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Appendix to Volume 3

Useful Facts

This volume assumes a fairly wide-ranging competence in analysis, a solid understanding of elementary set theory and some straightforward Boolean algebra. As in previous volumes, I start with a few pages of revision in set theory, but the absolutely essential material is in §3A2, on commutative rings, which is the basis of the treatment of Boolean rings in §311. I then give three sections of results in analysis: topological spaces (§3A3), uniform spaces (§3A4) and normed spaces (§3A5). Finally, I add six sentences on group theory (§3A6).

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3A1 Set Theory

3A1A The axioms of set theory This treatise is based on arguments within, or in principle reducible to, 'ZFC', meaning 'Zermelo-Fraenkel set theory, including the Axiom of Choice'. For discussions of this system, see, for instance, Krivine 71, Jech 03 or Kunen 80. As I remarked in §2A1, I believe that it is helpful, as a matter of general principle, to distinguish between results dependent on the axiom of choice and those which can be proved without it, or with some relatively weak axiom such as 'countable choice'. (See 134C. I will go much more deeply into this in Chapter 56 of Volume 5.) In Volumes 1 and 2, such a distinction is useful in appreciating the special features of different ideas. In the present volume, however, most of the principal theorems require something close to the full axiom of choice, and there are few areas where it seems at present appropriate to work with anything weaker. Indeed, at many points we shall approach questions which are, or may be, undecidable in ZFC; but with very few exceptions I postpone discussion of these to Volume 5. In particular, I specifically exclude, for the time being, results dependent on such axioms as the continuum hypothesis.

3A1B Definition Let X be a set. By an **enumeration** of X I mean a bijection $f : \kappa \to X$ where $\kappa = \#(X)$is the initial ordinal equipollent with X (2A1Kb); more often than not I shall express such a function in the form $\langle x_\xi \rangle_{\xi<\kappa}$. In this case I say that the function f, or the family $\langle x_\xi \rangle_{\xi<\kappa}$, **enumerates** X. You will see that I am tacitly assuming that $\#(X)$ is always defined, that is, that the axiom of choice is true.

3A1C Calculation of cardinalities The following formulae are basic.

(a) For any sets X and Y, $\#(X \times Y) \le \max(\omega, \#(X), \#(Y))$. (Enderton 77, p. 64; Jech 03, p. 51; Krivine 71, p. 33; Kunen 80, 10.13.)

(b) For any $r \in \mathbb{N}$ and any family $\langle X_i \rangle_{i\le r}$ of sets, $\#(\prod_{i=0}^r X_i) \le \max(\omega, \max_{i\le r} \#(X_i))$. (Induce on r.)

(c) For any family $\langle X_i \rangle_{i\in I}$ of sets, $\#(\bigcup_{i\in I} X_i) \le \max(\omega, \#(I), \sup_{i\in I} \#(X_i))$. (Jech 03, p. 52; Krivine 71, p. 33; Kunen 80, 10.21.)

(d) For any set X, the set $[X]^{<\omega}$ of finite subsets of X has cardinal at most $\max(\omega, \#(X))$. (There is a surjection from $\bigcup_{r\in\mathbb{N}} X^r$ onto $[X]^{<\omega}$. For the notation $[X]^{<\omega}$ see 3A1J below.)

3A1D Cardinal exponentiation For a cardinal κ, I write 2^κ for $\#(\mathcal{P}\kappa)$. So $2^\omega = \mathfrak{c}$, and $\kappa^+ \le 2^\kappa$ for every κ. (Enderton 77, p. 132; Lipschutz 64, p. 139; Jech 03, p. 29; Krivine 71, p. 25; Halmos 60, p. 93.)

3A1E Definition The class of infinite initial ordinals, or cardinals, is a subclass of the class On of all ordinals, so is itself well-ordered; being unbounded, it is a proper class; consequently there is a unique increasing enumeration of it as $\langle\omega_\xi\rangle_{\xi\in\mathrm{On}}$. We have $\omega_0 = \omega$, $\omega_{\xi+1} = \omega_\xi^+$ for every ξ (compare 2A1Fc), $\omega_\xi = \bigcup_{\eta<\xi}\omega_\eta$ for non-zero limit ordinals ξ. (Enderton 77, pp. 213-214; Jech 03, p. 30; Krivine 71, p. 31.)

3A1F Cofinal sets (a) If P is any partially ordered set (definition: 2A1Aa), a subset Q of P is **cofinal** with P if for every $p \in P$ there is a $q \in Q$ such that $p \le q$.

(b) If P is any partially ordered set, the **cofinality** of P, $\mathrm{cf}\, P$, is the least cardinal of any cofinal subset of P. Note that $\mathrm{cf}\, P = 0$ iff $P = \emptyset$, and that $\mathrm{cf}\, P = 1$ iff P has a greatest element.

(c) Observe that if P is upwards-directed and $\mathrm{cf}\, P$ is finite, then $\mathrm{cf}\, P$ is either 0 or 1; for if Q is a finite, non-empty cofinal set then it has an upper bound, which must be the greatest element of P.

(d) If P is a totally ordered set of cofinality κ, then there is a strictly increasing family $\langle p_\xi\rangle_{\xi<\kappa}$ in P such that $\{p_\xi : \xi < \kappa\}$ is cofinal with P. **P** If $\kappa = 0$ then $P = \emptyset$ and this is trivial. Otherwise, let Q be a cofinal subset of P with cardinal κ, and $\{q_\xi : \xi < \kappa\}$ an enumeration of Q. Define $\langle p_\xi\rangle_{\xi<\kappa}$ inductively, as follows. Start with $p_0 = q_0$. Given $\langle p_\eta\rangle_{\eta<\xi}$, where $\xi < \kappa$, then if $p_\eta < q_\xi$ for every $\eta < \xi$, take $p_\xi = q_\xi$; otherwise, because $\#(\xi) \le \xi < \kappa$, $\{p_\eta : \eta < \xi\}$ cannot be cofinal with P, so there is a $p_\xi \in P$ such that $p_\xi \not\le p_\eta$ for every $\eta < \xi$, that is, $p_\eta < p_\xi$ for every $\eta < \xi$. Note that there is some $\eta < \xi$ such that $q_\xi \le p_\eta$, so that $q_\xi \le p_\xi$. Continue.

Now $\langle p_\xi\rangle_{\xi<\kappa}$ is a strictly increasing family in P such that $q_\xi \le p_\xi$ for every ξ; it follows at once that $\{p_\xi : \xi < \kappa\}$ is cofinal with P. **Q**

(e) In particular, for a totally ordered set P, $\mathrm{cf}\, P = \omega$ iff there is a cofinal strictly increasing sequence in P.

3A1G Zorn's Lemma In Volume 2 I used Zorn's Lemma only once or twice, giving the arguments in detail. In the present volume I feel that continuing in such a manner would often be tedious; but nevertheless the arguments are not always quite obvious, at least until you have gained a good deal of experience. I therefore take a paragraph to comment on some of the standard forms in which they appear.

The statement of Zorn's Lemma, as quoted in 2A1M, refers to arbitrary partially ordered sets P. A large proportion of the applications can in fact be represented more or less naturally by taking P to be a family $\mathfrak{P}$ of sets ordered by $\subseteq$; in such a case, it will be sufficient to check that (i) $\mathfrak{P}$ is not empty (ii) $\bigcup\mathfrak{Q} \in \mathfrak{P}$ for every non-empty totally ordered $\mathfrak{Q} \subseteq \mathfrak{P}$. More often than not, this will in fact be true for all non-empty upwards-directed sets $\mathfrak{Q} \subseteq \mathfrak{P}$, and the line of the argument is sometimes clearer if phrased in this form.

Within this class of partially ordered sets, we can distinguish a special subclass. If A is any set and $\perp$ any relation on A, we can consider the collection $\mathfrak{P}$ of sets $I \subseteq A$ such that $a \perp b$ for all distinct a, $b \in I$. In this case we need look no farther before declaring '$\mathfrak{P}$ has a maximal element'; for $\emptyset$ necessarily belongs to $\mathfrak{P}$, and if $\mathfrak{Q}$ is any upwards-directed subset of $\mathfrak{P}$, then $\bigcup\mathfrak{Q} \in \mathfrak{P}$. **P** If a, b are distinct elements of $\bigcup\mathfrak{Q}$, there are I_1, $I_2 \in \mathfrak{Q}$ such that $a \in I_1$, $b \in I_2$; because $\mathfrak{Q}$ is upwards-directed, there is an $I \in \mathfrak{Q}$ such that $I_1 \cup I_2 \subseteq I$, so that a, b are distinct members of $I \in \mathfrak{P}$, and $a \perp b$. **Q** So $\bigcup\mathfrak{Q}$ is an upper bound of $\mathfrak{Q}$ in $\mathfrak{P}$; as $\mathfrak{Q}$ is arbitrary, $\mathfrak{P}$ satisfies the conditions of Zorn's Lemma, and must have a maximal element.

Another important type of partially ordered set in this context is a family Φ of functions, ordered by saying that $f \le g$ if g is an extension of f. In this case, for any non-empty upwards-directed $\Psi \subseteq \Phi$, we shall have a function h defined by saying that

$$\mathrm{dom}\, h = \bigcup_{f\in\Psi} \mathrm{dom}\, f, \quad h(x) = f(x) \text{ whenever } f \in \Psi,\ x \in \mathrm{dom}\, f,$$

and the usual attack is to seek to prove that any such h belongs to Φ.

I find that at least once I wish to use Zorn's Lemma 'upside down': that is, I have a non-empty partially ordered set P in which every non-empty totally ordered subset has a *lower* bound. In this case, of course, P has a *minimal* element. The point is that the definition of 'partial order' is symmetric, so that $(P,\ge)$ is a partially ordered set whenever $(P,\le)$ is; and we can seek to apply Zorn's Lemma to either.

3A1H Natural numbers and finite ordinals I remarked in 2A1De that the first few ordinals

$$\emptyset,\quad \{\emptyset\},\quad \{\emptyset,\{\emptyset\}\},\quad \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\quad \ldots$$

may be identified with the natural numbers $0,1,2,3,\ldots$; the idea being that $n=\{0,1,\ldots,n-1\}$ is a set with n elements. If we do this, then the set $\mathbb{N}$ of natural numbers becomes identified with the first infinite ordinal ω. This convention makes it possible to present a number of arguments in a particularly elegant form. A typical example is in 344H. There I wish to describe an inductive construction for a family $\langle K_\sigma\rangle_{\sigma\in S}$ where $S=\bigcup_{n\in\mathbb{N}}\{0,1\}^n$. If we think of n as the set of its predecessors, then $\sigma\in\{0,1\}^n$ becomes a function from n to $\{0,1\}$; since the set n has just n members, this corresponds well to the idea of σ as the list of its n coordinates, except that it would now be natural to list them as $\sigma(0),\ldots,\sigma(n-1)$ rather than as $x_1,\ldots,x_n$, which was the language I favoured in Volume 2. An extension of σ to a member of $\{0,1\}^{n+1}$ is of the form $\tau=\sigma^\frown\!<\!i\!>$ where $\tau(k)=\sigma(k)$ for $k<n$ ('$\tau\restriction n=\sigma$') and $\tau(n)=i$. If $w\in\{0,1\}^{\mathbb{N}}$, then we can identify the initial segment $(w(0),w(1),\ldots,w(n-1))$ of its first n coordinates with the restriction $w\restriction n$ of w to the set $n=\{0,\ldots,n-1\}$.

3A1I Definitions (a) If P and Q are lattices (2A1Ad), a **lattice homomorphism** from P to Q is a function $f:P\to Q$ such that $f(p\wedge p')=f(p)\wedge f(p')$ and $f(p\vee p')=f(p)\vee f(p')$ for all p, $p'\in P$. Such a homomorphism is surely order-preserving (313H), for if $p\le p'$ in P then $f(p')=f(p\vee p')=f(p)\vee f(p')$ and $f(p)\le f(p')$.

(b) If P is a lattice, a **sublattice** of P is a set $Q\subseteq P$ such that $p\vee q$ and $p\wedge q$ belong to Q for all p, $q\in Q$.

(c)(i) A lattice P is **distributive** if

$$(p\wedge q)\vee r=(p\vee r)\wedge(q\vee r),\qquad (p\vee q)\wedge r=(p\wedge r)\vee(q\wedge r)$$

for all p, q, $r\in P$.

(ii) In a distributive lattice we have a **median function** of three variables

$$\begin{aligned}\operatorname{med}(p,q,r)&=(p\wedge q)\vee(p\wedge r)\vee(q\wedge r)\\&=((p\wedge(q\vee r))\vee(q\wedge r)=(p\vee q)\wedge(p\vee r)\wedge(q\vee r).\end{aligned}$$

If P and Q are distributive lattices and $f:P\to Q$ is a lattice homomorphism, $f(\operatorname{med}(p,q,r))=\operatorname{med}(f(p),f(q),f(r))$ for all p, q, $r\in P$.

(iii) If P is a distributive lattice and $I\subseteq P$ is finite, then the sublattice of P generated by I is finite. **P** If $J=\{\sup I_0:\emptyset\ne I_0\subseteq I\}$ and $K=\{\inf J_0:\emptyset\ne J_0\subseteq J\}$ then K is a sublattice of P. **Q**

3A1J Subsets of given size The following concepts are used often enough for a special notation to be helpful. If X is a set and κ is a cardinal, write

$$[X]^\kappa=\{A:A\subseteq X,\ \#(A)=\kappa\},$$

$$[X]^{\le\kappa}=\{A:A\subseteq X,\ \#(A)\le\kappa\},$$

$$[X]^{<\kappa}=\{A:A\subseteq X,\ \#(A)<\kappa\}.$$

Thus

$$[X]^0=[X]^{\le 0}=[X]^{<1}=\{\emptyset\},$$

$[X]^2$ is the set of doubleton subsets of X, $[X]^{<\omega}$ is the set of finite subsets of X, $[X]^{\le\omega}$ is the set of countable subsets of X, and so on.

3A1K The next result is one of the fundamental theorems of combinatorics. In this volume it is used in the proofs of Ornstein's theorem (§387) and the Kalton-Roberts theorem (§392).

Hall's Marriage Lemma Suppose that X and Y are finite sets and $R \subseteq X \times Y$ is a relation such that $\#(R[I]) \geq \#(I)$ for every $I \subseteq X$. Then there is an injective function $f : X \to Y$ such that $(x, f(x)) \in R$ for every $x \in X$.

Remark Recall that $R[I]$ is the set $\{y : \exists\, x \in I, (x, y) \in R\}$ (1A1Bc). If we identify a function with its graph, then '$(x, f(x)) \in R$ for every $x \in X$' becomes '$f \subseteq R$'.

proof Bollobás 79, p. 54, Theorem 7; Anderson 87, 2.2.1; Bose & Manvel 84, §10.2.

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3A2 Rings

I give a very brief outline of the indispensable parts of the elementary theory of (commutative) rings. I assume that you have seen at least a little group theory.

3A2A Definition A **ring** is a triple $(R, +, .)$ such that

$(R, +)$ is an abelian group; its identity will always be denoted 0 or 0_R;

$(R, .)$ is a semigroup, that is, $ab \in R$ for all a, $b \in R$ and $a(bc) = (ab)c$ for all a, b, $c \in R$;

$a(b + c) = ab + ac$, $(a + b)c = ac + bc$ for all a, b, $c \in R$.

A **commutative ring** is one in which multiplication is commutative, that is, $ab = ba$ for all a, $b \in R$.

3A2B Elementary facts Let R be a ring.

(a) $a0 = 0a = 0$ for every $a \in R$. **P**

$$a0 = a(0 + 0) = a0 + a0, \quad 0a = (0 + 0)a = 0a + 0a;$$

because $(R, +)$ is a group, we may subtract $a0$ or $0a$ from each side of the appropriate equation to see that $0 = a0$, $0 = 0a$. **Q**

(b) $(-a)b = a(-b) = -(ab)$ for all a, $b \in R$. **P**

$$ab + ((-a)b) = (a + (-a))b = 0b = 0 = a0 = a(b + (-b)) = ab + a(-b);$$

subtracting ab from each term, we get $(-a)b = -(ab) = a(-b)$. **Q**

3A2C Subrings If R is a ring, a **subring** of R is a set $S \subseteq R$ such that $0 \in S$ and $a + b$, ab, $-a$ belong to S for all a, $b \in S$. In this case S, together with the addition and multiplication induced by those of R, is a ring in its own right.

3A2D Homomorphisms (a) Let R, S be two rings. A function $\phi : R \to S$ is a **ring homomorphism** if $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all a, $b \in R$. The **kernel** of ϕ is $\{a : a \in R, \phi(a) = 0_S\}$.

(b) Note that if $\phi : R \to S$ is a ring homomorphism, then it is also a group homomorphism from $(R, +)$ to $(S, +)$, so that $\phi(0_R) = 0_S$ and $\phi(-a) = -\phi(a)$ for every $a \in R$; moreover, $\phi[R]$ is a subring of S, and ϕ is injective iff its kernel is $\{0_R\}$.

(c) If R, S and T are rings, and $\phi : R \to S$, $\psi : S \to T$ are ring homomorphisms, then $\psi\phi : R \to T$ is a ring homomorphism, because

$$(\psi\phi)(a * b) = \psi(\phi(a * b)) = \psi(\phi(a) * \phi(b)) = \psi(\phi(a)) * \psi(\phi(b))$$

for all a, $b \in R$, taking $*$ to be either addition or multiplication. If ϕ is bijective, then $\phi^{-1} : S \to R$ is a ring homomorphism, because

$$\phi^{-1}(c * d) = \phi^{-1}(\phi(\phi^{-1}(c)) * \phi(\phi^{-1}(d))) = \phi^{-1}\phi(\phi^{-1}(c) * \phi^{-1}(d)) = \phi^{-1}(c) * \phi^{-1}(d)$$

for all c, $d \in S$, again taking $*$ to be either addition or multiplication.

3A2E Ideals (a) Let R be a ring. An **ideal** of R is a subring I of R such that $ab \in I$ and $ba \in I$ whenever $a \in I$ and $b \in R$. In this case we write $I \lhd R$.

Note that R and $\{0\}$ are always ideals of R.

(b) If R and S are rings and $\phi : R \to S$ is a ring homomorphism, then the kernel I of ϕ is an ideal of R. **P** (i) Because ϕ is a group homomorphism, I is a subgroup of $(R, +)$. (ii) If $a \in I$, $b \in R$ then

$$\phi(ab) = \phi(a)\phi(b) = 0_S\phi(b) = 0_S, \quad \phi(ba) = \phi(b)\phi(a) = \phi(b)0_S = 0_S$$

so ab, $ba \in I$. **Q**

3A2F Quotient rings (a) Let R be a ring and I an ideal of R. A **coset** of I is a set of the form $a + I = \{a + x : x \in I\}$ where $a \in R$. (Because $+$ is commutative, we do not need to distinguish between 'left cosets' $a + I$ and 'right cosets' $I + a$.) Let R/I be the set of cosets of I in R.

(b) For A, $B \in R/I$, set

$$A + B = \{x + y : x \in A,\ y \in B\}, \quad A \cdot B = \{xy + z : x \in A,\ y \in B,\ z \in I\}.$$

Then $A + B$, $A \cdot B$ both belong to R/I; moreover, if $A = a + I$ and $B = b + I$, then $A + B = (a + b) + I$ and $A \cdot B = ab + I$. **P** (i)

$$\begin{aligned} A + B &= (a + I) + (b + I) \\ &= \{(a + x) + (b + y) : x,\ y \in I\} \\ &= \{(a + b) + (x + y) : x,\ y \in I\} \end{aligned}$$

(because addition is associative and commutative)

$$\subseteq \{(a + b) + z : z \in I\} = (a + b) + I$$

(because $I + I \subseteq I$)

$$\begin{aligned} &= \{(a + b) + (z + 0) : z \in I\} \\ &\subseteq (a + I) + (b + I) = A + B \end{aligned}$$

because $0 \in I$. (ii)

$$\begin{aligned} A \cdot B &= \{(a + x)(b + y) + z : x,\ y,\ z \in I\} \\ &= \{ab + (ay + xb + z) : x,\ y,\ z \in I\} \\ &\subseteq \{ab + w : w \in I\} = ab + I \end{aligned}$$

(because ay, $xb \in I$ for all x, $y \in I$, and I is closed under addition)

$$\begin{aligned} &= \{(a + 0)(b + 0) + w : w \in I\} \\ &\subseteq A \cdot B. \end{aligned}$$ **Q**

(c) It is now an elementary exercise to check that $(R/I, +, \cdot)$ is a ring, with zero $0 + I = I$ and additive inverses $-(a + I) = (-a) + I$.

(d) Moreover, the map $a \mapsto a + I : R \to R/I$ is a ring homomorphism.

(e) Note that for a, $b \in R$, the following are equiveridical: (i) $a \in b+I$; (ii) $b \in a+I$; (iii) $(a+I)\cap(b+I) \neq \emptyset$; (iv) $a+I = b+I$; (v) $a-b \in I$. Thus the cosets of I are just the equivalence classes in R under the equivalence relation $a \sim b \iff a + I = b + I$; accordingly I shall generally write $a^\bullet$ for $a + I$, if there seems no room for confusion. In particular, the kernel of the canonical map from R to R/I is just $\{a : a + I = I\} = I = 0^\bullet$.

(f) If R is commutative so is R/I, since

$$a^\bullet b^\bullet = (ab)^\bullet = (ba)^\bullet = b^\bullet a^\bullet$$

for all a, $b \in R$.

3A2G Factoring homomorphisms through quotient rings: Proposition Let R and S be rings, I an ideal of R, and $\phi : R \to S$ a homomorphism such that I is included in the kernel of ϕ. Then we have a ring homomorphism $\pi : R/I \to S$ such that $\pi(a^\bullet) = \phi(a)$ for every $a \in R$. π is injective iff I is precisely the kernel of ϕ.

proof If a, $b \in R$ and $a^\bullet = b^\bullet$ in R/I, then $a-b \in I$ (3A2Fe), so $\phi(a)-\phi(b) = \phi(a-b) = 0$, and $\phi(a) = \phi(b)$. This means that the formula offered does indeed define a function π from R/I to S. Now if a, $b \in R$ and $*$ is either multiplication or addition,

$$\pi(a^\bullet * b^\bullet) = \pi((a*b)^\bullet) = \phi(a*b) = \phi(a)*\phi(b) = \pi(a^\bullet)*\pi(b^\bullet).$$

So π is a ring homomorphism.

The kernel of π is $\{a^\bullet : \phi(a) = 0\}$, which is $\{0\}$ iff $\phi(a) = 0 \iff a^\bullet = 0 \iff a \in I$.

3A2H Product rings (a) Let $\langle R_i\rangle_{i\in I}$ be any family of rings. Set $R = \prod_{i\in I} R_i$ and for a, $b \in R$ define $a+b$, $ab \in R$ by setting

$$(a+b)(i) = a(i)+b(i), \quad (ab)(i) = a(i)b(i)$$

for every $i \in I$. It is easy to check from the definition in 3A2A that R is a ring; its zero is given by the formula

$$0_R(i) = 0_{R_i} \text{ for every } i \in I,$$

and its additive inverses by the formula

$$(-a)(i) = -a(i) \text{ for every } i \in I.$$

(b) Now let S be any other ring. Then it is easy to see that a function $\phi : S \to R$ is a ring homomorphism iff $s \mapsto \phi(s)(i) : S \to R_i$ is a ring homomorphism for every $i \in I$.

(c) Note that R is commutative iff R_i is commutative for every i.

Version of 14.12.07

3A3 General topology

In §2A3, I looked at a selection of topics in general topology in some detail, giving proofs; the point was that an ordinary elementary course in the subject would surely go far beyond what we needed there, and at the same time might omit some of the results I wished to quote. It seemed therefore worth taking a bit of space to cover the requisite material, giving readers the option of delaying a proper study of the subject until a convenient opportunity arose. In the context of the present volume, this approach is probably no longer appropriate, since we need a much greater proportion of the fundamental ideas, and by the time you have reached familiarity with the topics here you will be well able to find your way about one of the many excellent textbooks on the subject. This time round, therefore, I give most of the results without proofs (as in §§2A1 and 3A1), hoping that some of the references I offer will be accessible in all senses. I do, however, give a full set of definitions, partly to avoid ambiguity (since even in this relatively mature subject, there are some awkward divergences remaining in the usage of different authors), and partly because many of the proofs are easy enough for even a novice to fill in with a bit of thought, once the meaning of the words is clear. In fact this happens so often that I will mark with a $*$ those points where a proof needs an idea not implicit in the preceding work.

3A3A Taxonomy of topological spaces I begin with the handful of definitions we need in order to classify the different types of topological space used in this volume. A couple have already been introduced in Volume 2, but I repeat them because the list would look so odd without them.

Definitions Let $(X, \mathfrak{T})$ be a topological space.

(a) X is $\mathbf{T_1}$ if singleton subsets of X are closed.

(b) X is **Hausdorff** if for any distinct points x, $y \in X$ there are disjoint open sets G, $H \subseteq X$ such that $x \in G$ and $y \in H$.

(c) X is **regular** if whenever $F \subseteq X$ is closed and $x \in X \setminus F$ there are disjoint open sets G, $H \subseteq X$ such that $x \in G$ and $F \subseteq H$. (Note that in this definition I do not require X to be Hausdorff, following James 87 but not Engelking 89, Bourbaki 66, Dugundji 66, Schubert 68 or Gaal 64.)

(d) X is **completely regular** if whenever $F \subseteq X$ is closed and $x \in X \setminus F$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$ for every $y \in F$. (Note that many authors restrict the phrase 'completely regular' to Hausdorff spaces.)

(e) X is **zero-dimensional** if whenever $G \subseteq X$ is an open set and $x \in G$ then there is an open-and-closed set H such that $x \in H \subseteq G$.

(f) X is **extremally disconnected** if the closure of every open set in X is open.

(g) X is **compact** if every open cover of X has a finite subcover.

(h) X is **locally compact** if for every $x \in X$ there is a set $K \subseteq X$ such that $x \in \operatorname{int} K$ and K is compact (in its subspace topology, as defined in 2A3C).

(i) If every subset of X is open, we call $\mathfrak{T}$ the **discrete topology** on X.

3A3B Elementary relationships (a) A completely regular space is regular. (Engelking 89, p. 39; Dugundji 66, p. 154; Schubert 68, p. 104.)

(b) A locally compact Hausdorff space is completely regular, therefore regular. * (Engelking 89, 3.3.1; Dugundji 66, p. 238; Gaal 64, p. 149.)

(c) A compact Hausdorff space is locally compact, therefore completely regular and regular.

(d) A regular extremally disconnected space is zero-dimensional. (Engelking 89, 6.2.25.)

(e) Any topology defined by pseudometrics (2A3F), in particular the weak topology of a normed space (2A5I), is completely regular, therefore regular. (Bourbaki 66, IX.1.5; Dugundji 66, p. 200.)

(f) If X is a completely regular Hausdorff space (in particular, if X is (locally) compact and Hausdorff), and x, y are distinct points in X, then there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$. (Apply 3A3Ad with $F = \{y\}$, which is closed because X is Hausdorff.)

(g) An open set in a locally compact Hausdorff space is locally compact in its subspace topology. (Engelking 89, 3.3.8; Bourbaki 66, I.9.7.)

(h) Any subspace of a Hausdorff space is Hausdorff; any subspace of a completely regular space is completely regular. (Engelking 89, 2.1.6).

3A3C Continuous functions Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological spaces.

(a) If $f : X \to Y$ is a function and $x \in X$, we say that f is **continuous at** x if $x \in \operatorname{int} f^{-1}[H]$ whenever $H \subseteq Y$ is an open set containing $f(x)$.

(b) Now a function from X to Y is continuous iff it is continuous at every point of X. (Bourbaki 66, I.2.1; Dugundji 66, p. 80; Schubert 68, p. 24; Gaal 64, p. 183; James 87, p. 26.)

(c) If $f : X \to Y$ is continuous at $x \in X$, and $A \subseteq X$ is such that $x \in \overline{A}$, then $f(x) \in \overline{f[A]}$. (Bourbaki 66, I.2.1; Schubert 68, p. 23.)

(d) If $f : X \to Y$ is continuous, then $f[\overline{A}] \subseteq \overline{f[A]}$ for every $A \subseteq X$. (Engelking 89, 1.4.1; Bourbaki 66, I.2.1; Dugundji 66, p. 80; Schubert 68, p. 24; Gaal 64, p. 184; James 87, p. 27.)

(e) A function $f : X \to Y$ is a **homeomorphism** if it is a continuous bijection and its inverse is also continuous; that is, if $\mathfrak{S} = \{f[G] : G \in \mathfrak{T}\}$ and $\mathfrak{T} = \{f^{-1}[H] : H \in \mathfrak{S}\}$.

(f) A function $f : X \to [-\infty, \infty]$ is **lower semi-continuous** if $\{x : x \in X,\ f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$. (Cf. 225H.)

3A3D Compact spaces Any extended series of applications of general topology is likely to involve some new features of compactness. I start with the easy bits, continuing from 2A3Nb.

(a) The first is just a definition of compactness in terms of closed sets instead of open sets. A family $\mathcal{F}$ of sets has the **finite intersection property** if $\bigcap \mathcal{F}_0$ is non-empty for every finite $\mathcal{F}_0 \subseteq \mathcal{F}$. Now a topological space X is compact iff $\bigcap \mathcal{F} \neq \emptyset$ whenever $\mathcal{F}$ is a family of closed subsets of X with the finite intersection property. (Engelking 89, 3.1.1; Bourbaki 66, I.9.1; Dugundji 66, p., 223; Schubert 68, p. 68; Gaal 64, p. 127.)

(b) A marginal generalization of this is the following. Let X be a topological space and $\mathcal{F}$ a family of closed subsets of X with the finite intersection property. If $\mathcal{F}$ contains a compact set then $\bigcap \mathcal{F} \neq \emptyset$. (Apply (a) to $\{K \cap F : F \in \mathcal{F}\}$ where $K \in \mathcal{F}$ is compact.)

(c) In a Hausdorff space, compact subsets are closed. (Engelking 89, 3.1.8; Bourbaki 66, I.9.4; Dugundji 66, p. 226; Schubert 68, p. 70; Gaal 64, p. 138; James 87, p. 77.)

(d) If X is compact, Y is Hausdorff and $\phi : X \to Y$ is continuous and injective, then ϕ is a homeomorphism between X and $\phi[X]$ (where $\phi[X]$ is given the subspace topology). (Engelking 89, 3.1.13; Bourbaki 66, I.9.4; Dugundji 66, p. 226; Schubert 68, p. 71; Gaal 64, p. 207.)

(e) Let X be a regular topological space and A a subset of X. Then the following are equiveridical: (i) A is relatively compact in X (that is, A is included in some compact subset of X, as in 2A3Na); (ii) $\overline{A}$ is compact; (iii) every ultrafilter on X which contains A has a limit in X. **P**(ii)$\Rightarrow$(i) is trivial, and (i)$\Rightarrow$(iii) is a consequence of 2A3R; neither of these requires X to be regular. Now assume (iii) and let $\mathcal{F}$ be an ultrafilter on X containing $\overline{A}$. Set

$$\mathcal{H} = \{B : B \subseteq X, \text{ there is an open set } G \in \mathcal{F} \text{ such that } A \cap G \subseteq B\}.$$

Then $\mathcal{H}$ does not contain $\emptyset$ and $B_1 \cap B_2 \in \mathcal{H}$ whenever B_1, $B_2 \in \mathcal{H}$, so $\mathcal{H}$ is a filter on X, and it contains A. Let $\mathcal{H}^* \supseteq \mathcal{H}$ be an ultrafilter (2A1O). By hypothesis, $\mathcal{H}^*$ has a limit x say. Because $A \in \mathcal{H}^*$, $X \setminus \overline{A}$ is an open set not belonging to $\mathcal{H}^*$, and cannot be a neighbourhood of x; thus x must belong to $\overline{A}$. Let G be an open set containing x. Then there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$ (this is where I use the hypothesis that X is regular). Because $\mathcal{H}^* \to x$, $H \in \mathcal{H}^*$ so $X \setminus \overline{H}$ does not belong to $\mathcal{H}^*$ and therefore does not belong to $\mathcal{H}$. But $X \setminus \overline{H}$ is open, so by the definition of $\mathcal{H}$ it cannot belong to $\mathcal{F}$. As $\mathcal{F}$ is an ultrafilter, $\overline{H} \in \mathcal{F}$ and $G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to x$. As $\mathcal{F}$ is arbitrary, $\overline{A}$ is compact (2A3R). Thus (iii)$\Rightarrow$(ii). **Q**

3A3E Dense sets Recall that a set D in a topological space X is **dense** if $\overline{D} = X$, and that X is **separable** if it has a countable dense subset (2A3Ud).

(a) If X is a topological space, $D \subseteq X$ is dense and $G \subseteq X$ is dense and open, then $G \cap D$ is dense. (Engelking 89, 1.3.6.) Consequently the intersection of finitely many dense open sets is always dense.

(b) If X and Y are topological spaces, $D \subseteq A \subseteq X$, D is dense in A and $f : X \to Y$ is a continuous function, then $f[D]$ is dense in $f[A]$. (Use 3A3Cd.)

3A3F Meager sets Let X be a topological space.

(a) A set $A \subseteq X$ is **nowhere dense** if $\operatorname{int}\overline{A} = \emptyset$, that is, $\operatorname{int}(X \setminus A) = X \setminus \overline{A}$ is dense, that is, for every non-empty open set G there is a non-empty open set $H \subseteq G \setminus A$.

(b) A set $M \subseteq X$ is **meager** if it is expressible as the union of a sequence of nowhere dense sets. A subset of X is **comeager** if its complement is meager.

(c) Any subset of a nowhere dense set is nowhere dense; the union of finitely many nowhere dense sets is nowhere dense. (3A3Ea.)

(d) Any subset of a meager set is meager; the union of countably many meager sets is meager. (314L.)

3A3G Baire's theorem for locally compact Hausdorff spaces Let X be a locally compact Hausdorff space and $\langle G_n\rangle_{n\in\mathbb{N}}$ a sequence of dense open subsets of X. Then $\bigcap_{n\in\mathbb{N}} G_n$ is dense. $*$ (Engelking 89, 3.9.4; Bourbaki 66, IX.5.3; Dugundji 66, p. 249; Schubert 68, p. 148.) Consequently every comeager subset of X is dense.

3A3H Corollary (a) Let X be a compact Hausdorff space. Then a non-empty open subset of X cannot be meager. (Dugundji 66, p. 250; Schubert 68, p. 147.)

(b) Let X be a non-empty locally compact Hausdorff space. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of sets covering X, then there is some $n \in \mathbb{N}$ such that $\operatorname{int}\overline{A}_n$ is non-empty. (Dugundji 66, p. 250.)

3A3I Product spaces (a) Definition Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, and $X = \prod_{i\in I} X_i$ their Cartesian product. We say that a set $G \subseteq X$ is open for the **product topology** if for every $x \in G$ there are a finite $J \subseteq I$ and a family $\langle G_j\rangle_{j\in J}$ such that every G_j is an open set in the corresponding X_j and

$$\{y : y \in X,\, y(j) \in G_j \text{ for every } j \in J\}$$

contains x and is included in G.

(Of course we must check that this does indeed define a topology; see Engelking 89, 2.3.1; Bourbaki 66, I.4.1; Schubert 68, p. 38; Gaal 64, p. 144.)

(b) If $\langle X_i\rangle_{i\in I}$ is a family of topological spaces, with product X, and Y another topological space, a function $\phi : Y \to X$ is continuous iff $\pi_i\phi$ is continuous for every $i \in I$, where $\pi_i(x) = x(i)$ for $x \in X$ and $i \in I$. (Engelking 89, 2.3.6; Bourbaki 66, I.4.1; Dugundji 66, p. 101; Schubert 68, p. 62; James 87, p. 31.)

(c) Let $\langle X_i\rangle_{i\in I}$ be any family of non-empty topological spaces, with product X. If $\mathcal{F}$ is a filter on X and $x \in X$, then $\mathcal{F} \to x$ iff $\pi_i[[\mathcal{F}]] \to x(i)$ for every i, where $\pi_i(y) = y(i)$ for $y \in X$, and $\pi_i[[\mathcal{F}]]$ is the image filter on X_i (2A1Ib). (Bourbaki 66, I.7.6; Schubert 68, p. 61; James 87, p. 32.

(d) The product of any family of Hausdorff spaces is Hausdorff. (Engelking 89, 2.3.11; Bourbaki 66, I.8.2; Schubert 68, p. 62; James 87, p. 87.)

(e) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. If D_i is a dense subset of X_i for each i, then $\prod_{i\in I} D_i$ is dense in $\prod_{i\in I} X_i$. (Engelking 89, 2.3.5.).

(f) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. If F_i is a closed subset of X_i for each i, then $\prod_{i\in I} F_i$ is closed in $\prod_{i\in I} X_i$. (Engelking 89, 2.3.4; Bourbaki 66, I.4.3.)

(g) Let $\langle (X_i, \mathfrak{T}_i)\rangle_{i\in I}$ be a family of topological spaces with product $(X, \mathfrak{T})$. Suppose that each $\mathfrak{T}_i$ is defined by a family P_i of pseudometrics on X_i (2A3F). Then $\mathfrak{T}$ is defined by the family $\mathrm{P} = \{\tilde\rho_i : i \in I, \rho \in \mathrm{P}_i\}$ of pseudometrics on X, where I write $\tilde\rho_i(x, y) = \rho(\pi_i(x), \pi_i(y))$ whenever $i \in I$, $\rho \in \mathrm{P}_i$ and x, $y \in X$, taking

π_i to be the coordinate map from X to X_i, as in (b)-(c). **P** (Compare 2A3Tb). (i) It is easy to check that every $\tilde{\rho}_i$ is a pseudometric on X. Write $\mathfrak{T}_{\mathrm{P}}$ for the topology generated by P. (ii) If $x \in G \in \mathfrak{T}_{\mathrm{P}}$, let $\mathrm{P}' \subseteq \mathrm{P}$ and $\delta > 0$ be such that P' is finite and $\{y : \tau(y,x) \le \delta$ for every $\tau \in \mathrm{P}'\}$ is included in G. Express P' as $\{\tilde{\rho}_j : j \in J,\, \rho \in \mathrm{P}'_j\}$ where $J \subseteq I$ is finite and $\mathrm{P}'_j \subseteq \mathrm{P}_j$ is finite for each $j \in J$. Set

$$G_j = \{t : t \in X_j,\ \rho(t, \pi_j(x)) < \delta \text{ for every } \rho \in \mathrm{P}'_j\}$$

for every $j \in J$. Then $G' = \{y : \pi_j(y) \in G_j$ for every $j \in J\}$ contains x, belongs to $\mathfrak{T}$ and is included in G. As x is arbitrary, $G \in \mathfrak{T}$; as G is arbitrary, $\mathfrak{T}_{\mathrm{P}} \subseteq \mathfrak{T}$. (iii) Every π_i is $(\mathfrak{T}_{\mathrm{P}}, \mathfrak{T}_i)$-continuous, by 2A3H; by (b) above, the identity map from X to itself is $(\mathfrak{T}_{\mathrm{P}}, \mathfrak{T})$-continuous, that is, $\mathfrak{T}_{\mathrm{P}} \subseteq \mathfrak{T}$ and $\mathfrak{T}_{\mathrm{P}} = \mathfrak{T}$, as claimed. **Q**

(h) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X, and Y another topological space. Then a function $f : X \to Y$ is **separately continuous** if for every $j \in I$ and $z \in \prod_{i \in I \setminus \{j\}} X_i$ the function $t \mapsto f(z^\frown{<}t{>}) : X_j \to Y$ is continuous, where $z^\frown{<}t{>}$ is the member of X extending z and such that $(z^\frown{<}t{>})(j) = t$.

3A3J Tychonoff's theorem The product of any family of compact topological spaces is compact.

proof Engelking 89, 3.2.4; Bourbaki 66, I.9.5; Dugundji 66, p. 224; Schubert 68, p. 72; Gaal 64, p. 146 and p. 272; James 87, p. 67.

3A3K The spaces $\{0,1\}^I$, $\mathbb{R}^I$ For any set I, we can think of $\{0,1\}^I$ as the product $\prod_{i \in I} X_i$ where $X_i = \{0,1\}$ for each i. If we endow each X_i with its discrete topology, the product topology is the **usual topology** on $\{0,1\}^I$. Being a product of Hausdorff spaces, it is Hausdorff; by Tychonoff's theorem, it is compact. A subset G of $\{0,1\}^I$ is open iff for every $x \in G$ there is a finite $J \subseteq I$ such that $\{y : y \in \{0,1\}^I,\ y{\restriction}J = x{\restriction}J\} \subseteq G$.

Similarly, the 'usual topology' of $\mathbb{R}^I$ is the product topology when each factor is given its Euclidean topology (cf. 2A3Tc).

3A3L Cluster points of filters (a) Let X be a topological space and $\mathcal{F}$ a filter on X. A point x of X is a **cluster point** of $\mathcal{F}$ if $x \in \overline{A}$ for every $A \in \mathcal{F}$.

(b) For any topological space X, filter $\mathcal{F}$ on X and $x \in X$, x is a cluster point of $\mathcal{F}$ iff there is a filter $\mathcal{G} \supseteq \mathcal{F}$ such that $\mathcal{G} \to x$. (Engelking 89, 1.6.8; Bourbaki 66, I.7.2; Gaal 64, p. 260; James 87, p. 22.)

(c) If $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathbb{R}$, $\alpha \in \mathbb{R}$ and $\lim_{n \to \mathcal{H}} \alpha_n = \alpha$ for every non-principal ultrafilter $\mathcal{H}$ on $\mathbb{N}$ (definition: 2A3Sb), then $\lim_{n \to \infty} \alpha_n = \alpha$. **P?** If $\langle \alpha_n \rangle_{n \in \mathbb{N}} \not\to \alpha$, there is some $\epsilon > 0$ such that $I = \{n : |\alpha_n - \alpha| \ge \epsilon\}$ is infinite. Now $\mathcal{F}_0 = \{F : F \subseteq \mathbb{N},\ I \setminus F$ is finite$\}$ is a filter on $\mathbb{N}$, so there is an ultrafilter $\mathcal{F} \supseteq \mathcal{F}_0$. But now α cannot be $\lim_{n \to \mathcal{F}} \alpha_n$. **XQ**

3A3M Topology bases (a) If X is a set and $\mathbb{T}$ is any non-empty family of topologies on X, $\bigcap \mathbb{T}$ is a topology on X. So if $\mathcal{A}$ is any family of subsets of X, the intersection of all the topologies on X including $\mathcal{A}$ is a topology on X; this is the **topology generated by** $\mathcal{A}$.

(b) If X is a set and $\mathfrak{T}$ is a topology on X, a **base** for $\mathfrak{T}$ is a set $\mathcal{U} \subseteq \mathfrak{T}$ such that whenever $x \in G \in \mathfrak{T}$ there is a $U \in \mathcal{U}$ such that $x \in U \subseteq G$; that is, such that $\mathfrak{T} = \{\bigcup \mathcal{G} : \mathcal{G} \subseteq \mathcal{U}\}$. In this case, of course, $\mathcal{U}$ generates $\mathfrak{T}$.

(c) If X is a set and $\mathcal{E}$ is a family of subsets of X, then $\mathcal{E}$ is a base for a topology on X iff (i) whenever E_1, $E_2 \in \mathcal{E}$ and $x \in E_1 \cap E_2$ then there is an $E \in \mathcal{E}$ such that $x \in E \subseteq E_1 \cap E_2$ (ii) $\bigcup \mathcal{E} = X$. (Engelking 89, p. 12.)

3A3N Uniform convergence (a) Let X be a set, (Y, ρ) a metric space and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of functions from X to Y. We say that $\langle f_n \rangle_{n \in \mathbb{N}}$ converges **uniformly** to a function $f : X \to Y$ if for every $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $\rho(f_n(x), f(x)) \le \epsilon$ whenever $n \ge n_0$ and $x \in X$.

(b) Let X be a topological space and (Y,ρ) a metric space. Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of continuous functions from X to Y converging uniformly to $f:X\to Y$. Then f is continuous. $*$ (Engelking 89, 1.4.7/4.2.19; Gaal 64, p. 202.)

3A3O One-point compactifications Let $(X,\mathfrak{T})$ be a locally compact Hausdorff space. Take any object x_∞ not belonging to X and set $X^*=X\cup\{x_\infty\}$. Let $\mathfrak{T}^*$ be the family of those sets $H\subseteq X^*$ such that $H\cap X\in\mathfrak{T}$ and either $x_\infty\notin H$ or $X\setminus H$ is compact (for $\mathfrak{T}$). Then $\mathfrak{T}^*$ is the unique compact Hausdorff topology on X^* inducing $\mathfrak{T}$ as the subspace topology on X; $(X^*,\mathfrak{T}^*)$ is the **one-point compactification** of $(X,\mathfrak{T})$. (Engelking 89, 3.5.11; Bourbaki 66, I.9.8; Dugundji 66, p. 246.)

3A3P Topologies defined from a sequential convergence: Proposition (a) Let X be a set and $\to^*$ a relation between $X^{\mathbb{N}}$ and X such that whenever $\langle x_n\rangle_{n\in\mathbb{N}}\in X^{\mathbb{N}}$, $x\in X$, $\langle x_n\rangle_{n\in\mathbb{N}}\to^* x$ and $\langle x'_n\rangle_{n\in\mathbb{N}}\in X^{\mathbb{N}}$ is a subsequence of $\langle x_n\rangle_{n\in\mathbb{N}}$ then $\langle x'_n\rangle_{n\in\mathbb{N}}\to^* x$. Then there is a unique topology on X for which a set $F\subseteq X$ is closed iff $x\in F$ whenever $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in F and $\langle x_n\rangle_{n\in\mathbb{N}}\to^* x$. Moreover, if $\langle x_n\rangle_{n\in\mathbb{N}}\to^* x$ then $\langle x_n\rangle_{n\in\mathbb{N}}$ converges to x for this topology.

(b) Let X and Y be sets, and suppose that $\to^*_X\subseteq X^{\mathbb{N}}\times X$, $\to^*_Y\subseteq Y^{\mathbb{N}}\times Y$ are relations with the subsequence property described in (a). Give X and Y the corresponding topologies. If $f:X\to Y$ is a function such that $\langle f(x_n)\rangle_{n\in\mathbb{N}}\to^*_Y f(x)$ whenever $\langle x_n\rangle_{n\in\mathbb{N}}\to^*_X x$, then f is continuous.

proof (a)(i) Let $\mathcal{F}$ be the family of those $F\subseteq X$ which are closed under *-convergence, that is, such that $x\in F$ whenever $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in F and $\langle x_n\rangle_{n\in\mathbb{N}}\to^* x$. Of course $\emptyset$ and X belong to $\mathcal{F}$; also the intersection of any non-empty subset of $\mathcal{F}$ belongs to $\mathcal{F}$. The point is that the union of two members of $\mathcal{F}$ belongs to $\mathcal{F}$. **P** Suppose that F_1, $F_2\in\mathcal{F}$ and that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in $F_1\cup F_2$ *-converging to x. Then there is a subsequence $\langle x'_n\rangle_{n\in\mathbb{N}}$ of $\langle x_n\rangle_{n\in\mathbb{N}}$ which lies entirely in one of the sets; say $x'_n\in F_j$ for every n. By hypothesis, $\langle x'_n\rangle_{n\in\mathbb{N}}\to^* x$, so $x\in F_j\subseteq F_1\cup F_2$. As $\langle x_n\rangle_{n\in\mathbb{N}}$ and x are arbitrary, $F_1\cup F_2\in\mathcal{F}$. **Q** Taking complements, we see that $\{X\setminus F:F\in\mathcal{F}\}$ is a topology on X for which $\mathcal{F}$ is the family of closed sets; and of course there can be only one such topology.

(ii) Now suppose that $\langle x_n\rangle_{n\in\mathbb{N}}\to^* x$. **?** If $\langle x_n\rangle_{n\in\mathbb{N}}$ does not converge topologically to x, then there is an open set G containing x such that $\{n:x_n\notin G\}$ is infinite, that is, there is a subsequence $\langle x'_n\rangle_{n\in\mathbb{N}}$ of $\langle x_n\rangle_{n\in\mathbb{N}}$ such that $x'_n\notin G$ for every n. Now $\langle x'_n\rangle_{n\in\mathbb{N}}\to^* x$; but $X\setminus G$ is supposed to be closed under *-convergence, so this is impossible. **X**

(b) Let $H\subseteq Y$ be open; set $F=Y\setminus H$ and let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a sequence in $f^{-1}[F]$ such that $\langle x_n\rangle_{n\in\mathbb{N}}\to^*_X x$. Then $\langle f(x_n)\rangle_{n\in\mathbb{N}}\to^*_Y f(x)$, so $f(x)\in F$ and $x\in f^{-1}[F]$, As $\langle x_n\rangle_{n\in\mathbb{N}}$ and x are arbitrary, $f^{-1}[F]$ is closed and $f^{-1}[H]$ is open; as H is arbitrary, f is continuous.

3A3Q Miscellaneous definitions Let X be a topological space.

(a) A subset of X is a **zero set** if it is of the form $f^{-1}[\{0\}]$ for some continuous function $f:X\to\mathbb{R}$. A subset of X is a **cozero set** if its complement is a zero set. A subset of X is a **G_δ set** if it is expressible as the intersection of a sequence of open sets.

(b) An **isolated point** of X is a point $x\in X$ such that the singleton set $\{x\}$ is open.

Version of 30.1.08

3A4 Uniformities

I continue the work of §3A3 with some notes on uniformities, so as to be able to discuss completeness and the extension of uniformly continuous functions in non-metrizable contexts (3A4F-3A4H). As in §3A3, most of the individual steps are elementary; I mark exceptions with a $*$.

3A4A Uniformities (a) Let X be a set. A **uniformity** on X is a filter $\mathcal{W}$ on $X\times X$ such that

(i) $(x, x) \in W$ for every $x \in X$, $W \in \mathcal{W}$;
(ii) for every $W \in \mathcal{W}$, $W^{-1} = \{(y, x) : (x, y) \in W\} \in \mathcal{W}$;
(iii) for every $W \in \mathcal{W}$, there is a $V \in \mathcal{W}$ such that

$$V \circ V = \{(x, z) : \exists\, y,\, (x, y) \in V \,\&\, (y, z) \in V\} \subseteq W.$$

It is convenient to allow the special case $X = \emptyset$, $\mathcal{W} = \{\emptyset\}$, even though this is not properly speaking a filter.

The pair $(X, \mathcal{W})$ is now a **uniform space**.

(b) If $\mathcal{W}$ is a uniformity on a set X, it induces a topology on X, the family of sets $G \subseteq X$ such that for every $x \in G$ there is a $W \in \mathcal{W}$ such that $W[\{x\}] = \{y : (x, y) \in W\}$ is included in G. (Engelking 89, 8.1.1; Bourbaki 66, II.1.2; Gaal 64, p. 48; Schubert 68, p. 115; James 87, p. 101.)

(c) We say that a uniformity is **Hausdorff** if it induces a Hausdorff topology.

(d) If U is a linear topological space, then it has an associated uniformity

$$\mathcal{W} = \{W : W \subseteq U \times U, \text{ there is an open set } G \text{ containing } 0 \\ \text{such that } (u, v) \in W \text{ whenever } u - v \in G\},$$

and $\mathcal{W}$ induces the topology of U in the sense of (b) above (Schaefer 66, I.1.4).

3A4B Uniformities and pseudometrics (a) If P is a family of pseudometrics on a set X, then the associated uniformity is the smallest uniformity on X containing all the sets $W(\rho; \epsilon) = \{(x, y) : \rho(x, y) < \epsilon\}$ as ρ runs over P, ϵ over $]0, \infty[$. (Engelking 89, 8.1.18; Bourbaki 66, IX.1.2.)

(b) If $\mathcal{W}$ is the uniformity defined by a family P of pseudometrics, then the topology induced by $\mathcal{W}$ is the topology defined from P (2A3F). (Dugundji 66, p. 203.)

(c) A uniformity $\mathcal{W}$ is **metrizable** if it can be defined by a single metric.

(d) If U is a linear space with a topology defined from a family T of F-seminorms (definition: 2A5B[1]), the uniformity defined from the topology (3A4Ad) coincides with the uniformity defined from the pseudometrics $\rho_\tau(u, v) = \tau(u - v)$ as τ runs over T. (Immediate from the definitions.)

3A4C Uniform continuity (a) If $(X, \mathcal{W})$ and $(Y, \mathcal{V})$ are uniform spaces, a function $\phi : X \to Y$ is **uniformly continuous** if $\{(x, y) : (\phi(x), \phi(y)) \in V\}$ belongs to $\mathcal{W}$ for every $V \in \mathcal{V}$.

(b) The composition of uniformly continuous functions is uniformly continuous. (Bourbaki 66, II.2.1; Schubert 68, p. 118.)

(c) If uniformities $\mathcal{W}$, $\mathcal{V}$ on sets X, Y are defined by non-empty families P, Θ of pseudometrics, then a function $\phi : X \to Y$ is uniformly continuous iff for every $\theta \in \Theta$, $\epsilon > 0$ there are $\rho_0, \dots, \rho_n \in$ P and $\delta > 0$ such that $\theta(\phi(x), \phi(y)) \le \epsilon$ whenever x, $y \in X$ and $\max_{i \le n} \rho_i(x, y) \le \delta$. (Elementary verification.)

(d) A uniformly continuous function is continuous for the induced topologies. (Bourbaki 66, II.2.1; Schubert 68, p. 118; James 87, p. 102.)

(e) Two metrics ρ, σ on a set X are **uniformly equivalent** if they give rise to the same uniformity; that is, if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\rho(x, y) \le \delta \Rightarrow \sigma(x, y) \le \epsilon, \quad \sigma(x, y) \le \delta \Rightarrow \rho(x, y) \le \epsilon.$$

(f) If U and V are linear topological spaces, and $T : U \to V$ is a continuous linear operator, then T is uniformly continuous for the uniformities associated with the topologies of U and V (3A4Ad). (Schaefer 66,).

[1] Later editions only.

3A4D Subspaces (a) If $(X,\mathcal{W})$ is a uniform space and Y is any subset of X, then $\mathcal{W}_Y=\{W\cap(Y\times Y):W\in\mathcal{W}\}$ is a uniformity on Y; it is the **subspace uniformity**. (Bourbaki 66, II.2.4; Schubert 68, p. 122.)

(b) If $\mathcal{W}$ defines a topology $\mathfrak{T}$ on X, then the topology defined by $\mathcal{W}_Y$ is the subspace topology on Y, as defined in 2A3C. (Schubert 68, p. 122; James 87, p. 103.)

(c) If $\mathcal{W}$ is defined by a family P of pseudometrics on X, then $\mathcal{W}_Y$ is defined by $\{\rho\restriction Y\times Y:\rho\in\mathrm{P}\}$. (Elementary verification.)

3A4E Product uniformities (a) If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are uniform spaces, the **product uniformity** is the smallest uniformity $\mathcal{W}$ on $X\times Y$ containing all sets of the form

$$\{((x,y),(x',y')):(x,x')\in U,\,(y,y')\in V\}$$

as U runs over $\mathcal{U}$ and V over $\mathcal{V}$. (Engelking 89, §8.2; Bourbaki 66, II.2.6; Schubert 68, p. 124; James 87, p. 93.)

(b) If $\mathcal{U}$, $\mathcal{V}$ are defined from families P, Θ of pseudometrics, then $\mathcal{W}$ will be defined by the family $\{\tilde\rho:\rho\in\mathrm{P}\}\cup\{\bar\theta:\theta\in\Theta\}$, writing

$$\tilde\rho((x,y),(x',y'))=\rho(x,x'),\quad \bar\theta((x,y),(x',y'))=\theta(y,y')$$

as in 2A3Tb. (Elementary verification.)

(c) If $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ are uniform spaces, a map $\phi:Z\to X\times Y$ is uniformly continuous iff the coordinate maps $\phi_1:Z\to X$ and $\phi_2:Z\to Y$ are uniformly continuous. (Engelking 89, 8.2.1; Bourbaki 66, II.2.6; Schubert 68, p. 125; James 87, p. 93.)

3A4F Completeness (a) If $\mathcal{W}$ is a uniformity on a set X, a filter $\mathcal{F}$ on X is **Cauchy** if for every $W\in\mathcal{W}$ there is an $F\in\mathcal{F}$ such that $F\times F\subseteq W$.

Any convergent filter in a uniform space is Cauchy. (Bourbaki 66, II.3.1; Gaal 64, p. 276; Schubert 68, p. 134; James 87, p. 109.)

(b) A uniform space is **complete** if every Cauchy filter is convergent.

(c) If $\mathcal{W}$ is defined from a family P of pseudometrics, then a filter $\mathcal{F}$ on X is Cauchy iff for every $\rho\in\mathrm{P}$ and $\epsilon>0$ there is an $F\in\mathcal{F}$ such that $\rho(x,y)\le\epsilon$ for all x, $y\in F$; equivalently, for every $\rho\in\mathrm{P}$, $\epsilon>0$ there is an $x\in X$ such that $U(x;\rho;\epsilon)\in\mathcal{F}$. (Elementary verification.)

(d) A complete subspace of a Hausdorff uniform space is closed. (Engelking 89, 8.3.6; Bourbaki 66, II.3.4; Schubert 68, p. 135; James 87, p. 148.) A closed subspace of a complete uniform space is complete under the subspace uniformity (references).

(e) A metric space is complete iff every Cauchy sequence converges (cf. 2A5H). (Schubert 68, p. 141; Gaal 64, p. 276; James 87, p. 150.)

(f) If (X,ρ) is a complete metric space, $D\subseteq X$ a dense subset, (Y,σ) a metric space and $f:X\to Y$ is an **isometry** (that is, $\sigma(f(x),f(x'))=\rho(x,x')$ for all x, $x'\in X$), then $f[X]$ is precisely the closure of $f[D]$ in Y. (For $f[X]$ must be complete, and we can use (d).)

(g) If U is a linear space with a linear space topology and the associated uniformity (3A4Ad), then a filter $\mathcal{F}$ on U is Cauchy iff for every open set G containing 0 there is an $F\in\mathcal{F}$ such that $F-F\subseteq G$ (cf. 2A5F). (Immediate from the definitions.)

3A4G Extension of uniformly continuous functions: Theorem If $(X,\mathcal{W})$ is a uniform space, $(Y,\mathcal{V})$ is a complete uniform space, $D\subseteq X$ is a dense subset of X, and $\phi:D\to Y$ is uniformly continuous (for the subspace uniformity of D), then there is a uniformly continuous $\hat\phi:X\to Y$ extending ϕ. If Y is Hausdorff, the extension is unique. $*$ (Engelking 89, 8.3.10; Bourbaki 66, II.3.6; Gaal 64, p. 300; Schubert 68, p. 137; James 87, p. 152.)

In particular, if (X,ρ) is a metric space, (Y,σ) is a complete metric space, $D\subseteq X$ is a dense subset, and $\phi:D\to Y$ is an isometry, then there is a unique isometry $\hat\phi:X\to Y$ extending ϕ.

3A4H Completions (a) Theorem If $(X,\mathcal{W})$ is any Hausdorff uniform space, then we can find a complete Hausdorff uniform space $(\hat{X},\hat{\mathcal{W}})$ in which X is embedded as a dense subspace; moreover, any two such spaces are essentially unique. $*$ (Engelking 89, 8.3.12; Bourbaki 66, II.3.7; Gaal 64, p. 297 & p. 300; Schubert 68, p. 139; James 87, p. 156.)

(b) Such a space $(\hat{X},\hat{\mathcal{W}})$ is called a **completion** of $(X,\mathcal{W})$. Because it is unique up to isomorphism as a uniform space, we may call it 'the' completion.

(c) If $\mathcal{W}$ is the uniformity defined by a metric ρ on a set X, then there is a unique extension of ρ to a metric $\hat{\rho}$ on $\hat{X}$ defining the uniformity $\hat{\mathcal{W}}$. (Bourbaki 66, IX.1.3.)

3A4I A note on metric spaces I mention some elementary facts. Let (X,ρ) be a metric space. If $x\in X$ and $A\subseteq X$ is non-empty, set

$$\rho(x,A)=\inf_{y\in A}\rho(x,y).$$

Then $\rho(x,A)=0$ iff $x\in\overline{A}$ (2A3Kb). If $B\subseteq X$ is another non-empty set, then

$$\rho(x,B)\le\rho(x,A)+\sup_{y\in A}\rho(y,B).$$

In particular, $\rho(x,\overline{A})=\rho(x,A)$. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of non-empty sets with union A, then

$$\rho(x,A)=\lim_{n\to\infty}\rho(x,A_n).$$

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3A5 Normed spaces

I run as quickly as possible over the results, nearly all of them standard elements of any introductory course in functional analysis, which I find myself calling on in this volume. As in the corresponding section of Volume 2 (§2A4), a large proportion of these are valid for both real and complex normed spaces, but as the present volume is almost exclusively concerned with real linear spaces I leave this unsaid, except in 3A5M, and if in doubt you may suppose for the time being that scalars belong to the field $\mathbb{R}$. A couple of the most basic results will be used in their complex forms in Volume 4.

3A5A The Hahn-Banach theorem: analytic forms The Hahn-Banach theorem is one of the central ideas of functional analysis, both finite- and infinite-dimensional, and appears in a remarkable variety of forms. I list those formulations which I wish to quote, starting with those which are more or less 'analytic', according to the classification of Bourbaki 87. Recall that if U is a normed space I write U^* for the Banach space of bounded linear functionals on U.

(a) Let U be a linear space and $p:U\to[0,\infty[$ a functional such that $p(u+v)\le p(u)+p(v)$ and $p(\alpha u)=\alpha p(u)$ whenever u, $v\in U$ and $\alpha\ge 0$. Then for any $u_0\in U$ there is a linear functional $f:U\to\mathbb{R}$ such that $f(u_0)=p(u_0)$ and $f(u)\le p(u)$ for every $u\in U$. (Rudin 91, 3.2; Dunford & Schwartz 57, II.3.10.)

(b) Let U be a normed space and V a linear subspace of U. Then for any $f\in V^*$ there is a $g\in U^*$, extending f, with $\|g\|=\|f\|$. (363R; Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.11; Lang 93, p. 69; Wilansky 64, p. 66; Taylor 64, 3.7-B & 4.3-A.)

(c) If U is a normed space and $u\in U$ there is an $f\in U^*$ such that $\|f\|\le 1$ and $f(u)=\|u\|$. (Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.14; Wilansky 64, p. 67; Taylor 64, 3.7-C & 4.3-B.)

(d) If U is a normed space and $V\subseteq U$ is a linear subspace which is not dense, then there is a non-zero $f\in U^*$ such that $f(v)=0$ for every $v\in V$. (Rudin 91, 3.5; Dunford & Schwartz 57, II.3.12; Taylor 64, 4.3-D.)

(e) If U is a normed space, U^* separates the points of U. (Rudin 91, 3.4; Lang 93, p. 70; Dunford & Schwartz 57, II.3.14.)

3A5B Cones (a) Let U be a linear space. A **convex cone** (with apex 0) is a set $C \subseteq U$ such that $\alpha u + \beta v \in C$ whenever $u, v \in C$ and $\alpha, \beta \geq 0$. The intersection of any family of convex cones is a convex cone, so for every subset A of U there is a smallest convex cone including A.

(b) Let U be a normed space. Then the closure of a convex cone is a convex cone. (Bourbaki 87, II.2.6; Dunford & Schwartz 57, V.2.1.)

3A5C Hahn-Banach theorem: geometric forms (a) Let U be a normed space and $C \subseteq U$ a convex set such that $\|u\| \geq 1$ for every $u \in C$. Then there is an $f \in U^*$ such that $\|f\| \leq 1$ and $f(u) \geq 1$ for every $u \in C$. (Dunford & Schwartz 57, V.1.12.)

(b) Let U be a normed space and $C \subseteq U$ a non-empty convex set such that $0 \notin \overline{C}$. Then there is an $f \in U^*$ such that $\inf_{u \in C} f(u) > 0$. (Bourbaki 87, II.4.1; Rudin 91, 3.4; Lang 93, p. 70; Dunford & Schwartz 57, V.2.12.)

(c) Let U be a normed space, C a closed convex subset of U containing 0, and u a point of $U \setminus C$. Then there is an $f \in U^*$ such that $f(u) > 1$ and $f(v) \leq 1$ for every $v \in C$. (Apply (b) to $C - u$ to find a $g \in U^*$ such that $g(u) < \inf_{v \in C} g(v)$ and now set $f = -\frac{1}{\alpha} g$ where $g(u) < \alpha < \inf_{v \in C} g(v)$).

3A5D Separation from finitely-generated cones Let U be a linear space over $\mathbb{R}$ and $u, v_0, \dots, v_n$ points of U such that u does not belong to the convex cone generated by $\{v_0, \dots, v_n\}$. Then there is a linear functional $f : U \to \mathbb{R}$ such that $f(v_i) \geq 0$ for every i and $f(u) < 0$.

proof (a) If U is finite-dimensional this is covered by Gale 60, p. 56.

(b) For the general case, let V be the linear subspace of U generated by $u, v_0, \dots, v_n$. Then there is a linear functional $f_0 : V \to \mathbb{R}$ such that $f_0(u) < 0 \leq f_0(v_i)$ for every i. By Zorn's Lemma, there is a maximal linear subspace $W \subseteq U$ such that $W \cap V = \{0\}$. Now $W + V = U$ (for if $u \notin W + V$, the linear subspace W' generated by $W \cup \{u\}$ still has trivial intersection with V), so we have an extension of f_0 to a linear functional $f : U \to \mathbb{R}$ defined by setting $f(v + w) = f_0(v)$ whenever $v \in V$ and $w \in W$. Now $f(u) < 0 \leq \min_{i \leq n} f(v_i)$, as required.

3A5E Weak topologies (a) Let U be any linear space over $\mathbb{R}$ and W a subset of the space U' of all linear functionals from U to $\mathbb{R}$. Then I write $\mathfrak{T}_s(U, W)$ for the linear space topology defined by the method of 2A5B from the seminorms $u \mapsto |f(u)|$ as f runs over W. (Bourbaki 87, II.6.2; Rudin 91, 3.10; Dunford & Schwartz 57, V.3.2; Taylor 64, 3.81.)

(b) I note that the weak topology of a normed space U (2A5Ia) is $\mathfrak{T}_s(U, U^*)$, while the weak* topology of U^* (2A5Ig) is $\mathfrak{T}_s(U^*, W)$ where W is the canonical image of U in U^{**}. (Rudin 91, 3.14.)

(c) Let U and V be linear spaces over $\mathbb{R}$ and $T : U \to V$ a linear operator. If $W \subseteq U'$ and $Z \subseteq V'$ are such that $gT \in W$ for every $g \in Z$, then T is continuous for $\mathfrak{T}_s(U, W)$ and $\mathfrak{T}_s(V, Z)$. (Bourbaki 87, II.6.4.)

(d) If U and V are normed spaces and $T : U \to V$ is a bounded linear operator then we have an **adjoint** operator $T' : V^* \to U^*$ defined by saying that $T'g = gT$ for every $g \in V^*$. T' is linear and is continuous for the weak* topologies of U^* and V^*. (Bourbaki 87, II.6.4; Dunford & Schwartz 57, §VI.2; Taylor 64, 4.5.)

(e) If U is a normed space and $A \subseteq U$ is convex, then the closure of A for the norm topology is the same as the closure of A for the weak topology of U. In particular, norm-closed convex subsets (for instance, norm-closed linear subspaces) of U are closed for the weak topology. (Rudin 91, 3.12; Lang 93, p. 88; Dunford & Schwartz 57, V.3.13.)

3A5F Weak* topologies: Theorem If U is a normed space, the unit ball of U^* is compact and Hausdorff for the weak* topology. (Rudin 91, 3.15; Lang 93, p. 71; Dunford & Schwartz 57, V.4.2; Taylor 64, 4.61-A.)

3A5G Reflexive spaces (a) A normed space U is **reflexive** if every member of U^{**} is of the form $f \mapsto f(u)$ for some $u \in U$.

(b) A normed space is reflexive iff bounded sets are relatively weakly compact. (Dunford & Schwartz 57, V.4.8; Taylor 64, 4.61-C.)

(c) If U is a reflexive space, $\langle u_n \rangle_{n\in\mathbb{N}}$ is a bounded sequence in U and $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$, then $\lim_{n\to\mathcal{F}} u_n$ is defined in U for the weak topology. (Use (b) and 2A3Se.)

3A5H (a) Uniform Boundedness Theorem Let U be a Banach space, V a normed space, and $A \subseteq \mathsf{B}(U;V)$ a set such that $\{Tu : T \in A\}$ is bounded in V for every $u \in U$. Then A is bounded in $\mathsf{B}(U;V)$. (Rudin 91, 2.6; Dunford & Schwartz 57, II.3.21; Taylor 64, 4.4-E.)

(b) Corollary If U is a normed space and $A \subseteq U$ is such that $f[A]$ is bounded for every $f \in U^*$, then A is bounded. (Wilansky 64, p. 117; Taylor 64, 4.4-AS.) Consequently any relatively weakly compact set in U is bounded. (Rudin 91, 3.18.)

***3A5I Strong operator topologies** If U and V are normed spaces, the **strong operator topology** on $\mathsf{B}(U;V)$ is that defined by the seminorms $T \mapsto \|Tu\|$ as u runs over U. If U is a Banach space, V is a normed space and $A \subseteq \mathsf{B}(U;V)$, then A is relatively compact for the strong operator topology iff $\{Tu : T \in A\}$ is relatively compact in V for every $u \in U$. (Put 3A5Ha and 2A3R together.)

3A5J Completions Let U be a normed space.

(a) Recall that U has a metric ρ associated with the norm (2A4Bb), and that the topology defined by ρ is a linear space topology (2A5D, 2A5B). This topology defines a uniformity $\mathcal{W}$ (3A4Ad) which is also the uniformity defined by ρ (3A4Bd). The norm itself is a uniformly continuous function from U to $\mathbb{R}$ (because $|\|u\| - \|v\|| \le \|u - v\|$ for all u, $v \in U$).

(b) Let $(\hat{U}, \hat{\mathcal{W}})$ be the uniform space completion of $(U, \mathcal{W})$ (3A4H). Then addition and scalar multiplication and the norm extend uniquely to make $\hat{U}$ a Banach space. (Schaefer 66, I.1.5; Lang 93, p. 78.)

(c) If U and V are Banach spaces with dense linear subspaces U_0 and V_0, then any norm-preserving isomorphism between U_0 and V_0 extends uniquely to a norm-preserving isomorphism between U and V (use 3A4G).

3A5K Normed algebras If U is a normed algebra (2A4J), its multiplication, regarded as a function from $U \times U$ to U, is continuous. (Wilansky 64, p. 259.)

3A5L Compact operators Let U and V be Banach spaces.

(a) A linear operator $T : U \to V$ is **compact** if $\{Tu : \|u\| \le 1\}$ is relatively compact in V for the topology defined by the norm of V.

(b) A linear operator $T : U \to V$ is **weakly compact** if $\{Tu : \|u\| \le 1\}$ is relatively weakly compact in V. Of course compact operators are weakly compact; because weakly compact sets must be norm-bounded (3A5Hb), weakly compact operators are bounded.

3A5M Hilbert spaces I mentioned the phrases 'inner product space', 'Hilbert space' briefly in 244N and 244P, without explanation, as I did not there rely on any of the abstract theory of these spaces. For the main result of §396 we need one of their fundamental properties, so I now skim over the definitions.

(a) An **inner product space** is a linear space U over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ together with an operator $(\ |\ ) : U \times U \to \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$ such that

$$(u_1 + u_2|v) = (u_1|v) + (u_2|v), \quad (\alpha u|v) = \alpha(u|v), \quad (u|v) = \overline{(v|u)}$$

(the complex conjugate of $(v|u)$),

$$(u|u) \ge 0, \quad u = 0 \text{ whenever } (u|u) = 0$$

for all u, u_1, u_2, $v \in U$ and $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

(b) If U is any inner product space, we have a norm on U defined by setting $\|u\| = \sqrt{(u|u)}$ for every $u \in U$, and $|(u|v)| \le \|u\|\|v\|$ for all u, $v \in U$. (Taylor 64, 3.2-B.)

(c) A **Hilbert space** is an inner product space which is a Banach space under the norm of (b) above, that is, is complete in the metric defined from its norm.

(d) If U is a Hilbert space, $C \subseteq U$ is a non-empty closed convex set, and $u \in U$, then there is a unique $v \in C$ such that $\|u - v\| = \inf_{w \in C} \|u - w\|$. (Taylor 64, 4.81-A; compare 244Yn.)

(e) If U is an inner product space, $C \subseteq U$ is a convex set, u, $u' \in U$ and v, $v' \in C$ are such that that $\|u - v\| = \inf_{w \in C} \|u - w\|$ and $\|u' - v'\| = \inf_{w \in C} \|u' - w\|$, then $\|v' - v\| \le \|u' - u\|$. **P** If $v' = v$ this is trivial. Otherwise, set $\gamma = \|v' - v\|$ and $e = \frac{1}{\gamma}(v' - v)$. If $0 < \alpha \le \gamma$, $v + \alpha e \in C$ so

$$\|u - v\|^2 \le \|u - v - \alpha e\|^2 = \|u - v\|^2 - 2\alpha\,\mathcal{R}\mathrm{e}(u - v|e) + \alpha^2$$

and $2\,\mathcal{R}\mathrm{e}(u - v|e) \le \alpha$; it follows that $\mathcal{R}\mathrm{e}(u - v|e) \le 0$. Similarly, $\mathcal{R}\mathrm{e}(u' - v'|e) \ge 0$, and

$$\|v' - v\| = (v' - v|e) \le \mathcal{R}\mathrm{e}(u' - u|e) \le \|u' - u\|\|e\| = \|u' - u\|$$

as claimed. **Q**

***3A5N Bounded sets in linear topological spaces** There is a point in §377 where a concept from the general theory of linear topological spaces helps an idea to flow more freely. Let U be a linear topological space over $\genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$.

(a) A set $A \subseteq U$ is **bounded** if for every neighbourhood G of 0 there is an $n \in \mathbb{N}$ such that $A \subseteq nG$.

(b) If $A \subseteq U$ is bounded, then
(i) every subset of A is bounded;
(ii) the closure of A is bounded;
(iii) αA is bounded for every $\alpha \in \genfrac{}{}{0pt}{}{\mathbb{R}}{\mathbb{C}}$;
(iv) $A \cup B$ and $A + B$ are bounded for every bounded $B \subseteq U$;
(v) if V is another linear topological space, and $T : U \to V$ is a continuous linear operator, then $T[A]$ is bounded.

(c) If $A \subseteq U$ is relatively compact, it is bounded.

(d) If U is a normed space, and $A \subseteq U$, then the following are equiveridical:
(i) A is bounded in the sense of (a) above for the norm topology of U;
(ii) A is bounded in the sense of 2A4Bc, that is, $\{\|u\| : u \in A\}$ is bounded above in $\mathbb{R}$;
(iii) A is bounded for the weak topology of U.

proof Schaefer 66, §I.5; Köthe 69, §15.6. For (d-iii), use 3A5Hb.

Version of 6.8.08

3A6 Group Theory

For Chapter 38 we need four definitions and two results from elementary abstract group theory.

3A6A Definition If G is a group, I will say that an element g of G is an **involution** if its order is 2, that is, $g^2 = e$, the identity of G, but $g \neq e$.

3A6B Definition If G is a group, the set $\mathrm{Aut}\, G$ of **automorphisms** of G (that is, bijective homomorphisms from G to itself) is a group. For $g \in G$ define $\hat{g} : G \to G$ by writing $\hat{g}(h) = ghg^{-1}$ for every $h \in G$; then $\hat{g} \in \mathrm{Aut}\, G$, and the map $g \mapsto \hat{g}$ is a homomorphism from G onto a normal subgroup J of $\mathrm{Aut}\, G$ (Rotman 84, p. 130). We call J the group of **inner automorphisms** of G. Members of $(\mathrm{Aut}\, G) \setminus J$ are called **outer automorphisms**.

3A6C Normal subgroups For any group G, the family of normal subgroups of G, ordered by $\subseteq$, is a Dedekind complete lattice, with $H \vee K = HK$ and $H \wedge K = H \cap K$. (Davey & Priestley 90, 2.8 & 2.19.)

Version of 5.3.08

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

3A3P Definitions This paragraph, referred to in the 2003 edition of Volume 4, is now 3A3Q.

3A5K Compact operators This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 3A5L.

3A5L Inner product spaces This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 3A5M.

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References for Volume 3

Anderson I. [87] *Combinatorics of Finite Sets.* Oxford U.P., 1987. [332Xk, 3A1K.]

Anzai H. [51] 'On an example of a measure-preserving transformation which is not conjugate to its inverse', Proc. Japanese Acad. Sci. 27 (1951) 517-522. [§382 *notes*.]

Balcar B., Główczyński W. & Jech T. [98] 'The sequential topology on complete Boolean algebras', Fundamenta Math. 155 (1998) 59-78. [393L, 393Q.]

Balcar B., Jech T. & Pazák T. [05] 'Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann', Bull. London Math. Soc. 37 (2005) 885-898. [393L, 393Q.]

Becker H. & Kechris A.S. [96] *The descriptive set theory of Polish group actions.* Cambridge U.P., 1996 (London Math. Soc. Lecture Note Series 232). [§395 *notes*.]

Bekkali M. & Bonnet R. [89] 'Rigid Boolean algebras', pp. 637-678 in Monk 89. [384L.]

Bellow A. & Kölzow D. [76] *Measure Theory, Oberwolfach 1975.* Springer, 1976 (Lecture Notes in Mathematics 541).

Bhaskara Rao, K.P.S. & Bhaskara Rao, M. *Theory of Charges.* Academic, 1983.

Billingsley P. [65] *Ergodic Theory and Information.* Wiley, 1965. [§372 *notes*.]

Bollobás B. [79] *Graph Theory.* Springer, 1979. [332Xk, 3A1K.]

Bose R.C. & Manvel B. [84] *Introduction to Combinatorial Theory.* Wiley, 1984. [3A1K.]

Bourbaki N. [66] *General Topology.* Hermann/Addison-Wesley, 1968. [§3A3, §3A4.]

Bourbaki N. [68] *Theory of Sets.* Hermann/Addison-Wesley, 1968. [§315 *notes*.]

Bourbaki N. [87] *Topological Vector Spaces.* Springer, 1987. [§3A5.]

Bukhvalov A.V. [95] 'Optimization without compactness, and its applications', pp. 95-112 in Huijsmans Kaashoek Luxemburg & Pagter 95. [367U.]

Burke M.R. [93] 'Liftings for Lebesgue measure', pp. 119-150 in Judah 93. [341L, 345F.]

Burke M.R. [n95] 'Consistent liftings', privately circulated, 1995. [346Ya.]

Burnside W. [1911] *Theory of Groups of Finite Order.* Cambridge U.P., 1911 (reprinted by Dover, 1955). [§384 *notes*.]

Chacon R.V. [69] 'Weakly mixing transformations which are not strongly mixing', Proc. Amer. Math. Soc. 22 (1969) 559-562. [372R.]

Chacon R.V. & Krengel U. [64] 'Linear modulus of a linear operator', Proc. Amer. Math. Soc. 15 (1964) 553-559. [§371 *notes*.]

Choksi J.R. & Prasad V.S. [82] 'Ergodic theory of homogeneous measure algebras', pp. 367-408 of Kölzow & Maharam-Stone 82. [383I.]

Cohn H. [06] 'A short proof of the simple continued fraction expansion of e', Amer. Math. Monthly 113 (2006) 57-62; `arXiv:math.NT/0601660`. [372L.]

Coleman A.J. & Ribenboim P. [67] (eds.) *Proceedings of the Symposium in Analysis, Queen's University, June 1967.* Queen's University, Kingston, Ontario, 1967.

Comfort W.W. & Negrepontis S. [82] *Chain Conditions in Topology.* Cambridge U.P., 1982. [§391 *notes*.]

Cziszár I. [67] 'Information-type measures of difference of probability distributions and indirect observations', Studia Scientiarum Math. Hungarica 2 (1967) 299-318. [386G.]

Davey B.A. & Priestley H.A. [90] 'Introduction to Lattices and Order', Cambridge U.P., 1990. [3A6C.]

Dugundji J. [66] *Topology.* Allyn & Bacon, 1966. [§3A3, 3A4Bb.]

Dunford N. [1936] 'Integration and linear operators', Trans. Amer. Math. Soc. 40 (1936) 474-494. [§376 *notes*.]

Dunford N. & Schwartz J.T. [57] *Linear Operators I.* Wiley, 1957 (reprinted 1988). [§356 *notes*, §371 *notes*, §376 *notes*, §3A5.]

Dye H.A. [59] 'On groups of measure preserving transformations I', Amer. J. Math. 81 (1959) 119-159. [§388 *notes*.]

Eigen S.J. [82] 'The group of measure-preserving transformations of $[0,1]$ has no outer automorphisms', Math. Ann. 259 (1982) 259-270. [384D.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [§3A3, §3A4.]

Enderton H.B. [77] *Elements of Set Theory.* Academic, 1977. [§3A1.]

Erdős P. & Oxtoby J.C. [55] 'Partitions of the plane into sets having positive measure in every non-null product set', Trans. Amer. Math. Soc. 79 (1955) 91-102. [§325 *notes*.]

Fathi A. [78] 'Le groupe des transformations de $[0,1]$ qui préservent la mesure de Lebesgue est un groupe simple', Israel J. Math. 29 (1978) 302-308. [§382 *notes*, 383I.]

Fremlin D.H. [74a] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [Chap. 35 *intro*., 354Yb, §355 *notes*, §356 *notes*, §363 *notes*, §365 *notes*, §371 *notes*, §376 *notes*.]

Fremlin D.H. [74b] 'A characterization of L-spaces', Indag. Math. 36 (1974) 270-275. [§371 *notes*.]

Fremlin D.H., de Pagter B. & Ricker W.J. [05] 'Sequential closedness of Boolean algebras of projections in Banach spaces', Studia Math. 167 (2005) 45-62. [§323 *notes*.]

Frolík Z. [68] 'Fixed points of maps of extremally disconnected spaces and complete Boolean algebras', Bull. Acad. Polon. Sci. 16 (1968) 269-275. [382E.]

Gaal S.A. [64] *Point Set Topology.* Academic, 1964. [§3A3, §3A4.]

Gaifman H. [64] 'Concerning measures on Boolean algebras', Pacific J. Math. 14 (1964) 61-73. [§391 *notes*.]

Gale D. [60] *The theory of linear economic models.* McGraw-Hill, 1960. [3A5D.]

Gnedenko B.V. & Kolmogorov A.N. [54] *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, 1954. [§342 *notes*.]

Graf S. & Weizsäcker H.von [76] 'On the existence of lower densities in non-complete measure spaces', pp. 155-158 in Bellow & Kölzow 76. [341L.]

Hajian A. & Ito Y. [69] 'Weakly wandering sets and invariant measures for a group of transformations', J. of Math. and Mech. 18 (1969) 1203-1216. [396B.]

Hajian A., Ito Y. & Kakutani S. [75] 'Full groups and a theorem of Dye', Advances in Math. 17 (1975) 48-59. [§388 *notes*.]

Halmos P.R. [1948] 'The range of a vector measure', Bull. Amer. Math. Soc. 54 (1948) 416-421. [326Yk.]

Halmos P.R. [60] *Naive Set Theory.* Van Nostrand, 1960. [3A1D.]

Huijsmans C.B., Kaashoek M.A., Luxemburg W.A.J. & de Pagter B. [95] (eds.) *Operator Theory in Function Spaces and Banach Lattices.* Birkhäuser, 1995.

Ionescu Tulcea C. & Ionescu Tulcea A. [69] *Topics in the Theory of Lifting.* Springer, 1969. [§341 *notes*.]

James I.M. [87] *Topological and Uniform Spaces.* Springer, 1987. [§3A3, §3A4.]

Jech T. [03] *Set Theory, Millennium Edition.* Springer, 2002. [§3A1.]

Jech T. [08] 'Algebraic characterizations of measure algebras', Proc. Amer. Math. Soc. 136 (2008) 1285-1294. [393Xj.]

Johnson R.A. [80] 'Strong liftings which are not Borel liftings', Proc. Amer. Math. Soc. 80 (1980) 234-236. [345F.]

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Kakutani S. [1941] 'Concrete representation of abstract L-spaces and the mean ergodic theorem', Annals of Math. 42 (1941) 523-537. [369E.]

Kalton N.J., Peck N.T. & Roberts J.W. [84] 'An F-space sampler', Cambridge U.P., 1984. [§375 *notes*.]

Kalton N.J. & Roberts J.W. [83] 'Uniformly exhaustive submeasures and nearly additive set functions', Trans. Amer. Math. Soc. 278 (1983) 803-816. [392D, §392 *notes*.]

Kantorovich L.V., Vulikh B.Z. & Pinsker A.G. [50] *Functional Analysis in Partially Ordered Spaces*, Gostekhizdat, 1950. [391D.]

Kawada Y. [1944] 'Über die Existenz der invarianten Integrale', Jap. J. Math. 19 (1944) 81-95. [§395 *notes*.]

Kelley J.L. [59] 'Measures on Boolean algebras', Pacific J. Math. 9 (1959) 1165-1177. [§391 *notes*.]

Kolmogorov A.N. [58] 'New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces', Dokl. Akad. Nauk SSSR 119 (1958) 861-864. [385P.]

Kölzow D. & Maharam-Stone D. [82] (eds.) *Measure Theory Oberwolfach 1981.* Springer, 1982 (Lecture Notes in Math. 945).

Koppelberg S. [89] *General Theory of Boolean Algebras*, vol. 1 of Monk 89. [Chap. 31 *intro*., §332 *notes*.]

Köthe G. [69] *Topological Vector Spaces I.* Springer, 1969. [§356 *notes*, 3A5N.]

Kranz P. & Labuda I. [93] (eds.) *Proceedings of the Orlicz Memorial Conference, 1991*, unpublished manuscript, available from first editor (`mmkranz@olemiss.edu`).

Krengel, U. [63] '"Uber den Absolutbetrag stetiger linearer Operatoren und seine Anwendung auf ergodische Zerlegungen', Math. Scand. 13 (1963) 151-187. [371Xb.]

Krieger W. [76] 'On ergodic flows and the isomorphism of factors', Math. Ann. 223 (1976) 19-70. [§388 *notes*.]

Krivine J.-L. [71] *Introduction to Axiomatic Set Theory.* D.Reidel, 1971. [§3A1.]

Kullback S. [67] 'A lower bound for discrimination information in terms of variation', IEEE Trans. on Information Theory 13 (1967) 126-127. [386G.]

Kunen K. [80] *Set Theory.* North-Holland, 1980. [§3A1.]

Kwapien S. [73] 'On the form of a linear operator on the space of all measurable functions', Bull. Acad. Polon. Sci. 21 (1973) 951-954. [§375 *notes*.]

Lang S. [93] *Real and Functional Analysis.* Springer, 1993. [§3A5.]

Liapounoff A.A. [1940] 'Sur les fonctions-vecteurs complètement additives', Bull. Acad. Sci. URSS (Izvestia Akad. Nauk SSSR) 4 (1940) 465-478. [326H.]

Lindenstrauss J. & Tzafriri L. [79] *Classical Banach Spaces II.* Springer, 1979, reprinted in Lindenstrauss & Tzafriri 96. [§354 *notes*, 374Xj.]

Lindenstrauss J. & Tzafriri L. [96] *Classical Banach Spaces I & II.* Springer, 1996.

Lipschutz S. [64] *Set Theory and Related Topics.* McGraw-Hill, 1964 (Schaum's Outline Series). [3A1D.]

Luxemburg W.A.J. [67a] 'Is every integral normal?', Bull. Amer. Math. Soc. 73 (1967) 685-688. [363S.]

Luxemburg W.A.J. [67b] 'Rearrangement-invariant Banach function spaces', pp. 83-144 in Coleman & Ribenboim 67. [§374 *notes*.]

Luxemburg W.A.J. & Zaanen A.C. [71] *Riesz Spaces I.* North-Holland, 1971. [Chap. 35 *intro*.]

Macheras N.D., Musiał K. & Strauss W. [99] 'On products of admissible liftings and densities', J. for Analysis and its Applications 18 (1999) 651-668. [346G.]

Macheras N.D. & Strauss W. [95] 'Products of lower densities', J. for Analysis and its Applications 14 (1995) 25-32. [346Xf.]

Macheras N.D. & Strauss W. [96a] 'On products of almost strong liftings', J. Australian Math. Soc. (A) 60 (1996) 1-23. [346Yc.]

Macheras N.D. & Strauss W. [96b] 'The product lifting for arbitrary products of complete probability spaces', Atti Sem. Math. Fis. Univ. Modena 44 (1996) 485-496. [346H, 346Yd.]

Maharam D. [1942] 'On homogeneous measure algebras', Proc. Nat. Acad. Sci. U.S.A. 28 (1942) 108-111. [331F, 332B.]

Maharam D. [1947] 'An algebraic characterization of measure algebras', Ann. Math. 48 (1947) 154-167. [393J.]

Maharam D. [58] 'On a theorem of von Neumann', Proc. Amer. Math. Soc. 9 (1958) 987-994. [§341 *notes*, §346 *notes*.]

Marczewski E. [53] 'On compact measures', Fund. Math. 40 (1953) 113-124. [342A.]

McCune W. [97] 'Solution of the Robbins problem', J. Automated Reasoning 19 (1997) 263-276. [311Yc.]

Miller B.D. [04] PhD Thesis, University of California, Berkeley, 2004. [382Xc, §382 *notes*.]

Monk J.D. [89] (ed.) *Handbook of Boolean Algebra.* North-Holland, 1989.

Nadkarni M.G. [90] 'On the existence of a finite invariant measure', Proc. Indian Acad. Sci., Math. Sci. 100 (1990) 203-220. [§395 *notes*.]

Ornstein D.S. [74] *Ergodic Theory, Randomness and Dynamical Systems.* Yale U.P., 1974. [§387 *notes*.]

Ornstein D.S. & Shields P.C. [73] 'An uncountable family of K-automorphisms', Advances in Math. 10 (1973) 63-88. [§382 *notes*.]

Perović Ž. & Veličković B. [18] 'Ranks of Maharam algebras', Advances in Math. 330 (2018) 253-279. [394A.]

Petersen K. [83] *Ergodic Theory.* Cambridge U.P., 1983. [328Xa, 385C, §385 *notes*, 386E.]

Roberts J.W. [93] 'Maharam's problem', in Kranz & Labuda 93. [§394 *notes*.]

Rotman J.J. [84] 'An Introduction to the Theory of Groups', Allyn & Bacon, 1984. [§384 *notes*, 3A6B.]

Rudin W. [91] *Functional Analysis.* McGraw-Hill, 1991. [§3A5.]

Ryzhikov V.V. [93] 'Factorization of an automorphism of a complete Boolean algebra into a product of three involutions', Mat. Zametki (=Math. Notes of Russian Acad. Sci.) 54 (1993) 79-84. [§382 *notes*.]

Sazonov V.V. [66] 'On perfect measures', A.M.S. Translations (2) 48 (1966) 229-254. [§342 *notes*.]

Schaefer H.H. [66] *Topological Vector Spaces.* MacMillan, 1966; reprinted with corrections Springer, 1971. [3A4A, 3A5J, 3A5N.]

Schaefer H.H. [74] *Banach Lattices and Positive Operators.* Springer, 1974. [Chap. 35 *intro*., §354 *notes*.]

Schubert H. [68] *Topology.* Allyn & Bacon, 1968. [§3A3, §3A4.]

Shelah S. [98] 'The lifting problem with the full ideal', J. Applied Analysis 4 (1998) 1-17. [341L.]

Sikorski R. [64] *Boolean Algebras.* Springer, 1964. [Chap. 31 *intro*.]

Sinaĭ Ya.G. [59] 'The notion of entropy of a dynamical system', Dokl. Akad. Nauk SSSR 125 (1959) 768-771. [385P.]

Sinaĭ Ya.G. [62] 'Weak isomorphism of transformations with an invariant measure', Soviet Math. 3 (1962) 1725-1729. [387E.]

Smorodinsky M. [71] *Ergodic Theory, Entropy.* Springer, 1971 (Lecture Notes in Math., 214). [§387 *notes*.]

Štěpánek P. & Rubin M. [89] 'Homogeneous Boolean algebras', pp. 679-715 in Monk 89. [382S, §382 *notes*.]

Talagrand M. [82a] 'Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations', Ann. Institut Fourier (Grenoble) 32 (1982) 39-69. [§346 *notes*.]

Talagrand M. [82b] 'La pathologie des relèvements invariants', Proc. Amer. Math. Soc. 84 (1982) 379-382. [345F.]

Talagrand M. [84] *Pettis integral and measure theory.* Mem. Amer. Math. Soc. 307 (1984). [§346 *notes*.]

Talagrand M. [08] 'Maharam's problem', Annals of Math. 168 (2008) 981-1009. [§394.]

Taylor A.E. [64] *Introduction to Functional Analysis.* Wiley, 1964. [§3A5.]

Todorčević S. [04] 'A problem of von Neumann and Maharam about algebras supporting continuous submeasures', Fund. Math. 183 (2004) 169-183. [393S.]

Truss J.K. [89] 'Infinite permutation groups I: products of conjugacy classes', J. Algebra 120 (1989) 454-493. [§382 *notes*.]

Vladimirov D.A. [02] *Boolean algebras in analysis.* Kluwer, 2002 (Math. and its Appl. 540). [367L.]

Vulikh B.C. [67] *Introduction to the Theory of Partially Ordered Vector Spaces.* Wolters-Noordhoff, 1967. [§364 *notes*.]

Wagon S. [85] *The Banach-Tarski Paradox.* Cambridge U.P., 1985. [§395 *notes*.]
Wilansky A. [64] *Functional Analysis.* Blaisdell, 1964. [§3A5.]

Zaanen A.C. [83] *Riesz Spaces II.* North-Holland, 1983. [Chap. 35 *intro*., 376K, §376 *notes*.]