Version of 14.12.06

I return in this volume to the study of measure *spaces* rather than measure *algebras*. For fifty years now measure theory has been intimately connected with general topology. Not only do a very large proportion of the measure spaces arising in applications carry topologies related in interesting ways to their measures, but many questions in abstract measure theory can be effectively studied by introducing suitable topologies. Consequently any course in measure theory at this level must be frankly dependent on a substantial knowledge of topology. With this proviso, I hope that the present volume will be accessible to graduate students, and will lead them to the most important ideas of modern abstract measure theory.

The first and third chapters of the volume seek to provide a thorough introduction into the ways in which topologies and measures can interact. They are divided by a short chapter on descriptive set theory, on the borderline between set theory, logic, real analysis and general topology, which I single out for detailed exposition because I believe that it forms an indispensable part of the background of any measure theorist. Chapter 41 is dominated by the concepts of inner regularity and τ-additivity, coming together in Radon measures (§416). Chapter 43 concentrates rather on questions concerning properties of a topological space which force particular relationships with measures on that space. But plenty of side-issues are treated in both, such as Lusin measurability (§418), the definition of measures from linear functionals (§436) and measure-free cardinals (§438). Chapters 45 and 46 continue some of the same themes, with particular investigations into 'disintegrations' or regular conditional probabilities (§§452-453), stochastic processes (§§454-456), Talagrand's theory of stable sets (§465) and the theory of measures on normed spaces (§§466-467).

In contrast with the relatively amorphous structure of Chapters 41, 43, 45 and 46, four chapters of this volume have definite topics. I have already said that Chapter 42 is an introduction to descriptive set theory; like Chapters 31 and 35 in the preceding volume, it is a kind of appendix brought into the main stream of the argument. Chapter 44 deals with topological groups. Most of it is of course devoted to Haar measure, giving the Pontryagin-van Kampen duality theorem (§445) and the Ionescu Tulcea theorem on the existence of translation-invariant liftings (§447). But there are also sections on Polish groups (§448) and amenable groups (§449), and some of the general theory of measures on measurable groups (§444). Chapter 47 is a second excursion, after Chapter 26, into geometric measure theory. It starts with Hausdorff measures (§471), gives a proof of the Di Giorgio-Federer Divergence Theorem (§475), and then examines a number of examples of 'concentration of measure' (§476). In the second half of the chapter, §§477-479, I decribe Brownian motion and use it as a basis of the theory of Newtonian capacity. In Chapter 48, I set out the elementary theory of gauge integrals, with sections on the Henstock and Pfeffer integrals (§§483-484). Finally, in Chapter 49, I give notes on seven special topics: equidistributed sequences (§491), combinatorial forms of concentration of measure (§492), extremely amenable groups and groups of measure-preserving automorphisms (§§493-494), Poisson point processes (§495), submeasures (§496), Szemerédi's theorem (§497) and subproducts in product spaces (§498).

I had better mention prerequisites, as usual. To embark on this material you will certainly need a solid foundation in measure theory. Since I do of course use my own exposition as my principal source of references to the elementary ideas, I advise readers to ensure that they have easy access to all three previous volumes before starting serious work on this one. But you may not need to read very much of them. It might be prudent to glance through the detailed contents of Volume 1 and the first five chapters of Volume 2 to check that most of the material there is more or less familiar. I think §417 might be difficult to read without at least the results-only version of Chapter 25 to hand. But Volume 3, and the last three chapters of Volume 2, can probably be left on one side for the moment. Of course you will need the Lifting Theorem (Chapter 34) for §§447, 452 and 453, and Chapter 26 is essential background for Chapter 47, while Chapter 28 (on Fourier analysis) may help to make sense of Chapter 44, and parts of Chapter 27 (on probability theory) are necessary for §§455-456 and 458-459. You will certainly need some Fourier analysis for §479. And measure algebras are mentioned in every chapter except (I think) Chapter 48; but I hope that the cross-references are precise enough to lead you to what you need to know at any particular point. Even Maharam's theorem is hardly used in this volume.

What you will need, apart from any knowledge of measure theory, is a sound background in general topology. This volume calls on a great many miscellaneous facts from general topology, and the list in §4A2 is not a good place to start if continuity and compactness and the separation axioms are unfamiliar. My primary reference for topology is Engelking 89. I do not insist that you should have read this book (though of course I hope you will do so sometime); but I do think you should make sure that you can use it.

In the general introduction to this treatise, I wrote 'I make no attempt to describe the history of the subject', and I have generally been casual – some would say negligent – in my attributions of results to their discoverers. Through much of the first three volumes I did at least have the excuse that the history exists in print in far more detail than I am qualified to describe. In the present volume I find my position more uncomfortable, in that I have been watching the evolution of the subject relatively closely over the last forty years, and ought to be able to say something about it. Nevertheless I remain reluctant to make definite statements crediting one person rather than another with originating an idea. My more intimate knowledge of the topic makes me even more conscious than elsewhere of the danger of error and of the breadth of reading that would be necessary to produce a balanced account. In some cases I do attach a result to a specific published paper, but these attributions should never be regarded as an assertion that any particular author has priority; at most, they declare that a historian should examine the source cited before coming to any decision. I assure my friends and colleagues that my omissions are not intended to slight either them or those we all honour. What I have tried to do is to include in the bibliography to this volume all the published work which (as far as I am consciously aware) has influenced me while writing it, so that those who wish to go into the matter will have somewhere to start their investigations.

Note on second printing

I fear that there were even more errors, not all of them trivial, in the first printing of this volume than there were in previous volumes. I have tried to correct those which I have noticed; many surely remain. Apart from these, there are many minor expansions and elaborations, and a couple of new results, but few new ideas and no dramatic rearrangements. Details may be found in `http://www1.essex.ac.uk/maths/people/fremlin/mterr4.03.pdf`.

Both printers and readers found that the 945-page format of the first printing was hard to handle. I have therefore divided the volume into two parts for the second printing. I hope you will find that the additional convenience is worth the the increase in cost.

Note on second ('Lulu') edition

I was right about many errors remaining (particularly in §458, on relative independence), and I hope I have cleared some of them out of the way. There are substantial additions in the new edition, the most important being a vastly expanded §455 on Lévy processes, an account of Brownian motion and Newtonian potential in §§477-479, and Tao's proof of Szemerédi's theorem in §497. I have included theorems of A.Törnquist and G.W.Mackey on the realization of group actions on measure algebras, some material on a version of the Kantorovich-Rubinstein distance between two measures, and a section on Maharam submeasures (§496).

Version of 17.4.10

Chapter 41

Topologies and Measures I

I begin this volume with an introduction to some of the most important ways in which topologies and measures can interact, and with a description of the forms which such constructions as subspaces and product spaces take in such contexts. By far the most important concept is that of Radon measure (411Hb, §416). In Radon measure spaces we find both the richest combinations of ideas and the most important applications. But, as usual, we are led both by analysis of these ideas and by other interesting examples to consider wider classes of topological measure space, and the greater part of the chapter, by volume, is taken up by a description of the many properties of Radon measures individually and in partial combinations.

I begin the chapter with a short section of definitions (§411), including a handful of more or less elementary examples. The two central properties of a Radon measure are 'inner regularity' (411B) and 'τ-additivity' (411C). The former is an idea of great versatility which I look at in an abstract setting in §412. I take a section (§413) to describe some methods of constructing measure spaces, extending the rather limited range of constructions offered in earlier volumes. There are two sections on τ-additive measures, §§414 and 417; the former covers the elementary ideas, and the latter looks at product measures, where it turns out that we need a new technique to supplement the purely measure-theoretic constructions of Chapter 25. On the way to Radon measures in §416, I pause over 'quasi-Radon' measures (411Ha, §415), where inner regularity and τ-additivity first come effectively together.

The possible interactions of a topology and a measure on the same space are so varied that even a brief account makes a long chapter; and this is with hardly any mention of results associated with particular types of topological space, most of which must wait for later chapters. But I include one section on the two most important classes of functions acting between topological measure spaces (§418), and another describing some examples to demonstrate special phenomena (§419).

Version of 31.12.08

411 Definitions

In something of the spirit of §211, but this time without apologising, I start this volume with a list of definitions. The rest of Chapter 41 will be devoted to discussing these definitions and relationships between them, and integrating the new ideas into the concepts and constructions of earlier volumes; I hope that by presenting the terminology now I can give you a sense of the directions the following sections will take. I ought to remark immediately that there are many cases in which the exact phrasing of a definition is important in ways which may not be immediately apparent.

411A Definition A **topological measure space** is a quadruple $(X,\mathfrak{T},\Sigma,\mu)$ where (X,Σ,μ) is a measure space and $\mathfrak{T}$ is a topology on X such that $\mathfrak{T}\subseteq\Sigma$.

411B Definition Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of sets. I say that μ is **inner regular with respect to $\mathcal{K}$** if

$$\mu E=\sup\{\mu K: K\in\Sigma\cap\mathcal{K},\, K\subseteq E\}$$

for every $E\in\Sigma$.

411C Definition Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X. I say that μ is **τ-additive** (the phrase **τ-regular** has also been used) if whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets such that $\mathcal{G}\subseteq\Sigma$ and $\bigcup\mathcal{G}\in\Sigma$ then $\mu(\bigcup\mathcal{G})=\sup_{G\in\mathcal{G}}\mu G$.

411D Definition Let (X,Σ,μ) be a measure space and $\mathcal{H}$ a family of subsets of X. Then μ is **outer regular with respect to $\mathcal{H}$** if

$$\mu E = \inf\{\mu H : H \in \Sigma \cap \mathcal{H},\ H \supseteq E\}$$

for every $E \in \Sigma$.

Note that a totally finite measure on a topological space is inner regular with respect to the family of closed sets iff it is outer regular with respect to the family of open sets.

411E Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. If μ is inner regular with respect to the compact sets, it is τ-additive.

411F Definitions Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) I say that μ is **locally finite** if every point of X has a neighbourhood of finite measure.

(b) I say that μ is **effectively locally finite** if for every non-negligible measurable set $E \subseteq X$ there is a measurable open set $G \subseteq X$ such that $\mu G < \infty$ and $E \cap G$ is not negligible.

(c) A real-valued function f defined on a subset of X is **locally integrable** if for every $x \in X$ there is an open set G containing x such that $\int_G f$ is defined and finite.

411G Elementary facts (a) If μ is a locally finite measure on a topological space X, then $\mu^* K < \infty$ for every compact set $K \subseteq X$.

(b) A measure μ on $\mathbb{R}^r$ is locally finite iff every bounded set has finite outer measure.

(d) An effectively locally finite measure must be semi-finite.

(e) A locally finite measure on a Lindelöf space X is σ-finite.

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space such that μ is locally finite and inner regular with respect to the compact sets. Then μ is effectively locally finite.

(g) If μ is a measure on a topological space and $f \in \mathcal{L}^0(\mu)$ is locally integrable, then $\int_K f d\mu$ is finite for every compact $K \subseteq X$.

(h) If μ is a locally finite measure on a topological space X, and $f \in \mathcal{L}^p(\mu)$ for some $p \in [1, \infty]$, then f is locally integrable.

(i) If $(X, \mathfrak{T})$ is a completely regular space and μ is a locally finite topological measure on X, then the collection of open sets with negligible boundaries is a base for $\mathfrak{T}$.

(j) Let X and Y be topological spaces, $f : X \to Y$ a continuous function, μ a measure on X and μf^{-1} the image measure on Y. Then if μ is a topological measure, so is μf^{-1}, and if μ is τ-additive, so is μf^{-1}.

411H Definitions (a) A **quasi-Radon measure space** is a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is complete and locally determined (ii) μ is τ-additive, inner regular with respect to the closed sets and effectively locally finite.

(b) A **Radon measure space** is a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is complete and locally determined (ii) $\mathfrak{T}$ is Hausdorff (iii) μ is locally finite and inner regular with respect to the compact sets.

411I Remarks Note that a measure on Euclidean space $\mathbb{R}^r$ is a Radon measure on the definition above iff it is a Radon measure as described in 256Ad.

411J Definitions (a) If $(X, \mathfrak{T})$ is a topological space, I will say that a measure μ on X is **tight** if it is inner regular with respect to the closed compact sets.

(b) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a topological measure space, I will say that μ is **completion regular** if it is inner regular with respect to the zero sets.

411K Borel and Baire measures If $(X, \mathfrak{T})$ is a topological space, I will call a measure with domain the Borel σ-algebra of X a **Borel measure** on X, and a measure with domain the Baire σ-algebra of X a **Baire measure** on X.

411L Definition Let X be a set, Σ a σ-algebra of subsets of X and $(Y, \mathfrak{S})$ a topological space. I will say that a function $f : X \to Y$ is **measurable** if $f^{-1}[G] \in \Sigma$ for every open set $G \subseteq Y$.

411M Definition Let (X, Σ, μ) be a measure space, $\mathfrak{T}$ a topology on X, and $(Y, \mathfrak{S})$ another topological space. I will say that a function $f : X \to Y$ is **almost continuous** if μ is inner regular with respect to the family of subsets A of X such that $f \restriction A$ is continuous.

411N Definitions Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) I will call a set $A \subseteq X$ **self-supporting** if $\mu^*(A \cap G) > 0$ for every open set G such that $A \cap G$ is non-empty.

(b) A **support** of μ is a closed self-supporting set F such that $X \setminus F$ is negligible.

(c) μ can have at most one support.

(d) If μ is a τ-additive topological measure it has a support.

(e) Let X and Y be topological spaces with topological measures μ, ν respectively and a continuous inverse-measure-preserving function $f : X \to Y$. Suppose that μ has a support E. Then $\overline{f[E]}$ is the support of ν.

(f) μ is **strictly positive** if $\mu^* G > 0$ for every non-empty open set $G \subseteq X$.

***(g)** If $(X, \mathfrak{T})$ is a topological space, and μ is a strictly positive σ-finite measure on X such that the domain of μ includes a π-base for $\mathfrak{T}$, then X is ccc.

411O Example Lebesgue measure on $\mathbb{R}^r$ is a Radon measure; it is locally finite and tight. It is τ-additive and effectively locally finite. It is completion regular, outer regular with respect to the open sets and strictly positive.

411P Example: Stone spaces (a) Let $(Z, \mathfrak{T}, \Sigma, \mu)$ be the Stone space of a semi-finite measure algebra $(\mathfrak{A}, \bar{\mu})$, so that $(Z, \mathfrak{T})$ is a zero-dimensional compact Hausdorff space, (Z, Σ, μ) is complete and semi-finite, the open-and-closed sets are measurable, the negligible sets are the nowhere dense sets, and every measurable set differs by a nowhere dense set from an open-and-closed set.

(b) μ is inner regular with respect to the open-and-closed sets; it is completion regular and tight. Consequently it is τ-additive.

(c) μ is strictly positive. μ is effectively locally finite.

(d) The following are equiveridical, that is, if one is true so are the others:
(i) $(\mathfrak{A}, \bar{\mu})$ is localizable;
(ii) μ is strictly localizable;
(iii) μ is locally determined;
(iv) μ is a quasi-Radon measure.

(e) The following are equiveridical:

(i) μ is a Radon measure;

(ii) μ is totally finite;

(iii) μ is locally finite;

(iv) μ is outer regular with respect to the open sets.

(f) Let $W \subseteq Z$ be the union of all the open subsets of Z with finite measure. W has full outer measure, so $(\mathfrak{A}, \bar{\mu})$ can be identified with the measure algebra of the subspace measure μ_W. μ_W is locally finite. If $(\mathfrak{A}, \bar{\mu})$ is localizable, then μ_W is a Radon measure.

411Q Example: Dieudonné's measure Recall that a set $E \subseteq \omega_1$ is a Borel set iff either E or its complement includes a cofinal closed set. So we may define a Borel measure μ on ω_1 by saying that $\mu E = 1$ if E includes a cofinal closed set and $\mu E = 0$ if E is disjoint from a cofinal closed set. μ is complete. μ is a purely atomic probability measure.

μ is a topological measure; it is locally finite and effectively locally finite. It is inner regular with respect to the closed sets, therefore outer regular with respect to the open sets. It is not τ-additive.

μ is not completion regular.

The only self-supporting subset of ω_1 is the empty set. μ does not have a support.

411R Example: The Baire σ-algebra of ω_1 The Baire σ-algebra of ω_1 is the countable-cocountable algebra. The countable-cocountable measure μ on ω_1 is a Baire measure. μ is inner regular with respect to the zero sets and outer regular with respect to the cozero sets. μ is not τ-additive.

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412 Inner regularity

As will become apparent as the chapter progresses, the concepts introduced in §411 are synergic; their most interesting manifestations are in combinations of various kinds. Any linear account of their properties will be more than usually like a space-filling curve. But I have to start somewhere, and enough results can be expressed in terms of inner regularity, more or less by itself, to be a useful beginning.

After a handful of elementary basic facts (412A) and a list of standard applications (412B), I give some useful sufficient conditions for inner regularity of topological and Baire measures (412D, 412E, 412G), based on an important general construction (412C). The rest of the section amounts to a review of ideas from Volume 2 and Chapter 32 in the light of the new concept here. I touch on completions (412H), c.l.d. versions and complete locally determined spaces (412H, 412J, 412M), strictly localizable spaces (412I), inverse-measure-preserving functions (412K, 412L), measure algebras (412N), subspaces (412O, 412P), indefinite-integral measures (412Q) and product measures (412R-412V), with a brief mention of outer regularity (412W); most of the hard work has already been done in Chapters 21 and 25.

412A Lemma (a) Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets such that

whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$.

Then whenever $E \in \Sigma$ there is a countable disjoint family $\langle K_i \rangle_{i \in I}$ in $\mathcal{K} \cap \Sigma$ such that $K_i \subseteq E$ for every i and $\sum_{i \in I} \mu K_i = \mu E$. If moreover

(†) $K \cup K' \in \mathcal{K}$ whenever K, K' are disjoint members of $\mathcal{K}$,

then μ is inner regular with respect to $\mathcal{K}$. If $\bigcup_{i \in I} K_i \in \mathcal{K}$ for every countable disjoint family $\langle K_i \rangle_{i \in I}$ in $\mathcal{K}$, then for every $E \in \Sigma$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K = \mu E$.

(b) Let (X, Σ, μ) be a measure space, T a σ-subalgebra of Σ, and $\mathcal{K}$ a family of sets. If μ is inner regular with respect to T and $\mu{\restriction}\mathrm{T}$ is inner regular with respect to $\mathcal{K}$, then μ is inner regular with respect to $\mathcal{K}$.

(c) Let (X, Σ, μ) be a semi-finite measure space and $\langle \mathcal{K}_n \rangle_{n \in \mathbb{N}}$ a sequence of families of sets such that μ is inner regular with respect to $\mathcal{K}_n$ and

(‡) if $\langle K_i \rangle_{i \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}_n$, then $\bigcap_{i \in \mathbb{N}} K_i \in \mathcal{K}_n$

for every $n \in \mathbb{N}$. Then μ is inner regular with respect to $\bigcap_{n \in \mathbb{N}} \mathcal{K}_n$.

412B Corollary Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. Suppose that $\mathcal{K}$ is

either the family of Borel subsets of X

or the family of closed subsets of X

or the family of compact subsets of X

or the family of zero sets in X,

and suppose that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$. Then μ is inner regular with respect to $\mathcal{K}$.

412C Lemma Let (X, Σ, μ) be a semi-finite measure space, and suppose that $\mathcal{A} \subseteq \Sigma$ is such that

$\emptyset \in \mathcal{A} \subseteq \Sigma$,

$X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$.

Let T be the σ-subalgebra of Σ generated by $\mathcal{A}$. Let $\mathcal{K}$ be a family of subsets of X such that

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

whenever $A \in \mathcal{A}$, $F \in \Sigma$ and $\mu(A \cap F) > 0$, there is a $K \in \mathcal{K} \cap \mathrm{T}$ such that $K \subseteq A$ and $\mu(K \cap F) > 0$.

Then $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$.

412D Theorem Let $(X, \mathfrak{T})$ be a topological space and μ a semi-finite Baire measure on X. Then μ is inner regular with respect to the zero sets.

412E Theorem Let $(X, \mathfrak{T})$ be a perfectly normal topological space. Then any semi-finite Borel measure on X is inner regular with respect to the closed sets.

412F Lemma Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X such that μ is effectively locally finite with respect to $\mathfrak{T}$. Then

$$\mu E = \sup\{\mu(E \cap G) : G \text{ is a measurable open set of finite measure}\}$$

for every $E \in \Sigma$.

412G Theorem Let (X, Σ, μ) be a measure space with a topology $\mathfrak{T}$ such that μ is effectively locally finite with respect to $\mathfrak{T}$ and Σ is the σ-algebra generated by $\mathfrak{T} \cap \Sigma$. If

$$\mu G = \sup\{\mu F : F \in \Sigma \text{ is closed, } F \subseteq G\}$$

for every measurable open set G of finite measure, then μ is inner regular with respect to the closed sets.

412H Proposition Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets.

(a) If μ is inner regular with respect to $\mathcal{K}$, so are its completion $\hat{\mu}$ and c.l.d. version $\tilde{\mu}$.

(b) Now suppose that μ is semi-finite and that

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$.

If either $\hat{\mu}$ or $\tilde{\mu}$ is inner regular with respect to $\mathcal{K}$ then μ is inner regular with respect to $\mathcal{K}$.

412I Lemma Let (X, Σ, μ) be a strictly localizable measure space and $\mathcal{K}$ a family of sets such that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$.

(a) There is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that at most one X_i does not belong to $\mathcal{K}$, and that exceptional one, if any, is negligible.

(b) There is a disjoint family $\mathcal{L} \subseteq \mathcal{K} \cap \Sigma$ such that $\mu^* A = \sum_{L\in\mathcal{L}} \mu^*(A \cap L)$ for every $A \subseteq X$.

(c) If μ is σ-finite then the family $\langle X_i\rangle_{i\in I}$ of (a) and the set $\mathcal{L}$ of (b) can be taken to be countable.

412J Proposition Let (X, Σ, μ) be a complete locally determined measure space, and $\mathcal{K}$ a family of sets such that μ is inner regular with respect to $\mathcal{K}$.

(a) If $E \subseteq X$ is such that $E \cap K \in \Sigma$ for every $K \in \mathcal{K} \cap \Sigma$, then $E \in \Sigma$.

(b) If $E \subseteq X$ is such that $E \cap K$ is negligible for every $K \in \mathcal{K} \cap \Sigma$, then E is negligible.
(c) For any $A \subseteq X$, $\mu^* A = \sup_{K \in \mathcal{K} \cap \Sigma} \mu^*(A \cap K)$.
(d) Let f be a non-negative $[0, \infty]$-valued function defined on a subset of X. If $\int_K f$ is defined in $[0, \infty]$ for every $K \in \mathcal{K}$, then $\int f$ is defined and equal to $\sup_{K \in \mathcal{K}} \int_K f$.
(e) If f is a μ-integrable function and $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $\int_{X \setminus K} |f| \leq \epsilon$.

412K Proposition Let (X, Σ, μ) be a complete locally determined measure space, (Y, T, ν) a measure space and $f : X \to Y$ a function. Suppose that $\mathcal{K} \subseteq \mathrm{T}$ is such that
(i) ν is inner regular with respect to $\mathcal{K}$;
(ii) $f^{-1}[K] \in \Sigma$ and $\mu f^{-1}[K] = \nu K$ for every $K \in \mathcal{K}$;
(iii) whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $\nu K < \infty$ and $\mu(E \cap f^{-1}[K]) > 0$.
Then f is inverse-measure-preserving for μ and ν.

412L Corollary Let (X, Σ, μ) be a complete probability space, (Y, T, ν) a probability space and $f : X \to Y$ a function. Suppose that whenever $F \in \mathrm{T}$ and $\nu F > 0$ there is a $K \in \mathrm{T}$ such that $K \subseteq F$, $\nu K > 0$, $f^{-1}[K] \in \Sigma$ and $\mu f^{-1}[K] \geq \nu K$. Then f is inverse-measure-preserving.

412M Proposition Let X be a set and $\mathcal{K}$ a family of subsets of X. Suppose that μ and ν are two complete locally determined measures on X, with domains including $\mathcal{K}$, and both inner regular with respect to $\mathcal{K}$.
(a) If $\mu K \leq \nu K$ for every $K \in \mathcal{K}$, then $\mu \leq \nu$ in the sense of 234P.
(b) If $\mu K = \nu K$ for every $K \in \mathcal{K}$, then $\mu = \nu$.

412N Lemma Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$. Then

$$E^\bullet = \sup\{K^\bullet : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$$

in the measure algebra $\mathfrak{A}$ of μ, for every $E \in \Sigma$. $\{K^\bullet : K \in \mathcal{K} \cap \Sigma\}$ is order-dense in $\mathfrak{A}$; and if $\mathcal{K}$ is closed under finite unions, then $\{K^\bullet : K \in \mathcal{K} \cap \Sigma\}$ is topologically dense in $\mathfrak{A}$ for the measure-algebra topology.

412O Lemma Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$.
(a) If $E \in \Sigma$, then the subspace measure μ_E is inner regular with respect to $\mathcal{K}$.
(b) Let $Y \subseteq X$ be any set such that the subspace measure μ_Y is semi-finite. Then μ_Y is inner regular with respect to $\mathcal{K}_Y = \{K \cap Y : K \in \mathcal{K}\}$.

412P Proposition Let (X, Σ, μ) be a measure space, $\mathfrak{T}$ a topology on X and Y a subset of X; write $\mathfrak{T}_Y$ for the subspace topology of Y and μ_Y for the subspace measure on Y. Suppose that *either* $Y \in \Sigma$ *or* μ_Y is semi-finite.
(a) If μ is a topological measure, so is μ_Y.
(b) If μ is inner regular with respect to the Borel sets, so is μ_Y.
(c) If μ is inner regular with respect to the closed sets, so is μ_Y.
(d) If μ is inner regular with respect to the zero sets, so is μ_Y.
(e) If μ is effectively locally finite, so is μ_Y.

412Q Proposition Let (X, Σ, μ) be a measure space, and ν an indefinite-integral measure over μ. If μ is inner regular with respect to a family $\mathcal{K}$ of sets, so is ν.

412R Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Suppose that $\mathcal{K} \subseteq \mathcal{P}X$, $\mathcal{L} \subseteq \mathcal{P}Y$ and $\mathcal{M} \subseteq \mathcal{P}(X \times Y)$ are such that
(i) μ is inner regular with respect to $\mathcal{K}$;
(ii) ν is inner regular with respect to $\mathcal{L}$;
(iii) $K \times L \in \mathcal{M}$ for all $K \in \mathcal{K}$, $L \in \mathcal{L}$;
(iv) $M \cup M' \in \mathcal{M}$ whenever $M, M' \in \mathcal{M}$;
(v) $\bigcap_{n \in \mathbb{N}} M_n \in \mathcal{M}$ for every sequence $\langle M_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{M}$.
Then λ is inner regular with respect to $\mathcal{M}$.

412S Proposition Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Let $\mathfrak{T}$, $\mathfrak{S}$ be topologies on X and Y respectively, and give $X\times Y$ the product topology.

(a) If μ and ν are inner regular with respect to the closed sets, so is λ.

(b) If μ and ν are tight, so is λ.

(c) If μ and ν are inner regular with respect to the zero sets, so is λ.

(d) If μ and ν are inner regular with respect to the Borel sets, so is λ.

(e) If μ and ν are effectively locally finite, so is λ.

412T Lemma Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X,Λ,λ). Suppose that $\mathcal{K}_i\subseteq\mathcal{P}X_i$, $\mathcal{M}\subseteq\mathcal{P}X$ are such that

(i) μ_i is inner regular with respect to $\mathcal{K}_i$ for each $i\in I$;

(ii) $\pi_i^{-1}[K]\in\mathcal{M}$ for every $i\in I$ and $K\in\mathcal{K}_i$, writing $\pi_i(x)=x(i)$ for $x\in X$;

(iii) $M\cup M'\in\mathcal{M}$ whenever M, $M'\in\mathcal{M}$;

(iv) $\bigcap_{n\in\mathbb{N}}M_n\in\mathcal{M}$ for every sequence $\langle M_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}$.

Then λ is inner regular with respect to $\mathcal{M}$.

412U Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X,Λ,λ). Suppose that we are given a topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the product topology on X.

(a) If every μ_i is inner regular with respect to the closed sets, so is λ.

(b) If every μ_i is inner regular with respect to the zero sets, so is λ.

(c) If every μ_i is inner regular with respect to the Borel sets, so is λ.

412V Corollary Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X,Λ,λ). Suppose that we are given a Hausdorff topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the product topology on X. Suppose that every μ_i is tight, and that X_i is compact for all but countably many $i\in I$. Then λ is tight.

***412W Outer regularity: Proposition** Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) Suppose that μ is outer regular with respect to the open sets. Then for any integrable function $f:X\to[0,\infty]$ and $\epsilon>0$, there is a lower semi-continuous measurable function $g:X\to[0,\infty]$ such that $f\le g$ and $\int g\le\epsilon+\int f$.

(b) Now suppose that there is a sequence of measurable open sets of finite measure covering X. Then the following are equiveridical:

(i) μ is inner regular with respect to the closed sets;

(ii) μ is outer regular with respect to the open sets;

(iii) for any measurable set $E\subseteq X$ and $\epsilon>0$, there are a measurable closed set $F\subseteq E$ and a measurable open set $H\supseteq E$ such that $\mu(H\setminus F)\le\epsilon$;

(iv) for every measurable function $f:X\to[0,\infty[$ and $\epsilon>0$, there is a lower semi-continuous measurable function $g:X\to[0,\infty]$ such that $f\le g$ and $\int g-f\le\epsilon$;

(v) for every measurable function $f:X\to\mathbb{R}$ and $\epsilon>0$, there is a lower semi-continuous measurable function $g:X\to\,]-\infty,\infty]$ such that $f\le g$ and $\mu\{x:g(x)\ge f(x)+\epsilon\}\le\epsilon$.

Version of 1.1.17

413 Inner measure constructions

I now turn in a different direction, giving some basic results on the construction of inner regular measures. The first step is to describe 'inner measures' (413A) and a construction corresponding to the Carathéodory construction of measures from outer measures (413C). Just as every measure gives rise to an outer measure, it gives rise to an inner measure (413D). Inner measures form an effective tool for studying complete locally determined measures (413F).

The most substantial results of the section concern the construction of measures as extensions of functionals defined on various classes $\mathcal{K}$ of sets. Typically, $\mathcal{K}$ is closed under finite unions and countable intersections, though we can sometimes relax the hypotheses a bit. The methods here make it possible to distinguish arguments which produce finitely additive functionals (413I, 413O, 413R, 413S) from the succeeding steps to countably additive measures (413J, 413P, 413U). 413I-413N investigate conditions on a functional $\phi : \mathcal{K} \to [0,\infty[$ sufficient to produce a measure extending ϕ, necessarily unique, which is inner regular with respect to $\mathcal{K}$ or $\mathcal{K}_\delta$, the set of intersections of sequences in $\mathcal{K}$. 413O-413P look instead at functionals defined on sublattices of the class $\mathcal{K}$ of interest, and at sufficient conditions to ensure the existence of a measure, not normally unique, defined on the whole of $\mathcal{K}$, inner regular with respect to $\mathcal{K}$ and extending the given functional. Finally, 413R-413U are concerned with majorizations rather than extensions; we seek a measure μ such that $\mu K \ge \phi K$ for $K \in \mathcal{K}$, while μX is as small as possible.

413A Definition Let X be a set. An **inner measure** on X is a functional $\phi : \mathcal{P}X \to [0,\infty]$ such that

$\phi\emptyset = 0$;

(α) $\phi(A \cup B) \ge \phi A + \phi B$ for all disjoint A, $B \subseteq X$;

(β) if $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X and $\phi A_0 < \infty$ then $\phi(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \phi A_n$;

($*$) $\phi A = \sup\{\phi B : B \subseteq A,\ \phi B < \infty\}$ for every $A \subseteq X$.

413B Lemma Let X be a set and $\phi : \mathcal{P}X \to [0,\infty]$ any functional such that $\phi\emptyset = 0$. Then

$$\Sigma = \{E : E \subseteq X,\ \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}$$

is an algebra of subsets of X, and $\phi(E \cup F) = \phi E + \phi F$ for all disjoint E, $F \in \Sigma$.

413C Measures from inner measures: Theorem Let X be a set and $\phi : X \to [0,\infty]$ an inner measure. Set

$$\Sigma = \{E : E \subseteq X,\ \phi(A \cap E) + \phi(A \setminus E) = \phi A \text{ for every } A \subseteq X\}.$$

Then $(X, \Sigma, \phi{\restriction}\Sigma)$ is a complete measure space.

413D The inner measure defined by a measure Let (X, Σ, μ) be any measure space. μ gives rise to an inner measure μ_* defined by the formula

$$\mu_* A = \sup\{\mu E : E \in \Sigma^f,\ E \subseteq A\},$$

where I write Σ^f for $\{E : E \in \Sigma,\ \mu E < \infty\}$.

413E Proposition Let (X, Σ, μ) be a measure space. Write Σ^f for $\{E : E \in \Sigma,\ \mu E < \infty\}$.

(a) For every $A \subseteq X$ there is an $E \in \Sigma$ such that $E \subseteq A$ and $\mu E = \mu_* A$.

(b) $\mu_* A \le \mu^* A$ for every $A \subseteq X$.

(c) If $E \in \Sigma$ and $A \subseteq X$, then $\mu_*(E \cap A) + \mu^*(E \setminus A) \le \mu E$, with equality if either (i) $\mu E < \infty$ or (ii) μ is semi-finite.

(d) $\mu_* E \le \mu E$ for every $E \in \Sigma$, with equality if either $\mu E < \infty$ or μ is semi-finite.

(e) If μ is inner regular with respect to $\mathcal{K}$, then (counting the supremum of the empty set as 0) $\mu_* A = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma^f,\ K \subseteq A\}$ for every $A \subseteq X$.

(f) If $A \subseteq X$ is such that $\mu_* A = \mu^* A < \infty$, then A is measured by the completion of μ.

(g) If $\hat\mu$, $\tilde\mu$ are the completion and c.l.d. version of μ, then $\hat\mu_* = \tilde\mu_* = \mu_*$.

(h) If (Y, T, ν) is another measure space, and $f : X \to Y$ is an inverse-measure-preserving function, then

$$\mu^*(f^{-1}[B]) \le \nu^* B, \quad \mu_*(f^{-1}[B]) \ge \nu_* B$$

for every $B \subseteq Y$, and

$$\nu^*(f[A]) \ge \mu^* A$$

for every $A \subseteq X$.

(i) Suppose that μ is semi-finite. If $A \subseteq E \in \Sigma$, then E is a measurable envelope of A iff $\mu_*(E \setminus A) = 0$.

413F Lemma Let (X, Σ, μ) be a complete locally determined measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$. Then for $E \subseteq X$ the following are equiveridical:

(i) $E \in \Sigma$;
(ii) $E \cap K \in \Sigma$ whenever $K \in \Sigma \cap \mathcal{K}$;
(iii) $\mu^*(K \cap E) + \mu^*(K \setminus E) = \mu^* K$ for every $K \in \mathcal{K}$;
(iv) $\mu_*(K \cap E) + \mu_*(K \setminus E) = \mu_* K$ for every $K \in \mathcal{K}$;
(v) $\mu^*(E \cap K) = \mu_*(E \cap K)$ for every $K \in \mathcal{K} \cap \Sigma$;
(vi) $\min(\mu^*(K \cap E), \mu^*(K \setminus E)) < \mu K$ whenever $K \in \mathcal{K} \cap \Sigma$ and $0 < \mu K < \infty$;
(vii) $\max(\mu_*(K \cap E), \mu_*(K \setminus E)) > 0$ whenever $K \in \mathcal{K} \cap \Sigma$ and $\mu K > 0$.

413G Lemma Let (X, Σ, μ) be a complete locally determined measure space and suppose that μ is inner regular with respect to $\mathcal{K} \subseteq \Sigma$. Suppose that $f : X \to \mathbb{R}$ is a function, and for $\alpha \in \mathbb{R}$ set $E_\alpha = \{x : f(x) \le \alpha\}$, $F_\alpha = \{x : f(x) \ge \beta\}$. Then f is Σ-measurable iff

$$\min(\mu^*(E_\alpha \cap K), \mu^*(F_\beta \cap K)) < \mu K$$

whenever $K \in \mathcal{K}$, $0 < \mu K < \infty$ and $\alpha < \beta$.

413H Proposition Let (X, Σ, μ) be a complete totally finite measure space, (Y, T, ν) a measure space, and $\mathfrak{S}$ a Hausdorff topology on Y such that ν is inner regular with respect to the closed sets. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of inverse-measure-preserving functions from X to Y. If $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in X$, then f is inverse-measure-preserving.

413I Lemma Let X be a set and $\mathcal{K}$ a family of subsets of X such that
$\emptyset \in \mathcal{K}$,
(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,
(‡) $K \cap K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$.
Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that
(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$.
Set

$$\phi A = \sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq A\} \text{ for } A \subseteq X,$$

$$\Sigma = \{E : E \subseteq X,\ \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then Σ is an algebra of subsets of X, including $\mathcal{K}$, and $\phi \restriction \Sigma : \Sigma \to [0, \infty]$ is an additive functional extending ϕ_0.

413J Theorem Let X be a set and $\mathcal{K}$ a family of subsets of X such that
$\emptyset \in \mathcal{K}$,
(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,
(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$.
Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that
(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,
(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.
Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}$.

413K Theorem Let X be a set and $\mathcal{K}$ a family of subsets of X such that
$\emptyset \in \mathcal{K}$,
(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,
(‡) $K \cap K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$.
Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that
(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$.

413L Corollary (a) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu : \Sigma \to [0,\infty[$ a non-negative finitely additive functional such that $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a complete locally determined measure on X which is inner regular with respect to the family Σ_δ of intersections of sequences in Σ.

(b) Let X be a set, Σ a subalgebra of $\mathcal{P}X$, and $\nu : \Sigma \to [0,\infty[$ a non-negative finitely additive functional such that $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a measure defined on the σ-algebra of subsets of X generated by Σ.

413M Definition A **countably compact class** is a family $\mathcal{K}$ of sets such that $\bigcap_{n\in\mathbb{N}} K_n \ne \emptyset$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{i\le n} K_i \ne \emptyset$ for every $n \in \mathbb{N}$.

413N Corollary Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0,\infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$.

413O Theorem Let X be a set, T_0 a subring of $\mathcal{P}X$, and $\nu_0 : \mathrm{T}_0 \to [0,\infty[$ a finitely additive functional. Suppose that $\mathcal{K} \subseteq \mathcal{P}X$ is a family of sets such that

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

every member of $\mathcal{K}$ is included in some member of T_0,

and ν_0 is inner regular with respect to $\mathcal{K}$ in the sense that

(α) $\nu_0 E = \sup\{\nu_0 K : K \in \mathcal{K} \cap \mathrm{T}_0,\ K \subseteq E\}$ for every $E \in \mathrm{T}_0$.

Then ν_0 has an extension to a non-negative finitely additive functional ν_1, defined on a subring T_1 of $\mathcal{P}X$ including $\mathrm{T}_0 \cup \mathcal{K}$, inner regular with respect to $\mathcal{K}$, and such that whenever $E \in \mathrm{T}_1$ and $\epsilon > 0$ there is an $E_0 \in \mathrm{T}_0$ such that $\nu_1(E \triangle E_0) \le \epsilon$.

413P Corollary Let (X, T, ν) be a measure space and $\mathcal{K}$ a countably compact class of subsets of X such that

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

$\nu^* K < \infty$ for every $K \in \mathcal{K}$,

ν is inner regular with respect to $\mathcal{K}$.

Then ν has an extension to a complete locally determined measure μ, defined on every member of $\mathcal{K}$, inner regular with respect to $\mathcal{K}$, and such that whenever $E \in \operatorname{dom}\mu$ and $\mu E < \infty$ there is an $F \in \mathrm{T}$ such that $\mu(E \triangle F) = 0$.

413Q Definitions Let P be a lattice and $f : P \to [-\infty,\infty[$ a function.

(a) f is **supermodular** if $f(p \vee q) + f(p \wedge q) \ge f(p) + f(q)$ for all p, $q \in P$.

(b) f is **submodular** if $f(p \vee q) + f(p \wedge q) \le f(p) + f(q)$ for all p, $q \in P$.

(c) f is **modular** if $f(p \vee q) + f(p \wedge q) = f(p) + f(q)$ for all p, $q \in P$.

413R Lemma Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\phi : \mathcal{K} \to \mathbb{R}$ be a bounded supermodular functional such that $\phi\emptyset = 0$. Then there is a finitely additive functional $\nu : \mathcal{P}X \to [0, \infty[$ such that

$$\nu X = \sup_{K \in \mathcal{K}} \phi K, \quad \nu K \ge \phi K \text{ for every } K \in \mathcal{K}.$$

413S Theorem Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let Σ be the algebra of subsets of X generated by $\mathcal{K}$, and $\nu_0 : \Sigma \to [0, \infty[$ a finitely additive functional. Then there is a finitely additive functional $\nu : \Sigma \to [0, \infty[$ such that

(i) $\nu X = \sup_{K \in \mathcal{K}} \nu_0 K$,

(ii) $\nu K \ge \nu_0 K$ for every $K \in \mathcal{K}$,

(iii) ν is inner regular with respect to $\mathcal{K}$ in the sense that $\nu E = \sup\{\nu K : K \in \mathcal{K}, K \subseteq E\}$ for every $E \in \Sigma$.

413T Lemma Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X. Then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K} \cup \{\emptyset, X\}$ such that $K \cup L \in \mathcal{K}^*$ and $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}^*$ whenever K, $L \in \mathcal{K}^*$ and $\langle K_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{K}^*$.

413U Corollary Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X. Let T be a subalgebra of $\mathcal{P}X$ and $\nu : \mathrm{T} \to \mathbb{R}$ a non-negative finitely additive functional.

(a) There is a complete measure μ on X such that $\mu X \le \nu X$, $\mathcal{K} \subseteq \operatorname{dom} \mu$ and $\mu K \ge \nu K$ for every $K \in \mathcal{K} \cap \mathrm{T}$.

(b) If $\emptyset \in \mathcal{K}$ and

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$,

(‡) $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$,

we may arrange that μ is inner regular with respect to $\mathcal{K}$.

Version of 26.1.10

414 τ-additivity

The second topic I wish to treat is that of 'τ-additivity'. Here I collect results which do not depend on any strong kind of inner regularity. I begin with what I think of as the most characteristic feature of τ-additivity, its effect on the properties of semi-continuous functions (414A), with a variety of corollaries, up to the behaviour of subspace measures (414K). A very important property of τ-additive topological measures is that they are often strictly localizable (414J).

The theory of inner regular τ-additive measures belongs to the next section, but here I give two introductory results: conditions under which a τ-additive measure will be inner regular with respect to closed sets (414M) and conditions under which a measure which is inner regular with respect to closed sets will be τ-additive (414N). I end the section with notes on 'density' and 'lifting' topologies (414P-414R).

414A Theorem Let $(X, \mathfrak{T})$ be a topological space and μ an effectively locally finite τ-additive measure on X with domain Σ and measure algebra $\mathfrak{A}$.

(a) Suppose that $\mathcal{G}$ is a non-empty family in $\Sigma \cap \mathfrak{T}$ such that $H = \bigcup \mathcal{G}$ also belongs to Σ. Then $\sup_{G \in \mathcal{G}} G^\bullet = H^\bullet$ in $\mathfrak{A}$.

(b) Write $\mathcal{L}$ for the family of Σ-measurable lower semi-continuous functions from X to $\mathbb{R}$. Suppose that $\emptyset \ne A \subseteq \mathcal{L}$ and set $g(x) = \sup_{f \in A} f(x)$ for every $x \in X$. If g is Σ-measurable and finite almost everywhere, then $\tilde{g}^\bullet = \sup_{f \in A} f^\bullet$ in $L^0(\mu)$, where $\tilde{g}(x) = g(x)$ whenever $g(x)$ is finite.

(c) Suppose that $\mathcal{F}$ is a non-empty family of measurable closed sets such that $\bigcap \mathcal{F} \in \Sigma$. Then $\inf_{F \in \mathcal{F}} F^\bullet = (\bigcap \mathcal{F})^\bullet$ in $\mathfrak{A}$.

(d) Write $\mathfrak{U}$ for the family of Σ-measurable upper semi-continuous functions from X to $\mathbb{R}$. Suppose that $A \subseteq \mathfrak{U}$ is non-empty and set $g(x) = \inf_{f \in A} f(x)$ for every $x \in X$. If g is Σ-measurable and finite almost everywhere, then $\tilde{g}^\bullet = \inf_{f \in A} f^\bullet$ in $L^0(\mu)$, where $\tilde{g}(x) = g(x)$ whenever $g(x)$ is finite.

414B Corollary Let X be a topological space and μ an effectively locally finite τ-additive topological measure on X.

(a) Suppose that A is a non-empty upwards-directed family of lower semi-continuous functions from X to $[0, \infty]$. Set $g(x) = \sup_{f \in A} f(x)$ in $[0, \infty]$ for every $x \in X$. Then $\int g = \sup_{f \in A} \int f$ in $[0, \infty]$.

(b) Suppose that A is a non-empty downwards-directed family of non-negative continuous real-valued functions on X, and that $g(x) = \inf_{x \in A} f(x)$ for every $x \in X$. If any member of A is integrable, then $\int g = \inf_{f \in A} \int f$.

414C Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space and $\mathcal{F}$ a non-empty downwards-directed family of closed sets. If $\inf_{F \in \mathcal{F}} \mu F$ is finite, this is the measure of $\bigcap \mathcal{F}$.

414D Corollary Let μ be an effectively locally finite τ-additive measure on a topological space X. If ν is a totally finite measure with the same domain as μ, truly continuous with respect to μ, then ν is τ-additive. In particular, if μ is σ-finite and ν is absolutely continuous with respect to μ, then ν is τ-additive.

414E Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space. Suppose that $\mathcal{G} \subseteq \mathfrak{T}$ is non-empty and upwards-directed, and $H = \bigcup \mathcal{G}$. Then

(a) $\mu(E \cap H) = \sup_{G \in \mathcal{G}} \mu(E \cap G)$ for every $E \in \Sigma$;

(b) if f is a non-negative virtually measurable real-valued function defined almost everywhere in X, then $\int_H f = \sup_{G \in \mathcal{G}} \int_G f$ in $[0, \infty]$.

414F Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space. Then for every $E \in \Sigma$ there is a unique relatively closed self-supporting set $F \subseteq E$ such that $\mu(E \setminus F) = 0$.

414G Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Hausdorff effectively locally finite τ-additive topological measure space and $E \in \Sigma$ is an atom for μ, then there is an $x \in E$ such that $E \setminus \{x\}$ is negligible.

414H Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is an effectively locally finite τ-additive topological measure space and ν is an indefinite-integral measure over μ, then ν is a τ-additive topological measure.

414I Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. If $E \subseteq X$ and $\mathcal{G} \subseteq \mathfrak{T}$ are such that $E \subseteq \bigcup \mathcal{G}$ and $E \cap G \in \Sigma$ for every $G \in \mathcal{G}$, then $E \in \Sigma$.

414J Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. Then μ is strictly localizable.

414K Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X, and $Y \subseteq X$ a subset such that the subspace measure μ_Y is semi-finite. If μ is an effectively locally finite τ-additive topological measure, so is μ_Y.

414L Lemma Let $(X, \mathfrak{T})$ be a topological space, and μ, ν two effectively locally finite Borel measures on X which agree on the open sets. Then they are equal.

414M Proposition Let (X, Σ, μ) be a measure space with a regular topology $\mathfrak{T}$ such that μ is effectively locally finite and τ-additive and Σ includes a base for $\mathfrak{T}$.

(a) $\mu G = \sup\{\mu F : F \in \Sigma \text{ is closed}, F \subseteq G\}$ for every open set $G \in \Sigma$.

(b) If μ is inner regular with respect to the σ-algebra generated by $\mathfrak{T} \cap \Sigma$, it is inner regular with respect to the closed sets.

414N Proposition Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X. Suppose that (i) μ is semi-finite and inner regular with respect to the closed sets (ii) whenever $\mathcal{F}$ is a non-empty downwards-directed family of measurable closed sets with empty intersection and $\inf_{F\in\mathcal{F}}\mu F<\infty$, then $\inf_{F\in\mathcal{F}}\mu F=0$. Then μ is τ-additive.

414O Proposition If X is a hereditarily Lindelöf space then every measure on X is τ-additive.

414P Density topologies: Proposition Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}X=X$. Set

$$\mathfrak{T}=\{E:E\in\Sigma,\ E\subseteq\underline{\phi}E\}.$$

Then $\mathfrak{T}$ is a topology on X, the **density topology** associated with $\underline{\phi}$, and $(X,\mathfrak{T},\Sigma,\mu)$ is an effectively locally finite τ-additive topological measure space; μ is strictly positive and inner regular with respect to the open sets.

414Q Lifting topologies Let (X,Σ,μ) be a measure space and $\phi:\Sigma\to\Sigma$ a lifting. The **lifting topology** associated with ϕ is the topology generated by $\{\phi E:E\in\Sigma\}$.

414R Proposition Let (X,Σ,μ) be a complete locally determined measure space and $\phi:\Sigma\to\Sigma$ a lifting with lifting topology $\mathfrak{S}$ and density topology $\mathfrak{T}$. Then $\mathfrak{S}\subseteq\mathfrak{T}\subseteq\Sigma$, and μ is τ-additive, effectively locally finite and strictly positive with respect to $\mathfrak{S}$. Moreover, $\mathfrak{S}$ is zero-dimensional.

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415 Quasi-Radon measure spaces

We are now I think ready to draw together the properties of inner regularity and τ-additivity. Indeed, this section will unite several of the themes which have been running through the treatise so far: (strict) localizability, subspaces and products as well as the new concepts of this chapter. In these terms, the principal results are that a quasi-Radon space is strictly localizable (415A), any subspace of a quasi-Radon space is quasi-Radon (415B), and the product of a family of strictly positive quasi-Radon probability measures on separable metrizable spaces is quasi-Radon (415E). I describe a basic method of constructing quasi-Radon measures (415K), with details of one of the standard ways of applying it (415L, 415N) and some notes on how to specify a quasi-Radon measure uniquely (415H-415I). I spell out useful results on indefinite-integral measures (415O) and L^p spaces (415P), and end the section with a discussion of the Stone space Z of a localizable measure algebra $\mathfrak{A}$ and an important relation in $Z\times X$ when $\mathfrak{A}$ is the measure algebra of a quasi-Radon measure space X (415Q-415R).

It would be fair to say that the study of quasi-Radon spaces for their own sake is a minority interest. If you are not already well acquainted with Radon measure spaces, it would make good sense to read this section in parallel with the next. In particular, the constructions of 415K and 415L derive much of their importance from the corresponding constructions in §416.

415A Theorem A quasi-Radon measure space is strictly localizable.

415B Theorem Any subspace of a quasi-Radon measure space is quasi-Radon.

415C Proposition Let $(X,\mathfrak{T})$ be a regular topological space.

(a) If μ is a complete locally determined effectively locally finite τ-additive topological measure on X, inner regular with respect to the Borel sets, then it is a quasi-Radon measure.

(b) If μ is an effectively locally finite τ-additive Borel measure on X, its c.l.d. version is a quasi-Radon measure.

415D Proposition Let $(X, \mathfrak{T})$ be a regular hereditarily Lindelöf topological space.

(i) If μ is a complete effectively locally finite measure on X, inner regular with respect to the Borel sets, and its domain includes a base for $\mathfrak{T}$, then it is a quasi-Radon measure.

(ii) If μ is an effectively locally finite Borel measure on X, then its completion is a quasi-Radon measure.

(iii) Any quasi-Radon measure on X is σ-finite.

(iv) Any quasi-Radon measure on X is completion regular.

415E Theorem Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of separable metrizable quasi-Radon probability spaces such that every μ_i is strictly positive, and λ the product measure on $X = \prod_{i\in I} X_i$. Then

(i) λ is a completion regular quasi-Radon measure;

(ii) if $F \subseteq X$ is a closed self-supporting set, there is a countable set $J \subseteq I$ such that F is determined by coordinates in J, so F is a zero set.

415F Corollary (a) If Y is either $[0,1[$ or $]0,1[$, endowed with Lebesgue measure, and I is any set, then Y^I, with the product topology and measure, is a quasi-Radon measure space.

(b) If $\langle\nu_i\rangle_{i\in I}$ is a family of probability distributions on $\mathbb{R}$, and every ν_i is strictly positive, then the product measure on $\mathbb{R}^I$ is a quasi-Radon measure.

415G Comparing quasi-Radon measures: Proposition Let X be a topological space, and μ, ν two quasi-Radon measures on X. Then the following are equiveridical:

(i) $\mu F \le \nu F$ for every closed set $F \subseteq X$;

(ii) $\mu \le \nu$ in the sense of 234P.

If ν is locally finite, we can add

(iii) $\mu G \le \nu G$ for every open set $G \subseteq X$;

(iv) there is a base $\mathcal{U}$ for the topology of X such that $G \cup H \in \mathcal{U}$ for all G, $H \in \mathcal{U}$ and $\mu G \le \nu G$ for $G \in \mathcal{U}$.

415H Uniqueness of quasi-Radon measures: Proposition Let $(X, \mathfrak{T})$ be a topological space and μ, ν two quasi-Radon measures on X. Then the following are equiveridical:

(i) $\mu = \nu$;

(ii) $\mu F = \nu F$ for every closed set $F \subseteq X$;

(iii) $\mu G = \nu G$ for every open set $G \subseteq X$;

(iv) there is a base $\mathcal{U}$ for the topology of X such that $G \cup H \in \mathcal{U}$ for every G, $H \in \mathcal{U}$ and $\mu{\restriction}\mathcal{U} = \nu{\restriction}\mathcal{U}$;

(v) there is a base $\mathcal{U}$ for the topology of X such that $G \cap H \in \mathcal{U}$ for every G, $H \in \mathcal{U}$ and $\mu{\restriction}\mathcal{U} = \nu{\restriction}\mathcal{U}$.

415I Proposition Let X be a completely regular topological space and μ, ν two quasi-Radon measures on X such that $\int f d\mu = \int f d\nu$ whenever $f : X \to \mathbb{R}$ is a bounded continuous function integrable with respect to both measures. Then $\mu = \nu$.

415J Proposition Let X be a regular topological space, Y a subspace of X, and ν a quasi-Radon measure on Y. Then there is a quasi-Radon measure μ on X such that $\mu E = \nu(E \cap Y)$ whenever μ measures E, that is, Y has full outer measure in X and ν is the subspace measure on Y.

415K Theorem Let X be a topological space and $\mathcal{K}$ a family of closed subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $F \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $F \subseteq K$ is closed.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{K\in\mathcal{K}'} \phi_0 K = 0$ whenever $\mathcal{K}'$ is a non-empty downwards-directed subset of $\mathcal{K}$ with empty intersection,

(γ) whenever $K \in \mathcal{K}$ and $\phi_0 K > 0$, there is an open set G such that the supremum $\sup_{K'\in\mathcal{K}, K'\subseteq G} \phi_0 K'$ is finite, while $\phi_0 K' > 0$ for some $K' \in \mathcal{K}$ such that $K' \subseteq K \cap G$.

Then there is a unique quasi-Radon measure on X extending ϕ_0 and inner regular with respect to $\mathcal{K}$.

415L Proposition Let (X, Σ_0, μ_0) be a measure space and $\mathfrak{T}$ a topology on X such that Σ_0 includes a base for $\mathfrak{T}$ and μ_0 is τ-additive, effectively locally finite and inner regular with respect to the closed sets. Then μ_0 has a unique extension to a quasi-Radon measure μ on X. Moreover,

(i) $\mu F = \mu_0^* F$ whenever $F \subseteq X$ is closed and $\mu_0^* F < \infty$,

(ii) $\mu G = (\mu_0)_* G$ whenever $G \subseteq X$ is open,

(iii) the embedding $\Sigma_0 \subsetneqq \Sigma$ identifies the measure algebra $(\mathfrak{A}_0, \bar{\mu}_0)$ of μ_0 with an order-dense subalgebra of the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ, so that the subrings $\mathfrak{A}_0^f$, $\mathfrak{A}^f$ of elements of finite measure coincide, and $L^p(\mu_0)$ may be identified with $L^p(\mu)$ for $1 \le p < \infty$,

(iv) whenever $E \in \Sigma$ and $\mu E < \infty$, there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$,

(v) for every μ-integrable real-valued function f there is a μ_0-integrable function g such that $f = g$ μ-a.e.

If μ_0 is complete and locally determined, then we have

(i)$'$ $\mu F = \mu_0^* F$ for every closed $F \subseteq X$.

If μ_0 is localizable, then we have

(iii)$'$ $\mathfrak{A}_0 = \mathfrak{A}$, so that $L^0(\mu) \cong L^0(\mu_0)$ and $L^\infty(\mu) \cong L^\infty(\mu_0)$,

(iv)$'$ for every $E \in \Sigma$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$,

(v)$'$ for every Σ-measurable real-valued function f there is a Σ_0-measurable real-valued function g such that $f = g$ μ-a.e.

415M Corollary Let $(X, \mathfrak{T})$ be a regular topological space and μ_0 an effectively locally finite τ-additive measure on X, defined on the σ-algebra generated by a base for $\mathfrak{T}$. Then μ_0 has a unique extension to a quasi-Radon measure on X.

415N Corollary Let $(X, \mathfrak{T})$ be a completely regular topological space, and μ_0 a τ-additive effectively locally finite Baire measure on X. Then μ_0 has a unique extension to a quasi-Radon measure on X.

415O Proposition (a) Let $(X, \mathfrak{T})$ be a topological space, and μ, ν two quasi-Radon measures on X. Then ν is an indefinite-integral measure over μ iff $\nu F = 0$ whenever $F \subseteq X$ is closed and $\mu F = 0$.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and ν an indefinite-integral measure over μ. If ν is effectively locally finite it is a quasi-Radon measure.

415P Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space.

(a) Suppose that $(X, \mathfrak{T})$ is completely regular. If $1 \le p < \infty$ and $f \in \mathcal{L}^p(\mu)$, then for any $\epsilon > 0$ there is a bounded continuous function $g : X \to \mathbb{R}$ such that $\mu\{x : g(x) \ne 0\} < \infty$ and $\|f - g\|_p \le \epsilon$.

(b) Suppose that $(X, \mathfrak{T})$ is regular and Lindelöf. Let $f \in \mathcal{L}^0(\mu)$ be locally integrable. Then for any $\epsilon > 0$ there is a continuous function $g : X \to \mathbb{R}$ such that $\|f - g\|_1 \le \epsilon$.

415Q Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ be the Stone space of $(\mathfrak{A}, \bar{\mu})$. For $E \in \Sigma$ let $E^* \subseteq Z$ be the open-and-closed set corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Define $R \subseteq Z \times X$ by saying that $(z, x) \in R$ iff $x \in F$ whenever $F \subseteq X$ is closed and $z \in F^*$. Set $Q = R^{-1}[X]$.

(a) R is a closed subset of $Z \times X$.

(b) For any $E \in \Sigma$, $R[E^*]$ is the smallest closed set such that $\mu(E \setminus R[E^*]) = 0$. In particular, if $F \subseteq X$ is closed then $R[F^*]$ is the self-supporting closed set included in F such that $\mu(F \setminus R[F^*]) = 0$; and $R[Z]$ is the support of μ.

(c) Q has full outer measure for ν.

(d) For any $E \in \Sigma$, $R^{-1}[E] \triangle (Q \cap E^*)$ is negligible; consequently $\nu^* R^{-1}[E] = \mu E$ and $R^{-1}[E] \cap R^{-1}[X \setminus E]$ is negligible.

(e) For any $A \subseteq X$, $\nu^* R^{-1}[A] = \mu^* A$.

(f) If $(X, \mathfrak{T})$ is regular, then $R^{-1}[G]$ is relatively open in Q for every open set $G \subseteq X$, $R^{-1}[F]$ is relatively closed in Q for every closed set $F \subseteq X$ and $R^{-1}[X \setminus E] = Q \setminus R^{-1}[E]$ for every Borel set $E \subseteq X$.

415R Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Hausdorff quasi-Radon measure space and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of its measure algebra. Let $R \subseteq Z \times X$ be the relation described in 415Q. Then

(a) R is (the graph of) a function f;

(b) f is inverse-measure-preserving for the subspace measure ν_Q on $Q = \operatorname{dom} f$, and in fact μ is the image measure $\nu_Q f^{-1}$;

(c) if $(X, \mathfrak{T})$ is regular, then f is continuous.

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416 Radon measure spaces

We come now to the results for which the chapter so far has been preparing. The centre of topological measure theory is the theory of 'Radon' measures, measures inner regular with respect to compact sets. Most of the section is devoted to pulling the earlier work together, and in particular to re-stating theorems on quasi-Radon measures in the new context. Of course this has to begin with a check that Radon measures are quasi-Radon (416A). It follows immediately that Radon measures are (strictly) localizable (416B). After presenting a miscellany of elementary facts, I turn to the constructions of §413, which take on simpler and more dramatic forms in this context (416J-416P). I proceed to investigate subspace measures (416R-416T) and some special product measures (416U). I end the section with further notes on the forms which earlier theorems on Stone spaces (416V) and compact measure spaces (416W) take when applied to Radon measure spaces.

416A Proposition A Radon measure space is quasi-Radon.

416B Corollary A Radon measure space is strictly localizable.

416C Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally finite Hausdorff quasi-Radon measure space. Then the following are equiveridical:

(i) μ is a Radon measure;

(ii) whenever $E \in \Sigma$ and $\mu E > 0$ there is a compact set K such that $\mu(E \cap K) > 0$;

(iii) $\sup\{K^\bullet : K \subseteq X \text{ is compact}\} = 1$ in the measure algebra of μ.

If μ is totally finite we can add

(iv) $\sup\{\mu K : K \subseteq X \text{ is compact}\} = \mu X$.

416D Lemma (a) In a Radon measure space, every compact set has finite measure.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $E \subseteq X$ a set such that $E \cap K \in \Sigma$ for every compact $K \subseteq X$. Then $E \in \Sigma$.

(c) A Radon measure is inner regular with respect to the self-supporting compact sets.

(d) Let X be a Hausdorff space and μ a tight locally finite complete locally determined measure on X. If μ measures every compact set, μ is a Radon measure.

(e) Let X be a Hausdorff space and $\langle \mu_i \rangle_{i \in I}$ a family of Radon measures on X. Let $\mu = \sum_{i \in I} \mu_i$ be their sum. If μ is locally finite, it is a Radon measure.

416E Specification of Radon measures: Proposition Let X be a Hausdorff space and μ, ν two Radon measures on X.

(a) The following are equiveridical:

(i) $\nu \leq \mu$ in the sense of 234P;

(ii) $\mu K \leq \nu K$ for every compact set $K \subseteq X$;

(iii) $\mu G \leq \nu G$ for every open set $G \subseteq X$;

(iv) $\mu F \leq \nu F$ for every closed set $F \subseteq X$.

If X is locally compact, we can add

(v) $\int f d\mu \leq \int f d\nu$ for every non-negative continuous function $f : X \to \mathbb{R}$ with compact support.

(b) The following are equiveridical:

(i) $\mu = \nu$;

(ii) $\mu K = \nu K$ for every compact set $K \subseteq X$;

(iii) $\mu G = \nu G$ for every open set $G \subseteq X$;

(iv) $\mu F = \nu F$ for every closed set $F \subseteq X$.

If X is locally compact, we can add

(v) $\int f d\mu = \int f d\nu$ for every continuous function $f : X \to \mathbb{R}$ with compact support.

416F Proposition Let X be a Hausdorff space and μ a Borel measure on X. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) μ is locally finite and tight;

(iii) μ is locally finite and effectively locally finite, and $\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\}$ for every open set $G \subseteq X$;

(iv) μ is locally finite, effectively locally finite and τ-additive, and $\mu G = \sup\{\mu(G \cap K) : K \subseteq X \text{ is compact}\}$ for every open set $G \subseteq X$.

In this case the extension is unique; it is the c.l.d. version of μ.

416G Proposition Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space and μ a locally finite quasi-Radon measure on X. Then μ is a Radon measure.

416H Corollary Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, and μ a locally finite, effectively locally finite, τ-additive Borel measure on X. Then μ is tight and its c.l.d. version is a Radon measure, the unique Radon measure on X extending μ.

416I Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally compact Radon measure space. Write C_k for the space of continuous real-valued functions on X with compact supports. If $1 \leq p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a $g \in C_k$ such that $\|f - g\|_p \leq \epsilon$.

416J Theorem Let X be a Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(γ) for every $x \in X$ there is an open set G containing x such that $\sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq G\}$ is finite.

Then there is a unique Radon measure on X extending ϕ_0.

416K Proposition Let X be a Hausdorff space, T a subring of $\mathcal{P}X$ such that $\mathcal{H} = \{G : G \in \mathrm{T} \text{ is open}\}$ covers X, and $\nu : \mathrm{T} \to [0, \infty[$ a finitely additive functional. Then there is a Radon measure μ on X such that $\mu K \geq \nu K$ for every compact $K \in \mathrm{T}$ and $\mu G \leq \nu G$ for every open $G \in \mathrm{T}$.

416L Proposition Let X be a regular Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X, and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

(α_1) $\phi_0 K \leq \phi_0(K \cup L) \leq \phi_0 K + \phi_0 L$ for all K, $L \in \mathcal{K}$,

(α_2) $\phi_0(K \cup L) = \phi_0 K + \phi_0 L$ whenever K, $L \in \mathcal{K}$ and $K \cap L = \emptyset$,

(γ) for every $x \in X$ there is an open set G containing x such that $\sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq G\}$ is finite.

Then there is a unique Radon measure μ on X such that

$$\mu K = \inf_{G \subseteq X \text{ is open}, K \subseteq G} \sup_{L \subseteq G \text{ is compact}} \phi_0 L$$

for every $K \in \mathcal{K}$.

416M Corollary Let X be a locally compact Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X, and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

$\phi_0 K \le \phi_0(K \cup L) \le \phi_0 K + \phi_0 L$ for all K, $L \in \mathcal{K}$,

$\phi_0(K \cup L) = \phi_0 K + \phi_0 L$ whenever K, $L \in \mathcal{K}$ and $K \cap L = \emptyset$.

Then there is a unique Radon measure μ on X such that

$$\mu K = \inf\{\phi_0 K' : K' \in \mathcal{K},\, K \subseteq \operatorname{int} K'\}$$

for every $K \in \mathcal{K}$.

416N Henry's theorem Let X be a Hausdorff space and μ_0 a measure on X which is locally finite and tight. Then μ_0 has an extension to a Radon measure μ on X, and the extension may be made in such a way that whenever $\mu E < \infty$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$.

416O Theorem Let X be a Hausdorff space and T a subalgebra of $\mathcal{P}X$. Let $\nu : \mathrm{T} \to [0, \infty[$ be a finitely additive functional such that

$$\nu E = \sup\{\nu F : F \in \mathrm{T},\, F \subseteq E,\, F \text{ is closed}\} \text{ for every } E \in \mathrm{T},$$

$$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{F \in \mathrm{T}, F \supseteq K} \nu F.$$

Then there is a Radon measure μ on X extending ν.

416P Theorem Let X be a Hausdorff space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) for every non-negligible measurable set $E \subseteq X$ there is a compact set $K \subseteq E$ such that $\mu^* K > 0$.

If μ is totally finite, we can add

(iii) $\sup\{\mu^* K : K \subseteq X \text{ is compact}\} = \mu X$.

416Q Proposition (a) Let X be a compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Then any non-negative finitely additive functional from $\mathcal{E}$ to $\mathbb{R}$ has an extension to a Radon measure on X. If X is zero-dimensional then the extension is unique.

(b) Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Then there is a one-to-one correspondence between non-negative additive functionals ν on $\mathfrak{A}$ and Radon measures μ on Z given by the formula

$$\nu a = \mu \widehat{a} \text{ for every } a \in \mathfrak{A},$$

where for $a \in \mathfrak{A}$ I write $\widehat{a}$ for the corresponding open-and-closed subset of Z.

416R Theorem (a) Any subspace of a Radon measure space is a quasi-Radon measure space.

(b) A measurable subspace of a Radon measure space is a Radon measure space.

(c) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Hausdorff complete locally determined topological measure space, and $Y \subseteq X$ is such that the subspace measure μ_Y on Y is a Radon measure, then $Y \in \Sigma$.

416S Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space.

(a) If ν is a locally finite indefinite-integral measure over μ, it is a Radon measure.

(b) If ν is a Radon measure on X and $\nu K = 0$ whenever $K \subseteq X$ is compact and $\mu K = 0$, then ν is an indefinite-integral measure over μ.

416T Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally finite completely regular Hausdorff quasi-Radon measure space. Then it is isomorphic, as topological measure space, to a subspace of a locally compact Radon measure space.

416U Theorem (a) If $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is a family of compact metrizable Radon probability spaces such that every μ_i is strictly positive, the product measure on $X=\prod_{i\in I}X_i$ is a completion regular Radon measure.

(b) In particular, the usual measures on $\{0,1\}^I$ and $[0,1]^I$ and $\mathcal{P}I$ are completion regular Radon measures, for any set I.

416V Stone spaces: Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, and $(Z,\mathfrak{S},\mathrm{T},\nu)$ the Stone space of its measure algebra $(\mathfrak{A},\bar\mu)$. For $E\in\Sigma$ let E^* be the open-and-closed set in Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Define $R\subseteq Z\times X$ by saying that $(z,x)\in R$ iff $x\in F$ whenever $F\subseteq X$ is closed and $z\in F^*$.

(a) R is the graph of a function $f:Q\to X$, where $Q=R^{-1}[X]$. If we set $W=\bigcup\{K^*:K\subseteq X$ is compact$\}$, then $W\subseteq Q$ is a ν-conegligible open set, and the subspace measure ν_W on W is a Radon measure.

(b) Setting $g=f\restriction W$, g is continuous and μ is the image measure $\nu_W g^{-1}$.

(c) If X is compact, $W=Q=Z$ and $\mu=\nu g^{-1}$.

416W Compact measure spaces: Proposition (a) Any Radon measure space is a compact measure space, therefore perfect.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, with measure algebra $(\mathfrak{A},\bar\mu)$, and (Y,T,ν) a complete strictly localizable measure space, with measure algebra $(\mathfrak{B},\bar\nu)$. If $\pi:\mathfrak{A}\to\mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a function $f:Y\to X$ such that $f^{-1}[E]\in\mathrm{T}$ and $f^{-1}[E]^\bullet=\pi E^\bullet$ for every $E\in\Sigma$. If π is measure-preserving, f is inverse-measure-preserving.

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417 τ-additive product measures

The 'ordinary' product measures introduced in Chapter 25 have served us well for a volume and a half. But we come now to a fundamental obstacle. If we start with two Radon measure spaces, their product measure, as defined in §251, need not be a Radon measure (419E). Furthermore, the counterexample is one of the basic compact measure spaces of the theory; and while it is dramatically non-metrizable, there is no other reason to set it aside. Consequently, if we wish (as we surely do) to create Radon measure spaces as products of Radon measure spaces, we need a new construction. This is the object of the present section. It turns out that the construction can be adapted to work well beyond the special context of Radon measure spaces; the methods here apply to general effectively locally finite τ-additive topological measures (for the product of finitely many factors) and to τ-additive topological probability measures (for the product of infinitely many factors).

The fundamental theorems are 417C and 417E, listing the essential properties of what I call 'τ-additive product measures', which are extensions of the c.l.d. product measures and product probability measures of Chapter 25. They depend on a straightforward lemma on the extension of a measure to make every element of a given class of sets negligible (417A). We still have Fubini's theorem for the new product measures (417G), and the basic operations from §254 still apply (417J, 417K, 417M).

It is easy to check that if we start with quasi-Radon measures, then the τ-additive product measure is again quasi-Radon (417N, 417O). The τ-additive product of two Radon measures is Radon (417P), and the τ-additive product of Radon probability measures with compact supports is Radon (417Q).

In the last part of the section I look at continuous real-valued functions and Baire σ-algebras; it turns out that for these the ordinary product measures are adequate (417U, 417V).

417A Lemma Let (X,Σ,μ) be a semi-finite measure space, and $\mathcal{A}\subseteq\mathcal{P}X$ a family of sets such that the inner measure $\mu_*(\bigcup_{n\in\mathbb{N}}A_n)$ is 0 for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$. Then there is a measure μ' on X, extending μ, such that

(i) $\mu' A$ is defined and zero for every $A\in\mathcal{A}$;

(ii) μ' is complete if μ is;

(iii) for every F in the domain Σ' of μ' there is an $E \in \Sigma$ such that $\mu'(F \triangle E) = 0$;

(iv) whenever $\mathcal{K}$, $\mathcal{G}$ are families of sets such that

(α) μ is inner regular with respect to $\mathcal{K}$,

(β) $K \cup K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

(γ) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

(δ) for every $A \in \mathcal{A}$ there is a $G \in \mathcal{G}$, including A, such that $G \setminus A \in \Sigma$,

(ϵ) $K \setminus G \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $G \in \mathcal{G}$,

then μ' is inner regular with respect to $\mathcal{K}$.

In particular, μ and μ' have isomorphic measure algebras, so that μ' is localizable if μ is.

417B Lemma Let X and Y be topological spaces, and ν a τ-additive topological measure on Y.

(a) If $W \subseteq X \times Y$ is open, then $x \mapsto \nu W[\{x\}] : X \to [0, \infty]$ is lower semi-continuous.

(b) If ν is effectively locally finite and σ-finite and $W \subseteq X \times Y$ is a Borel set, then $x \mapsto \nu W[\{x\}]$ is Borel measurable.

(c) If $f : X \times Y \to [0, \infty]$ is a lower semi-continuous function, then $x \mapsto \int f(x, y)\nu(dy) : X \to [0, \infty]$ is lower semi-continuous.

(d) If ν is totally finite and $f : X \times Y \to \mathbb{R}$ is a bounded continuous function, then $x \mapsto \int f(x, y)\nu(dy)$ is continuous.

(e) If ν is totally finite and $W \subseteq X \times Y$ is a Baire set, then $x \mapsto \nu W[\{x\}]$ is Baire measurable.

417C Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces.

(a) There is a unique complete locally determined effectively locally finite τ-additive topological measure $\tilde{\lambda}$ on $X \times Y$ which is inner regular with respect to the σ-algebra $\tilde{\Lambda}^0 = (\Sigma \widehat{\otimes} \mathrm{T}) \vee \mathcal{B}(X \times Y)$ generated by $\{E \times F : E \in \Sigma,\ F \in \mathrm{T}\} \cup \{W : W \subseteq X \times Y \text{ is open}\}$ and is such that $\tilde{\lambda}(E \times F)$ is defined and equal to $\mu E \cdot \nu F$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$.

(b)(i) $\tilde{\lambda}$ extends the c.l.d. product measure λ on $X \times Y$, and if $\tilde{Q}$ is measured by $\tilde{\lambda}$, there is a Q measured by λ such that $\tilde{\lambda}(\tilde{Q} \triangle Q) = 0$.

(ii) The support of $\tilde{\lambda}$ is the product of the supports of μ and ν.

(iii) If $\tilde{Q}$ is measured by $\tilde{\lambda}$,

$$\tilde{\lambda}\tilde{Q} = \sup\{\tilde{\lambda}(\tilde{Q} \cap (G \times H)) : G \in \mathfrak{T},\ H \in \mathfrak{S},\ \mu G < \infty,\ \mu H < \infty\}.$$

(iv) If $\Sigma' \subseteq \Sigma$ and $\mathrm{T}' \subseteq \mathrm{T}$ are σ-algebras such that μ is inner regular with respect to Σ' and ν is inner regular with respect to T', then $\tilde{\lambda}$ is inner regular with respect to $(\Sigma' \widehat{\otimes} \mathrm{T}') \vee \mathcal{B}(X \times Y)$.

(v) If $W \subseteq X \times Y$ is open, then

(α) there is an open set $W' \in \Lambda$ such that $W' \subseteq W$ and $\lambda W' = \tilde{\lambda} W$, so $\tilde{\lambda} W = \lambda_* W$,

(β) if $E \in \Sigma$ and $F \in \mathrm{T}$ then

$$\tilde{\lambda}(W \cap (E \times F)) = \int_E \nu(W[\{x\}] \cap F)\mu(dx) = \int_F \mu(W^{-1}[\{y\}] \cap E)\nu(dy).$$

(vi) If μ and ν are inner regular with respect to the Borel sets, so is $\tilde{\lambda}$.

(vii) If μ and ν are inner regular with respect to the closed sets, so is $\tilde{\lambda}$.

(viii) If μ and ν are tight, so is $\tilde{\lambda}$.

(ix) If μ and ν are locally finite, so is $\tilde{\lambda}$.

417D Multiple products: Proposition (a) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a non-empty finite family of effectively locally finite τ-additive topological measure spaces. Then there is a unique complete locally determined effectively locally finite τ-additive topological measure $\tilde{\lambda}$ on $X = \prod_{i\in I} X_i$ which is inner regular with respect to the σ-algebra $(\widehat{\bigotimes}_{i\in I}\Sigma_i) \vee \mathcal{B}(X)$ generated by $\{\prod_{i\in I} E_i : E_i \in \Sigma_i \text{ for every } i \in I\} \cup \{W : W \subseteq X \text{ is open}\}$ and is such that $\tilde{\lambda}(\prod_{i\in I} E_i)$ is defined and equal to $\prod_{i\in I} \mu_i E_i$ whenever $E_i \in \Sigma_i$ for every $i \in I$.

(b) If now $\langle I_k\rangle_{k\in K}$ is a partition of I into non-empty sets, and $\tilde\lambda_k$ is the product measure defined by the construction of (a) on $Z_k=\prod_{i\in I_k}X_i$ for each $k\in K$, then the natural bijection between X and $\prod_{k\in K}Z_k$ identifies $\tilde\lambda$ with the product of the $\tilde\lambda_k$ defined by the construction of (a).

417E Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, with product probability space (X,Λ,λ).

(a) There is a unique complete τ-additive topological probability measure $\tilde\lambda$ on X which is inner regular with respect to $\tilde\Lambda^0=(\widehat{\bigotimes}_{i\in I}\Sigma_i)\vee\mathcal{B}(X)$ and is such that $\tilde\lambda\{x:x\in X,\ x(i)\in E_i\text{ for every }i\in J\}$ is defined and equal to $\prod_{i\in J}\mu_iE_i$ whenever $J\subseteq I$ is finite and $E_i\in\Sigma_i$ for every $i\in J$.

(b)(i) If $\tilde Q$ is measured by $\tilde\lambda$, there is a $Q\in\Lambda$ such that $\tilde\lambda(\tilde Q\triangle Q)=0$.

(ii) $\tilde\lambda W=\lambda_*W$ for every open set $W\subseteq X$, and $\tilde\lambda F=\lambda^*F$ for every closed set $F\subseteq X$.

(iii) The support of $\tilde\lambda$ is the product of the supports of the μ_i.

(iv) If for each $i\in I$ we are given a σ-subalgebra $\Sigma_i'\subseteq\Sigma_i$ such that μ_i is inner regular with respect to Σ_i', then $\tilde\lambda$ is inner regular with respect to $(\widehat{\bigotimes}_{i\in I}\Sigma_i')\vee\mathcal{B}(X)$.

(v) If every μ_i is inner regular with respect to the Borel sets, so is $\tilde\lambda$.

(vi) If every μ_i is inner regular with respect to the closed sets, so is $\tilde\lambda$.

417F Notation In the context of 417C, 417D or 417E, I will call $\tilde\lambda$ the **τ-additive product measure** on $\prod_{i\in I}X_i$.

417G Fubini's theorem for τ-additive product measures Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces. Let $\tilde\lambda$ be the τ-additive product measure on $X\times Y$, and $\tilde\Lambda$ its domain.

(a) Let f be a $[-\infty,\infty]$-valued function such that $\int f\,d\tilde\lambda$ is defined in $[-\infty,\infty]$ and $(X\times Y)\setminus\{(x,y):(x,y)\in\operatorname{dom}f,\ f(x,y)=0\}$ can be covered by a set of the form $X\times\bigcup_{n\in\mathbb{N}}Y_n$ where $\nu Y_n<\infty$ for every $n\in\mathbb{N}$. Then the repeated integral $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\tilde\lambda$.

(b) Let $f:X\times Y\to[0,\infty]$ be lower semi-continuous. Then

$$\iint f(x,y)\nu(dy)\mu(dx)=\iint f(x,y)\mu(dx)\nu(dy)=\int fd\tilde\lambda$$

in $[0,\infty]$.

(c) Let f be a $\tilde\Lambda$-measurable real-valued function defined $\tilde\lambda$-a.e. on $X\times Y$. If either $\iint|f(x,y)|\nu(dy)\mu(dx)$ or $\iint|f(x,y)|\mu(dx)\nu(dy)$ is defined and finite, then f is $\tilde\lambda$-integrable.

417H Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces. Let $\tilde\lambda$ be the τ-additive product measure on $X\times Y$, and $\tilde\Lambda$ its domain. If $A\subseteq X$, $B\subseteq Y$ are non-negligible sets such that $A\times B\in\tilde\Lambda$, then $A\in\Sigma$ and $B\in\mathrm{T}$.

417I Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces, and $\tilde\lambda$ the τ-additive product measure on $X\times Y$. Suppose that $A\subseteq X$ and $B\subseteq Y$, and write μ_A, ν_B for the corresponding subspace measures; assume that both μ_A and ν_B are semi-finite. Then these are also effectively locally finite and τ-additive, and the subspace measure $\tilde\lambda_{A\times B}$ induced by $\tilde\lambda$ on $A\times B$ is just the τ-additive product measure of μ_A and ν_B.

417J Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces and $\mathcal{K}$ a partition of I. For $J\subseteq I$ let $\tilde\lambda_J$ be the τ-additive product measure on $Z_J=\prod_{i\in J}X_i$, and write Z for $\prod_{K\in\mathcal{K}}Z_K$. Then we have a natural bijection $\phi:Z\to Z_I$ defined by setting

$$\phi(\langle z_K\rangle_{K\in\mathcal{K}})=\bigcup_{K\in\mathcal{K}}z_K$$

which identifies the τ-additive product $\tilde\lambda$ of the family $\langle\tilde\lambda_K\rangle_{K\in\mathcal{K}}$ with $\tilde\lambda_I$.

In particular, if $K\subseteq I$ is any set, then $\tilde\lambda_I$ can be identified with the τ-additive product of the τ-additive product measures on $\prod_{i\in K}X_i$ and $\prod_{i\in I\setminus K}X_i$.

417K Corollary Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces and $(X, \tilde{\Lambda}, \tilde{\lambda})$ their τ-additive product. For $J \subseteq I$ let $\tilde{\lambda}_J$ be the τ-additive product measure on $X_J = \prod_{i\in J} X_i$, and $\tilde{\Lambda}_J$ its domain; let $\pi_J : X \to X_J$ be the canonical map. Then $\tilde{\lambda}_J$ is the image measure $\tilde{\lambda}\pi_J^{-1}$. In particular, if $W \in \tilde{\Lambda}$ is determined by coordinates in $J \subseteq I$, then $\pi_J[W] \in \tilde{\Lambda}_J$ and $\tilde{\lambda}_J\pi_J[W] = \tilde{\lambda}W$.

417L Corollary Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, and $(X, \tilde{\Lambda}, \tilde{\lambda})$ their τ-additive product. Let $\langle K_j\rangle_{j\in J}$ be a disjoint family of subsets of I, and for $j \in J$ write $\tilde{\Lambda}_j$ for the σ-algebra of members of $\tilde{\Lambda}$ determined by coordinates in K_j. Then $\langle\tilde{\Lambda}_j\rangle_{j\in J}$ is a stochastically independent family of σ-algebras.

417M Proposition Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is strictly positive. For $J \subseteq I$ let π_J be the canonical map from X onto $X_J = \prod_{i\in J} X_i$; write λ_J, $\tilde{\lambda}_J$ for the ordinary and τ-additive product measures on X_J, and Λ_J, $\tilde{\Lambda}_J$ for their domains. Set $\tilde{\lambda} = \tilde{\lambda}_I$, $\tilde{\Lambda} = \tilde{\Lambda}_I$, $\lambda = \lambda_I$, $\Lambda = \Lambda_I$.

(a) Let $F \subseteq X$ be a closed self-supporting set, and J the smallest subset of I such that F is determined by coordinates in J. Then

(i) if $W \in \tilde{\Lambda}$ is such that $W\triangle F$ is $\tilde{\lambda}$-negligible and W is determined by coordinates in $K \subseteq I$, then $K \supseteq J$;

(ii) J is countable;

(iii) there is a $W \in \Lambda$, determined by coordinates in J, such that $W\triangle F$ is $\tilde{\lambda}$-negligible.

(b) $\tilde{\lambda}$ is inner regular with respect to the family of sets of the form $\bigcap_{n\in\mathbb{N}} V_n$ where each $V_n \in \tilde{\Lambda}$ is determined by coordinates in a finite set.

(c) If $W \in \tilde{\Lambda}$, there are a countable $J \subseteq I$ and sets W', $W'' \in \tilde{\Lambda}$, determined by coordinates in J, such that $W' \subseteq W \subseteq W''$ and $\tilde{\lambda}(W'' \setminus W') = 0$. Consequently $\tilde{\lambda}\pi_J^{-1}[\pi_J[W]] = \tilde{\lambda}W$.

417N Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two quasi-Radon measure spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X \times Y$ is a quasi-Radon measure, the unique quasi-Radon measure on $X \times Y$ such that $\tilde{\lambda}(E \times F) = \mu E \cdot \nu F$ for every $E \in \Sigma$ and $F \in \mathrm{T}$.

417O Theorem Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X = \prod_{i\in I} X_i$ is a quasi-Radon measure, the unique quasi-Radon measure on X extending the ordinary product measure.

417P Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be Radon measure spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X \times Y$ is a Radon measure, the unique Radon measure on $X \times Y$ such that $\tilde{\lambda}(E \times F) = \mu E \cdot \nu F$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$.

417Q Theorem Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of Radon probability spaces, and $\tilde{\lambda}$ the τ-additive product measure on $X = \prod_{i\in I} X_i$. For each $i \in I$, let $Z_i \subseteq X_i$ be the support of μ_i. Suppose that $J = \{i : i \in I,\ Z_i \text{ is not compact}\}$ is countable. Then $\tilde{\lambda}$ is a Radon measure, the unique Radon measure on X extending the ordinary product measure.

417R Notation I will use the phrase **quasi-Radon product measure** for a τ-additive product measure which is in fact a quasi-Radon measure; similarly, a **Radon product measure** is a τ-additive product measure which is a Radon measure.

417S Proposition (a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces and λ the c.l.d. product measure on $X \times Y$. If every open subset of $X \times Y$ is measured by λ, then λ is the τ-additive product measure on $X \times Y$.

(b) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces and λ the ordinary product measure on $X = \prod_{i\in I} X_i$. If every open subset of X is measured by λ, then λ is the τ-additive product measure on X.

(c) In (b), let λ_J be the ordinary product measure on $X_J = \prod_{i\in J} X_i$ for each $J \subseteq I$, and $\tilde\lambda_J$ the τ-additive product measure. If $\lambda_J = \tilde\lambda_J$ for every finite $J \subseteq I$, and every μ_i is strictly positive, then $\lambda = \tilde\lambda_I$ is the τ-additive product measure on X.

417T Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces and λ the c.l.d. product measure on $X \times Y$. If X has a conegligible subset with a countable network, then λ is the τ-additive product measure on $X \times Y$.

417U Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces. Let λ be the ordinary product probability measure on $X = \prod_{i\in I} X_i$ and Λ its domain. Then every continuous function $f : X \to \mathbb{R}$ is Λ-measurable, so Λ includes the Baire σ-algebra of X.

417V Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces, and $(X \times Y, \Lambda, \lambda)$ their c.l.d. product. Then every continuous function $f : X \times Y \to \mathbb{R}$ is Λ-measurable, and the Baire σ-algebra of $X \times Y$ is included in Λ.

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418 Measurable functions and almost continuous functions

In this section I work through the basic properties of measurable and almost continuous functions, as defined in 411L and 411M. I give the results in the full generality allowed by the terminology so far introduced, but most of the ideas are already required even if you are interested only in Radon measure spaces as the domains of the functions involved. Concerning the codomains, however, there is a great difference between metrizable spaces and others, and among metrizable spaces separability is of essential importance.

I start with the elementary properties of measurable functions (418A-418C) and almost continuous functions (418D). Under mild conditions on the domain space, almost continuous functions are measurable (418E); for a separable metrizable codomain, we can expect that measurable functions should be almost continuous (418J). Before coming to this, I spend a couple of paragraphs on image measures: a locally finite image measure under a measurable function is Radon if the measure on the domain is Radon and the function is almost continuous (418I).

418L-418Q are important results on expressing given Radon measures as image measures associated with continuous functions, first dealing with ordinary functions $f : X \to Y$ (418L) and then coming to Prokhorov's theorem on projective limits of probability spaces (418M).

The machinery of the first part of the section can also be used to investigate representations of vector-valued functions in terms of product spaces (418R-418T).

418A Proposition Let X be a set, Σ a σ-algebra of subsets of X, Y a topological space and $f : X \to Y$ a measurable function.

(a) $f^{-1}[F] \in \Sigma$ for every Borel set $F \subseteq Y$.

(b) If $A \subseteq X$ is any set, endowed with the subspace σ-algebra, then $f\restriction A : A \to Y$ is measurable.

(c) Let $(Z, \mathfrak{T})$ be another topological space. Then $gf : X \to Z$ is measurable for every Borel measurable function $g : Y \to Z$; in particular, for every continuous function $g : Y \to Z$.

418B Proposition Let X be a set and Σ a σ-algebra of subsets of X.

(a) If Y is a metrizable space and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions from X to Y such that $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in X$, then $f : X \to Y$ is measurable.

(b) If Y is a topological space, Z is a separable metrizable space and $f : X \to Y$, $g : X \to Z$ are functions, then $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable iff f and g are measurable.

(c) If Y is a hereditarily Lindelöf space, $\mathcal{U}$ a family of open sets generating its topology, and $f : X \to Y$ a function such that $f^{-1}[U] \in \Sigma$ for every $U \in \mathcal{U}$, then f is measurable.

(d) If $\langle Y_i\rangle_{i\in I}$ is a countable family of separable metrizable spaces, with product Y, then a function $f : X \to Y$ is measurable iff $\pi_i f : X \to Y_i$ is measurable for every i, writing $\pi_i(y) = y(i)$ for $y \in Y$ and $i \in \mathbb{N}$.

418C Proposition Let (X,Σ,μ) be a measure space and Y a Polish space. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable functions from X to Y. Then

$$\{x : x\in X,\ \lim_{n\to\infty} f_n(x) \text{ is defined in } Y\}$$

belongs to Σ.

418D Proposition Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) Suppose that Y is a topological space. Then any continuous function from X to Y is almost continuous.

(b) Suppose that Y and Z are topological spaces, $f : X\to Y$ is almost continuous and $g : Y\to Z$ is continuous. Then $gf : X\to Z$ is almost continuous.

(c) Suppose that $(Y,\mathfrak{S},\mathrm{T},\nu)$ is a σ-finite topological measure space, Z is a topological space, $g : Y\to Z$ is almost continuous and $f : X\to Y$ is inverse-measure-preserving and almost continuous. Then $gf : X\to Z$ is almost continuous.

(d) Suppose that μ is semi-finite, and that $\langle Y_i\rangle_{i\in I}$ is a countable family of topological spaces with product Y. Then a function $f : X\to Y$ is almost continuous iff $f_i=\pi_i f$ is almost continuous for every $i\in I$, writing $\pi_i(y)=y(i)$ for $i\in I$ and $y\in Y$.

418E Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined topological measure space, Y a topological space, and $f : X\to Y$ an almost continuous function. Then f is measurable.

418F Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a semi-finite topological measure space, Y a metrizable space, and $f : X\to Y$ a function. Suppose there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of almost continuous functions from X to Y such that $f(x)=\lim_{n\to\infty} f_n(x)$ for almost every $x\in X$. Then f is almost continuous.

418V Proposition Let (X,Σ,μ) be a σ-finite measure space, $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the Borel sets, $(Y,\mathfrak{S})$ a topological space and $f : X\to Y$ an almost continuous function. Then there is a Borel measurable function $g : X\to Y$ which is equal almost everywhere to f.

418G Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite quasi-Radon measure space, Y a metrizable space and $f : X\to Y$ an almost continuous function. Then there is a conegligible set $X_0\subseteq X$ such that $f[X_0]$ is separable.

418H Proposition (a) Let X and Y be topological spaces, μ an effectively locally finite τ-additive topological measure on X, and $f : X\to Y$ an almost continuous function. Then the image measure μf^{-1} is τ-additive.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a totally finite quasi-Radon measure space, $(Y,\mathfrak{S})$ a regular topological space, and $f : X\to Y$ an almost continuous function. Then there is a unique quasi-Radon measure ν on Y such that f is inverse-measure-preserving for μ and ν.

418I Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, Y a Hausdorff space, and $f : X\to Y$ an almost continuous function. If the image measure $\nu=\mu f^{-1}$ is locally finite, it is a Radon measure.

418J Theorem Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a second-countable space and $f : X\to Y$ is measurable. Then f is almost continuous.

418K Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space and Y a separable metrizable space. Then a function $f : X\to Y$ is measurable iff it is almost continuous.

418L Theorem Let $(X,\mathfrak{T})$ be a Hausdorff space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a Radon measure space and $f : X\to Y$ a continuous function such that whenever $F\in\mathrm{T}$ and $\nu F>0$ there is a compact set $K\subseteq X$ such that $\nu(F\cap f[K])>0$. Then there is a Radon measure μ on X such that ν is the image measure μf^{-1} and the inverse-measure-preserving function f induces an isomorphism between the measure algebras of ν and μ.

418M Prokhorov's theorem Suppose that $(I, \leq)$, $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\leq j\in I}$, $(X, \mathfrak{T})$ and $\langle g_i\rangle_{i\in I}$ are such that

(α) $(I, \leq)$ is a non-empty upwards-directed partially ordered set,

(β) every $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ is a Radon probability space,

(γ) $f_{ij} : X_j \to X_i$ is an inverse-measure-preserving function whenever $i \leq j$ in I,

(δ) $(X, \mathfrak{T})$ is a Hausdorff space,

(ϵ) $g_i : X \to X_i$ is a continuous function for every $i \in I$,

(ζ) $g_i = f_{ij}g_j$ whenever $i \leq j$ in I.

(η) for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu_i g_i[K] \geq 1 - \epsilon$ for every $i \in I$.

Then there is a Radon probability measure μ on X such that every g_i is inverse-measure-preserving for μ. If moreover

(θ) the family $\langle g_i\rangle_{i\in I}$ separates the points of X,

then μ is uniquely defined.

418O Proposition Suppose that $(I, \leq)$, $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\leq j\in I}$ are such that

$(I, \leq)$ is a non-empty upwards-directed partially ordered set,

every $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ is a compact Radon measure space,

$f_{ij} : X_j \to X_i$ is a continuous inverse-measure-preserving function whenever $i \leq j$ in I,

$f_{ij}f_{jk} = f_{ik}$ whenever $i \leq j \leq k$ in I.

Then there are a compact Hausdorff space X and a family $\langle g_i\rangle_{i\in I}$ such that I, $\langle X_i\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\leq j\in I}$, X and $\langle g_i\rangle_{i\in I}$ satisfy all the hypotheses (α)-(θ) of 418M.

418P Proposition Let $(I, \leq)$, $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\leq j\in I}$ be such that

$(I, \leq)$ is a countable non-empty upwards-directed partially ordered set,

every $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ is a Radon probability space,

$f_{ij} : X_j \to X_i$ is an inverse-measure-preserving almost continuous function whenever $i \leq j$ in I,

$f_{ij}f_{jk} = f_{ik}$ μ_k-a.e. whenever $i \leq j \leq k$ in I.

Then there are a Radon probability space $(X, \mathfrak{T}, \Sigma, \mu)$ and continuous inverse-measure-preserving functions $g_i : X \to X_i$, separating the points of X, such that $g_i = f_{ij}g_j$ whenever $i \leq j$ in I.

418Q Corollary Let $\langle(X_n, \mathfrak{T}_n, \Sigma_n, \mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability spaces, and suppose we are given an inverse-measure-preserving almost continuous function $f_n : X_{n+1} \to X_n$ for each n. Set

$$X = \{x : x \in \prod_{n\in\mathbb{N}} X_n,\ f_n(x(n + 1)) = x(n) \text{ for every } n \in \mathbb{N}\}.$$

Then there is a unique Radon probability measure μ on X such that all the coordinate maps $x \mapsto x(n) : X \to X_n$ are inverse-measure-preserving.

418R Theorem Let X be a set, Σ a σ-algebra of subsets of X, and (Y, T, ν) a σ-finite measure space. Give $L^0(\nu)$ the topology of convergence in measure. Write $\mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$ for the space of $\Sigma\widehat{\otimes}\mathrm{T}$-measurable real-valued functions on $X \times Y$. Then for a function $f : X \to L^0(\nu)$ the following are equiveridical:

(i) $f[X]$ is separable and f is measurable;

(ii) there is an $h \in \mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$ such that $f(x) = h^\bullet_x$ for every $x \in X$, where $h_x(y) = h(x, y)$ for $x \in X$, $y \in Y$.

418S Corollary Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Give $L^0(\nu)$ the topology of convergence in measure.

(a) If $h \in \mathcal{L}^0(\lambda)$, set $h_x(y) = h(x, y)$ whenever this is defined. Then

$$\{x : f(x) = h^\bullet_x \text{ is defined in } L^0(\nu)\}$$

is μ-conegligible, and includes a conegligible set X_0 such that $f : X_0 \to L^0(\nu)$ is measurable and $f[X_0]$ is separable.

(b) If $f : X \to L^0(\nu)$ is measurable and there is a conegligible set $X_0 \subseteq X$ such that $f[X_0]$ is separable, then there is an $h \in \mathcal{L}^0(\lambda)$ such that $f(x) = h_x^\bullet$ for almost every $x \in X$.

418T Corollary Let (Y, T, ν) be a σ-finite measure space, and $(\mathfrak{B}, \bar{\nu})$ its measure algebra, with its measure-algebra topology.

(a) Let X be a set, Σ a σ-algebra of subsets of X, and $f : X \to \mathfrak{B}$ a function. Then the following are equiveridical:

(i) $f[X]$ is separable and f is measurable;

(ii) there is a $W \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $f(x) = W[\{x\}]^\bullet$ for every $x \in X$.

(b) Let (X, Σ, μ) be a σ-finite measure space and Λ the domain of the c.l.d. product measure λ on $X \times Y$.

(i) Suppose that ν is complete. If $W \in \Lambda$, then

$$\{x : f(x) = W[\{x\}]^\bullet \text{ is defined in } \mathfrak{B}\}$$

is μ-conegligible, and includes a conegligible set X_0 such that $f : X_0 \to \mathfrak{B}$ is measurable and $f[X_0]$ is separable.

(ii) If $f : X \to \mathfrak{B}$ is measurable and there is a conegligible set $X_0 \subseteq X$ such that $f[X_0]$ is separable, then there is a $W \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $f(x) = W[\{x\}]^\bullet$ for almost every $x \in X$.

***418U Independent families of measurable functions** In §455 we shall have occasion to look at independent families of random variables taking values in spaces other than $\mathbb{R}$. We can use the same principle as in §272: a family $\langle X_i\rangle_{i\in I}$ of random variables is independent if $\langle\Sigma_i\rangle_{i\in I}$ is independent, where Σ_i is the σ-subalgebra defined by X_i for each i. Let (X, Σ, μ) be a probability space, Y a topological space, and f a Y-valued function defined on a conegligible subset $\operatorname{dom} f$ of X, which is μ-virtually measurable, that is, such that f is measurable with respect to the subspace σ-algebra on $\operatorname{dom} f$ induced by $\hat{\Sigma} = \operatorname{dom}\hat{\mu}$, where $\hat{\mu}$ is the completion of μ. The 'σ-algebra defined by f' will be

$$\{f^{-1}[F] : F \in \mathcal{B}(Y)\} \cup \{(\Omega \setminus \operatorname{dom} f) \cup f^{-1}[F] : F \in \mathcal{B}(Y)\} \subseteq \hat{\Sigma},$$

where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y.

Now, given a family $\langle(f_i, Y_i)\rangle_{i\in I}$ where each Y_i is a topological space and each f_i is a $\hat{\Sigma}$-measurable Y_i-valued function defined on a conegligible subset of X, I will say that $\langle f_i\rangle_{i\in I}$ is **independent** if $\langle\Sigma_i\rangle_{i\in I}$ is independent (with respect to $\hat{\mu}$), where Σ_i is the σ-algebra defined by f_i for each i.

$\langle f_i\rangle_{i\in I}$ is independent iff

$$\hat{\mu}(\bigcap_{j\le n} f_{i_j}^{-1}[G_j]) = \prod_{j\le n} \hat{\mu} f_{i_j}^{-1}[G_j]$$

whenever $i_0, \dots, i_n \in I$ are distinct and $G_j \subseteq Y_{i_j}$ is open for every $j \le n$.

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419 Examples

In §216, I went much of the way to describing examples of spaces with all the possible combinations of the properties considered in Chapter 21. When we come to topological measure spaces, the number of properties involved makes it unreasonable to seek any such comprehensive list. I therefore content myself with seven examples to indicate some of the boundaries of the theory developed here.

The first example (419A) is supposed to show that the hypothesis 'effectively locally finite' which appears in so many of the theorems of this chapter cannot as a rule be replaced by 'locally finite'. The next two (419C-419D) address technical questions concerning the definition of 'Radon measure', and show how small variations in the definition can lead to very different kinds of measure space. The fourth example (419E) shows that the τ-additive product measures of §417 are indeed new constructions. 419J is there to show that extension theorems of the types proved in §415 and §417 cannot be taken for granted. The classic example 419K exhibits one of the obstacles to generalizations of Prokhorov's theorem (418M, 418Q). Finally, I return to the split interval (419L) to describe its standard topology and its relation to the measure introduced in 343J.

419A Example There is a locally compact Hausdorff space X with a complete, σ-finite, locally finite, τ-additive topological measure μ, inner regular with respect to the closed sets, which has a closed subset Y, of measure 1, such that the subspace measure μ_Y on Y is not τ-additive. In particular, μ is not effectively locally finite.

419B Lemma For any non-empty set I, there is a dense G_δ set in $[0,1]^I$ which is negligible for the usual measure on $[0,1]^I$.

419C Example There is a completion regular Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$ such that

(i) there is an $E\in\Sigma$ such that $\mu(F\triangle E)>0$ for every Borel set $F\subseteq X$, that is, not every element of the measure algebra of μ can be represented by a Borel set;

(ii) μ is not outer regular with respect to the Borel sets;

(iii) writing ν for the restriction of μ to the Borel σ-algebra of X, ν is a locally finite, effectively locally finite, tight τ-additive completion regular topological measure, and there is a set $Y\subseteq X$ such that the subspace measure ν_Y is not semi-finite.

419D Example There is a complete locally determined τ-additive completion regular topological measure space $(X,\mathfrak{T},\Sigma,\mu)$ in which μ is tight and compact sets have finite measure, but μ is not localizable.

419E Example Let $(Z,\mathfrak{S},\mathrm{T},\nu)$ be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, so that ν is a strictly positive completion regular Radon probability measure. Then the c.l.d. product measure on $Z\times Z$ is not a topological measure, so is not equal to the τ-additive product measure $\tilde\lambda$, and $\tilde\lambda$ is not completion regular.

419F Theorem $\mathcal{P}(\omega_1\times\omega_1)=\mathcal{P}\omega_1\widehat{\otimes}\mathcal{P}\omega_1$.

419G Corollary Let Y be a set of cardinal at most ω_1 and μ a semi-finite measure with domain $\mathcal{P}Y$. Then μ is point-supported; in particular, if μ is σ-finite there is a countable conegligible set $A\subseteq Y$.

419H Lemma If $(X,\mathfrak{T},\Sigma,\mu)$ is an atomless Radon measure space and $E\in\Sigma$ has non-zero measure, then $\#(E)\ge\mathfrak{c}$.

419I Lemma Let μ be Lebesgue measure on $\mathbb{R}$, and H any measurable subset of $\mathbb{R}$. Then there is a disjoint family $\langle A_\alpha\rangle_{\alpha<\mathfrak{c}}$ of subsets of H such that H is a measurable envelope of every A_α; in particular, $\mu_*A_\alpha=0$ and $\mu^*A_\alpha=\mu H$ for every $\alpha<\mathfrak{c}$.

419J Example There is a complete probability space (X,Σ,μ) with a Hausdorff topology $\mathfrak{T}$ on X such that μ is τ-additive and inner regular with respect to the Borel sets, $\mathfrak{T}$ is generated by $\mathfrak{T}\cap\Sigma$, but μ has no extension to a topological measure.

419K Example (Blackwell 56) There are sequences $\langle X_n\rangle_{n\in\mathbb{N}}$, $\langle\mathfrak{T}_n\rangle_{n\in\mathbb{N}}$ and $\langle\nu_n\rangle_{n\in\mathbb{N}}$ such that (i) for each n, $(X_n,\mathfrak{T}_n)$ is a separable metrizable space and ν_n is a quasi-Radon probability measure on $Z_n=\prod_{i\le n}X_i$ (ii) for $m\le n$ the canonical map $\pi_{mn}:Z_n\to Z_m$ is inverse-measure-preserving (iii) there is no probability measure on $Z=\prod_{i\in\mathbb{N}}X_i$ such that all the canonical maps from Z to Z_n are inverse-measure-preserving.

419L The split interval again (a) Let $I^\parallel$ be the set $\{a^+:a\in[0,1]\}\cup\{a^-:a\in[0,1]\}$. Order it by saying that

$$a^+\le b^+\iff a^-\le b^+\iff a^-\le b^-\iff a\le b,\quad a^+\le b^-\iff a<b.$$

Then $I^\parallel$ is a totally ordered space, and is Dedekind complete. Its greatest element is 1^+ and its least element is 0^-. Consequently the order topology on $I^\parallel$ is a compact Hausdorff topology. $Q=\{q^+:q\in[0,1]\cap\mathbb{Q}\}\cup\{q^-:q\in[0,1]\cap\mathbb{Q}\}$ is dense. $I^\parallel$ is ccc and hereditarily Lindelöf.

(b) If we define $h : I^{\parallel} \to [0,1]$ by writing $h(a^+) = h(a^-) = a$ for every $a \in [0,1]$, then h is continuous. Now a set $E \subseteq I^{\parallel}$ is Borel iff there is a Borel set $F \subseteq [0,1]$ such that $E \triangle h^{-1}[F]$ is countable.

(c) In 343J I described the standard measure μ on $I^{\parallel}$; its domain is the set $\Sigma = \{h^{-1}[F] \triangle M : F \in \Sigma_L,\ M \subseteq I^{\parallel},\ \mu_L h[M] = 0\}$, where Σ_L is the set of Lebesgue measurable subsets of $[0,1]$ and μ_L is Lebesgue measure, and $\mu E = \mu_L h[E]$ for $E \in \Sigma$. h is inverse-measure-preserving for μ and μ_L.

μ is a completion regular Radon measure.