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Appendix to Volume 3

Useful Facts

This volume assumes a fairly wide-ranging competence in analysis, a solid understanding of elementary set theory and some straightforward Boolean algebra. As in previous volumes, I start with a few pages of revision in set theory, but the absolutely essential material is in §3A2, on commutative rings, which is the basis of the treatment of Boolean rings in §311. I then give three sections of results in analysis: topological spaces (§3A3), uniform spaces (§3A4) and normed spaces (§3A5). Finally, I add six sentences on group theory (§3A6).

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3A1 Set Theory

3A1B Definition Let X be a set. By an **enumeration** of X I mean a bijection $f:\kappa\to X$ where $\kappa=\#(X)$; more often than not I shall express such a function in the form $\langle x_\xi\rangle_{\xi<\kappa}$. In this case I say that the function f, or the family $\langle x_\xi\rangle_{\xi<\kappa}$, **enumerates** X.

3A1C Calculation of cardinalities

(a) For any sets X and Y, $\#(X\times Y)\le\max(\omega,\#(X),\#(Y))$.

(b) For any $r\in\mathbb{N}$ and any family $\langle X_i\rangle_{i\le r}$ of sets, $\#(\prod_{i=0}^r X_i)\le\max(\omega,\max_{i\le r}\#(X_i))$.

(c) For any family $\langle X_i\rangle_{i\in I}$ of sets, $\#(\bigcup_{i\in I}X_i)\le\max(\omega,\#(I),\sup_{i\in I}\#(X_i))$.

(d) For any set X, $[X]^{<\omega}$ has cardinal at most $\max(\omega,\#(X))$.

3A1D Cardinal exponentiation For a cardinal κ, I write 2^κ for $\#(\mathcal{P}\kappa)$. So $2^\omega=\mathfrak{c}$, and $\kappa^+\le 2^\kappa$ for every κ.

3A1E Definition $\omega_0=\omega$, $\omega_{\xi+1}=\omega_\xi^+$ for every ξ, $\omega_\xi=\bigcup_{\eta<\xi}\omega_\eta$ for non-zero limit ordinals ξ.

3A1F Cofinal sets (a) If P is any partially ordered set, a subset Q of P is **cofinal** with P if for every $p\in P$ there is a $q\in Q$ such that $p\le q$.

(b) If P is any partially ordered set, the **cofinality** of P, $\operatorname{cf}P$, is the least cardinal of any cofinal subset of P. $\operatorname{cf}P=0$ iff $P=\emptyset$, and $\operatorname{cf}P=1$ iff P has a greatest element.

(c) Observe that if P is upwards-directed and $\operatorname{cf}P$ is finite, then $\operatorname{cf}P$ is either 0 or 1.

(d) If P is a totally ordered set of cofinality κ, then there is a strictly increasing family $\langle p_\xi\rangle_{\xi<\kappa}$ in P such that $\{p_\xi:\xi<\kappa\}$ is cofinal with P.

(e) In particular, for a totally ordered set P, $\operatorname{cf}P=\omega$ iff there is a cofinal strictly increasing sequence in P.

3A1I Definitions (a) If P and Q are lattices, a **lattice homomorphism** from P to Q is a function $f : P \to Q$ such that $f(p \wedge p') = f(p) \wedge f(p')$ and $f(p \vee p') = f(p) \vee f(p')$ for all p, $p' \in P$. Such a homomorphism is order-preserving.

(b) If P is a lattice, a **sublattice** of P is a set $Q \subseteq P$ such that $p \vee q$ and $p \wedge q$ belong to Q for all p, $q \in Q$.

(c)(i) A lattice P is **distributive** if

$$(p \wedge q) \vee r = (p \vee r) \wedge (q \vee r), \quad (p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$$

for all p, q, $r \in P$.

(ii) In a distributive lattice we have a **median function** of three variables

$$\operatorname{med}(p, q, r) = (p \wedge q) \vee (p \wedge r) \vee (q \wedge r).$$

If P and Q are distributive lattices and $f : P \to Q$ is a lattice homomorphism, $f(\operatorname{med}(p, q, r)) = \operatorname{med}(f(p), f(q), f(r))$ for all p, q, $r \in P$.

(iii) If P is a distributive lattice and $I \subseteq P$ is finite, then the sublattice of P generated by I is finite.

3A1J Subsets of given size If X is a set and κ is a cardinal, write

$$[X]^\kappa = \{A : A \subseteq X, \#(A) = \kappa\},$$

$$[X]^{\le\kappa} = \{A : A \subseteq X, \#(A) \le \kappa\},$$

$$[X]^{<\kappa} = \{A : A \subseteq X, \#(A) < \kappa\}.$$

3A1K Hall's Marriage Lemma Suppose that X and Y are finite sets and $R \subseteq X \times Y$ is a relation such that $\#(R[I]) \ge \#(I)$ for every $I \subseteq X$. Then there is an injective function $f : X \to Y$ such that $(x, f(x)) \in R$ for every $x \in X$.

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3A2 Rings

I give a very brief outline of the indispensable parts of the elementary theory of (commutative) rings. I assume that you have seen at least a little group theory.

3A2A Definition A **ring** is a triple $(R, +, .)$ such that

$(R, +)$ is an abelian group; its identity will be denoted 0 or 0_R;

$(R, .)$ is a semigroup, that is, $ab \in R$ for all a, $b \in R$ and $a(bc) = (ab)c$ for all a, b, $c \in R$;

$a(b + c) = ab + ac$, $(a + b)c = ac + bc$ for all a, b, $c \in R$.

A **commutative ring** is one in $ab = ba$ for all a, $b \in R$.

3A2B Elementary facts Let R be a ring.

(a) $a0 = 0a = 0$ for every $a \in R$.

(b) $(-a)b = a(-b) = -(ab)$ for all a, $b \in R$.

3A2C Subrings If R is a ring, a **subring** of R is a set $S \subseteq R$ such that $0 \in S$ and $a + b$, ab, $-a$ belong to S for all a, $b \in S$. In this case S, together with the addition and multiplication induced by those of R, is a ring in its own right.

3A2D Homomorphisms (a) Let R, S be two rings. A function $\phi : R \to S$ is a **ring homomorphism** if $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all a, $b \in R$. The **kernel** of ϕ is $\{a : a \in R, \phi(a) = 0_S\}$.

(b) Note that if $\phi : R \to S$ is a ring homomorphism, then it is also a group homomorphism from $(R, +)$ to $(S, +)$, so that $\phi(0_R) = 0_S$ and $\phi(-a) = -\phi(a)$ for every $a \in R$; moreover, $\phi[R]$ is a subring of S, and ϕ is injective iff its kernel is $\{0_R\}$.

(c) If R, S and T are rings, and $\phi : R \to S$, $\psi : S \to T$ are ring homomorphisms, then $\psi\phi : R \to T$ is a ring homomorphism. If ϕ is bijective, then $\phi^{-1} : S \to R$ is a ring homomorphism.

3A2E Ideals (a) Let R be a ring. An **ideal** of R is a subring I of R such that $ab \in I$ and $ba \in I$ whenever $a \in I$ and $b \in R$. In this case we write $I \lhd R$.

(b) If R and S are rings and $\phi : R \to S$ is a ring homomorphism, then the kernel I of ϕ is an ideal of R.

3A2F Quotient rings (a) Let R be a ring and I an ideal of R. A **coset** of I is a set of the form $a + I = \{a + x : x \in I\}$ where $a \in R$. Let R/I be the set of cosets of I in R.

(b) For A, $B \in R/I$, set

$$A + B = \{x + y : x \in A, y \in B\}, \quad A \cdot B = \{xy + z : x \in A, y \in B, z \in I\}.$$

Then $A + B$, $A \cdot B$ both belong to R/I; moreover, if $A = a + I$ and $B = b + I$, then $A + B = (a + b) + I$ and $A \cdot B = ab + I$.

(c) $(R/I, +, \cdot)$ is a ring, with zero $0 + I = I$ and additive inverses $-(a + I) = (-a) + I$.

(d) Moreover, the map $a \mapsto a + I : R \to R/I$ is a ring homomorphism.

(e) Note that for a, $b \in R$, the following are equiveridical: (i) $a \in b+I$; (ii) $b \in a+I$; (iii) $(a+I) \cap (b+I) \neq \emptyset$; (iv) $a+I = b+I$; (v) $a-b \in I$. Thus the cosets of I are just the equivalence classes in R under the equivalence relation $a \sim b \iff a + I = b + I$; accordingly I shall generally write $a^\bullet$ for $a + I$. In particular, the kernel of the canonical map from R to R/I is just $\{a : a + I = I\} = I = 0^\bullet$.

(f) If R is commutative so is R/I.

3A2G Factoring homomorphisms through quotient rings: Proposition Let R and S be rings, I an ideal of R, and $\phi : R \to S$ a homomorphism such that I is included in the kernel of ϕ. Then we have a ring homomorphism $\pi : R/I \to S$ such that $\pi(a^\bullet) = \phi(a)$ for every $a \in R$. π is injective iff I is precisely the kernel of ϕ.

3A2H Product rings (a) Let $\langle R_i \rangle_{i \in I}$ be any family of rings. Set $R = \prod_{i \in I} R_i$ and for a, $b \in R$ define $a + b$, $ab \in R$ by setting

$$(a + b)(i) = a(i) + b(i), \quad (ab)(i) = a(i)b(i)$$

for every $i \in I$. R is a ring; its zero is given by the formula

$$0_R(i) = 0_{R_i} \text{ for every } i \in I,$$

and its additive inverses by the formula

$$(-a)(i) = -a(i) \text{ for every } i \in I.$$

(b) Now let S be any other ring. Then a function $\phi : S \to R$ is a ring homomorphism iff $s \mapsto \phi(s)(i) : S \to R_i$ is a ring homomorphism for every $i \in I$.

(c) R is commutative iff R_i is commutative for every i.

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3A3 General topology

In §2A3, I looked at a selection of topics in general topology in some detail, giving proofs; the point was that an ordinary elementary course in the subject would surely go far beyond what we needed there, and at the same time might omit some of the results I wished to quote. It seemed therefore worth taking a bit of space to cover the requisite material, giving readers the option of delaying a proper study of the subject until a convenient opportunity arose. In the context of the present volume, this approach is probably no longer appropriate, since we need a much greater proportion of the fundamental ideas, and by the time you have reached familiarity with the topics here you will be well able to find your way about one of the many excellent textbooks on the subject. This time round, therefore, I give most of the results without proofs (as in §§2A1 and 3A1), hoping that some of the references I offer will be accessible in all senses. I do, however, give a full set of definitions, partly to avoid ambiguity (since even in this relatively mature subject, there are some awkward divergences remaining in the usage of different authors), and partly because many of the proofs are easy enough for even a novice to fill in with a bit of thought, once the meaning of the words is clear.

3A3A Taxonomy of topological spaces: Definitions Let $(X,\mathfrak{T})$ be a topological space.

(a) X is $\mathbf{T_1}$ if singleton subsets of X are closed.

(b) X is **Hausdorff** if for any distinct points x, $y\in X$ there are disjoint open sets G, $H\subseteq X$ such that $x\in G$ and $y\in H$.

(c) X is **regular** if whenever $F\subseteq X$ is closed and $x\in X\setminus F$ there are disjoint open sets G, $H\subseteq X$ such that $x\in G$ and $F\subseteq H$.

(d) X is **completely regular** if whenever $F\subseteq X$ is closed and $x\in X\setminus F$ there is a continuous function $f:X\to[0,1]$ such that $f(x)=1$ and $f(y)=0$ for every $y\in F$.

(e) X is **zero-dimensional** if whenever $G\subseteq X$ is an open set and $x\in G$ then there is an open-and-closed set H such that $x\in H\subseteq G$.

(f) X is **extremally disconnected** if the closure of every open set in X is open.

(g) X is **compact** if every open cover of X has a finite subcover.

(h) X is **locally compact** if for every $x\in X$ there is a set $K\subseteq X$ such that $x\in\operatorname{int}K$ and K is compact.

(i) If every subset of X is open, we call $\mathfrak{T}$ the **discrete topology** on X.

3A3B Elementary relationships (a) A completely regular space is regular.

(b) A locally compact Hausdorff space is completely regular.

(c) A compact Hausdorff space is locally compact.

(d) A regular extremally disconnected space is zero-dimensional.

(e) Any topology defined by pseudometrics is completely regular.

(f) If X is a completely regular Hausdorff space and x, y are distinct points in X, then there is a continuous function $f:X\to\mathbb{R}$ such that $f(x)\neq f(y)$.

(g) An open set in a locally compact Hausdorff space is locally compact in its subspace topology.

(h) Any subspace of a Hausdorff space is Hausdorff; any subspace of a completely regular space is completely regular.

3A3C Continuous functions Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces.

(a) If $f : X \to Y$ is a function and $x \in X$, we say that f is **continuous at** x if $x \in \operatorname{int} f^{-1}[H]$ whenever $H \subseteq Y$ is an open set containing $f(x)$.

(b) Now a function from X to Y is continuous iff it is continuous at every point of X.

(c) If $f : X \to Y$ is continuous at $x \in X$, and $A \subseteq X$ is such that $x \in \overline{A}$, then $f(x) \in \overline{f[A]}$.

(d) If $f : X \to Y$ is continuous, then $f[\overline{A}] \subseteq \overline{f[A]}$ for every $A \subseteq X$.

(e) A function $f : X \to Y$ is a **homeomorphism** if it is a continuous bijection and its inverse is also continuous.

(f) A function $f : X \to [-\infty, \infty]$ is **lower semi-continuous** if $\{x : x \in X,\ f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$.

3A3D Compact spaces (a) A family $\mathcal{F}$ of sets has the **finite intersection property** if $\bigcap \mathcal{F}_0$ is non-empty for every finite $\mathcal{F}_0 \subseteq \mathcal{F}$. Now a topological space X is compact iff $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F}$ is a family of closed subsets of X with the finite intersection property.

(b) Let X be a topological space and $\mathcal{F}$ a family of closed subsets of X with the finite intersection property. If $\mathcal{F}$ contains a compact set then $\bigcap \mathcal{F} \neq \emptyset$.

(c) In a Hausdorff space, compact subsets are closed.

(d) If X is compact, Y is Hausdorff and $\phi : X \to Y$ is continuous and injective, then ϕ is a homeomorphism between X and $\phi[X]$.

(e) Let X be a regular topological space and A a subset of X. Then the following are equiveridical: (i) A is relatively compact in X; (ii) $\overline{A}$ is compact; (iii) every ultrafilter on X which contains A has a limit in X.

3A3E Dense sets (a) If X is a topological space, $D \subseteq X$ is dense and $G \subseteq X$ is dense and open, then $G \cap D$ is dense. Consequently the intersection of finitely many dense open sets is always dense.

(b) If X and Y are topological spaces, $D \subseteq A \subseteq X$, D is dense in A and $f : X \to Y$ is a continuous function, then $f[D]$ is dense in $f[A]$.

3A3F Meager sets Let X be a topological space.

(a) A set $A \subseteq X$ is **nowhere dense** if $\operatorname{int} \overline{A} = \emptyset$.

(b) A set $M \subseteq X$ is **meager** if it is expressible as the union of a sequence of nowhere dense sets. A subset of X is **comeager** if its complement is meager.

(c) Any subset of a nowhere dense set is nowhere dense; the union of finitely many nowhere dense sets is nowhere dense.

(d) Any subset of a meager set is meager; the union of countably many meager sets is meager.

3A3G Baire's theorem for locally compact Hausdorff spaces Let X be a locally compact Hausdorff space and $\langle G_n\rangle_{n\in\mathbb{N}}$ a sequence of dense open subsets of X. Then $\bigcap_{n\in\mathbb{N}} G_n$ is dense. Consequently every comeager subset of X is dense.

3A3H Corollary (a) Let X be a compact Hausdorff space. Then a non-empty open subset of X cannot be meager.

(b) Let X be a non-empty locally compact Hausdorff space. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of sets covering X, then there is some $n\in\mathbb{N}$ such that $\operatorname{int}\overline{A}_n$ is non-empty.

3A3I Product spaces (a) Definition Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, and $X=\prod_{i\in I}X_i$ their Cartesian product. We say that a set $G\subseteq X$ is open for the **product topology** if for every $x\in G$ there are a finite $J\subseteq I$ and a family $\langle G_j\rangle_{j\in J}$ such that every G_j is an open set in the corresponding X_j and

$$\{y : y\in X,\ y(j)\in G_j \text{ for every } j\in J\}$$

contains x and is included in G.

(b) If $\langle X_i\rangle_{i\in I}$ is a family of topological spaces, with product X, and Y another topological space, a function $\phi:Y\to X$ is continuous iff $\pi_i\phi$ is continuous for every $i\in I$, where $\pi_i(x)=x(i)$ for $x\in X$ and $i\in I$.

(c) Let $\langle X_i\rangle_{i\in I}$ be any family of non-empty topological spaces, with product X. If $\mathcal{F}$ is a filter on X and $x\in X$, then $\mathcal{F}\to x$ iff $\pi_i[[\mathcal{F}]]\to x(i)$ for every i, where $\pi_i(y)=y(i)$ for $y\in X$.

(d) The product of any family of Hausdorff spaces is Hausdorff.

(e) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. If D_i is a dense subset of X_i for each i, then $\prod_{i\in I}D_i$ is dense in $\prod_{i\in I}X_i$. .

(f) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. If F_i is a closed subset of X_i for each i, then $\prod_{i\in I}F_i$ is closed in $\prod_{i\in I}X_i$.

(g) Let $\langle(X_i,\mathfrak{T}_i)\rangle_{i\in I}$ be a family of topological spaces with product $(X,\mathfrak{T})$. Suppose that each $\mathfrak{T}_i$ is defined by a family P_i of pseudometrics on X_i. Then $\mathfrak{T}$ is defined by the family $\mathrm{P}=\{\tilde\rho_i : i\in I, \rho\in\mathrm{P}_i\}$ of pseudometrics on X, where I write $\tilde\rho_i(x,y)=\rho(\pi_i(x),\pi_i(y))$ whenever $i\in I$, $\rho\in\mathrm{P}_i$ and x, $y\in X$.

(h) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with product X, and Y another topological space. Then a function $f:X\to Y$ is **separately continuous** if for every $j\in I$ and $z\in\prod_{i\in I\setminus\{j\}}X_i$ the function $t\mapsto f(z^\frown{<}t{>}):X_j\to Y$ is continuous, where $z^\frown{<}t{>}$ is the member of X extending z and such that $(z^\frown{<}t{>})(j)=t$.

3A3J Tychonoff's theorem The product of any family of compact topological spaces is compact.

3A3K The spaces $\{0,1\}^I$, $\mathbb{R}^I$ For any set I, we can think of $\{0,1\}^I$ as the product $\prod_{i\in I}X_i$ where $X_i=\{0,1\}$ for each i. If we endow each X_i with its discrete topology, the product topology is the **usual topology** on $\{0,1\}^I$. Being a product of Hausdorff spaces, it is Hausdorff; by Tychonoff's theorem, it is compact. A subset G of $\{0,1\}^I$ is open iff for every $x\in G$ there is a finite $J\subseteq I$ such that $\{y : y\in\{0,1\}^I,\ y{\restriction}J=x{\restriction}J\}\subseteq G$.

Similarly, the 'usual topology' of $\mathbb{R}^I$ is the product topology when each factor is given its Euclidean topology.

3A3L Cluster points of filters (a) Let X be a topological space and $\mathcal{F}$ a filter on X. A point x of X is a **cluster point** of $\mathcal{F}$ if $x\in\overline{A}$ for every $A\in\mathcal{F}$.

(b) For any topological space X, filter $\mathcal{F}$ on X and $x\in X$, x is a cluster point of $\mathcal{F}$ iff there is a filter $\mathcal{G}\supseteq\mathcal{F}$ such that $\mathcal{G}\to x$.

(c) If $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$, $\alpha \in \mathbb{R}$ and $\lim_{n\to\mathcal{H}} \alpha_n = \alpha$ for every non-principal ultrafilter $\mathcal{H}$ on $\mathbb{N}$, then $\lim_{n\to\infty} \alpha_n = \alpha$.

3A3M Topology bases (a) If X is a set and $\mathbb{T}$ is any non-empty family of topologies on X, $\bigcap \mathbb{T}$ is a topology on X. So if $\mathcal{A}$ is any family of subsets of X, the intersection of all the topologies on X including $\mathcal{A}$ is a topology on X; this is the **topology generated by** $\mathcal{A}$.

(b) If X is a set and $\mathfrak{T}$ is a topology on X, a **base** for $\mathfrak{T}$ is a set $\mathcal{U} \subseteq \mathfrak{T}$ such that whenever $x \in G \in \mathfrak{T}$ there is a $U \in \mathcal{U}$ such that $x \in U \subseteq G$. In this case $\mathcal{U}$ generates $\mathfrak{T}$.

(c) If X is a set and $\mathcal{E}$ is a family of subsets of X, then $\mathcal{E}$ is a base for a topology on X iff (i) whenever E_1, $E_2 \in \mathcal{E}$ and $x \in E_1 \cap E_2$ then there is an $E \in \mathcal{E}$ such that $x \in E \subseteq E_1 \cap E_2$ (ii) $\bigcup \mathcal{E} = X$.

3A3N Uniform convergence (a) Let X be a set, (Y, ρ) a metric space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of functions from X to Y. We say that $\langle f_n \rangle_{n\in\mathbb{N}}$ converges **uniformly** to a function $f : X \to Y$ if for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) \le \epsilon$ whenever $n \ge n_0$ and $x \in X$.

(b) Let X be a topological space and (Y, ρ) a metric space. Suppose that $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of continuous functions from X to Y converging uniformly to $f : X \to Y$. Then f is continuous.

3A3O One-point compactifications Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space. Take any object x_∞ not belonging to X and set $X^* = X \cup \{x_\infty\}$. Let $\mathfrak{T}^*$ be the family of those sets $H \subseteq X^*$ such that $H \cap X \in \mathfrak{T}$ and either $x_\infty \notin H$ or $X \setminus H$ is compact (for $\mathfrak{T}$). Then $\mathfrak{T}^*$ is the unique compact Hausdorff topology on X^* inducing $\mathfrak{T}$ as the subspace topology on X; $(X^*, \mathfrak{T}^*)$ is the **one-point compactification** of $(X, \mathfrak{T})$.

3A3P Topologies defined from a sequential convergence: Proposition (a) Let X be a set and $\to^*$ a relation between $X^\mathbb{N}$ and X such that whenever $\langle x_n \rangle_{n\in\mathbb{N}} \in X^\mathbb{N}$, $x \in X$, $\langle x_n \rangle_{n\in\mathbb{N}} \to^* x$ and $\langle x'_n \rangle_{n\in\mathbb{N}} \in X^\mathbb{N}$ is a subsequence of $\langle x_n \rangle_{n\in\mathbb{N}}$ then $\langle x'_n \rangle_{n\in\mathbb{N}} \to^* x$. Then there is a unique topology on X for which a set $F \subseteq X$ is closed iff $x \in F$ whenever $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in F and $\langle x_n \rangle_{n\in\mathbb{N}} \to^* x$. Moreover, if $\langle x_n \rangle_{n\in\mathbb{N}} \to^* x$ then $\langle x_n \rangle_{n\in\mathbb{N}}$ converges to x for this topology.

(b) Let X and Y be sets, and suppose that $\to^*_X \subseteq X^\mathbb{N} \times X$, $\to^*_Y \subseteq Y^\mathbb{N} \times Y$ are relations with the subsequence property described in (a). Give X and Y the corresponding topologies. If $f : X \to Y$ is a function such that $\langle f(x_n) \rangle_{n\in\mathbb{N}} \to^*_Y f(x)$ whenever $\langle x_n \rangle_{n\in\mathbb{N}} \to^*_X x$, then f is continuous.

3A3Q Miscellaneous definitions Let X be a topological space.

(a) A subset of X is a **zero set** if it is of the form $f^{-1}[\{0\}]$ for some continuous function $f : X \to \mathbb{R}$. A subset of X is a **cozero set** if its complement is a zero set. A subset of X is a **$\mathbf{G}_\delta$ set** if it is expressible as the intersection of a sequence of open sets.

(b) An **isolated point** of X is a point $x \in X$ such that the singleton set $\{x\}$ is open.

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3A4 Uniformities

I continue the work of §3A3 with some notes on uniformities, so as to be able to discuss completeness and the extension of uniformly continuous functions in non-metrizable contexts (3A4F-3A4H).

3A4A Uniformities (a) Let X be a set. A **uniformity** on X is a filter $\mathcal{W}$ on $X \times X$ such that

(i) $(x, x) \in W$ for every $x \in X$, $W \in \mathcal{W}$;

(ii) for every $W \in \mathcal{W}$, $W^{-1} = \{(y, x) : (x, y) \in W\} \in \mathcal{W}$;

(iii) for every $W \in \mathcal{W}$, there is a $V \in \mathcal{W}$ such that

$$V \circ V = \{(x, z) : \exists\, y,\ (x, y) \in V \,\&\, (y, z) \in V\} \subseteq W.$$

It is convenient to allow the special case $X = \emptyset$, $\mathcal{W} = \{\emptyset\}$.

The pair $(X, \mathcal{W})$ is now a **uniform space**.

(b) If $\mathcal{W}$ is a uniformity on a set X, it induces a topology on X, the family of sets $G \subseteq X$ such that for every $x \in G$ there is a $W \in \mathcal{W}$ such that $W[\{x\}]$ is included in G.

(c) We say that a uniformity is **Hausdorff** if it induces a Hausdorff topology.

(d) If U is a linear topological space, then it has an associated uniformity

$$\mathcal{W} = \{W : W \subseteq U \times U, \text{ there is an open set } G \text{ containing } 0 \text{ such that } (u,v) \in W \text{ whenever } u - v \in G\},$$

and $\mathcal{W}$ induces the topology of U in the sense of (b) above.

3A4B Uniformities and pseudometrics (a) If P is a family of pseudometrics on a set X, then the associated uniformity is the smallest uniformity on X containing all the sets $W(\rho;\epsilon) = \{(x,y) : \rho(x,y) < \epsilon\}$ as ρ runs over P, ϵ over $]0,\infty[$.

(b) If $\mathcal{W}$ is the uniformity defined by a family P of pseudometrics, then the topology induced by $\mathcal{W}$ is the topology defined from P.

(c) A uniformity $\mathcal{W}$ is **metrizable** if it can be defined by a single metric.

(d) If U is a linear space with a topology defined from a family T of F-seminorms, the uniformity defined from the topology coincides with the uniformity defined from the pseudometrics $\rho_\tau(u,v) = \tau(u-v)$ as τ runs over T.

3A4C Uniform continuity (a) If $(X,\mathcal{W})$ and $(Y,\mathcal{V})$ are uniform spaces, a function $\phi : X \to Y$ is **uniformly continuous** if $\{(x,y) : (\phi(x),\phi(y)) \in V\}$ belongs to $\mathcal{W}$ for every $V \in \mathcal{V}$.

(b) The composition of uniformly continuous functions is uniformly continuous.

(c) If uniformities $\mathcal{W}$, $\mathcal{V}$ on sets X, Y are defined by non-empty families P, Θ of pseudometrics, then a function $\phi : X \to Y$ is uniformly continuous iff for every $\theta \in \Theta$, $\epsilon > 0$ there are $\rho_0,\dots,\rho_n \in$ P and $\delta > 0$ such that $\theta(\phi(x),\phi(y)) \le \epsilon$ whenever x, $y \in X$ and $\max_{i \le n} \rho_i(x,y) \le \delta$.

(d) A uniformly continuous function is continuous for the induced topologies.

(e) Two metrics ρ, σ on a set X are **uniformly equivalent** if they give rise to the same uniformity

(f) If U and V are linear topological spaces, and $T : U \to V$ is a continuous linear operator, then T is uniformly continuous for the uniformities associated with the topologies of U and V.

3A4D Subspaces (a) If $(X,\mathcal{W})$ is a uniform space and Y is any subset of X, then $\mathcal{W}_Y = \{W \cap (Y \times Y) : W \in \mathcal{W}\}$ is a uniformity on Y; it is the **subspace uniformity**.

(b) If $\mathcal{W}$ defines a topology $\mathfrak{T}$ on X, then the topology defined by $\mathcal{W}_Y$ is the subspace topology on Y.

(c) If $\mathcal{W}$ is defined by a family P of pseudometrics on X, then $\mathcal{W}_Y$ is defined by $\{\rho \restriction Y \times Y : \rho \in \text{P}\}$.

3A4E Product uniformities (a) If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces, the **product uniformity** is the smallest uniformity $\mathcal{W}$ on $X \times Y$ containing all sets of the form

$$\{((x,y),(x',y')) : (x,x') \in U,\ (y,y') \in V\}$$

as U runs over $\mathcal{U}$ and V over $\mathcal{V}$.

(b) If $\mathcal{U}$, $\mathcal{V}$ are defined from families P, Θ of pseudometrics, then $\mathcal{W}$ will be defined by the family $\{\tilde\rho : \rho \in \text{P}\} \cup \{\bar\theta : \theta \in \Theta\}$, writing

$$\tilde\rho((x,y),(x',y')) = \rho(x,x'), \quad \bar\theta((x,y),(x',y')) = \theta(y,y')$$

as in 2A3Tb.

(c) If $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ are uniform spaces, a map $\phi : Z \to X \times Y$ is uniformly continuous iff the coordinate maps $\phi_1 : Z \to X$ and $\phi_2 : Z \to Y$ are uniformly continuous.

3A4F Completeness (a) If $\mathcal{W}$ is a uniformity on a set X, a filter $\mathcal{F}$ on X is **Cauchy** if for every $W \in \mathcal{W}$ there is an $F \in \mathcal{F}$ such that $F \times F \subseteq W$.

Any convergent filter in a uniform space is Cauchy.

(b) A uniform space is **complete** if every Cauchy filter is convergent.

(c) If $\mathcal{W}$ is defined from a family P of pseudometrics, then a filter $\mathcal{F}$ on X is Cauchy iff for every $\rho \in \mathrm{P}$ and $\epsilon > 0$ there is an $F \in \mathcal{F}$ such that $\rho(x, y) \le \epsilon$ for all x, $y \in F$; equivalently, for every $\rho \in \mathrm{P}$, $\epsilon > 0$ there is an $x \in X$ such that $U(x; \rho; \epsilon) \in \mathcal{F}$.

(d) A complete subspace of a Hausdorff uniform space is closed. A closed subspace of a complete uniform space is complete under the subspace uniformity (references).

(e) A metric space is complete iff every Cauchy sequence converges.

(f) If (X, ρ) is a complete metric space, $D \subseteq X$ a dense subset, (Y, σ) a metric space and $f : X \to Y$ is an **isometry** (that is, $\sigma(f(x), f(x')) = \rho(x, x')$ for all x, $x' \in X$), then $f[X]$ is precisely the closure of $f[D]$ in Y.

(g) If U is a linear space with a linear space topology and the associated uniformity, then a filter $\mathcal{F}$ on U is Cauchy iff for every open set G containing 0 there is an $F \in \mathcal{F}$ such that $F - F \subseteq G$.

3A4G Extension of uniformly continuous functions: Theorem If $(X, \mathcal{W})$ is a uniform space, $(Y, \mathcal{V})$ is a complete uniform space, $D \subseteq X$ is a dense subset of X, and $\phi : D \to Y$ is uniformly continuous, then there is a uniformly continuous $\hat{\phi} : X \to Y$ extending ϕ. If Y is Hausdorff, the extension is unique.

In particular, if (X, ρ) is a metric space, (Y, σ) is a complete metric space, $D \subseteq X$ is a dense subset, and $\phi : D \to Y$ is an isometry, then there is a unique isometry $\hat{\phi} : X \to Y$ extending ϕ.

3A4H Completions (a) Theorem If $(X, \mathcal{W})$ is any Hausdorff uniform space, then we can find a complete Hausdorff uniform space $(\hat{X}, \hat{\mathcal{W}})$ in which X is embedded as a dense subspace; moreover, any two such spaces are essentially unique.

(b) Such a space $(\hat{X}, \hat{\mathcal{W}})$ is called a **completion** of $(X, \mathcal{W})$. Because it is unique up to isomorphism as a uniform space, we may call it 'the' completion.

(c) If $\mathcal{W}$ is the uniformity defined by a metric ρ on a set X, then there is a unique extension of ρ to a metric $\hat{\rho}$ on $\hat{X}$ defining the uniformity $\hat{\mathcal{W}}$.

3A4I A note on metric spaces Let (X, ρ) be a metric space. If $x \in X$ and $A \subseteq X$ is non-empty, set

$$\rho(x, A) = \inf_{y \in A} \rho(x, y).$$

Then $\rho(x, A) = 0$ iff $x \in \overline{A}$. If $B \subseteq X$ is another non-empty set, then

$$\rho(x, B) \le \rho(x, A) + \sup_{y \in A} \rho(y, B).$$

In particular, $\rho(x, \overline{A}) = \rho(x, A)$. If $\langle A_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of non-empty sets with union A, then

$$\rho(x, A) = \lim_{n \to \infty} \rho(x, A_n).$$

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3A5 Normed spaces

I run as quickly as possible over the results, nearly all of them standard elements of any introductory course in functional analysis, which I find myself calling on in this volume. As in the corresponding section of Volume 2 (§2A4), a large proportion of these are valid for both real and complex normed spaces, but as the present volume is almost exclusively concerned with real linear spaces I leave this unsaid, except in 3A5M, and if in doubt you may suppose for the time being that scalars belong to the field $\mathbb{R}$. A couple of the most basic results will be used in their complex forms in Volume 4.

3A5A The Hahn-Banach theorem: analytic forms (a) Let U be a linear space and $p : U \to [0, \infty[$ a functional such that $p(u + v) \le p(u) + p(v)$ and $p(\alpha u) = \alpha p(u)$ whenever $u, v \in U$ and $\alpha \ge 0$. Then for any $u_0 \in U$ there is a linear functional $f : U \to \mathbb{R}$ such that $f(u_0) = p(u_0)$ and $f(u) \le p(u)$ for every $u \in U$.

(b) Let U be a normed space and V a linear subspace of U. Then for any $f \in V^*$ there is a $g \in U^*$, extending f, with $\|g\| = \|f\|$.

(c) If U is a normed space and $u \in U$ there is an $f \in U^*$ such that $\|f\| \le 1$ and $f(u) = \|u\|$.

(d) If U is a normed space and $V \subseteq U$ is a linear subspace which is not dense, then there is a non-zero $f \in U^*$ such that $f(v) = 0$ for every $v \in V$.

(e) If U is a normed space, U^* separates the points of U.

3A5B Cones (a) Let U be a linear space. A **convex cone** (with apex 0) is a set $C \subseteq U$ such that $\alpha u + \beta v \in C$ whenever $u, v \in C$ and $\alpha, \beta \ge 0$. The intersection of any family of convex cones is a convex cone, so for every subset A of U there is a smallest convex cone including A.

(b) Let U be a normed space. Then the closure of a convex cone is a convex cone.

3A5C Hahn-Banach theorem: geometric forms (a) Let U be a normed space and $C \subseteq U$ a convex set such that $\|u\| \ge 1$ for every $u \in C$. Then there is an $f \in U^*$ such that $\|f\| \le 1$ and $f(u) \ge 1$ for every $u \in C$.

(b) Let U be a normed space and $C \subseteq U$ a non-empty convex set such that $0 \notin \overline{C}$. Then there is an $f \in U^*$ such that $\inf_{u \in C} f(u) > 0$.

(c) Let U be a normed space, C a closed convex subset of U containing 0, and u a point of $U \setminus C$. Then there is an $f \in U^*$ such that $f(u) > 1$ and $f(v) \le 1$ for every $v \in C$.

3A5D Separation from finitely-generated cones Let U be a linear space over $\mathbb{R}$ and $u, v_0, \dots, v_n$ points of U such that u does not belong to the convex cone generated by $\{v_0, \dots, v_n\}$. Then there is a linear functional $f : U \to \mathbb{R}$ such that $f(v_i) \ge 0$ for every i and $f(u) < 0$.

3A5E Weak topologies (a) Let U be any linear space over $\mathbb{R}$ and W a subset of the space U' of all linear functionals from U to $\mathbb{R}$. Then I write $\mathfrak{T}_s(U, W)$ for the linear space topology defined by the method of 2A5B from the seminorms $u \mapsto |f(u)|$ as f runs over W.

(c) Let U and V be linear spaces over $\mathbb{R}$ and $T : U \to V$ a linear operator. If $W \subseteq U'$ and $Z \subseteq V'$ are such that $gT \in W$ for every $g \in Z$, then T is continuous for $\mathfrak{T}_s(U, W)$ and $\mathfrak{T}_s(V, Z)$.

(d) If U and V are normed spaces and $T : U \to V$ is a bounded linear operator then we have an **adjoint** operator $T' : V^* \to U^*$ defined by saying that $T'g = gT$ for every $g \in V^*$. T' is linear and is continuous for the weak* topologies of U^* and V^*.

(e) If U is a normed space and $A \subseteq U$ is convex, then the closure of A for the norm topology is the same as the closure of A for the weak topology of U. In particular, norm-closed convex subsets (for instance, norm-closed linear subspaces) of U are closed for the weak topology.

3A5F Weak* topologies: Theorem If U is a normed space, the unit ball of U^* is compact and Hausdorff for the weak* topology.

3A5G Reflexive spaces (a) A normed space U is **reflexive** if every member of U^{**} is of the form $f \mapsto f(u)$ for some $u \in U$.

(b) A normed space is reflexive iff bounded sets are relatively weakly compact.

(c) If U is a reflexive space, $\langle u_n \rangle_{n \in \mathbb{N}}$ is a bounded sequence in U and $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$, then $\lim_{n \to \mathcal{F}} u_n$ is defined in U for the weak topology.

3A5H (a) Uniform Boundedness Theorem Let U be a Banach space, V a normed space, and $A \subseteq \mathsf{B}(U;V)$ a set such that $\{Tu : T \in A\}$ is bounded in V for every $u \in U$. Then A is bounded in $\mathsf{B}(U;V)$.

(b) Corollary If U is a normed space and $A \subseteq U$ is such that $f[A]$ is bounded for every $f \in U^*$, then A is bounded. Consequently any relatively weakly compact set in U is bounded.

***3A5I Strong operator topologies** If U and V are normed spaces, the **strong operator topology** on $\mathsf{B}(U;V)$ is that defined by the seminorms $T \mapsto \|Tu\|$ as u runs over U. If U is a Banach space, V is a normed space and $A \subseteq \mathsf{B}(U;V)$, then A is relatively compact for the strong operator topology iff $\{Tu : T \in A\}$ is relatively compact in V for every $u \in U$.

3A5J Completions Let U be a normed space.

(a) U has a metric ρ associated with the norm, and the topology defined by ρ is a linear space topology. This topology defines a uniformity $\mathcal{W}$ which is also the uniformity defined by ρ. The norm itself is a uniformly continuous function from U to $\mathbb{R}$.

(b) Let $(\hat{U}, \hat{\mathcal{W}})$ be the uniform space completion of $(U, \mathcal{W})$. Then addition and scalar multiplication and the norm extend uniquely to make $\hat{U}$ a Banach space.

(c) If U and V are Banach spaces with dense linear subspaces U_0 and V_0, then any norm-preserving isomorphism between U_0 and V_0 extends uniquely to a norm-preserving isomorphism between U and V.

3A5K Normed algebras If U is a normed algebra, its multiplication, regarded as a function from $U \times U$ to U, is continuous.

3A5L Compact operators Let U and V be Banach spaces.

(a) A linear operator $T : U \to V$ is **compact** if $\{Tu : \|u\| \le 1\}$ is relatively compact in V for the topology defined by the norm of V.

(b) A linear operator $T : U \to V$ is **weakly compact** if $\{Tu : \|u\| \le 1\}$ is relatively weakly compact in V. Of course compact operators are weakly compact; weakly compact operators are bounded.

3A5M Hilbert spaces (a) An **inner product space** is a linear space U over $\substack{\mathbb{R}\\ \mathbb{C}}$ together with an operator $(\ |\ ) : U \times U \to \substack{\mathbb{R}\\ \mathbb{C}}$ such that

$$(u_1 + u_2|v) = (u_1|v) + (u_2|v), \quad (\alpha u|v) = \alpha(u|v), \quad (u|v) = \overline{(v|u)}$$

(the complex conjugate of $(v|u)$),

$$(u|u) \ge 0, \quad u = 0 \text{ whenever } (u|u) = 0$$

for all u, u_1, u_2, $v \in U$ and $\alpha \in \substack{\mathbb{R}\\ \mathbb{C}}$.

(b) If U is any inner product space, we have a norm on U defined by setting $\|u\| = \sqrt{(u|u)}$ for every $u \in U$, and $|(u|v)| \le \|u\|\|v\|$ for all u, $v \in U$.

(c) A **Hilbert space** is an inner product space which is complete in the metric defined from its norm.

(d) If U is a Hilbert space, $C \subseteq U$ is a non-empty closed convex set, and $u \in U$, then there is a unique $v \in C$ such that $\|u - v\| = \inf_{w \in C} \|u - w\|$.

(e) If U is an inner product space, $C \subseteq U$ is a convex set, u, $u' \in U$ and v, $v' \in C$ are such that that $\|u - v\| = \inf_{w \in C} \|u - w\|$ and $\|u' - v'\| = \inf_{w \in C} \|u' - w\|$, then $\|v' - v\| \le \|u' - u\|$.

***3A5N Bounded sets in linear topological spaces** Let U be a linear topological space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

(a) A set $A \subseteq U$ is **bounded** if for every neighbourhood G of 0 there is an $n \in \mathbb{N}$ such that $A \subseteq nG$.

(b) If $A \subseteq U$ is bounded, then

(i) every subset of A is bounded;

(ii) the closure of A is bounded;

(iii) αA is bounded for every $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$;

(iv) $A \cup B$ and $A + B$ are bounded for every bounded $B \subseteq U$;

(v) if V is another linear topological space, and $T : U \to V$ is a continuous linear operator, then $T[A]$ is bounded.

(c) If $A \subseteq U$ is relatively compact, it is bounded.

(d) If U is a normed space, and $A \subseteq U$, then the following are equiveridical:

(i) A is bounded in the sense of (a) above for the norm topology of U;

(ii) A is bounded in the sense of 2A4Bc;

(iii) A is bounded for the weak topology of U.

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3A6 Group Theory

For Chapter 38 we need four definitions and two results from elementary abstract group theory.

3A6A Definition If G is a group, an element g of G is an **involution** if its order is 2.

3A6B Definition If G is a group, the set $\operatorname{Aut} G$ of **automorphisms** of G (that is, bijective homomorphisms from G to itself) is a group. For $g \in G$ define $\hat{g} : G \to G$ by writing $\hat{g}(h) = ghg^{-1}$ for every $h \in G$; then $\hat{g} \in \operatorname{Aut} G$, and the map $g \mapsto \hat{g}$ is a homomorphism from G onto a normal subgroup J of $\operatorname{Aut} G$. We call J the group of **inner automorphisms** of G. Members of $(\operatorname{Aut} G) \setminus J$ are called **outer automorphisms**.

3A6C Normal subgroups For any group G, the family of normal subgroups of G, ordered by $\subseteq$, is a Dedekind complete lattice, with $H \vee K = HK$ and $H \wedge K = H \cap K$.