Version of 17.11.10

Chapter 39

Measurable algebras

In the final chapter of this volume, I present results connected with the following question: which algebras can appear as the underlying Boolean algebras of measure algebras? Put in this form, there is a trivial answer (391A). The proper question is rather: which algebras can appear as the underlying Boolean algebras of semi-finite measure algebras? This is easily reducible to the question: which algebras can appear as the underlying Boolean algebras of probability algebras? Now in one sense Maharam's theorem (§332) gives us the answer exactly: they are the countable simple products of the measure algebras of $\{0,1\}^\kappa$ for cardinals κ. But if we approach from another direction, things are more interesting. Probability algebras share a very large number of very special properties. Can we find a selection of these properties which will be sufficient to force an abstract Boolean algebra to be a probability algebra when endowed with a suitable functional?

No fully satisfying answer to this question is known. But in exploring the possibilities we encounter some interesting and important ideas. In §391 I discuss algebras which have strictly positive additive real-valued functionals; for such algebras, weak (σ,∞)-distributivity is necessary and sufficient for the existence of a measure; so we are led to look for conditions sufficient to ensure that there is a strictly positive additive functional. A slightly different approach lies through the concept of 'submeasure'. Submeasures arise naturally in the theories of topological Boolean algebras (393J), topological Riesz spaces (393K) and vector measures (394P), and on any given algebra there is a strictly positive 'uniformly exhaustive' submeasure iff there is a strictly positive additive functional; this is the Kalton-Roberts theorem (392F).

Submeasures in general are common, but correspondingly limited in what they can tell us about a structure in the absence of further properties. Uniformly exhaustive submeasures are not far from additive functionals. An intermediate class, the 'exhaustive' submeasures, has been intensively studied, originally in the hope that they might lead to characterizations of measurable algebras, but more recently for their own sake. Just as additive functionals lead to measurable algebras, totally finite exhaustive submeasures lead to 'Maharam algebras' (§393). For many years it was not known whether every exhaustive submeasure was uniformly exhaustive (equivalently, whether every Maharam algebra was a measurable algebra); an example was eventually found by M.Talagrand, and is presented in §394.

In §395, I look at a characterization of measurable algebras in terms of the special properties which the automorphism group of a measure algebra must have (Kawada's theorem, 395Q). §396 complements the previous section by looking briefly at the subgroups of an automorphism group $\operatorname{Aut}\mathfrak{A}$ which can appear as groups of measure-preserving automorphisms.

Version of 5.9.07

391 Kelley's theorem

In this section I introduce the notion of 'measurable algebra' (391B), which will be the subject of the whole chapter once the trivial construction of 391A has been dealt with. I show that for weakly (σ,∞)-distributive algebras countable additivity can be left to look after itself, and all we need to find is a strictly positive finitely additive functional (391D). I give Kelley's criterion for the existence of such a functional (391H-391J).

391A Proposition Let $\mathfrak{A}$ be any Dedekind σ-complete Boolean algebra. Then there is a function $\bar\mu:\mathfrak{A}\to[0,\infty]$ such that $(\mathfrak{A},\bar\mu)$ is a measure algebra.

391B Definition (a) I will call a Boolean algebra $\mathfrak{A}$ **measurable** if there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty[$ such that $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra.

(b) I will call a Boolean algebra $\mathfrak{A}$ **chargeable** if there is an additive functional $\nu : \mathfrak{A} \to [0, \infty[$ which is **strictly positive**, that is, $\nu a > 0$ for every non-zero $a \in \mathfrak{A}$.

(c) I will call a Boolean algebra **nowhere measurable** if none of its non-zero principal ideals are measurable algebras.

391C Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) The following are equiveridical: (i) there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ such that $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra; (ii) $\mathfrak{A}$ is Dedekind σ-complete and $\{a : a \in \mathfrak{A}, \mathfrak{A}_a \text{ is measurable}\}$ is order-dense in $\mathfrak{A}$.

(b) The following are equiveridical: (i) there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ such that $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra; (ii) $\mathfrak{A}$ is Dedekind complete and $\{a : a \in \mathfrak{A}, \mathfrak{A}_a \text{ is measurable}\}$ is order-dense in $\mathfrak{A}$.

391D Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is measurable;

(ii) $\mathfrak{A}$ is Dedekind σ-complete, weakly (σ, ∞)-distributive and chargeable.

391E Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\phi : \mathfrak{A} \to [0, 1]$ a functional. Then the following are equiveridical:

(i) there is a finitely additive functional $\nu : \mathfrak{A} \to [0, 1]$ such that $\nu 1 = 1$ and $\nu a \le \phi a$ for every $a \in \mathfrak{A}$;

(ii) whenever $\langle a_i \rangle_{i \in I}$ is a finite indexed family in $\mathfrak{A}$, $m \in \mathbb{N}$ and $\sum_{i \in I} \chi a_i \ge m\chi 1$ in $S = S(\mathfrak{A})$, then $\sum_{i \in I} \phi a_i \ge m$.

391F Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\psi : A \to [0, 1]$ a functional, where $A \subseteq \mathfrak{A}$. Then the following are equiveridical:

(i) there is a finitely additive functional $\nu : \mathfrak{A} \to [0, 1]$ such that $\nu 1 = 1$ and $\nu a \ge \psi a$ for every $a \in A$;

(ii) whenever $\langle a_i \rangle_{i \in I}$ is a finite indexed family in A, there is a set $J \subseteq I$ such that $\#(J) \ge \sum_{i \in I} \psi a_i$ and $\inf_{i \in J} a_i \ne 0$.

391G Corollary Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\nu_0 : \mathfrak{B} \to \mathbb{R}$ a non-negative finitely additive functional. Then there is a non-negative finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ extending ν_0.

391H Definition Let $\mathfrak{A}$ be a Boolean algebra, and $A \subseteq \mathfrak{A} \setminus \{0\}$ any non-empty set. The **intersection number** of A is the largest $\delta \ge 0$ such that whenever $\langle a_i \rangle_{i \in I}$ is a finite family in A, with $I \ne \emptyset$, there is a $J \subseteq I$ such that $\#(J) \ge \delta\#(I)$ and $\inf_{i \in J} a_i \ne 0$.

391I Proposition Let $\mathfrak{A}$ be a Boolean algebra and $A \subseteq \mathfrak{A} \setminus \{0\}$ any non-empty set. Write C for the set of non-negative finitely additive functionals $\nu : \mathfrak{A} \to [0, 1]$ such that $\nu 1 = 1$. Then the intersection number of A is precisely $\max_{\nu \in C} \inf_{a \in A} \nu a$.

391J Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is chargeable;

(ii) either $\mathfrak{A} = \{0\}$ or $\mathfrak{A} \setminus \{0\}$ is expressible as a countable union of sets with non-zero intersection numbers.

391K Corollary Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is measurable iff it is Dedekind σ-complete and weakly (σ, ∞)-distributive and either $\mathfrak{A} = \{0\}$ or $\mathfrak{A} \setminus \{0\}$ is expressible as a countable union of sets with non-zero intersection numbers.

391L Proposition (a) If $\mathfrak{A}$ is a measurable algebra, all its principal ideals and σ-subalgebras are, in themselves, measurable algebras.

(b) The simple product of countably many measurable algebras is a measurable algebra.

(c) If $\mathfrak{A}$ is a measurable algebra, $\mathfrak{B}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\mathfrak{B}$ is a measurable algebra, isomorphic to a principal ideal of $\mathfrak{A}$.

Version of 11.2.08

392 Submeasures

In §391 I looked at what we can deduce if a Boolean algebra carries a strictly positive finitely additive functional. There are important contexts in which we find ourselves with subadditive, rather than additive, functionals, and these are what I wish to investigate here. It turns out that, once we have found the right hypotheses, such functionals can also provide a criterion for measurability of an algebra (392G below). The argument runs through a new idea, using a result in finite combinatorics (392D).

At the end of the section I include notes on metrics associated with submeasures (392H) and on products of submeasures (392K).

392A Definition Let $\mathfrak{A}$ be a Boolean algebra. A **submeasure** on $\mathfrak{A}$ is a functional $\nu : \mathfrak{A} \to [0,\infty]$ such that

$\nu 0 = 0$,
$\nu a \le \nu b$ whenever $a \subseteq b$,
$\nu(a \cup b) \le \nu a + \nu b$ for all a, $b \in \mathfrak{A}$.

392B Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty]$ a submeasure.

(a) ν is **strictly positive** if $\nu a > 0$ for every $a \ne 0$.

(b) ν is **exhaustive** if $\lim_{n\to\infty} \nu a_n = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

(c) ν is **uniformly exhaustive** if for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that there is no disjoint family $a_0, \dots , a_n$ with $\nu a_i \ge \epsilon$ for every $i \le n$.

(d) ν is **totally finite** if $\nu 1 < \infty$.

(e) ν is **unital** if $\nu 1 = 1$.

(f) ν is **atomless** if whenever $a \in \mathfrak{A}$ and $\nu a > 0$ there is a $b \subseteq a$ such that $\nu b > 0$ and $\nu(a \setminus b) > 0$.

(g) If ν' is another submeasure on $\mathfrak{A}$, then ν' is **absolutely continuous** with respect to ν if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\nu' a \le \epsilon$ whenever $\nu a \le \delta$.

392C Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If there is an exhaustive strictly positive submeasure on $\mathfrak{A}$, then $\mathfrak{A}$ is ccc.

(b) A uniformly exhaustive submeasure on $\mathfrak{A}$ is exhaustive.

(c) Any non-negative additive functional on $\mathfrak{A}$ is a uniformly exhaustive submeasure.

392D Lemma Suppose that k, l, $m \in \mathbb{N}$ are such that $3 \le k \le l \le m$ and $18mk \le l^2$. Let L, M be sets with l, m members respectively. Then there is a set $R \subseteq M \times L$ such that (i) each vertical section of R has just three members (ii) $\#(R[E]) \ge \#(E)$ whenever $E \in [M]^{\le k}$; so that for every $E \in [M]^{\le k}$ there is an injective function $f : E \to L$ such that $(x, f(x)) \in R$ for every $x \in E$.

392E Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty]$ a uniformly exhaustive submeasure. Then for any $\epsilon \in \,]0, \nu 1]$ the set $A = \{a : \nu a \ge \epsilon\}$ has intersection number greater than 0.

392F Theorem Let $\mathfrak{A}$ be a Boolean algebra with a strictly positive uniformly exhaustive submeasure. Then $\mathfrak{A}$ is chargeable.

392G Corollary Let $\mathfrak{A}$ be a Boolean algebra. Then it is measurable iff it is weakly (σ,∞)-distributive and Dedekind σ-complete and has a strictly positive uniformly exhaustive submeasure.

392H Metrics from submeasures: Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a strictly positive totally finite submeasure on $\mathfrak{A}$.

(a) We have a metric ρ on $\mathfrak{A}$ defined by the formula

$$\rho(a, b) = \nu(a \triangle b)$$

for all a, $b \in \mathfrak{A}$.

(b) The Boolean operations $\cup$, $\cap$, $\triangle$, $\setminus$ and the function $\nu : \mathfrak{A} \to \mathbb{R}$ are all uniformly continuous for ρ.

(c) The metric space completion $(\widehat{\mathfrak{A}}, \hat{\rho})$ of $(\mathfrak{A}, \rho)$ is a Boolean algebra under the natural continuous extensions of the Boolean operations, and ν has a unique continuous extension $\hat{\nu}$ to $\widehat{\mathfrak{A}}$ which is again a strictly positive submeasure.

(d) If ν is additive, then $(\widehat{\mathfrak{A}}, \hat{\nu})$ is a totally finite measure algebra.

392I Corollary Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative additive functional on $\mathfrak{A}$. Then there are a totally finite measure algebra $(\mathfrak{C}, \bar{\mu})$ and a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\nu a = \bar{\mu}(\pi a)$ for every $a \in \mathfrak{A}$.

392J Proposition Let $\mathfrak{A}$ be a Boolean algebra, ν an exhaustive submeasure on $\mathfrak{A}$, and $\langle a_n \rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$. Then there is an infinite $I \subseteq \mathbb{N}$ such that $\nu(\inf_{i\in I\cap n} a_i) > 0$ for every $n \in \mathbb{N}$.

***392K Products of submeasures (a)** Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with submeasures μ, ν respectively. On the free product $\mathfrak{A} \otimes \mathfrak{B}$, we have a functional $\mu \ltimes \nu$ defined by saying that whenever $c \in \mathfrak{A} \otimes \mathfrak{B}$ is of the form $\sup_{i\in I} a_i \otimes b_i$ where $\langle a_i \rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$, then

$$\begin{aligned}(\mu \ltimes \nu)(c) &= \min_{J\subseteq I} \max(\{\mu(\sup_{i\in J} a_i)\} \cup \{\nu b_i : i \in I \setminus J\}) \\ &= \min\{\epsilon : \epsilon \in [0, \infty],\ \mu(\sup\{a_i : i \in I,\ \nu b_i > \epsilon\}) \le \epsilon\}.\end{aligned}$$

$(\mu \ltimes \nu)(a \otimes b) = \min(\mu a, \nu b)$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$.

(b) In the context of (a), $\mu \ltimes \nu$ is a submeasure.

(c) I note that only in exceptional cases will $\mu \ltimes \nu$ be matched with $\nu \ltimes \mu$ by the canonical isomorphism between $\mathfrak{A} \otimes \mathfrak{B}$ and $\mathfrak{B} \otimes \mathfrak{A}$; this product is not 'commutative'. It is however 'associative', in the following sense. Let $(\mathfrak{A}_1, \mu_1)$, $(\mathfrak{A}_2, \mu_2)$, $(\mathfrak{A}_3, \mu_3)$ be Boolean algebras endowed with submeasures. Set

$$\lambda_{12} = \mu_1 \ltimes \mu_2, \quad \lambda_{(12)3} = \lambda_{12} \ltimes \mu_3, \quad \lambda_{23} = \mu_2 \ltimes \mu_3, \quad \lambda_{1(23)} = \mu_1 \ltimes \lambda_{23}.$$

Then the canonical isomorphisms between $(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \otimes \mathfrak{A}_3$, $\mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \mathfrak{A}_3$ and $\mathfrak{A}_1 \otimes (\mathfrak{A}_2 \otimes \mathfrak{A}_3)$ identify $\lambda_{(12)3}$ with $\lambda_{1(23)}$.

(d) If μ, μ' are submeasures on $\mathfrak{A}$, ν and ν' are submeasures on $\mathfrak{B}$, μ is absolutely continuous with respect to μ' and ν is absolutely continuous with respect to ν', then $\mu \ltimes \nu$ is absolutely continuous with respect to $\mu' \otimes \nu'$.

(e) If μ and ν are exhaustive, so is $\mu \ltimes \nu$.

(f) We can extend the construction to infinite products, as follows. Let I be a totally ordered set and $\langle (\mathfrak{A}_i, \mu_i) \rangle_{i\in I}$ a family of Boolean algebras endowed with unital submeasures. For a finite set $J = \{i_0, \dots, i_n\}$ where $i_0 < \dots < i_n$ in I, let λ_J be the product submeasure $(.(\mu_{i_0} \ltimes \mu_{i_1}) \ltimes \dots) \ltimes \mu_{i_n}$ on $\mathfrak{C}_J = \bigotimes_{j\in J} \mathfrak{A}_j$; for definiteness, on $\mathfrak{C}_\emptyset = \{0, 1\}$ take $\lambda_\emptyset$ to be the unital submeasure, while $\mathfrak{C}_{\{i\}} = \mathfrak{A}_i$ and $\lambda_{\{i\}} = \mu_i$ for each $i \in I$. Using (c) repeatedly, we see that if J, $K \in [I]^{<\omega}$ and $j < k$ for every $j \in J$, $k \in K$, then the identification of $\mathfrak{C}_{J\cup K}$ with $\mathfrak{C}_J \otimes \mathfrak{C}_K$ matches $\lambda_{J\cup K}$ with $\lambda_J \ltimes \lambda_K$. Moreover, if $K \in [I]^{<\omega}$ and J is any subset of K and $\varepsilon_{JK} : \mathfrak{C}_J \to \mathfrak{C}_K$ is the canonical embedding corresponding to the identification of $\mathfrak{C}_K$ with $\mathfrak{C}_J \otimes \mathfrak{C}_{K\setminus J}$, then $\lambda_J = \lambda_K \varepsilon_{JK}$; this also is an easy induction on $\#(K)$. What this means is that for any subset M of I we have a submeasure λ_M on $\mathfrak{C}_M = \bigcup\{\varepsilon_{JM}\mathfrak{C}_J : J \in [M]^{<\omega}\}$, being the unique functional such that $\lambda_M \varepsilon_{JM} = \lambda_J$ for every $J \in [M]^{<\omega}$. Finally, if L, M are subsets of I with $l < m$ for every $l \in L$ and $m \in M$, then $\lambda_{L\cup M}$ can be identified with $\lambda_L \ltimes \lambda_M$.

(g) I should perhaps have remarked already that if μ and ν, in (a), are additive and unital, then we have an additive function λ' on $\mathfrak{A} \otimes \mathfrak{B}$ such that $\lambda'(a \otimes b) = \mu a \cdot \nu b$ for every $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Now, setting $\lambda = \mu \ltimes \nu$, each of λ, λ' is absolutely continuous with respect to the other.

Version of 11.5.08

393 Maharam submeasures

Continuing our exploration of variations on the idea of 'measurable algebra', we come to sequentially order-continuous submeasures. These are associated with 'Maharam algebras' (393E), which share a great many properties with measurable algebras; for instance, the existence of a standard topology defined by the algebraic structure (393G). This topology is intimately connected with the order*-convergence of sequences introduced in §367 (393L). We can indeed characterize Maharam algebras in terms of properties of the order-sequential topology defined by this convergence (393Q).

393A Definition Let $\mathfrak{A}$ be a Boolean algebra. A **Maharam submeasure** or **continuous outer measure** on $\mathfrak{A}$ is a totally finite submeasure $\nu : \mathfrak{A} \to [0, \infty[$ such that $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

393B Lemma Let $\mathfrak{A}$ be a Boolean algebra and ν a Maharam submeasure on $\mathfrak{A}$.

(a) ν is sequentially order-continuous.

(b) ν is 'countably subadditive', that is, whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $a \in \mathfrak{A}$ is such that $a = \sup_{n\in\mathbb{N}} a \cap a_n$, then $\nu a \le \sum_{n=0}^{\infty} \nu a_n$.

(c) If $\mathfrak{A}$ is Dedekind σ-complete, then ν is exhaustive.

393C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν a strictly positive Maharam submeasure on $\mathfrak{A}$. Then $\mathfrak{A}$ is ccc, Dedekind complete and weakly (σ, ∞)-distributive, and ν is order-continuous.

393D Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then it is measurable iff it is Dedekind σ-complete and carries a uniformly exhaustive strictly positive Maharam submeasure.

393E Maharam algebras (a) Definition A **Maharam algebra** is a Dedekind σ-complete Boolean algebra $\mathfrak{A}$ such that there is a strictly positive Maharam submeasure on $\mathfrak{A}$.

(b) Every measurable algebra is a Maharam algebra, while every Maharam algebra is ccc and weakly (σ, ∞)-distributive, therefore Dedekind complete. A Maharam algebra $\mathfrak{A}$ is measurable iff there is a strictly positive uniformly exhaustive submeasure on $\mathfrak{A}$.

(c)(i) A principal ideal in a Maharam algebra is a Maharam algebra; an order-closed subalgebra of a Maharam algebra is a Maharam algebra.

(ii) The simple product of a countable family of Maharam algebras is a Maharam algebra.

393F Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν, ν' two Maharam submeasures on $\mathfrak{A}$ such that $\nu a = 0$ whenever $\nu' a = 0$. Then ν is absolutely continuous with respect to ν'.

393G Proposition Let $\mathfrak{A}$ be a Maharam algebra, and ν and ν' two strictly positive Maharam submeasures on $\mathfrak{A}$. Then the metrics they induce on $\mathfrak{A}$ are uniformly equivalent, so we have a topology and uniformity on $\mathfrak{A}$ which we may call the **Maharam-algebra topology** and the **Maharam-algebra uniformity**.

393H Proposition Let $\mathfrak{A}$ be a Boolean algebra, and ν an exhaustive strictly positive totally finite submeasure on $\mathfrak{A}$. Let $\widehat{\mathfrak{A}}$ be the metric completion of $\mathfrak{A}$, and $\hat\nu$ the continuous extension of ν to $\widehat{\mathfrak{A}}$. Then $\hat\nu$ is a Maharam submeasure, so $\widehat{\mathfrak{A}}$ is a Maharam algebra.

393I Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν an atomless Maharam submeasure on $\mathfrak{A}$. Then for every $\epsilon > 0$ there is a finite partition C of unity in $\mathfrak{A}$ such that $\nu c \le \epsilon$ for every $c \in C$.

393J Lemma Let $\mathfrak{A}$ be a ccc Boolean algebra with a T_1 topology $\mathfrak{T}$ such that (i) $\cup : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is continuous at $(0, 0)$ (ii) whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\langle a_n \rangle_{n\in\mathbb{N}} \to 0$ for $\mathfrak{T}$. Then $\mathfrak{A}$ has a strictly positive Maharam submeasure.

***393K Theorem** Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra. Then $\mathfrak{A}$ is a Maharam algebra iff there is a Hausdorff linear space topology $\mathfrak{T}$ on $L^0(\mathfrak{A})$ such that for every neighbourhood G of 0 there is a neighbourhood H of 0 such that $u \in G$ whenever $v \in H$ and $|u| \le |v|$.

393L Definition Let P be a lattice, and consider the relation '$\langle p_n \rangle_{n\in\mathbb{N}}$ order*-converges to p' as a relation between $P^{\mathbb{N}}$ and P. There is a unique topology on P for which a set $F \subseteq P$ is closed iff $a \in F$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in F which order*-converges to a in P. I will call this topology the **order-sequential topology** of P.

393M Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) A sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ order*-converges to $a \in \mathfrak{A}$ iff there is a partition B of unity in $\mathfrak{A}$ such that $\{n : n \in \mathbb{N},\, (a_n \triangle a) \cap b \ne 0\}$ is finite for every $b \in B$.

(b) If $\langle a_n \rangle_{n\in\mathbb{N}}$ order*-converges to a and $c \in \mathfrak{A}$, then $\langle a_n \cup c \rangle_{n\in\mathbb{N}}$, $\langle a_n \cap c \rangle_{n\in\mathbb{N}}$ and $\langle a_n \triangle c \rangle_{n\in\mathbb{N}}$ order*-converge to $a \cup c$, $a \cap c$ and $a \triangle c$ respectively.

(c) The operations $\cap$, $\cup$ and $\triangle$ are separately continuous for the order-sequential topology.

(d) Every disjoint sequence in $\mathfrak{A}$ is order*-convergent to 0.

393N Proposition Let $\mathfrak{A}$ be a Maharam algebra. Then the Maharam-algebra topology on $\mathfrak{A}$ is the order-sequential topology.

393O Proposition Let $\mathfrak{A}$ be a ccc Dedekind σ-complete Boolean algebra, with its order-sequential topology, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then the topological closure of $\mathfrak{B}$ is the smallest order-closed set including $\mathfrak{B}$; $\mathfrak{B}$ is order-closed iff it is topologically closed.

393P Lemma Let $\mathfrak{A}$ be a ccc weakly (σ, ∞)-distributive Boolean algebra, endowed with its order-sequential topology.

(a) If $\langle a_{mn} \rangle_{m,n\in\mathbb{N}}$, $\langle a_m \rangle_{m\in\mathbb{N}}$ and a are such that $\langle a_{mn} \rangle_{n\in\mathbb{N}}$ order*-converges to a_m for each m, while $\langle a_m \rangle_{m\in\mathbb{N}}$ order*-converges to a, then there is a sequence $\langle k(m) \rangle_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $\langle a_{m,k(m)} \rangle_{m\in\mathbb{N}}$ order*-converges to a.

(b) If $A \subseteq \mathfrak{A}$ and $a \in \overline{A}$, there is a sequence in A which order*-converges to a.

(c) If G is a neighbourhood of 0 in $\mathfrak{A}$ then there is an open neighbourhood H of 0, included in G, such that $[0, a] \subseteq H$ for every $a \in H$.

(d) For $A \subseteq \mathfrak{A}$, set $\bigvee_0(A) = \{0\}$ and $\bigvee_{n+1}(A) = \{a \cup b : a \in \bigvee_n(A),\, b \in A\}$ for $n \in \mathbb{N}$.

(i) If $A \subseteq \mathfrak{A}$ is such that $[0, a] \subseteq A$ for every $a \in A$, and $n \in \mathbb{N}$, then $[0, a] \subseteq \bigvee_n(A)$ for every $a \in \bigvee_n(A)$.

(ii) If $H \subseteq \mathfrak{A}$ is an open set containing 0 such that $[0, a] \subseteq H$ for every $a \in H$, then $\bigvee_{n+1}(H)$ is open and $\overline{\bigvee_n(H)} \subseteq \bigvee_{n+1}(H)$ for every $n \in \mathbb{N}$.

(e) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. Then for every open set G containing 0 there is an open set H containing 0 such that $\bigvee_3(H) \subseteq \bigvee_2(G)$.

393Q Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is a Maharam algebra;

(ii) $\mathfrak{A}$ is ccc and the order-sequential topology is Hausdorff;

(iii) $\mathfrak{A}$ is weakly (σ, ∞)-distributive and $\{0\}$ is a G_δ set for the order-sequential topology of $\mathfrak{A}$;

(iv) $\mathfrak{A}$ is ccc and there is a T_1 topology on $\mathfrak{A}$ such that (α) $\cup : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is continuous at $(0, 0)$ (β) whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with infimum 0, then $\langle a_n \rangle_{n\in\mathbb{N}} \to 0$.

393R Definition Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is **σ-finite-cc** if $\mathfrak{A}$ can be expressed as $\bigcup_{n\in\mathbb{N}} A_n$ where no A_n includes any infinite disjoint set.

393S Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is a Maharam algebra iff it is σ-finite-cc, weakly (σ,∞)-distributive and Dedekind σ-complete.

Version of 13.6.11/30.8.18

394 Talagrand's example

I rewrite the construction in Talagrand 08 of an exhaustive submeasure which is not uniformly exhaustive, generalized as in Perović & Veličković 18.

394A PV norms (a) I will say that a **PV norm** is a function $\|\ \| : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ such that

— $\|\emptyset\| = 0$, $\|I\| = 1$ if $\#(I) = 1$,
— $\|I \cup J\| \le \|I\| + \|J\|$ for all I, $J \in [\mathbb{N}]^{<\omega}$,
— $\|I\| \le \|J\|$ whenever I, $J \in [\mathbb{N}]^{<\omega}$ and $\#(I \cap n) \le \#(J \cap n)$ for every $n \in \mathbb{N}$,
— $\lim_{n\to\infty} \|A \cap n\| = \infty$ for every infinite $A \subseteq \mathbb{N}$

(Perović & Veličković 18, 2.2).

(b) Note that if $\|\ \|$ is a PV norm then $\|I\| \le \|J\| \le \#(J)$ whenever $I \subseteq J \in [\mathbb{N}]^{<\omega}$. We see also that if $I \in [\mathbb{N}]^{<\omega}$ and $k < \|I\|$ there is an $n \in I$ such that $\|I \cap n\| = k$.

(c) The version of Talagrand's example in the 2012 edition of Volume 3 corresponds to the case in which $\|I\| = \#(I)$ for every $I \in [\mathbb{N}]^{<\omega}$. For the work of this section there is no need to consider any other, and some of the formulae in 394D become more readable if you make this simplification; but it makes no real difference to the ideas required.

394B Definitions (a) I shall work throughout with $X = \prod_{n\in\mathbb{N}} T_n$ where $\langle T_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty finite sets and $\sup_{n\in\mathbb{N}} \#(T_n)$ is infinite. X may be regarded as a compact Hausdorff space with the product of the discrete topologies on the T_n. For each $n \in \mathbb{N}$, $\mathfrak{B}_n$ will be the algebra of subsets of X determined by coordinates less than n and $\mathcal{A}_n$ the set of its atoms. $\mathfrak{B}$ will be the algebra of open-and-closed subsets of X. For $I \subseteq \mathbb{N}$ and $z \in \prod_{n\in I} T_n$, Y_z will be $\{x : z \subseteq x \in X\}$. Finally, $\|\ \|$ will be a PV norm on $[\mathbb{N}]^{<\omega}$.

(b) We shall need a sequence $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{R}$ and a sequence $\langle N_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$. I declare the properties they must have.

(i) $\alpha_k > 0$ and $(2^{k+4})^{\alpha_k} \le 2$ for every $k \in \mathbb{N}$, $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ is non-increasing, and $\sum_{k=0}^{\infty} \alpha_k \le \frac{1}{2}$.

(ii) $N_k \in \mathbb{N}$ and $2^{-k}(2^{-2k-12}N_k)^{\alpha_k} \ge 2^4$ for every $k \in \mathbb{N}$.

(c) For a set $\mathcal{I} \subseteq \mathcal{P}X \times \mathcal{P}\mathbb{N} \times [0,\infty[$, define $\operatorname{spr}\mathcal{I}$ to be $\bigcup_{(E,I,w)\in\mathcal{I}} E$ and $\operatorname{wt}\mathcal{I}$ to be $\sum_{(E,I,w)\in\mathcal{I}} w$.

(d) For any family $\mathcal{E} \subseteq \mathcal{P}X \times \mathcal{P}\mathbb{N} \times [0,\infty[$ define $\phi_{\mathcal{E}} : \mathfrak{B} \to [0,\infty]$ by setting

$$\phi_{\mathcal{E}}E = \inf\{\operatorname{wt}\mathcal{I} : \mathcal{I} \subseteq \mathcal{E} \text{ is finite, } E \subseteq \operatorname{spr}\mathcal{I}\},$$

counting $\inf\emptyset$ as ∞.

(e) For $D \subseteq X$ and $I \subseteq \mathbb{N}$ set

$$\theta_I(D) = \{y : y \in X,\ y{\restriction}I = x{\restriction}I \text{ for some } x \in D\}.$$

(f)(i) If $m < n$ in $\mathbb{N}$, $\phi : \mathfrak{B} \to [0,\infty]$ is a function and $E \in \mathfrak{B}$, then E is **ϕ-thin between m and n** if $\phi(X \setminus \theta_{n\setminus m}(A \cap E)) \ge 1$ for every $A \in \mathcal{A}_m$.

(ii) If $I \subseteq \mathbb{N}$, $\phi : \mathfrak{B} \to [0,\infty]$ is a function and $E \in \mathfrak{B}$, then E is **ϕ-thin along I** if it is ϕ-thin between m and n whenever m, $n \in I$ and $m < n$.

(g) For $k \le p \in \mathbb{N}$ define $\mathcal{C}_{kp}$ and ν_{kp} by downwards induction on k. $\mathcal{C}_{pp} = \emptyset$ for every p. $\nu_{kp} = \phi_{\mathcal{C}_{kp}}$. Given that $k < p$ and $\mathcal{C}_{k+1,p}$ and $\nu_{k+1,p} = \phi_{\mathcal{C}_{k+1,p}}$ have been defined, set

$$\mathcal{E}_{kp} = \{(E, I, w) : E \in \mathfrak{B},\, I \in [\mathbb{N}]^{<\omega},\, 1 \le \|I\| \le N_k,$$
$$w \ge 2^{-k}\bigl(\tfrac{N_k}{\|I\|}\bigr)^{\alpha_k},\, E \text{ is } \nu_{k+1,p}\text{-thin along } I\},$$
$$\mathcal{C}_{kp} = \mathcal{E}_{kp} \cup \mathcal{C}_{k+1,p}$$

and continue.

(h) Define $\langle c_k\rangle_{k\in\mathbb{N}}$ by setting $c_0 = 8$, $c_{k+1} = 2^{2\alpha_k} c_k$ for every k.

394C Very elementary facts (a) $\phi_{\mathcal{E}} : \mathfrak{B} \to [0, \infty]$ is a submeasure for any $\mathcal{E} \subseteq \mathcal{P}X \times \mathcal{P}\mathbb{N} \times [0, \infty[$.

(b) If I, $J \subseteq \mathbb{N}$ then $\theta_I\theta_J = \theta_{I\cap J}$. If $I \subseteq J \subseteq \mathbb{N}$ then $\theta_I(D) = \theta_I\theta_J(D) \supseteq \theta_J(D)$ for all $D \subseteq X$. If $I \subseteq \mathbb{N}$ then $\theta_I(D \cap \theta_I(E)) = \theta_I(E \cap \theta_I(D))$ for all D, $E \subseteq X$. For any $I \subseteq \mathbb{N}$ and any family $\mathcal{D}$ of subsets of X, $\theta_I(\bigcup \mathcal{D}) = \bigcup_{D\in\mathcal{D}} \theta_I(D)$.

For $n \in \mathbb{N}$ and $D \subseteq X$, $D \in \mathfrak{B}_n$ iff $\theta_n(D) = D$. If $E \in \mathfrak{B}$ and $I \subseteq \mathbb{N}$, $\theta_I(E) \in \mathfrak{B}$. If $m \le n$ in $\mathbb{N}$, $A \in \mathcal{A}_m$ and $A_1 \in \mathcal{A}_n$, then $A \cap \theta_{n\setminus m}(A_1) \in \mathcal{A}_n$. If $m \in \mathbb{N}$ and $A \in \mathcal{A}_m$ then $E \mapsto \theta_{\mathbb{N}\setminus m}(A \cap E) : \mathfrak{B} \to \mathfrak{B}$ is a Boolean homomorphism.

(c) If $m < n$, $\phi : \mathfrak{B} \to [0, \infty]$ is a non-decreasing function, $E \in \mathfrak{B}$ is ϕ-thin between m and n and $E' \in \mathfrak{B}$ is included in E, then E' is ϕ-thin between m and n' for every $n' \ge n$.

(d) $\mathcal{E}_{kp}$, $\mathcal{C}_{kp}$ are closed under increases in the scalar variable and decreases in the first variable, that is,
— if $k < p$, $(E, I, w) \in \mathcal{E}_{kp}$, $E' \in \mathfrak{B}$, $E' \subseteq E$ and $w' \ge w$ then $(E', I, w') \in \mathcal{E}_{kp}$,
— if $k \le p$, $(E, I, w) \in \mathcal{C}_{kp}$, $E' \in \mathfrak{B}$, $E' \subseteq E$ and $w' \ge w$ then $(E', I, w') \in \mathcal{C}_{kp}$.

(e) If $k \le p$, $\mathcal{C}_{kp} = \bigcup_{k\le l<p} \mathcal{E}_{lp}$.

(f) If $k < p$, $\nu_{kp} \le \nu_{k+1,p}$.

(g) $8 \le c_k \le 16$ for every $k \in \mathbb{N}$.

(h) If $k < p$, then $\nu_{kp}X \le 2^{-k}N_k^{\alpha_k}$ and ν_{kp} is totally finite.

394D Lemma Suppose that $\mathcal{K}$ is a non-empty finite family of subsets of $\mathbb{N}$ and $r \in \mathbb{N}$ is such that $\|K\| \ge r\#(\mathcal{K})$ for every $K \in \mathcal{K}$. Then we have an enumeration $\langle K_i\rangle_{i<s}$ of $\mathcal{K}$ and a non-decreasing family $\langle n_i\rangle_{i\le s}$ such that $\|K_i \cap n_{i+1} \setminus n_i\| = r$ for every $i < s$.

(b) Suppose that $\langle K_i\rangle_{i<s}$ is a family of finite subsets of $\mathbb{N}$ such that $\|K_i\| \ge n \ge 3$ for every $i < s$ and $\max K_i < \min K_{i+1}$ for $i \le s-2$, and that $A \in [\mathbb{N}]^{<\omega}$ is such that $\|A\| \le 1$. Let $\mathcal{J}$ be a finite subset of $\mathcal{P}X \times ([\mathbb{N}]^{<\omega} \setminus \{\emptyset\}) \times [0, \infty[$. Then we can find $\langle u_i\rangle_{i<s}$ and $\langle v_i\rangle_{i<s}$ such that u_i, $v_i \in K_i$ and $u_i < v_i$ for each $i < s$ and, setting $W = \bigcup_{i<s} v_i \setminus u_i$, $A \cap W = \emptyset$ and

$$\operatorname{wt}\{(E, I, w) : (E, I, W) \in \mathcal{J},\, \|I \setminus W\| < \tfrac{1}{2}\|I\|\} \le \frac{1}{n-2}\operatorname{wt}\mathcal{J}.$$

394E Lemma Suppose that $k \le p$, $m < n$, $A \in \mathcal{A}_m$, $(E, I, w) \in \mathcal{C}_{kp}$ and $I' = I \cap n \setminus m$ is non-empty. If $E' = \theta_{n\setminus m}(E \cap A)$ and $w' \ge \bigl(\frac{\|I\|}{\|I'\|}\bigr)^{\alpha_k} w$, then $(E', I', w') \in \mathcal{C}_{kp}$.

394F Corollary (a) Suppose that $n \in \mathbb{N}$ and $k \le p$ and that $\mathcal{I} \subseteq \mathcal{C}_{kp}$ is a finite set such that $\|I \cap n\| \ge \frac{1}{4}\|I\|$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_n(\operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$.

(b) Suppose that $m \in \mathbb{N}$, $k \le p$ and $A \in \mathcal{A}_m$. Let $\mathcal{I}$ be a finite subset of $\mathcal{C}_{kp}$ such that $\|I \setminus m\| \ge \frac{1}{4}\|I\|$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_{\mathbb{N}\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$.

(c) Suppose that $m < n$ in $\mathbb{N}$, $k \le p$ and $A \in \mathcal{A}_m$. Let $\mathcal{I}$ be a finite subset of $\mathcal{C}_{kp}$ such that $\|I \cap n \setminus m\| \ge 2^{-k-4}\|I\|$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_{n\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$.

394G Lemma Suppose that $L \in [\mathbb{N}]^{<\omega}$ is such that $\|L\| \le 1$, and $z \in \prod_{r\in L} T_r$. Then $\nu_{kp}Y_z \ge c_k$ whenever $k \le p$ in $\mathbb{N}$.

394H Definitions Fix on a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. For $k \in \mathbb{N}$, set

$$\nu_k E = \lim_{p\to\mathcal{F}} \nu_{kp}E \in [0,\infty]$$

for every $E \in \mathfrak{B}$; write ν for ν_0.

394I Proposition (a) For every $k \in \mathbb{N}$, ν_k is a totally finite submeasure and $\nu_k X \ge 8$.

(b) ν is not uniformly exhaustive.

394J Lemma Suppose that $k \in \mathbb{N}$, $E \in \mathfrak{B}$, $I \in [\mathbb{N}]^{<\omega}$ and E is $\frac{1}{2}\nu_k$-thin along I. Then

$$\{p : p \ge k,\ E \text{ is } \nu_{kp}\text{-thin along } I\} \in \mathcal{F}.$$

If $k \ge 1$ and $\|I\| = N_{k-1}$, then $\nu_{k-1}E \le 2^{-k+1}$.

394K Lemma Let m, $k \in \mathbb{N}$ and let $\langle E_i\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$ such that

every E_i is determined by coordinates in $\mathbb{N}\setminus m$,

$\nu_k(\bigcup_{i\le n} E_i) < 2$ for every $n \in \mathbb{N}$.

Then for every $\eta > 0$ there is a $C \in \mathfrak{B}$, determined by coordinates in $\mathbb{N}\setminus m$, such that $\nu_k C \le 4$ and $\nu_k(E_i \setminus C) \le \eta$ for each i.

394L Lemma Suppose that $k \in \mathbb{N}$, $\epsilon > 0$, $m \in \mathbb{N}$, $B \in \mathfrak{B}_m$ and that $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$. Then there are $n > m$ and $B' \in \mathfrak{B}_n$ such that $B' \subseteq B$, B' is $\frac{1}{2}\nu_k$-thin between m and n and $\limsup_{i\to\infty} \nu_k(E_i \cap B \setminus B') \le \epsilon$.

394M Theorem ν is exhaustive.

394N Remarks (a) Note that the whole construction is invariant under the action of the group $\prod_{n\in\mathbb{N}} G_n$ where G_n is the group of all permutations of T_n for each n. In particular, if we give each T_n a group structure and X the product group structure, then ν is translation-invariant.

(b) ν is strictly positive.

(c) We can form the metric completion $\widehat{\mathfrak{B}}$ of $\mathfrak{B}$, and $\widehat{\mathfrak{B}}$ will be a Maharam algebra, with a strictly positive Maharam submeasure $\hat\nu$ continuously extending ν. $\widehat{\mathfrak{B}}$ is not measurable.

***394O Control measures** Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and U a Hausdorff linear topological space. A function $\theta : \mathfrak{A} \to U$ is a **vector measure** if $\sum_{n=0}^{\infty} \theta a_n = \lim_{n\to\infty} \sum_{i=0}^{n} \theta a_i$ is defined in U and equal to $\theta(\sup_{n\in\mathbb{N}} a_n)$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. In this case, a non-negative countably additive functional $\mu : \mathfrak{A} \to [0,\infty[$ is a **control measure** for θ if $\theta a = 0$ whenever $\mu a = 0$.

***394P Example** There are a metrizable linear topological space U and a vector measure $\theta : \Sigma \to U$, where Σ is a σ-algebra of sets, such that θ has no control measure.

***394Q Theorem** Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, U a normed space and $\theta : \mathfrak{A} \to U$ a vector measure. Then θ has a control measure.

394Z Problems Suppose that $\|\ \|$, $\langle T_n\rangle_{n\in\mathbb{N}}$, $\mathfrak{B}$, $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ and $\langle N_k\rangle_{k\in\mathbb{N}}$ satisfy the conditions of 394Ba-394Bb. Let ν be the exhaustive submeasure on $\mathfrak{B}$ constructed by the method of 394B and 394H, and $\widehat{\mathfrak{B}}$ the corresponding Maharam algebra.

(a) Does $\widehat{\mathfrak{B}}$ have an order-closed subalgebra isomorphic to the measure algebra of Lebesgue measure? In particular, if we take $\mathfrak{C} \subseteq \mathfrak{B}$ to be the algebra of sets generated by sets of the form $\{x : x \in X,\ x(n) = 0\}$ for $n \in \mathbb{N}$, is $\nu\restriction\mathfrak{C}$ uniformly exhaustive?

(b) Suppose that instead of taking large sets T_n, we simply set $T_n = \{0, 1\}$ for every n, but otherwise used the same construction. Should we then find that ν was uniformly exhaustive?

(c) Is the Boolean algebra $\widehat{\mathfrak{B}}$ homogeneous?

Version of 15.6.08

395 Kawada's theorem

I now describe a completely different characterization of (homogeneous) measurable algebras, based on the special nature of their automorphism groups. The argument depends on the notion of 'non-paradoxical' group of automorphisms; this is an idea of great importance in other contexts, and I therefore aim at a fairly thorough development, with proofs which are adaptable to other circumstances.

395A Definitions Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For a, $b \in \mathfrak{A}$ I will say that an isomorphism $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ between the corresponding principal ideals belongs to the **full local semigroup generated by** G if there are a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \pi_i\rangle_{i\in I}$ in G such that $\phi c = \pi_i c$ whenever $i \in I$ and $c \subseteq a_i$. If such an isomorphism exists I will say that a and b are **G-τ-equidecomposable**.

I will write $a \preccurlyeq^\tau_G b$ to mean that there is a $b' \subseteq b$ such that a and b' are G-τ-equidecomposable.

For any function f with domain $\mathfrak{A}$, I will say that f is **G-invariant** if $f(\pi a) = f(a)$ whenever $a \in \mathfrak{A}$ and $\pi \in G$.

395B Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_τ for the full local semigroup generated by G.

(a) Suppose that a, $b \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism. Then the following are equiveridical:

(i) $\phi \in G^*_\tau$;

(ii) for every non-zero $c_0 \subseteq a$ there are a non-zero $c_1 \subseteq c_0$ and a $\pi \in G$ such that $\phi c = \pi c$ for every $c \subseteq c_1$;

(iii) for every non-zero $c_0 \subseteq a$ there are a non-zero $c_1 \subseteq c_0$ and a $\psi \in G^*_\tau$ such that $\phi c = \psi c$ for every $c \subseteq c_1$.

(b) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_τ, then $\phi^{-1} : \mathfrak{A}_b \to \mathfrak{A}_a$ also belongs to G^*_τ.

(c) Suppose that a, b, a', $b' \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_{a'}$, $\psi : \mathfrak{A}_b \to \mathfrak{A}_{b'}$ belong to G^*_τ. Then $\psi\phi \in G^*_\tau$; its domain is $\mathfrak{A}_c$ where $c = \phi^{-1}(b \cap a')$, and its set of values is $\mathfrak{A}_{c'}$ where $c' = \psi(b \cap a')$.

(d) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_τ, then $\phi\restriction\mathfrak{A}_c \in G^*_\tau$ for any $c \subseteq a$.

(e) Suppose that a, $b \in \mathfrak{A}$ and that $\psi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism such that there are a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \phi_i\rangle_{i\in I}$ in G^*_τ such that $\psi c = \phi_i c$ whenever $i \in I$ and $c \subseteq a_i$. Then $\psi \in G^*_\tau$.

395C Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_τ for the full local semigroup generated by G.

(a) For a, $b \in \mathfrak{A}$, $a \preccurlyeq^\tau_G b$ iff there is a $\phi \in G^*_\tau$ such that $a \in \operatorname{dom}\phi$ and $\phi a \subseteq b$.

(b)(i) $\preccurlyeq^\tau_G$ is transitive and reflexive;

(ii) if $a \preccurlyeq^\tau_G b$ and $b \preccurlyeq^\tau_G a$ then a and b are G-τ-equidecomposable.

(c) G-τ-equidecomposability is an equivalence relation on $\mathfrak{A}$.

(d) If $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ are families in $\mathfrak{A}$, of which $\langle b_i\rangle_{i\in I}$ is disjoint, and $a_i \preccurlyeq^\tau_G b_i$ for every $i \in I$, then $\sup_{i\in I} a_i \preccurlyeq^\tau_G \sup_{i\in I} b_i$.

395D Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is an $a \neq 1$ such that a is G-τ-equidecomposable with 1;
(ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-τ-equidecomposable;
(iii) there are non-zero G-τ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;
(iv) there are G-τ-equidecomposable a, $b \in \mathfrak{A}$ such that $a \subset b$.

395E Definition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. I will say that G is **fully non-paradoxical** if the statements of 395D are false; that is, if one of the following equiveridical statements is true:

(i) if a is G-τ-equidecomposable with 1 then $a = 1$;

(ii) there is no disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-τ-equidecomposable;

(iii) there are no non-zero G-τ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) if $a \subseteq b \in \mathfrak{A}$ and a, b are G-τ-equidecomposable then $a = b$.

Note that if G is fully non-paradoxical, and H is a subgroup of G, then H also is fully non-paradoxical.

395F Proposition Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and $G = \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ the group of all measure-preserving automorphisms of $\mathfrak{A}$. Then G is fully non-paradoxical.

395G The fixed-point subalgebra of a group Let $\mathfrak{A}$ be a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$.

(a) By the **fixed-point subalgebra** of G I mean

$$\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c \text{ for every } \pi \in G\}.$$

This is a subalgebra of $\mathfrak{A}$, and is order-closed.

(b) Now suppose that $\mathfrak{A}$ is Dedekind complete. In this case $\mathfrak{C}$ is Dedekind complete, and we have, for any $a \in \mathfrak{A}$, an upper envelope $\operatorname{upr}(a, \mathfrak{C})$ of $\mathfrak{C}$, defined by setting

$$\operatorname{upr}(a, \mathfrak{C}) = \inf\{c : a \subseteq c \in \mathfrak{C}\}.$$

Now $\operatorname{upr}(a, \mathfrak{C}) = \sup\{\pi a : \pi \in G\}$.

(c) Again supposing that $\mathfrak{A}$ is Dedekind complete, write G^*_τ for the full local semigroup generated by G. Then $\phi(a \cap c) = \phi a \cap c$ whenever $\phi \in G^*_\tau$, $a \in \operatorname{dom}\phi$ and $c \in \mathfrak{C}$.
$\operatorname{upr}(\phi a, \mathfrak{C}) = \operatorname{upr}(a, \mathfrak{C})$ whenever $\phi \in G^*_\tau$ and $a \in \operatorname{dom}\phi$.
$\operatorname{upr}(a, \mathfrak{C}) \subseteq \operatorname{upr}(b, \mathfrak{C})$ whenever $a \preccurlyeq^\tau_G b$.

(d) Still supposing that $\mathfrak{A}$ is Dedekind complete, we also find that if $a \preccurlyeq^\tau_G b$ and $c \in \mathfrak{C}$ then $a \cap c \preccurlyeq^\tau_G b \cap c$. Hence $a \cap c$ and $b \cap c$ are G-τ-equidecomposable whenever a and b are G-τ-equidecomposable and $c \in \mathfrak{C}$.

(e) I will say that G is **ergodic** if $\sup_{\pi\in G} \pi a = 1$ for every non-zero $a \in \mathfrak{A}$.

(f) If G is ergodic, then $\mathfrak{C} = \{0, 1\}$. If $\mathfrak{A}$ is Dedekind complete and $\mathfrak{C} = \{0, 1\}$ then G is ergodic.

395H Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $\mathfrak{C}$ for the fixed-point subalgebra of G. Take any a, $b \in \mathfrak{A}$. Set $c_0 = \sup\{c : c \in \mathfrak{C},\ a \cap c \preccurlyeq^\tau_G b\}$; then $a \cap c_0 \preccurlyeq^\tau_G b$ and $b \setminus c_0 \preccurlyeq^\tau_G a$.

395I Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Suppose that a, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a, \mathfrak{C}) = 1$. Then there are u, $v \in L^0 = L^0(\mathfrak{C})$ such that

$$[\![u \ge n]\!] = \max\{c : c \in \mathfrak{C},\text{ there is a disjoint family }\langle d_i\rangle_{i<n}\text{ such that }c \cap a \preccurlyeq^\tau_G d_i \subseteq b\text{ for every }i<n\},$$

$$[\![v \le n]\!] = \max\{c : c \in \mathfrak{C},\text{ there is a family }\langle d_i\rangle_{i<n}\text{ such that }d_i \preccurlyeq^\tau_G a\text{ for every }i<n\text{ and }b \cap c \subseteq \sup_{i<n} d_i\}$$

for every $n \in \mathbb{N}$. Moreover, we can arrange that

(i) $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$,

(ii) $[\![v > 0]\!] = \operatorname{upr}(b, \mathfrak{C})$,

(iii) $u \le v \le u + \chi 1$.

395J Notation Observe that the specification of $[\![u \ge n]\!]$ and $[\![v \le n]\!]$, together with the declaration that $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$, determine u and v uniquely. So we can write $\lfloor b : a \rfloor$ for u and $\lceil b : a \rceil$ for v.

395K Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a, b, b_1, $b_2 \in \mathfrak{A}$ and that $\operatorname{upr}(a, \mathfrak{C}) = 1$.

(a) $\lfloor 0 : a \rfloor = \lceil 0 : a \rceil = 0$, $\lfloor 1 : a \rfloor \ge \chi 1$ and $\lfloor 1 : 1 \rfloor = \chi 1$.

(b) If $b_1 \preccurlyeq^\tau_G b_2$ then $\lfloor b_1 : a \rfloor \le \lfloor b_2 : a \rfloor$ and $\lceil b_1 : a \rceil \le \lceil b_2 : a \rceil$.

(c) $\lceil b_1 \cup b_2 : a \rceil \le \lceil b_1 : a \rceil + \lceil b_2 : a \rceil$.

(d) If $b_1 \cap b_2 = 0$, $\lfloor b_1 : a \rfloor + \lfloor b_2 : a \rfloor \le \lfloor b_1 \cup b_2 : a \rfloor$.

(e) If $c \in \mathfrak{C}$ is such that $a \cap c$ is a relative atom over $\mathfrak{C}$, then $c \subseteq [\![\lceil b : a \rceil - \lfloor b : a \rfloor = 0]\!]$.

395L Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a_1, a_2, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a_1, \mathfrak{C}) = \operatorname{upr}(a_2, \mathfrak{C}) = 1$. Then

$$\lfloor b : a_2 \rfloor \ge \lfloor b : a_1 \rfloor \times \lfloor a_1 : a_2 \rfloor, \quad \lceil b : a_2 \rceil \le \lceil b : a_1 \rceil \times \lceil a_1 : a_2 \rceil.$$

395M Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$.

(a) For any $a \in \mathfrak{A}$, there is a $b \subseteq a$ such that $b \preccurlyeq^\tau_G a \setminus b$ and $a \setminus \operatorname{upr}(b, \mathfrak{C})$ is a either 0 or a relative atom over $\mathfrak{C}$.

(b) Now suppose that G is fully non-paradoxical. Then for any $\epsilon > 0$ there is an $a \in \mathfrak{A}$ such that $\operatorname{upr}(a, \mathfrak{C}) = 1$ and $\lceil b : a \rceil \le \lfloor b : a \rfloor + \epsilon \lfloor 1 : a \rfloor$ for every $b \in \mathfrak{A}$.

395N Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Then there is a unique function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ such that

(i) θ is additive, non-negative and order-continuous;

(ii) $[\![\theta a > 0]\!] = \operatorname{upr}(a, \mathfrak{C})$ for every $a \in \mathfrak{A}$; in particular, $\theta a = 0$ iff $a = 0$;

(iii) $\theta 1 = \chi 1$;

(iv) $\theta(a \cap c) = \theta a \times \chi c$ for every $a \in \mathfrak{A}$, $c \in \mathfrak{C}$; in particular, $\theta c = \chi c$ for every $c \in \mathfrak{C}$;

(v) If a, $b \in \mathfrak{A}$ are G-τ-equidecomposable, then $\theta a = \theta b$; in particular, θ is G-invariant.

395O Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Then there is a G-invariant additive functional $\nu : \mathfrak{A} \to [0, 1]$ such that $\nu 1 = 1$.

395P Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Then the following are equiveridical:

(i) $\mathfrak{A}$ is a measurable algebra;

(ii) $\mathfrak{C}$ is a measurable algebra;

(iii) there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

395Q Corollary: Kawada's theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $\operatorname{Aut}\mathfrak{A}$ has a subgroup which is ergodic and fully non-paradoxical. Then $\mathfrak{A}$ is measurable.

395R Proposition If $(\mathfrak{A},\bar{\mu})$ is a homogeneous totally finite measure algebra, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is ergodic.

395Z Problem Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, not $\{0\}$, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ such that whenever $\langle a_i\rangle_{i\le n}$ is a finite partition of unity in $\mathfrak{A}$ and we are given π_i, $\pi'_i\in G$ for every $i\le n$, then the elements π_0a_0, π'_0a_0, π_1a_1, $\pi'_1a_1,\ldots,\pi'_na_n$ are not all disjoint. Must there be a non-zero non-negative G-invariant finitely additive functional θ on $\mathfrak{A}$?

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396 The Hajian-Ito theorem

In the notes to the last section, I said that the argument there short-circuits if we are told that we are dealing with a measurable algebra. The point is that in this case there is a much simpler criterion for the existence of a G-invariant measure (396B(ii)), with a proof which is independent of §395 in all its non-trivial parts, which makes it easy to prove that non-paradoxicality is sufficient as well as necessary.

396A Lemma Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra.

(a) Let $\pi\in\operatorname{Aut}\mathfrak{A}$ be a Boolean automorphism, and T_π the corresponding Riesz homomorphism from $L^0=L^0(\mathfrak{A})$ to itself. Then there is a unique $w_\pi\in(L^0)^+$ such that $\int w_\pi\times v=\int T_\pi v$ for every $v\in(L^0)^+$.

(b) If ϕ, $\pi\in\operatorname{Aut}\mathfrak{A}$ then $w_{\pi\phi}=w_\phi\times T_{\phi^{-1}}w_\pi$.

(c) For each $\pi\in\operatorname{Aut}\mathfrak{A}$ we have a norm-preserving isomorphism U_π from $L^2=L^2(\mathfrak{A},\bar{\mu})$ to itself defined by setting

$$U_\pi v=T_\pi v\times\sqrt{w_{\pi^{-1}}}$$

for every $v\in L^2$, and $U_{\pi\phi}=U_\pi U_\phi$ for all π, $\phi\in\operatorname{Aut}\mathfrak{A}$.

396B Theorem Let $\mathfrak{A}$ be a measurable algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is a G-invariant functional $\bar{\nu}$ such that $(\mathfrak{A},\bar{\nu})$ is a totally finite measure algebra;

(ii) whenever $a\in\mathfrak{A}\setminus\{0\}$ and $\langle\pi_n\rangle_{n\in\mathbb{N}}$ is a sequence in G, $\langle\pi_na\rangle_{n\in\mathbb{N}}$ is not disjoint;

(iii) G is fully non-paradoxical.

Version of 21.10.07

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

393B The association of a metric with a strictly positive submeasure, used in the 2003 and 2006 editions of Volume 4, is now in 392H and 393H.

393C The result that a non-negative additive functional on a Boolean algebra can be factored through a measure algebra, used in the 2003 and 2006 editions of Volume 4, is now in 392I.

393O The note on control measures for vector measures, referred to in the 2003 and 2006 editions of Volume 4, is now in 394Q.

§**394** Kawada's theorem, referred to in the 2003 and 2006 editions of Volume 4, is now in §395.