\in V$, and $\theta(x)(E)\in S_{EVn}$. Otherwise, $V\cap P_E(x)$ is non-meager. But

$$E=\bigcup\{F:F\in\mathcal{E},\,F\subseteq E,\,\operatorname{diam}_\tau(F)\le 2^{-n}\},$$

so

$$P_E(x)=\bigcup\{P_F(x):F\in\mathcal{E},\,F\subseteq E,\,\operatorname{diam}_\tau(F)\le 2^{-n}\};$$

because $\mathcal{E}$ is countable, this is a countable union and there is an $F\in\mathcal{E}$ such that $F\subseteq E$, $\operatorname{diam}_\tau(F)\le 2^{-n}$ and $P_F(x)\cap V$ is non-meager. In this case $Q_F(x)\cap V$ is non-empty and $\theta(x)(F)=f_F(x)$ is non-zero at some point of V; thus again $\theta(x)\in S_{EVn}$. As E, V and n are arbitrary, we have the result. **Q**

(l) (The magic bit.) $Y\subseteq\theta[X]$. **P** Take any $\phi\in Y$. Choose $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ and $\langle\tilde g_n\rangle_{n\in\mathbb{N}}$ in G as follows. $E_0=X$ and $V_0=G$. Given that $\phi(E_n)$ is non-zero at some point of V_n, then, because $\phi\in S_{E_nV_nn}$, there is an $E_{n+1}\in\mathcal{E}$ such that $E_{n+1}\subseteq E_n$, $\operatorname{diam}_\tau(E_{n+1})\le 2^{-n}$ and $\phi(E_{n+1})$ is non-zero at some point of V_n; say $\tilde g_n\in V_n$ is such that $\phi(E_{n+1})(\tilde g_n)>0$. Now we can find a $V_{n+1}\in\mathcal{V}$ such that $\tilde g_n\in V_{n+1}\subseteq V_n$ and $\operatorname{diam}_\sigma(V_{n+1})\le 2^{-n}$. Continue.

We are supposing also that $\phi\in\overline{\theta[X]}$, so we have a sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ in X such that $\langle\theta(x_i)\rangle_{i\in\mathbb{N}}\to\phi$, that is, $\langle f_E(x_i)\rangle_{i\in\mathbb{N}}\to\phi(E)$ for every $E\in\mathcal{E}$. In particular,

$$\lim_{i\to\infty}f_{E_{n+1}}(x_i)(\tilde g_n)=\phi(E_{n+1})(\tilde g_n)>0$$

for every n. Let $\langle i_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $f_{E_{n+1}}(x_{i_n})(\tilde g_n)>0$ for every n. Then

$$\tilde g_n\in V_{n+1}\cap Q_{E_{n+1}}(x_{i_n})\subseteq V_{n+1}\cap\overline{P_{E_{n+1}}(x_{i_n})};$$

there is therefore some $g_n\in V_{n+1}\cap P_{E_{n+1}}(x_{i_n})$, so that $g_n\bullet x_{i_n}\in E_{n+1}$.

$\langle V_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets with σ-diameters converging to 0, so $\langle g_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for the complete metric σ. Similarly, $\langle g_n\bullet x_{i_n}\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for the complete metric τ, because $\operatorname{diam}_\tau(E_{n+1})\le 2^{-n}$. We therefore have $g\in G$, $y\in X$ such that $\langle g_n\rangle_{n\in\mathbb{N}}\to g$ for $\mathfrak{S}$ and $\langle g_n\bullet x_{i_n}\rangle_{n\in\mathbb{N}}\to y$ for $\mathfrak{T}^*$. In this case, $\langle g_n\bullet x_{i_n}\rangle_{n\in\mathbb{N}}\to y$ for the coarser topology $\mathfrak{T}'$, while $\langle g_n^{-1}\rangle_{n\in\mathbb{N}}\to g^{-1}$ for $\mathfrak{S}$. Because the action is $(\mathfrak{S}\times\mathfrak{T}',\mathfrak{T}')$-continuous,

$$\langle x_{i_n}\rangle_{n\in\mathbb{N}}=\langle g_n^{-1}\bullet(g_n\bullet x_{i_n})\rangle_{n\in\mathbb{N}}\to g^{-1}\bullet y$$

for $\mathfrak{T}'$. But θ is continuous for $\mathfrak{T}'$, by the definition of $\mathfrak{T}'$, so

$$\theta(g^{-1}\bullet y)=\lim_{n\to\infty}\theta(x_{i_n})=\phi,$$

and $\phi\in\theta[X]$. As ϕ is arbitrary, $Y\subseteq\theta[X]$. **Q**

(m) Thus $\theta[X]=Y$ is a G_δ set in the compact metric space $Z^{\mathcal{E}}$, and is a Polish space in its induced topology (4A2Qd). But this means that $(X,\mathfrak{T}')$, which is homeomorphic to $\theta[X]$, is also Polish.

(n) I have still to check that $\mathfrak{T}'$ has the same Borel sets as $\mathfrak{T}$. But $\mathfrak{T}$, $\mathfrak{T}^*$ and $\mathfrak{T}'$ are all Polish topologies and $\mathfrak{T}^*$ is finer than both the other two. By 423Fb, $\mathfrak{T}^*$ has the same Borel sets as either of the others.

This completes the proof.

424X Basic exercises >(a) Let $\langle(X_i,\Sigma_i)\rangle_{i\in I}$ be a countable family of standard Borel spaces, and (X,Σ) their direct sum, that is, $X=\{(x,i):i\in I,\,x\in X_i\}$, $\Sigma=\{E:E\subseteq X,\,\{x:(x,i)\in E\}\in\Sigma_i$ for every $i\}$. Show that (X,Σ) is a standard Borel space.

>(b) Let (X,Σ) be a standard Borel space and T a countably generated σ-subalgebra of Σ. Show that there is an analytic Hausdorff space Z such that T is isomorphic to the Borel σ-algebra of Z. (*Hint*: by 4A3I, we can suppose that X is a Polish space and T is generated by a sequence of open-and-closed sets, corresponding to a continuous function from X to $\{0,1\}^{\mathbb{N}}$.)

>(c) Let (X,Σ) be a standard Borel space and T_1, T_2 two countably generated σ-subalgebras of Σ which separate the same points, in the sense that if $x,\,y\in X$ then there is an $E\in\mathrm{T}_1$ such that $x\in E$ and $y\in X\setminus E$ iff there is an $E'\in\mathrm{T}_2$ such that $x\in E'$ and $y\in X\setminus E'$. Show that $\mathrm{T}_1=\mathrm{T}_2$. (*Hint*: 424Xb, 423Fb.) In particular, if T_1 separates the points of X then $\mathrm{T}_1=\Sigma$.

(d) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\mathfrak{A}$ is isomorphic to the Borel σ-algebra of an analytic Hausdorff space iff it is isomorphic to a countably generated σ-subalgebra of the Borel σ-algebra of $[0,1]$.

>(e) Let U be a separable Banach space. Show that its Borel σ-algebra is generated, as σ-algebra, by the sets of the form $\{u:h(u)\le\alpha\}$ as h runs over the dual U^* and α runs over $\mathbb{R}$.

>(f) Let X be a compact metrizable space. Show that the Borel σ-algebra of $C(X)$ (the Banach space of continuous real-valued functions on X) is generated, as σ-algebra, by the sets $\{u:u\in C(X),\,u(x)\ge\alpha\}$ as x runs over X and α runs over $\mathbb{R}$.

(g) Let (X,Σ) be a standard Borel space, Y any set, and T a σ-algebra of subsets of Y. Write T^* for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathrm{T})$, where $\mathcal{S}$ is Souslin's operation. Let $W\in\mathcal{S}(\Sigma\widehat{\otimes}\mathrm{T})$. Show that $W[X]\in\mathcal{S}(\mathrm{T})$ and that there is a (T^*,Σ)-measurable function $f:W[X]\to X$ such that $(f(y),y)\in W$ for every $y\in W[X]$. (*Hint*: 423O.)

(h) Let (X,Σ) be a standard Borel space, Y any set, and T a countably generated σ-algebra of subsets of Y. Let $f:X\to Y$ be a (Σ,T)-measurable function, and write T^* for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathrm{T})$, where $\mathcal{S}$ is Souslin's operation. Show that there is a (T^*,Σ)-measurable function $g:f[X]\to X$ such that gf is the identity on X. (*Hint*: start with the case in which T separates the points of Y, so that the graph of f belongs to $\Sigma\widehat{\otimes}\mathrm{T}$.)

>(i) Show that 424Xc and 424Xh are both false if we omit the phrase 'countably generated' from the hypotheses. (*Hint*: consider (i) the countable-cocountable algebra of $\mathbb{R}$ (ii) the split interval.)

(j) Let (X,Σ,μ) be a σ-finite measure space in which Σ is countably generated. Let $\mathcal{A}$ be the set of atoms A of the Boolean algebra Σ such that $\mu A>0$, and set $H=X\setminus\bigcup\mathcal{A}$. Show that the subspace measure on H is atomless.

424Y Further exercises (a) Let $(X,\mathfrak{T})$ be a Polish space and $\mathcal{C}$ the family of closed subsets of X. Let Σ be the σ-algebra of subsets of $\mathcal{C}$ generated by the sets $\mathcal{E}_H=\{F:F\in\mathcal{C},\,F\cap H\ne\emptyset\}$ as H runs over the open subsets of X. (i) Show that $(\mathcal{C},\Sigma)$ is a standard Borel space. (*Hint*: take a complete metric ρ defining the topology of X. Set $S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$ and choose a family $\langle U_\sigma\rangle_{\sigma\in S}$ of open sets in X such that $U_\emptyset=X$, $\operatorname{diam}U_\sigma\le 2^{-n}$ whenever $\#(\sigma)=n+1$, $U_\sigma=\bigcup_{i\in\mathbb{N}}U_{\sigma^\frown<i>}$ for every σ, $\overline{U}_{\sigma^\frown<i>}\subseteq U_\sigma$ for every σ, i. Define $f:\mathcal{C}\to\{0,1\}^S$ by setting $f(F)(\sigma)=1$ if $F\cap U_\sigma\ne\emptyset$, 0 otherwise. Show that $Z=f[\mathcal{C}]$ is a Borel set and that f is an isomorphism between Σ and the Borel σ-algebra of Z.) (This is the **Effros Borel structure** on $\mathcal{C}$.) (ii) Show that $[X]^n\in\Sigma$ for every $n\in\mathbb{N}$. (iii) Show that Σ is the Borel σ-algebra of $\mathcal{C}$ when $\mathcal{C}$ is given its Fell topology.

(b) Let (X,Σ) be a standard Borel space. Let $\mathbb{T}$ be the family of Polish topologies on X for which Σ is the Borel σ-algebra. Show that any sequence in $\mathbb{T}$ has an upper bound in $\mathbb{T}$, and that any sequence with a lower bound has a least upper bound.

(c) Let (X,Σ) be a standard Borel space. Say that $C\subseteq X$ is **coanalytic** if its complement belongs to $\mathcal{S}(\Sigma)$. Show that for any such C the partially ordered set $\Sigma\cap\mathcal{P}C$ has cofinality 1 if $C\in\Sigma$ and cofinality ω_1 otherwise. (*Hint*: 423R.)

(d) Let $I^{\parallel}$ be the split interval. Show that there is a σ-algebra Σ of subsets of $I^{\parallel}$ such that $(I^{\parallel},\Sigma)$ is a standard Borel space and $\{(x,y):x,\,y\in I^{\parallel},\,x\le y\}\in\Sigma\widehat{\otimes}\Sigma$.

(e) Let (X,Σ) be a standard Borel space. Show that if X is uncountable, Σ has a countably generated σ-subalgebra not isomorphic either to Σ or to $\mathcal{P}I$ for any set I.

(f) Let (X,Σ,μ) be a σ-finite countably separated perfect measure space (definition: 342K/451Ad). Show that there is a standard Borel space (Y,T) such that $Y\in\Sigma$, $\mathrm{T}\subseteq\Sigma$ and μ is inner regular with respect to T.

424 Notes and comments In this treatise I have generally indulged my prejudice in favour of 'complete' measures. Consequently Borel σ-algebras, as such, have taken subordinate roles. But important parts of the theory of Lebesgue measure, and Radon measures on Polish spaces in general, are associated with the fact that these are completions of measures defined on standard Borel spaces. Moreover, such spaces provide a suitable framework for a large part of probability theory. Of course they become deficient in contexts where we need to look at uncountable independent families of random variables, and there are also difficulties with σ-subalgebras, even countably generated ones, since these can correspond to the Borel algebras of general analytic spaces, which will not always be standard Borel structures (424F, 423M). 424Xf and 424Ya suggest the ubiquity of standard Borel structures; the former shows that they are not always presented as countably generated algebras, while the latter is an example in which we have to make a special construction in order to associate a topology with the algebra. The theory is of course dominated by the results of §423, especially 423Fb and 423I.

I include 424H in this section because there is no other convenient place for it, but I have an excuse: the idea of 'Borel measurable action' can, in this context, be described entirely in terms of σ-algebras, since the Borel algebra of $G\times X$ is just the σ-algebra product of the Borel algebras of the factors (as in 424Bb). Of course for the theorem as expressed here we do need to know that G has a Polish group structure; but X could be presented just as a standard Borel space. The result is a dramatic expression of the fact that, given a standard Borel space (X,Σ), we have a great deal of freedom in defining a corresponding Polish topology on X.

Version of 9.8.13

*425 Realization of automorphisms

In §344 I presented some results on the representation of a countable semigroup of Boolean homomorphisms in a measure algebra by a semigroup of functions on the measure space underlying the algebra. §424 provides us with the tools needed for a remarkable extension, in the case of the Lebesgue measure algebra, to groups with cardinal ω_1 (Theorem 425D). The expression of the ideas is made smoother by using the language of group actions (4A5B-4A5C).

425A I begin with what amounts to a special case of the main theorem, with some refinements which will be useful elsewhere.

Proposition (a) Let (X,Σ) and (Y,T) be non-empty standard Borel spaces, and $\mathcal{I}$, $\mathcal{J}$ σ-subalgebras of Σ, T respectively; write $\mathfrak{A}=\Sigma/\mathcal{I}$ and $\mathfrak{B}=\mathrm{T}/\mathcal{J}$ for the quotient algebras. For $E\in\Sigma$, $F\in\mathrm{T}$ write Σ_E, T_F for the subspace σ-algebras on E, F respectively.

(a) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, there is a (T,Σ)-measurable $f:Y\to X$ which represents π in the sense that $\pi E^\bullet=f^{-1}[E]^\bullet$ for every $E\in\Sigma$.

(b) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a Boolean isomorphism, there are $G\in\mathcal{I}$, $H\in\mathcal{J}$ and a bijection $h:Y\setminus H\to X\setminus G$ which is an isomorphism between $(Y\setminus H,\mathrm{T}_{Y\setminus H})$ and $(X\setminus G,\Sigma_{X\setminus G})$, and represents π in the sense that $\pi E^\bullet=h^{-1}[E\setminus G]^\bullet$ for every $E\in\Sigma$.

(c) If $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean automorphism, there is a bijection $h:X\to X$ which is an automorphism of (X,Σ) and represents π in the sense of (a).

(d) If $\#(X)=\#(Y)=\mathfrak{c}$, $\mathfrak{A}$ and $\mathfrak{B}$ are ccc, and $\pi:\mathfrak{A}\to\mathfrak{B}$ is a Boolean isomorphism, there is a bijection $h:Y\to X$ which is an isomorphism between (Y,T) and (X,Σ), and represents π in the sense of (a).

proof (a)(i) If either $\mathfrak{A}$ or $\mathfrak{B}$ is $\{0\}$, so is the other, and we can take f to be a constant function. So henceforth let us suppose that $X\notin\mathcal{I}$ and $Y\notin\mathcal{J}$.

(ii) If $X\subseteq\mathbb{N}$ and $\Sigma=\mathcal{P}X$, set $Z=\{n:n\in X,\ \{n\}\notin\mathcal{I}\}$. For $n\in Z$, set $a_n=\{n\}^\bullet$ and $b_n=\pi a_n$, and choose $F_n\in\mathrm{T}$ such that $F_n^\bullet=b_n$. Since $\sup_{n\in Z}a_n=1$ in $\mathfrak{A}$, $\sup_{n\in Z}b_n=1$ in $\mathfrak{B}$ and $Y\setminus\bigcup_{n\in Z}F_n\in\mathcal{J}$. Define $f:Y\to X$ by saying that

$$\begin{aligned}f(y)&=\min\{n:n\in Z,\ y\in F_n\}\text{ if }y\in\bigcup_{n\in Z}F_n,\\&=\min\ \text{otherwise}.\end{aligned}$$

Then f represents π in the required sense.

(iii) If $X=\{0,1\}^{\mathbb{N}}$ and Σ is its Borel σ-algebra, then for each $n\in\mathbb{N}$ set $E_n=\{x:x(n)=1\}$ and $e_n=E_n^\bullet$ and choose $F_n\in\mathrm{T}$ such that $F_n^\bullet=\pi e_n$. Set $f(y)=\langle\chi F_n(y)\rangle_{n\in\mathbb{N}}$ for $y\in Y$. Then $f^{-1}[E_n]^\bullet=\pi E_n^\bullet$ for every n; as $\{E:f^{-1}[E]^\bullet=\pi E^\bullet\}$ is a σ-subalgebra of Σ containing every E_n, it is the whole of Σ, and again f represents π.

(iv) By 424C, any standard Borel space is isomorphic to either that in (iii) or one of those in (ii), so these cases together are sufficient to prove the general result.

(b) By (a), we have $f:Y\to X$ and $g:X\to Y$ representing π, π^{-1} respectively. Now we see that $gf:Y\to Y$ represents $\pi\pi^{-1}:\mathfrak{B}\to\mathfrak{B}$, that is, $F\triangle(gf)^{-1}[F]\in\mathcal{J}$ for every $F\in\mathrm{T}$. Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in T which separates the points of Y, and set $H_0=\bigcup_{n\in\mathbb{N}}F_n\triangle(gf)^{-1}[F_n]$; then $H_0\in\mathcal{J}$ and $g(f(y))=y$ for every $y\in Y\setminus H_0$.

Similarly, there is a $G_0\in\mathcal{I}$ such that $f(g(x))=x$ for every $x\in X\setminus G_0$. Set $G=G_0\cup g^{-1}[H_0]$ and $H=H_0\cup f^{-1}[G_0]$. Then $g[X\setminus G]\subseteq Y\setminus H$. **P** If $x\in X\setminus G$ then $g(x)\notin H_0$; moreover, $f(g(x))=x\notin G_0$ so $g(x)\notin f^{-1}[G_0]$ and $g(x)\notin H$. **Q** Similarly, $f(y)\in X\setminus G$ for every $y\in Y\setminus H$. As $f(g(x))=x$ for $x\in X\setminus G$ and $g(f(y))=y$ for $y\in Y\setminus H$, $h=f\restriction Y\setminus H$ is a bijection with inverse $g\restriction X\setminus G$. Because f is (T,Σ)-measurable, h is $(\mathrm{T}_{Y\setminus H},\Sigma_{X\setminus G})$-measurable; because g is (Σ,T)-measurable, h^{-1} is $(\Sigma_{X\setminus G},\mathrm{T}_{Y\setminus H})$-measurable, and h is an isomorphism between $(Y\setminus H,\mathrm{T}_{Y\setminus H})$ and $(X\setminus G,\Sigma_{X\setminus G})$. Finally, if $E\in\Sigma$,

$$\pi(E^\bullet) = \pi((E \setminus G)^\bullet) = (f^{-1}[E \setminus G])^\bullet = (h^{-1}[E \setminus G])^\bullet,$$

so h represents π in the sense declared.

(c) We can repeat the proof of (b) with an additional idea. The point is that there is an element E_0 of $\mathcal{I}$ with maximal cardinality. **P** Because (G, Σ_G) is a standard Borel space (424G), $\#(G)$ is either $\mathfrak{c}$ or countable (424Db), for every $G \in \mathcal{I}$. If $\mathcal{I}$ contains arbitrarily large finite sets, it must contain an infinite set, because it is a σ-algebra. So the supremum $\sup\{\#(G) : G \in \mathcal{I}\}$ is attained. **Q**

Now, in (b), take $(Y, \mathrm{T}) = (X, \Sigma)$ and $\mathfrak{B} = \mathfrak{A}$, and choose f, H_0, g and G_0 as before; but this time set $G = (G_0 \cup E_0) \cup g^{-1}[H_0 \cup E_0]$ and $H = (H_0 \cup E_0) \cup f^{-1}[G_0 \cup E_0]$. The same arguments as before tell us that $h_0 = f \restriction Y \setminus H$ is an isomorphism between $(Y \setminus H, \mathrm{T}_{Y \setminus H})$ and $(X \setminus G, \Sigma_{X \setminus G})$ representing π. We now, however, have G, $H \in \mathcal{I}$ and both include E_0; so we must have $\#(G) = \#(E_0) = \#(H)$. By 424Da, there is an isomorphism h_1 between (H, Σ_H) and (G, Σ_G). So if we set

$$\begin{aligned} h(y) &= h_0(y) \text{ if } y \in X \setminus H, \\ &= h_1(y) \text{ if } y \in H, \end{aligned}$$

then $h : X \to X$ is a bijection which is an automorphism of (X, Σ), and

$$h^{-1}[E]^\bullet = (H \cap h^{-1}[E])^\bullet = (h^{-1}[E \cap G])^\bullet = (h_0^{-1}[E \cap G])^\bullet = \pi E^\bullet$$

for every $E \in \Sigma$, as required.

(d) An adaptation of the ideas of (c) works in this case too. First note that as $\#(X) = \mathfrak{c}$, $(X, \Sigma) \cong (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and there is a partition of X into $\mathfrak{c}$ members of Σ all with cardinal $\mathfrak{c}$. As $\mathfrak{A}$ is ccc, all but countably many of these must belong to $\mathcal{I}$, and we have an $E_0 \in \mathcal{I}$ with $\#(E_0) = \mathfrak{c}$. Similarly, there is an $F_0 \in \mathcal{J}$ with $\#(F_0) = \mathfrak{c}$.

Now choose f, H_0, g and G_0 as in (b); but this time set $G = (G_0 \cup E_0) \cup g^{-1}[H_0 \cup F_0]$ and $H = (H_0 \cup F_0) \cup f^{-1}[G_0 \cup E_0]$. Again, $h_0 = f \restriction Y \setminus H$ is an isomorphism between $(Y \setminus H, \mathrm{T}_{Y \setminus H})$ and $(X \setminus G, \Sigma_{X \setminus G})$ representing π. As $\#(G) = \mathfrak{c} = \#(H)$, there is an isomorphism h_1 between (H, T_H) and (G, Σ_G). So if we set

$$\begin{aligned} h(y) &= h_0(y) \text{ if } y \in Y \setminus H, \\ &= h_1(y) \text{ if } y \in H, \end{aligned}$$

we get a bijection $h : Y \to X$ which is an isomorphism between (Y, T) and (X, Σ), and represents π, just as in (c).

425B Lemma Let G be a group, G_0 a subgroup of G, H another group, and X, Z sets; let $\bullet_r$ be the right shift action of H on Z^H (4A5C(c-ii)). Suppose we are given a group homomorphism $\theta : G \to H$, an injective function $f : \mathbb{N} \times Z^H \to X$ and an action $\bullet_0$ of G_0 on X such that $\pi \bullet_0 f(n, z) = f(n, \theta(\pi) \bullet_r z)$ whenever $n \in \mathbb{N}$ and $z \in Z^H$.

(a) If $\#(X \setminus f[\mathbb{N} \times Z^H]) \le \#(Z)$, there is an action $\bullet$ of G on X extending $\bullet_0$.

(b) Suppose moreover that H is countable, X and Z are Polish spaces, and f is Borel measurable when $\mathbb{N} \times Z^H$ is given the product topology. If $x \mapsto \pi \bullet_0 x$ is Borel measurable for every $\pi \in G_0$, then $\bullet$ can be chosen in such a way that $x \mapsto \psi \bullet x$ is Borel measurable for every $\psi \in G$.

proof (a)(i) Let $D \subseteq H$ be a selector for the left cosets of the subgroup $\theta[G_0]$, so that every member of H is uniquely expressible as $\psi\theta(\pi)$ where $\psi \in D$ and $\pi \in G_0$. Observe that if $\pi \in G_0$ and $x \in f[\mathbb{N} \times Z^H]$, then $\pi^{-1} \bullet_0 x \in f[\mathbb{N} \times Z^H]$; consequently, setting $Y = X \setminus f[\mathbb{N} \times Z^H]$, $\pi \bullet_0 y \in Y$ for every $y \in Y$, because $\pi^{-1} \bullet_0 (\pi \bullet_0 y) = y$. Let $g_0 : Y \to Z$ be any injection, and define $g : Y \to Z^H$ by setting $g(y)(\psi\theta(\pi)) = g_0(\pi \bullet_0 y)$ whenever $\psi \in D$, $\pi \in G_0$ and $y \in Y$. In this case, $\theta(\pi) \bullet_r g(y) = g(\pi \bullet_0 y)$ whenever $y \in Y$ and $\pi \in G_0$. **P** Take any $\psi \in D$ and $\phi \in G_0$. Then

$$\begin{aligned} (\theta(\pi) \bullet_r g(y))(\psi\theta(\phi)) &= g(y)(\psi\theta(\phi)\theta(\pi)) = g(y)(\psi\theta(\phi\pi)) \\ &= g_0(\phi\pi \bullet_0 y) = g_0(\phi \bullet_0 (\pi \bullet_0 y)) = g(\pi \bullet_0 y)(\psi\theta(\phi)). \end{aligned}$$

As $D\theta[G_0] = H$, this shows that $\theta(\pi) \bullet_r g(y) = g(\pi \bullet_0 y)$. **Q**

Note that as $g(y)(\psi)=g_0(y)$ whenever $\psi\in D$ and $y\in Y$, g also is injective.

(ii) Now define $h:\mathbb{N}\times Z^H\to X$ by setting

$$\begin{aligned} h(n,z)&=g^{-1}(z) \text{ if } n=0 \text{ and } z\in g[Y]\\ &\qquad\qquad \text{matching } \{0\}\times g[Y] \text{ with } Y,\\ &=f(n-1,z) \text{ if } n\ge 1 \text{ and } z\in g[Y]\\ &\qquad\qquad \text{matching } (\mathbb{N}\setminus\{0\})\times g[Y] \text{ with } f[\mathbb{N}\times g[Y]],\\ &=f(n,z) \text{ if } n\in\mathbb{N} \text{ and } z\notin g[Y]\\ &\qquad\qquad \text{matching } \mathbb{N}\times(Z^H\setminus g[Y]) \text{ with } f[\mathbb{N}\times(Z^H\setminus g[Y])]. \end{aligned}$$

Clearly h is a bijection. If $n\in\mathbb{N}$, $z\in Z^H$ and $\pi\in G_0$, then $h(n,\theta(\pi)\bullet_r z)=\pi\bullet_0 h(n,z)$. **P** If $z\in g[Y]$, then

$$\theta(\pi)\bullet_r z=\theta(\pi)\bullet_r g(g^{-1}(z))=g(\pi\bullet_0 g^{-1}(z))\in g[Y],$$

so

$$h(0,\theta(\pi)\bullet_r z)=\pi\bullet_0 g^{-1}(z)=\pi\bullet_0 h(0,z),$$

while if $n\ge 1$, then

$$h(n,\theta(\pi)\bullet_r z)=f(n-1,\theta(\pi)\bullet_r z)=\pi\bullet_0 f(n-1,z)=\pi\bullet_0 h(n,z).$$

On the other hand, if $n\in\mathbb{N}$ and $z\in Z^H\setminus g[Y]$, then $\theta(\pi)\bullet_r z\notin g[Y]$, because $\theta(\pi^{-1})\bullet_r(\theta(\pi)\bullet_r z)\notin g[Y]$; so

$$h(n,\theta(\pi)\bullet_r z)=f(n,\theta(\pi)\bullet_r z)=\pi\bullet_0 f(n,z)=\pi\bullet_0 h(n,z).\ \mathbf{Q}$$

(iii) Try

$$\psi\bullet x=h(n,\theta(\psi)\bullet_r z)$$

whenever $x\in X$, $h^{-1}(x)=(n,z)$ and $\psi\in G$. If $x\in X$, ψ, $\psi'\in G$ and $\pi\in G_0$, express $h^{-1}(x)$ as (n,z); then, writing ι for the identity of G,

$$\begin{aligned} \iota\bullet x&=h(n,\theta(\iota)\bullet_r z)=h(n,z)=x,\\ \psi'\psi\bullet x&=h(n,\theta(\psi'\psi)\bullet_r z)=h(n,\theta(\psi')\bullet_r(\theta(\psi)\bullet_r z))=\psi'\bullet h(n,\theta(\psi)\bullet_r z)=\psi'\bullet(\psi\bullet x),\\ \pi\bullet x&=h(n,\theta(\pi)\bullet_r z)=\pi\bullet_0 h(n,z)=\pi\bullet_0 x, \end{aligned}$$

so we have an action of G on X extending $\bullet_0$, as required.

(b) Under the topological hypotheses, we follow the same line of argument, but taking time for checks at each stage. Because Z, and therefore $\mathbb{N}\times Z^H$, are Polish spaces, and f is a measurable injection, $f[\mathbb{N}\times Z^H]$ and Y are Borel sets (423Ib). In (i), we must of course take g_0 to be Borel measurable; since all the functions $x\mapsto\pi\bullet_0 x$ are Borel measurable, all the functions $y\mapsto g(y)(\psi)$ will be Borel measurable, and $g:Y\to Z^H$ will be Borel measurable. Consequently all the sets $\{n\}\times g[Y]$ will be Borel, and h will be Borel measurable, therefore a Borel isomorphism between $\mathbb{N}\times Z^H$ and X. Finally, because $z\mapsto\theta(\psi)\bullet_r z:Z^H\to Z^H$ is Borel measurable for every $\psi\in G$, $(n,z)\mapsto(n,\theta(\psi)\bullet_r z)$ is Borel measurable for every n, and $x\mapsto\psi\bullet x$ is Borel measurable, for every $\psi\in G$. So $\bullet$ is measurable in the required sense.

425C Master actions (a) For each $R\subseteq\mathbb{N}^2$, consider the family F_R of injective functions f from countable ordinals to $\mathbb{N}$ such that

for every $\beta\in\operatorname{dom}f$, $f(\beta)$ is the unique member of $\mathbb{N}$ such that $R[\{f(\beta)\}]=f[\beta]$.

If f, $g\in F_R$ have domains α, α' respectively where $\alpha\le\alpha'$, then $f=g\restriction\alpha$. **P** If $\beta<\alpha$ is such that $f\restriction\beta=g\restriction\beta$, then both $f(\beta)$ and $g(\beta)$ are the unique $m\in\mathbb{N}$ such that $R[\{m\}]=f[\beta]$. **Q** Consequently $f_R=\bigcup F_R$ is the unique maximal element of F_R. (Compare 2A1B.)

For a countable ordinal α, let $\mathcal{R}_\alpha$ be the set of those $R\subseteq\mathbb{N}^2$ such that $\alpha\le\operatorname{dom}f_R$. Note that if $f:\alpha\to\mathbb{N}$ is injective, then there is an $R\in\mathcal{R}_\alpha$ such that $f=f_R\restriction\alpha$ (set $R=\{(f(\beta),f(\gamma)):\gamma<\beta<\alpha\}\cup((\mathbb{N}\setminus f[\alpha])\times\mathbb{N})$).

(b) Let $\maltese$ be the family of group operations $\star$ on $\mathbb{N}$. We are going to need the natural Borel structure on $\maltese$ corresponding to the identification of each $\star \in \maltese$, which is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, with the set

$$\{(i,j,k) : i \star j = k\} \subseteq \mathbb{N}^3.$$

So we can think of $\maltese$ as the set of subsets $\star$ of $\mathbb{N}^3$ such that

for all i, $j \in \mathbb{N}$ there is just one $k \in \mathbb{N}$ such that $(i,j,k) \in \star$,

if i, j, k, l, m, $n \in \mathbb{N}$, and (i,j,l), (l,k,n), (j,k,m) belong to $\star$, then $(i,m,n) \in \star$,

$(0,i,i)$, $(i,0,i) \in \star$ for every $i \in \mathbb{N}$,

for every $i \in \mathbb{N}$ there is a $j \in \mathbb{N}$ such that $(i,j,0)$ and $(j,i,0)$ belong to $\star$.

For $\star \in \maltese$, let $\bullet_r^\star$ be the corresponding right shift action of $\mathbb{N}$ on $\mathbb{R}^{\mathbb{N}}$, so that $(m\bullet_r^\star z)(i) = z(i \star m)$ whenever $z \in \mathbb{R}^{\mathbb{N}}$ and i, $m \in \mathbb{N}$.

(c) Let G be a group, with cardinal ω_1, with identity ι; let $\langle \pi_\alpha \rangle_{\alpha<\omega_1}$ enumerate G, with $\pi_0 = \iota$. Let $F \subseteq \omega_1$ be the set of those α such that $G_\alpha = \{\pi_\beta : \beta < \alpha\}$ is a subgroup of G. For $\alpha \in F$, set

$$\begin{aligned}\mathcal{S}_\alpha = \{(R,\star) : R \in \mathcal{R}_\alpha,\ \star \in \maltese \text{ and } f_R(\beta) \star f_R(\gamma) = f_R(\delta)\\ \text{whenever } \beta,\ \gamma,\ \delta < \alpha \text{ and } \pi_\beta\pi_\gamma = \pi_\delta\};\end{aligned}$$

so that if $(R,\star) \in \mathcal{S}_\alpha$, $f_R{\restriction}\alpha$ codes a group homomorphism from G_α to $(\mathbb{N},\star)$.

(d) For $\alpha \in F$, set

$$\mathcal{M}_\alpha = \{(R,\star,z) : (R,\star) \in \mathcal{S}_\alpha,\ z \in \mathbb{R}^{\mathbb{N}}\}.$$

Then we have an action $\bullet'_\alpha$ of G_α on $\mathcal{M}_\alpha$ defined by saying that

$$\pi_\beta \bullet'_\alpha (R,\star,z) = (R,\star,f_R(\beta)\bullet_r^\star z)$$

whenever $(R,\star) \in \mathcal{S}_\alpha$, $z \in \mathbb{R}^{\mathbb{N}}$ and $\beta < \alpha$. **P** $\bullet'_\alpha$ is well-defined as a function on $G_\alpha \times \mathcal{M}_\alpha$ because $f_R(\beta) = f_R(\gamma)$ whenever $(R,\star) \in \mathcal{S}_\alpha$ and $\pi_\beta = \pi_\gamma$. If β, γ, $\delta < \alpha$ and $\pi_\delta = \pi_\beta\pi_\gamma$, then

$$\begin{aligned}\pi_\beta\bullet'_\alpha(\pi_\gamma\bullet'_\alpha(R,\star,z)) &= \pi_\beta\bullet'_\alpha(R,\star,f_R(\gamma)\bullet_r^\star z)\\ &= (R,\star,f_R(\beta)\bullet_r^\star(f_R(\gamma)\bullet_r^\star z)) = (R,\star,f_R(\delta)\bullet_r^\star z)\end{aligned}$$

(because $(f_R(\beta), f_R(\gamma), f_R(\delta)) \in \star$)

$$\begin{aligned}&= \pi_\delta\bullet'_\alpha(R,\star,z),\\ \iota\bullet'_\alpha(R,\star,z) &= \pi_0\bullet'_\alpha(R,\star,z) = (R,\star,f_R(0)\bullet_r^\star z) = (R,\star,z).\ \mathbf{Q}\end{aligned}$$

(e) If α, $\beta \in F$ and $\alpha \le \beta$, then it is elementary to check that $\mathcal{R}_\beta \subseteq \mathcal{R}_\alpha$, $\mathcal{S}_\beta \subseteq \mathcal{S}_\alpha$, $\mathcal{M}_\beta \subseteq \mathcal{M}_\alpha$ and $\pi\bullet'_\beta(R,\star,z) = \pi\bullet'_\alpha(R,\star,z)$ whenever $(R,\star,z) \in \mathcal{M}_\beta$ and $\pi \in G_\alpha$. If $\beta \in F$ and $\beta = \sup(\beta \cap F)$, then $\mathcal{R}_\beta = \bigcap_{\alpha\in\beta\cap F}\mathcal{R}_\alpha$ and $\mathcal{M}_\beta = \bigcap_{\alpha\in\beta\cap F}\mathcal{M}_\alpha$.

(f)(i) For $\alpha < \omega_1$, $\mathcal{R}_\alpha$ belongs to the Borel σ-algebra $\mathcal{B}(\mathcal{P}\mathbb{N}^2)$ when $\mathcal{P}(\mathbb{N}^2)$ is given its usual compact Hausdorff topology (4A2Ud), and moreover $\{(R,m) : R \subseteq \mathbb{N}^2,\ (\beta,m) \in f_R\} \in \mathcal{B}(\mathcal{P}\mathbb{N}^2 \times \mathbb{N})$ whenever $m \in \mathbb{N}$ and $\beta < \alpha$. **P** Induce on α. We start with $\mathcal{R}_0 = \mathcal{P}(\mathbb{N}^2)$. For the inductive step to a successor ordinal $\alpha+1$, given $R \subseteq \mathbb{N}^2$ and $m \in \mathbb{N}$, then

$$\begin{aligned}(\alpha,m) \in f_R \iff{}& R \in \mathcal{R}_\alpha \text{ and } m \text{ is the unique member of } \mathbb{N}\\ &\text{such that } R[\{m\}] = f_R[\alpha]\\ \iff{}& R \in \mathcal{R}_\alpha,\ R[\{m\}] \ne R[\{n\}] \text{ for every } n \ne m,\\ &\forall\, i \in \mathbb{N},\ (m,i) \in R \iff \exists\, \beta < \alpha,\ (\beta,i) \in f_R.\end{aligned}$$

So $\{(R,m) : (\alpha,m) \in f_R\}$ is

$$(\mathcal{R}_\alpha \times \mathbb{N}) \cap \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \{(R,m) : n = m \text{ or } (m,i) \in R \,\&\, (n,i) \notin R \text{ or } (m,i) \notin R \,\&\, (n,i) \in R\}$$

$$\cap \bigcap_{i\in\mathbb{N}} \bigcup_{\beta<\alpha} \{(R,m) : (m,i) \notin R \text{ or } (\beta,i) \in f_R\}$$

$$\cap \bigcap_{i\in\mathbb{N}} \bigcap_{\beta<\alpha} \{(R,m) : (m,i) \in R \text{ or } (\beta,i) \notin f_R\},$$

and is a Borel set. Now

$$\mathcal{R}_{\alpha+1} = \bigcup_{m\in\mathbb{N}} \{R : (\alpha,m) \in f_R\} \in \mathcal{B}(\mathcal{P}\mathbb{N}^2).$$

For the inductive step to a countable limit ordinal $\alpha > 0$, $\mathcal{R}_\alpha = \bigcap_{\gamma<\alpha} \mathcal{R}_\gamma$. **Q**

(ii) ✠ is a Borel subset of $\mathcal{P}(\mathbb{N}^3)$, so $\mathcal{S}_\alpha$ is a Borel subset of $\mathcal{P}(\mathbb{N}^2) \times \mathcal{P}(\mathbb{N}^3)$, and $\mathcal{M}_\alpha$ is a Borel subset of $\mathcal{P}(\mathbb{N}^2) \times \mathcal{P}(\mathbb{N}^3) \times \mathbb{R}^\mathbb{N}$, for every $\alpha \in F$. Next, $(R,\star,z) \mapsto \pi\bullet'_\alpha(R,\star,z) : \mathcal{M}_\alpha \to \mathcal{M}_\alpha$ is Borel measurable whenever $\alpha \in F$ and $\pi \in G_\alpha$. **P** Let $\beta < \alpha$ be such that $\pi = \pi_\beta$. If $E_0 \subseteq \mathcal{P}(\mathbb{N}^2) \times \mathcal{P}(\mathbb{N}^3)$ is a Borel set, and $E = \{(R,\star,z) : (R,\star,z) \in \mathcal{M}_\alpha,\ (R,\star) \in E_0\}$, then

$$\{(R,\star,z) : \pi\bullet'_\alpha(R,\star,z) \in E\} = E$$

is Borel; if $n \in \mathbb{N}$ and $E_1 \subseteq \mathbb{R}$ is a Borel set, and $E = \{(R,\star,z) : (R,\star,z) \in \mathcal{M}_\alpha,\ z(n) \in E_1\}$, then

$$\{(R,\star,z) : \pi\bullet'_\alpha(R,\star,z) \in E\} = \{(R,\star,z) : (f_R(\beta)\bullet^\star_r z)(n) \in E_1\}$$

$$= \bigcup_{i,j\in\mathbb{N}} \{(R,\star,z) : f_R(\beta) = i,\ (n,i,j) \in \star,\ z(j) \in E_1\}$$

is Borel. Since $\mathcal{B}(\mathcal{P}\mathbb{N}^2 \times \mathcal{P}\mathbb{N}^3 \times \mathbb{R}^\mathbb{N})$ is the product σ-algebra

$$\mathcal{B}(\mathcal{P}\mathbb{N}^2)\widehat{\otimes}\mathcal{B}(\mathcal{P}\mathbb{N}^3)\widehat{\otimes}\widehat{\bigotimes}_\mathbb{N}\mathcal{B}(\mathbb{R}),$$

$(R,\star,z) \mapsto \pi\bullet'_\alpha(R,\star,z)$ is Borel measurable. **Q**

425D Törnquist's theorem (Törnquist 11) Let (X,Σ) be a standard Borel space and $\mathcal{I}$ a σ-ideal of Σ containing an uncountable set. Let $\mathfrak{A}$ be the quotient algebra $\Sigma/\mathcal{I}$, and $G \subseteq \operatorname{Aut}\mathfrak{A}$ a subgroup with cardinal at most ω_1. Then there is an action $\bullet$ of G on X which represents G in the sense that $\pi\bullet E = \{\pi\bullet x : x \in E\}$ belongs to Σ, and $(\pi\bullet E)^\bullet = \pi(E^\bullet)$, for every $E \in \Sigma$ and $\pi \in G$.

proof (a) It may help if I try to describe the line of argument I mean to follow. The important case is when G has an enumeration $\langle\pi_\alpha\rangle_{\alpha<\omega_1}$; let F be the set of those $\alpha < \omega_1$ such that $G_\alpha = \{\pi_\beta : \beta < \alpha\}$ is a subgroup of G. For each $\pi \in G$, choose $g_\pi : X \to X$ representing π. For $\alpha \in F$, set

$$Y_\alpha = \{x : x \in X,\ g_\phi(g_\pi(x)) = g_{\pi\phi}(x) \in X \setminus M \text{ for all } \pi,\ \phi \in G_\alpha\},$$

where M is an uncountable member of $\mathcal{I}$. Choose $\langle\bullet_\alpha\rangle_{\alpha\in F}$ inductively such that $\bullet_\alpha$ is an action of G_α on X, $\pi\bullet_\alpha x = g_{\pi^{-1}}(x)$ whenever $\pi \in G_\alpha$ and $x \in Y_\alpha$, and $\pi\bullet_\alpha x = \pi\bullet_\beta x$ whenever $\beta < \alpha$, $\pi \in G_\beta$ and $x \in X$. At the end of the construction, set $\bullet = \bigcup_{\alpha\in F} \bullet_\alpha$.

It is straightforward to show that $X \setminus Y_\alpha$ always belongs to $\mathcal{I}$ (part (c) of the proof); consequently $\bullet$ will represent G. The non-trivial part of the proof is in the extension of a given action of G_α to an action of G_β where β is the next element of F above α, and this is where we shall need 425B-425C.

Now for the details.

(b) Give X a Polish topology for which Σ is the Borel σ-algebra $\mathcal{B}(X)$. For nearly the whole of the proof (down to the end of (g) below), suppose that $\#(G) = \omega_1$, that $X = \{0,1\}^\mathbb{N}$ and that $\Sigma = \mathcal{B}(X)$ is the Borel σ-algebra of X. Enumerate G as $\langle\pi_\alpha\rangle_{\alpha<\omega_1}$ starting with π_0 equal to the identity ι. We need to know that, setting $G_\alpha = \{\pi_\beta : \beta < \alpha\}$, the set $F = \{\alpha : \alpha < \omega_1,\ G_\alpha \text{ is a subgroup of } G\}$ is a closed cofinal subset of ω_1. **P** This is elementary. If $\alpha \in \overline{F}$, then $\{G_\beta : \beta \in F,\ \beta \le \alpha\}$ is a non-empty upwards-directed family of subgroups of G, so $G_\alpha = \bigcup_{\beta\in F,\beta\le\alpha} G_\beta$ is a subgroup of G, and $\alpha \in F$. If $\alpha < \omega_1$, let $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in ω_1 such that $\alpha_0 = \max(1,\alpha)$ and, for each $n \in \mathbb{N}$,

— for all β, $\gamma < \alpha_n$ there is a $\delta < \alpha_{n+1}$ such that $\pi_\delta = \pi_\beta\pi_\gamma$,

—— for every $\beta < \alpha_n$ there is a $\delta < \alpha_{n+1}$ such that $\pi_\delta = \pi_\beta^{-1}$.

Setting $\alpha^* = \sup_{n\in\mathbb{N}} \alpha_n$, we see that $\alpha \le \alpha^* \in F$. So F is cofinal with ω_1. **Q**

Of course $1 = \min F$ and $G_1 = \{\iota\}$, because we started with $\pi_0 = \iota$.

(c) Next, for every $\pi \in \operatorname{Aut}\mathfrak{A}$, we can choose a Borel measurable $g_\pi : X \to X$ representing π in the sense that $\pi E^\bullet = g_\pi^{-1}[E]^\bullet$ for every $E \in \mathcal{B}(X)$ (425Ac). Of course when $\pi = \iota$ we take $g_\iota(x) = x$ for every $x \in X$. Fix an uncountable $M \in \mathcal{I}$, and for $\alpha \in F$ set

$$Y_\alpha = \{x : x \in X,\ g_\phi(g_\pi(x)) = g_{\pi\phi}(x) \in X \setminus M \text{ for all } \pi,\ \phi \in G_\alpha\},$$

as declared in (a). If $\alpha \in F$ and $\pi \in G_\alpha$, $X \setminus Y_\alpha \in \mathcal{I}$ and $g_\pi \restriction Y_\alpha$ is a permutation of Y_α. **P** (Compare the proof of 344B.) There is a sequence $\langle E_k \rangle_{k\in\mathbb{N}}$ in $\mathcal{B}(X)$ separating the points of X. So, for any ψ, $\phi \in G_\alpha$, the set

$$\{x : g_\psi g_\phi(x) \ne g_{\phi\psi}(x)\} = \bigcup_{k\in\mathbb{N}} g_\phi^{-1}[g_\psi^{-1}[E_k]] \triangle g_{\phi\psi}^{-1}[E_k]$$

is Borel, and moreover, transferring the formulae to the quotient algebra,

$$\begin{aligned}\{x : g_\psi g_\phi(x) \ne g_{\phi\psi}(x)\}^\bullet &= \sup_{k\in\mathbb{N}} g_\phi^{-1}[g_\psi^{-1}[E_k]]^\bullet \triangle g_{\phi\psi}^{-1}[E_k]^\bullet \\ &= \sup_{k\in\mathbb{N}} \phi(\psi E_k^\bullet) \triangle (\phi\psi)E_k^\bullet = 0,\end{aligned}$$

so $\{x : g_\psi g_\phi(x) \ne g_{\phi\psi}(x)\} \in \mathcal{I}$. Accordingly

$$X \setminus Y_\alpha = \bigcup_{\psi,\phi\in G_\alpha} \{x : g_\psi g_\phi(x) \ne g_{\phi\psi}(x)\} \cup \bigcup_{\psi\in G_\alpha} g_\psi^{-1}[M]$$

belongs to $\mathcal{I}$.

If $\pi \in G_\alpha$ and $x \in Y_\alpha$, then

$$g_\psi g_\phi g_\pi(x) = g_\psi g_{\pi\phi}(x) = g_{\pi\phi\psi}(x) = g_{\phi\psi} g_\pi(x)$$

and $g_{\pi\phi\psi}(x) \notin M$, for all ϕ, $\psi \in G_\alpha$; so $g_\pi(x) \in Y_\alpha$. Similarly, $g_{\pi^{-1}}[Y_\alpha] \subseteq Y_\alpha$. Moreover,

$$g_\pi g_{\pi^{-1}}(x) = g_\iota(x) = x = g_{\pi^{-1}} g_\pi(x)$$

for every $x \in Y_\alpha$, so that $g_\pi \restriction Y_\alpha$ must be a permutation of Y_α. **Q**

(d) From the group G and the enumeration $\langle \pi_\alpha \rangle_{\alpha<\omega_1}$, construct $\maltese$ and families $\langle f_R \rangle_{R\in\mathcal{P}\mathbb{N}^2}$, $\langle \mathcal{R}_\alpha \rangle_{\alpha<\omega_1}$, $\langle \bullet_r^\star \rangle_{\star\in\maltese}$ and $\langle (\mathcal{S}_\alpha, \mathcal{M}_\alpha, \bullet'_\alpha) \rangle_{\alpha\in F}$ as in 425C. Let h be a Borel isomorphism from $\mathbb{R} \times \mathbb{N} \times \mathcal{M}_1$ to M (424Da again). Fix a family $\langle t_\delta \rangle_{\delta\in F}$ of distinct members of $\mathbb{R}$, and set $J_\alpha = \mathbb{R} \setminus \{t_\delta : \delta < \alpha\}$ for $\alpha \in F$.

(e) (The key. Some readers may wish at this point to provide themselves with coffee and a large scratch-pad.) Let $\alpha < \beta < \gamma$ be members of F, and suppose that $\bullet_0$ is an action of G_α on X such that

$x \mapsto \pi \bullet_0 x$ is Borel measurable for every $\pi \in G_\alpha$,

$\pi \bullet_0 x = g_{\pi^{-1}}(x)$ whenever $x \in Y_\alpha$ and $\pi \in G_\alpha$,

$\pi \bullet_0 h(t, n, q) = h(t, n, \pi \bullet'_\beta q)$ whenever $t \in J_\alpha$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\beta$ and $\pi \in G_\alpha$.

Then there is an action $\bullet_1$ of G_β on X such that

$x \mapsto \pi \bullet_1 x$ is Borel measurable for every $\pi \in G_\beta$,

$\pi \bullet_1 x = g_{\pi^{-1}}(x)$ whenever $x \in Y_\beta$ and $\pi \in G_\beta$,

$\pi \bullet_1 h(t, n, q) = h(t, n, \pi \bullet'_\gamma q)$ whenever $t \in J_\beta$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\gamma$ and $\pi \in G_\beta$,

$\pi \bullet_1 x = \pi \bullet_0 x$ whenever $\pi \in G_\alpha$ and $x \in X$.

P Choose $\star \in \maltese$ such that there is an injective group homomorphism θ from G_β to $(\mathbb{N}, \star)$, and set $f'(\delta) = \theta(\pi_\delta)$ for $\delta < \beta$; let $R \in \mathcal{R}_\beta$ be such that $f_R \restriction \beta = f'$. (In the normal case, when $\beta \ge \omega$, we can choose f' first, as an arbitrary bijection from β to $\mathbb{N}$, and use this to define θ, $\star$ and R. If $\beta < \omega$, the first step is to take an injective group homomorphism from G_β to a countably infinite group, e.g., $G_\beta \times \mathbb{Z}$.) Then $(R, \star) \in \mathcal{S}_\beta$. Define $h_0 : \mathbb{N} \times \mathbb{R}^\mathbb{N} \to M$ by setting $h_0(n, z) = h(t_\alpha, n, (R, \star, z))$ for $n \in \mathbb{N}$ and $z \in \mathbb{R}^\mathbb{N}$. Then h_0 is injective and Borel measurable, and if $\pi \in G_\alpha$ then

$$h_0(n, \theta(\pi) \bullet_r^\star z) = h(t_\alpha, n, (R, \star, \theta(\pi) \bullet_r^\star z)) = h(t_\alpha, n, (R, \star, f_R(\delta) \bullet_r^\star z))$$

(where $\delta < \alpha$ is such that $\pi = \pi_\delta$)

$$= h(t_\alpha, n, \pi\bullet'_\alpha(R, \star, z)) = h(t_\alpha, n, \pi\bullet'_\beta(R, \star, z))$$
$$= \pi\bullet_0 h(t_\alpha, n, (R, \star, z)) = \pi\bullet_0 h_0(n, z)$$

for every $n \in \mathbb{N}$ and $z \in \mathbb{R}^{\mathbb{N}}$. Set

$$V = h[J_\beta \times \mathbb{N} \times \mathcal{M}_\gamma] \subseteq M \subseteq X \setminus Y_\beta, \quad X' = X \setminus (Y_\beta \cup V).$$

Note that if $\pi \in G_\alpha$, $t \in J_\beta$, $n \in \mathbb{N}$ and $q \in \mathcal{M}_\gamma$, then

$$\pi\bullet_0 h(t, n, q) = h(t, n, \pi\bullet'_\beta q) = h(t, n, \pi\bullet'_\gamma q) \in V;$$

thus V is invariant under the action $\bullet_0$. The same is true of Y_β, because $g_{\pi^{-1}}\restriction Y_\beta$ is a permutation of Y_β for every $\pi \in G_\alpha$, therefore also of X'. Note that as $t_\alpha \notin J_\beta$, $h_0[\mathbb{N} \times \mathbb{R}^{\mathbb{N}}] \subseteq X'$, while there is certainly a Borel measurable injection from $X' \setminus h_0[\mathbb{N} \times \mathbb{R}^{\mathbb{N}}]$ into $\mathbb{R}$. So 425Bb tells us that there is an action $\hat\bullet_1$ of G_β on X', extending $\bullet_0\restriction G_\alpha \times X'$, such that $x \mapsto \pi\hat\bullet_1 x : X' \to X'$ is Borel measurable for every $\pi \in G_\beta$.

We can therefore define $\bullet_1$ by setting

$$\begin{aligned}\pi\bullet_1 x &= g_{\pi^{-1}}(x) \text{ if } x \in Y_\beta,\\ &= h(t, n, \pi\bullet'_\gamma q) \text{ whenever } t \in J_\beta,\ n \in \mathbb{N},\ q \in \mathcal{M}_\gamma \text{ and } x = h(t, n, q),\\ &= \pi\hat\bullet_1 x \text{ if } x \in X'\end{aligned}$$

for every $\pi \in G_\beta$. It is easy to check that $\bullet_1$ is a function from $G_\beta \times X$ to X extending $\bullet_0$, and that $x \mapsto \pi\bullet_1 x$ is Borel measurable for every $\pi \in G_\beta$. If π, $\phi \in G_\beta$ and $x \in X$, then

$$\begin{aligned}\pi\bullet_1(\phi\bullet_1 x) &= \pi\bullet_1 g_{\phi^{-1}}(x) = g_{\pi^{-1}} g_{\phi^{-1}}(x) = g_{(\pi\phi)^{-1}}(x) = (\pi\phi)\bullet_1 x \text{ if } x \in Y_\beta,\\ &= \pi\bullet_1 h(t, n, \phi\bullet'_\gamma q) = h(t, n, \pi\bullet'_\gamma(\phi\bullet'_\gamma q)) = h(t, n, (\pi\phi)\bullet'_\gamma q)\\ &\qquad = (\pi\phi)\bullet_1 h(t, n, q) = (\pi\phi)\bullet_1 x\\ &\qquad \text{whenever } t \in J_\beta,\ n \in \mathbb{N},\ q \in \mathcal{M}_\gamma \text{ and } x = h(t, n, q),\\ &= \pi\hat\bullet_1(\phi\hat\bullet_1 x) = (\pi\phi)\hat\bullet_1 x = (\pi\phi)\bullet_1 x \text{ if } x \in X'.\end{aligned}$$

So $\bullet_1$ is an action of G_β on X, as required. **Q**

(f) Accordingly we can build $\langle\bullet_\alpha\rangle_{\alpha\in F}$ inductively, as follows. The inductive hypothesis will be that, for each $\alpha \in F$,

$\bullet_\alpha$ is an action of G_α on X,
$x \mapsto \pi\bullet_\alpha x$ is Borel measurable for every $\pi \in G_\alpha$,
$\pi\bullet_\alpha x = g_{\pi^{-1}}(x)$ whenever $x \in Y_\alpha$ and $\pi \in G_\alpha$,
$\pi\bullet_\alpha h(t, n, q) = h(t, n, \pi\bullet'_\beta q)$ whenever $t \in J_\alpha$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\beta$, $\beta \in F$, $\beta > \alpha$ and $\pi \in G_\alpha$,
$\bullet_\delta = \bullet_\alpha\restriction G_\delta \times X$ whenever $\delta \in F \cap \alpha$.

The induction starts with $G_1 = \{\iota\}$ and $\iota\bullet_1 x = x$ for every $x \in X$.

Given $\alpha \in F$ and $\bullet_\alpha$, let β be the next element of F above α and γ the next element of F above β. By (d), we have an action $\bullet_\beta$ of G_β on X such that

$x \mapsto \pi\bullet_\beta x$ is Borel measurable for every $\pi \in G_\beta$,
$\pi\bullet_\beta x = g_{\pi^{-1}}(x)$ whenever $x \in Y_\beta$ and $\pi \in G_\beta$,
$\pi\bullet_\beta h(t, n, q) = h(t, n, \pi\bullet'_\gamma q)$ whenever $t \in J_\beta$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\gamma$ and $\pi \in G_\alpha$,
$\bullet_\alpha = \bullet_\beta\restriction G_\alpha \times X$.

It follows at once that if $\delta \in F$ and $\delta \le \alpha$,

$$\pi\bullet_\delta x = \pi\bullet_\alpha x = \pi\bullet_\beta x$$

whenever $\pi \in G_\delta$ and $x \in X$; on the other side, if $\delta \in F$ and $\delta \ge \gamma$,

$$\pi\bullet_\beta h(t, n, q) = h(t, n, \pi\bullet'_\gamma q) = h(t, n, \pi\bullet'_\delta q)$$

whenever $t \in J_\beta$, $n \in \mathbb{N}$, $q \in \mathcal{M}_\delta$ and $\pi \in G_\beta$. So the induction continues to the next step.

If $\alpha \in F$ and $\alpha = \sup(F \cap \alpha)$, then $G_\alpha = \bigcup_{\beta\in F\cap\alpha} G_\beta$, so we have an action $\bullet_\alpha = \bigcup_{\beta\in F\cap\alpha} \bullet_\beta$ of G_α on X; and it is elementary to check that the inductive hypothesis is satisfied at the new level.

(g) At the end of the induction, $\bullet = \bigcup_{\alpha \in F} \bullet_\alpha$ will be an action of G on X such that $x \mapsto \pi \bullet x$ is Borel measurable for every $\pi \in G$. Moreover, $\pi E^\bullet = (\pi \bullet E)^\bullet$ for every Borel set $E \subseteq X$ and $\pi \in G$. **P** Let $\alpha \in F$ be such that $\pi \in G_\alpha$. Then

$$\pi(E^\bullet) = \pi((E \cap Y_\alpha)^\bullet)$$

(because $X \setminus Y_\alpha \in \mathcal{I}$)

$$= (g_\pi^{-1}[E \cap Y_\alpha])^\bullet = (Y_\alpha \cap g_\pi^{-1}[E \cap Y_\alpha])^\bullet = (g_{\pi^{-1}}[E \cap Y_\alpha])^\bullet$$

(because $g_\pi \restriction Y_\alpha$ is a permutation with inverse $g_{\pi^{-1}} \restriction Y_\alpha$)

$$= (\pi \bullet (E \cap Y_\alpha))^\bullet \subseteq (\pi \bullet E)^\bullet.$$

(Because $x \mapsto \pi \bullet x$ is a Borel measurable permutation of X, $\pi \bullet E$ is certainly a Borel set.) Since equally we must have

$$\pi((X \setminus E)^\bullet) \subseteq (\pi \bullet (X \setminus E))^\bullet,$$

while $\pi(E^\bullet) \cup \pi((X \setminus E)^\bullet) = 1_{\mathfrak{A}}$ and $(\pi \bullet E)^\bullet \cap (\pi \bullet (X \setminus E))^\bullet = 0_{\mathfrak{A}}$, both the inclusions here are equalities, and $\pi E^\bullet = (\pi \bullet E)^\bullet$. **Q**

(h) As for the elementary case in which G is countable, we can use arguments already presented, as follows. For each $\pi \in G$, choose g_π representing π. This time, go straight to $Y = \{x : g_{\pi\phi}(x) = g_\phi g_\pi(x)$ for all $\pi, \phi \in G\}$; as in (c), $X \setminus Y \in \mathcal{I}$ and $g_\pi \restriction Y$ is a permutation of Y for every $\pi \in G$. So if we set

$$\pi \bullet x = g_{\pi^{-1}}(x) \text{ for } x \in Y,$$
$$= x \text{ for } x \in X \setminus Y,$$

$\bullet$ will be an appropriate action of G on X.

425E Scholium The theorem here applies to groups with cardinal at most ω_1. So it is worth noting that in the context of 425D the whole group $\operatorname{Aut}\mathfrak{A}$ has cardinal at most $\mathfrak{c}$. **P** Since Σ is countably σ-generated, there is a countable set $D \subseteq \mathfrak{A}$ σ-generating $\mathfrak{A}$. If $\pi, \phi \in \operatorname{Aut}\mathfrak{A}$ and $\pi \restriction D = \phi \restriction D$, then $\pi = \phi$; so $\#(\operatorname{Aut}\mathfrak{A}) \le \#(\mathfrak{A}^D)$. As $\#(\mathfrak{A}) \le \#(\Sigma) \le \mathfrak{c}$ (424Db), $\#(\operatorname{Aut}\mathfrak{A})$ is at most $\mathfrak{c}$ (4A1A(c-ii)). **Q**

We therefore have a corollary of 425D, as follows:

> Suppose the continuum hypothesis is true. Let (X, Σ) be a standard Borel space and $\mathcal{I}$ a σ-ideal of subsets of Σ containing an uncountable set. Then there is an action $\bullet$ of $\operatorname{Aut}(\Sigma/\mathcal{I})$ on X such that $\pi E^\bullet = (\pi \bullet E)^\bullet$ whenever $E \in \Sigma$ and $\pi \in \operatorname{Aut}(\Sigma/\mathcal{I})$.

425X Basic exercises (a) (Cf. 382Xc.) Let (X, Σ) be a standard Borel space, $\mathcal{I}$ a σ-ideal of subsets of X and $\mathfrak{A} = \Sigma/\mathcal{I}$ the quotient algebra. (i) Show that every member of $\operatorname{Aut}\mathfrak{A}$ has a separator (definition: 382Aa). (ii) Show that if G is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$, then every member of G is expressible as the product of at most three involutions belonging to G.

(b) Let (X, Σ) be a standard Borel space and set $\mathfrak{A} = \Sigma/[X]^{\le\omega}$. (i) Show that the Boolean algebra $\mathfrak{A}$ is homogeneous (definition: 316N). (ii) Show that $\operatorname{Aut}\mathfrak{A}$ is simple. (*Hint*: 382Yc.)

(c) Let (X, Σ) be a standard Borel space and $\mathcal{I}$ a σ-subalgebra of Σ with associated quotient algebra $\mathfrak{A} = \Sigma/\mathcal{I}$. Suppose that G is a countable semigroup of sequentially order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Show that there is a family $\langle f_\pi \rangle_{\pi \in G}$ of (Σ, Σ)-measurable functions from X to itself such that (α) $\pi E^\bullet = f_\pi^{-1}[E]^\bullet$ for every $\pi \in G$ and $E \in \Sigma$ (β) $f_{\pi\phi} = f_\phi f_\pi$ for all $\pi, \phi \in G$.

(d) Let X be a set, Σ a σ-algebra of subsets of X, $\mathcal{I}$ a σ-ideal of Σ and $\mathfrak{A}$ the quotient $\Sigma/\mathcal{I}$. Suppose that (X, Σ) is countably separated in the sense that there is a countable subset of Σ separating the points of X. Let G be a countable subsemigroup of the semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that for every $\pi \in G$ there is a (Σ, Σ)-measurable $g : X \to X$ such that $\pi E^\bullet = g^{-1}[E]^\bullet$ for every $E \in \Sigma$. Show that there is a family $\langle f_\pi \rangle_{\pi \in G}$ of (Σ, Σ)-measurable functions from X to itself such that $f_\pi^{-1}[E]^\bullet = \pi E^\bullet$ and $f_{\pi\phi} = f_\phi f_\pi$ whenever $\pi, \phi \in G$ and $E \in \Sigma$. (*Hint*: 344B.)

(e) Show that the set ✠ in 425Cb is a G_δ set in the compact metrizable space $\mathcal{P}(\mathbb{N}^3)$.

(f) Give an example of a standard Borel space (X,Σ), a σ-ideal $\mathcal{I}$ of Σ, a finite subgroup G of $\mathrm{Aut}(\Sigma/\mathcal{I})$, a subgroup H of G and an action $\bullet_0$ of H on X such that $\pi E^\bullet = (\pi\bullet_0 E)^\bullet$ whenever $\pi \in H$ and $x \in X$, but there is no action $\bullet$ of G on X, extending $\bullet_0$, such that $\pi E^\bullet = (\pi\bullet E)^\bullet$ whenever $\pi \in G$ and $x \in X$. (*Hint*: $\#(X) = 6$, $\#(H) = 2$.)

(g) Let $I^\parallel$ be the split interval with its usual topology and measure, and $\mathfrak{A}$ its measure algebra. Let G be a subgroup of $\mathrm{Aut}\,\mathfrak{A}$ with cardinal at most ω_1. Show that there is an action $\bullet$ of G on $I^\parallel$ such that $\pi\bullet E$ is a Borel set and $(\pi\bullet E)^\bullet = \pi E^\bullet$ for every $E \in \mathcal{B}(I^\parallel)$ and $\pi \in G$.

425Y Further exercises (a) (Törnquist 11) Let X be a set, Σ a σ-algebra of subsets of X, $\mathcal{I}$ a σ-ideal of Σ and $\mathfrak{A}$ the quotient $\Sigma/\mathcal{I}$. Suppose that $(X,\Sigma,\mathcal{I})$ is countably separated in the sense that there is a countable subset of Σ separating the points of X, and complete in the sense that $A \in \mathcal{I}$ whenever $A \subseteq B \in \mathcal{I}$. Let $G \subseteq \mathrm{Aut}\,\mathfrak{A}$ be a group with cardinal at most ω_1 such that for every $\pi \in G$ there is a (Σ,Σ)-measurable function $g : X \to X$ such that $\pi E^\bullet = g^{-1}[E]^\bullet$ for every $E \in \Sigma$. Show that if $\mathcal{I}$ contains a set with cardinal $\mathfrak{c}$ there is an action $\bullet$ of G on X such that $\pi E^\bullet = (\pi\bullet E)^\bullet$ for every $\pi \in G$ and $E \in \Sigma$.

(b) Let (X,Σ,μ) be a countably separated perfect complete strictly localizable measure space, $\mathfrak{A}$ its measure algebra and G a subgroup of $\mathrm{Aut}\,\mathfrak{A}$ of cardinal at most ω_1. Show that there is an action $\bullet$ of G on X such that $\pi\bullet E \in \Sigma$ and $(\pi\bullet E)^\bullet = \pi(E^\bullet)$ whenever $\pi \in G$ and $E \in \Sigma$.

425Z Problems (a) Suppose that (X,Σ) is an uncountable standard Borel space and $\mathcal{I}$ the ideal $[X]^{\le\omega}$ of countable subsets of X. Which subgroups G of $\mathrm{Aut}(\Sigma/\mathcal{I})$ can be represented by actions of G on X?

(b) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure μ on $[0,1]$, and G a semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself with $\#(G) = \omega_1$. Must there be a family $\langle f_\pi\rangle_{\pi\in G}$ of inverse-measure-preserving functions from $[0,1]$ to itself such that $f_{\pi\phi} = f_\phi f_\pi$ for all π, $\phi \in G$ and f_π represents π, in the sense of 425A, for every $\pi \in G$? (See 344B.)

(c) Let $(\mathfrak{B}_{\omega_1},\bar\nu_{\omega_1})$ be the measure algebra of the usual measure on $\{0,1\}^{\omega_1}$, and G a group of measure-preserving automorphisms of $\mathfrak{B}_{\omega_1}$ with $\#(G) = \omega_1$. Must there be a family $\langle f_\pi\rangle_{\pi\in G}$ of inverse-measure-preserving functions from $\{0,1\}^{\omega_1}$ to itself such that $f_{\pi\phi} = f_\phi f_\pi$ for all π, $\phi \in G$ and f_π represents π for every $\pi \in G$? (See 344E.)

425 Notes and comments I have starred this section because (apart from 425A) it deals with a very special topic. From the point of view of measure theory, 425D applies only to copies of Borel measures on $\mathbb{R}$ (and not quite all of those), and the limitation to groups of cardinal ω_1 means that we need to assume the continuum hypothesis, or at least $\mathrm{add}\,\mathcal{N} = \mathfrak{p} = \mathfrak{c}$ (535Yc), to get a theorem we really want. However the result connects naturally with an important theme from Chapter 34, and the general question of simultaneous representation of many automorphisms has significant implications for the ergodic theory treated in Chapter 38 and §494 of this volume.

What makes 425D difficult is the ambitious target: we want to represent the automorphisms in G by a consistent family of Borel measurable functions. We know from Chapter 34 that we can hope to handle countable groups, so it is natural to start by expressing G as an inductive limit of a family $\langle G_\alpha\rangle_{\alpha\in F}$, as in parts (a)-(b) of the proof of 425D, and to try to define the action of G from actions of the G_α. Since any $\pi \in G$ must eventually determine a Borel measurable function on X, we are going to have to freeze its action at some point; we could afford to change it once or twice, or even countably often, as the induction continued, but sometime we must stop tinkering, and really we want to have $\pi\bullet x = \pi\bullet_\alpha x$ whenever $x \in X$ and $\pi \in G_\alpha$. In this case, $\bullet_\beta$ will have to be a direct extension of $\bullet_\alpha$ whenever $\beta > \alpha$. We are in a context in which arbitrary actions are not always extensible (425Xf), and something like Lemma 425B is going to be needed. This demands a plentiful supply of copies of shift actions, at the very least including representations of all the shift actions on X^{G_α}, which will have to be built in from the very beginning. The trouble is that these have to be assembled in a way which will give us Borel measurable functions. Now while X can be taken to be a fixed Polish space, the groups G_α are more or less arbitrary countable groups. They can all be represented by group structures on $\mathbb{N}$, but if we go by that road, we seem to need to choose injective functions from countable ordinals into $\mathbb{N}$. (Of course each G_α comes with a bijection between it and the ordinal α.) No direct enumeration of these is going to lead to Borel structures. Instead, we have to look at the whole set of group structures on $\mathbb{N}$, the set ✠ of 425Cb, and nearly all injective functions from countable ordinals to $\mathbb{N}$, coded by subsets of $\mathbb{N}^2$, as in 425Ca. Fortunately it does not matter that there is a great deal of redundancy in this coding; it can all be fitted naturally into a standard Borel structure, and we just need to include, in the hypotheses of 425D, a negligible set of size $\mathfrak{c}$. When $\mathcal{I} \subseteq [X]^{\le\omega}$, the problem changes (425Za).

In order to ensure that each of the actions $\bullet_\alpha$ correctly represents the action of G_α on $\mathfrak{A}$, we can use essentially the same method as that of 344B (part (c) of the proof of 425D). This means, of course, that 425B has to be applied to a carefully chosen fragment of X, the set X' of part (d) of the proof of 425D. There is an awkward shift here between the representation of an automorphism π by the function g_π, where I follow the conventions used in Chapter 34 and 425A here with the contravariant formula $\pi(E^\bullet) = (g_\pi^{-1}[E])^\bullet$, and the representation in the statement of this theorem, with the covariant formula $\pi E^\bullet = (\pi\bullet E)^\bullet$. The latter is forced by the rule of 4A5Ba that $(\pi\phi)\bullet x = \pi\bullet(\phi\bullet x)$. By allowing 'reverse actions', in which $(\pi\phi)\bullet x = \phi\bullet(\pi\bullet x)$, we could escape this conflict, and in the formulae of parts (d)-(e) of the proof of 425D we should have $\pi\bullet_\alpha x = g_\pi(x)$ for $x \in Y_\alpha$ and $\pi \in G_\alpha$. This would be essential if we wanted to use the formulae here on semigroups of Boolean homomorphisms, as in §344, so that the g_π were no longer injective on conegligible sets. But there seem to be more substantial obstacles (425Zb).

The proof of 425D appeals repeatedly to the special properties of standard Borel spaces. But conceivably enough of it can be applied to the Baire σ-algebras of powers of $\{0,1\}$ to give a similar result for other important probability spaces (425Zc). If we change the rules, and assume that $\mathcal{I}$ is an ideal of $\mathcal{P}X$ as well as of Σ, we can dispense with the ideas of 425C and work directly from 425Ba, using copies of $\mathbb{N} \times X^{G_\alpha}$ inside a negligible set M (425Ya); this gives us an approach to use on complete measure spaces (425Yb, 535Yc).

Another way of looking at 425D is to think of it as a kind of lifting theorem. Let Φ be the group of (Σ,Σ)-bimeasurable $\mathcal{I}$-invariant permutations of X. Then each $f \in \Phi$ induces an automorphism f° of $\mathfrak{A}$ defined by saying that $f^\circ(E^\bullet) = (f[E])^\bullet$ for each $E \in \Sigma$. (I am using the push-forward rather than the pull-back representation here.) 425Ac is enough to show that the group homomorphism $f \mapsto f^\circ : \Phi \to \operatorname{Aut}\mathfrak{A}$ is surjective. Under the conditions of 425D, the action $\bullet$ corresponds to a group homomorphism $\theta : G \to \Phi$ such that $(\theta\pi)^\circ = \pi$ for every $\pi \in G$; and subject to the continuum hypothesis, we have a lifting for the whole of $\operatorname{Aut}\mathfrak{A}$. The word 'lifting' in this context should remind you of the Lifting Theorem of measure theory (341K). That theorem demands a complete measure space, and does not ordinarily apply to Borel measures. However, subject to the continuum hypothesis, there is a lifting theorem applicable to a variety of non-complete measure spaces, including any (X,Σ,μ) where (X,Σ) is a standard Borel space and μ is σ-finite (535E(b-i) of Volume 5). I am not sure that it is really helpful to think of 425E and 535E together; certainly the manoeuvres of 425C have no analogues in the lifting theorems of measure theory. But the correspondence is striking and suggests directions of enquiry which may be worth exploring.

Version of 30.11.16

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

423J Extra results have been interpolated into §423, so the second half of that section (423J-423R), referred to in the 2008 and 2015 editions of Volume 5, is now 423K-423T.

Version of 9.2.13

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