I return in this volume to the study of measure *spaces* rather than measure *algebras*. For fifty years now measure theory has been intimately connected with general topology. Not only do a very large proportion of the measure spaces arising in applications carry topologies related in interesting ways to their measures, but many questions in abstract measure theory can be effectively studied by introducing suitable topologies. Consequently any course in measure theory at this level must be frankly dependent on a substantial knowledge of topology. With this proviso, I hope that the present volume will be accessible to graduate students, and will lead them to the most important ideas of modern abstract measure theory.

The first and third chapters of the volume seek to provide a thorough introduction into the ways in which topologies and measures can interact. They are divided by a short chapter on descriptive set theory, on the borderline between set theory, logic, real analysis and general topology, which I single out for detailed exposition because I believe that it forms an indispensable part of the background of any measure theorist. Chapter 41 is dominated by the concepts of inner regularity and τ-additivity, coming together in Radon measures (§416). Chapter 43 concentrates rather on questions concerning properties of a topological space which force particular relationships with measures on that space. But plenty of side-issues are treated in both, such as Lusin measurability (§418), the definition of measures from linear functionals (§436) and measure-free cardinals (§438). Chapters 45 and 46 continue some of the same themes, with particular investigations into 'disintegrations' or regular conditional probabilities (§§452-453), stochastic processes (§§454-456), Talagrand's theory of stable sets (§465) and the theory of measures on normed spaces (§§466-467).

In contrast with the relatively amorphous structure of Chapters 41, 43, 45 and 46, four chapters of this volume have definite topics. I have already said that Chapter 42 is an introduction to descriptive set theory; like Chapters 31 and 35 in the preceding volume, it is a kind of appendix brought into the main stream of the argument. Chapter 44 deals with topological groups. Most of it is of course devoted to Haar measure, giving the Pontryagin-van Kampen duality theorem (§445) and the Ionescu Tulcea theorem on the existence of translation-invariant liftings (§447). But there are also sections on Polish groups (§448) and amenable groups (§449), and some of the general theory of measures on measurable groups (§444). Chapter 47 is a second excursion, after Chapter 26, into geometric measure theory. It starts with Hausdorff measures (§471), gives a proof of the Di Giorgio-Federer Divergence Theorem (§475), and then examines a number of examples of 'concentration of measure' (§476). In the second half of the chapter, §§477-479, I decribe Brownian motion and use it as a basis of the theory of Newtonian capacity. In Chapter 48, I set out the elementary theory of gauge integrals, with sections on the Henstock and Pfeffer integrals (§§483-484). Finally, in Chapter 49, I give notes on seven special topics: equidistributed sequences (§491), combinatorial forms of concentration of measure (§492), extremely amenable groups and groups of measure-preserving automorphisms (§§493-494), Poisson point processes (§495), submeasures (§496), Szemerédi's theorem (§497) and subproducts in product spaces (§498).

I had better mention prerequisites, as usual. To embark on this material you will certainly need a solid foundation in measure theory. Since I do of course use my own exposition as my principal source of references to the elementary ideas, I advise readers to ensure that they have easy access to all three previous volumes before starting serious work on this one. But you may not need to read very much of them. It might be prudent to glance through the detailed contents of Volume 1 and the first five chapters of Volume 2 to check that most of the material there is more or less familiar. I think §417 might be difficult to read without at least the results-only version of Chapter 25 to hand. But Volume 3, and the last three chapters of Volume 2, can probably be left on one side for the moment. Of course you will need the Lifting Theorem (Chapter 34) for §§447, 452 and 453, and Chapter 26 is essential background for Chapter 47, while Chapter 28 (on Fourier analysis) may help to make sense of Chapter 44, and parts of Chapter 27 (on probability theory) are necessary for §§455-456 and 458-459. You will certainly need some Fourier analysis for §479. And measure algebras are mentioned in every chapter except (I think) Chapter 48; but I hope that the cross-references are precise enough to lead you to what you need to know at any particular point. Even Maharam's theorem is hardly used in this volume.

What you will need, apart from any knowledge of measure theory, is a sound background in general topology. This volume calls on a great many miscellaneous facts from general topology, and the list in §4A2 is not a good place to start if continuity and compactness and the separation axioms are unfamiliar. My primary reference for topology is Engelking 89. I do not insist that you should have read this book (though of course I hope you will do so sometime); but I do think you should make sure that you can use it.

In the general introduction to this treatise, I wrote 'I make no attempt to describe the history of the subject', and I have generally been casual – some would say negligent – in my attributions of results to their discoverers. Through much of the first three volumes I did at least have the excuse that the history exists in print in far more detail than I am qualified to describe. In the present volume I find my position more uncomfortable, in that I have been watching the evolution of the subject relatively closely over the last forty years, and ought to be able to say something about it. Nevertheless I remain reluctant to make definite statements crediting one person rather than another with originating an idea. My more intimate knowledge of the topic makes me even more conscious than elsewhere of the danger of error and of the breadth of reading that would be necessary to produce a balanced account. In some cases I do attach a result to a specific published paper, but these attributions should never be regarded as an assertion that any particular author has priority; at most, they declare that a historian should examine the source cited before coming to any decision. I assure my friends and colleagues that my omissions are not intended to slight either them or those we all honour. What I have tried to do is to include in the bibliography to this volume all the published work which (as far as I am consciously aware) has influenced me while writing it, so that those who wish to go into the matter will have somewhere to start their investigations.

Note on second printing

I fear that there were even more errors, not all of them trivial, in the first printing of this volume than there were in previous volumes. I have tried to correct those which I have noticed; many surely remain. Apart from these, there are many minor expansions and elaborations, and a couple of new results, but few new ideas and no dramatic rearrangements. Details may be found in `http://www1.essex.ac.uk/maths/people/fremlin/mterr4.03.pdf`.

Both printers and readers found that the 945-page format of the first printing was hard to handle. I have therefore divided the volume into two parts for the second printing. I hope you will find that the additional convenience is worth the the increase in cost.

Note on second ('Lulu') edition

I was right about many errors remaining (particularly in §458, on relative independence), and I hope I have cleared some of them out of the way. There are substantial additions in the new edition, the most important being a vastly expanded §455 on Lévy processes, an account of Brownian motion and Newtonian potential in §§477-479, and Tao's proof of Szemerédi's theorem in §497. I have included theorems of A.Törnquist and G.W.Mackey on the realization of group actions on measure algebras, some material on a version of the Kantorovich-Rubinstein distance between two measures, and a section on Maharam submeasures (§496).

Version of 17.4.10

Chapter 41

Topologies and Measures I

I begin this volume with an introduction to some of the most important ways in which topologies and measures can interact, and with a description of the forms which such constructions as subspaces and product spaces take in such contexts. By far the most important concept is that of Radon measure (411Hb, §416). In Radon measure spaces we find both the richest combinations of ideas and the most important applications. But, as usual, we are led both by analysis of these ideas and by other interesting examples to consider wider classes of topological measure space, and the greater part of the chapter, by volume, is taken up by a description of the many properties of Radon measures individually and in partial combinations.

I begin the chapter with a short section of definitions (§411), including a handful of more or less elementary examples. The two central properties of a Radon measure are 'inner regularity' (411B) and 'τ-additivity' (411C). The former is an idea of great versatility which I look at in an abstract setting in §412. I take a section (§413) to describe some methods of constructing measure spaces, extending the rather limited range of constructions offered in earlier volumes. There are two sections on τ-additive measures, §§414 and 417; the former covers the elementary ideas, and the latter looks at product measures, where it turns out that we need a new technique to supplement the purely measure-theoretic constructions of Chapter 25. On the way to Radon measures in §416, I pause over 'quasi-Radon' measures (411Ha, §415), where inner regularity and τ-additivity first come effectively together.

The possible interactions of a topology and a measure on the same space are so varied that even a brief account makes a long chapter; and this is with hardly any mention of results associated with particular types of topological space, most of which must wait for later chapters. But I include one section on the two most important classes of functions acting between topological measure spaces (§418), and another describing some examples to demonstrate special phenomena (§419).

Version of 31.12.08

411 Definitions

In something of the spirit of §211, but this time without apologising, I start this volume with a list of definitions. The rest of Chapter 41 will be devoted to discussing these definitions and relationships between them, and integrating the new ideas into the concepts and constructions of earlier volumes; I hope that by presenting the terminology now I can give you a sense of the directions the following sections will take. I ought to remark immediately that there are many cases in which the exact phrasing of a definition is important in ways which may not be immediately apparent.

411A I begin with a phrase which will be a useful shorthand for the context in which most, but not all, of the theory here will be developed.

Definition A **topological measure space** is a quadruple $(X, \mathfrak{T}, \Sigma, \mu)$ where (X, Σ, μ) is a measure space and $\mathfrak{T}$ is a topology on X such that $\mathfrak{T} \subseteq \Sigma$, that is, every open set (and therefore every Borel set) is measurable.

411B Now I come to what are in my view the two most important concepts to master; jointly they will dominate the chapter.

Definition Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets. I say that μ is **inner regular with respect to** $\mathcal{K}$ if

$$\mu E = \sup\{\mu K : K \in \Sigma \cap \mathcal{K}, K \subseteq E\}$$

for every $E \in \Sigma$. (Cf. 256Ac, 342Aa.)

Remark Note that in this definition I do not assume that $\mathcal{K} \subseteq \Sigma$, nor even that $\mathcal{K} \subseteq \mathcal{P}X$. But of course μ will be inner regular with respect to $\mathcal{K}$ iff it is inner regular with respect to $\mathcal{K} \cap \Sigma$.

It is convenient in this context to interpret $\sup \emptyset$ as 0, so that we have to check the definition only when $\mu E > 0$, and need not insist that $\emptyset \in \mathcal{K}$.

411C Definition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. I say that μ is **τ-additive** (the phrase **τ-regular** has also been used) if whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets such that $\mathcal{G} \subseteq \Sigma$ and $\bigcup \mathcal{G} \in \Sigma$ then $\mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G$.

Remark Note that in this definition I do not assume that every open set is measurable. Consequently we cannot take it for granted that an extension of a τ-additive measure will be τ-additive; on the other hand, the restriction of a τ-additive measure to any σ-subalgebra will be τ-additive.

411D Complementary to 411B we have the following.

Definition Let (X, Σ, μ) be a measure space and $\mathcal{H}$ a family of subsets of X. Then μ is **outer regular with respect to $\mathcal{H}$** if

$$\mu E = \inf\{\mu H : H \in \Sigma \cap \mathcal{H},\ H \supseteq E\}$$

for every $E \in \Sigma$.

Note that a totally finite measure on a topological space is inner regular with respect to the family of closed sets iff it is outer regular with respect to the family of open sets.

411E I delay discussion of most of the relationships between the concepts here to later in the chapter. But it will be useful to have a basic fact set out immediately.

Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. If μ is inner regular with respect to the compact sets, it is τ-additive.

proof Let $\mathcal{G}$ be a non-empty upwards-directed family of measurable open sets such that $H = \bigcup \mathcal{G} \in \Sigma$. If $\gamma < \mu H$, there is a compact set $K \subseteq H$ such that $\mu K \ge \gamma$; now there must be a $G \in \mathcal{G}$ which includes K, so that $\mu G \ge \gamma$. As γ is arbitrary, $\sup_{G \in \mathcal{G}} \mu G = \mu H$.

411F In order to deal efficiently with measures which are not totally finite, I think we need the following ideas.

Definitions Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) I say that μ is **locally finite** if every point of X has a neighbourhood of finite measure, that is, if the open sets of finite outer measure cover X.

(b) I say that μ is **effectively locally finite** if for every non-negligible measurable set $E \subseteq X$ there is a measurable open set $G \subseteq X$ such that $\mu G < \infty$ and $E \cap G$ is not negligible.

Note that an effectively locally finite measure must measure many open sets, while a locally finite measure need not.

(c) This seems a convenient moment at which to introduce the following term. A real-valued function f defined on a subset of X is **locally integrable** if for every $x \in X$ there is an open set G containing x such that $\int_G f$ is defined (in the sense of 214D) and finite.

411G Elementary facts (a) If μ is a locally finite measure on a topological space X, then $\mu^* K < \infty$ for every compact set $K \subseteq X$. **P** The family $\mathcal{G}$ of open sets of finite outer measure is upwards-directed and covers X, so there must be some $G \in \mathcal{G}$ including K, in which case $\mu^* K \le \mu^* G$ is finite. **Q**

(b) A measure μ on $\mathbb{R}^r$ is locally finite iff every bounded set has finite outer measure (cf. 256Ab). **P** (i) If every bounded set has finite outer measure then, in particular, every open ball has finite outer measure, so that μ is locally finite. (ii) If μ is locally finite and $A \subseteq \mathbb{R}^r$ is bounded, then its closure $\overline{A}$ is compact (2A2F), so that $\mu^* A \le \mu^* \overline{A}$ is finite, by (a) above. **Q**

(c) I should perhaps remark immediately that a locally finite topological measure need not be effectively locally finite (419A), and an effectively locally finite measure need not be locally finite (411P).

(d) An effectively locally finite measure must be semi-finite.

(e) A locally finite measure on a Lindelöf space X is σ-finite. **P** Let $\mathcal{G}$ be the family of open sets of finite outer measure. Because μ is locally finite, $\mathcal{G}$ is a cover of X. Because X is Lindelöf, there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ covering X. For each $n \in \mathbb{N}$, there is a measurable set $E_n \supseteq G_n$ of finite measure, and now $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of sets of finite measure covering X. **Q**

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space such that μ is locally finite and inner regular with respect to the compact sets. Then μ is effectively locally finite. **P** Suppose that $\mu E > 0$. Then there is a measurable compact set $K \subseteq E$ such that $\mu K > 0$. As in the argument for (a) above, there is an open set G of finite measure including K, so that $\mu(E \cap G) > 0$. **Q**

(g) Corresponding to (a) above, we have the following fact. If μ is a measure on a topological space and $f \in \mathcal{L}^0(\mu)$ is locally integrable, then $\int_K f d\mu$ is finite for every compact $K \subseteq X$, because K can be covered by a finite family of open sets G such that $\int_G |f| d\mu < \infty$.

(h) If μ is a locally finite measure on a topological space X, and $f \in \mathcal{L}^p(\mu)$ for some $p \in [1, \infty]$, then f is locally integrable; this is because $\int_G |f| \le \int_E |f| \le \|f\|_p \|\chi E\|_q$ is finite whenever $G \subseteq E$ and $\mu E < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality (244Eb).

(i) If $(X, \mathfrak{T})$ is a completely regular space and μ is a locally finite topological measure on X, then the collection of open sets with negligible boundaries is a base for $\mathfrak{T}$. **P** If $x \in G \in \mathfrak{T}$, let $H \subseteq G$ be an open set of finite measure containing x, and $f : X \to [0, 1]$ a continuous function such that $f(x) = 1$ and $f(y) = 0$ for $y \in X \setminus H$. Then $\{f^{-1}[\{\alpha\}] : 0 < \alpha < 1\}$ is an uncountable disjoint family of measurable subsets of H, so there must be some $\alpha \in]0, 1[$ such that $f^{-1}[\{\alpha\}]$ is negligible. Set $U = \{y : f(y) > \alpha\}$; then U is an open neighbourhood of x included in G and its boundary $\partial U \subseteq f^{-1}[\{\alpha\}]$ is negligible. **Q**

(j) Let X and Y be topological spaces, $f : X \to Y$ a continuous function, μ a measure on X and μf^{-1} the image measure on Y. Then if μ is a topological measure, so is μf^{-1}, and if μ is τ-additive, so is μf^{-1}. (Immediate from the definitions.)

411H Two particularly important combinations of the properties above are the following.

Definitions (a) A **quasi-Radon measure space** is a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is complete and locally determined (ii) μ is τ-additive, inner regular with respect to the closed sets and effectively locally finite.

(b) A **Radon measure space** is a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is complete and locally determined (ii) $\mathfrak{T}$ is Hausdorff (iii) μ is locally finite and inner regular with respect to the compact sets.

411I Remarks (a) You may like to seek your own proof that a Radon measure space is always quasi-Radon, before looking it up in §416 below.

(b) Note that a measure on Euclidean space $\mathbb{R}^r$ is a Radon measure on the definition above iff it is a Radon measure as described in 256Ad. **P** In 256Ad, I said that a measure μ on $\mathbb{R}^r$ is 'Radon' if it is a locally finite complete topological measure, inner regular with respect to the compact sets. (The definition of 'locally finite' in 256A was not the same as the one above, but I have already covered this point in 411Gb.) So the only thing to add is that μ is necessarily locally determined, because it is σ-finite (256Ba). **Q**

411J The following special types of inner regularity are of sufficient importance to have earned separate names.

Definitions (a) If $(X, \mathfrak{T})$ is a topological space, I will say that a measure μ on X is **tight** if it is inner regular with respect to the closed compact sets.

(b) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a topological measure space, I will say that μ is **completion regular** if it is inner regular with respect to the zero sets (definition: 3A3Qa).

411K Borel and Baire measures If $(X, \mathfrak{T})$ is a topological space, I will call a measure with domain (exactly) the Borel σ-algebra of X (4A3A) a **Borel measure** on X, and a measure with domain (exactly) the Baire σ-algebra of X (4A3K) a **Baire measure** on X.

Of course a Borel measure is a topological measure in the sense of 411A. On a metrizable space, the Borel and Baire measures coincide (4A3Kb). The most important measures in this chapter will be c.l.d. versions of Borel measures.

411L When we come to look at functions defined on a topological measure space, we shall have to relate ideas of continuity and measurability. Two basic concepts are the following.

Definition Let X be a set, Σ a σ-algebra of subsets of X and $(Y, \mathfrak{S})$ a topological space. I will say that a function $f : X \to Y$ is **measurable** if $f^{-1}[G] \in \Sigma$ for every open set $G \subseteq Y$.

Remarks (a) Note that a function $f : X \to \mathbb{R}$ is measurable on this definition (when $\mathbb{R}$ is given its usual topology) iff it is measurable according to the familiar definition in 121C, which asks only that sets of the form $\{x : f(x) < \alpha\}$ should be measurable (121Ef).

(b) For any topological space $(Y, \mathfrak{S})$, a function $f : X \to Y$ is measurable iff f is $(\Sigma, \mathcal{B}(Y))$-measurable, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y (4A3Cb).

411M Definition Let (X, Σ, μ) be a measure space, $\mathfrak{T}$ a topology on X, and $(Y, \mathfrak{S})$ another topological space. I will say that a function $f : X \to Y$ is **almost continuous** or **Lusin measurable** if μ is inner regular with respect to the family of subsets A of X such that $f \restriction A$ is continuous.

411N Finally, I introduce some terminology to describe ways in which (sometimes) measures can be located in one part of a topological space rather than another.

Definitions Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) I will call a set $A \subseteq X$ **self-supporting** if $\mu^*(A \cap G) > 0$ for every open set G such that $A \cap G$ is non-empty. (Such sets are sometimes called **of positive measure everywhere**.)

(b) A **support** of μ is a closed self-supporting set F such that $X \setminus F$ is negligible.

(c) Note that μ can have at most one support. **P** If F_1, F_2 are supports then $\mu^*(F_1 \setminus F_2) \le \mu^*(X \setminus F_2) = 0$ so $F_1 \setminus F_2$ must be empty. Similarly, $F_2 \setminus F_1 = \emptyset$, so $F_1 = F_2$. **Q**

(d) If μ is a τ-additive topological measure it has a support. **P** Let $\mathcal{G}$ be the family of negligible open sets, and F the closed set $X \setminus \bigcup \mathcal{G}$. Then $\mathcal{G}$ is an upwards-directed family in $\mathfrak{T} \cap \Sigma$ and $\bigcup \mathcal{G} \in \mathfrak{T} \cap \Sigma$, so

$$\mu(X \setminus F) = \mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G = 0.$$

If G is open and $G \cap F \ne \emptyset$ then $G \notin \mathcal{G}$ so $\mu^*(G \cap F) = \mu(G \cap F) = \mu G > 0$; thus F is self-supporting and is the support of μ. **Q**

(e) Let X and Y be topological spaces with topological measures μ, ν respectively and a continuous inverse-measure-preserving function $f : X \to Y$. Suppose that μ has a support E. Then $\overline{f[E]}$ is the support of ν. **P** We have only to observe that for an open set $H \subseteq Y$

$$\begin{aligned} \nu H > 0 &\iff \mu f^{-1}[H] > 0 \iff f^{-1}[H] \cap E \ne \emptyset \\ &\iff H \cap f[E] \ne \emptyset \iff H \cap \overline{f[E]} \ne \emptyset. \textbf{ Q} \end{aligned}$$

(f) μ is **strictly positive** (with respect to $\mathfrak{T}$) if $\mu^* G > 0$ for every non-empty open set $G \subseteq X$, that is, X itself is the support of μ.

***(g)** If $(X,\mathfrak{T})$ is a topological space, and μ is a strictly positive σ-finite measure on X such that the domain Σ of μ includes a π-base $\mathcal{U}$ for $\mathfrak{T}$, then X is ccc. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of finite measure covering X. Let $\mathcal{G}$ be a disjoint family of non-empty open sets. For each $G\in G$, take $U_G\in\mathcal{U}\setminus\{\emptyset\}$ such that $U_G\subseteq G$; then $\mu U_G>0$, so there is an $n(G)$ such that $\mu(E_{n(G)}\cap U_G)>0$. Now $\sum_{G\in G,n(G)=k}\mu(E_k\cap U_G)\le\mu E_k$ is finite for every k, so $\{G:n(G)=k\}$ must be countable and $\mathcal{G}$ is countable. **Q**

411O Example Lebesgue measure on $\mathbb{R}^r$ is a Radon measure (256Ha); in particular, it is locally finite and tight. It is therefore τ-additive and effectively locally finite (411E, 411Gf). It is completion regular (because every compact set is a zero set, see 4A2Lc), outer regular with respect to the open sets (134Fa) and strictly positive.

411P Example: Stone spaces (a) Let $(Z,\mathfrak{T},\Sigma,\mu)$ be the Stone space of a semi-finite measure algebra $(\mathfrak{A},\bar\mu)$, so that $(Z,\mathfrak{T})$ is a zero-dimensional compact Hausdorff space, (Z,Σ,μ) is complete and semi-finite, the open-and-closed sets are measurable, the negligible sets are the nowhere dense sets, and every measurable set differs by a nowhere dense set from an open-and-closed set (311I, 321K, 322Bd, 322Ra[1]).

(b) μ is inner regular with respect to the open-and-closed sets (322Ra); in particular, it is completion regular and tight. Consequently it is τ-additive (411E).

(c) μ is strictly positive, because the open-and-closed sets form a base for $\mathfrak{T}$ (311I) and a non-empty open-and-closed set has non-zero measure. μ is effectively locally finite. **P** Suppose that $E\in\Sigma$ is not negligible. There is a measurable set $F\subseteq E$ such that $0<\mu F<\infty$; now there is a non-empty open-and-closed set G included in F, in which case $\mu G<\infty$ and $\mu(E\cap G)>0$. **Q**

(d) The following are equiveridical, that is, if one is true so are the others:

(i) $(\mathfrak{A},\bar\mu)$ is localizable;

(ii) μ is strictly localizable;

(iii) μ is locally determined;

(iv) μ is a quasi-Radon measure.

P The equivalence of (i)-(iii) is Theorem 322O[2]. (iv)$\Rightarrow$(iii) is trivial. If one, therefore all, of (i)-(iii) are true, then μ is a topological measure, because if $G\subseteq Z$ is open, then $\overline{G}$ is open-and-closed, by 314S, therefore measurable, and $\overline{G}\setminus G$ is nowhere dense, therefore also measurable. We know already that μ is complete, effectively locally finite and τ-additive, so that if it is also locally determined it is a quasi-Radon measure. **Q**

(e) The following are equiveridical:

(i) μ is a Radon measure;

(ii) μ is totally finite;

(iii) μ is locally finite;

(iv) μ is outer regular with respect to the open sets.

P (ii)$\Rightarrow$(iv) If μ is totally finite and $E\in\Sigma$, then for any $\epsilon>0$ there is a closed set $F\subseteq Z\setminus E$ such that $\mu F\ge\mu(Z\setminus E)-\epsilon$, and now $G=Z\setminus F$ is an open set including E with $\mu G\le\mu E+\epsilon$. (iv)$\Rightarrow$(iii) Suppose that μ is outer regular with respect to the open sets, and $z\in Z$. Because Z is Hausdorff, $\{z\}$ is closed. If it is open it is measurable, and because μ is semi-finite it must have finite measure. Otherwise it is nowhere dense, therefore negligible, and must be included in open sets of arbitrarily small measure. Thus in both cases z belongs to an open set of finite measure; as z is arbitrary, μ is locally finite. (iii)$\Rightarrow$(ii) Because Z is compact, this is a consequence of 411Ga. (i)$\Rightarrow$(iii) is part of the definition of 'Radon measure'. Finally, (ii)$\Rightarrow$(i), again directly from the definition and the facts set out in (a)-(b) above. **Q**

(f) Let $W\subseteq Z$ be the union of all the open subsets of Z with finite measure. Because μ is effectively locally finite, W has full outer measure, so $(\mathfrak{A},\bar\mu)$ can be identified with the measure algebra of the subspace measure μ_W (322Jb). By the definition of W, μ_W is locally finite. If $(\mathfrak{A},\bar\mu)$ is localizable, then μ_W is a

[1]Formerly 322Qa.

[2]Formerly 322N.

Radon measure. **P** Every open subset of W belongs to Σ, by (d), and therefore to the domain of μ_W, and μ_W is a topological measure. By 214Ka, μ_W is complete and locally determined. Because μ is inner regular with respect to the compact sets, so is μ_W. **Q**

411Q Example: Dieudonné's measure Recall that a set $E \subseteq \omega_1$ is a Borel set iff either E or its complement includes a cofinal closed set (4A3J). So we may define a Borel measure μ on ω_1 by saying that $\mu E = 1$ if E includes a cofinal closed set and $\mu E = 0$ if E is disjoint from a cofinal closed set. If E is disjoint from some cofinal closed set, so is any subset of E, so μ is complete. Since μ takes only the values 0 and 1, it is a purely atomic probability measure.

μ is a topological measure; being totally finite, it is surely locally finite and effectively locally finite. It is inner regular with respect to the closed sets (because if $\mu E > 0$, there is a cofinal closed set $F \subseteq E$, and now F is a closed set with $\mu F = \mu E$), therefore outer regular with respect to the open sets. It is not τ-additive (because $\xi = [0, \xi[$ is an open set of zero measure for every $\xi < \omega_1$, and the union of these sets is a measurable open set of measure 1).

μ is not completion regular, because the set of countable limit ordinals is a closed set (4A1Bb) which does not include any uncountable zero set (see 411R below).

The only self-supporting subset of ω_1 is the empty set (because there is a cover of ω_1 by negligible open sets). In particular, μ does not have a support.

Remark There is a measure of this type on any ordinal of uncountable cofinality; see 411Xj.

411R Example: The Baire σ-algebra of ω_1 The Baire σ-algebra $\mathcal{B}\mathfrak{a}(\omega_1)$ of ω_1 is the countable-cocountable algebra (4A3P). The countable-cocountable measure μ on ω_1 is therefore a Baire measure on the definition of 411K. Since all sets of the form $]\xi, \omega_1[$ are zero sets, μ is inner regular with respect to the zero sets and outer regular with respect to the cozero sets. Since sets of the form $[0, \xi[$ $(= \xi)$ form a cover of ω_1 by measurable open sets of zero measure, μ is not τ-additive.

411X Basic exercises >(a) Let (X, Σ, μ) be a totally finite measure space and $\mathfrak{T}$ a topology on X. Show that μ is inner regular with respect to the closed sets iff it is outer regular with respect to the open sets, and is inner regular with respect to the zero sets iff it is outer regular with respect to the cozero sets.

(b) Let μ be a Radon measure on $\mathbb{R}^r$, where $r \geq 1$, and $f \in \mathcal{L}^0(\mu)$. Show that f is locally integrable in the sense of 411Fc iff it is locally integrable in the sense of 256E, that is, $\int_E f d\nu < \infty$ for every bounded set $E \subseteq \mathbb{R}^r$.

(c) Let μ be a measure on a topological space, $\hat{\mu}$ its completion and $\tilde{\mu}$ its c.l.d. version. Show that μ is locally finite iff $\hat{\mu}$ is locally finite, and in this case $\tilde{\mu}$ is locally finite.

>(d) Let μ be an effectively locally finite measure on a topological space X. (i) Show that the completion and c.l.d. version of μ are effectively locally finite. (ii) Show that if μ is complete and locally determined, then the union of the measurable open sets of finite measure is conegligible. (iii) Show that if X is hereditarily Lindelöf then μ must be σ-finite.

(e) Let X be a topological space and μ a measure on X. Let $U \subseteq L^0(\mu)$ be the set of equivalence classes of locally integrable functions in $\mathcal{L}^0(\mu)$. Show that U is a solid linear subspace of $L^0(\mu)$. Show that if μ is locally finite then U includes $L^p(\mu)$ for every $p \in [1, \infty]$.

(f) Let X be a topological space. (i) Let μ, ν be two totally finite Borel measures which agree on the closed sets. Show that they are equal. (*Hint*: 136C.) (ii) Let μ, ν be two totally finite Baire measures which agree on the zero sets. Show that they are equal.

(g) Let $(X, \mathfrak{T})$ be a topological space, μ a measure on X, and Y a subset of X; let $\mathfrak{T}_Y$, μ_Y be the subspace topology and measure. Show that if μ is a topological measure, or locally finite, or a Borel measure, so is μ_Y.

(h) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X, Σ, μ); suppose that we are given a topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the disjoint union topology on X (definition: 4A2A). Show that μ is a topological measure, or locally finite, or effectively locally finite, or a Borel measure, or a Baire measure, or strictly positive, iff every μ_i is.

(i) Let (X, Σ, μ) and (Y, T, ν) be two measure spaces, with c.l.d. product measure λ on $X \times Y$. Suppose we are given topologies $\mathfrak{T}$, $\mathfrak{S}$ on X, Y respectively, and give $X \times Y$ the product topology. Show that λ is locally finite, or effectively locally finite, if μ and ν are.

(j) Let α be any ordinal of uncountable cofinality with its order topology (definitions: 3A1Fb, 4A2A). Show that there is a complete topological probability measure μ on α defined by saying that $\mu E = 1$ if E includes a cofinal closed set in α, 0 if E is disjoint from some cofinal closed set. Show that μ is inner regular with respect to the closed sets but is not completion regular.

(k) Let $\langle (X_i, \mathfrak{T}_i)\rangle_{i\in I}$ be a family of topological spaces, and μ_i a strictly positive probability measure on X_i for each i. Show that the product measure on $\prod_{i\in I} X_i$ is strictly positive.

411Y Further exercises (a) Let r, $s \geq 1$ be integers. Show that a function $f : \mathbb{R}^r \to \mathbb{R}^s$ is measurable iff it is almost continuous (where $\mathbb{R}^r$ is endowed with Lebesgue measure and its usual topology, of course). (*Hint*: 256F.)

(b) Let (X, ρ) be a metric space, $r \geq 0$, and write μ_{Hr} for r-dimensional Hausdorff measure on X (264K, 471A). (i) Show that μ_{Hr} is a topological measure, outer regular with respect to the Borel sets. (ii) Show that the c.l.d. version $\tilde{\mu}_{Hr}$ of μ_{Hr} is inner regular with respect to the closed totally bounded sets. (iii) Show that $\tilde{\mu}_{Hr}$ is completion regular. (iv) Show that if X is complete then $\tilde{\mu}_{Hr}$ is tight.

(c) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space. Set $\mathcal{E} = \{E : E \subseteq X,\ \mu(\partial E) = 0\}$, where ∂E is the boundary of A. (i) Show that $\mathcal{E}$ is a subalgebra of $\mathcal{P}X$, and that every member of $\mathcal{E}$ is measured by the completion of μ. ($\mathcal{E}$ is sometimes called the **Jordan algebra** of $(X, \mathfrak{T}, \Sigma, \mu)$. Do not confuse with the 'Jordan algebras' of abstract algebra.) (ii) Suppose that μ is totally finite and inner regular with respect to the closed sets, and that $\mathfrak{T}$ is normal. Show that $\{E^\bullet : E \in \mathcal{E} \cap \Sigma\}$ is dense in the measure algebra of μ endowed with its usual topology. (iii) Suppose that μ is a quasi-Radon measure and $\mathfrak{T}$ is completely regular. Show that $\{E^\bullet : E \in \mathcal{E}\}$ is dense in the measure algebra of μ. (*Hint*: 414Aa.)

(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a second-countable atomless topological probability space with a strictly positive measure, $\mathcal{E}$ the Jordan algebra of μ as defined in 411Yc, $(\mathfrak{A}, \bar{\mu})$ the measure algebra of μ and $\mathfrak{C}$ the image $\{E^\bullet : E \in \mathcal{E}\} \subseteq \mathfrak{A}$. Let $\mathfrak{B}$ be a Boolean algebra and $\nu : \mathfrak{B} \to [0, 1]$ a finitely additive functional. Show that $(\mathfrak{B}, \nu) \cong (\mathfrak{C}, \bar{\mu}\restriction\mathfrak{C})$ iff (α) ν is strictly positive and properly atomless in the sense of 326F[3], and $\nu 1 = 1$ (β) there is a countable subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ such that $\nu b = \sup\{\nu c : c \in \mathfrak{B}_0,\ c \subseteq b\}$ for every $b \in \mathfrak{B}$ (γ) whenever A, $B \subseteq \mathfrak{B}$ are upwards-directed sets such that $a \cap b = 0$ for every $a \in A$ and $b \in B$ and $\sup\{\nu(a \cup b) : a \in A,\ b \in B\} = 1$, then $\sup A$ is defined in $\mathfrak{B}$.

411 Notes and comments Of course the list above can give only a rough idea of the ways in which topologies and measures can interact. In particular I have rather arbitrarily given a sort of priority to three particular relationships between the domain Σ of a measure and the topology: 'topological measure space' (in which Σ includes the Borel σ-algebra), 'Borel measure' (in which Σ is precisely the Borel σ-algebra) and 'Baire measure' (in which Σ is the Baire σ-algebra).

Abstract topological measure theory is a relatively new subject, and there are many technical questions on which different authors take different views. For instance, the phrase 'Radon measure' is commonly used to mean what I would call a 'tight locally finite Borel measure' (cf. 416F); and some writers enlarge the definition of 'topological measure' to include Baire measures as defined above.

I give very few examples at this stage, two drawn from the constructions of Volumes 1-3 (Lebesgue measure and Stone spaces, 411O-411P) and one new one ('Dieudonné's measure', 411Q), with a glance at

[3] Formerly 326Ya.

the countable-cocountable measure of ω_1 (411R). The most glaring omission is that of the product measures on $\{0,1\}^I$ and $[0,1]^I$. I pass these by at the moment because a proper study of them requires rather more preparation than can be slipped into a parenthesis. (I return to them in 416U.) I have also omitted any discussion of 'measurable' and 'almost continuous' functions, except for a reference to a theorem in Volume 2 (411Ya), which will have to be repeated and amplified later on (418K). There is an obvious complementarity between the notions of 'inner' and 'outer' regularity (411B, 411D), but it works well only for totally finite spaces (411Xa); in other cases it may not be obvious what will happen (411O, 411Pe, 412W).

Version of 17.6.16

412 Inner regularity

As will become apparent as the chapter progresses, the concepts introduced in §411 are synergic; their most interesting manifestations are in combinations of various kinds. Any linear account of their properties will be more than usually like a space-filling curve. But I have to start somewhere, and enough results can be expressed in terms of inner regularity, more or less by itself, to be a useful beginning.

After a handful of elementary basic facts (412A) and a list of standard applications (412B), I give some useful sufficient conditions for inner regularity of topological and Baire measures (412D, 412E, 412G), based on an important general construction (412C). The rest of the section amounts to a review of ideas from Volume 2 and Chapter 32 in the light of the new concept here. I touch on completions (412H), c.l.d. versions and complete locally determined spaces (412H, 412J, 412M), strictly localizable spaces (412I), inverse-measure-preserving functions (412K, 412L), measure algebras (412N), subspaces (412O, 412P), indefinite-integral measures (412Q) and product measures (412R-412V), with a brief mention of outer regularity (412W); most of the hard work has already been done in Chapters 21 and 25.

412A I begin by repeating a lemma from Chapter 34, with some further straightforward facts.

Lemma (a) Let (X,Σ,μ) be a measure space and $\mathcal{K}$ a family of sets such that

whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\mu K>0$.

Then whenever $E\in\Sigma$ there is a countable disjoint family $\langle K_i\rangle_{i\in I}$ in $\mathcal{K}\cap\Sigma$ such that $K_i\subseteq E$ for every i and $\sum_{i\in I}\mu K_i=\mu E$. If moreover

(†) $K\cup K'\in\mathcal{K}$ whenever K, K' are disjoint members of $\mathcal{K}$,

then μ is inner regular with respect to $\mathcal{K}$. If $\bigcup_{i\in I}K_i\in\mathcal{K}$ for every countable disjoint family $\langle K_i\rangle_{i\in I}$ in $\mathcal{K}$, then for every $E\in\Sigma$ there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\mu K=\mu E$.

(b) Let (X,Σ,μ) be a measure space, T a σ-subalgebra of Σ, and $\mathcal{K}$ a family of sets. If μ is inner regular with respect to T and $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$, then μ is inner regular with respect to $\mathcal{K}$.

(c) Let (X,Σ,μ) be a semi-finite measure space and $\langle\mathcal{K}_n\rangle_{n\in\mathbb{N}}$ a sequence of families of sets such that μ is inner regular with respect to $\mathcal{K}_n$ and

(‡) if $\langle K_i\rangle_{i\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}_n$, then $\bigcap_{i\in\mathbb{N}}K_i\in\mathcal{K}_n$

for every $n\in\mathbb{N}$. Then μ is inner regular with respect to $\bigcap_{n\in\mathbb{N}}\mathcal{K}_n$.

proof (a) This is 342B-342C.

(b) If $E\in\Sigma$ and $\gamma<\mu E$, there are an $F\in\mathrm{T}$ such that $F\subseteq E$ and $\mu F>\gamma$, and a $K\in\mathcal{K}\cap\mathrm{T}$ such that $K\subseteq F$ and $\mu K\geq\gamma$.

(c) Suppose that $E\in\Sigma$ and that $0\leq\gamma<\mu E$. Because μ is semi-finite, there is an $F\in\Sigma$ such that $F\subseteq E$ and $\gamma<\mu F<\infty$ (213A). Choose $\langle K_i\rangle_{i\in\mathbb{N}}$ inductively, as follows. Start with $K_0=F$. Given that $K_i\in\Sigma$ and $\gamma<\mu K_i$, then let $n_i\in\mathbb{N}$ be such that $2^{-n_i}(i+1)$ is an odd integer, and choose $K_{i+1}\in\mathcal{K}_{n_i}\cap\Sigma$ such that $K_{i+1}\subseteq K_i$ and $\mu K_{i+1}>\gamma$; this will be possible because μ is inner regular with respect to $\mathcal{K}_{n_i}$. Consider $K=\bigcap_{i\in\mathbb{N}}K_i$. Then $K\subseteq E$ and $\mu K=\lim_{i\to\infty}\mu K_i\geq\gamma$. But also

$$K=\bigcap_{j\in\mathbb{N}}K_{2^n(2j+1)}\in\mathcal{K}_n$$

because $\langle K_{2^n(2j+1)}\rangle_{j\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}_n$, for each n. So $K\in\bigcap_{n\in\mathbb{N}}\mathcal{K}_n$. As E and γ are arbitrary, μ is inner regular with respect to $\bigcap_{n\in\mathbb{N}}\mathcal{K}_n$.

412B Corollary Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X. Suppose that $\mathcal{K}$ is

either the family of Borel subsets of X

or the family of closed subsets of X

or the family of compact subsets of X

or the family of zero sets in X,

and suppose that whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\mu K>0$. Then μ is inner regular with respect to $\mathcal{K}$.

proof In every case, $\mathcal{K}$ satisfies the condition (†) of 412Aa.

412C The next lemma provides a particularly useful method of proving that measures are inner regular with respect to 'well-behaved' families of sets.

Lemma Let (X,Σ,μ) be a semi-finite measure space, and suppose that $\mathcal{A}\subseteq\Sigma$ is such that

$\emptyset\in\mathcal{A}\subseteq\Sigma$,

$X\setminus A\in\mathcal{A}$ for every $A\in\mathcal{A}$.

Let T be the σ-subalgebra of Σ generated by $\mathcal{A}$. Let $\mathcal{K}$ be a family of subsets of X such that

(†) $K\cup K'\in\mathcal{K}$ whenever K, $K'\in\mathcal{K}$,

(‡) $\bigcap_{n\in\mathbb{N}}K_n\in\mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

whenever $A\in\mathcal{A}$, $F\in\Sigma$ and $\mu(A\cap F)>0$, there is a $K\in\mathcal{K}\cap\mathrm{T}$ such that $K\subseteq A$ and $\mu(K\cap F)>0$.

Then $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$.

proof (a) Write $\mathfrak{A}$ for the measure algebra of (X,Σ,μ), and $\mathcal{L}=\mathcal{K}\cap\mathrm{T}$, so that $\mathcal{L}$ also is closed under finite unions and countable intersections. Set

$$\mathcal{H}=\{E:E\in\Sigma,\ \sup_{L\in\mathcal{L},L\subseteq E}L^\bullet=E^\bullet\}\text{ in }\mathfrak{A},$$

$$\mathrm{T}'=\{E:E\in\mathcal{H},\ X\setminus E\in\mathcal{H}\},$$

so that the last condition tells us that $\mathcal{A}\subseteq\mathcal{H}$ and therefore that $\mathcal{A}\subseteq\mathrm{T}'$.

(b) The intersection of any sequence in $\mathcal{H}$ belongs to $\mathcal{H}$. **P** Let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ with intersection H. Write A_n for $\{L^\bullet:L\in\mathcal{L},\ L\subseteq H_n\}\subseteq\mathfrak{A}$ for each $n\in\mathbb{N}$. Since μ is semi-finite, $\mathfrak{A}$ is weakly (σ,∞)-distributive (322F). As A_n is upwards-directed and $\sup A_n=H_n^\bullet$ for each $n\in\mathbb{N}$,

$$H^\bullet=\inf_{n\in\mathbb{N}}H_n^\bullet$$

(because $F\mapsto F^\bullet:\Sigma\to\mathfrak{A}$ is sequentially order-continuous, by 321H)

$$=\inf_{n\in\mathbb{N}}\sup A_n=\sup\{\inf_{n\in\mathbb{N}}a_n:a_n\in A_n\text{ for every }n\in\mathbb{N}\}$$

(316H(iv))

$$=\sup\{(\bigcap_{n\in\mathbb{N}}L_n)^\bullet:L_n\in\mathcal{L},\ L_n\subseteq H_n\text{ for every }n\in\mathbb{N}\}$$

$$\subseteq\sup\{L^\bullet:L\in\mathcal{L},\ L\subseteq H\}$$

(by (‡))

$$\subseteq H^\bullet,$$

and $H\in\mathcal{H}$. **Q**

(c) The union of any sequence in $\mathcal{H}$ belongs to $\mathcal{H}$. **P** If $\langle H_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ with union H then

$$\sup_{L\in\mathcal{L},L\subseteq H}L^\bullet\supseteq\sup_{n\in\mathbb{N}}\sup_{L\in\mathcal{L},L\subseteq E_n}L^\bullet=\sup_{n\in\mathbb{N}}H_n^\bullet=H^\bullet,$$

so $H\in\mathcal{H}$. **Q**

(d) T' is a σ-subalgebra of Σ. **P** (i) $\emptyset$ and X belong to $\mathcal{A} \subseteq \mathcal{H}$, so $\emptyset \in \mathrm{T}'$. (ii) Obviously $X \setminus E \in \mathrm{T}'$ whenever $E \in \mathrm{T}'$. (iii) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in T' with union E then $E \in \mathcal{H}$, by (c); but also $X \setminus E = \bigcap_{n\in\mathbb{N}}(X \setminus E_n)$ belongs to $\mathcal{H}$, by (b). So $E \in \mathrm{T}'$. **Q**

(e) Accordingly $\mathrm{T} \subseteq \mathrm{T}'$, and $E^\bullet = \sup_{L\in\mathcal{L},L\subseteq E} L^\bullet$ for every $E \in \mathrm{T}$. It follows at once that if $E \in \mathrm{T}$ and $\mu E > 0$, there must be an $L \in \mathcal{L}$ such that $L \subseteq E$ and $\mu L > 0$; since (†) is true, and $\mathcal{L} \subseteq \mathrm{T}$, we can apply 412Aa to see that $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{L}$, therefore with respect to $\mathcal{K}$.

412D As corollaries of the last lemma I give two-and-a-half basic theorems.

Theorem Let $(X, \mathfrak{T})$ be a topological space and μ a semi-finite Baire measure on X. Then μ is inner regular with respect to the zero sets.

proof Write Σ for the Baire σ-algebra of X, the domain of μ, $\mathcal{K}$ for the family of zero sets, and $\mathcal{A}$ for $\mathcal{K} \cup \{X \setminus K : K \in \mathcal{K}\}$. Since the union of two zero sets is a zero set (4A2C(b-ii)), the intersection of a sequence of zero sets is a zero set (4A2C(b-iii)), and the complement of a zero set is the union of a sequence of zero sets (4A2C(b-vi)), the conditions of 412C are satisfied; and as the σ-algebra generated by $\mathcal{A}$ is just Σ, μ is inner regular with respect to $\mathcal{K}$.

412E Theorem Let $(X, \mathfrak{T})$ be a perfectly normal topological space (e.g., any metrizable space). Then any semi-finite Borel measure on X is inner regular with respect to the closed sets.

proof Because the Baire and Borel σ-algebras are the same (4A3Kb), this is a special case of 412D.

412F Lemma Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X such that μ is effectively locally finite with respect to $\mathfrak{T}$. Then

$$\mu E = \sup\{\mu(E \cap G) : G \text{ is a measurable open set of finite measure}\}$$

for every $E \in \Sigma$.

proof Apply 412Aa with $\mathcal{K}$ the family of subsets of measurable open sets of finite measure.

412G Theorem Let (X, Σ, μ) be a measure space with a topology $\mathfrak{T}$ such that μ is effectively locally finite with respect to $\mathfrak{T}$ and Σ is the σ-algebra generated by $\mathfrak{T} \cap \Sigma$. If

$$\mu G = \sup\{\mu F : F \in \Sigma \text{ is closed}, F \subseteq G\}$$

for every measurable open set G of finite measure, then μ is inner regular with respect to the closed sets.

proof In 412C, take $\mathcal{K}$ to be the family of measurable closed subsets of X, and $\mathcal{A}$ to be the family of measurable sets which are *either* open *or* closed. If $G \in \Sigma \cap \mathfrak{T}$, $F \in \Sigma$ and $\mu(G \cap F) > 0$, then there is an open set H of finite measure such that $\mu(H \cap G \cap F) > 0$, because μ is effectively locally finite; now there is a $K \in \mathcal{K}$ such that $K \subseteq H \cap G$ and $\mu K > \mu(H \cap G) - \mu(H \cap G \cap F)$, so that $\mu(K \cap F) > 0$. This is the only non-trivial item in the list of hypotheses in 412C, so we can conclude that $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$, where T is the σ-algebra generated by $\mathcal{A}$; but of course this is just Σ.

Remark There is a similar result in 416F(iii) below.

412H Proposition Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets.

(a) If μ is inner regular with respect to $\mathcal{K}$, so are its completion $\hat{\mu}$ (212C) and c.l.d. version $\tilde{\mu}$ (213E).

(b) Now suppose that μ is semi-finite and that

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$.

If either $\hat{\mu}$ or $\tilde{\mu}$ is inner regular with respect to $\mathcal{K}$ then μ is inner regular with respect to $\mathcal{K}$.

proof (a) If F belongs to the domain of $\hat{\mu}$, then there is an $E \in \Sigma$ such that $E \subseteq F$ and $\hat{\mu}(F \setminus E) = 0$. So if $0 \le \gamma < \hat{\mu}F = \mu E$, there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E \subseteq F$ and $\hat{\mu}K = \mu K \ge \gamma$.

If H belongs to the domain of $\tilde{\mu}$ and $0 \le \gamma < \tilde{\mu}H$, there is an $E \in \Sigma$ such that $\mu E < \infty$ and $\hat{\mu}(E \cap H) > \gamma$ (213D). Now there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E \cap H$ and $\mu K \ge \gamma$. As $\mu K < \infty$, $\tilde{\mu}K = \mu K \ge \gamma$.

(b) Write $\check{\mu}$ for whichever of $\hat{\mu}$, $\tilde{\mu}$ is supposed to be inner regular with respect to $\mathcal{K}$. Then $\check{\mu}$ is inner regular with respect to Σ (212Ca, 213Fc), so is inner regular with respect to $\mathcal{K}\cap\Sigma$ (412Ac). Also $\check{\mu}$ extends μ (212D, 213Hc). Take $E\in\Sigma$ and $\gamma<\mu E=\check{\mu}E$. Then there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\gamma<\check{\mu}K=\mu K$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

412I Lemma Let (X,Σ,μ) be a strictly localizable measure space and $\mathcal{K}$ a family of sets such that whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\mu K>0$.

(a) There is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that at most one X_i does not belong to $\mathcal{K}$, and that exceptional one, if any, is negligible.

(b) There is a disjoint family $\mathcal{L}\subseteq\mathcal{K}\cap\Sigma$ such that $\mu^*A=\sum_{L\in\mathcal{L}}\mu^*(A\cap L)$ for every $A\subseteq X$.

(c) If μ is σ-finite then the family $\langle X_i\rangle_{i\in I}$ of (a) and the set $\mathcal{L}$ of (b) can be taken to be countable.

proof (a) Let $\langle E_j\rangle_{j\in J}$ be any decomposition of X. For each $j\in J$, let $\mathcal{K}_j$ be a maximal disjoint subset of

$$\{K:K\in\mathcal{K}\cap\Sigma,\ K\subseteq E_j,\ \mu K>0\}.$$

Because $\mu E_j<\infty$, $\mathcal{K}_j$ must be countable. Set $E'_j=E_j\setminus\bigcup\mathcal{K}_j$. By the maximality of $\mathcal{K}_j$, E'_j cannot include any non-negligible set in $\mathcal{K}\cap\Sigma$; but this means that $\mu E'_j=0$. Set $X'=\bigcup_{j\in J}E'_j$. Then

$$\mu X'=\sum_{j\in J}\mu(X'\cap E_j)=\sum_{j\in J}\mu E'_j=0.$$

Note that if j, $j'\in J$ are distinct, $K\in\mathcal{K}_j$ and $K'\in\mathcal{K}_{j'}$, then $K\cap K'=\emptyset$; thus $\mathcal{L}=\bigcup_{j\in J}\mathcal{K}_j$ is disjoint. Let $\langle X_i\rangle_{i\in I}$ be any indexing of $\{X'\}\cup\mathcal{L}$. This is a partition (that is, disjoint cover) of X into sets of finite measure. If $E\subseteq X$ and $E\cap X_i\in\Sigma$ for every $i\in I$, then for every $j\in J$

$$E\cap E_j=(E\cap X'\cap E_j)\cup\bigcup_{K\in\mathcal{K}_j}E\cap K$$

belongs to Σ, so that $E\in\Sigma$ and

$$\mu E=\sum_{j\in J}\mu(E\cap E_j)=\sum_{j\in J}\sum_{K\in\mathcal{K}_j}\mu(E\cap K)=\sum_{i\in I}\mu(E\cap X_i).$$

Thus $\langle X_i\rangle_{i\in I}$ is a decomposition of X, and it is of the right type because every X_i but one belongs to $\mathcal{L}\subseteq\mathcal{K}$.

(b) If now $A\subseteq X$ is any set,

$$\mu^*A=\mu_AA=\sum_{i\in I}\mu_A(A\cap X_i)=\sum_{i\in I}\mu^*(A\cap X_i)$$

by 214Ia, writing μ_A for the subspace measure on A. So we have

$$\mu^*A=\mu^*(A\cap X')+\sum_{L\in\mathcal{L}}\mu^*(A\cap L)=\sum_{L\in\mathcal{L}}\mu^*(A\cap L),$$

while $\mathcal{L}\subseteq\mathcal{K}$ is disjoint.

(c) If μ is σ-finite we can take J to be countable, so that I and $\mathcal{L}$ will also be countable.

412J Proposition Let (X,Σ,μ) be a complete locally determined measure space, and $\mathcal{K}$ a family of sets such that μ is inner regular with respect to $\mathcal{K}$.

(a) If $E\subseteq X$ is such that $E\cap K\in\Sigma$ for every $K\in\mathcal{K}\cap\Sigma$, then $E\in\Sigma$.

(b) If $E\subseteq X$ is such that $E\cap K$ is negligible for every $K\in\mathcal{K}\cap\Sigma$, then E is negligible.

(c) For any $A\subseteq X$, $\mu^*A=\sup_{K\in\mathcal{K}\cap\Sigma}\mu^*(A\cap K)$.

(d) Let f be a non-negative $[0,\infty]$-valued function defined on a subset of X. If $\int_K f$ is defined in $[0,\infty]$ for every $K\in\mathcal{K}$, then $\int f$ is defined and equal to $\sup_{K\in\mathcal{K}}\int_K f$.

(e) If f is a μ-integrable function and $\epsilon>0$, there is a $K\in\mathcal{K}$ such that $\int_{X\setminus K}|f|\le\epsilon$.

Remark In (c), we must interpret $\sup\emptyset$ as 0 if $\mathcal{K}\cap\Sigma=\emptyset$.

proof (a) If $F\in\Sigma$ and $\mu F<\infty$, then $E\cap F\in\Sigma$. **P** If $\mu F=0$, this is trivial, because μ is complete and $E\cap F$ is negligible. Otherwise, there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}\cap\Sigma$ such that $K_n\subseteq F$ for each n and $\sup_{n\in\mathbb{N}}\mu K_n=\mu F$. Now $E\cap F\setminus\bigcup_{n\in\mathbb{N}}K_n$ is negligible, therefore measurable, while $E\cap K_n$ is measurable for every $n\in\mathbb{N}$, by hypothesis; so $E\cap F$ is measurable. **Q** As μ is locally determined, $E\in\Sigma$, as claimed.

(b) By (a), $E\in\Sigma$; and because μ is inner regular with respect to $\mathcal{K}$, μE must be 0.

(c) Let μ_A be the subspace measure on A. Because μ is complete and locally determined, μ_A is semi-finite (214Id). So if $0 \le \gamma < \mu^* A = \mu_A A$, there is an $H \subseteq A$ such that $\mu_A H$ is defined, finite and greater than γ. Let $E \in \Sigma$ be a measurable envelope of H (132Ee), so that $\mu E = \mu^* H > \gamma$. Then there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K \ge \gamma$. In this case

$$\mu^*(A \cap K) \ge \mu^*(H \cap K) = \mu(E \cap K) = \mu K \ge \gamma.$$

As γ is arbitrary,

$$\mu^* A \le \sup_{K \in \mathcal{K} \cap \Sigma} \mu^*(A \cap K);$$

but the reverse inequality is trivial, so we have the result.

(d) Applying (b) with $E = X \setminus \operatorname{dom} f$, we see that f is defined almost everywhere in X. Applying (a) with $E = \{x : x \in \operatorname{dom} f,\ f(x) \ge \alpha\}$ for each $\alpha \in \mathbb{R}$, we see that f is measurable. So $\int f$ is defined in $[0, \infty]$, and of course $\int f \ge \sup_{K \in \mathcal{K}} \int_K f$. If $\gamma < \int f$, there is a non-negative simple function g such that $g \le_{\text{a.e.}} f$ and $\int g > \gamma$; taking $E = \{x : g(x) > 0\}$, there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu(E \setminus K)\|g\|_\infty \le \int g - \gamma$, so that $\int_K f \ge \int_K g \ge \gamma$. As γ is arbitrary, $\int f = \sup_{K \in \mathcal{K}} \int_K f$.

(e) By (d), there is a $K \in \mathcal{K}$ such that $\int_K |f| \ge \int |f| - \epsilon$.

Remark See also 413F below.

412K Proposition Let (X, Σ, μ) be a complete locally determined measure space, (Y, T, ν) a measure space and $f : X \to Y$ a function. Suppose that $\mathcal{K} \subseteq \mathrm{T}$ is such that

(i) ν is inner regular with respect to $\mathcal{K}$;

(ii) $f^{-1}[K] \in \Sigma$ and $\mu f^{-1}[K] = \nu K$ for every $K \in \mathcal{K}$;

(iii) whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $\nu K < \infty$ and $\mu(E \cap f^{-1}[K]) > 0$.

Then f is inverse-measure-preserving for μ and ν.

proof (a) If $F \in \mathrm{T}$, $E \in \Sigma$ and $\mu E < \infty$, then $E \cap f^{-1}[F] \in \Sigma$. **P** Let H_1, $H_2 \in \Sigma$ be measurable envelopes for $E \cap f^{-1}[F]$ and $E \setminus f^{-1}[F]$ respectively. **?** If $\mu(H_1 \cap H_2) > 0$, there is a $K \in \mathcal{K}$ such that νK is finite and $\mu(H_1 \cap H_2 \cap f^{-1}[K]) > 0$. Because ν is inner regular with respect to $\mathcal{K}$, there are K_1, $K_2 \in \mathcal{K}$ such that $K_1 \subseteq K \cap F$, $K_2 \subseteq K \setminus F$ and

$$\begin{aligned}\nu K_1 + \nu K_2 &> \nu(K \cap F) + \nu(K \setminus F) - \mu(H_1 \cap H_2 \cap f^{-1}[K]) \\ &= \nu K - \mu(H_1 \cap H_2 \cap f^{-1}[K]).\end{aligned}$$

Now

$$\mu(H_1 \cap f^{-1}[K_2]) = \mu^*(E \cap f^{-1}[F] \cap f^{-1}[K_2]) = 0,$$

$$\mu(H_2 \cap f^{-1}[K_1]) = \mu^*(E \cap f^{-1}[K_1] \setminus f^{-1}[F]) = 0,$$

so $\mu(H_1 \cap H_2 \cap f^{-1}[K_1 \cup K_2]) = 0$ and

$$\begin{aligned}\mu(H_1 \cap H_2 \cap f^{-1}[K]) &\le \mu(f^{-1}[K] \setminus f^{-1}[K_1 \cup K_2]) \\ &= \mu f^{-1}[K] - \mu f^{-1}[K_1] - \mu f^{-1}[K_2] \\ &= \nu K - \nu K_1 - \nu K_2 < \mu(H_1 \cap H_2 \cap f^{-1}[K]),\end{aligned}$$

which is absurd. **X**

Now $(E \cap H_1) \setminus (E \cap f^{-1}[F]) \subseteq H_1 \cap H_2$ is negligible, therefore measurable (because μ is complete), and $E \cap f^{-1}[F] \in \Sigma$, as claimed. **Q**

(b) It follows (because μ is locally determined) that $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$.

(c) If $F \in \mathrm{T}$ and $\nu F = 0$ then $\mu f^{-1}[F] = 0$. **P?** Otherwise, there is a $K \in \mathcal{K}$ such that $\nu K < \infty$ and

$$0 < \mu(f^{-1}[F] \cap f^{-1}[K]) = \mu f^{-1}[F \cap K].$$

Let $K' \in \mathcal{K}$ be such that $K' \subseteq K \setminus F$ and $\nu K' > \nu K - \mu f^{-1}[F \cap K]$. Then $f^{-1}[K'] \cap f^{-1}[F \cap K] = \emptyset$, so

$$\nu K = \mu f^{-1}[K] \ge \mu f^{-1}[K'] + \mu f^{-1}[F \cap K] > \nu K' + \nu K - \nu K' = \nu K,$$

which is absurd. **XQ**

(d) Finally, $\mu f^{-1}[F] = \nu F$ for every $F \in \mathrm{T}$. **P** Let $\langle K_i \rangle_{i \in I}$ be a countable disjoint family in $\mathcal{K}$ such that $K_i \subseteq F$ for every i and $\sum_{i \in I} \nu K_i = \nu F$ (412Aa). Set $F' = F \setminus \bigcup_{i \in I} K_i$. Then

$$\mu f^{-1}[F] = \mu f^{-1}[F'] + \sum_{i \in I} \mu f^{-1}[K_i] = \mu f^{-1}[F'] + \sum_{i \in I} \nu K_i = \mu f^{-1}[F'] + \nu F.$$

If $\nu F = \infty$ then surely $\mu f^{-1}[F] = \infty = \nu F$. Otherwise, $\nu F' = 0$ so $\mu f^{-1}[F'] = 0$ (by (c)) and again $\mu f^{-1}[F] = \nu F$. **Q**

Thus f is inverse-measure-preserving.

412L Corollary Let (X, Σ, μ) be a complete probability space, (Y, T, ν) a probability space and $f : X \to Y$ a function. Suppose that whenever $F \in \mathrm{T}$ and $\nu F > 0$ there is a $K \in \mathrm{T}$ such that $K \subseteq F$, $\nu K > 0$, $f^{-1}[K] \in \Sigma$ and $\mu f^{-1}[K] \ge \nu K$. Then f is inverse-measure-preserving.

proof Set $\mathcal{K}^* = \{K : K \in \mathrm{T},\ f^{-1}[K] \in \Sigma,\ \mu f^{-1}[K] \ge \nu K\}$. Then $\mathcal{K}^*$ is closed under countable disjoint unions and includes $\mathcal{K}$, so for every $F \in \mathrm{T}$ there is a $K \in \mathcal{K}^*$ such that $K \subseteq F$ and $\nu K = \nu F$, by 412Aa. But this means that $\mu f^{-1}[K] = \nu K$ for every $K \in \mathcal{K}^*$. **P** There is a $K' \in \mathcal{K}^*$ such that $K' \subseteq Y \setminus K$ and $\nu K' = 1 - \nu K$; but in this case

$$\mu f^{-1}[K'] + \mu f^{-1}[K] \le 1 = \nu K' + \nu K,$$

so $\mu f^{-1}[K]$ must be equal to νK. **Q** Moreover, there is a $K^* \in \mathcal{K}^*$ such that $\nu K^* = \nu Y = 1$, so $\mu f^{-1}[K^*] = \mu X = 1$ and $\mu(E \cap f^{-1}[K^*]) > 0$ whenever $\mu E > 0$. Applying 412K to $\mathcal{K}^*$ we have the result.

412M Proposition Let X be a set and $\mathcal{K}$ a family of subsets of X. Suppose that μ and ν are two complete locally determined measures on X, with domains including $\mathcal{K}$, and both inner regular with respect to $\mathcal{K}$.

(a) If $\mu K \le \nu K$ for every $K \in \mathcal{K}$, then $\mu \le \nu$ in the sense of 234P, that is, $\operatorname{dom} \nu \subseteq \operatorname{dom} \mu$ and $\mu E \le \nu E$ for every $E \in \operatorname{dom} \nu$.

(b) If $\mu K = \nu K$ for every $K \in \mathcal{K}$, then $\mu = \nu$.

proof (a)(i) Write Σ for the domain of μ and T for the domain of ν. If $E \in \Sigma \cap \mathrm{T}$ then of course

$$\mu E = \sup_{K \in \mathcal{K}, K \subseteq E} \mu K \le \sup_{K \in \mathcal{K}, K \subseteq E} \nu K = \nu E.$$

So we have only to show that $\mathrm{T} \subseteq \Sigma$.

(ii) Suppose that $F \in \mathrm{T}$, $E \in \Sigma$ and $\mu E < \infty$. Let E', $E'' \subseteq E$ be measurable envelopes of $E \cap F$, $E \setminus F$ with respect to μ. Take any $K \in \mathcal{K}$ such that $K \subseteq E' \cap E''$, Then $K \cap F$ and $K \setminus F$ belong to T. Let $\langle L_n \rangle_{n \in \mathbb{N}}$, $\langle L'_n \rangle_{n \in \mathbb{N}}$ be sequences in $\mathcal{K}$ such that $L_n \subseteq K \cap F$ and $L'_n \subseteq K \setminus F$ for every n, $\lim_{n \to \infty} \nu L_n = \nu(K \cap F)$ and $\lim_{n \to \infty} \nu L'_n = \nu(K \setminus F)$. Because $\nu K < \infty$, $\nu(K \setminus \bigcup_{n \in \mathbb{N}}(L_n \cup L'_n)) = 0$ and therefore $\mu(K \setminus \bigcup_{n \in \mathbb{N}}(L_n \cup L'_n)) = 0$, since $K \setminus \bigcup_{n \in \mathbb{N}}(L_n \cup L'_n) \in \Sigma \cap \mathrm{T}$. On the other hand, for each n, $L_n \in \Sigma$ and $L_n \subseteq E'' \cap F$ so $\mu L_n = \mu^*(L_n \setminus F) = 0$; similarly, $\mu L'_n = 0$. We conclude that $\mu K = 0$. As K is arbitrary, $\mu(E' \cap E'') = 0$; as μ is complete, $F \cap E''$ and $F \cap E = (E \setminus E'') \cup (F \cap E'')$ belong to Σ; as E is arbitrary and μ is locally determined, $F \in \Sigma$; as F is arbitrary, $\mathrm{T} \subseteq \Sigma$ and $\mu \le \nu$.

(b) By (a), we have $\mu \le \nu$ and $\nu \le \mu$, so $\mu = \nu$ (234Qa).

412N Lemma Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$. Then

$$E^\bullet = \sup\{K^\bullet : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$$

in the measure algebra $\mathfrak{A}$ of μ, for every $E \in \Sigma$. In particular, $\{K^\bullet : K \in \mathcal{K} \cap \Sigma\}$ is order-dense in $\mathfrak{A}$; and if $\mathcal{K}$ is closed under finite unions, then $\{K^\bullet : K \in \mathcal{K} \cap \Sigma\}$ is topologically dense in $\mathfrak{A}$ for the measure-algebra topology.

proof ? If $E^\bullet \ne \sup\{K^\bullet : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$, there is a non-zero $a \in \mathfrak{A}$ such that $a \subseteq E^\bullet \setminus K^\bullet$ whenever $K \in \mathcal{K} \cap \Sigma$ and $K \subseteq E$. Express a as $F^\bullet$ where $F \subseteq E$. Then $\mu F > 0$, so there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq F$ and $\mu K > 0$. But in this case $0 \ne K^\bullet \subseteq a$, while $K \subseteq E$. **X**

It follows at once that $D = \{K^\bullet : K \in \mathcal{K} \cap \Sigma\}$ is order-dense. If $\mathcal{K}$ is closed under finite unions, and $a \in \mathfrak{A}$, then $D_a = \{d : d \in D,\ d \subseteq a\}$ is upwards-directed and has supremum a, so $a \in \overline{D}_a \subseteq \overline{D}$ (323D(a-ii)).

412O Lemma Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$.

(a) If $E \in \Sigma$, then the subspace measure μ_E (131B) is inner regular with respect to $\mathcal{K}$.

(b) Let $Y \subseteq X$ be any set such that the subspace measure μ_Y (214B) is semi-finite. Then μ_Y is inner regular with respect to $\mathcal{K}_Y = \{K \cap Y : K \in \mathcal{K}\}$.

proof (a) This is elementary.

(b) Suppose that F belongs to the domain Σ_Y of μ_Y and $0 \le \gamma < \mu_Y F$. Because μ_Y is semi-finite there is an $F' \in \Sigma_Y$ such that $F' \subseteq F$ and $\gamma < \mu_Y F' < \infty$. Let $G \in \Sigma$ be such that $F' = G \cap Y$, and let $E \subseteq G$ be a measurable envelope for F' with respect to μ, so that

$$\mu E = \mu^* F' = \mu_Y F' > \gamma.$$

There is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K \ge \gamma$, in which case $K \cap Y \in \mathcal{K}_Y \cap \Sigma_Y$ and

$$\mu_Y(K \cap Y) = \mu^*(K \cap Y) = \mu^*(K \cap F') = \mu(K \cap E) = \mu K \ge \gamma.$$

As F and γ are arbitrary, μ_Y is inner regular with respect to $\mathcal{K}_Y$.

Remark Recall from 214Ic that if (X, Σ, μ) has locally determined negligible sets (in particular, is either strictly localizable or complete and locally determined), then all its subspaces are semi-finite.

412P Proposition Let (X, Σ, μ) be a measure space, $\mathfrak{T}$ a topology on X and Y a subset of X; write $\mathfrak{T}_Y$ for the subspace topology of Y and μ_Y for the subspace measure on Y. Suppose that *either* $Y \in \Sigma$ *or* μ_Y is semi-finite.

(a) If μ is a topological measure, so is μ_Y.

(b) If μ is inner regular with respect to the Borel sets, so is μ_Y.

(c) If μ is inner regular with respect to the closed sets, so is μ_Y.

(d) If μ is inner regular with respect to the zero sets, so is μ_Y.

(e) If μ is effectively locally finite, so is μ_Y.

proof (a) is an immediate consequence of the definitions of 'subspace measure', 'subspace topology' and 'topological measure'. The other parts follow directly from 412O if we recall that

(i) a subset of Y is Borel for $\mathfrak{T}_Y$ whenever it is expressible as $Y \cap E$ for some Borel set $E \subseteq X$ (4A3Ca);

(ii) a subset of Y is closed in Y whenever it is expressible as $Y \cap F$ for some closed set $F \subseteq X$;

(iii) a subset of Y is a zero set in Y whenever it is expressible as $Y \cap F$ for some zero set $F \subseteq X$ (4A2C(b-v));

(iv) μ is effectively locally finite iff it is inner regular with respect to subsets of open sets of finite measure.

412Q Proposition Let (X, Σ, μ) be a measure space, and ν an indefinite-integral measure over μ (definition: 234J). If μ is inner regular with respect to a family $\mathcal{K}$ of sets, so is ν.

proof Because μ and its completion $\hat{\mu}$ give the same integrals, ν is an indefinite-integral measure over $\hat{\mu}$ (234Ke); and as $\hat{\mu}$ is still inner regular with respect to $\mathcal{K}$ (412Ha), we may suppose that μ itself is complete. Let f be a Radon-Nikodým derivative of ν with respect to μ; by 234Ka, we may suppose that $f : X \to [0, \infty[$ is Σ-measurable.

Suppose that $F \in \operatorname{dom} \nu$ and that $\gamma < \nu F$. Set $G = \{x : f(x) > 0\}$, so that $F \cap G \in \Sigma$ (234La). For $n \in \mathbb{N}$, set $H_n = \{x : x \in F,\ 2^{-n} \le f(x) \le 2^n\}$, so that $H_n \in \Sigma$ and

$$\nu F = \int f \times \chi F d\mu = \lim_{n\to\infty} \int f \times \chi H_n d\mu.$$

Let $n \in \mathbb{N}$ be such that $\int f \times \chi H_n d\mu > \gamma$.

If $\mu H_n = \infty$, there is a $K \in \mathcal{K}$ such that $K \subseteq H_n$ and $\mu K \ge 2^n \gamma$, so that $\nu K \ge \gamma$. If μH_n is finite, there is a $K \in \mathcal{K}$ such that $K \subseteq H_n$ and $2^n(\mu H_n - \mu K) \le \int f \times \chi H_n d\mu - \gamma$, so that $\int f \times \chi(H_n \setminus K) d\mu + \gamma \le \int f \times \chi H_n d\mu$ and $\nu K = \int f \times \chi K\, d\mu \ge \gamma$. Thus in either case we have a $K \in \mathcal{K}$ such that $K \subseteq F$ and $\nu K \ge \gamma$; as F and γ are arbitrary, ν is inner regular with respect to $\mathcal{K}$.

412R Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$ (251F). Suppose that $\mathcal{K} \subseteq \mathcal{P}X$, $\mathcal{L} \subseteq \mathcal{P}Y$ and $\mathcal{M} \subseteq \mathcal{P}(X \times Y)$ are such that

(i) μ is inner regular with respect to $\mathcal{K}$;
(ii) ν is inner regular with respect to $\mathcal{L}$;
(iii) $K \times L \in \mathcal{M}$ for all $K \in \mathcal{K}$, $L \in \mathcal{L}$;
(iv) $M \cup M' \in \mathcal{M}$ whenever M, $M' \in \mathcal{M}$;
(v) $\bigcap_{n\in\mathbb{N}} M_n \in \mathcal{M}$ for every sequence $\langle M_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}$.

Then λ is inner regular with respect to $\mathcal{M}$.

proof Write $\mathcal{A} = \{E \times Y : E \in \Sigma\} \cup \{X \times F : F \in \mathrm{T}\}$. Then the σ-algebra of subsets of $X \times Y$ generated by $\mathcal{A}$ is $\Sigma\widehat{\otimes}\mathrm{T}$. If $V \in \mathcal{A}$, $W \in \Lambda$ and $\lambda(W \cap V) > 0$, there is an $M \in \mathcal{M} \cap (\Sigma\widehat{\otimes}\mathrm{T})$ such that $M \subseteq W$ and $\lambda(M \cap V) > 0$. **P** Suppose that $V = E \times Y$ where $E \in \Sigma$. There must be $E_0 \in \Sigma$ and $F_0 \in \mathrm{T}$, both of finite measure, such that $\lambda(W \cap V \cap (E_0 \times F_0)) > 0$ (251F). Now there are $K \in \mathcal{K} \cap \Sigma$ and $L \in \mathcal{L} \cap \mathrm{T}$ such that $K \subseteq E \cap E_0$, $L \subseteq F \cap F_0$ and

$$\mu((E \cap E_0) \setminus K) \cdot \nu F_0 + \mu E_0 \cdot \nu((F \cap F_0) \setminus L) < \lambda(W \cap V \cap (E_0 \times F_0));$$

but this means that $M = K \times L$ is included in V and $\mu(W \cap M) > 0$, while $M \in \mathcal{M} \cap (\Sigma\widehat{\otimes}\mathrm{T})$. Reversing the roles of the coordinates, the same argument deals with the case in which $V = X \times F$ for some $F \in \mathrm{T}$. **Q**

By 412C, $\lambda\restriction\Sigma\widehat{\otimes}\mathrm{T}$ is inner regular with respect to $\mathcal{M}$. But λ is inner regular with respect to $\Sigma\widehat{\otimes}\mathrm{T}$ (251Ib) so is also inner regular with respect to $\mathcal{M}$ (412Ab).

412S Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Let $\mathfrak{T}$, $\mathfrak{S}$ be topologies on X and Y respectively, and give $X \times Y$ the product topology.

(a) If μ and ν are inner regular with respect to the closed sets, so is λ.
(b) If μ and ν are tight (that is, inner regular with respect to the closed compact sets), so is λ.
(c) If μ and ν are inner regular with respect to the zero sets, so is λ.
(d) If μ and ν are inner regular with respect to the Borel sets, so is λ.
(e) If μ and ν are effectively locally finite, so is λ.

proof We have only to read the conditions (i)-(v) of 412R carefully and check that they apply in each case. (In part (e), recall that 'effectively locally finite' is the same thing as 'inner regular with respect to the subsets of open sets of finite measure'.)

412T Lemma Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X, Λ, λ) (§254). Suppose that $\mathcal{K}_i \subseteq \mathcal{P}X_i$, $\mathcal{M} \subseteq \mathcal{P}X$ are such that

(i) μ_i is inner regular with respect to $\mathcal{K}_i$ for each $i \in I$;
(ii) $\pi_i^{-1}[K] \in \mathcal{M}$ for every $i \in I$ and $K \in \mathcal{K}_i$, writing $\pi_i(x) = x(i)$ for $x \in X$;
(iii) $M \cup M' \in \mathcal{M}$ whenever M, $M' \in \mathcal{M}$;
(iv) $\bigcap_{n\in\mathbb{N}} M_n \in \mathcal{M}$ for every sequence $\langle M_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}$.

Then λ is inner regular with respect to $\mathcal{M}$.

proof (Compare 412R.) Write $\mathcal{A} = \{\pi_i^{-1}[E] : i \in I,\ E \in \Sigma_i\}$. If $V \in \mathcal{A}$, $W \in \Lambda$ and $\lambda(W \cap V) > 0$, express V as $\pi_i^{-1}[E]$, where $i \in I$ and $E \in \Sigma_i$, and take $K \in \mathcal{K}_i$ such that $K \subseteq E$ and $\mu_i(E \setminus K) < \lambda(W \cap V)$; then $M = \pi_i^{-1}[K]$ belongs to $\mathcal{M} \cap \mathcal{A}$, is included in V, and meets W in a non-negligible set. So the conditions of 412C are met by $\mathcal{A}$ and $\mathcal{M}$.

It follows that $\lambda_0 = \lambda\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$ is inner regular with respect to $\mathcal{M}$. But λ is the completion of λ_0 (254Fd, 254Ff), so is also inner regular with respect to $\mathcal{M}$ (412Ha).

412U Proposition Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X, Λ, λ). Suppose that we are given a topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the product topology on X.

(a) If every μ_i is inner regular with respect to the closed sets, so is λ.
(b) If every μ_i is inner regular with respect to the zero sets, so is λ.
(c) If every μ_i is inner regular with respect to the Borel sets, so is λ.

proof This follows from 412T just as 412S follows from 412R.

412V Corollary Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X, Λ, λ). Suppose that we are given a Hausdorff topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the product topology on X. Suppose that every μ_i is tight, and that X_i is compact for all but countably many $i \in I$. Then λ is tight.

proof By 412Ua, λ is inner regular with respect to the closed sets. If $W \in \Lambda$ and $\gamma < \lambda W$, let $V \subseteq W$ be a measurable closed set such that $\lambda V > \gamma$. Let J be the set of those $i \in I$ such that X_i is not compact; we are supposing that J is countable. Let $\langle \epsilon_i \rangle_{i \in J}$ be a family of strictly positive real numbers such that $\sum_{i \in J} \epsilon_j \le \lambda V - \gamma$ (4A1P). For each $i \in J$, let $K_i \subseteq X_i$ be a compact measurable set such that $\mu_i(X_i \setminus K_i) \le \epsilon_i$; and for $i \in I \setminus J$, set $K_i = X_i$. Then $K = \prod_{i \in I} K_i$ is a compact measurable subset of X, and

$$\lambda(X \setminus K) \le \textstyle\sum_{i \in J} \mu_i(X_i \setminus K_i) \le \lambda V - \gamma,$$

so $\lambda(K \cap V) \ge \gamma$; while $K \cap V$ is a compact measurable subset of W. As W and γ are arbitrary, λ is tight.

***412W Outer regularity** I have already mentioned the complementary notion of 'outer regularity' (411D). In this book it will not be given much prominence. It is however a useful tool when dealing with Lebesgue measure (see, for instance, the proof of 225K), for reasons which the next proposition will make clear.

Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) Suppose that μ is outer regular with respect to the open sets. Then for any integrable function $f : X \to [0, \infty]$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to [0, \infty]$ such that $f \le g$ and $\int g \le \epsilon + \int f$.

(b) Now suppose that there is a sequence of measurable open sets of finite measure covering X. Then the following are equiveridical:

(i) μ is inner regular with respect to the closed sets;

(ii) μ is outer regular with respect to the open sets;

(iii) for any measurable set $E \subseteq X$ and $\epsilon > 0$, there are a measurable closed set $F \subseteq E$ and a measurable open set $H \supseteq E$ such that $\mu(H \setminus F) \le \epsilon$;

(iv) for every measurable function $f : X \to [0, \infty[$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to [0, \infty]$ such that $f \le g$ and $\int g - f \le \epsilon$;

(v) for every measurable function $f : X \to \mathbb{R}$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to]-\infty, \infty]$ such that $f \le g$ and $\mu\{x : g(x) \ge f(x) + \epsilon\} \le \epsilon$.

proof (a) Let $\eta \in]0, 1]$ be such that $\eta(9 + \int f d\mu) \le \epsilon$. For $n \in \mathbb{Z}$, set $E_n = \{x : (1 + \eta)^n \le f(x) < (1 + \eta)^{n+1}\}$, and let $E'_n \in \Sigma$ be a measurable envelope of E_n; let $G_n \supseteq E'_n$ be a measurable open set such that $\mu G_n \le 3^{-|n|}\eta + \mu E'_n$. Next, let E be a measurable envelope of the negligible set $\{x : f(x) = \infty\}$, and for each n let $H_n \supseteq E$ be a measurable open set such that $\mu H_n \le 2^{-n}$. Set

$$g = \textstyle\sum_{n=-\infty}^{\infty} (1 + \eta)^{n+1} \chi G_n + \sum_{n=0}^{\infty} \chi H_n.$$

Then g is lower semi-continuous (4A2B(d-iii), 4A2B(d-v)), $f \le g$ and

$$\begin{aligned} \int g \, d\mu &= \sum_{n=-\infty}^{\infty} (1 + \eta)^{n+1} \mu G_n + \sum_{n=0}^{\infty} \mu H_n \\ &\le (1 + \eta) \sum_{n=-\infty}^{\infty} (1 + \eta)^n \mu E'_n + \sum_{n=-\infty}^{\infty} (1 + \eta)^{n+1} 3^{-|n|} \eta + \sum_{n=0}^{\infty} 2^{-n} \eta \\ &\le (1 + \eta) \int f d\mu + 9\eta \le \int f d\mu + \epsilon, \end{aligned}$$

as required.

(b) Let $\langle G_n \rangle_{n \in \mathbb{N}}$ be a sequence of open sets of finite measure covering X; replacing it by $\langle \bigcup_{i<n} G_i \rangle_{n \in \mathbb{N}}$ if necessary, we may suppose that $\langle G_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and that $G_0 = \emptyset$.

(i)$\Rightarrow$(iii) Suppose that μ is inner regular with respect to the closed sets, and that $E \in \Sigma$, $\epsilon > 0$. For each $n \in \mathbb{N}$ let $F_n \subseteq G_n \setminus E$ be a measurable closed set such that $\mu F_n \ge \mu(G_n \setminus E) - 2^{-n-2}\epsilon$. Then

$H=\bigcup_{n\in\mathbb{N}}(G_n\setminus F_n)$ is a measurable open set including E and $\mu(H\setminus E)\le\frac{1}{2}\epsilon$. Applying the same argument to $X\setminus E$, we get a closed set $F\subseteq E$ such that $\mu(E\setminus F)\le\frac{1}{2}\epsilon$, so that $\mu(H\setminus F)\le\epsilon$.

(ii)⇒(iii) The same idea works. Suppose that μ is outer regular with respect to the open sets, and that $E\in\Sigma$, $\epsilon>0$. For each $n\in\mathbb{N}$, let $H_n\supseteq G_n\cap E$ be an open set such that $\mu(H_n\setminus E)\le 2^{-n-2}\epsilon$; then $H=\bigcup_{n\in\mathbb{N}}H_n$ is a measurable open set including E, and $\mu(H\setminus E)\le\frac{1}{2}\epsilon$. Now repeat the argument on $X\setminus E$ to find a measurable closed set $F\subseteq E$ such that $\mu(E\setminus F)\le\frac{1}{2}\epsilon$.

(iii)⇒(iv) Assume (iii), and let $f:X\to[0,\infty[$ be a measurable function, $\epsilon>0$. Set $\eta_n=2^{-n}\epsilon/(16+4\mu G_n)$ for each $n\in\mathbb{N}$. For $k\in\mathbb{N}$ set $E_k=\bigcup_{n\in\mathbb{N}}\{x:x\in G_n,\,k\eta_n\le f(x)<(k+1)\eta_n\}$, and choose an open set $H_k\supseteq E_k$ such that $\mu(H_k\setminus E_k)\le 2^{-k}$. Set

$$g=\sup_{k,n\in\mathbb{N}}(k+1)\eta_n\chi(G_n\cap H_k).$$

Then $g:X\to[0,\infty]$ is lower semi-continuous (4A2B(d-v) again). Since

$$\sup_{k,n\in\mathbb{N}}k\eta_n\chi(G_n\cap E_k)\le f\le\sup_{k,n\in\mathbb{N}}(k+1)\eta_n\chi(G_n\cap E_k),$$

$f\le g$ and

$$g-f\le\sup_{k,n\in\mathbb{N}}(k+1)\eta_n\chi(G_n\cap H_k\setminus E_k)+\sup_{k,n\in\mathbb{N}}\eta_n\chi(G_n\cap E_k)$$

has integral at most

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(k+1)\eta_n 2^{-k}+\sum_{n=0}^{\infty}\eta_n\mu G_n\le\epsilon.$$

(i)⇒(v) Assume (i), and suppose that $f:X\to\mathbb{R}$ is measurable and $\epsilon>0$. For each $n\in\mathbb{N}$, let $\alpha_n\ge 0$ be such that $\mu E_n<2^{-n-1}\epsilon$, where $E_n=\{x:x\in G_{n+1}\setminus G_n,\,f(x)\le-\alpha_n\}$. Let $F_n\subseteq(G_{n+1}\setminus G_n)\setminus E_n$ be a measurable closed set such that $\mu((G_{n+1}\setminus G_n)\setminus F_n)\le 2^{-n-2}\epsilon$. Because $\langle F_n\rangle_{n\in\mathbb{N}}$ is disjoint, $h=\sum_{n=0}^{\infty}\alpha_n\chi F_n$ is defined as a function from X to $[0,\infty[$. $\{F_n:n\in\mathbb{N}\}$ is locally finite, so $\{x:h(x)\ge\alpha\}=\bigcup_{n\in\mathbb{N},\alpha_n\ge\alpha}F_n$ is closed for every $\alpha>0$ (4A2B(h-i)), and h is upper semi-continuous. Now $f_1=f+h$ is a real-valued measurable function. Since (i)⇒(iii)⇒(iv), there is a measurable lower semi-continuous function $g_1:X\to[0,\infty]$ such that $f_1^+\le g_1$ and $\int g_1-f_1^+\le\frac{1}{2}\epsilon^2$, where $f_1^+=\max(0,f_1)$. But if we now set $g=g_1-h$, g is lower semi-continuous, $f\le g$ and

$$\begin{aligned}\{x:f(x)+\epsilon\le g(x)\}&\subseteq\{x:f_1^+(x)+\epsilon\le g_1(x)\}\cup\{x:f_1(x)<0\}\\&\subseteq\{x:f_1^+(x)+\epsilon\le g_1(x)\}\cup\bigcup_{n\in\mathbb{N}}(G_{n+1}\setminus G_n)\setminus F_n\end{aligned}$$

has measure at most ϵ, as required.

(iv)⇒(ii) and **(v)⇒(ii)** Suppose that either (iv) or (v) is true, and that $E\in\Sigma$, $\epsilon>0$. Then there is a measurable lower semi-continuous function $g:X\to\,]0,\infty]$ such that $\chi E\le g$ and $\mu\{x:\chi E(x)+\frac{1}{2}\le g(x)\}\le\epsilon$, since this is certainly true if $\int g-\chi E\le\frac{1}{2}\epsilon$. Set $G=\{x:g(x)>\frac{1}{2}\}$; then $E\subseteq G$ and $\mu(G\setminus E)\le\epsilon$.

(iii)⇒(i) is trivial. Assembling these fragments, the proof is complete.

412X Basic exercises (a) Let (X,Σ,μ) be a semi-finite measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^0(\mu)$ which converges almost everywhere to $f\in\mathcal{L}^0(\mu)$. Show that μ is inner regular with respect to $\{E:\langle f_n\restriction E\rangle_{n\in\mathbb{N}}$ is uniformly convergent$\}$. (Cf. 215Yb.)

(b) Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets *and* with respect to the compact sets. Show that μ is tight.

(c) Explain how 213A is a special case of 412Aa.

(d) Let X be a set and $\mathcal{K}$ a family of sets. Suppose that μ and ν are two semi-finite measures on X with the same domain and the same null ideal. Show that if one is inner regular with respect to $\mathcal{K}$, so is the other. (*Hint*: show that if $\nu F<\infty$ then $\nu F=\sup\{\nu E:E\subseteq F,\,\mu E<\infty\}$.)

>(e) Let (X,Σ,μ) be a measure space, and Σ_0 a σ-subalgebra of Σ such that μ is inner regular with respect to Σ_0. Show that if $1\le p<\infty$ then every member of $L^p(\mu)$ is of the form $f^\bullet$ for some Σ_0-measurable $f:X\to\mathbb{R}$.

(f) Let $\langle\mu_i\rangle_{i\in I}$ be a family of measures on a set X, with sum μ (234G). Suppose that $\mathcal{K}\subseteq\operatorname{dom}\mu$ is a family of sets such that $K\cup K'\in\mathcal{K}$ whenever K, $K'\in\mathcal{K}$ are disjoint and every μ_i is inner regular with respect to $\mathcal{K}$. Show that μ is inner regular with respect to $\mathcal{K}$.

(g) Let X be a topological space, and μ a tight topological measure on X. Suppose that $\mathcal{F}$ is a non-empty downwards-directed family of closed compact subsets of X with intersection F_0, and that $\gamma=\inf_{F\in\mathcal{F}}\mu F$ is finite. Show that $\mu F_0=\gamma$.

>(h) Let (X,Σ,μ) be a semi-finite measure space and $\mathcal{A}\subseteq\Sigma$ an algebra of sets such that the σ-algebra generated by $\mathcal{A}$ is Σ. Write $\mathcal{K}$ for $\{\bigcap_{n\in\mathbb{N}}E_n:E_n\in\mathcal{A}$ for every $n\in\mathbb{N}\}$. Show that μ is inner regular with respect to $\mathcal{K}$.

(i) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an effectively locally finite Hausdorff topological measure space such that μ is inner regular with respect to the Borel sets. Suppose that $\mu G=\sup\{\mu K:K\subseteq G$ is compact$\}$ for every open set $G\subseteq X$. Show that μ is tight.

(j) Let $(X,\mathfrak{T})$ be a topological space such that every open set is an F_σ set. Show that any effectively locally finite Borel measure on X is inner regular with respect to the closed sets.

(k) Let $(X,\mathfrak{T})$ be a normal topological space and μ a topological measure on X which is inner regular with respect to the closed sets. Show that $\mu G=\max\{\mu H:H\subseteq G$ is a cozero set$\}$ for every open set $G\subseteq X$. Show that if μ is totally finite, then $\mu F=\min\{\mu H:H\supseteq F$ is a zero set$\}$ for every closed set $F\subseteq X$.

(l) Let (X,Σ,μ) be a complete locally determined measure space, and suppose that μ is inner regular with respect to a family $\mathcal{K}$ of sets. Let Σ_0 be the σ-algebra of subsets of X generated by $\mathcal{K}\cap\Sigma$. (i) Show that μ is the c.l.d. version of $\mu\restriction\Sigma_0$. (*Hint*: 412J-412M.) (ii) Show that if μ is σ-finite, it is the completion of $\mu\restriction\Sigma_0$.

>(m)(i) Let (X,Σ,μ) be a σ-finite measure space and T a σ-subalgebra of Σ. Show that if μ is inner regular with respect to T then the completion of $\mu\restriction\mathrm{T}$ extends μ, so that μ and $\mu\restriction\mathrm{T}$ have the same negligible sets. (ii) Show that if μ is a σ-finite topological measure which is inner regular with respect to the Borel sets, then every μ-negligible set is included in a μ-negligible Borel set.

(n) Devise a direct proof of 412Mb by (i) showing that $\mu^*(A\cap K)=\nu^*(A\cap K)$ whenever $A\subseteq X$ and $K\in\mathcal{K}$ (ii) showing that $\mu^*=\nu^*$ (iii) quoting 213C.

(o) Let (X,Σ,μ) be a complete locally determined measure space, Y a set and $f:X\to Y$ a function. Show that the following are equiveridical: (i) μ is inner regular with respect to $\{f^{-1}[B]:B\subseteq Y\}$ (ii) $f^{-1}[f[E]]\setminus E$ is negligible for every $E\in\Sigma$.

(p) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X,Σ,μ). Suppose that for each $i\in I$ we are given a topology $\mathfrak{T}_i$ on X_i, and let $\mathfrak{T}$ be the corresponding disjoint union topology on X. Show that (i) μ is inner regular with respect to the closed sets iff every μ_i is (ii) μ is inner regular with respect to the compact sets iff every μ_i is (iii) μ is inner regular with respect to the zero sets iff every μ_i is (iv) μ is inner regular with respect to the Borel sets iff every μ_i is.

(q) Use 412M and 412Q to shorten the proof of 253I.

(r) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and suppose that we are given, for each $i\in I$, a σ-algebra Σ_i of subsets of X_i and a topology $\mathfrak{T}_i$ on X_i. Let $\mathfrak{T}$ be the product topology on $X=\prod_{i\in I}X_i$, and $\Sigma=\widehat{\bigotimes}_{i\in I}\Sigma_i$. Let μ be a totally finite measure with domain Σ, and set $\mu_i=\mu\pi_i^{-1}$ for each $i\in I$, where $\pi_i(x)=x(i)$ for $i\in I$,

$x \in X$. (i) Show that μ is inner regular with respect to the family $\mathcal{K}$ of sets expressible as $X \setminus \bigcup_{n\in\mathbb{N}} \prod_{i\in I} E_{ni}$ where $E_{ni} \in \Sigma_i$ for every n, i and $\{i : E_{ni} \neq X_i\}$ is finite for each n. (ii) Show that if every μ_i is inner regular with respect to the closed sets, so is μ. (iii) Show that if every μ_i is inner regular with respect to the zero sets, so is μ. (iv) Show that if every μ_i is inner regular with respect to the Borel sets, so is μ. (v) Show that if every μ_i is tight, and all but countably many of the X_i are compact, then μ is tight.

(s) Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a Lindelöf topology on X such that μ is locally finite. (i) Show that μ is σ-finite. (ii) Show that μ is inner regular with respect to the closed sets iff it is outer regular with respect to the open sets.

(t) Let X be a topological space and μ a measure on X which is outer regular with respect to the open sets. Show that for any $Y \subseteq X$ the subspace measure on Y is outer regular with respect to the open sets.

(u) Let X be a topological space and μ a measure on X which is inner regular with respect to the closed sets and outer regular with respect to the open sets. Show that if $f : X \to [-\infty, \infty]$ is integrable and $\epsilon > 0$ then there is an integrable lower semi-continuous $g : X \to \,]-\infty, \infty]$ such that $f \le g$ and $\int g \le \epsilon + \int f$.

>(v) Let X be a topological space and μ a measure on X which is effectively locally finite and inner regular with respect to the closed sets. (i) Show that if $\mu E < \infty$ and $\epsilon > 0$ there is a measurable open set G such that $\mu(E \triangle G) \le \epsilon$. (ii) Show that if f is a non-negative integrable function and $\epsilon > 0$ there is a measurable lower semi-continuous function $g : X \to [0, \infty[$ such that $\int |f - g| \le \epsilon$. (iii) Show that if f is an integrable real-valued function there are measurable lower semi-continuous functions g_1, $g_2 : X \to [0, \infty]$ such that $f =_{\text{a.e.}} g_1 - g_2$ and $\int g_1 + g_2 \le \int |f| + \epsilon$. (iv) Now suppose that μ is σ-finite. Show that for every measurable $f : X \to \mathbb{R}$ there are measurable lower semi-continuous functions g_1, $g_2 : X \to [0, \infty]$ such that $f =_{\text{a.e.}} g_1 - g_2$.

412Y Further exercises (a) In 216E, give $\{0, 1\}^I$ its usual compact Hausdorff topology. Show that the measure μ described there is inner regular with respect to the zero sets.

(b) Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}$ which are homeomorphic to the Cantor set. Show that Lebesgue measure is inner regular with respect to $\mathcal{K}$.

(c)(i) Show that if X is a perfectly normal space then any semi-finite topological measure on X which is inner regular with respect to the Borel sets is inner regular with respect to the closed sets. (ii) Show that any subspace of a perfectly normal space is perfectly normal. (iii) Show that ω_1, with its order topology, is completely regular, normal and Hausdorff, but not perfectly normal. (iv) Show that $[0, 1]^I$ is perfectly normal iff I is countable.

(d) Give an example of a measure space (X, Σ, μ) and a family $\mathcal{K}$ of sets such that

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$

and the completion of μ is inner regular with respect to $\mathcal{K}$, but μ is not.

(e) Let (X, Σ, μ) be a measure space, and suppose that μ is inner regular with respect to $\mathcal{K} \subseteq \mathcal{P}X$. Write Σ^f for $\{E : E \in \Sigma, \, \mu E < \infty\}$. Show that $\{E^\bullet : E \in \mathcal{K} \cap \Sigma^f\}$ is dense in $\{E^\bullet : E \in \Sigma^f\}$ for the strong measure-algebra topology.

(f) Let (X, Σ, μ) be $[0, 1]$ with Lebesgue measure, and $Y = [0, 1]$ with counting measure ν; give X its usual topology and Y its discrete topology, and let λ be the c.l.d. product measure on $X \times Y$. (i) Show that μ, ν and λ are all tight (for the appropriate topologies) and therefore completion regular. (ii) Let λ_0 be the primitive product measure on $X \times Y$ (definition: 251C). Show that λ_0 is not tight. (*Hint*: 252Yk.) *Remark*: it is undecidable in ZFC whether λ_0 is inner regular with respect to the closed sets.

(g) Give an example of a Hausdorff topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that μ is complete, strictly localizable and outer regular with respect to the open sets, but not inner regular with respect to the closed sets.

412 Notes and comments In this volume we are returning to considerations which have been left on one side for almost the whole of Volume 3 – the exceptions being in Chapter 34, where I looked at realization of homomorphisms of measure algebras by functions between measure spaces, and was necessarily dragged into an investigation of measure spaces which had enough points to be adequate codomains (343B). The idea of 'inner regularity' is to distinguish families $\mathcal{K}$ of sets which will be large enough to describe the measure entirely, but whose members will be of familiar types. For an example of this principle see 412Yb. Of course we cannot always find a single type of set adequate to fill a suitable family $\mathcal{K}$, though this happens oftener than one might expect, but it is surely easier to think about an arbitrary zero set (for instance) than an arbitrary measurable set, and whenever a measure is inner regular with respect to a recognisable class it is worth knowing about it.

I have tried to use the symbols † and ‡ (412A, 412C) consistently enough for them to act as a guide to some of the ideas which will be used repeatedly in this chapter. Note the emphasis on disjoint unions and countable intersections; I mentioned similar conditions in 136Xi-136Xj. You will recognise 412Aa as an exhaustion principle; observe that it is enough to use disjoint unions, as in 313K. In the examples of this section this disjointness is not important. Of course inner regularity has implications for the measure algebra (412N), but it is important to recognise that 'μ is inner regular with respect to $\mathcal{K}$' is saying much more than '$\{K^\bullet : K \in \mathcal{K}\}$ is order-dense in the measure algebra'; the latter formulation tells us only that whenever $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \setminus E$ is negligible and $\mu K > 0$, while the former tells us that we can take K to be actually a subset of E.

412D, 412E and 412G are all of great importance. 412D looks striking, but of course the reason it works is just that the Baire σ-algebra is very small. In 412E the Baire and Borel σ-algebras coincide, so it is nothing but a special case of 412D; but as metric spaces are particularly important it is worth having it spelt out explicitly. In 412D and 412E the hypothesis 'semi-finite' is sufficient, while in 412G we need 'effectively locally finite'; this is because in both 412D and 412E the open sets we are looking at are countable unions of measurable closed sets. There are interesting non-metrizable spaces in which the same thing happens (412Yc). As you know, I am strongly biased in favour of complete and locally determined measures, and the Baire and Borel measures dealt with in these three results are rarely complete; but they can still be applied to completions and c.l.d. versions of these measures, using 412Ab or 412H.

412O-412V are essentially routine. For subspace measures, the only problem we need to come to terms with is the fact that subspaces of semi-finite measure spaces need not be semi-finite (216Xa). For product measures the point is that the c.l.d. product of two measure spaces, and the product of any family of probability spaces, as I defined them in Chapter 25, are inner regular with respect to the σ-algebra of sets generated by the cylinder sets. This is not in general true of the 'primitive' product measure (412Yf), which is one of my reasons for being prejudiced against it. I should perhaps warn you of a trap in the language I use here. I say that if the factor measures are inner regular with respect to the closed sets, so is the c.l.d. product measure. But I do not say that all closed sets in the product are measured by the product measure, even if closed sets in the factors are measured by the factor measures. So the path is open for a different product measure to exist, still inner regular with respect to the closed sets; and indeed I shall be going down that path in §417. The uniqueness result in 412M specifically refers to complete locally determined measures defined on all sets of the family $\mathcal{K}$.

There is one special difficulty in 412V: in order to ensure that there are enough compact measurable sets in $X = \prod_{i \in I} X_i$, we need to know that all but countably many of the X_i are actually compact. When we come to look more closely at products of Radon probability spaces we shall need to consider this point again (417Q, 417Xq).

In fact some of the ideas of 412U-412V are not restricted to the product measures considered there. Other measures on the product space will have inner regularity properties if their images on the factors, their 'marginals' in the language of probability theory, are inner regular; see 412Xr. I will return to this in §454.

This section is almost exclusively concerned with *inner* regularity. The complementary notion of *outer* regularity is not much use except in σ-finite spaces (415Xi), and not always then (416Ya). In totally finite spaces, of course, and some others, any version of inner regularity corresponds to a version of outer regularity, as in 412Wb(i)-(ii); and when we have something as strong as 412Wb(iii) available it is worth knowing about it.

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413 Inner measure constructions

I now turn in a different direction, giving some basic results on the construction of inner regular measures. The first step is to describe 'inner measures' (413A) and a construction corresponding to the Carathéodory construction of measures from outer measures (413C). Just as every measure gives rise to an outer measure, it gives rise to an inner measure (413D). Inner measures form an effective tool for studying complete locally determined measures (413F).

The most substantial results of the section concern the construction of measures as extensions of functionals defined on various classes $\mathcal{K}$ of sets. Typically, $\mathcal{K}$ is closed under finite unions and countable intersections, though we can sometimes relax the hypotheses a bit. The methods here make it possible to distinguish arguments which produce finitely additive functionals (413I, 413O, 413R, 413S) from the succeeding steps to countably additive measures (413J, 413P, 413U). 413I-413N investigate conditions on a functional $\phi : \mathcal{K} \to [0, \infty[$ sufficient to produce a measure extending ϕ, necessarily unique, which is inner regular with respect to $\mathcal{K}$ or $\mathcal{K}_\delta$, the set of intersections of sequences in $\mathcal{K}$. 413O-413P look instead at functionals defined on sublattices of the class $\mathcal{K}$ of interest, and at sufficient conditions to ensure the existence of a measure, not normally unique, defined on the whole of $\mathcal{K}$, inner regular with respect to $\mathcal{K}$ and extending the given functional. Finally, 413R-413U are concerned with majorizations rather than extensions; we seek a measure μ such that $\mu K \ge \phi K$ for $K \in \mathcal{K}$, while μX is as small as possible.

413A I begin with some material from the exercises of earlier volumes.

Definition Let X be a set. An **inner measure** on X is a functional $\phi : \mathcal{P}X \to [0, \infty]$ such that

$\phi\emptyset = 0$;

(α) $\phi(A \cup B) \ge \phi A + \phi B$ for all disjoint A, $B \subseteq X$;

(β) if $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X and $\phi A_0 < \infty$ then $\phi(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \phi A_n$;

($*$) $\phi A = \sup\{\phi B : B \subseteq A,\ \phi B < \infty\}$ for every $A \subseteq X$.

413B The following fact will be recognised as an element of Carathéodory's method. There will be an application later in which it will be useful to know that it is not confined to proving countable additivity.

Lemma Let X be a set and $\phi : \mathcal{P}X \to [0, \infty]$ any functional such that $\phi\emptyset = 0$. Then

$$\Sigma = \{E : E \subseteq X,\ \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}$$

is an algebra of subsets of X, and $\phi(E \cup F) = \phi E + \phi F$ for all disjoint E, $F \in \Sigma$.

proof The symmetry of the definition of Σ ensures that $X \setminus E \in \Sigma$ whenever $E \in \Sigma$. If E, $F \in \Sigma$ and $A \subseteq X$, then

$$\begin{aligned}\phi(A \cap (E{\cup}F)) &+ \phi(A \setminus (E \cup F))\\ &= \phi(A \cap (E \cup F) \cap E) + \phi(A \cap (E \cup F) \setminus E) + \phi(A \setminus (E \cup F))\\ &= \phi(A \cap E) + \phi((A \setminus E) \cap F) + \phi((A \setminus E) \setminus F)\\ &= \phi(A \cap E) + \phi(A \setminus E) = \phi A.\end{aligned}$$

As A is arbitrary, $E \cup F \in \Sigma$. Finally, if $A \subseteq X$,

$$\phi(A \cap \emptyset) + \phi(A \setminus \emptyset) = \phi\emptyset + \phi A = \phi A$$

because $\phi\emptyset = 0$; so $\emptyset \in \Sigma$.

Thus Σ is an algebra of sets. If E, $F \in \Sigma$ and $E \cap F = \emptyset$, then

$$\phi(E \cup F) = \phi((E \cup F) \cap E) + \phi((E \cup F) \setminus E) = \phi E + \phi F.$$

413C Measures from inner measures I come now to a construction corresponding to Carathéodory's method of defining measures from outer measures.

Theorem Let X be a set and $\phi : X \to [0,\infty]$ an inner measure. Set

$$\Sigma = \{E : E \subseteq X,\, \phi(A \cap E) + \phi(A \setminus E) = \phi A \text{ for every } A \subseteq X\}.$$

Then $(X, \Sigma, \phi\restriction\Sigma)$ is a complete measure space.

proof (Compare 113C.)

(a) The first step is to note that if $A \subseteq B \subseteq X$ then

$$\phi B \ge \phi A + \phi(B \setminus A) \ge \phi A.$$

Next, a subset E of X belongs to Σ iff $\phi A \le \phi(A \cap E) + \phi(A \setminus E)$ whenever $A \subseteq X$ and $\mu A < \infty$. **P** Of course any element of Σ satisfies the condition. If E satisfies the condition and $A \subseteq X$, then

$$\begin{aligned}\phi A &= \sup\{\phi B : B \subseteq A,\, \phi B < \infty\}\\ &\le \sup\{\phi(B \cap E) + \phi(B \setminus E) : B \subseteq A\}\\ &= \phi(A \cap E) + \phi(A \setminus E) \le \phi A,\end{aligned}$$

so $E \in \Sigma$. **Q**

(b) By 413B, Σ is an algebra of subsets of X. Now suppose that $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ, with union E. If $A \subseteq X$ and $\phi A < \infty$, then

$$\phi(A \setminus E) = \inf_{n\in\mathbb{N}} \phi(A \setminus E_n) = \lim_{n\to\infty} \phi(A \setminus E_n)$$

because $\langle A \setminus E_n\rangle_{n\in\mathbb{N}}$ is non-increasing and $\phi(A \setminus E_0)$ is finite; so

$$\phi(A \cap E) + \phi(A \setminus E) \ge \lim_{n\to\infty} \phi(A \cap E_n) + \phi(A \setminus E_n) = \phi A.$$

By (a), $E \in \Sigma$. So Σ is a σ-algebra.

(c) If E, $F \in \Sigma$ and $E \cap F = \emptyset$ then $\phi(E \cup F) = \phi E + \phi F$, by 413B. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ with union E, then

$$\mu E \ge \mu(\bigcup_{i\le n} E_i) = \sum_{i=0}^{n} \mu E_i$$

for every n, so $\mu E \ge \sum_{i=0}^{\infty} \mu E_i$. **?** If $\mu E > \sum_{i=0}^{\infty} \mu E_i$, there is an $A \subseteq E$ such that $\sum_{i=0}^{\infty} \mu E_i < \phi A < \infty$. But now, setting $F_n = \bigcup_{i\le n} E_i$ for each n, we have $\lim_{n\to\infty} \phi(A \setminus F_n) = 0$, so that

$$\phi A = \lim_{n\to\infty} \phi(A \cap F_n) + \phi(A \setminus F_n) = \sum_{i=0}^{\infty} \phi(A \cap E_i) < \phi A,$$

which is absurd. **X** Thus $\mu E = \sum_{i=0}^{\infty} \mu E_i$. As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, μ is a measure.

(d) Finally, suppose that $B \subseteq E \in \Sigma$ and $\mu E = 0$. Then for any $A \subseteq X$ we must have

$$\phi(A \cap B) + \phi(A \setminus B) \ge \phi(A \setminus E) = \phi(A \cap E) + \phi(A \setminus E) = \phi A,$$

so $B \in \Sigma$. Thus μ is complete.

Remark For a simple example see 213Yd.

413D The inner measure defined by a measure Let (X, Σ, μ) be any measure space. Just as μ has an associated outer measure μ^* defined by the formula

$$\mu^* A = \inf\{\mu E : A \subseteq E \in \Sigma\}$$

(132A-132B), it gives rise to an inner measure μ_* defined by the formula

$$\mu_* A = \sup\{\mu E : E \in \Sigma^f,\, E \subseteq A\},$$

where I write Σ^f for $\{E : E \in \Sigma,\, \mu E < \infty\}$. **P** $\mu_*\emptyset = \mu\emptyset = 0$. (α) If $A \cap B = \emptyset$, and $E \subseteq A$, $F \subseteq B$ belong to Σ^f, then $E \cup F \subseteq A \cup B$ also has finite measure, so

$$\mu_*(A \cup B) \ge \mu(E \cup F) = \mu E + \mu F;$$

taking the supremum over E and F, $\mu_*(A\cup B)\ge\mu_*A+\mu_*B$. ($\beta$) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets with intersection A and $\mu_*A_0<\infty$, then for each $n\in\mathbb{N}$ we can find an $E_n\subseteq A_n$ such that $\mu E_n\ge\mu_*A_n-2^{-n}$. In this case,

$$\mu(\bigcup_{n\in\mathbb{N}}E_n)\le\mu_*A_0<\infty.$$

Set

$$E=\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}E_m\subseteq A.$$

Then $E\in\Sigma^f$, so

$$\mu_*A\ge\mu E\ge\limsup_{n\to\infty}\mu E_n=\lim_{n\to\infty}\mu_*A_n\ge\mu_*A.$$

($*$) If $A\subseteq X$ and $\mu_*A=\infty$ then

$$\sup\{\mu_*B:B\subseteq A,\,\mu_*B<\infty\}\ge\sup\{\mu E:E\in\Sigma^f,\,E\subseteq A\}=\infty.\ \mathbf{Q}$$

Warning Many authors use the formula

$$\mu_*A=\sup\{\mu E:A\supseteq E\in\Sigma\}.$$

In ‘ordinary’ cases, when (X,Σ,μ) is semi-finite, this agrees with my usage (413Ed); but for non-semi-finite spaces there is a difference. See 413Yd.

413E I note the following simple facts concerning inner measures defined from measures.

Proposition Let (X,Σ,μ) be a measure space. Write Σ^f for $\{E:E\in\Sigma,\,\mu E<\infty\}$.

(a) For every $A\subseteq X$ there is an $E\in\Sigma$ such that $E\subseteq A$ and $\mu E=\mu_*A$.

(b) $\mu_*A\le\mu^*A$ for every $A\subseteq X$.

(c) If $E\in\Sigma$ and $A\subseteq X$, then $\mu_*(E\cap A)+\mu^*(E\setminus A)\le\mu E$, with equality if either (i) $\mu E<\infty$ or (ii) μ is semi-finite.

(d) In particular, $\mu_*E\le\mu E$ for every $E\in\Sigma$, with equality if either $\mu E<\infty$ or μ is semi-finite.

(e) If μ is inner regular with respect to $\mathcal{K}$, then (counting the supremum of the empty set as 0) $\mu_*A=\sup\{\mu K:K\in\mathcal{K}\cap\Sigma^f,\,K\subseteq A\}$ for every $A\subseteq X$.

(f) If $A\subseteq X$ is such that $\mu_*A=\mu^*A<\infty$, then A is measured by the completion of μ.

(g) If $\hat\mu$, $\tilde\mu$ are the completion and c.l.d. version of μ, then $\hat\mu_*=\tilde\mu_*=\mu_*$.

(h) If (Y,T,ν) is another measure space, and $f:X\to Y$ is an inverse-measure-preserving function, then

$$\mu^*(f^{-1}[B])\le\nu^*B,\quad \mu_*(f^{-1}[B])\ge\nu_*B$$

for every $B\subseteq Y$, and

$$\nu^*(f[A])\ge\mu^*A$$

for every $A\subseteq X$.

(i) Suppose that μ is semi-finite. If $A\subseteq E\in\Sigma$, then E is a measurable envelope of A iff $\mu_*(E\setminus A)=0$.

proof (a) There is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ^f such that $E_n\subseteq A$ for each n and $\lim_{n\to\infty}\mu E_n=\mu_*A$; now set $E=\bigcup_{n\in\mathbb{N}}E_n$.

(b) If E, $F\in\Sigma$ and $E\subseteq A\subseteq F$ we must have $\mu E\le\mu F$.

(c) If $F\subseteq E\cap A$ and $F\in\Sigma^f$, then

$$\mu F+\mu^*(E\setminus A)\le\mu F+\mu(E\setminus F)=\mu E;$$

taking the supremum over F, $\mu_*(E\cap A)+\mu^*(E\setminus A)\le\mu E$. If $\mu E<\infty$, then

$$\begin{aligned}\mu_*(E\cap A)&=\sup\{\mu F:F\in\Sigma,\,F\subseteq E\cap A\}\\&=\mu E-\inf\{\mu(E\setminus F):F\in\Sigma,\,F\subseteq E\cap A\}\\&=\mu E-\inf\{\mu F:F\in\Sigma,\,E\setminus A\subseteq F\subseteq E\}=\mu E-\mu^*(E\setminus A).\end{aligned}$$

If μ is semi-finite, then

$$\mu_*(E \cap A) + \mu^*(E \setminus A) \ge \sup\{\mu_*(F \cap A) + \mu^*(F \setminus A) : F \in \Sigma^f,\ F \subseteq E\}$$
$$= \sup\{\mu F : F \in \Sigma^f,\ F \subseteq E\} = \mu E.$$

(d) Take $A = E$ in (c).

(e)

$$\mu_* A = \sup\{\mu E : E \in \Sigma^f,\ E \subseteq A\}$$
$$= \sup\{\mu K : K \in \mathcal{K} \cap \Sigma,\ \exists\, E \in \Sigma^f,\ K \subseteq E \subseteq A\}$$
$$= \sup\{\mu K : K \in \mathcal{K} \cap \Sigma^f,\ K \subseteq A\}.$$

(f) By (a) above and 132Aa, there are E, $F \in \Sigma$ such that $E \subseteq A \subseteq F$ and

$$\mu E = \mu_* A = \mu^* A = \mu F < \infty;$$

now $\mu(F \setminus E) = 0$, so $F \setminus A$ and A are measured by the completion of μ.

(g) Write $\check\mu$ for either $\hat\mu$ or $\tilde\mu$, and $\check\Sigma$ for its domain, and let $A \subseteq X$. (i) If $\gamma < \mu_* A$, there is an $E \in \Sigma$ such that $E \subseteq A$ and $\gamma \le \mu E < \infty$; now $\check\mu E = \mu E$ (212D, 213Fa), so $\check\mu_* A \ge \gamma$. As γ is arbitrary, $\mu_* A \le \check\mu_* A$. (ii) If $\gamma < \check\mu_* A$, there is an $E \in \check\Sigma$ such that $E \subseteq A$ and $\gamma \le \check\mu E < \infty$. Now there is an $F \in \Sigma$ such that $F \subseteq E$ and $\mu F = \check\mu E$ (212Cb, 213Fc), so that $\mu_* A \ge \gamma$. As γ is arbitrary, $\mu_* A \ge \check\mu_* A$.

(h) This is elementary; all we have to note is that if F, $F' \in \mathrm{T}$ and $F \subseteq B \subseteq F'$, then $f^{-1}[F] \subseteq f^{-1}[B] \subseteq f^{-1}[F']$, so that

$$\nu F = \mu f^{-1}[F] \le \mu_* f^{-1}[B] \le \mu^* f^{-1}[B] \le \mu f^{-1}[F'] = \nu F'.$$

As F and F' are arbitrary,

$$\nu_* B \le\le \mu_* f^{-1}[B], \quad \mu^* f^{-1}[B] \le \nu^* B.$$

Now, for $A \subseteq X$,

$$\mu^* A \le \mu^*(f^{-1}[f[A]]) \le \nu^*(f[A]).$$

(i)(i) If E is a measurable envelope of A and $F \in \Sigma$ is included in $E \setminus A$, then

$$\mu F = \mu(F \cap E) = \mu^*(F \cap A) = 0;$$

as F is arbitrary, $\mu_*(E \setminus A) = 0$. (ii) If E is not a measurable envelope of A, there is an $F \in \Sigma$ such that $\mu^*(F \cap A) < \mu(F \cap E)$. Let $G \in \Sigma$ be such that $F \cap A \subseteq G$ and $\mu G = \mu^*(F \cap A)$. Then $\mu(F \cap E \setminus G) > 0$; because μ is semi-finite, $\mu_*(E \setminus A) \ge \mu_*(F \cap E \setminus G) > 0$.

413F The language of 413D makes it easy to express some useful facts about complete locally determined measure spaces, complementing 412J.

Lemma Let (X, Σ, μ) be a complete locally determined measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$. Then for $E \subseteq X$ the following are equiveridical:
(i) $E \in \Sigma$;
(ii) $E \cap K \in \Sigma$ whenever $K \in \Sigma \cap \mathcal{K}$;
(iii) $\mu^*(K \cap E) + \mu^*(K \setminus E) = \mu^* K$ for every $K \in \mathcal{K}$;
(iv) $\mu_*(K \cap E) + \mu_*(K \setminus E) = \mu_* K$ for every $K \in \mathcal{K}$;
(v) $\mu^*(E \cap K) = \mu_*(E \cap K)$ for every $K \in \mathcal{K} \cap \Sigma$;
(vi) $\min(\mu^*(K \cap E), \mu^*(K \setminus E)) < \mu K$ whenever $K \in \mathcal{K} \cap \Sigma$ and $0 < \mu K < \infty$;
(vii) $\max(\mu_*(K \cap E), \mu_*(K \setminus E)) > 0$ whenever $K \in \mathcal{K} \cap \Sigma$ and $\mu K > 0$.

proof (a) Assume (i). Then of course $E \cap K \in \Sigma$ for every $K \in \Sigma \cap \mathcal{K}$, and (ii) is true. For any $K \in \mathcal{K}$ there is an $F \in \Sigma$ such that $F \supseteq K$ and $\mu F = \mu^* K$ (132Aa again); now

$$\mu^* K \le \mu^*(K \cap E) + \mu^*(K \setminus E) \le \mu(F \cap E) + \mu(F \setminus E) = \mu F = \mu^* K,$$

so (iii) is true. Next, for any $K \in \mathcal{K}$,

$$\mu_*(K \cap E) + \mu_*(K \setminus E) \le \mu_* K = \sup\{\mu F : F \in \Sigma^f,\, F \subseteq K\}$$

(writing Σ^f for $\{F : F \in \Sigma,\, \mu F < \infty\}$)

$$= \sup\{\mu(F \cap E) + \mu(F \setminus E) : F \in \Sigma^f,\, F \subseteq K\}$$
$$\le \mu_*(K \cap E) + \mu_*(K \setminus E).$$

So (iv) is true. If $K \in \mathcal{K} \cap \Sigma$, then

$$\mu_*(E \cap K) = \sup\{\mu F : F \in \Sigma^f,\, F \subseteq E \cap K\} = \mu(E \cap K) = \mu^*(E \cap K)$$

because μ is semi-finite. So (v) is true. Since (iii)$\Rightarrow$(vi) and (iv)$\Rightarrow$(vii), we see that all the conditions are satisfied.

(b) Now suppose that $E \notin \Sigma$; I have to show that (ii)-(vii) are all false. Because μ is locally determined, there is an $F \in \Sigma^f$ such that $E \cap F \notin \Sigma$. Take measurable envelopes H, H' of $F \cap E$ and $F \setminus E$ respectively (132Ee). Then $F \setminus H' \subseteq F \cap E \subseteq F \cap H$, so

$$G = (F \cap H) \setminus (F \setminus H') = F \cap H \cap H'$$

cannot be negligible. Take $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq G$ and $\mu K > 0$. As $G \subseteq F$, $\mu K < \infty$. Now

$$\mu^*(K \cap E) = \mu^*(K \cap F \cap E) = \mu(K \cap H) = \mu K,$$

$$\mu^*(K \setminus E) = \mu^*(K \cap F \setminus E) = \mu(K \cap H') = \mu K.$$

But this means that

$$\mu_*(K \cap E) = \mu K - \mu^*(K \setminus E) = 0, \quad \mu_*(K \setminus E) = \mu K - \mu^*(K \cap E) = 0$$

by 413Ec. Now we see that this K witnesses that (ii)-(vii) are all false.

413G The ideas of 413F can be used to give criteria for measurability of real-valued functions. I spell out one which is particularly useful.

Lemma Let (X, Σ, μ) be a complete locally determined measure space and suppose that μ is inner regular with respect to $\mathcal{K} \subseteq \Sigma$. Suppose that $f : X \to \mathbb{R}$ is a function, and for $\alpha \in \mathbb{R}$ set $E_\alpha = \{x : f(x) \le \alpha\}$, $F_\alpha = \{x : f(x) \ge \beta\}$. Then f is Σ-measurable iff

$$\min(\mu^*(E_\alpha \cap K), \mu^*(F_\beta \cap K)) < \mu K$$

whenever $K \in \mathcal{K}$, $0 < \mu K < \infty$ and $\alpha < \beta$.

proof (a) If f is measurable, then

$$\mu^*(E_\alpha \cap K) + \mu^*(F_\beta \cap K) = \mu(E_\alpha \cap K) + \mu(F_\beta \cap K) \le \mu K$$

whenever $K \in \Sigma$ and $\alpha < \beta$, so if $0 < \mu K < \infty$ then we must have $\min(\mu^*(E_\alpha \cap K), \mu^*(F_\beta \cap K)) < \mu K$.

(b) If f is not measurable, then there is some $\alpha \in \mathbb{R}$ such that E_α is not measurable. 413F(vi) tells us that there is a $K \in \mathcal{K}$ such that $0 < \mu K < \infty$ and $\mu^*(E_\alpha \cap K) = \mu^*(K \setminus E_\alpha) = \mu K$. Note that K is a measurable envelope of $K \cap E_\alpha$ (132Eb). Now $\langle K \cap F_{\alpha+2^{-n}} \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union $K \setminus E_\alpha$, so there is some $\beta > \alpha$ such that $K \cap F_\beta$ is not negligible. Let $H \subseteq K$ be a measurable envelope of $K \cap F_\beta$, and $K' \in \mathcal{K}$ such that $K' \subseteq H$ and $\mu K' > 0$; then

$$\mu^*(K' \cap E_\alpha) = \mu^*(K' \cap K \cap E_\alpha) = \mu(K' \cap K) = \mu K',$$

$$\mu^*(K' \cap F_\beta) = \mu^*(K' \cap H \cap F_\beta) = \mu(K' \cap H) = \mu K',$$

so K', α and β witness that the condition is not satisfied.

413H The following fact is interesting and not quite obvious.

Proposition Let (X,Σ,μ) be a complete totally finite measure space, (Y,T,ν) a measure space, and $\mathfrak{S}$ a Hausdorff topology on Y such that ν is inner regular with respect to the closed sets. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of inverse-measure-preserving functions from X to Y. If $f(x)=\lim_{n\to\infty}f_n(x)$ is defined in Y for every $x\in X$, then f is inverse-measure-preserving.

proof Let μ_* be the inner measure associated with μ (413D). If $F\in\mathrm{T}$ is closed then $\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}f_m^{-1}[F]\subseteq f^{-1}[F]$, so

$$\mu_*f^{-1}[F]\ge\mu(\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}f_m^{-1}[F])\ge\liminf_{n\to\infty}\mu f_n^{-1}[F]=\nu F.$$

So if H is any member of T,

$$\begin{aligned}\mu_*f^{-1}[H]&\ge\sup\{\mu_*f^{-1}[F]:F\in\mathrm{T},\ F\subseteq H\text{ and }F\text{ is closed}\}\\&\ge\sup\{\nu F:F\in\mathrm{T},\ F\subseteq H\text{ and }F\text{ is closed}\}=\nu H.\end{aligned}$$

Taking complements,

$$\mu^*f^{-1}[H]=\mu X-\mu_*f^{-1}[Y\setminus H]$$

(413Ec)

$$\le\nu Y-\nu(Y\setminus H)=\nu H$$

(of course $\nu Y=\mu f_0^{-1}[Y]=\mu X$). So $\mu^*f^{-1}[H]=\mu_*f^{-1}[H]=\nu H$. Because μ is complete, $\mu f^{-1}[H]$ is defined and equal to νH (413Ef). As H is arbitrary, f is inverse-measure-preserving.

413I Inner measure constructions based on 413C are important because they offer an efficient way of setting up measures which are inner regular with respect to given families of sets. Two of the fundamental results are 413J and 413K. I proceed by means of a lemma on finitely additive functionals.

Lemma Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset\in\mathcal{K}$,

(†) $K\cup K'\in\mathcal{K}$ whenever $K,\ K'\in\mathcal{K}$ are disjoint,

(‡) $K\cap K'\in\mathcal{K}$ for all $K,\ K'\in\mathcal{K}$.

Let $\phi_0:\mathcal{K}\to[0,\infty[$ be a functional such that

(α) $\phi_0K=\phi_0L+\sup\{\phi_0K':K'\in\mathcal{K},\ K'\subseteq K\setminus L\}$ whenever $K,\ L\in\mathcal{K}$ and $L\subseteq K$.

Set

$$\phi A=\sup\{\phi_0K:K\in\mathcal{K},\ K\subseteq A\}\text{ for }A\subseteq X,$$

$$\Sigma=\{E:E\subseteq X,\ \phi A=\phi(A\cap E)+\phi(A\setminus E)\text{ for every }A\subseteq X\}.$$

Then Σ is an algebra of subsets of X, including $\mathcal{K}$, and $\phi\restriction\Sigma:\Sigma\to[0,\infty]$ is an additive functional extending ϕ_0.

proof (a) To see that Σ is an algebra of subsets and $\phi\restriction\Sigma$ is additive, all we need to know is that $\phi\emptyset=0$ (413B); and this is because, applying hypothesis (α) with $K=L=\emptyset$, $\phi_0\emptyset=\phi_0\emptyset+\phi_0\emptyset$, so $\phi_0\emptyset=0$. (α) also assures us that $\phi_0L\le\phi_0K$ whenever $K,\ L\in\mathcal{K}$ and $L\subseteq K$, so $\phi K=\phi_0K$ for every $K\in\mathcal{K}$.

(b) To check that $\mathcal{K}\subseteq\Sigma$, we have a little more work to do. First, observe that (†) and (α) together tell us that $\phi_0(K\cup K')=\phi_0K+\phi_0K'$ for all disjoint $K,\ K'\in\mathcal{K}$. So if $A,\ B\subseteq X$ and $A\cap B=\emptyset$ then

$$\phi A+\phi B=\sup_{\substack{K\in\mathcal{K}\\K\subseteq A}}\phi_0K+\sup_{\substack{L\in\mathcal{K}\\L\subseteq B}}\phi_0L=\sup_{\substack{K,L\in\mathcal{K}\\K\subseteq A\\L\subseteq B}}\phi_0(K\cup L)\le\phi(A\cup B).$$

(c) $\mathcal{K}\subseteq\Sigma$. **P** Take $K\in\mathcal{K}$ and $A\subseteq X$. If $L\in\mathcal{K}$ and $L\subseteq A$, then

$$\phi_0L=\phi_0(K\cap L)+\sup\{\phi_0L':L'\in\mathcal{K},\ L'\subseteq L\setminus K\}\le\phi(A\cap K)+\phi(A\setminus K).$$

(Note the use of the hypothesis (‡).) As L is arbitrary, $\phi A\le\phi(A\cap K)+\phi(A\setminus K)$. We already know that $\phi(A\cap K)+\phi(A\setminus K)\le\phi A$; as A is arbitrary, $K\in\Sigma$. **Q**

This completes the proof.

413J Theorem (Topsøe 70a) Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0,\infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}$.

proof (a) Set

$$\phi A = \sup\{\phi_0 K : K \in \mathcal{K},\, K \subseteq A\} \text{ for } A \subseteq X,$$

$$\Sigma = \{E : E \subseteq X,\, \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then 413I tells us that Σ is an algebra of subsets of X, including $\mathcal{K}$, and $\mu = \phi\restriction\Sigma$ is an additive functional extending ϕ_0.

(b) Now $\mu(\bigcap_{n\in\mathbb{N}} K_n) = \inf_{n\in\mathbb{N}} \mu K_n$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$. **P** Set $L = \bigcap_{n\in\mathbb{N}} K_n$. Of course $\mu L \le \inf_{n\in\mathbb{N}} \mu K_n$. For the reverse inequality, take $\epsilon > 0$. Then (α) tells us that there is a $K' \in \mathcal{K}$ such that $K' \subseteq K_0 \setminus L$ and $\mu K_0 \le \mu L + \mu K' + \epsilon$. Since $\langle K_n \cap K'\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, (β) tells us that there is an $n \in \mathbb{N}$ such that $\mu(K_n \cap K') \le \epsilon$. Now

$$\begin{aligned}\mu K_0 - \mu L = \mu(K_0 \setminus L) &= \mu(K_0 \setminus (K' \cup L)) + \mu K' \\ &\le \epsilon + \mu(K_n \cap K') + \mu(K' \setminus K_n) \le 2\epsilon + \mu(K_0 \setminus K_n) = 2\epsilon + \mu K_0 - \mu K_n.\end{aligned}$$

(These calculations depend, of course, on the additivity of μ and the finiteness of μK_0.) So $\mu L \ge \mu K_n - 2\epsilon$. As ϵ is arbitrary, $\mu L = \inf_{n\in\mathbb{N}} \mu K_n$. **Q**

(c) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X, with intersection A, and $\phi A_0 < \infty$, then $\phi A = \inf_{n\in\mathbb{N}} \phi A_n$. **P** Of course $\phi A \le \phi A_n$ for every n. Given $\epsilon > 0$, then for each $n \in \mathbb{N}$ choose $K_n \in \mathcal{K}$ such that $K_n \subseteq A_n$ and $\phi_0 K_n \ge \phi A_n - 2^{-n}\epsilon$ (this is where I use the hypothesis that ϕA_0 is finite); set $L_n = \bigcap_{i\le n} K_i$ for each n, and $L = \bigcap_{n\in\mathbb{N}} L_n$. Then we have

$$\begin{aligned}\phi A_{n+1} - \mu L_{n+1} &= \phi A_{n+1} - \mu(K_{n+1} \cap L_n) \\ &= \phi A_{n+1} - \mu K_{n+1} - \mu L_n + \mu(K_{n+1} \cup L_n) \\ &\le 2^{-n-1}\epsilon - \mu L_n + \phi A_n\end{aligned}$$

because $K_{n+1} \subseteq A_{n+1} \subseteq A_n$ and $L_n \subseteq K_n \subseteq A_n$. Inducing on n, we see that $\mu L_n \ge \phi A_n - 2\epsilon + 2^{-n}\epsilon$ for every n. So

$$\phi A \ge \mu L = \inf_{n\in\mathbb{N}} \mu L_n \ge \inf_{n\in\mathbb{N}} \phi A_n - 2\epsilon,$$

using (b) above for the middle equality. As ϵ is arbitrary, $\phi A = \inf_{n\in\mathbb{N}} \phi A_n$. **Q**

(d) It follows that ϕ is an inner measure. **P** The arguments of parts (a) and (b) of the proof of 413I tell us that $\phi\emptyset = 0$ and $\phi(A \cup B) \le \phi A + \phi B$ whenever A, $B \subseteq X$ are disjoint. We have just seen that $\phi(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \phi A_n$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets and $\phi A_0 < \infty$. Finally, $\phi K = \phi_0 K$ is finite for every $K \in \mathcal{K}$, so $\phi A = \sup\{\phi B : B \subseteq A,\, \phi B < \infty\}$ for every $A \subseteq X$. Putting these together, ϕ is an inner measure. **Q**

(e) So 413C tells us that μ is a complete measure, and of course it is inner regular with respect to $\mathcal{K}$, by the definition of ϕ. It is semi-finite because $\mu K = \phi_0 K$ is finite for every $K \in \mathcal{K}$. Now suppose that $E \subseteq X$ and that $E \cap F \in \Sigma$ whenever $\mu F < \infty$. Take any $A \subseteq X$. If $L \in \mathcal{K}$ and $L \subseteq A$, we have $L \in \Sigma$ and $\mu L < \infty$, so

$$\phi_0 L = \mu L = \mu(L \cap E) + \mu(L \setminus E) = \phi(L \cap E) + \phi(L \setminus E) \le \phi(A \cap E) + \phi(A \setminus E);$$

taking the supremum over L, $\phi A \le \phi(A \cap E) + \phi(A \setminus E)$. As A is arbitrary, $E \in \Sigma$; as E is arbitrary, μ is locally determined.

(f) Finally, μ is unique by 412Mb.

413K Theorem Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$.

proof (a) Set

$$\psi A = \sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq A\} \text{ for } A \subseteq X,$$

$$\mathrm{T} = \{E : E \subseteq X,\ \psi A = \psi(A \cap E) + \psi(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then 413I tells us that T is an algebra of subsets of X, including $\mathcal{K}$, and $\nu = \psi\restriction\mathrm{T}$ is an additive functional extending ϕ_0.

(b) Write T^f for $\{E : E \in \mathrm{T},\ \nu E < \infty\}$. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, $\lim_{n\to\infty} \nu E_n = 0$. **P** Given $\epsilon > 0$, we can choose a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $K_n \subseteq E_n$ and

$$\nu K_n = \phi_0 K_n \ge \nu E_n - 2^{-n}\epsilon$$

for each n. Set $L_n = \bigcap_{i\le n} K_i$ for each n; then

$$\lim_{n\to\infty} \nu L_n = \lim_{n\to\infty} \phi_0 L_n = 0$$

by hypothesis (β). But also, for each n,

$$\nu E_n \le \nu L_n + \sum_{i=0}^{n} \nu(E_i \setminus K_i) \le \nu L_n + 2\epsilon,$$

because ν is additive and non-negative and $E_n \subseteq L_n \cup \bigcup_{i\le n}(E_i \setminus K_i)$. So $\limsup_{n\to\infty} \nu E_n \le 2\epsilon$; as ϵ is arbitrary, $\lim_{n\to\infty} \nu E_n = 0$. **Q**

(c) Write T^f_δ for the family of sets expressible as intersections of sequences in T^f, and for $H \in \mathrm{T}^f_\delta$ set $\phi_1 H = \inf\{\nu E : H \subseteq E \in \mathrm{T}\}$. Note that because $E \cap F \in \mathrm{T}^f$ whenever E, $F \in \mathrm{T}^f$, every member of T^f_δ can be expressed as the intersection of a non-increasing sequence in T^f.

(i) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with intersection $H \in \mathrm{T}^f_\delta$, $\phi_1 H = \lim_{n\to\infty} \nu E_n$. **P** Of course

$$\phi_1 H \le \inf_{n\in\mathbb{N}} \nu E_n = \lim_{n\to\infty} \nu E_n.$$

On the other hand, if $H \subseteq E \in \mathrm{T}$, then $\langle E_n \setminus E\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, and

$$\nu E \ge \lim_{n\to\infty} \nu(E_n \cap E) = \lim_{n\to\infty} \nu E_n - \lim_{n\to\infty} \nu(E_n \setminus E) = \lim_{n\to\infty} \nu E_n$$

by (b) above. As E is arbitrary, $\phi_1(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \nu E_n$. **Q**

(ii) Because $\mathcal{K} \subseteq \mathrm{T}^f$, $\mathcal{K}_\delta \subseteq \mathrm{T}^f_\delta$. Now for any $H \in \mathrm{T}^f_\delta$, $\phi_1 H = \sup\{\phi_1 L : L \in \mathcal{K}_\delta,\ L \subseteq H\}$. **P** Express H as $\bigcap_{n\in\mathbb{N}} E_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f. Given $\epsilon > 0$, we can choose a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $K_n \subseteq E_n$ and $\nu K_n \ge \nu E_n - 2^{-n}\epsilon$ for each n. Setting $L_n = \bigcap_{i\le n} K_i$ for each n and $L = \bigcap_{n\in\mathbb{N}} L_n$, we have $L \in \mathcal{K}_\delta$, $L \subseteq H$ and

$$\phi_1 H = \lim_{n\to\infty} \nu E_n \le \lim_{n\to\infty}(\nu L_n + \sum_{i=0}^{n} \nu(E_i \setminus K_i)) \le \phi_1 L + 2\epsilon.$$

As ϵ is arbitrary, this gives the result. **Q**

(d) We find that T^f_δ and ϕ_1 satisfy the conditions of 413J. **P** Of course $\emptyset \in \mathrm{T}^f_\delta$. If G, $H \in \mathrm{T}^f_\delta$ and $G \cap H = \emptyset$, express them as $\bigcap_{n\in\mathbb{N}} E_n$, $\bigcap_{n\in\mathbb{N}} F_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are non-increasing sequences in T^f. Then

$$G \cup H = \bigcap_{n\in\mathbb{N}} E_n \cup F_n$$

belongs to T^f_δ, and

$$\begin{aligned}\phi_1(G\cup H) &= \lim_{n\to\infty} \nu(E_n \cup F_n) = \lim_{n\to\infty} \nu E_n + \nu F_n - \nu(E_n \cap F_n)\\ &= \lim_{n\to\infty} \nu E_n + \nu F_n\end{aligned}$$

(by (b))

$$= \phi_1 G + \phi_1 H.$$

The definition of T^f_δ as the set of intersections of sequences in T^f ensures that the intersection of any sequence in T^f_δ will belong to T^f_δ.

Now suppose that G, $H \in \mathrm{T}^f_\delta$ and that $G \subseteq H$. Express them as intersections $\bigcap_{n\in\mathbb{N}} E_n$, $\bigcap_{n\in\mathbb{N}} F_n$ of non-increasing sequences in T^f, so that $\phi_1 G = \lim_{n\to\infty} \nu E_n$ and $\phi_1 H = \lim_{n\to\infty} \nu F_n$. For each n, set $H_n = \bigcap_{m\in\mathbb{N}} F_m \setminus E_n$, so that $H_n \in \mathrm{T}^f_\delta$, $H_n \subseteq H \setminus G$, and

$$\begin{aligned}\phi_1 H_n &= \lim_{m\to\infty} \nu(F_m \setminus E_n) = \lim_{m\to\infty} \nu F_m - \nu(F_m \cap E_n)\\ &\ge \lim_{m\to\infty} \nu F_m - \nu E_n = \phi_1 H - \nu E_n.\end{aligned}$$

Accordingly

$$\sup\{\phi_1 G' : G' \in \mathrm{T}^f_\delta,\ G' \subseteq H \setminus G\} \ge \sup_{n\in\mathbb{N}} \phi_1 H - \nu E_n = \phi_1 H - \phi_1 G.$$

On the other hand, if $G' \in \mathrm{T}^f_\delta$ and $G' \subseteq H \setminus G$, then

$$\phi_1 G + \phi_1 G' = \phi_1(G \cup G') \le \phi_1 H$$

because of course ϕ_1 is non-decreasing, as well as being additive on disjoint sets. So

$$\sup\{\phi_1 G' : G' \in \mathrm{T}^f_\delta,\ G' \subseteq H \setminus G\} = \phi_1 H - \phi_1 G$$

as required by condition (α) of 413J. Finally, suppose that $\langle H_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f_δ with empty intersection. For each $n \in \mathbb{N}$, let $\langle E_{ni}\rangle_{i\in\mathbb{N}}$ be a non-increasing sequence in T^f with intersection H_n, and set $F_m = \bigcap_{i,j\le m} E_{ji}$ for each m. Then $\langle F_m\rangle_{m\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, while $H_m \subseteq F_m$ for each m, so

$$\lim_{m\to\infty} \phi_1 H_m \le \lim_{m\to\infty} \nu F_m = 0.$$

Thus condition 413J(β) is satisfied, and we have the full list. **Q**

(e) By 413J, we have a complete locally determined measure μ, extending ϕ_1, and inner regular with respect to T^f_δ. Since $\phi_1 K = \nu K = \phi_0 K$ for $K \in \mathcal{K}$, μ extends ϕ_0. If G belongs to the domain of μ, and $\gamma < \mu G$, there is an $H \in \mathrm{T}^f_\delta$ such that $H \subseteq G$ and $\gamma < \mu H = \phi_1 H$; by (c-ii), there is an $L \in \mathcal{K}_\delta$ such that $L \subseteq H$ and $\gamma \le \phi_1 L = \mu L$. Thus μ is inner regular with respect to $\mathcal{K}_\delta$. To see that μ is unique, observe that if μ' is any other measure with these properties, and $L \in \mathcal{K}_\delta$, then L is expressible as $\bigcap_{n\in\mathbb{N}} K_n$ where $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$. Now

$$\mu L = \lim_{n\to\infty} \mu(\bigcap_{i\le n} K_i) = \lim_{n\to\infty} \phi_0(\bigcap_{i\le n} K_i) = \mu' L.$$

So μ and μ' must agree on $\mathcal{K}_\delta$, and by 412Mb again they are identical.

413L Corollary (a) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu : \Sigma \to [0, \infty[$ a non-negative finitely additive functional such that $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a complete locally determined measure on X which is inner regular with respect to the family Σ_δ of intersections of sequences in Σ.

(b) Let X be a set, Σ a subalgebra of $\mathcal{P}X$, and $\nu : \Sigma \to [0, \infty[$ a non-negative finitely additive functional such that $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a measure defined on the σ-algebra of subsets of X generated by Σ.

proof (a) Take Σ, ν in place of $\mathcal{K}$, ϕ_0 in 413K.

(b) Let ν_1 be the complete extension as in (a), and let ν_1' be the restriction of ν_1 to the σ-algebra Σ' generated by Σ; this is the extension required here. To see that ν_1' is unique, use the Monotone Class Theorem (136C).

Remark These are versions of the **Hahn extension theorem**. You will sometimes see (b) above stated as 'an additive functional on an algebra of sets extends to a measure iff it is countably additive'. But this formulation depends on a different interpretation of the phrase 'countably additive' from the one used in this book; see the note after the definition in 326I.

413M It will be useful to have a definition extending an idea in §342.

Definition A **countably compact class** (or **semicompact paving**) is a family $\mathcal{K}$ of sets such that $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ whenever $\langle K_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{i\le n} K_i \neq \emptyset$ for every $n \in \mathbb{N}$.

413N Corollary Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K}, K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$.

proof The point is that the hypothesis (β) of 413K is necessarily satisfied: if $\langle K_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then, because $\mathcal{K}$ is countably compact, there must be some n such that $K_n = \emptyset$. Since hypothesis (α) here is already enough to ensure that $\phi_0\emptyset = 0$ and $\phi_0 K \ge 0$ for every $K \in \mathcal{K}$, we must have $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$. So we apply 413K to get the result.

413O I now turn to constructions of a different kind, being extension theorems in which the extension is not uniquely defined. Again I start with a theorem on finitely additive functionals.

Theorem Let X be a set, T_0 a subring of $\mathcal{P}X$, and $\nu_0 : \mathrm{T}_0 \to [0, \infty[$ a finitely additive functional. Suppose that $\mathcal{K} \subseteq \mathcal{P}X$ is a family of sets such that

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

every member of $\mathcal{K}$ is included in some member of T_0,

and ν_0 is inner regular with respect to $\mathcal{K}$ in the sense that

(α) $\nu_0 E = \sup\{\nu_0 K : K \in \mathcal{K} \cap \mathrm{T}_0, K \subseteq E\}$ for every $E \in \mathrm{T}_0$.

Then ν_0 has an extension to a non-negative finitely additive functional ν_1, defined on a subring T_1 of $\mathcal{P}X$ including $\mathrm{T}_0 \cup \mathcal{K}$, inner regular with respect to $\mathcal{K}$, and such that whenever $E \in \mathrm{T}_1$ and $\epsilon > 0$ there is an $E_0 \in \mathrm{T}_0$ such that $\nu_1(E \triangle E_0) \le \epsilon$.

proof (a) Let P be the set of all non-negative additive real-valued functionals ν, defined on subrings of $\mathcal{P}X$, inner regular with respect to $\mathcal{K}$, and such that

($*$) whenever $E \in \operatorname{dom}\nu$ and $\epsilon > 0$ there is an $E_0 \in \mathrm{T}_0$ such that $\nu(E \triangle E_0) \le \epsilon$.

Order P by extension of functions, so that P is a partially ordered set.

(b) It will be convenient to borrow some notation from the theory of countably additive functionals. If T is a subring of $\mathcal{P}X$ and $\nu : \mathrm{T} \to [0, \infty[$ is a non-negative additive functional, set

$$\nu^* A = \inf\{\nu E : A \subseteq E \in \mathrm{T}\}, \quad \nu_* A = \sup\{\nu E : A \supseteq E \in \mathrm{T}\}$$

for every $A \subseteq X$ (interpreting $\inf \emptyset$ as ∞ if necessary). Now if $A \subseteq X$ and E, $F \in \mathrm{T}$ are disjoint,

$$\nu^*(A \cap (E \cup F)) = \nu^*(A \cap E) + \nu^*(A \cap F),$$

$$\nu_*(A \cap (E \cup F)) = \nu_*(A \cap E) + \nu_*(A \cap F).$$

P
$$\begin{aligned}\nu^*(A \cap (E \cup F)) &= \inf\{\nu G : G \in \mathrm{T},\, A \cap (E \cup F) \subseteq G\} \\ &= \inf\{\nu G : G \in \mathrm{T},\, A \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\ &= \inf\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T},\, A \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\ &= \inf\{\nu G_1 + \nu G_2 : G_1,\, G_2 \in \mathrm{T},\, A \cap E \subseteq G_1 \subseteq E,\, A \cap F \subseteq G_2 \subseteq F\} \\ &= \inf\{\nu G_1 : G_1 \in \mathrm{T},\, A \cap E \subseteq G_1 \subseteq E\} \\ &\qquad + \inf\{\nu G_2 : G_2 \in \mathrm{T},\, A \cap F \subseteq G_2 \subseteq F\} \\ &= \nu^*(E \cap A) + \nu^*(F \cap A),\end{aligned}$$

$$\begin{aligned}\nu_*(A \cap (E \cup F)) &= \sup\{\nu G : G \in \mathrm{T},\, A \cap (E \cup F) \supseteq G\} \\ &= \sup\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T},\, A \cap (E \cup F) \supseteq G\} \\ &= \sup\{\nu G_1 + \nu G_2 : G_1,\, G_2 \in \mathrm{T},\, A \cap E \supseteq G_1,\, A \cap F \supseteq G_2\} \\ &= \sup\{\nu G_1 : G_1 \in \mathrm{T},\, A \cap E \supseteq G_1\} \\ &\qquad + \sup\{\nu G_2 : G_2 \in \mathrm{T},\, A \cap F \supseteq G_2\} \\ &= \nu_*(E \cap A) + \nu_*(F \cap A).\end{aligned}$$
Q

(c) The key to the proof is the following fact: if $\nu \in P$ and $M \in \mathcal{K}$, there is a $\nu' \in P$ such that ν' extends ν and $M \in \operatorname{dom} \nu'$. **P** Set $\mathrm{T} = \operatorname{dom} \nu$, $\mathrm{T}' = \{(E \cap M) \cup (F \setminus M) : E,\, F \in \mathrm{T}\}$. For $H \in \mathrm{T}'$, set

$$\nu' H = \nu^*(H \cap M) + \nu_*(H \setminus M).$$

Now we have to check the following.

(i) T' is a subring of $\mathcal{P}X$, because if E, F, E', $F' \in \mathrm{T}$ then

$$((E \cap M) \cup (F \setminus M)) \star ((E' \cap M) \cup (F' \setminus M)) = ((E \star E') \cap M) \cup ((F \star F') \setminus M)$$

for both the Boolean operations $\star = \triangle$ and $\star = \cap$. $\mathrm{T}' \supseteq \mathrm{T}$ because $E = (E \cap M) \cup (E \setminus M)$ for every $E \in \mathrm{T}$. (Cf. 312N.) $M \in \mathrm{T}'$ because there is some $E \in \mathrm{T}_0$ such that $M \subseteq E$, so that $M = (E \cap M) \cup (\emptyset \setminus M) \in \mathrm{T}'$.

(ii) ν' is finite-valued because if $H = (E \cap M) \cup (F \setminus M)$, where E, $F \in \mathrm{T}$, then $\nu' H \le \nu E + \nu F$. If H, $H' \in \mathrm{T}$ are disjoint, they can be expressed as $(E \cap M) \cup (F \setminus M)$, $(E' \cap M) \cup (F' \setminus M)$ where E, F, E', F' belong to T; replacing E', F' by $E' \setminus E$ and $F' \setminus F$ if necessary, we may suppose that $E \cap E' = F \cap F' = \emptyset$. Now

$$\begin{aligned}\nu'(H \cup H') &= \nu^*((E \cup E') \cap M) + \nu_*((F \cup F') \cap (X \setminus M)) \\ &= \nu^*(E \cap M) + \nu^*(E' \cap M) + \nu_*(F \cap (X \setminus M)) + \nu_*(F' \cap (X \setminus M))\end{aligned}$$

(by (b) above)

$$= \nu' H + \nu' H'.$$

Thus ν' is additive.

(iii) If $E \in \mathrm{T}$, then

$$\begin{aligned}\nu_*(E\setminus M)&=\sup\{\nu F:F\in\mathrm{T},\,F\subseteq E\setminus M\}\\&=\sup\{\nu E-\nu(E\setminus F):F\in\mathrm{T},\,F\subseteq E\setminus M\}\\&=\sup\{\nu E-\nu F:F\in\mathrm{T},\,E\cap M\subseteq F\subseteq E\}\\&=\nu E-\inf\{\nu F:F\in\mathrm{T},\,E\cap M\subseteq F\subseteq E\}=\nu E-\nu^*(E\cap M).\end{aligned}$$

So

$$\nu' E=\nu^*(E\cap M)+\nu_*(E\setminus M)=\nu E.$$

Thus ν' extends ν.

(iv) If $H\in\mathrm{T}'$ and $\epsilon>0$, express H as $(E\cap M)\cup(F\setminus M)$, where E, $F\in\mathrm{T}$. Then we can find (α) a $K\in\mathcal{K}\cap\mathrm{T}$ such that $K\subseteq E$ and $\nu(E\setminus K)\le\epsilon$ (β) an $F'\in\mathrm{T}$ such that $F'\subseteq F\setminus M$ and $\nu F'\ge\nu_*(F\setminus M)-\epsilon$ (γ) a $K'\in\mathcal{K}\cap\mathrm{T}$ such that $K'\subseteq F'$ and $\nu K'\ge\nu F'-\epsilon$. Set $L=(K\cap M)\cup K'\in\mathrm{T}'$; by the hypotheses (†) and (‡), $L\in\mathcal{K}$. Now $L\subseteq H$ and

$$\begin{aligned}\nu' L&=\nu'(K\cap M)+\nu' K'=\nu'(E\cap M)-\nu'((E\setminus K)\cap M)+\nu K'\\&=\nu^*(H\cap M)-\nu^*((E\setminus K)\cap M)+\nu K'\ge\nu^*(H\cap M)-\nu(E\setminus K)+\nu F'-\epsilon\\&\ge\nu^*(H\cap M)+\nu_*(F\setminus M)-3\epsilon=\nu' H-3\epsilon.\end{aligned}$$

As H and ϵ are arbitrary, ν is inner regular with respect to $\mathcal{K}$.

(v) Finally, given $H\in\mathrm{T}'$ and $\epsilon>0$, take E, $F\in\mathrm{T}$ such that $H\cap M\subseteq E$, $F\subseteq H\setminus M$, $\nu E\le\nu^*(H\cap M)+\epsilon$ and $\nu F\ge\nu_*(H\setminus M)-\epsilon$. In this case,

$$\nu'(E\setminus(H\cap M))=\nu' E-\nu'(H\cap M)=\nu E-\nu^*(H\cap M)\le\epsilon,$$

$$\nu'((H\setminus M)\setminus F)=\nu'(H\setminus M)-\nu' F=\nu_*(H\setminus M)-\nu F\le\epsilon.$$

But as

$$H\triangle(E\cup F)\subseteq(E\setminus(H\cap M))\cup((H\setminus M)\setminus F),$$

$\nu'(H\triangle(E\cup F))\le 2\epsilon$. Now ν satisfies the condition $(*)$, so there is an $E_0\in\mathrm{T}_0$ such that $\nu((E\cup F)\triangle E_0)\le\epsilon$, and $\nu'(H\triangle E_0)\le 3\epsilon$. As H and ϵ are arbitrary, ν' satisfies $(*)$.

This completes the proof that ν' is a member of P extending ν. **Q**

(d) It is easy to check that if $Q\subseteq P$ is a non-empty totally ordered subset, the smallest common extension ν' of the functions in Q belongs to P. (To see that ν' is inner regular with respect to $\mathcal{K}$, observe that if $E\in\operatorname{dom}\nu'$ and $\gamma<\nu' E$, there is some $\nu\in Q$ such that $E\in\operatorname{dom}\nu$; now there is a $K\in\mathcal{K}\cap\operatorname{dom}\nu$ such that $K\subseteq E$ and $\nu K\ge\gamma$, so that $K\in\mathcal{K}\cap\operatorname{dom}\nu'$ and $\nu' K\ge\gamma$.) And of course P is not empty, because $\nu_0\in P$. So by Zorn's Lemma P has a maximal element ν_1 say; write T_1 for the domain of ν_1. If $M\in\mathcal{K}$ there is an element of P, with a domain containing M, extending ν_1; as ν_1 is maximal, this must be ν_1 itself, so $M\in\mathrm{T}_1$. Thus $\mathcal{K}\subseteq\mathrm{T}_1$, and ν_1 has all the required properties.

413P Corollary Let (X,T,ν) be a measure space and $\mathcal{K}$ a countably compact class of subsets of X such that

(†) $K\cup K'\in\mathcal{K}$ whenever K, $K'\in\mathcal{K}$ are disjoint,

(‡) $\bigcap_{n\in\mathbb{N}}K_n\in\mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

$\nu^*K<\infty$ for every $K\in\mathcal{K}$,

ν is inner regular with respect to $\mathcal{K}$.

Then ν has an extension to a complete locally determined measure μ, defined on every member of $\mathcal{K}$, inner regular with respect to $\mathcal{K}$, and such that whenever $E\in\operatorname{dom}\mu$ and $\mu E<\infty$ there is an $F\in\mathrm{T}$ such that $\mu(E\triangle F)=0$.

proof (a) Set $\mathrm{T}^f=\{E:E\in\mathrm{T},\,\nu E<\infty\}$. Then T^f and $\nu\restriction\mathrm{T}^f$ satisfy the conditions of 413O; take ν_1 extending ν to $\mathrm{T}_1\supseteq\mathrm{T}^f\cup\mathcal{K}$ as in 413O. If K, $L\in\mathcal{K}$ and $L\subseteq K$, then

$$\nu_1 L+\sup\{\nu_1 K' : K'\in\mathcal{K},\,K'\subseteq K\setminus L\}=\nu_1 L+\nu_1(K\setminus L)=\nu_1 K.$$

So $\nu_1\restriction\mathcal{K}$ satisfies the conditions of 413N and there is a complete locally determined measure μ, extending $\nu_1\restriction\mathcal{K}$, and inner regular with respect to $\mathcal{K}$.

(b) Write Σ for the domain of μ. Then $\mathrm{T}_1\subseteq\Sigma$. **P** If $E\in\mathrm{T}_1$ and $K\in\mathcal{K}$,

$$\begin{aligned}&\mu_*(K\cap E)+\mu_*(K\setminus E)\\&\qquad\ge\sup\{\mu K' : K'\in\mathcal{K},\,K'\subseteq K\cap E\}+\sup\{\mu K' : K'\in\mathcal{K},\,K'\subseteq K\setminus E\}\\&\qquad=\sup\{\nu_1 K' : K'\in\mathcal{K},\,K'\subseteq K\cap E\}+\sup\{\nu_1 K' : K'\in\mathcal{K},\,K'\subseteq K\setminus E\}\\&\qquad=\nu_1(K\cap E)+\nu_1(K\setminus E)=\nu_1 K=\mu K.\end{aligned}$$

By 413F(iv), $E\in\Sigma$. **Q** It follows at once that μ extends ν_1, since if $E\in\mathrm{T}_1$

$$\nu_1 E=\sup\{\nu_1 K : K\in\mathcal{K},\,K\subseteq E\}=\sup\{\mu K : K\in\mathcal{K},\,K\subseteq E\}=\mu E.$$

(c) In particular, μ agrees with ν on T^f. Now in fact μ extends ν. **P** Take $E\in\mathrm{T}$. If $K\in\mathcal{K}$, there is an $F\in\mathrm{T}$ such that $K\subseteq F$ and $\nu F<\infty$. Since $E\cap F\in\mathrm{T}^f\subseteq\Sigma$, $E\cap K=E\cap F\cap K\in\Sigma$. As K is arbitrary, $E\in\Sigma$, by 413F(ii). Next, because every member of $\mathcal{K}$ is included in a member of T^f,

$$\begin{aligned}\nu E&=\sup\{\nu K : K\in\mathcal{K}\cap\mathrm{T},\,K\subseteq E\}=\sup\{\nu(E\cap F) : F\in\mathrm{T}^f\}\\&=\sup\{\mu(E\cap F) : F\in\mathrm{T}^f\}=\sup\{\mu K : K\in\mathcal{K},\,K\subseteq E\}=\mu E.\ \mathbf{Q}\end{aligned}$$

(d) Finally, suppose that $E\in\Sigma$ and $\mu E<\infty$. For each $n\in\mathbb{N}$ we can find $K_n\in\mathcal{K}$ and $F_n\in\mathrm{T}$ such that $K_n\subseteq E$, $\mu(E\setminus K_n)\le 2^{-n}$ and $\nu_1(K_n\triangle F_n)\le 2^{-n}$. In this case $\sum_{n=0}^{\infty}\mu(E\triangle F_n)<\infty$, so $\mu(E\triangle F)=0$, where $F=\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}F_m\in\mathrm{T}$.

Thus μ has all the required properties.

413Q Definitions Let P be a lattice and $f:P\to[-\infty,\infty[$ a function.

(a) f is **supermodular** if $f(p\vee q)+f(p\wedge q)\ge f(p)+f(q)$ for all p, $q\in P$.

(b) f is **submodular** if $f(p\vee q)+f(p\wedge q)\le f(p)+f(q)$ for all p, $q\in P$. The phrase 'strongly subadditive' is used by many authors in similar contexts.

(c) f is **modular** if $f(p\vee q)+f(p\wedge q)=f(p)+f(q)$ for all p, $q\in P$.

413R I now describe an alternative route to some of the applications of 413O. As before, I do as much as possible in the context of finitely additive functionals.

Lemma Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\phi:\mathcal{K}\to\mathbb{R}$ be a bounded supermodular functional such that $\phi\emptyset=0$. Then there is a finitely additive functional $\nu:\mathcal{P}X\to[0,\infty[$ such that

$$\nu X=\sup_{K\in\mathcal{K}}\phi K,\quad \nu K\ge\phi K\text{ for every }K\in\mathcal{K}.$$

proof (a) Let us consider first the case in which $\mathcal{K}$ is finite and ϕ is non-decreasing. I induce on $n=\#(\mathcal{K})$. If $n=1$ then $\mathcal{K}=\{\emptyset\}$ and ν must be the zero functional. For the inductive step to $n>1$, let K_0 be a minimal member of $\mathcal{K}\setminus\{\emptyset\}$. If $K\in\mathcal{K}$ then $K\cap K_0$ is a member of $\mathcal{K}$ included in K_0, so is either empty or K_0, that is, either $K\cap K_0=\emptyset$ or $K\supseteq K_0$. Set $Y=X\setminus K_0$ and $\mathcal{L}=\{K\setminus K_0 : K\in\mathcal{K}\}$. Then $\mathcal{L}$ is a sublattice of $\mathcal{P}Y$ containing $\emptyset$, and $K\mapsto K\setminus K_0:\mathcal{K}\to\mathcal{L}$ is surjective but not injective, so $\#(\mathcal{L})<n$.

For $L\in\mathcal{L}$, observe that $L\cup K_0\in\mathcal{K}$. **P** There is a $K\in\mathcal{K}$ such that $L=K\setminus K_0$. Now If K is disjoint from K_0, then $L\cup K_0=K\cup K_0$ belongs to $\mathcal{K}$; if K includes K_0 then $L\cup K_0=K$ belongs to $\mathcal{K}$. **Q**

We can therefore define $\phi':\mathcal{L}\to[0,\infty[$ by setting

$$\phi' L=\phi(L\cup K_0)-\phi K_0$$

for every $L\in\mathcal{L}$. Of course $\phi'\emptyset=0$ and $\phi' L\le\phi' L'$ whenevery $L\subseteq L'$. If L, $L'\in\mathcal{L}$ then

$$\begin{aligned}\phi'(L\cup L')+\phi'(L\cap L') &= \phi(L\cup L'\cup K_0)+\phi((L\cap L')\cup K_0)-2\phi K_0\\ &= \phi((L\cup K_0)\cup(L'\cup K_0))+\phi((L\cup K_0)\cap(L'\cup K_0))-2\phi K_0\\ &\ge \phi(L\cup K_0)+\phi(L'\cup K_0)-2\phi K_0=\phi' L+\phi' L'.\end{aligned}$$

So ϕ' is supermodular, and by the inductive hypothesis there is a finitely additive functional $\nu':\mathcal{P}Y\to[0,\infty[$ such that

$$\nu' Y=\sup_{L\in\mathcal{L}}\phi' L,\quad \nu' L\ge\phi' L \text{ for every } L\in\mathcal{L}.$$

Fix any $x_0\in K_0$ and define $\nu:\mathcal{P}X\to[0,\infty[$ by setting

$$\begin{aligned}\nu A &= \phi K_0+\nu'(A\cap Y) \text{ if } x_0\in A\subseteq X,\\ &= \nu'(A\cap Y) \text{ for other } A\subseteq X.\end{aligned}$$

Then ν is additive. If $K\in\mathcal{K}$ is disjoint from K_0 then

$$\nu K=\nu' K\ge\phi' K=\phi(K\cup K_0)-\phi K_0\ge\phi K-\phi(K\cap K_0)=\phi K.$$

If $K\in\mathcal{K}$ includes K_0 then

$$\nu K=\phi K_0+\nu'(K\setminus K_0)\ge\phi K_0+\phi'(K\setminus K_0)=\phi K_0+\phi K-\phi K_0=\phi K.$$

Finally,

$$\begin{aligned}\nu X &= \phi K_0+\nu' Y=\phi K_0+\sup_{L\in\mathcal{L}}\phi' L\\ &= \phi K_0+\sup_{K\in\mathcal{K}}(\phi(K\cup K_0)-\phi K_0)=\sup_{K\in\mathcal{K}}\phi(K\cup K_0)=\sup_{K\in\mathcal{K}}\phi K.\end{aligned}$$

So ν has the required properties and the induction continues.

(b) Now suppose only that ϕ is non-decreasing. Set $\gamma=\sup_{K\in\mathcal{K}}\phi K$. We need to know that every finite subset of $\mathcal{K}$ is included in a finite sublattice of $\mathcal{K}$; this is because it is included in a finite subalgebra $\mathcal{E}$ of $\mathcal{P}X$ and $\mathcal{K}\cap\mathcal{E}$ is a sublattice. Let N be the set of all finitely additive functionals $\nu:\mathcal{P}X\to[0,\gamma]$. Then N is a closed subset of $[0,\gamma]^{\mathcal{P}X}$, so is compact. For each $K\in\mathcal{K}$ set $\mathrm{N}_K=\{\nu:\nu\in\mathrm{N},\ \nu K\ge\phi K\}$. Then N_K is a closed subset of N. If $\mathcal{K}_0\subseteq\mathcal{K}$ is finite, there is a finite sublattice $\mathcal{L}$ of $\mathcal{K}$ including $\mathcal{K}_0\cup\{\emptyset\}$, and now (a) tells us that there is a $\nu\in\bigcap_{K\in\mathcal{L}}\mathrm{N}_K$. Thus $\{\mathrm{N}_K:K\in\mathcal{K}\}$ has the finite intersection property and there is a $\nu\in\bigcap_{K\in\mathcal{K}}\mathrm{N}_K$. In this case, $\nu:\mathcal{P}X\to[0,\gamma]$ is a finitely additive functional dominating ϕ; it follows that $\nu X=\gamma$ and the proof is complete.

(c) Finally, for the general case, set $\phi' K=\sup_{L\in\mathcal{K},L\subseteq K}\phi L$ for $K\in\mathcal{K}$. Then ϕ' is non-decreasing, $\phi'\emptyset=0$ and $\sup_{K\in\mathcal{K}}\phi' K=\sup_{K\in\mathcal{K}}\phi K$ is finite. If K, $K'\in\mathcal{K}$ then

$$\begin{aligned}\phi' K+\phi' K' &= \sup_{\substack{L\in\mathcal{K}\\ L\subseteq K}}\phi L+\sup_{\substack{L'\in\mathcal{K}\\ L'\subseteq K'}}\phi L'=\sup_{\substack{L,L'\in\mathcal{K}\\ L\subseteq K\\ L'\subseteq K'}}\phi L+\phi L'\\ &\le \sup_{\substack{L,L'\in\mathcal{K}\\ L\subseteq K\\ L'\subseteq K'}}\phi(L\cup L')+\phi(L\cap L')\le\phi'(K\cup K')+\phi'(K\cap K'),\end{aligned}$$

so ϕ' is supermodular. By (b), there is an additive $\nu:\mathcal{P}X\to[0,\infty[$ such that $\nu K\ge\phi' K\ge\phi K$ for every $K\in\mathcal{K}$ and $\nu X=\sup_{K\in\mathcal{K}}\phi' K=\sup_{K\in\mathcal{K}}\phi K$.

413S Theorem Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let Σ be the algebra of subsets of X generated by $\mathcal{K}$, and $\nu_0:\Sigma\to[0,\infty[$ a finitely additive functional. Then there is a finitely additive functional $\nu:\Sigma\to[0,\infty[$ such that

(i) $\nu X=\sup_{K\in\mathcal{K}}\nu_0 K$,

(ii) $\nu K\ge\nu_0 K$ for every $K\in\mathcal{K}$,

(iii) ν is inner regular with respect to $\mathcal{K}$ in the sense that $\nu E=\sup\{\nu K:K\in\mathcal{K},\ K\subseteq E\}$ for every $E\in\Sigma$.

proof (a) Set $\gamma=\sup_{K\in\mathcal{K}}\nu_0K$. Let P be the set of all supermodular functionals $\phi:\mathcal{K}\to[0,\gamma]$. Give P the natural partial order inherited from $\mathbb{R}^{\mathcal{K}}$. Note that $\nu_0\restriction\mathcal{K}$ is actually modular, so belongs to P. If $Q\subseteq P$ is non-empty and upwards-directed, then $\sup Q$, taken in $\mathbb{R}^{\mathcal{K}}$, belongs to P; so there is a maximal $\phi\in P$ such that $\nu_0\restriction\mathcal{K}\le\phi$. By 413R, there is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu K\ge\phi K$ for every $K\in\mathcal{K}$ and $\nu X=\gamma$. Since $\nu\restriction\mathcal{K}$ also belongs to P, we must have $\nu K=\phi K$ for every $K\in\mathcal{K}$.

(b) Now for any $K_0\in\mathcal{K}$,

$$\nu K_0+\sup\{\nu L:L\in\mathcal{K},\,L\subseteq X\setminus K_0\}=\gamma.$$

P (i) Set $\mathcal{L}=\{L:L\in\mathcal{K},\,L\subseteq X\setminus K_0\}$. For $A\subseteq X$, set $\theta_0A=\sup_{L\in\mathcal{L}}\nu(A\cap L)$. Because $\mathcal{L}$ is upwards-directed, $\theta_0:\mathcal{P}X\to\mathbb{R}$ is additive, and of course $0\le\theta_0\le\nu$. Set $\theta_1=\nu-\theta_0$, so that θ_1 is another additive functional, and write

$$\phi'K=\theta_0K+\sup\{\theta_1M:M\in\mathcal{K},\,M\cap K_0\subseteq K\}$$

for $K\in\mathcal{K}$.

(ii) If K, $K'\in\mathcal{K}$ and $\epsilon>0$, there are M, $M'\in\mathcal{K}$ such that $M\cap K_0\subseteq K$, $M'\cap K_0\subseteq K'$ and

$$\theta_0K+\theta_1M\ge\phi'K-\epsilon,\quad\theta_0K'+\theta_1M'\ge\phi'K'-\epsilon.$$

Now

$$M\cup M'\in\mathcal{K},\quad M\cap M'\in\mathcal{K},$$

$$(M\cup M')\cap K_0\subseteq K\cup K',\quad(M\cap M')\cap K_0\subseteq K\cap K',$$

so

$$\begin{aligned}\phi'(K\cup K')+\phi'(K\cap K')&\ge\theta_0(K\cup K')+\theta_1(M\cup M')+\theta_0(K\cap K')+\theta_1(M\cap M')\\&=\theta_0K+\theta_1M+\theta_0K'+\theta_1M'\ge\phi'K+\phi'K'-2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\phi'(K\cup K')+\phi'(K\cap K')\ge\phi K+\phi K'$.

(iii) Suppose that K, $M\in\mathcal{K}$ are such that $M\cap K_0\subseteq K$. If $L\in\mathcal{L}$, then

$$\begin{aligned}\nu(K\cap L)+\theta_1M&=\nu(K\cap L)+\nu M-\theta_0M\\&=\nu(M\cap K\cap L)+\nu(M\cup(K\cap L))-\theta_0M\le\gamma\end{aligned}$$

because $K\cap L\in\mathcal{L}$; taking the supremum over L and M, $\phi'K\le\gamma$. As K is arbitrary, $\phi'\in P$.

(iv) If $K\in\mathcal{K}$, then of course $K\cap K_0\subseteq K$, so

$$\phi'K\ge\theta_0K+\theta_1K=\nu K=\phi K.$$

Thus $\phi'\ge\phi$. Because ϕ is maximal, $\phi'=\phi$. But this means that

$$\phi K_0=\phi'K_0=\theta_0K_0+\sup\{\theta_1M:M\in\mathcal{K},\,M\cap K_0\subseteq K_0\}=\sup_{M\in\mathcal{K}}\theta_1M.$$

Now given $\epsilon>0$ there is an $M\in\mathcal{K}$ such that

$$\gamma-\epsilon\le\nu_0M\le\phi M=\nu M,$$

so that

$$\nu K_0=\phi K_0\ge\theta_1M=\nu M-\theta_0M\ge\gamma-\epsilon-\theta_0M\ge\gamma-\epsilon-\sup_{L\in\mathcal{L}}\nu L,$$

and $\nu K_0+\sup_{L\in\mathcal{L}}\nu L\ge\gamma-\epsilon$. As ϵ is arbitrary, $\nu K_0+\sup_{L\in\mathcal{L}}\nu L\ge\gamma$. But of course $\nu K_0+\nu L\le\nu X=\gamma$ for every $L\in\mathcal{L}$, so $\nu K_0+\sup_{L\in\mathcal{L}}\nu L=\gamma$, as claimed. **Q**

(c) It follows that if K, $L\in\mathcal{K}$ and $L\subseteq K$,

$$\nu K=\nu L+\sup\{\nu K':K'\in\mathcal{K},\,K'\subseteq K\setminus L\}.$$

P Because ν is additive and non-negative, we surely have

$$\nu K\ge\nu L+\sup\{\nu K':K'\in\mathcal{K},\,K'\subseteq K\setminus L\}.$$

On the other hand, given $\epsilon > 0$, there is an $M \in \mathcal{K}$ such that $M \subseteq X \setminus L$ and $\nu L + \nu M \ge \gamma - \epsilon$, so that $M \cap K \in \mathcal{K}$, $M \cap K \subseteq K \setminus L$ and

$$\nu L + \nu(M \cap K) = \nu L + \nu K + \nu M - \nu(M \cup K) \ge \nu K + \gamma - \epsilon - \gamma = \nu K - \epsilon.$$

As ϵ is arbitrary,

$$\nu K \le \nu L + \sup\{\nu K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\}$$

and we have equality. **Q**

(d) By 413I, we have an additive functional $\nu' : \Sigma \to [0, \infty[$ such that $\nu' E = \sup\{\nu K : K \in \mathcal{K},\, K \subseteq E\}$ for every $E \in \Sigma$. Using 313Ga, it is easy to show that ν' and ν must agree on Σ, but even without doing so we can see that ν' has the properties (i)-(iii) required in the theorem.

413T The following lemma on countably compact classes, corresponding to 342Db, will be useful.

Lemma (Marczewski 53) Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X. Then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K} \cup \{\emptyset, X\}$ such that $K \cup L \in \mathcal{K}^*$ and $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}^*$ whenever $K, L \in \mathcal{K}^*$ and $\langle K_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}^*$.

proof (a) Of course $\mathcal{K} \cup \{\emptyset, X\}$ is still a countably compact class of sets, so we can suppose from the beginning that $\emptyset$ and X belong to $\mathcal{K}$. Write $\mathcal{K}_s$ for $\{K_0 \cup \ldots \cup K_n : K_0, \ldots, K_n \in \mathcal{K}\}$. Then $\mathcal{K}_s$ is countably compact. **P** Let $\langle L_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_s$ such that $\bigcap_{i\le n} L_i \ne \emptyset$ for each $n \in \mathbb{N}$. Then there is an ultrafilter $\mathcal{F}$ on X containing every L_n. For each n, L_n is a finite union of members of $\mathcal{K}$, so there must be a $K_n \in \mathcal{K}$ such that $K_n \subseteq L_n$ and $K_n \in \mathcal{F}$. Now $\bigcap_{i\le n} K_i \ne \emptyset$ for every n, so $\bigcap_{n\in\mathbb{N}} K_n \ne \emptyset$ and $\bigcap_{n\in\mathbb{N}} L_n \ne \emptyset$. As $\langle L_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}_s$ is countably compact. **Q**

Note that $L \cup L' \in \mathcal{K}_s$ for all $L, L' \in \mathcal{K}_s$.

(b) Write $\mathcal{K}^*$ for

$$\{\bigcap \mathcal{L}_0 : \mathcal{L}_0 \subseteq \mathcal{K}_s \text{ is non-empty and countable}\}.$$

Then $\mathcal{K}^*$ is countably compact. **P** If $\langle M_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{K}^*$ such that $\bigcap_{i\le n} M_i \ne \emptyset$ for every $n \in \mathbb{N}$, then for each $n \in \mathbb{N}$ let $\mathcal{L}_n \subseteq \mathcal{K}_s$ be a countable non-empty set such that $M_n = \bigcap \mathcal{L}_n$. Let $\langle L_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\bigcup_{n\in\mathbb{N}} \mathcal{L}_n$; then $\bigcap_{i\le n} L_i \ne \emptyset$ for every n, so $\bigcap_{n\in\mathbb{N}} L_n = \bigcap_{n\in\mathbb{N}} M_n$ is non-empty. As $\langle M_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}^*$ is countably compact. **Q**

(c) Of course $\mathcal{K} \subseteq \mathcal{K}_s \subseteq \mathcal{K}^*$. It is immediate from the definition of $\mathcal{K}^*$ that it is closed under countable intersections. Finally, if $M_1, M_2 \in \mathcal{K}^*$, let $\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathcal{K}_s$ be countable sets such that $M_1 = \bigcap \mathcal{L}_1$ and $M_2 = \bigcap \mathcal{L}_2$; then $\mathcal{L} = \{L_1 \cup L_2 : L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\}$ is a countable subset of $\mathcal{K}_s$, so $M_1 \cup M_2 = \bigcap \mathcal{L}$ belongs to $\mathcal{K}^*$.

413U Corollary Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X. Let T be a subalgebra of $\mathcal{P}X$ and $\nu : \mathrm{T} \to \mathbb{R}$ a non-negative finitely additive functional.

(a) There is a complete measure μ on X such that $\mu X \le \nu X$, $\mathcal{K} \subseteq \operatorname{dom}\mu$ and $\mu K \ge \nu K$ for every $K \in \mathcal{K} \cap \mathrm{T}$.

(b) If $\emptyset \in \mathcal{K}$ and

(†) $K \cup K' \in \mathcal{K}$ whenever $K, K' \in \mathcal{K}$,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

we may arrange that μ is inner regular with respect to $\mathcal{K}$.

proof By 413T, there is always a countably compact class $\mathcal{K}^* \supseteq \mathcal{K} \cup \{\emptyset\}$ satisfying (†) and (‡); for case (b), take $\mathcal{K}^* = \mathcal{K}$. By 391G, there is an extension of ν to an additive functional $\nu' : \mathcal{P}X \to [0, \infty[$. Let T_1 be the subalgebra of $\mathcal{P}X$ generated by $\mathcal{K}^*$. By 413S, there is a non-negative additive functional $\nu_1 : \mathrm{T}_1 \to \mathbb{R}$ such that $\nu_1 X \le \nu' X = \nu X$, $\nu_1 K \ge \nu' K = \nu K$ for every $K \in \mathcal{K}^* \cap \mathrm{T}$ and $\nu_1 E = \sup\{\nu_1 K : K \in \mathcal{K}^*, K \subseteq E\}$ for every $E \in \mathrm{T}_1$. In particular, if $K, L \in \mathcal{K}^*$,

$$\nu_1 L + \sup\{\nu_1 K' : K' \in \mathcal{K}^*,\, K' \subseteq K \setminus L\} = \nu_1 L + \nu_1(K \setminus L) = \nu_1 K.$$

So $\mathcal{K}^*$ and $\nu_1{\restriction}\mathcal{K}^*$ satisfy the hypotheses of 413N. Accordingly we have a complete measure μ extending $\nu_1{\restriction}\mathcal{K}^*$ and inner regular with respect to $\mathcal{K}^* = \mathcal{K}^*_\delta$; in which case

$$\mu K = \nu_1 K \geq \nu K$$

for every $K \in \mathcal{K} \cap \mathrm{T}$, and

$$\mu X = \sup_{K\in\mathcal{K}^*} \mu K = \sup_{K\in\mathcal{K}^*} \nu_1 K \leq \nu_1 X \leq \nu X,$$

as required.

413X Basic exercises (a) Define $\phi : \mathcal{P}\mathbb{N} \to [0,\infty[$ by setting $\phi A = 0$ if A is finite, ∞ otherwise. Check that ϕ satisfies conditions (α) and (β) of 413A, but that if we attempt to reproduce the construction of 413C then we obtain $\Sigma = \mathcal{P}\mathbb{N}$ and $\mu = \phi$, so that μ is not countably additive.

(b) Let ϕ_1, ϕ_2 be two inner measures on a set X, inducing measures μ_1 and μ_2 by the method of 413C. (i) Show that $\phi = \phi_1 + \phi_2$ is an inner measure. (ii) Show that the measure μ induced by ϕ extends the measure $\mu_1 + \mu_2$ defined on $\operatorname{dom}\mu_1 \cap \operatorname{dom}\mu_2$.

>(c) Let X be a set, ϕ an inner measure on X, and μ the measure constructed from it by the method of 413C. (i) Let Y be a subset of X. Show that $\phi{\restriction}\mathcal{P}Y$ is an inner measure on Y, and that the measure on Y defined from it extends the subspace measure μ_Y induced on Y by μ. (ii) Now suppose that ϕX is finite. Let Y be a set and $f : X \to Y$ a function. Show that $B \mapsto \phi f^{-1}[B]$ is an inner measure on Y, and that it defines a measure on Y which extends the image measure μf^{-1}.

(d) Let (X,Σ,μ) be a measure space. Set $\theta A = \frac{1}{2}(\mu^* A + \mu_* A)$ for every $A \subseteq X$. Show that θ is an outer measure on X, and that if μ is semi-finite then the measure defined from θ by Carathéodory's method extends μ. (See also 438Ym below and 543Xd in the next volume.)

>(e) Show that there is a partition $\langle A_n\rangle_{n\in\mathbb{N}}$ of $[0,1]$ such that $\mu_*(\bigcup_{i\leq n} A_i) = 0$ for every n, where μ_* is Lebesgue inner measure. (*Hint*: set $A_n = (A + q_n) \cap [0,1]$ where $\langle q_n\rangle_{n\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$ and A is a suitable set; cf. 134B.)

(f) Let (X,Σ,μ) be a measure space. (i) Show that $\mu_*{\restriction}\Sigma$ is the semi-finite version μ_{sf} of μ as constructed in 213Xc. (ii) Show that if A is any subset of X, and Σ_A the subspace σ-algebra, then $\mu_*{\restriction}\Sigma_A$ is a semi-finite measure on A.

>(g) Let (X,Σ,μ) and (Y,T,ν) be two measure spaces, and λ the c.l.d. product measure on $X \times Y$. Show that $\lambda_*(A \times B) = \mu_* A \cdot \nu_* B$ for all $A \subseteq X$ and $B \subseteq Y$. (*Hint*: use Fubini's theorem to show that $\lambda_*(A \times B) \leq \mu_* A \cdot \nu_* B$.)

(h)(i) Let (X,Σ,μ) be a σ-finite measure space and $f : X \to \mathbb{R}$ a function such that $\overline{\int} f d\mu$ is finite. Show that for every $\epsilon > 0$ there is a measure ν on X extending μ such that $\int f d\nu \geq \overline{\int} f d\mu - \epsilon$. (*Hint*: 215B(viii), 133Ja, 417Xa.) (ii) Let (X,Σ,μ) be a totally finite measure space and $f : X \to \mathbb{R}$ a bounded function. Show that there is a finitely additive functional $\nu : \mathcal{P}X \to [0,\infty[$, extending μ, such that $\oint f d\nu$, defined as in 363Lf, is equal to $\overline{\int} f d\mu$.

(i) Let X be a set and μ, ν two complete locally determined measures on X with domains Σ, T respectively, both inner regular with respect to $\mathcal{K} \subseteq \Sigma \cap \mathrm{T}$. Suppose that, for $K \in \mathcal{K}$, $\mu K = 0$ iff $\nu K = 0$. Show that $\Sigma = \mathrm{T}$ and that μ and ν have the same null ideals.

>(j) Let (X,Σ,μ) be a measure space. (i) Show that the measure constructed by the method of 413C from the inner measure μ_* is the c.l.d. version of μ. (ii) Set $\mathcal{K} = \{E : E \in \Sigma,\ \mu E < \infty\}$, $\phi_0 = \mu{\restriction}\mathcal{K}$. Show that $\mathcal{K}$ and ϕ_0 satisfy the conditions of 413J, and that the measure constructed by the method there is again the c.l.d. version of μ.

>(k) Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}$ expressible as disjoint finite unions of bounded closed intervals. (i) Show from first principles that there is a unique functional $\phi_0 : \mathcal{K} \to [0,\infty[$ such that $\phi_0[\alpha,\beta] = \beta - \alpha$ whenever $\alpha \leq \beta$ and ϕ_0 satisfies the conditions of 413K. (ii) Show that the measure on $\mathbb{R}$ constructed from ϕ_0 by the method of 413K is Lebesgue measure.

(l) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu : \Sigma \to [0, \infty[$ a non-negative additive functional such that $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection, as in 413L. Define $\theta : \mathcal{P}X \to [0, \infty]$ by setting

$$\theta A = \inf\{\sum_{n=0}^{\infty} \nu E_n : \langle E_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \Sigma \text{ covering } A\}$$

for $A \subseteq X$, interpreting $\inf \emptyset$ as ∞ if necessary. Show that θ is an outer measure. Let μ_θ be the measure defined from θ by Carathéodory's method. Show that the measure defined from ν by the process of 413L is the c.l.d. version of μ_θ. (*Hint*: the c.l.d. version of μ_θ is inner regular with respect to Σ_δ.)

>(m) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu : \Sigma \to [0, \infty[$ a non-negative additive functional. Show that the following are equiveridical: (i) ν has an extension to a measure on X; (ii) $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection; (iii) $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \nu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ such that $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$.

>(n) Let $\langle(X_n, \Sigma_n, \mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces, and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. For x, $y \in \prod_{n\in\mathbb{N}} X_n$, write $x \sim y$ if $\{n : x(n) = y(n)\} \in \mathcal{F}$. (i) Show that $\sim$ is an equivalence relation; write X for the set of equivalence classes, and $x^\bullet \in X$ for the equivalence class of $x \in \prod_{n\in\mathbb{N}} X_n$. (Compare 351M.) (ii) Let Σ be the set of subsets of X expressible in the form $Q(\langle E_n\rangle_{n\in\mathbb{N}}) = \{x^\bullet : x \in \prod_{n\in\mathbb{N}} E_n\}$, where $E_n \in \Sigma_n$ for each $n \in \mathbb{N}$. Show that Σ is an algebra of subsets of X, and that there is a well-defined additive functional $\nu : \Sigma \to [0, 1]$ defined by setting $\nu(Q(\langle E_n\rangle_{n\in\mathbb{N}})) = \lim_{n\to\mathcal{F}} \mu_n E_n$. (iii) Show that for any non-increasing sequence $\langle H_i\rangle_{i\in\mathbb{N}}$ in Σ there is an $H \in \Sigma$ such that $H \subseteq \bigcap_{i\in\mathbb{N}} H_i$ and $\nu H = \lim_{n\to\infty} \nu H_n$. (*Hint*: express each H_i as $Q(\langle E_{in}\rangle_{n\in\mathbb{N}})$. Do this in such a way that $E_{i+1,n} \subseteq E_{in}$ for all i, n. Take a decreasing sequence $\langle J_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{F}$, with empty intersection, such that $\nu H_i \le \mu E_{in} + 2^{-i}$ for $n \in J_i$. Set $E_n = E_{in}$ for $n \in J_i \setminus J_{i+1}$.) (iv) Show that there is a unique extension of ν to a complete probability measure μ on X which is inner regular with respect to Σ. (This is a kind of **Loeb measure**. Compare 328B.)

(o) Let $\mathfrak{A}$ be a Boolean algebra and $K \subseteq \mathfrak{A}$ a sublattice containing 0. Suppose that $\phi : K \to [0, \infty[$ is a bounded supermodular functional such that $\phi 0 = 0$. Show that there is a non-negative additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu a \ge \phi a$ for every $a \in K$ and $\nu 1 = \sup_{a\in K} \phi a$.

(p) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\phi : \mathcal{K} \to \mathbb{R}$ be an order-preserving modular function . Show that there is a non-negative additive functional $\nu : \mathcal{P}X \to [0, \infty]$ extending ϕ. (*Hint*: start with the case $X \in \mathcal{K}$.)

(q) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$. Let $\phi : \mathcal{K} \to [0, 1]$ be a submodular functional such that

$$\phi K \le \phi K' \text{ whenever } K,\ K' \in \mathcal{K} \text{ and } K \subseteq K', \quad \inf_{K\in\mathcal{K}} \phi K = 0.$$

Show that there is a finitely additive functional $\nu : \mathcal{P}X \to [0, 1]$ such that

$$\nu X = \sup_{K\in\mathcal{K}} \phi K, \quad \nu K \le \phi K \text{ for every } K \in \mathcal{K}.$$

(r) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\phi : \mathcal{K} \to [0, \infty[$ be such that $\phi K \le \sum_{n=0}^{\infty}(\phi K_n - \phi L_n)$ whenever $K \in \mathcal{K}$ and $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle L_n\rangle_{n\in\mathbb{N}}$ are sequences in $\mathcal{K}$ such that $L_n \subseteq K_n$ for every n and $\langle K_n \setminus L_n\rangle_{n\in\mathbb{N}}$ is a disjoint cover of K. Show that there is a measure on X extending ϕ. (*Hint*: Show that ϕ is modular. Show that if T is the ring of subsets of X generated by $\mathcal{K}$, every member of T is a finite union of differences of members of $\mathcal{K}$. Now apply 413La. See Kelley & Srinivasan 71.)

413Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra, $(S, +)$ a commutative semigroup with identity e and $\phi : \mathfrak{A} \to S$ a function such that $\phi 0 = e$. Show that

$$\mathfrak{B} = \{b : b \in \mathfrak{A},\ \phi a = \phi(a \cap b) + \phi(a \setminus b) \text{ for every } a \in \mathfrak{A}\}$$

is a subalgebra of of $\mathfrak{A}$, and that $\phi(a \cup b) = \phi a + \phi b$ for all disjoint a, $b \in \mathfrak{B}$.

(b) Give an example of two inner measures ϕ_1, ϕ_2 on a set X such that the measure defined by $\phi_1 + \phi_2$ strictly extends the sum of the measures defined by ϕ_1 and ϕ_2.

(c) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of probability spaces, and λ the product measure on $X=\prod_{i\in I}X_i$. Show that $\lambda_*(\prod_{i\in I}A_i)\le\prod_{i\in I}(\mu_i)_*A_i$ whenever $A_i\subseteq X_i$ for every i, with equality if I is countable.

(d) Find a measure space (X,Σ,μ), with $\mu X>0$, and a sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of subsets of X, covering X, such that whenever $E\in\Sigma$, $n\in\mathbb{N}$ and $\mu E>0$, there is an $F\in\Sigma$ such that $F\subseteq E\setminus X_n$ and $\mu F=\mu E$. (ii) For $A\subseteq X$ set $\phi A=\sup\{\mu E:E\in\Sigma,\,E\subseteq A\}$. Set $\mathrm{T}=\{G:G\subseteq X,\,\phi A=\phi(A\cap G)+\phi(A\setminus G)$ for every $A\subseteq X\}$. Show that $\phi\restriction\mathrm{T}$ is not a measure.

(e) Let (X,Σ,μ) be a totally finite measure space, and Z the Stone space of the Boolean algebra Σ. For $E\in\Sigma$ write $\widehat{E}$ for the corresponding open-and-closed subset of Z. Show that there is a unique function $f:X\to Z$ such that $f^{-1}[\widehat{E}]=E$ for every $E\in\Sigma$. Show that there is a measure ν on Z, inner regular with respect to the open-and-closed sets, such that f is inverse-measure-preserving with respect to μ and ν, and that f represents an isomorphism between the measure algebras of μ and ν. Use this construction to prove (vi)$\Rightarrow$(i) in Theorem 343B without appealing to the Lifting Theorem.

(f) Let X be a set, T a subalgebra of $\mathcal{P}X$, and $\nu:\mathrm{T}\to[0,\infty[$ a finitely additive functional. Suppose that there is a set $\mathcal{K}\subseteq\mathrm{T}$, containing $\emptyset$, such that (i) $\mu F=\sup\{\mu K:K\in\mathcal{K},\,K\subseteq F\}$ for every $F\in\mathrm{T}$ (ii) $\mathcal{K}$ is **monocompact**, that is, $\bigcap_{n\in\mathbb{N}}K_n\ne\emptyset$ for every non-increasing sequence in $\mathcal{K}$. Show that ν extends to a measure on X.

(g)(i) Let X be a topological space. Show that the family of closed countably compact subsets of X is a countably compact class. (ii) Let X be a Hausdorff space. Show that the family of sequentially compact subsets of X is a countably compact class.

(h) Let $\mathfrak{A}$ be a Boolean algebra and ν a totally finite submeasure on $\mathfrak{A}$ which is *either* supermodular *or* exhaustive and submodular. Show that ν is uniformly exhaustive.

(i) Let X be a set, $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$, and $f:\mathcal{K}\to\mathbb{R}$ a modular functional such that $f(\emptyset)=0$. Show that there is an additive functional $\nu:\mathcal{P}X\to\mathbb{R}$ extending f.

(j) Let (X,Σ,μ) be a semi-finite measure space, λ the c.l.d. product measure on $X\times\mathbb{R}$ when $\mathbb{R}$ is given Lebesgue measure, and λ_* the associated inner measure. Show that for any $f:X\to[0,\infty[$,

$$\underline{\int}f\,d\lambda=\lambda_*\{(x,\alpha):x\in X,\,0\le\alpha<f(x)\}=\lambda_*\{(x,\alpha):x\in X,\,0\le\alpha\le f(x)\}.$$

413 Notes and comments I gave rather few methods of constructing measures in the first three volumes of this treatise; in the present volume I shall have to make up for lost time. In particular I used Carathéodory's construction for Lebesgue measure (Chapter 11), product measures (Chapter 25) and Hausdorff measures (Chapter 26). The first two, at least, can be tackled in quite different ways if we choose. The first alternative approach I offer is the 'inner measure' method of 413C. Note the exact definition in 413A; I do not think it is an obvious one. In particular, while (α) seems to have something to do with subadditivity, and (β) is a kind of sequential order-continuity, there is no straightforward way in which to associate an outer measure with an inner measure, unless they both happen to be derived from measures (132B, 413D), even when they are finite-valued; and for an inner measure which is allowed to take the value ∞ we have to add the semi-finiteness condition $(*)$ of 413A (see 413Xa).

Once we have got these points right, however, we have a method which rivals Carathéodory's in scope, and in particular is especially well adapted to the construction of inner regular measures. As an almost trivial example, we have a route to the c.l.d. version of a measure μ (413Xj(i)), which can be derived from the inner measure μ_* defined from μ (413D). Henceforth μ_* will be a companion to the familiar outer measure μ^*, and many calculations will be a little easier with both available, as in 413E-413F.

The intention behind 413J-413K is to find a minimal set of properties of a functional ϕ_0 which will ensure that it has an extension to a measure. Indeed it is easy to see that, in the context of 413J, given a family $\mathcal{K}$ with the properties (†) and (‡) there, a functional ϕ_0 on $\mathcal{K}$ can have an extension to an inner regular measure only if it satisfies the conditions (α) and (β), so in this sense 413J is the best possible result. Note that while Carathéodory's construction is liable to produce wildly infinite measures (like Hausdorff measures, or primitive product measures), the construction here always gives us locally determined measures, provided only that ϕ_0 is finite-valued.

We have to work rather hard for the step from 413J to 413K. Of course 413J is a special case of 413K, and I could have saved a little space by giving a direct proof of the latter result. But I do not think that this would have made it easier; 413K really does require an extra step, because somehow we have to extend the functional ϕ_0 from $\mathcal{K}$ to $\mathcal{K}_\delta$. The method I have chosen uses 413B and 413I to cast as much of the argument as possible into the context of algebras of sets with additive functionals, where I hope the required manipulations will seem natural. (But perhaps I should insist that you must not take them too much for granted, as some of the time we have a finitely additive functional taking infinite values, and must take care not to subtract illegally, as well as not to take limits in the wrong places.) Note that the progression $\phi_0 \to \phi_1 \to \mu$ in the proof of 413K involves first an approximation from outside (if $K \in \mathcal{K}_\delta$, then $\phi_1 K$ will be $\inf\{\phi_0 K' : K \subseteq K' \in \mathcal{K}\}$) and then an approximation from inside (if $E \in \Sigma$, then $\mu E = \sup\{\phi_1 K : K \in \mathcal{K}_\delta, K \subseteq E\}$). The essential difficulty in the proof is just that we have to take successive non-exchangeable limits. I have slipped 413L in as a corollary of 413K; but it can be regarded as one of the fundamental results of measure theory. A non-negative finitely additive functional ν on an algebra Σ of sets can be extended to a countably additive measure iff it is 'relatively countably additive' in the sense that $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ such that $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$ (413Xm). Of course the same result can easily be got from an outer measure construction (413Xl). Note that the outer measure construction also has repeated limits, albeit simpler ones: in the formula

$$\theta A = \inf\{\textstyle\sum_{n=0}^{\infty} \nu E_n : \langle E_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \Sigma \text{ covering } A\}$$

the sum $\sum_{n=0}^{\infty} \nu E_n = \sup_{n\in\mathbb{N}} \sum_{i=0}^{n} \nu E_i$ can be regarded as a crude approximation from inside, while the infimum is an approximation from outside. To get the result as stated in 413L, of course, the outer measure construction needs a third limiting process, to obtain the c.l.d. version automatically provided by the inner measure method, and the inner regularity with respect to Σ_δ, while easily checked, also demands a few words of argument.

Many applications of the method of 413J-413K pass through 413N; if the family $\mathcal{K}$ is a countably compact class then the sequential order-continuity hypothesis (β) of 413J or 413K becomes a consequence of the other hypotheses. The essence of the method is the inner regularity hypothesis (α). I have tried to use the labels †, ‡, α and β consistently enough to suggest the currents which I think are flowing in this material.

In 413O we strike out in a new direction. The object here is to build an extension which is not going to be unique, and for which choices will have to be made. As with any such argument, the trick is to specify the allowable intermediate stages, that is, the partially ordered set P to which we shall apply Zorn's Lemma. But here the form of the theorem makes it easy to guess what P should be: it is the set of functionals satisfying the hypotheses of the theorem which have not wandered outside the boundary set by the conclusion, that is, which satisfy the condition (*) of part (a) of the proof of 413O. The finitistic nature of the hypotheses makes it easy to check that totally ordered subsets of P have upper bounds (that is to say, if we did this by transfinite induction there would be no problem at limit stages), and all we have to prove is that maximal elements of P are defined on adequately large domains; which amounts to showing that a member of P not defined on every element of $\mathcal{K}$ has a proper extension, that is, setting up a construction for the step to a successor ordinal in the parallel transfinite induction (part (c) of the proof).

Of course the principal applications of 413O in this book will be in the context of countably additive functionals, as in 413P.

It is clear that 413O and 413S overlap to some extent. I include both because they have different virtues. 413O provides actual extensions of functionals in a way that 413S, as given, does not; but its chief advantage, from the point of view of the work to come, is the approximation of members of T_1, in measure, by members of T_0. This will eventually enable us to retain control of the Maharam types of measures constructed by the method of 413P. In 413U we have a different kind of control; we can specify a lower bound for the measure of each member of our basic class $\mathcal{K}$, provided only that our specifications are consistent with some *finitely* additive functional.

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414 τ-additivity

The second topic I wish to treat is that of 'τ-additivity'. Here I collect results which do not depend on any strong kind of inner regularity. I begin with what I think of as the most characteristic feature of τ-additivity, its effect on the properties of semi-continuous functions (414A), with a variety of corollaries, up to the behaviour of subspace measures (414K). A very important property of τ-additive topological measures is that they are often strictly localizable (414J).

The theory of inner regular τ-additive measures belongs to the next section, but here I give two introductory results: conditions under which a τ-additive measure will be inner regular with respect to closed sets (414M) and conditions under which a measure which is inner regular with respect to closed sets will be τ-additive (414N). I end the section with notes on 'density' and 'lifting' topologies (414P-414R).

414A Theorem Let $(X,\mathfrak{T})$ be a topological space and μ an effectively locally finite τ-additive measure on X with domain Σ and measure algebra $\mathfrak{A}$.

(a) Suppose that $\mathcal{G}$ is a non-empty family in $\Sigma\cap\mathfrak{T}$ such that $H=\bigcup\mathcal{G}$ also belongs to Σ. Then $\sup_{G\in\mathcal{G}}G^\bullet=H^\bullet$ in $\mathfrak{A}$.

(b) Write $\mathcal{L}$ for the family of Σ-measurable lower semi-continuous functions from X to $\mathbb{R}$. Suppose that $\emptyset\ne A\subseteq\mathcal{L}$ and set $g(x)=\sup_{f\in A}f(x)$ for every $x\in X$. If g is Σ-measurable and finite almost everywhere, then $\tilde g^\bullet=\sup_{f\in A}f^\bullet$ in $L^0(\mu)$, where $\tilde g(x)=g(x)$ whenever $g(x)$ is finite.

(c) Suppose that $\mathcal{F}$ is a non-empty family of measurable closed sets such that $\bigcap\mathcal{F}\in\Sigma$. Then $\inf_{F\in\mathcal{F}}F^\bullet=(\bigcap\mathcal{F})^\bullet$ in $\mathfrak{A}$.

(d) Write $\mathfrak{U}$ for the family of Σ-measurable upper semi-continuous functions from X to $\mathbb{R}$. Suppose that $A\subseteq\mathfrak{U}$ is non-empty and set $g(x)=\inf_{f\in A}f(x)$ for every $x\in X$. If g is Σ-measurable and finite almost everywhere, then $\tilde g^\bullet=\inf_{f\in A}f^\bullet$ in $L^0(\mu)$, where $\tilde g(x)=g(x)$ whenever $g(x)$ is finite.

proof (a) ? If $H^\bullet\ne\sup_{G\in\mathcal{G}}G^\bullet$, there is a non-zero $a\in\mathfrak{A}$ such that $a\subseteq H^\bullet$ but $a\cap G^\bullet=0$ for every $G\in\mathcal{G}$. Express a as $E^\bullet$ where $E\in\Sigma$ and $E\subseteq H$. Because μ is effectively locally finite, there is a measurable open set H_0 of finite measure such that $\mu(H_0\cap E)>0$. Now $\{H_0\cap G:G\in\mathcal{G}\}$ is an upwards-directed family of measurable open sets with union $H_0\cap H\supseteq H_0\cap E$; as μ is τ-additive, there is a $G\in\mathcal{G}$ such that $\mu(H_0\cap G)>\mu H_0-\mu(H_0\cap E)$. But in this case $\mu(G\cap E)>0$, which is impossible, because $G^\bullet\cap E^\bullet=0$. **X**

(b) For any $\alpha\in\mathbb{R}$,

$$\{x:g(x)>\alpha\}=\bigcup_{f\in A}\{x:f(x)>\alpha\},$$

and these are all measurable open sets. Identifying $\{x:g(x)>\alpha\}^\bullet\in\mathfrak{A}$ with $[\![\tilde g^\bullet>\alpha]\!]$ (364Ib[4]), we see from (a) that $[\![\tilde g^\bullet>\alpha]\!]=\sup_{f\in A}[\![f^\bullet>\alpha]\!]$ for every α. But this means that $\tilde g^\bullet=\sup_{f\in A}f^\bullet$, by 364L(a-ii)[5].

(c) Apply (a) to $\mathcal{G}=\{X\setminus F:F\in\mathcal{F}\}$.

(d) Apply (b) to $\{-f:f\in A\}$.

414B Corollary Let X be a topological space and μ an effectively locally finite τ-additive topological measure on X.

(a) Suppose that A is a non-empty upwards-directed family of lower semi-continuous functions from X to $[0,\infty]$. Set $g(x)=\sup_{f\in A}f(x)$ in $[0,\infty]$ for every $x\in X$. Then $\int g=\sup_{f\in A}\int f$ in $[0,\infty]$.

(b) Suppose that A is a non-empty downwards-directed family of non-negative continuous real-valued functions on X, and that $g(x)=\inf_{x\in A}f(x)$ for every $x\in X$. If any member of A is integrable, then $\int g=\inf_{f\in A}\int f$.

proof (a) Of course all the $f\in A$, and also g, are measurable functions. Set $g_n=g\wedge n\chi X$ for every $n\in\mathbb{N}$. Then

$$g_n(x)=\sup_{f\in A}(f\wedge n\chi X)(x)$$

[4]Formerly 364Jb.

[5]Formerly 364Mb.

for every $x \in X$, so $g_n^\bullet = \sup_{f\in A}(f \wedge n\chi X)^\bullet$, by 414Ab, and

$$\int g_n = \int g_n^\bullet = \sup_{f\in A} \int (f \wedge n\chi X)^\bullet = \sup_{f\in A} \int f \wedge n\chi X$$

by 365Dh. But now, of course,

$$\int g = \sup_{n\in\mathbb{N}} \int g_n = \sup_{n\in\mathbb{N}, f\in A} \int f \wedge n\chi X = \sup_{f\in A} \int f,$$

as claimed.

(b) Take an integrable $f_0 \in A$, and apply (a) to $\{(f_0 - f)^+ : f \in A\}$.

414C Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space and $\mathcal{F}$ a non-empty downwards-directed family of closed sets. If $\inf_{F\in\mathcal{F}} \mu F$ is finite, this is the measure of $\bigcap \mathcal{F}$.

proof Setting $F_0 = \bigcap \mathcal{F}$, then $F_0^\bullet = \inf_{F\in\mathcal{F}} F^\bullet$, by 414Ac; now

$$\mu F_0 = \bar\mu F_0^\bullet = \inf_{F\in\mathcal{F}} \bar\mu F^\bullet = \inf_{F\in\mathcal{F}} \mu F$$

by 321F.

414D Corollary Let μ be an effectively locally finite τ-additive measure on a topological space X. If ν is a totally finite measure with the same domain as μ, truly continuous with respect to μ, then ν is τ-additive. In particular, if μ is σ-finite and ν is absolutely continuous with respect to μ, then ν is τ-additive.

proof We have a functional $\bar\nu : \mathfrak{A} \to [0, \infty[$, where $\mathfrak{A}$ is the measure algebra of μ, such that $\bar\nu E^\bullet = \nu E$ for every E in the common domain Σ of μ and ν. Now $\bar\nu$ is continuous for the measure-algebra topology of $\mathfrak{A}$ (327Cd), therefore completely additive (327Ba), therefore order-continuous (326Oc[6]). So if $\mathcal{G}$ is an upwards-directed family of open sets belonging to Σ with union $G_0 \in \Sigma$,

$$\sup_{G\in\mathcal{G}} \nu G = \sup_{G\in\mathcal{G}} \bar\nu G^\bullet = \bar\nu G_0^\bullet = \nu G_0$$

because $G_0^\bullet = \sup_{G\in\mathcal{G}} G^\bullet$.

The last sentence follows at once, because on a σ-finite space an absolutely continuous countably additive functional is truly continuous (232Bc).

414E Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space. Suppose that $\mathcal{G} \subseteq \mathfrak{T}$ is non-empty and upwards-directed, and $H = \bigcup \mathcal{G}$. Then

(a) $\mu(E \cap H) = \sup_{G\in\mathcal{G}} \mu(E \cap G)$ for every $E \in \Sigma$;

(b) if f is a non-negative virtually measurable real-valued function defined almost everywhere in X, then $\int_H f = \sup_{G\in\mathcal{G}} \int_G f$ in $[0, \infty]$.

proof (a) In the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ,

$$\begin{aligned}(E \cap H)^\bullet = E^\bullet \cap H^\bullet &= E^\bullet \cap \sup_{G\in\mathcal{G}} G^\bullet \\ &= \sup_{G\in\mathcal{G}} E^\bullet \cap G^\bullet = \sup_{G\in\mathcal{G}} (E \cap G)^\bullet,\end{aligned}$$

using 414Aa and the distributive law 313Ba. So

$$\mu(E \cap H) = \bar\mu(E \cap H)^\bullet = \sup_{G\in\mathcal{G}} \bar\mu(E \cap G)^\bullet = \sup_{G\in\mathcal{G}} \mu(E \cap G)$$

by 321D, because $\mathcal{G}$ and $\{(E \cap G)^\bullet : G \in \mathcal{G}\}$ are upwards-directed.

(b) For each $G \in \mathcal{G}$,

$$\int_G f = \int f \times \chi G = \int (f \times \chi G)^\bullet = \int f^\bullet \times \chi G^\bullet,$$

where $\chi G^\bullet$ can be interpreted either as $(\chi G)^\bullet$ (in $L^0(\mu)$) or as $\chi(G^\bullet)$ (in $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ); see 364J[7]. Now $H^\bullet = \sup_{G\in\mathcal{G}} G^\bullet$ (414Aa); since χ and $\times$ are order-continuous (364Jc, 364N[8]), $f^\bullet \times \chi H^\bullet = \sup_{G\in\mathcal{G}} f^\bullet \times \chi G^\bullet$; so

[6] Formerly 326Kc.
[7] Formerly 364K.
[8] Formerly 364P.

$$\int_H f = \int f^\bullet \times \chi H^\bullet = \sup_{G \in \mathcal{G}} \int f^\bullet \times \chi G^\bullet = \sup_{G \in \mathcal{G}} \int_G f$$

by 365Dh again.

414F Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space. Then for every $E \in \Sigma$ there is a unique relatively closed self-supporting set $F \subseteq E$ such that $\mu(E \setminus F) = 0$.

proof Let $\mathcal{G}$ be the set $\{G : G \in \mathfrak{T}, \mu(G \cap E) = 0\}$. Then $\mathcal{G}$ is upwards-directed, so $\mu(E \cap G^*) = \sup_{G \in \mathcal{G}} \mu(E \cap G) = 0$, where $G^* = \bigcup \mathcal{G}$. Set $F = E \setminus G^*$. Then $F \subseteq E$ is relatively closed, and $\mu(E \setminus F) = 0$. If $H \in \mathfrak{T}$ and $H \cap F \neq \emptyset$, then $H \notin \mathcal{G}$ so $\mu(F \cap H) = \mu(E \cap H) > 0$; thus F is self-supporting. If $F' \subseteq E$ is another self-supporting relatively closed set such that $\mu(E \setminus F') = 0$, then $\mu(F \setminus F') = \mu(F' \setminus F) = 0$; but as $F \setminus F'$ is relatively open in F, and $F' \setminus F$ is relatively open in F', these must both be empty, and $F = F'$.

414G Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Hausdorff effectively locally finite τ-additive topological measure space and $E \in \Sigma$ is an atom for μ (definition: 211I), then there is an $x \in E$ such that $E \setminus \{x\}$ is negligible.

proof Let $F \subseteq E$ be a self-supporting set such that $\mu(E \setminus F) = 0$. Since $\mu F = \mu E > 0$, F is not empty; take $x \in F$. **?** If $F \neq \{x\}$, let $y \in F \setminus \{x\}$. Because $\mathfrak{T}$ is Hausdorff, there are disjoint open sets G, H containing x, y respectively; and in this case $\mu(E \cap G) = \mu(F \cap G)$ and $\mu(E \cap H) = \mu(F \cap H)$ are both non-zero, which is impossible, since E is an atom. **X**

So $F = \{x\}$ and $E \setminus \{x\}$ is negligible.

414H Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is an effectively locally finite τ-additive topological measure space and ν is an indefinite-integral measure over μ (definition: 234J[9]), then ν is a τ-additive topological measure.

proof Because ν measures every set in Σ (234La[10]), it is a topological measure. To see that it is τ-additive, apply 414Eb to a Radon-Nikodým derivative of ν.

414I Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. If $E \subseteq X$ and $\mathcal{G} \subseteq \mathfrak{T}$ are such that $E \subseteq \bigcup \mathcal{G}$ and $E \cap G \in \Sigma$ for every $G \in \mathcal{G}$, then $E \in \Sigma$.

proof Set $\mathcal{K} = \{K : K \in \Sigma, E \cap K \in \Sigma\}$. Then whenever $F \in \Sigma$ and $\mu F > 0$ there is a $K \in \mathcal{K}$ included in F with $\mu K > 0$. **P** Set $K_1 = F \setminus \bigcup \mathcal{G}$. Then K_1 is a member of $\mathcal{K}$ included in F. If $\mu K_1 > 0$ then we can stop. Otherwise, $\mathcal{G}^* = \{G_0 \cup \ldots \cup G_n : G_0, \ldots, G_n \in \mathcal{G}\}$ is an upwards-directed family of open sets, and

$$\sup_{G \in \mathcal{G}^*} \mu(F \cap G) = \mu(F \cap \bigcup \mathcal{G}^*) = \mu F > 0,$$

by 414Ea. So there is a $G \in \mathcal{G}^*$ such that $\mu(F \cap G) > 0$; but now $E \cap G \in \Sigma$ so $F \cap G \in \mathcal{K}$. **Q**

By 412Aa, μ is inner regular with respect to $\mathcal{K}$; by 412Ja, $E \in \Sigma$.

414J Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. Then μ is strictly localizable.

proof Let $\mathcal{F}$ be a maximal disjoint family of self-supporting measurable sets of finite measure. Then whenever $E \in \Sigma$ and $\mu E > 0$, there is an $F \in \mathcal{F}$ such that $\mu(E \cap F) > 0$. **P?** Otherwise, let G be an open set of finite measure such that $\mu(G \cap E) > 0$, and set $\mathcal{F}_0 = \{F : F \in \mathcal{F}, F \cap G \neq \emptyset\}$. Then $\mu(F \cap G) > 0$ for every $F \in \mathcal{F}_0$, while $\mu G < \infty$ and $\mathcal{F}_0$ is disjoint, so $\mathcal{F}_0$ is countable and $\bigcup \mathcal{F}_0 \in \Sigma$. Set $E' = E \setminus \bigcup \mathcal{F}_0$; then $E \setminus E' = E \cap \bigcup \mathcal{F}_0$ is negligible, so $\mu(G \cap E') > 0$. By 414F, there is a self-supporting set $F' \subseteq G \cap E'$ such that $\mu F' > 0$. But in this case $F' \cap F = \emptyset$ for every $F \in \mathcal{F}$, so we ought to have added F' to $\mathcal{F}$. **XQ**

This means that $\mathcal{F}$ satisfies the criterion of 213Oa. Because (X, Σ, μ) is complete and locally determined, it is strictly localizable.

414K Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X, and $Y \subseteq X$ a subset such that the subspace measure μ_Y is semi-finite (see the remark following 412O). If μ is an effectively locally finite τ-additive topological measure, so is μ_Y.

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[9]Formerly 234B.
[10]Formerly 234D.

proof By 412Pe, μ_Y is an effectively locally finite topological measure. Now suppose that $\mathcal{H}$ is a non-empty upwards-directed family in $\mathfrak{T}_Y$ with union H^*. Set

$$\mathcal{G} = \{G : G \in \mathfrak{T},\, G \cap Y \in \mathcal{H}\}, \quad G^* = \bigcup \mathcal{G},$$

so that $\mathcal{G}$ is upwards-directed and $H^* = Y \cap G^*$. Let $\mathcal{K}$ be the family of sets $K \subseteq X$ such that $K \cap G^* \setminus G = \emptyset$ for some $G \in \mathcal{G}$. If $E \in \Sigma$,

$$\mu E = \mu(E \setminus G^*) + \mu(E \cap G^*) = \mu(E \setminus G^*) + \sup_{G \in \mathcal{G}} \mu(E \cap G)$$

(414Ea)

$$= \sup_{G \in \mathcal{G}} \mu(E \setminus (G^* \setminus G)),$$

so μ is inner regular with respect to $\mathcal{K}$. By 412Ob, μ_Y is inner regular with respect to $\{K \cap Y : K \in \mathcal{K}\}$. So if $\gamma < \mu_Y H^*$, there is a $K \in \mathcal{K}$ such that $K \cap Y \subseteq H^*$ and $\mu_Y(K \cap Y) \ge \gamma$. But now there is a $G \in \mathcal{G}$ such that $K \cap G^* \setminus G = \emptyset$, so that $K \cap Y \subseteq G \cap Y \in \mathcal{H}$ and $\sup_{H \in \mathcal{H}} \mu H \ge \gamma$. As $\mathcal{H}$ and γ are arbitrary, μ is τ-additive.

Remarks Recall from 214Ic that if (X, Σ, μ) has locally determined negligible sets (in particular, is either strictly localizable or complete and locally determined), then all its subspaces are semi-finite. In 419C below I describe a tight locally finite Borel measure with a subset on which the subspace measure is not semi-finite, therefore not effectively locally finite or τ-additive. In 419A I describe a σ-finite locally finite τ-additive topological measure, inner regular with respect to the closed sets, with a closed subset on which the subspace measure is totally finite but not τ-additive.

414L Lemma Let $(X, \mathfrak{T})$ be a topological space, and μ, ν two effectively locally finite Borel measures on X which agree on the open sets. Then they are equal.

proof Write $\mathfrak{T}^f$ for the family of open sets of finite measure. (I do not need to specify which measure I am using here.) For $G \in \mathfrak{T}^f$, set $\mu_G E = \mu(G \cap E)$, $\nu_G E = \nu(G \cap E)$ for every Borel set E. Then μ_G and ν_G are totally finite Borel measures which agree on $\mathfrak{T}$. By the Monotone Class Theorem (136C), μ_G and ν_G agree on the σ-algebra generated by $\mathfrak{T}$, that is, the Borel σ-algebra $\mathcal{B}$. Now, for any $E \in \mathcal{B}$,

$$\mu E = \sup_{G \in \mathfrak{T}^f} \mu_G E = \sup_{G \in \mathfrak{T}^f} \nu_G E = \nu E,$$

by 412F. So $\mu = \nu$.

414M Proposition Let (X, Σ, μ) be a measure space with a regular topology $\mathfrak{T}$ such that μ is effectively locally finite and τ-additive and Σ includes a base for $\mathfrak{T}$.

(a) $\mu G = \sup\{\mu F : F \in \Sigma \text{ is closed}, F \subseteq G\}$ for every open set $G \in \Sigma$.

(b) If μ is inner regular with respect to the σ-algebra generated by $\mathfrak{T} \cap \Sigma$, it is inner regular with respect to the closed sets.

proof (a) For $U \in \Sigma \cap \mathfrak{T}$, the set

$$\mathcal{H}_U = \{H : H \in \Sigma \cap \mathfrak{T},\, \overline{H} \subseteq U\}$$

is an upwards-directed family of open sets, and $\bigcup \mathcal{H}_U = U$ because $\mathfrak{T}$ is regular and Σ includes a base for $\mathfrak{T}$. Because μ is τ-additive, $\mu U = \sup\{\mu H : H \in \mathcal{H}_U\}$. Now, given $\gamma < \mu G$, we can choose $\langle U_n \rangle_{n \in \mathbb{N}}$ in $\Sigma \cap \mathfrak{T}$ inductively, as follows. Start by taking $U_0 \subseteq G$ such that $\gamma < \mu U_0 < \infty$ (using the hypothesis that μ is effectively locally finite). Given $U_n \in \Sigma \cap \mathfrak{T}$ and $\mu U_n > \gamma$, take $U_{n+1} \in \Sigma \cap \mathfrak{T}$ such that $\overline{U}_{n+1} \subseteq U_n$ and $\mu U_{n+1} > \gamma$. On completing the induction, set

$$F = \bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} \overline{U}_n;$$

then F is a closed set belonging to Σ, $F \subseteq G$ and $\mu F \ge \gamma$. As γ is arbitrary, we have the result.

(b) Let Σ_0 be the σ-algebra generated by $\Sigma \cap \mathfrak{T}$ and set $\mu_0 = \mu \restriction \Sigma_0$. Then $\Sigma_0 \cap \mathfrak{T} = \Sigma \cap \mathfrak{T}$ is still a base for $\mathfrak{T}$ and μ_0 is still τ-additive and effectively locally finite, so by (a) and 412G it is inner regular with respect to the closed sets. Now we are supposing that μ is inner regular with respect to Σ_0, so μ is inner regular with respect to the closed sets, by 412Ab.

414N Proposition Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X. Suppose that (i) μ is semi-finite and inner regular with respect to the closed sets (ii) whenever $\mathcal{F}$ is a non-empty downwards-directed family of measurable closed sets with empty intersection and $\inf_{F\in\mathcal{F}}\mu F<\infty$, then $\inf_{F\in\mathcal{F}}\mu F=0$. Then μ is τ-additive.

proof Let $\mathcal{G}$ be a non-empty upwards-directed family of measurable open sets with measurable union H. Take any $\gamma<\mu H$. Because μ is semi-finite, there is a measurable set $E\subseteq H$ such that $\gamma<\mu E<\infty$. Now there is a measurable closed set $F\subseteq E$ such that $\mu F\ge\gamma$. Consider $\mathcal{F}=\{F\setminus G:G\in\mathcal{G}\}$. This is a downwards-directed family of closed sets of finite measure with empty intersection. So $\inf_{G\in\mathcal{G}}\mu(F\setminus G)=0$, that is,

$$\gamma\le\mu F=\sup_{G\in\mathcal{G}}\mu(F\cap G)\le\sup_{G\in\mathcal{G}}\mu G.$$

As γ is arbitrary, $\mu H=\sup_{G\in\mathcal{G}}\mu G$; as $\mathcal{G}$ is arbitrary, μ is τ-additive.

414O The following elementary result is worth noting.

Proposition If X is a hereditarily Lindelöf space (e.g., if it is separable and metrizable) then every measure on X is τ-additive.

proof If μ is a measure on X, with domain Σ, and $\mathcal{G}\subseteq\Sigma$ is a non-empty upwards-directed family of measurable open sets, then there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ such that $\bigcup\mathcal{G}=\bigcup_{n\in\mathbb{N}}G_n$. Now

$$\mu(\bigcup\mathcal{G})=\lim_{n\to\infty}\mu(\bigcup_{i\le n}G_i)\le\sup_{G\in\mathcal{G}}\mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive.

414P Density topologies Recall that a lower density for a measure space (X,Σ,μ) is a function $\underline{\phi}:\Sigma\to\Sigma$ such that $\underline{\phi}E=\underline{\phi}F$ whenever E, $F\in\Sigma$ and $\mu(E\triangle F)=0$, $\mu(E\triangle\underline{\phi}E)=0$ for every $E\in\Sigma$, $\underline{\phi}\emptyset=\emptyset$ and $\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F$ for all E, $F\in\Sigma$ (341C).

Proposition Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}X=X$. Set

$$\mathfrak{T}=\{E:E\in\Sigma,\ E\subseteq\underline{\phi}E\}.$$

Then $\mathfrak{T}$ is a topology on X, the **density topology** associated with $\underline{\phi}$, and $(X,\mathfrak{T},\Sigma,\mu)$ is an effectively locally finite τ-additive topological measure space; μ is strictly positive and inner regular with respect to the open sets.

proof (a)(i) For any $E\in\Sigma$, $\underline{\phi}(E\cap\underline{\phi}E)=\underline{\phi}E$ because $E\setminus\underline{\phi}E$ is negligible; consequently $E\cap\underline{\phi}E\in\mathfrak{T}$. In particular, $\emptyset=\emptyset\cap\underline{\phi}\emptyset$ and $X=X\cap\underline{\phi}X$ belong to $\mathfrak{T}$. If E, $F\in\mathfrak{T}$ then

$$\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F\supseteq E\cap F,$$

so $E\cap F\in\mathfrak{T}$.

(ii) Suppose that $\mathcal{G}\subseteq\mathfrak{T}$ and $H=\bigcup\mathcal{G}$. By 341M, μ is (strictly) localizable, so $\mathcal{G}$ has an essential supremum $F\in\Sigma$ such that $F^\bullet=\sup_{G\in\mathcal{G}}G^\bullet$ in the measure algebra $\mathfrak{A}$ of μ; that is, for $E\in\Sigma$, $\mu(G\setminus E)=0$ for every $G\in\mathcal{G}$ iff $\mu(F\setminus E)=0$. Now $F\setminus H$ is negligible, by 213K. On the other hand,

$$G\subseteq\underline{\phi}G=\underline{\phi}(G\cap F)\subseteq\underline{\phi}F$$

for every $G\in\mathcal{G}$, so $H\subseteq\underline{\phi}F$, and $H\setminus F\subseteq\underline{\phi}F\setminus F$ is negligible. But as μ is complete, this means that $H\in\Sigma$. Also $\underline{\phi}H=\underline{\phi}F\supseteq H$, so $H\in\mathfrak{T}$. Thus $\mathfrak{T}$ is closed under arbitrary unions and is a topology.

(b) By its definition, $\mathfrak{T}$ is included in Σ, so μ is a topological measure. If $E\in\Sigma$ then $E\cap\underline{\phi}E$ belongs to $\mathfrak{T}$, is included in E and has the same measure as E; so μ is inner regular with respect to the open sets. As μ is semi-finite, it is inner regular with respect to the open sets of finite measure, and is effectively locally finite. If $E\in\mathfrak{T}$ is non-empty, then $\underline{\phi}E\supseteq E$ is non-empty, so $\mu E>0$; thus μ is strictly positive. Finally, if $\mathcal{G}$ is a non-empty upwards-directed family in $\mathfrak{T}$, then the argument of (a-ii) shows that $(\bigcup\mathcal{G})^\bullet=\sup_{G\in\mathcal{G}}G^\bullet$ in $\mathfrak{A}$, so that $\mu(\bigcup\mathcal{G})=\sup_{G\in\mathcal{G}}\mu G$. Thus μ is τ-additive.

414Q Lifting topologies Let (X,Σ,μ) be a measure space and $\phi:\Sigma\to\Sigma$ a lifting, that is, a Boolean homomorphism such that $\phi E=\emptyset$ whenever $\mu E=0$ and $\mu(E\triangle\phi E)=0$ for every $E\in\Sigma$ (341A). The **lifting topology** associated with ϕ is the topology generated by $\{\phi E:E\in\Sigma\}$. Note that $\{\phi E:E\in\Sigma\}$ is a topology base, so is a base for the lifting topology.

414R Proposition Let (X,Σ,μ) be a complete locally determined measure space and $\phi:\Sigma\to\Sigma$ a lifting with lifting topology $\mathfrak{S}$ and density topology $\mathfrak{T}$. Then $\mathfrak{S}\subseteq\mathfrak{T}\subseteq\Sigma$, and μ is τ-additive, effectively locally finite and strictly positive with respect to $\mathfrak{S}$. Moreover, $\mathfrak{S}$ is zero-dimensional.

proof Of course ϕ is a lower density, so we can talk of its density topology, and since $\phi^2E=\phi E$, $\phi E\in\mathfrak{T}$ for every $E\in\Sigma$, so $\mathfrak{S}\subseteq\mathfrak{T}$. Because μ is τ-additive and strictly positive with respect to $\mathfrak{T}$, it must also be τ-additive and strictly positive with respect to $\mathfrak{S}$. If $E\in\Sigma$ and $\mu E>0$ there is an $F\subseteq E$ such that $0<\mu F<\infty$, and now ϕF is an $\mathfrak{S}$-open set of finite measure meeting E in a non-negligible set; so μ is effectively locally finite with respect to $\mathfrak{S}$. Of course $\mathfrak{S}$ is zero-dimensional because $\phi[\Sigma]$ is a base for $\mathfrak{S}$ consisting of open-and-closed sets.

414X Basic exercises >(a) Let (X,Σ,μ) and (Y,T,ν) be measure spaces with topologies $\mathfrak{T}$ and $\mathfrak{S}$, and $f:X\to Y$ a continuous inverse-measure-preserving function. Show that if μ is τ-additive with respect to $\mathfrak{T}$ then ν is τ-additive with respect to $\mathfrak{S}$. Show that if ν is locally finite, so is μ.

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X,Σ,μ); suppose that we are given a topology $\mathfrak{T}_i$ on X_i for each i, and let $\mathfrak{T}$ be the disjoint union topology on X. Show that μ is τ-additive iff every μ_i is.

>(c) Let $(X,\mathfrak{T})$ be a topological space and μ a totally finite measure on X which is inner regular with respect to the closed sets. Suppose that $\mu X=\sup_{G\in\mathcal{G}}\mu G$ whenever $\mathcal{G}$ is an upwards-directed family of measurable open sets covering X. Show that μ is τ-additive.

(d) Let μ be an effectively locally finite τ-additive σ-finite measure on a topological space X, and $\nu:\operatorname{dom}\mu\to[0,\infty[$ a countably additive functional which is absolutely continuous with respect to μ. Show from first principles that ν is τ-additive.

(e) Give an example of an indefinite-integral measure over Lebesgue measure on $\mathbb{R}$ which is not effectively locally finite. (*Hint*: arrange for every non-trivial interval to have infinite measure.)

(f) Let $(X,\mathfrak{T})$ be a topological space and μ a complete locally determined effectively locally finite τ-additive topological measure on X. Show that if f is a real-valued function, defined on a subset of X, which is locally integrable in the sense of 411Fc, then f is measurable.

(g) Let $(X,\mathfrak{T})$ be a topological space and μ an effectively locally finite τ-additive measure on X. Let $\mathcal{G}$ be a cover of X consisting of measurable open sets, and $\mathcal{K}$ the ideal of subsets of X generated by $\mathcal{G}$. Show that μ is inner regular with respect to $\mathcal{K}$.

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space, and A a subset of X. Suppose that for every $x\in A$ there is an open set G containing x such that $A\cap G$ is negligible. Show that A is negligible.

(i) Give an alternative proof of 414K based on the fact that the canonical map from the measure algebra of μ to the measure algebra of μ_Y is order-continuous (322Yd).

>(j)(i) If μ is an effectively locally finite τ-additive Borel measure on a regular topological space, show that the c.l.d. version of μ is a quasi-Radon measure. (ii) If μ is a locally finite, effectively locally finite τ-additive Borel measure on a locally compact Hausdorff space, show that μ is tight, so that the c.l.d. version of μ is a Radon measure.

>(k) Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}$ a lower density for μ such that $\underline{\phi}X=X$; let $\mathfrak{T}$ be the corresponding density topology. (i) Show that a dense open subset of X must be conegligible. (ii) Show that a subset of X is nowhere dense for $\mathfrak{T}$ iff it is negligible iff it is meager for $\mathfrak{T}$. (iii) Show that a function $f : X \to \mathbb{R}$ is Σ-measurable iff it is $\mathfrak{T}$-continuous at almost every point of X. (*Hint*: if f is measurable, set $E_q = \{x : f(x) > q\}$, $F_q = \{x : f(x) < q\}$; show that f is continuous at every point of $X \setminus \bigcup_{q\in\mathbb{Q}}((E_q \setminus \underline{\phi}E_q) \cup (F_q \setminus \underline{\phi}F_q))$.) (iv) Show that Σ is both the Borel σ-algebra of $(X, \mathfrak{T})$ and the Baire-property algebra of $(X, \mathfrak{T})$. (v) Show that $(X, \mathfrak{T})$ is a Baire space.

(l) Let (X, Σ, μ) be a complete locally determined measure space and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}X = X$, with density topology $\mathfrak{T}$. Show that if $A \subseteq X$ and E is a measurable envelope of A then the $\mathfrak{T}$-closure of A is just $A \cup (X \setminus \underline{\phi}(X \setminus E))$.

(m) Let μ be Lebesgue measure on $\mathbb{R}^r$, Σ its domain, $\mathrm{int}^* : \Sigma \to \Sigma$ lower Lebesgue density (341E) and $\mathfrak{T}$ the corresponding density topology. (i) Show that $\mathfrak{T}$ is finer than the usual Euclidean topology of $\mathbb{R}^r$. (ii) Show that for any $A \subseteq \mathbb{R}$, the closure of A for $\mathfrak{T}$ is just $A \cup \{x : \limsup_{\delta\downarrow 0} \frac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)} > 0\}$, and the interior is $A \cap \{x : \lim_{\delta\downarrow 0} \frac{\mu_*(A\cap B(x,\delta))}{\mu B(x,\delta)} = 1\}$.

(n) Let (X, Σ, μ) be a complete locally determined measure space and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}X = X$; let $\mathfrak{T}$ be the associated density topology. Let A be a subset of X and E a measurable envelope of A; let Σ_A be the subspace σ-algebra and μ_A the subspace measure on A. (i) Show that we have a lower density $\underline{\phi}_A : \Sigma_A \to \Sigma_A$ defined by setting $\underline{\phi}_A(F \cap A) = A \cap \underline{\phi}(E \cap F)$ for every $F \in \Sigma$. (ii) Show that $\underline{\phi}_A A = A$ iff $A \subseteq \underline{\phi}E$, and that in this case the density topology on A derived from $\underline{\phi}_A$ is just the subspace topology.

(o) Let (X, Σ, μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting, with density topology $\mathfrak{T}$ and lifting topology $\mathfrak{S}$. (i) Show that

$$\mathfrak{T} = \{H \cap G : G \in \mathfrak{S},\ H \text{ is conegligible}\} = \{H \cap \phi E : E \in \Sigma,\ H \text{ is conegligible}\}.$$

(ii) Show that if $A \subseteq X$ and E is a measurable envelope of A then the $\mathfrak{T}$-closure of A is $A \cup \phi E$.

(p) Let (X, Σ, μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting; let $\mathfrak{S}$ be its lifting topology. Let A be a subset of X such that $A \subseteq \phi E$ for some (therefore any) measurable envelope E of A. Let Σ_A be the subspace σ-algebra and μ_A the subspace measure on A. (i) Show that we have a lifting $\phi_A : \Sigma_A \to \Sigma_A$ defined by setting $\phi_A(F \cap A) = A \cap \phi F$ for every $F \in \Sigma$. (ii) Show that the lifting topology on A derived from ϕ_A is just the subspace topology.

(q) Let (X, Σ, μ) and (Y, T, ν) be complete locally determined measure spaces and $f : X \to Y$ an inverse-measure-preserving function. Suppose that we have lower densities $\underline{\phi} : \Sigma \to \Sigma$ and $\underline{\psi} : \mathrm{T} \to \mathrm{T}$ such that $\underline{\phi}X = X$, $\underline{\psi}Y = Y$ and $\underline{\phi}f^{-1}[F] = f^{-1}[\underline{\psi}F]$ for every $F \in \mathrm{T}$. (i) Show that f is continuous for the density topologies of $\underline{\phi}$ and $\underline{\psi}$. (ii) Show that if $\underline{\phi}$ and $\underline{\psi}$ are liftings then f is continuous for the lifting topologies.

(r) Let (X, Σ, μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting, with associated lifting topology $\mathfrak{S}$. Show that a function $f : X \to \mathbb{R}$ is Σ-measurable iff there is a conegligible set H such that $f\restriction H$ is $\mathfrak{S}$-continuous. (Compare 414Xk, 414Xt.)

(s) Let (X, Σ, μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting. Let (Z, T, ν) be the Stone space of the measure algebra of μ, and $f : X \to Z$ the inverse-measure-preserving function associated with ϕ (341P). Show that the lifting topology on X is just $\{f^{-1}[G] : G \subseteq Z \text{ is open}\}$.

(t) Let (X, Σ, μ) be a strictly localizable measure space and $\phi : \Sigma \to \Sigma$ a lifting. Write $\mathcal{L}^\infty$ for the Banach lattice of bounded Σ-measurable real-valued functions on X, identified with $\mathcal{L}^\infty(\Sigma)$ (363H); let $T : \mathcal{L}^\infty \to \mathcal{L}^\infty$ be the Riesz homomorphism associated with ϕ (363F). (i) Show that $T^2 = T$. (ii) Show that if X is given the lifting topology $\mathfrak{S}$ defined by ϕ, then $T[\mathcal{L}^\infty]$ is precisely the space of bounded continuous real-valued functions on X. (iii) Show that if $f \in \mathcal{L}^\infty$, $x \in X$ and $\epsilon > 0$ there is an $\mathfrak{S}$-open set U containing x such that $|(Tf)(x) - \frac{1}{\mu V}\int_V f d\mu| \le \epsilon$ for every non-negligible measurable set V included in U.

(u) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. Show that there is a decomposition $\langle X_i\rangle_{i\in I}$ for μ in which every X_i is expressible as the intersection of a closed set with an open set. (*Hint*: enumerate the open sets of finite measure as $\langle G_\xi\rangle_{\xi<\kappa}$, and set $\mathcal{F}=\{G_\xi\setminus\bigcup_{\eta<\xi}G_\eta:\xi<\kappa\}$.)

414Y Further exercises (a) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a totally finite topological measure space. For $E\in\Sigma$ set

$$\mu_\tau E=\inf\{\sup_{G\in\mathcal{G}}\mu(E\cap G):\mathcal{G}\subseteq\mathfrak{T}\text{ is an upwards-directed set with union }X\}.$$

Suppose *either* that μ is inner regular with respect to the closed sets *or* that $\mathfrak{T}$ is regular. Show that μ_τ is a τ-additive measure, the largest τ-additive measure with domain Σ which is dominated by μ.

(b) Let X be a set, Σ an algebra of subsets of X, and $\mathfrak{T}$ a topology on X. Let M be the L-space of bounded finitely additive real-valued functionals on Σ (362B). Let $N\subseteq M$ be the set of those functionals ν such that $\inf_{G\in\mathcal{G}}|\nu|(H\setminus G)=0$ whenever $\mathcal{G}\subseteq\mathfrak{T}\cap\Sigma$ is a non-empty upwards-directed family with union $H\in\Sigma$. Show that N is a band in M. (Cf. 362Xi.)

(c) Find a probability space (X,Σ,μ) and a topology $\mathfrak{T}$ on X such that Σ includes a base for $\mathfrak{T}$ and μ is τ-additive, but there is a set $E\in\Sigma$ such that the subspace measure μ_E is not τ-additive.

(d) Let int* be lower Lebesgue density on $\mathbb{R}^r$, and $\mathfrak{T}$ the associated density topology. Show that every $\mathfrak{T}$-Borel set is an F_σ set for $\mathfrak{T}$.

(e) Let (X,ρ) be a metric space and μ a strictly positive locally finite quasi-Radon measure on X; write $\mathfrak{T}$ for the topology of X and Σ for the domain of μ. For $E\in\Sigma$ set $\underline{\phi}(E)=\{x:x\in X,\lim_{\delta\downarrow0}\frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)}=1\}$. Suppose that $E\setminus\underline{\phi}(E)$ is negligible for every $E\in\Sigma$ (cf. 261D, 472D). (i) Show that $\underline{\phi}$ is a lower density for μ, with $\underline{\phi}(X)=X$. Let $\mathfrak{T}_d$ be the associated density topology. (ii) Suppose that $H\in\mathfrak{T}_d$ and that $K\subseteq H$ is $\mathfrak{T}$-closed and ρ-totally bounded. Show that there is a $\mathfrak{T}$-closed, ρ-totally bounded $K'\subseteq H$ such that K is included in the $\mathfrak{T}_d$-interior of K'. (iii) Show that $\mathfrak{T}_d$ is completely regular. (*Hint*: Lukeš Malý & Zajíček 86.)

(f) Show that the density topology on $\mathbb{R}$ associated with lower Lebesgue density is not normal.

(g) Let μ be Lebesgue measure on $\mathbb{R}^r$, Σ its domain, $\text{int}^*:\Sigma\to\Sigma$ lower Lebesgue density and $\mathfrak{T}$ the corresponding density topology. (i) Show that if $f:\mathbb{R}^r\to\mathbb{R}^r$ is a permutation such that f and f^{-1} are both differentiable everywhere, with continuous derivatives, then f is a homeomorphism for $\mathfrak{T}$. (*Hint*: 263D.) (ii) Show that if $\phi:\Sigma\to\Sigma$ is a lifting and $\mathfrak{S}$ the corresponding lifting topology, then $x\mapsto-x$ is not a homeomorphism for $\mathfrak{S}$. (*Hint*: 345Xc.)

414 Notes and comments I have remarked before that it is one of the abiding frustrations of measure theory, at least for anyone ambitious to apply the power of modern general topology to measure-theoretic problems, that the basic convergence theorems are irredeemably confined to sequences. In Volume 3 I showed that if we move to measure algebras and function spaces, we can hope that the countable chain condition or the countable sup property will enable us to replace arbitrary directed sets with monotonic sequences, thereby giving theorems which apply to apparently more general types of convergence. In 414A and its corollaries we come to a quite different context in which a measure, or integral, behaves like an order-continuous functional. Of course the theorems here depend directly on the hypothesis of τ-additivity, which rather begs the question; but we shall see in the rest of the chapter that this property does indeed often appear. For the moment, I remark only that as Lebesgue measure is τ-additive we certainly have a non-trivial example to work with.

The hypotheses of the results above move a touch awkwardly between those with the magic phrase 'topological measure' and those without. The point is that (as in 412G, for instance) it is sometimes useful to be able to apply these ideas to Baire measures on completely regular spaces, which are defined on a base for the topology but may not be defined on every open set. The device I have used in the definition of τ-additivity (411C) makes this possible, at the cost of occasional paradoxical phenomena like 414Yc.

I hope that no confusion will arise between the two topologies associated with a lifting on a complete locally determined space. I have called them the 'density topology' and the 'lifting topology' because the former can be defined directly from a lower density; but it would be equally reasonable to call them the 'fine' and 'coarse' lifting topologies. The density topology has the apparent advantage of giving us a measure which is inner regular with respect to the Borel sets, but at the cost of being rather odd regarded as a topological space (414P, 414Xk, 414Ye, 414Yf). It has the important advantage that there are densities (like the Lebesgue lower density) which have some claim to be called canonical, and others with useful special properties, as in §346, while liftings are always arbitrary and invariance properties for them sometimes unachievable. So, for instance, the Lebesgue density topology on $\mathbb{R}^r$ is invariant under diffeomorphisms, which no lifting topology can be (414Yg). The lifting topology is well-behaved as a topology, but only in special circumstances (as in 453Xd) is the measure inner regular with respect to its Borel sets, and even the closure of a set can be difficult to determine.

As with inner regularity, τ-additivity can be associated with the band structure of the space of bounded additive functionals on an algebra (414Yb); there will therefore be corresponding decompositions of measures into τ-additive and 'purely non-τ-additive' parts (cf. 414Ya).

Version of 16.5.17

415 Quasi-Radon measure spaces

We are now I think ready to draw together the properties of inner regularity and τ-additivity. Indeed, this section will unite several of the themes which have been running through the treatise so far: (strict) localizability, subspaces and products as well as the new concepts of this chapter. In these terms, the principal results are that a quasi-Radon space is strictly localizable (415A), any subspace of a quasi-Radon space is quasi-Radon (415B), and the product of a family of strictly positive quasi-Radon probability measures on separable metrizable spaces is quasi-Radon (415E). I describe a basic method of constructing quasi-Radon measures (415K), with details of one of the standard ways of applying it (415L, 415N) and some notes on how to specify a quasi-Radon measure uniquely (415H-415I). I spell out useful results on indefinite-integral measures (415O) and L^p spaces (415P), and end the section with a discussion of the Stone space Z of a localizable measure algebra $\mathfrak{A}$ and an important relation in $Z \times X$ when $\mathfrak{A}$ is the measure algebra of a quasi-Radon measure space X (415Q-415R).

It would be fair to say that the study of quasi-Radon spaces for their own sake is a minority interest. If you are not already well acquainted with Radon measure spaces, it would make good sense to read this section in parallel with the next. In particular, the constructions of 415K and 415L derive much of their importance from the corresponding constructions in §416.

415A Theorem A quasi-Radon measure space is strictly localizable.

proof This is a special case of 414J.

415B Theorem Any subspace of a quasi-Radon measure space is quasi-Radon.

proof Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(Y, \mathfrak{T}_Y, \Sigma_Y, \mu_Y)$ a subspace with the induced topology and measure. Because μ is complete, locally determined and localizable (by 415A), so is μ_Y (214Ie). Because μ_Y is semi-finite and μ is an effectively locally finite τ-additive topological measure, so is μ_Y (414K). Because μ is inner regular with respect to the closed sets and μ_Y is semi-finite, μ_Y is inner regular with respect to the relatively closed subsets of Y (412Pc). So μ_Y is a quasi-Radon measure.

415C In regular topological spaces, the condition 'inner regular with respect to the closed sets' in the definition of 'quasi-Radon measure' can be weakened or omitted.

Proposition Let $(X, \mathfrak{T})$ be a regular topological space.

(a) If μ is a complete locally determined effectively locally finite τ-additive topological measure on X, inner regular with respect to the Borel sets, then it is a quasi-Radon measure.

(b) If μ is an effectively locally finite τ-additive Borel measure on X, its c.l.d. version is a quasi-Radon measure.

proof (a) By 414Mb, μ is inner regular with respect to the closed sets, which is the only feature missing from the given hypotheses.

(b) The c.l.d. version of μ satisfies the hypotheses of (a).

415D In separable metrizable spaces, among others, we can even omit τ-additivity.

Proposition Let $(X,\mathfrak{T})$ be a regular hereditarily Lindelöf topological space; e.g., a separable metrizable space (4A2P(a-iii)), indeed any regular space with a countable network (4A2Nb).

(i) If μ is a complete effectively locally finite measure on X, inner regular with respect to the Borel sets, and its domain includes a base for $\mathfrak{T}$, then it is a quasi-Radon measure.

(ii) If μ is an effectively locally finite Borel measure on X, then its completion is a quasi-Radon measure.

(iii) Any quasi-Radon measure on X is σ-finite.

(iv) Any quasi-Radon measure on X is completion regular.

proof (a) The basic fact we need is that if $\mathcal{G}$ is any family of open sets in X, then there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup\mathcal{G}_0 = \bigcup\mathcal{G}$ (4A2H(c-i)). Consequently any effectively locally finite measure μ on X is σ-finite. **P** Let $\mathcal{G}$ be the family of measurable open sets of finite measure. Let $\mathcal{G}_0 \subseteq \mathcal{G}$ be a countable set with the same union as $\mathcal{G}$. Then $E = X \setminus \bigcup\mathcal{G}_0$ is measurable, and $E \cap G = \emptyset$ for every $G \in \mathcal{G}$, so $\mu E = 0$; accordingly $\mathcal{G}_0 \cup \{E\}$ is a countable cover of X by sets of finite measure, and μ is σ-finite. **Q**

Moreover, any measure on X is τ-additive. **P** If $\mathcal{G}$ is a non-empty upwards-directed family of open measurable sets, there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ with union $\bigcup\mathcal{G}$. If $n \in \mathbb{N}$ there is a $G \in \mathcal{G}$ such that $\bigcup_{i\le n} G_i \subseteq G$, so

$$\mu(\bigcup\mathcal{G}) = \mu(\bigcup_{n\in\mathbb{N}} G_n) = \sup_{n\in\mathbb{N}} \mu(\bigcup_{i\le n} G_i) \le \sup_{G\in\mathcal{G}} \mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. **Q**

(b)(i) Now let μ be a complete effectively locally finite measure on X, inner regular with respect to the Borel sets, and with domain Σ including a base for the topology of X. If $H \in \mathfrak{T}$, then $\mathcal{G} = \{G : G \in \Sigma\cap\mathfrak{T},\, G \subseteq H\}$ has union H, because $\Sigma\cap\mathfrak{T}$ is a base for $\mathfrak{T}$; but in this case there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $H = \bigcup\mathcal{G}_0$, so that $H \in \Sigma$. Thus μ is a topological measure. We know also from (a) that it is τ-additive and σ-finite, therefore locally determined. By 415Ca, it is a quasi-Radon measure.

(ii) If μ is an effectively locally finite Borel measure on X, then its completion $\hat\mu$ satisfies the conditions of (i), so is a quasi-Radon measure.

(iii) If μ is a quasi-Radon measure on X, it is surely effectively locally finite, therefore σ-finite.

(iv) Every closed set is a zero set (4A2H(c-ii)), so any measure which is inner regular with respect to the closed sets is completion regular.

415E I am delaying most of the theory of products of (quasi-)Radon measures to §417. However, there is one result which is so important that I should like to present it here, even though some of the ideas will have to be repeated later.

Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of separable metrizable quasi-Radon probability spaces such that every μ_i is strictly positive, and λ the product measure on $X = \prod_{i\in I} X_i$. Then

(i) λ is a completion regular quasi-Radon measure;

(ii) if $F \subseteq X$ is a closed self-supporting set, there is a countable set $J \subseteq I$ such that F is determined by coordinates in J, so F is a zero set.

proof (a) Write Λ for the domain of λ, and $\mathcal{U}$ for the family of subsets of X of the form $\prod_{i\in I} G_i$ where $G_i \in \mathfrak{T}_i$ for every $i \in I$ and $\{i : G_i \ne X_i\}$ is finite. Then $\mathcal{U}$ is a base for the topology of X, included in Λ. For $J \subseteq I$ let λ_J be the product measure on $X_J = \prod_{i\in J} X_i$ and Λ_J its domain.

(b) Consider first the case in which I is countable. In this case X also is separable and metrizable (4A2P(a-v)), while Λ includes a base for its topology. Also λ is a complete probability measure and inner regular with respect to the closed sets (412Ua), so must be a quasi-Radon measure, by 415D(i).

(c) Now consider uncountable I. The key to the proof is the following fact: if $\mathcal{V} \subseteq \mathcal{U}$ has union W, then $W \in \Lambda$ and $\lambda W = \sup_{V\in\mathcal{V}^*} \lambda V$, where $\mathcal{V}^*$ is the set of unions of finite subsets of $\mathcal{V}$.

P(i) By 215B(iv), there is a countable set $\mathcal{V}_1 \subseteq \mathcal{V}$ such that $\lambda(U \setminus W_1) = 0$ for every $U \in \mathcal{V}$, where $W_1 = \bigcup \mathcal{V}_1$. Every member of $\mathcal{U}$ is determined by coordinates in some finite set (see 254M for this concept), so there is a countable set $J \subseteq I$ such that every member of $\mathcal{V}_1$ is determined by coordinates in J, and W_1 also is determined by coordinates in J. Let $\pi_J : X \to X_J$ be the canonical map. Because it is an open map (4A2B(f-i)), $\pi_J[W]$ and $\pi_J[W_1]$ are open in X_J, and belong to Λ_J, by (b).

(ii) ? Suppose, if possible, that $\lambda_J\pi_J[W] > \lambda_J\pi_J[W_1]$. Since $\pi_J[W] = \bigcup\{\pi_J[U] : U \in \mathcal{V}\}$, while λ_J is quasi-Radon and all the sets $\pi_J[U]$ are open, there must be some $U \in \mathcal{V}$ such that $\lambda_J(\pi_J[U] \setminus \pi_J[W_1]) > 0$ (414Ea). Now π_J is inverse-measure-preserving (254Oa), so

$$0 < \lambda\pi_J^{-1}[\pi_J[U] \setminus \pi_J[W_1]] = \lambda(\pi_J^{-1}[\pi_J[U]] \setminus \pi_J^{-1}[\pi_J[W_1]]) = \lambda(\pi_J^{-1}[\pi_J[U]] \setminus W_1),$$

because W_1 is determined by coordinates in J.

At this point note that U is of the form $\prod_{i\in I} G_i$, where $G_i \in \mathfrak{T}_i$ for each I, so we can express U as $U' \cap U''$, where $U' = \pi_J^{-1}[\pi_J[U]]$ and $U'' = \pi_{I\setminus J}^{-1}[\pi_{I\setminus J}[U]]$. U' is determined by coordinates in J and U'' is determined by coordinates in $I \setminus J$. In this case

$$\lambda(U \setminus W_1) = \lambda(U'' \cap U' \setminus W_1) = \lambda U'' \cdot \lambda(U' \setminus W_1),$$

because U'' is determined by coordinates in $I \setminus J$ and $U' \setminus W_1$ is determined by coordinates in J, and we can identify λ with the product $\lambda_{I\setminus J} \times \lambda_J$ (254N). But now recall that every μ_i is strictly positive. Since U is surely not empty, no G_i can be empty and no $\mu_i G_i$ can be 0. Consequently $\prod_{i\in I} \mu_i G_i > 0$ (because only finitely many terms in the product are less than 1) and $\lambda U > 0$; more to the point, $\lambda U'' > 0$. Since we chose U so that $\lambda(U' \setminus W_1) > 0$, we have $\lambda(U \setminus W_1) > 0$. But this contradicts the first sentence of (i) just above. **X**

(iii) Thus $\lambda_J\pi_J[W] = \lambda_J\pi_J[W_1]$. But this means that $\lambda\pi_J^{-1}[\pi_J[W]] = \lambda W_1$. Since λ is complete and $W_1 \subseteq W \subseteq \pi_J^{-1}[\pi_J[W]]$, λW is defined and equal to λW_1.

Taking $\langle V_n\rangle_{n\in\mathbb{N}}$ to be a sequence running over $\mathcal{V}_1 \cup \{\emptyset\}$, we have

$$\lambda W = \lambda W_1 = \lambda(\bigcup_{n\in\mathbb{N}} V_n) = \sup_{n\in\mathbb{N}} \lambda(\bigcup_{i\le n} V_i) \le \sup_{V\in\mathcal{V}^*} \lambda V \le \lambda W,$$

so $\lambda W = \sup_{V\in\mathcal{V}^*} \lambda V$, as required. **Q**

(d) Thus we see that λ is a topological measure. But it is also τ-additive. **P** If $\mathcal{W}$ is an upwards-directed family of open sets in X with union W^*, set

$$\mathcal{V} = \{U : U \in \mathcal{U}, \exists\, W \in \mathcal{W}, U \subseteq W\}.$$

Then $W^* = \bigcup \mathcal{V}$, so $\lambda W^* = \sup_{V\in\mathcal{V}^*} \lambda V$, where $\mathcal{V}^*$ is the set of finite unions of members of $\mathcal{V}$. But because $\mathcal{W}$ is upwards-directed, every member of $\mathcal{V}^*$ is included in some member of $\mathcal{W}$, so

$$\lambda W^* = \sup_{V\in\mathcal{V}^*} \lambda V \le \sup_{W\in\mathcal{W}} \lambda W \le \lambda W^*.$$

As $\mathcal{W}$ is arbitrary, λ is τ-additive. **Q**

(e) As in (b) above, we know that λ is a complete probability measure and is inner regular with respect to the closed sets, so it is a quasi-Radon measure. Because λ is inner regular with respect to the zero sets (412Ub), it is completion regular.

(f) Now suppose that $F \subseteq X$ is a closed self-supporting set. By 254Oc, there is a set $W \subseteq X$, determined by coordinates in some countable set $J \subseteq I$, such that $W \triangle F$ is negligible. **?** Suppose, if possible, that $x \in F$ and $y \in X \setminus F$ are such that $x{\restriction}J = y{\restriction}J$. Then there is a $U \in \mathcal{U}$ such that $y \in U \subseteq X \setminus F$. As in (b-ii) above, we can express U as $U' \cap U''$ where U', $U'' \in \mathcal{U}$ are determined by coordinates in J and $I \setminus J$ respectively. In this case,

$$\lambda(F\cap U)=\lambda(W\cap U)=\lambda(W\cap U')\cdot\lambda U''$$
$$=\lambda(F\cap U')\cdot\lambda U''>0,$$

because $x\in F\cap U'$ and F is self-supporting, while $U''\neq\emptyset$ and λ is strictly positive. But $F\cap U=\emptyset$, so this is impossible. **X**

Thus F is determined by coordinates in the countable set J. Consequently it is of the form $\pi_J^{-1}[\pi_J[F]]$. But $\pi_J[X\setminus F]$ is open (4A2B(f-i) again), so its complement $\pi_J[F]$ is closed. Now X_J is metrizable (4A2P(a-v)), so $\pi_J[F]$ is a zero set (4A2Lc) and F is a zero set (4A2C(b-iv)).

415F Corollary (a) If Y is either $[0,1[$ or $]0,1[$, endowed with Lebesgue measure, and I is any set, then Y^I, with the product topology and measure, is a quasi-Radon measure space.

(b) If $\langle\nu_i\rangle_{i\in I}$ is a family of probability distributions on $\mathbb{R}$, in the sense of §271 (that is, Radon probability measures), and every ν_i is strictly positive, then the product measure on $\mathbb{R}^I$ is a quasi-Radon measure.

Remark See also 416U below, and 453I, where there is an alternative proof of the main step in 415E, applicable to some further cases. Yet another approach, most immediately applicable to $[0,1[^I$, is in 443Xp. For further facts about these product measures, see §417, particularly 417M.

415G Comparing quasi-Radon measures: Proposition Let X be a topological space, and μ, ν two quasi-Radon measures on X. Then the following are equiveridical:

(i) $\mu F\le\nu F$ for every closed set $F\subseteq X$;

(ii) $\mu\le\nu$ in the sense of 234P.

If ν is locally finite, we can add

(iii) $\mu G\le\nu G$ for every open set $G\subseteq X$;

(iv) there is a base $\mathcal{U}$ for the topology of X such that $G\cup H\in\mathcal{U}$ for all G, $H\in\mathcal{U}$ and $\mu G\le\nu G$ for $G\in\mathcal{U}$.

proof (a) Of course (ii)⇒(i), and (i)⇒(ii) by 412Ma.

(b) Evidently (ii)⇒(iii), and (iii)⇒(iv) is trivial. If (iv) is true and $G\subseteq X$ is open, then $\mathcal{V}=\{V:V\in\mathcal{U},\,V\subseteq G\}$ is upwards-directed and has union G, so

$$\mu G=\sup_{V\in\mathcal{V}}\mu V\le\sup_{V\in\mathcal{V}}\nu V=\nu G.$$

Thus (iv)⇒(iii).

Now assume that ν is locally finite and that (iii) is true. **?** Suppose, if possible, that $F\subseteq X$ is a closed set such that $\nu F<\mu F$. Then $\mathcal{H}$, as defined in part (a) of the proof, is upwards-directed and has union X, so there is an $H\in\mathcal{H}$ such that $\nu F<\mu(F\cap H)$. Now there is a closed set $F'\subseteq H\setminus F$ such that

$$\nu F'>\nu(H\setminus F)-\mu(F\cap H)+\nu F\ge\nu H-\mu(F\cap H).$$

Set $G=H\setminus F'$, so that $F\cap H\subseteq G$ and

$$\nu G=\nu H-\nu F'<\mu(F\cap H)\le\mu G,$$

which is impossible. **X**

This shows that (provided that ν is locally finite) (iii)⇒(i).

415H Uniqueness of quasi-Radon measures: Proposition Let $(X,\mathfrak{T})$ be a topological space and μ, ν two quasi-Radon measures on X. Then the following are equiveridical:

(i) $\mu=\nu$;

(ii) $\mu F=\nu F$ for every closed set $F\subseteq X$;

(iii) $\mu G=\nu G$ for every open set $G\subseteq X$;

(iv) there is a base $\mathcal{U}$ for the topology of X such that $G\cup H\in\mathcal{U}$ for every G, $H\in\mathcal{U}$ and $\mu\restriction\mathcal{U}=\nu\restriction\mathcal{U}$;

(v) there is a base $\mathcal{U}$ for the topology of X such that $G\cap H\in\mathcal{U}$ for every G, $H\in\mathcal{U}$ and $\mu\restriction\mathcal{U}=\nu\restriction\mathcal{U}$.

proof Of course (i) implies all the others. (ii)⇒(i) is immediate from 415G (see also 412M). If (iii) is true, then, for any closed set $F\subseteq X$,

$$\begin{aligned}\mu F &= \sup\{\mu(G \cap F) : G \in \mathfrak{T},\, \mu G < \infty\}\\ &= \sup\{\mu G - \mu(G \setminus F) : G \in \mathfrak{T},\, \mu G < \infty\}\\ &= \sup\{\nu G - \nu(G \setminus F) : G \in \mathfrak{T},\, \nu G < \infty\} = \nu F;\end{aligned}$$

so (iii)⇒(ii). (iv)⇒(iii) by the argument of (iv)⇒(iii) in the proof of 415G.

Finally, suppose that (v) is true. Then $\mu(G_0 \cup \ldots \cup G_n) = \nu(G_0 \cup \ldots \cup G_n)$ for all $G_0, \ldots, G_n \in \mathcal{U}$. **P** Induce on n. For the inductive step to $n \ge 1$, if any G_i has infinite measure (for either measure) the result is trivial. Otherwise,

$$\begin{aligned}\mu(G_0 \cup \ldots \cup G_n) &= \mu(\bigcup_{i<n} G_i) + \mu G_n - \mu(G_n \cap \bigcup_{i<n} G_i)\\ &= \nu(\bigcup_{i<n} G_i) + \nu G_n - \nu(G_n \cap \bigcup_{i<n} G_i) = \nu(G_0 \cup \ldots \cup G_n). \text{ \textbf{Q}}\end{aligned}$$

So μ and ν agree on the base $\{G_0 \cup \ldots \cup G_n : G_0, \ldots, G_n \in \mathcal{U}\}$, and (iv) is true.

415I Proposition Let X be a completely regular topological space and μ, ν two quasi-Radon measures on X such that $\int f d\mu = \int f d\nu$ whenever $f : X \to \mathbb{R}$ is a bounded continuous function integrable with respect to both measures. Then $\mu = \nu$.

proof ? Otherwise, there is an open set $G \subseteq X$ such that $\mu G \ne \nu G$; suppose $\mu G < \nu G$. Because ν is effectively locally finite, there is an open set $G' \subseteq G$ such that $\mu G < \nu G' < \infty$. Now the cozero sets form a base for the topology of X, so $\mathcal{H} = \{H : H \subseteq G'$ is a cozero set$\}$ has union G'; as ν is τ-additive, there is an $H \in \mathcal{H}$ such that $\nu H > \mu G$. Express H as $\{x : g(x) > 0\}$ where $g : X \to [0, \infty[$ is continuous. For each $n \in \mathbb{N}$, set $f_n = ng \wedge \chi X$; then $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with limit χH, so there is an $n \in \mathbb{N}$ such that $\int f_n d\nu > \mu G \ge \int f_n d\mu$. But f_n is both μ-integrable and ν-integrable because μG and νH are both finite. **X**

415J Proposition Let X be a regular topological space, Y a subspace of X, and ν a quasi-Radon measure on Y. Then there is a quasi-Radon measure μ on X such that $\mu E = \nu(E \cap Y)$ whenever μ measures E, that is, Y has full outer measure in X and ν is the subspace measure on Y.

proof Write $\mathcal{B}$ for the Borel σ-algebra of X, and set $\mu_0 E = \nu(E \cap Y)$ for every $E \in \mathcal{B}$. Then it is easy to see that μ_0 is a τ-additive Borel measure on X. Moreover, μ_0 is effectively locally finite. **P** If $E \in \mathcal{B}$ and $\mu_0 E > 0$, there is a relatively open set $H \subseteq Y$ such that $\nu H < \infty$ and $\nu(H \cap E \cap Y) > 0$. Now H is of the form $G \cap Y$ where $G \subseteq X$ is open, and we have $\mu_0 G = \nu H < \infty$, $\mu_0(E \cap G) = \nu(H \cap E \cap Y) > 0$. **Q**

By 415Cb, the c.l.d. version μ of μ_0 is a quasi-Radon measure on X. Because μ_0 is semi-finite (411Gd), μ extends μ_0 (213Hc), and $\mu F = \mu_0 F = \nu(F \cap Y)$ for every closed $F \subseteq X$. In particular, $\mu F = 0$ whenever $F \subseteq X$ is closed and $F \cap Y = \emptyset$; as μ is inner regular with respect to the closed sets, Y has full outer measure in X. So if $H \subseteq Y$ is relatively closed,

$$\nu H = \nu(\bar{H} \cap Y) = \mu \bar{H} = \mu^*(\bar{H} \cap Y) = \mu_Y H$$

where μ_Y is the subspace measure on Y induced by μ. Now we know that μ_Y is a quasi-Radon measure on Y (415B); as it agrees with the quasi-Radon measure ν on the relatively closed subsets of Y, $\mu_Y = \nu$ (415H(ii)), as required.

415K I come now to a couple of basic results on the construction of quasi-Radon measures. The first follows 413K.

Theorem Let X be a topological space and $\mathcal{K}$ a family of closed subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $F \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $F \subseteq K$ is closed.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever $K, L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{K \in \mathcal{K}'} \phi_0 K = 0$ whenever $\mathcal{K}'$ is a non-empty downwards-directed subset of $\mathcal{K}$ with empty intersection,

(γ) whenever $K \in \mathcal{K}$ and $\phi_0 K > 0$, there is an open set G such that the supremum $\sup_{K' \in \mathcal{K}, K' \subseteq G} \phi_0 K'$ is finite, while $\phi_0 K' > 0$ for some $K' \in \mathcal{K}$ such that $K' \subseteq K \cap G$.

Then there is a unique quasi-Radon measure on X extending ϕ_0 and inner regular with respect to $\mathcal{K}$.

proof By 413K, there is a complete locally determined measure μ on X, inner regular with respect to $\mathcal{K}$, and extending ϕ_0; write Σ for the domain of μ. If $F \subseteq X$ is closed, then $K \cap F \in \mathcal{K} \subseteq \Sigma$ for every $K \in \mathcal{K}$, so $F \in \Sigma$, by 413F(ii); accordingly every open set is measurable. Because μ is inner regular with respect to $\mathcal{K}$ it is surely inner regular with respect to the closed sets. If $E \in \Sigma$ and $\mu E > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$; now (γ) tells us that there is an open set G such that $\mu G < \infty$ and $\mu(G \cap K) > 0$, so that $\mu(G \cap E) > 0$. As E is arbitrary, μ is effectively locally finite. Now suppose that $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H, and that $\gamma < \mu H$. Then there is a $K \in \mathcal{K}$ such that $K \subseteq H$ and $\mu K > \gamma$. Applying the hypothesis (β) to $\mathcal{K}' = \{K \setminus G : G \in \mathcal{G}\}$, we see that $\inf_{G \in \mathcal{G}} \mu(K \setminus G) = 0$, so that

$$\sup_{G \in \mathcal{G}} \mu G \ge \sup_{G \in \mathcal{G}} \mu(K \cap G) = \mu K \ge \gamma.$$

As $\mathcal{G}$ and γ are arbitrary, μ is τ-additive. So μ is a quasi-Radon measure.

415L Proposition Let (X, Σ_0, μ_0) be a measure space and $\mathfrak{T}$ a topology on X such that Σ_0 includes a base for $\mathfrak{T}$ and μ_0 is τ-additive, effectively locally finite and inner regular with respect to the closed sets. Then μ_0 has a unique extension to a quasi-Radon measure μ on X. Moreover,

(i) $\mu F = \mu_0^* F$ whenever $F \subseteq X$ is closed and $\mu_0^* F < \infty$,

(ii) $\mu G = (\mu_0)_* G$ whenever $G \subseteq X$ is open,

(iii) the embedding $\Sigma_0 \subsetneq \Sigma$ identifies the measure algebra $(\mathfrak{A}_0, \bar\mu_0)$ of μ_0 with an order-dense subalgebra of the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ, so that the subrings $\mathfrak{A}_0^f$, $\mathfrak{A}^f$ of elements of finite measure coincide, and $L^p(\mu_0)$ may be identified with $L^p(\mu)$ for $1 \le p < \infty$,

(iv) whenever $E \in \Sigma$ and $\mu E < \infty$, there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$,

(v) for every μ-integrable real-valued function f there is a μ_0-integrable function g such that $f = g$ μ-a.e.

If μ_0 is complete and locally determined, then we have

(i)$'$ $\mu F = \mu_0^* F$ for every closed $F \subseteq X$.

If μ_0 is localizable, then we have

(iii)$'$ $\mathfrak{A}_0 = \mathfrak{A}$, so that $L^0(\mu) \cong L^0(\mu_0)$ and $L^\infty(\mu) \cong L^\infty(\mu_0)$,

(iv)$'$ for every $E \in \Sigma$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$,

(v)$'$ for every Σ-measurable real-valued function f there is a Σ_0-measurable real-valued function g such that $f = g$ μ-a.e.

proof (a) Let $\mathcal{K}$ be the set of closed subsets of X of finite outer measure for μ_0. Note that μ_0 is inner regular with respect to $\mathcal{K}$, because it is inner regular with respect to the closed sets and also with respect to the sets of finite measure.

It is obvious from its definition that $\mathcal{K}$ satisfies (†) and (‡) of 415K. For $K \in \mathcal{K}$, set $\phi_0 K = \mu_0^* K$. Then ϕ_0 satisfies (α)-(γ) of 415K.

P (α) If $K, L \in \mathcal{K}$ and $L \subseteq K$, take measurable envelopes $E_0, E_1 \in \Sigma_0$ of K, L respectively. (i) Let $\epsilon > 0$. Because μ_0 is inner regular with respect to the closed sets, there is a closed set $F \in \Sigma_0$ such that $F \subseteq E_0 \setminus E_1$ and $\mu_0 F \ge \mu_0(E_0 \setminus E_1) - \epsilon$. Set $K' = F \cap K$. Then $K' \in \mathcal{K}$ and

$$\phi_0 K' = \mu_0^*(F \cap K) = \mu_0(F \cap E_0) = \mu_0 F \ge \mu_0 E_0 - \mu_0 E_1 - \epsilon = \phi_0 K - \phi_0 L - \epsilon.$$

As ϵ is arbitrary, we have

$$\phi_0 K \le \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}.$$

(ii) On the other hand, **?** suppose, if possible, that there is a closed set $K' \subseteq K \setminus L$ such that $\mu_0^*L + \mu_0^*K' > \mu_0^*K$. Let E_2 be a measurable envelope of K', so that $\mu_0 E_1 + \mu_0 E_2 > \mu_0 E_0$; since

$$\mu_0(E_1 \setminus E_0) = \mu_0^*(L \setminus E_0) = \mu_0^*\emptyset = 0, \quad \mu_0(E_2 \setminus E_0) = \mu_0^*(K' \setminus E_0) = 0,$$

$\mu_0(E_1 \cap E_2) > 0$. Because μ_0 is effectively locally finite, there is a measurable open set G_0, of finite measure, such that $\mu_0(G_0 \cap E_1 \cap E_2) > 0$. Set

$$\mathcal{G} = \{G \cup G' : G, G' \in \Sigma_0 \cap \mathfrak{T}, G \subseteq G_0 \setminus L, G' \subseteq G_0 \setminus K'\}.$$

Then $\mathcal{G}$ is an upwards-directed family of measurable open sets, and because Σ_0 includes a base for the topology of X, its union is $(G_0 \setminus L) \cup (G_0 \setminus K') = G_0$. So there is an $H \in \mathcal{G}$ such that $\mu_0 H > \mu_0 G_0 - \mu_0(E_1 \cap E_2)$, that is, there are open sets $G, G' \in \Sigma_0$ such that $G \subseteq G_0 \setminus L$, $G' \subseteq G_0 \setminus K'$ and $\mu_0((G \cup G') \cap E_1 \cap E_2)) > 0$. But we must have

$$\mu_0(G \cap E_1) = \mu_0^*(G \cap L) = 0, \quad \mu_0(G' \cap E_2) = \mu_0^*(G' \cap K') = 0,$$

so this is impossible. **X**

Accordingly

$$\phi_0 K \ge \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K}, K' \subseteq K \setminus L\},$$

so that ϕ_0 satisfies condition (α) of 415K.

(β) Let $\mathcal{K}' \subseteq \mathcal{K}$ be a non-empty downwards-directed family with empty intersection. Fix $K_0 \in \mathcal{K}'$ and $\epsilon > 0$. Let E_0 be a measurable envelope of K_0 and G_0 a measurable open set of finite measure such that $\mu_0(G_0 \cap E_0) \ge \mu_0 E_0 - \epsilon$. Then

$$\mathcal{G} = \{G : G \in \Sigma_0 \cap \mathfrak{T}, G \subseteq G_0 \setminus K \text{ for some } K \in \mathcal{K}' \text{ such that } K \subseteq K_0\}$$

is an upwards-directed family of measurable open sets, and its union is $G_0 \setminus \bigcap \mathcal{K}' = G_0$, again because Σ_0 includes a base for the topology $\mathfrak{T}$. So there is a $G \in \mathcal{G}$ such that $\mu_0 G \ge \mu_0 G_0 - \epsilon$. Let $K \in \mathcal{K}'$ be such that $K \subseteq K_0$ and $G \cap K = \emptyset$; then

$$\phi_0 K = \mu_0^* K \le \mu_0(E_0 \setminus G) \le \mu_0(E_0 \setminus G_0) + \mu_0(G_0 \setminus G) \le 2\epsilon.$$

As ϵ is arbitrary, $\inf_{K \in \mathcal{K}'} \phi_0 K = 0$.

(γ) If $K \in \mathcal{K}$ and $\phi_0 K > 0$, let E_0 be a measurable envelope of K. Then there is a measurable open set G of finite measure such that $\mu_0(G \cap E_0) > 0$. Of course $\sup_{K' \in \mathcal{K}, K' \subseteq G} \phi_0 K' \le \mu_0 G < \infty$; but also there is a measurable closed set $K' \subseteq G \cap E_0$ such that $\mu_0 K' > 0$, in which case $\phi_0(K \cap K') = \mu_0(E_0 \cap K') > 0$. So ϕ_0 satisfies condition (γ). **Q**

(b) By 415K, ϕ_0 has an extension to a quasi-Radon measure μ on X which is inner regular with respect to $\mathcal{K}$. Write Σ for the domain of μ. Note that, for $K \in \mathcal{K}$,

$$\mu K = \phi_0 K = \mu_0^* K,$$

so we can already be sure that the conclusion (i) of this theorem is satisfied. Now μ extends μ_0.

P (i) Take any $K \in \mathcal{K}$. Let $E_0 \in \Sigma_0$ be a measurable envelope of K for the measure μ_0. If $E \in \Sigma_0$, then surely

$$\begin{aligned}\mu_*(K \cap E) &= \sup\{\mu K' : K' \in \mathcal{K}, K' \subseteq K \cap E\} \\ &= \sup\{\mu_0^* K' : K' \in \mathcal{K}, K' \subseteq K \cap E\} \le \mu_0^*(K \cap E).\end{aligned}$$

On the other hand, given $\gamma < \mu_0^*(K \cap E) = \mu_0(E_0 \cap E)$, there is a closed set $F \in \Sigma_0$ such that $F \subseteq E_0 \cap E$ and $\mu_0 F \ge \gamma$, so that

$$\mu_*(K \cap E) \ge \mu(K \cap F) = \mu_0^*(K \cap F) = \mu_0(E_0 \cap F) \ge \gamma.$$

Thus $\mu_*(K \cap E) = \mu_0^*(K \cap E)$ for every $K \in \mathcal{K}$ and $E \in \Sigma_0$.

(ii) If $K \in \mathcal{K}$ and $E \in \Sigma_0$ then

$$\mu_*(K \cap E) + \mu_*(K \setminus E) = \mu_0^*(K \cap E) + \mu_0^*(K \setminus E) = \mu_0^* K = \mu K.$$

Because μ is complete and locally determined and inner regular with respect to $\mathcal{K}$, $E \in \Sigma$ (413F(iv)). Thus $\Sigma_0 \subseteq \Sigma$.

(iii) For any $E \in \Sigma_0$, we now have

$$\begin{aligned}\mu E &= \sup\{\mu K : K \in \mathcal{K},\, K \subseteq E\} = \sup\{\mu_0^* K : K \in \mathcal{K},\, K \subseteq E\}\\ &\le \mu_0 E = \sup\{\mu_0 K : K \in \mathcal{K} \cap \Sigma_0,\, K \subseteq E\} \le \mu E.\end{aligned}$$

As E is arbitrary, μ extends μ_0. **Q**

(c) Because $\Sigma_0 \cap \mathfrak{T}$ is a base for $\mathfrak{T}$, closed under finite unions, μ is unique, by 415H(iv).

(d) Now for the conditions (i)-(v). I have already noted that (i) is guaranteed by the construction. Concerning (ii), if $G \subseteq X$ is open, we surely have $(\mu_0)_* G \le \mu_* G = \mu G$ because μ extends μ_0. On the other hand, writing $\mathcal{G} = \{G' : G' \in \Sigma_0 \cap \mathfrak{T},\, G' \subseteq G\}$, $\mathcal{G}$ is upwards-directed and has union G, so

$$\mu G = \sup_{G' \in \mathcal{G}} \mu G' = \sup_{G' \in \mathcal{G}} \mu_0 G' \le (\mu_0)_* G.$$

So (ii) is true.

Because μ extends μ_0, the embedding $\Sigma_0 \subsetneqq \Sigma$ corresponds to a measure-preserving embedding of $\mathfrak{A}_0$ as a σ-subalgebra of $\mathfrak{A}$. To see that $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$, take any non-zero $a \in \mathfrak{A}$. This is expressible as $E^\bullet$ for some $E \in \Sigma$ with $\mu E > 0$. Now there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. There is an $E_0 \in \Sigma_0$ which is a measurable envelope for K with respect to μ_0, so that

$$\mu E_0 = \mu_0 E_0 = \mu_0^* K = \mu K.$$

But this means that

$$0 \ne E_0^\bullet = K^\bullet \subseteq E^\bullet = a$$

in $\mathfrak{A}$, while $E_0^\bullet \in \mathfrak{A}_0$. As a is arbitrary, $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$.

If $a \in \mathfrak{A}^f$, then $B = \{b : b \in \mathfrak{A}_0,\, b \subseteq a\}$ is upwards-directed and $\sup_{b \in B} \bar{\mu}_0 b \le \bar{\mu} a$ is finite; accordingly B has a supremum in $\mathfrak{A}_0$ (321C), which must also be its supremum in $\mathfrak{A}$, which is a (313O, 313K). So $a \in \mathfrak{A}_0$. Thus $\mathfrak{A}^f$ can be identified with $\mathfrak{A}_0^f$. But this means that, for any $p \in [1, \infty[$, $L^p(\mu) \cong L^p(\mathfrak{A}, \bar{\mu})$ is identified with $L^p(\mathfrak{A}_0, \bar{\mu}_0) \cong L^p(\mu)$ (366H(d-iv)). This proves (iii).

Of course (iv) and (v) are just translations of this. If $E \in \Sigma$ and $\mu E < \infty$, then $E^\bullet \in \mathfrak{A}^f \subseteq \mathfrak{A}_0$, that is, there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$. If f is μ-integrable, then $f^\bullet \in L^1(\mu) = L^1(\mu_0)$, that is, there is a μ_0-integrable function f_0 such that $f = f_0$ μ-a.e.

(e) If μ_0 is complete and locally determined and $F \subseteq X$ is an arbitrary closed set, then

$$\mu_0^* F = \sup_{K \in \mathcal{K}} \mu_0^*(F \cap K) = \sup_{K \in \mathcal{K}} \mu(F \cap K) = \sup_{K \in \mathcal{K}, K \subseteq F} \mu K = \mu F$$

by 412Jc, because μ and μ_0 are both inner regular with respect to $\mathcal{K}$.

(f) If μ_0 is localizable, $\mathfrak{A}_0$ is Dedekind complete; as it is order-dense in $\mathfrak{A}$, the two must coincide (314Ib). Consequently

$$L^0(\mu) \cong L^0(\mathfrak{A}) = L^0(\mathfrak{A}_0) \cong L^0(\mu_0), \quad L^\infty(\mu) \cong L^\infty(\mathfrak{A}) = L^\infty(\mathfrak{A}_0) \cong L^\infty(\mu_0).$$

So (iii)$'$ is true; now (iv)$'$ and (v)$'$ follow at once.

415M Corollary Let $(X, \mathfrak{T})$ be a regular topological space and μ_0 an effectively locally finite τ-additive measure on X, defined on the σ-algebra generated by a base for $\mathfrak{T}$. Then μ_0 has a unique extension to a quasi-Radon measure on X.

proof By 414Mb, μ_0 is inner regular with respect to the closed sets. So 415L gives the result.

415N Corollary Let $(X, \mathfrak{T})$ be a completely regular topological space, and μ_0 a τ-additive effectively locally finite Baire measure on X. Then μ_0 has a unique extension to a quasi-Radon measure on X.

proof This is a special case of 415M, because the domain $\mathcal{B}\mathfrak{a}(X)$ of μ_0 is generated by the family of cozero sets, which form a base for $\mathfrak{T}$ (4A2Fc).

415O Proposition (a) Let $(X,\mathfrak{T})$ be a topological space, and μ, ν two quasi-Radon measures on X. Then ν is an indefinite-integral measure over μ (definition: 234J) iff $\nu F=0$ whenever $F\subseteq X$ is closed and $\mu F=0$.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, and ν an indefinite-integral measure over μ. If ν is effectively locally finite it is a quasi-Radon measure.

proof (a) If ν is an indefinite-integral measure over μ, then of course it is zero on all μ-negligible closed sets. So let us suppose that the condition is satisfied. Write $\Sigma=\operatorname{dom}\mu$ and $\mathrm{T}=\operatorname{dom}\nu$.

(i) If $E\subseteq X$ is a μ-negligible Borel set it is ν-negligible, because every closed subset of E must be μ-negligible, therefore ν-negligible, and ν is inner regular with respect to the closed sets. In particular, taking U^* to be the union of the family $\mathcal{U}=\{U:U\in\mathfrak{T},\,\mu U<\infty\}$, $\nu(X\setminus U^*)=\mu(X\setminus U^*)=0$ because μ is effectively locally finite. Also, of course, taking V^* to be the union of the family $\mathcal{V}=\{V:V\in\mathfrak{T},\,\nu V<\infty\}$, $\nu(X\setminus V^*)=0$ because ν is effectively locally finite. Setting $\mathcal{G}=\mathcal{U}\cap\mathcal{V}$ and $G^*=\bigcup\mathcal{G}$, we have $G^*=U^*\cap V^*$, so G^* is ν-conegligible.

(ii) In fact, every μ-negligible set E is ν-negligible. **P?** Otherwise, $\nu^*(E\cap G^*)>0$. Because the subspace measure ν_E is quasi-Radon (415B), there is a $G\in\mathcal{G}$ such that $\nu^*(E\cap G)>0$. But there is an F_σ set $H\subseteq G\setminus E$ such that $\mu H=\mu(G\setminus E)$, and now $E\cap G$ is included in the μ-negligible Borel set $G\setminus H$, so that $\nu(E\cap G)=\nu(G\setminus H)=0$. **XQ**

(iii) Let $\mathcal{K}$ be the family of closed subsets F of X such that either F is included in some member of $\mathcal{G}$ or $F\cap G^*=\emptyset$. If $E\in\operatorname{dom}\mu$ and $\mu E>0$, then there is an $F\in\mathcal{K}$ such that $F\subseteq E$ and $\mu F>0$. **P** If $\mu(E\setminus G^*)>0$ take any closed set $F\subseteq E\setminus G^*$ with $\mu F>0$. Otherwise, $\mu(E\cap G^*)>0$. Because the subspace measure μ_E is quasi-Radon, there is a $G\in\mathcal{G}$ such that $\mu(E\cap G)>0$; and now we can find a closed set $F\subseteq E\cap G$ with $\mu F>0$, and $F\in\mathcal{K}$. **Q**

(iv) By 412Ia, there is a decomposition $\langle X_i\rangle_{i\in I}$ for μ such that every X_i except perhaps one belongs to $\mathcal{K}$ and that exceptional one, if any, is μ-negligible. Now $\langle X_i\rangle_{i\in I}$ is a decomposition for ν. **P** Every X_i is measured by ν because it is either closed or μ-negligible, and of finite measure for ν because it is either ν-negligible or included in a member of $\mathcal{G}$. If $E\subseteq X$ and $\nu E>0$, then $\nu(E\cap G^*)>0$, so there must be some $G\in\mathcal{G}$ such that $\nu(E\cap G)>0$. Now $J=\{i:i\in I,\,\mu(X_i\cap G)>0\}$ is countable, and $\nu(G\setminus\bigcup_{i\in J}X_i)=\mu(G\setminus\bigcup_{i\in J}X_i)=0$, so there is an $i\in J$ such that $\nu(X_i\cap E)>0$. By 213Ob, $\langle X_i\rangle_{i\in I}$ is a decomposition for ν. **Q**

(v) It follows that $\Sigma\subseteq\mathrm{T}$. **P** If $E\in\Sigma$, then for every $i\in I$ there is an F_σ set $H\subseteq E\cap X_i$ such that $E\cap X_i\setminus H$ is μ-negligible, therefore ν-negligible, and $E\cap X_i\in\mathrm{T}$. As i is arbitrary, $E\in\mathrm{T}$. **Q** In fact, ν is the completion of $\nu{\restriction}\Sigma$. **P** If $F\in\mathrm{T}$, then for every $i\in I$ there is an F_σ set $H_i\subseteq F\cap X_i$ such that $F\cap X_i\setminus H_i$ is ν-negligible. Set $H=\bigcup_{i\in I}H_i$; because $H\cap X_i=H_i$ belongs to Σ for every i, $H\in\Sigma$; and $\nu(F\setminus H)=\sum_{i\in I}\nu(F\cap X_i\setminus H)=0$. Similarly, there is an $H'\in\Sigma$ such that $H'\subseteq X\setminus F$ and $\nu((X\setminus F)\setminus H')=0$, so that $H\subseteq F\subseteq X\setminus H'$ and $\nu((X\setminus H')\setminus H)=0$. So F is measured by the completion of $\nu{\restriction}\Sigma$. Since ν itself is complete, it must be the completion of $\nu{\restriction}\Sigma$. **Q**

(vi) By (iv), ν is inner regular with respect to $\{E:E\in\Sigma,\,\mu E<\infty\}$. By 234O, ν is an indefinite-integral measure over μ.

(b) Let $f\in\mathcal{L}^0(\mu)$ be a non-negative function such that $\nu F=\int f\times\chi F\,d\mu$ whenever this is defined. Because μ is complete and locally determined, so is ν. Because μ is an effectively locally finite τ-additive topological measure, ν is a τ-additive topological measure (414H). Because μ is inner regular with respect to the closed sets, so is ν (412Q). Since we are assuming in the hypotheses that ν is effectively locally finite, it is a quasi-Radon measure.

415P Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space.

(a) Suppose that $(X,\mathfrak{T})$ is completely regular. If $1\le p<\infty$ and $f\in\mathcal{L}^p(\mu)$, then for any $\epsilon>0$ there is a bounded continuous function $g:X\to\mathbb{R}$ such that $\mu\{x:g(x)\ne0\}<\infty$ and $\|f-g\|_p\le\epsilon$.

(b) Suppose that $(X,\mathfrak{T})$ is regular and Lindelöf. Let $f\in\mathcal{L}^0(\mu)$ be locally integrable. Then for any $\epsilon>0$ there is a continuous function $g:X\to\mathbb{R}$ such that $\|f-g\|_1\le\epsilon$.

proof (a) Write $\mathfrak{C}$ for the set of bounded continuous functions $g : X \to \mathbb{R}$ such that $\{x : g(x) \neq 0\}$ has finite measure. Then $\mathfrak{C}$ is a linear subspace of $\mathbb{R}^X$ included in $\mathcal{L}^p = \mathcal{L}^p(\mu)$. Let $\mathfrak{U}$ be the closure of $\mathfrak{C}$ in $\mathcal{L}^p$, that is, the set of $h \in \mathcal{L}^p$ such that for every $\epsilon > 0$ there is a $g \in \mathfrak{C}$ such that $\|h - g\|_p \leq \epsilon$. Then $\mathfrak{U}$ is closed under addition and scalar multiplication. Also $\chi E \in \mathfrak{U}$ whenever $\mu E < \infty$. **P** Let $\epsilon > 0$. Set $\delta = \frac{1}{4}\epsilon^{1/p}$. Write $\mathcal{G}$ for the family of open sets of finite measure. Because μ is effectively locally finite, there is a $G \in \mathcal{G}$ such that $\mu(E \setminus G) \leq \delta$ (412Aa). Let $F \subseteq G \setminus E$ be a closed set such that $\mu F \geq \mu(G \setminus E) - \delta$; then $\mu(E\triangle(G \setminus F)) \leq 2\delta$. Write $\mathcal{H}$ for the family of cozero sets. Because $\mathfrak{T}$ is completely regular, $\mathcal{H}$ is a base for $\mathfrak{T}$; because $\mathcal{H}$ is closed under finite unions (4A2C(b-iii)) and μ is τ-additive, there is an $H \in \mathcal{H}$ such that $H \subseteq G \setminus F$ and $\mu H \geq \mu(G \setminus F) - \delta$, so that $\mu(E\triangle H) \leq 3\delta$. Express H as $\{x : g(x) > 0\}$ where $g : X \to \mathbb{R}$ is a continuous function. For each $n \in \mathbb{N}$, set $g_n = ng \wedge \chi X \in \mathfrak{C}$; then

$$|\chi E - g_n|^p \leq \chi(E\triangle H) + (\chi H - g_n)^p$$

for every n, so

$$\int |\chi E - g_n|^p \leq \mu(E\triangle H) + \int (\chi H - g_n)^p \to \mu(E\triangle H)$$

as $n \to \infty$, because $g_n \to \chi H$. So there is an $n \in \mathbb{N}$ such that $\int |\chi E - g_n|^p \leq 4\delta$, that is, $\|\chi E - g_n\|_p \leq \epsilon$. As ϵ is arbitrary, $\chi E \in \mathfrak{U}$. **Q**

Accordingly every simple function belongs to $\mathfrak{U}$. But if $f \in \mathcal{L}^p$ and $\epsilon > 0$, there is a simple function h such that $\|f - h\|_p \leq \frac{1}{2}\epsilon$ (244Ha); now there is a $g \in \mathfrak{C}$ such that $\|h - g\|_p \leq \frac{1}{2}\epsilon$ and $\|f - g\|_p \leq \epsilon$, as claimed.

(b) This time, write $\mathcal{G}$ for the family of open subsets of X such that $\int_G f$ is finite, so that $\mathcal{G}$ is an open cover of X. As X is paracompact (4A2H(b-i)), there is a locally finite family $\mathcal{G}_0 \subseteq \mathcal{G}$ covering X, which must be countable (4A2H(b-ii)).

Let $\langle\epsilon_G\rangle_{G\in\mathcal{G}_0}$ be a family of strictly positive real numbers such that $\sum_{G\in\mathcal{G}_0} \epsilon_G \leq \epsilon$ (4A1P). Since X is completely regular (4A2H(b-i)), we can apply (a) to see that, for each $G \in \mathcal{G}_0$, there is a continuous function $g_G : X \to \mathbb{R}$ such that $\int |g_G - f \times \chi G| \leq \epsilon_G$. Next, because X is normal (4A2H(b-i)), there is a family $\langle h_G\rangle_{G\in\mathcal{G}_0}$ of continuous functions from X to $[0, 1]$ such that $h_G \leq \chi G$ for every $G \in \mathcal{G}_0$ and $\sum_{G\in\mathcal{G}_0} h_G(x) = 1$ for every $x \in X$ (4A2F(d-viii)).

Set $g(x) = \sum_{G\in\mathcal{G}_0} g_G(x)h_G(x)$ for every $x \in X$. Because $\mathcal{G}_0$ is locally finite, $g : X \to \mathbb{R}$ is continuous (4A2Bh). Now

$$\begin{aligned}\int |f - g| &= \int |\sum_{G\in\mathcal{G}_0} (f - g_G) \times h_G| \leq \sum_{G\in\mathcal{G}_0} \int |(f - g_G) \times h_G| \\ &\leq \sum_{G\in\mathcal{G}_0} \int_G |f - g_G| \leq \sum_{G\in\mathcal{G}_0} \epsilon_G \leq \epsilon,\end{aligned}$$

as required.

415Q Recall (411P) that if $(\mathfrak{A}, \bar\mu)$ is a localizable measure algebra, with Stone space $(Z, \mathfrak{S}, \mathrm{T}, \nu)$, then ν is a strictly positive completion regular quasi-Radon measure, inner regular with respect to the open-and-closed sets (which are all compact). Moreover, subsets of Z are negligible iff they are nowhere dense, and every measurable set differs by a nowhere dense set from an open-and-closed set. The following construction is primarily important for Radon measure spaces (see 416V), but is also of interest for general quasi-Radon spaces.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(\mathfrak{A}, \bar\mu)$ its measure algebra. Let $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ be the Stone space of $(\mathfrak{A}, \bar\mu)$. For $E \in \Sigma$ let $E^* \subseteq Z$ be the open-and-closed set corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Define $R \subseteq Z \times X$ by saying that $(z, x) \in R$ iff $x \in F$ whenever $F \subseteq X$ is closed and $z \in F^*$. Set $Q = R^{-1}[X]$.

(a) R is a closed subset of $Z \times X$.

(b) For any $E \in \Sigma$, $R[E^*]$ is the smallest closed set such that $\mu(E \setminus R[E^*]) = 0$. In particular, if $F \subseteq X$ is closed then $R[F^*]$ is the self-supporting closed set included in F such that $\mu(F \setminus R[F^*]) = 0$; and $R[Z]$ is the support of μ.

(c) Q has full outer measure for ν.

(d) For any $E \in \Sigma$, $R^{-1}[E] \triangle (Q \cap E^*)$ is negligible; consequently $\nu^* R^{-1}[E] = \mu E$ and $R^{-1}[E] \cap R^{-1}[X \setminus E]$ is negligible.

(e) For any $A \subseteq X$, $\nu^* R^{-1}[A] = \mu^* A$.

(f) If $(X, \mathfrak{T})$ is regular, then $R^{-1}[G]$ is relatively open in Q for every open set $G \subseteq X$, $R^{-1}[F]$ is relatively closed in Q for every closed set $F \subseteq X$ and $R^{-1}[X \setminus E] = Q \setminus R^{-1}[E]$ for every Borel set $E \subseteq X$.

proof (a)

$$R = \bigcap_{F \subseteq X \text{ is closed}} ((Z \setminus F^*) \times X) \cup (Z \times F)$$

is an intersection of closed sets, therefore closed.

(b) Let $\mathcal{G}$ be the family of open sets $G \subseteq X$ such that $\mu(E \cap G) = 0$, and $G_0 = \bigcup \mathcal{G}$; then $G_0 \in \mathcal{G}$ (414Ea). Set $F_0 = X \setminus G_0$, so that F_0 is the smallest closed set such that $E \setminus F_0$ is negligible, and $F_0^* \supseteq E^*$. If $(z, x) \in R$ and $z \in E^*$ we must have $x \in F_0$. Thus $R[E^*] \subseteq F_0$. On the other hand, if $x \in F_0$, and G is an open set containing x, then $G \notin \mathcal{G}$ so $\mu(G \cap E) > 0$ and $(E \cap G)^* \neq \emptyset$. Accordingly $\{(G \cap E)^* : x \in G \in \mathfrak{T}\}$ is a downwards-directed family of non-empty open-and-closed sets in the compact space Z and has non-empty intersection, containing a point z say. If $H \subseteq X$ is closed and $z \in H^*$, then $X \setminus H$ is open and $z \notin (X \setminus H)^*$, so x cannot belong to $X \setminus H$, that is, $x \in H$; as H is arbitrary, $(z, x) \in R$ and $x \in R[E^*]$; as x is arbitrary, $R[E^*] = F_0$, as claimed.

Of course, when E is actually closed, $R[E^*] = F_0 \subseteq E$. Taking $E = X$ we see that $R[Z] = R[X^*]$ is the support of μ.

(c) If $W \in \mathrm{T}$ and $\nu W > 0$, there is a non-empty open-and-closed set $U \subseteq W$, by 322Ra, which must be of the form E^* for some $E \in \Sigma$. By (b), $R[E^*]$ cannot be empty; but $E^* \subseteq W$, so $R[W] \neq \emptyset$, that is, $W \cap Q \neq \emptyset$. As W is arbitrary, Q has full outer measure in Z.

(d)(i) Let $\mathcal{F}$ be the set of closed subsets of X included in E. Then $\sup_{F \in \mathcal{F}} F^\bullet = E^\bullet$ in $\mathfrak{A}$ (412N), so $E^* \setminus \bigcup_{F \in \mathcal{F}} F^*$ is nowhere dense and negligible. Now for each $F \in \mathcal{F}$, $R[F^*] \subseteq F$, so $Q \cap F^* \subseteq R^{-1}[F] \subseteq R^{-1}[E]$. Accordingly

$$Q \cap E^* \setminus R^{-1}[E] \subseteq E^* \setminus \bigcup_{F \in \mathcal{F}} F^*$$

is nowhere dense and negligible.

(ii) ? Suppose, if possible, that $\nu^*(R^{-1}[E] \setminus E^*) > 0$. Then there is an open-and-closed set U of finite measure such that $\nu^*(R^{-1}[E] \cap U \setminus E^*) > 0$ (use 412Jc). Express U as H^*, where $\mu H < \infty$, and let $F \subseteq H \setminus E$ be a closed set such that $\mu((H \setminus E) \setminus F) < \nu^*(R^{-1}[E] \cap H^* \setminus E^*)$. Then we must have $\nu^*(R^{-1}[E] \cap F^*) > 0$. But $R[F^*] \subseteq F \subseteq X \setminus E$ so $F^* \cap R^{-1}[E] = \emptyset$, which is impossible. **X**

(iii) Putting these together, $(Q \cap E^*) \triangle R^{-1}[E]$ is negligible.

(iv) It follows at once that (because Z is a measurable envelope for Q)

$$\nu^* R^{-1}[E] = \nu^*(Q \cap E^*) = \nu E^* = \mu E.$$

Moreover, applying the result to $X \setminus E$,

$$R^{-1}[X \setminus E] \cap R^{-1}[E] \subseteq (R^{-1}[X \setminus E] \triangle (Q \cap (X \setminus E)^*)) \cup (R^{-1}[E] \triangle (Q \cap E^*))$$

is negligible.

(e)(i) Take $E \in \Sigma$ such that $A \subseteq E$ and $\mu E = \mu^* A$; then $R^{-1}[A] \subseteq R^{-1}[E]$, so

$$\nu^* R^{-1}[A] \leq \nu^* R^{-1}[E] = \mu E = \mu^* A.$$

(ii) ? Suppose, if possible, that $\nu^* R^{-1}[A] < \mu^* A$. Let $W \in \mathrm{T}$ be a measurable envelope of $R^{-1}[A]$, and take $F \in \Sigma$ such that $\nu(W \triangle F^*) = 0$. Since

$$\mu F = \nu F^* = \nu W < \mu^* A,$$

$\mu^*(A \setminus F) > 0$; let G be a measurable envelope of $A \setminus F$ disjoint from F (213L). Then $G^* \cap F^* = \emptyset$ so

$$\nu(G^* \setminus W) = \nu G^* = \mu G > 0$$

and there is a non-empty open-and-closed $V \subseteq G^* \setminus W$; let $H \in \Sigma$ be such that $H \subseteq G$ and $V = H^*$. In this case, $R[V]$ is closed and $\mu(H \setminus R[V]) = 0$, by (b), so that $H \cap R[V]$ is measurable, not negligible, and included in G. But $H \cap R[V] \cap A$ is empty, because $V \cap R^{-1}[A]$ is empty, so $\mu^*(H \cap R[V] \cap A) < \mu(H \cap R[V])$, and G cannot be a measurable envelope of $A \setminus F$. **X**

Thus $\nu^* R^{-1}[A] = \mu^* A$, as claimed.

(f) Suppose now that $(X, \mathfrak{T})$ is regular.

(i) If $G \subseteq X$ is open, $R^{-1}[G] \cap R^{-1}[X \setminus G] = \emptyset$. **P** If $z \in R^{-1}[G]$, then there is an $x \in G$ such that $(z, x) \in R$. Let H be an open set containing x such that $\overline{H} \subseteq G$. Then $x \notin X \setminus H$ so $z \notin (X \setminus H)^*$, that is, $z \in H^*$. But

$$R[H^*] \subseteq R[\overline{H}^*] \subseteq \overline{H} \subseteq G,$$

so $H^* \cap R^{-1}[X \setminus G] = \emptyset$ and $z \notin R^{-1}[X \setminus G]$. **Q**

(ii) It is easy to check that

$$\begin{aligned} \mathcal{E} &= \{E : E \subseteq X,\ R^{-1}[E] \cap R^{-1}[X \setminus E] = \emptyset\} \\ &= \{E : E \subseteq X,\ R^{-1}[X \setminus E] = Q \setminus R^{-1}[E]\} \end{aligned}$$

is a σ-algebra of subsets of X (indeed, an algebra closed under arbitrary unions), just because $R \subseteq Z \times X$ and $R^{-1}[X] = Q$. Because it contains all open sets, $\mathcal{E}$ must contain all Borel sets.

(iii) Now suppose once again that $G \subseteq X$ is open and that $z \in R^{-1}[G]$. As in (i) above, there is an open set $H \subseteq G$ such that $z \in H^* \subseteq Z \setminus R^{-1}[X \setminus G]$, so that $z \in H^* \cap Q \subseteq R^{-1}[G]$. As z is arbitrary, $R^{-1}[G]$ is relatively open in Q.

(iv) Finally, if $F \subseteq X$ is closed, $R^{-1}[F] = Q \setminus R^{-1}[X \setminus F]$ is relatively closed in Q.

415R Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Hausdorff quasi-Radon measure space and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of its measure algebra. Let $R \subseteq Z \times X$ be the relation described in 415Q. Then

(a) R is (the graph of) a function f;

(b) f is inverse-measure-preserving for the subspace measure ν_Q on $Q = \operatorname{dom} f$, and in fact μ is the image measure $\nu_Q f^{-1}$;

(c) if $(X, \mathfrak{T})$ is regular, then f is continuous.

proof (a) If $z \in Z$ and x, $y \in X$ are distinct, let G, H be disjoint open sets containing x, y respectively. Then

$$(X \setminus G)^* \cup (X \setminus H)^* = ((X \setminus G) \cup (X \setminus H))^* = Z,$$

defining * as in 415Q, so z must belong to at least one of $(X \setminus G)^*$, $(X \setminus H)^*$. In the former case $(z, x) \notin R$ and in the latter case $(z, y) \notin R$. This shows that R is a function; to remind us of its new status I will henceforth call it f. The domain of f is just $Q = R^{-1}[X]$.

(b) By 415Qd, f is inverse-measure-preserving for ν_Q and μ. Suppose that $A \subseteq X$ and $f^{-1}[A]$ is in the domain T_Q of ν_Q, that is, is of the form $Q \cap U$ for some $U \in \mathrm{T}$. Take any $E \in \Sigma$ such that $\mu E > 0$; then either $\nu(E^* \cap U) > 0$ or $\nu(E^* \setminus U) > 0$. (α) Suppose that $\nu(E^* \cap U) > 0$. Because ν is inner regular with respect to the open-and-closed sets, there is an $H \in \Sigma$ such that $H^* \subseteq E^* \cap U$ and $\mu H = \nu H^* > 0$. Now there is a closed set $F \subseteq E \cap H$ with $\mu F > 0$. In this case, $f[F^*] \subseteq F \subseteq E$, by 415Qb, while $F^* \cap Q \subseteq U \cap Q = f^{-1}[A]$, so $f[F^*] \subseteq E \cap A$. But this means that

$$\mu_*(E \cap A) \geq \mu f[F^*] = \mu F > 0.$$

(β) If $\nu(E^* \setminus U) > 0$, then the same arguments show that $\mu_*(E \setminus A) > 0$. (γ) Thus $\mu_*(E \cap A) + \mu_*(E \setminus A) > 0$ whenever $\mu E > 0$. Because μ is complete and locally determined, $A \in \Sigma$ (413F(vii)).

Thus we see that $\{A : A \subseteq X,\ f^{-1}[A] \in \mathrm{T}_Q\}$ is included in Σ, and μ is the image measure $\nu_Q f^{-1}$.

(c) If $\mathfrak{T}$ is regular, then 415Qf tells us that f is continuous.

415X Basic exercises >(a) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space and $E\in\Sigma$ an atom for the measure. Show that there is a closed set $F\subseteq E$ such that $\mu F>0$ and F is an atom of Σ, in the sense that the only measurable subsets of F are $\emptyset$ and F. (*Hint*: 414G.) Show that μ is atomless iff all countable subsets of X are negligible.

(b) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of quasi-Radon measure spaces. Show that the direct sum measure on $X=\{(x,i):i\in I,\,x\in X_i\}$ is a quasi-Radon measure when X is given its disjoint union topology.

>(c) The **right-facing Sorgenfrey topology** or **lower limit topology** on $\mathbb{R}$ is the topology generated by the half-open intervals of the form $[\alpha,\beta[$; I will use the phrase **Sorgenfrey line** to mean $\mathbb{R}$ with this topology. Show that Lebesgue measure on the Sorgenfrey line is completion regular and quasi-Radon. (*Hint*: 114Yj or 221Yb, or 419L.)

(d) Let X be a topological space and μ a complete measure on X, and suppose that there is a conegligible closed measurable set $Y\subseteq X$ such that the subspace measure on Y is quasi-Radon. Show that μ is quasi-Radon.

(e) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space. Show that μ is inner regular with respect to the family of self-supporting closed sets included in open sets of finite measure.

(f) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space. Show that whenever $E\in\Sigma$ and $\epsilon>0$ there is an open set G such that $\mu G\le\mu E+\epsilon$ and $E\setminus G$ is negligible.

(g) Describe a compact Hausdorff quasi-Radon measure space which is not σ-finite.

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, $(\mathfrak{A},\bar\mu)$ its measure algebra, and $\mathfrak{A}^f$ the ideal $\{a:a\in\mathfrak{A},\,\bar\mu a<\infty\}$. Show that $\{G^\bullet:G\in\mathfrak{T},\,\mu G<\infty\}$ is dense in $\mathfrak{A}^f$ for the strong measure-algebra topology (323Ad).

(i) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless quasi-Radon measure space which is outer regular with respect to the open sets. Show that it is σ-finite. (*Hint*: take a decomposition $\langle X_i\rangle_{i\in I}$ in which every X_i except one is self-supporting, and a set A meeting every X_i in just one point.)

(j) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite quasi-Radon measure space with $\mu X>0$. Show that there is a quasi-Radon probability measure on X with the same measurable sets and the same negligible sets as μ.

(k) Let (X,Σ,μ) be a σ-finite measure space in which Σ is countably generated as a σ-algebra. Show that, for a suitable topology on X, the completion of μ is a quasi-Radon measure. (*Hint*: take the topology generated by a countable subalgebra of Σ, and use the arguments of 415D.)

(l) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces such that every μ_i is strictly positive, and λ the product measure on $X=\prod_{i\in I}X_i$. Show that if every $\mathfrak{T}_i$ has a countable network, λ is a quasi-Radon measure.

(m) Let $\langle X_i\rangle_{i\in I}$ be a family of separable metrizable spaces, and μ a quasi-Radon measure on $X=\prod_{i\in I}X_i$. Show that μ is completion regular iff every self-supporting closed set in X is determined by coordinates in a countable set. (*Hint*: 4A2Eb.)

(n) Find two quasi-Radon measures μ, ν on the unit interval such that $\mu G\le\nu G$ for every open set G but there is a closed set F such that $\nu F<\mu F$.

(o) Let X be a topological space and μ, ν two quasi-Radon measures on X. (i) Suppose that $\mu F=\nu F$ whenever $F\subseteq X$ is closed and both μF and νF are finite. Show that $\mu=\nu$. (ii) Suppose that $\mu G=\nu G$ whenever $G\subseteq X$ is open and both μG and νG are finite. Show that $\mu=\nu$.

(p) Find a second-countable Hausdorff topological space X with a τ-additive Borel probability measure which is not inner regular with respect to the closed sets. (*Hint*: starting from a Radon probability measure, declare a set with full outer measure and zero inner measure to be open and conegligible.)

(q) Find a second-countable Hausdorff space X, a subset Y and a quasi-Radon probability measure on Y which is not the subspace measure induced by any quasi-Radon measure on X.

(r) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite paracompact Hausdorff quasi-Radon measure space, and $f\in\mathcal{L}^0(\mu)$ a locally integrable function. Show that for any $\epsilon>0$ there is a continuous function $g:X\to\mathbb{R}$ such that $\int|f-g|\le\epsilon$.

>(s) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite completely regular quasi-Radon measure space. (i) Show that for every $E\in\Sigma$ there is an F in the Baire σ-algebra $\mathcal{B}\mathfrak{a}(X)$ of X such that $\mu(E\triangle F)=0$. (*Hint*: start with an open set E of finite measure.) (ii) Show that for every Σ-measurable function $f:X\to\mathbb{R}$ there is a $\mathcal{B}\mathfrak{a}(X)$-measurable function equal almost everywhere to f.

(t) Let (X,Σ,μ) be a measure space and f a μ-integrable real-valued function. Show that there is a unique quasi-Radon measure λ on $\mathbb{R}$ such that $\lambda\{0\}=0$ and $\lambda\,[\alpha,\infty[\,=\mu^*\{x:x\in\operatorname{dom}f,\ f(x)\ge\alpha\}$, $\lambda\,]-\infty,-\alpha]=\mu^*\{x:x\in\operatorname{dom}f,\ f(x)\le-\alpha\}$ whenever $\alpha>0$; and that $\int h\,d\lambda=\int hf d\mu$ whenever $h\in\mathcal{L}^0(\lambda)$ and $h(0)=0$ and either integral is defined in $[-\infty,\infty]$. (*Hint*: set $\lambda E=\mu^*f^{-1}[E\setminus\{0\}]$ for Borel sets $E\subseteq\mathbb{R}$, and use 414Mb, 414O and 235Gb.)

415Y Further exercises (a) Give an example of two quasi-Radon measures μ, ν on $\mathbb{R}$ such that their sum, as defined in 234G, is not effectively locally finite, therefore not a quasi-Radon measure.

(b) Show that any quasi-Radon measure space is isomorphic, as topological measure space, to a subspace of a compact quasi-Radon measure space. (*Hint*: if X is a T_1 quasi-Radon measure space, let $\hat{X}$ be its Wallman compactification (Engelking 89, 3.6.21).)

(c) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space. Show that the following are equiveridical: (i) μ is outer regular with respect to the open sets; (ii) every negligible subset of X is included in an open set of finite measure; (iii) $\{x:\mu\{x\}=0\}$ can be covered by a sequence of open sets of finite measure.

(d) Show that $+:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous for the right-facing Sorgenfrey topology.

(e) Let $r\ge1$. On $\mathbb{R}^r$ let $\mathfrak{S}$ be the topology generated by the half-open intervals $[a,b[$ where a, $b\in\mathbb{R}^r$ (definition: 115Ab). (i) Show that $\mathfrak{S}$ is the product topology if each factor is given the right-facing Sorgenfrey topology. (ii) Show that Lebesgue measure is quasi-Radon for $\mathfrak{S}$. (*Hint*: induce on r. See also 417Yi.)

(f) Find a base $\mathcal{U}$ for the topology of $X=\{0,1\}^{\mathbb{N}}$ and two totally finite (quasi-)Radon measures μ, ν on X such that $G\cap H\in\mathcal{U}$ for all G, $H\in\mathcal{U}$, $\mu G\le\nu G$ for every $G\in\mathcal{U}$, but $\nu X<\mu X$.

(g) Let X be a topological space and $\mathcal{G}$ an open cover of X. Suppose that for each $G\in\mathcal{G}$ we are given a quasi-Radon measure μ_G on G such that $\mu_G(U)=\mu_H(U)$ whenever G, $H\in\mathcal{G}$ and $U\subseteq G\cap H$ is open. Show that there is a unique quasi-Radon measure on X such that each μ_G is the subspace measure on G. (*Hint*: if $\langle\mu_G\rangle_{G\in\mathcal{G}}$ is a maximal family with the given properties, then $\mathcal{G}$ is upwards-directed.)

(h) Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets, and suppose that there is a family $\mathcal{U}\subseteq\Sigma\cap\mathfrak{T}$ such that

$\mu U<\infty$ for every $U\in\mathcal{U}$,

for every $U\in\mathcal{U}$, $\mathfrak{T}\cap\Sigma\cap\mathcal{P}U$ is a base for the subspace topology of U,

if $\mathcal{G}$ is an upwards-directed family in $\mathfrak{T}\cap\Sigma$ and $\bigcup\mathcal{G}\in\mathcal{U}$, then $\mu(\bigcup\mathcal{G})=\sup_{G\in\mathcal{G}}\mu G$,

if $E\in\Sigma$ and $\mu E>0$ then there is a $U\in\mathcal{U}$ such that $\mu(E\cap U)>0$.

Show that μ has an extension to a quasi-Radon measure on X.

(i) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space such that $\mathfrak{T}$ is normal (but not necessarily Hausdorff or regular). Show that if $1 \le p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a bounded continuous function $g : X \to \mathbb{R}$ such that $\|f - g\|_p \le \epsilon$ and $\{x : g(x) \ne 0\}$ has finite measure.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular quasi-Radon measure space and suppose that we are given a uniformity defining the topology $\mathfrak{T}$. Show that if $1 \le p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a bounded uniformly continuous function $g : X \to \mathbb{R}$ such that $\|f - g\|_p \le \epsilon$ and $\{x : g(x) \ne 0\}$ has finite measure.

(k) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular quasi-Radon measure space and τ an extended Fatou norm on $L^0(\mu)$ (definition: 369F) such that (i) $\tau \restriction L^\tau$ is an order-continuous norm (ii) whenever $E \in \Sigma$ and $\mu E > 0$ there is an open set G such that $\mu(E \cap G) > 0$ and $\tau(\chi G^\bullet) < \infty$. Show that $L^\tau \cap \{f^\bullet : f : X \to \mathbb{R}$ is continuous$\}$ is norm-dense in L^τ.

(l) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Show that μ is a compact measure (definition: 342Ac or 451Ab) iff there is a locally compact topology $\mathfrak{S}$ on X such that $(X, \mathfrak{S}, \Sigma, \mu)$ is quasi-Radon.

415 Notes and comments 415B is particularly important because a very high proportion of the quasi-Radon measure spaces we study are actually subspaces of Radon measure spaces. I would in fact go so far as to say that when you have occasion to wonder whether all quasi-Radon measure spaces have a property, you should as a matter of habit look first at subspaces of Radon measure spaces; if the answer is affirmative for them, you will have most of what you want, even if the generalization to arbitrary quasi-Radon spaces gives difficulties. Of course the reverse phenomenon can also occur. Stone spaces (411P) can be thought of as quasi-Radon compactifications of Radon measure spaces (416V). But this is relatively rare. Indeed the reason why I give so few examples of quasi-Radon spaces at this point is just that the natural ones arise from Radon measure spaces. Note however that the quasi-Radon product of an uncountable family of Radon probability spaces need not be Radon (see 417Xq), so that 415E here and 417O below are sources of non-Radon quasi-Radon measure spaces. Density and lifting topologies can also provide us with quasi-Radon measure spaces (453Xd, 453Xg).

415K is the second in a series of inner-regular-extension theorems; there will be a third in 416J.

I have been saying since Volume 1 that the business of measure theory, since Lebesgue's time, has been to measure as many sets and integrate as many functions as possible. I therefore take seriously any theorem offering a canonical extension of a measure. 415L and its corollaries can be regarded as improvement theorems, showing that a good measure can be made even better. We have already had such improvement theorems in Chapter 21: the completion and c.l.d. version of a measure (212C, 213E). In all such theorems we need to know exactly what effect our improvement is having on the other constructions we are interested in; primarily, the measure algebra and the function spaces. The machinery of Chapter 36 shows that if we understand the measure algebra(s) involved then the function spaces will give us no further surprises. Completion of a measure does not affect the measure algebra at all (322Da). Taking the c.l.d. version does not change $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$ or L^1 (213Fc, 213G, 322D(b-i), 366H), but can affect the rest of the measure algebra and therefore L^0 and L^∞. In this respect, what we might call the 'quasi-Radon version' behaves like the c.l.d. version (as could be expected, since the quasi-Radon version must itself be complete and locally determined). The archetypal application of 415L is 415N. We shall see later how Baire measures arise naturally when studying Banach spaces of continuous functions (436E). 415N will be one of the keys to applying the general theory of topological measure spaces in such contexts. A virtue of Baire measures is that inner regularity with respect to closed sets comes almost free (412D); but there can be unsurmountable difficulties if we wish to extend them to Borel measures (439M), and it is important to know that τ-additivity, even in the relatively weak form allowed by the definition I use here (411C), is often enough to give a canonical extension to a well-behaved measure defined on every Borel set. In 415C we have inner regularity for a different reason, and the measure is already known to be defined on every Borel set, so in fact the quasi-Radon version of the measure is just the c.l.d. version.

One interpretation of the Lifting Theorem is that for a complete strictly localizable measure space (X, Σ, μ) there is a function $g : X \to Z$, where Z is the Stone space of the measure algebra of μ, such that $E \triangle g^{-1}[E^*]$ is negligible for every $E \in \Sigma$, where $E^* \subseteq Z$ is the open-and-closed set corresponding to the image of E in the measure algebra (341Q). For a Hausdorff quasi-Radon measure space we have a function $f : Q \to X$, where Q is a dense subset of Z, such that $(Q \cap E^*) \triangle f^{-1}[E]$ is negligible for every $E \in \Sigma$ (415Qd, 415R); moreover, there is a canonical construction for this function. For completeness' sake, I have given the result for general, not necessarily Hausdorff, spaces X (415Q); but evidently it will be of greatest interest for regular Hausdorff spaces (415Rc). Perhaps I should remark that in the most important applications, Q is the whole of Z (416Vc). Of course the question arises, whether fg can be the identity. (Z typically has larger cardinal than X, so asking for gf to be the identity is a bit optimistic.) This is in fact an important question; I will return to it in 453M-453N.

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416 Radon measure spaces

We come now to the results for which the chapter so far has been preparing. The centre of topological measure theory is the theory of 'Radon' measures (411Hb), measures inner regular with respect to compact sets. Most of the section is devoted to pulling the earlier work together, and in particular to re-stating theorems on quasi-Radon measures in the new context. Of course this has to begin with a check that Radon measures are quasi-Radon (416A). It follows immediately that Radon measures are (strictly) localizable (416B). After presenting a miscellany of elementary facts, I turn to the constructions of §413, which take on simpler and more dramatic forms in this context (416J-416P). I proceed to investigate subspace measures (416R-416T) and some special product measures (416U). I end the section with further notes on the forms which earlier theorems on Stone spaces (416V) and compact measure spaces (416W) take when applied to Radon measure spaces.

416A Proposition A Radon measure space is quasi-Radon.

proof Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Because $\mathfrak{T}$ is Hausdorff, every compact set is closed, so μ is inner regular with respect to the closed sets. By 411E, μ is τ-additive; by 411Gf, it is effectively locally finite. Thus all parts of condition (ii) of 411Ha are satisfied, and μ is a quasi-Radon measure.

416B Corollary A Radon measure space is strictly localizable.

proof Put 416A and 415A or 414J together.

416C In order to use the results of §415 effectively, it will be helpful to spell out elementary conditions ensuring that a quasi-Radon measure is Radon.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally finite Hausdorff quasi-Radon measure space. Then the following are equiveridical:

(i) μ is a Radon measure;

(ii) whenever $E \in \Sigma$ and $\mu E > 0$ there is a compact set K such that $\mu(E \cap K) > 0$;

(iii) $\sup\{K^\bullet : K \subseteq X \text{ is compact}\} = 1$ in the measure algebra of μ.

If μ is totally finite we can add

(iv) $\sup\{\mu K : K \subseteq X \text{ is compact}\} = \mu X$.

proof (i)$\Rightarrow$(ii) and (ii)$\Leftrightarrow$(iii) are trivial. For (ii)$\Rightarrow$(i), observe that if $E \in \Sigma$ and $\mu E > 0$ there is a compact set $K \subseteq E$ such that $\mu K > 0$. **P** There is a compact set K' such that $\mu(E \cap K') > 0$, by hypothesis; now there is a closed set $K \subseteq E \cap K'$ such that $\mu K > 0$, because μ is inner regular with respect to the closed sets, and K is compact. **Q** By 412Aa, μ is inner regular with respect to the compact sets. Being a complete, locally determined, locally finite topological measure, it is a Radon measure.

When $\mu X < \infty$, of course, we also have (iii)$\Leftrightarrow$(iv).

416D Some further elementary facts are worth writing out plainly.

Lemma (a) In a Radon measure space, every compact set has finite measure.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $E \subseteq X$ a set such that $E \cap K \in \Sigma$ for every compact $K \subseteq X$. Then $E \in \Sigma$.

(c) A Radon measure is inner regular with respect to the self-supporting compact sets.

(d) Let X be a Hausdorff space and μ a tight locally finite complete locally determined measure on X. If μ measures every compact set, μ is a Radon measure.

(e) Let X be a Hausdorff space and $\langle \mu_i \rangle_{i \in I}$ a family of Radon measures on X. Let $\mu = \sum_{i \in I} \mu_i$ be their sum (definition: 234G). If μ is locally finite, it is a Radon measure.

proof (a) 411Ga.

(b) We have only to remember that μ is complete, locally determined and tight, and apply 413F(ii).

(c) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space, $E \in \Sigma$ and $\gamma < \mu E$, there is a compact set $K \subseteq E$ such that $\mu K \geq \gamma$. By 414F, there is a self-supporting relatively closed set $L \subseteq K$ such that $\mu L = \mu K$; but now of course L is compact, while $L \subseteq E$ and $\mu L \geq \gamma$.

(d) Let $\mathcal{K}$ be the family of compact subsets of X; write Σ for the domain of μ. If $F \subseteq X$ is closed, then $F \cap K \in \mathcal{K} \subseteq \Sigma$ for every $K \in \mathcal{K}$; accordingly $F \in \Sigma$, by 412Ja. But this means that every closed set, therefore every open set, belongs to Σ, and μ is a Radon measure.

(e) Because every μ_i is a topological measure, so is μ; because every μ_i is complete, so is μ (234Ha). By hypothesis, μ is locally finite. If $\mu E > 0$, then there is some $i \in I$ such that $\mu_i E > 0$; now there is a compact $K \subseteq E$ such that $0 < \mu_i K \leq \mu K$. So μ is inner regular with respect to the compact sets.

Now suppose that $E \subseteq X$ is such that μ measures $E \cap F$ whenever $\mu F < \infty$. Then, in particular, $E \cap K$ is measured by μ, therefore measured by every μ_i, whenever $K \subseteq X$ is compact. By 413F(ii) again, μ_i measures E for every i, so μ measures E. As E is arbitrary, μ is locally determined and is a Radon measure.

Remark In (e) above, note that if I is finite then μ is necessarily locally finite.

416E Specification of Radon measures In 415H I described some conditions which enable us to be sure that two quasi-Radon measures on a given topological space are the same. In the case of Radon measures we have a similar list. This time I include a note on the natural ordering of Radon measures.

Proposition Let X be a Hausdorff space and μ, ν two Radon measures on X.

(a) The following are equiveridical:

(i) $\nu \leq \mu$ in the sense of 234P, that is, νE is defined and $\nu E \leq \mu E$ whenever μ measures E;

(ii) $\mu K \leq \nu K$ for every compact set $K \subseteq X$;

(iii) $\mu G \leq \nu G$ for every open set $G \subseteq X$;

(iv) $\mu F \leq \nu F$ for every closed set $F \subseteq X$.

If X is locally compact, we can add

(v) $\int f d\mu \leq \int f d\nu$ for every non-negative continuous function $f : X \to \mathbb{R}$ with compact support.

(b) The following are equiveridical:

(i) $\mu = \nu$;

(ii) $\mu K = \nu K$ for every compact set $K \subseteq X$;

(iii) $\mu G = \nu G$ for every open set $G \subseteq X$;

(iv) $\mu F = \nu F$ for every closed set $F \subseteq X$.

If X is locally compact, we can add

(v) $\int f d\mu = \int f d\nu$ for every continuous function $f : X \to \mathbb{R}$ with compact support.

proof (a)(i)⇒(iv)⇒(ii) and **(i)⇒(iii)** are trivial, if we recall that $\nu \leq \mu$ when $\operatorname{dom} \nu \supseteq \operatorname{dom} \mu$ and $\nu E \leq \mu E$ for every $E \in \operatorname{dom} \mu$.

(ii)⇒(i) is a special case of 412Mb.

(iii)⇒(ii) The point is that if $K \subseteq X$ is compact, then $\mu K = \inf\{\mu G : G \subseteq X \text{ is open}, K \subseteq G\}$. **P** Because $X = \bigcup\{\mu G : G \subseteq X \text{ is open}, \mu G < \infty\}$, there is an open set G_0 of finite measure including K. Now, for any $\gamma > \mu K$, there is a compact set $L \subseteq G_0 \setminus K$ such that $\mu L \geq \mu G_0 - \gamma$, so that $\mu G \leq \gamma$, where $G = G_0 \setminus L$ is an open set including K. **Q**

The same is true for ν. So, if (iii) is true,

$$\mu K = \inf_{G\supseteq K \text{ is open}} \mu G \le \inf_{G\supseteq K \text{ is open}} \nu G = \nu K$$

for every compact $K \subseteq X$, and (ii) is true.

(iii)⇒(v) If (iii) is true and $f : X \to [0,\infty[$ is a non-negative continuous function, then

$$\int f d\mu = \int_0^\infty \mu\{x : f(x) > t\}dt$$

(252O)

$$\le \int_0^\infty \nu\{x : f(x) > t\}dt = \int f d\nu.$$

(v)⇒(iii) If X is locally compact and (v) is true, take any open set $G \subseteq X$, and consider the set A of continuous functions $f : X \to [0,1]$ with compact support such that $f \le \chi G$. Then A is upwards-directed and $\sup_{f\in A} f(x) = \chi G(x)$ for every $x \in X$, by 4A2G(e-i). So

$$\mu G = \sup_{f\in A} \int f d\mu \le \sup_{f\in A} \int f d\nu = \nu G$$

by 414Ba. As G is arbitrary, (iii) is true.

(b) now follows at once, or from 415H-415I.

416F Proposition Let X be a Hausdorff space and μ a Borel measure on X. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) μ is locally finite and tight;

(iii) μ is locally finite and effectively locally finite, and $\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\}$ for every open set $G \subseteq X$;

(iv) μ is locally finite, effectively locally finite and τ-additive, and $\mu G = \sup\{\mu(G \cap K) : K \subseteq X \text{ is compact}\}$ for every open set $G \subseteq X$.

In this case the extension is unique; it is the c.l.d. version of μ.

proof (a)(i)⇒(iv) If $\mu = \tilde\mu{\restriction}\mathcal{B}(X)$ where $\tilde\mu$ is a Radon measure and $\mathcal{B}(X)$ is the Borel σ-algebra of X, then of course μ is locally finite and effectively locally finite and τ-additive because $\tilde\mu$ is (see 416A) and every open set belongs to $\mathcal{B}(X)$. Also

$$\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\} \le \sup\{\mu(G \cap K) : K \subseteq X \text{ is compact}\} \le \mu G$$

for every open set $G \subseteq X$, because $\tilde\mu$ is tight and compact sets belong to $\mathcal{B}(X)$.

(b)(iv)⇒(iii) Suppose that (iv) is true. Of course μ is locally finite and effectively locally finite. Suppose that $G \subseteq X$ is open and that $\gamma < \mu G$. Then there is a compact $K \subseteq X$ such that $\mu(G \cap K) > \gamma$. By 414K, the (totally finite) subspace measure μ_K is τ-additive. Now K is a compact Hausdorff space, therefore regular. By 414Ma there is a closed set $F \subseteq G \cap K$ such that $\mu_K F \ge \gamma$. Now F is compact, $F \subseteq G$ and $\mu F \ge \gamma$. As G and γ are arbitrary, (iii) is true.

(c)(iii)⇒(ii) I have to show that if μ satisfies the conditions of (iii) it is tight. Let $\mathcal{K}$ be the family of compact subsets of X and $\mathcal{A}$ the family of subsets of X which are either open or closed. Then whenever $A \in \mathcal{A}$, $F \in \Sigma$ and $\mu(A \cap F) > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq A$ and $\mu(K \cap F) > 0$. **P** Because μ is effectively locally finite, there is an open set G of finite measure such that $\mu(G \cap A \cap F) > 0$. (α) If A is open, then there will be a compact set $K \subseteq G \cap A$ such that $\mu K > \mu(G \cap A) - \mu(G \cap A \cap F)$, so that $\mu(K \cap F) > 0$. (β) If A is closed, then let $L \subseteq G$ be a compact set such that $\mu L > \mu G - \mu(G \cap A \cap F)$; then $K = L \cap A$ is compact and $\mu(K \cap F) > 0$. **Q**

By 412C, μ is inner regular with respect to $\mathcal{K}$, as required.

(d)(ii)⇒(i) If μ is locally finite and tight, let $\tilde\mu$ be the c.l.d. version of μ. Then $\tilde\mu$ is complete, locally determined, locally finite (because μ is), a topological measure (because μ is) and tight (because μ is, using

412Ha); so is a Radon measure. Every compact set has finite measure for μ, so μ is semi-finite and $\tilde{\mu}$ extends μ (213Hc).

(e) By 416Eb there can be at most one Radon measure extending μ, and we have observed in (d) above that in the present case it is the c.l.d. version of μ.

416G One of the themes of §434 will be the question: on which Hausdorff spaces is every locally finite quasi-Radon measure a Radon measure? I do not think we are ready for a general investigation of this, but I can give one easy special result.

Proposition Let $(X,\mathfrak{T})$ be a locally compact Hausdorff space and μ a locally finite quasi-Radon measure on X. Then μ is a Radon measure.

proof μ satisfies condition (ii) of 416C. **P** Take $E\in\operatorname{dom}\mu$ such that $\mu E>0$. Let $\mathcal{G}$ be the family of relatively compact open subsets of X; then $\mathcal{G}$ is upwards-directed and has union X. By 414Ea, there is a $G\in\mathcal{G}$ such that $\mu(E\cap G)>0$. But now $\overline{G}$ is compact and $\mu(E\cap\overline{G})>0$. **Q** By 416C, μ is a Radon measure.

416H Corollary Let $(X,\mathfrak{T})$ be a locally compact Hausdorff space, and μ a locally finite, effectively locally finite, τ-additive Borel measure on X. Then μ is tight and its c.l.d. version is a Radon measure, the unique Radon measure on X extending μ.

proof By 415Cb, the c.l.d. version $\tilde{\mu}$ of μ is a quasi-Radon measure extending μ. Because μ is locally finite, so is $\tilde{\mu}$; by 416G, $\tilde{\mu}$ is a Radon measure. By 416Eb, the extension is unique. Now

$$\mu E=\tilde{\mu}E=\sup_{K\subseteq E\text{ is compact}}\tilde{\mu}K=\sup_{K\subseteq E\text{ is compact}}\mu K$$

for every Borel set $E\subseteq X$, so μ itself is tight.

416I While on the subject of locally compact spaces, I mention an important generalization of a result from Chapter 24.

Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a locally compact Radon measure space. Write C_k for the space of continuous real-valued functions on X with compact supports. If $1\le p<\infty$, $f\in\mathcal{L}^p(\mu)$ and $\epsilon>0$, there is a $g\in C_k$ such that $\|f-g\|_p\le\epsilon$.

proof By 415Pa, there is a bounded continuous function $h_1:X\to\mathbb{R}$ such that $G=\{x:h_1(x)\ne0\}$ has finite measure and $\|f-h_1\|_p\le\frac{1}{2}\epsilon$. Let $K\subseteq G$ be a compact set such that $\|h_1\|_\infty(\mu(G\setminus K))^{1/p}\le\frac{1}{2}\epsilon$, and let $h_2\in C_k$ be such that $\chi K\le h_2\le\chi G$ (4A2G(e-i) again). Set $g=h_1\times h_2$. Then $g\in C_k$ and

$$\int|h_1-g|^p\le\int_{G\setminus K}|h_1|^p\le\mu(G\setminus K)\|h_1\|_\infty^p,$$

so $\|h_1-g\|_p\le\frac{1}{2}\epsilon$ and $\|f-g\|_p\le\epsilon$, as required.

416J I turn now to constructions of Radon measures based on ideas in §413.

Theorem Let X be a Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X and $\phi_0:\mathcal{K}\to[0,\infty[$ a functional such that

(α) $\phi_0K=\phi_0L+\sup\{\phi_0K':K'\in\mathcal{K},\ K'\subseteq K\setminus L\}$ whenever K, $L\in\mathcal{K}$ and $L\subseteq K$,

(γ) for every $x\in X$ there is an open set G containing x such that $\sup\{\phi_0K:K\in\mathcal{K},\ K\subseteq G\}$ is finite.

Then there is a unique Radon measure on X extending ϕ_0.

proof By 413N, there is a unique complete locally determined measure μ on X, extending ϕ_0, which is inner regular with respect to $\mathcal{K}$. By (γ), μ is locally finite; by 416Dd, it is a Radon measure.

416K Proposition (see Topsøe 70a) Let X be a Hausdorff space, T a subring of $\mathcal{P}X$ such that $\mathcal{H}=\{G:G\in\mathrm{T}\text{ is open}\}$ covers X, and $\nu:\mathrm{T}\to[0,\infty[$ a finitely additive functional. Then there is a Radon measure μ on X such that $\mu K\ge\nu K$ for every compact $K\in\mathrm{T}$ and $\mu G\le\nu G$ for every open $G\in\mathrm{T}$.

proof (a) For $H \in \mathcal{H}$ set $\mathrm{T}_H = \mathrm{T} \cap \mathcal{P}H$; then T_H is an algebra of subsets of H, and $\nu_H = \nu \restriction \mathrm{T}_H$ is additive. By 391G, there is an additive functional $\tilde{\nu}_H : \mathcal{P}H \to [0, \infty[$ extending ν_H. Let $\mathfrak{F}$ be an ultrafilter on $\mathcal{H}$ containing $\{H : H_0 \subseteq H \in \mathcal{H}\}$ for every $H_0 \in \mathcal{H}$, and $\tilde{\mathrm{T}}$ the ideal of subsets of X generated by $\mathcal{H}$. If $A \in \tilde{\mathrm{T}}$ then there is an $H_0 \in \mathcal{H}$ including A, and now

$$\tilde{\nu}_H(A \cap H) = \tilde{\nu}_H(A \cap H_0) \le \nu_H(H \cap H_0) = \nu(H \cap H_0) \le \nu H_0$$

for every $H \in \mathcal{H}$, so $\tilde{\nu}A = \lim_{H \to \mathfrak{F}} \nu_H(A \cap H)$ is defined in $[0, \infty[$. Note that if $A \in \mathrm{T} \cap \tilde{\mathrm{T}}$ then there is an $H_0 \in \mathcal{H}$ including A, so that $\nu_H A = \nu A$ whenever $H \in \mathcal{H}$ and $H \supseteq H_0$, and $\tilde{\nu}A = \nu A$. Also $\tilde{\nu} : \tilde{\mathrm{T}} \to [0, \infty[$ is additive because all the functionals $A \mapsto \tilde{\nu}_H(A \cap H)$ are.

(b) Let $\mathcal{K}$ be the family of compact subsets of X. Because $X = \bigcup \mathcal{H}$, $\mathcal{K} \subseteq \tilde{\mathrm{T}}$. For $K \in \mathcal{K}$, set

$$\phi_0 K = \inf\{\tilde{\nu}G : G \in \tilde{\mathrm{T}} \text{ is open, } K \subseteq G\}.$$

Then ϕ_0 satisfies the conditions of 416J. **P** (α) Take K, $L \in \mathcal{K}$ such that $L \subseteq K$, and $\epsilon > 0$. Then there are open sets G_0, $H_0 \in \tilde{\mathrm{T}}$ such that

$$K \subseteq G_0, \quad \tilde{\nu}G_0 \le \phi_0 K + \epsilon, \quad L \subseteq H_0, \quad \tilde{\nu}H_0 \le \phi_0 L + \epsilon.$$

(i) If $K' \in \mathcal{K}$ is such that $K' \subseteq K \setminus L$, there are disjoint open sets H, $H' \subseteq X$ such that $L \subseteq H$ and $K' \subseteq H'$ (4A2F(h-i)). So

$$\phi_0 L + \phi_0 K' \le \tilde{\nu}(G_0 \cap H) + \tilde{\nu}(G_0 \cap H') \le \tilde{\nu}G_0 \le \phi_0 K + \epsilon.$$

(ii) In the other direction, consider $K_1 = K \setminus H_0$. Then there is an open set $H_1 \in \tilde{\mathrm{T}}$ such that $K_1 \subseteq H_1$ and $\tilde{\nu}H_1 \le \phi_0 K_1 + \epsilon$, so that

$$\phi_0 K \le \tilde{\nu}(H_0 \cup H_1) \le \tilde{\nu}H_0 + \tilde{\nu}H_1 \le \phi_0 L + \phi_0 K_1 + 2\epsilon.$$

As ϵ is arbitrary,

$$\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$$

as required by 416J(α). (γ) If $x \in X$ there is an $H_0 \in \mathcal{H}$ containing x, and now

$$\sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq H_0\} \le \tilde{\nu}H_0$$

is finite. So the second hypothesis also is satisfied. **Q**

(c) By 416J, we have a Radon measure μ on X extending ϕ_0. If $K \in \mathrm{T}$ is compact, then $\mu K \ge \nu K$. **P** Since $K \in \mathrm{T} \cap \tilde{\mathrm{T}}$, $\tilde{\nu}K = \nu K$. Now

$$\mu K = \phi_0 K = \inf\{\tilde{\nu}G : G \in \tilde{\mathrm{T}} \text{ is open, } K \subseteq G\} \ge \tilde{\nu}K = \nu K. \text{ **Q**}$$

If $G \in \mathrm{T}$ is open, then $\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\}$. But if $K \subseteq G$ is compact then

$$\mu K = \phi_0 K \le \tilde{\nu}G = \nu G,$$

so $\mu G \le \nu G$.

Thus μ has the required properties.

416L Proposition Let X be a regular Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X, and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

(α_1) $\phi_0 K \le \phi_0(K \cup L) \le \phi_0 K + \phi_0 L$ for all K, $L \in \mathcal{K}$,

(α_2) $\phi_0(K \cup L) = \phi_0 K + \phi_0 L$ whenever K, $L \in \mathcal{K}$ and $K \cap L = \emptyset$,

(γ) for every $x \in X$ there is an open set G containing x such that $\sup\{\phi_0 K : K \in \mathcal{K},\ K \subseteq G\}$ is finite.

Then there is a unique Radon measure μ on X such that

$$\mu K = \inf_{G \subseteq X \text{ is open}, K \subseteq G} \sup_{L \subseteq G \text{ is compact}} \phi_0 L$$

for every $K \in \mathcal{K}$.

proof (a) For open sets $G \subseteq X$ set

$$\psi G = \sup_{L \in \mathcal{K}, L \subseteq G} \phi_0 L,$$

and for compact sets $K \subseteq X$ set

$$\phi_1 K = \inf\{\psi G : G \subseteq X \text{ is open, } K \subseteq G\}.$$

Evidently $\psi G \le \psi H$ whenever $G \subseteq H$. We need to know that $\psi(G \cup H) \le \psi G + \psi H$ for all open sets G, $H \subseteq X$. **P** If $L \subseteq G \cup H$ is compact, then the disjoint compact sets $L \setminus G$, $L \setminus H$ can be separated by disjoint open sets H', G' (4A2F(h-i) again); now $L \setminus G' \subseteq H$, $L \setminus H' \subseteq G$ are compact and cover L, so

$$\phi_0 L \le \phi_0(L \setminus G') + \phi_0(L \setminus H') \le \psi H + \psi G.$$

As L is arbitrary, $\psi(G \cup H) \le \psi G + \psi H$. **Q**

Moreover, $\psi(G \cup H) = \psi G + \psi H$ if $G \cap H = \emptyset$. **P** If $K \subseteq G$, $L \subseteq H$ are compact, then

$$\phi_0 K + \phi_0 L = \phi_0(K \cup L) \le \psi(G \cup H).$$

As K and L are arbitrary, $\psi G + \psi H \le \psi(G \cup H)$. **Q**

(b) It follows that $\phi_1 K$ is finite for every compact $K \subseteq X$. **P** Set $\mathcal{G} = \{G : G \subseteq X \text{ is open, } \psi G < \infty\}$. Then (a) tells us that $\mathcal{G}$ is upwards-directed. But also we are supposing that $\mathcal{G}$ covers X, by (γ). So if $K \subseteq X$ is compact there is a member of $\mathcal{G}$ including K and $\phi_1 K < \infty$. **Q**

(c) Now ϕ_1 satisfies the conditions of 416J.

P $(\boldsymbol{\alpha})$ Suppose that K, $L \in \mathcal{K}$ and $L \subseteq K$. Set $\gamma = \sup\{\phi_1 M : M \in \mathcal{K},\, M \subseteq K \setminus L\}$. Take any $\epsilon > 0$.

Let G be an open set such that $K \subseteq G$ and $\psi G \le \phi_1 K + \epsilon$. If $M \in \mathcal{K}$ and $M \subseteq K \setminus L$, there are disjoint open sets U, V such that $L \subseteq U$ and $M \subseteq V$ (4A2F(h-i) once more); we can arrange that $U \cup V \subseteq G$. In this case,

$$\phi_1 L + \phi_1 M \le \psi U + \psi V = \psi(U \cup V)$$

(by the second part of (a) above)

$$\le \psi G \le \phi_1 K + \epsilon.$$

As M is arbitrary, $\gamma \le \phi_1 K - \phi_1 L + \epsilon$.

On the other hand, there is an open set H such that $L \subseteq H$ and $\psi H \le \phi_1 L + \epsilon$. Set $F = K \setminus H$, so that F is a compact subset of $K \setminus L$. Then there is an open set V such that $F \subseteq V$ and $\psi V \le \phi_1 F + \epsilon$. In this case $K \subseteq H \cup V$, so

$$\begin{aligned}\phi_1 K &\le \psi(H \cup V) \le \psi H + \psi V \\ &\le \phi_1 L + \phi_1 F + 2\epsilon \le \phi_1 L + \gamma + 2\epsilon,\end{aligned}$$

so $\gamma \ge \phi_1 K - \phi_1 L - 2\epsilon$.

As ϵ is arbitrary, $\gamma = \phi_1 K - \phi_1 L$; as K and L are arbitrary, ϕ_1 satisfies condition (α) of 416J.

$(\boldsymbol{\gamma})$ Any $x \in X$ is contained in an open set G such that $\psi G < \infty$; but now $\sup\{\phi_1 K : K \in \mathcal{K},\, K \subseteq G\} \le \psi G$ is finite. So ϕ_1 satisfies condition (γ) of 416J. **Q**

(d) By 416J, there is a unique Radon measure on X extending ϕ_1, as claimed.

416M Corollary Let X be a locally compact Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X, and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

$\phi_0 K \le \phi_0(K \cup L) \le \phi_0 K + \phi_0 L$ for all K, $L \in \mathcal{K}$,

$\phi_0(K \cup L) = \phi_0 K + \phi_0 L$ whenever K, $L \in \mathcal{K}$ and $K \cap L = \emptyset$.

Then there is a unique Radon measure μ on X such that

$$\mu K = \inf\{\phi_0 K' : K' \in \mathcal{K},\, K \subseteq \operatorname{int} K'\}$$

for every $K \in \mathcal{K}$.

proof Observe that ϕ_0 satisfies the conditions of 416L; 416L(γ) is true because X is locally compact. Define ψ, ϕ_1 as in the proof of 416L, and set

$$\phi_1'K = \inf\{\phi_0 K' : K' \in \mathcal{K},\, K \subseteq \operatorname{int} K'\}$$

for every $K \in \mathcal{K}$. Then $\phi_1' = \phi_1$. **P** Suppose that $K \in \mathcal{K}$ and $\epsilon > 0$. (i) There is an open set $G \subseteq X$ such that $K \subseteq G$ and $\psi G \le \phi_1 K + \epsilon$. Now the relatively compact open subsets with closures included in G form an upwards-directed cover of K, so there is a $K' \in \mathcal{K}$ such that $K \subseteq \operatorname{int} K'$ and $K' \subseteq G$. Accordingly

$$\phi_1'K \le \phi_0 K' \le \psi G \le \phi_1 K + \epsilon.$$

(ii) There is an $L \in \mathcal{K}$ such that $K \subseteq \operatorname{int} L$ and $\phi_0 L \le \phi_1' K + \epsilon$, so that

$$\phi_1 K \le \psi(\operatorname{int} L) \le \phi_0 L \le \phi_1' K + \epsilon.$$

(iii) As K and ϵ are arbitrary, $\phi_1' = \phi_1$. **Q**

Now 416L tells us that there is a unique Radon measure extending ϕ_1, and this is the measure we seek.

416N The extension theorems in the second half of §413 also have important applications to Radon measures.

Henry's theorem (Henry 69) Let X be a Hausdorff space and μ_0 a measure on X which is locally finite and tight. Then μ_0 has an extension to a Radon measure μ on X, and the extension may be made in such a way that whenever $\mu E < \infty$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$.

proof All the work has been done in §413; here we need to check only that the family $\mathcal{K}$ of compact subsets of X and the measure μ_0 satisfy the hypotheses of 413P. But (†) and (‡) there are elementary, and $\mu_0^* K < \infty$ for every $K \in \mathcal{K}$ by 411Ga.

Now take the measure μ from 413P. It is complete, locally determined and inner regular with respect to $\mathcal{K}$; also $\mathcal{K} \subseteq \operatorname{dom}\mu$. Because μ_0 is locally finite and μ extends μ_0, μ is locally finite. By 416Dd, μ is a Radon measure. And the construction of 413P ensures that every set of finite measure for μ differs from a member of Σ_0 by a μ-negligible set.

416O Theorem Let X be a Hausdorff space and T a subalgebra of $\mathcal{P}X$. Let $\nu : \mathrm{T} \to [0, \infty[$ be a finitely additive functional such that

$$\nu E = \sup\{\nu F : F \in \mathrm{T},\, F \subseteq E,\, F \text{ is closed}\} \text{ for every } E \in \mathrm{T},$$

$$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{F \in \mathrm{T}, F \supseteq K} \nu F.$$

Then there is a Radon measure μ on X extending ν.

proof (a) For $A \subseteq X$, write

$$\nu^* A = \inf_{F \in \mathrm{T}, F \supseteq A} \nu F.$$

Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence of compact subsets of X such that $\lim_{n\to\infty} \nu^* K_n = \nu X$; replacing K_n by $\bigcup_{i<n} K_i$ if necessary, we may suppose that $\langle K_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and that $K_0 = \emptyset$. For each $n \in \mathbb{N}$, set

$$\nu_n' E = \nu^*(E \cap K_n)$$

for every $E \in \mathrm{T}$. Then ν_n' is additive. **P** (I copy from the proof of 413O.) If $E, F \in \mathrm{T}$ and $E \cap F = \emptyset$,

$$\begin{aligned}
\nu_n'(E \cup F) &= \inf\{\nu G : G \in \mathrm{T},\, K_n \cap (E \cup F) \subseteq G\} \\
&= \inf\{\nu G : G \in \mathrm{T},\, K_n \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T},\, K_n \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu G_1 + \nu G_2 : G_1, G_2 \in \mathrm{T},\, K_n \cap E \subseteq G_1 \subseteq E,\, K_n \cap F \subseteq G_2 \subseteq F\} \\
&= \inf\{\nu G_1 : G_1 \in \mathrm{T},\, K_n \cap E \subseteq G_1 \subseteq E\} \\
&\qquad\qquad + \inf\{\nu G_2 : G_2 \in \mathrm{T},\, K_n \cap F \subseteq G_2 \subseteq F\} \\
&= \nu_n' E + \nu_n' F.
\end{aligned}$$

As E and F are arbitrary, ν_n' is additive. **Q**

(b) For each $n \in \mathbb{N}$, set $\nu_n E = \nu'_{n+1}E - \nu'_n E$ for every $E \in \mathrm{T}$; then ν_n is additive. Because $K_{n+1} \supseteq K_n$, ν_n is non-negative.

If $E \in \mathrm{T}$ and $E \cap K_{n+1} = \emptyset$, then $\nu_n E = \nu'_{n+1}E = 0$. So if we set $\mathrm{T}_n = \{E \cap K_{n+1} : E \in \mathrm{T}\}$, we have an additive functional $\tilde{\nu}_n : \mathrm{T}_n \to [0, \infty[$ defined by setting $\tilde{\nu}_n(E \cap K_{n+1}) = \nu_n E$ for every $E \in \mathrm{T}$. Also $\tilde{\nu}_n H = \sup\{\tilde{\nu}_n K : K \in \mathrm{T}_n,\ K \subseteq H,\ K \text{ is compact}\}$ for every $H \in \mathrm{T}_n$. **P** Express H as $E \cap K_{n+1}$ where $E \in \mathrm{T}$. Given $\epsilon > 0$, there is a closed set $F \in \mathrm{T}$ such that $F \subseteq E$ and $\nu F \geq \nu E - \epsilon$; but now $K = F \cap K_{n+1} \in \mathrm{T}_n$ is a compact subset of H, and

$$\tilde{\nu}_n(H \setminus K) = \nu_n(E \setminus F) \leq \nu'_{n+1}(E \setminus F) \leq \nu(E \setminus F) \leq \epsilon,$$

so $\tilde{\nu}_n K \geq \tilde{\nu}_n H - \epsilon$. **Q**

(c) For each $n \in \mathbb{N}$, we have a Radon measure μ_n on K_{n+1}, with domain Σ_n say, such that $\mu_n K_{n+1} \leq \tilde{\nu}_n K_{n+1}$ and $\mu_n K \geq \tilde{\nu}_n K$ for every compact set $K \subseteq K_{n+1}$ (416K). Since K_{n+1} is itself compact, we must have $\mu_n K_{n+1} = \tilde{\nu}_n K_{n+1}$. But this means that μ_n extends $\tilde{\nu}_n$. **P** If $H \in \mathrm{T}_n$ and $\epsilon > 0$ there is a compact set $K \in \mathrm{T}_n$ such that $K \subseteq H$ and $\tilde{\nu}_n K \geq \tilde{\nu}_n H - \epsilon$, so that $(\mu_n)_* H \geq \mu_n K \geq \tilde{\nu}_n H - \epsilon$; as ϵ is arbitrary, $(\mu_n)_* H \geq \tilde{\nu}_n H$. So there is an $F_1 \in \Sigma_n$ such that $F_1 \subseteq H$ and $\mu_n F_1 \geq \tilde{\nu}_n H$. Similarly, there is an $F_2 \in \Sigma_n$ such that $F_2 \subseteq K_{n+1} \setminus H$ and $\mu_n F_2 \geq \tilde{\nu}_n(K_{n+1} \setminus H)$. But in this case $H \setminus F_1 \subseteq K_{n+1} \setminus (F_1 \cup F_2)$ is μ_n-negligible, because

$$\mu_n F_1 + \mu_n F_2 \geq \tilde{\nu}_n H + \tilde{\nu}_n(K_{n+1} \setminus H) = \tilde{\nu}_n K_{n+1} = \mu_n K_{n+1}.$$

So $H \setminus F_1$ and H belong to Σ_n and $\mu_n H = \mu_n F_1 = \tilde{\nu}_n H$. **Q**

(d) Set

$$\Sigma = \{E : E \subseteq X,\ E \cap K_{n+1} \in \Sigma_n \text{ for every } n \in \mathbb{N}\},$$

$$\mu E = \textstyle\sum_{n=0}^{\infty} \mu_n(E \cap K_{n+1}) \text{ for every } E \in \Sigma.$$

Then μ is a Radon measure on X extending ν.

P (i) It is easy to check that Σ is a σ-algebra of subsets of X including T, just because each Σ_n is a σ-algebra of subsets of K_{n+1} including T_n; and that μ is a complete measure because every μ_n is.

(ii) If $E \in \mathrm{T}$, then

$$\begin{aligned}\mu E &= \sum_{n=0}^{\infty} \mu_n(E \cap K_{n+1}) = \sum_{n=0}^{\infty} \tilde{\nu}_n(E \cap K_{n+1}) = \sum_{n=0}^{\infty} \nu_n E = \lim_{n\to\infty} \sum_{i=0}^{n} \nu_i E \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} \nu^*(E \cap K_{i+1}) - \nu^*(E \cap K_i) = \lim_{n\to\infty} \nu^*(E \cap K_n) \leq \nu E.\end{aligned}$$

On the other hand,

$$\mu X = \lim_{n\to\infty} \nu^* K_{n+1} = \nu X,$$

so in fact $\mu E = \nu E$ for every $E \in \mathrm{T}$, that is, μ extends ν. In particular, μ is totally finite, therefore locally determined and locally finite.

(iii) If $G \subseteq X$ is open, then $G \cap K_{n+1} \in \Sigma_n$ for every n, so $G \in \Sigma$; thus μ is a topological measure. If $\mu E > 0$, there is some $n \in \mathbb{N}$ such that $\mu_n(E \cap K_{n+1}) > 0$; now there is a compact set $K \subseteq E \cap K_{n+1}$ such that $\mu_n K > 0$, so that $\mu K > 0$. This shows that μ is tight, so is a Radon measure, as required. **Q**

Remark Observe that in this construction

$$\mu K_{n+1} = \sum_{i=0}^{\infty} \mu_i(K_{n+1} \cap K_{i+1}) = \sum_{i=0}^{\infty} \tilde{\nu}_i(K_{n+1} \cap K_{i+1}) = \sum_{i=0}^{\infty} \nu_i(K_{n+1} \cap K_{i+1})$$
$$= \sum_{i=0}^{\infty} \nu'_{i+1}(K_{n+1} \cap K_{i+1}) - \nu'_i(K_{n+1} \cap K_{i+1})$$
$$= \sum_{i=0}^{\infty} \nu^*(K_{n+1} \cap K_{i+1}) - \nu^*(K_{n+1} \cap K_i)$$
$$= \sum_{i=0}^{n} \nu^*(K_{n+1} \cap K_{i+1}) - \nu^*(K_{n+1} \cap K_i) = \nu^* K_{n+1}$$

for every $n \in \mathbb{N}$. What this means is that if instead of the hypothesis

$$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{F \in \mathfrak{T}, F \supseteq K} \nu F$$

we are presented with a specified non-decreasing sequence $\langle L_n \rangle_{n\in\mathbb{N}}$ of compact subsets of X such that $\nu X = \sup_{n\in\mathbb{N}} \nu^* L_n$, then we can take $K_{n+1} = L_n$ in the argument above and we shall have $\mu L_n = \nu^* L_n$ for every n.

416P Theorem Let X be a Hausdorff space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) for every non-negligible measurable set $E \subseteq X$ there is a compact set $K \subseteq E$ such that $\mu^* K > 0$.

If μ is totally finite, we can add

(iii) $\sup\{\mu^* K : K \subseteq X \text{ is compact}\} = \mu X$.

proof Write Σ for the domain of μ.

(a)(i)⇒(ii) If λ is a Radon measure extending μ, and $\mu E > 0$, then $\lambda E > 0$, so there is a compact set $K \subseteq E$ such that $\lambda K > 0$; but now, because λ is an extension of μ,

$$\mu^* K \geq \lambda^* K = \lambda K > 0.$$

(b)(ii)⇒(i) & (iii)($\boldsymbol{\alpha}$) Let $\mathcal{E}$ be the family of measurable envelopes of compact sets. Then $\mu E < \infty$ for every $E \in \mathcal{E}$. **P** If $E \in \mathcal{E}$, there is a compact set K such that E is a measurable envelope of K. Now $\mu E = \mu^* K$ is finite by 411Ga, as usual. **Q**

Next, $\mathcal{E}$ is closed under finite unions, by 132Ed. The hypothesis (ii) tells us that if $\mu E > 0$ then there is some $F \in \mathcal{E}$ such that $F \subseteq E$ and $\mu F > 0$; for there is a compact set $K \subseteq E$ such that $\mu^* K > 0$, K has a measurable envelope F_0, and $F = E \cap F_0$ is still a measurable envelope of K. So in fact μ is inner regular with respect to $\mathcal{E}$ (412Aa). In particular, μ is semi-finite.

If $\gamma < \mu X$ there is an $F \in \mathcal{E}$ such that $\mu F \geq \gamma$, and now there is a compact set K such that F is a measurable envelope of K, so that $\mu^* K = \mu F \geq \gamma$. As γ is arbitrary, (iii) is true.

($\boldsymbol{\beta}$) Because μ is inner regular with respect to $\mathcal{E}$, $D = \{E^\bullet : E \in \mathcal{E}\}$ is order-dense in the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ (412N), so there is a family $\langle d_i \rangle_{i\in I}$ in D which is a partition of unity in $\mathfrak{A}$ (313K). For each $i \in I$, take $E_i \in \mathcal{E}$ such that $E_i^\bullet = d_i$. Then

$$\sum_{i\in I} \mu(E \cap E_i) = \sum_{i\in I} \bar{\mu}(E^\bullet \cap d_i) = \bar{\mu} E^\bullet$$

(321E)

$$= \mu E$$

for every $E \in \Sigma$.

(γ) For each $i \in I$, let μ_i be the subspace measure on E_i. Then there is a Radon measure λ_i on E_i extending μ_i. **P** Because μ is inner regular with respect to the closed sets, μ_i is inner regular with respect to the relatively closed subsets of E_i (412Oa). Also there is a compact subset $K \subseteq E_i$ such that

$$\mu_i E_i = \mu E_i = \mu^* K = \mu_i^* K,$$

so μ_i satisfies the conditions of 416O and has an extension to a Radon measure. **Q**

(δ) Define

$$\lambda E = \sum_{i \in I} \lambda_i (E \cap E_i)$$

whenever $E \subseteq X$ is such that λ_i measures $E \cap E_i$ for every $i \in I$. Then λ is a Radon measure on X extending μ. **P** It is easy to check that it is a measure, just because every λ_i is a measure, and it extends μ by (β) above. If $G \subseteq X$ is open, then $G \cap E_i$ is relatively open for every $i \in I$, so λ measures G; thus λ is a topological measure. If $\lambda E = 0$ and $A \subseteq E$, then $\lambda_i(A \cap E_i) \le \lambda(E \cap E_i) = 0$ for every i, so $\lambda A = 0$; thus λ is complete. For all distinct i, $j \in I$,

$$\lambda_i(E_i \cap E_j) = \mu_i(E_i \cap E_j) = \mu(E_i \cap E_j) = \bar{\mu}(d_i \cap d_j) = 0,$$

so $\lambda E_i = \lambda_i E_i = \mu_i E_i$ is finite. This means that if $E \subseteq X$ is such that λ measures $E \cap F$ whenever $\lambda F < \infty$, then λ must measure $E \cap E_i$ for every i, and λ measures E; thus λ is locally determined. If $\lambda E > 0$ there are an $i \in I$ such that $\lambda_i(E \cap E_i) > 0$ and a compact $K \subseteq E \cap E_i$ such that $0 < \lambda_i K = \lambda K$; consequently λ is tight. Finally, if $x \in X$, there is an $E \in \Sigma$ such that $x \in \operatorname{int} E$ and $\lambda E = \mu E < \infty$, so λ is locally finite. Thus λ is a Radon measure. **Q**

So (i) is true.

(c) Finally, suppose that μ is totally finite and (iii) is true. Then we can appeal directly to 416O to see that (i) is true.

416Q Proposition (a) Let X be a compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Then any non-negative finitely additive functional from $\mathcal{E}$ to $\mathbb{R}$ has an extension to a Radon measure on X. If X is zero-dimensional then the extension is unique.

(b) Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Then there is a one-to-one correspondence between non-negative additive functionals ν on $\mathfrak{A}$ and Radon measures μ on Z given by the formula

$$\nu a = \mu \widehat{a} \text{ for every } a \in \mathfrak{A},$$

where for $a \in \mathfrak{A}$ I write $\widehat{a}$ for the corresponding open-and-closed subset of Z.

proof (a) Let $\nu : \mathcal{E} \to [0, \infty[$ be a non-negative additive functional. Then ν satisfies the conditions of 416O (because every member of $\mathcal{E}$ is closed, while X is compact), so has an extension to a Radon measure μ. If X is zero-dimensional, $\mathcal{E}$ is a base for the topology of X closed under finite unions and intersections, so μ is unique, by 415H(iv) or 415H(v).

(b) The map $a \mapsto \widehat{a}$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra $\mathcal{E}$ of open-and-closed subsets of Z, so we have a one-to-one correspondence between non-negative additive functionals ν on $\mathfrak{A}$ and non-negative additive functionals ν' on $\mathcal{E}$, defined by the formula $\nu' \widehat{a} = \nu a$. Now Z is compact, Hausdorff and zero-dimensional, so ν' has a unique extension to a Radon measure on Z, by part (a). And of course every Radon measure μ on Z gives us a non-negative additive functional $\mu \restriction \mathcal{E}$ on $\mathcal{E}$, corresponding to a non-negative additive functional on $\mathfrak{A}$.

416R Theorem (a) Any subspace of a Radon measure space is a quasi-Radon measure space.

(b) A measurable subspace of a Radon measure space is a Radon measure space.

(c) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Hausdorff complete locally determined topological measure space, and $Y \subseteq X$ is such that the subspace measure μ_Y on Y is a Radon measure, then $Y \in \Sigma$.

proof (a) Put 416A and 415B together.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $(E, \mathfrak{T}_E, \Sigma_E, \mu_E)$ a member of Σ with the induced topology and measure. Because μ is complete and locally determined, so is μ_E (214Ka). Because $\mathfrak{T}$ is

Hausdorff, so is $\mathfrak{T}_E$ (3A3Bh). Because μ is locally finite, so is μ_E. Because μ is tight (and a subset of E is compact for $\mathfrak{T}_E$ whenever it is compact for $\mathfrak{T}$), μ_E is tight (412Oa again).

(c) ? If $Y \notin \Sigma$, then there is a set $F \in \Sigma$ such that $\mu_*(Y \cap F) < \mu^*(Y \cap F)$ (413F(v)). But now $\mu^*(Y \cap F) = \mu_Y(Y \cap F)$, so there is a compact set $K \subseteq Y \cap F$ such that $\mu_Y K > \mu_*(Y \cap F)$. When regarded as a subset of X, K is still compact; because $\mathfrak{T}$ is Hausdorff, K is closed, so belongs to Σ, and

$$\mu_*(Y \cap F) \geq \mu K = \mu_Y K > \mu_*(Y \cap F),$$

which is absurd. **X**

416S Corresponding to 415O, we have the following.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space.

(a) If ν is a locally finite indefinite-integral measure over μ, it is a Radon measure.

(b) If ν is a Radon measure on X and $\nu K = 0$ whenever $K \subseteq X$ is compact and $\mu K = 0$, then ν is an indefinite-integral measure over μ.

proof (a) Because μ is complete and locally determined, so is ν (234Nb). Because μ is tight, so is ν (412Q). So if ν is also locally finite, it is a Radon measure.

(b) Write T for the domain of ν.

(i) If $E \in \Sigma \cap \mathrm{T}$ and $\mu E = 0$, then $\nu K = \mu K = 0$ for every compact $K \subseteq E$, so $\nu E = 0$.

(ii) $\mathrm{T} \supseteq \Sigma$. **P** If $E \in \Sigma$ and $K \subseteq X$ is compact, there are Borel sets F, F' such that $F \subseteq E \cap K \subseteq F' \subseteq K$ and $\mu(F' \setminus F) = 0$. Consequently $\nu(F' \setminus F) = 0$ and $E \cap K \in \mathrm{T}$, because ν is complete. By 416Db, $E \in \mathrm{T}$; as E is arbitrary, $\Sigma \subseteq \mathrm{T}$. **Q**

(iii) If $E \in \mathrm{T}$, there is an $F \in \Sigma$ such that $E \subseteq F$ and $\nu(F \setminus E) = 0$. **P** By 416Dc and 412Ia, there is a decomposition $\langle X_i \rangle_{i \in I}$ of X for the measure μ such that at most one of the X_i is not a compact μ-self-supporting set, and that exceptional one, if any, is μ-negligible. For each i, let F_i be such that

— if X_i is a compact μ-self-supporting set, then F_i is a Borel subset of X_i, $F_i \supseteq E \cap X_i$ and $\nu(F_i \setminus E) = 0$,

— if X_i is not a compact μ-self-supporting set, $F_i = X_i$.

Then $F_i \in \Sigma$ for every i so $F = \bigcup_{i \in I} F_i$ belongs to Σ. We also have $\nu(F_i \setminus E) = 0$ for every i, because if X_i is not compact and μ-self-supporting then $\nu X_i = \mu X_i = 0$. Of course $E \subseteq F$. If $K \subseteq X$ is compact, there is an open set $G \supseteq K$ such that $\mu G < \infty$; consequently $\{i : \mu(X_i \cap G) \geq \epsilon\}$ is finite for every $\epsilon > 0$, so $\{i : X_i \cap K \neq \emptyset\}$ is countable and $\nu(K \cap F \setminus E) = 0$. By 412Jb, $\nu(F \setminus E) = 0$. **Q**

Applying the same argument to $X \setminus E$, we can get an $F' \in \Sigma$ such that $F' \subseteq E$ and $\nu(E \setminus F') = 0$. As E is arbitrary, ν is the completion of its restriction to Σ.

(iv) Now look at the conditions of 234O. We know that μ is localizable and ν is semi-finite. We saw in (ii) above that $\mathrm{T} \supseteq \Sigma$ and in (i) that ν is zero on μ-negligible sets. In (iii) we saw that ν is the completion of $\nu \restriction \Sigma$. And if $\nu E > 0$ there is a compact $K \subseteq E$ such that $\nu K > 0$, while $\mu K < \infty$. So 234O tells us that ν is an indefinite-integral measure over μ.

416T I said in the notes to §415 that the most important quasi-Radon measure spaces are subspaces of Radon measure spaces. I do not know of a useful necessary and sufficient condition, but the following deals with completely regular spaces.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally finite completely regular Hausdorff quasi-Radon measure space. Then it is isomorphic, as topological measure space, to a subspace of a locally compact Radon measure space.

proof (a) Write βX for the Stone-Čech compactification of X (4A2I); I will take it that X is actually a subspace of βX. Let $\mathcal{U}$ be the set of those open subsets U of βX such that $\mu(U \cap X) < \infty$; then $\mathcal{U}$ is upwards-directed and covers X, because μ is locally finite. Set $W = \bigcup \mathcal{U} \supseteq X$. Then W is an open subset of βX, so is locally compact.

(b) Let $\mathcal{B}(W)$ be the Borel σ-algebra of W. Then $V \cap X$ is a Borel subset of X for every $V \in \mathcal{B}(W)$ (4A3Ca), so we have a measure $\nu : \mathcal{B}(W) \to [0, \infty]$ defined by setting $\nu V = \mu(X \cap V)$ for every $V \in \mathcal{B}(W)$. Now ν satisfies the conditions of 415Cb. **P** (α) If $\nu V > 0$, then, because μ is effectively locally finite, there is an open set $G \subseteq X$ such that $\mu(G \cap V) > 0$ and $\mu G < \infty$. There is an open set $U \subseteq \beta X$ such that $U \cap X = G$, in which case $U \subseteq W$, $\nu U < \infty$ and $\nu(U \cap V) > 0$. Thus ν is effectively locally finite. (β) If $\mathcal{U}$ is an upwards-directed family of open subsets of W, then $\{U \cap X : U \in \mathcal{U}\}$ is an upwards-directed family of open subsets of X, so

$$\begin{aligned}\nu(\bigcup \mathcal{U}) &= \mu(X \cap \bigcup \mathcal{U}) = \mu(\bigcup\{U \cap X : U \in \mathcal{U}\}) \\ &= \sup_{U \in \mathcal{U}} \mu(X \cap U) = \sup_{U \in \mathcal{U}} \nu U.\end{aligned}$$

Thus ν is τ-additive. **Q** So the c.l.d. version $\tilde{\nu}$ of ν is a quasi-Radon measure on W (415Cb).

(c) The construction of W ensures that ν and $\tilde{\nu}$ are locally finite. By 416G, $\tilde{\nu}$ is a Radon measure. So the subspace measure $\tilde{\nu}_X$ is a quasi-Radon measure on X (416Ra). But $\tilde{\nu}_X G = \mu G$ for every open set $G \subseteq X$. **P** Note first that as ν effectively locally finite, therefore semi-finite, $\tilde{\nu}$ extends ν (213Hc again). If $K \subseteq W$ is a compact set not meeting X, then

$$\tilde{\nu} K = \nu K = \mu(K \cap X) = 0;$$

accordingly $\tilde{\nu}_*(W \setminus X) = 0$, by 413Ee. Now there is an open set $U \subseteq W$ such that $G = X \cap U$, and

$$\begin{aligned}\tilde{\nu}_X G = \tilde{\nu}^* G &\le \tilde{\nu} U = \nu U = \mu(U \cap X) = \mu G \\ &= \tilde{\nu}^*(U \cap X) + \tilde{\nu}_*(U \setminus X)\end{aligned}$$

(by 413E(c-ii), because $\tilde{\nu}$ is semi-finite)

$$\le \tilde{\nu}^*(U \cap X) + \tilde{\nu}_*(W \setminus X) = \tilde{\nu}^* G. \ \mathbf{Q}$$

So 415H(iii) tells us that $\mu = \tilde{\nu}_X$ is the subspace measure induced by ν.

416U Theorem (a) If $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ is a family of compact metrizable Radon probability spaces such that every μ_i is strictly positive, the product measure on $X = \prod_{i \in I} X_i$ is a completion regular Radon measure.

(b) In particular, the usual measures on $\{0,1\}^I$ and $[0,1]^I$ and $\mathcal{P}I$ are completion regular Radon measures, for any set I.

proof (a) By 415E, it is a completion regular quasi-Radon probability measure; but X is a compact Hausdorff space, so it is a Radon measure, by 416G or otherwise.

(b) follows at once. (I suppose it is obvious that by the 'usual measure on $[0,1]^I$' I mean the product measure when each copy of $[0,1]$ is given Lebesgue measure. Recall also that the 'usual measure on $\mathcal{P}I$' is just the copy of the usual measure on $\{0,1\}^I$ induced by the standard bijection $A \leftrightarrow \chi A$ (254Jb), which is a homeomorphism (4A2Ud).

416V Stone spaces The results of 415Q-415R become simpler and more striking in the present context.

Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of its measure algebra $(\mathfrak{A}, \bar{\mu})$. For $E \in \Sigma$ let E^* be the open-and-closed set in Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Define $R \subseteq Z \times X$ by saying that $(z, x) \in R$ iff $x \in F$ whenever $F \subseteq X$ is closed and $z \in F^*$.

(a) R is the graph of a function $f : Q \to X$, where $Q = R^{-1}[X]$. If we set $W = \bigcup\{K^* : K \subseteq X$ is compact$\}$, then $W \subseteq Q$ is a ν-conegligible open set, and the subspace measure ν_W on W is a Radon measure.

(b) Setting $g = f \restriction W$, g is continuous and μ is the image measure $\nu_W g^{-1}$.

(c) If X is compact, $W = Q = Z$ and $\mu = \nu g^{-1}$.

proof (a) By 415Ra, R is the graph of a function. If $K \subseteq X$ is compact and $z \in K^*$, then $\mathcal{F} = \{F : F \subseteq X$ is closed, $z \in F^*\}$ is a family of non-empty closed subsets of X, closed under finite intersections, and containing the compact set K; so it has non-empty intersection, and there is an $x \in K$ such that $(z, x) \in R$, that is, $z \in Q$ and $f(z) \in K$. Thus $W \subseteq Q$. Of course W is an open set, being the union of a family of open-and-closed sets; but it is also conegligible, because $\sup\{K^\bullet : K \subseteq X \text{ is compact}\} = 1$ in $\mathfrak{A}$ (412N again), so $Z \setminus W$ must be nowhere dense, therefore negligible. Now the subspace measure ν_W is quasi-Radon because ν is (411P(d-iv), 415B); but W is a union of compact open sets of finite measure, so ν_W is locally finite and W is locally compact; by 416G, ν_W is a Radon measure.

(b) g is continuous. **P** Let $G \subseteq X$ be an open set and $z \in g^{-1}[G]$. Let $K \subseteq X$ be a compact set such that $z \in K^*$. As remarked above, $g(z) = f(z)$ belongs to K. K, being a compact Hausdorff space, is regular (3A3Bb), so there is an open set H containing $g(z)$ such that $L = \overline{H \cap K}$ is included in G. Note that L is compact, so $L^* \subseteq W$. Now $g(z)$ does not belong to the closed set $X \setminus H$, so $z \notin (X \setminus H)^*$ and $z \in H^*$; accordingly $z \in (H \cap K)^* \subseteq L^*$. If $w \in L^*$, $g(w) \in L \subseteq G$; so $L^* \subseteq g^{-1}[G]$, and $z \in \operatorname{int} g^{-1}[G]$. As z is arbitrary, $g^{-1}[G]$ is open; as G is arbitrary, g is continuous. **Q**

By 415Rb, we know that $\mu = \nu_Q f^{-1}$, where ν_Q is the subspace measure on Q. But as ν is complete and both Q and W are conegligible, we have

$$\nu_Q f^{-1}[A] = \nu f^{-1}[A] = \nu g^{-1}[A] = \nu_W g^{-1}[A]$$

whenever $A \subseteq X$ and any of the four terms is defined, so that $\mu = \nu_Q f^{-1} = \nu_W g^{-1}$.

(c) If X is compact, then $Z = X^* \subseteq W$, so $W = Q = Z$ and $\nu g^{-1} = \nu_W g^{-1} = \mu$.

416W Compact measure spaces Recall that a semi-finite measure space (X, Σ, μ) is 'compact' (as a measure space) if there is a family $\mathcal{K} \subseteq \Sigma$ such that μ is inner regular with respect to $\mathcal{K}$ and $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property (342Ac); while (X, Σ, μ) is 'perfect' if whenever $f : X \to \mathbb{R}$ is measurable and $\mu E > 0$, there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$ (342K). In §342 I introduced these concepts in order to study the realization of homomorphisms between measure algebras. The following result is now very easy.

Proposition (a) Any Radon measure space is a compact measure space, therefore perfect.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$, and (Y, T, ν) a complete strictly localizable measure space, with measure algebra $(\mathfrak{B}, \bar{\nu})$. If $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a function $f : Y \to X$ such that $f^{-1}[E] \in \mathrm{T}$ and $f^{-1}[E]^\bullet = \pi E^\bullet$ for every $E \in \Sigma$. If π is measure-preserving, f is inverse-measure-preserving.

proof (a) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space, μ is inner regular with respect to the compact class consisting of the compact subsets of X, so (X, Σ, μ) is a compact measure space. By 342L, it is perfect.

(b) Use (i)$\Rightarrow$(v) of Theorem 343B. (Of course f is inverse-measure-preserving iff π is measure-preserving.)

416X Basic exercises >(a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $E \in \Sigma$ an atom for the measure. Show that there is a point $x \in E$ such that $\mu\{x\} = \mu E$.

(b) Let X be a Hausdorff space and μ a point-supported measure on X, as described in 112Bd. (i) Show that μ is tight, so is a Radon measure iff it is locally finite. In particular, show that if X has its discrete topology then counting measure on X is a Radon measure. (ii) Show that every purely atomic Radon measure is of this type.

>(c) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of Radon measure spaces, with direct sum (X, Σ, μ) (214L). Give X its disjoint union topology. Show that μ is a Radon measure.

(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite Radon measure space with $\mu X > 0$. Show that there is a Radon probability measure on X with the same measurable sets and the same negligible sets as μ.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Show that μ has a decomposition $\langle X_i \rangle_{i \in I}$ in which every X_i except at most one is a self-supporting compact set, and the exceptional one, if any, is negligible.

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Show that $\alpha\mu$, defined on Σ, is a Radon measure for any $\alpha > 0$.

(g) Let X be a Hausdorff space and μ, ν two Radon measures on X such that $\nu G = \mu G$ whenever $G \subseteq X$ is open and of finite measure for both. Show that $\mu = \nu$.

(h) Let $(X, \mathfrak{T})$ be a completely regular Hausdorff space and μ a locally finite topological measure on X which is inner regular with respect to the closed sets. Show that

$$\begin{aligned}\mu K &= \inf\{\mu G : G \supseteq K \text{ is a cozero set}\}\\ &= \inf\{\mu F : F \supseteq K \text{ is a zero set}\} = \inf\{\int f d\mu : \chi K \le f \in C(X)\}\end{aligned}$$

for every compact set $K \subseteq X$.

(i) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally compact Radon measure space, and C_k the space of continuous real-valued functions on X with compact supports. Show that $\{f^\bullet : f \in C_k\}$ is dense in $L^0(\mu)$ for the topology of convergence in measure.

(j) Let $(X, \mathfrak{T})$ be a Hausdorff space, $\Sigma \supseteq \mathfrak{T}$ a σ-algebra of subsets of X, and $\nu : \Sigma \to [0, \infty[$ a finitely additive functional such that $\nu E = \sup\{\nu K : K \subseteq E \text{ is compact}\}$ for every $E \in \Sigma$. Show that ν is countably additive and that its completion is a Radon measure on X.

(k) Let X be a completely regular Hausdorff space and ν a locally finite Baire measure on X. (i) Show that $\nu^* K = \inf\{\nu G : G \subseteq X \text{ is a cozero set}, K \subseteq G\}$ for every compact set $K \subseteq X$. (ii) Show that there is a Radon measure μ on X such that $\mu K = \nu^* K$ for every compact set $K \subseteq X$.

(l) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence of Hausdorff spaces with product X; write $\mathcal{B}(X_n)$ for the Borel σ-algebra of X_n. Let T be the σ-algebra $\widehat{\bigotimes}_{n \in \mathbb{N}} \mathcal{B}(X_n)$ (definition: 254E). Let $\nu : \mathrm{T} \to [0, \infty[$ be a finitely additive functional such that $E \mapsto \nu \pi_n^{-1}[E] : \mathcal{B}(X_n) \to [0, \infty[$ is countably additive and tight for each $n \in \mathbb{N}$, writing $\pi_n(x) = x(n)$ for $x \in X$ and $n \in \mathbb{N}$. Show that there is a unique Radon measure on X extending ν. (*Hint*: 416O.)

(m) Set $S_2 = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$, and let $\phi : S_2 \to [0, \infty[$ be a functional such that $\phi(\sigma) = \phi(\sigma^\frown{<}0{>}) + \phi(\sigma^\frown{<}1{>})$ for every $\sigma \in S_2$, writing $\sigma^\frown{<}0{>}$ and $\sigma^\frown{<}1{>}$ for the two members of $\{0, 1\}^{n+1}$ extending any $\sigma \in \{0, 1\}^n$. Show that there is a unique Radon measure μ on $\{0, 1\}^{\mathbb{N}}$ such that $\mu\{x : x \restriction \{0, \dots, n-1\} = \sigma\} = \phi(\sigma)$ whenever $n \in \mathbb{N}$, $\sigma \in \{0, 1\}^{\mathbb{N}}$.

(n) Let X be a Hausdorff space, μ a complete locally finite measure on X, and Y a conegligible subset of X. Show that μ is a Radon measure iff the subspace measure on Y is a Radon measure.

(o) Let X be a Hausdorff space, Y a subset of X, and ν a Radon measure on Y. Define a measure μ on X by setting $\mu E = \nu(E \cap Y)$ whenever ν measures $E \cap Y$. Show that if *either* Y is closed *or* ν is totally finite, μ is a Radon measure on X. (Cf. 418I.)

(p) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and $\mathcal{E} \subseteq \Sigma$ a non-empty upwards-directed family. Set $\nu F = \sup_{E \in \mathcal{E}} \mu(E \cap F)$ whenever $F \subseteq X$ is such that μ measures $E \cap F$ for every $E \in \mathcal{E}$. Show that ν is a Radon measure on X.

(q) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Show that a measure ν on X is an indefinite-integral measure over μ iff (α) ν is a complete locally determined tight topological measure (β) $\nu K = 0$ whenever $K \subseteq X$ is compact and $\mu K = 0$.

(r) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a compact Hausdorff quasi-Radon measure space. Let $W \subseteq X$ be the union of the open subsets of X of finite measure. Show that the subspace measure on W is a Radon measure.

(s) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular Radon measure space. Show that it is isomorphic, as topological measure space, to a measurable subspace of a locally compact Radon measure space.

(t) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a compact Radon measure space and $(Z,\mathfrak{S},\mathrm{T},\nu)$ the Stone space of its measure algebra. For $E\in\Sigma$ let E^* be the corresponding open-and-closed subset of Z, as in 416V. Show that the function described in 416V is the unique continuous function $h:Z\to X$ such that $\nu(E^*\triangle h^{-1}[E])=0$ for every $E\in\Sigma$. (*Hint*: 415Qd.)

(u) Show that the Sorgenfrey line (415Xc, 439Q), with Lebesgue measure, is a quasi-Radon measure space which, regarded as a measure space, is compact, but, regarded as a topological measure space, is not a Radon measure space.

(v) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $(Z,\mathfrak{T},\Sigma,\mu)$ its Stone space. Show that if ν is a strictly positive Radon measure on Z then μ is an indefinite-integral measure over ν.

416Y Further exercises (a) Let $X\subseteq\beta\mathbb{N}$ be the union of all those open sets $G\subseteq\beta\mathbb{N}$ such that $\sum_{n\in G\cap\mathbb{N}}\frac{1}{n+1}$ is finite. For $E\subseteq X$ set $\mu E=\sum_{n\in E\cap\mathbb{N}}\frac{1}{n+1}$. Show that μ is a σ-finite Radon measure on the locally compact Hausdorff space X. Show that μ is not outer regular with respect to the open sets.

(b) Let X be a Hausdorff space and ν a countably additive real-valued functional defined on a σ-algebra Σ of subsets of X. Show that the following are equiveridical: (i) $|\nu|:\Sigma\to[0,\infty[$, defined as in 362B, is a Radon measure on X; (ii) ν is expressible as $\mu_1-\mu_2$, where μ_1, μ_2 are Radon measures on X and $\Sigma=\operatorname{dom}\mu_1\cap\operatorname{dom}\mu_2$.

(c) Let X be a topological space and μ_0 a semi-finite measure on X which is inner regular with respect to the family $\mathcal{K}_{\mathrm{ccc}}$ of closed countably compact sets. Show that μ_0 has an extension to a complete locally determined topological measure μ on X, still inner regular with respect to $\mathcal{K}_{\mathrm{ccc}}$; and that the extension may be done in such a way that whenever $\mu E<\infty$ there is an $E_0\in\Sigma_0$ such that $\mu(E\triangle E_0)=0$.

(d) Let X be a topological space and μ_0 a semi-finite measure on X which is inner regular with respect to the family $\mathcal{K}_{\mathrm{sc}}$ of sequentially compact sets. Show that μ_0 has an extension to a complete locally determined topological measure μ on X, still inner regular with respect to $\mathcal{K}_{\mathrm{sc}}$; and that the extension may be done in such a way that whenever $\mu E<\infty$ there is an $E_0\in\Sigma_0$ such that $\mu(E\triangle E_0)=0$.

(e) Set $S=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$, and let $\phi:S\to[0,\infty[$ be a functional such that $\phi(\sigma)=\sum_{i=0}^{\infty}\phi(\sigma^\frown{<}i{>})$ for every $\sigma\in S$, writing $\sigma^\frown{<}i{>}$ for the members of $\mathbb{N}^{n+1}$ extending any $\sigma\in\mathbb{N}^n$. Show that there is a unique Radon measure μ on $\mathbb{N}^{\mathbb{N}}$ such that $\mu I_\sigma=\phi(\sigma)$ for every $\sigma\in S$, where $I_\sigma=\{x:x\restriction\{0,\dots,n-1\}=\sigma\}$ for any $n\in\mathbb{N}$, $\sigma\in\mathbb{N}^{\mathbb{N}}$.

(f) In 416Qb, show that μ is atomless iff ν is properly atomless in the sense of 326F.

(g) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space. Show that a measure ν on X is an indefinite-integral measure over μ iff (i) there is a topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, such that ν is a Radon measure with respect to $\mathfrak{S}$ (ii) $\nu K=0$ whenever K is a $\mathfrak{T}$-compact set and $\mu K=0$.

(h) Let $\langle x_n\rangle_{n\in\mathbb{N}}$ enumerate a dense subset of $X=\{0,1\}^{\mathfrak{c}}$ (4A2B(e-ii)). Let $\nu_{\mathfrak{c}}$ be the usual measure on X, and set $\mu E=\frac{1}{2}\nu_{\mathfrak{c}}E+\sum_{n=0}^{\infty}2^{-n-2}\chi E(x_n)$ for $E\in\operatorname{dom}\nu_{\mathfrak{c}}$. (i) Show that μ is a strictly positive Radon probability measure on X with Maharam type $\mathfrak{c}$. (ii) Let $I\in[\mathfrak{c}]^{\le\omega}$ be such that $x_m\restriction I\ne x_n\restriction I$ whenever $m\ne n$. Set $Z=\{0,1\}^I$ and let $\pi:X\to Z$ be the canonical map. Show that if $f\in C(X)$ is such that $\int f\times g\pi\,d\mu=0$ for every $g\in C(Z)$, then $f=0$. (*Hint*: otherwise, take $n\in\mathbb{N}$ such that $|f(x_n)|\ge\frac{1}{2}\|f\|_\infty$, and let $g\ge0$ be such that $g(\pi x_n)=1$ and $\int g\,d(\mu\pi^{-1})<3\cdot2^{-n-3}$; show that $\int f\times g\pi\,d\mu>0$.) (iii) Show that there is no orthonormal basis for $L^2(\mu)$ in $\{f^\bullet:f\in C(X)\}$. (See Hart & Kunen 99.)

(i) In 254Ye, show that if we start from a continuous inverse-measure-preserving $f:[0,1]\to[0,1]^2$, as in 134Yl, we get a continuous inverse-measure-preserving surjection $g:[0,1]\to[0,1]^{\mathbb{N}}$.

(j) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and $\mathcal{A}\subseteq\Sigma$ a countable set. Let $\mathfrak{S}$ be the topology generated by $\mathfrak{T}\cup\mathcal{A}$. Show that μ is $\mathfrak{S}$-Radon.

416 Notes and comments The original measures studied by Radon (Radon 1913) were, in effect, what I call differences of Radon measures on $\mathbb{R}^r$, as introduced in §256. Successive generalizations moved first to Radon measures on general compact Hausdorff spaces, then to locally compact Hausdorff spaces, and finally to arbitrary Hausdorff spaces, as presented in this section. I ought perhaps to remark that, following Bourbaki 65, many authors use the term 'Radon measure' to describe a linear functional on a space of continuous functions; I will discuss the relationship between such functionals and the measures of this chapter in §436. For the moment, observe that by 415I a Radon measure on a completely regular space can be determined from the integrals it assigns to continuous functions. It is also common for the phrase 'Radon measure' to be used for what I would call a tight Borel measure; you have to check each author to see whether local finiteness is also assumed. In my usage, a Radon measure is necessarily the c.l.d. version of a Borel measure. The Borel measures which correspond to Radon measures are described in 416F.

In §256, I discussed Radon measures on $\mathbb{R}^r$ as a preparation for a discussion of convolutions of measures. It should now be coming clear why I felt that it was impossible, in that context, to give you a proper idea of what a Radon measure, in the modern form, 'really' is. In Euclidean space, too many concepts coincide. As a trivial example, the simplest definition of 'local finiteness' (256Ab) is not the right formulation in other spaces (411Fa). Next, because every closed set is a countable union of compact sets, there is no distinction between 'inner regular with respect to closed sets' and 'inner regular with respect to compact sets', so one cannot get any intuition for which is important in which arguments. (When we come to subspace measures on non-measurable subsets, of course, this changes; quasi-Radon measures on subsets of Euclidean space are important and interesting.) Third, the fact that the c.l.d. product of two Radon measures on Euclidean space is already a Radon measure (256K) leaves us with no idea of what to do with a general product of Radon measures. (There are real difficulties at this point, which I will attack in the next section. For the moment I offer just 416U.) And finally, we simply cannot represent a product of uncountably many Radon probability measures on Euclidean spaces as a measure on Euclidean space.

As you would expect, a very large proportion of the results of this chapter, and many theorems from earlier volumes, were originally proved for compact Radon measure spaces. The theory of general totally finite Radon measures is, in effect, the theory of measurable subspaces of compact Radon measure spaces, while the theory of quasi-Radon measures is pretty much the theory of non-measurable subspaces of Radon measure spaces. Thus the theorem that every quasi-Radon measure space is strictly localizable is almost a consequence of the facts that every Radon measure space is strictly localizable and any subspace of a strictly localizable space is strictly localizable.

The cluster of results between 416J and 416Q form only a sample, I hope a reasonably representative sample, of the many theorems on construction of Radon measures from functionals on algebras or lattices of sets. (See also 416Ye.) The essential simplification, compared with the theorems in §413 and §415, is that we do not need to mention any σ- or τ-additivity condition of the type 413J(β) or 415K(β), because we are dealing with a 'compact class', the family of compact subsets of a Hausdorff space. We can use this even at some distance, as in 416O (where the hypotheses do not require any non-empty compact set to belong to the domain of the original functional). The particular feature of 416O which makes it difficult to prove from such results as 413K and 413P above is that we have to retain control of the outer measures of a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of non-measurable sets. In general this is hard to do, and is possible here principally because the sequence is non-decreasing, so that we can make sense of the functionals $E\mapsto\nu^*(E\cap K_{n+1})-\nu^*(E\cap K_n)$; compare 214P.

In 416De and 416Ea I suggest elements of an algebraic structure on a space of Radon measures; for more about this, see 436Xs and 437Yi below.

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417 τ-additive product measures

The 'ordinary' product measures introduced in Chapter 25 have served us well for a volume and a half. But we come now to a fundamental obstacle. If we start with two Radon measure spaces, their product measure, as defined in §251, need not be a Radon measure (419E). Furthermore, the counterexample is one of the basic compact measure spaces of the theory; and while it is dramatically non-metrizable, there is no other reason to set it aside. Consequently, if we wish (as we surely do) to create Radon measure spaces as products of Radon measure spaces, we need a new construction. This is the object of the present section. It turns out that the construction can be adapted to work well beyond the special context of Radon measure spaces; the methods here apply to general effectively locally finite τ-additive topological measures (for the product of finitely many factors) and to τ-additive topological probability measures (for the product of infinitely many factors).

The fundamental theorems are 417C and 417E, listing the essential properties of what I call 'τ-additive product measures', which are extensions of the c.l.d. product measures and product probability measures of Chapter 25. They depend on a straightforward lemma on the extension of a measure to make every element of a given class of sets negligible (417A). We still have Fubini's theorem for the new product measures (417G), and the basic operations from §254 still apply (417J, 417K, 417M).

It is easy to check that if we start with quasi-Radon measures, then the τ-additive product measure is again quasi-Radon (417N, 417O). The τ-additive product of two Radon measures is Radon (417P), and the τ-additive product of Radon probability measures with compact supports is Radon (417Q).

In the last part of the section I look at continuous real-valued functions and Baire σ-algebras; it turns out that for these the ordinary product measures are adequate (417U, 417V).

417A Lemma Let (X,Σ,μ) be a semi-finite measure space, and $\mathcal{A}\subseteq\mathcal{P}X$ a family of sets such that the inner measure $\mu_*(\bigcup_{n\in\mathbb{N}}A_n)$ is 0 for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$. Then there is a measure μ' on X, extending μ, such that

(i) $\mu'A$ is defined and zero for every $A\in\mathcal{A}$;

(ii) μ' is complete if μ is;

(iii) for every F in the domain Σ' of μ' there is an $E\in\Sigma$ such that $\mu'(F\triangle E)=0$;

(iv) whenever $\mathcal{K}$, $\mathcal{G}$ are families of sets such that

(α) μ is inner regular with respect to $\mathcal{K}$,

(β) $K\cup K'\in\mathcal{K}$ for all K, $K'\in\mathcal{K}$,

(γ) $\bigcap_{n\in\mathbb{N}}K_n\in\mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

(δ) for every $A\in\mathcal{A}$ there is a $G\in\mathcal{G}$, including A, such that $G\setminus A\in\Sigma$,

(ϵ) $K\setminus G\in\mathcal{K}$ whenever $K\in\mathcal{K}$ and $G\in\mathcal{G}$,

then μ' is inner regular with respect to $\mathcal{K}$.

In particular, μ and μ' have isomorphic measure algebras, so that μ' is localizable if μ is.

proof (a) Let $\mathcal{A}^*$ be the collection of subsets of X which can be covered by a countable subfamily of $\mathcal{A}$. Then $\mathcal{A}^*$ is a σ-ideal of subsets of X and $\mu_*A=0$ for every $A\in\mathcal{A}^*$. Set

$$\Sigma'=\{E\triangle A:E\in\Sigma,\ A\in\mathcal{A}^*\}.$$

Then Σ' is a σ-algebra of subsets of X. **P** (i) $\emptyset=\emptyset\triangle\emptyset\in\Sigma'$. (ii) If $E\in\Sigma$, $A\in\mathcal{A}^*$ then $X\setminus(E\triangle A)=(X\setminus E)\triangle A\in\Sigma'$. (iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle A_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ, $\mathcal{A}^*$ respectively, then

$$E=\bigcup_{n\in\mathbb{N}}E_n\in\Sigma,\quad A=E\triangle\bigcup_{n\in\mathbb{N}}(E_n\triangle A_n)\subseteq\bigcup_{n\in\mathbb{N}}A_n\in\mathcal{A}^*,$$

so $\bigcup_{n\in\mathbb{N}}(E_n\triangle A_n)=E\triangle A\in\Sigma'$. **Q**

(b) If E, $E'\in\Sigma$, A, $A'\in\mathcal{A}^*$ and $E\triangle A=E'\triangle A'$, then $E\triangle E'=A\triangle A'\in\mathcal{A}^*$ and $\mu_*(E\triangle E')=0$; because μ is semi-finite, $\mu(E\triangle E')=0$ and $\mu E=\mu E'$. Accordingly we can define $\mu':\Sigma'\to[0,\infty]$ by setting

$$\mu'(E\triangle A)=\mu E\text{ whenever }E\in\Sigma,\ A\in\mathcal{A}^*.$$

Evidently μ' extends μ and $\mu'A=0$ for every $A\in\mathcal{A}$. Also μ' is a measure. **P** (i) $\mu'\emptyset=\mu\emptyset=0$. (ii) If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ', with union F, express each F_n as $E_n\triangle A_n$ where $E_n\in\Sigma$, $A_n\in\mathcal{A}^*$; set $E=\bigcup_{n\in\mathbb{N}}E_n$, so that $F\triangle E\in\mathcal{A}^*$ and $\mu'F=\mu E$. If $m\neq n$, then $E_m\cap E_n\subseteq A_m\cup A_n$, so $\mu(E_m\cap E_n)=0$; accordingly

$$\mu'F=\mu E=\sum_{n=0}^{\infty}\mu E_n=\sum_{n=0}^{\infty}\mu'F_n.\ \textbf{Q}$$

(c) A subset of X is μ'-negligible iff it can be included in a set of the form $E\triangle A$ where $\mu E=0$ and $A\in\mathcal{A}^*$, so μ' is complete if μ is. The embedding $\Sigma\subsetneqq\Sigma'$ induces a measure-preserving homomorphism from the measure algebra of μ to the measure algebra of μ' which is an isomorphism just because every member of Σ' is the symmetric difference of a member of Σ and a μ'-negligible set.

(d) This deals with (i)-(iii) in the statement of the lemma. Now suppose that $\mathcal{K}$ and $\mathcal{G}$ are as in (iv). Take $F \in \Sigma'$ and $\gamma < \mu' F$. Take $E \in \Sigma$ and a sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $E \triangle F \subseteq \bigcup_{n\in\mathbb{N}} A_n$. Then $\mu E = \mu' F > \gamma$ so (again because μ is semi-finite) there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\gamma < \mu K < \infty$; set $\epsilon = \frac{1}{2}(\mu K - \gamma) > 0$. For each $n \in \mathbb{N}$, choose $G_n \in \mathcal{G}$ and $K_n \in \mathcal{K} \cap \Sigma$ such that

$$A_n \subseteq G_n, \quad E_n = G_n \setminus A_n \text{ belongs to } \Sigma,$$

$$K_n \subseteq K \cap E_n, \quad \mu(K \cap E_n \setminus K_n) \le 2^{-n}\epsilon.$$

Set $L = \bigcap_{n\in\mathbb{N}} (K \setminus G_n) \cup K_n$. Putting the hypotheses (iv-ϵ), (iv-β) and (iv-γ) together, we see that $L \in \mathcal{K}$; moreover, since $G_n = E_n \cup A_n$ belongs to Σ' for every n, $L \in \Sigma'$. Next, setting $H = \bigcap_{n\in\mathbb{N}} (K \setminus E_n) \cup K_n$, $L = H \setminus \bigcup_{n\in\mathbb{N}} A_n$, so $\mu' L = \mu H$ and $L \subseteq F$. But $K \setminus H \subseteq \bigcup_{n\in\mathbb{N}} (K \cap E_n \setminus K_n)$ so

$$\mu' L = \mu H \ge \mu K - \textstyle\sum_{n=0}^{\infty} 2^{-n}\epsilon = \gamma.$$

As F and γ are arbitrary, μ' is inner regular with respect to $\mathcal{K}$.

417B Lemma Let X and Y be topological spaces, and ν a τ-additive topological measure on Y.

(a) If $W \subseteq X \times Y$ is open, then $x \mapsto \nu W[\{x\}] : X \to [0,\infty]$ is lower semi-continuous.

(b) If ν is effectively locally finite and σ-finite and $W \subseteq X \times Y$ is a Borel set, then $x \mapsto \nu W[\{x\}]$ is Borel measurable.

(c) If $f : X \times Y \to [0,\infty]$ is a lower semi-continuous function, then $x \mapsto \int f(x,y)\nu(dy) : X \to [0,\infty]$ is lower semi-continuous.

(d) If ν is totally finite and $f : X \times Y \to \mathbb{R}$ is a bounded continuous function, then $x \mapsto \int f(x,y)\nu(dy)$ is continuous.

(e) If ν is totally finite and $W \subseteq X \times Y$ is a Baire set, then $x \mapsto \nu W[\{x\}]$ is Baire measurable.

proof (a) If $x \in X$ and $\nu W[\{x\}] > \alpha$, then

$$\mathcal{H} = \{H : H \subseteq Y \text{ is open, there is an open set } G \text{ containing } x \text{ such that } G \times H \subseteq W\}$$

is an upwards-directed family of open sets with union $W[\{x\}]$, so there is an $H \in \mathcal{H}$ such that $\nu H \ge \alpha$. Now there is an open set G containing x such that $G \times H \subseteq W$, so that $\nu W[\{x'\}] \ge \alpha$ for every $x' \in G$.

(b)(i) Suppose to begin with that ν is totally finite. In this case, the set

$$\{W : W \subseteq X \times Y,\ x \mapsto \nu W[\{x\}] \text{ is a Borel measurable function defined everywhere on } X\}$$

is a Dynkin class containing every open set, so contains every Borel set, by the Monotone Class Theorem (136B).

(ii) For the general case, let $\langle Y_n \rangle_{n\in\mathbb{N}}$ be a disjoint sequence of sets of finite measure covering Y, and for $n \in \mathbb{N}$ let ν_n be the subspace measure on Y_n. Then ν_n is effectively locally finite and τ-additive (414K). If $W \subseteq X \times Y$ is a Borel set, then $W_n = W \cap (X \times Y_n)$ is a relatively Borel set for each n, so that $x \mapsto \nu_n W_n[\{x\}]$ is Borel measurable, by (i). Since $\nu W[\{x\}] = \sum_{n=0}^{\infty} \nu_n W_n[\{x\}]$ for every x, $x \mapsto \nu W[\{x\}]$ is Borel measurable.

(c) For i, $n \in \mathbb{N}$ set $W_{ni} = \{(x,y) : f(x,y) > 2^{-n} i\}$, so that $W_{ni} \subseteq X \times Y$ is open. Set $f_n = 2^{-n} \sum_{i=1}^{4^n} \chi W_{ni}$; then $\langle f_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum f. For $n \in \mathbb{N}$ and $x \in X$, $\int f_n(x,y)\nu(dy) = 2^{-n} \sum_{i=1}^{4^n} \nu W_{ni}[\{x\}]$, so $x \mapsto \int f_n(x,y)\nu(dy)$ is lower semi-continuous, by (a) and 4A2B(d-iii). By 414Ba, $\int f(x,y)\nu(dy)$ is the supremum $\sup_{n\in\mathbb{N}} \int f_n(x,y)\nu(dy)$ for every x, so $x \mapsto \int f(x,y)\nu(dy)$ is lower semi-continuous (4A2B(d-v)).

(d) Applying (c) to $f + \|f\|_\infty \chi(X \times Y)$, we see that $x \mapsto \int f(x,y)\nu(dy)$ is lower semi-continuous. Similarly, $x \mapsto -\int f(x,y)\nu(dy)$ is lower semi-continuous, so $x \mapsto \int f(x,y)\nu(dy)$ is continuous (4A2B(d-vi)).

(e) Suppose first that W is a cozero set; let $f : X \times Y \to [0,1]$ be a continuous function such that $W = \{(x,y) : f(x,y) > 0\}$. For $n \in \mathbb{N}$ set $f_n = nf \wedge \chi(X \times Y)$. Then $\langle f_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of continuous functions with supremum χW. By (d), all the functions $x \mapsto \int f_n(x,y)\nu(dy)$ are continuous, so their limit $x \mapsto \nu W[\{x\}]$ is Baire measurable.

Now

$$\{W : W \subseteq X \times Y,\, x \mapsto \nu W[\{x\}] \text{ is a Baire measurable function defined everywhere on } X\}$$

is a Dynkin class containing every cozero set, so contains every Baire set, by the Monotone Class Theorem again.

417C Theorem (Ressel 77) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces.

(a) There is a unique complete locally determined effectively locally finite τ-additive topological measure $\tilde\lambda$ on $X \times Y$ which is inner regular with respect to the σ-algebra $\tilde\Lambda^0 = (\Sigma\widehat{\otimes}\mathrm{T}) \vee \mathcal{B}(X \times Y)$ generated by $\{E \times F : E \in \Sigma,\, F \in \mathrm{T}\} \cup \{W : W \subseteq X \times Y \text{ is open}\}$ and is such that $\tilde\lambda(E \times F)$ is defined and equal to $\mu E \cdot \nu F$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$.

(b)(i) $\tilde\lambda$ extends the c.l.d. product measure λ on $X \times Y$, and if $\tilde Q$ is measured by $\tilde\lambda$, there is a Q measured by λ such that $\tilde\lambda(\tilde Q \triangle Q) = 0$.

(ii) The support of $\tilde\lambda$ is the product of the supports of μ and ν.

(iii) If $\tilde Q$ is measured by $\tilde\lambda$,

$$\tilde\lambda\tilde Q = \sup\{\tilde\lambda(\tilde Q \cap (G \times H)) : G \in \mathfrak{T},\, H \in \mathfrak{S},\, \mu G < \infty,\, \mu H < \infty\}.$$

(iv) If $\Sigma' \subseteq \Sigma$ and $\mathrm{T}' \subseteq \mathrm{T}$ are σ-algebras such that μ is inner regular with respect to Σ' and ν is inner regular with respect to T', then $\tilde\lambda$ is inner regular with respect to $(\Sigma'\widehat{\otimes}\mathrm{T}') \vee \mathcal{B}(X \times Y)$.

(v) If $W \subseteq X \times Y$ is open, then

(α) there is an open set $W' \in \Lambda$ such that $W' \subseteq W$ and $\lambda W' = \tilde\lambda W$, so $\tilde\lambda W = \lambda_* W$,

(β) if $E \in \Sigma$ and $F \in \mathrm{T}$ then

$$\tilde\lambda(W \cap (E \times F)) = \int_E \nu(W[\{x\}] \cap F)\mu(dx) = \int_F \mu(W^{-1}[\{y\}] \cap E)\nu(dy).$$

(vi) If μ and ν are inner regular with respect to the Borel sets, so is $\tilde\lambda$.

(vii) If μ and ν are inner regular with respect to the closed sets, so is $\tilde\lambda$.

(viii) If μ and ν are tight (that is, inner regular with respect to the closed compact sets), so is $\tilde\lambda$.

(ix) If μ and ν are locally finite, so is $\tilde\lambda$.

Notation For the rest of this section, if Σ, Σ' are σ-algebras of subsets of a set X, $\Sigma \vee \Sigma'$ will be the σ-algebra generated by $\Sigma \cup \Sigma'$; and if X is a topological space, $\mathcal{B}(X)$ will be its Borel σ-algebra.

proof Write

$$\Sigma^f = \{E : E \in \Sigma,\, \mu E < \infty\}, \quad \mathrm{T}^f = \{F : F \in \mathrm{T},\, \nu F < \infty\},$$

$$\mathfrak{T}^f = \mathfrak{T} \cap \Sigma^f, \quad \mathfrak{S}^f = \mathfrak{S} \cap \mathrm{T}^f.$$

(a)(i) Let λ be the c.l.d. product of μ and ν, and Λ its domain. Write $\mathcal{U}$ for $\{G \times H : G \in \mathfrak{T}^f,\, H \in \mathfrak{S}^f\}$. Because $\mathfrak{T} \subseteq \Sigma$ and $\mathfrak{S} \subseteq \mathrm{T}$, $\mathcal{U} \subseteq \Lambda$. $\mathcal{U}$ need not be a base for the topology of $X \times Y$, unless μ and ν are locally finite, but if an open subset of $X \times Y$ is included in a member of $\mathcal{U}$ it is the union of the members of $\mathcal{U}$ it includes. Moreover, if $Q \in \Lambda$, then $\lambda Q = \sup_{U \in \mathcal{U}} \lambda(Q \cap U)$. **P** By 412R, λ is inner regular with respect to $\bigcup_{U \in \mathcal{U}} \mathcal{P}U$. **Q**

Write $\mathcal{U}_s$ for the set of finite unions of members of $\mathcal{U}$, and $\mathfrak{V}$ for the set of non-empty upwards-directed families $\mathcal{V} \subseteq \mathcal{U}_s$ such that $\sup_{V \in \mathcal{V}} \lambda V < \infty$. For each $\mathcal{V} \in \mathfrak{V}$, fix on a countable $\mathcal{V}' \subseteq \mathcal{V}$ such that $\sup_{V \in \mathcal{V}'} \lambda V = \sup_{V \in \mathcal{V}} \lambda V$; because $\mathcal{V}$ is upwards-directed, we may suppose that $\mathcal{V}' = \{V_n : n \in \mathbb{N}\}$ for some non-decreasing sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{V}$. Set $A(\mathcal{V}) = \bigcup \mathcal{V} \setminus \bigcup \mathcal{V}'$.

(ii)(α) For $V \in \mathcal{U}_s$, set $f_V(x) = \nu V[\{x\}]$ for every $x \in X$. This is always defined because V is open; moreover, f_V is lower semi-continuous, by 417Ba. Because V is a finite union of products of sets of finite measure, $\int f_V d\mu = \lambda V$.

(β) The key to the proof is the following fact: for any $\mathcal{V} \in \mathfrak{V}$, almost every vertical section of $A(\mathcal{V})$ is negligible. **P** $\langle f_V \rangle_{V \in \mathcal{V}}$ is a non-empty upwards-directed set of lower semi-continuous functions. Set

$$g(x) = \nu(\bigcup_{V \in \mathcal{V}} V[\{x\}]), \quad h(x) = \nu(\bigcup_{V \in \mathcal{V}'} V[\{x\}])$$

for every $x \in X$. Because $\mathcal{V}$ is upwards-directed and ν is τ-additive,

$$g(x) = \sup_{V \in \mathcal{V}} \nu V[\{x\}] = \sup_{V \in \mathcal{V}} f_V(x)$$

in $[0, \infty]$ for each x, so, by 414Ba again,

$$\int g\,d\mu = \sup_{V \in \mathcal{V}} \int f_V d\mu = \sup_{V \in \mathcal{V}} \lambda V = \sup_{V \in \mathcal{V}'} \lambda V = \int h\,d\mu.$$

Since $h \le g$ and $\sup_{V \in \mathcal{V}} \lambda V$ is finite, $g(x) = h(x) < \infty$ for μ-almost every x. But for any such x, we must have

$$\nu(\bigcup \mathcal{V})[\{x\}] = \nu(\bigcup \mathcal{V}')[\{x\}] < \infty,$$

so that

$$A(\mathcal{V})[\{x\}] = (\bigcup \mathcal{V})[\{x\}] \setminus (\bigcup \mathcal{V}')[\{x\}]$$

is negligible. **Q**

(iii) ? Suppose, if possible, that there is a sequence $\langle \mathcal{V}_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{V}$ such that $\lambda_*(\bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n)) > 0$. Take $W \in \Lambda$ such that $W \subseteq \bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n)$ and $\lambda W > 0$. Because almost every vertical section of every $A(\mathcal{V}_n)$ is negligible, almost every vertical section of W is negligible. But this contradicts Fubini's theorem (252F). **X**

(iv) Setting $\mathcal{A} = \{A(\mathcal{V}) : \mathcal{V} \in \mathfrak{V}\}$, let λ' be the corresponding extension of λ as described in 417A, $\tilde{\lambda}$ the c.l.d. version of λ' (213E), and Λ', $\tilde{\Lambda}$ the domains of λ', $\tilde{\lambda}$ respectively.

(α) If $W \in \Lambda$, then $W \in \Lambda' \subseteq \tilde{\Lambda}$. Also, because λ is semi-finite,

$$\begin{aligned} \lambda' W = \lambda W &= \sup\{\lambda W' : W' \subseteq W,\, W \in \Lambda,\, \lambda W' < \infty\} \\ &\le \sup\{\lambda' W' : W' \subseteq W,\, W \in \Lambda',\, \lambda' W' < \infty\} = \tilde{\lambda} W \le \lambda' W. \end{aligned}$$

Thus $\lambda W = \tilde{\lambda} W$; as W is arbitrary, $\tilde{\lambda}$ extends λ. Because μ and ν are semi-finite (411Gd),

$$\tilde{\lambda}(E \times F) = \lambda(E \times F) = \mu E \cdot \nu F$$

whenever $E \in \Sigma$ and $\nu \in F$ (251J).

(β) If $\tilde{Q} \in \tilde{\Lambda}$ and $\gamma < \tilde{\lambda}\tilde{Q}$, there is a $U \in \mathcal{U}$ such that $\tilde{\lambda}(\tilde{Q} \cap U) \ge \gamma$. **P** There is a $Q' \in \Lambda'$ such that $Q' \subseteq \tilde{Q}$ and $\gamma < \lambda' Q' < \infty$. By 417A(iii), there is a $Q \in \Lambda$ such that $\lambda'(Q \triangle Q') = 0$, so that

$$\lambda Q = \lambda' Q = \lambda' Q' > \gamma.$$

There is a $U \in \mathcal{U}$ such that $\lambda(Q \cap U) \ge \gamma$, by (i). Now

$$\tilde{\lambda}(\tilde{Q} \cap U) \ge \lambda'(Q' \cap U) = \lambda'(Q \cap U) = \lambda(Q \cap U) \ge \gamma. \ \textbf{Q}$$

Consequently $\tilde{\lambda}$ is effectively locally finite.

(γ) $\tilde{\lambda}$ is a topological measure. **P** Let $W \subseteq X \times Y$ be an open set. Suppose that $\tilde{Q} \in \tilde{\Lambda}$ and $\tilde{\lambda}\tilde{Q} > 0$. By ($\beta$), there is a $U \in \mathcal{U}$ such that $\tilde{\lambda}(\tilde{Q} \cap U) > 0$. Let $\mathcal{V}$ be $\{V : V \in \mathcal{U}_s,\, V \subseteq W \cap U\}$. Then $\mathcal{V} \in \mathfrak{V}$, so $\tilde{\lambda} A(\mathcal{V}) = \lambda' A(\mathcal{V}) = 0$, by 417A(i); since $\bigcup \mathcal{V}' \in \Lambda$,

$$W \cap U = \bigcup \mathcal{V} \in \Lambda' \subseteq \tilde{\Lambda}.$$

But this means that $\tilde{Q} \cap U \cap W$ and $\tilde{Q} \cap U \setminus W = (\tilde{Q} \cap U) \setminus (W \cap U)$ belong to $\tilde{\Lambda}$ and

$$\tilde{\lambda}_*(\tilde{Q} \cap W) + \tilde{\lambda}_*(\tilde{Q} \setminus W) \ge \tilde{\lambda}(\tilde{Q} \cap U \cap W) + \tilde{\lambda}(\tilde{Q} \cap U \setminus W) = \tilde{\lambda}(\tilde{Q} \cap U) > 0.$$

Because $\tilde{\lambda}$ is complete and locally determined, and $\tilde{Q}$ is arbitrary, this is enough to ensure that $W \in \tilde{\Lambda}$ (413F(vii)). **Q**

(δ) $\tilde{\lambda}$ is τ-additive. **P?** Suppose, if possible, otherwise; that there is a non-empty upwards-directed family $\mathcal{W}$ of open sets in $X \times Y$ such that $\tilde{\lambda}W^* > \gamma = \sup_{W \in \mathcal{W}} \tilde{\lambda}W$, where $W^* = \bigcup \mathcal{W}$. In this case, we can find a $Q' \in \Lambda'$ such that $Q' \subseteq W^*$ and $\lambda' Q' > \gamma$, a $Q \in \Lambda$ such that $\lambda'(Q' \triangle Q) = 0$, and a $U \in \mathcal{U}$ such that $\lambda(Q \cap U) > \gamma$ (using (i) again). Let $\mathcal{V} \in \mathfrak{V}$ be the set of those $V \in \mathcal{U}_s$ such that $V \subseteq W \cap U$ for some $W \in \mathcal{W}$. Then $\bigcup \mathcal{V} = W^* \cap U$, so

$$\begin{aligned}\gamma < \lambda(Q \cap U) &= \lambda'(Q \cap U) = \lambda'(Q' \cap U) \\ &\le \tilde{\lambda}(W^* \cap U) = \tilde{\lambda}(\bigcup \mathcal{V}) = \tilde{\lambda}(\bigcup \mathcal{V}')\end{aligned}$$

(because $\tilde{\lambda}A(\mathcal{V}) = 0$)

$$= \sup_{V \in \mathcal{V}'} \tilde{\lambda}V$$

(because $\mathcal{V}'$ is countable and upwards-directed)

$$\le \sup_{W \in \mathcal{W}} \tilde{\lambda}W \le \gamma,$$

which is absurd. **XQ**

(ϵ) $\tilde{\lambda}$ is inner regular with respect to $\tilde{\Lambda}^0$. **P** Applying 412R with $\mathcal{K} = \Sigma$ and $\mathcal{L} = \mathrm{T}$, we see that λ is inner regular with respect to $\Sigma \widehat{\otimes} \mathrm{T}$ and therefore with respect to $\tilde{\Lambda}^0$. If $\mathcal{V} \in \mathfrak{V}$, then

$$A(\mathcal{V}) \subseteq \bigcup \mathcal{V} \in \tilde{\Lambda}^0,$$

$$\bigcup \mathcal{V} \setminus A(\mathcal{V}) = \bigcup \mathcal{V}' \in \Sigma \widehat{\otimes} \mathrm{T} \subseteq \tilde{\Lambda}^0,$$

so 417A(iv) with $\mathcal{K} = \Sigma \widehat{\otimes} \mathrm{T}$ and $\mathcal{G} = \{\bigcup \mathcal{V} : \mathcal{V} \in \mathfrak{V}\}$ assures us that λ' is inner regular with respect to $\tilde{\Lambda}^0$, so that $\tilde{\lambda}$ also is (412Ha). **Q**

(v) Thus $\tilde{\lambda}$ satisfies the conditions listed. To see that it is unique, suppose that $\tilde{\lambda}'$ is another measure on $X \times Y$ with the same properties. Then

$$\tilde{\lambda}(E \times F) = \mu E \cdot \nu F = \tilde{\lambda}'(E \times F)$$

whenever $E \in \Sigma^f$ and $F \in \mathrm{T}^f$, so $\tilde{\lambda}((E \times F) \cap U) = \tilde{\lambda}'((E \times F) \cap U)$ whenever $E \in \Sigma^f$, $F \in \mathrm{T}^f$ and $U \in \mathcal{U}_s$. Consequently $\tilde{\lambda}((E \times F) \cap W) = \tilde{\lambda}'((E \times F) \cap W)$ whenever $E \in \Sigma^f$, $F \in \mathrm{T}^f$ and $W \subseteq X \times Y$ is open. **P** Set $X_0 = \bigcup \mathfrak{T}^f$, $Y_0 = \bigcup \mathfrak{S}^f$; because μ is an effectively locally finite topological measure, $X_0 \in \Sigma$ is μ-conegligible; similarly, $Y_0 \in \mathrm{T}$ is ν-conegligible. Consequently $X_0 \times Y_0$ is both $\tilde{\lambda}$-conegligible and $\tilde{\lambda}'$-conegligible. So, setting $\mathcal{V} = \{U : U \in \mathcal{U}_s, U \subseteq W\}$,

$$\begin{aligned}\tilde{\lambda}((E \times F) \cap W) &= \tilde{\lambda}((E \times F) \cap (W \cap (X_0 \times Y_0))) \\ &= \tilde{\lambda}((E \times F) \cap \bigcup \mathcal{V}) = \sup_{V \in \mathcal{V}} \tilde{\lambda}((E \times F) \cap V)\end{aligned}$$

(by 414Ea, because $\tilde{\lambda}$ is an effectively locally finite τ-additive topological measure)

$$= \sup_{V \in \mathcal{V}} \tilde{\lambda}'((E \times F) \cap V) = \tilde{\lambda}'((E \times F) \cap W). \textbf{ Q}$$

If $E_0 \in \Sigma^f$ and $F_0 \in \mathrm{T}^f$, then the subspace measures $\tilde{\lambda}_{E_0 \times F_0}$ and $\tilde{\lambda}'_{E_0 \times F_0}$ agree on

$$\mathcal{I} = \{((E \cap E_0) \times (F \cap F_0)) \cap W : E \in \Sigma, F \in \mathrm{T}, W \subseteq X \times Y \text{ is open}\}$$

which is closed under $\cap$, so by the Monotone Class Theorem (136C) they agree on the σ-algebra of subsets of $E_0 \times F_0$ generated by $\mathcal{I}$, which is $\tilde{\Lambda}^0 \cap \mathcal{P}(E_0 \times F_0)$.

Next, if $Q \in \tilde{\Lambda}^0$,

$$\tilde{\lambda}' Q = \tilde{\lambda}'(Q \cap (X_0 \times Y_0)) = \sup_{U \in \mathcal{U}} \tilde{\lambda}'(Q \cap U)$$

(414Ea again)

$$= \sup_{E\in\Sigma^f,F\in\mathrm{T}^f} \tilde\lambda'(Q\cap(E\times F)) = \sup_{E\in\Sigma^f,F\in\mathrm{T}^f} \tilde\lambda(Q\cap(E\times F)) = \tilde\lambda Q.$$

So $\tilde\lambda$ and $\tilde\lambda'$ agree on $\tilde\Lambda^0$. By 412Mb they are equal.

(b)(i)(α) In the construction of $\tilde\lambda$ described in (a-iv) above, we have $\lambda = \lambda'\restriction\Lambda = \tilde\lambda\restriction\Lambda$. So $\tilde\lambda$ extends λ.

(β) Note that λ is (strictly) localizable. **P** Let $\tilde\mu$, $\tilde\nu$ be the c.l.d. versions of μ and ν. These are τ-additive topological measures (because μ and ν are), complete and locally determined (by construction), and are still effectively locally finite (cf. 412Ha), so are strictly localizable (414J again). Now λ is the c.l.d. product of $\tilde\mu$ and $\tilde\nu$ (251T), therefore strictly localizable (251O). **Q**

By 417A, λ' is localizable. By 213Hb, there is a $Q'\in\Lambda'$ such that $\tilde\lambda(\tilde Q\triangle Q') = 0$; by 417A(iii), there is a $Q\in\Lambda$ such that $\lambda'(Q'\triangle Q) = 0$; so that $0 = \tilde\lambda(Q'\triangle Q) = \tilde\lambda(\tilde Q\triangle Q)$.

(ii) For an open set $W\subseteq X\times Y$,

$$\begin{aligned} W\cap\operatorname{supp}\tilde\lambda = \emptyset &\iff \tilde\lambda W = 0\\ &\iff \tilde\lambda(G\times H) = 0 \text{ whenever } G\in\mathfrak{T},\ H\in\mathfrak{S} \text{ and } G\times H\subseteq W \end{aligned}$$

(because $\tilde\lambda$ is τ-additive)

$$\begin{aligned} &\iff \mu G\cdot\nu H = 0 \text{ whenever } G\in\mathfrak{T},\ H\in\mathfrak{S} \text{ and } G\times H\subseteq W\\ &\iff (G\times H)\cap(\operatorname{supp}\mu\times\operatorname{supp}\nu) = \emptyset\\ &\qquad\qquad \text{whenever } G\in\mathfrak{T},\ H\in\mathfrak{S} \text{ and } G\times H\subseteq W\\ &\iff W\cap(\operatorname{supp}\mu\times\operatorname{supp}\nu) = \emptyset, \end{aligned}$$

so $\operatorname{supp}\tilde\lambda = \operatorname{supp}\mu\times\operatorname{supp}\nu$.

(iii) This is dealt with in (a-iv-β).

(iv) By 412R, λ is inner regular with respect to $\Sigma'\widehat\otimes\mathrm{T}'$. Applying 417A(iv) with $\mathcal{G} = \{\sup\mathcal{V} : \mathcal{V}\in\mathfrak{V}\}$ as defined in (a-i), we see that λ' is inner regular with respect to $(\Sigma'\widehat\otimes\mathrm{T}')\vee\mathcal{B}(X\times Y)$; by 412Ha again, so is its c.l.d. version $\tilde\lambda$.

(v)(α) Set $\mathcal{V} = \{V : V\in\mathcal{U}_s,\ V\subseteq W\}$. Then $\bigcup\mathcal{V} = W\cap(X_0\times Y_0)$, where $X_0 = \bigcup\mathfrak{T}^f$ and $Y_0 = \bigcup\mathfrak{S}^f$, as in (a-v) above. Because $X_0\times Y_0$ is λ-conegligible, therefore $\tilde\lambda$-conegligible,

$$\tilde\lambda W = \tilde\lambda(W\cap(X_0\times Y_0)) = \sup_{V\in\mathcal{V}}\tilde\lambda V = \sup_{V\in\mathcal{V}}\lambda V \le \tilde\lambda W.$$

Next, if we take a countable upwards-directed $\mathcal{V}'\subseteq\mathcal{V}$ such that

$$\sup_{V\in\mathcal{V}'}\lambda V = \sup_{V\in\mathcal{V}}\lambda V = \tilde\lambda W,$$

and set $W' = \bigcup\mathcal{V}'$, then W' is an open set belonging to Λ and included in W, and

$$\lambda W' = \tilde\lambda W' = \sup_{V\in\mathcal{V}'}\tilde\lambda V = \sup_{V\in\mathcal{V}'}\lambda V = \tilde\lambda W.$$

And because $\tilde\lambda$ extends λ,

$$\tilde\lambda_*W \ge \lambda_*W \ge \sup_{V\in\mathcal{V}}\lambda V = \tilde\lambda W = \tilde\lambda_*W$$

so $\tilde\lambda W = \lambda_*W$.

(β) Because μ and ν are semi-finite (411Gd) and E and F are measurable, the subspace measures μ_E and ν_F are semi-finite (214Ka) therefore effectively locally finite and τ-additive (414K). For open $V\subseteq X\times Y$, set $f_V(x) = \nu(V[\{x\}]\cap F) = \nu_F(V[\{x\}]\cap F)$ for $x\in E$; then f_V is lower semi-continuous (417Ba). Now $\int_E f_V d\mu = \int_E f_V d\mu_E = \lambda(V\cap(E\times F))$ for every $V\in\mathcal{V}$, just because $V\cap(E\times F)$ is a union of measurable rectangles and μ and ν are semi-finite (251J); and setting $g(x) = \nu(W[\{x\}]\cap F)$, we have $g(x) = \sup_{V\in\mathcal{V}} f_V(x)$ for every $x\in X_0$, by 414Ea. Once more, 414Ba tells us that

$$\begin{aligned} \int_E \nu(W[\{x\}]\cap F)\mu(dx) &= \int_E g\,d\mu_E = \sup_{V\in\mathcal{V}}\int_E f_V d\mu_E\\ &= \sup_{V\in\mathcal{V}}\lambda(V\cap(E\times F)) = \tilde\lambda(W\cap(E\times F)). \end{aligned}$$

Similarly,

$$\int_F \mu(W^{-1}[\{y\}]\cap E)\nu(dy) = \tilde\lambda(W\times(E\times F)).$$

(The point here is that while some of the arguments of this proof give different roles to μ and ν, the asserted properties of the extension in part (a), and the following deductions, are symmetric between the two factors.)

(vi) In this case, μ is inner regular with respect to $\mathcal{B}(X)$ and ν is inner regular with respect to $\mathcal{B}(Y)$, so (iv) tells us that $\tilde\lambda$ is inner regular with respect to $(\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y))\vee\mathcal{B}(X\times Y)=\mathcal{B}(X\times Y)$.

(vii) If μ and ν are inner regular with respect to the closed sets, λ also is, by 412Sa. This time, we can apply 417A(iv) with $\mathcal{K}$ the family of closed subsets of $X\times Y$ and $\mathcal{G}=\{\sup\mathcal{V}:\mathcal{V}\in\mathfrak{V}\}$ to see that λ' and $\tilde\lambda$ are inner regular with respect to $\mathcal{K}$.

(viii) Repeat the argument of (vii) with $\mathcal{K}$ the family of closed compact subsets of $X\times Y$, using 412Sb.

(ix) $\{G\times H: G\in\mathfrak{T}^f,\ H\in\mathfrak{S}^f\}$ is a cover of $X\times Y$ by open sets of finite measure for $\tilde\lambda$.

417D Multiple products Just as with the c.l.d. product measure (see 251W), we can apply the construction of 417C repeatedly to obtain measures on the products of finite families of τ-additive measure spaces.

Proposition (a) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a non-empty finite family of effectively locally finite τ-additive topological measure spaces. Then there is a unique complete locally determined effectively locally finite τ-additive topological measure $\tilde\lambda$ on $X=\prod_{i\in I}X_i$ which is inner regular with respect to the σ-algebra $(\widehat{\bigotimes}_{i\in I}\Sigma_i)\vee\mathcal{B}(X)$ generated by $\{\prod_{i\in I}E_i: E_i\in\Sigma_i$ for every $i\in I\}\cup\{W: W\subseteq X$ is open$\}$ and is such that $\tilde\lambda(\prod_{i\in I}E_i)$ is defined and equal to $\prod_{i\in I}\mu_iE_i$ whenever $E_i\in\Sigma_i$ for every $i\in I$.

(b) If now $\langle I_k\rangle_{k\in K}$ is a partition of I into non-empty sets, and $\tilde\lambda_k$ is the product measure defined by the construction of (a) on $Z_k=\prod_{i\in I_k}X_i$ for each $k\in K$, then the natural bijection between X and $\prod_{k\in K}Z_k$ identifies $\tilde\lambda$ with the product of the $\tilde\lambda_k$ defined by the construction of (a).

proof (a) Of course the idea is to induce on $\#(I)$, but there are some wrinkles to take care of.

(i) Suppose that $I=\{j\}$ is a singleton. Then $\tilde\lambda$ must be the c.l.d. version of μ_j; this is surely a topological measure, and it is effectively locally finite by 412Ha, taking $\mathcal{K}$ there to be the family of subsets of open sets of finite measure for μ_j.

(ii) If $\#(I)>1$, take $j\in I$ and set $J=I\setminus\{j\}$. By the inductive hypothesis, we have a complete locally determined effectively locally finite τ-additive topological measure $\tilde\lambda_J$ on $Z=\prod_{i\in J}X_i$ which is inner regular with respect to $(\widehat{\bigotimes}_{i\in J}\Sigma_i)\vee\mathcal{B}(Z)$ and is such that $\tilde\lambda_J(\prod_{i\in I}E_i)=\prod_{i\in J}\mu_iE_i$ whenever $E_i\in\Sigma_i$ for every $i\in J$. Write $\tilde\Lambda_J$ for the domain of $\tilde\lambda_J$. By 417C, there is a complete locally determined effectively locally finite τ-additive topological measure $\tilde\lambda$ on $Z\times X_j$ which is inner regular with respect to the σ-algebra $\tilde\Lambda^{(0)}=(\tilde\Lambda_J\widehat{\otimes}\Sigma_j)\vee\mathcal{B}(X)$ and is such that $\tilde\lambda(W\times F)=\tilde\lambda_JW\times\mu_jF$ whenever $W\in\tilde\Lambda_J$ and $F\in\Sigma_j$. Now 417C(b-iv) tells us that $\tilde\lambda$ is inner regular with respect to the σ-algebra

$$(((\widehat{\bigotimes}_{i\in J}\Sigma_i)\vee\mathcal{B}(Z))\widehat{\otimes}\Sigma_j)\vee\mathcal{B}(X)$$

generated by

$$\begin{aligned}\{E\times X_j:&E\in\widehat{\bigotimes}_{i\in J}\Sigma_i\}\cup\{V\times X_j: V\in\mathcal{B}(Z)\}\cup\{Z\times F: F\in\Sigma_j\}\cup\mathcal{B}(X)\\ &\subseteq(\widehat{\bigotimes}_{i\in I}\Sigma_i)\cup\mathcal{B}(X),\end{aligned}$$

so $\tilde\lambda$ is inner regular with respect to $(\widehat{\bigotimes}_{i\in I}\Sigma_i)\vee\mathcal{B}(X)$. And of course

$$\tilde\lambda(\prod_{i\in I}E_i)=\tilde\lambda_J(\prod_{i\in J}E_i)\cdot\mu_jE_j=\prod_{i\in J}\mu_iE_i\cdot\mu_jE_j=\prod_{i\in I}\mu_iE_i$$

whenever $E_i\in\Sigma_i$ for $i\in I$.

(iii) As for the uniqueness of $\tilde{\lambda}$, we can use the same argument as in (a-v) of the proof of 417C. Suppose that $\tilde{\lambda}'$ is another measure with the given properties. Taking

$$\mathcal{U} = \{\prod_{i\in I} G_i : G_i \in \mathfrak{T}_i^f \text{ for } i \in I\},$$

$$\mathcal{U}_s = \{\bigcup \mathcal{V} : \mathcal{V} \subseteq \mathcal{U}_s \text{ is finite}\},$$

$\bigcup \mathcal{U} = \prod_{i\in I} \bigcup \mathfrak{T}_i^f$ is conegligible for both $\tilde{\lambda}$ and $\tilde{\lambda}'$, while $\tilde{\lambda}$ and $\tilde{\lambda}'$ agree on

$$\{(\prod_{i\in I} E_i) \cap U : U \in \mathcal{U}_s,\ E_i \in \Sigma_i \text{ for } i \in I\}$$

and therefore on

$$\{(\prod_{i\in I} E_i) \cap W : W \subseteq \bigcup \mathcal{U} \text{ is open},\ E_i \in \Sigma_i \text{ for } i \in I\}$$

and on

$$\{(\prod_{i\in I} E_i) \cap W : W \subseteq X \text{ is open},\ E_i \in \Sigma_i^f \text{ for } i \in I\}.$$

By the Monotone Class Theorem they agree on $\tilde{\Lambda}^0 \cap \mathcal{P}(\prod_{i\in I} E_i)$ whenever $E_i \in \Sigma_i^f$ for every i, and by 414Ea once more they agree on $\tilde{\Lambda}^0$, so must be equal.

(b)(i) I begin with something to match 417C(b-iv): if, for each $i \in I$, $\Sigma_i' \subseteq \Sigma_i$ is a σ-algebra such that μ_i is inner regular with respect to Σ_i', then $\tilde{\lambda}$ will be inner regular with respect to $(\widehat{\bigotimes}_{i\in I}\Sigma_i') \vee \mathcal{B}(X)$. **P** Induce on $\#(I)$ as in (i)-(ii) of the proof of (a) above. If $I = \{j\}$ is a singleton, $\tilde{\lambda}$ is the c.l.d. version of μ_j and is inner regular with respect to Σ_j' by 412Ha, as always. For the inductive step, with $J = I \setminus \{j\}$, the inductive hypothesis tells us that $\tilde{\lambda}_J$ is inner regular with respect to $(\widehat{\bigotimes}_{i\in J}\Sigma_i') \vee \mathcal{B}(Z)$; by 417C(b-iv), $\tilde{\lambda}$, as constructed in (a), is inner regular with respect to

$$(((\widehat{\bigotimes}_{i\in J}\Sigma_i') \vee \mathcal{B}(Z))\widehat{\otimes}\Sigma_j') \vee \mathcal{B}(X) \subseteq (\widehat{\bigotimes}_{i\in I}\Sigma_i') \vee \mathcal{B}(X). \text{ } \mathbf{Q}$$

(ii) Now, given a partition $\langle I_k\rangle_{k\in K}$ into non-empty sets and associated effectively locally finite τ-additive product measures $\tilde{\lambda}_k$ on $Z_k = \prod_{i\in I_k} X_i$ for $k \in I$, let $\tilde{\lambda}'$ be the corresponding effectively locally finite τ-additive product measure on $Z = \prod_{k\in K} Z_k$. Then (i) here tells us that $\tilde{\lambda}'$ is inner regular with respect to

$$\widehat{\bigotimes}_{k\in K}((\widehat{\bigotimes}_{i\in I_k}\Sigma_i) \vee \mathcal{B}(Z_i)) \vee \mathcal{B}(Z) = (\widehat{\bigotimes}_{k\in K}\widehat{\bigotimes}_{i\in I_k}\Sigma_i) \vee \mathcal{B}(Z).$$

Copying this into X, we get a c.l.d. measure which is inner regular with respect to $\tilde{\Lambda}^0 = (\widehat{\bigotimes}_{i\in I}\Sigma_i) \vee \mathcal{B}(X)$, defined on the whole of $\tilde{\Lambda}^0$ and agreeing with $\tilde{\lambda}$ on products of measurable sets, so it must be $\tilde{\lambda}$.

417E Theorem Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, with product probability space (X, Λ, λ).

(a) There is a unique complete τ-additive topological probability measure $\tilde{\lambda}$ on X which is inner regular with respect to $\tilde{\Lambda}^0 = (\widehat{\bigotimes}_{i\in I}\Sigma_i) \vee \mathcal{B}(X)$ and is such that $\tilde{\lambda}\{x : x \in X,\ x(i) \in E_i \text{ for every } i \in J\}$ is defined and equal to $\prod_{i\in J} \mu_i E_i$ whenever $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for every $i \in J$.

(b)(i) If $\tilde{Q}$ is measured by $\tilde{\lambda}$, there is a $Q \in \Lambda$ such that $\tilde{\lambda}(\tilde{Q}\triangle Q) = 0$.

(ii) $\tilde{\lambda}W = \lambda_*W$ for every open set $W \subseteq X$, and $\tilde{\lambda}F = \lambda^*F$ for every closed set $F \subseteq X$.

(iii) The support of $\tilde{\lambda}$ is the product of the supports of the μ_i.

(iv) If for each $i \in I$ we are given a σ-subalgebra $\Sigma_i' \subseteq \Sigma_i$ such that μ_i is inner regular with respect to Σ_i', then $\tilde{\lambda}$ is inner regular with respect to $(\widehat{\bigotimes}_{i\in I}\Sigma_i') \vee \mathcal{B}(X)$.

(v) If every μ_i is inner regular with respect to the Borel sets, so is $\tilde{\lambda}$.

(vi) If every μ_i is inner regular with respect to the closed sets, so is $\tilde{\lambda}$.

proof The strategy of the proof is the same as in 417C, subject to some obviously necessary modifications. The key step, showing that every union $\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)$ has zero inner measure, is harder, but we escape a little work because we no longer have to worry about sets of infinite measure.

(a)(i) I begin by setting up some machinery. Let $\mathcal{C}$ be the family of subsets of X expressible in the form $\prod_{i\in I} E_i$, where $E_i \in \Sigma_i$ for every i and $\{i : E_i \neq X_i\}$ is finite. Let $\mathcal{U} \subseteq \mathcal{C}$ be the standard basis for the topology $\mathfrak{T}$ of X, consisting of sets expressible as $\prod_{i\in I} G_i$ where $G_i \in \mathfrak{T}_i$ for every $i \in I$ and $\{i : G_i \neq X_i\}$ is finite. Write $\mathcal{U}_s$ for the set of finite unions of members of $\mathcal{U}$, and $\mathfrak{V}$ for the set of non-empty upwards-directed families in $\mathcal{U}_s$. Note that every member of $\mathcal{U}_s$ is determined by coordinates in some finite subset of I (definition: 254M).

If $J \subseteq I$, write λ_J for the product measure on $\prod_{i\in J} X_i$; we shall need $\lambda_\emptyset$, which is the unique probability measure on the single-point set $\{\emptyset\} = \prod_{i\in\emptyset} X_i$. For $J \subseteq I$, $v \in \prod_{i\in J} X_i$ and $W \subseteq X$ set

$$f_W(v) = \lambda_{I\setminus J}\{w : (v, w) \in W\}$$

if this is defined, identifying $\prod_{i\in J} X_i \times \prod_{i\in I\setminus J} X_i$ with X.

(ii) We need two easy facts.

(α) $f_W(v) = \int f_W(v^\frown\langle t\rangle)\mu_j(dt)$ whenever $W \in \widehat{\bigotimes}_{i\in I}\Sigma_i$, $J \subseteq I$, $v \in \prod_{i\in J} X_i$ and $j \in I \setminus J$, writing $v^\frown\langle t\rangle$ for the member $v \cup \{(j, t)\}$ of $\prod_{i\in J\cup\{j\}} X_i$ extending v and taking the value t at the coordinate j. **P** Let $\mathcal{A}$ be the family of sets W satisfying the property. Then $\mathcal{A}$ is a Dynkin class including $\mathcal{C}$, so includes the σ-algebra generated by $\mathcal{C}$, which is $\widehat{\bigotimes}_{i\in I}\Sigma_i$. **Q**

(β) If $J \subseteq I$, $v \in \prod_{i\in J} X_i$, $j \in I \setminus J$ and $V \in \mathcal{U}_s$, and we set $g(t) = f_V(v^\frown\langle t\rangle)$ for $t \in X_j$, then g is lower semi-continuous. **P** We can express V as $\bigcup_{n\le m}\prod_{i\in I} G_{ni}$, where $G_{ni} \subseteq X_i$ is open for every $n \le m$, $i \in I$. Now if $t \in X_j$, we shall have

$$\{w : (v^\frown\langle t\rangle, w) \in V\} \subseteq \{w : (v^\frown\langle t'\rangle, w) \in V\}$$

whenever

$$t' \in H = X_j \cap \bigcap\{G_{nj} : n \le m,\ t \in G_{nj}\}.$$

So $g(t') \ge g(t)$ for every $t' \in H$, which is an open neighbourhood of t. As t is arbitrary, g is lower semi-continuous. **Q**

(iii) For each $\mathcal{V} \in \mathfrak{V}$, fix, for the remainder of this proof, a countable $\mathcal{V}' \subseteq \mathcal{V}$ such that $\sup_{V\in\mathcal{V}'} \lambda V = \sup_{V\in\mathcal{V}} \lambda V$; because $\mathcal{V}$ is upwards-directed, we may suppose that $\mathcal{V}' = \{V_n : n \in \mathbb{N}\}$ for some non-decreasing sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$. Set $A(\mathcal{V}) = \bigcup\mathcal{V} \setminus \bigcup\mathcal{V}'$.

? Suppose, if possible, that there is a sequence $\langle\mathcal{V}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{V}$ such that $\lambda_*(\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) > 0$.

(α) We have $\lambda^*(X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) < 1$; let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ such that

$$X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n) \subseteq \bigcup_{n\in\mathbb{N}} C_n, \quad \sum_{n=0}^\infty \lambda C_n = \gamma_0 < 1$$

(see 254A-254C). For each $n \in \mathbb{N}$, express $\mathcal{V}'_n$ as $\{V_{nr} : r \in \mathbb{N}\}$ where $\langle V_{nr}\rangle_{r\in\mathbb{N}}$ is non-decreasing, and set $W_n = \bigcup\mathcal{V}'_n = \bigcup_{r\in\mathbb{N}} V_{nr}$. Let $J \subseteq I$ be a countable set such that every C_n and every V_{nr} is determined by coordinates in J. Express J as $\bigcup_{k\in\mathbb{N}} J_k$ where $J_0 = \emptyset$ and, for each k, J_{k+1} is equal either to J_k or to J_k with one point added. (As in the proof of 254Fa, I am using a formulation which will apply equally to finite and infinite I, though of course the case of finite I is elementary once we have 417C.)

(β) For each $n \in \mathbb{N}$, set

$$W'_n = \bigcup_{k\in\mathbb{N}}\{x : x \in X,\ f_{W_n}(x{\restriction}J_k) = 1\}.$$

Then $\lambda(W'_n \setminus W_n) = 0$. **P** For any $k \in \mathbb{N}$, if we think of λ as the product of λ_{J_k} and $\lambda_{I\setminus J_k}$ and of f_{W_n} as a measurable function on $\prod_{i\in J_k} X_i$, we see that $\{x : f_{W_n}(x{\restriction}J_k) = 1\}$ is of the form $F_k \times \prod_{i\in I\setminus J_k} X_i$, where $F_k \subseteq \prod_{i\in J_k} X_i$ is measurable; and

$$\lambda((F_k \times \prod_{i\in I\setminus J_k} X_i) \setminus W_n) = \int_{F_k}(1 - f_{W_n}(v))\lambda_{J_k}(dv) = 0.$$

Summing over k, we see that $W'_n \setminus W_n$ is negligible. **Q**

Observe that every W'_n, like W_n, is determined by coordinates in J. So $\bigcup_{n\in\mathbb{N}} W'_n \setminus W_n$ is of the form $E \times \prod_{i\in I\setminus J} X_i$ where $\lambda_J E = 0$ (254Ob). There is therefore a sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of measurable cylinders in $\prod_{i\in J} X_i$ such that $E \subseteq \bigcup_{n\in\mathbb{N}} D_n$ and $\sum_{n=0}^\infty \lambda_J D_n < 1 - \gamma_0$. Set $C'_n = \{x : x \in X,\ x{\restriction}J \in D_n\} \in \mathcal{C}$ for each n. Then $\bigcup_{n\in\mathbb{N}} W'_n \setminus W_n \subseteq \bigcup_{n\in\mathbb{N}} C'_n$, so

$$(X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) \cup \bigcup_{n\in\mathbb{N}} W'_n \setminus W_n \subseteq \bigcup_{n\in\mathbb{N}} C_n \cup \bigcup_{n\in\mathbb{N}} C'_n,$$

$$\gamma = \sum_{n=0}^\infty \lambda C_n + \sum_{n=0}^\infty \lambda C'_n < 1,$$

while each C_n and each C'_n is determined by coordinates in a finite subset of J.

(γ) For $k \in \mathbb{N}$, let P_k be the set of those $v \in \prod_{i\in J_k} X_i$ such that

$$\sum_{n=0}^\infty f_{C_n}(v) + f_{C'_n}(v) \le \gamma, \quad f_V(v) \le f_{W_n}(v) \text{ whenever } n \in \mathbb{N} \text{ and } V \in \mathcal{V}_n.$$

Our hypothesis is that

$$\sum_{n=0}^\infty f_{C_n}(\emptyset) + f_{C'_n}(\emptyset) = \sum_{n=0}^\infty \lambda C_n + \lambda C'_n \le \gamma,$$

and the $\mathcal{V}'_n$ were chosen such that

$$f_V(\emptyset) = \lambda V \le \lambda W_n = f_{W_n}(\emptyset)$$

for every $n \in \mathbb{N}$, $V \in \mathcal{V}_n$; that is, $\emptyset \in P_0$.

(δ) Now if $k \in \mathbb{N}$ and $v \in P_k$ there is a $v' \in P_{k+1}$ extending v. **P** If $J_{k+1} = J_k$ we can take $v' = v$. Otherwise, $J_{k+1} = J_k \cup \{j\}$ for some $j \in I \setminus J_k$. Now

$$\gamma \ge \sum_{n=0}^\infty f_{C_n}(v) + f_{C'_n}(v) = \sum_{n=0}^\infty \int f_{C_n}(v^\frown{<}t{>}) + f_{C'_n}(v^\frown{<}t{>})\,\mu_j(dt)$$

((α) above)

$$= \int \sum_{n=0}^\infty f_{C_n}(v^\frown{<}t{>}) + f_{C'_n}(v^\frown{<}t{>})\,\mu_j(dt),$$

so

$$H = \{t : t \in X_j,\ \sum_{n=0}^\infty f_{C_n}(v^\frown{<}t{>}) + f_{C'_n}(v^\frown{<}t{>})\mu_j(dt) \le \gamma\}$$

has positive measure.

Next, for $V \in \mathcal{U}_s$, set $g_V(t) = f_V(v^\frown{<}t{>})$ for each $t \in X_j$. Then g_V is lower semi-continuous, by (β) above. For each $n \in \mathbb{N}$, $\{g_V : V \in \mathcal{V}_n\}$ is an upwards-directed family of lower semi-continuous functions, so its supremum g^*_n is lower semi-continuous, and because μ_j is τ-additive,

$$\int g^*_n d\mu_j = \sup_{V\in\mathcal{V}_n} \int g_V d\mu_j = \sup_{V\in\mathcal{V}_n} f_V(v) \le f_{W_n}(v) = \int f_{W_n}(v^\frown{<}t{>})\mu_j(dt)$$

(using 414B and (α) again). But also, because $\langle V_{nr}\rangle_{r\in\mathbb{N}}$ is non-decreasing and has union W_n,

$$f_{W_n}(v^\frown{<}t{>}) = \sup_{r\in\mathbb{N}} f_{V_{nr}}(v^\frown{<}t{>}) \le g^*_n(t)$$

for every $t \in X_j$. So we must have

$$f_{W_n}(v^\frown{<}t{>}) = g^*_n(t) \text{ a.e.}(t).$$

And this is true for every $n \in \mathbb{N}$.

There is therefore a $t \in H$ such that

$$f_{W_n}(v^\frown{<}t{>}) = g^*_n(v^\frown{<}t{>}) \text{ for every } n \in \mathbb{N}.$$

Fix on such a t and set $v' = v^\frown{<}t{>} \in \prod_{i\in J_{k+1}} X_i$; then $v' \in P_{k+1}$, as required. **Q**

(ϵ) We can therefore choose a sequence $\langle v_k\rangle_{k\in\mathbb{N}}$ such that $v_k \in P_k$ and v_{k+1} extends v_k for each k. Choose $x \in X$ such that $x(i) = v_k(i)$ whenever $k \in \mathbb{N}$ and $i \in J_k$, and $x(i)$ belongs to the support of μ_i whenever $i \in I \setminus J$. (Once again, 411Nd tells us that every μ_i has a support.)

We need to know that if $k, n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ then $f_{V\setminus W_n}(v_k) = 0$. **P** For any $r \in \mathbb{N}$ there is a $V' \in \mathcal{V}_n$ such that $V \cup V_{nr} \subseteq V'$, so

$$f_{V\cup V_{nr}}(v_k) \le f_{V'}(v_k) \le f_{W_n}(v_k),$$

and

$$f_{V\setminus W_n}(v_k) \le f_{V\setminus V_{nr}}(v_k) = f_{V\cup V_{nr}}(v_k) - f_{V_{nr}}(v_k) \le f_{W_n}(v_k) - f_{V_{nr}}(v_k) \to 0$$

as $r \to \infty$. **Q**

(ζ) If $n \in \mathbb{N}$, then $x \notin C_n \cup C'_n$. **P** C_n and C'_n are determined by coordinates in a finite subset of J, so must be determined by coordinates in J_k for some $k \in \mathbb{N}$. Now $f_{C_n}(v_k) + f_{C'_n}(v_k) \le \gamma < 1$, so $\{y : y{\restriction}J_k = v_k\}$ cannot be included in $C_n \cup C'_n$, and must be disjoint from it; accordingly $x \notin C_n \cup C'_n$. **Q**

(η) Because

$$(X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) \cup \bigcup_{n\in\mathbb{N}} W'_n \setminus W_n \subseteq \bigcup_{n\in\mathbb{N}} C_n \cup \bigcup_{n\in\mathbb{N}} C'_n,$$

there is some $n \in \mathbb{N}$ such that

$$x \in A(\mathcal{V}_n) \setminus (W'_n \setminus W_n) \subseteq (\bigcup \mathcal{V}_n) \setminus W'_n,$$

that is, there is some $V \in \mathcal{V}_n$ such that $x \in V \setminus W'_n$. Let $U \in \mathcal{U}$ be such that $x \in U \subseteq V$. Express U as $U' \cap U''$ where $U' \in \mathcal{U}$ is determined by coordinates in a finite subset of J and $U'' \in \mathcal{U}$ is determined by coordinates in a finite subset of $I \setminus J$. Let $k \in \mathbb{N}$ be such that U' is determined by coordinates in J_k. Then

$$f_{U\setminus W_n}(v_k) \le f_{V\setminus W_n}(v_k) = 0$$

by (ϵ) above. Now

$$\{w : w \in \prod_{i\in I\setminus J_k} X_i,\ (v_k, w) \in U \setminus W_n\} = \{w : (v_k, w) \in U'' \setminus W_n\}$$

(because $(v_k, w) = (x{\restriction}J_k, w) \in U'$ for every w), while

$$\{w : (v_k, w) \in U''\}, \quad \{w : (v_k, w) \in W_n\}$$

are stochastically independent because the former is determined by coordinates in $I \setminus J$, while the latter is determined by coordinates in $J \setminus J_k$. So we must have

$$\begin{aligned} 0 = f_{U\setminus W_n}(v_k) &= \lambda_{I\setminus J_k}\{w : (v_k, w) \in U \setminus W_n\} \\ &= \lambda_{I\setminus J_k}\{w : (v_k, w) \in U'' \setminus W_n\} \\ &= \lambda_{I\setminus J_k}\{w : (v_k, w) \in U''\}(1 - \lambda_{I\setminus J_k}\{w : (v_k, w) \in W_n\}). \end{aligned}$$

At this point, recall that $x(i)$ belongs to the support of μ_i for every $i \in I \setminus J$, while $x \in U''$. So if $U'' = \{y : y(i) \in H_i \text{ for } i \in K\}$, where $K \subseteq I \setminus J$ is finite and $H_i \subseteq X_i$ is open for every i, we must have $\mu_i H_i > 0$ for every i, and

$$\lambda_{I\setminus J_k}\{w : (v_k, w) \in U''\} = \prod_{i\in K} \mu_i H_i > 0.$$

On the other hand, we are also supposing that $x \notin W'_n$, so

$$\lambda_{I\setminus J_k}\{w : (v_k, w) \in W_n\} = f_{W_n}(v_k) = f_{W_n}(x{\restriction}J_k) < 1.$$

But this means that we have expressed 0 as the product of two non-zero numbers, which is absurd. **X**

(iv) Thus $\lambda_*(\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) = 0$ for every sequence $\langle \mathcal{V}_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{V}$. Accordingly there is an extension of λ to a measure $\tilde\lambda$ on X as in 417A. By 417A(ii), $\tilde\lambda$ is complete.

Now $\tilde\lambda$ is a topological measure. **P** If $W \subseteq X$ is open, then $\mathcal{V} = \{V : V \in \mathcal{U}_s,\ V \subseteq W\}$ belongs to $\mathfrak{V}$, and $\bigcup \mathcal{V} = W$. Since $\bigcup \mathcal{V}' \in \Lambda$ (because $\mathcal{V}'$ is countable),

$$W = \bigcup \mathcal{V}' \cup A(\mathcal{V})$$

is measured by $\tilde\lambda$. **Q**

Also, $\tilde\lambda$ is τ-additive. **P** Let $\mathcal{W}$ be a non-empty upwards-directed family of open subsets of X with union W^*. Set

$$\mathcal{V} = \{V : V \in \mathcal{U}_s,\ \exists\, W \in \mathcal{W},\ V \subseteq W\}.$$

Then $\mathcal{V} \in \mathfrak{V}$ and $\bigcup \mathcal{V} = W^*$, so $\tilde\lambda A(\mathcal{V}) = 0$ and

$$\tilde\lambda W^* = \tilde\lambda(\bigcup \mathcal{V}') = \sup_{V\in\mathcal{V}'} \tilde\lambda V \le \sup_{W\in\mathcal{W}} \tilde\lambda W \le \tilde\lambda W^*$$

(using the fact that $\mathcal{V}'$ is upwards-directed). As $\mathcal{W}$ is arbitrary, $\tilde\lambda$ is τ-additive. **Q**

(v) As for the uniqueness of $\tilde{\lambda}$, if $\tilde{\lambda}'$ is another measure with the same properties, then $\tilde{\lambda}'(U \cap C) = \tilde{\lambda}(U \cap C)$ whenever $U \in \mathcal{U}_s$ and $C \in \mathcal{C}$. Since $\tilde{\lambda}$ and $\tilde{\lambda}'$ are both τ-additive, they agree on sets of the form $W \cap C$ where $W \subseteq X$ is open and $C \in \mathcal{C}$; by the Monotone Class Theorem, they agree on $\tilde{\Lambda}^0$ and are therefore both the completion of their common restriction to $\tilde{\Lambda}^0$.

(b)(i) Immediate from 417A(iii).

(ii) Let $W \subseteq X$ be an open set. Set $\mathcal{V} = \{V : V \in \mathcal{U}_s,\, V \subseteq W\}$. Then

$$\tilde{\lambda}W = \sup_{V \in \mathcal{V}} \tilde{\lambda}V = \sup_{V \in \mathcal{V}} \lambda V \le \lambda_* W \le \tilde{\lambda}W$$

just because $\tilde{\lambda}$ is a τ-additive extension of λ. Now if $F \subseteq X$ is closed,

$$\tilde{\lambda}F = 1 - \tilde{\lambda}(X \setminus F) = 1 - \lambda_*(X \setminus F) = \lambda^* F.$$

(iii) For each $i \in I$ write Z_i for the support of μ_i, and set $Z = \prod_{i \in I} Z_i$. This is closed because every Z_i is. Its complement is covered by the negligible open sets $\{x : x \in X,\, x(i) \in X_i \setminus Z_i\}$ as i runs over I; as $\tilde{\lambda}$ is τ-additive, the union of the negligible open sets is negligible, and Z is conegligible. If $W \subseteq X$ is open and $x \in Z \cap W$, let $U \in \mathcal{U}$ be such that $x \in U \subseteq W$. Express U as $\prod_{i \in I} G_i$ where $G_i \in \mathfrak{T}_i$ for every $i \in I$ and $J = \{i : G_i \ne X_i\}$ is finite. Then $x(i) \in G_i \cap Z_i$, so $\mu_i G_i > 0$, for every i. Accordingly

$$\tilde{\lambda}(W \cap Z) = \tilde{\lambda}W \ge \lambda U = \prod_{i \in J} \mu_i G_i > 0.$$

Thus Z is self-supporting and is the support of $\tilde{\lambda}$.

(iv) (See (b-iv) of the proof of 417C). By 412T, λ is inner regular with respect to $\widehat{\bigotimes}_{i \in I} \Sigma_i'$. Applying 417A(iv) with $\mathcal{G} = \{\sup \mathcal{V} : \mathcal{V} \in \mathfrak{V}\}$, we see that $\tilde{\lambda}$ is inner regular with respect to $(\widehat{\bigotimes}_{i \in I} \Sigma_i') \vee \mathcal{B}(X)$.

(v), (vi) As in the proof of 417Cb, apply 417A(iv) with $\mathcal{G}$ the family of open subsets of X and $\mathcal{K}$ either the Borel σ-algebra of X or the family of closed subsets of X, this time using 412U to confirm that λ is inner regular with respect to $\mathcal{K}$.

417F Notation In the context of 417C, 417D or 417E, I will call $\tilde{\lambda}$ the **τ-additive product measure** on $\prod_{i \in I} X_i$.

Remarks (a) Note that the uniqueness assertions in 417D and 417E mean that for the products of finitely many probability spaces we do not need to distinguish between the two constructions. The latter also shows that we can relate 415E to the new method: if every $\mathfrak{T}_i$ is separable and metrizable and every μ_i is strictly positive, then the 'ordinary' product measure λ is a complete topological measure. Since it is also inner regular with respect to the Borel sets (412Uc), and τ-additive (because we now know that it has an extension to a τ-additive measure) it must be exactly the τ-additive product measure as described here.

(b) In 417D it seemed simpler to restrict the concept of 'product' to non-empty families; in 417E, I omitted any reference to the possiblity that the set I might be empty. This is because I regard a product $X = \prod_{i \in I} X_i$ as a set of functions defined on I, and if $I = \emptyset$ there is just one such function, itself the empty set; so we have $X = \{\emptyset\}$, with just one topology on X and just one probability measure defined on X, which will do very nicely for the required $\tilde{\lambda}$.

417G Fubini's theorem for τ-additive product measures Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces. Let $\tilde{\lambda}$ be the τ-additive product measure on $X \times Y$, and $\tilde{\Lambda}$ its domain.

(a) Let f be a $[-\infty, \infty]$-valued function such that $\int f\, d\tilde{\lambda}$ is defined in $[-\infty, \infty]$ and $(X \times Y) \setminus \{(x, y) : (x, y) \in \operatorname{dom} f,\, f(x, y) = 0\}$ can be covered by a set of the form $X \times \bigcup_{n \in \mathbb{N}} Y_n$ where $\nu Y_n < \infty$ for every $n \in \mathbb{N}$. Then the repeated integral $\iint f(x, y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\tilde{\lambda}$.

(b) Let $f : X \times Y \to [0, \infty]$ be lower semi-continuous. Then

$$\iint f(x, y)\nu(dy)\mu(dx) = \iint f(x, y)\mu(dx)\nu(dy) = \int f d\tilde{\lambda}$$

in $[0, \infty]$.

(c) Let f be a $\tilde{\Lambda}$-measurable real-valued function defined $\tilde{\lambda}$-a.e. on $X \times Y$. If either $\iint |f(x,y)|\nu(dy)\mu(dx)$ or $\iint |f(x,y)|\mu(dx)\nu(dy)$ is defined and finite, then f is $\tilde{\lambda}$-integrable.

proof As in 417C, set $\tilde{\Lambda}^0 = (\Sigma\widehat{\otimes}\mathrm{T}) \vee \mathcal{B}(X \times Y)$.

(a) I use 252B.

(i) Write $\mathcal{W}$ for the set of those $W \in \tilde{\Lambda}$ such that $\int \nu W[\{x\}]\mu(dx)$ is defined in $[0,\infty]$ and equal to $\tilde{\lambda}W$. Then open sets belong to $\mathcal{W}$, by 417C(b-v-β). Next, if $W \in \tilde{\Lambda}^0$ is included in an open set W_0 of finite measure, $W \in \mathcal{W}$. **P** If W_0 is an open set of finite measure, then $\{W : W \subseteq X \times Y,\, W \cap W_0 \in \mathcal{W}\}$ is a Dynkin class, and by 417C(b-v-β), it contains $W \cap (E \times F)$ whenever $E \in \Sigma$, $F \in \mathrm{T}$ and $W \subseteq X \times Y$ is open. By the Monotone Class Theorem it includes $\tilde{\Lambda}^0$. **Q**

Now suppose that $W \subseteq X \times Y$ is $\tilde{\lambda}$-negligible and included in $X \times \bigcup_{n\in\mathbb{N}} Y_n$, where $\nu Y_n < \infty$ for every n. Then $W \in \mathcal{W}$. **P** Set $A = \{x : x \in X,\, \nu^* W[\{x\}] > 0\}$. For each n, let $H_n \subseteq Y$ be an open set of finite measure such that $\nu(Y_n \setminus H_n) \le 2^{-n}$; we may arrange that $H_{n+1} \supseteq H_n$ for each n. Set $H = \bigcup_{n\in\mathbb{N}} H_n$, so that $W[\{x\}] \setminus H \subseteq \bigcup_{n\in\mathbb{N}} Y_n \setminus H$ is negligible for every $x \in X$.

Fix an open set $G \subseteq X$ of finite measure and $n \in \mathbb{N}$ for the moment. Because $\tilde{\lambda}$ is inner regular with respect to $\tilde{\Lambda}^0$, there is a $V \in \tilde{\Lambda}^0$ such that $V \subseteq (G \times H_n) \setminus W$ and $\tilde{\lambda}V = \tilde{\lambda}((G \times H_n) \setminus W)$, that is, $\tilde{\lambda}V' = 0$, where $V' = (G \times H_n) \setminus V \supseteq (G \times H_n) \cap W$. We know that $V' \in \mathcal{W}$, so

$$\int \nu V'[\{x\}]dx = \tilde{\lambda}V' = 0,$$

and $\nu V'[\{x\}] = 0$ for almost every $x \in X$; but this means that $H_n \cap W[\{x\}]$ is negligible for almost every $x \in G$.

At this point, recall that n was arbitrary, so $H \cap W[\{x\}]$ and $W[\{x\}]$ are negligible for almost every $x \in G$, that is, $A \cap G$ is negligible. This is true for every open set $G \subseteq X$ of finite measure. Because μ is inner regular with respect to subsets of open sets of finite measure, and is complete and locally determined, A is negligible (412Jb). But this means that $\int \nu W[\{x\}]\mu(dx)$ is defined and equal to zero, so that $W \in \mathcal{W}$. **Q**

(ii) Now suppose that $\int f d\tilde{\lambda}$ is defined in $[-\infty,\infty]$ and that there is a sequence $\langle Y_n\rangle_{n\in\mathbb{N}}$ of sets of finite measure in Y such that $f(x,y)$ is defined and zero whenever $x \in X$ and $y \in Y \setminus \bigcup_{n\in\mathbb{N}} Y_n$. Set $Z = \bigcup_{n\in\mathbb{N}} Y_n$. Write λ for the c.l.d. product measure on $X \times Y$ and Λ for its domain. Then there is a Λ-measurable function $g : X \times Y \to [-\infty,\infty]$ which is equal $\tilde{\lambda}$-almost everywhere to f. **P** For $q \in \mathbb{Q}$ set $W_q = \{(x,y) : (x,y) \in \operatorname{dom} f,\, f(x,y) \ge q\} \in \tilde{\Lambda}$, and choose $V_q \in \Lambda$ such that $\tilde{\lambda}(W_q \triangle V_q) = 0$ (417C(b-i)); set $g(x,y) = \sup\{q : q \in \mathbb{Q},\, (x,y) \in V_q\}$ for $x \in X$ and $y \in Y$, interpreting $\sup\emptyset$ as $-\infty$. **Q** Adjusting g if necessary, we may suppose that it is zero on $X \times (Y \setminus Z)$. Set

$$A = (X \times Y) \setminus \{(x,y) : f(x,y) = g(x,y)\},$$

so that A is $\tilde{\lambda}$-negligible and included in $X \times Z$. By (i), $\nu A[\{x\}] = 0$, that is, $y \mapsto f(x,y)$ and $y \mapsto g(x,y)$ are equal ν-a.e., for μ-almost every x. Write $\lambda_{X\times Z}$ for the subspace measure induced by λ on $X \times Z$; note that this is the c.l.d. product of μ with the subspace measure ν_Z on Z, by 251Q(ii-α).

Now we have

$$\int f d\tilde{\lambda} = \int g\, d\tilde{\lambda} = \int g\, d\lambda$$

(by 235Gb, because the identity map from $(X \times Y, \tilde{\lambda})$ to $(X \times Y, \lambda)$ is inverse-measure-preserving)

$$= \int_{X\times Z} g\, d\lambda = \int_{X\times Z} g\, d\lambda_{X\times Z} = \iint_Z g(x,y)\nu_Z(dy)\mu(dx)$$

(by 252B, because ν_Z is σ-finite)

$$= \iint g(x,y)\nu(dy)\mu(dx)$$

(because $g(x,y) = 0$ if $y \in Y \setminus Z$)

$$= \iint f(x,y)\nu(dy)\mu(dx).$$

(b) If f is non-negative and lower semi-continuous, set

$$W_{ni} = \{(x,y) : f(x,y) > 2^{-n}i\}$$

for n, $i \in \mathbb{N}$, and

$$f_n = 2^{-n}\sum_{i=1}^{4^n} \chi W_{ni}$$

for $n \in \mathbb{N}$. Applying 417C(b-v) we see that

$$\int f_n d\tilde{\lambda} = \iint f_n(x,y)dydx = \iint f_n(x,y)dxdy$$

in $[0,\infty]$ for every n; taking the limit as $n \to \infty$,

$$\int f d\tilde{\lambda} = \iint f(x,y)dydx = \iint f(x,y)dxdy,$$

because $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit f.

(c) ? Suppose, if possible, that $\gamma = \iint |f(x,y)|dydx$ is finite, but that f is not integrable. Because $\tilde{\lambda}$ is semi-finite, there must be a non-negative $\tilde{\lambda}$-simple function g such that $g \le_{\text{a.e.}} |f|$ and $\int g\,d\tilde{\lambda} > \gamma$ (213B). For each $n \in \mathbb{N}$, there are open sets $G_n \subseteq X$, $H_n \subseteq Y$ of finite measure such that $\tilde{\lambda}(\{(x,y) : g(x,y) \ge 2^{-n}\} \setminus (G_n \times H_n)) \le 2^{-n}$, by 417C(b-iii); now $\langle g \times \chi(G_n \times H_n)\rangle_{n\in\mathbb{N}} \to g$ a.e., so there is some n such that $\int_{G_n\times H_n} g\,d\tilde{\lambda} > \gamma$. In this case, setting $g'(x,y) = \min(g(x,y), |f(x,y)|)$ for $(x,y) \in (G_n \times H_n) \cap \operatorname{dom} f$, 0 otherwise, we have $g = g'$ a.e. on $G_n \times H_n$, so that $\int g' d\tilde{\lambda} > \gamma$. But we can apply (a) to g' to see that

$$\gamma < \int g' d\tilde{\lambda} = \iint g'(x,y)dydx \le \iint |f(x,y)|dydx \le \gamma,$$

which is absurd. **X**

So if $\iint |f(x,y)|dydx$ is finite, f must be $\tilde{\lambda}$-integrable. Of course the same arguments, reversing the roles of X and Y, show that f is $\tilde{\lambda}$-integrable if $\iint |f(x,y)|dxdy$ is defined and finite.

417H Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces. Let $\tilde{\lambda}$ be the τ-additive product measure on $X \times Y$, and $\tilde{\Lambda}$ its domain. If $A \subseteq X$, $B \subseteq Y$ are non-negligible sets such that $A \times B \in \tilde{\Lambda}$, then $A \in \Sigma$ and $B \in \mathrm{T}$.

proof (Cf. 252Xc.) If $F \in \mathrm{T}$ and $\nu F < \infty$, $B \cap F \in \mathrm{T}$. **P** If $\nu F = 0$ then $B \cap F \in \mathrm{T}$ because ν is complete. Otherwise, set $f = \chi(A \times (B \cap F))$. Then $\int f\,d\tilde{\lambda} = \tilde{\lambda}((A \times B) \cap (X \times F))$ is defined and f is zero outside $X \times F$. By 417Ga, $\iint f(x,y)\nu(dy)\mu(dx)$ is defined, that is, $\int \chi A(x) \times \nu(B \cap F)\mu(dx)$ is defined and $x \mapsto \chi A(x) \times \nu(B \cap F)$ is defined μ-a.e. As A is not negligible, $\nu(B \cap F)$ is defined. **Q** As ν is locally determined, $B \in \mathrm{T}$.

As the specification of $\tilde{\lambda}$ in 417Ca is symmetric between the two factors, we must also have $A \in \Sigma$.

417I The constructions here have most of the properties one would hope for. I give several in the exercises (417Xd-417Xf, 417Xj). One fact which is particularly useful, and also has a trap in it, is the following.

Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces, and $\tilde{\lambda}$ the τ-additive product measure on $X \times Y$. Suppose that $A \subseteq X$ and $B \subseteq Y$, and write μ_A, ν_B for the corresponding subspace measures; assume that both μ_A and ν_B are semi-finite. Then these are also effectively locally finite and τ-additive, and the subspace measure $\tilde{\lambda}_{A\times B}$ induced by $\tilde{\lambda}$ on $A \times B$ is just the τ-additive product measure of μ_A and ν_B.

proof (a) To check that μ_A and ν_B are effectively locally finite and τ-additive, see 414K. Next, because $\tilde{\lambda}$ is complete, locally determined, effectively locally finite and τ-additive (417Ca), it is strictly localizable (414J), so $\tilde{\lambda}_{A\times B}$ is also effectively locally finite and τ-additive, by 414K again.

(b) Writing $\tilde{\Lambda}^0$ for $(\Sigma\widehat{\otimes}\mathrm{T}) \vee \mathcal{B}(X \times Y)$, as in 417C, we know that $\tilde{\lambda}$ is inner regular with respect to $\tilde{\Lambda}^0$, so $\tilde{\lambda}_{A\times B}$ is inner regular with respect to $\{Q \cap (A \times B) : Q \in \tilde{\Lambda}^0\}$, by 412Ob. But this will be the σ-algebra $\tilde{\Lambda}^0_{A\times B}$ of subsets of $A \times B$ generated by

$$\{(E\times F)\cap(A\times B):E\in\Sigma,\,F\in\mathrm{T}\}\cup\{W\cap(A\times B):W\in\mathcal{B}(X\times Y)\}$$
$$=\{E\times F:E\in\operatorname{dom}\mu_A,\,F\in\operatorname{dom}\nu_B\}\cup\mathcal{B}(A\times B)\}.$$

(c) Now if $C\in\operatorname{dom}\mu_A$ and $D\in\operatorname{dom}\nu_B$, then $\tilde\lambda^*(C\times D)=\mu_AC\cdot\nu_BD$. **P** ($\alpha$) There are $E\in\Sigma$, $F\in\mathrm{T}$ such that $C\subseteq E$, $D\subseteq F$, $\mu E=\mu^*C$ and $\nu F=\nu^*D$; in which case

$$\tilde\lambda^*(C\times D)\le\tilde\lambda(E\times F)=\lambda(E\times F)=\mu E\cdot\nu F$$

(251J)

$$=\mu^*C\cdot\nu^*D=\mu_AC\cdot\nu_BD.$$

(β) If $\gamma<\mu_AC\cdot\nu_BD$ then, because μ_A and ν_B are semi-finite, there are $C'\subseteq C$, $D'\subseteq D$ such that both have finite outer measure and $\mu^*C'\cdot\nu^*D'\ge\gamma$. In this case, take $E'\in\Sigma$, $F'\in\mathrm{T}$ such that $C'\subseteq E'$, $D'\subseteq F'$ and both E' and F' have finite measure. Now if $W\in\operatorname{dom}\tilde\lambda$ and $C\times D\subseteq W$, we have $C'\times D'\subseteq W\cap(E\times F)$, so that $\nu(W\cap(E\times F))[\{x\}]\ge\nu^*D'$ for every $x\in C'$, and

$$\tilde\lambda W\ge\int_E\nu(W\cap(E\times F))[\{x\}]\mu(dx)\ge\mu^*C'\cdot\nu^*D'\ge\gamma,$$

by 417Ga. As W is arbitrary, $\tilde\lambda^*(C\times D)\ge\gamma$; as γ is arbitrary, $\tilde\lambda^*(C\times D)\ge\mu_AC\cdot\nu_BD$. **Q**

(d) Thus $\tilde\lambda_{A\times B}$ agrees with the product of μ_A and μ_B on measurable rectangles, as well as being inner regular with respect to $\tilde\Lambda^0_{A\times B}$. So the uniqueness assertion in 417Ca tells us that $\tilde\lambda_{A\times B}$ is the τ-additive product measure of μ_A and μ_B.

417J In order to use 417G effectively in the theory of infinite products, we need a generalized associative law corresponding to 254N and 417Db.

Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces and $\mathcal{K}$ a partition of I. For $J\subseteq I$ let $\tilde\lambda_J$ be the τ-additive product measure on $Z_J=\prod_{i\in J}X_i$, and write Z for $\prod_{K\in\mathcal{K}}Z_K$. Then we have a natural bijection $\phi:Z\to Z_I$ defined by setting

$$\phi(\langle z_K\rangle_{K\in\mathcal{K}})=\bigcup_{K\in\mathcal{K}}z_K$$

which identifies the τ-additive product $\tilde\lambda$ of the family $\langle\tilde\lambda_K\rangle_{K\in\mathcal{K}}$ with $\tilde\lambda_I$.

In particular, if $K\subseteq I$ is any set, then $\tilde\lambda_I$ can be identified with the τ-additive product of the τ-additive product measures on $\prod_{i\in K}X_i$ and $\prod_{i\in I\setminus K}X_i$.

proof (a) I am claiming that $\tilde\lambda_I$ is precisely the image measure $\tilde\lambda\phi^{-1}$ where $\phi(\langle z_K\rangle_{K\in\mathcal{K}})$ is the common extension of the z_K to a function on I. Now we know that $\tilde\lambda$ is a complete τ-additive topological measure and that ϕ is continuous (indeed, a homeomorphism), so $\tilde\lambda\phi^{-1}$ is also a complete τ-additive topological measure (234Eb, 411Gj).

(b) Suppose that $\langle E_i\rangle_{i\in I}\in\prod_{i\in I}\Sigma_i$ is such that $\{i:i\in I,\,E_i\ne X_i\}$ is finite. For $K\in\mathcal{K}$, set $H_K=\prod_{i\in K}E_i$; then H_K belongs to the domain $\tilde\Lambda_K$ of $\tilde\lambda_K$,

$$\phi^{-1}[\prod_{i\in I}E_i]=\{\langle z_K\rangle_{K\in\mathcal{K}}:z_K(i)\in E_i\text{ whenever }i\in K\in\mathcal{K}\}=\prod_{K\in\mathcal{K}}H_K,$$

while

$$\{K:K\in\mathcal{K},\,H_K\ne Z_K\}=\{K:K\in\mathcal{K},\,K\cap\{i:E_i\ne X_i\}\ne\emptyset\}$$

is finite, so

$$\tilde\lambda\phi^{-1}(\prod_{i\in I}E_i)=\tilde\lambda(\prod_{K\in\mathcal{K}}H_K)=\prod_{K\in\mathcal{K}}\tilde\lambda_KH_K$$
$$=\prod_{K\in\mathcal{K}}\prod_{i\in K}\mu_iE_i=\prod_{i\in I}\mu_iE_i.$$

(c) For each $K\in\mathcal{K}$, $\tilde\lambda_K$ is inner regular with respect to $\tilde\Lambda^0_K=(\widehat{\bigotimes}_{i\in K}\Sigma_i)\vee\mathcal{B}(Z_K)$. By 417E(b-iv), $\tilde\lambda$ is inner regular with respect to

$$\tilde\Lambda'=(\widehat{\bigotimes}_{K\in\mathcal{K}}((\widehat{\bigotimes}_{i\in K}\Sigma_i)\vee\mathcal{B}(Z_K)))\vee\mathcal{B}(Z)=(\widehat{\bigotimes}_{K\in\mathcal{K}}(\widehat{\bigotimes}_{i\in K}\Sigma_i))\vee\mathcal{B}(Z).$$

Now ϕ identifies $(\widehat{\bigotimes}_{K\in\mathcal{K}}(\widehat{\bigotimes}_{i\in K}\Sigma_i))$ with $\widehat{\bigotimes}_{i\in I}\Sigma_i$ in the sense that for $W\subseteq Z_I$, $W\in\widehat{\bigotimes}_{i\in I}\Sigma_i$ iff $\phi^{-1}[W]\in(\widehat{\bigotimes}_{K\in\mathcal{K}}(\widehat{\bigotimes}_{i\in K}\Sigma_i))$, while similarly (because ϕ is a homeomorphism) $W\in\mathcal{B}(Z_I)$ iff $\phi^{-1}[W]\in\mathcal{B}(Z)$. It follows that $W\in\tilde\Lambda^0_I$ iff $\phi^{-1}[W]\in\tilde\Lambda'$. Now if W is measured by $\tilde\lambda\phi^{-1}$, there is a $V\in\tilde\Lambda'$ such that $V\subseteq\phi^{-1}[W]$ and $\tilde\lambda W=\tilde\lambda\phi^{-1}[W]$; in which case $\phi[V]\in\tilde\Lambda_I$, $\phi[V]\subseteq W$ and $\tilde\lambda\phi^{-1}[\phi[V]]=\tilde\lambda\phi^{-1}[W]$.

This shows that $\tilde\lambda\phi^{-1}$ is inner regular with respect to $\tilde\Lambda^0_I$. Together with (a) and (b) and the uniqueness promised in 417E, this shows that $\tilde\lambda\phi^{-1}=\tilde\lambda_I$, as claimed.

417K Corollary Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces and $(X,\tilde\Lambda,\tilde\lambda)$ their τ-additive product. For $J\subseteq I$ let $\tilde\lambda_J$ be the τ-additive product measure on $X_J=\prod_{i\in J}X_i$, and $\tilde\Lambda_J$ its domain; let $\pi_J:X\to X_J$ be the canonical map. Then $\tilde\lambda_J$ is the image measure $\tilde\lambda\pi_J^{-1}$. In particular, if $W\in\tilde\Lambda$ is determined by coordinates in $J\subseteq I$, then $\pi_J[W]\in\tilde\Lambda_J$ and $\tilde\lambda_J\pi_J[W]=\tilde\lambda W$.

proof By 417J, we can identify $\tilde\lambda$ with the τ-additive product of $\tilde\lambda_J$ and $\tilde\lambda_{I\setminus J}$. If $A\subseteq X_J$, then $\pi_J^{-1}[A]=A\times X_{I\setminus J}$. If $A\in\tilde\Lambda_J$, $\pi_J^{-1}[A]\in\tilde\Lambda$ by the definition in 417C; if $\pi_J^{-1}[A]\in\tilde\Lambda$ then $A\in\tilde\Lambda_J$ by 417H. And in either case

$$\tilde\lambda\pi_J^{-1}[A]=\tilde\lambda_JA\cdot\tilde\lambda_{I\setminus J}X_{I\setminus J}=\tilde\lambda_JA.$$

So $\tilde\lambda_J=\tilde\lambda\pi_J^{-1}$.

417L Corollary Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, and $(X,\tilde\Lambda,\tilde\lambda)$ their τ-additive product. Let $\langle K_j\rangle_{j\in J}$ be a disjoint family of subsets of I, and for $j\in J$ write $\tilde\Lambda_j$ for the σ-algebra of members of $\tilde\Lambda$ determined by coordinates in K_j. Then $\langle\tilde\Lambda_j\rangle_{j\in J}$ is a stochastically independent family of σ-algebras (definition: 272Ab).

proof It is enough to consider the case in which J is finite (272Bb), no K_j is empty (since if $K_j=\emptyset$ then $\tilde\Lambda_j=\{\emptyset,X\}$) and $\bigcup_{j\in J}K_j=I$ (adding an extra term if necessary). In this case, if $W_j\in\tilde\Lambda_j$ for each j, then the identification between X and $\prod_{j\in J}\prod_{i\in K_j}X_i$, as described in 417J, matches $\bigcap_{j\in J}W_j$ with $\prod_{j\in J}\pi_{K_j}[W_j]$, writing $\pi_{K_j}(x)$ for $x{\restriction}K_j$. Now if $\tilde\lambda_j$ is the τ-additive product measure on $Z_j=\prod_{i\in K_j}X_i$, we have $\tilde\lambda_j\pi_{K_j}[W_j]=\tilde\lambda W_j$, by 417K. Since $\tilde\lambda$ can be identified with the τ-additive product of $\langle\tilde\lambda_j\rangle_{j\in J}$ (417J),

$$\tilde\lambda(\bigcap_{j\in J}W_j)=\prod_{j\in J}\tilde\lambda_j\pi_{K_j}[W_j]=\prod_{j\in J}\tilde\lambda W_j.$$

As $\langle W_j\rangle_{j\in J}$ is arbitrary, $\langle\tilde\Lambda_j\rangle_{j\in J}$ is independent.

417M Proposition Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is strictly positive. For $J\subseteq I$ let π_J be the canonical map from X onto $X_J=\prod_{i\in J}X_i$; write λ_J, $\tilde\lambda_J$ for the ordinary and τ-additive product measures on X_J, and Λ_J, $\tilde\Lambda_J$ for their domains. Set $\tilde\lambda=\tilde\lambda_I$, $\tilde\Lambda=\tilde\Lambda_I$, $\lambda=\lambda_I$, $\Lambda=\Lambda_I$.

(a) Let $F\subseteq X$ be a closed self-supporting set, and J the smallest subset of I such that F is determined by coordinates in J (4A2B(g-ii)). Then

(i) if $W\in\tilde\Lambda$ is such that $W\triangle F$ is $\tilde\lambda$-negligible and W is determined by coordinates in $K\subseteq I$, then $K\supseteq J$;

(ii) J is countable;

(iii) there is a $W\in\Lambda$, determined by coordinates in J, such that $W\triangle F$ is $\tilde\lambda$-negligible.

(b) $\tilde\lambda$ is inner regular with respect to the family of sets of the form $\bigcap_{n\in\mathbb{N}}V_n$ where each $V_n\in\tilde\Lambda$ is determined by coordinates in a finite set.

(c) If $W\in\tilde\Lambda$, there are a countable $J\subseteq I$ and sets W', $W''\in\tilde\Lambda$, determined by coordinates in J, such that $W'\subseteq W\subseteq W''$ and $\tilde\lambda(W''\setminus W')=0$. Consequently $\tilde\lambda\pi_J^{-1}[\pi_J[W]]=\tilde\lambda W$.

proof (a)(i) ? Suppose, if possible, otherwise. Then F is not determined by coordinates in K, so there are $x \in F$, $y \in X \setminus F$ such that $x\restriction K = y\restriction K$. Let U be an open set containing y, disjoint from F, and of the form $\prod_{i\in I} G_i$, where $G_i \in \mathfrak{T}_i$ for every i and $L = \{i : G_i \neq X_i\}$ is finite. Set

$$U' = \{z : z \in X,\, z(i) \in G_i \text{ for every } i \in L \cap K\},$$

$$U'' = \{z : z(i) \in G_i \text{ for every } i \in L \setminus K\}.$$

Then $U' \cap W$ is determined by coordinates in K, while U'' is determined by coordinates in $I \setminus K$, so

$$0 = \tilde\lambda(F \cap U) = \tilde\lambda(W \cap U) = \tilde\lambda(W \cap U' \cap U'') = \tilde\lambda(W \cap U') \cdot \tilde\lambda U''$$

(by 417L)

$$= \tilde\lambda(F \cap U') \cdot \tilde\lambda U'' = \tilde\lambda(F \cap U') \cdot \prod_{i\in L\setminus K} \mu_i G_i.$$

But $y \in U'$, and $x\restriction K = y\restriction K$, so $x \in F \cap U'$; as F is self-supporting, $\tilde\lambda(F \cap U') > 0$. Because every μ_i is strictly positive, and no G_i is empty, $\prod_{i\in L\setminus K} \mu_i G_i > 0$; and this is impossible. **X**

(ii) By 417E(b-i), there is a $W_0 \in \Lambda$ such that $\tilde\lambda(F \triangle W_0) = 0$. By 254Oc there is a $W_1 \in \Lambda$, determined by coordinates in a countable subset K of I, such that $\lambda(W_0 \triangle W_1) = 0$. Now $\tilde\lambda(F \triangle W_1) = 0$, so (i) tells us that $J \subseteq K$ is countable.

(iii) By 417K, $\pi_J[F] \in \tilde\Lambda_J$. By 417E(b-i) again, there is a $V \in \Lambda_J$ such that $V \triangle \pi_J[F]$ is $\tilde\lambda_J$-negligible. Set $W = \pi_J^{-1}[V]$. Then $W \in \Lambda$, W is determined by coordinates in J, and $W \triangle F = \pi_J^{-1}[V \triangle \pi_J[F]]$ is $\tilde\lambda$-negligible.

(b)(i) Write $\mathcal{V}$ for the set of those members of $\tilde\Lambda$ which are determined by coordinates in a finite set, and $\mathcal{V}_\delta$ for the set of intersections of sequences in $\mathcal{V}$. Because $\mathcal{V}$ is closed under finite unions, so is $\mathcal{V}_\delta$; $\mathcal{V}_\delta$ is surely closed under countable intersections, and $\emptyset$, X belong to $\mathcal{V}_\delta$.

(ii) We need to know that every self-supporting closed set $F \subseteq X$ belongs to $\mathcal{V}_\delta$. **P** By (a), F is determined by coordinates in a countable set J. Express J as the union of a non-decreasing sequence $\langle J_n\rangle_{n\in\mathbb{N}}$ of finite sets. Then $F_n = \pi_{J_n}^{-1}[\overline{\pi_{J_n}[F]}] \in \mathcal{V}$ for each n, and $F = \bigcap_{n\in\mathbb{N}} F_n \in \mathcal{V}_\delta$. **Q**

(iii) Set $\tilde\Lambda^0 = (\widehat{\bigotimes}_{i\in I}\Sigma_i) \vee \mathcal{B}(X)$ as in 417E, and let $\mathcal{A}$ be the family of subsets of X which are finite disjoint unions of sets of the form $B \cap \prod_{i\in I} E_i$ where $B \subseteq X$ is either open or closed, $E_i \in \Sigma_i$ for every $i \in I$, and $\{i : E_i \neq X_i\}$ is finite. Then $X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$, and $\tilde\Lambda^0$ is the σ-algebra generated by $\mathcal{A}$. Now if $A \in \mathcal{A}$, $V \in \tilde\Lambda$ and $\tilde\lambda(A \cap V) > 0$, there is a $K \in \mathcal{V}_\delta \cap \mathcal{A}$ such that $K \subseteq A$ and $\tilde\lambda(K \cap V) > 0$. **P** There is a set A' of the form $B \cap \prod_{i\in I} E_i$ where $B \subseteq X$ is either open or closed, $E_i \in \Sigma_i$ for every $i \in I$, and $\{i : E_i \neq X_i\}$ is finite, such that $A' \subseteq A$ and $\tilde\lambda(A' \cap V) > 0$. (α) If B is open, set

$$\mathcal{U} = \{U : U \in \mathcal{V} \text{ is open},\, U \subseteq B\}.$$

Because $\mathcal{V}$ includes a base for the topology of X, $\bigcup\mathcal{U} = B$; because $\tilde\lambda$ is τ-additive and $\mathcal{V}$ is closed under finite unions, there is a $U \in \mathcal{U}$ such that $U \subseteq B$ and $\tilde\lambda U > \tilde\lambda B - \tilde\lambda(B \cap V)$, so that $\tilde\lambda(U \cap V) > 0$, while $U \in \mathcal{V}$. (β) If B is closed, then it includes a self-supporting closed set F of the same measure (414F), which belongs to $\mathcal{V}_\delta$, by (ii) just above, and now $F \cap \prod_{i\in I} E_i \in \mathcal{V}_\delta$, $F \cap \prod_{i\in I} E_i \subseteq A$ and $\tilde\lambda(F \cap \prod_{i\in I} E_i \cap V) > 0$. **Q**

(iv) By 412C, $\tilde\lambda\restriction\tilde\Lambda^0$ is inner regular with respect to $\mathcal{V}_\delta$. But $\tilde\lambda$ is just the completion of $\tilde\lambda\restriction\tilde\Lambda^0$, so it also is inner regular with respect to $\mathcal{V}_\delta$ (412Ha once more).

(c) By (b), we have sequences $\langle V_n\rangle_{n\in\mathbb{N}}$, $\langle V'_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}_\delta$ such that $V_n \subseteq W$, $V'_n \subseteq X \setminus W$, $\tilde\lambda V_n \geq \tilde\lambda W - 2^{-n}$ and $\tilde\lambda V'_n \geq \tilde\lambda(X \setminus W) - 2^{-n}$ for every $n \in \mathbb{N}$. Each V_n, V'_n is determined by a coordinates in a countable set, so there is a single countable set $J \subseteq I$ such that every V_n and every V'_n is determined by coordinates in J. Set $W' = \bigcup_{n\in\mathbb{N}} V_n$, $W'' = X \setminus \bigcup_{n\in\mathbb{N}} V'_n$; then W', W'' are both determined by coordinates in J, $W' \subseteq W \subseteq W''$ and $\tilde\lambda(W'' \setminus W') = 0$, as required.

417N Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two quasi-Radon measure spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X\times Y$ is a quasi-Radon measure, the unique quasi-Radon measure on $X\times Y$ such that $\tilde{\lambda}(E\times F)=\mu E\cdot\nu F$ for every $E\in\Sigma$ and $F\in\mathrm{T}$.

proof By 417C(b-vii) it is inner regular with respect to the closed sets; being also a complete, locally determined, τ-additive and effectively locally finite topological measure, it is quasi-Radon. By 417Ca it is the only quasi-Radon measure on $X\times Y$ with the right values on measurable rectangles.

417O Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X=\prod_{i\in I}X_i$ is a quasi-Radon measure, the unique quasi-Radon measure on X extending the ordinary product measure.

proof By 417E(b-vi), $\tilde{\lambda}$ is inner regular with respect to the closed sets, so is a quasi-Radon measure, which is unique by 417Ea.

417P Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X\times Y$ is a Radon measure, the unique Radon measure on $X\times Y$ such that $\tilde{\lambda}(E\times F)=\mu E\cdot\nu F$ whenever $E\in\Sigma$ and $F\in\mathrm{T}$.

proof By 417C(b-viii) and (b-ix), $\tilde{\lambda}$ is tight and locally finite; being also a quasi-Radon measure (417N), it is a Radon measure; as in 417N, it is uniquely defined by its values on measurable rectangles.

417Q Theorem Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of Radon probability spaces, and $\tilde{\lambda}$ the τ-additive product measure on $X=\prod_{i\in I}X_i$. For each $i\in I$, let $Z_i\subseteq X_i$ be the support of μ_i. Suppose that $J=\{i:i\in I,\ Z_i\text{ is not compact}\}$ is countable. Then $\tilde{\lambda}$ is a Radon measure, the unique Radon measure on X extending the ordinary product measure.

proof Of course X, being a product of Hausdorff spaces, is Hausdorff, and $\tilde{\lambda}$, being totally finite, is locally finite. Now, given $\epsilon\in\,]0,1]$, let $\langle\epsilon_j\rangle_{j\in J}$ be a family of strictly positive numbers such that $\sum_{j\in J}\epsilon_j\le\epsilon$, and for $j\in J$ choose a compact set $K_j\subseteq X_j$ such that $\mu_jK_j\ge 1-\epsilon_j$; for $i\in I\setminus J$, set $K_i=Z_i$, so that K_i is compact and $\mu_iK_i=1$. Consider $K=\prod_{i\in I}K_i$. Then, using 417E(b-ii) and 254Lb for the two equalities,

$$\tilde{\lambda}K=\lambda^*K=\prod_{i\in I}\mu_iK_i\ge\prod_{j\in J}1-\epsilon_j\ge 1-\epsilon,$$

where λ is the ordinary product measure on X. As ϵ is arbitrary, $\tilde{\lambda}$ satisfies the condition (iii) of 416C, and is a Radon measure. By 417Ea, it is the unique Radon measure on X extending λ.

417R Notation I will use the phrase **quasi-Radon product measure** for a τ-additive product measure which is in fact a quasi-Radon measure; similarly, a **Radon product measure** is a τ-additive product measure which is a Radon measure.

417S Later I will give an example in which a τ-additive product measure is different from the corresponding c.l.d. product measure (419E). In 415E-415F, 415Ye and 416U I have described cases in which c.l.d. measures are τ-additive product measures. It remains very unclear when to expect this to happen. I can however give a couple of results which show that sometimes, at least, we can be sure that the two measures coincide.

Proposition (a) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces and λ the c.l.d. product measure on $X\times Y$. If every open subset of $X\times Y$ is measured by λ, then λ is the τ-additive product measure on $X\times Y$.

(b) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces and λ the ordinary product measure on $X=\prod_{i\in I}X_i$. If every open subset of X is measured by λ, then λ is the τ-additive product measure on X.

(c) In (b), let λ_J be the ordinary product measure on $X_J=\prod_{i\in J}X_i$ for each $J\subseteq I$, and $\tilde{\lambda}_J$ the τ-additive product measure. If $\lambda_J=\tilde{\lambda}_J$ for every finite $J\subseteq I$, and every μ_i is strictly positive, then $\lambda=\tilde{\lambda}_I$ is the τ-additive product measure on X.

proof (a), (b) In both cases, λ is a complete locally determined effectively locally finitemeasure (assembling facts from 251I, 254F and 412Se). We know also from 417C and 417E that λ has an extension to a τ-additive measure $\tilde{\lambda}$ and is therefore itself τ-additive (411C), while it is inner regular with respect to $\Sigma\widehat{\otimes}\mathrm{T}$ or $\widehat{\bigotimes}_{i\in I}\Sigma_i$ (251Ib, 254Ff) and therefore with respect to $\tilde{\Lambda}_0$ as defined in 417C or 417E. The additional hypothesis here is that λ is a topological measure. But that means that it satisfies all the conditions required of a τ-additive product measure and is equal to $\tilde{\lambda}$.

(c)(i) The first step is to note that $\lambda_J = \tilde{\lambda}_J$ for every countable $J \subseteq I$. **P** Express J as $\bigcup_{n\in\mathbb{N}} J_n$ where $\langle J_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of finite sets. If $F \subseteq X_J$ is closed, then it is $\bigcap_{n\in\mathbb{N}} \pi_n^{-1}[\overline{\pi_n[F]}]$, where $\pi_n : X_J \to X_{J_n}$ is the canonical map for each n. But every $\overline{\pi_n[F]}$ is a closed subset of X_{J_n}, therefore measured by λ_{J_n}; because π_n is inverse-measure-preserving (417K), $\pi_n^{-1}[\overline{\pi_n[F]}] \in \operatorname{dom}\lambda_J$ for each n, and $F \in \operatorname{dom}\lambda_J$. Thus every closed set, therefore every open set is measured by λ_J, and λ_J is a topological measure; by (b), $\lambda_J = \tilde{\lambda}_J$. **Q**

(ii) Suppose that $W \subseteq X$ is open. By 417M, there are W', W'' measured by $\tilde{\lambda}$ such that $W' \subseteq W \subseteq W''$, both W'' and W' are determined by coordinates in a countable set, and $\tilde{\lambda}_I(W''\setminus W') = 0$. Let $J \subseteq I$ be a countable set such that W' and W'' are determined by coordinates in J. Then $\lambda_J = \tilde{\lambda}_J$ measures $\pi_J[W']$, by 417K, so λ measures $W' = \pi_J^{-1}[\pi_J[W]]$, by 254Oa. Similarly, λ measures W''. Now $\lambda(W''\setminus W') = \tilde{\lambda}_I(W''\setminus W') = 0$, so λ measures W. As W is arbitrary, λ is a topological measure and must be the τ-additive product measure, by (b).

417T Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces and λ the c.l.d. product measure on $X\times Y$. If X has a conegligible subset with a countable network (e.g., if X is metrizable and μ is σ-finite), then λ is the τ-additive product measure on $X\times Y$.

proof (a) Suppose to begin with that μ and ν are totally finite, and that X itself has a countable network; let $\langle A_n\rangle_{n\in\mathbb{N}}$ run over a network for X. Let $\hat{\mu}$ be the completion of μ and $\hat{\Sigma}$ its domain. Let $\tilde{\lambda}$ be the τ-additive product measure on $X\times Y$. (We are going to need Fubini's theorem both for λ and for $\tilde{\lambda}$. I will use a sprinkling of references to §§251-252 to indicate which parts of the argument below depend on the properties of λ.)

Let $W \subseteq X\times Y$ be an open set. For each $n\in\mathbb{N}$, set

$$H_n = \bigcup\{H : H\in\mathfrak{S},\ A_n\times H\subseteq W\},$$

so that H_n is open. Then $W = \bigcup_{n\in\mathbb{N}} A_n\times H_n$. **P** Of course $A_n\times H_n\subseteq W$ for every $n\in\mathbb{N}$. If $(x,y)\in W$, there are open sets $G\subseteq X$, $H\subseteq Y$ such that $(x,y)\in G\times H\subseteq W$; now there is an $n\in\mathbb{N}$ such that $x\in A_n\subseteq G$, so that $H\subseteq H_n$ and $(x,y)\in A_n\times H_n$. **Q**

By 417C(b-v-α), there is an open set W_0 in the domain Λ of λ such that $W_0\subseteq W$ and $\tilde{\lambda}(W\setminus W_0) = 0$. By 417Ga, applied to $\chi(W\setminus W_0)$, $A = \{x : \nu(W[\{x\}]\setminus W_0[\{x\}]) > 0\}$ is μ-negligible. For each $n\in\mathbb{N}$, $x\in X$ set $f_n(x) = \nu(H_n\cap W_0[\{x\}])$; then 252B tells us that $\int f_n d\mu$ is defined and equal to $\lambda(W_0\cap(X\times H_n))$. In particular, f_n is $\hat{\Sigma}$-measurable. Set $E_n = \{x : f_n(x) = \nu H_n\}\in\hat{\Sigma}$. If $x\in A_n$, then $H_n\subseteq W[\{x\}]$, so $A_n\setminus E_n\subseteq A$.

Now, by 252B again,

$$\begin{aligned}\lambda((E_n\times H_n)\setminus W_0) &= \int_{E_n}\nu(H_n\setminus W_0[\{x\}])\mu(dx)\\ &= \int_{E_n}\nu H_n - \nu(H_n\cap W_0[\{x\}])\mu(dx) = 0.\end{aligned}$$

So if we set $W_1 = \bigcup_{n\in\mathbb{N}} E_n\times H_n$, $W_1\setminus W\subseteq W_1\setminus W_0$ is λ-negligible. On the other hand,

$$W\setminus W_1\subseteq\bigcup_{n\in\mathbb{N}}(A_n\setminus E_n)\times H_n\subseteq A\times Y$$

is also λ-negligible. Because λ is complete, $W\in\Lambda$. As W is arbitrary, λ is a topological measure and is equal to $\tilde{\lambda}$, by 417Sa.

(b) Now consider the general case. Let Z be a conegligible subset of X with a countable network; since any subset of a space with a countable network again has a countable network (4A2Na), we may suppose that $Z \in \Sigma$. Again let W be an open set in $X \times Y$. This time, take arbitrary $E \in \Sigma$, $F \in \mathrm{T}$ of finite measure, and consider the subspace measures $\mu_{E\cap Z}$ and ν_F. These are still effectively locally finite and τ-additive (414K), and are now totally finite. Also $E \cap Z$ has a countable network. So (a) tells us that the relatively open set $W \cap ((E \cap Z) \times F)$ is measured by the c.l.d. product of $\mu_{E\cap Z}$ and ν_F, which is the subspace measure on $(E \cap Z) \times F$ induced by λ (251Q). Since λ surely measures $E \times F$, it measures $W \cap (Z \times Y) \cap (E \times F)$. As E and F are arbitrary, λ measures $W \cap (Z \times Y)$ (251H). But $\lambda((X \setminus Z) \times Y) = \mu(X \times Z) \cdot \nu Y = 0$ (251Ia), so λ also measures W. As W is arbitrary, λ is the τ-additive product measure.

417U Proposition Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces. Let λ be the ordinary product probability measure on $X = \prod_{i\in I} X_i$ and Λ its domain. Then every continuous function $f : X \to \mathbb{R}$ is Λ-measurable, so Λ includes the Baire σ-algebra of X.

proof (a) Let $\tilde\lambda$ be a τ-additive topological measure extending λ (417E), and $\tilde\Lambda$ its domain; then f is $\tilde\Lambda$-measurable, just because $\tilde\lambda$ is a topological measure. For $\alpha \in \mathbb{R}$, set

$$G_\alpha = \{x : x \in X,\ f(x) < \alpha\}, \quad H_\alpha = \{x : x \in X,\ f(x) > \alpha\},$$

$$F_\alpha = \{x : x \in X,\ f(x) = \alpha\}.$$

Then $\langle F_\alpha\rangle_{\alpha\in\mathbb{R}}$ is disjoint, so $A = \{\alpha : \alpha \in \mathbb{R},\ \tilde\lambda F_\alpha > 0\}$ is countable, and $A' = \mathbb{R} \setminus A$ is dense in $\mathbb{R}$; let $Q \subseteq A'$ be a countable dense set.

For each $q \in Q$, let $V_q \subseteq G_q$, $W_q \subseteq H_q$ be such that

$$\lambda V_q = \lambda_* G_q = \tilde\lambda G_q, \quad \lambda W_q = \lambda_* H_q = \tilde\lambda H_q$$

(413Ea, 417E(b-ii)). Then

$$\lambda^*(G_q \setminus V_q) \le \lambda(X \setminus (V_q \cup W_q)) = 1 - \lambda V_q - \lambda W_q = \tilde\lambda(X \setminus (G_q \cup H_q)) = 0.$$

Because λ is complete, $G_q \setminus V_q$ and G_q belong to Λ. But now, if $\alpha \in \mathbb{R}$,

$$\{x : f(x) < \alpha\} = \bigcup_{q\in Q, q<\alpha} G_q \in \Lambda,$$

so f is Λ-measurable.

(b) It follows that every zero set belongs to Λ, so that Λ must include the Baire σ-algebra of X.

417V Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces, and $(X \times Y, \Lambda, \lambda)$ their c.l.d. product. Then every continuous function $f : X \times Y \to \mathbb{R}$ is Λ-measurable, and the Baire σ-algebra of $X \times Y$ is included in Λ.

proof Let $Z \subseteq X \times Y$ be a zero set. If $E \in \Sigma$, $F \in \mathrm{T}$ are sets of finite measure, then $Z \cap (E \times F)$ is a zero set for the relative topology of $E \times F$. Now the subspace measures μ_E and ν_F are τ-additive topological measures (414K), so $Z \cap (E \times F)$ is measured by the c.l.d. product $\mu_E \times \nu_F$ of μ_E and ν_F. **P** If either μE or νF is zero, this is trivial. Otherwise, they have scalar multiples μ'_E, ν'_F which are probability measures, and of course are still τ-additive topological measures. By 417U, $Z \cap (E \times F)$ is measured by $\mu'_E \times \nu'_F$. Since $\mu_E \times \nu_F$ is just a scalar multiple of $\mu'_E \times \nu'_F$, $Z \cap (E \times F)$ is measured by $\mu_E \times \nu_F$. **Q** But $\mu_E \times \nu_F$ is the subspace measure $\lambda_{E\times F}$ (251Q), so $Z \cap (E \times F) \in \Lambda$. As E and F are arbitrary, $Z \in \Lambda$ (251H).

Thus every zero set belongs to Λ; accordingly Λ must include the Baire σ-algebra, and every continuous function must be Λ-measurable.

417X Basic exercises (a) Let (X, Σ, μ) be a semi-finite measure space and $\mathcal{A}$ a family of subsets of X. Show that the following are equiveridical: (i) there is a measure μ' on X, extending μ, such that $\mu' A = 0$ for every $A \in \mathcal{A}$; (ii) $\mu_*(\bigcup_{n\in\mathbb{N}} A_n) = 0$ for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$.

>(c) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be topological measure spaces such that μ and ν are both effectively locally finite τ-additive Borel measures. Show that there is a unique effectively locally finite τ-additive Borel measure λ' on $X \times Y$ such that $\lambda'(G \times H) = \mu G \cdot \nu H$ for all open sets $G \subseteq X$, $H \subseteq Y$.

>(d) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of topological probability spaces in which every μ_i is a τ-additive Borel measure. Show that there is a unique τ-additive Borel measure λ' on $X = \prod_{i\in I} X_i$ such that $\lambda'(\prod_{i\in I} F_i) = \prod_{i\in I} \mu_i F_i$ whenever $F_i \subseteq X_i$ is closed for every $i \in I$.

(e) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces and $\tilde\lambda$ the τ-additive product measure on $X\times Y$. Let $\langle X_i\rangle_{i\in I}$, $\langle Y_j\rangle_{j\in J}$ be decompositions for μ, ν respectively (definition: 211E). Show that $\langle X_i\times Y_j\rangle_{i\in I,j\in J}$ is a decomposition for $\tilde\lambda$. (Cf. 251O.)

>(f) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that μ_i is inner regular with respect to the Borel sets for every i, and $\tilde\lambda$ the τ-additive product measure on $X=\prod_{i\in I}X_i$. Take $A_i\subseteq X_i$ for each $i\in I$. (i) Show that if $\mu_i^*A_i=1$ for every i, then the subspace measure induced by $\tilde\lambda$ on $A=\prod_{i\in I}A_i$ is just the τ-additive product $\tilde\lambda^\#$ of the subspace measures on the A_i. (*Hint*: show that if we set $\lambda'W=\tilde\lambda^\#(W\cap A)$ for Borel sets $W\subseteq X$, then λ' satisfies the conditions of 417Xd.) (ii) Show that in any case $\tilde\lambda^*A=\prod_{i\in I}\mu_i^*A_i$. (Cf. 254L.)

(g) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle(Y_i,\mathfrak{S}_i,\mathrm{T}_i,\nu_i)\rangle_{i\in I}$ be two families of τ-additive topological probability spaces in which every μ_i and every ν_i is inner regular with respect to the Borel sets. Let $\tilde\lambda$, $\tilde\lambda'$ be the τ-additive product measures on $X=\prod_{i\in I}X_i$ and $Y=\prod_{i\in I}Y_i$ respectively. Suppose that for each $i\in I$ we are given a continuous inverse-measure-preserving function $\phi_i:X_i\to Y_i$. Show that the function $\phi:X\to Y$ defined by setting $\phi(x)(i)=\phi_i(x(i))$ for $x\in X$, $i\in I$ is inverse-measure-preserving.

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets. Let $\tilde\lambda$ be the τ-additive product measure on $X\times Y$, and $\tilde\Lambda$ its domain. Suppose that ν is σ-finite. Show that for any $W\in\tilde\Lambda$, $W[\{x\}]\in\mathrm{T}$ for almost every $x\in X$, and $x\mapsto\nu W[\{x\}]$ is measurable.

>(i) Let $(X,\mathfrak{T},\Sigma,\mu)$ be $[0,1]$ with its usual topology and Lebesgue measure, and let $(Y,\mathfrak{S},\mathrm{T},\nu)$ be $[0,1]$ with its discrete topology and counting measure. (i) Show that both are Radon measure spaces. (ii) Show that the c.l.d. product measure on $X\times Y$ is a Radon measure. (*Hint*: 252Kc, or use 417T and 417P.) (iii) Show that 417Ga can fail if we omit the hypothesis on $\{(x,y):f(x,y)\ne 0\}$.

(j) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two effectively locally finite τ-additive topological measure spaces. Let λ be the c.l.d. product measure and $\tilde\lambda$ the τ-additive product measure on $X\times Y$. Show that $\lambda^*(A\times B)=\tilde\lambda^*(A\times B)$ for all sets $A\subseteq X$, $B\subseteq Y$. (*Hint*: start with A, B of finite outer measure, so that 417I applies.)

(k) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces with strictly positive measures, all inner regular with respect to the Borel sets, and $(X,\mathfrak{T},\tilde\Lambda,\tilde\lambda)$ their τ-additive product. For $J\subseteq I$ let $\tilde\lambda_J$ be the τ-additive product measure on $X_J=\prod_{i\in J}X_i$, and $\tilde\Lambda_J$ its domain. (i) Show that if f is a real-valued $\tilde\Lambda$-measurable function defined $\tilde\lambda$-almost everywhere on X, we can find a countable set $J\subseteq I$ and a $\tilde\Lambda_J$-measurable function g, defined $\tilde\lambda_J$-almost everywhere on X_J, such that f extends $g\pi_J$. (ii) In (i), show that $\int f d\tilde\lambda=\int g\,d\tilde\lambda_J$ if either is defined in $[-\infty,\infty]$.

(l) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X,\mathfrak{T},\tilde\Lambda,\tilde\lambda)$ their τ-additive product. Show that for any $W\in\tilde\Lambda$ there is a smallest set $J\subseteq I$ for which there is a $W'\in\tilde\Lambda$, determined by coordinates in J, with $\tilde\lambda(W\triangle W')=0$. (*Hint*: 254R.)

(m) What needs to be added to 417M and 415Xm to complete a proof of 415E?

(n) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless τ-additive topological probability space such that μ is inner regular with respect to the Borel sets, and I a set with cardinal at most that of the support of μ. Show that the set of injective functions from I to X has full outer measure for the τ-additive product measure on X^I.

>(o) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces. Show that the Radon product measure on $X\times Y$ is the unique Radon measure $\tilde\lambda$ such that $\tilde\lambda(K\times L)=\mu K\cdot\nu L$ for all compact sets $K\subseteq X$, $L\subseteq Y$.

>(p) Let I be an uncountable set, and λ, $\tilde\lambda$ be the ordinary and τ-additive product measures on $X=\{0,1\}^I$ when each factor is given its usual topology and the Dirac measure concentrated at 1. Show that $\tilde\lambda$ properly extends λ, and that the support of $\tilde\lambda$ is not determined by coordinates in any countable set. Find a $\tilde\lambda$-negligible open set $W\subseteq X$ such that its projection onto $\{0,1\}^J$ is conegligible for every proper subset J of I.

(q) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of Radon probability spaces, and $\tilde\lambda$ the quasi-Radon product measure on $X=\prod_{i\in I}X_i$. For each $i\in I$, let $Z_i\subseteq X_i$ be the support of μ_i. Show that $\tilde\lambda$ is a Radon measure iff $\{i:i\in I,\ Z_i$ is not compact$\}$ is countable. In particular, show that the ordinary product measure on $[0,1[^I$, where I is uncountable and each copy of $[0,1[$ is given Lebesgue measure, is a quasi-Radon measure, but not a Radon measure.

(r) Let $\langle(X_n,\mathfrak{T}_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability spaces. Show that the Radon product measure on $X=\prod_{n\in\mathbb{N}}X_n$ is the unique Radon measure $\tilde\lambda$ on X such that $\tilde\lambda(\prod_{n\in\mathbb{N}}K_n)=\prod_{n=0}^\infty\mu_nK_n$ whenever $K_n\subseteq X_n$ is compact for every n.

(s) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces and λ, $\tilde\lambda$ the ordinary and τ-additive product measures on $X=\prod_{i\in I}X_i$. Show that if $A\subseteq X$ has $\tilde\lambda$-negligible boundary, then A is measured by λ.

(t) Let us say that a topological space X is **chargeable** if there is an additive functional $\nu:\mathcal{P}X\to[0,\infty[$ such that $\nu G>0$ for every non-empty open set $G\subseteq X$. (i) Show that if there is a σ-finite measure μ on X such that $\mu_*G>0$ for every non-empty open set G, then X is chargeable. (*Hint*: 215B(vii), 391G.) (ii) Show that any separable space is chargeable. (iii) Show that X is chargeable iff its regular open algebra is chargeable in the sense of 391Bb. (*Hint*: see the proof of 314P.) (iv) Show that any open subspace of a chargeable space is chargeable. (v) Show that if $Y\subseteq X$ is dense, then X is chargeable iff Y is chargeable. (vi) Show that if X is expressible as the union of countably many chargeable subspaces, then it is chargeable. (vii) Show that any product of chargeable spaces is chargeable. (Cf. 391Xb(iii).) (viii) Show that if $\langle X_i\rangle_{i\in I}$ is a family of chargeable spaces with product X, then all regular open subsets of X and all Baire subsets of X are determined by coordinates in countable sets. (*Hint*: 4A2Eb, 4A3Mb.) (ix) Show that a continuous image of a chargeable space is chargeable. (x) Show that a compact Hausdorff space is chargeable iff it carries a strictly positive Radon measure. (*Hint*: 416K.)

(u) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be quasi-Radon measure spaces such that $\mu X\cdot\nu Y>0$. Show that the quasi-Radon product measure on $X\times Y$ is completion regular iff it is equal to the c.l.d. product measure and μ and ν are both completion regular. (*Hint*: 412Sc; if μE, νF are finite and $Z\subseteq E\times F$ is a zero set of positive measure, use Fubini's theorem to show that Z has sections of positive measure.)

(v) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces. Show that the quasi-Radon product measure on $\prod_{i\in I}X_i$ is completion regular iff it is equal to the ordinary product measure and every μ_i is completion regular.

(w) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $\tilde\lambda$ the τ-additive product measure on $X=\prod_{i\in I}X_i$; write $\tilde\Lambda$ for its domain. (i) Show that if $W\in\tilde\Lambda$, $\tilde\lambda W>0$ and $\epsilon>0$ then there are a finite $J\subseteq I$ and a $W'\in\tilde\Lambda$ such that $\tilde\lambda W'\ge1-\epsilon$ and for every $x\in W'$ there is a $y\in W$ such that $x{\restriction}I\setminus J=y{\restriction}I\setminus J$. (Cf. 254Sb.) (ii) Show that if $A\subseteq X$ is determined by coordinates in $I\setminus\{i\}$ for every $i\in I$ then $\tilde\lambda^*A\in\{0,1\}$. (Cf. 254Sa.)

417Y Further exercises (a) (i) Show that if, in 417A, μ is strictly localizable, then it has a strictly localizable extension μ' with the properties (i)-(iv) there. (ii) Give an example to show that the construction offered in 417A may not immediately achieve this result.

(b) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be effectively locally finite τ-additive topological measure spaces such that μ and ν are both inner regular with respect to the Borel sets.

(i) Fix open sets $G\subseteq X$, $H\subseteq Y$ of finite measure. Let $\mathcal{W}_{GH}$ be the set of those $W\subseteq X\times Y$ such that $\theta_{GH}(W)=\int_G\hat\nu(W[\{x\}]\cap H)dx$ is defined, where $\hat\nu$ is the completion of ν. (α) Show that every open set belongs to $\mathcal{W}_{GH}$. (β) Show that θ_{GH} is countably additive in the sense that $\theta_{GH}(\bigcup_{n\in\mathbb{N}}W_n)=\sum_{n=0}^\infty\theta_{GH}(W_n)$ for every disjoint sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}_{GH}$, and τ-additive in the sense that $\theta_{GH}(\bigcup\mathcal{V})=\sup_{V\in\mathcal{V}}\theta_{GH}(V)$ for every non-empty upwards-directed family $\mathcal{V}$ of open sets in $X\times Y$. (γ) Show that every Borel set belongs to $\mathcal{W}_{GH}$. (*Hint*: Monotone Class Theorem.) (δ) Show that $\theta_{GH}{\restriction}\mathcal{B}(X\times Y)$ is a τ-additive Borel measure; let λ_{GH} be its completion. (ϵ) Show that $\lambda_{GH}=\theta_{GH}{\restriction}\Lambda_{GH}$, where $\Lambda_{GH}=\operatorname{dom}\lambda_{GH}$. ($\zeta$)

Show that $\lambda_{GH}(E\times F)$ is defined and equal to $\mu E\cdot\nu F$ whenever $E\in\Sigma$, $F\in\mathrm{T}$, $E\subseteq G$ and $F\subseteq H$. (*Hint*: start with open E and F, move to Borel E and F with the Monotone Class Theorem.) (η) Writing λ for the c.l.d. product measure on $X\times Y$, show that $\lambda_{GH}(W)$ is defined and equal to $\lambda(W\cap(G\times H))$ whenever $W\in\operatorname{dom}\lambda$.

(ii) Now take $\tilde\Lambda$ to be $\bigcap\{\Lambda_{GH}:G\in\mathfrak{T},\,H\in\mathfrak{S},\,\mu G<\infty,\,\nu H<\infty\}$ and $\tilde\lambda W=\sup_{G,H}\lambda_{GH}(W)$ for $W\in\tilde\Lambda$. Show that $\tilde\lambda$ is an extension of λ to a complete locally determined effectively locally finite τ-additive topological measure on $X\times Y$ which is inner regular with respect to the Borel sets, so is the τ-additive product measure as defined in 417F.

(c) Let (X,Σ,μ) and (Y,T,ν) be complete measure spaces with topologies $\mathfrak{T}$, $\mathfrak{S}$. Suppose that μ and ν are effectively locally finite and τ-additive and moreover that their domains include bases for the two topologies. Show that the c.l.d. product measure on $X\times Y$ has the same properties. (*Hint*: start by assuming that μX and νY are both finite. If $\mathcal{V}$ is an upwards-directed family of measurable open sets with measurable open union W, look at $g_V(x)=\nu V[\{x\}]$ for $V\in\mathcal{V}$.)

(d) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X,\mathfrak{T},\tilde\Lambda,\tilde\lambda)$ their τ-additive product. (i) Show that the following are equiveridical: (α) μ_i is strictly positive for all but countably many $i\in I$; (β) whenever $W\in\tilde\Lambda$ there are a countable $J\subseteq I$ and W_1, $W_2\in\tilde\Lambda$, determined by coordinates in J, such that $W_1\subseteq W\subseteq W_2$ and $\tilde\lambda(W_2\setminus W_1)=0$. (ii) Show that when these are false, $\tilde\lambda$ cannot be equal to the ordinary product measure on X.

(e) Let (X,Σ,μ) and (Y,T,ν) be measure spaces with Hausdorff topologies $\mathfrak{T}$, $\mathfrak{S}$ such that both μ and ν are inner regular with respect to the families of sequentially compact sets in each space. Show that the c.l.d. product measure λ on $X\times Y$ is also inner regular with respect to the sequentially compact sets, so has an extension to a topological measure which is inner regular with respect to the sequentially compact sets. (*Hint*: 412R, 416Yd.)

(f) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces with topologies $\mathfrak{T}_i$ such that every μ_i is inner regular with respect to the family of closed countably compact sets in X_i and every X_i is compact. Show that the ordinary product measure λ on $X=\prod_{i\in I}X_i$ is also inner regular with respect to the closed countably compact sets, so has an extension to a topological measure $\tilde\lambda$ which is inner regular with respect to the closed countably compact sets in X. Show that this can be done in such a way that for every $W\in\operatorname{dom}\tilde\lambda$ there is a $V\in\operatorname{dom}\lambda$ such that $\tilde\lambda(W\triangle V)=0$. (*Hint*: 412T, 416Yc.)

(g) Let $\langle(X_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces with Hausdorff topologies $\mathfrak{T}_n$ such that every μ_n is inner regular with respect to the family of sequentially compact sets in X_n. Show that the ordinary product measure λ on $X=\prod_{n\in\mathbb{N}}X_n$ is also inner regular with respect to the sequentially compact sets, so has an extension to a topological measure $\tilde\lambda$ which is inner regular with respect to the sequentially compact sets in X. Show that this can be done in such a way that for every $W\in\operatorname{dom}\tilde\lambda$ there is a $V\in\operatorname{dom}\lambda$ such that $\tilde\lambda(W\triangle V)=0$.

(h) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces, and λ, $\tilde\lambda$ the ordinary and quasi-Radon product measures on $X=\prod_{i\in I}X_i$. Suppose that all but *one* of the $\mathfrak{T}_i$ have countable networks and all but *countably many* of the μ_i are strictly positive. Show that $\lambda=\tilde\lambda$.

(i) Let us say that a quasi-Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$ has the **simple product property** if the c.l.d. product measure on $X\times Y$ is equal to the quasi-Radon product measure for every quasi-Radon measure space $(Y,\mathfrak{S},\mathrm{T},\nu)$. (i) Show that if $(X,\mathfrak{T})$ has a countable network then $(X,\mathfrak{T},\Sigma,\mu)$ has the simple product property. (ii) Show that if a quasi-Radon measure space has the simple product property so do all its subspaces. (iii) Show that the quasi-Radon product of two quasi-Radon measure spaces with the simple product property has the simple product property. (iv) Show that the quasi-Radon product of any family of quasi-Radon probability spaces with the simple product property has the simple product property. (v) Show that the Sorgenfrey line (415Xc, 439Q) with Lebesgue measure has the simple product property.

417 Notes and comments The general problem of determining just when a measure can be extended to a measure with given properties is one which will recur throughout this volume. I have more than once mentioned the Banach-Ulam problem; if you like, this is the question of whether there can ever be an extension of the countable-cocountable measure on an uncountable set X to a measure defined on the whole algebra $\mathcal{P}X$. This particular question appears to be undecidable from the ordinary axioms of set theory; but for many sets (for instance, if $X = \omega_1$) it is known that the answer is 'no'. (See 419G and 438C.) This being so, we have to take each manifestation of the general question on its own merits. In 417C and 417E the challenge is to take a product measure λ defined in terms of the factor measures alone, disregarding their topological properties, and extend it to a topological measure, preferably τ-additive. Of course there are important cases in which λ is itself already a topological measure; for instance, we know that the c.l.d. product of Lebesgue measure on $\mathbb{R}$ with itself is Lebesgue measure on $\mathbb{R}^2$ (251N), and other examples are in 415E, 415Ye, 416U, 417S-417T, 417Yh and 453I. But in general not every open set in the product belongs to the domain of λ, even when we have the product of two Radon measures on compact Hausdorff spaces (419E).

Once we have resolved to grasp the nettle, however, there is a natural strategy for the proof. It is easy to see that if λ, in 417C or 417E, is to have an extension to a τ-additive topological measure $\tilde{\lambda}$, then we must have $\tilde{\lambda}A(\mathcal{V}) = 0$ for every $\mathcal{V}$ belonging to the class $\mathfrak{V}$. Now 417A descibes a sufficient (and obviously necessary) condition for there to be an extension of λ with this property. So all we have to do is check. The check is not perfectly straightforward; in 417E it uses all the resources of the original proof that there is a product measure on an arbitrary product of probability spaces (which I suppose is to be expected), with 414B (of course) to apply the hypothesis that the factor measures are τ-additive, and a couple of extra wrinkles (the W'_n and C'_n of part (a-iii-β) of the proof of 417E, and the use of supports in part (a-iii-ϵ)). I take the opportunity to say that the precise form of the results in 417C and 417E, leading to definitions of 'τ-additive product measure' in terms of the σ-algebras $\tilde{\Lambda}_0$, was devised in response to a remark by M.R.Burke.

It is worth noting that (both for finite and for infinite products) the measure algebras of λ and $\tilde{\lambda}$ are identical (417C(b-i), 417E(b-i)), so there is no new work to do in identifying the measure algebra of $\tilde{\lambda}$ and the associated function spaces.

An obstacle we face in 417C-417E is the fact that *not* every τ-additive measure μ has an extension to a τ-additive topological measure, even when μ is totally finite and its domain includes a base for the topology. (I give an example in 419J.) Consequently it is not enough, in 417C or 417E, to show that the ordinary product measure λ is τ-additive. But perhaps I should remark that if λ is inner regular with respect to the closed sets, this obstacle evaporates (415L). Accordingly, for the principal applications (to quasi-Radon and Radon product measures, and in particular whenever the topological spaces involved are regular) we have rather easier proofs available, based on the constructions of §415. For completely regular spaces, there is yet another approach, because the product measures can be described in terms of the integrals of continuous functions (415I), which by 417U and 417V can be calculated from the ordinary product measures. Of course the proof that λ itself is τ-additive is by no means trivial, especially in the case of infinite products, corresponding to 417E; but for finite products there are relatively direct arguments, applying indeed to slightly more general situations (417Yc). If we have measures which are inner regular with respect to countably compact classes of sets, then there may be other ways of approaching the extension, using theorems from §413 (see 417Ye-417Yg), and for compact Radon measure spaces, λ becomes tight (412Sb, 412V), so its τ-additivity is elementary.

The arguments of 417C and 417E depend on specific constructions of the new product measures $\tilde{\lambda}$. It is therefore important to develop descriptions in respect of which they may be considered canonical, as in 417Ca and 417Ea. With these in hand, we can reasonably expect 'commutative' and 'associative' and 'distributive' laws, as in 417Db, 417J and 417Xe. Subspaces mostly behave themselves (417I, 417Xf). If you prefer to restrict your measures to Borel algebras, you again get canonical product Borel measures (417Xc-417Xd).

Of course extending the product measure means that we get new integrable functions on the product, so that Fubini's theorem has to be renegotiated. Happily, it remains valid, at least in the contexts in which it was effective before (417Ga); we still need, in effect, one of the measures to be σ-finite. The theorem still fails for arbitrary integrable functions on products of Radon measure spaces, and the same example works as before (417Xi). In fact this means that we have an alternative route to the construction of the

τ-additive product of two measures (417Yb). But note that on this route 'commutativity', the identification of the product measure on $X\times Y$ with that on $Y\times X$, becomes something which can no longer be taken for granted, because if we *define* $\tilde{\lambda}W$ to be $\int\nu W[\{x\}]dx$ we have to worry about when, and why, this will be equal to $\int\mu W^{-1}[\{y\}]dy$.

A version of Tonelli's theorem follows from Fubini's theorem, as before (417Gc). We also have results corresponding to most of the theorems of §254. But note that there are two traps. In the theorem that a measurable set can be described in terms of a projection onto a countable subproduct (254O, 417M) we need to suppose that the factor measures are strictly positive, and in the theorem that a product of Radon measures is a Radon measure (417Q) we need to suppose that the factor measures have compact supports. The basic examples to note in this context are 417Xp and 417Xq.

It is not well understood when we can expect c.l.d. product measures to be topological measures, even in the case of compact Radon probability spaces. Example 419E remains a rather special case, but of course much more effort has gone into seeking positive results. Note that the ordinary product measures of this section are always effectively locally finite and τ-additive (417C, 417E), so that they will be equal to the τ-additive products iff they measure every open set (417S). Regarding infinite products, the τ-additive product measure can fail to be the ordinary product measure in just two ways: if one of the *finite* product measures is not a topological measure, or if uncountably many of the factor measures are not strictly positive (417Sc, 417Xp, 417Yd). So it is finite products which need to be studied.

Whenever we have a subset F of an infinite product $X=\prod_{i\in I}X_i$, it is important to know when F is determined by coordinates in a proper subset of I; in measure theory, we are particularly interested in sets determined by coordinates in countable subsets of I (254Mb). It may happen that there is a *smallest* set J such that F is determined by coordinates in J; for instance, when we have a topological product and F is closed (4A2Bg). When we have a product of probability spaces, we sometimes wish to identify sets J such that F is 'essentially' determined by coordinates in J, in the sense that there is an F', determined by coordinates in J, such that $F\triangle F'$ is negligible. In this context, again, there is a smallest such set (254Rd), which can be identified in terms of the probability algebra free product of the measure algebras (325Mb). In 417Ma the two ideas come together: under the conditions there, we get the same smallest J by either route.

In 417Ma, we have a product of strictly positive τ-additive topological probability measures. If we keep the 'strictly positive' but abandon everything else, we still have very striking results just because the product topology is ccc, so that we can apply 4A2Eb. An abstract expression of this idea is in 417Xt.

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418 Measurable functions and almost continuous functions

In this section I work through the basic properties of measurable and almost continuous functions, as defined in 411L and 411M. I give the results in the full generality allowed by the terminology so far introduced, but most of the ideas are already required even if you are interested only in Radon measure spaces as the domains of the functions involved. Concerning the codomains, however, there is a great difference between metrizable spaces and others, and among metrizable spaces separability is of essential importance.

I start with the elementary properties of measurable functions (418A-418C) and almost continuous functions (418D). Under mild conditions on the domain space, almost continuous functions are measurable (418E); for a separable metrizable codomain, we can expect that measurable functions should be almost continuous (418J). Before coming to this, I spend a couple of paragraphs on image measures: a locally finite image measure under a measurable function is Radon if the measure on the domain is Radon and the function is almost continuous (418I).

418L-418Q are important results on expressing given Radon measures as image measures associated with continuous functions, first dealing with ordinary functions $f:X\to Y$ (418L) and then coming to Prokhorov's theorem on projective limits of probability spaces (418M).

The machinery of the first part of the section can also be used to investigate representations of vector-valued functions in terms of product spaces (418R-418T).

418A Proposition Let X be a set, Σ a σ-algebra of subsets of X, Y a topological space and $f:X\to Y$ a measurable function.

(a) $f^{-1}[F] \in \Sigma$ for every Borel set $F \subseteq Y$.

(b) If $A \subseteq X$ is any set, endowed with the subspace σ-algebra, then $f{\restriction}A : A \to Y$ is measurable.

(c) Let $(Z, \mathfrak{T})$ be another topological space. Then $gf : X \to Z$ is measurable for every Borel measurable function $g : Y \to Z$; in particular, for every continuous function $g : Y \to Z$.

proof (a) The set $\{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of Y containing every open set, so contains every Borel subset of Y.

(b) is obvious from the definition of 'subspace σ-algebra' (121A).

(c) If $H \subseteq Z$ is open, then $g^{-1}[H]$ is a Borel subset of Y so $(gf)^{-1}[H] = f^{-1}[g^{-1}[H]]$ belongs to Σ.

418B Proposition Let X be a set and Σ a σ-algebra of subsets of X.

(a) If Y is a metrizable space and $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions from X to Y such that $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in X$, then $f : X \to Y$ is measurable.

(b) If Y is a topological space, Z is a separable metrizable space and $f : X \to Y$, $g : X \to Z$ are functions, then $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable iff f and g are measurable.

(c) If Y is a hereditarily Lindelöf space, $\mathcal{U}$ a family of open sets generating its topology, and $f : X \to Y$ a function such that $f^{-1}[U] \in \Sigma$ for every $U \in \mathcal{U}$, then f is measurable.

(d) If $\langle Y_i \rangle_{i\in I}$ is a countable family of separable metrizable spaces, with product Y, then a function $f : X \to Y$ is measurable iff $\pi_i f : X \to Y_i$ is measurable for every i, writing $\pi_i(y) = y(i)$ for $y \in Y$ and $i \in \mathbb{N}$.

proof (a) Let ρ be a metric defining the topology of Y. Let $G \subseteq Y$ be any open set, and for each $n \in \mathbb{N}$ set

$$F_n = \{y : y \in Y,\, \rho(y, z) \ge 2^{-n} \text{ for every } z \in Y \setminus G\}.$$

Then F_n is closed, so $f_i^{-1}[F_n] \in \Sigma$ for every n, $i \in \mathbb{N}$. But this implies that

$$f^{-1}[G] = \bigcup_{n\in\mathbb{N}} \bigcap_{i\ge n} f_i^{-1}[F_i] \in \Sigma.$$

As G is arbitrary, f is measurable.

(b)(i) The functions $(y, z) \mapsto y$, $(y, z) \mapsto z$ are continuous, so if $x \mapsto (f(x), g(x))$ is measurable, so are f and g, by 418Ac.

(ii) Now suppose that f and g are measurable, and that $W \subseteq Y \times Z$ is open. By 4A2P(a-i), the topology of Z has a countable base $\mathcal{H}$; let $\langle H_n \rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{H} \cup \{\emptyset\}$. For each n, set

$$G_n = \bigcup\{G : G \subseteq Y \text{ is open}, G \times H_n \subseteq W\};$$

then G_n is open and $G_n \times H_n \subseteq W$. Accordingly $W \supseteq \bigcup_{n\in\mathbb{N}} G_n \times H_n$. But in fact $W = \bigcup_{n\in\mathbb{N}} G_n \times H_n$. **P** If $(y, z) \in W$, there are open sets $G \subseteq Y$, $H \subseteq Z$ such that $(y, z) \in G \times H \subseteq W$. Now there is an $n \in \mathbb{N}$ such that $z \in H_n \subseteq H$, in which case $G \times H_n \subseteq W$ and $G \subseteq G_n$ and $(y, z) \in G_n \times H_n$. **Q**

Accordingly

$$\{x : (f(x), g(x)) \in W\} = \bigcup_{n\in\mathbb{N}} f^{-1}[G_n] \cap g^{-1}[H_n] \in \Sigma.$$

As W is arbitrary, $x \mapsto (f(x), g(x))$ is measurable.

(c) This is just 4A3Db.

(d) If f is measurable, so is every $\pi_i f$, by 418Ac. If every $\pi_i f$ is measurable, set

$$\mathcal{U} = \{\pi_i^{-1}[H] : i \in I,\, H \subseteq Y_i \text{ is open}\}.$$

Then $\mathcal{U}$ generates the topology of Y, and if $U = \pi_i^{-1}[H]$ then $f^{-1}[U] = (\pi_i f)^{-1}[H]$, so $f^{-1}[U] \in \Sigma$ for every U. Also Y is hereditarily Lindelöf (4A2P(a-iii)), so f is measurable, by (c).

418C Proposition Let (X, Σ, μ) be a measure space and Y a Polish space. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of measurable functions from X to Y. Then

$$\{x : x \in X,\, \lim_{n\to\infty} f_n(x) \text{ is defined in } Y\}$$

belongs to Σ.

proof (Compare 121H.) Let ρ be a complete metric on Y defining the topology of Y.

(a) For m, $n \in \mathbb{N}$ and $\delta > 0$, the set $\{x : \rho(f_m(x), f_n(x)) \le \delta\}$ belongs to Σ. **P** The function $x \mapsto (f_m(x), f_n(x)) : X \to Y^2$ is measurable, by 418Bb, and the function $\rho : Y^2 \to \mathbb{R}$ is continuous, so $x \mapsto \rho(f_m(x), f_n(x))$ is measurable and $\{x : \rho(f_m(x), f_n(x)) \le \delta\} \in \Sigma$. **Q**

(b) Now $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is convergent iff it is Cauchy, because Y is complete. But

$$\{x : x \in X, \langle f_n(x)\rangle_{n\in\mathbb{N}} \text{ is Cauchy}\} = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{i\ge m} \{x : \rho(f_i(x), f_m(x)) \le 2^{-n}\}$$

belongs to Σ.

418D Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) Suppose that Y is a topological space. Then any continuous function from X to Y is almost continuous.

(b) Suppose that Y and Z are topological spaces, $f : X \to Y$ is almost continuous and $g : Y \to Z$ is continuous. Then $gf : X \to Z$ is almost continuous.

(c) Suppose that $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ is a σ-finite topological measure space, Z is a topological space, $g : Y \to Z$ is almost continuous and $f : X \to Y$ is inverse-measure-preserving and almost continuous. Then $gf : X \to Z$ is almost continuous.

(d) Suppose that μ is semi-finite, and that $\langle Y_i\rangle_{i\in I}$ is a countable family of topological spaces with product Y. Then a function $f : X \to Y$ is almost continuous iff $f_i = \pi_i f$ is almost continuous for every $i \in I$, writing $\pi_i(y) = y(i)$ for $i \in I$ and $y \in Y$.

proof (a) is trivial.

(b) The set $\{A : A \subseteq X, gf\restriction A \text{ is continuous}\}$ includes $\{A : A \subseteq X, f\restriction A \text{ is continuous}\}$; so if μ is inner regular with respect to the latter, it is inner regular with respect to the former.

(c) Take $E \in \Sigma$, $\gamma < \mu E$ and $\epsilon > 0$. We have a cover of Y by a non-decreasing sequence $\langle Y_n\rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure; now $\langle f^{-1}[Y_n]\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence covering E, so there is an $n \in \mathbb{N}$ such that $\mu(E \cap f^{-1}[Y_n]) \ge \gamma$. Because f is inverse-measure-preserving, $E \cap f^{-1}[Y_n]$ has finite measure. Now we can find measurable sets $F \subseteq Y_n$, $E_1 \subseteq E \cap f^{-1}[Y_n]$ such that $f\restriction E_1$, $g\restriction F$ are continuous and $\nu F \ge \nu Y_n - \epsilon$, $\mu E_1 \ge \mu(E \cap f^{-1}[Y_n]) - \epsilon$. In this case $E_0 = E_1 \cap f^{-1}[F]$ has measure at least $\gamma - 2\epsilon$ and $gf\restriction E_0$ is continuous. As E, γ and ϵ are arbitrary, gf is almost continuous.

(d)(i) If f is almost continuous, every f_i must be almost continuous, by (b).

(ii) Now suppose that every f_i is almost continuous. Take $E \in \Sigma$ and $\gamma < \mu E$. There is an $E_0 \subseteq E$ such that $E_0 \in \Sigma$ and $\gamma < \mu E_0 < \infty$. Let $\langle \epsilon_i\rangle_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \epsilon_i \le \mu E_0 - \gamma$. For each $i \in I$ choose a measurable set $F_i \subseteq E_0$ such that $\mu F_i \ge \mu E_0 - \epsilon_i$ and $f_i\restriction F_i$ is continuous. Then $F = E_0 \cap \bigcap_{i\in I} F_i$ is a subset of E with measure at least γ, and $f\restriction F$ is continuous because $f_i\restriction F$ is continuous for every i (3A3Ib).

418E Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space, Y a topological space, and $f : X \to Y$ an almost continuous function. Then f is measurable.

proof Set $\mathcal{K} = \{K : K \in \Sigma, f\restriction K \text{ is continuous}\}$; then μ is inner regular with respect to $\mathcal{K}$. If $H \subseteq Y$ is open and $K \in \mathcal{K}$, then $K \cap f^{-1}[H]$ is relatively open in K, that is, there is an open set $G \subseteq X$ such that $K \cap f^{-1}[H] = K \cap G$. Because μ is a topological measure, $G \in \Sigma$ so $K \cap f^{-1}[H] \in \Sigma$. As K is arbitrary, and μ is complete and locally determined, $f^{-1}[H] \in \Sigma$ (412Ja). As H is arbitrary, f is measurable.

418F Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a semi-finite topological measure space, Y a metrizable space, and $f : X \to Y$ a function. Suppose there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of almost continuous functions from X to Y such that $f(x) = \lim_{n\to\infty} f_n(x)$ for almost every $x \in X$. Then f is almost continuous.

proof Suppose that $E \in \Sigma$, $\gamma < \mu E$ and $\epsilon > 0$. Then there is a measurable set $F \subseteq E$ such that $\gamma \le \mu F < \infty$; discarding a negligible set if necessary, we may arrange that $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in F$. Let ρ be a metric on Y defining its topology. For each $n \in \mathbb{N}$, let $F_n \subseteq F$ be a measurable set such that $f_n\restriction F_n$ is continuous and $\mu(F_n \setminus F) \le 2^{-n}\epsilon$; set $G = \bigcap_{n\in\mathbb{N}} F_n$, so that $\mu G \ge \gamma - 2\epsilon$ and $f_n\restriction G$ is continuous for every $n \in \mathbb{N}$.

For m, $n \in \mathbb{N}$, the functions $x \mapsto (f_m(x), f_n(x)) : G \to Y^2$ and $x \mapsto \rho(f_m(x), f_n(x)) : G \to \mathbb{R}$ are continuous, therefore measurable, because μ is a topological measure. Also $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in G$. So if we set $G_{kn} = \{x : x \in G,\ \rho(f_i(x), f_j(x)) \le 2^{-k}$ for all $i, j \ge n\}$, $\langle G_{kn}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of measurable sets with union G for each $k \in \mathbb{N}$, and we can find a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that $\mu(G\setminus G_{kn_k}) \le 2^{-k}\epsilon$ for every k. Setting $H = \bigcap_{k\in\mathbb{N}} G_{kn_k}$, $\mu H \ge \mu G - 2\epsilon \ge \gamma - 4\epsilon$ and $\rho(f_i(x), f_{n_k}(x)) \le 2^{-k}$ whenever $x \in H$ and $i \ge n_k$; consequently $\rho(f(x), f_{n_k}(x)) \le 2^{-k}$ whenever $x \in H$ and $k \in \mathbb{N}$. But this means that $\langle f_{n_k}\rangle_{k\in\mathbb{N}}$ converges to f uniformly on H, while every f_{n_k} is continuous on H, so $f\restriction H$ is continuous (3A3Nb). And of course $H \subseteq E$.

As E, γ and ϵ are arbitrary, f is almost continuous.

418V Proposition Let (X, Σ, μ) be a σ-finite measure space, $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the Borel sets, $(Y, \mathfrak{S})$ a topological space and $f : X \to Y$ an almost continuous function. Then there is a Borel measurable function $g : X \to Y$ which is equal almost everywhere to f.

proof If Y is empty, so is X, and the result is trivial; so suppose that we have a point $y_0 \in Y$. Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of X of finite measure covering X. For each $n \in \mathbb{N}$, there is a measurable set $E_n \subseteq X_n$ such that $\mu E_n \ge \mu X_n - 2^{-n}$ and $f\restriction E_n$ is continuous, and a measurable Borel set $E'_n \subseteq E_n$ such that $\mu E'_n \ge \mu E_n - 2^{-n}$ and E'_n is Borel. Now $E = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} E'_n$ is a coneglible Borel set. Set $g(x) = f(x)$ for $x \in E$, y_0 for $x \in X \setminus E$; then $g =_{\text{a.e.}} f$. If $G \subseteq Y$ is open and $n \in \mathbb{N}$, then $E_n \cap f^{-1}[G]$ is relatively open in E_n so $E'_n \cap f^{-1}[G]$ is a Borel subset of X. Accordingly $E \cap f^{-1}[G] = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} E'_n \cap f^{-1}[G]$ is a Borel subset of X. But now $g^{-1}[G]$ is either $E \cap f^{-1}[G]$ or $(E \cap f^{-1}[G]) \cup (X \setminus E)$, and in either case is a Borel set. As G is arbitrary, g is Borel measurable.

418G Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite quasi-Radon measure space, Y a metrizable space and $f : X \to Y$ an almost continuous function. Then there is a conegligible set $X_0 \subseteq X$ such that $f[X_0]$ is separable.

proof (a) Let $\mathcal{K}$ be the family of self-supporting measurable sets K of finite measure such that $f\restriction K$ is continuous. Then μ is inner regular with respect to $\mathcal{K}$. **P** If $E \in \Sigma$ and $\gamma < \mu E$, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$; there is an $H \in \Sigma$ such that $H \subseteq F$, $\gamma \le \mu H$ and $f\restriction H$ is continuous; and there is a measurable self-supporting $K \subseteq H$ with the same measure as H (414F), in which case $K \in \mathcal{K}$ and $K \subseteq E$ and $\mu K \ge \gamma$. **Q**

(b) Now $f[K]$ is ccc for every $K \in \mathcal{K}$. **P** If $\mathcal{G}$ is a disjoint family of non-empty relatively open subsets of $f[K]$, then $\langle K \cap f^{-1}[G]\rangle_{G\in\mathcal{G}}$ is a disjoint family of non-empty relatively open subsets of K, because $f\restriction K$ is continuous, and $\sum_{G\in\mathcal{G}} \mu(K \cap f^{-1}[G]) \le \mu K$. Because K is self-supporting, $\mu(K \cap f^{-1}[G]) > 0$ for every $G \in \mathcal{G}$; because μK is finite, $\mathcal{G}$ is countable. As $\mathcal{G}$ is arbitrary, $f[K]$ is ccc. **Q**

Because Y is metrizable, $f[K]$ must be separable (4A2Pd).

(c) Because μ is σ-finite, there is a countable family $\mathcal{L} \subseteq \mathcal{K}$ such that $X_0 = \bigcup \mathcal{L}$ is conegligible (412Ic). Now $f[X_0] = \bigcup_{L\in\mathcal{L}} f[L]$ is a countable union of separable spaces, so is separable (4A2B(e-i)).

418H Proposition (a) Let X and Y be topological spaces, μ an effectively locally finite τ-additive topological measure on X, and $f : X \to Y$ an almost continuous function. Then the image measure μf^{-1} is τ-additive.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a totally finite quasi-Radon measure space, $(Y, \mathfrak{S})$ a regular topological space, and $f : X \to Y$ an almost continuous function. Then there is a unique quasi-Radon measure ν on Y such that f is inverse-measure-preserving for μ and ν.

proof (a) Let $\mathcal{H}$ be an upwards-directed family of open subsets of Y, all measured by μf^{-1}, and suppose that $H^* = \bigcup \mathcal{H}$ also is measurable. Take any $\gamma < (\mu f^{-1})(H^*) = \mu f^{-1}[H^*]$. Then there is a measurable set $E \subseteq f^{-1}[H^*]$ such that $\mu E \ge \gamma$ and $f\restriction E$ is continuous. Consider $\{E \cap f^{-1}[H] : H \in \mathcal{H}\}$. This is an upwards-directed family of relatively open measurable subsets of E with measurable union E. By 414K, the subspace measure on E is τ-additive, so

$$\gamma \le \mu E \le \sup_{H\in\mathcal{H}} \mu(E\cap f^{-1}[H]) \le \sup_{H\in\mathcal{H}} \mu f^{-1}[H].$$

As γ is arbitrary, $\mu f^{-1}[H^*] \le \sup_{H\in\mathcal{H}} \mu f^{-1}[H]$; as $\mathcal{H}$ is arbitrary, μf^{-1} is τ-additive.

(b) By 418E, f is measurable. Let ν_0 be the restriction of μf^{-1} to the Borel σ-algebra of Y; by (a), ν_0 is τ-additive, and f is inverse-measure-preserving with respect to μ and ν_0. Because Y is regular, the completion ν of ν_0 is a quasi-Radon measure (415Cb). Because μ is complete, f is still inverse-measure-preserving with respect to μ and ν (234Ba).

To see that ν is unique, observe that its values on Borel sets are determined by the requirement that f be inverse-measure-preserving, so that 415H gives the result.

418I The next theorem is one of the central properties of Radon measures. I have already presented what amounts to a special case in 256G.

Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, Y a Hausdorff space, and $f : X \to Y$ an almost continuous function. If the image measure $\nu = \mu f^{-1}$ is locally finite, it is a Radon measure.

proof (a) By 418E, f is measurable, that is, $f^{-1}[H] \in \Sigma$ for every open set $H \subseteq Y$; but this means that the domain T of ν contains every open set, and ν is a topological measure.

(b) ν is inner regular with respect to the compact sets. **P** If $F \in \mathrm{T}$ and $\nu F > 0$, then $\mu f^{-1}[F] > 0$, so there is an $E \subseteq f^{-1}[F]$ such that $\mu E > 0$ and $f\restriction E$ is continuous. Next, there is a compact set $K \subseteq E$ such that $\mu K > 0$. In this case, $L = f[K]$ is a compact subset of F, and

$$\nu L = \mu f^{-1}[L] \ge \mu K > 0.$$

As Y is Hausdorff, this is enough to prove that ν is tight. **Q** Note that because ν is locally finite, $\nu L < \infty$ for every compact $L \subseteq Y$ (411Ga).

(c) Because μ is complete, so is ν (234Eb). Next, ν is locally determined. **P** Suppose that $H \subseteq Y$ is such that $H \cap F \in \mathrm{T}$ whenever $\nu F < \infty$. Then, in particular, $H \cap f[K] \in \mathrm{T}$ whenever $K \subseteq X$ is compact and $f\restriction K$ is continuous. But setting

$$\mathcal{K} = \{K : K \subseteq X \text{ is compact, } f\restriction K \text{ is continuous}\},$$

μ is inner regular with respect to $\mathcal{K}$ (412Ac). And if $K \in \mathcal{K}$,

$$K \cap f^{-1}[H] = K \cap f^{-1}[H \cap f[K]] \in \Sigma.$$

Because μ is complete and locally determined, this is enough to show that $f^{-1}[H] \in \Sigma$ (412Ja again), that is, $H \in \mathrm{T}$. As H is arbitrary, ν is locally determined. **Q**

(d) Thus ν is a complete locally determined locally finite topological measure which is inner regular with respect to the compact sets; that is, it is a Radon measure.

418J Theorem Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a second-countable space (for instance, Y might be separable and metrizable), and $f : X \to Y$ is measurable. Then f is almost continuous.

proof Let $\mathcal{H}$ be a countable base for the topology of Y, and $\langle H_n\rangle_{n\in\mathbb{N}}$ a sequence running over $\mathcal{H}\cup\{\emptyset\}$. Take $E \in \Sigma$ and $\gamma < \mu E$. Choose $\langle E_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. There is an $E_0 \in \Sigma$ such that $E_0 \subseteq E$ and $\gamma < \mu E_0 < \infty$. Given $E_n \in \Sigma$ with $\gamma < \mu E_n < \infty$, $E_n \setminus f^{-1}[H_n] \in \Sigma$, so there is a closed set $F_n \in \Sigma$ such that

$$F_n \subseteq E_n \setminus f^{-1}[H_n], \quad \mu((E_n \setminus f^{-1}[H_n]) \setminus F_n) < \mu E_n - \gamma;$$

set $E_{n+1} = (E_n \cap f^{-1}[H_n]) \cup F_n$, so that

$$E_{n+1} \in \Sigma, \quad E_{n+1} \subseteq E_n, \quad \mu E_{n+1} > \gamma, \quad E_{n+1} \setminus f^{-1}[H_n] = F_n.$$

Continue.

At the end of the induction, set $F = \bigcap_{n\in\mathbb{N}} E_n$. Then $F \subseteq E$, $\mu F \ge \gamma$, and for every $n \in \mathbb{N}$

$$F \cap f^{-1}[H_n] = F \cap E_{n+1} \cap f^{-1}[H_n] = F \cap E_{n+1} \setminus F_n = F \setminus F_n$$

is relatively open in F. It follows that $f{\restriction}F$ is continuous (4A2B(a-ii)). As E, γ are arbitrary, f is almost continuous.

Remark For variations on this idea, see 418Yf, 433E and 434Yb; also 418Yg.

418K Corollary Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space and Y a separable metrizable space. Then a function $f:X\to Y$ is measurable iff it is almost continuous.

proof Put 418E and 418J together.

Remark This generalizes 256F.

418L In all the results above, the measure starts on the left of the diagram $f:X\to Y$; in 418H-418I, it is transferred to an image measure on Y. If X has enough compact sets, a measure can move in the reverse direction, as follows.

Theorem Let $(X,\mathfrak{T})$ be a Hausdorff space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ a Radon measure space and $f:X\to Y$ a continuous function such that whenever $F\in\mathrm{T}$ and $\nu F>0$ there is a compact set $K\subseteq X$ such that $\nu(F\cap f[K])>0$. Then there is a Radon measure μ on X such that ν is the image measure μf^{-1} and the inverse-measure-preserving function f induces an isomorphism between the measure algebras of ν and μ.

proof (a) Note first that ν is inner regular with respect to $\mathcal{L}=\{f[K]:K\in\mathcal{K}\}$, where $\mathcal{K}$ is the family of compact subsets of X. **P** If $\nu F>0$, there is a $K\in\mathcal{K}$ such that $\nu(F\cap f[K])>0$; now there is a closed set $F'\subseteq F\cap f[K]$ such that $\nu F'>0$, and $K'=K\cap f^{-1}[F']$ is compact, while $f[K']\subseteq F$ has non-zero measure. As $\mathcal{L}$ is closed under finite unions, this is enough to show that ν is inner regular with respect to $\mathcal{L}$ (412Aa). **Q**

(b) Consequently there is a disjoint set $\mathcal{L}_0\subseteq\mathcal{L}$ such that every non-negligible $F\in\mathrm{T}$ meets some member of $\mathcal{L}_0$ in a non-negligible set (412Ib). We can express $\mathcal{L}_0$ as $\{f[K]:K\in\mathcal{K}_0\}$ where $\mathcal{K}_0\subseteq\mathcal{K}$ is disjoint. Set $X_0=\bigcup\mathcal{K}_0$.

(c) Set

$$\Sigma_0=\{X_0\cap f^{-1}[F]:F\in\mathrm{T}\}.$$

Then Σ_0 is a σ-algebra of subsets of X_0. If F, $F'\in\mathrm{T}$ and $\nu F\neq\nu F'$, then there must be some $K\in\mathcal{K}_0$ such that $f[K]\cap(F\triangle F')\neq\emptyset$, so that $X_0\cap f^{-1}[F]\neq X_0\cap f^{-1}[F']$; we therefore have a functional $\mu_0:\Sigma_0\to[0,\infty]$ defined by setting $\mu_0(X_0\cap f^{-1}[F])=\nu F$ whenever $F\in\mathrm{T}$. It is easy to check that μ_0 is a measure on X_0. Now μ_0 is inner regular with respect to $\mathcal{K}$. **P** If $E\in\Sigma_0$ and $\mu E>0$, there is an $F\in\mathrm{T}$ such that $E=X_0\cap f^{-1}[F]$ and $\nu F>0$. There are a $K\in\mathcal{K}_0$ such that $\nu(F\cap f[K])>0$, and a closed set $F'\subseteq F\cap f[K]$ such that $\nu F'>0$; now $K\cap f^{-1}[F']=X_0\cap f^{-1}[F']$ belongs to $\Sigma_0\cap\mathcal{K}$, is included in E and has measure greater than 0. Because $\mathcal{K}$ is closed under finite unions, this is enough to show that μ_0 is inner regular with respect to $\mathcal{K}$. **Q**

(d) Set

$$\Sigma_1=\{E:E\subseteq X,\,E\cap X_0\in\Sigma_0\},\quad \mu_1E=\mu_0(E\cap X_0)\text{ for every }E\in\Sigma_1.$$

Then μ_1 is a measure on X (being the image measure $\mu_0\iota^{-1}$, where $\iota:X_0\to X$ is the identity map), and is inner regular with respect to $\mathcal{K}$. If $F\in\mathrm{T}$, then

$$\mu_1f^{-1}[F]=\mu_0(X_0\cap f^{-1}[F])=\nu F,$$

so f is inverse-measure-preserving for μ_1 and ν. Consequently μ_1 is locally finite. **P** If $x\in X$, there is an open set $H\subseteq Y$ such that $f(x)\in H$ and $\nu H<\infty$; now $f^{-1}[H]$ is an open subset of X of finite measure containing x. **Q** In particular, $\mu_1^*K<\infty$ for every compact $K\subseteq X$ (411Ga again).

(e) By 413P, there is an extension of μ_1 to a complete locally determined measure μ on X which is inner regular with respect to $\mathcal{K}$, defined on every member of $\mathcal{K}$, and such that whenever E belongs to the domain Σ of μ and $\mu E<\infty$, there is an $E_1\in\Sigma_1$ such that $\mu(E\triangle E_1)=0$. Now μ is locally finite because μ_1 is, so μ is a Radon measure; and f is inverse-measure-preserving for μ and ν because it is inverse-measure-preserving for μ_1 and ν.

(f) The image measure μf^{-1} extends ν, so is locally finite, and is therefore a Radon measure (418I); since it agrees with ν on the compact subsets of Y, it must be identical with ν.

(g) I have still to check that the corresponding measure-preserving homomorphism π from the measure algebra $\mathfrak{B}$ of ν to the measure algebra $\mathfrak{A}$ of μ is actually an isomorphism, that is, is surjective. If $a \in \mathfrak{A}$ and $\bar{\mu}a < \infty$, we can find $E \in \Sigma$ such that $E^\bullet = a$ and $E_1 \in \Sigma_1$ such that $\mu(E \triangle E_1) = 0$. Now $E_1 \cap X_0 = f^{-1}[F] \cap X_0$ for some $F \in \mathrm{T}$; but in this case

$$\mu(E_1 \triangle f^{-1}[F]) = \mu_1(E_1 \triangle f^{-1}[F]) = 0, \quad a = E_1^\bullet = (f^{-1}[F])^\bullet = \pi F^\bullet.$$

Accordingly $\pi[\mathfrak{B}]$ includes $\{a : \bar{\mu}a < \infty\}$, and is order-dense in $\mathfrak{A}$. But as π is injective and $\mathfrak{B}$ is Dedekind complete (being the measure algebra of a Radon measure, which is strictly localizable), it follows that $\pi[\mathfrak{B}] = \mathfrak{A}$ (314Ia). Thus π is an isomorphism, as required.

Remarks Of course this result is most commonly applied when X and Y are both compact and f is a surjection, in which case the condition

(*) whenever $F \in \mathrm{T}$ and $\nu F > 0$ there is a compact set $K \subseteq X$ such that $\nu(F \cap f[K]) > 0$

is trivially satisfied.

Evidently (*) is necessary if there is to be any Radon measure on X for which f is inverse-measure-preserving, so in this sense the result is best possible. In 433D, however, there is a version of the theorem in which f is not required to be continuous.

418M Prokhorov's theorem Suppose that $(I, \leq)$, $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\leq j\in I}$, $(X, \mathfrak{T})$ and $\langle g_i\rangle_{i\in I}$ are such that

(α) $(I, \leq)$ is a non-empty upwards-directed partially ordered set,

(β) every $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ is a Radon probability space,

(γ) $f_{ij} : X_j \to X_i$ is an inverse-measure-preserving function whenever $i \leq j$ in I,

(δ) $(X, \mathfrak{T})$ is a Hausdorff space,

(ϵ) $g_i : X \to X_i$ is a continuous function for every $i \in I$,

(ζ) $g_i = f_{ij}g_j$ whenever $i \leq j$ in I.

(η) for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu_i g_i[K] \geq 1 - \epsilon$ for every $i \in I$.

Then there is a Radon probability measure μ on X such that every g_i is inverse-measure-preserving for μ. If moreover

(θ) the family $\langle g_i\rangle_{i\in I}$ separates the points of X,

then μ is uniquely defined.

proof (a) Set

$$\mathrm{T} = \{g_i^{-1}[E] : i \in I, E \in \Sigma_i\} \subseteq \mathcal{P}X.$$

Then T is a subalgebra of $\mathcal{P}X$. **P** (i) There is an $i \in I$, so $\emptyset = g_i^{-1}[\emptyset]$ belongs to T. (ii) If $H \in \mathrm{T}$ there are $i \in I$, $E \in \Sigma_i$ such that $H = g_i^{-1}[E]$; now $X \setminus H = g_i^{-1}[X_i \setminus E]$ belongs to T. (iii) If G, $H \in \mathrm{T}$, there are i, $j \in I$ and $E \in \Sigma_i$, $F \in \Sigma_j$ such that $G = g_i^{-1}[E]$ and $H = g_j^{-1}[F]$. Now I is upwards-directed, so there is a $k \in I$ such that $i \leq k$ and $j \leq k$. Because f_{ik} and f_{jk} are inverse-measure-preserving, $f_{ik}^{-1}[E]$ and $f_{jk}^{-1}[F]$ belong to Σ_k, so that

$$\begin{aligned} G \cap H &= g_i^{-1}[E] \cap g_j^{-1}[F] = (f_{ik}g_k)^{-1}[E] \cap (f_{jk}g_k)^{-1}[F] \\ &= g_k^{-1}[f_{ik}^{-1}[E] \cap f_{jk}^{-1}[F]] \in \mathrm{T}. \ \mathbf{Q} \end{aligned}$$

(b) There is an additive functional $\nu : \mathrm{T} \to [0, 1]$ defined by writing $\nu g_i^{-1}[E] = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$.

P (i) Suppose that i, $j \in I$ and $E \in \Sigma_i$, $F \in \Sigma_j$ are such that $g_i^{-1}[E] = g_j^{-1}[F]$. Let $k \in I$ be such that $i \leq k$ and $j \leq k$. Then

$$g_k^{-1}[f_{ik}^{-1}[E] \triangle f_{jk}^{-1}[F]] = g_i^{-1}[E] \triangle g_j^{-1}[F] = \emptyset,$$

so $g_k[X] \cap (f_{ik}^{-1}[E] \triangle f_{jk}^{-1}[F]) = \emptyset$. But now remember that for every $\epsilon > 0$ there is a set $K \subseteq X$ such that $\mu_k g_k[K] \geq 1 - \epsilon$. This means that $\mu_k g_k[X]$ must be 1, so that $f_{ik}^{-1}[E] \triangle f_{jk}^{-1}[F]$ must be negligible, and

$$\mu_i E = \mu_k f_{ik}^{-1}[E] = \mu_k f_{jk}^{-1}[F] = \mu_j F.$$

Thus the proposed formula for ν defines a function on T.

(ii) Now suppose that G, $H \in$ T are disjoint. Again, take i, $j \in I$ and $E \in \Sigma_i$, $F \in \Sigma_j$ such that $G = g_i^{-1}[E]$ and $H = g_j^{-1}[F]$, and $k \in I$ such that $i \leq k$ and $j \leq k$. Then

$$\begin{aligned} \nu G + \nu H &= \mu_i E + \mu_j F = \mu_k f_{ik}^{-1}[E] + \mu_k f_{jk}^{-1}[F] \\ &= \mu_k (f_{ik}^{-1}[E] \cup f_{jk}^{-1}[F]) + \mu_k (f_{ik}^{-1}[E] \cap f_{jk}^{-1}[F]) \\ &= \nu g_k^{-1}[f_{ik}^{-1}[E] \cup f_{jk}^{-1}[F]] + \nu g_k^{-1}[f_{ik}^{-1}[E] \cap f_{jk}^{-1}[F]] \\ &= \nu(G \cup H) + \nu(G \cap H). \end{aligned}$$

But as $\nu\emptyset$ is certainly 0, we get $\nu(G \cup H) = \nu G + \nu H$. As G, H are arbitrary, ν is additive. **Q**

Note that $\nu X = 1$.

(c) $\nu G = \sup\{\nu H : H \in \mathrm{T}, H \subseteq G, H \text{ is closed}\}$ for every $G \in$ T. **P** If $\gamma < \nu G$, there are an $i \in I$ and an $E \in \Sigma_i$ such that $G = g_i^{-1}[E]$. In this case $\mu_i E = \nu G > \gamma$; let $L \subseteq E$ be a compact set such that $\mu_i L \geq \gamma$; then $H = g_i^{-1}[L]$ is a closed subset of G and $\nu H = \mu_i L \geq \gamma$. **Q**

$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{G \in \mathrm{T}, G \supseteq K} \nu G$. **P** If $\epsilon > 0$, there is a compact $K \subseteq X$ such that $\mu_i g_i[K] \geq 1 - \epsilon$ for every $i \in I$, by the final hypothesis of this theorem. If $G \in$ T and $G \supseteq K$, there are an $i \in I$ and an $E \in \Sigma_i$ such that $G = g_i^{-1}[E]$, in which case $g_i[K] \subseteq E$, so that

$$\nu G = \mu_i E \geq \mu_i g_i[K] \geq 1 - \epsilon.$$

Thus $\inf_{G \in \mathrm{T}, G \supseteq K} \nu G \geq 1 - \epsilon$; as ϵ is arbitrary, we have the result. **Q**

This means that the conditions of 416O are satisfied, and there is a Radon measure μ on X extending ν. Of course this means that every g_i is inverse-measure-preserving.

(d) Now suppose that $\langle g_i \rangle_{i \in I}$ separates the points of X. Then whenever K, $L \subseteq X$ are disjoint there is an $i \in I$ such that $g_i[K] \cap g_i[L] = \emptyset$. **P** Set $V_i = \{(x, y) : x \in K, y \in L, g_i(x) = g_i(y)\}$ for $i \in I$. Because g_i is continuous and $\mathfrak{T}_i$ is Hausdorff, V_i is closed. If $i \leq j$ in I, then $g_i = f_{ij} g_i$ so $V_j \subseteq V_i$; accordingly $\langle V_i \rangle_{i \in I}$ is downwards-directed. Because $\langle g_i \rangle_{i \in I}$ separates the points of X, $\bigcap_{i \in I} V_i$ is empty. As $K \times L$ is compact, there is an $i \in I$ such that $V_i = \emptyset$, that is, $g_i[K]$ and $g_i[L]$ are disjoint. **Q**

Let ν be any Radon probability measure on X such that g_i is inverse-measure-preserving for ν and μ_i for every $i \in I$. Let $K \subseteq X$ be compact. **?** If $\mu K < \nu K$ then there is a compact $L \subseteq X \setminus K$ such that $\mu L + \nu K > 1$. Let $i \in I$ be such that $g_i[K] \cap g_i[L] = \emptyset$; then

$$1 < \mu L + \nu K \leq \mu g_i^{-1}[g_i[L]] + \nu g_i^{-1}[g_i[K]] = \mu_i g_i[L] + \mu_i g_i[K] \leq 1,$$

which is impossible. **X** So $\nu K \leq \mu K$. Similarly, $\mu K \leq \nu K$. By 416Eb, $\mu = \nu$. Thus μ is uniquely determined.

418N Remarks (a) Taking I to be a singleton, we get a version of 418L in which Y is a probability space, and omitting the check that the function g induces an isomorphism of the measure algebras. Taking I to be the family of finite subsets of a set T, and every X_i to be a product $\prod_{t \in i} Z_t$ of Radon probability spaces with its product Radon measure, we obtain a method of constructing products of arbitrary families of compact probability spaces from finite products.

(b) In the hypotheses of 418M, I asked only that the f_{ij} should be measurable, and omitted any check on the compositions $f_{ij} f_{jk}$ when $i \leq j \leq k$. But it is easy to see that every f_{ij} must in fact be almost continuous, and that $f_{ij} f_{jk}$ must be equal almost everywhere to f_{ik} (418Xw), just as in 418P below.

(c) In the theorem as written out above, the space X and the functions $g_i : X \to X_i$ are part of the data. Of course in many applications we start with a structure

$$(\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}, \langle f_{ij} \rangle_{i \leq j \in I}),$$

and the first step is to find suitable X and g_i, as in 418O and 418P.

(d) There are important questions concerning possible relaxations of the hypotheses in 418M, especially in the special case already mentioned, in which $X_i = \prod_{t\in i} Z_t$, $f_{ij}(x) = x\restriction i$ when $i \subseteq j \in [T]^{<\omega}$, $X = \prod_{t\in T} Z_t$, and $g_i(x) = x\restriction i$ for $x \in X$ and $i \in I$, but there is no suggestion that the μ_i are product measures. For a case in which we can dispense with any mention of auxiliary topologies on the X_i, see 454G.

(e) A typical class of applications of Prokhorov's theorem is in the theory of stochastic processes, in which we have large families $\langle X_t\rangle_{t\in T}$ of random variables; for definiteness, imagine that $T = [0,\infty[$, so that we are looking at a system evolving over time. Not infrequently our intuition leads us to a clear description of the joint distributions ν_J of finite subfamilies $\langle X_t\rangle_{t\in J}$ without providing any suggestion of a measure space on which the whole family $\langle X_t\rangle_{t\in T}$ might be defined. (As I tried to explain in the introduction to Chapter 27, probability spaces themselves are often very shadowy things in true probability theory.) Each ν_J can be thought of as a Radon measure on $\mathbb{R}^J$, and for $I \subseteq J \in [T]^{<\omega}$ we have a natural map $f_{IJ} : \mathbb{R}^J \to \mathbb{R}^I$, setting $f_{IJ}(y) = y\restriction I$ for $y \in \mathbb{R}^J$. If our distributions ν_J mean anything at all, every f_{IJ} will surely be inverse-measure-preserving; this is simply saying that ν_I is the joint distribution of a subfamily of $\langle X_t\rangle_{t\in J}$. If we can find a Hausdorff space Ω and a continuous function $g : \Omega \to \mathbb{R}^T$ such that, for every finite $J \subseteq T$ and $\epsilon > 0$, there is a compact set $K \subseteq \Omega$ such that $\nu_J g_J[K] \ge 1 - \epsilon$ (where $g_J(x) = g(x)\restriction J$), then Prokhorov's theorem will give us a measure μ on Ω which will then provide us with a suitable realization of $\langle X_t\rangle_{t\in T}$ as a family of random variables on a genuine probability space, writing $X_t(\omega) = g(\omega)(t)$. That they become continuous functions on a Radon measure space is a valuable shield against irrelevant complications.

Clearly, if this can be done at all it can be done with $\Omega = \mathbb{R}^T$; but some of the central results of probability theory are specifically concerned with the possibility of using other sets Ω (e.g., Ω a set of càllàl functions, as in 455H, or continuous functions, as in 477B).

(f) In (e) above, we do always have the option of regarding each ν_J as a measure on the compact space $[-\infty,\infty]^J$. In this case, by 418O below or otherwise, we can be sure of finding a measure on $[-\infty,\infty]^T$ to support functions X_t, at the cost of either allowing the values $\pm\infty$ or (as I should myself ordinarily do) accepting that each X_t would be undefined on a negligible set. The advantage of this is just that it gives us confidence in applying the Kolmogorov-Lebesgue theory to the whole family $\langle X_t\rangle_{t\in T}$ at once, rather than to finite or countable subfamilies. For an example of what can happen if we try to do similar things with non-compact measures, see 419K. For an example of the problems which can arise with uncountable families, see 418Xx.

418O I mention two cases in which we can be sure that the projective limit $(X, \langle g_i\rangle_{i\in I})$ required in Prokhorov's theorem will exist.

Proposition Suppose that $(I,\le)$, $\langle (X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\le j\in I}$ are such that

$(I,\le)$ is a non-empty upwards-directed partially ordered set,

every $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ is a compact Radon measure space,

$f_{ij} : X_j \to X_i$ is a continuous inverse-measure-preserving function whenever $i \le j$ in I,

$f_{ij}f_{jk} = f_{ik}$ whenever $i \le j \le k$ in I.

Then there are a compact Hausdorff space X and a family $\langle g_i\rangle_{i\in I}$ such that I, $\langle X_i\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\le j\in I}$, X and $\langle g_i\rangle_{i\in I}$ satisfy all the hypotheses (α)-(θ) of 418M.

proof For each i, let F_i be the support of μ_i; because X_i is compact, so is F_i. If $i \le j$, then $F_i = \overline{f_{ij}[F_j]} = f_{ij}[F_j]$, by 411Ne. Set

$$X = \{x : x \in \textstyle\prod_{i\in I} F_i,\ f_{ij}x(j) = x(i) \text{ whenever } i \le j \in I\},$$

$$g_i(x) = x(i) \text{ for } x \in X,\ i \in I.$$

Of course $g_i = f_{ij}g_j$ whenever $i \le j$. Also $g_i[X] \supseteq F_i$ for every $i \in I$. **P** Take any $y \in F_i$. For each finite set $J \subseteq I$,

$$H_J = \{x : x \in \textstyle\prod_{J\in I} F_j,\ x(i) = y,\ f_{jk}x(k) = x(j) \text{ whenever } j \le k \in J\}$$

is a closed set. H_J is always non-empty, because if k is an upper bound of $J \cup \{i\}$ there is a $z \in F_k$ such that $f_{ik}(z) = y$, in which case $x \in H_J$ whenever $x \in X$ and $x(j) = f_{jk}(z)$ for every $j \in J \cup \{i\}$. Now

$\{H_J : J \in [I]^{<\omega}\}$ is a downwards-directed family of non-empty closed sets in the compact space $\prod_{j\in I} F_j$, so has non-empty intersection, and if x is any point of the intersection then $x \in X$ and $g_i(x) = y$. **Q**

Accordingly $\mu_i g_i[X] = \mu_i X_i$ for every i; as X itself is compact, 418M(η) is satisfied. Finally, because $X \subseteq \prod_{i\in I} X_i$, $\langle g_i\rangle_{i\in I}$ separates the points of X.

418P Proposition Let $(I, \le)$, $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\le j\in I}$ be such that

$(I, \le)$ is a countable non-empty upwards-directed partially ordered set,

every $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ is a Radon probability space,

$f_{ij} : X_j \to X_i$ is an inverse-measure-preserving almost continuous function whenever $i \le j$ in I,

$f_{ij}f_{jk} = f_{ik}$ μ_k-a.e. whenever $i \le j \le k$ in I.

Then there are a Radon probability space $(X, \mathfrak{T}, \Sigma, \mu)$ and continuous inverse-measure-preserving functions $g_i : X \to X_i$, separating the points of X, such that $g_i = f_{ij}g_j$ whenever $i \le j$ in I.

proof (a) We can use nearly the same formula as in 418O:

$$X = \{x : x \in \textstyle\prod_{i\in I} X_i,\ f_{ij}x(j) = x(i) \text{ whenever } i \le j \in I\},$$

$$g_i(x) = x(i) \text{ for } x \in X,\ i \in I.$$

As before, the consistency relation $g_i = f_{ij}g_j$ is a trivial consequence of the definition of X. For the rest, we have to check that 418M(η) is satisfied. Fix $\epsilon \in \,]0, 1[$. Start by taking a family $\langle\epsilon_{ij}\rangle_{i\le j\in I}$ of strictly positive numbers such that $\sum_{i\le j\in I} \epsilon_{ij} \le \frac{1}{2}\epsilon$. (This is where we need to know that I is countable.) Set $\epsilon_j = \sum_{i\le j} \epsilon_{ij}$ for each j, so that $\sum_{j\in I} \epsilon_j \le \frac{1}{2}\epsilon$.

For $i \le j \le k$ in I, set

$$E_{ijk} = \{x : x \in X_k,\ f_{ik}(x) = f_{ij}f_{jk}(x)\},$$

so that E_{ijk} is μ_k-conegligible; set $E_k = \bigcap_{i\le j\le k} E_{ijk}$, so that E_k is μ_k-conegligible. For $i \le j \in I$, choose compact sets $K_{ij} \subseteq E_j$ such that $\mu_j K_{ij} \ge 1 - \epsilon_{ij}$ and $f_{ij}{\restriction}K_{ij}$ is continuous. Now we seem to need a three-stage construction, as follows:

for $j \in I$, set $K_j = \bigcap_{i\le j} K_{ij}$;

for $j \in I$, set $K_j^* = K_j \cap \bigcap_{i\le j} f_{ij}^{-1}[K_i]$;

finally, set $K = X \cap \prod_{j\in I} K_j^*$.

Let us trace the properties of these sets stage by stage.

(b) For each $j \in I$, $K_j \subseteq K_{jj} \subseteq E_j$ is compact and

$$\mu_j(X_j \setminus K_j) \le \textstyle\sum_{i\le j} \mu_j(X_j \setminus K_{ij}) \le \sum_{i\le j} \epsilon_{ij} = \epsilon_j,$$

so that $\mu_j K_j \ge 1 - \epsilon_j$. Note that f_{ik} agrees with $f_{ij}f_{jk}$ on K_k whenever $i \le j \le k$, and that $f_{ij}{\restriction}K_j$ is continuous whenever $i \le j$.

(c) Every K_j^* is compact, and if $i \le j \le k$ then f_{ik} agrees with $f_{ij}f_{jk}$ on K_k^*, while $f_{ij}{\restriction}K_j^*$ is always continuous. Also

$$\mu_j(X_j \setminus K_j^*) \le \mu_j(X_j \setminus K_j) + \sum_{i\le j} \mu_j(X_j \setminus f_{ij}^{-1}[K_i])$$
$$\le \epsilon_j + \sum_{i\le j} \epsilon_i \le \epsilon,$$

so $\mu_j K_j^* \ge 1 - \epsilon$, for every $j \in I$.

The point of moving from K_j to K_j^* is that $f_{jk}[K_k^*] \subseteq K_j^*$ whenever $j \le k$ in I. **P** $K_k^* \subseteq f_{jk}^{-1}[K_j]$, so $f_{jk}[K_k^*] \subseteq K_j$. If $i \le j$, then

$$K_k^* = K_k^* \cap f_{ik}^{-1}[K_i] = K_k^* \cap f_{jk}^{-1}[f_{ij}^{-1}[K_i]]$$

because $f_{ij}f_{jk}$ agrees with f_{ik} on K_k^*. So $f_{jk}[K_k^*] \subseteq f_{ij}^{-1}[K_i]$. As i is arbitrary, $f_{jk}[K_k^*] \subseteq K_j^*$. **Q**

Again because f_{ik} agrees with $f_{ij}f_{jk}$ on K^*_k, we have $f_{ik}[K^*_k]=f_{ij}[f_{jk}[K^*_k]]\subseteq f_{ij}[K^*_j]$ whenever $i\le j\le k$. And because $f_{ij}\restriction K^*_j$ is always continuous, all the sets $f_{ij}[K^*_j]$ are compact.

(d)(i) K is compact. **P**

$$K=\{x:x\in\prod_{i\in I}K^*_i,\ f_{ij}x(j)=x(i)\text{ whenever }i\le j\in I\}$$

is closed in $\prod_{i\in I}K^*_i$ because $f_{ij}\restriction K^*_j$ is always continuous (and every X_i is Hausdorff). Since $\prod_{i\in I}K^*_i$ is compact, so is K. **Q**

(ii) $\mu_i g_i[K]\ge 1-\epsilon$ for every $i\in I$. **P** By (c), $f_{ik}[K^*_k]\subseteq f_{ij}[K^*_j]$ whenever $i\le j\le k$. So $\{f_{ij}[K^*_j]:j\ge i\}$ is a downwards-directed family of compact sets; write L for their intersection. Since

$$\mu_i f_{ij}[K^*_j]=\mu_j f^{-1}_{ij}[f_{ij}[K^*_j]]\ge\mu_j K^*_j\ge 1-\epsilon$$

for every $j\ge i$, $\mu_i L\ge 1-\epsilon$ (414C). If $z\in L$, then for every $k\ge i$ the set

$$F_k=\{x:x\in\prod_{j\in I}K^*_j,\ x(i)=z,\ f_{jk}x(k)=x(j)\text{ whenever }j\le k\}$$

is a closed set in $\prod_{j\in I}K^*_j$, while $F_k\subseteq F_j$ when $j\le k$. Also F_k is non-empty, because there is a $t\in K^*_k$ such that $f_{ik}(t)=z$, and now if we take any $x\in\prod_{j\in I}K^*_j$ such that $x(j)=f_{jk}(t)$ for every $j\le k$, we shall have $x\in F_k$. So $\{F_k:k\ge i\}$ is a downwards-directed family of non-empty closed sets in a compact space, and has non-empty intersection. But if $x\in\bigcap_{k\ge i}F_k$, then $x\in K$ and $x(i)=z$, so $z\in g_i[K]$. Thus $g_i[K]\supseteq L$ and $\mu_i g_i[K]\ge 1-\epsilon$. **Q**

(e) As ϵ is arbitrary, 418M(η) is satisfied. But now 418M gives the result.

418Q Corollary Let $\langle(X_n,\mathfrak{T}_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability spaces, and suppose we are given an inverse-measure-preserving almost continuous function $f_n:X_{n+1}\to X_n$ for each n. Set

$$X=\{x:x\in\prod_{n\in\mathbb{N}}X_n,\ f_n(x(n+1))=x(n)\text{ for every }n\in\mathbb{N}\}.$$

Then there is a unique Radon probability measure μ on X such that all the coordinate maps $x\mapsto x(n):X\to X_n$ are inverse-measure-preserving.

proof For $i\le j\in\mathbb{N}$, define $f_{ij}:X_j\to X_i$ by writing

$$f_{ii}(x)=x\text{ for every }x\in X_i,$$
$$f_{i,j+1}=f_{ij}f_j\text{ for every }j\ge i.$$

It is easy to check that $f_{ij}f_{jk}=f_{ik}$ whenever $i\le j\le k$, and that every f_{ij} is inverse-measure-preserving and almost continuous (using 418Dc). So we are exactly in the situation of 418P, and we know that there is a Radon probability measure on X for which every g_i is inverse-measure-preserving; moreover, the coordinate functionals g_i separate the points of X, so μ is unique, by the last remark in 418M.

418R I turn now to a special kind of measurable function, corresponding to a new view of product spaces.

Theorem Let X be a set, Σ a σ-algebra of subsets of X, and (Y,T,ν) a σ-finite measure space. Give $L^0(\nu)$ the topology of convergence in measure (§245). Write $\mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$ for the space of $\Sigma\widehat{\otimes}\mathrm{T}$-measurable real-valued functions on $X\times Y$, where $\Sigma\widehat{\otimes}\mathrm{T}$ is the σ-algebra of subsets of $X\times Y$ generated by $\{E\times F:E\in\Sigma,\ F\in\mathrm{T}\}$. Then for a function $f:X\to L^0(\nu)$ the following are equiveridical:

(i) $f[X]$ is separable and f is measurable;

(ii) there is an $h\in\mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$ such that $f(x)=h^\bullet_x$ for every $x\in X$, where $h_x(y)=h(x,y)$ for $x\in X$, $y\in Y$.

proof Let $\langle Y_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of Y of finite measure covering Y.

(a)(i)⇒(ii) For each $n\in\mathbb{N}$, let ρ_n be the continuous pseudometric on $L^0(\nu)$ defined by saying that $\rho_n(g^\bullet_1,g^\bullet_2)=\int_{Y_n}\min(1,|g_1-g_2|)d\nu$ for g_1, $g_2\in\mathcal{L}^0_{\mathrm{T}}$, writing $\mathcal{L}^0_{\mathrm{T}}$ for the space of T-measurable real-valued functions on Y. Then $\{\rho_n:n\in\mathbb{N}\}$ defines the topology of $L^0(\nu)$ (see the proof of 245Eb). Because $f[X]$ is separable, there is a sequence $\langle v_k\rangle_{k\in\mathbb{N}}$ in $L^0(\nu)$ such that $f[X]\subseteq\overline{\{v_k:k\in\mathbb{N}\}}$. For each k, choose $g_k\in\mathcal{L}^0_{\mathrm{T}}$ such that $g^\bullet_k=v_k$. For n, $k\in\mathbb{N}$ set

$$E_{nk} = \{x : x \in X,\ \rho_n(f(x), v_k) < 2^{-n}\},$$

$$H_{nk} = E_{nk} \setminus \bigcup_{i<k} E_{ni}.$$

Then every E_{nk} belongs to Σ (because f is measurable) and $\bigcup_{k\in\mathbb{N}} E_{nk} = X$ (because $\{v_k : k \in \mathbb{N}\}$ is dense); so $\langle H_{nk}\rangle_{k\in\mathbb{N}}$ is a partition of X into measurable sets. Set $h^{(n)}(x,y) = g_k(y)$ whenever $k \in \mathbb{N}$, $x \in H_{nk}$ and $y \in Y$; then $h^{(n)} \in \mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$.

Fix $x \in X$ for the moment. Then for each $n \in \mathbb{N}$ there is a unique k_n such that $x \in H_{nk_n}$, and $\rho_n(f(x), v_{k_n}) \le 2^{-n}$. So if $n \le m$,

$$\begin{aligned}\int_{Y_n} \min(1, |h_x^{(m+1)} - h_x^{(m)}|) = \int_{Y_n} \min(1, |g_{k_{m+1}} - g_{k_m}|) &= \rho_n(g^\bullet_{k_{m+1}}, g^\bullet_{k_m})\\ &\le \rho_n(g^\bullet_{k_{m+1}}, f(x)) + \rho_n(f(x), g^\bullet_{k_m})\\ &\le \rho_{m+1}(g^\bullet_{k_{m+1}}, f(x)) + \rho_m(f(x), g^\bullet_{k_m}) \le 3\cdot 2^{-m-1}.\end{aligned}$$

But this means that $\sum_{m=0}^\infty \int_{Y_n} \min(1, |h_x^{(m+1)} - h_x^{(m)}|)$ is finite, so that $\langle h_x^{(m)}\rangle_{m\in\mathbb{N}}$ must be convergent almost everywhere in Y_n. As this is true for every n, $\langle h_x^{(m)}\rangle_{m\in\mathbb{N}}$ is convergent a.e. on Y. Moreover,

$$\lim_{m\to\infty}(h_x^{(m)})^\bullet = \lim_{m\to\infty} g^\bullet_{k_m} = f(x)$$

in $L^0(\nu)$.

Since this is true for every x,

$$W = \{(x,y) : \langle h^{(m)}(x,y)\rangle_{m\in\mathbb{N}} \text{ converges in } \mathbb{R}\}$$

has conegligible vertical sections, while of course $W \in \Sigma\widehat{\otimes}\mathrm{T}$ because every $h^{(m)}$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable (418C). If we set $h(x,y) = \lim_{m\to\infty} h^{(m)}(x,y)$ for $(x,y) \in W$, 0 for other $(x,y) \in X \times Y$, then $h \in \mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$, while (by 245Ca)

$$h_x^\bullet = \lim_{m\to\infty}(h_x^{(m)})^\bullet = f(x)$$

in $L^0(\nu)$ for every $x \in X$. So we have a suitable h.

(b)(ii)⇒(i) Let Φ be the set of those $h \in \mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$ such that (i) is satisfied; that is, $x \mapsto h_x^\bullet$ is measurable, and $\{h_x^\bullet : x \in X\}$ is separable.

(α) Φ is closed under addition. **P** If h, $\tilde h$ belong to Φ, set $A = \{h_x^\bullet : x \in X\}$, $\tilde A = \{\tilde h_x^\bullet : x \in X\}$. Then both A and $\tilde A$ are separable metrizable spaces, so $A \times \tilde A$ is separable and metrizable and $x \mapsto (h_x^\bullet, \tilde h_x^\bullet) : X \to A \times \tilde A$ is measurable (418Bb). But addition on $L^0(\nu)$ is continuous (245Da), so

$$x \mapsto h_x^\bullet + \tilde h_x^\bullet = (h + \tilde h)_x^\bullet$$

is measurable (418Ac), and

$$\{(h+\tilde h)_x^\bullet : x \in X\} \subseteq \{u + \tilde u : u \in A,\ \tilde u \in \tilde A\}$$

is separable (4A2B(e-iii), 4A2P(a-iv)). Thus $h + \tilde h \in \Phi$. **Q**

(β) Φ is closed under scalar multiplication, just because $u \mapsto \alpha u : L^0(\nu) \to L^0(\nu)$ is always continuous.

(γ) If $\langle h^{(n)}\rangle_{n\in\mathbb{N}}$ is a sequence in Φ and $h(x,y) = \lim_{n\to\infty} h^{(n)}(x,y)$ for all $x \in X$, $y \in Y$, then $h \in \Phi$. **P** Setting $A_n = \{(h_x^{(n)})^\bullet : x \in X\}$ for each n, then $A = \{h_x^\bullet : x \in X\}$ is included in $\overline{\bigcup_{n\in\mathbb{N}} A_n}$, which is separable (4A2B(e-i) again), so A is separable (4A2P(a-iv) again); moreover, $h_x^\bullet = \lim_{n\to\infty}(h_x^{(n)})^\bullet$ for every $x \in X$, so $x \mapsto h_x^\bullet$ is measurable, by 418Ba. **Q**

(δ) What this means is that if we set $\mathcal{W} = \{W : W \in \Sigma\widehat{\otimes}\mathrm{T},\ \chi W \in \Phi\}$, then $W \setminus W' \in \mathcal{W}$ whenever $W, W' \in \mathcal{W}$ and $W' \subseteq W$, and $\bigcup_{n\in\mathbb{N}} W_n \in \mathcal{W}$ whenever $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$. Also, it is easy to see that $E \times F \in \mathcal{W}$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$. By the Monotone Class Theorem (136B), $\mathcal{W}$ includes the σ-algebra generated by $\{E \times F : E \in \Sigma,\ F \in \mathrm{T}\}$, that is, is equal to $\Sigma\widehat{\otimes}\mathrm{T}$. It follows at once, from (α) and (β), that $\sum_{i=0}^n \alpha_i \chi W_i \in \Phi$ whenever $W_0, \dots, W_n \in \Sigma\widehat{\otimes}\mathrm{T}$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$, and hence (using ($\gamma$)) that $\mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}} \subseteq \Phi$, which is what we had to prove.

418S Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Give $L^0(\nu)$ the topology of convergence in measure. Write $\mathcal{L}^0(\lambda)$ for the space of Λ-measurable real-valued functions defined λ-a.e. on $X\times Y$, as in 241A.

(a) If $h\in\mathcal{L}^0(\lambda)$, set $h_x(y)=h(x,y)$ whenever this is defined. Then

$$\{x:f(x)=h_x^\bullet\text{ is defined in }L^0(\nu)\}$$

is μ-conegligible, and includes a conegligible set X_0 such that $f:X_0\to L^0(\nu)$ is measurable and $f[X_0]$ is separable.

(b) If $f:X\to L^0(\nu)$ is measurable and there is a conegligible set $X_0\subseteq X$ such that $f[X_0]$ is separable, then there is an $h\in\mathcal{L}^0(\lambda)$ such that $f(x)=h_x^\bullet$ for almost every $x\in X$.

proof (a) The point is that λ is just the completion of its restriction to $\Sigma\widehat{\otimes}\mathrm{T}$ (251K). So there is a conegligible set $W\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $h\restriction W$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable (212Fa). Setting $\tilde{h}(x,y)=h(x,y)$ for $(x,y)\in W$, 0 otherwise, and setting $\tilde{f}(x)=\tilde{h}_x^\bullet$ for every $x\in X$, we see from 418R that $\tilde{f}$ is measurable and that $\tilde{f}[X]$ is separable. But 252D tells us that

$$X_0=\{x:((X\times Y)\setminus W)[\{x\}]\text{ is negligible}\}$$

is conegligible; and if $x\in X_0$ then $h_x=\tilde{h}_x$ ν-a.e., so that $f(x)$ is defined and equal to $\tilde{f}(x)$. This proves the result.

(b) We can suppose that $X_0\in\Sigma$. Setting $f_1(x)=f(x)$ for $x\in X_0$, 0 otherwise, f_1 satisfies 418R(i). So there is some $h\in\mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ such that $f_1(x)=h_x^\bullet$ for every x, in which case $f(x)=h_x^\bullet$ for almost every x, and (ii) is true.

418T Corollary (Mauldin & Stone 81) Let (Y,T,ν) be a σ-finite measure space, and $(\mathfrak{B},\bar{\nu})$ its measure algebra, with its measure-algebra topology (§323).

(a) Let X be a set, Σ a σ-algebra of subsets of X, and $f:X\to\mathfrak{B}$ a function. Then the following are equiveridical:

(i) $f[X]$ is separable and f is measurable;

(ii) there is a $W\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $f(x)=W[\{x\}]^\bullet$ for every $x\in X$.

(b) Let (X,Σ,μ) be a σ-finite measure space and Λ the domain of the c.l.d. product measure λ on $X\times Y$.

(i) Suppose that ν is complete. If $W\in\Lambda$, then

$$\{x:f(x)=W[\{x\}]^\bullet\text{ is defined in }\mathfrak{B}\}$$

is μ-conegligible, and includes a conegligible set X_0 such that $f:X_0\to\mathfrak{B}$ is measurable and $f[X_0]$ is separable.

(ii) If $f:X\to\mathfrak{B}$ is measurable and there is a conegligible set $X_0\subseteq X$ such that $f[X_0]$ is separable, then there is a $W\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $f(x)=W[\{x\}]^\bullet$ for almost every $x\in X$.

proof Everything follows directly from 418R and 418S if we observe that $\mathfrak{B}$ is homeomorphically embedded in $L^0(\nu)$ by the function $F^\bullet\mapsto(\chi F)^\bullet$ for $F\in\mathrm{T}$ (323Xa, 367Ra). We do need to check, for (i)$\Rightarrow$(ii) of part (a), that if $h\in\mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$ and $h_x^\bullet$ is always of the form $(\chi F)^\bullet$, then there is some $W\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $h_x^\bullet=(\chi W[\{x\}])^\bullet$ for every x; but of course this is true if we just take $W=\{(x,y):h(x,y)=1\}$. Now (b-ii) follows from (a) just as 418Sb followed from 418R.

***418U Independent families of measurable functions** In §455 we shall have occasion to look at independent families of random variables taking values in spaces other than $\mathbb{R}$. We can use the same principle as in §272: a family $\langle X_i\rangle_{i\in I}$ of random variables is independent if $\langle\Sigma_i\rangle_{i\in I}$ is independent, where Σ_i is the σ-subalgebra defined by X_i for each i (272D). Of course this depends on agreement about the definition of Σ_i. The natural thing to do, in the context of this section, is to follow 272C, as follows. Let (X,Σ,μ) be a probability space, Y a topological space, and f a Y-valued function defined on a conegligible subset $\operatorname{dom} f$ of X, which is μ-virtually measurable, that is, such that f is measurable with respect to the subspace σ-algebra on $\operatorname{dom} f$ induced by $\hat{\Sigma}=\operatorname{dom}\hat{\mu}$, where $\hat{\mu}$ is the completion of μ. (Note that if Y is not second-countable this may not imply that $f\restriction D$ is Σ-measurable for a conegligible subset D of X.) The 'σ-algebra defined by f' will be

$$\{f^{-1}[F] : F \in \mathcal{B}(Y)\} \cup \{(\Omega \setminus \operatorname{dom} f) \cup f^{-1}[F] : F \in \mathcal{B}(Y)\} \subseteq \hat{\Sigma},$$

where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y; that is, the σ-algebra of subsets of X generated by $\{f^{-1}[G] : G \subseteq Y$ is open$\}$.

Now, given a family $\langle(f_i, Y_i)\rangle_{i\in I}$ where each Y_i is a topological space and each f_i is a $\hat{\Sigma}$-measurable Y_i-valued function defined on a conegligible subset of X, I will say that $\langle f_i\rangle_{i\in I}$ is **independent** if $\langle\Sigma_i\rangle_{i\in I}$ is independent (with respect to $\hat{\mu}$), where Σ_i is the σ-algebra defined by f_i for each i.

Corresponding to 272D, we can use the Monotone Class Theorem to show that $\langle f_i\rangle_{i\in I}$ is independent iff

$$\hat{\mu}(\bigcap_{j\le n} f_{i_j}^{-1}[G_j]) = \prod_{j\le n} \hat{\mu} f_{i_j}^{-1}[G_j]$$

whenever $i_0, \dots , i_n \in I$ are distinct and $G_j \subseteq Y_{i_j}$ is open for every $j \le n$.

418X Basic exercises >(a) Let (X, Σ, μ) be a measure space, and $(X, \hat{\Sigma}, \hat{\mu})$ its completion. (i) Show that if Y is a second-countable topological space, a function $f : X \to Y$ is $\hat{\Sigma}$-measurable iff there is a Σ-measurable $g : X \to Y$ such that $f =_{\text{a.e.}} g$. (ii) Show that if X is endowed with a topology, and Y is a topological space, then a function from X to Y is μ-almost continuous iff it is $\hat{\mu}$-almost continuous.

(b) Let (X, Σ, μ) be a measure space, Y a set and $h : X \to Y$ a function; give Y the image measure μh^{-1}. Show that for any function g from Y to a topological space Z, g is measurable iff $gh : X \to Z$ is measurable.

>(c) Let X be a set, Σ a σ-algebra of subsets of X, $\langle Y_n\rangle_{n\in\mathbb{N}}$ a sequence of topological spaces with product Y, and $f : X \to Y$ a function. Show that f is measurable iff $\psi_n f : X \to \prod_{i\le n} Y_i$ is measurable for every $n \in \mathbb{N}$, where $\psi_n(y) = (y(0), \dots , y(n))$ for $y \in Y$ and $n \in \mathbb{N}$.

(d) Let (X, Σ, μ) be a semi-finite measure space, $(Y, \mathfrak{S})$ a metrizable space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of measurable functions from X to Y such that $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is convergent for almost every $x \in X$. Show that μ is inner regular with respect to $\{E : \langle f_n{\restriction}E\rangle_{n\in\mathbb{N}}$ is uniformly convergent$\}$. (Cf. 412Xa.)

>(e) Set $Y = [0, 1]^{[0,1]}$, with the product topology. For $n \in \mathbb{N}$ and $x \in [0, 1]$ define $f_n(x) \in Y$ by saying that $f_n(x)(t) = \max(0, 1-2^n|x-t|)$ for $t \in [0, 1]$. Check that (i) each f_n is continuous, therefore measurable; (ii) $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in [0, 1]$; (iii) for each $t \in [0, 1]$, the coordinate functional $x \mapsto f(x)(t)$ is continuous except at t, and in particular is almost continuous and measurable; (iv) $f{\restriction}F$ is not continuous for any infinite closed set $F \subseteq [0, 1]$, and in particular f is not almost continuous; (v) every subset of $[0, 1]$ is of the form $f^{-1}[H]$ for some open set $H \subseteq Y$; (vi) f is not measurable; (vii) the image measure μf^{-1}, where μ is Lebesgue measure on $[0, 1]$, is neither a topological measure nor tight.

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, Y a topological space, and $f : X \to Y$ a function. Suppose that for every $x \in X$ there is an open set G containing x such that $f{\restriction}G$ is almost continuous with respect to the subspace measure on G. Show that f is almost continuous.

(g) For $i = 1$, 2 let $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ and $(Y_i, \mathfrak{S}_i, \mathrm{T}_i, \nu_i)$ be quasi-Radon measure spaces, and $f_i : X_i \to Y_i$ an almost continuous inverse-measure-preserving function. Show that $(x_1, x_2) \mapsto (f_1(x_1), f_2(x_2))$ is almost continuous and inverse-measure-preserving for the product topologies and quasi-Radon product measures.

(h) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ and $\langle(Y_i, \mathfrak{T}_i, \Sigma_i, \nu_i)\rangle_{i\in I}$ be two families of topological spaces with τ-additive Borel probability measures, and let μ, ν be the τ-additive product measures on $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$ (417E, 417F). Show that if $f_i : X_i \to Y_i$ is almost continuous and inverse-measure-preserving for each i, then $x \mapsto \langle f_i(x(i))\rangle_{i\in I} : X \to Y$ is inverse-measure-preserving, but need not be almost continuous.

(i) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be quasi-Radon measure spaces, $(Z, \mathfrak{U})$ a topological space and $f : X \times Y \to Z$ a function which is almost continuous with respect to the quasi-Radon product measure on $X \times Y$. Suppose that ν is σ-finite. Show that $y \mapsto f(x, y)$ is almost continuous for almost every $x \in X$.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space, Y a topological space and $f : X \to Y$ an almost continuous function. (i) Show that the image measure μf^{-1} is τ-additive. (ii) Show that if μ is a totally finite quasi-Radon measure and the topology on Y is regular, then μf^{-1} is quasi-Radon.

(k) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a topological measure space and U a linear topological space. Show that if $f:X\to U$ and $g:X\to U$ are almost continuous, then $f+g:X\to U$ is almost continuous.

(l) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be topological measure spaces, and $(Z,\mathfrak{U})$ a topological space; let $f:X\to Y$ be almost continuous and inverse-measure-preserving, and $g:Y\to Z$ almost continuous. Show that if *either* ν is strictly localizable *or* μ is a Radon measure and ν is locally finite *or* μ is τ-additive and effectively locally finite and ν is effectively locally finite, then $gf:X\to Z$ is almost continuous. (*Hint*: show that if $\mu E>0$ there is a set F such that $\nu F<\infty$ and $\mu(E\cap f^{-1}[F])>0$.)

(m) Let μ be Lebesgue measure on $\mathbb{R}^r$, where $r\ge 1$, X a Hausdorff space and $f:\mathbb{R}^r\to X$ an almost continuous function. Show that for almost every $x\in\mathbb{R}^r$ there is a measurable set $E\subseteq\mathbb{R}^r$ such that x is a density point of E and $\lim_{y\in E,y\to x}f(y)=f(x)$.

(n) Let (X,Σ,μ) be a complete locally determined measure space, $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}X=X$, and $\mathfrak{T}$ the associated density topology on X (414P). Let $f:X\to\mathbb{R}$ be a function. Show that the following are equiveridical: (i) f is measurable; (ii) f is almost continuous; (iii) f is continuous at almost every point; (iv) there is a conegligible set $H\subseteq X$ such that $f\restriction H$ is continuous. (Cf. 414Xk.)

(o) Let (X,Σ,μ) be a complete locally determined measure space, $\phi:\Sigma\to\Sigma$ a lifting, and $\mathfrak{S}$ the lifting topology on X (414Q). Let $f:X\to\mathbb{R}$ be a function. Show that the following are equiveridical: (i) f is measurable; (ii) f is almost continuous; (iii) there is a conegligible set $H\subseteq X$ such that $f\restriction H$ is continuous. (Cf. 414Xr.)

(p) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, $(Y,\mathfrak{S})$ a regular topological space and $f:X\to Y$ an almost continuous function. Show that there is a quasi-Radon measure ν on Y such that f is inverse-measure-preserving for μ and ν iff $\bigcup\{f^{-1}[H]:H\subseteq Y$ is open, $\mu f^{-1}[H]<\infty\}$ is conegligible in X.

(q) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, $(Y,\mathfrak{S})$ and $(Z,\mathfrak{U})$ Hausdorff spaces, $f:X\to Y$ an almost continuous function such that $\nu=\mu f^{-1}$ is locally finite, and $g:Y\to Z$ a function. Show that g is almost continuous with respect to ν iff gf is almost continuous with respect to μ.

(r) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be topological probability spaces, and $f:X\to Y$ a measurable function such that $\mu f^{-1}[H]\ge\nu H$ for every $H\in\mathfrak{S}$. Show that (i) $\int gf\,d\mu=\int g\,d\nu$ for every $g\in C_b(Y)$ (ii) $\mu f^{-1}[F]=\nu F$ for every Baire set $F\subseteq Y$ (iii) if μ is a Radon measure and f is almost continuous, then $\mu f^{-1}[F]=\nu F$ for every Borel set $F\subseteq Y$, so that if in addition ν is complete and inner regular with respect to the Borel sets then it is a Radon measure.

(s) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite topological measure space in which the topology $\mathfrak{T}$ is normal and μ is outer regular with respect to the open sets. Show that if $f:X\to\mathbb{R}$ is a measurable function and $\epsilon>0$ there is a continuous $g:X\to\mathbb{R}$ such that $\mu\{x:g(x)\ne f(x)\}\le\epsilon$. (*Hint*: 4A2Fd.)

(t) Let X and Y be Hausdorff spaces, ν a totally finite Radon measure on Y, and $f:X\to Y$ an injective continuous function. Show that the following are equiveridical: (i) there is a Radon measure μ on X such that f is inverse-measure-preserving; (ii) $f[X]$ is conegligible and $f^{-1}:f[X]\to X$ is almost continuous.

(u) Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be Radon measure spaces and $f:X\to Y$ an almost continuous inverse-measure-preserving function. Show that (i) $\mu_*A\le\nu_*f[A]$ for every $A\subseteq X$ (ii) ν is precisely the image measure μf^{-1}.

(v) Let X be a compact Hausdorff space. Show that there is an atomless Radon probability measure on X iff X is not scattered. (*Hint*: 4A2Gj.)

(w) In 418M, show that all the f_{ij} must be almost continuous. Show that if $i\le j\le k$ then $f_{ij}f_{jk}=f_{ik}$ almost everywhere in X_k.

>(x) Let $\mathcal{I}$ be the family of finite subsets of $[0,1]$, and for each $I\in\mathcal{I}$ let $(X_I,\mathfrak{T}_I,\Sigma_I,\mu_I)$ be $[0,1]\setminus I$ with its subspace topology and measure induced by Lebesgue measure. For $I\subseteq J\in\mathcal{I}$ and $y\in X_J$ set $f_{IJ}(y)=y$. Show that these X_I, f_{IJ} satisfy nearly all the hypotheses of 418O, but that there are no X, g_I which satisfy the hypotheses of 418M.

(y) Let T be any set, and X the set of total orders on T. (i) Regarding each member of X as a subset of $T\times T$, show that X is a closed subset of $\mathcal{P}(T\times T)$. (ii) Show that there is a unique Radon measure μ on X such that $\Pr(t_1\le t_2\le\ldots\le t_n)=\frac{1}{n!}$ for all distinct $t_1,\ldots,t_n\in T$. (*Hint*: for $I\in[T]^{<\omega}$, let X_I be the set of total orders on I with the uniform probability measure giving the same measure to each singleton; show that the natural map from X_I to X_J is inverse-measure-preserving whenever $J\subseteq I$.)

(z) In 418Sb, suppose that $f_1:X\to L^0(\nu)$ and $f_2:X\to L^0(\nu)$ correspond to h_1, $h_2\in\mathcal{L}^0(\lambda)$. Show that $f_1(x)\le f_2(x)$ μ-a.e.(x) iff $h_1\le h_2$ λ-a.e. Hence show that (if we assign appropriate algebraic operations to the space of functions from X to $L^0(\nu)$) we have an f-algebra isomorphism between $L^0(\lambda)$ and the space of equivalence classes of measurable functions from X to $L^0(\nu)$ with separable ranges.

418Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, Y a topological space and $f:X\to Y$ a function. Set $\mathrm{T}=\{F:F\subseteq Y,\,f^{-1}[F]\in\Sigma\}$. Suppose that Y is hereditarily Lindelöf and its topology is generated by some subset of T. Show that f is measurable.

(b) Let (X,Σ,μ) be a measure space, Y and Z topological spaces and $f:X\to Y$, $g:X\to Z$ measurable functions. Show that if Z has a countable network consisting of Borel sets (e.g., Z is second-countable, or Z is regular and has a countable network), then $x\mapsto(f(x),g(x)):X\to Y\times Z$ is measurable.

(c) Let X be a set, Σ a σ-algebra of subsets of X, and $\langle Y_i\rangle_{i\in I}$ a countable family of topological spaces with product Y. Suppose that every Y_i has a countable network, and that $f:X\to Y$ is a function such that $\pi_i f$ is measurable for every $i\in I$, writing $\pi_i(y)=y(i)$. Show that f is measurable.

(d) Let (X,Σ,μ) be a σ-finite measure space and $\mathfrak{T}$ a topology on X such that μ is effectively locally finite and τ-additive. Let Y be a topological space and $f:X\to Y$ an almost continuous function. Show that there is a conegligible subset X_0 of X such that $f[X_0]$ is ccc.

(e) Show that if μ is Lebesgue measure on $\mathbb{R}$, $\mathfrak{T}$ is the usual topology on $\mathbb{R}$ and $\mathfrak{S}$ is the right-facing Sorgenfrey topology (415Xc), then the identity map from $(\mathbb{R},\mathfrak{T},\mu)$ to $(\mathbb{R},\mathfrak{S})$ is measurable, but not almost continuous, and the image measure is not a Radon measure.

(f) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a topological space with a countable network consisting of Borel sets, and that $f:X\to Y$ is measurable. Show that f is almost continuous.

(g) Find a topological probability space $(X,\mathfrak{T},\Sigma,\mu)$ in which μ is inner regular with respect to the closed sets, a topological space Y with a countable network and a measurable function $f:X\to Y$ which is not almost continuous.

(h) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Let $A\subseteq L^\infty(\mu)$ be a norm-compact set. Show that there is a set B of bounded real-valued measurable functions on X such that (i) $A=\{f^\bullet:f\in B\}$ (ii) B is norm-compact in $\ell^\infty(X)$ (iii) μ is inner regular with respect to $\{E:f\restriction E$ is continuous for every $f\in B\}$.

(i) Let μ be Lebesgue measure on $[0,1]$. For $t\in[0,1]$ set $u_t=\chi[0,t]^\bullet\in L^0(\mu)$. Show that $A=\{u_t:t\in[0,1]\}$ is norm-compact in $L^p(\mu)$ for every $p\in[1,\infty[$ and also compact for the topology of convergence in measure on $L^0(\mu)$. Show that if B is a set of measurable functions such that $A=\{f^\bullet:f\in B\}$ then μ is not inner regular with respect to $\{E:f\restriction E$ is continuous for every $f\in B\}$.

(j) Let μ be Lebesgue measure on $[0,1]$, and $A\subseteq[0,1]$ a set with inner measure 0 and outer measure 1; let $\mathfrak{T}$ be the usual topology on $[0,1]$. Let $\mathcal{I}$ be the family of sets $I\subseteq A$ such that every point of A has a neighbourhood containing at most one point of I. Show that $\mathfrak{S}=\{G\setminus I:G\in\mathfrak{T},\,I\in\mathcal{I}\}$ is a topology on $[0,1]$ with a countable network. Show that the identity map from $[0,1]$ to itself, regarded as a map from $([0,1],\mathfrak{T},\mu)$ to $([0,1],\mathfrak{S})$, is measurable but not almost continuous.

(k) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y and Z are separable metrizable spaces, $f:X\times Y\to Z$ is a function such that $x\mapsto f(x,y)$ is measurable for every $y\in Y$, and $y\mapsto f(x,y)$ is continuous for every $x\in X$. Show that μ is inner regular with respect to $\{F:F\subseteq X,\ f{\restriction}F\times Y$ is continuous$\}$.

(l) Suppose that $(I,\le)$, $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\le j\in I}$ are such that (α) $(I,\le)$ is a non-empty upwards-directed partially ordered set (β) every $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ is a completely regular Hausdorff quasi-Radon probability space (γ) $f_{ij}:X_j\to X_i$ is a continuous inverse-measure-preserving function whenever $i\le j$ in I (δ) $f_{ij}f_{jk}=f_{ik}$ whenever $i\le j\le k$ in I. Let X'_i be the support of μ_i for each i; show that $f_{ij}[X'_j]$ is a dense subset of X'_i whenever $i\le j$. Let Z_i be the Stone-Čech compactification of X'_i and let $\tilde f_{ij}:Z_j\to Z_i$ be the continuous extension of $f_{ij}{\restriction}X'_j$ for $i\le j$; let $\tilde\mu_i$ be the Radon probability measure on Z_i corresponding to $\mu_i{\restriction}\mathcal{P}X'_i$ (416V). Show that Z_i, $\tilde f_{ij}$ satisfy the conditions of 418O, so that we have a projective limit Z, $\langle g_i\rangle_{i\in I}$, μ as in 418M.

(m) Let X be a set, Σ a σ-algebra of subsets of X and (Y,T,ν) a σ-finite measure space with countable Maharam type. (i) Let $f:X\to L^1(\nu)$ be a function such that $x\mapsto\int_F f(x)d\nu$ is Σ-measurable for every $F\in\mathrm{T}$. Show that f is Σ-measurable for the norm topology on $L^1(\nu)$. (ii) Let $g:X\times Y\to\mathbb{R}$ be a function such that $\int g(x,y)\nu(dy)$ is defined for every $x\in X$, and $x\mapsto\int_F g(x,y)\nu(dy)$ is Σ-measurable for every $F\in\mathrm{T}$. Show that there is an $h\in\mathcal{L}^0_{\Sigma\widehat{\otimes}\mathrm{T}}$ such that, for every $x\in X$, $g(x,y)=h(x,y)$ for ν-almost every y.

(n) Use 418M and 418O to prove 328H.

(o) Let X be a set, Σ a σ-algebra of subsets of X, (Y,T,ν) a σ-finite measure space and $W\in\Sigma\widehat{\otimes}\mathrm{T}$. Show that there is a $V\subseteq W$ such that $V\in\Sigma\widehat{\otimes}\mathrm{T}$, $W[\{x\}]\setminus V[\{x\}]$ is negligible for every $x\in X$, and $\bigcap_{x\in I}V[\{x\}]$ is either empty or non-negligible for every non-empty finite $I\subseteq X$. (*Hint*: 341Xb.)

(p) Let X be a compact Hausdorff space, Y a Hausdorff space, ν a Radon probability measure on Y and $R\subseteq X\times Y$ a closed set such that $\nu^*R[X]=1$. Show that there is a Radon probability measure μ on X such that $\mu R^{-1}[F]\ge\nu F$ for every closed set $F\subseteq Y$.

418 Notes and comments The message of this section is that measurable functions are dangerous, but that almost continuous functions behave themselves. There are two fundamental problems with measurable functions: a function $x\mapsto(f(x),g(x))$ may not be measurable when the components f and g are measurable (419Xg), and an image measure under a measurable function can lose tightness, even when both domain and codomain are Radon measure spaces and the function is inverse-measure-preserving (419Xh). (This is the 'image measure catastrophe' mentioned in 235H and the notes to §343.) Consequently, as long as we are dealing with measurable functions, we often have to impose strong conditions on the range spaces – commonly, we have to restrict ourselves to separable metrizable spaces (418B, 418C), or something similar, which indeed often means that a measurable function is actually almost continuous (418J, 433E). Indeed, for functions taking values in metrizable spaces, 'almost continuous' is very close to 'measurable with essentially separable range' (418G, 418J). The condition 'separable and metrizable' is a little stronger than is strictly necessary (418Yb, 418Yc, 418Yf), but covers the principal applications other than 433E. If we keep the 'metrizable' we can very substantially relax the 'separable' (438E, 438F), and it is in fact the case that a measurable function from a Radon measure space to any metrizable space is almost continuous (451T). These extensions apply equally to the results in 418R-418T (438Xi-438Xj, 451Xp). But both take us deeper into set theory than seems appropriate at the moment.

For almost continuous functions, the two problems mentioned above do not arise (418Dd, 418I, 418Xu). Indeed we rather expect almost continuous functions to behave as if they were continuous. But we still have to be careful. The limit of a sequence of almost continuous functions need not be almost continuous (418Xe), unless the codomain is metrizable (418F); and if we have a function f from a topological measure space to an uncountable product of topological spaces, it can happen that every coordinate of f is an almost continuous function while f is not (418Xe again). But for many purposes, intuitions gained from the study of measurable functions between Euclidean spaces can be transferred to general almost continuous functions.

Theorems 418L and 418M are of a quite different kind, but seem to belong here as well as anywhere. Even in the simplest application of 418L (when $Y = [0,1]$ with Lebesgue measure, and $X \subseteq [0,1]^2$ is a closed set meeting every vertical line) it is not immediately obvious that there will be a measure with the right projection onto the horizontal axis, though there are at least two proofs which are easier than the general case treated in 413O-413P-418L.

As I explain in 418N, the really interesting question concerning 418M is when, given the projective system $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\le j\in I}$, we can expect to find X and $\langle g_i\rangle_{i\in I}$ satisfying the rest of the hypotheses, and once past the elementary results 418O-418Q this can be hard to determine. I describe a method in 418Yl which can sometimes be used, but (like the trick in 418Nf) it is too easy and too abstract to be often illuminating. See 454G below for something rather deeper.

The results of 418R-418T stand somewhat aside from anything else considered in this chapter, but they form part of an important technique. A special case has already been mentioned in 253Yg. I do not discuss vector-valued measurable functions in this book, except incidentally, but 418R is one of the fundamental results on their representation; it means, for instance, that if V is any of the Banach function spaces of Chapter 36 we can expect to represent Bochner integrable V-valued functions (253Yf) in terms of functions on product spaces, because V will be continuously embedded in an L^0 space (367O). The measure-algebra version in 418T will be useful in Volume 5 when establishing relationships between properties of measure spaces and corresponding properties of measure algebras.

Version of 2.12.05

419 Examples

In §216, I went much of the way to describing examples of spaces with all the possible combinations of the properties considered in Chapter 21. When we come to topological measure spaces, the number of properties involved makes it unreasonable to seek any such comprehensive list. I therefore content myself with seven examples to indicate some of the boundaries of the theory developed here.

The first example (419A) is supposed to show that the hypothesis 'effectively locally finite' which appears in so many of the theorems of this chapter cannot as a rule be replaced by 'locally finite'. The next two (419C-419D) address technical questions concerning the definition of 'Radon measure', and show how small variations in the definition can lead to very different kinds of measure space. The fourth example (419E) shows that the τ-additive product measures of §417 are indeed new constructions. 419J is there to show that extension theorems of the types proved in §415 and §417 cannot be taken for granted. The classic example 419K exhibits one of the obstacles to generalizations of Prokhorov's theorem (418M, 418Q). Finally, I return to the split interval (419L) to describe its standard topology and its relation to the measure introduced in 343J.

419A Example There is a locally compact Hausdorff space X with a complete, σ-finite, locally finite, τ-additive topological measure μ, inner regular with respect to the closed sets, which has a closed subset Y, of measure 1, such that the subspace measure μ_Y on Y is not τ-additive. In particular, μ is not effectively locally finite.

proof (a) Let Q be a countably infinite set, not containing any ordinal. Fix an enumeration $\langle q_n\rangle_{n\in\mathbb{N}}$ of Q, and for $A \subseteq Q$ set $\nu A = \sum\{\frac{1}{n+1} : q_n \in A\}$. Let $\mathcal{I}$ be the ideal $\{A : A \subseteq Q, \nu A < \infty\}$. For any sets I, J, say that $I \subseteq^* J$ if $I \setminus J$ is finite; then $\subseteq^*$ is a reflexive transitive relation.

Let κ be the smallest cardinal of any family $\mathcal{K} \subseteq \mathcal{I}$ for which there is no $I \in \mathcal{I}$ such that $K \subseteq^* I$ for every $K \in \mathcal{K}$. Then κ is uncountable. **P** (i) Of course κ is infinite. (ii) If $\langle I_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{I}$, then for each $n \in \mathbb{N}$ we can find a finite $I'_n \subseteq I_n$ such that $\nu(I_n \setminus I'_n) \le 2^{-n}$; setting $I = \bigcup_{n\in\mathbb{N}} I_n \setminus I'_n$, we have $\nu I \le 2 < \infty$, while $I_n \subseteq^* I$ for every n. Thus $\kappa > \omega$. **Q**

(b) There is a family $\langle I_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ such that (i) $I_\eta \subseteq^* I_\xi$ whenever $\eta \le \xi < \kappa$ (ii) there is no $I \in \mathcal{I}$ such that $I_\xi \subseteq^* I$ for every $\xi < \kappa$. **P** Take a family $\langle K_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ such that there is no $I \in \mathcal{I}$ such that $K_\xi \subseteq^* I$ for every $\xi < \kappa$. Choose $\langle I_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ inductively in such a way that

$$K_\xi \subseteq^* I_\xi, \quad I_\eta \subseteq^* I_\xi \text{ for every } \eta < \xi.$$

(This can be done because $\{K_\xi\} \cup \{I_\eta : \eta < \xi\}$ will always be a subset of $\mathcal{I}$ with cardinal less than κ.) If now $I_\xi \subseteq^* I$ for every $\xi < \kappa$, then $K_\xi \subseteq^* I$ for every $\xi < \kappa$, so $I \notin \mathcal{I}$. **Q**

Hence, or otherwise, we see that κ is regular. **P** If $A \subseteq \kappa$ and $\#(A) < \kappa$, then there is an $I \in \mathcal{I}$ such that $I_\zeta \subseteq^* I$ for every $\zeta \in A$; now there must be a $\xi < \kappa$ such that $I_\xi \not\subseteq^* I$, in which case $\zeta < \xi$ for every $\zeta \in A$, and A is not cofinal with κ. **Q**

(c) Set $X = Q \cup \kappa$. (This is where it is helpful to have arranged at the start that no ordinal belongs to Q, so that $Q \cap \kappa = \emptyset$.) Let $\mathfrak{T}$ be the family of sets $G \subseteq X$ such that

$G \cap \kappa$ is open for the order topology of κ,
for every $\xi \in G \cap \kappa \setminus \{0\}$ there is an $\eta < \xi$ such that $I_\xi \setminus I_\eta \subseteq^* G$,
if $0 \in G$ then $I_0 \subseteq^* G$.

(i) This is a Hausdorff topology on X. **P** (α) It is easy to check that $X \in \mathfrak{T}$, $\emptyset \in \mathfrak{T}$ and $\bigcup \mathcal{G} \in \mathfrak{T}$ for every $\mathcal{G} \subseteq \mathfrak{T}$. (β) Suppose that G, $H \in \mathfrak{T}$. Then $(G \cap H) \cap \kappa = (G \cap \kappa) \cap (H \cap \kappa)$ is open for the order topology of κ. If $\xi \in G \cap H \cap \kappa \setminus \{0\}$ there are η, $\zeta < \xi$ such that $I_\xi \setminus I_\eta \subseteq^* G$ and $I_\xi \setminus I_\zeta \subseteq^* H$, and now

$$\alpha = \max(\eta, \zeta) < \xi, \quad I_\eta \cup I_\zeta \subseteq^* I_\alpha,$$

so

$$I_\xi \setminus I_\alpha \subseteq^* (I_\xi \setminus I_\eta) \cap (I_\xi \setminus I_\zeta) \subseteq^* G \cap H.$$

Finally, if $0 \in G \cap H$ then $I_0 \subseteq^* G \cap H$. So $G \cap H \in \mathfrak{T}$. Thus $\mathfrak{T}$ is a topology on X. (γ) For any $\xi < \kappa$, the set $E_\xi = (\xi + 1) \cup I_\xi$ is open-and-closed for $\mathfrak{T}$; for any $q \in Q$, $\{q\}$ is open-and-closed. Since these sets separate the points of X, $\mathfrak{T}$ is Hausdorff. **Q**

(ii) The sets E_ξ of the last paragraph are all compact for $\mathfrak{T}$. **P** Let $\mathcal{F}$ be an ultrafilter on X containing E_ξ. (α) If a finite set K belongs to $\mathcal{F}$, then $\mathcal{F}$ must contain $\{x\}$ for some $x \in K$, and converges to x. So suppose henceforth that $\mathcal{F}$ contains no finite set. (β) If $E_0 \in \mathcal{F}$, then for any open set G containing 0, $E_0 \setminus G$ is finite, so does not belong to $\mathcal{F}$, and $G \in \mathcal{F}$; as G is arbitrary, $\mathcal{F} \to 0$. (γ) If $E_0 \notin \mathcal{F}$, let $\eta \le \xi$ be the least ordinal such that $E_\eta \in \mathcal{F}$. If G is an open set containing η, there are ζ', $\zeta'' < \eta$ such that $I_\eta \setminus I_{\zeta'} \subseteq^* G$ and $]\zeta'', \eta] \subseteq G$; so that $E_\eta \setminus E_\zeta \subseteq^* G$, where $\zeta = \max(\zeta', \zeta'') < \eta$. Now $E_\eta \in \mathcal{F}$, $E_\zeta \notin \mathcal{F}$ and $(E_\eta \setminus E_\zeta) \setminus G \notin \mathcal{F}$, so that $G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to \eta$. (δ) As $\mathcal{F}$ is arbitrary, E_ξ is compact. **Q**

(iii) It follows that $\mathfrak{T}$ is locally compact. **P** For $q \in Q$, $\{q\}$ is a compact open set containing q; for $\xi < \kappa$, E_ξ is a compact open set containing ξ. **Q**

(iv) The definition of $\mathfrak{T}$ makes it clear that $Q \in \mathfrak{T}$, that is, that κ is a closed subset of X. We need also to check that the subspace topology $\mathfrak{T}_\kappa$ on κ induced by $\mathfrak{T}$ is just the order topology of κ. **P** (α) By the definition of $\mathfrak{T}$, $G \cap \kappa$ is open for the order topology of κ for every $G \in \mathfrak{T}$. (β) For any $\xi < \kappa$, E_ξ is open-and-closed for $\mathfrak{T}$ so $\xi + 1 = E_\xi \cap \kappa$ is open-and-closed for $\mathfrak{T}_\kappa$. But this means that all sets of the forms $[0, \xi[\, = \bigcup_{\eta < \xi} \eta + 1$ and $]\xi, \kappa[\, = \kappa \setminus (\xi + 1)$ belong to $\mathfrak{T}_\kappa$; as these generate the order topology, every open set for the order topology belongs to $\mathfrak{T}_\kappa$, and the two topologies are equal. **Q**

(d) Now let $\mathcal{F}$ be the filter on X generated by the cofinal closed sets in κ. Because the intersection of any sequence of closed cofinal sets in κ is another (4A1Bd), the intersection of any sequence in $\mathcal{F}$ belongs to $\mathcal{F}$. So

$$\Sigma = \mathcal{F} \cup \{X \setminus F : F \in \mathcal{F}\}$$

is a σ-algebra of subsets of X, and we have a measure $\mu_1 : \Sigma \to \{0, 1\}$ defined by saying that $\mu_1 F = 1$, $\mu_1(X \setminus F) = 0$ if $F \in \mathcal{F}$.

(e) Set $\mu E = \nu(E \cap Q) + \mu_1 E$ for $E \in \Sigma$. Then μ is a measure. Let us work through the properties called for.

(i) If $\mu E = 0$ and $A \subseteq E$, then $X \setminus A \supseteq X \setminus E \in \mathcal{F}$, so $A \in \Sigma$. Thus μ is complete.

(ii) $\mu\kappa = 1$ and $\mu\{q\}$ is finite for every $q \in Q$, so μ is σ-finite.

(iii) If $G \subseteq X$ is open, then $\kappa \setminus G$ is closed, in the order topology of κ; if it is cofinal with κ, it belongs to $\mathcal{F}$; otherwise, $\kappa \cap G \in \mathcal{F}$. Thus in either case $G \in \Sigma$, and μ is a topological measure.

(iv) The next thing to note is that $\mu G = \nu(G \cap Q)$ for every open set $G \subseteq X$. **P** If $G \notin \mathcal{F}$ this is trivial. If $G \in \mathcal{F}$, then $\kappa \setminus G$ cannot be cofinal with κ, so there is a $\xi < \kappa$ such that $\kappa \setminus \xi \subseteq G$. **?** If $G \cap Q \in \mathcal{I}$, then $(G \cap Q) \cup I_\xi \in \mathcal{I}$. There must be a least $\eta < \kappa$ such that $I_\eta \not\subseteq^* (G \cap Q) \cup I_\xi$; of course $\eta > \xi$, so $\eta \in G$. There is some $\zeta < \eta$ such that $I_\eta \setminus I_\zeta \subseteq^* G$; but as $I_\zeta \subseteq^* G \cup I_\xi$, by the choice of η, we must also have $I_\eta \subseteq^* G \cup I_\xi$, which is impossible. **X** Thus $G \cap Q \notin \mathcal{I}$ and $\mu G = \nu(G \cap Q) = \infty$. **Q**

(v) It follows that μ is τ-additive. **P** Suppose that $\mathcal{G} \subseteq \mathfrak{T}$ is a non-empty upwards-directed set with union H. Then

$$\mu H = \nu(H \cap Q) = \sup_{G \in \mathcal{G}} \nu(G \cap Q) = \sup_{G \in \mathcal{G}} \mu G$$

because ν is τ-additive (indeed, is a Radon measure) with respect to the discrete topology on Q. **Q**

(vi) μ is inner regular with respect to the closed sets. **P** Take $E \in \Sigma$ and $\gamma < \mu E$. Then $\gamma - \mu_1 E < \nu(E \cap Q)$, so there is a finite $I \subseteq E \cap Q$ such that $\nu I > \gamma - \mu_1 E$. If $\mu_1 E = 0$, then $I \subseteq E$ is already a closed set with $\mu I > \gamma$. Otherwise, $E \cap \kappa \in \mathcal{F}$, so there is a cofinal closed set $F \subseteq \kappa$ such that $F \subseteq E$; now F is closed in X (because κ is closed in X and the subspace topology on κ is the order topology), so $I \cup F$ is closed, and $\mu(I \cup F) > \gamma$. As E and γ are arbitrary, μ is inner regular with respect to the closed sets. **Q**

(vii) μ is locally finite. **P** For any $\xi < \kappa$, E_ξ is an open set containing ξ, and $\mu E_\xi = \nu I_\xi$ is finite. For any $q \in Q$, $\{q\}$ is an open set containing q, and $\mu\{q\} = \nu\{q\}$ is finite. **Q**

(viii) Now consider $Y = \kappa$. This is surely a closed set, and $\mu\kappa = 1$. I noted in (c-iv) above that the subspace topology $\mathfrak{T}_\kappa$ is just the order topology of κ. But this means that $\{\xi : \xi < \kappa\}$ is an upwards-directed family of negligible relatively open sets with union κ, so that the subspace measure $\mu_\kappa = \mu_1$ is not τ-additive.

(ix) It follows from 414K that μ cannot be effectively locally finite; but it is also obvious from the work above that κ is a measurable set, of non-zero measure, such that $\mu(\kappa \cap G) = 0$ whenever G is an open set of finite measure.

419B Lemma For any non-empty set I, there is a dense G_δ set in $[0,1]^I$ which is negligible for the usual measure on $[0,1]^I$.

proof Fix on some $i_0 \in I$, and set $\pi(x) = x(i_0)$ for each $x \in [0,1]^I$, so that π is continuous and inverse-measure-preserving for the usual topologies and measures on $[0,1]^I$ and $[0,1]$. For each $n \in \mathbb{N}$ let $G_n \supseteq [0,1] \cap \mathbb{Q}$ be an open subset of $[0,1]$ with measure at most 2^{-n}, so that $\pi^{-1}[G_n]$ is an open set of measure at most 2^{-n}, and $E = \bigcap_{n \in \mathbb{N}} \pi^{-1}[G_n]$ is a G_δ set of measure 0. If $H \subseteq [0,1]^I$ is any non-empty open set, its image $\pi[H]$ is open in $[0,1]$, so contains some rational number, and meets $\bigcap_{n \in \mathbb{N}} G_n$; but this means that $H \cap E \ne \emptyset$, so E is dense.

419C Example (Fremlin 75b) There is a completion regular Radon measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that

(i) there is an $E \in \Sigma$ such that $\mu(F \triangle E) > 0$ for every Borel set $F \subseteq X$, that is, not every element of the measure algebra of μ can be represented by a Borel set;

(ii) μ is not outer regular with respect to the Borel sets;

(iii) writing ν for the restriction of μ to the Borel σ-algebra of X, ν is a locally finite, effectively locally finite, tight (that is, inner regular with respect to the compact sets) τ-additive completion regular topological measure, and there is a set $Y \subseteq X$ such that the subspace measure ν_Y is not semi-finite.

proof (a) For each $\xi < \omega_1$ set $X_\xi = [0,1]^{\omega_1 \setminus \xi}$, and take μ_ξ to be the usual measure on X_ξ; write Σ_ξ for its domain. Note that μ_ξ is a completion regular Radon measure for the usual topology $\mathfrak{T}_\xi$ of X_ξ (416U). Set $X = \bigcup_{\xi < \omega_1} X_\xi$, and let μ be the direct sum measure on X (214L), that is, write

$$\Sigma = \{E : E \subseteq X,\ E \cap X_\xi \in \Sigma_\xi \text{ for every } \xi < \omega_1\},$$

$$\mu E = \sum_{\xi < \omega_1} \mu_\xi(E \cap X_\xi) \text{ for every } E \in \Sigma.$$

Then μ is a complete locally determined (in fact, strictly localizable) measure on X. Write Σ for its domain.

(b) For each $\eta < \omega_1$ let $\langle \beta_{\xi\eta} \rangle_{\xi \le \eta}$ be a summable family of strictly positive real numbers with $\beta_{\eta\eta} = 1$ (4A1P). Define $g_\eta : X \to \mathbb{R}$ by setting

$$g_\eta(x) = \frac{1}{\beta_{\xi\eta}}x(\eta) \text{ if } x \in X_\xi \text{ where } \xi \le \eta,$$
$$= 0 \text{ if } x \in X_\xi \text{ where } \xi > \eta.$$

Now define $f : X \to \omega_1 \times \mathbb{R}^{\omega_1}$ by setting

$$f(x) = (\xi, \langle g_\eta(x)\rangle_{\eta<\omega_1})$$

if $x \in X_\xi$. Note that f is injective. Let $\mathfrak{T}$ be the topology on X defined by f, that is, the family $\{f^{-1}[W] : W \subseteq \omega_1 \times \mathbb{R}^{\omega_1}$ is open$\}$, where ω_1 and $\mathbb{R}^{\omega_1}$ are given their usual topologies (4A2S, 3A3K), and their product is given its product topology. Because f is injective, $\mathfrak{T}$ can be identified with the subspace topology on $f[X]$; it is Hausdorff and completely regular.

(c) For ξ, $\eta < \omega_1$, $g_\eta\restriction X_\xi$ is continuous for the compact topology $\mathfrak{T}_\xi$. Consequently $f\restriction X_\xi$ is continuous, and the subspace topology on X_ξ induced by $\mathfrak{T}$ must be $\mathfrak{T}_\xi$ exactly. It follows that μ is a Radon measure for $\mathfrak{T}$. **P** (i) We know already that μ is complete and locally determined. (ii) If $G \in \mathfrak{T}$ then $G \cap X_\xi \in \mathfrak{T}_\xi \subseteq \Sigma_\xi$ for every $\xi < \omega_1$, so $G \in \Sigma$; thus μ is a topological measure. (iii) If $E \in \Sigma$ and $\mu E > 0$, there is a $\xi < \omega_1$ such that $\mu_\xi(E \cap X_\xi) > 0$. Because μ_ξ is a Radon measure, there is a $\mathfrak{T}_\xi$-compact set $F \subseteq E \cap X_\xi$ such that $\mu_\xi F > 0$. Now F is $\mathfrak{T}$-compact and $\mu F > 0$. As E is arbitrary, μ is tight (using 412B). (iv) If $x \in X$, take that $\xi < \omega_1$ such that $x \in X_\xi$, and consider

$$G = f^{-1}[(\xi + 1) \times \{w : w \in \mathbb{R}^{\omega_1},\, w(\xi) < 2\}].$$

Because $\xi + 1$ is open in ω_1, $G \in \mathfrak{T}$. Because $g_\xi(x) = x(\xi) \le 1$, $x \in G$. Now for $\zeta \le \xi$,

$$\mu_\zeta(G \cap X_\zeta) = \mu_\zeta\{x : x \in X_\zeta,\, g_\xi(x) < 2\}$$
$$= \mu_\zeta\{x : x \in X_\zeta,\, \beta_{\zeta\xi}^{-1}x(\xi) < 2\}$$
$$= \mu_\zeta\{x : x \in X_\zeta,\, x(\xi) < 2\beta_{\zeta\xi}\} \le 2\beta_{\zeta\xi},$$

so

$$\mu G = \sum\nolimits_{\zeta\le\xi} \mu_\zeta(G \cap X_\zeta) \le 2\sum\nolimits_{\zeta\le\xi} \beta_{\zeta\xi} < \infty.$$

As x is arbitrary, μ is locally finite, therefore a Radon measure. **Q**

We also find that μ is completion regular. **P** If $E \subseteq X$ and $\mu E > 0$, then there is a $\xi < \omega_1$ such that $\mu(E \cap X_\xi) > 0$. Because μ_ξ is completion regular, there is a set $F \subseteq E \cap X_\xi$, a zero set for $\mathfrak{T}_\xi$, such that $\mu F > 0$. Now X_ξ is a G_δ set in X (being the intersection of the open sets $\bigcup_{\eta<\zeta<\xi+1} X_\zeta$ for $\eta < \xi$, unless $\xi = 0$, in which case X_ξ is actually open), so F is a G_δ set in X (4A2C(a-iv)); being a compact G_δ set in a completely regular space, it is a zero set (4A2F(h-v)).

Thus every set of positive measure includes a zero set of positive measure. So μ is inner regular with respect to the zero sets (412B). **Q**

(d) The key to the example is the following fact: if $G \subseteq X$ is open, then *either* there is a cofinal closed set $V \subseteq \omega_1$ such that $G \cap X_\xi = \emptyset$ for every $\xi \in V$ *or* $\{\xi : \mu(G \cap X_\xi) \ne 1\}$ is countable. **P** Suppose that $A = \{\xi : G \cap X_\xi \ne \emptyset\}$ meets every cofinal closed set, that is, is stationary (4A1C). Then $B = A \cap \Omega$ is stationary, where Ω is the set of non-zero countable limit ordinals (4A1Bb, 4A1Cb). Let $H \subseteq \omega_1 \times \mathbb{R}^{\omega_1}$ be an open set such that $G = f^{-1}[H]$.

For each $\xi \in B$ choose $x_\xi \in G \cap X_\xi$. Then $f(x_\xi) \in H$, so there must be a $\zeta_\xi < \xi$, a finite set $I_\xi \subseteq \omega_1$, and a $\delta_\xi > 0$ such that $z \in H$ whenever $z = (\gamma, \langle t_\eta\rangle_{\eta<\omega_1}) \in \omega_1 \times \mathbb{R}^{\omega_1}$, $\zeta_\xi < \gamma \le \xi$ and $|t_\eta - g_\eta(x)| < \delta_\xi$ for every $\eta \in I_\xi$. Because ξ is a non-zero limit ordinal, $\zeta'_\xi = \sup(\{\zeta_\xi\} \cup (I_\xi \cap \xi)) < \xi$.

By the Pressing-Down Lemma (4A1Cc), there is a $\zeta < \omega_1$ such that $C = \{\xi : \xi \in B,\, \zeta'_\xi = \zeta\}$ is uncountable. **?** Suppose, if possible, that $\zeta < \eta < \omega_1$ and $\mu(G \cap X_\eta) < 1$. Then there is a measurable subset F of $X_\eta \setminus G$, determined by coordinates in a countable set $J \subseteq \omega_1 \setminus \eta$, such that $\mu F = \mu_\eta F > 0$ (254Ff). Let $\xi \in C$ be such that $\eta < \xi$ and $J \subseteq \xi$, and take any $y \in F$. If we define $y' \in X_\eta$ by setting

$$y'(\gamma) = y(\gamma) \text{ for } \gamma \in \xi \setminus \eta$$
$$= x_\xi(\gamma) \text{ for } \gamma \in \omega_1 \setminus \xi,$$

then $y' \in F$. But also $\zeta_\xi \le \zeta'_\xi = \zeta < \eta < \xi$ and $\xi \setminus \eta \subseteq \xi \setminus \zeta'_\xi$ is disjoint from I_ξ, so $g_\gamma(y') = g_\gamma(x_\xi)$ for every $\gamma \in I_\xi$, since both are zero if $\gamma < \eta$ and otherwise $y'(\gamma) = x_\xi(\gamma)$. By the choice of ζ_ξ and I_ξ we must have $f(y') \in H$ and $y' \in F \cap G$; which is impossible. **X**

Thus $\mu(G \cap X_\eta) = 1$ for every $\eta > \zeta$, as required by the second alternative. **Q**

(e) For each $\xi < \omega_1$, let $\mathcal{I}_\xi$ be the family of negligible meager subsets of X_ξ. Then $\mathcal{I}_\xi$ is a σ-ideal; note that it contains every closed negligible set, because μ_ξ is strictly positive. Set

$$\mathrm{T}_\xi = \mathcal{I}_\xi \cup \{X_\xi \setminus F : F \in \mathcal{I}_\xi\},$$

so that T_ξ is a σ-algebra of subsets of X_ξ, containing every conegligible open set, and $\mu_\xi F \in \{0,1\}$ for every $F \in \mathrm{T}_\xi$. Set

$$\mathrm{T} = \{E : E \in \Sigma,\ \{\xi : E \cap X_\xi \notin \mathrm{T}_\xi\} \text{ is non-stationary}\}.$$

Then T is a σ-subalgebra of Σ (because the non-stationary sets form a σ-ideal of subsets of ω_1, 4A1Cb), and contains every open set, by (d); so includes the Borel σ-algebra $\mathcal{B}$ of X.

If we set

$$E_\xi = \{x : x \in X_\xi,\ x(\xi) \le \tfrac{1}{2}\} \text{ for each } \xi < \omega_1, \quad E = \bigcup_{\xi<\omega_1} E_\xi,$$

then $E \in \Sigma$. But if $F \subseteq X$ is a Borel set, $F \in \mathrm{T}$ so $\mu(E \triangle F) = \infty$. This proves the property (i) claimed for the example.

(f) Next, for each $\xi < \omega_1$, take a negligible dense G_δ set $E'_\xi \subseteq X_\xi$ (419B). Set $Y = \bigcup_{\xi<\omega_1} E'_\xi$, so that $\mu Y = 0$. If $F \supseteq Y$ is a Borel set, then $F \cap X_\xi \supseteq E_\xi \notin \mathcal{I}_\xi$ for every $\xi < \omega_1$, while $F \in \mathrm{T}$, so $\{\xi : \mu_\xi(F \cap X_\xi) = 0\}$ is non-stationary and $\mu F = \infty$. Thus μ is not outer regular with respect to the Borel sets. Taking $\nu = \mu \restriction \mathcal{B}$, the subspace measure ν_Y is not semi-finite. **P** We have just seen that $\nu_Y Y = \nu^* Y$ is infinite. If $F \in \mathcal{B}$ and $\nu F < \infty$, then $A = \{\xi : \mu_\xi(F \cap X_\xi) > 0\}$ is countable, so $F_0 = \bigcup_{\xi \in A} E'_\xi$ and $F_1 = F \setminus \bigcup_{\xi \in A} X_\xi$ are negligible Borel sets; since $F \cap Y \subseteq F_0 \cup F_1$, $\nu_Y(F \cap Y) = 0$. But this means that ν_Y takes no values in $]0, \infty[$ and is not semi-finite. **Q**

Remark X here is not locally compact. But as it is Hausdorff and completely regular, it can be embedded as a subspace of a locally compact Radon measure space $(X', \mathfrak{T}', \Sigma', \mu')$ (416T). Now μ' still has the properties (i)-(iii).

419D Example (Fremlin 75b) There is a complete locally determined τ-additive completion regular topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ in which μ is tight and compact sets have finite measure, but μ is not localizable.

proof (a) Let I be a set with cardinal greater than $\mathfrak{c}$. Set $X = [0,1]^I$. For $i \in I$, $t \in [0,1]$ set $X_{it} = \{x : x \in X,\ x(i) = t\}$. Give X_{it} its natural topology $\mathfrak{T}_{it}$ and measure μ_{it}, with domain Σ_{it}, defined from the expression of X_{it} as $[0,1]^{I \setminus \{i\}} \times \{t\}$, each factor $[0,1]$ being given its usual topology and Lebesgue measure, and the singleton factor $\{t\}$ being given its unique (discrete) topology and (atomic) probability measure. By 416U, μ_{it} is a completion regular Radon measure. Set

$$\mathfrak{T} = \{G : G \subseteq X,\ G \cap X_{it} \in \mathfrak{T}_{it} \text{ for all } i \in I,\ t \in [0,1]\},$$

$$\Sigma = \{E : E \subseteq X,\ E \cap X_{it} \in \Sigma_{it} \text{ for all } i \in I,\ t \in [0,1]\},$$

$$\mu E = \sum_{i \in I, t \in [0,1]} \mu_{it}(E \cap X_{it}) \text{ for every } E \in \Sigma.$$

(Compare 216D.) Then it is easy to check that $\mathfrak{T}$ is a topology. $\mathfrak{T}$ is Hausdorff because it is finer(= larger) than the usual topology $\mathfrak{S}$ on X; because each $\mathfrak{T}_{it}$ is the subspace topology induced by $\mathfrak{S}$, it is also the subspace topology induced by $\mathfrak{T}$. Next, the definition of μ makes it a locally determined measure; it is a tight complete topological measure because every μ_{it} is.

(b) If $K \subseteq X$ is compact, $\mu K < \infty$. **P?** Otherwise, $M = \{(i,t) : i \in I,\ t \in [0,1],\ \mu_{it}(K \cap X_{it}) > 0\}$ must be infinite. Take any sequence $\langle (i_n, t_n) \rangle_{n \in \mathbb{N}}$ of distinct elements of M. Choose a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in K inductively, as follows. Given $\langle x_m \rangle_{m<n}$, then set $C_{ni} = \{x_m(i) : m < n\}$ for each $i \in I$, and

$$A_n = \{x : x \in X_{i_n t_n},\ x(i) \notin C_{ni} \text{ for } i \in I \setminus \{i_n\}\};$$

then $\mu^*_{i_nt_n}A_n = 1$ (254Lb). Since $\mu_{i_nt_n}(X_{it} \cap X_{ju}) = 0$ whenever $(i,t) \neq (j,u)$, there must be some $x_n \in K \cap A_n \setminus \bigcup_{m\neq n} X_{i_m,t_m}$. Continue.

This construction ensures that if $i \in I$ and $m < n$, either $i \neq i_n$ so $x_n(i) \notin C_{ni}$ and $x_n(i) \neq x_m(i)$, or $i = i_n \neq i_m$ and $x_m \notin X_{i_nt_n}$ so $x_n(i) = t_n \neq x_m(i)$, or $i = i_m = i_n$ and $x_n(i) = t_n \neq t_m = x_m(i)$. But this means that $\{x_n : n \in \mathbb{N}\}$ is an infinite set meeting each X_{it} in at most one point, and is closed for $\mathfrak{T}$; so $\langle x_n\rangle_{n\in\mathbb{N}}$ has no cluster point for $\mathfrak{T}$, which is impossible. **XQ**

(c) μ is not localizable. **P** Fix on any $k \in I$ and consider $\mathcal{E} = \{X_{kt} : t \in [0,1]\}$. **?** If $E \in \Sigma$ is an essential supremum for $\mathcal{E}$, then $E \cap X_{kt}$ must be μ_{kt}-conegligible for every $t \in [0,1]$. We can therefore find a countable set $J_t \subseteq I$ and a μ_{kt}-conegligible set $F_t \subseteq E \cap X_{kt}$, determined by coordinates in J_t. At this point recall that $\#(I) > \mathfrak{c}$, so there is some $j \in I \setminus (\{k\} \cup \bigcup_{t\in[0,1]} J_t)$. Since $X_{j0} \cap X_{kt}$ is negligible for every $t \in [0,1]$, $X_{j0} \cap E$ must be negligible, and $\int_0^1 \nu H_t dt = 0$, where

$$H_t = \{y : y \in [0,1]^{I\setminus\{j,k\}},\ (y,0,t) \in E\}$$

and ν is the usual measure on $[0,1]^{I\setminus\{j,k\}}$, identifying X with $[0,1]^{I\setminus\{j,k\}} \times [0,1] \times [0,1]$. But because F_t is determined by coordinates in $I \setminus \{j\}$, we can identify it with $F'_t \times [0,1] \times \{t\}$ where F'_t is a ν-conegligible subset of $[0,1]^{I\setminus\{j,k\}}$, and $F'_t \subseteq H_t$, so $\nu H_t = 1$ for every t, which is absurd. **X**

Thus $\mathcal{E}$ has no essential supremum in Σ, and μ cannot be localizable. **Q**

(d) I have still to check that μ is completion regular. **P** If $E \in \Sigma$ and $\mu E > 0$, there are $i \in I$, $t \in [0,1]$ such that $\mu_{it}(E \cap X_{it}) > 0$, and an $F \subseteq E \cap X_{it}$, a zero set for the subspace topology of X_{it}, such that $\mu_{it}F > 0$. But now observe that X_{it} is a zero set in X for the usual topology $\mathfrak{S}$, so that F is a zero set for $\mathfrak{S}$ (4A2G(c-i)) and therefore for the finer topology $\mathfrak{T}$. By 412B, this is enough to show that μ is inner regular with respect to the zero sets. **Q**

Remark It may be worth noting that the topology $\mathfrak{T}$ here is not regular. See Fremlin 75b, p. 106.

419E Example (Fremlin 76) Let $(Z,\mathfrak{S},\mathrm{T},\nu)$ be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, so that ν is a strictly positive completion regular Radon probability measure (411P). Then the c.l.d. product measure λ on $Z \times Z$ is not a topological measure, so is not equal to the τ-additive product measure $\tilde\lambda$, and $\tilde\lambda$ is not completion regular.

proof Consider the sets W, $\tilde W$ described in 346K. We have $W \in \Lambda = \operatorname{dom}\lambda$ and $\tilde W = \bigcup \mathcal{V}$, where

$$\mathcal{V} = \{G \times H : G,\ H \subseteq Z \text{ are open-and-closed, } (G \times H) \setminus W \text{ is negligible}\}.$$

$\tilde W$ is a union of open sets, therefore must be open in Z^2. And $\lambda_*\tilde W \leq \lambda W$. **P?** Otherwise, there is a $V \in \Lambda$ such that $V \subseteq \tilde W$ and $\lambda V > \lambda W$. Now λ is tight, by 412Sb, so there is a compact set $K \subseteq V$ such that $K \in \Lambda$ and $\lambda K > \lambda W$. There must be $U_0,\dots,U_n \in \mathcal{V}$ such that $K \subseteq \bigcup_{i\leq n} U_i$. But $\lambda(U_i \setminus W) = 0$ for every i, so $\lambda(K \setminus W) = 0$ and $\lambda K \leq \lambda W$. **XQ**

However, the construction of 346K arranged that $\lambda^*\tilde W$ should be 1 and λW strictly less than 1. So $\lambda_*\tilde W < \lambda^*\tilde W$ and $\tilde W \notin \Lambda$. Accordingly λ is not a topological measure and cannot be equal to the Radon measure $\tilde\lambda$ of 417P.

We know that λ is inner regular with respect to the zero sets (412Sc) and is defined on every zero set (417V), while $\tilde\lambda$ properly extends λ. But this means that $\tilde\lambda$ cannot be inner regular with respect to the zero sets, by 412Mb, that is, cannot be completion regular.

419F Theorem (Rao 69) $\mathcal{P}(\omega_1\times\omega_1) = \mathcal{P}\omega_1\widehat{\otimes}\mathcal{P}\omega_1$, the σ-algebra of subsets of ω_1 generated by $\{E\times F : E,\ F \subseteq \omega_1\}$.

proof (a) Because $\omega_1 \leq \mathfrak{c}$, there is an injection $h : \omega_1 \to \{0,1\}^{\mathbb{N}}$; set $E_i = \{\xi : h(\xi)(i) = 1\}$ for each $i \in \mathbb{N}$.

(b) Suppose that $A \subseteq \omega_1$ has countable vertical sections. Then $A \in \mathcal{P}\omega_1\widehat{\otimes}\mathcal{P}\omega_1$. **P** Set $B = A^{-1}[\omega_1]$ and for $\xi \in B$ choose a surjection $f_\xi : \mathbb{N} \to A[\{\xi\}]$. Set $g_n(\xi) = f_\xi(n)$ for $\xi \in B$ and $n \in \mathbb{N}$, and $A_n = \{(\xi, f_\xi(n)) : \xi \in B\}$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}A_n &= \{(\xi,\eta) : \xi \in B,\, \eta = g_n(\xi)\}\\ &= \{(\xi,\eta) : \xi \in B,\, \eta < \omega_1,\, h(g_n(\xi)) = h(\eta)\}\\ &= \bigcap_{i\in\mathbb{N}} \big(\{(\xi,\eta) : \xi \in g_n^{-1}[E_i],\, \eta \in E_i\} \cup \{(\xi,\eta) : \xi \in B \setminus g_n^{-1}[E_i],\, \eta \in \omega_1 \setminus E_i\}\big)\\ &\in \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1.\end{aligned}$$

So

$$A = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1. \quad \mathbf{Q}$$

(c) Similarly, if a subset of $\omega_1 \times \omega_1$ has countable horizontal sections, it belongs to $\mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$. But for any $A \subseteq \omega_1 \times \omega_1$, $A = A' \cup A''$ where

$$A' = \{(\xi,\eta) : (\xi,\eta) \in A,\, \eta \le \xi\} \text{ has countable vertical sections,}$$
$$A'' = \{(\xi,\eta) : (\xi,\eta) \in A,\, \xi \le \eta\} \text{ has countable horizontal sections,}$$

so both A' and A'' belong to $\mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$ and A also does.

419G Corollary (Ulam 1930) Let Y be a set of cardinal at most ω_1 and μ a semi-finite measure with domain $\mathcal{P}Y$. Then μ is point-supported; in particular, if μ is σ-finite there is a countable conegligible set $A \subseteq Y$.

proof ? Suppose, if possible, otherwise.

(a) We can suppose that Y is either countable or actually equal to ω_1. Let μ_0 be the point-supported part of μ, that is, $\mu_0 A = \sum_{y\in A} \mu\{y\}$ for every $A \subseteq Y$; then μ_0 is a point-supported measure (112Bd), so is not equal to μ. Let $A \subseteq Y$ be such that $\mu_0 A \ne \mu A$. Then $\mu_0 A < \mu A$; because μ is semi-finite, there is a set $B \subseteq A$ such that $\mu_0 A < \mu B < \infty$. Set $\nu C = \mu(B \cap C) - \mu_0(B \cap C)$ for $C \subseteq Y$; then ν is a non-zero totally finite measure with domain $\mathcal{P}Y$, and is zero on singletons.

(b) As $\nu C = 0$ for every countable $C \subseteq Y$, Y is uncountable and $Y = \omega_1$. Let $\lambda = \nu \times \nu$ be the product measure on $\omega_1 \times \omega_1$. By 419F, the domain of λ is the whole of $\mathcal{P}(\omega_1 \times \omega_1)$; in particular, it contains the set $V = \{(\xi,\eta) : \xi \le \eta < \omega_1\}$. Now by Fubini's theorem

$$\lambda V = \int \nu V[\{\xi\}]\nu(d\xi) = \int \nu(\omega_1 \setminus \xi)\nu(d\xi) = (\nu\omega_1)^2 > 0,$$

and also

$$\lambda V = \int \nu V^{-1}[\{\eta\}]\nu(d\eta) = \int \nu(\eta + 1)\nu(d\eta) = 0. \quad \mathbf{X}$$

Remark I ought to remark that this result, though not 419F, is valid for many other cardinals besides ω_1; see, in particular, 438C below. There will be more on this topic in Chapter 54 of Volume 5.

419H For the next two examples it will be helpful to know some basic facts about Lebesgue measure which seemed a little advanced for Volume 1 and for which I have not found a suitable place since.

Lemma If $(X, \mathfrak{T}, \Sigma, \mu)$ is an atomless Radon measure space and $E \in \Sigma$ has non-zero measure, then $\#(E) \ge \mathfrak{c}$.

proof The subspace measure on E is a Radon measure (416Rb), therefore compact and perfect (416Wa), and is not purely atomic; by 344H, there is in fact a negligible subset of E with cardinal $\mathfrak{c}$.

419I The next result is a strengthening of 134D.

Lemma Let μ be Lebesgue measure on $\mathbb{R}$, and H any measurable subset of $\mathbb{R}$. Then there is a disjoint family $\langle A_\alpha\rangle_{\alpha<\mathfrak{c}}$ of subsets of H such that H is a measurable envelope of every A_α; in particular, $\mu_* A_\alpha = 0$ and $\mu^* A_\alpha = \mu H$ for every $\alpha < \mathfrak{c}$.

proof If $\mu H = 0$, we can take every A_α to be empty; so suppose that $\mu H > 0$. Let $\mathcal{E}$ be the family of closed subsets of H of non-zero measure. By 4A3Fa, $\#(\mathcal{E}) \le \mathfrak{c}$; enumerate $\mathcal{E} \times \mathfrak{c}$ as $\langle (F_\xi, \alpha_\xi)\rangle_{\xi<\mathfrak{c}}$ (3A1Ca).

Choose $\langle x_\xi\rangle_{\xi<\mathfrak{c}}$ inductively, as follows. Given $\langle x_\eta\rangle_{\eta<\xi}$, where $\xi<\mathfrak{c}$, F_ξ has cardinal (at least) $\mathfrak{c}$, by 419H, so cannot be included in $\{x_\eta:\eta<\xi\}$; take any $x_\xi\in F_\xi\setminus\{x_\eta:\eta<\xi\}$, and continue.

At the end of the induction, set

$$A_\alpha=\{x_\xi:\xi<\mathfrak{c},\,\alpha_\xi=\alpha\}.$$

Then the A_α are disjoint just because the x_ξ are distinct.

? Suppose, if possible, that H is not a measurable envelope of A_α for some α. Then $\mu_*(H\setminus A_\alpha)>0$ (413Ei), so there is a non-negligible measurable set $E\subseteq H\setminus A_\alpha$. Now there is an $F\in\mathcal{E}$ such that $F\subseteq E$. Let $\xi<\mathfrak{c}$ be such that $F=F_\xi$ and $\alpha=\alpha_\xi$; then $x_\xi\in A_\alpha\cap F$, which is impossible. **X**

Thus H is always a measurable envelope of A_α. It follows from the definition of 'measurable envelope' that $\mu^*A_\alpha=\mu H$. But also, if $\alpha<\mathfrak{c}$, $\mu_*A_\alpha\le\mu_*(H\setminus A_{\alpha+1})$, which is 0, as we have just seen. So we have a suitable family.

419J Example There is a complete probability space (X,Σ,μ) with a Hausdorff topology $\mathfrak{T}$ on X such that μ is τ-additive and inner regular with respect to the Borel sets, $\mathfrak{T}$ is generated by $\mathfrak{T}\cap\Sigma$, but μ has no extension to a topological measure.

proof (a) Set $Y=\omega_1+1=\omega_1\cup\{\omega_1\}$. Let T be the σ-algebra of subsets of Y generated by $\{\{\xi\}:\xi<\omega_1\}$. Let ν be the probability measure with domain T defined by the formula

$$\nu F=\frac{1}{2}\#(F\cap\{0,\omega_1\})\text{ for every }F\in\mathrm{T}.$$

Set

$$\mathfrak{S}=\{\emptyset,Y\}\cup\{H:0\in H\subseteq\omega_1\}.$$

This is a topology on Y, and every subset of Y is a Borel set for $\mathfrak{S}$; so ν is surely inner regular with respect to the Borel sets.

Note that

$$\{\{0,\alpha\}:\alpha<\omega_1\}\cup\{Y\}$$

is a base for $\mathfrak{S}$ included in T.

(b) Set $Z=Y^{\mathbb{N}}\times[0,1]$. Let λ be the product probability measure on Z when each copy of Y is given the measure ν and $[0,1]$ is given Lebesgue measure μ_L; let $\mathfrak{U}$ be the product topology when each copy of Y is given the topology $\mathfrak{S}$ and $[0,1]$ its usual topology. Let Λ be the domain of λ and Λ_0 the σ-algebra generated by sets of the form $\{(x,t):x(i)\in F,\,t\in[0,1],\,t<q\}$ where $i\in\mathbb{N}$, $F\in\mathrm{T}$ and $q\in\mathbb{Q}$; then ν is inner regular with respect to Λ_0 (see 254Ff). Note that $\mathfrak{U}\cap\Lambda_0$ is a base for $\mathfrak{U}$ because $\mathfrak{S}\cap\mathrm{T}$ is a base for $\mathfrak{S}$, and λ is inner regular with respect to the $\mathfrak{U}$-Borel sets (412Uc, or otherwise).

Define $\phi:\{0,1\}^{\mathbb{N}}\to Y^{\mathbb{N}}$ by setting

$$\begin{aligned}\phi(u,t)(n)&=0\text{ if }u(n)=0,\\&=\omega_1\text{ if }u(n)=1,\end{aligned}$$

and set $\psi(u,t)=(\phi(u),t)$ for $u\in\{0,1\}^{\mathbb{N}}$, $t\in[0,1]$. Then ψ is continuous, just because $\mathfrak{U}$ is a product topology. Let $\nu_\omega\times\mu_L$ be the product measure on $\{0,1\}^{\mathbb{N}}\times[0,1]$; then ψ is inverse-measure-preserving for $\nu_\omega\times\mu_L$ and λ, by 254H.

If $V\in\Lambda_0$ there is an $\alpha<\omega_1$ such that

if $x,\,y\in Y^{\mathbb{N}}$, $t\in[0,1]$ and $y(i)=x(i)$ whenever $\min(x(i),y(i))<\alpha$, then $(x,t)\in V$ iff $(y,t)\in V$.

P Let $\mathcal{W}$ be the family of sets $V\subseteq Z$ with this property. Then $\mathcal{W}$ is a σ-algebra of subsets of Z containing every measurable cylinder, so includes Λ_0. **Q**

(c) $\#(\Lambda_0)\le\mathfrak{c}$. **P** Set

$$A_{i\xi}=\{(x,t):x\in Y^{\mathbb{N}},\,t\in[0,1],\,x(i)=\xi\},\quad A'_q=\{(x,t):x\in Y^{\mathbb{N}},\,t\in[0,1],\,t\le q\};$$

for $i\in\mathbb{N}$, $\xi<\omega_1$ and $q\in\mathbb{Q}$, and

$$\mathcal{A}=\{A_{i\xi}:i\in\mathbb{N},\,\xi<\omega_1\}\cup\{A'_q:q\in\mathbb{Q}\}.$$

Then Λ_0 is the σ-algebra of subsets of Z generated by $\mathcal{A}$, and $\#(\mathcal{A}) = \omega_1 \le \mathfrak{c}$, so $\#(\Lambda_0) \le \mathfrak{c}$ (4A1O). **Q**

(d) There is a family $\langle z_\xi \rangle_{\xi<\mathfrak{c}}$ in Z such that

(α) whenever $W \in \Lambda$ and $\lambda W > 0$ there is a $\xi < \mathfrak{c}$ such that $z_\xi \in W$ and $\lambda(H \cap W) > 0$ whenever H is a measurable open subset of Z containing z_ξ,

(β) setting $z_\xi = (x_\xi, t_\xi)$ for each ξ, there is for every $\xi < \mathfrak{c}$ a $j \in \mathbb{N}$ such that $0 < x_\xi(j) < \omega_1$,

(γ) $t_\xi \ne t_\eta$ if $\eta < \xi < \mathfrak{c}$.

P By 4A3Fa, the set of closed subsets of $\{0,1\}^{\mathbb{N}} \times [0,1]$ has cardinal at most $\mathfrak{c}$, so there is a family $\langle (K_\xi, V_\xi) \rangle_{\xi<\mathfrak{c}}$ running over all pairs (K, V) such that $V \in \Lambda_0$ and $K \subseteq \psi^{-1}[V]$ is a non-negligible compact set. Choose $\langle (x_\xi, t_\xi) \rangle_{\xi<\mathfrak{c}}$ inductively, as follows. Given $\langle t_\eta \rangle_{\eta<\xi}$ where $\xi < \mathfrak{c}$, then

$$\{t : t \in [0,1],\, \nu_\omega K_\xi^{-1}[\{t\}] > 0\}$$

is a non-negligible measurable subset of $[0,1]$, so has cardinal $\mathfrak{c}$ (419H); let t_ξ be a point of this set distinct from every t_η for $\eta < \xi$. Now Lemma 345E tells us that there are points u, $u' \in K_\xi^{-1}[\{t_\xi\}]$ which differ at exactly one coordinate $j \in \mathbb{N}$; suppose that $u(j) = 1$ and $u'(j) = 0$.

Let $\alpha < \omega_1$ be such that if x, $y \in Y^{\mathbb{N}}$, $t \in [0,1]$ and $y(i) = x(i)$ whenever $\min(x(i), y(i)) < \alpha$, then $(x,t) \in V_\xi$ iff $(y,t) \in V_\xi$. Define $x_\xi \in Y^{\mathbb{N}}$ by setting $x_\xi(j) = \alpha$ and $x_\xi(i) = \phi(u)(i)$ for $i \ne j$. Then $z_\xi = (x_\xi, t_\xi)$ belongs to V_ξ. If $H \subseteq Z$ is any open set containing z_ξ, we have a sequence $\langle H_i \rangle_{i\in\mathbb{N}}$ in $\mathfrak{S}$ such that $x_\xi \in \prod_{i\in\mathbb{N}} H_i$ and $\prod_{i\in\mathbb{N}} H_i \times \{t_\xi\} \subseteq H$; now $H_j \ne \emptyset$ so $0 \in H_j$ and $\phi(u') \in \prod_{i\in\mathbb{N}} H_i$, so that $(u', t_\xi) \in K_\xi \cap \psi^{-1}[H]$. Continue.

The construction ensures that (β) and (γ) are satisfied. Now, if $\lambda W > 0$, let $V \in \Lambda_0$ be such that $V \subseteq W$ and $\lambda V > 0$. In this case, $(\nu_\omega \times \mu_L)(\psi^{-1}[V]) > 0$; let $K \subseteq \psi^{-1}[V]$ be a self-supporting non-negligible compact set. Let $\xi < \mathfrak{c}$ be such that $(K, V) = (K_\xi, V_\xi)$. Then $z_\xi \in V_\xi = V \subseteq W$. If H is a measurable open subset of Z containing z_ξ, then $K \cap \psi^{-1}[H]$ is not empty; as ψ is continuous and inverse-measure-preserving and K is self-supporting,

$$0 < (\nu_\omega \times \mu_L)(K \cap \psi^{-1}[H]) \le (\nu_\omega \times \mu_L)\psi^{-1}[V \cap H] = \lambda(V \cap H) \le \lambda(W \cap H).$$

So (α) is satisfied. **Q**

(e) Set $X = \{z_\xi : \xi < \mathfrak{c}\}$ and let μ be the subspace measure on X induced by λ; let $\mathfrak{T}$ be the subspace topology on X.

(i) λ is complete, so μ also is. Next, $\mu X = \lambda^* X = 1$. **P?** Otherwise, there is a $W \in \Lambda$ such that $\lambda W > 0$ and $X \cap W = \emptyset$. But we know that there is now some $\xi < \mathfrak{c}$ such that $z_\xi \in W$. **XQ**

(ii) $\mathfrak{T}$ is Hausdorff because the projection from X to $[0,1]$ is injective and continuous. $\mathfrak{T}$ is generated by $\mathfrak{T} \cap \Sigma$ because $\mathfrak{U}$ is generated by $\mathfrak{U} \cap \Lambda$. μ is inner regular with respect to the Borel sets because λ is (412Pb).

(iii) μ is τ-additive. **P?** Suppose, if possible, otherwise. Then there is an upwards-directed family $\mathcal{G}$ of measurable open subsets of X such that $G^* = \bigcup \mathcal{G}$ is measurable and $\mu G^* > \sup_{G\in\mathcal{G}} \mu G$. Let $\mathcal{H}$ be the family of sets $H \in \Lambda \cap \mathfrak{U}$ such that $H \cap X$ is included in some member of $\mathcal{G}$; because $\mathfrak{U}$ is generated by $\mathfrak{U} \cap \Lambda$, $G^* = W \cap X$, where $W = \bigcup \mathcal{H}$. At the same time, there is a $V \in \Lambda$ such that $G^* = X \cap V$.

Because $\mathcal{G}$ is upwards-directed, so is $\mathcal{H}$. Because X has full outer measure,

$$\sup_{H\in\mathcal{H}} \lambda H = \sup_{H\in\mathcal{H}} \mu(X \cap H) \le \sup_{G\in\mathcal{G}} \mu G < \mu G^* = \lambda V.$$

Let $\langle H_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{H}$ such that $\sup_{n\in\mathbb{N}} \lambda H_n = \sup_{H\in\mathcal{H}} \lambda H$, and set $W_0 = \bigcup_{n\in\mathbb{N}} H_n$; then $\lambda W_0 < \lambda V$ and $\lambda(H \setminus W_0) = 0$ for every $H \in \mathcal{H}$. However, $\lambda(V \setminus W_0) > 0$, so there is a $z \in X \cap V \setminus W_0$ such that $\lambda(H \cap V \setminus W_0) > 0$ for every measurable open set H containing z. As $z \in X \cap V = X \cap W$, there must be an $H \in \mathcal{H}$ containing z, so this is impossible. **XQ**

(iv) **?** Suppose, if possible, that there is a topological measure $\tilde{\mu}$ on X agreeing with μ on every open set in the domain of μ. For each $i \in \mathbb{N}$, set $\pi_i(x) = x(i)$ for $(x,t) \in X$. Every subset of Y is a Borel set for $\mathfrak{S}$; because π_i is continuous for $\mathfrak{T}$ and $\mathfrak{S}$, the image measure $\tilde{\mu}\pi_i^{-1}$ has domain $\mathcal{P}Y$. Now $\#(Y) = \omega_1$, so there must be a countable conegligible set (419G), and there must be some $\alpha_i < \omega_1$ such that $\tilde{\mu}\pi_i^{-1}(\omega_1 \setminus \alpha_i) = 0$. On the other hand,

$$\tilde{\mu}\pi_i^{-1}(\alpha_i \setminus \{0\}) = \mu\pi_i^{-1}(\alpha_i \setminus \{0\}) = \lambda\{(x,t) : 0 < x(i) < \alpha_i\} = \nu(\alpha_i \setminus \{0\}) = 0,$$

so $\tilde{\mu}\pi_i^{-1}(\omega_1 \setminus \{0\}) = 0$.

But (d-β) ensures that

$$X = \bigcup_{i\in\mathbb{N}} \pi_i^{-1}(\omega_1 \setminus \{0\}),$$

so this is impossible. **X**

Thus we have the required example.

Remark I note that the topology of X is not regular. Of course the phenomenon here cannot arise with regular spaces, by 415M.

419K Example (Blackwell 56) There are sequences $\langle X_n\rangle_{n\in\mathbb{N}}$, $\langle\mathfrak{T}_n\rangle_{n\in\mathbb{N}}$ and $\langle\nu_n\rangle_{n\in\mathbb{N}}$ such that (i) for each n, $(X_n, \mathfrak{T}_n)$ is a separable metrizable space and ν_n is a quasi-Radon probability measure on $Z_n = \prod_{i\le n} X_i$ (ii) for $m \le n$ the canonical map $\pi_{mn} : Z_n \to Z_m$ is inverse-measure-preserving (iii) there is no probability measure on $Z = \prod_{i\in\mathbb{N}} X_i$ such that all the canonical maps from Z to Z_n are inverse-measure-preserving.

proof Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of subsets of $[0,1]$ such that $\mu_*([0,1]\setminus A_n) = 0$, that is, $\mu^*A_n = 1$ for every n, where μ is Lebesgue measure (using 419I). Set $X_n = \bigcup_{i\ge n} A_i$, so that $\langle X_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets of outer measure 1 with empty intersection. For each $n \ge 1$, we have a map $f_n : X_n \to Z_n$ defined by setting $f_n(x)(i) = x$ for every $i \le n$, $x \in X_n$. Let ν_n be the image measure $\mu_{X_n} f_n^{-1}$, where μ_{X_n} is the subspace measure on X_n induced by μ. Note that f_n is a homeomorphism between X_n and the diagonal $\Delta_n = \{z : z \in Z_n,\ z(i) = z(j) \text{ for all } i,\ j \le n\}$, which is a closed subset of Z_n; so that ν_n, like μ_{X_n}, is a quasi-Radon probability measure.

If $m \le n$, then π_{mn} is inverse-measure-preserving, where $\pi_{mn}(z)(i) = z(i)$ for $z \in Z_n$ and $i \le m$. **P** If $W \subseteq Z_m$ is measured by ν_m, then $f_m^{-1}[W]$ is measured by μ_{X_m}, so is of the form $X_m \cap E$ where E is Lebesgue measurable. But in this case $f_n^{-1}[\pi_{mn}^{-1}[W]] = X_n \cap E$, so that

$$\nu_n(\pi_{mn}^{-1}[W]) = \mu_{X_n}(f_n^{-1}[\pi_{mn}^{-1}[W]]) = \mu^*(X_n \cap E) = \mu E = \mu^*(X_m \cap E) = \nu_m W. \quad \mathbf{Q}$$

? But suppose, if possible, that there is a probability measure ν on $Z = \prod_{i\in\mathbb{N}} X_i$ such that $\pi_n : Z \to Z_n$ is inverse-measure-preserving for every n, where $\pi_n(z)(i) = z(i)$ for $z \in Z$ and $i \le n$. Then

$$\nu\pi_n^{-1}[\Delta_n] = \nu_n\Delta_n = \mu_{X_n} f_n^{-1}[\Delta_n] = 1$$

for each n, so

$$1 = \nu(\bigcap_{n\in\mathbb{N}} \pi_n^{-1}[\Delta_n]) = \nu\{z : z \in Z,\ z(i) = z(j) \text{ for all } i,\ j \in \mathbb{N}\} = \nu\emptyset,$$

because $\bigcap_{n\in\mathbb{N}} X_n = \emptyset$; which is impossible. **X**

419L The split interval again (a) For the sake of an example in §343, I have already introduced the 'split interval' or 'double arrow space'. As this construction gives us a topological measure space of great interest, I repeat it here. Let $I^{\|}$ be the set $\{a^+ : a \in [0,1]\} \cup \{a^- : a \in [0,1]\}$. Order it by saying that

$$a^+ \le b^+ \iff a^- \le b^+ \iff a^- \le b^- \iff a \le b, \quad a^+ \le b^- \iff a < b.$$

Then it is easy to check that $I^{\|}$ is a totally ordered space, and that it is Dedekind complete. (If $A \subseteq [0,1]$ is a non-empty set, then $\sup_{a\in A} a^- = (\sup A)^-$, while $\sup_{a\in A} a^+$ is either $(\sup A)^+$ or $(\sup A)^-$, depending on whether $\sup A$ belongs to A or not.) Its greatest element is 1^+ and its least element is 0^-. Consequently the order topology on $I^{\|}$ is a compact Hausdorff topology (4A2Ri, Alexandroff & Urysohn 1929). Note that $Q = \{q^+ : q \in [0,1] \cap \mathbb{Q}\} \cup \{q^- : q \in [0,1] \cap \mathbb{Q}\}$ is dense, because it meets every non-trivial interval in $I^{\|}$. By 4A2E(a-ii) and 4A2Rn, $I^{\|}$ is ccc and hereditarily Lindelöf.

(b) If we define $h : I^{\|} \to [0,1]$ by writing $h(a^+) = h(a^-) = a$ for every $a \in [0,1]$, then h is continuous, because $\{x : h(x) < a\} = \{x : x < a^-\}$, $\{x : h(x) > a\} = \{x : x > a^+\}$ for every $a \in [0,1]$. Now we can describe the Borel sets of $I^{\|}$, as follows: a set $E \subseteq I^{\|}$ is Borel iff there is a Borel set $F \subseteq [0,1]$ such that $E\triangle h^{-1}[F]$ is countable. **P** Write Σ_0 for the family of subsets E of $I^{\|}$ such that $E\triangle h^{-1}[F]$ is countable for some Borel set $F \subseteq [0,1]$. It is easy to check that Σ_0 is a σ-algebra of subsets of $I^{\|}$. (If $E\triangle h^{-1}[F]$ is countable, so is $(I^{\|} \setminus E)\triangle h^{-1}[[0,1] \setminus F]$; if $E_n\triangle h^{-1}[F_n]$ is countable for every n, so is $(\bigcup_{n\in\mathbb{N}} E_n)\triangle h^{-1}[\bigcup_{n\in\mathbb{N}} F_n]$.) Because the topology of $I^{\|}$ is Hausdorff, every singleton set is closed, so every countable set is Borel. Also $h^{-1}[F]$ is Borel for every Borel set $F \subseteq [0,1]$, because h is continuous (4A3Cd). So if $E \triangle h^{-1}[F]$ is countable for some Borel set $F \subseteq [0,1]$, $E = h^{-1}[F] \triangle (E \triangle h^{-1}[F])$ is a Borel set in $I^{\parallel}$. Thus Σ_0 is included in the Borel σ-algebra $\mathcal{B}$ of $I^{\parallel}$. On the other hand, if $J \subseteq I^{\parallel}$ is an interval, $h[J]$ also is an interval, therefore a Borel set, and $h^{-1}[h[J]] \setminus J$ can contain at most two points, so $J \in \Sigma_0$. If $G \subseteq I^{\parallel}$ is open, it is expressible as $\bigcup_{i \in I} J_i$, where $\langle J_i \rangle_{i \in I}$ is a disjoint family of non-empty open intervals (4A2Rj). As X is ccc, I must be countable. Thus G is expressed as a countable union of members of Σ_0 and belongs to Σ_0. But this means that the Borel σ-algebra $\mathcal{B}$ must be included in Σ_0, by the definition of 'Borel algebra'. So $\mathcal{B} = \Sigma_0$, as claimed. **Q**

(c) In 343J I described the standard measure μ on $I^{\parallel}$; its domain is the set $\Sigma = \{h^{-1}[F] \triangle M : F \in \Sigma_L,\ M \subseteq I^{\parallel},\ \mu_L h[M] = 0\}$, where Σ_L is the set of Lebesgue measurable subsets of $[0,1]$ and μ_L is Lebesgue measure, and $\mu E = \mu_L h[E]$ for $E \in \Sigma$. h is inverse-measure-preserving for μ and μ_L.

The new fact I wish to mention is: μ is a completion regular Radon measure. **P** I noted in 343Ja that it is a complete probability measure; *a fortiori*, it is locally determined and locally finite. If $G \subseteq I^{\parallel}$ is open, then we can express it as $h^{-1}[F] \triangle C$ for some Borel set $F \subseteq [0,1]$ and countable $C \subseteq I^{\parallel}$ ((b) above), so it belongs to Σ; thus μ is a topological measure. If $E \in \Sigma$ and $\mu E > \gamma$, then $F = [0,1] \setminus h[I^{\parallel} \setminus E]$ is Lebesgue measurable, and $\mu E = \mu_L F$. So there is a compact set $L \subseteq F$ such that $\mu_L L \geq \gamma$. But now $K = h^{-1}[L] \subseteq E$ is closed, therefore compact, and $\mu K \geq \gamma$. Moreover, L is a zero set, being a closed set in a metrizable space (4A2Lc), so K is a zero set (4A2C(b-iv)). As E and γ are arbitrary, μ is inner regular with respect to the compact zero sets, and is a completion regular Radon measure. **Q**

419X Basic exercises (a) Show that the topological space X of 419A is zero-dimensional.

(b) Give an example of a compact Radon probability space in which every dense G_δ set is conegligible. (*Hint*: 411P.)

(c) In 419E, show that we can start from any atomless probability measure in place of Lebesgue measure on $[0,1]$.

>(d)(i) Show that if $E \subseteq \mathbb{R}^2$ is Lebesgue measurable, with non-zero measure, then it cannot be covered by fewer than $\mathfrak{c}$ lines. (*Hint*: if $H = \{t : \mu_1 E[\{t\}] > 0\}$, where μ_1 is Lebesgue measure on $\mathbb{R}$, then $\mu_1 H > 0$, so $\#(H) = \mathfrak{c}$. So if we have a family $\mathcal{L}$ of lines, with $\#(\mathcal{L}) < \mathfrak{c}$, there must be a $t \in H$ such that $L_t = \{t\} \times \mathbb{R}$ does not belong to $\mathcal{L}$. Now $\#(L_t \cap E) = \mathfrak{c}$ and each member of $\mathcal{L}$ meets $L_t \cap E$ in at most one point.) (ii) Show that there is a subset A of $\mathbb{R}^2$, of full outer measure, which meets every vertical line and every horizontal line in exactly one point. (*Hint*: enumerate $\mathbb{R}$ as $\langle t_\xi \rangle_{\xi<\mathfrak{c}}$ and the closed sets of non-zero measure as $\langle F_\xi \rangle_{\xi<\mathfrak{c}}$. Choose $\langle I_\xi \rangle_{\xi<\mathfrak{c}}$ such that every I_ξ is finite, no two points of $I_\xi \cup \bigcup_{\eta<\xi} I_\eta$ lie on any horizontal or vertical line, the lines $\{t_\xi\} \times \mathbb{R}$ and $\mathbb{R} \times \{t_\xi\}$ both meet $I_\xi \cup \bigcup_{\eta<\xi} I_\eta$, and I_ξ meets F_ξ.) (iii) Show that there is a subset B of $\mathbb{R}^2$, of full outer measure, such that every straight line meets B in exactly two points. (*Hint*: enumerate the straight lines in $\mathbb{R}^2$ as $\langle L_\xi \rangle_{\xi<\mathfrak{c}}$. Choose $\langle J_\xi \rangle_{\xi<\mathfrak{c}}$ such that every J_ξ is finite, no three points of $J_\xi \cup \bigcup_{\eta<\xi} J_\eta$ lie on any line, $L_\xi \cap (J_\xi \cup \bigcup_{\eta<\xi} J_\eta)$ has just two points and $J_\xi \cap F_\xi \neq \emptyset$.)

(e) Show that there is a subset A of the Cantor set C (134G) such that $A+A$ is not Lebesgue measurable. (*Hint*: enumerate the closed non-negligible subsets of $C + C = [0,2]$ as $\langle F_\xi \rangle_{\xi<\mathfrak{c}}$. Choose $x_\xi \in C$, $y_\xi \in C$, $z_\xi \in F_\xi$ so that $z_\xi \notin A_\xi + A_\xi$ and $x_\xi + y_\xi \in F_\xi$ and $A_{\xi+1} + A_{\xi+1}$ does not meet $\{z_\eta : \eta \leq \xi\}$, where $A_\xi = \{x_\eta : \eta < \xi\} \cup \{y_\eta : \eta < \xi\}$.)

(f) Let $\mathbb{R}^{\parallel}$ be the **split line**, that is, the set $\{a^+ : a \in \mathbb{R}\} \cup \{a^- : a \in \mathbb{R}\}$, ordered by the rules in 419L. Show that $\mathbb{R}^{\parallel}$ is a Dedekind complete totally ordered set, so that its order topology $\mathfrak{T}$ is locally compact. Write μ_L for Lebesgue measure on $\mathbb{R}$ and Σ_L for its domain. Set $h(a^+) = h(a^-) = a$ for $a \in \mathbb{R}$, $\Sigma = \{E : E \subseteq X,\ h[E] \in \Sigma_L,\ \mu_L(h[E] \cap h[X \setminus E]) = 0\}$, $\mu E = \mu_L h[E]$ for $E \in \Sigma$. Show that μ is a completion regular Radon measure on $\mathbb{R}^{\parallel}$ and that h is continuous and inverse-measure-preserving for μ and μ_L. Show that the set $\{a^+ : a \in \mathbb{R}\}$, with the induced topology and measure, is isomorphic, as quasi-Radon measure space, to the Sorgenfrey line (415Xc) with Lebesgue measure. Show that $\mathbb{R}^{\parallel}$ and the Sorgenfrey line are hereditarily Lindelöf.

(g) Let μ be Lebesgue measure on $[0,1]$ and Σ its domain. Let $I^{\|}$ be the split interval. (i) Show that the functions $x \mapsto x^+ : [0,1] \to I^{\|}$ and $x \mapsto x^- : [0,1] \to I^{\|}$ are measurable. (*Hint*: 419Lb.) (ii) Show that the function $x \mapsto (x^+, x^-) : [0,1] \to (I^{\|})^2$ is not measurable. (*Hint*: the subspace topology on $\{(x^+, x^-) : x \in [0,1]\}$ is discrete.)

>(h)(i) Again writing $I^{\|}$ for the split interval, show that the function which exchanges x^+ and x^- for every $x \in [0,1]$ is a Borel automorphism and an automorphism for the usual Radon measure ν on $I^{\|}$, but is not almost continuous. (ii) Show that if we set $f(x) = x^+$ for $x \in [0,1]$, then f is inverse-measure-preserving for Lebesgue measure μ_L on $[0,1]$, but the image measure $\mu_L f^{-1}$ is not ν (nor, indeed, a Radon measure).

(i) Show that the split interval $I^{\|}$ is perfectly normal, but that $I^{\|} \times I^{\|}$ is not perfectly normal.

419Y Further exercises (a) In the example of 419E, show that there is a Borel set $V \subseteq Z^2$ such that $\tilde{\lambda} V = 0$ and $\lambda^* V = 1$.

(b) Show that if $\mathcal{A} \subseteq \mathcal{P}\omega_1$ is any family with $\#(\mathcal{A}) \le \omega_1$, there is a countably generated σ-algebra Σ of subsets of ω_1 such that $\mathcal{A} \subseteq \Sigma$.

(c) Show that the split interval with its usual topology and measure has the simple product property (417Yi).

(d) Give an example of a Lebesgue measurable function $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $\operatorname{dom} \frac{\partial\phi}{\partial\xi_1}$ is not measurable. (Compare 222Yd, 225Yg, 262Yc.)

419 Notes and comments The construction of the locally compact space X in 419A from the family $\langle I_\xi \rangle_{\xi<\kappa}$ is a standard device which has been used many times. The relation $\subseteq^*$ also appears in many contexts. In effect, part of the argument is taking place in the quotient algebra $\mathfrak{A} = \mathcal{P}Q/[Q]^{<\omega}$, since $I \subseteq^* J$ iff $I^\bullet \subseteq J^\bullet$ in $\mathfrak{A}$; setting $\mathcal{I}^\# = \{I^\bullet : I \in \mathcal{I}\}$, the cardinal κ is $\min\{\#(A) : A \subseteq \mathcal{I}^\#$ has no upper bound in $\mathcal{I}^\#\}$, the 'additivity' of the partially ordered set $\mathcal{I}^\#$. Additivities of partially ordered sets will be one of the important concerns of Volume 5. I remark that we do not need to know whether (for instance) $\kappa = \omega_1$ or $\kappa = \mathfrak{c}$. This is an early taste of the kind of manoeuvre which has become a staple of set-theoretic analysis. It happens that the cardinal κ here is one of the most important cardinals of set-theoretic measure theory; it is 'the additivity of Lebesgue measure' (529Xe[11]), and under that name will appear repeatedly in Volume 5.

Observe that the measure μ of 419A only just fails to be a quasi-Radon measure; it is locally finite instead of being effectively locally finite. And it would be a Radon measure if it were inner regular with respect to the compact sets, rather than just with respect to the closed sets.

419C and 419D are relevant to the question: have I given the 'right' definition of Radon measure space? 419C is perhaps more important. Here we have a Radon measure space (on my definition) for which the associated Borel measure is not localizable. (If $\mathfrak{A}$ is the measure algebra of the measure μ, and $\mathfrak{B}$ the measure algebra of $\mu\restriction\mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of X, then the embedding $\mathcal{B} \subsetneqq \Sigma$ induces an embedding of $\mathfrak{B}$ in $\mathfrak{A}$ which represents $\mathfrak{B}$ as an order-dense subalgebra of $\mathfrak{A}$, just because μ is inner regular with respect to $\mathcal{B}$. Property (i) of 419C shows that $\mathfrak{B} \ne \mathfrak{A}$, so $\mathfrak{B}$ cannot be Dedekind complete in itself, by 314Ib.) Since (I believe) localizable versions of measure spaces should almost always be preferred, I take this as strong support for my prejudice in favour of insisting that 'Radon' measure spaces should be locally determined as well as complete. Property (ii) of 419C is not I think of real significance, but is further evidence, to be added to 415Xi, that outer regularity is like an exoskeleton: it may inhibit growth above a certain size.

In 419D I explore the consequences of omitting the condition 'locally finite' from the definition of Radon measure. Even if we insist instead that compact sets should have finite measure, we are in danger of getting a non-localizable measure. Of course this particular space is pathological in terms of most of the criteria of this chapter – for instance, every non-empty open set has infinite measure, and the topology is not regular.

[11] Formerly 529Xc.

Perhaps the most important example in the section is 419E. The analysis of τ-additive product measures in §417 was long and difficult, and if these were actually equal to the familiar product measures in all important cases the structure of the theory would be very different. But we find that for one of the standard compact Radon probability spaces of the theory, the c.l.d. product measure on its square is not a Radon measure, and something has to be done about it.

I present 419J here to indicate one of the obstacles to any simplification of the arguments in 417C and 417E. It is not significant in itself, but it offers a welcome excuse to describe some fundamental facts about ω_1 (419F-419G). Similarly, 419K asks for some elementary facts about Lebesgue measure (419H-419I) which seem to have got left out. This example really is important in itself, as it touches on the general problem of representing stochastic processes, to which I will return in Chapter 45.

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Concordance to Chapter 41

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

411I Completion regular measures The definition of 'completion regular' measure, referred to in Fremlin 00, has been moved to 411J.

412L Uniqueness of measures Corollary 412L, referred to in the 2008 and 2015 printings of Volume 5, is now 412Ma.

413R Countably compact classes Lemma 413R, referred to in Fremlin 00, is now 413T.

413I Inner measure The constructions in 413H and 413J, referred to in König p09b and the 2008 and 2015 editions of Volume 5, are now 413I and 413K.

413Yf Uniform exhaustivity Exercise 413Yf, referred to in the 2008 and 2015 editions of Volume 5, is now 413Yh.

414N Density topologies The note on 'density topologies', referred to in the 2001 edition of Volume 2, has been moved to 414P.

415Xp This exercise, referred to in the 2008 and 2015 printings of Volume 5, has been moved to 415Xs.

415Yd Sorgenfrey line This exercise, referred to in the 2001 edition of Volume 2, has been moved to 415Ye.

416M Henry's theorem, mentioned in König 04, has been re-named 416N.

416P The algebra of open-and-closed subsets This paragraph, referred to in the 2002 edition of Volume 3, has been moved to 416Q.

416T Kakutani's theorem The description of the usual measure on $\{0,1\}^\kappa$, referred to in Fremlin & Plebanek 03, has been moved to 416U.

417E τ-additive product measures The reference in Fremlin 00 to Kakutani's theorem that the product measure on $\{0,1\}^I$ is completion regular should be directed to 415E or 416U rather than 417E.

419H The example 419H of a measure, inner regular with respect to the Borel sets but with no extension to a topological measure, mentioned in König p09, is now 419J.

419J Partitions into sets of full outer measure Lemma 419J, mentioned in the 2004 edition of Volume 1, has been moved to 419I.

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