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Chapter 39

Measurable algebras

In the final chapter of this volume, I present results connected with the following question: which algebras can appear as the underlying Boolean algebras of measure algebras? Put in this form, there is a trivial answer (391A). The proper question is rather: which algebras can appear as the underlying Boolean algebras of semi-finite measure algebras? This is easily reducible to the question: which algebras can appear as the underlying Boolean algebras of probability algebras? Now in one sense Maharam's theorem (§332) gives us the answer exactly: they are the countable simple products of the measure algebras of $\{0,1\}^\kappa$ for cardinals κ. But if we approach from another direction, things are more interesting. Probability algebras share a very large number of very special properties. Can we find a selection of these properties which will be sufficient to force an abstract Boolean algebra to be a probability algebra when endowed with a suitable functional?

No fully satisfying answer to this question is known. But in exploring the possibilities we encounter some interesting and important ideas. In §391 I discuss algebras which have strictly positive additive real-valued functionals; for such algebras, weak (σ,∞)-distributivity is necessary and sufficient for the existence of a measure; so we are led to look for conditions sufficient to ensure that there is a strictly positive additive functional. A slightly different approach lies through the concept of 'submeasure'. Submeasures arise naturally in the theories of topological Boolean algebras (393J), topological Riesz spaces (393K) and vector measures (394P), and on any given algebra there is a strictly positive 'uniformly exhaustive' submeasure iff there is a strictly positive additive functional; this is the Kalton-Roberts theorem (392F).

Submeasures in general are common, but correspondingly limited in what they can tell us about a structure in the absence of further properties. Uniformly exhaustive submeasures are not far from additive functionals. An intermediate class, the 'exhaustive' submeasures, has been intensively studied, originally in the hope that they might lead to characterizations of measurable algebras, but more recently for their own sake. Just as additive functionals lead to measurable algebras, totally finite exhaustive submeasures lead to 'Maharam algebras' (§393). For many years it was not known whether every exhaustive submeasure was uniformly exhaustive (equivalently, whether every Maharam algebra was a measurable algebra); an example was eventually found by M.Talagrand, and is presented in §394.

In §395, I look at a characterization of measurable algebras in terms of the special properties which the automorphism group of a measure algebra must have (Kawada's theorem, 395Q). §396 complements the previous section by looking briefly at the subgroups of an automorphism group $\operatorname{Aut}\mathfrak{A}$ which can appear as groups of measure-preserving automorphisms.

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391 Kelley's theorem

In this section I introduce the notion of 'measurable algebra' (391B), which will be the subject of the whole chapter once the trivial construction of 391A has been dealt with. I show that for weakly (σ,∞)-distributive algebras countable additivity can be left to look after itself, and all we need to find is a strictly positive finitely additive functional (391D). I give Kelley's criterion for the existence of such a functional (391H-391J).

391A Proposition Let $\mathfrak{A}$ be any Dedekind σ-complete Boolean algebra. Then there is a function $\bar{\mu}:\mathfrak{A}\to[0,\infty]$ such that $(\mathfrak{A},\bar{\mu})$ is a measure algebra.

proof Set $\bar{\mu}0=0$, $\bar{\mu}a=\infty$ for $a\in\mathfrak{A}\setminus\{0\}$.

391B Definition (a) I will call a Boolean algebra $\mathfrak{A}$ **measurable** if there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty[$ such that $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra.

In this case, if $\bar{\mu} \neq 0$, then it has a scalar multiple with total mass 1. So a Boolean algebra $\mathfrak{A}$ is measurable iff either it is $\{0\}$ or there is a functional $\bar{\mu}$ such that $(\mathfrak{A}, \bar{\mu})$ is a probability algebra.

(b) I will call a Boolean algebra $\mathfrak{A}$ **chargeable** if there is an additive functional $\nu : \mathfrak{A} \to [0, \infty[$ which is **strictly positive**, that is, $\nu a > 0$ for every non-zero $a \in \mathfrak{A}$.

Of course a measurable algebra is chargeable.

(c) I will call a Boolean algebra **nowhere measurable** if none of its non-zero principal ideals are measurable algebras.

391C Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) The following are equiveridical: (i) there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ such that $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra; (ii) $\mathfrak{A}$ is Dedekind σ-complete and $\{a : a \in \mathfrak{A}, \mathfrak{A}_a$ is measurable$\}$ is order-dense in $\mathfrak{A}$, writing $\mathfrak{A}_a$ for the principal ideal generated by a.

(b) The following are equiveridical: (i) there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ such that $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra; (ii) $\mathfrak{A}$ is Dedekind complete and $\{a : a \in \mathfrak{A}, \mathfrak{A}_a$ is measurable$\}$ is order-dense in $\mathfrak{A}$.

proof (a) (i)$\Rightarrow$(ii): if $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, then $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$ is order-dense in $\mathfrak{A}$ and $\mathfrak{A}_a$ is measurable for every $a \in \mathfrak{A}^f$.

(ii)$\Rightarrow$(i): setting $D = \{a : a \in \mathfrak{A}, \mathfrak{A}_a$ is measurable$\}$, D is order-dense, so there is a partition of unity $C \subseteq D$ (313K). For each $c \in C$, choose $\bar{\mu}_c$ such that $(\mathfrak{A}_c, \bar{\mu}_c)$ is a totally finite measure algebra. Set $\bar{\mu}a = \sum_{c \in C} \bar{\mu}_c(a \cap c)$ for every $a \in \mathfrak{A}$; then it is easy to check that $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra.

(b) Follows immediately.

391D Theorem (Kantorovich Vulikh & Pinsker 50) Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is measurable;

(ii) $\mathfrak{A}$ is Dedekind σ-complete, weakly (σ, ∞)-distributive and chargeable.

proof (i)$\Rightarrow$(ii) Put the definition together with 322C(b)-(c) (for Dedekind completeness) and 322F (for weak (σ, ∞)-distributivity).

(ii)$\Rightarrow$(i) Given that (ii) is satisfied, let M be the L-space of bounded additive functionals on $\mathfrak{A}$, $M_\tau \subseteq M$ the band of completely additive functionals, and $P_\tau : M \to M_\tau$ the band projection (362Bd). Let $\nu : \mathfrak{A} \to [0, \infty[$ be a strictly positive additive functional, and set $\bar{\mu} = P_\tau(\nu)$. Then $\bar{\mu}$ is strictly positive. **P** If $c \in \mathfrak{A}$ is non-zero, there is an upwards-directed set A, with supremum c, such that $\bar{\mu}c = \sup_{a \in A} \nu c$ (362D); as ν is strictly positive and A contains a non-zero element, $\bar{\mu}c > 0$. **Q** Of course $\bar{\mu}$ is countably additive, so witnesses that $\mathfrak{A}$ is measurable.

391E Thus we are led naturally to the question: which Boolean algebras carry strictly positive *finitely* additive functionals? The Hahn-Banach theorem, suitably applied, gives some sort of answer to this question. For the sake of applications later on, I give two general results on the existence of additive functionals related to given functionals.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\phi : \mathfrak{A} \to [0, 1]$ a functional. Then the following are equiveridical:

(i) there is a finitely additive functional $\nu : \mathfrak{A} \to [0, 1]$ such that $\nu 1 = 1$ and $\nu a \leq \phi a$ for every $a \in \mathfrak{A}$;

(ii) whenever $\langle a_i \rangle_{i \in I}$ is a finite indexed family in $\mathfrak{A}$, $m \in \mathbb{N}$ and $\sum_{i \in I} \chi a_i \geq m\chi 1$ in $S = S(\mathfrak{A})$ (definition: 361A), then $\sum_{i \in I} \phi a_i \geq m$.

proof (a)(i)$\Rightarrow$(ii) If $\nu : \mathfrak{A} \to [0, 1]$ is a finitely additive functional such that $\nu 1 = 1$ and $\nu a \leq \phi a$ for every $a \in \mathfrak{A}$, let $h : S \to \mathbb{R}$ be the positive linear functional corresponding to ν (361G). Now if $\langle a_i \rangle_{i \in I}$ is a finite family in $\mathfrak{A}$ and $\sum_{i \in I} \chi a_i \geq m\chi 1$, then

$$\sum_{i\in I}\phi a_i \ge \sum_{i\in I}\nu a_i = \sum_{i\in I}h(\chi a_i)$$
$$= h(\sum_{i\in I}\chi a_i) \ge h(m\chi 1) = m.$$

As $\langle a_i\rangle_{i\in I}$ is arbitrary, (ii) is true.

(b)(ii)⇒(i) Now suppose that ϕ satisfies (ii). For $u \in S$, set

$$p(u) = \inf\{\sum_{i=0}^n \alpha_i\phi a_i : a_0,\dots,a_n \in \mathfrak{A},\ \alpha_0,\dots,\alpha_n \ge 0,\ \sum_{i=0}^n \alpha_i\chi a_i \ge u\}.$$

Then it is easy to check that $p(u+v) \le p(u)+p(v)$ for all u, $v \in S$, and that $p(\alpha u) = \alpha p(u)$ for all $u \in S$, $\alpha \ge 0$. Also $p(\chi 1) \ge 1$. **P?** If not, there are $a_0,\dots,a_n \in \mathfrak{A}$ and $\alpha_0,\dots,\alpha_n \ge 0$ such that $\chi 1 \le \sum_{i=0}^n \alpha_i\chi a_i$ but $\sum_{i=0}^n \alpha_i\phi a_i < 1$. Increasing each α_i slightly if necessary, we may suppose that every α_i is rational; let $m \ge 1$ and $k_0,\dots,k_n \in \mathbb{N}$ be such that $\alpha_i = k_i/m$ for each $i \le n$.

Set $K = \{(i,j) : 0 \le i \le n,\ 1 \le j \le k_i\}$, and for $(i,j) \in K$ set $a_{ij} = a_i$. Then

$$\sum_{(i,j)\in K}\chi a_{ij} = \sum_{i=0}^n k_i\chi a_i = m\sum_{i=0}^n \alpha_i\chi a_i \ge m\chi 1,$$

but

$$\sum_{(i,j)\in K}\phi a_{ij} = \sum_{i=0}^n k_i\phi a_i = m\sum_{i=0}^n \alpha_i\phi a_i < m,$$

which is supposed to be impossible. **XQ**

By the Hahn-Banach theorem, in the form 3A5Aa, there is a linear functional $h : S \to \mathbb{R}$ such that $h(\chi 1) = p(\chi 1) \ge 1$ and $h(u) \le p(u)$ for every $u \in S$. In particular, $h(\chi a) \le \phi b$ whenever $a \subseteq b \in \mathfrak{A}$. Set $\nu a = h(\chi a)$ for $a \in \mathfrak{A}$; then $\nu : \mathfrak{A} \to [0,\infty[$ is an additive functional, $\nu 1 \ge 1$ and $\nu a \le \phi b$ whenever $a \subseteq b$ in $\mathfrak{A}$. We do not know whether ν is positive, but if we define ν^+ as in 362Ab, we shall have a non-negative additive functional such that

$$\nu^+ a = \sup_{b\subseteq a}\nu b \le \phi a$$

for every $a \in \mathfrak{A}$, and

$$1 \le \nu 1 \le \nu^+ 1 \le \phi 1 \le 1,$$

so ν^+ witnesses that (i) is true.

391F Theorem Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\psi : A \to [0,1]$ a functional, where $A \subseteq \mathfrak{A}$. Then the following are equiveridical:

(i) there is a finitely additive functional $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$ and $\nu a \ge \psi a$ for every $a \in A$;

(ii) whenever $\langle a_i\rangle_{i\in I}$ is a finite indexed family in A, there is a set $J \subseteq I$ such that $\#(J) \ge \sum_{i\in I}\psi a_i$ and $\inf_{i\in J} a_i \ne 0$.

Remark In (ii) here, we may have to interpret the infimum of the empty set in $\mathfrak{A}$ as 1.

proof (a) We apply 391E to ϕ, where

$$\phi a = 1 - \psi(1\setminus a) \text{ if } a \in \mathfrak{A} \text{ and } 1\setminus a \in A,$$
$$= 1 \text{ for other } a \in \mathfrak{A}.$$

(b) Suppose that (i) here is true of ψ. Then 391E(i) is true of ϕ. **P** Let $\nu : \mathfrak{A} \to [0,1]$ be an additive functional such that $\nu 1 = 1$ and $\nu a \ge \psi a$ for every $a \in \mathfrak{A}$. If $a \in \mathfrak{A}$ and $1\setminus a \in A$, then

$$\nu a = 1 - \nu(1\setminus a) \le 1 - \psi(1\setminus a) = \phi a;$$

for other $a \in \mathfrak{A}$, $\nu a \le 1 = \phi a$. **Q**

(c) Suppose that 391E(i) is true of ϕ. Then (i) here is true of ψ. **P** There is an additive functional $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$ and $\nu a \le \phi a$ for every $a \in \mathfrak{A}$; in this case, for $a \in A$,

$$\nu a = 1 - \nu(1\setminus a) \ge 1 - \phi(1\setminus a) = \psi a. \text{ **Q**}$$

(d) Suppose that (ii) here is true of ψ, and that $\langle a_i\rangle_{i\in I}$ is a finite family in $\mathfrak{A}$ such that $\sum_{i\in I}\chi a_i\ge m\chi 1$, while $\sum_{i\in I}\phi a_i=\beta$. Set $K=\{i:i\in I,\,1\setminus a_i\in A\}$.

$$\sum_{i\in K}\psi(1\setminus a_i)=\sum_{i\in K}(1-\phi a_i)=\#(K)-\sum_{i\in I}\phi a_i+\#(I\setminus K)=\#(I)-\beta,$$

so there is a set $J\subseteq K$ such that $\#(J)\ge\#(I)-\beta$ and $\inf_{i\in J}(1\setminus a_i)=c\ne 0$. Now $c\cap a_i=0$ for $i\in J$, so

$$m\chi c\le\sum_{i\in I}\chi(a_i\cap c)=\sum_{i\in I\setminus J}\chi(a_i\cap c)\le\#(I\setminus J)\chi c$$

and $m\le\#(I)-\#(J)\le\beta$. As $\langle a_i\rangle_{i\in I}$ is arbitrary, 391E(ii) is true of ϕ.

(e) Suppose that 391E(ii) is true of ϕ, and that $\langle a_i\rangle_{i\in I}$ is a family in A. Set

$$\beta=\sum_{i\in I}\phi(1\setminus a_i)=\#(I)-\sum_{i\in I}\psi a_i$$

and let k be the least integer greater than β. Since $\sum_{i\in I}\phi(1\setminus a_i)<k$, $\sum_{i\in I}\chi(1\setminus a_i)\not\ge k\chi 1$, that is, $\sum_{i\in I}\chi a_i\not\le(\#(I)-k)\chi 1$. But this means that there must be some $J\subseteq I$ such that $\#(J)>\#(I)-k$ and $\inf_{i\in J}a_i\ne 0$. Now

$$\sum_{i\in I}\psi a_i=\#(I)-\beta\le\#(I)-(k-1)\le\#(J).$$

As $\langle a_i\rangle_{i\in I}$ is arbitrary, (ii) here is true of ψ.

(f) Since we know that 391E(i)$\Leftrightarrow$391E(ii), we can conclude that (i) and (ii) here are equiveridical.

391G Corollary Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\nu_0:\mathfrak{B}\to\mathbb{R}$ a non-negative finitely additive functional. Then there is a non-negative finitely additive functional $\nu:\mathfrak{A}\to\mathbb{R}$ extending ν_0.

proof (a) Suppose first that $\nu_0 1=1$. Set $\psi b=\nu_0 b$ for every $b\in\mathfrak{B}$. Then ψ must satisfy the condition (ii) of 391F when regarded as a functional defined on a subset of $\mathfrak{B}$; but this means that it satisfies the same condition when regarded as a functional defined on a subset of $\mathfrak{A}$. So there is a non-negative finitely additive functional $\nu:\mathfrak{A}\to\mathbb{R}$ such that $\nu 1=1$ and $\nu b\ge\nu_0 b$ for every $b\in\mathfrak{B}$. In this case

$$\nu b=1-\nu(1\setminus b)\le 1-\nu_0(1\setminus b)=\nu_0 b\le\nu b$$

for every $b\in\mathfrak{B}$, so ν extends ν_0.

(b) For the general case, if $\nu_0 1=0$ then ν_0 must be the zero functional on $\mathfrak{B}$, so we can take ν to be the zero functional on $\mathfrak{A}$; and if $\nu_0 1=\gamma>0$, we apply (a) to $\gamma^{-1}\nu_0$.

391H Definition Let $\mathfrak{A}$ be a Boolean algebra, and $A\subseteq\mathfrak{A}\setminus\{0\}$ any non-empty set. The **intersection number** of A is the largest $\delta\ge 0$ such that whenever $\langle a_i\rangle_{i\in I}$ is a finite family in A, with $I\ne\emptyset$, there is a $J\subseteq I$ such that $\#(J)\ge\delta\#(I)$ and $\inf_{i\in J}a_i\ne 0$.

Remarks (a) It is essential to note that in the definition here the $\langle a_i\rangle_{i\in I}$ are indexed families, with repetitions allowed; see 391Xi.

(b) I spoke perhaps rather glibly of 'the largest δ such that . . . '; you may prefer to write

$$\delta=\inf\{\sup_{\emptyset\ne J\subseteq\{0,\dots,n\},\inf_{j\in J}a_j\ne 0}\frac{\#(J)}{n+1}:a_0,\dots,a_n\in A\}.$$

391I Proposition Let $\mathfrak{A}$ be a Boolean algebra and $A\subseteq\mathfrak{A}\setminus\{0\}$ any non-empty set. Write C for the set of non-negative finitely additive functionals $\nu:\mathfrak{A}\to[0,1]$ such that $\nu 1=1$. Then the intersection number of A is precisely $\max_{\nu\in C}\inf_{a\in A}\nu a$.

proof Write δ for the intersection number of A, and δ' for $\sup_{\nu\in C}\inf_{a\in A}\nu a$.

(a) For any $\gamma<\delta'$, we can find a $\nu\in C$ such that $\nu a\ge\gamma$ for every $a\in A$. So if we set $\psi a=\gamma$ for every $a\in A$, ψ satisfies condition (i) of 391F. But this means that if $\langle a_i\rangle_{i\in I}$ is any finite family in A, there must be a $J\subseteq I$ such that $\inf_{i\in J}a_i\ne 0$ and $\#(J)\ge\gamma\#(I)$. Accordingly $\gamma\le\delta$; as γ is arbitrary, $\delta'\le\delta$.

(b) Define $\psi:A\to[0,1]$ by setting $\psi a=\delta$ for every $a\in A$. If $\langle a_i\rangle_{i\in I}$ is a finite indexed family in A, there is a $J\subseteq I$ such that $\#(J)\ge\delta\#(I)$ and $\inf_{i\in J}a_i\ne 0$; but $\delta\#(I)=\sum_{i\in I}\psi a_i$, so this means that condition (ii) of 391F is satisfied. So there is a $\nu\in C$ such that $\nu a\ge\delta$ for every $a\in A$; and ν witnesses not only that $\delta'\ge\delta$, but that the supremum is a maximum.

391J Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:
(i) $\mathfrak{A}$ is chargeable;
(ii) either $\mathfrak{A}=\{0\}$ or $\mathfrak{A}\setminus\{0\}$ is expressible as a countable union of sets with non-zero intersection numbers.

proof (i)⇒(ii) If there is a strictly positive finitely additive functional ν on $\mathfrak{A}$, and $\mathfrak{A}\ne\{0\}$, set $A_n=\{a:\nu a\ge 2^{-n}\nu 1\}$ for every $n\in\mathbb{N}$; then (applying 391I to the functional $\frac{1}{\nu 1}\nu$) we see that every A_n has intersection number at least 2^{-n}, while $\mathfrak{A}\setminus\{0\}=\bigcup_{n\in\mathbb{N}}A_n$ because ν is strictly positive, so (ii) is satisfied.

(ii)⇒(i) If $\mathfrak{A}\setminus\{0\}$ is expressible as $\bigcup_{n\in\mathbb{N}}A_n$, where each A_n has intersection number $\delta_n>0$, then for each n choose a finitely additive functional ν_n on $\mathfrak{A}$ such that $\nu_n 1=1$ and $\nu_n a\ge\delta_n$ for every $a\in A_n$. Setting $\nu a=\sum_{n=0}^{\infty}2^{-n}\nu_n a$ for every $a\in\mathfrak{A}$, ν is a strictly positive additive functional on $\mathfrak{A}$, and (i) is true.

391K Corollary Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is measurable iff it is Dedekind σ-complete and weakly (σ,∞)-distributive and either $\mathfrak{A}=\{0\}$ or $\mathfrak{A}\setminus\{0\}$ is expressible as a countable union of sets with non-zero intersection numbers.

proof Put 391D and 391J together.

391L When we come to study the structure of measurable algebras in later volumes, it will be convenient to have the following facts on the table.

Proposition (a) If $\mathfrak{A}$ is a measurable algebra, all its principal ideals and σ-subalgebras are, in themselves, measurable algebras.
(b) The simple product of countably many measurable algebras is a measurable algebra.
(c) If $\mathfrak{A}$ is a measurable algebra, $\mathfrak{B}$ is a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\mathfrak{B}$ is a measurable algebra, isomorphic to a principal ideal of $\mathfrak{A}$.

proof (a) Use 322H and 322Na.

(b) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a countable family of measurable algebras with simple product $\mathfrak{A}$. For each $i\in I$ let $\bar\mu_i$ be such that $(\mathfrak{A}_i,\bar\mu_i)$ is a measure algebra and $\bar\mu_i 1_{\mathfrak{A}_i}\le 1$. Let $f:I\to\mathbb{N}$ be an injection. For $a=\langle a_i\rangle_{i\in I}\in\mathfrak{A}$, set $\bar\mu a=\sum_{i\in I}2^{-f(i)}\bar\mu_i a_i$. Then $(\mathfrak{A},\bar\mu)$ is a measure algebra (see 322La); as $\bar\mu 1_{\mathfrak{A}}\le 2$, $\mathfrak{A}$ is a measurable algebra.

(c) Consider the kernel $I=\{a:\pi a=0\}$ of π. By 313Pa, I is order-closed. Because $\mathfrak{A}$ is Dedekind complete, $c=\sup I$ is defined in $\mathfrak{A}$; as I is upwards-directed, $c\in I$ and I is the principal ideal generated by c. Let $\mathfrak{A}_{1\setminus c}$ be the principal ideal generated by $1\setminus c$. Then $I\cap\mathfrak{A}_{1\setminus c}=\{0\}$ so $\pi\restriction\mathfrak{A}_{1\setminus c}$ is injective. We are supposing that

$$\mathfrak{B}=\{\pi a:a\in\mathfrak{A}\}=\{\pi(a\cap c)\cup\pi(a\setminus c):a\in\mathfrak{A}\}=\{\pi(a\setminus c):a\in\mathfrak{A}\}=\pi[\mathfrak{A}_{1\setminus c}].$$

So $\pi\restriction\mathfrak{A}_{1\setminus c}$ is an isomorphism between $\mathfrak{A}_{1\setminus c}$ and $\mathfrak{B}$. But $\mathfrak{A}_{1\setminus c}$ is a measurable algebra, by (a), so $\mathfrak{B}$ is a measurable algebra.

391X Basic exercises (a) Show that a chargeable Boolean algebra is ccc, so is Dedekind complete iff it is Dedekind σ-complete.

(b) Show (i) that any subalgebra of a chargeable Boolean algebra is chargeable (ii) that a countable simple product of chargeable Boolean algebras is chargeable (iii) that any free product of chargeable Boolean algebras is chargeable.

(c)(i) Let $\mathfrak{A}$ be a Boolean algebra with a chargeable order-dense subalgebra. Show that $\mathfrak{A}$ is chargeable. (ii) Show that the Dedekind completion of a chargeable Boolean algebra is chargeable.

(d)(i) Show that the algebra of open-and-closed subsets of $\{0,1\}^I$ is chargeable for any set I. (ii) Show that the regular open algebra of $\mathbb{R}$ is chargeable.

(e)(i) Show that any principal ideal of a chargeable Boolean algebra is chargeable. (ii) Let $\mathfrak{A}$ be a chargeable Boolean algebra and $\mathcal{I}$ an order-closed ideal of $\mathfrak{A}$. Show that $\mathfrak{A}/\mathcal{I}$ is chargeable.

>(f) Show that a Boolean algebra is chargeable iff it is isomorphic to a subalgebra of a measurable algebra. (*Hint*: 324O, 392H.)

(g) Let $\mathfrak{A}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is chargeable and weakly (σ,∞)-distributive; (ii) there is a strictly positive countably additive functional on $\mathfrak{A}$; (iii) there is a strictly positively completely additive functional on $\mathfrak{A}$.

(h) Explain how to use the Hahn-Banach theorem to prove 391G directly, without passing through 391F. (*Hint*: $S(\mathfrak{B})$ can be regarded as a subspace of $S(\mathfrak{A})$.)

>(i) Take $X=\{0,1,2,3\}$, $\mathfrak{A}=\mathcal{P}X$, $A=\{\{0,1\},\{0,2\},\{0,3\},\{1,2,3\}\}$. Show that the intersection number of A is $\frac{3}{5}$. (*Hint*: use 391I.) Show that if $a_0,\dots,a_n$ are *distinct* members of A then there is a set $J\subseteq\{0,\dots,n\}$, with $\#(J)\ge\frac{2}{3}(n+1)$, such that $\inf_{j\in J}a_j\ne 0$.

(j) Let $\mathfrak{A}$ be a Boolean algebra. For non-empty $A\subseteq\mathfrak{A}\setminus\{0\}$ write $\delta(A)$ for the intersection number of A. Show that for any non-empty $A\subseteq\mathfrak{A}\setminus\{0\}$, $\delta(A)=\inf\{\delta(I):I$ is a non-empty finite subset of $A\}$.

(k) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$. For $a_0,\dots,a_n\in\mathfrak{A}$ set $t(a_0,\dots,a_n)=\max\{m:m\in\mathbb{N}$, $m\chi 1\le\sum_{i=0}^n\chi a_i\}$. Let $A\subseteq\mathfrak{A}$ be non-empty. Show that

$$\begin{aligned}\sup\{\frac{1}{n+1}t(a_0,\dots,a_n):a_0,\dots,a_n\in A\}\\ =\min\{\sup_{a\in A}\nu a:\nu\text{ is a non-negative additive functional on }\mathfrak{A},\ \nu 1=1\}.\end{aligned}$$

(This is the **Kelley covering number** of A.)

(l) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the following are equiveridical: (α) there is a functional $\bar\mu$ such that $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra; (β) $L^\infty(\mathfrak{A})$ is a perfect Riesz space (definition: 356J). (ii) Show that in this case $\mathfrak{A}$ is a measurable algebra iff it is ccc.

391Y Further exercises (a) Show that in 391D and 391K we can replace 'weakly (σ,∞)-distributive' by 'weakly σ-distributive'.

(b) Show that $\mathcal{P}\mathbb{N}$ is chargeable but that the quotient algebra $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is not ccc, therefore not chargeable.

(c)(i) Show that if X is a separable topological space, then its regular open algebra is chargeable. (ii) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces with chargeable regular open algebras. Show that their product has a chargeable regular open algebra.

(d) Let μ be Lebesgue measure on $[0,1]$, and Σ its domain. Let $\mathcal{A}$ be a non-empty family of non-empty subsets of X, with intersection number δ, and let $\mathcal{W}$ be the family of those sets $W\in\mathcal{P}X\widehat{\otimes}\Sigma$ such that $W^{-1}[\{t\}]\in\mathcal{A}$ for every $t\in[0,1]$. Set $\alpha=\inf_{W\in\mathcal{W}}\sup_{x\in X}\mu W[\{x\}]$. (i) Show that $\alpha\le\delta$. (ii) Give an example in which $\alpha<\delta$.

(e) Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, U a linear space and $\nu_0:\mathfrak{B}\to U$ an additive functional. Show that there is an additive functional $\nu:\mathfrak{A}\to U$ extending ν_0. (*Hint*: 361F.)

391 Notes and comments By the standards of this volume, this is an easy section; I note that I have hardly called on anything after Chapter 32, except for a reference to the construction $S(\mathfrak{A})$ in §361. I do ask for a bit of functional analysis (the Hahn-Banach theorem) in 391E.

391J-391K are due to Kelley 59; condition (ii) of 391J is called **Kelley's criterion**. It provides some sort of answer to the question 'which Boolean algebras carry strictly positive finitely additive functionals?', but leaves quite open the possibility that there is some more abstract criterion which is also necessary and sufficient. It is indeed a non-trivial exercise to find any ccc Boolean algebra which does not carry a strictly positive finitely additive functional. The first example published seems to have been that of Gaifman 64, which is described in Comfort & Negrepontis 82. But for the purposes of this book Gaifman's example has been superseded by Talagrand's example, presented in §394.

Kelley's criterion is a little unsatisfying. It is undoubtedly important (see 392F below), but at the same time the structure of the criterion – a special sequence of subsets of $\mathfrak{A}$ – is rather close to the structure of the conclusion; after all, one is, or can be represented by, a function from $\mathfrak{A}\setminus\{0\}$ to $\mathbb{N}$, while the other is a function from $\mathfrak{A}$ to $\mathbb{R}$. Also the actual intersection number of a family $A\subseteq\mathfrak{A}\setminus\{0\}$ can be hard to calculate; as often as not, the best method is to look at the additive functionals on $\mathfrak{A}$ (see 391Xi).

Version of 11.2.08

392 Submeasures

In §391 I looked at what we can deduce if a Boolean algebra carries a strictly positive finitely additive functional. There are important contexts in which we find ourselves with subadditive, rather than additive, functionals, and these are what I wish to investigate here. It turns out that, once we have found the right hypotheses, such functionals can also provide a criterion for measurability of an algebra (392G below). The argument runs through a new idea, using a result in finite combinatorics (392D).

At the end of the section I include notes on metrics associated with submeasures (392H) and on products of submeasures (392K).

392A Definition Let $\mathfrak{A}$ be a Boolean algebra. A **submeasure** on $\mathfrak{A}$ is a functional $\nu:\mathfrak{A}\to[0,\infty]$ such that

$\nu 0=0$,

$\nu a\le\nu b$ whenever $a\subseteq b$,

$\nu(a\cup b)\le\nu a+\nu b$ for all a, $b\in\mathfrak{A}$.

392B The following list mostly repeats ideas we have already used in the context of measures; but (b) and (c) are new, and will be the basis of this section.

Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\nu:\mathfrak{A}\to[0,\infty]$ a submeasure.

(a) ν is **strictly positive** if $\nu a>0$ for every $a\ne0$.

(b) ν is **exhaustive** if $\lim_{n\to\infty}\nu a_n=0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

(c) ν is **uniformly exhaustive** if for every $\epsilon>0$ there is an $n\in\mathbb{N}$ such that there is no disjoint family $a_0,\dots,a_n$ with $\nu a_i\ge\epsilon$ for every $i\le n$.

(d) ν is **totally finite** if $\nu 1<\infty$.

(e) ν is **unital** if $\nu 1=1$.

(f) ν is **atomless** if whenever $a\in\mathfrak{A}$ and $\nu a>0$ there is a $b\subseteq a$ such that $\nu b>0$ and $\nu(a\setminus b)>0$.

(g) If ν' is another submeasure on $\mathfrak{A}$, then ν' is **absolutely continuous** with respect to ν if for every $\epsilon>0$ there is a $\delta>0$ such that $\nu'a\le\epsilon$ whenever $\nu a\le\delta$.

392C Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If there is an exhaustive strictly positive submeasure on $\mathfrak{A}$, then $\mathfrak{A}$ is ccc.

(b) A uniformly exhaustive submeasure on $\mathfrak{A}$ is exhaustive.

(c) Any non-negative additive functional on $\mathfrak{A}$ is a uniformly exhaustive submeasure.

proof These are all elementary. If $\nu:\mathfrak{A}\to[0,\infty]$ is an exhaustive strictly positive submeasure, and $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}\setminus\{0\}$, then $\{i:\nu a_i\ge2^{-n}\}$ must be finite for each n, so I is countable. (Cf. 322G.) If $\nu:\mathfrak{A}\to[0,\infty]$ is a uniformly exhaustive submeasure and $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint in $\mathfrak{A}$, then $\{i:\nu a_i\ge2^{-n}\}$ is finite for each n, so $\lim_{i\to\infty}\nu a_i=0$. If $\nu:\mathfrak{A}\to[0,\infty[$ is a non-negative additive functional, it is a submeasure, by 326Ba and 326Bf. If $\epsilon>0$, then take $n\ge\frac{1}{\epsilon}\nu 1$; if $a_0,\dots,a_n$ are disjoint, then $\sum_{i=0}^n\nu a_i\le\nu 1$, so $\min_{i\le n}\nu a_i<\epsilon$.

392D Lemma Suppose that k, l, $m\in\mathbb{N}$ are such that $3\le k\le l\le m$ and $18mk\le l^2$. Let L, M be sets with l, m members respectively. Then there is a set $R\subseteq M\times L$ such that (i) each vertical section of

R has just three members (ii) $\#(R[E]) \ge \#(E)$ whenever $E \in [M]^{\le k}$; so that for every $E \in [M]^{\le k}$ there is an injective function $f : E \to L$ such that $(x, f(x)) \in R$ for every $x \in E$.

Recall that $[M]^{\le k} = \{I : I \subseteq M, \#(I) \le k\}$, $[M]^k = \{I : I \subseteq M, \#(I) = k\}$ (3A1J).

proof (a) We need to know that $n! \ge 3^{-n}n^n$ for every $n \in \mathbb{N}$; this is immediate from the inequality

$$\sum_{i=2}^n \ln i \ge \int_1^n \ln x\,dx = n\ln n - n + 1 \text{ for every } n \ge 2.$$

(b) Let Ω be the set of those $R \subseteq M \times L$ such that each vertical section of R has just three members, so that

$$\#(\Omega) = \#([L]^3)^m = \big(\tfrac{l!}{3!(l-3)!}\big)^m.$$

Let us regard Ω as a probability space with the uniform probability.

If $F \in [L]^n$, where $3 \le n \le k$, and $x \in M$, then

$$\Pr(R[\{x\}] \subseteq F) = \tfrac{\#([F]^3)}{\#([L]^3)}$$

(because $R[\{x\}]$ is a random member of $[L]^3$)

$$= \tfrac{n(n-1)(n-2)}{l(l-1)(l-2)} \le \tfrac{n^3}{l^3}$$

as $n < l$. So if $E \in [M]^n$ and $F \in [L]^n$, then

$$\Pr(R[E] \subseteq F) = \prod_{x\in E} \Pr(R[\{x\}] \subseteq F)$$

(because the sets $R[\{x\}]$ are chosen independently)

$$\le \tfrac{n^{3n}}{l^{3n}}.$$

Accordingly

$$\Pr(\text{there is an } E \subseteq M \text{ such that } \#(R[E]) < \#(E) \le k)$$
$$\le \Pr(\text{there is a non-empty } E \subseteq M \text{ such that } \#(R[E]) \le \#(E) \le k)$$
$$= \Pr(\text{there is an } E \subseteq M \text{ such that } 3 \le \#(R[E]) \le \#(E) \le k)$$

(because if $E \ne \emptyset$ then $\#(R[E]) \ge 3$)

$$\le \sum_{n=3}^k \sum_{E\in[M]^n} \sum_{F\in[L]^n} \Pr(R[E] \subseteq F) \le \sum_{n=3}^k \#([M]^n)\#([L]^n)\frac{n^{3n}}{l^{3n}}$$
$$= \sum_{n=3}^k \frac{m!}{n!(m-n)!}\frac{l!}{n!(l-n)!}\frac{n^{3n}}{l^{3n}} \le \sum_{n=3}^k \frac{m^n l^n n^{3n}}{n!n!l^{3n}} \le \sum_{n=3}^k \frac{m^n n^n 3^{2n}}{l^{2n}}$$

(using (a))

$$= \sum_{n=3}^k \big(\frac{9mn}{l^2}\big)^n \le \sum_{n=3}^k \frac{1}{2^n} < 1.$$

There must therefore be some $R \in \Omega$ such that $\#(R[E]) \ge \#(E)$ whenever $E \subseteq M$ and $\#(E) \le k$.

(c) If now $E \in [M]^{\le k}$, the restriction $R_E = R \cap (E \times L)$ has the property that $\#(R_E[I]) \ge \#(I)$ for every $I \subseteq E$. By Hall's Marriage Lemma (3A1K) there is an injective function $f : E \to L$ such that $(x, f(x)) \in R_E \subseteq R$ for every $x \in E$.

Remark Of course this argument can be widely generalized; see references in Kalton & Roberts 83.

392E Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0, \infty]$ a uniformly exhaustive submeasure. Then for any $\epsilon \in]0, \nu 1]$ the set $A = \{a : \nu a \ge \epsilon\}$ has intersection number greater than 0.

proof (a) To begin with (down to the end of (d) below), suppose that $\nu 1 = 1$. Because ν is uniformly exhaustive, there is an $r \ge 1$ such that whenever $\langle c_i \rangle_{i \in I}$ is a disjoint family in $\mathfrak{A}$ then $\#(\{i : \nu c_i > \frac{1}{5}\epsilon\}) \le r$, so that $\sum_{i \in I} \nu c_i \le r + \frac{1}{5}\epsilon\#(I)$. Set $\delta = \epsilon/5r$, $\eta = \frac{1}{74}\delta^2$, so that

$$\delta - \eta \ge \tfrac{1}{18}(\delta - \eta)^2 \ge \tfrac{1}{18}(\delta^2 - 2\eta) = 4\eta.$$

(b) Let $\langle a_i \rangle_{i \in I}$ be a non-empty finite family in A. Let m be any multiple of $\#(I)$ greater than or equal to $1/\eta$. Then there are integers k, l such that

$$3\eta \le \frac{k}{m} \le 4\eta \le \frac{1}{18}(\delta - \eta)^2, \quad \delta - \eta \le \frac{l}{m} \le \delta,$$

in which case

$$3 \le k \le l \le m, \quad 18mk \le m^2(\delta - \eta)^2 \le l^2.$$

(c) Take a set M of the form $I \times S$ where $\#(S) = m/\#(I)$, so that $\#(M) = m$. For $x = (i, s) \in M$ set $d_x = a_i$. Let L be a set with l members. By 392D, there is a set $R \subseteq M \times L$ such that every vertical section of R has just three members and whenever $E \in [M]^{\le k}$ there is an injective function $f_E : E \to L$ such that $(x, f_E(x)) \in R$ for every $x \in E$.

For $E \subseteq M$ set

$$b_E = \inf_{x \in E} d_x \setminus \sup_{x \in M \setminus E} d_x,$$

so that $\langle b_E \rangle_{E \subseteq M}$ is a partition of unity in $\mathfrak{A}$. For $x \in M$ and $j \in L$ set

$$c_{xj} = \sup\{b_E : x \in E \in [M]^{\le k}, \, f_E(x) = j\}.$$

If x, y are distinct members of M and $j \in L$ then

$$c_{xj} \cap c_{yj} = \sup\{b_E : x, \, y \in E \in [M]^{\le k}, \, f_E(x) = f_E(y) = j\} = 0,$$

because every f_E is injective. Set

$$m_j = \#(\{x : x \in M, \, c_{xj} \ne 0\})$$

for each $j \in L$. Note that $c_{xj} = 0$ if $(x, j) \notin R$, so $\sum_{j \in L} m_j \le \#(R) = 3m$.

We have

$$\textstyle\sum_{x \in M} \nu c_{xj} \le r + \frac{1}{5}\epsilon m_j$$

for each j, by the choice of r; so

$$\sum_{x \in M, j \in L} \nu c_{xj} \le rl + \frac{1}{5}\epsilon \sum_{j \in L} m_j \le rl + \frac{3}{5}m\epsilon$$

$$\le (r\delta + \frac{3}{5}\epsilon)m = \frac{4}{5}\epsilon m < \epsilon m$$

by the choice of l and δ. There must therefore be some $x \in M$ such that

$$\nu(\textstyle\sup_{j \in L} c_{xj}) \le \sum_{j \in L} \nu c_{xj} < \epsilon \le \nu d_x,$$

and d_x cannot be included in

$$\textstyle\sup_{j \in L} c_{xj} = \sup\{b_E : x \in E \in [M]^{\le k}\}.$$

But as $\sup\{b_E : x \in E \subseteq M\}$ is just d_x, there must be an $E \subseteq M$, with cardinal greater than k, such that $b_E \ne 0$.

Recall now that $M = I \times S$, and that

$$k \ge 3\eta m = 3\eta\#(I)\#(S).$$

The set $J = \{i : \exists \, s, \, (i, s) \in E\}$ must therefore have more than $3\eta\#(I)$ members, since $E \subseteq J \times S$. But also $d_{(i,s)} = a_i$ for each $(i, s) \in E$, so that $\inf_{i \in J} a_i \supseteq b_E \ne 0$.

(d) As $\langle a_i\rangle_{i\in I}$ is arbitrary, the intersection number of A is at least $3\eta > 0$.

(e) This completes the proof in the case in which $\nu 1 = 1$. If $\nu 1 = 0$ the result is vacuous. If $\nu 1 > 0$, set $\nu' a = \frac{\min(\nu a,1)}{\min(\nu 1,1)}$ for each a; then it is easy to check that ν' is a uniformly exhaustive submeasure with $\nu' 1 = 1$, and

$$\{a : \nu a \ge \epsilon\} \subseteq \{a : \nu' a \ge \frac{\min(\epsilon,1)}{\min(\nu 1,1)}\}$$

has non-zero intersection number for any $\epsilon \in \,]0, \nu 1]$. So the result is true in the generality stated.

392F Theorem Let $\mathfrak{A}$ be a Boolean algebra with a strictly positive uniformly exhaustive submeasure. Then $\mathfrak{A}$ is chargeable, that is, has a strictly positive finitely additive functional.

proof If $\mathfrak{A} = \{0\}$ this is trivial. Otherwise, let $\nu : \mathfrak{A} \to [0,\infty]$ be a strictly positive uniformly exhaustive submeasure. For each n, $A_n = \{a : \nu a \ge \min(2^{-n}, \nu 1)\}$ has intersection number greater than 0, and $\bigcup_{n\in\mathbb{N}} A_n = \mathfrak{A} \setminus \{0\}$ because ν is strictly positive; so $\mathfrak{A}$ has a strictly positive finitely additive functional, by Kelley's theorem (391J).

392G Corollary Let $\mathfrak{A}$ be a Boolean algebra. Then it is measurable iff it is weakly (σ,∞)-distributive and Dedekind σ-complete and has a strictly positive uniformly exhaustive submeasure.

proof Put 391D and 392F together.

392H This completes the main work of this section. However it will be convenient later to have some more facts available which belong to the same group of ideas.

Metrics from submeasures: Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a strictly positive totally finite submeasure on $\mathfrak{A}$.

(a) We have a metric ρ on $\mathfrak{A}$ defined by the formula

$$\rho(a, b) = \nu(a \,\triangle\, b)$$

for all a, $b \in \mathfrak{A}$.

(b) The Boolean operations $\cup$, $\cap$, $\triangle$, $\setminus$ and the function $\nu : \mathfrak{A} \to \mathbb{R}$ are all uniformly continuous for ρ.

(c) The metric space completion $(\widehat{\mathfrak{A}}, \hat\rho)$ of $(\mathfrak{A}, \rho)$ is a Boolean algebra under the natural continuous extensions of the Boolean operations, and ν has a unique continuous extension $\hat\nu$ to $\widehat{\mathfrak{A}}$ which is again a strictly positive submeasure.

(d) If ν is additive, then $(\widehat{\mathfrak{A}}, \hat\nu)$ is a totally finite measure algebra.

proof (a)-(b) This is just a generalization of 323A-323B; essentially the same formulae can be used. For the triangle inequality for ρ, we have $a \,\triangle\, c \subseteq (a \,\triangle\, b) \cup (b \,\triangle\, c)$, so

$$\rho(a, c) = \nu(a \,\triangle\, c) \le \nu(a \,\triangle\, b) + \nu(b \,\triangle\, c) = \rho(a, b) + \rho(b, c).$$

For the uniform continuity of the Boolean operations, we have

$$(b \star c) \,\triangle\, (b' \star c') \subseteq (b \,\triangle\, b') \cup (c \,\triangle\, c')$$

so that

$$\rho(b \star c, b' \star c') \le \rho(b, b') + \rho(c, c')$$

for each of the operations $\star = \cup$, $\cap$, $\setminus$ and $\triangle$ and all b, c, b', $c' \in \mathfrak{A}$. For the uniform continuity of the function ν itself, we have

$$\nu b \le \nu c + \nu(b \setminus c) \le \nu c + \rho(b, c),$$

so that $|\nu b - \nu c| \le \rho(b, c)$.

(c) $\mathfrak{A} \times \mathfrak{A}$ is a dense subset of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, so the Boolean operations on $\mathfrak{A}$, regarded as uniformly continuous functions from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A} \subseteq \widehat{\mathfrak{A}}$, have unique extensions to continuous binary operations on $\widehat{\mathfrak{A}}$ (3A4G). If we look at

$$A = \{(a,b,c) : a \triangle (b \triangle c) = (a \triangle b) \triangle c\},$$

this is a closed subset of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, because the maps $(a,b,c) \mapsto a \triangle (b \triangle c)$, $(a,b,c) \mapsto (a \triangle b) \triangle c$ are continuous and the topology of $\widehat{\mathfrak{A}}$ is Hausdorff; since A includes the dense set $\mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$, it is the whole of $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, that is, $a \triangle (b \triangle c) = (a \triangle b) \triangle c$ for all a, b, $c \in \widehat{\mathfrak{A}}$. All the other identities we need to show that $\widehat{\mathfrak{A}}$ is a Boolean algebra can be confirmed by the same method. Of course $\mathfrak{A}$ is now a subalgebra of $\widehat{\mathfrak{A}}$.

Because $\nu : \mathfrak{A} \to [0,\infty[$ is uniformly continuous, it has a unique continuous extension $\hat\nu : \widehat{\mathfrak{A}} \to [0,\infty[$. We have

$$\hat\nu 0 = 0, \quad \hat\nu a \le \hat\nu(a \cup b) \le \hat\nu a + \hat\nu b, \quad \hat\nu a = \hat\rho(a,0)$$

for every a, $b \in \mathfrak{A}$ and therefore for every a, $b \in \widehat{\mathfrak{A}}$, so $\hat\nu$ is a submeasure on $\widehat{\mathfrak{A}}$, and

$$\hat\nu a = 0 \Longrightarrow \hat\rho(a,0) = 0 \Longrightarrow a = 0,$$

so $\hat\nu$ is strictly positive.

(d) We have $\nu(a \cup b) + \nu(a \cap b) = \nu a + \nu b$ for all a, $b \in \mathfrak{A}$; because all the operations are continuous, $\hat\nu(a \cup b) + \hat\nu(a \cap b) = \hat\nu a + \hat\nu b$ for all a, $b \in \widehat{\mathfrak{A}}$. In particular, since $\hat\nu 0 = 0$, $\hat\nu$ is additive. Next, if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\widehat{\mathfrak{A}}$, $\hat\rho(a_m \triangle a_n) = |\hat\nu a_m - \hat\nu a_n|$ for all m, $n \in \mathbb{N}$, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is $\hat\rho$-Cauchy, therefore convergent to some $a \in \widehat{\mathfrak{A}}$. Since

$$a \cap a_n = \lim_{m\to\infty} a_m \cap a_n = a_n$$

for each n, $a \supseteq a_n$ for every n. If $b \in \widehat{\mathfrak{A}}$ is any upper bound for $\{a_n : n \in \mathbb{N}\}$, then

$$b \cap a = \lim_{n\to\infty} b \cap a_n = \lim_{n\to\infty} a_n = a$$

and $b \supseteq a$; thus a is the least upper bound of $\{a_n : n \in \mathbb{N}\}$.

So, first, if $\langle b_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\widehat{\mathfrak{A}}$, and we set $a_n = \sup_{i\le n} b_i$ for each n, $\sup_{n\in\mathbb{N}} a_n$ is defined and must be equal to $\sup_{n\in\mathbb{N}} b_n$; accordingly $\widehat{\mathfrak{A}}$ is Dedekind σ-complete. Next, if $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\widehat{\mathfrak{A}}$, and again we set $a_n = \sup_{i\le n} b_i$ for each n, $a = \sup_{n\in\mathbb{N}} a_n = \sup_{n\in\mathbb{N}} b_n$, we shall have

$$\hat\nu a = \lim_{n\to\infty} \hat\nu a_n = \lim_{n\to\infty} \sum_{i=0}^n \hat\nu b_i = \sum_{n=0}^\infty \hat\nu b_n;$$

which means that $\hat\nu$ is countably additive, and $(\widehat{\mathfrak{A}}, \hat\nu)$ is a measure algebra.

392I Corollary Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative additive functional on $\mathfrak{A}$. Then there are a totally finite measure algebra $(\mathfrak{C}, \bar\mu)$ and a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\nu a = \bar\mu(\pi a)$ for every $a \in \mathfrak{A}$.

proof Set $I = \{a : \nu a = 0\}$; then $I \triangleleft \mathfrak{A}$, so we can form the quotient algebra $\mathfrak{B} = \mathfrak{A}/I$ (312L); let $\pi : \mathfrak{A} \to \mathfrak{B}$ be the canonical map. As in part (b) of the proof of 321H, we have an additive functional $\mu : \mathfrak{B} \to [0,\infty[$ such that $\mu(\pi a) = \nu a$ for every $a \in \mathfrak{A}$, and (as in 321H) μ is strictly positive. Take $(\mathfrak{C}, \bar\mu)$ to be $(\widehat{\mathfrak{B}}, \hat\mu)$ as in 392Hd, so that $(\mathfrak{C}, \bar\mu)$ is a totally finite measure algebra. If we now think of π as a map from $\mathfrak{A}$ to $\mathfrak{C}$, it will still be a Boolean homomorphism, and

$$\nu a = \mu(\pi a) = \bar\mu(\pi a)$$

for every $a \in \mathfrak{A}$.

392J Proposition Let $\mathfrak{A}$ be a Boolean algebra, ν an exhaustive submeasure on $\mathfrak{A}$, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$. Then there is an infinite $I \subseteq \mathbb{N}$ such that $\nu(\inf_{i\in I\cap n} a_i) > 0$ for every $n \in \mathbb{N}$.

Remark In the formula $I \cap n$ I am identifying n with the set of its predecessors, as in 3A1H.

proof For finite $J \subseteq \mathbb{N}$ set $b_J = \inf_{i\in J} a_i$. Let $\mathcal{J}$ be the family of those $J \in [\mathbb{N}]^{<\omega}$ such that $\limsup_{n\to\infty} \nu(a_n \cap b_J) > 0$.

? Suppose, if possible, that there is no strictly increasing sequence in $\mathcal{J}$. Then $\mathcal{J}$ must have a maximal element J say. Set $a'_n = a_n \cap b_J$ for $n \in \mathbb{N}$ and $\delta = \limsup_{n\to\infty} \nu a'_n > 0$. For any $n \in \mathbb{N} \setminus J$, $J \cup \{n\} \notin \mathcal{J}$ so

$$\lim_{m\to\infty} a'_m \cap a'_n = \lim_{m\to\infty} a_m \cap b_{J\cup\{n\}} = 0.$$

We can therefore choose inductively a sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ such that

$$k_n > \sup J,\quad \nu a'_{k_n} \ge \frac{3}{4}\delta,\quad \nu(a'_{k_n}\cap a'_{k_i}) \le 2^{-i-2}\delta \text{ for every } i<n$$

for each $n\in\mathbb{N}$. Now set $b_n = a_{k_n}\setminus\sup_{i<n}a_{k_i}$ for each n. Then $\langle b_n\rangle_{n\in\mathbb{N}}$ is disjoint. Also

$$\frac{3}{4}\delta \le \nu a_{k_n} \le \nu b_n + \textstyle\sum_{i=0}^{n-1}\nu(a_{k_n}\cap a_{k_i}) \le \nu b_n + \sum_{i=0}^{n-1}2^{-i-2}\delta \le \nu b_n + \frac{1}{2}\delta$$

and $\nu b_n \ge \frac{1}{4}\delta$ for every n; which is impossible. **X**

There must therefore be a strictly increasing sequence $\langle J_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{J}$. Set $I=\bigcup_{n\in\mathbb{N}}J_n$. If $n\in\mathbb{N}$, there is an $m\in\mathbb{N}$ such that $I\cap n\subseteq J_m$ and $\nu(\inf_{i\in I\cap n}a_i)\ge \nu b_{J_m}>0$. So we have an appropriate I.

***392K Products of submeasures** There seems to be no fully satisfying general construction for products of submeasures. However the following method has some interesting features.

(a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with submeasures μ, ν respectively. On the free product $\mathfrak{A}\otimes\mathfrak{B}$ (§315), we have a functional $\mu\ltimes\nu$ defined by saying that whenever $c\in\mathfrak{A}\otimes\mathfrak{B}$ is of the form $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$, then

$$\begin{aligned}(\mu\ltimes\nu)(c) &= \min_{J\subseteq I}\max(\{\mu(\sup_{i\in J}a_i)\}\cup\{\nu b_i : i\in I\setminus J\})\\ &= \min\{\epsilon:\epsilon\in[0,\infty],\ \mu(\sup\{a_i : i\in I,\ \nu b_i>\epsilon\})\le\epsilon\}.\end{aligned}$$

P Every $c\in\mathfrak{A}\otimes\mathfrak{B}$ can be expressed in this form (315Oa). Of course this can be done in many different ways. But if $c=\sup_{j\in J}a'_j\otimes b'_j$ is another expression of the same kind, then $b_i=b'_j$ whenever $a_i\cap a'_j\ne 0$. So

$$\begin{aligned}\sup\{a_i : i\in I,\ \nu b_i>\epsilon\} &= \sup\{a_i\cap a'_j : i\in I,\ j\in J,\ a_i\cap a'_j\ne 0,\ \nu b_i>\epsilon\}\\ &= \sup\{a_i\cap a'_j : i\in I,\ j\in J,\ a_i\cap a'_j\ne 0,\ \nu b'_j>\epsilon\}\\ &= \sup\{a'_j : j\in J,\ \nu b'_j>\epsilon\}\end{aligned}$$

for any ϵ, and the two calculations for $\mu\ltimes\nu$ give the same result. **Q**

Note that $(\mu\ltimes\nu)(a\otimes b)=\min(\mu a,\nu b)$ for all $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$.

(b) In the context of (a), $\mu\ltimes\nu$ is a submeasure.

P By definition, $(\mu\ltimes\nu)c\ge 0$ for every $c\in\mathfrak{A}\otimes\mathfrak{B}$; and if $c=0$ then $c=1\otimes 0$ and $(\mu\ltimes\nu)c=0$.

If c, c' are two members of $\mathfrak{A}\otimes\mathfrak{B}$, express them in the forms $c=\sup_{i\in I}a_i\otimes b_i$ and $c'=\sup_{j\in J}a'_j\otimes b'_j$ where $\langle a_i\rangle_{i\in I}$ and $\langle a'_j\rangle_{j\in J}$ are partitions of unity in $\mathfrak{A}$. Set $K=\{(i,j): a_i\cap a'_j\ne 0\}\subseteq I\times J$, $a''_{ij}=a_i\cap a'_j$ for $(i,j)\in K$; then $\langle a''_{ij}\rangle_{(i,j)\in K}$ is a partition of unity in $\mathfrak{A}$, $c=\sup_{(i,j)\in K}a''_{ij}\otimes b_i$ and $c'=\sup_{(i,j)\in K}a''_{ij}\otimes b'_j$. Set $\alpha=(\mu\ltimes\nu)c$, $\beta=(\mu\ltimes\nu)c'$, $L=\{(i,j):(i,j)\in K,\ \nu b_i>\alpha\}$, $L'=\{(i,j):(i,j)\in K,\ \nu b'_j>\beta\}$, $e=\sup\{a_{ij}:(i,j)\in L\}$ and $e'=\sup\{a_{ij}:(i,j)\in L'\}$; then $\mu e\le\alpha$ and $\mu e'\le\beta$. So $\mu(e\cup e')\le\alpha+\beta$; but $e\cup e'=\sup_{(i,j)\in L\cup L'}a''_{ij}$ and

$$\nu(b_i\cup b'_j)\le\nu b_i+\nu b'_j\le\alpha+\beta$$

for all $(i,j)\in K\setminus(L\cup L')$. So $(\mu\ltimes\nu)(c\cup c')\le\alpha+\beta$.

If $c\subseteq c'$, then $b_i\subseteq b'_j$ for every $(i,j)\in K$. So $\nu b_i\le\beta$ for every $(i,j)\in K\setminus L'$ and $(\mu\ltimes\nu)c\le\beta$.

Thus $\mu\ltimes\nu$ is subadditive and order-preserving and is a submeasure. **Q**

(c) I note that only in exceptional cases will $\mu\ltimes\nu$ be matched with $\nu\ltimes\mu$ by the canonical isomorphism between $\mathfrak{A}\otimes\mathfrak{B}$ and $\mathfrak{B}\otimes\mathfrak{A}$; this product is not 'commutative'. (See 392Yc.) It is however 'associative', in the following sense. Let $(\mathfrak{A}_1,\mu_1)$, $(\mathfrak{A}_2,\mu_2)$, $(\mathfrak{A}_3,\mu_3)$ be Boolean algebras endowed with submeasures. Set

$$\lambda_{12}=\mu_1\ltimes\mu_2,\quad \lambda_{(12)3}=\lambda_{12}\ltimes\mu_3,\quad \lambda_{23}=\mu_2\ltimes\mu_3,\quad \lambda_{1(23)}=\mu_1\ltimes\lambda_{23}.$$

Then the canonical isomorphisms between $(\mathfrak{A}_1\otimes\mathfrak{A}_2)\otimes\mathfrak{A}_3$, $\mathfrak{A}_1\otimes\mathfrak{A}_2\otimes\mathfrak{A}_3$ and $\mathfrak{A}_1\otimes(\mathfrak{A}_2\otimes\mathfrak{A}_3)$ (315L) identify $\lambda_{(12)3}$ with $\lambda_{1(23)}$.

P Take any $d\in\mathfrak{A}_1\otimes\mathfrak{A}_2\otimes\mathfrak{A}_3$. Express d as $\sup_{i\in I}a_i\otimes e_i$ where $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}_1$ and $e_i\in\mathfrak{A}_2\otimes\mathfrak{A}_3$ for each i; express each e_i as $\sup_{j\in J_i}b_{ij}\otimes c_{ij}$ where $\langle b_{ij}\rangle_{j\in J_i}$ is a partition of unity in

$\mathfrak{A}_2$ and $c_{ij} \in \mathfrak{A}_3$ for $i \in I$, $j \in J_i$. In this case, $\langle a_i \otimes b_{ij}\rangle_{i\in I,j\in J_i}$ is a partition of unity in $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ and $d = \sup_{i\in I,j\in J_i} a_i \otimes b_{ij} \otimes c_{ij}$.

Let $\epsilon > 0$. For $i \in I$, set $J'_i = \{j : j \in J_i,\ \mu_3 c_{ij} > \epsilon\}$, $e'_i = \sup_{j\in J'_i} b_{ij}$. Then $\lambda_{23}(\sup_{j\in J_i} b_{ij} \otimes c_{ij}) \le \epsilon$ iff $\mu_2 e'_i \le \epsilon$. Set $I' = \{i : \mu_2 e'_i > \epsilon\}$; then $\lambda_{1(23)}d \le \epsilon$ iff $\mu_1(\sup_{i\in I'} a_i) \le \epsilon$. From the other direction, set $f = \sup\{a_i \otimes b_{ij} : i \in I,\ j \in J'_i\}$; then $\lambda_{(12)3}d \le \epsilon$ iff $\lambda_{12}f \le \epsilon$. But $f = \sup_{i\in I} a_i \otimes e'_i$, so $\lambda_{12}f \le \epsilon$ iff $\mu_1(\sup_{i\in I'} a_i) \le \epsilon$.

As ϵ and d are arbitrary, $\lambda_{(12)3} = \lambda_{1(23)}$, as claimed. **Q**

(d) If μ, μ' are submeasures on $\mathfrak{A}$, ν and ν' are submeasures on $\mathfrak{B}$, μ is absolutely continuous with respect to μ' and ν is absolutely continuous with respect to ν', then $\mu \ltimes \nu$ is absolutely continuous with respect to $\mu' \otimes \nu'$. **P** For any $\epsilon > 0$ there is a $\delta > 0$ such that $\mu a \le \epsilon$ whenever $\mu' a \le \delta$ and $\nu b \le \epsilon$ whenever $\nu' b \le \delta$. If now $c \in \mathfrak{A} \otimes \mathfrak{B}$ and $(\mu' \ltimes \nu')(c) \le \delta$, we have an expression $c = \sup_{i\in I} a_i \otimes b_i$ and a set $J \subseteq I$ such that $\langle a_i\rangle_{i\in I}$ is a partition of unity, $\mu'(\sup_{i\in J} a_i) \le \delta$ and $\nu' b_i \le \delta$ for every $i \in I \setminus J$; so $\mu(\sup_{i\in J} a_i) \le \epsilon$, $\nu b_i \le \epsilon$ for every $i \in I \setminus J$ and $(\mu \ltimes \nu)(c) \le \epsilon$. **Q**

(e) If μ and ν are exhaustive, so is $\mu \ltimes \nu$. **P** Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A} \otimes \mathfrak{B}$ such that $(\mu \ltimes \nu)c_n > \epsilon > 0$ for every n. For each n, express c_n as $\sup_{i\in I_n} a_{ni} \otimes b_{ni}$ where $\langle a_{ni}\rangle_{i\in I_n}$ is a partition of unity; set $I'_n = \{i : i \in I_n,\ \nu b_{ni} > \epsilon\}$, $a_n = \sup_{i\in I'_n} a_{ni}$; then $\mu a_n > \epsilon$. By 392J, there is an infinite $J \subseteq \mathbb{N}$ such that $\inf_{i\in J\cap n} a_i \ne 0$ for every $n \in \mathbb{N}$. Let Z be the Stone space of $\mathfrak{A}$, and write $\widehat{a} \subseteq Z$ for the open-and-closed set corresponding to $a \in \mathfrak{A}$; then there is a $z \in \bigcap_{n\in J} \widehat{a}_n$. For every $n \in J$ there is an $i_n \in I'_n$ such that $z \in \widehat{a}_{n,i_n}$. But now observe that $\nu b_{n,i_n} > \epsilon$ for every $n \in J$, so there must be distinct m, $n \in J$ such that $b_{m,i_m} \cap b_{n,i_n} \ne 0$; as $a_{m,i_m} \cap a_{n,i_n}$ is also non-zero, $c_m \cap c_n \ne 0$. As $\langle c_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mu \ltimes \nu$ is exhaustive. **Q**

(f) We can extend the construction to infinite products, as follows. Let I be a totally ordered set and $\langle(\mathfrak{A}_i, \mu_i)\rangle_{i\in I}$ a family of Boolean algebras endowed with unital submeasures. For a finite set $J = \{i_0, \dots, i_n\}$ where $i_0 < \ldots < i_n$ in I, let λ_J be the product submeasure $(.(\mu_{i_0} \ltimes \mu_{i_1}) \ltimes \ldots) \ltimes \mu_{i_n}$ on $\mathfrak{C}_J = \bigotimes_{j\in J} \mathfrak{A}_j$; for definiteness, on $\mathfrak{C}_\emptyset = \{0, 1\}$ take $\lambda_\emptyset$ to be the unital submeasure, while $\mathfrak{C}_{\{i\}} = \mathfrak{A}_i$ and $\lambda_{\{i\}} = \mu_i$ for each $i \in I$. Using (c) repeatedly, we see that if J, $K \in [I]^{<\omega}$ and $j < k$ for every $j \in J$, $k \in K$, then the identification of $\mathfrak{C}_{J\cup K}$ with $\mathfrak{C}_J \otimes \mathfrak{C}_K$ (315L) matches $\lambda_{J\cup K}$ with $\lambda_J \ltimes \lambda_K$. Moreover, if $K \in [I]^{<\omega}$ and J is any subset of K (not necessarily an initial segment) and $\varepsilon_{JK} : \mathfrak{C}_J \to \mathfrak{C}_K$ is the canonical embedding corresponding to the identification of $\mathfrak{C}_K$ with $\mathfrak{C}_J \otimes \mathfrak{C}_{K\setminus J}$, then $\lambda_J = \lambda_K \varepsilon_{JK}$; this also is an easy induction on $\#(K)$. What this means is that for any subset M of I we have a submeasure λ_M on $\mathfrak{C}_M = \bigcup\{\varepsilon_{JM}\mathfrak{C}_J : J \in [M]^{<\omega}\}$, being the unique functional such that $\lambda_M \varepsilon_{JM} = \lambda_J$ for every $J \in [M]^{<\omega}$. Finally, if L, M are subsets of I with $l < m$ for every $l \in L$ and $m \in M$, then $\lambda_{L\cup M}$ can be identified with $\lambda_L \ltimes \lambda_M$.

(g) I should perhaps have remarked already that if μ and ν, in (a), are additive and unital, then we have an additive function λ' on $\mathfrak{A} \otimes \mathfrak{B}$ such that $\lambda'(a \otimes b) = \mu a \cdot \nu b$ for every $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ (326E). Now, setting $\lambda = \mu \ltimes \nu$, each of λ, λ' is absolutely continuous with respect to the other. **P** If $c \in \mathfrak{A} \otimes \mathfrak{B}$, express c as $\sup_{i\in I} a_i \otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity. Then $\mu(\sup\{a_i : \nu b_i > \lambda c\}) \le \lambda c$, so $\lambda' c = \sum_{i\in I} \mu a_i \cdot \nu b_i$ is at most $2\lambda c$. On the other hand, $\mu(\sup\{a_i : \nu b_i > \sqrt{\lambda' c}\}) \le \sqrt{\lambda' c}$, so $\lambda c \le \sqrt{\lambda' c}$. **Q**

392X Basic exercises (a) Show that the first two clauses of the definition 392A can be replaced by '$\nu a \le \nu(a \cup b) \le \nu a + \nu b$ whenever $a \cap b = 0$'.

(b) Let $\mathfrak{A}$ be any Boolean algebra and ν a finite-valued submeasure on $\mathfrak{A}$. (i) Show that ν is order-continuous iff whenever $A \subseteq \mathfrak{A}$ is non-empty, downwards-directed and has infimum 0, then $\inf_{a\in A} \nu a = 0$. (ii) Show that in this case ν is exhaustive. (*Hint*: if $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint, then $\bigcup_{n\in\mathbb{N}}\{b : b \supseteq a_i$ for every $i \ge n\}$ has infimum 0.)

(c) Let $\mathfrak{A}$ be a Boolean algebra and μ, ν two strictly positive submeasures on $\mathfrak{A}$, each of which is absolutely continuous with respect to the other. Show that they induce uniformly equivalent metrics on $\mathfrak{A}$ (392H), so that both give the same metric completion of $\mathfrak{A}$.

(d) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras with uniformly exhaustive submeasures μ, ν respectively. Show that $\mu \ltimes \nu$ is uniformly exhaustive.

392Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and $\lambda : \mathfrak{A} \to [0,1]$ a functional such that $\lambda 0 = 0$ and $\lambda a \le \lambda(a \cup b) \le 2\max(\lambda a, \lambda b)$ for all a, $b \in \mathfrak{A}$. Show that there is a submeasure ν on $\mathfrak{A}$ such that $\frac{1}{2}\lambda \le \nu \le \lambda$.

(b) (T.Jech) Show that a Boolean algebra $\mathfrak{A}$ is chargeable iff there are sequences $\langle A_n\rangle_{n\in\mathbb{N}}$ and $\langle k_n\rangle_{n\in\mathbb{N}}$ such that (α) $\bigcup_{n\in\mathbb{N}} A_n = \mathfrak{A} \setminus \{0\}$ (β) whenever a, $b \in \mathfrak{A}$, $n \in \mathbb{N}$ and $a \cup b \in A_n$ then at least one of a, b belongs to A_{n+1} (γ) if $n \in \mathbb{N}$ then $k_n \in \mathbb{N}$ and if $a_0, \dots, a_{k_n} \in \mathfrak{A}$ are disjoint then some a_j does not belong to A_n.

(c) I will say that a submeasure ν on a Boolean algebra $\mathfrak{A}$ is **properly atomless** if for every $\epsilon > 0$ there is a finite partition A of unity in $\mathfrak{A}$ such that $\nu a \le \epsilon$ for every $a \in A$. (Compare 326F.) (i) Show that if $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras with submeasures μ, ν respectively, we have a functional $\mu \rtimes \nu : \mathfrak{A} \otimes \mathfrak{B} \to [0,\infty]$ defined by saying that

$$(\mu \rtimes \nu)(\sup_{i\in I} a_i \otimes b_i) = \min_{J\subseteq I} \max(\{\nu(\sup_{i\in J} b_i)\} \cup \{\mu a_i : i \in I \setminus J\})$$

whenever $\langle b_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{B}$ and $a_i \in \mathfrak{A}$ for each $i \in I$. (ii) Show that if μ is a non-zero properly atomless submeasure, ν is a submeasure, and $\mu \ltimes \nu$ is absolutely continuous with respect to $\mu \rtimes \nu$, then ν is uniformly exhaustive.

(d) (See 328H.) Let $(I, \le)$ be a non-empty upwards-directed partially ordered set, and $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ a family of probability algebras; suppose that $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$ is a measure-preserving Boolean homomorphism whenever $i \le j$, and that $\pi_{ki} = \pi_{kj}\pi_{ji}$ whenever $i \le j \le k$. (i) Let $\mathcal{F}$ be the filter

$$\{A : A \subseteq I, \text{ there is some } i \in I \text{ such that } j \in A \text{ whenever } i \le j\},$$

and set $\nu\langle a_i\rangle_{i\in I} = \limsup_{i\to\mathcal{F}} \bar\mu_i a_i$ for $\langle a_i\rangle_{i\in I} \in \prod_{i\in I} \mathfrak{A}_i$. Show that ν is a submeasure on $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i$. (ii) Let $\mathcal{J}$ be the ideal $\{d : \nu d = 0\}$ of $\mathfrak{A}$, and $\mathfrak{D}$ the quotient algebra $\mathfrak{A}/\mathcal{J}$. Show that we have a strictly positive unital submeasure $\bar\nu$ on $\mathfrak{D}$ such that $\bar\nu d^\bullet = \nu d$ for every $d \in \mathfrak{A}$, and that $\mathfrak{D}$ is complete under the metric defined by $\bar\nu$. (iii) Show that for each $i \in I$ we have a Boolean homomorphism $\pi_i : \mathfrak{A}_i \to \mathfrak{D}$ defined by setting $\pi_i a = \langle a_j\rangle^\bullet_{j\in I}$, where $a_j = \pi_{ji} a$ if $j \ge i$, $0_{\mathfrak{A}_j}$ otherwise, and that $\bar\nu\pi_i = \bar\mu_i$. Show that $\pi_i = \pi_j\phi_{ji}$ whenever $i \le j$. (iv) Show that $\mathfrak{D}_0 = \bigcup_{i\in I} \pi_i[\mathfrak{A}_i]$ is a subalgebra of $\mathfrak{D}$, and that $\bar\nu\restriction\mathfrak{D}_0$ is additive. (v) Let $\mathfrak{C}$ be the closure of $\mathfrak{D}_0$ in $\mathfrak{D}$, and set $\bar\lambda = \bar\nu\restriction\mathfrak{C}$. Show that $(\mathfrak{C}, \bar\lambda)$ is a probability algebra. (vi) Now suppose that $(\mathfrak{B}, \bar\nu)$ is a probability algebra, and that for each $i \in I$ we are given a measure-preserving Boolean homomorphism $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ such that $\phi_i = \phi_j\pi_{ji}$ whenever $i \le j$. Show that there is a unique measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\pi_i = \phi_i$ for every $i \in I$.

(e) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty]$ a submeasure. Show that the following are equiveridical: (i) ν is uniformly exhaustive; (ii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$, there is a set $I \subseteq \mathbb{N}$, not of zero asymptotic density, such that $a_i \cap a_j \ne 0$ for all i, $j \in I$; (iii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$, there is a set $I \subseteq \mathbb{N}$, not of zero asymptotic density, such that $\nu(\inf_{i\in I, i\le n} a_i) > 0$ for every $n \in \mathbb{N}$.

392 Notes and comments Much of the first part of this section is a matter of generalizing earlier arguments. Thus 392C ought by now to be very easy, while 392Xb recalls the elementary theory of τ-additive functionals.

The new ideas are in the combinatorics of 392D-392E. I have cast 392D in the form of an argument in probability theory. Of course there is nothing here but simple counting, since the probability measure simply puts the same mass on each point of Ω, and every statement of the form 'Pr$(R \dots) \le \dots$' is just a matter of counting the elements R of Ω with the given property. But I think many of us find that the probabilistic language makes the calculations more natural; in particular, we can use intuitions associated with the notion of independence of events. Indeed I strongly recommend the method. It has been used to very great effect in the last sixty years in a wide variety of combinatorial problems. 392F and 392G together constitute the **Kalton-Roberts theorem** (Kalton & Roberts 83).

Version of 11.5.08

393 Maharam submeasures

Continuing our exploration of variations on the idea of 'measurable algebra', we come to sequentially order-continuous submeasures. These are associated with 'Maharam algebras' (393E), which share a great many properties with measurable algebras; for instance, the existence of a standard topology defined by the algebraic structure (393G). This topology is intimately connected with the order*-convergence of sequences introduced in §367 (393L). We can indeed characterize Maharam algebras in terms of properties of the order-sequential topology defined by this convergence (393Q).

393A Definition Let $\mathfrak{A}$ be a Boolean algebra. A **Maharam submeasure** or **continuous outer measure** on $\mathfrak{A}$ is a totally finite submeasure $\nu : \mathfrak{A} \to [0, \infty[$ such that $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

393B Lemma Let $\mathfrak{A}$ be a Boolean algebra and ν a Maharam submeasure on $\mathfrak{A}$.

(a) ν is sequentially order-continuous.

(b) ν is 'countably subadditive', that is, whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $a \in \mathfrak{A}$ is such that $a = \sup_{n\in\mathbb{N}} a \cap a_n$, then $\nu a \le \sum_{n=0}^{\infty} \nu a_n$.

(c) If $\mathfrak{A}$ is Dedekind σ-complete, then ν is exhaustive.

proof (a) (Of course ν is an order-preserving function, by the definition of 'submeasure'; so we can apply the ordinary definition of 'sequentially order-continuous' in 313Hb.) (i) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a, then $\langle a_n \setminus a \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, so $\lim_{n\to\infty} \nu(a_n \setminus a) = 0$; but as

$$\nu a_n \le \nu a \le \nu a_n + \nu(a \setminus a_n)$$

for every n, it follows that $\nu a = \lim_{n\to\infty} \nu a_n$. (ii) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum a, then

$$\nu a \le \nu a_n \le \nu a + \nu(a_n \setminus a) \to \nu a$$

as $n \to \infty$.

(b) Set $b_n = \sup_{i\le n} a \cap a_i$; then $\nu b_n \le \sum_{i=0}^{n} \nu a_i$ for each n (inducing on n), so that

$$\nu a = \lim_{n\to\infty} \nu b_n \le \sum_{i=0}^{\infty} \nu a_i.$$

(c) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, set $b_n = \sup_{i\ge n} a_i$ for each n; then $\inf_{n\in\mathbb{N}} b_n = 0$, so

$$\limsup_{n\to\infty} \nu a_n \le \lim_{n\to\infty} \nu b_n = 0.$$

393C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν a strictly positive Maharam submeasure on $\mathfrak{A}$. Then $\mathfrak{A}$ is ccc, Dedekind complete and weakly (σ, ∞)-distributive, and ν is order-continuous.

proof By 393Bc, ν is exhaustive; by 392Ca, $\mathfrak{A}$ is ccc; by 316Fa, $\mathfrak{A}$ is Dedekind complete; by 316Fc and 393Ba, ν is order-continuous

Now suppose that we have a sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Let B be the set

$$\{b : b \in \mathfrak{A},\ \forall\ n \in \mathbb{N}\ \exists\ a \in A_n \text{ such that } a \subseteq b\}.$$

As ν is order-continuous, $\inf_{a\in A_n} \nu a = 0$ for each n. Given $\epsilon > 0$, we can choose $\langle a_n \rangle_{n\in\mathbb{N}}$ such that $a_n \in A_n$ and $\nu a_n \le 2^{-n}\epsilon$ for each n; now $b = \sup_{n\in\mathbb{N}} a_n$ belongs to B and $\nu b \le \sum_{n=0}^{\infty} \nu a_n \le 2\epsilon$. Thus $\inf_{b\in B} \nu b = 0$. Since ν is strictly positive, $\inf B = 0$. As $\langle A_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ, ∞)-distributive.

393D Theorem Let $\mathfrak{A}$ be a Boolean algebra. Then it is measurable iff it is Dedekind σ-complete and carries a uniformly exhaustive strictly positive Maharam submeasure.

proof If $\mathfrak{A}$ is measurable, it surely satisfies the conditions, since any totally finite measure on $\mathfrak{A}$ is also a uniformly exhaustive strictly positive Maharam submeasure. If $\mathfrak{A}$ satisfies the conditions, then it is weakly (σ,∞)-distributive, by 393C, so 392G gives the result.

393E Maharam algebras (a) Definition A **Maharam algebra** is a Dedekind σ-complete Boolean algebra $\mathfrak{A}$ such that there is a strictly positive Maharam submeasure on $\mathfrak{A}$.

(b) Every measurable algebra is a Maharam algebra, while every Maharam algebra is ccc and weakly (σ,∞)-distributive (393C), therefore Dedekind complete. A Maharam algebra $\mathfrak{A}$ is measurable iff there is a strictly positive uniformly exhaustive submeasure on $\mathfrak{A}$. (Put 393C and 392G together again.)

(c)(i) A principal ideal in a Maharam algebra is a Maharam algebra; an order-closed subalgebra of a Maharam algebra is a Maharam algebra. **P** Let $\mathfrak{A}$ be a Maharam algebra and $\mathfrak{B}$ *either* a principal ideal of $\mathfrak{A}$ *or* an order-closed subalgebra of $\mathfrak{A}$. Because $\mathfrak{A}$ is Dedekind complete, so is $\mathfrak{B}$ (314Ea). Let $\nu:\mathfrak{A}\to[0,\infty[$ be a strictly positive Maharam submeasure. Then $\nu\restriction\mathfrak{B}$ is a strictly positive Maharam submeasure on $\mathfrak{A}$, so $\mathfrak{B}$ is a Maharam algebra. **Q**

(ii) The simple product of a countable family of Maharam algebras is a Maharam algebra. **P** Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a countable family of Maharam algebras and $\mathfrak{A}$ its simple product. Then $\mathfrak{A}$ is Dedekind complete (315De). For each $i\in I$, let ν_i be a strictly positive Maharam submeasure on $\mathfrak{A}_i$. Let $\langle\epsilon_i\rangle_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I}\epsilon_i$ is finite. Set $\nu a=\sum_{i\in I}\min(\epsilon_i,\nu_i a(i))$ for $a\in\mathfrak{A}$; then ν is a strictly positive Maharam submeasure on $\mathfrak{A}$, so $\mathfrak{A}$ is a Maharam algebra. **Q**

393F Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν, ν' two Maharam submeasures on $\mathfrak{A}$ such that $\nu a=0$ whenever $\nu' a=0$. Then ν is absolutely continuous with respect to ν'.

proof (Compare 232Ba.) **?** Otherwise, we can find a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\nu' a_n\le 2^{-n}$ for every n, but $\epsilon=\inf_{n\in\mathbb{N}}\nu a_n>0$. Set $b_n=\sup_{i\ge n}a_i$ for each n, $b=\inf_{n\in\mathbb{N}}b_n$. Then $\nu' b_n\le\sum_{i=n}^{\infty}2^{-i}\le 2^{-n+1}$ for each n (393Bb), so $\nu' b=0$; but $\nu b_n\ge\epsilon$ for each n, so $\nu b\ge\epsilon$ (393Ba), contrary to the hypothesis. **X**

393G Proposition Let $\mathfrak{A}$ be a Maharam algebra, and ν and ν' two strictly positive Maharam submeasures on $\mathfrak{A}$. Then the metrics they induce on $\mathfrak{A}$ are uniformly equivalent, so we have a topology and uniformity on $\mathfrak{A}$ which we may call the **Maharam-algebra topology** and the **Maharam-algebra uniformity**.

proof By 393F, ν and ν' are mutually absolutely continuous; translating this with the formula of 392Ha, we see that the metrics are uniformly equivalent, so induce the same topology and uniformity. As $\mathfrak{A}$ does have a strictly positive Maharam submeasure, we may use it to define the Maharam-algebra topology and uniformity of $\mathfrak{A}$.

393H Proposition Let $\mathfrak{A}$ be a Boolean algebra, and ν an exhaustive strictly positive totally finite submeasure on $\mathfrak{A}$. Let $\widehat{\mathfrak{A}}$ be the metric completion of $\mathfrak{A}$, as described in 392H, and $\hat\nu$ the continuous extension of ν to $\widehat{\mathfrak{A}}$. Then $\hat\nu$ is a Maharam submeasure, so $\widehat{\mathfrak{A}}$ is a Maharam algebra.

proof (Compare 392Hd.)

(a) The point is that any non-increasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\widehat{\mathfrak{A}}$ is a Cauchy sequence for the metric $\hat\rho$. **P** Let $\epsilon>0$. For each $n\in\mathbb{N}$, choose $b_n\in\mathfrak{A}$ such that $\hat\rho(a_n,b_n)\le 2^{-n}\epsilon$, and set $c_n=\inf_{i\le n}b_i$. Then

$$\hat\rho(a_n,c_n)=\hat\rho(\inf_{i\le n}a_i,\inf_{i\le n}b_i)\le\sum_{i=0}^{n}\hat\rho(a_i,b_i)\le 2\epsilon$$

for every n. Choose $\langle n(k)\rangle_{k\in\mathbb{N}}$ inductively so that, for each $k\in\mathbb{N}$, $n(k+1)\ge n(k)$ and

$$\nu(c_{n(k)}\setminus c_{n(k+1)})\ge\sup_{i\ge n(k)}\nu(c_{n(k)}\setminus c_i)-\epsilon.$$

Then $\langle c_{n(k)}\setminus c_{n(k+1)}\rangle_{k\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, so

$$\begin{aligned}\limsup_{k\to\infty}\sup_{i\ge n(k)}\hat\rho(a_{n(k)},a_i) &\le 4\epsilon+\limsup_{k\to\infty}\sup_{i\ge n(k)}\hat\rho(c_{n(k)},c_i)\\ &=4\epsilon+\limsup_{k\to\infty}\sup_{i\ge n(k)}\nu(c_{n(k)}\setminus c_i)\\ &\le 4\epsilon+\limsup_{k\to\infty}\nu(c_{n(k)}\setminus c_{n(k+1)})+\epsilon=5\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\langle a_n\rangle_{n\in\mathbb{N}}$ is Cauchy. **Q**

(b) It follows that $\widehat{\mathfrak{A}}$ is Dedekind σ-complete. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\widehat{\mathfrak{A}}$, $\langle b_n\rangle_{n\in\mathbb{N}}=\langle\inf_{i\le n}a_i\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence with a limit $b\in\widehat{\mathfrak{A}}$. For any $k\in\mathbb{N}$,

$$\hat\nu(b\setminus a_k)=\lim_{n\to\infty}\hat\nu(b_n\setminus a_k)=0,$$

so $b\subseteq a_k$, because $\hat\nu$ is strictly positive. While if $c\in\widehat{\mathfrak{A}}$ is a lower bound for $\{a_n:n\in\mathbb{N}\}$, we have $c\subseteq b_n$ for every n, so

$$\hat\nu(c\setminus b)=\lim_{n\to\infty}\hat\nu(c\setminus b_n)=0$$

and $c\subseteq b$. Thus $b=\inf_{n\in\mathbb{N}}a_n$; as $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\widehat{\mathfrak{A}}$ is Dedekind σ-complete (314Bc). **Q**

(c) We find also that $\hat\nu$ is a Maharam submeasure, because if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\widehat{\mathfrak{A}}$ with infimum 0, it must have a limit a which (as in (b) above) must be its infimum, that is, $a=0$; consequently

$$\lim_{n\to\infty}\hat\nu a_n=\hat\nu a=0.$$

(d) It follows at once that $\hat\nu$ is exhaustive (393Bc), so that $\widehat{\mathfrak{A}}$ is ccc (392Ca) and Dedekind complete (316Fa).

393I Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν an atomless Maharam submeasure on $\mathfrak{A}$. Then for every $\epsilon>0$ there is a finite partition C of unity in $\mathfrak{A}$ such that $\nu c\le\epsilon$ for every $c\in C$.

proof Let $A\subseteq\mathfrak{A}$ be a maximal disjoint set such that $0<\nu a\le\epsilon$ for every $a\in A$. As ν is exhaustive (393Bc), A is countable. Set $c=1\setminus\sup A$. **?** If $\nu c>0$, then (because ν is atomless) we can choose inductively a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ such that $b_0=c$, $b_{n+1}\subseteq b_n$, $\nu b_{n+1}>0$ and $\nu(b_n\setminus b_{n+1})>0$ for every $n\in\mathbb{N}$. But now $\langle b_n\setminus b_{n+1}\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of elements of non-zero submeasure, so one of them has submeasure in $]0,\epsilon]$ and ought to have been added to A. **X**

If A is finite, we can set $C=A\cup\{c\}$ and stop. Otherwise, enumerate A as $\langle a_n\rangle_{n\in\mathbb{N}}$ and set $c_n=\sup_{i\ge n}a_i$ for each n; then $\lim_{n\to\infty}\nu c_n=0$, so there is an n such that $\nu c_n\le\epsilon$, and we can set $C=\{a_i:i<n\}\cup\{c_n\cup c\}$.

393J Lemma (Maharam 1947) Let $\mathfrak{A}$ be a ccc Boolean algebra with a T_1 topology $\mathfrak{T}$ such that (i) $\cup:\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ is continuous at $(0,0)$ (ii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\langle a_n\rangle_{n\in\mathbb{N}}\to 0$ for $\mathfrak{T}$. Then $\mathfrak{A}$ has a strictly positive Maharam submeasure.

proof (a) For any $e\in\mathfrak{A}\setminus\{0\}$, there is a Maharam submeasure ν on $\mathfrak{A}$ such that $\nu e>0$.

P (i) Choose a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0, as follows. Because $\mathfrak{T}$ is T_1, $G_0=\mathfrak{A}\setminus\{0\}$ is a neighbourhood of 0 not containing e. Given G_n, choose a neighbourhood G_{n+1} of 0 such that $G_{n+1}\subseteq G_n$ and $a\cup b\cup c\in G_n$ whenever a, b, $c\in G_{n+1}$. (Take neighbourhoods H, H' of 0 such that $a\cup b\in G_n$ for a, $b\in H$, $b\cup c\in H$ for b, $c\in H'$ and set $G_{n+1}=H\cap H'\cap G_n$.) Define $\nu_0:\mathfrak{A}\to[0,1]$ by setting

$$\begin{aligned}\nu_0 a&=1\text{ if }a\notin G_0,\\ &=2^{-n}\text{ if }a\in G_n\setminus G_{n+1},\\ &=0\text{ if }a\in\bigcap_{n\in\mathbb{N}}G_n.\end{aligned}$$

Then whenever $a_0,\ldots,a_r\in\mathfrak A$, $n\in\mathbb N$ and $\sum_{i=0}^r\nu_0a_i<2^{-n}$, $\sup_{i\le r}a_i\in G_n$. To see this, induce on r. If $r=0$ then we have $\nu_0a_0<2^{-n}$ so $a_0\in G_{n+1}\subseteq G_n$. For the inductive step to $r\ge1$, there must be a $k\le r$ such that $\sum_{i<k}\nu_0a_i<2^{-n-1}$ and $\sum_{k<i\le n}\nu_0a_i<2^{-n-1}$ (allowing $k=0$ or $k=n$, in which case one of the sums will be zero). (If $\sum_{i=0}^r\nu_0a_i<2^{-n-1}$, take $k=n$; otherwise, take k to be the least number such that $\sum_{i=0}^k\nu_0a_i\ge2^{-n-1}$.) By the inductive hypothesis, and because 0 certainly belongs to G_{n+1}, $b=\sup_{i<k}a_i$ and $c=\sup_{k<i\le r}a_i$ both belong to G_{n+1}; but also $\nu_0a_k<2^{-n}$ so $a_k\in G_{n+1}$. Accordingly, by the choice of G_{n+1},

$$\sup_{i\le r}a_i=b\cup a_k\cup c$$

belongs to G_n, and the induction continues.

(ii) Set

$$\nu_1a=\inf\{\textstyle\sum_{i=0}^r\nu_0a_i:a_0,\ldots,a_r\in\mathfrak A,\ a=\sup_{i\le r}a_i\}$$

for every $a\in\mathfrak A$. It is easy to see that $\nu_1(a\cup b)\le\nu_1a+\nu_1b$ for all a, $b\in\mathfrak A$; also $a\in G_n$ whenever $\nu_1a<2^{-n}$, so, in particular, $\nu_1e\ge1$, because $e\notin G_0$.

Set

$$\nu a=\inf\{\nu_1b:a\cap e\subseteq b\subseteq e\}$$

for every $a\in\mathfrak A$. Then of course $0\le\nu a\le\nu b$ whenever $a\subseteq b$, and

$$\nu0\le\nu_10\le\nu_00=0,$$

so $\nu0=0$. If a, $b\in\mathfrak A$ and $\epsilon>0$, there are a', b' such that $a\cap e\subseteq a'\subseteq e$, $b\cap e\subseteq b'\subseteq e$, $\nu_1a'\le\nu a+\epsilon$ and $\nu_1b'\le\nu b+\epsilon$; so that $(a\cup b)\cap e\subseteq a'\cup b'\subseteq e$ and

$$\nu(a\cup b)\le\nu_1(a'\cup b')\le\nu_1a'+\nu_1b'\le\nu a+\nu b+2\epsilon.$$

As ϵ, a and b are arbitrary, ν is a submeasure. Next, if $\langle a_i\rangle_{i\in\mathbb N}$ is any non-increasing sequence in $\mathfrak A$ with infimum 0, $\langle a_i\cap e\rangle_{i\in\mathbb N}$ is another, so converges to 0 for $\mathfrak T$. If $n\in\mathbb N$ there is an m such that $a_i\cap e\in G_n$ for every $i\ge m$, so that

$$\nu a_i\le\nu_1(a_i\cap e)\le\nu_0(a_i\cap e)\le2^{-n}$$

for every $i\ge m$. As n is arbitrary, $\lim_{i\to\infty}\nu a_i=0$; as $\langle a_i\rangle_{i\in\mathbb N}$ is arbitrary, ν is a Maharam submeasure. Finally,

$$\nu e=\nu_1e\ge1,$$

so $\nu e\ne0$. **Q**

(b) Write C for the set of those $c\in\mathfrak A$ such that there is a strictly positive Maharam submeasure on the principal ideal $\mathfrak A_c$. Then C is order-dense in $\mathfrak A$. **P** Take any $e\in\mathfrak A\setminus\{0\}$. By (a), there is a Maharam submeasure ν such that $\nu e>0$. Set $A=\{e\setminus a:\nu a=0\}$. Because ν is a submeasure, A is downwards-directed. **?** If $\inf A=0$ then, because $\mathfrak A$ is ccc, there is a non-increasing sequence $\langle a_n\rangle_{n\in\mathbb N}$ in A with infimum 0; because ν is a Maharam submeasure,

$$\nu e\le\inf_{n\in\mathbb N}\nu a_n+\nu(e\setminus a_n)=\inf_{n\in\mathbb N}\nu a_n=0.\ \textbf{X}$$

Thus A has a non-zero lower bound c, and $\nu\restriction\mathfrak A_c$ is a strictly positive Maharam submeasure, while $c\subseteq e$. **Q**

(c) Because $\mathfrak A$ is ccc, there is a sequence $\langle c_n\rangle_{n\in\mathbb N}$ in C with supremum 1. For each n, let ν_n be a strictly positive Maharam submeasure on $\mathfrak A_{c_n}$; multiplying by a scalar if necessary, we may suppose that $\nu_nc_n\le2^{-n}$. We can therefore define $\nu:\mathfrak A\to[0,2]$ by setting $\nu a=\sum_{n=0}^\infty\nu_n(a\cap c_n)$ for every $a\in\mathfrak A$, and it is easy to check that ν is a strictly positive Maharam submeasure on $\mathfrak A$.

***393K Theorem** Let $\mathfrak A$ be a ccc Dedekind complete Boolean algebra. Then $\mathfrak A$ is a Maharam algebra iff there is a Hausdorff linear space topology $\mathfrak T$ on $L^0(\mathfrak A)$ such that for every neighbourhood G of 0 there is a neighbourhood H of 0 such that $u\in G$ whenever $v\in H$ and $|u|\le|v|$.

proof (a) Suppose that $\mathfrak A$ is a Maharam algebra; let ν be a strictly positive Maharam submeasure on $\mathfrak A$.

(i) For $u\in L^0=L^0(\mathfrak A)$ set

$$\tau(u)=\inf\{\alpha:\alpha\ge 0,\,\nu[\![|u|>\alpha]\!]\le\alpha\}.$$

Then τ is an F-seminorm (definition: 2A5B[1]). **P** (i) It will save a moment if we observe that whenever $\beta>\tau(u)$ there is an $\alpha\le\beta$ such that $\nu[\![|u|>\alpha]\!]\le\alpha$, so that

$$\nu[\![|u|>\beta]\!]\le\nu[\![|u|>\alpha]\!]\le\alpha\le\beta.$$

Also, because ν is sequentially order-continuous,

$$\nu[\![|u|>\tau(u)]\!]=\lim_{n\to\infty}\nu[\![|u|>\tau(u)+2^{-n}]\!]\le\lim_{n\to\infty}\tau(u)+2^{-n}=\tau(u).$$

(ii) So

$$\begin{aligned}\nu[\![|u+v|>\tau(u)+\tau(v)]\!]&\le\nu[\![|u|+|v|>\tau(u)+\tau(v)]\!])\\&\le\nu([\![|u|>\tau(u)]\!]\cup[\![|v|>\tau(v)]\!])\end{aligned}$$

(364Ea)

$$\le\nu[\![|u|>\tau(u)]\!]+\nu[\![|v|>\tau(v)]\!]\le\tau(u)+\tau(v),$$

and $\tau(u+v)\le\tau(u)+\tau(v)$. (iii) If $|\alpha|\le 1$ then

$$\nu[\![|\alpha u|>\tau(u)]\!]\le\nu[\![|u|>\tau(u)]\!]\le\tau(u),$$

and $\tau(\alpha u)\le\tau(u)$. (iv) $\lim_{n\to\infty}\nu[\![|u|>n]\!]=0$ because $\langle[\![|u|>n]\!]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0. So if $\epsilon>0$, there is an $n\ge 1$ such that $\nu[\![|u|>n\epsilon]\!]\le\epsilon$, in which case $\nu[\![|\alpha u|>\epsilon]\!]\le\epsilon$ whenever $|\alpha|\le\frac{1}{n}$, so that $\tau(\alpha u)\le\epsilon$ whenever $|\alpha|\le\frac{1}{n}$. As ϵ is arbitrary, $\lim_{\alpha\to 0}\tau(\alpha u)=0$. Thus all the conditions of 2A5B are satisfied and τ is an F-seminorm. **Q**

(ii) Accordingly we have a pseudometric $(u,v)\mapsto\tau(u-v)$ which defines a linear space topology $\mathfrak{T}$ on L^0 (2A5B). In fact this is a metric, because if $\tau(u-v)=0$ then $\nu[\![|u-v|>0]\!]=0$ and (since ν is strictly positive) $u=v$. So $\mathfrak{T}$ is Hausdorff. Now let G be an open set containing 0. Then there is an $\epsilon>0$ such that $H=\{u:\tau(u)<\epsilon\}$ is included in G. If $v\in H$ and $|u|\le|v|$, then

$$\nu[\![|u|>\tau(v)]\!]\le\nu[\![|v|>\tau(v)]\!]\le\tau(v),$$

so $\tau(u)\le\tau(v)$ and $u\in H\subseteq G$. So $\mathfrak{T}$ satisfies all the conditions.

(b) Given such a topology $\mathfrak{T}$ on L^0, let $\mathfrak{S}$ be the topology on $\mathfrak{A}$ induced by $\mathfrak{T}$ and the function $\chi:\mathfrak{A}\to L^0$; that is, $\mathfrak{S}=\{\chi^{-1}[G]:G\in\mathfrak{T}\}$. Then $\mathfrak{S}$ satisfies the conditions of 393J. **P** (i) Because $\mathfrak{T}$ is Hausdorff and χ is injective, $\mathfrak{S}$ is Hausdorff, therefore T_1. (ii) If $0\in G\in\mathfrak{S}$, there is an $H\in\mathfrak{T}$ such that $G=\chi^{-1}[H]$. Now 0 (the zero of L^0) belongs to H, so there is an open set H_1 containing 0 such that $u\in H$ whenever $v\in H_1$ and $|u|\le|v|$. Next, addition on L^0 is continuous for $\mathfrak{T}$, so there is an open set H_2 containing 0 such that $u+v\in H_1$ whenever u, $v\in H_2$. Consider $G'=\chi^{-1}[H_2]$. This is an open set in $\mathfrak{A}$ containing $0_{\mathfrak{A}}$, and if a, $b\in G'$ then

$$|\chi(a\cup b)|\le\chi a+\chi b\in H_2+H_2\subseteq H_1,$$

so $\chi(a\cup b)\in H$ and $a\cup b\in G$. As G is arbitrary, $\cup$ is continuous at $(0,0)$. (iii) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, $u_0=\sup_{n\in\mathbb{N}}n\chi a_n$ is defined in L^0 (use the criterion of 364L(a-i):

$$\inf_{m\in\mathbb{N}}\sup_{n\in\mathbb{N}}[\![n\chi a_n>m]\!]=\inf_{m\in\mathbb{N}}a_{m+1}=0.)$$

If $0\in G\in\mathfrak{S}$, take $H\in\mathfrak{T}$ such that $G=\chi^{-1}[H]$, and $H_1\in\mathfrak{T}$ such that $0\in H_1$ and $u\in H$ whenever $v\in H_1$ and $|u|\le|v|$. Because scalar multiplication is continuous for $\mathfrak{T}$, there is a $k\ge 1$ such that $\frac{1}{k}u_0\in H_1$. For any $n\ge k$, $\chi a_n\le\frac{1}{k}u_0$ so $\chi a_n\in H$ and $a_n\in G$. As G is arbitrary, $\langle a_n\rangle_{n\in\mathbb{N}}\to 0$ for $\mathfrak{S}$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, condition (ii) in the statement of 393J is satisfied. **Q**

So 393J tells us that $\mathfrak{A}$ has a strictly positive Maharam submeasure, and is a Maharam algebra.

393L I now turn to some very remarkable ideas relating the order*-convergence of §367 to the questions here.

[1]Later editions only.

Definition Let P be a lattice, and consider the relation '$\langle p_n\rangle_{n\in\mathbb{N}}$ order*-converges to p' as a relation between $P^{\mathbb{N}}$ and P. By 367Bc, this satisfies the hypothesis of 3A3Pa, so there is a unique topology on P for which a set $F\subseteq P$ is closed iff $a\in F$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in F which order*-converges to a in P. I will call this topology the **order-sequential topology** of P.

Warning! For the next few paragraphs I shall be closely following the papers Balcar Głowczyński & Jech 98 and Balcar Jech & Pazák 05. I should therefore note explicitly that if $\mathfrak{A}$ is a Boolean algebra which is neither Dedekind σ-complete nor ccc, my 'order-sequential topology' on $\mathfrak{A}$ may not be identical to theirs.

393M Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) A sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ order*-converges to $a\in\mathfrak{A}$ iff there is a partition B of unity in $\mathfrak{A}$ such that $\{n:n\in\mathbb{N},\,(a_n\triangle a)\cap b\ne 0\}$ is finite for every $b\in B$.

(b) If $\langle a_n\rangle_{n\in\mathbb{N}}$ order*-converges to a and $c\in\mathfrak{A}$, then $\langle a_n\cup c\rangle_{n\in\mathbb{N}}$, $\langle a_n\cap c\rangle_{n\in\mathbb{N}}$ and $\langle a_n\triangle c\rangle_{n\in\mathbb{N}}$ order*-converge to $a\cup c$, $a\cap c$ and $a\triangle c$ respectively.

(c) The operations $\cap$, $\cup$ and $\triangle$ are separately continuous for the order-sequential topology.

(d) Every disjoint sequence in $\mathfrak{A}$ is order*-convergent to 0.

proof (a) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ and $a\in\mathfrak{A}$; set

$$C=\{c:\exists\,n\in\mathbb{N},\,c\subseteq a_i\text{ for every }i\ge n\},$$

$$D=\{d:\exists\,n\in\mathbb{N},\,a_i\subseteq d\text{ for every }i\ge n\}.$$

(i) If $\langle a_n\rangle_{n\in\mathbb{N}}$ order*-converges to a, then $a=\sup C=\inf D$ (367Be). Since

$$\inf\{d\setminus a:d\in D\}=\inf\{a\setminus c:c\in C\}=0,$$

$$E=\{(d\setminus a)\cup(a\setminus c):c\in C,\,d\in D\}$$

also has infimum 0 (313A, 313B). So there is a partition B of unity such that for every $b\in B$ there is an $e\in E$ such that $b\cap e=0$. Now, given $b\in B$, there are $c\in C$ and $d\in D$ such that $b\cap(d\setminus c)=0$; there are n_1, $n_2\in\mathbb{N}$ such that $c\subseteq a_n$ for $n\ge n_1$ and $a_n\subseteq d$ for $n\ge n_2$; so that $\{n:(a_n\triangle a)\cap b\ne 0\}$ is bounded above by $\max(n_1,n_2)$ and is finite. So B witnesses that the condition is satisfied.

(ii) Now suppose that B is a partition of unity such that $\{n:(a_n\triangle a)\cap b\ne 0\}$ is finite for every $b\in B$. Then $a\cup(1\setminus b)\in D$ for every $b\in B$, because $\{n:a_n\not\subseteq a\cup(1\setminus b)\}\subseteq\{n:(a_n\triangle a)\cap b\ne 0\}$ is finite. So any lower bound for D is also a lower bound for $\{a\cup(1\setminus b):b\in B\}$ and is included in a. Similarly, any upper bound for C includes a; as $c\subseteq d$ whenever $c\in C$ and $d\in D$, $a=\sup C=\inf D$ and $\langle a_n\rangle_{n\in\mathbb{N}}$ order*-converges to a.

(b) These are all immediate from (a), because

$$(a_n\cup c)\triangle(a\cup c)\subseteq a_n\triangle a,\quad (a_n\cap c)\triangle(a\cap c)\subseteq a_n\triangle a,$$

$$(a_n\triangle c)\triangle(a\triangle c)=a_n\triangle a$$

for every n.

(c) By (b), we can apply 3A3Pb to each of the functions $a\mapsto a\cap b=b\cap a$, $a\mapsto a\cup b=b\cup a$ and $a\mapsto a\triangle b=b\triangle a$ to see that these are all continuous for every $b\in\mathfrak{A}$.

(d) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, there is a partition B of unity in $\mathfrak{A}$ containing every a_n (311Gd); now B witnesses that the condition of (a) is satisfied.

393N Proposition Let $\mathfrak{A}$ be a Maharam algebra. Then the Maharam-algebra topology on $\mathfrak{A}$ is the order-sequential topology.

proof Let $\mathfrak{T}_o$ be the order-sequential topology on $\mathfrak{A}$, ν a strictly positive Maharam submeasure on $\mathfrak{A}$, ρ the metric defined from ν (392H) and $\mathfrak{T}_M$ the Maharam-algebra topology induced by ρ (393G).

(a) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\langle a_n\rangle_{n\in\mathbb{N}} \to^* a$ in $\mathfrak{A}$, then $\lim_{n\to\infty}\rho(a_n,a)=0$. **P** By 393Mb, $\langle a_n \triangle a\rangle_{n\in\mathbb{N}} \to^* 0$; by 367Bf, $0=\inf_{n\in\mathbb{N}}\sup_{i\ge n}(a_i\triangle a)$, so

$$\rho(a_n,a)\le\nu(\sup_{i\ge n}a_i\triangle a)\to 0$$

as $n\to\infty$. **Q**

It follows that every $\mathfrak{T}_M$-closed set is $\mathfrak{T}_o$-closed, and $\mathfrak{T}_o\subseteq\mathfrak{T}_M$.

(b) Conversely, suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ converging for $\mathfrak{T}_M$ to $a\in\mathfrak{A}$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ has a subsequence $\langle a'_n\rangle_{n\in\mathbb{N}}$ such that $\rho(a'_n,a)\le 2^{-n}$ for every $n\in\mathbb{N}$. In this case, setting $b_m=\sup_{n\ge m}a'_n\triangle a$ for each m, $\nu b_m\le 2^{-m+1}$ for every m (393Bb), so $\inf_{m\in\mathbb{N}}b_m=0$, and $\langle a'_n\triangle a\rangle_{n\in\mathbb{N}}\to^* 0$, that is, $\langle a'_n\rangle_{n\in\mathbb{N}}\to^* a$.

Thus every $\mathfrak{T}_M$-convergent sequence has an order*-convergent subsequence with the same limit; it follows that every $\mathfrak{T}_o$-closed set is $\mathfrak{T}_M$-closed, that is, $\mathfrak{T}_M\subseteq\mathfrak{T}_o$.

393O Proposition Let $\mathfrak{A}$ be a ccc Dedekind σ-complete Boolean algebra, with its order-sequential topology, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then the topological closure of $\mathfrak{B}$ is the smallest order-closed set including $\mathfrak{B}$; in particular, $\mathfrak{B}$ is order-closed iff it is topologically closed.

proof (a) Let $\overline{\mathfrak{B}}$ be the topological closure of $\mathfrak{B}$, and $\mathfrak{B}^\sim$ the smallest order-closed set including $\mathfrak{B}$.

(i) Suppose that $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\overline{\mathfrak{B}}$ with supremum b in $\mathfrak{A}$; then $\langle b_n\rangle_{n\in\mathbb{N}}\to^* b$, by 367Bf or 367Xa. So $b\in\overline{\mathfrak{B}}$. Similarly, $\inf_{n\in\mathbb{N}}b_n\in\overline{\mathfrak{B}}$ for every non-increasing sequence in $\overline{\mathfrak{B}}$. Thus $\overline{\mathfrak{B}}$ is sequentially order-closed. But this means that it is order-closed, by 316Fb. So $\overline{\mathfrak{B}}\supseteq\mathfrak{B}^\sim$.

(ii) By 313Fc, $\mathfrak{B}^\sim$ is a subalgebra of $\mathfrak{A}$. Now suppose that $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}^\sim$ which order*-converges to $a\in\mathfrak{A}$. Then $c_{mn}=\sup_{m\le i\le n}b_i$ belongs to $\mathfrak{B}^\sim$ whenever $m\le n$; as $\langle c_{mn}\rangle_{n\ge m}$ is non-decreasing, $c_m=\sup_{i\ge m}b_i=\sup_{n\ge m}c_{mn}$ belongs to $\mathfrak{B}^\sim$ for every $m\in\mathbb{N}$; as $\langle c_m\rangle_{m\in\mathbb{N}}$ is non-increasing, $\inf_{m\in\mathbb{N}}c_m\in\mathfrak{B}^\sim$. But $c=b$ (367Bf). As $\langle b_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{B}^\sim$ is closed for the order-sequential topology, and must include $\overline{\mathfrak{B}}$.

Thus $\overline{\mathfrak{B}}=\mathfrak{B}^\sim$, as claimed.

(b) Now

$$\mathfrak{B}\text{ is order-closed}\iff\mathfrak{B}=\mathfrak{B}^\sim\iff\mathfrak{B}=\overline{\mathfrak{B}}\iff\mathfrak{B}\text{ is topologically closed.}$$

393P Lemma Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Boolean algebra, endowed with its order-sequential topology.

(a) If $\langle a_{mn}\rangle_{m,n\in\mathbb{N}}$, $\langle a_m\rangle_{m\in\mathbb{N}}$ and a are such that $\langle a_{mn}\rangle_{n\in\mathbb{N}}$ order*-converges to a_m for each m, while $\langle a_m\rangle_{m\in\mathbb{N}}$ order*-converges to a, then there is a sequence $\langle k(m)\rangle_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $\langle a_{m,k(m)}\rangle_{m\in\mathbb{N}}$ order*-converges to a.

(b) If $A\subseteq\mathfrak{A}$ and $a\in\overline{A}$, there is a sequence in A which order*-converges to a.

(c) If G is a neighbourhood of 0 in $\mathfrak{A}$ then there is an open neighbourhood H of 0, included in G, such that $[0,a]\subseteq H$ for every $a\in H$.

(d) For $A\subseteq\mathfrak{A}$, set $\bigvee_0(A)=\{0\}$ and $\bigvee_{n+1}(A)=\{a\cup b: a\in\bigvee_n(A),\ b\in A\}$ for $n\in\mathbb{N}$.

(i) If $A\subseteq\mathfrak{A}$ is such that $[0,a]\subseteq A$ for every $a\in A$, and $n\in\mathbb{N}$, then $[0,a]\subseteq\bigvee_n(A)$ for every $a\in\bigvee_n(A)$.

(ii) If $H\subseteq\mathfrak{A}$ is an open set containing 0 such that $[0,a]\subseteq H$ for every $a\in H$, then $\bigvee_{n+1}(H)$ is open and $\overline{\bigvee_n(H)}\subseteq\bigvee_{n+1}(H)$ for every $n\in\mathbb{N}$.

(e) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. Then for every open set G containing 0 there is an open set H containing 0 such that $\bigvee_3(H)\subseteq\bigvee_2(G)$.

proof (a) Let C_m, for $m\in\mathbb{N}$, be partitions of unity in $\mathfrak{A}$ such that

$$\{m:(a_m\triangle a)\cap c\ne 0\}\text{ is finite for every }c\in C_0,$$

$$\{n:(a_{mn}\triangle a_m)\cap c\ne 0\}\text{ is finite whenever }m\in\mathbb{N}\text{ and }c\in C_{m+1}$$

(393Ma). Because $\mathfrak{A}$ is weakly (σ,∞)-distributive, there is a partition B of unity such that $\{c: c\in C_m, c\cap b\ne 0\}$ is finite whenever $m\in\mathbb{N}$ and $b\in B$ (316H(ii)). Because $\mathfrak{A}$ is ccc, there is a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$

running over $B\cup\{0\}$. Now, for each m, any sufficiently large $k(m)$ will be such that $(a_{m,k(m)}\triangle a_m)\cap b_i=0$ for every $i\le m$. In this case, for any i,

$$\{m:(a_{m,k(m)}\triangle a)\cap b_i\ne 0\}\subseteq\{m:m<i\}\cup\{m:(a_m\triangle a)\cap b_i\ne 0\}$$

is finite, so B witnesses that $\langle a_{m,k(m)}\rangle_{m\in\mathbb{N}}\to^* a$ (393Ma, in the other direction).

(b) Let $A^\sim$ be the set of order*-limits of sequences in A. Of course $A^\sim$ must be included in $\overline{A}$. But from (a) we see that the limit of any order*-convergent sequence in $A^\sim$ belongs to $A^\sim$. So $A^\sim$ is closed and is equal to $\overline{A}$. Turning this round, we see that $\overline{A}$ is just the set of order*-limits of sequences in A, as claimed.

(c) Set $D=\{d:d\in\mathfrak{A},\ [0,d]\not\subseteq G\}$, $H=\mathfrak{A}\setminus\overline{D}$. Since $D\supseteq\mathfrak{A}\setminus G$, H is an open subset of G.

? If $0\in\overline{D}$, then (b) tells us that there is a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in D order*-converging to 0. Now there is for each $n\in\mathbb{N}$ a $c_n\subseteq d_n$ such that $c_n\notin G$. By 367Be or 393Ma, $\langle c_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, and $0\in\overline{\mathfrak{A}\setminus G}$; but G is supposed to be a neighbourhood of 0. **X** Thus $0\in H$ and H is a neighbourhood of 0.

? If $a\in H$ and $b\in[0,a]\setminus H$, then $b\in\overline{D}$, so there is a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in D order*-converging to b. In this case, $\langle d_n\cup a\rangle_{n\in\mathbb{N}}$ order*-converges to $b\cup a=a$, by 393Mb. But also $[0,d_n\cup a]\supseteq[0,d_n]$ is not included in G, so $d_n\cup a\in D$ for each n, and $a\in\overline{D}$; which is impossible. **X** Thus $[0,a]\subseteq H$ for every $a\in H$, and H has the properties declared.

(d)(i) This is an elementary induction on n.

(ii) The point is that $\bigvee_{n+1}(H)=\{a\triangle b:a\in\bigvee_n(H),\ b\in H\}$. **P** If $a\in\bigvee_n(H)$ and $b\in H$, then $a\setminus b\in\bigvee_n(H)$, by (i), and $b\setminus a\in H$, so $a\triangle b\in\bigvee_{n+1}(H)$. On the other hand, if $c\in\bigvee_{n+1}(H)$, it is expressible as $a\cup b=a\triangle(b\setminus a)$ where $a\in\bigvee_n(H)$ and b and $b\setminus a$ belong to H. **Q**

Since $\triangle$ is separately continuous, it follows at once that

$$\bigvee_{n+1}(H)=\bigcup_{a\in\bigvee_n(H)}\{a\triangle b:b\in H\}=\bigcup_{a\in\bigvee_n(H)}\{b:a\triangle b\in H\}$$

is open, because $\triangle$ is separately continuous (393Mc). Next, if $d\in\overline{\bigvee_n(H)}$, then there is a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\bigvee_n(H)$ order*-converging to d, by (b). Now $\langle d_n\triangle d\rangle_{n\in\mathbb{N}}\to^* 0$, by 393Mb, so $\langle d_n\triangle d\rangle_{n\in\mathbb{N}}$ converges topologically to 0, by 3A3Pa, and there is an $n\in\mathbb{N}$ such that $d_n\triangle d\in H$; in which case $d=d_n\triangle(d_n\triangle d)$ belongs to $\bigvee_{n+1}(H)$. As d is arbitrary, $\overline{\bigvee_n(H)}\subseteq\bigvee_{n+1}(H)$.

(e) ? Suppose, if possible, otherwise.

(i) Choose H_n, a_n, b_n and c_n inductively, as follows. $H_0\subseteq G$ is to be an open neighbourhood of 0 such that $[0,a]\subseteq H_0$ whenever $a\in H_0$ ((c) above). Given that H_n is an open set containing 0 and including $[0,a]$ whenever it contains a, we are supposing that $\bigvee_3(H_n)\not\subseteq\bigvee_2(G)$; choose a_n, b_n, $c_n\in H_n$ such that $a_n\cup b_n\cup c_n\notin\bigvee_2(G)$, and set

$$H_{n+1}=\{a:a,\ a\cup a_n,\ a\cup b_n\text{ and }a\cup c_n\text{ all belong to }H_n\},$$

so that H_{n+1} is an open set containing 0, and $[0,a]\subseteq H_{n+1}$ for every $a\in H_{n+1}$. Continue.

(ii) At the end of the induction, set $F=\bigcap_{n\in\mathbb{N}}\overline{H}_n$ and $a^*=\inf_{n\in\mathbb{N}}\sup_{i\ge n}a_i$. Then $a^*\cup d\in F$ for every $d\in F$. **P** For $m\le n\in\mathbb{N}$, $\sup_{m\le i\le n}a_i\cup d\in H_m$ for every $d\in H_{n+1}$ (induce downwards on m). Because $\cup$ is separately continuous, $\sup_{m\le i\le n}a_i\cup d\in\overline{H}_m$ for every $d\in F$. Letting $n\to\infty$, $d\cup\sup_{i\ge m}a_i\in\overline{H}_m$ whenever $d\in F$ and $m\in\mathbb{N}$. Next, for any $b\in\mathfrak{A}$, $\{a:a\cap b\in\overline{H}_m\}$ is a closed set including H_m, so $a\cap b\in\overline{H}_m$ for every $a\in\overline{H}_m$; that is, $[0,a]\subseteq\overline{H}_m$ for every $a\in\overline{H}_m$. As $a^*\subseteq\sup_{i\ge m}a_i$, $d\cup a^*\in\overline{H}_m$ for every $d\in F$. As m is arbitrary, $d\cup a^*\in F$ for every $d\in F$. **Q**

Similarly, setting $b^*=\inf_{n\in\mathbb{N}}\sup_{i\ge n}b_i$ and $c^*=\inf_{n\in\mathbb{N}}\sup_{i\ge n}c_i$, $d\cup b^*$ and $d\cup c^*$ belong to F for every $d\in F$; and of course $0\in F$. So $e=a^*\cup b^*\cup c^*$ belongs to F. For each $n\in\mathbb{N}$, $a_n\cup b_n\cup c_n\notin\bigvee_2(H_0)$; but $[0,a]\subseteq\bigvee_2(H_0)$ for every $a\in\bigvee_2(H_0)$, by (d-i), so $\sup_{i\ge n}a_i\cup b_i\cup c_i\notin\bigvee_2(H_0)$. Accordingly $e=\inf_{n\in\mathbb{N}}\sup_{i\ge n}a_i\cup b_i\cup c_i$ does not belong to the open set $\bigvee_2(H_0)$, and $e\notin\overline{H}_0$, by (d-ii). So $e\in F\setminus\overline{H}_0$; which is impossible. **X**

393Q Theorem (Balcar Głowczyński & Jech 98, Balcar Jech & Pazák 05) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is a Maharam algebra;

(ii) $\mathfrak{A}$ is ccc and the order-sequential topology is Hausdorff;

(iii) $\mathfrak{A}$ is weakly (σ,∞)-distributive and $\{0\}$ is a G_δ set for the order-sequential topology of $\mathfrak{A}$;

(iv) $\mathfrak{A}$ is ccc and there is a T_1 topology on $\mathfrak{A}$ such that (α) $\cup : \mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ is continuous at $(0,0)$ (β) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with infimum 0, then $\langle a_n\rangle_{n\in\mathbb{N}}\to 0$.

proof (a)(i)$\Rightarrow$(ii) By 393Eb, $\mathfrak{A}$ is ccc. By 393N, the order-sequential topology is metrizable, therefore Hausdorff.

(b)(ii)$\Rightarrow$(iii) Suppose that the conditions of (ii) are satisfied. In the following argument, all topological terms will refer to the order-sequential topology on $\mathfrak{A}$.

(α) $\mathfrak{A}$ is weakly (σ,∞)-distributive. **P** Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of partitions of unity in $\mathfrak{A}$, and set

$$D=\{d:d\in\mathfrak{A},\ \{a:a\in A_n,\ a\cap d\ne 0\}\text{ is finite for every }n\in\mathbb{N}\}.$$

Take any $c\in\mathfrak{A}^+$. Let G, H be disjoint open sets containing 0, c respectively. Choose $\langle c_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $c_0=c$. Given $c_n\in H$, let $\langle a_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence running over A_n, and set $c_{nj}=\sup_{i\le j}c_n\cap a_{ni}$; then $\langle c_{nj}\rangle_{j\in\mathbb{N}}$ order*-converges to c_n (367Bf/367Xa), so there is a j_n such that $c_{nj_n}\in H$; set $c_{n+1}=c_{nj_n}$, and continue.

This gives us a non-increasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in H. Set $d=\inf_{n\in\mathbb{N}}c_n$; then $d\notin G$ so $d\ne 0$, while $d\subseteq\sup_{i\le j_n}a_{ni}$ for each n, so $d\in D$.

As c is arbitrary, D is order-dense in $\mathfrak{A}$ and includes a partition of unity. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive (316H). **Q**

(β) For any $a\in\mathfrak{A}^+$ there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $a\not\subseteq\sup(\bigcap_{n\in\mathbb{N}}H_n)$. **P** For $A\subseteq\mathfrak{A}$ and $n\in\mathbb{N}$, define $\bigvee_n(A)$ as in 393Pd. Let G, G' be disjoint neighbourhoods of 0 and a respectively, and set $G_0=G\cap\{a\mathbin{\triangle}b:b\in G'\}$; then G_0 is a neighbourhood of 0 (393Mc). By 393Pc, we can find a neighbourhood H_0 of 0 such that $H_0\subseteq G_0$ and $[0,b]\subseteq H_0$ for every $b\in H_0$, in which case $[0,b]\subseteq\bigvee_2(H_0)$ for every $b\in\bigvee_2(H_0)$, while $a\notin\bigvee_2(H_0)$. By 393Pe, we can choose neighbourhoods H_n of 0 such that $H_n\subseteq H_{n-1}$ and $\bigvee_3(H_n)\subseteq\bigvee_2(H_{n-1})$ for every $n\ge 1$; by 393Pc, we can suppose that $[0,b]\subseteq H_n$ whenever $b\in H_n$. But this will ensure that $\bigvee_4(H_{n+2})\subseteq\bigvee_2(H_n)$ for every n, so that $\bigvee_{2^k}(H_{2k})\subseteq\bigvee_2(H_2)$ for every $k\ge 1$. Set $F=\bigcap_{n\in\mathbb{N}}H_n$. Then

$$\bigvee_{2^k}(F)\subseteq\bigvee_{2^k}(H_{2k})\subseteq\bigvee_2(H_2)$$

for every $k\ge 1$. Since $\sup F$ is the limit of a sequence in $\bigcup_{k\ge 1}\bigvee_{2^k}(F)$,

$$\sup F\in\overline{\bigvee_2(H_2)}\subseteq\bigvee_3(H_2)\subseteq\bigvee_2(H_0)$$

(using 393P(d-ii) for the first inequality) and cannot include a. **Q**

(γ) Now consider the set D of those $d\in\mathfrak{A}$ such that there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $d\cap\sup(\bigcap_{n\in\mathbb{N}}H_n)=0$. By (β), D is order-dense, so includes a partition of unity A. A is countable, so there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $d\cap\sup(\bigcap_{n\in\mathbb{N}}H_n)=0$ for every $d\in A$; but this means that $\bigcap_{n\in\mathbb{N}}H_n=\{0\}$. So (iii) is true.

(c)(iii)$\Rightarrow$(iv) Now suppose that the conditions in (iii) are satisfied. As in (b), all topological terms will refer to the order-sequential topology on $\mathfrak{A}$.

(α) There is a non-increasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open neighbourhoods of 0 such that $\bigcap_{n\in\mathbb{N}}\overline{G}_n=\{0\}$. **P** Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a sequence of open sets with intersection $\{0\}$. Set $G_0=\mathfrak{A}$, and for $n\in\mathbb{N}$ choose an open neighbourhood G_{n+1} of 0, included in $U_n\cap G_n$, such that $[0,a]\subseteq G_{n+1}$ for every $a\in G_{n+1}$ (393P). **?** If $0\ne d\in\bigcap_{n\in\mathbb{N}}\overline{G}_n$, then for each $n\in\mathbb{N}$ we can find a sequence $\langle a_{ni}\rangle_{i\in\mathbb{N}}$ in G_n order*-converging to d (393Pc). By 393Pa, there is a sequence $\langle k(n)\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $\langle a_{n,k(n)}\rangle_{n\in\mathbb{N}}$ order*-converges to d. Now $d=\sup_{n\in\mathbb{N}}\inf_{i\ge n}a_{i,k(i)}$ (367Bf), so there is an $n\in\mathbb{N}$ such that $c=\inf_{i\ge n}a_{i,k(i)}$ is non-zero. But in this case we must have $c\le a_{i,k(i)}\in G_i$ and $c\in G_i\subseteq U_j$ whenever $i\ge\max(n,j+1)$, so $c=0$. **X** Thus $\bigcap_{n\in\mathbb{N}}\overline{G}_n=\{0\}$, as required. **Q**

(β) For every neighbourhood G of 0 there is a neighbourhood H of 0 such that $a\cup b\in G$ for all a, $b\in H$. **P?** Otherwise, choose $\langle H_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with an open neighbourhood H_0 of 0 such that $H_0\subseteq G$ and $[0,a]\subseteq H_0$ for every $a\in H_0$. Given that H_n is an open neighbourhood of 0, let a_n, $b_n \in H_n$ be such that $a_n \cup b_n \notin G$. Because the maps $a \mapsto a \cup a_n$ and $a \mapsto a \cup b_n$ are continuous, there is an open neighbourhood H_{n+1} of 0 such that $a \cup a_n$ and $a \cup b_n$ belong to H_n for every $a \in H_{n+1}$; and we may suppose that $H_{n+1} \subseteq G_n$. Continue.

An easy induction on k shows that $a \cup \sup_{n \le i \le n+k} a_i$ and $a \cup \sup_{n \le i \le n+k} b_i$ belong to H_n whenever $k \in \mathbb{N}$ and $a \in H_{n+k+1}$. In particular, $\sup_{n \le i \le n+k} a_i \in H_n$ for every k; since $\langle \sup_{n \le i \le n+k} a_i \rangle_{k \in \mathbb{N}}$ is order*-convergent to $\sup_{i \ge n} a_i$, $\sup_{i \ge n} a_i \in \overline{H}_n \subseteq \overline{G}_n$ for every n. Set $a^* = \inf_{n \in \mathbb{N}} \sup_{i \ge n} a_i$. Then $\langle \sup_{i \ge n} a_i \rangle_{n \in \mathbb{N}} \to^* a^*$, and $\sup_{i \ge n} a_i \in \overline{G}_m$ whenever $n \ge m$, so $a^* \in \overline{G}_m$ for every m, and $a^* = 0$.

In the same way, $\inf_{n \in \mathbb{N}} \sup_{i \ge n} b_i = 0$. It follows that $\inf_{n \in \mathbb{N}} c_n = 0$, where $c_n = \sup_{i \ge n} a_i \cup \sup_{i \ge n} b_i$ for each n. But now $\langle c_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence with infimum 0, so order*-converges to 0, and there must be an n such that $c_n \in H_0$. Since $a_n \cup b_n \subseteq c_n$, $a_n \cup b_n \in H_0 \subseteq G$, contrary to the choice of a_n and b_n. **XQ**

(γ) $\mathfrak{A}$ is ccc. **P** Let $\langle U_n \rangle_{n \in \mathbb{N}}$ be a sequence of open sets with intersection $\{0\}$, and $A \subseteq \mathfrak{A} \setminus \{0\}$ a partition of unity. If $\langle a_i \rangle_{i \in \mathbb{N}}$ is a sequence of distinct elements of A, then $\langle a_i \rangle_{i \in \mathbb{N}} \to^* 0$ (393Md); so $A \setminus U_n$ is finite for every n, and A is countable. **Q**

(δ) (β) means just that $\cup$ is continuous at $(0,0)$. Also a non-increasing sequence with infimum 0 order*-converges to 0, so converges topologically to 0 (3A3Pa); and the topology is certainly T_1. So all the conditions of (iv) are satisfied by the order-sequential topology.

(d)(iv)⇒(i) By 393J, there is a strictly positive Maharam submeasure on $\mathfrak{A}$; as $\mathfrak{A}$ is Dedekind σ-complete, it is a Maharam algebra.

393R Definition Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is **σ-finite-cc** if $\mathfrak{A}$ can be expressed as $\bigcup_{n \in \mathbb{N}} A_n$ where no A_n includes any infinite disjoint set.

393S Theorem (Todorčević 04) Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is a Maharam algebra iff it is σ-finite-cc, weakly (σ, ∞)-distributive and Dedekind σ-complete.

proof (B.Balcar)**(a)** If $\mathfrak{A}$ is a Maharam algebra, then of course it is Dedekind σ-complete, and we have known since 393C that it is weakly (σ, ∞)-distributive. Also it carries a strictly positive exhaustive submeasure, so is σ-finite-cc.

Of course $\{0\}$ is a Maharam algebra. For the rest of the proof, therefore, I suppose that $\mathfrak{A}$ is a non-trivial algebra satisfying the conditions, and seek to show that it is a Maharam algebra.

(b)(i) Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of sets, with union $\mathfrak{A}^+$, such that no A_n includes any infinite disjoint set. For each n, set $B_n = \bigcup_{m \le n} \bigcup_{a \in A_m} [a, 1]$, so that B_n includes no infinite disjoint subset. Now there is an n such that 1 is in the interior of B_n for the order-sequential topology. **P?** Otherwise, of course $\mathfrak{A}$ is ccc, so there is for each $n \in \mathbb{N}$ a sequence $\langle b_{ni} \rangle_{i \in \mathbb{N}}$ in $\mathfrak{A} \setminus B_n$ which is order*-convergent to 1 (393Pb). By 393Pa, there is a sequence $\langle k(n) \rangle_{n \in \mathbb{N}}$ in $\mathbb{N}$ such that $\langle b_{n,k(n)} \rangle_{n \in \mathbb{N}}$ order*-converges to 1. As $1 \ne 0$, there must be an $m \in \mathbb{N}$ such that $c = \inf_{i \ge m} b_{i,k(i)} \ne 0$. There is an n such that $c \in A_n$, in which case $b_{i,k(i)} \in B_n \subseteq B_i$ for every $i \ge \max(m, n)$. **XQ**

(ii) Set $H = \operatorname{int} B_n$. Then there is a $c \in H$ such that for every $d \in \mathfrak{A}$ one of $c \cap d$, $c \setminus d \notin H$. **P ?** Otherwise, we can choose a sequence $\langle c_i \rangle_{i \in \mathbb{N}}$ in H such that $c_0 = 1$ and, for each $i \in \mathbb{N}$, $c_{i+1} \subseteq c_i$ and $c_i \setminus c_{i+1} \in H$. But in this case $\langle c_i \setminus c_{i+1} \rangle_{i \in \mathbb{N}}$ is a disjoint sequence in B_n, which is impossible. **XQ**

(iii) 0 and 1 can be separated by open sets. **P** Take H and c from (ii). Then $G_0 = \{d : c \setminus d \in H\}$ and $G_1 = \{d : c \cap d \in H\}$ are disjoint open sets containing 0 and 1 respectively. **Q**

(b) It follows that $\mathfrak{A}$ is actually Hausdorff in the order-sequential topology. **P** Let a_0, $a_1 \in \mathfrak{A}$ be such that $b = a_1 \setminus a_0$ is non-zero. Consider the principal ideal $\mathfrak{A}_b$. Like $\mathfrak{A}$, this is σ-finite-cc, weakly (σ, ∞)-distributive and Dedekind σ-complete. By (a), there are disjoint subsets U, V of $\mathfrak{A}_b$, open for the order-sequential topology of $\mathfrak{A}_b$, such that $0 \in U$ and $b \in V$. The function $a \mapsto a \cap b : \mathfrak{A} \to \mathfrak{A}_b$ is continuous for the order-sequential topologies (3A3Pb), so $G = \{a : a \cap b \in U\}$ and $H = \{a : a \cap b \in V\}$ are open. Now G and H are disjoint open sets in $\mathfrak{A}$ containing a_0, a_1 respectively. As a_0 and a_1 are arbitrary, $\mathfrak{A}$ is Hausdorff. **Q**

By 393Q, $\mathfrak{A}$ is a Maharam algebra.

393X Basic exercises >(a) Let $\mathfrak{A}$ be the finite-cofinite algebra on an uncountable set (316Yl). (i) Set $\nu_1 0 = 0$, $\nu_1 a = 1$ for $a \in \mathfrak{A} \setminus \{0\}$. Show that ν_1 is a strictly positive Maharam submeasure but is not exhaustive. (ii) Set $\nu_2 a = 0$ for finite a, 1 for cofinite a. Show that ν_2 is a uniformly exhaustive Maharam submeasure but is not order-continuous.

>(b) Let $\mathfrak{A}$ be a Boolean algebra and ν a submeasure on $\mathfrak{A}$. Set $I = \{a : \nu a = 0\}$. Show that (i) I is an ideal of $\mathfrak{A}$ (ii) there is a submeasure $\bar{\nu}$ on $\mathfrak{A}/I$ defined by setting $\bar{\nu} a^\bullet = \nu a$ for every $a \in \mathfrak{A}$ (iii) if ν is exhaustive, so is $\bar{\nu}$ (iv) if ν is uniformly exhaustive, so is $\bar{\nu}$ (v) if ν is a Maharam submeasure, I is a σ-ideal (vi) if ν is a Maharam submeasure and $\mathfrak{A}$ is Dedekind σ-complete, $\bar{\nu}$ is a Maharam submeasure.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and ν an order-continuous submeasure on $\mathfrak{A}$. Show that ν has a unique support $a \in \mathfrak{A}$ such that $\nu \restriction \mathfrak{A}_a$ is strictly positive and $\nu \restriction \mathfrak{A}_{1 \setminus a}$ is identically zero.

(d) Let $\mathfrak{A}$ be a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$ such that $\nu a = \lim_{n \to \infty} \nu a_n$ whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a. Show that ν is a Maharam submeasure.

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν a uniformly exhaustive Maharam submeasure on $\mathfrak{A}$. Show that there is a non-negative countably additive functional μ on $\mathfrak{A}$ such that $\{a : \mu a = 0\} = \{a : \nu a = 0\}$. (*Hint*: 393Xb(vi).)

(f) Let $\mathfrak{A}$ be a Maharam algebra with its Maharam-algebra topology and uniformity. (i) Let $B \subseteq \mathfrak{A}$ be a non-empty upwards-directed set. For $b \in B$ set $F_b = \{c : b \subseteq c \in B\}$. Show that $\{F_b : b \in B\}$ generates a Cauchy filter $\mathcal{F}(B\uparrow)$ on $\mathfrak{A}$ which converges to $\sup B$. (ii) Show that closed subsets of $\mathfrak{A}$ are order-closed. (iii) Show that an order-dense subalgebra of $\mathfrak{A}$ must be dense in the topological sense.

(g) Let $\mathfrak{A}$ be a Maharam algebra. Show that it is a measurable algebra iff for every $A \subseteq \mathfrak{A}$ including antichains of all finite sizes there is a sequence in A which is order*-convergent to 0.

(h) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $\langle a_n \rangle_{n \in \mathbb{N}}$, $\langle b_n \rangle_{n \in \mathbb{N}}$ are sequences in $\mathfrak{A}$ order*-converging to a, b respectively. Show that $\langle a_n \bigcirc b_n \rangle_{n \in \mathbb{N}} \to^* a \bigcirc b$ when $\bigcirc$ is any of the operations $\cup$, $\cap$, $\triangle$ or $\setminus$.

(i) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Write $\mathfrak{T}_{\mathrm{os}}$ for the order-sequential topology on $\mathfrak{A}$ and $\mathfrak{T}_{\mathrm{ma}}$ for the measure-algebra topology. Show that $\mathfrak{T}_{\mathrm{os}} \supseteq \mathfrak{T}_{\mathrm{ma}}$, with equality iff $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(j) (Jech 08) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle A_n \rangle_{n \in \mathbb{N}}$ a sequence of subsets of $\mathfrak{A}$ such that (α) for every $n \in \mathbb{N}$, any antichain in A_n has at most n elements (β) a sequence $\langle a_k \rangle_{k \in \mathbb{N}}$ in $\mathfrak{A}$ is order*-convergent to 0 iff $\{k : a_k \in A_n\}$ is finite for every $n \in \mathbb{N}$. (i) Show that $\mathfrak{A}$ is ccc. (ii) Show that $\mathfrak{A}$ is weakly (σ, ∞)-distributive. (*Hint*: if C_n is non-empty and downwards-directed with infimum 0 for each n, show that there is a sequence $\langle a_n \rangle_{n \in \mathbb{N}} \to^* 0$ such that $a_n \in C_n$ for every n.) (iii) Show that $\mathfrak{A}$ is a Maharam algebra. (*Hint*: 393S.) (iv) Show that any Maharam submeasure on $\mathfrak{A}$ is uniformly exhaustive. (v) Show that $\mathfrak{A}$ is a measurable algebra.

393Y Further exercises (a) Let $\mathfrak{A}$ be any Boolean algebra with a strictly positive Maharam submeasure. Show that $\mathfrak{A}$ is weakly σ-distributive.

(b) Let U be a Riesz space, with its order-sequential topology. (i) Show that addition and subtraction are separately continuous. (ii) Show that U is Archimedean iff scalar multiplication is separately continuous as a function from $\mathbb{R} \times U$ to U, and that in this case scalar multiplication is actually continuous.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and give $L^0 = L^0(\mathfrak{A})$ its order-sequential topology. Suppose $h : \mathbb{R} \to \mathbb{R}$ is continuous, and let $\bar{h} : L^0 \to L^0$ be the corresponding function as defined in 364H. Show that $\bar{h}$ is continuous.

(d) Let $\mathfrak{A}$ be a Maharam algebra. Show that a topology $\mathfrak{T}$ on $L^0(\mathfrak{A})$ defined by the method of 393K must be the order-sequential topology on $L^0(\mathfrak{A})$.

(e) Let U be a weakly (σ,∞)-distributive Riesz space with the countable sup property, with its order-sequential topology, and A a subset of U. Show that $\overline{A}$ is the set of order*-limits of sequences in A.

(f) Let U be a weakly (σ,∞)-distributive Dedekind complete Riesz space with the countable sup property, endowed with its order-sequential topology, and $\mathfrak{A}$ its band algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is a Maharam algebra; (ii) U is Hausdorff; (iii) addition on U is continuous at $(0,0)$; (iv) $\vee : U\times U \to U$ is continuous at $(0,0)$.

(g) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, with its order-sequential topology. (i) Show that if U, V are open sets in $\mathfrak{G}$ containing $0_{\mathfrak{G}} = \emptyset$ and $1_{\mathfrak{G}} = \mathbb{R}$ respectively, then $U \cap V \ne \emptyset$. (ii) Show that if U is an open set in $\mathfrak{G}$ containing $\emptyset$ then there are G, $H \in U$ such that $H = \mathbb{R} \setminus \overline{G}$. (iii) Show that $\{\emptyset\}$ is a G_δ set in $\mathfrak{G}$. (iv) Show that there is no non-zero Maharam submeasure on $\mathfrak{G}$. (v) Show that there is no non-zero countably additive functional on $\mathfrak{G}$.

(h) In 393Xj, show that each of the sets A_n must have non-zero intersection number.

(i) Let $\mathfrak{A}$ be an atomless Boolean algebra with countable Maharam type. Show that there is a submeasure μ on $\mathfrak{A}$, order-continuous on the left, such that whenever $a \in \mathfrak{A}\setminus\{0\}$ there is a $b \subseteq a$ such that $\mu b < \mu a$.

393 Notes and comments For many years it was not known whether there were any Maharam algebras which were not measurable algebras; this was the famous 'control measure problem', eventually solved by M.Talagrand. I will present his example in the next section. We now know that we have a larger class, but it remains very poorly understood, and the material presented here must be regarded as work in progress. As in §§391-392, the stimulus for these ideas has been the attempt to characterize measurable algebras in more or less algebraic terms. If we are prepared to allow order*-convergence of sequences to be an 'algebraic' notion, then 393Xj is such a characterization; but it shares with Kelley's criterion 391K the need for a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$, covering $\mathfrak{A}^+$, with defined properties. The advance, if any, is that the properties (α) and (β) of 393Xj are a good deal farther from any formula for a measure.

The first few results of this section, down to 393G, are concerned with checking that Maharam algebras share properties with measurable algebras, and the proofs use the same ideas, with occasional minor modifications. In 393H we have to think a little, since exhaustivity is less familiar, and harder to apply, than additivity. From this proposition we see that exhaustive submeasures are to uniformly exhaustive submeasures something like what Maharam algebras are to measurable algebras. 393K is a further example of a well-known construction – this time, convergence in measure – which has a version based on Maharam algebras.

In §367 I examined order*-convergence in Riesz spaces, without explicitly discussing the associated topology, and in 393L-393Q here I look at Boolean algebras. In both cases the usefulness of the idea starts with the fact that the algebraic operations are separately continuous (367Ca, 393M), which is itself a consequence of the strong distributive laws in 313A-313B and 352E. It is easy to see that in a Maharam algebra the order-sequential topology is the Maharam-algebra topology (393N). What is remarkable is that natural questions about the order-sequential topology lead to characterizations of Maharam algebras (393Q). This leads directly to an astonishing algebraic characterization of Maharam algebras (393S). (But once again we need to hypothesize the existence of a suitable sequence of sets covering $\mathfrak{A}^+$.)

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394 Talagrand's example

I rewrite the construction in Talagrand 08 of an exhaustive submeasure which is not uniformly exhaustive, generalized as in Perović & Veličković 18.

394A PV norms (a) I will say that a **PV norm** is a function $\|\ \| : [\mathbb{N}]^{<\omega} \to \mathbb{N}$ such that

— $\|\emptyset\| = 0$, $\|I\| = 1$ if $\#(I) = 1$,

—— $\|I \cup J\| \le \|I\| + \|J\|$ for all I, $J \in [\mathbb{N}]^{<\omega}$,

—— $\|I\| \le \|J\|$ whenever I, $J \in [\mathbb{N}]^{<\omega}$ and $\#(I \cap n) \le \#(J \cap n)$ for every $n \in \mathbb{N}$,

—— $\lim_{n\to\infty} \|A \cap n\| = \infty$ for every infinite $A \subseteq \mathbb{N}$

(Perović & Veličković 18, 2.2).

(b) Note that if $\| \, \|$ is a PV norm then $\|I\| \le \|J\| \le \#(J)$ whenever $I \subseteq J \in [\mathbb{N}]^{<\omega}$. We see also that if $I \in [\mathbb{N}]^{<\omega}$ and $k < \|I\|$ there is an $n \in I$ such that $\|I \cap n\| = k$.

(c) The version of Talagrand's example in the 2012 edition of Volume 3 corresponds to the case in which $\|I\| = \#(I)$ for every $I \in [\mathbb{N}]^{<\omega}$. For the work of this section there is no need to consider any other, and some of the formulae in 394D become more readable if you make this simplification; but it makes no real difference to the ideas required.

394B Definitions We are ready to begin work. The construction is complex and demands a large volume of special notation.

(a) I shall work throughout with $X = \prod_{n\in\mathbb{N}} T_n$ where $\langle T_n \rangle_{n\in\mathbb{N}}$ is a sequence of non-empty finite sets and $\sup_{n\in\mathbb{N}} \#(T_n)$ is infinite. X may be regarded as a compact Hausdorff space with the product of the discrete topologies on the T_n. For each $n \in \mathbb{N}$, $\mathfrak{B}_n$ will be the algebra of subsets of X determined by coordinates less than n and $\mathcal{A}_n$ the set of its atoms, that is, the family of sets of the form $\{x : z \subseteq x \in X\}$ for some $z \in \prod_{i<n} T_i$. $\mathfrak{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$ will be the algebra of open-and-closed subsets of X. For $I \subseteq \mathbb{N}$ and $z \in \prod_{n\in I} T_n$, Y_z will be $\{x : z \subseteq x \in X\}$. Finally, $\| \, \|$ will be a PV norm on $[\mathbb{N}]^{<\omega}$.

(b) We shall need a sequence $\langle \alpha_k \rangle_{k\in\mathbb{N}}$ in $\mathbb{R}$ and a sequence $\langle N_k \rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$. It is easy enough to give appropriate formulae but perhaps the ideas will be clearer if instead I declare the properties they must have.

(i) $\alpha_k > 0$ and $(2^{k+4})^{\alpha_k} \le 2$ for every $k \in \mathbb{N}$, $\langle \alpha_k \rangle_{k\in\mathbb{N}}$ is non-increasing, and $\sum_{k=0}^{\infty} \alpha_k \le \frac{1}{2}$.

(ii) $N_k \in \mathbb{N}$ and $2^{-k}(2^{-2k-12} N_k)^{\alpha_k} \ge 2^4$ for every $k \in \mathbb{N}$.

(c) Now we come to some of the key ideas. For a set $\mathcal{I} \subseteq \mathcal{P}X \times \mathcal{P}\mathbb{N} \times [0, \infty[$, define its 'spread' $\operatorname{spr} \mathcal{I}$ to be $\bigcup_{(E,I,w)\in\mathcal{I}} E$ and its 'weight' $\operatorname{wt} \mathcal{I}$ to be $\sum_{(E,I,w)\in\mathcal{I}} w$.

(d) For any family $\mathcal{E} \subseteq \mathcal{P}X \times \mathcal{P}\mathbb{N} \times [0, \infty[$ define $\phi_{\mathcal{E}} : \mathfrak{B} \to [0, \infty]$ by setting

$$\phi_{\mathcal{E}} E = \inf\{\operatorname{wt} \mathcal{I} : \mathcal{I} \subseteq \mathcal{E} \text{ is finite, } E \subseteq \operatorname{spr} \mathcal{I}\},$$

counting $\inf \emptyset$ as ∞. So $\phi_{\emptyset} \emptyset = 0$ and $\phi_{\emptyset} E = \infty$ for $E \in \mathfrak{B} \setminus \{\emptyset\}$.

(e) For $D \subseteq X$ and $I \subseteq \mathbb{N}$ set

$$\theta_I(D) = \{y : y \in X, \, y \restriction I = x \restriction I \text{ for some } x \in D\}.$$

(f)(i) If $m < n$ in $\mathbb{N}$, $\phi : \mathfrak{B} \to [0, \infty]$ is a function and $E \in \mathfrak{B}$, then E is **ϕ-thin between m and n** if $\phi(X \setminus \theta_{n\setminus m}(A \cap E)) \ge 1$ for every $A \in \mathcal{A}_m$.

(ii) If $I \subseteq \mathbb{N}$, $\phi : \mathfrak{B} \to [0, \infty]$ is a function and $E \in \mathfrak{B}$, then E is **ϕ-thin along I** if it is ϕ-thin between m and n whenever m, $n \in I$ and $m < n$.

(g) For $k \le p \in \mathbb{N}$ define $\mathcal{C}_{kp}$ and ν_{kp} by downwards induction on k, as follows. Start with $\mathcal{C}_{pp} = \emptyset$ for every p. Given $\mathcal{C}_{kp}$, set $\nu_{kp} = \phi_{\mathcal{C}_{kp}}$. Given that $k < p$ and $\mathcal{C}_{k+1,p}$ and $\nu_{k+1,p} = \phi_{\mathcal{C}_{k+1,p}}$ have been defined, set

$$\mathcal{E}_{kp} = \{(E, I, w) : E \in \mathfrak{B}, \, I \in [\mathbb{N}]^{<\omega}, \, 1 \le \|I\| \le N_k,$$
$$w \ge 2^{-k}\Bigl(\frac{N_k}{\|I\|}\Bigr)^{\alpha_k}, \, E \text{ is } \nu_{k+1,p}\text{-thin along } I\},$$

$$\mathcal{C}_{kp} = \mathcal{E}_{kp} \cup \mathcal{C}_{k+1,p}$$

and continue.

(h) Define $\langle c_k\rangle_{k\in\mathbb{N}}$ by setting $c_0=8$, $c_{k+1}=2^{2\alpha_k}c_k$ for every k.

394C Very elementary facts In the hope of aiding digestion of the definitions here, of which 394Bf and 394Bg are likely to be wholly obscure to anyone who has not worked through this proof before, I run over some obvious facts which will be used below.

(a) $\phi_{\mathcal{E}}:\mathfrak{B}\to[0,\infty]$ is a submeasure for any $\mathcal{E}\subseteq\mathcal{P}X\times\mathcal{P}\mathbb{N}\times[0,\infty[$. (Subadditivity and monotonicity are written into the definition.)

(b) If I, $J\subseteq\mathbb{N}$ then $\theta_I\theta_J=\theta_{I\cap J}$. If $I\subseteq J\subseteq\mathbb{N}$ then $\theta_I(D)=\theta_I\theta_J(D)\supseteq\theta_J(D)$ for all $D\subseteq X$. If $I\subseteq\mathbb{N}$ then $\theta_I(D\cap\theta_I(E))=\theta_I(E\cap\theta_I(D))$ for all D, $E\subseteq X$. For any $I\subseteq\mathbb{N}$ and any family $\mathcal{D}$ of subsets of X, $\theta_I(\bigcup\mathcal{D})=\bigcup_{D\in\mathcal{D}}\theta_I(D)$.

For $n\in\mathbb{N}$ and $D\subseteq X$, $D\in\mathfrak{B}_n$ iff $\theta_n(D)=D$. If $E\in\mathfrak{B}$ and $I\subseteq\mathbb{N}$, $\theta_I(E)\in\mathfrak{B}$. If $m\le n$ in $\mathbb{N}$, $A\in\mathcal{A}_m$ and $A_1\in\mathcal{A}_n$, then $A\cap\theta_{n\setminus m}(A_1)\in\mathcal{A}_n$. If $m\in\mathbb{N}$ and $A\in\mathcal{A}_m$ then $E\mapsto\theta_{\mathbb{N}\setminus m}(A\cap E):\mathfrak{B}\to\mathfrak{B}$ is a Boolean homomorphism.

(c) If $m<n$, $\phi:\mathfrak{B}\to[0,\infty]$ is a non-decreasing function, $E\in\mathfrak{B}$ is ϕ-thin between m and n and $E'\in\mathfrak{B}$ is included in E, then E' is ϕ-thin between m and n' for every $n'\ge n$.

(d) All the classes $\mathcal{E}_{kp}$, $\mathcal{C}_{kp}$ are closed under increases in the scalar variable and decreases in the first variable, that is,
— if $k<p$, $(E,I,w)\in\mathcal{E}_{kp}$, $E'\in\mathfrak{B}$, $E'\subseteq E$ and $w'\ge w$ then $(E',I,w')\in\mathcal{E}_{kp}$,
— if $k\le p$, $(E,I,w)\in\mathcal{C}_{kp}$, $E'\in\mathfrak{B}$, $E'\subseteq E$ and $w'\ge w$ then $(E',I,w')\in\mathcal{C}_{kp}$.

(e) If $k\le p$ in $\mathbb{N}$, $\mathcal{C}_{kp}=\bigcup_{k\le l<p}\mathcal{E}_{lp}$.

(f) If $k<p$ in $\mathbb{N}$, $\nu_{kp}\le\nu_{k+1,p}$, because $\mathcal{C}_{kp}\supseteq\mathcal{C}_{k+1,p}$.

(g) $8\le c_k\le 16$ for every $k\in\mathbb{N}$, because $\sum_{k=0}^{\infty}2\alpha_k\le 1$.

(h) If $k<p$ in $\mathbb{N}$, then $(X,\{0\},2^{-k}N_k^{\alpha_k})\in\mathcal{E}_{kp}$ so $\nu_{kp}X\le 2^{-k}N_k^{\alpha_k}$ and ν_{kp} is totally finite.

394D Moving up a gear, we have the following.

Lemma Suppose that $\mathcal{K}$ is a non-empty finite family of subsets of $\mathbb{N}$ and $r\in\mathbb{N}$ is such that $\|K\|\ge r\#(\mathcal{K})$ for every $K\in\mathcal{K}$. Then we have an enumeration $\langle K_i\rangle_{i<s}$ of $\mathcal{K}$ and a non-decreasing family $\langle n_i\rangle_{i\le s}$ such that $\|K_i\cap n_{i+1}\setminus n_i\|=r$ for every $i<s$.

(b) Suppose that $\langle K_i\rangle_{i<s}$ is a family of finite subsets of $\mathbb{N}$ such that $\|K_i\|\ge n\ge 3$ for every $i<s$ and $\max K_i<\min K_{i+1}$ for $i\le s-2$, and that $A\in[\mathbb{N}]^{<\omega}$ is such that $\|A\|\le 1$. Let $\mathcal{J}$ be a finite subset of $\mathcal{P}X\times([\mathbb{N}]^{<\omega}\setminus\{\emptyset\})\times[0,\infty[$. Then we can find $\langle u_i\rangle_{i<s}$ and $\langle v_i\rangle_{i<s}$ such that u_i, $v_i\in K_i$ and $u_i<v_i$ for each $i<s$ and, setting $W=\bigcup_{i<s}v_i\setminus u_i$, $A\cap W=\emptyset$ and

$$\operatorname{wt}\{(E,I,w):(E,I,W)\in\mathcal{J},\ \|I\setminus W\|<\tfrac12\|I\|\}\le\frac{1}{n-2}\operatorname{wt}\mathcal{J}.$$

proof (a) If $r=0$ we can take any enumeration of $\mathcal{K}$ and set $n_i=0$ for every i. Otherwise, write s for $\#(\mathcal{K})$ and choose n_i, K_i inductively, as follows. Start with $n_0=0$. Given $j<s$, $n_j\in\mathbb{N}$ and $\langle K_i\rangle_{i<j}$ such that $\|K\setminus n_j\|\ge r(s-j)$ for every $K\in\mathcal{K}_j=\mathcal{K}\setminus\{K_i:i<j\}$, set

$$n_{j+1}=\min\{n:\|K\cap n\setminus n_j\|\ge r\text{ for some }K\in\mathcal{K}_j\}$$

and choose $K_j\in\mathcal{K}_j$ such that $\|K_j\cap n_{j+1}\setminus n_j\|\ge r$. Observe that

$$\|K\cap n_{j+1}\setminus n_j\|\le\|K\cap(n_{j+1}-1)\setminus n_j\|+\|\{n_{j+1}-1\}\|\le(r-1)+1=r$$

for every $K\in\mathcal{K}_j$, so in fact $\|K_j\cap n_{j+1}\setminus n_j\|=r$ and also

$$\|K\setminus n_{j+1}\|\ge\|K\setminus n_j\|-\|K\cap n_{j+1}\setminus n_j\|\ge r(s-j)-r=r(s-(j+1))$$

for every $K\in\mathcal{K}_j$, so the induction will continue.

(b) For $i < s$ and $k \in K_i \setminus \{\min K_i\}$ write k^- for the greatest member of K_i less than k. Set

$$K_i' = \{k : k \in K_i \setminus \{\min K_i\},\ A \cap k \setminus k^- \neq \emptyset\}, \quad K_i'' = K_i \setminus (K_i' \cup \{\min K_i\}).$$

Then $\|K_i'\| \le 1$. **P** For $k \in K_i'$ set $g(k) = \min(A \cap k \setminus k^-)$. Then $g : K_i' \to A$ is injective and $g(k) \le k$ for every $k \in K_i'$. So $\#(K_i' \setminus m) \le \#(g[K_i'] \setminus m) \le \#(A \setminus m)$ for every $m \in \mathbb{N}$, and $\|K_i'\| \le \|A\| \le 1$. **Q**

Consequently

$$\#(K_i'') \ge \|K_i''\| \ge \|K_i\| - \|K_i'\| - \|\{\min K_i\}\| \ge n - 2$$

and we have a family $\langle k_{ij}\rangle_{j<n-2}$ of distinct elements of K_i'', so that $A \cap k_{ij} \setminus k_{ij}^- = 0$ for every $j < n-2$.

For $j < n - 2$, set

$$W_j = \bigcup_{i<s} k_{ij} \setminus k_{ij}^-, \quad \mathcal{J}_j = \{(E, I, w) : (E, I, w) \in \mathcal{J},\ \|I \setminus W_j\| < \tfrac{1}{2}\|I\|\}.$$

Then $W_0, \dots , W_{n-3}$ are disjoint and none of them meet A. Since $\|\ \|$ is subadditive and $I = (I\setminus W_j)\cup(I\setminus W_{j'})$ whenever j, j' are distinct, $\mathcal{J}_0, \dots , \mathcal{J}_{n-3}$ are disjoint. So $\operatorname{wt}\mathcal{J} \ge \sum_{j=0}^{n-3} \operatorname{wt}\mathcal{J}_j$ and there is a $j < n - 2$ such that $\operatorname{wt}\mathcal{J}_j \le \frac{1}{n-2}\operatorname{wt}\mathcal{J}$. Take $u_i = k_{ij}^-$ and $v_i = k_{ij}$ for each i, so that $W = \bigcup_{i<s} v_i \setminus u_i$ is W_j, and we have an appropriate pair of sequences.

394E Lemma Suppose that $k \le p$, $m < n$, $A \in \mathcal{A}_m$, $(E, I, w) \in \mathcal{C}_{kp}$ and $I' = I \cap n \setminus m$ is non-empty. If $E' = \theta_{n\setminus m}(E \cap A)$ and $w' \ge \bigl(\frac{\|I\|}{\|I'\|}\bigr)^{\alpha_k} w$, then $(E', I', w') \in \mathcal{C}_{kp}$.

proof There is an l such that $k \le l < p$ and $(E, I, w) \in \mathcal{E}_{lp}$. Now E' is $\nu_{l+1,p}$-thin along I'. **P** Suppose that i, $j \in I'$ and $i < j$, so that $m \le i < j < n$. Take any $A_1 \in \mathcal{A}_i$, and set $A_2 = A \cap \theta_{n\setminus m}(A_1)$, so that A_2 also belongs to $\mathcal{A}_i$. Then, using the list in 394Cb,

$$\begin{aligned}\theta_{j\setminus i}(E' \cap A_1) &= \theta_{j\setminus i}(A_1 \cap \theta_{n\setminus m}(E \cap A))\\ &= \theta_{j\setminus i}(\theta_{n\setminus m}(A_1 \cap \theta_{n\setminus m}(E \cap A)))\\ &= \theta_{j\setminus i}(\theta_{n\setminus m}(E \cap A \cap \theta_{n\setminus m}(A_1)))\\ &= \theta_{j\setminus i}(\theta_{n\setminus m}(E \cap A_2)) = \theta_{j\setminus i}(E \cap A_2).\end{aligned}$$

So

$$\nu_{l+1,p}(X \setminus \theta_{j\setminus i}(E' \cap A_1)) = \nu_{l+1,p}(X \setminus \theta_{j\setminus i}(E \cap A_2)) \ge 1$$

because E is $\nu_{l+1,p}$-thin between i and j. As i, j and A_1 are arbitrary, E' is $\nu_{l+1,p}$-thin along I'. **Q**

Of course $\|I'\| \le \|I\| \le N_l$. Finally, because $\alpha_l \le \alpha_k$, we have

$$w' \ge \bigl(\tfrac{\|I\|}{\|I'\|}\bigr)^{\alpha_k} w \ge \bigl(\tfrac{\|I\|}{\|I'\|}\bigr)^{\alpha_l} \cdot 2^{-l}\bigl(\tfrac{N_l}{\|I\|}\bigr)^{\alpha_l} = 2^{-l}\bigl(\tfrac{N_l}{\|I'\|}\bigr)^{\alpha_l},$$

and $(E', I', w') \in \mathcal{E}_{lp} \subseteq \mathcal{C}_{kp}$.

394F Corollary (a) Suppose that $n \in \mathbb{N}$ and $k \le p$ and that $\mathcal{I} \subseteq \mathcal{C}_{kp}$ is a finite set such that $\|I \cap n\| \ge \frac{1}{4}\|I\|$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_n(\operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$.

(b) Suppose that $m \in \mathbb{N}$, $k \le p$ and $A \in \mathcal{A}_m$. Let $\mathcal{I}$ be a finite subset of $\mathcal{C}_{kp}$ such that $\|I \setminus m\| \ge \frac{1}{4}\|I\|$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_{\mathbb{N}\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$.

(c) Suppose that $m < n$ in $\mathbb{N}$, $k \le p$ and $A \in \mathcal{A}_m$. Let $\mathcal{I}$ be a finite subset of $\mathcal{C}_{kp}$ such that $\|I \cap n \setminus m\| \ge 2^{-k-4}\|I\|$ whenever $(E, I, w) \in \mathcal{I}$. Then $\nu_{kp}(\theta_{n\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}$.

proof (a) For each $(E, I, w) \in \mathcal{I}$ set $E' = \theta_n(E) \in \mathcal{B}_n$, $I' = I \cap n$ and

$$w' = \bigl(\tfrac{\|I\|}{\|I'\|}\bigr)^{\alpha_k} w \le 4^{\alpha_k} w \le 2w.$$

By 394E, with $m = 0$ and $A = X$, $(E', I', w') \in \mathcal{C}_{kp}$. Set $\mathcal{J} = \{(E', I', w') : (E, I, w) \in \mathcal{I}\}$ and $B = \operatorname{spr}\mathcal{J}$. Then

$$B = \bigcup_{(E,I,w)\in\mathcal{I}} \theta_n(E) = \theta_n(\operatorname{spr}\mathcal{I})$$

and

$$\nu_{kp}B \le \operatorname{wt}\mathcal{J} = \sum_{(E,I,w)\in\mathcal{I}} w' \le 2\operatorname{wt}\mathcal{I},$$

as required.

(b) This time, take $n > m$ so large that $I \subseteq n$ whenever $(E,I,w) \in \mathcal{I}$. For $(E,I,w) \in \mathcal{I}$, set

$$E' = \theta_{n\setminus m}(A \cap E), \quad I' = I \setminus m = I \cap n \setminus m, \quad w' = \Bigl(\frac{\|I\|}{\|I'\|}\Bigr)^{\alpha_k} w \le 2w.$$

Then 394E tells us that $(E', I', w') \in \mathcal{C}_{kp}$. Setting $\mathcal{J} = \{(E', I', w') : (E,I,w) \in \mathcal{I}\}$,

$$\theta_{\mathbb{N}\setminus m}(A \cap \operatorname{spr}\mathcal{I}) = \bigcup_{(E,I,w)\in\mathcal{I}} \theta_{\mathbb{N}\setminus m}(A \cap E) \subseteq \bigcup_{(E,I,w)\in\mathcal{I}} E' = \operatorname{spr}\mathcal{J},$$

so

$$\nu_{kp}(\theta_{N\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le \operatorname{wt}\mathcal{J} \le 2\operatorname{wt}\mathcal{I}.$$

(c) For $(E,I,w) \in \mathcal{I}$ set

$$E' = \theta_{n\setminus m}(A \cap E), \quad I' = I \cap n \setminus m, \quad w' = \Bigl(\frac{\|I\|}{\|I'\|}\Bigr)^{\alpha_k} w \le (2^{k+4})^{\alpha_k} w \le 2w.$$

Then $(E', I', w') \in \mathcal{C}_{kp}$. Setting $\mathcal{J} = \{(E', I', w') : (E,I,w) \in \mathcal{I}\}$,

$$\theta_{n\setminus m}(A \cap \operatorname{spr}\mathcal{I}) = \bigcup_{(E,I,w)\in\mathcal{I}} \theta_{n\setminus m}(A \cap E) = \bigcup_{(E,I,w)\in\mathcal{I}} E' = \operatorname{spr}\mathcal{J},$$

so

$$\nu_{kp}(\theta_{n\setminus m}(A \cap \operatorname{spr}\mathcal{I})) \le \operatorname{wt}\mathcal{J} \le 2\operatorname{wt}\mathcal{I}.$$

394G We are at the centre of the argument.

Lemma Suppose that $L \in [\mathbb{N}]^{<\omega}$ is such that $\|L\| \le 1$, and $z \in \prod_{r\in L} T_r$. Then $\nu_{kp}Y_z \ge c_k$ whenever $k \le p$ in $\mathbb{N}$.

proof Induce on $p - k$.

(a) If $k = p$ then

$$\mathcal{C}_{pp} = \emptyset, \quad \nu_{pp}Y_z = \infty.$$

For the downwards step to $k < p$, given that $\nu_{k+1,p}Y_z \ge c_{k+1}$, take a finite set $\mathcal{I} \subseteq \mathcal{C}_{kp}$ such that $\operatorname{wt}\mathcal{I} < c_k$. The rest of this proof is devoted to showing that $Y_z \not\subseteq \operatorname{spr}\mathcal{I}$.

(b) It will help to get a trivial case out of the way. If $\mathcal{I} \subseteq \mathcal{C}_{k+1,p}$, then we have

$$\operatorname{wt}\mathcal{I} < c_k \le c_{k+1} \le \nu_{k+1,p}Y_z$$

by the inductive hypothesis, so certainly $Y_z \not\subseteq \operatorname{spr}\mathcal{I}$. Accordingly we may suppose henceforth that $\mathcal{I} \not\subseteq \mathcal{C}_{k+1,p}$.

A second elementary point is that $\|I\| \ge 2^{2k+12}$ whenever $(E,I,w) \in \mathcal{I}$. **P** We have an l such that $k \le l < p$ and $(E,I,w) \in \mathcal{E}_{kp}$, so

$$2^{-l}\Bigl(\frac{N_l}{\|I\|}\Bigr)^{\alpha_l} \le w \le c_k \le 2^4$$

and $\|I\| \ge 2^{2l+12} \ge 2^{2k+12}$, by the choice of N_l. **Q**

(c) Express $\mathcal{I}$ as $\mathcal{J} \cup \mathcal{K}$ where $\mathcal{J} \subseteq \mathcal{C}_{k+1,p}$ and $\mathcal{K} \subseteq \mathcal{E}_{kp}$. Set $s = \#(\mathcal{K}) > 0$. For $(E,I,w) \in \mathcal{K}$ we have $w \ge 2^{-k}$, so $s \le 2^k c_k \le 2^{k+4}$ (394Cg). Consequently $\|I\| \ge 2^{2k+12} \ge 2^{k+8}s$ whenever $(E,I,w) \in \mathcal{K}$. By 394Da, we can find $m_0 < m_1 < \ldots < m_s$ and an enumeration $\langle (E_i, K_i, w_i)\rangle_{i<s}$ of $\mathcal{K}$ such that $\|K_i \cap m_{i+1} \setminus m_i\| = 2^{k+8}$ for $i < s$. By 394Db we can find members u_i, v_i of $K_i \cap m_{i+1} \setminus m_i$ such that $u_i < v_i$, for $i < s$, and setting $W = \bigcup_{i<s} v_i \setminus u_i$, $L \cap W = \emptyset$ and

$$S = \text{wt}\{(E,I,w) : (E,I,w) \in \mathcal{J},\ \|I \setminus W\| < \frac{1}{2}\|I\|\}$$
$$\le \frac{1}{2^{k+8}-2}\,\text{wt}\,\mathcal{J} \le 2^{-k-6}c_k \le 2^{-k-2}.$$

(d) Set

$$\mathcal{J}_1 = \{(E,I,w) : (E,I,w) \in \mathcal{J},\ \|I \setminus W\| < \frac{1}{2}\|I\|\}, \quad \mathcal{J}_2 = \mathcal{J} \setminus \mathcal{J}_1;$$

then

$$\text{wt}\,\mathcal{J}_1 = S \le 2^{-k-2} \le \frac{1}{4}$$

and $\|I \cap W\| > \frac{1}{2}\|I\|$ whenever $(E,I,w) \in \mathcal{J}_1$. For $i < s$ set

$$\mathcal{J}_{1i} = \{(E,I,w) : (E,I,w) \in \mathcal{J}_1,\ \|I \cap v_i \setminus u_i\| \ge 2^{-k-5}\|I\|\}.$$

Since $s \le 2^{k+4}$, $\mathcal{J}_1 = \bigcup_{i<s}\mathcal{J}_{1i}$.

(e) Suppose that $i < s$ and $A \in \mathcal{A}_{u_i}$. Then there is an $A' \in \mathcal{A}_{v_i}$ such that $A' \subseteq A \setminus (E_i \cup \text{spr}\,\mathcal{J}_{1i})$. **P** Set $C = \theta_{v_i\setminus u_i}(A \cap \text{spr}\,\mathcal{J}_{1i}) \in \mathfrak{B}_{v_i}$. By 394Fc, applied in $\mathcal{C}_{k+1,p}$,

$$\nu_{k+1,p}C \le 2\,\text{wt}\,\mathcal{J}_{1i} \le 2\,\text{wt}\,\mathcal{J}_1 < 1.$$

As $(E_i,K_i,w_i) \in \mathcal{E}_{kp}$, E_i is $\nu_{k+1,p}$-thin between u_i and v_i, $\nu_{k+1,p}(X \setminus \theta_{v_i\setminus u_i}(A \cap E_i)) \ge 1$ and C does not include $X \setminus \theta_{v_i\setminus u_i}(A \cap E_i)$. Since these sets both belong to $\mathfrak{B}_{v_i}$ there is an $A_1 \in \mathcal{A}_{v_i}$ disjoint from both C and $\theta_{v_i\setminus u_i}(A \cap E_i)$, that is, disjoint from $\theta_{v_i\setminus u_i}(A \cap (E_i \cup \text{spr}\,\mathcal{J}_{1i}))$. Now $A' = A \cap \theta_{v_i\setminus u_i}(A_1)$ belongs to $\mathcal{A}_{v_i}$, is included in A and is disjoint from $E_i \cup \text{spr}\,\mathcal{J}_{1i}$. **Q**

(f) We can therefore find a function $\Gamma : X \to X$ such that $\Gamma[X]$ is disjoint from $\text{spr}(\mathcal{K} \cup \mathcal{J}_1)$, while $\Gamma(x){\restriction}m$ is determined by $x{\restriction}m$ for every $m \in \mathbb{N}$. **P** By (e) just above, we have for each $i < s$ a function $q_i : \mathcal{A}_{u_i} \to \mathcal{A}_{v_i}$ such that $q_i(A) \subseteq A \setminus (E_i \cup \text{spr}\,\mathcal{J}_{1i})$ for every $A \in \mathcal{A}_{u_i}$. We can re-interpret q_i as a function $h_i : \prod_{n<u_i} T_n \to \prod_{n<v_i} T_n$ defined by saying that if $A = \{x : x{\restriction}u_i = y\}$ then $q_i(A) = \{x : x{\restriction}v_i = h_i(y)\}$; note that $y = h_i(y){\restriction}u_i$ for every $y \in \prod_{n<u_i} T_n$. Now, for $x \in X$, define $\Gamma(x)(n)$ inductively by saying that

$$\Gamma(x)(n) = x(n) \text{ if } n \in \mathbb{N} \setminus W,$$
$$= h_i(\Gamma(x){\restriction}u_i)(n) \text{ if } i < s \text{ and } u_i \le n < v_i.$$

Of course this ensures that $\Gamma(x){\restriction}m$ is determined by $x{\restriction}m$ for every m. If $i < s$, $x \in X$, and $A \in \mathcal{A}_{u_i}$ is such that $\Gamma(x) \in A$, then $\Gamma(x) \in q_i(A)$, which is disjoint from $E_i \cup \text{spr}\,\mathcal{J}_{1i}$. Thus $\Gamma[X]$ is disjoint from $\bigcup_{i<s} E_i \cup \text{spr}\,\mathcal{J}_{1i} = \text{spr}(\mathcal{K} \cup \mathcal{J}_1)$. **Q**

(g) Take $(E,I,w) \in \mathcal{J}_2$ and consider $\nu_{k+1,p}(\Gamma^{-1}[E])$.

(i) There is an l such that $k \le l < p$ and $(E,I,w) \in \mathcal{E}_{lp}$. Now if m, $n \in I$ are such that $m < n$ and $n \setminus m$ is disjoint from W, $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin between m and n. **P** Take any $A \in \mathcal{A}_m$. Because $\Gamma(x){\restriction}m$ is determined by $x{\restriction}m$, we can find an $A' \in \mathcal{A}_m$ such that $\Gamma[A] \subseteq A'$. In this case,

$$A \cap \Gamma^{-1}[E] \subseteq \Gamma^{-1}[\Gamma[A] \cap E] \subseteq \Gamma^{-1}[A' \cap E] \subseteq \theta_{n\setminus m}(A' \cap E)$$

because $\Gamma(x)(i) = x(i)$ whenever $x \in X$ and $i \in n \setminus m$. So $\theta_{n\setminus m}(A \cap \Gamma^{-1}[E]) \subseteq \theta_{n\setminus m}(A' \cap E)$ and

$$\nu_{l+1,p}(X \setminus \theta_{n\setminus m}(A \cap \Gamma^{-1}[E])) \ge \nu_{l+1,p}(X \setminus \theta_{n\setminus m}(A' \cap E)) \ge 1$$

because E is $\nu_{l+1,p}$-thin between m and n. **Q**

(ii) As noted in (b), $\|I\| \ge 2^{2k+12} \ge 4s$. For each $i < s$ such that $\min I \le u_i$, let u_i^- be the largest element of I which is less than or equal to u_i. Set $I' = I \setminus (W \cup \{u_i^- : i < s,\ \min I \le u_i\})$. Then

$$\|I'\| \ge \frac{\|I\|}{2} - s \ge \frac{\|I\|}{4}.$$

Now $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin along I'. **P** Suppose that m, $n \in I'$ and $m < n$. Let m^+ be the least element of I such that $m < m^+$. Then $m^+ \le n$. **?** If $W \cap m^+ \setminus m \ne \emptyset$, there is an $i < s$ such that $m^+ \setminus m$ meets $v_i \setminus u_i$, that is, $m < v_i$ and $u_i < m^+$. Since $m \in I' \subseteq \mathbb{N} \setminus W$, $m \le u_i$ and u_i^- is defined; now $m \ne u_i^-$ so $m < u_i^- \in I$ and $m^+ \le u_i^- \le u_i$. **X** Thus $W \cap m^+ \setminus m$ is empty and (i) tells us that $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin between m and m^+, therefore $\nu_{l+1,p}$-thin between m and n (394Cc). As m and n are arbitrary, $\Gamma^{-1}[E]$ is $\nu_{l+1,p}$-thin along I'. **Q**

(iii) If we now set $w' = 4^{\alpha_l} w$, we see that

$$1 \le \|I'\| \le \|I\| \le N_l, \quad w' \ge 2^{-l} 4^{\alpha_l} \big(\frac{N_l}{\|I\|}\big)^{\alpha_l} \ge 2^{-l} \big(\frac{N_l}{\|I'\|}\big)^{\alpha_l},$$

so

$$(\Gamma^{-1}[E], I', w') \in \mathcal{E}_{lp} \subseteq \mathcal{C}_{k+1,p}$$

and

$$\nu_{k+1,p}(\Gamma^{-1}[E]) \le w' = 4^{\alpha_l} w \le 4^{\alpha_k} w.$$

(h) We are nearly done. Applying (g) to each member of $\mathcal{J}_2$,

$$\nu_{k+1,p}(\Gamma^{-1}[\operatorname{spr} \mathcal{J}_2]) \le 4^{\alpha_k} \operatorname{wt} \mathcal{J}_2 \le 4^{\alpha_k} \operatorname{wt} \mathcal{I} < 4^{\alpha_k} c_k = c_{k+1} \le \nu_{k+1,p} Y_z$$

by the inductive hypothesis in its full strength. So there is a $y \in Y_z \setminus \Gamma^{-1}[\operatorname{spr} \mathcal{J}_2]$. With (f), this means that $\Gamma(y)$ does not belong to

$$\operatorname{spr}(\mathcal{K} \cup \mathcal{J}_1) \cup \operatorname{spr}(\mathcal{J}_2) = \operatorname{spr} \mathcal{I}.$$

On the other hand, $\Gamma(y) \in Y_z$ because $L \cap W = \emptyset$. As $\mathcal{I}$ was arbitrary, $\nu_{kp} Y_z$ must be at least c_k, which is what we need to know to proceed with the induction.

394H Definitions I present the last two definitions required. Fix on a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. For $k \in \mathbb{N}$, set

$$\nu_k E = \lim_{p \to \mathcal{F}} \nu_{kp} E \in [0, \infty]$$

for every $E \in \mathfrak{B}$; finally, write ν for ν_0.

394I Proposition (a) For every $k \in \mathbb{N}$, ν_k is a totally finite submeasure and $\nu_k X \ge 8$.
(b) ν is not uniformly exhaustive.

proof (a) It follows directly from the definition in 392A that ν_k, being a limit of submeasures, is a submeasure. By 394Ch, $\nu_k X \le 2^{-k} N_k^{\alpha_k}$ is finite. By 394G and 394Cg,

$$\nu_k X = \lim_{p \to \mathcal{F}} \nu_{kp} X \ge c_k \ge 8.$$

(b) For any $n \in \mathbb{N}$ and $t \in T_n$,

$$\nu Y_{nt} = \lim_{p \to \mathcal{F}} \nu_{0p} Y_{nt} \ge 8$$

by 394G. As $\sup_{n \in \mathbb{N}} \#(T_n)$ is infinite, and $\langle Y_{nt} \rangle_{t \in T_n}$ is disjoint for every n, ν is not uniformly exhaustive.

394J Lemma Suppose that $k \in \mathbb{N}$, $E \in \mathfrak{B}$, $I \in [\mathbb{N}]^{<\omega}$ and E is $\frac{1}{2}\nu_k$-thin along I. Then

$$\{p : p \ge k,\ E \text{ is } \nu_{kp}\text{-thin along } I\} \in \mathcal{F}.$$

If $k \ge 1$ and $\|I\| = N_{k-1}$, then $\nu_{k-1} E \le 2^{-k+1}$.

proof If m, $n \in I$, $m < n$ and $A \in \mathcal{A}_m$, then $\nu_k(X \setminus \theta_{n \setminus m}(A \cap E)) \ge 2$. So

$$U_{An} = \{p : p \ge k,\ \nu_{kp}(X \setminus \theta_{n \setminus m}(A \cap E)) \ge 1\}$$

belongs to $\mathcal{F}$. Setting $U = \bigcap_{m<n \text{ in } I, A \in \mathcal{A}_m} U_{An}$, $U \in \mathcal{F}$ and E is ν_{kp}-thin along I for every $p \in U$.

If $k \ge 1$ and $\|I\| = N_{k-1}$, then $(E, I, 2^{-k+1}) \in \mathcal{E}_{k-1,p}$ for every $p \in U$, so $\nu_{k-1,p} E \le 2^{-k+1}$ for every $p \in U$ and $\nu_{k-1} E \le 2^{-k+1}$.

394K Lemma Let m, $k \in \mathbb{N}$ and let $\langle E_i\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$ such that

every E_i is determined by coordinates in $\mathbb{N}\setminus m$,

$\nu_k(\bigcup_{i\le n} E_i) < 2$ for every $n \in \mathbb{N}$.

Then for every $\eta > 0$ there is a $C \in \mathfrak{B}$, determined by coordinates in $\mathbb{N}\setminus m$, such that $\nu_k C \le 4$ and $\nu_k(E_i \setminus C) \le \eta$ for each i.

proof (a) For each $n > m$, set

$$\tilde E_n = \bigcup\{E_i : i \le n,\ E_i \in \mathfrak{B}_n\},$$

so that $\tilde E_n$ is determined by coordinates in $n\setminus m$ and $\nu_k \tilde E_n < 2$. Set

$$U_n = \{p : p \ge k,\ \nu_{kp}\tilde E_n < 2\} \in \mathcal{F}.$$

For $p \in U_n$ we can find a finite $\mathcal{I}_{np} \subseteq \mathcal{C}_{kp}$ such that $\tilde E_n \subseteq \operatorname{spr}\mathcal{I}_{np}$ and $\operatorname{wt}\mathcal{I}_{np} \le 2$. For $r > m$ set

$$\mathcal{I}_{npr} = \{(E,I,w) : (E,I,w) \in \mathcal{I}_{np},\ \|I\cap(r-1)\setminus m\| < \frac{1}{2}\|I\| \le \|I\cap r\setminus m\|\},$$

and set

$$\mathcal{I}'_{np} = \{(E,I,w) : (E,I,w) \in \mathcal{I}_{np},\ \|I\cap m\| \ge \frac{1}{4}\|I\|\}.$$

Set $B_{np} = \theta_m(\operatorname{spr}\mathcal{I}'_{np})$; then

$$\nu_{kp}B_{np} \le 2\operatorname{wt}\mathcal{I}'_{np} \le 4,$$

by 394Fa. Since $\nu_{kp}X \ge c_k \ge 8$ (394G, 394Cg again), $B_{np} \ne X$ and there is an $A_{np} \in \mathcal{A}_m$ disjoint from $\operatorname{spr}\mathcal{I}'_{np}$. Next, for $m < r \le n$ and $p \in U_n$ set

$$\mathcal{J}_{npr} = \{(\theta_{r\setminus m}(A_{np}\cap E), I\cap r\setminus m, 2w) : (E,I,w) \in \mathcal{I}_{npr}\}, \quad F_{npr} = \operatorname{spr}\mathcal{J}_{npr}.$$

By 394E, $\mathcal{J}_{npr} \subseteq \mathcal{C}_{kp}$, so $\nu_{kp}F_{npr} \le \operatorname{wt}\mathcal{J}_{npr} \le 2\operatorname{wt}\mathcal{I}_{npr}$. Note that F_{npr} is determined by coordinates in $r\setminus m$ and includes $A_{np}\cap\operatorname{spr}\mathcal{I}_{npr}$. Now if $m < j \le n$ and $p \in U_n$, $\tilde E_j \subseteq \bigcup_{m<r\le j} F_{npr}$. **P?** Otherwise, since both sets are determined by coordinates in $j\setminus m$, and since $A_{np} \in \mathcal{A}_m$, there is an $A \in \mathcal{A}_j$ with

$$A \subseteq A_{np}\cap\tilde E_j\setminus\bigcup_{m<r\le j} F_{npr} \subseteq A_{np}\cap\tilde E_j\setminus\bigcup_{m<r\le j}\operatorname{spr}\mathcal{I}_{npr}.$$

Since A is also disjoint from $\operatorname{spr}\mathcal{I}'_{np}$ and $A \subseteq \tilde E_j \subseteq \operatorname{spr}\mathcal{I}_{np}$, $A \subseteq \operatorname{spr}\mathcal{I}$, where

$$\begin{aligned}\mathcal{I} &= \mathcal{I}_{np}\setminus(\mathcal{I}'_{np}\cup\bigcup_{m<r\le j}\mathcal{I}_{npr})\\ &\subseteq \{(E,I,w) : (E,I,w) \in \mathcal{I}_{np},\ \|I\setminus j\| \ge \frac{1}{4}\|I\|\}.\end{aligned}$$

Since $\mathcal{I} \subseteq \mathcal{C}_{kp}$,

$$\begin{aligned}8 \le \nu_{kp}X = \nu_{kp}(\theta_{\mathbb{N}\setminus j}(A)) &= \nu_{kp}(\theta_{\mathbb{N}\setminus j}(A\cap\operatorname{spr}\mathcal{I})) \le 2\operatorname{wt}\mathcal{I}\\ &\le 4.\ \textbf{XQ}\end{aligned}$$

(394Fb)

(b) For $r > m$ we can find $F_r \in \mathfrak{B}$ such that

$\sum_{r=m+1}^{\infty} \nu_k F_r \le 4$,

$\tilde E_j \subseteq \bigcup_{m<r\le j} F_r$ for every $j > m$,

F_r is determined by coordinates in $r\setminus m$.

P If $n \ge r > m$, then, because $\mathfrak{B}_r$ is finite, there is a set $F_{nr} \in \mathfrak{B}_r$ such that $\{p : p \in U_n,\ F_{npr} = F_{nr}\}$ belongs to $\mathcal{F}$. Next, if $r > m$ there is an $F_r \in \mathfrak{B}_r$ such that $\{n : n \ge r,\ F_{nr} = F_r\}$ belongs to $\mathcal{F}$. Now

$$\nu_k F_r = \lim_{n\to\mathcal{F}} \nu_k F_{nr}$$

(because $\{n : \nu_k F_{nr} = \nu_k F_r\} \supseteq \{n : F_{nr} = F_r\} \in \mathcal{F}$)

$$= \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \nu_{kp} F_{nr} = \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \nu_{kp} F_{npr} \le 2 \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \operatorname{wt}\mathcal{I}_{npr}.$$

So, for $s > m$,

$$\sum_{r=m+1}^{s} \nu_k F_r \le 2 \sum_{r=m+1}^{s} \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \operatorname{wt}\mathcal{I}_{npr}$$

$$= 2 \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \sum_{r=m+1}^{s} \operatorname{wt}\mathcal{I}_{npr} \le 2 \lim_{n\to\mathcal{F}} \lim_{p\to\mathcal{F}} \operatorname{wt}\mathcal{I}_{np} \le 4.$$

As s is arbitrary, $\sum_{r=m+1}^{\infty} \nu_k F_r \le 4$.

If $n \ge j > m$, then we saw in (a) that $\tilde{E}_j \subseteq \bigcup_{m<r\le j} F_{npr}$ for every $p \in U_n$. Since there are many p such that $F_{nr} = F_{npr}$ whenever $m < r \le j$, $\tilde{E}_j \subseteq \bigcup_{m<r\le j} F_{nr}$. Now, given $j > m$, there are many n such that $F_{nr} = F_r$ whenever $m < r \le j$, so $\tilde{E}_j \subseteq \bigcup_{m<r\le j} F_r$.

Finally, take any $r > m$. Since F_{npr} is determined by coordinates in $r \setminus m$ whenever $n \ge r$ and $p \in U_n$, F_{nr} is determined by coordinates in $r \setminus m$ whenever $n \ge r$, and F_r also is determined by coordinates in $r \setminus m$. **Q**

(c) Let $r_0 \ge m$ be such that $\sum_{r=r_0+1}^{\infty} \nu_k F_r \le \eta$. Set $C = \bigcup_{m<r\le r_0} F_r$. Then C is determined by coordinates in $\mathbb{N} \setminus m$ and

$$\nu_k C \le \sum_{r=m+1}^{r_0} \nu_k F_r \le 4.$$

For any $i \in \mathbb{N}$, there is some $j > r_0$ such that $E_i \subseteq \tilde{E}_j$, in which case

$$E_i \setminus C \subseteq \bigcup_{r_0<r\le j} F_r$$

and

$$\nu_k(E_i \setminus C) \le \sum_{r=r_0+1}^{j} \nu_k F_r \le \eta,$$

as required.

394L Lemma Suppose that $k \in \mathbb{N}$, $\epsilon > 0$, $m \in \mathbb{N}$, $B \in \mathfrak{B}_m$ and that $\langle E_i \rangle_{i\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$. Then there are $n > m$ and $B' \in \mathfrak{B}_n$ such that $B' \subseteq B$, B' is $\frac{1}{2}\nu_k$-thin between m and n and $\limsup_{i\to\infty} \nu_k(E_i \cap B \setminus B') \le \epsilon$.

proof Set $\eta = \frac{\epsilon}{\#(\mathcal{A}_m)}$. (This is where we need to know that all the T_n are finite.) For those $A \in \mathcal{A}_m$ included in B define $C'_A \subseteq A$ as follows.

case 1 If there is some r such that $\nu_k(\theta_{\mathbb{N}\setminus m}(A \cap \bigcup_{i\le r} E_i)) \ge 2$, set $C'_A = A \setminus \bigcup_{i\le r} E_i$, so that $\frac{1}{2}\nu_k(\theta_{\mathbb{N}\setminus m}(A \setminus C'_A)) \ge 1$ and $E_i \cap A \setminus C'_A = \emptyset$ for $i > r$.

case 2 If $\nu_k(\theta_{\mathbb{N}\setminus m}(A \cap \bigcup_{i\le r} E_i)) < 2$ for every r, then by 394K, applied to the sequence $\langle \theta_{\mathbb{N}\setminus m}(A \cap E_i) \rangle_{i\in\mathbb{N}}$, we can find a $C \in \mathfrak{B}$, determined by coordinates in $\mathbb{N} \setminus m$, such that $\nu_k C \le 4$ and $\nu_k(\theta_{\mathbb{N}\setminus m}(A \cap E_i) \setminus C) \le \eta$ for every i. Set $C'_A = C \cap A$. Because C is determined by coordinates in $\mathbb{N} \setminus m$ and $A \in \mathcal{A}_m$, $\nu_k(\theta_{\mathbb{N}\setminus m}(C'_A)) = \nu_k C \le 4$. Also $E_i \cap A \setminus C'_A \subseteq \theta_{\mathbb{N}\setminus m}(A \cap E_i) \setminus C$ so $\nu_k(E_i \cap A \setminus C'_A) \le \eta$ for every i.

Set

$$B' = \bigcup\{C'_A : A \in \mathcal{A}_m,\ A \subseteq B\}.$$

Then $B' \in \mathfrak{B}$, $B' \subseteq B$ and

$$\limsup_{i\to\infty}\nu_k(E_i\cap B\setminus B')\le\sum_{A\in\mathcal{A}_m,A\subseteq B}\limsup_{i\to\infty}\nu_k(E_i\cap A\setminus C'_A)$$
$$\le\sum_{A\in\mathcal{A}_m,A\subseteq B}\eta\le\epsilon.$$

Let $n>m$ be such that $C'_A\in\mathfrak{B}_n$ whenever $A\in\mathcal{A}_m$ and $A\subseteq B$. Then B' is $\frac{1}{2}\nu_k$-thin between m and n. **P** Take any $A\in\mathcal{A}_m$ and set $\tilde{C}=\theta_{n\setminus m}(A\cap B')$. If $A\not\subseteq B$ then $A\cap B'$ and $\tilde{C}$ are empty and $\nu_k(X\setminus\tilde{C})\ge 8$ (394Ia). Otherwise, $A\cap B'=C'_A\in\mathfrak{B}_n$ so $\tilde{C}=\theta_{\mathbb{N}\setminus m}(C'_A)$ is disjoint from $\theta_{\mathbb{N}\setminus m}(A\setminus C'_A)$ (see the last remark in 394Cb). If C'_A was chosen as in case 1 above,

$$\nu_k(X\setminus\tilde{C})\ge\nu_k(\theta_{\mathbb{N}\setminus m}(A\setminus C'_A))\ge 2.$$

If C'_A was chosen as in case 2,

$$\nu_k(X\setminus\tilde{C})=\nu_k(X\setminus\theta_{\mathbb{N}\setminus m}(C'_A))\ge\nu_kX-\nu_k(\theta_{\mathbb{N}\setminus m}(C'_A))\ge 8-4.$$

So in all three cases we have $\frac{1}{2}\nu_k(X\setminus\tilde{C})\ge 1$, as required. **Q**

Thus we have an appropriate B'.

394M Theorem ν is exhaustive.

proof Let $\langle E_i\rangle_{i\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{B}$. Take any $k\in\mathbb{N}$ and $\epsilon>0$, and choose $\langle B_j\rangle_{j\in\mathbb{N}}$ and $\langle n_j\rangle_{j\in\mathbb{N}}$ inductively, as follows. $B_0=X$ and $n_0=0$. Given that $B_j\in\mathfrak{B}_{n_j}$, take $n_{j+1}>n_j$ and $B_{j+1}\in\mathfrak{B}_{n_{j+1}}$ such that $B_{j+1}\subseteq B_j$, B_{j+1} is $\frac{1}{2}\nu_{k+1}$-thin between n_j and n_{j+1}, and $\limsup_{i\to\infty}\nu_{k+1}(E_i\cap B_j\setminus B_{j+1})\le 2^{-j}\epsilon$ (394L). Continue. Note that $\limsup_{i\to\infty}\nu_{k+1}(E_i\setminus B_j)\le 2\epsilon$ for every j.

Let l be so large that $I=\{n_j:j<l\}$ has $\|I\|=N_k$. (This is where we need to know that $\lim_{l\to\infty}\|A\cap l\|=\infty$ for every infinite $A\subseteq\mathbb{N}$.) Set $B=B_{l-1}$. Then B is $\frac{1}{2}\nu_{k+1}$-thin along I (use 394Cc). By 394J, $\nu_kB\le 2^{-k}$.

Of course $\nu\le\nu_k\le\nu_{k+1}$ (394Cf). So

$$\limsup_{i\to\infty}\nu E_i\le\nu_kB+\limsup_{i\to\infty}\nu_{k+1}(E_i\setminus B)\le 2^{-k}+2\epsilon.$$

As k, ϵ and $\langle E_i\rangle_{i\in\mathbb{N}}$ are arbitrary, ν is exhaustive.

394N Remarks (a) Note that the whole construction is invariant under the action of the group $\prod_{n\in\mathbb{N}}G_n$ where G_n is the group of all permutations of T_n for each n. In particular, if we give each T_n a group structure and X the product group structure, then ν is translation-invariant.

(b) It follows that ν is strictly positive. **P** For each $n\in\mathbb{N}$, ν is constant on $\mathcal{A}_n$, so $\nu E\ge\nu X/\#(\mathcal{A}_n)>0$ for every non-empty $E\in\mathfrak{B}_n$. **Q**

(c) We can therefore form the metric completion $\widehat{\mathfrak{B}}$ of $\mathfrak{B}$, as in 392H, and $\widehat{\mathfrak{B}}$ will be a Maharam algebra, with a strictly positive Maharam submeasure $\hat{\nu}$ continuously extending ν (393H). Now $\widehat{\mathfrak{B}}$ is not measurable. **P?** Otherwise, let $\bar{\mu}$ be such that $(\widehat{\mathfrak{B}},\bar{\mu})$ is a probability algebra. Then $\bar{\mu}$ and $\hat{\nu}$ are strictly positive Maharam submeasures on $\widehat{\mathfrak{B}}$, so $\hat{\nu}$ is absolutely continuous with respect to $\bar{\mu}$ (393F). Let $n\ge 1$ be such that $\hat{\nu}b<8$ whenever $\bar{\mu}b\le 1/\#(T_n)$. Then there must be a $t\in T_n$ such that $\bar{\mu}Y_{nt}\le 1/\#(T_n)$; but $\hat{\nu}Y_{nt}=\nu Y_{nt}\ge 8$ (see the proof of 394Ib). **XQ**

In fact, $\widehat{\mathfrak{B}}$ is nowhere measurable (394Ya).

***394O Control measures** One of the original reasons for studying Maharam submeasures was their connexion with the following notion. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and U a Hausdorff linear topological space. (The idea is intended to apply, in particular, when $\mathfrak{A}$ is a σ-algebra of subsets of a set.) A function $\theta:\mathfrak{A}\to U$ is a **vector measure** if $\sum_{n=0}^{\infty}\theta a_n=\lim_{n\to\infty}\sum_{i=0}^{n}\theta a_i$ is defined in U and equal to $\theta(\sup_{n\in\mathbb{N}}a_n)$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. In this case, a non-negative countably additive functional $\mu:\mathfrak{A}\to[0,\infty[$ is a **control measure** for θ if $\theta a=0$ whenever $\mu a=0$.

***394P Example** There are a metrizable linear topological space U and a vector measure $\theta:\Sigma\to U$, where Σ is a σ-algebra of sets, such that θ has no control measure.

proof As in 394Nc, let $\widehat{\mathfrak{B}}$ be the metric completion of $\mathfrak{B}$, and $\hat{\nu}$ the continuous extension of ν to $\widehat{\mathfrak{B}}$. Give $L^0 = L^0(\widehat{\mathfrak{B}})$ the topology defined from $\hat{\nu}$ as in 393K, so that L^0 is a metrizable linear topological space. By 314M, we can identify $\widehat{\mathfrak{B}}$ with a quotient algebra $\Sigma/\mathcal{N}$ where Σ is a σ-algebra of subsets of a set Ω and $\mathcal{N}$ is a σ-ideal in Σ. Set $\theta E = \chi E^\bullet \in L^0$ for $E \in \Sigma$. Then θ is a vector measure. **P** If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ with union E, set $F_n = \bigcup_{i\le n} E_i$, so that $\chi F_n^\bullet = \sum_{i=0}^n \chi E_i^\bullet$ for each n. We have $\hat{\nu}(E^\bullet \setminus F_n^\bullet) \to 0$, so that

$$\tau(\theta E - \theta F_n) = \tau(\chi E^\bullet - \chi F_n^\bullet)) = \min(1, \hat{\nu}(E^\bullet \setminus F_n^\bullet)) \to 0,$$

where τ is the functional of the proof of 393K, and $\theta E = \sum_{i=0}^\infty \theta E_i$ in L^0. **Q**

If μ is a totally finite measure with domain Σ, set

$$\lambda a = \inf\{\mu E : E \in \Sigma,\ E^\bullet = a\}$$

for every $a \in \widehat{\mathfrak{B}}$. Note that the infimum is always attained. **P** If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ such that $E_n^\bullet = a$ for every $n \in \mathbb{N}$ and $\lambda a = \lim_{n\to\infty} \mu E_n$, set $E = \bigcap_{n\in\mathbb{N}} E_n$; then $E^\bullet = a$ and $\mu E = \lambda a$. **Q** Next, λ is countably additive. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\widehat{\mathfrak{B}}$ with supremum a, take $E_n \in \Sigma$ such that $E_n^\bullet = a_n$ and $\mu E_n = \lambda a_n$ for each n, and $E \in \Sigma$ such that $E^\bullet = a$ and $\mu E = \lambda a$. Set $F_n = E \cap E_n \setminus \bigcup_{i<n} E_i$ for each n, and $F = \bigcup_{n\in\mathbb{N}} F_n$. Then $F_n^\bullet = a_n$ and $F_n \subseteq E_n$, so $\mu F_n = \lambda a_n$ for each n; similarly, $F^\bullet = a$ and $F \subseteq E$, so $\mu F = \lambda a$. Also $\langle F_n\rangle_{n\in\mathbb{N}}$ is disjoint and has union F. Accordingly

$$\lambda a = \mu F = \textstyle\sum_{n=0}^\infty \mu F_n = \sum_{n=0}^\infty \lambda a_n. \ \textbf{Q}$$

Since $\widehat{\mathfrak{B}}$ is not a measurable algebra, λ cannot be strictly positive, and there is a non-zero $a \in \widehat{\mathfrak{B}}$ such that $\lambda a = 0$. Let $E \in \Sigma$ be such that $E^\bullet = a$ and $\mu E = 0$; then $\theta E = \chi a \ne 0$. So μ is not a control measure for θ.

***394Q** This is not a book about vector measures, but having gone so far I ought to note that the generality of the phrase 'metrizable linear topological space' in 394P is essential. If we look only at normed spaces the situation is very different.

Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, U a normed space and $\theta : \mathfrak{A} \to U$ a vector measure. Then θ has a control measure.

proof (a) Since U can certainly be embedded in a Banach space $\hat{U}$ (3A5Jb), and as θ will still be a vector measure when regarded as a map from $\mathfrak{A}$ to $\hat{U}$, we may assume from the beginning that U itself is complete.

(b) θ is bounded (that is, $\sup_{a\in\mathfrak{A}} \|\theta a\|$ is finite). **P?** (Cf. 326M.) Suppose, if possible, otherwise. Choose $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $a_0 = 1$. Given that $\sup_{a\subseteq a_n} \|\theta a\| = \infty$, choose $b \subseteq a_n$ such that $\|\theta b\| \ge \|\theta a_n\| + 1$. Then $\|\theta(a_n \setminus b)\| \ge 1$. Also

$$\sup_{a\subseteq a_n} \|\theta a\| \le \sup_{a\subseteq a_n} \|\theta(a \cap b)\| + \|\theta(a \setminus b)\|,$$

so at least one of $\sup_{a\subseteq b} \|\theta a\|$, $\sup_{a\subseteq a_n\setminus b} \|\theta a\|$ must be infinite. We may therefore take a_{n+1} to be either b or $a_n \setminus b$ and such that $\sup_{a\subseteq a_{n+1}} \|\theta a\| = \infty$. Observe that in either case we shall have $\|\theta(a_n \setminus a_{n+1})\| \ge 1$. Continue.

At the end of the induction we shall have a disjoint sequence $\langle a_n \setminus a_{n+1}\rangle_{n\in\mathbb{N}}$ such that $\|\theta(a_n \setminus a_{n+1})\| \ge 1$ for every n, so that $\sum_{n=0}^\infty \theta(a_n \setminus a_{n+1})$ cannot be defined in U; which is impossible. **XQ**

(c) Accordingly we have a bounded linear operator $T : L^\infty \to U$, where $L^\infty = L^\infty(\mathfrak{A})$, such that $T\chi = \theta$ (363Ea).

Now the key to the proof is the following fact: if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $(L^\infty)^+$, $\langle Tu_n\rangle_{n\in\mathbb{N}} \to 0$ in U. **P** Let γ be such that $u_n \le \gamma\chi 1$ for every n. Let $\epsilon > 0$, and let k be the integer part of γ/ϵ. For $n \in \mathbb{N}$, $i \le k$ set $a_{ni} = [\![u_n > \epsilon(i+1)]\!]$; then $\langle a_{ni}\rangle_{n\in\mathbb{N}}$ is disjoint for each i, and if we set $v_n = \epsilon\sum_{i=0}^k \chi a_{ni}$, we get $v_n \le u_n \le v_n + \epsilon\chi 1$, so $\|u_n - v_n\|_\infty \le \epsilon$.

Because $\langle a_{ni}\rangle_{n\in\mathbb{N}}$ is disjoint, $\sum_{n=0}^\infty \theta a_{ni}$ is defined in U, and $\langle \theta a_{ni}\rangle_{n\in\mathbb{N}} \to 0$, for each $i \le k$. Consequently

$$Tv_n = \epsilon\textstyle\sum_{i=0}^k \theta a_{ni} \to 0$$

as $n \to \infty$. But

$$\|Tu_n - Tv_n\| \le \|T\|\|u_n - v_n\|_\infty \le \epsilon\|T\|$$

for each n, so $\limsup_{n\to\infty}\|Tu_n\| \le \epsilon\|T\|$. As ϵ is arbitrary, $\lim_{n\to\infty}\|Tu_n\| = 0$. **Q**

(d) Consider the adjoint operator $T' : U^* \to (L^\infty)^*$. Recall that L^∞ is an M-space (363Ba) so that its dual is an L-space (356N). Write

$$A = \{T'g : g \in U^*,\, \|g\| \le 1\} \subseteq (L^\infty)^* = (L^\infty)^\sim.$$

If $u \in L^\infty$, then

$$\sup_{f\in A}|f(u)| = \sup_{\|g\|\le 1}|(T^*g)(u)| = \sup_{\|g\|\le 1}|g(Tu)| = \|Tu\|.$$

Now A is uniformly integrable. **P** I use the criterion of 356O. Of course $\|f\| \le \|T'\|$ for every $f \in A$, so A is norm-bounded. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded disjoint sequence in $(L^\infty)^+$, then

$$\sup_{f\in A}|f(u_n)| = \|Tu_n\| \to 0$$

as $n \to \infty$. So A is uniformly integrable. **Q**

(e) Next, $A \subseteq (L^\infty)^\sim_c$. **P** If $f \in A$, it is of the form $T'g$ for some $g \in U^*$, that is,

$$f(\chi a) = (T'g)(\chi a) = gT(\chi a) = g(\theta a)$$

for every $a \in \mathfrak{A}$. If now $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a,

$$f(\chi a) = g(\theta(\sup_{n\in\mathbb{N}} a_n)) = g(\sum_{n=0}^\infty \theta a_n) = \sum_{n=0}^\infty g(\theta a_n) = \sum_{n=0}^\infty f(\chi a_n).$$

So $f\chi$ is countably additive. By 363K, $f \in (L^\infty)^\sim_c$. **Q**

(f) Because A is uniformly integrable, there is for each $m \in \mathbb{N}$ an $f_m \ge 0$ in $(L^\infty)^*$ such that $\|(|f| - f_m)^+\| \le 2^{-m}$ for every $f \in A$; moreover, we can suppose that f_m is of the form $\sup_{i\le k_m}|f_{mi}|$ where every f_{mi} belongs to A (354R(b-iii)), so that $f_m \in (L^\infty)^\sim_c$ and $\mu_m = f_m\chi$ is countably additive. Set

$$\mu = \sum_{m=0}^\infty \frac{1}{2^m(1+\mu_m 1)}\mu_m;$$

then $\mu : \mathfrak{A} \to [0,\infty[$ is a non-negative countably additive functional.

Now μ is a control measure for θ. **P** If $\mu a = 0$, then $\mu_m a = 0$, that is, $f_m(\chi a) = 0$, for every $m \in \mathbb{N}$. But this means that if $g \in U^*$ and $\|g\| \le 1$,

$$|g(\theta a)| = |(T'g)(\chi a)| \le f_m(\chi a) + \|(|T'g| - f_m)^+\| \le 2^{-m}$$

for every m, by the choice of f_m; so that $g(\theta a) = 0$. As g is arbitrary, $\theta a = 0$; as a is arbitrary, μ is a control measure for θ. **Q**

394X Basic exercises (a) Show that the metric completion $\widehat{\mathfrak{B}}$ of $\mathfrak{B}$, as defined in 394N, always has many involutions (definition: 382O).

394Y Further exercises (a)(i) Show that if $r \in \mathbb{N}$, $k \le p$ and $E \in \mathfrak{B}_{r+1}$ are such that $\nu_{kp}E < c_k$, then $\nu_{kp}(\theta_r(E)) \le \frac{32}{c_k}\nu_{kp}E$. (ii) Show that if $E \in \mathcal{B}_r$ then $\nu(E \cap Y_{rt}) \ge \min(8, \frac{1}{4}\nu E)$ for every $t \in T_r$. (iii) Let $\widehat{\mathfrak{B}}$ be the metric completion of $\mathfrak{B}$ and $\hat\nu$ the continuous extension of ν to $\widehat{\mathfrak{B}}$. Show that for every $a \in \mathfrak{B}$ and $n \in \mathbb{N}$ there is a disjoint family $\langle c_i\rangle_{i\le n}$ such that $c_i \subseteq a$ and $\hat\nu c_i \ge \min(7, \frac{1}{5}\hat\nu a)$ for every $i \le n$. (iv) Show that the only countably additive real-valued functional on $\widehat{\mathfrak{B}}$ is the zero functional. (v) Show that $\widehat{\mathfrak{B}}$ is nowhere measurable. (vi) Show that if ν' is a uniformly exhaustive submeasure on $\mathfrak{B}$ which is absolutely continuous with respect to ν, then $\nu' = 0$.

394Z Problems Suppose that $\|\ \|$, $\langle T_n\rangle_{n\in\mathbb{N}}$, $\mathfrak{B}$, $\langle\alpha_k\rangle_{k\in\mathbb{N}}$ and $\langle N_k\rangle_{k\in\mathbb{N}}$ satisfy the conditions of 394Ba-394Bb. Let ν be the exhaustive submeasure on $\mathfrak{B}$ constructed by the method of 394B and 394H, and $\widehat{\mathfrak{B}}$ the corresponding Maharam algebra.

(a) Does $\widehat{\mathfrak{B}}$ have an order-closed subalgebra isomorphic to the measure algebra of Lebesgue measure? In particular, if we take $\mathfrak{C} \subseteq \mathfrak{B}$ to be the algebra of sets generated by sets of the form $\{x : x \in X,\, x(n) = 0\}$ for $n \in \mathbb{N}$, is $\nu\restriction\mathfrak{C}$ uniformly exhaustive?

(b) Suppose that instead of taking large sets T_n, we simply set $T_n = \{0, 1\}$ for every n, but otherwise used the same construction. Should we then find that ν was uniformly exhaustive? (This might be relevant to (a) above.)

(c) Is the Boolean algebra $\widehat{\mathfrak{B}}$ homogeneous?

394 Notes and comments 'Maharam's problem', or the 'control measure problem', was for fifty years one of the most vexing questions in abstract measure theory. To begin with, there were reasonable hopes that there was a positive answer – in the language of this book, that every Maharam algebra was a measurable algebra. If this had been the case, there would have been consequences all over the theories of topological Boolean algebras, topological Riesz spaces and vector measures. In the 1970s, it began to seem too much to ask for. In 1983 the Kalton-Roberts theorem gave new life to the conjecture for a moment, but Roberts 93 demonstrated a major obstacle, which Talagrand (building on some further ideas of I.Farah) eventually developed into the construction above. The ideas which for a generation were collected together by their association with the control measure problem no longer have this as a unifying principle, and (as after any successful revolution) are now more naturally grouped in other ways. There is a relic of this era in 394P.

Now that we know for sure that there are non-measurable Maharam algebras, it becomes possible to ask questions about their structure. Frustratingly, practically none of these questions has yet been answered even for the examples constructed by Talagrand's original method, in which $\|I\| = \#(I)$ for every I. (Of course this allows variations in the parameters $\langle T_n \rangle_{n\in\mathbb{N}}$, $\langle \alpha_k \rangle_{k\in\mathbb{N}}$ and $\langle N_k \rangle_{k\in\mathbb{N}}$ and the filter $\mathcal{F}$, and there is every reason to suppose that $\mathfrak{c}$ non-isomorphic examples can be constructed by the formulae set out above.) I will return briefly to such questions in Volumes 4 and 5, as I come to further properties of measure algebras which can be interpreted in Maharam algebras. In particular, following Perović & Veličković 18, I will show in §539 how different PV norms can give rise to distinguishable Maharam algebras.

Version of 15.6.08

395 Kawada's theorem

I now describe a completely different characterization of (homogeneous) measurable algebras, based on the special nature of their automorphism groups. The argument depends on the notion of 'non-paradoxical' group of automorphisms; this is an idea of great importance in other contexts, and I therefore aim at a fairly thorough development, with proofs which are adaptable to other circumstances.

395A Definitions Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For a, $b \in \mathfrak{A}$ I will say that an isomorphism $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ between the corresponding principal ideals belongs to the **full local semigroup generated by** G if there are a partition of unity $\langle a_i \rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle \pi_i \rangle_{i\in I}$ in G such that $\phi c = \pi_i c$ whenever $i \in I$ and $c \subseteq a_i$. If such an isomorphism exists I will say that a and b are **G-τ-equidecomposable**.

I will write $a \preccurlyeq^\tau_G b$ to mean that there is a $b' \subseteq b$ such that a and b' are G-τ-equidecomposable.

For any function f with domain $\mathfrak{A}$, I will say that f is **G-invariant** if $f(\pi a) = f(a)$ whenever $a \in \mathfrak{A}$ and $\pi \in G$.

395B The notion of 'full local semigroup' is of course an extension of the idea of 'full subgroup' (381Be; see also 381Yb). The word 'semigroup' is justified by (c) of the following lemma, and the word 'full' by (e).

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_τ for the full local semigroup generated by G.

(a) Suppose that a, $b \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism. Then the following are equiveridical:

(i) $\phi \in G^*_\tau$;

(ii) for every non-zero $c_0 \subseteq a$ there are a non-zero $c_1 \subseteq c_0$ and a $\pi \in G$ such that $\phi c = \pi c$ for every $c \subseteq c_1$;

(iii) for every non-zero $c_0 \subseteq a$ there are a non-zero $c_1 \subseteq c_0$ and a $\psi \in G^*_\tau$ such that $\phi c = \psi c$ for every $c \subseteq c_1$.

(b) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_τ, then $\phi^{-1} : \mathfrak{A}_b \to \mathfrak{A}_a$ also belongs to G^*_τ.

(c) Suppose that a, b, a', $b' \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_{a'}$, $\psi : \mathfrak{A}_b \to \mathfrak{A}_{b'}$ belong to G^*_τ. Then $\psi\phi \in G^*_\tau$; its domain is $\mathfrak{A}_c$ where $c = \phi^{-1}(b \cap a')$, and its set of values is $\mathfrak{A}_{c'}$ where $c' = \psi(b \cap a')$.

(d) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_τ, then $\phi{\restriction}\mathfrak{A}_c \in G^*_\tau$ for any $c \subseteq a$.

(e) Suppose that a, $b \in \mathfrak{A}$ and that $\psi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism such that there are a partition of unity $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}_a$ and a family $\langle \phi_i \rangle_{i \in I}$ in G^*_τ such that $\psi c = \phi_i c$ whenever $i \in I$ and $c \subseteq a_i$. Then $\psi \in G^*_\tau$.

proof (a) (Compare 381I.)

(i)⇒(iii) is trivial, since of course $G \subseteq G^*_\tau$.

(iii)⇒(ii) Suppose that ϕ satisfies (iii), and that $0 \ne c_0 \subseteq a$. Then we can find a $\psi \in G^*_\tau$ and a non-zero $c_1 \subseteq c_0$ such that ϕ agrees with ψ on $\mathfrak{A}_{c_1}$. Suppose that $\operatorname{dom}\psi = \mathfrak{A}_d$, where necessarily $d \supseteq c_1$. Then there are a partition of unity $\langle d_i \rangle_{i \in I}$ in $\mathfrak{A}_d$ and a family $\langle \pi_i \rangle_{i \in I}$ such that $\psi c = \pi_i c$ whenever $c \subseteq d_i$. There is some $i \in I$ such that $c_2 = c_1 \cap d_i \ne 0$, and we see that $\phi c = \psi c = \pi_i c$ for every $c \subseteq c_2$. As c_0 is arbitrary, ϕ satisfies (ii).

(ii)⇒(i) If ϕ satisfies (ii), set

$$D = \{d : d \subseteq a, \text{ there is some } \pi \in G \text{ such that } \pi c = \phi c \text{ for every } c \subseteq d\}.$$

The hypothesis is that D is order-dense in $\mathfrak{A}$, so there is a partition of unity $\langle a_i \rangle_{i \in I}$ of $\mathfrak{A}_a$ lying within D (313K); for each $i \in I$ take $\pi_i \in G$ such that $\phi c = \pi_i c$ for $c \subseteq a_i$; then $\langle a_i \rangle_{i \in I}$ and $\langle \pi_i \rangle_{i \in I}$ witness that $\phi \in G^*_\tau$.

(b) This is elementary; if $\langle a_i \rangle_{i \in I}$, $\langle \pi_i \rangle_{i \in I}$ witness that $\phi \in G^*_\tau$, then $\langle \phi a_i \rangle_{i \in I} = \langle \pi_i a_i \rangle_{i \in I}$, $\langle \pi_i^{-1} \rangle_{i \in I}$ witness that $\phi^{-1} \in G^*_\tau$.

(c) I ought to start by computing the domain of $\psi\phi$:

$$\begin{aligned} d \in \operatorname{dom}(\psi\phi) &\iff d \in \operatorname{dom}\phi,\ \phi d \in \operatorname{dom}\psi \\ &\iff d \subseteq a,\ \phi d \subseteq b \iff d \subseteq \phi^{-1}(a' \cap b) = c. \end{aligned}$$

So the domain of $\psi\phi$ is indeed $\mathfrak{A}_c$; now $\phi{\restriction}\mathfrak{A}_c$ is an isomorphism between $\mathfrak{A}_c$ and $\mathfrak{A}_{\phi c}$, where $\phi c = a' \cap b \in \mathfrak{A}_b$, so $\psi\phi$ is an isomorphism between $\mathfrak{A}_c$ and $\mathfrak{A}_{\psi\phi c} = \mathfrak{A}_{c'}$. Let $\langle a_i \rangle_{i \in I}$, $\langle b_j \rangle_{j \in J}$ be partitions of unity in $\mathfrak{A}_a$, $\mathfrak{A}_b$ respectively, and $\langle \pi_i \rangle_{i \in I}$, $\langle \theta_j \rangle_{j \in J}$ families in G such that $\phi d = \pi_i d$ for $d \subseteq a_i$, $\psi e = \theta_j e$ for $e \subseteq b_j$. Set $c_{ij} = a_i \cap \pi_i^{-1} b_j$; then $\langle c_{ij} \rangle_{i \in I, j \in J}$ is a partition of unity in $\mathfrak{A}_c$ and $\psi\phi d = \theta_j \pi_i d$ for $d \subseteq c_{ij}$, so $\psi\phi \in G^*_\tau$ (because all the $\theta_j \pi_i$ belong to G).

(d) This is nearly trivial; use the definition of G^*_τ or the criteria of (a), or apply (c) with the identity map on $\mathfrak{A}_c$ as one of the factors.

(e) This follows at once from the criterion (a-iii) above, or otherwise.

395C Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_τ for the full local semigroup generated by G.

(a) For a, $b \in \mathfrak{A}$, $a \preccurlyeq^\tau_G b$ iff there is a $\phi \in G^*_\tau$ such that $a \in \operatorname{dom}\phi$ and $\phi a \subseteq b$.

(b)(i) $\preccurlyeq^\tau_G$ is transitive and reflexive;

(ii) if $a \preccurlyeq^\tau_G b$ and $b \preccurlyeq^\tau_G a$ then a and b are G-τ-equidecomposable.

(c) G-τ-equidecomposability is an equivalence relation on $\mathfrak{A}$.

(d) If $\langle a_i \rangle_{i \in I}$ and $\langle b_i \rangle_{i \in I}$ are families in $\mathfrak{A}$, of which $\langle b_i \rangle_{i \in I}$ is disjoint, and $a_i \preccurlyeq^\tau_G b_i$ for every $i \in I$, then $\sup_{i \in I} a_i \preccurlyeq^\tau_G \sup_{i \in I} b_i$.

proof (a) This is immediate from the definition of 'G-τ-equidecomposable' and 395Bd.

(b)(i) $a \preccurlyeq^\tau_G a$ because the identity homomorphism belongs to G^*_τ. If $a \preccurlyeq^\tau_G b \preccurlyeq^\tau_G c$ there are ϕ, $\psi \in G^*_\tau$ such that $\phi a \subseteq b$, $\psi b \subseteq c$ so that $\psi\phi a \subseteq c$; as $\psi\phi \in G^*_\tau$ (395Bc), $a \preccurlyeq^\tau_G c$.

(ii) (This is of course a Schröder-Bernstein theorem, and the proof is the usual one.) Take ϕ, $\psi \in G^*_\tau$ such that $\phi a \subseteq b$, $\psi b \subseteq a$. Set $a_0 = a$, $b_0 = b$, $a_{n+1} = \psi b_n$ and $b_{n+1} = \phi a_n$ for each n. Then $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ are non-increasing sequences; set $a_\infty = \inf_{n\in\mathbb{N}} a_n$, $b_\infty = \inf_{n\in\mathbb{N}} b_n$. For each n,

$$\phi{\restriction}\mathfrak{A}_{a_{2n}\setminus a_{2n+1}} : \mathfrak{A}_{a_{2n}\setminus a_{2n+1}} \to \mathfrak{A}_{b_{2n+1}\setminus b_{2n+2}},$$

$$\psi{\restriction}\mathfrak{A}_{b_{2n}\setminus b_{2n+1}} : \mathfrak{A}_{b_{2n}\setminus b_{2n+1}} \to \mathfrak{A}_{a_{2n+1}\setminus a_{2n+2}}$$

are isomorphisms, while

$$\phi{\restriction}\mathfrak{A}_{a_\infty} : \mathfrak{A}_{a_\infty} \to \mathfrak{A}_{b_\infty}$$

is another. So we can define an isomorphism $\theta : \mathfrak{A}_a \to \mathfrak{A}_b$ by setting

$$\begin{aligned}\theta c &= \phi c \text{ if } c \subseteq a_\infty \cup \sup_{n\in\mathbb{N}} a_{2n} \setminus a_{2n+1},\\ &= \psi^{-1} c \text{ if } c \subseteq \sup_{n\in\mathbb{N}} a_{2n+1} \setminus a_{2n+2}.\end{aligned}$$

By 395Be, $\theta \in G^*_\tau$, so a and b are G-τ-equidecomposable.

(c) This is easy to prove directly from the results in 395B, but also follows at once from (b); any transitive reflexive relation gives rise to an equivalence relation.

(d) We may suppose that I is well-ordered by a relation $\le$. For $i \in I$, set $a'_i = a_i \setminus \sup_{j<i} a_j$. Set $a = \sup_{i\in I} a_i = \sup_{i\in I} a'_i$, $b = \sup_{i\in I} b_i$. For each $i \in I$, we have a $b'_i \subseteq b_i$ and a $\phi_i \in G^*_\tau$ such that $\phi_i a'_i = b'_i$. Set $b' = \sup_{i\in I} b'_i \subseteq b$; then we have an isomorphism $\psi : \mathfrak{A}_a \to \mathfrak{A}_{b'}$ defined by setting $\psi d = \phi_i d$ if $d \subseteq a'_i$, and $\psi \in G^*_\tau$, so a and b' are G-τ-equidecomposable and $a \preccurlyeq^\tau_G b$.

395D Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is an $a \ne 1$ such that a is G-τ-equidecomposable with 1;

(ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-τ-equidecomposable;

(iii) there are non-zero G-τ-equidecomposable a, b, $c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) there are G-τ-equidecomposable a, $b \in \mathfrak{A}$ such that $a \subset b$.

proof Write G^*_τ for the full local semigroup generated by G.

(i)⇒(ii) Assume (i). There is a $\phi \in G^*_\tau$ such that $\phi 1 = a$. Set $a_n = \phi^n(1 \setminus a)$ for each $n \in \mathbb{N}$; because every ϕ^n belongs to G^*_τ (counting ϕ^0 as the identity operator on $\mathfrak{A}$, and using 395Bc), with $\operatorname{dom}\phi^n = \mathfrak{A}$, a_n is G-τ-equidecomposable with $a_0 = 1 \setminus a$ for every n. Also $a_n = \phi^n 1 \setminus \phi^{n+1} 1$ for each n, while $\langle \phi^n 1\rangle_{n\in\mathbb{N}}$ is non-increasing, so $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint. Thus (ii) is true.

(ii)⇒(iii) Assume (ii). Set $a = \sup_{n\in\mathbb{N}} a_{2n}$, $b = \sup_{n\in\mathbb{N}} a_{2n+1}$, $c = \sup_{n\in\mathbb{N}} a_n$, so that $a \cap b = 0$ and $a \cup b = c$. For each n we have a $\phi_n \in G^*_\tau$ such that $\phi_n a_0 = a_n$. So if we set

$$\psi d = \sup\nolimits_{n\in\mathbb{N}} \phi_n \phi_{2n}^{-1}(d \cap a_{2n}) \text{ for } d \subseteq a,$$

ψ belongs to G^*_τ (using 395B) and witnesses that a and c are G-τ-equidecomposable. Similarly, b and c are G-τ-equidecomposable, so (iii) is true.

(iii)⇒(iv) is trivial.

(iv)⇒(i) Take $\phi \in G^*_\tau$ such that $\phi b = a$. Set

$$\psi d = \phi(d \cap b) \cup (d \setminus b)$$

for $d \in \mathfrak{A}$; then $\psi \in G^*_\tau$ witnesses that 1 is G-τ-equidecomposable with $a \cup (1 \setminus b) \ne 1$.

395E Definition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. I will say that G is **fully non-paradoxical** if the statements of 395D are false; that is, if one of the following equiveridical statements is true:

(i) if a is G-τ-equidecomposable with 1 then $a = 1$;

(ii) there is no disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-τ-equidecomposable;

(iii) there are no non-zero G-τ-equidecomposable a, b, $c\in\mathfrak{A}$ such that $a\cap b=0$ and $a\cup b\subseteq c$;

(iv) if $a\subseteq b\in\mathfrak{A}$ and a, b are G-τ-equidecomposable then $a=b$.

Note that if G is fully non-paradoxical, and H is a subgroup of G, then H also is fully non-paradoxical, because if $a\preccurlyeq^\tau_H b$ then $a\preccurlyeq^\tau_G b$, so that a and b are G-τ-equidecomposable whenever they are H-τ-equidecomposable.

395F Proposition Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and $G=\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ the group of all measure-preserving automorphisms of $\mathfrak{A}$. Then G is fully non-paradoxical.

proof If $\phi:\mathfrak{A}\to\mathfrak{A}_a$ belongs to the full local semigroup generated by G, then we have a partition of unity $\langle a_i\rangle_{i\in I}$ and a family $\langle\pi_i\rangle_{i\in I}$ in G such that $\phi a_i=\pi_i a_i$ for every i; but this means that

$$\bar\mu a=\sum_{i\in I}\bar\mu\phi_i a_i=\sum_{i\in I}\bar\mu\pi_i a_i=\sum_{i\in I}\bar\mu a_i=\bar\mu 1.$$

As $\bar\mu 1<\infty$, we can conclude that $a=1$, so that G satisfies the condition (i) of 395E.

395G The fixed-point subalgebra of a group Let $\mathfrak{A}$ be a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$.

(a) By the **fixed-point subalgebra** of G I mean

$$\mathfrak{C}=\{c:c\in\mathfrak{A},\ \pi c=c\text{ for every }\pi\in G\}.$$

(I looked briefly at this construction in 333R, and in the special case of a group generated by a single element it appeared at various points in Chapter 38.) This is a subalgebra of $\mathfrak{A}$, and is order-closed, because every $\pi\in G$ is order-continuous.

(b) Now suppose that $\mathfrak{A}$ is Dedekind complete. In this case $\mathfrak{C}$ is Dedekind complete (314Ea), and we have, for any $a\in\mathfrak{A}$, an upper envelope $\operatorname{upr}(a,\mathfrak{C})$ of $\mathfrak{C}$, defined by setting

$$\operatorname{upr}(a,\mathfrak{C})=\inf\{c:a\subseteq c\in\mathfrak{C}\}$$

(313S). Now $\operatorname{upr}(a,\mathfrak{C})=\sup\{\pi a:\pi\in G\}$. **P** Set $c_1=\operatorname{upr}(a,\mathfrak{C})$, $c_2=\sup\{\pi a:\pi\in G\}$. (i) Because $a\subseteq c_1\in\mathfrak{C}$, $\pi a\subseteq\pi c_1=c_1$ for every $\pi\in G$, and $c_2\subseteq c_1$. (ii) For any $\phi\in G$,

$$\phi c_2=\sup_{\pi\in G}\phi\pi a=\sup_{\pi\in G}\pi a=c_2$$

because $G=\{\phi\pi:\pi\in G\}$. So $c_2\in\mathfrak{C}$; since also $a\subseteq c_2$, $c_1\subseteq c_2$, and $c_1=c_2$, as claimed. **Q**

(c) Again supposing that $\mathfrak{A}$ is Dedekind complete, write G^*_τ for the full local semigroup generated by G. Then $\phi(a\cap c)=\phi a\cap c$ whenever $\phi\in G^*_\tau$, $a\in\operatorname{dom}\phi$ and $c\in\mathfrak{C}$. **P** We have $\phi a=\sup_{i\in I}\pi_i a_i$, where $a=\sup_{i\in I}a_i$ and $\pi_i\in G$ for every i. Now

$$\phi(a\cap c)=\sup_{i\in I}\pi_i(a_i\cap c)=\sup_{i\in I}\pi_i a_i\cap c=\phi a\cap c.\ \mathbf{Q}$$

Consequently $\operatorname{upr}(\phi a,\mathfrak{C})=\operatorname{upr}(a,\mathfrak{C})$ whenever $\phi\in G^*_\tau$ and $a\in\operatorname{dom}\phi$. **P** For $c\in\mathfrak{C}$,

$$a\subseteq c\iff a\cap c=a\iff\phi(a\cap c)=\phi a\iff\phi a\cap c=\phi a\iff\phi a\subseteq c.\ \mathbf{Q}$$

It follows that $\operatorname{upr}(a,\mathfrak{C})\subseteq\operatorname{upr}(b,\mathfrak{C})$ whenever $a\preccurlyeq^\tau_G b$.

(d) Still supposing that $\mathfrak{A}$ is Dedekind complete, we also find that if $a\preccurlyeq^\tau_G b$ and $c\in\mathfrak{C}$ then $a\cap c\preccurlyeq^\tau_G b\cap c$. **P** There is a $\phi\in G^*_\tau$ such that $\phi a\subseteq b$; now $\phi(a\cap c)=\phi a\cap c\subseteq b\cap c$. **Q** Hence, or otherwise, $a\cap c$ and $b\cap c$ are G-τ-equidecomposable whenever a and b are G-τ-equidecomposable and $c\in\mathfrak{C}$.

(e) By analogy with the notion of 'ergodic automorphism', I will say that G is **ergodic** if $\sup_{\pi\in G}\pi a=1$ for every non-zero $a\in\mathfrak{A}$. Thus an automorphism π is ergodic in the sense of 372Oa iff the group $\{\pi^n:n\in\mathbb{Z}\}$ it generates is ergodic (372Pb).

(f) If G is ergodic, then $\mathfrak{C}=\{0,1\}$. (If $c\in\mathfrak{C}\setminus\{0\}$, then $1=\sup_{\pi\in G}\pi c=c$.) If $\mathfrak{A}$ is Dedekind complete and $\mathfrak{C}=\{0,1\}$ then G is ergodic. (If $a\in\mathfrak{A}\setminus\{0\}$, then $1=\operatorname{upr}(a,\mathfrak{C})=\sup_{\pi\in G}\pi a$, by (b) above.) (Cf. 392Sa, 392Sc.)

395H I now embark on a series of lemmas leading to the main theorem (395N).

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $\mathfrak{C}$ for the fixed-point subalgebra of G. Take any a, $b\in\mathfrak{A}$. Set $c_0=\sup\{c:c\in\mathfrak{C},\,a\cap c\preccurlyeq_G^\tau b\}$; then $a\cap c_0\preccurlyeq_G^\tau b$ and $b\setminus c_0\preccurlyeq_G^\tau a$.

proof Enumerate G as $\langle\pi_\xi\rangle_{\xi<\kappa}$, where $\kappa=\#(G)$. Define $\langle a_\xi\rangle_{\xi<\kappa}$, $\langle b_\xi\rangle_{\xi<\kappa}$ inductively, setting

$$a_\xi=(a\setminus\sup_{\eta<\xi}a_\eta)\cap\pi_\xi^{-1}(b\setminus\sup_{\eta<\xi}b_\eta),\quad b_\xi=\pi_\xi a_\xi.$$

Then $\langle a_\xi\rangle_{\xi<\kappa}$ is a disjoint family in $\mathfrak{A}_a$ and $\langle b_\xi\rangle_{\xi<\kappa}$ is a disjoint family in $\mathfrak{A}_b$, and $\sup_{\xi<\kappa}a_\xi$ is G-τ-equidecomposable with $\sup_{\xi<\kappa}b_\xi$. Set $a'=a\setminus\sup_{\xi<\kappa}a_\xi$, $b'=b\setminus\sup_{\xi<\kappa}b_\xi$,

$$\tilde c_0=1\setminus\operatorname{upr}(a',\mathfrak{C})=\sup\{c:c\in\mathfrak{C},\,c\cap a'=0\}.$$

Then

$$a\cap\tilde c_0\subseteq\sup_{\xi<\kappa}a_\xi\preccurlyeq_G^\tau b,$$

so $\tilde c_0\subseteq c_0$.

Now $b'\subseteq\tilde c_0$. **P?** Otherwise, because $\tilde c_0=1\setminus\sup_{\xi<\kappa}\pi_\xi a'$ (395Gb), there must be a $\xi<\kappa$ such that $\pi_\xi a'\cap b'\ne0$. But in this case $d=a'\cap\pi_\xi^{-1}b'\ne0$, and we have

$$d\subseteq(a\setminus\sup_{\eta<\xi}a_\eta)\cap\pi_\xi^{-1}(b\setminus\sup_{\eta<\xi}b_\eta),$$

so that $d\subseteq a_\xi$, which is absurd. **XQ** Consequently

$$b\setminus\tilde c_0\subseteq\sup_{\xi<\kappa}b_\xi\preccurlyeq_G^\tau a.$$

Now take any $c\in\mathfrak{C}$ such that $a\cap c\preccurlyeq_G^\tau b$, and consider $c'=c\setminus\tilde c_0$. Then $b'\cap c'=0$, that is, $b\cap c'=\sup_{\xi<\kappa}b_\xi\cap c'$, which is G-τ-equidecomposable with $\sup_{\xi<\kappa}a_\xi\cap c'=(a\setminus a')\cap c'$ (395Gd). But now

$$a\cap c'=a\cap c\cap c'\preccurlyeq_G^\tau b\cap c'\preccurlyeq_G^\tau(a\cap c')\setminus(a'\cap c');$$

because G is fully non-paradoxical, $a'\cap c'$ must be 0, that is, $c'\subseteq\tilde c_0$ and $c'=0$. As c' is arbitrary, $c_0\subseteq\tilde c_0$ and $c_0=\tilde c_0$. So c_0 has the required properties.

Remark By analogy with the notation I used in discussing the Hahn decomposition of countably additive functionals (326S-326T), we might denote c_0 as '$[\![a\preccurlyeq_G^\tau b]\!]$', or perhaps '$[\![a\preccurlyeq_G^\tau b]\!]_{\mathfrak{C}}$', 'the region (in $\mathfrak{C}$) where $a\preccurlyeq_G^\tau b$'. The same notation would write $\operatorname{upr}(a,\mathfrak{C})$ as '$[\![a\ne0]\!]_{\mathfrak{C}}$'.

395I The construction I wish to use depends essentially on L^0 spaces as described in §364. The next step is the following.

Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Suppose that a, $b\in\mathfrak{A}$ and that $\operatorname{upr}(a,\mathfrak{C})=1$. Then there are u, $v\in L^0=L^0(\mathfrak{C})$ such that

$$\begin{aligned}[\![u\ge n]\!]=\max\{c:c\in\mathfrak{C},\ &\text{there is a disjoint family }\langle d_i\rangle_{i<n}\\ &\text{such that }c\cap a\preccurlyeq_G^\tau d_i\subseteq b\text{ for every }i<n\},\\ [\![v\le n]\!]=\max\{c:c\in\mathfrak{C},\ &\text{there is a family }\langle d_i\rangle_{i<n}\\ &\text{such that }d_i\preccurlyeq_G^\tau a\text{ for every }i<n\text{ and }b\cap c\subseteq\sup_{i<n}d_i\}\end{aligned}$$

for every $n\in\mathbb{N}$. Moreover, we can arrange that

(i) $[\![u\in\mathbb{N}]\!]=[\![v\in\mathbb{N}]\!]=1$,

(ii) $[\![v>0]\!]=\operatorname{upr}(b,\mathfrak{C})$,

(iii) $u\le v\le u+\chi1$.

Remark By writing 'max' in the formulae above, I mean to imply that the elements $[\![u \ge n]\!]$, $[\![v \le n]\!]$ belong to the sets described.

proof (a) Choose $\langle c_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ as follows. Given $\langle b_i\rangle_{i<n}$, set $b'_n = b\setminus \sup_{i<n} b_i$,

$$c_n = \sup\{c : c \in \mathfrak{C},\, a\cap c \preccurlyeq^\tau_G b'_n\},$$

so that $a\cap c_n \preccurlyeq^\tau_G b'_n$ (395H); choose $b_n \subseteq b'_n$ such that $a\cap c_n$ is G-τ-equidecomposable with b_n, and continue. Then $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_b$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{C}$.

For each n, we have $b'_n\setminus c_n \preccurlyeq^\tau_G a$, by 395H; while $a\cap c \not\preccurlyeq^\tau_G b'_n$ whenever $c\in\mathfrak{C}$ and $c\not\subseteq c_n$. Note also that, because $\operatorname{upr}(a,\mathfrak{C}) = 1$,

$$c_n = \operatorname{upr}(a\cap c_n,\mathfrak{C}) = \operatorname{upr}(b_n,\mathfrak{C}) \subseteq \operatorname{upr}(b'_n,\mathfrak{C})$$

(using 395Gc for the second equality).

(b) Now $\inf_{n\in\mathbb{N}} c_n = 0$. **P** Setting $c_\infty = \inf_{n\in\mathbb{N}} c_n$, $\langle b_n\cap c_\infty\rangle_{n\in\mathbb{N}}$ is a disjoint sequence, all G-τ-equidecomposable with $a\cap c_\infty$, so $a\cap c_\infty = 0$, because G is fully non-paradoxical; because $\operatorname{upr}(a,\mathfrak{C}) = 1$, it follows that $c_\infty = 0$. **Q** Accordingly, if we set $u = \sup_{n\in\mathbb{N}}(n+1)\chi c_n$, $u\in L^0$ and $[\![u\ge n]\!] = c_{n-1}$ for $n\ge 1$. The construction ensures that $[\![u\in\mathbb{N}]\!]$, as defined in 364G, is equal to 1.

(c) Consider next $c'_0 = \operatorname{upr}(b,\mathfrak{C})$, $c'_n = c_{n-1}\cap\operatorname{upr}(b'_n,\mathfrak{C})$ for $n\ge 1$. Then $\langle c'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with zero infimum, so again we can define $v\in L^0$ by setting $v = \sup_{n\in\mathbb{N}}(n+1)\chi c'_n$. Once again, $[\![v\in\mathbb{N}]\!] = 1$, and $[\![v\le n]\!] = 1\setminus c'_n$ for each n.

Of course $[\![v>0]\!] = c'_0 = \operatorname{upr}(b,\mathfrak{C})$. Because $c_n\subseteq c'_n\subseteq c_{n-1}$,

$$(n+1)\chi c_n \le (n+1)\chi c'_n \le n\chi c_{n-1} + \chi 1$$

for each $n\ge 1$, and $u\le v\le u+\chi 1$.

(d) Now set

$$C_n = \{c : c\in\mathfrak{C},\ \text{there is a disjoint family } \langle d_i\rangle_{i<n}$$
$$\text{such that } c\cap a \preccurlyeq^\tau_G d_i \subseteq b \text{ for every } i<n\}.$$

Then $c_n = \max C_{n+1}$.

P(α) Because $c_n\subseteq c_{n-1}\subseteq\ldots\subseteq c_0$, $a\cap c_n \preccurlyeq^\tau_G b_i$ for every $i\le n$, so that $\langle b_i\rangle_{i\le n}$ witnesses that $c_n\in C_{n+1}$.

(β) Suppose that $c\in C_{n+1}$; let $\langle d_i\rangle_{i\le n}$ be a disjoint family such that $c\cap a\preccurlyeq^\tau_G d_i\subseteq b$ for every i. Set $c' = c\setminus c_n$. For each $i<n$, $b_i \preccurlyeq^\tau_G a$, so

$$b_i\cap c' \preccurlyeq^\tau_G a\cap c' \preccurlyeq^\tau_G d_i\cap c',$$

while also

$$b'_n\cap c' \preccurlyeq^\tau_G a\cap c' \preccurlyeq^\tau_G d_n\cap c'.$$

Take $d\subseteq d_n\cap c'$ such that $b'_n\cap c'$ is G-τ-equidecomposable with d. Then

$$b\cap c' = (b'_n\cap c')\cup\sup_{i<n}(b_i\cap c') \preccurlyeq^\tau_G d\cup\sup_{i<n}(d_i\cap c')\subseteq b\cap c'.$$

Because G is fully non-paradoxical, $d\cup\sup_{i<n}(d_i\cap c')$ must be exactly $b\cap c'$, so d must be the whole of $d_n\cap c'$, and

$$a\cap c' \preccurlyeq^\tau_G d_n\cap c' = d \preccurlyeq^\tau_G b'_n.$$

But this means that $c'\subseteq c_n$. Thus $c'=0$ and $c\subseteq c_n$. So $c_n = \sup C_{n+1} = \max C_{n+1}$. **Q**

Accordingly

$$[\![u\ge n]\!] = c_{n-1} = \max C_n$$

for $n\ge 1$. For $n=0$ we have $[\![u\ge 0]\!] = 1 = \max C_0$. So $[\![u\ge n]\!] = \max C_n$ for every n, as required.

(e) Similarly, if we set

$$C'_n = \{c : c \in \mathfrak{C},\text{ there is a family } \langle d_i\rangle_{i<n}$$
$$\text{such that } d_i \preccurlyeq^\tau_G a \text{ for every } i<n \text{ and } b\cap c \subseteq \sup_{i<n} d_i\}$$

then $1\setminus c'_n = \max C'_n$ for every n.

P(α) If $n=0$, then of course (interpreting $\sup\emptyset$ as 0) $1\setminus c'_0 \in C'_0$ because $b\subseteq c'_0$. For each $n\in\mathbb{N}$, set

$$\tilde b_n = b_n \cup (b'_n\setminus c_n) = (b_n\cap c_n)\cup(b'_n\setminus c_n).$$

Because $b_n \preccurlyeq^\tau_G a$ and $b'_n\setminus c_n \preccurlyeq^\tau_G a$, we have $b_n\cap c_n \preccurlyeq^\tau_G a\cap c_n$ and $b'_n\setminus c_n \preccurlyeq^\tau_G a\setminus c_n$, so $\tilde b_n \preccurlyeq^\tau_G a$ (395Cd). If we look at

$$\sup_{i<n}\tilde b_i \supseteq \sup_{i<n} b_i \cup (b'_{n-1}\setminus c_{n-1}),$$

we see that, for $n\ge 1$,

$$b\setminus \sup_{i<n}\tilde b_i \subseteq b'_n\cap c_{n-1} \subseteq c'_n,$$

so that $b\setminus c'_n \subseteq \sup_{i<n}\tilde b_i$ and $\{\tilde b_i : i<n\}$ witnesses that $1\setminus c'_n \in C'_n$.

(β) Now take any $c\in C'_n$ and a corresponding family $\langle d_i\rangle_{i<n}$ such that $d_i \preccurlyeq^\tau_G a$ for every $i<n$ and $b\cap c\subseteq \sup_{i<n} d_i$.

Set $c' = c\cap c'_n$. For each $i<n$,

$$c'\cap d_i \preccurlyeq^\tau_G c'\cap a \preccurlyeq^\tau_G b_i$$

because $c'\subseteq c_i$. So (by 395Cd, as usual)

$$c'\cap b \preccurlyeq^\tau_G c'\cap \sup_{i<n} b_i \subseteq c'\cap b,$$

and (again because G is fully non-paradoxical) $c'\cap b = c'\cap\sup_{i<n} b_i$, that is, $c'\cap b'_n = 0$. But $c'\subseteq c'_n \subseteq \operatorname{upr}(b'_n,\mathfrak{C})$, so c' must be 0, which means that $c\subseteq 1\setminus c'_n$. As c is arbitrary, $1\setminus c'_n = \sup C'_n = \max C'_n$. **Q**

Thus $[\![v\le n]\!] = \max C'_n$, as declared.

395J Notation Observe that the specification of $[\![u\ge n]\!]$ and $[\![v\le n]\!]$, together with the declaration that $[\![u\in\mathbb{N}]\!] = [\![v\in\mathbb{N}]\!] = 1$, determine u and v uniquely, because $\langle[\![u=n]\!]\rangle_{n\in\mathbb{N}}$ and $\langle[\![v=n]\!]\rangle_{n\in\mathbb{N}}$ must be partitions of unity. So, in the context of 395I, we can write $\lfloor b:a\rfloor$ for u and $\lceil b:a\rceil$ for v.

395K Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a, b, b_1, $b_2\in\mathfrak{A}$ and that $\operatorname{upr}(a,\mathfrak{C})=1$.

(a) $\lfloor 0:a\rfloor = \lceil 0:a\rceil = 0$, $\lfloor 1:a\rfloor \ge \chi 1$ and $\lfloor 1:1\rfloor = \chi 1$.

(b) If $b_1 \preccurlyeq^\tau_G b_2$ then $\lfloor b_1:a\rfloor \le \lfloor b_2:a\rfloor$ and $\lceil b_1:a\rceil \le \lceil b_2:a\rceil$.

(c) $\lceil b_1\cup b_2:a\rceil \le \lceil b_1:a\rceil + \lceil b_2:a\rceil$.

(d) If $b_1\cap b_2=0$, $\lfloor b_1:a\rfloor + \lfloor b_2:a\rfloor \le \lfloor b_1\cup b_2:a\rfloor$.

(e) If $c\in\mathfrak{C}$ is such that $a\cap c$ is a relative atom over $\mathfrak{C}$ (definition: 331A), then $c\subseteq [\![\lceil b:a\rceil - \lfloor b:a\rfloor = 0]\!]$.

proof (a)-(b) are immediate from the definitions and the basic properties of $\preccurlyeq^\tau_G$, $\lceil\ldots\rceil$ and $\lfloor\ldots\rfloor$, as listed in 395C and 395I.

(c) For j, $k\in\mathbb{N}$, set $c_{jk} = [\![\lceil b_1:a\rceil = j]\!]\cap[\![\lceil b_2:a\rceil = k]\!]$. Then

$$c_{jk} \subseteq [\![\lceil b_1\cup b_2:a\rceil \le j+k]\!]\cap[\![\lceil b_1:a\rceil + \lceil b_2:a\rceil = j+k]\!].$$

P We may suppose that $c_{jk}\ne 0$. Of course

$$c_{jk}\subseteq [\![\lceil b_1:a\rceil + \lceil b_2:a\rceil = j+k]\!].$$

Next, there are sets J, $J'\subseteq\mathfrak{A}$ such that $d\preccurlyeq^\tau_G a$ for every $d\in J\cup J'$, $\#(J)\le j$, $\#(J')\le k$, $\sup J\supseteq b_1\cap c_{jk}$ and $\sup J'\supseteq b_2\cap c_{jk}$. So $\sup(J\cup J')\supseteq (b_1\cup b_2)\cap c_{jk}$ and $J\cup J'$ witnesses that $c_{jk}\subseteq [\![\lceil b_1\cup b_2:a\rceil \le j+k]\!]$. **Q**

Accordingly

$$c_{jk}\subseteq [\![\lceil b_1:a\rceil + \lceil b_2:a\rceil - \lceil b_1\cup b_2:a\rceil \ge 0]\!].$$

Now as $\sup_{j,k\in\mathbb{N}} c_{jk} = 1$, we must have $\lceil b_1 \cup b_2 : a\rceil \le \lceil b_1 : a\rceil + \lceil b_2 : a\rceil$.

(d) This time, set $c_{jk} = [\![\lfloor b_1 : a\rfloor = j]\!] \cap [\![\lfloor b_2 : a\rfloor = k]\!]$ for j, $k \in \mathbb{N}$. Then

$$c_{jk} \subseteq [\![\lfloor b_1 \cup b_2 : a\rfloor \ge j+k]\!] \cap [\![\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor = j+k]\!]$$

for every j, $k \in \mathbb{N}$. **P** Once again, we surely have

$$c_{jk} \subseteq [\![\lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor = j+k]\!].$$

Next, we can find a family $\langle d_i\rangle_{i<j+k}$ such that

$$\langle d_i\rangle_{i<j} \text{ is disjoint, } a \cap c_{jk} \preccurlyeq^\tau_G d_i \subseteq b_1 \text{ for every } i < k,$$

$$\langle d_i\rangle_{j\le i<j+k} \text{ is disjoint, } a \cap c_{jk} \preccurlyeq^\tau_G d_i \subseteq b_2 \text{ for } j \le i < j+k.$$

As $b_1 \cap b_2 = 0$, the whole family $\langle d_i\rangle_{i<j+k}$ is disjoint and witnesses that $c_{jk} \subseteq [\![\lfloor b_1 \cup b_2 : a\rfloor \ge j+k]\!]$. **Q**

So

$$c_{jk} \subseteq [\![\lfloor b_1 \cup b_2 : a\rfloor - \lfloor b_1 : a\rfloor - \lfloor b_2 : a\rfloor \ge 0]\!]$$

Since $\sup_{j,k\in\mathbb{N}} c_{jk} = 1$, as before, we must have $\lfloor b_1 \cup b_2 : a\rfloor \ge \lfloor b_1 : a\rfloor + \lfloor b_2 : a\rfloor$.

(e) ? Otherwise, there must be some $k \in \mathbb{N}$ such that

$$c_0 = c \cap [\![\lfloor b : a\rfloor = k]\!] \cap [\![\lceil b : a\rceil > k]\!] \ne 0.$$

Let $\langle d_i\rangle_{i<k}$ be a disjoint family in $\mathfrak{A}_b$ such that $a \cap c_0 \preccurlyeq^\tau_G d_i$ for each i; cutting the d_i down if necessary, we may suppose that $a \cap c_0$ is G-τ-equidecomposable with d_i for each i. As $c_0 \not\subseteq [\![\lceil b : a\rceil \le k]\!]$, $b \cap c_0 \not\subseteq \sup_{i<k} d_i$; set $d = b \cap c_0 \setminus \sup_{i<k} d_i \ne 0$. By 395H, there is a $c_1 \in \mathfrak{C}$ such that $d \cap c_1 \preccurlyeq^\tau_G a$ and $a \setminus c_1 \preccurlyeq^\tau_G d$. Setting $d_k = d$, $\langle d_i\rangle_{i\le k}$ witnesses that $c_0 \subseteq c_1 \subseteq [\![\lfloor b : a\rfloor \ge k+1]\!]$, so $c_0 \subseteq c_1$ must be 0 and $d \cap c_0 \preccurlyeq^\tau_G a$. There is therefore a non-zero $\tilde a \subseteq a \cap c_0$ such that $\tilde a \preccurlyeq^\tau_G d$. But now remember that $a \cap c$ is supposed to be a relative atom over $\mathfrak{C}$, so $\tilde a = a \cap \tilde c$ for some $\tilde c \in \mathfrak{C}$ such that $\tilde c \subseteq c_0$. In this case, $a \cap \tilde c \preccurlyeq^\tau_G d_i$ for every $i < k$ and also $a \cap \tilde c \preccurlyeq^\tau_G d$, so $0 \ne \tilde c \subseteq [\![\lfloor b : a\rfloor \ge k+1]\!]$, which is absurd. **X**

395L Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a_1, a_2, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a_1, \mathfrak{C}) = \operatorname{upr}(a_2, \mathfrak{C}) = 1$. Then

$$\lfloor b : a_2\rfloor \ge \lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor, \quad \lceil b : a_2\rceil \le \lceil b : a_1\rceil \times \lceil a_1 : a_2\rceil.$$

proof I use the same method as in 395K. As usual, write G^*_τ for the full local semigroup generated by G.

(a) For j, $k \in \mathbb{N}$ set

$$c_{j,k} = [\![\lfloor b : a_1\rfloor = j]\!] \cap [\![\lfloor a_1 : a_2\rfloor = k]\!].$$

Then

$$c_{j,k} \subseteq [\![\lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor = jk]\!] \cap [\![\lfloor b : a_2\rfloor \ge jk]\!].$$

P Write c for $c_{j,k}$. As in parts (c) and (d) of the proof of 395K, the fact that $c \subseteq [\![\lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor = jk]\!]$ is elementary; what we need to check is that $c \subseteq [\![\lfloor b : a_2\rfloor \ge jk]\!]$. Again, we may suppose that $c \ne 0$. There are families $\langle d_i\rangle_{i<j}$, $\langle d^*_l\rangle_{l<k}$ such that

$$\langle d_i\rangle_{i<j} \text{ is disjoint, } a_1 \cap c \preccurlyeq^\tau_G d_i \subseteq b \text{ for every } i < j,$$

$$\langle d^*_l\rangle_{l<k} \text{ is disjoint, } a_2 \cap c \preccurlyeq^\tau_G d^*_l \subseteq a_1 \text{ for every } l < k.$$

For each $i < j$, let $\phi_i \in G^*_\tau$ be such that $\phi_i(a_1 \cap c) \subseteq d_i$. If $i < j$ and $l < k$, then

$$a_2 \cap c \preccurlyeq^\tau_G d^*_l \cap c \preccurlyeq^\tau_G \phi_i(d^*_l \cap c) \subseteq \phi_i(a_1 \cap c) \subseteq d_i \subseteq b.$$

Also $\langle \phi_i(d^*_l \cap c)\rangle_{i<j,l<k}$ is disjoint because $\langle \phi_i(a_1 \cap c)\rangle_{i<j}$ and $\langle d^*_l\rangle_{l<k}$ are, so witnesses that $c \subseteq [\![\lfloor b : a_2\rfloor \ge jk]\!]$. **Q**

Now, just as in 395K, it follows from the fact that $\sup_{j,k\in\mathbb{N}} c_{j,k} = 1$ that $\lfloor b : a_1\rfloor \times \lfloor a_1 : a_2\rfloor \le \lfloor b : a_2\rfloor$.

(b) For j, $k \in \mathbb{N}$ set

$$c_{j,k}=[\![\lceil b:a_1\rceil=j]\!]\cap[\![\lceil a_1:a_2\rceil=k]\!].$$

Then

$$c_{j,k}\subseteq[\![\lceil b:a_1\rceil\times\lceil a_1:a_2\rceil=jk]\!]\cap[\![\lceil b:a_2\rceil\le jk]\!].$$

P Write c for $c_{j,k}$. Then $c\subseteq[\![\lceil b:a_1\rceil\times\lceil a_1:a_2\rceil=jk]\!]$. There are families $\langle d_i\rangle_{i<j}$, $\langle d_l^*\rangle_{l<k}$ such that $d_i\preccurlyeq_G^\tau a_1$ for every $i<j$, $d_l^*\preccurlyeq_G^\tau a_2$ for every $l<k$, $b\cap c\subseteq\sup_{i<j}d_i$ and $a_1\cap c\subseteq\sup_{l<k}d_l^*$. For each $i<j$, let $d_i'\subseteq a_1$ be G-τ-equidecomposable with d_i, and take $\phi_i\in G_\tau^*$ such that $\phi_i d_i'=d_i$. Then

$$\phi_i(d_i'\cap d_l^*)\preccurlyeq_G^\tau d_l^*\preccurlyeq_G^\tau a_2\text{ for every }i<j,\,l<k,$$

$$\begin{aligned}\sup_{i<j,l<k}\phi_i(d_i'\cap d_l^*)&=\sup_{i<j}\phi_i(d_i'\cap\sup_{l<k}d_l^*)\supseteq\sup_{i<j}\phi_i(d_i'\cap c)\\&=\sup_{i<j}d_i\cap c\supseteq b\cap c.\end{aligned}$$

So $\langle\phi_i(d_i'\cap d_l^*)\rangle_{i<j,l<k}$ witnesses that $c\subseteq[\![\lceil b:a_2\rceil\le jk]\!]$. **Q**

Once again, it follows easily that $\lceil b:a_1\rceil\times\lceil a_1:a_2\rceil\ge\lceil b:a_2\rceil$.

395M Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$.

(a) For any $a\in\mathfrak{A}$, there is a $b\subseteq a$ such that $b\preccurlyeq_G^\tau a\setminus b$ and $a\setminus\operatorname{upr}(b,\mathfrak{C})$ is a either 0 or a relative atom over $\mathfrak{C}$.

(b) Now suppose that G is fully non-paradoxical. Then for any $\epsilon>0$ there is an $a\in\mathfrak{A}$ such that $\operatorname{upr}(a,\mathfrak{C})=1$ and $\lceil b:a\rceil\le\lfloor b:a\rfloor+\epsilon\lfloor 1:a\rfloor$ for every $b\in\mathfrak{A}$.

proof (a) Set $B=\{d:d\subseteq a,\ d\preccurlyeq_G^\tau a\setminus d\}$ and let $D\subseteq B$ be a maximal subset such that $\operatorname{upr}(d,\mathfrak{C})\cap\operatorname{upr}(d',\mathfrak{C})=0$ for all distinct d, $d'\in\mathfrak{D}$. Set $b=\sup D$. For any $d\in D$, $d\preccurlyeq_G^\tau a\setminus d$, so

$$\begin{aligned}b\cap\operatorname{upr}(d,\mathfrak{C})&=\sup_{d'\in D}d'\cap\operatorname{upr}(d,\mathfrak{C})=\sup_{d'\in D}d'\cap\operatorname{upr}(d',\mathfrak{C})\cap\operatorname{upr}(d,\mathfrak{C})=d\cap\operatorname{upr}(d,\mathfrak{C})\\&\preccurlyeq_G^\tau(a\setminus d)\cap\operatorname{upr}(d,\mathfrak{C})=(a\setminus b)\cap\operatorname{upr}(d,\mathfrak{C})\subseteq a\setminus b\end{aligned}$$

by 395Gc. By 395H,

$$b=\sup\nolimits_{d\in D}b\cap\operatorname{upr}(d,\mathfrak{C})\preccurlyeq_G^\tau a\setminus b.$$

? Suppose, if possible, that $a'=a\setminus\operatorname{upr}(b,\mathfrak{C})$ is neither 0 nor a relative atom over $\mathfrak{C}$. Let $d_0\subseteq a'$ be an element not expressible as $a'\cap c$ for any $c\in\mathfrak{C}$; then $d_0\ne a\cap\operatorname{upr}(d_0,\mathfrak{C})$ and there must be a $\pi\in G$ such that $d_1=\pi d_0\cap a\setminus d_0$ is non-zero (395Gb). In this case

$$d_1\preccurlyeq_G^\tau\pi^{-1}d_1\subseteq d_0\subseteq a\setminus d_1,$$

so $d_1\in B$; but also

$$d_1\cap\operatorname{upr}(d,\mathfrak{C})\subseteq d_1\cap\operatorname{upr}(b,\mathfrak{C})=0,$$

so $\operatorname{upr}(d_1,\mathfrak{C})\cap\operatorname{upr}(d,\mathfrak{C})=0$, for every $d\in D$, and we ought to have put d_1 into D. **X**

Thus b has the required properties.

(b)(i) For every $n\in\mathbb{N}$ we can find $a_n\in\mathfrak{A}$ and $c_n\in\mathfrak{C}$ such that $\operatorname{upr}(a_n,\mathfrak{C})=1$, $a_n\setminus c_n$ is either 0 or a relative atom over $\mathfrak{C}$, and $\lfloor 1:a_n\rfloor\ge 2^n\chi c_n$. **P** Induce on n. The induction starts with $a_0=c_0=1$, because $\lfloor 1:1\rfloor=\chi 1$. For the inductive step, having found a_n and c_n, let $d\subseteq a_n\cap c_n$ be such that $d\preccurlyeq_G^\tau a_n\cap c_n\setminus d$ and $a_n\cap c_n\setminus\operatorname{upr}(d,\mathfrak{C})$ is either 0 or a relative atom over $\mathfrak{C}$, as in (a). Set $c_{n+1}=\operatorname{upr}(d,\mathfrak{C})$, $a_{n+1}=(a_n\setminus c_{n+1})\cup d$; then

$$\begin{aligned}\operatorname{upr}(a_{n+1},\mathfrak{C})&=\operatorname{upr}(a_n\setminus c_{n+1},\mathfrak{C})\cup\operatorname{upr}(d,\mathfrak{C})\\&=(\operatorname{upr}(a_n,\mathfrak{C})\setminus c_{n+1})\cup c_{n+1}=(1\setminus c_{n+1})\cup c_{n+1}=1\end{aligned}$$

by 313Sb-313Sc and the inductive hypothesis.

We have $c_{n+1}\cap d\preccurlyeq_G^\tau c_{n+1}\cap a_n\setminus d$, so

$$c_{n+1} \cap a_{n+1} = d \subseteq a_n, \quad c_{n+1} \cap a_{n+1} \preccurlyeq_G^\tau a_n \setminus d,$$

and $\lfloor a_n : a_{n+1} \rfloor \ge 2\chi c_{n+1}$; by 395L,

$$\lfloor 1 : a_{n+1} \rfloor \ge \lfloor 1 : a_n \rfloor \times \lfloor a_n : a_{n+1} \rfloor \ge 2^n \chi c_n \times 2\chi c_{n+1} = 2^{n+1}\chi c_{n+1}.$$

If

$$b \subseteq a_{n+1} \setminus c_{n+1} = (a_n \setminus c_n) \cup (a_n \cap c_n \setminus c_{n+1}),$$

then, because both terms on the right are either 0 or relative atoms over $\mathfrak{C}$, there are c', $c'' \in \mathfrak{C}$ such that

$$\begin{aligned} b &= (b \cap a_n \setminus c_n) \cup (b \cap a_n \cap c_n \setminus c_{n+1}) \\ &= (c' \cap a_n \setminus c_n) \cup (c'' \cap a_n \cap c_n \setminus c_{n+1}) = c \cap a_{n+1} \setminus c_{n+1} \end{aligned}$$

where $c = (c' \setminus c_n) \cup (c'' \cap c_n)$ belongs to $\mathfrak{C}$. So $a_{n+1} \setminus c_{n+1}$ is either 0 or a relative atom over $\mathfrak{C}$.

Thus the induction continues. **Q**

(ii) Now suppose that $\epsilon > 0$. Take n such that $2^{-n} \le \epsilon$, and consider a_n, c_n taken from (i) above. Let $b \in \mathfrak{A}$. Set

$$c = [\![\lceil b : a_n \rceil - \lfloor b : a_n \rfloor - \epsilon \lfloor 1 : a_n \rfloor > 0]\!] \in \mathfrak{C}.$$

Since we know that

$$\epsilon \lfloor 1 : a_n \rfloor \ge 2^{-n} 2^n \chi c_n = \chi c_n, \quad \lceil b : a_n \rceil \le \lfloor b : a_n \rfloor + \chi 1,$$

we must have $c \cap c_n = 0$. But this means that $a_n \cap c$ is either 0 or a relative atom over $\mathfrak{C}$. By 395Ke, c is included in $[\![\lceil b : a_n \rceil - \lfloor b : a_n \rfloor = 0]\!]$; as also $\lfloor 1 : a_n \rfloor \ge \chi 1$ (395Ka), c must be zero, that is, $\lceil b : a_n \rceil \le \lfloor b : a_n \rfloor + \epsilon \lfloor 1 : a_n \rfloor$.

395N We are at last ready for the theorem.

Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Then there is a unique function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ such that

(i) θ is additive, non-negative and order-continuous;

(ii) $[\![\theta a > 0]\!] = \operatorname{upr}(a, \mathfrak{C})$ for every $a \in \mathfrak{A}$; in particular, $\theta a = 0$ iff $a = 0$;

(iii) $\theta 1 = \chi 1$;

(iv) $\theta(a \cap c) = \theta a \times \chi c$ for every $a \in \mathfrak{A}$, $c \in \mathfrak{C}$; in particular, $\theta c = \chi c$ for every $c \in \mathfrak{C}$;

(v) If a, $b \in \mathfrak{A}$ are G-τ-equidecomposable, then $\theta a = \theta b$; in particular, θ is G-invariant.

proof If $\mathfrak{A} = \{0\}$ this is trivial; so I suppose henceforth that $\mathfrak{A} \ne \{0\}$.

(a) Set $A^* = \{a : a \in \mathfrak{A}, \operatorname{upr}(a, \mathfrak{C}) = 1\}$ and for $a \in A^*$, $b \in \mathfrak{A}$ set

$$\theta_a(b) = \frac{\lceil b : a \rceil}{\lfloor 1 : a \rfloor} \in L^0 = L^0(\mathfrak{C});$$

the first thing to note is that because $\lfloor 1 : a \rfloor \ge \chi 1$, we can always do the divisions to obtain elements $\theta_a(b)$ of $L^0(\mathfrak{A})$ (364N). Set

$$\theta b = \inf_{a \in A^*} \theta_a b$$

for $b \in \mathfrak{A}$. (Note that $L^0(\mathfrak{C})$ is Dedekind complete, by 364M, so the infimum is defined.)

(b) The formulae of 395K tell us that, for $a \in A^*$ and b_1, $b_2 \in \mathfrak{A}$,

$$\theta_a 0 = 0, \quad \theta_a b_1 \le \theta_a b_2 \text{ if } b_1 \subseteq b_2,$$

$$\theta_a(b_1 \cup b_2) \le \theta_a b_1 + \theta_a b_2,$$

$$\theta_a 1 \ge \chi 1.$$

It follows at once that

$$\theta 0 = 0, \quad \theta b_1 \le \theta b_2 \text{ if } b_1 \subseteq b_2,$$

$$\theta 1 \ge \chi 1.$$

(c) For each $n \in \mathbb{N}$ there is an $e_n \in A^*$ such that $\lceil b : e_n \rceil \le \lfloor b : e_n \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor$ for every $b \in \mathfrak{A}$ (395Mb). Now $\theta_{e_n} b \le \theta_a b + 2^{-n} \lceil b : a \rceil$ for every $a \in A^*$, $b \in \mathfrak{A}$. **P** $\lceil a : e_n \rceil \le \lfloor a : e_n \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor$, so

$$\begin{aligned} \lceil a : e_n \rceil \times \lfloor 1 : a \rfloor &\le \lfloor a : e_n \rfloor \times \lfloor 1 : a \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor \times \lfloor 1 : a \rfloor \\ &\le \lfloor 1 : e_n \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor \times \lfloor 1 : a \rfloor \end{aligned}$$

(by 395L); accordingly

$$\lceil b : e_n \rceil \times \lfloor 1 : a \rfloor \le \lceil b : a \rceil \times \lceil a : e_n \rceil \times \lfloor 1 : a \rfloor$$

(by the other half of 395L)

$$\le \lceil b : a \rceil \times \lfloor 1 : e_n \rfloor + 2^{-n} \lceil b : a \rceil \times \lfloor 1 : e_n \rfloor \times \lfloor 1 : a \rfloor$$

and, dividing by $\lfloor 1 : a \rfloor \times \lfloor 1 : e_n \rfloor$, we get $\theta_{e_n} b \le \theta_a b + 2^{-n} \lceil b : a \rceil$. **Q**

(d) Now θ is additive. **P** Taking $\langle e_n \rangle_{n \in \mathbb{N}}$ from (c), observe first that

$$\inf_{n \in \mathbb{N}} \theta_{e_n} b \le \theta_a b + \inf_{n \in \mathbb{N}} 2^{-n} \lceil b : a \rceil = \theta_a b$$

for every $a \in A^*$, $b \in \mathfrak{A}$, so that $\theta b = \inf_{n \in \mathbb{N}} \theta_{e_n} b$ for every b. Now suppose that b_1, $b_2 \in \mathfrak{A}$ and $b_1 \cap b_2 = 0$. Then, for any $n \in \mathbb{N}$,

$$\begin{aligned} \lceil b_1 : e_n \rceil + \lceil b_2 : e_n \rceil &\le \lfloor b_1 : e_n \rfloor + \lfloor b_2 : e_n \rfloor + 2^{-n+1} \lfloor 1 : e_n \rfloor \\ &\le \lfloor b_1 \cup b_2 : e_n \rfloor + 2^{-n+1} \lfloor 1 : e_n \rfloor \end{aligned}$$

(by 395Kd)

$$\le \lceil b_1 \cup b_2 : e_n \rceil + 2^{-n+1} \lfloor 1 : e_n \rfloor.$$

Dividing by $\lfloor 1 : e_n \rfloor$, we have

$$\theta b_1 + \theta b_2 \le \theta_{e_n} b_1 + \theta_{e_n} b_2 \le \theta_{e_n} (b_1 \cup b_2) + 2^{-n+1} \chi 1.$$

Taking the infimum over n, we get

$$\theta b_1 + \theta b_2 \le \theta (b_1 \cup b_2).$$

In the other direction, if a, $a' \in A^*$ and $n \in \mathbb{N}$,

$$\begin{aligned} \theta(b_1 \cup b_2) &\le \theta_{e_n}(b_1 \cup b_2) \le \theta_{e_n}(b_1) + \theta_{e_n}(b_2) \\ &\le \theta_a(b_1) + 2^{-n} \lceil b_1 : a \rceil + \theta_{a'}(b_2) + 2^{-n} \lceil b_2 : a' \rceil. \end{aligned}$$

As n is arbitrary, $\theta(b_1 \cup b_2) \le \theta_a(b_1) + \theta_{a'}(b_2)$; as a and a' are arbitrary, $\theta(b_1 \cup b_2) \le \theta b_1 + \theta b_2$ (using 351Dc). As b_1 and b_2 are arbitrary, θ is additive. **Q**

We see also that $\lceil 1 : e_n \rceil \le (1 + 2^{-n}) \lfloor 1 : e_n \rfloor$, so that $\theta_{e_n} 1 \le (1 + 2^{-n}) \chi 1$ for each n; since we already know that $\theta 1 \ge \chi 1$, we have $\theta 1 = \chi 1$ exactly.

(e) If $c \in \mathfrak{C}$ then

$$[\![\theta c > 0]\!] \subseteq [\![\theta_1 c > 0]\!] \subseteq [\![\lceil c : 1 \rceil > 0]\!] = \operatorname{upr}(c, \mathfrak{C}) = c$$

(395I(ii)). It follows that

$$\theta(b \cap c) \le \theta b \wedge \theta c \le \theta b \times \chi c$$

for any $b \in \mathfrak{A}$, $c \in \mathfrak{C}$. Similarly, $\theta(b \setminus c) \le \theta b \times \chi(1 \setminus c)$; adding, we must have equality in both, and $\theta(b \cap c) = \theta b \times \chi c$.

Rather late, I point out that

$$0 \le \theta a \le \theta 1 = \chi 1 \in L^\infty = L^\infty(\mathfrak{C})$$

for every $a \in \mathfrak{A}$, so that $\theta a \in L^\infty$ for every a.

(f) If $b \in \mathfrak{A} \setminus \{0\}$, then

$$[\![\theta b > 0]\!] \subseteq [\![\theta_1 b > 0]\!] \subseteq [\![\lceil b : 1\rceil > 0]\!] = \operatorname{upr}(b, \mathfrak{C})$$

by 395I(ii) again. **?** Suppose, if possible, that $[\![\theta b > 0]\!] \ne \operatorname{upr}(b, \mathfrak{C})$. Set $c_0 = \operatorname{upr}(b, \mathfrak{C}) \setminus [\![\theta b > 0]\!]$, $a_0 = b \cup (1 \setminus \operatorname{upr}(b, \mathfrak{C})) \in A^*$. Let $k \ge 1$ be such that $c_1 = c_0 \cap [\![\lceil 1 : a_0\rceil \le k]\!] \ne 0$. Then $a_0 \cap c_1 = b \cap c_1$, so

$$\theta a_0 \times \chi c_1 = \theta(a_0 \cap c_1) = \theta(b \cap c_1) = \theta b \times \chi c_1 = 0.$$

By 364L(b-ii), there is an $a \in A^*$ such that $c_1 \not\subseteq [\![\theta_a a_0 \times \chi c_1 \ge \frac{1}{k}]\!]$, that is, $c_2 = c_1 \cap [\![\theta_a a_0 < \frac{1}{k}]\!] \ne 0$. Now

$$c_2 \subseteq [\![\lfloor 1 : a\rfloor - k\lceil a_0 : a\rceil > 0]\!] \subseteq [\![\lceil 1 : a_0\rceil \times \lceil a_0 : a\rceil - k\lceil a_0 : a\rceil > 0]\!] \subseteq [\![\lceil 1 : a_0\rceil > k]\!],$$

which is impossible, as $c_2 \subseteq c_1$. **X**

Thus $[\![\theta b > 0]\!] = \operatorname{upr}(b, \mathfrak{C})$. In particular, $\theta b = 0$ iff $b = 0$.

(g) If b, $b' \in \mathfrak{A}$ and $b \preccurlyeq^\tau_G b'$, then $\theta b \le \theta b'$. **P** For every $a \in A^*$, $\lceil b : a\rceil \le \lceil b' : a\rceil$ (395Kb) so $\theta_a b \le \theta_a b'$. **Q** So if b, $b' \in \mathfrak{A}$ and $c = [\![\theta b - \theta b' > 0]\!]$, $b' \cap c \preccurlyeq^\tau_G b$. **P?** Otherwise, by 395H, there is a non-zero $c' \subseteq c$ such that $b \cap c' \preccurlyeq^\tau_G b'$. But in this case $\theta b \times \chi c' = \theta(b \cap c') \le \theta b'$ and $c' \subseteq [\![\theta b' - \theta b \ge 0]\!]$. **XQ**

(h) If $\langle a_i\rangle_{i \in I}$ is any disjoint family in $\mathfrak{A}$ with supremum a, $\theta a = \sum_{i \in I} \theta a_i$, where the sum is to be interpreted as $\sup_{J \subseteq I \text{ is finite}} \sum_{i \in J} \theta a_i$. **P** Induce on $\#(I)$. If $\#(I)$ is finite, this is just finite additivity ((d) above). For the inductive step to $\#(I) = \kappa \ge \omega$, we may suppose that I is actually equal to the cardinal κ. Of course

$$\theta a \ge \theta(\sup_{\xi \in J} a_\xi) = \textstyle\sum_{\xi \in J} \theta a_\xi$$

for every finite $J \subseteq \kappa$, so (because $L^\infty(\mathfrak{C})$ is Dedekind complete) $u = \sum_{\xi < \kappa} \theta a_\xi$ is defined, and $u \le \theta a$.

For $\zeta < \kappa$, set $b_\zeta = \sup_{\xi < \zeta} a_\xi$. By the inductive hypothesis,

$$\theta b_\zeta = \textstyle\sum_{\xi < \zeta} \theta a_\xi = \sup_{J \subseteq \zeta \text{ is finite}} \sum_{\xi \in J} \theta a_\xi \le u.$$

At the same time, if $J \subseteq \kappa$ is finite, there is some $\zeta < \kappa$ such that $J \subseteq \zeta$, so that $\sum_{\xi \in J} \theta a_\xi \le \theta b_\zeta$; accordingly $\sup_{\zeta < \kappa} \theta b_\zeta = u$.

? Suppose, if possible, that $u < \theta a$; set $v = \theta a - u$. Take $\delta > 0$ such that $c_0 = [\![v > \delta]\!] \ne 0$. Let $\zeta < \kappa$ be such that $c_1 = c_0 \setminus [\![u - \theta b_\zeta > \delta]\!]$ is non-zero (cf. 364L(b-ii)). Now $v = \theta a - u \le \theta(a \setminus b_\zeta)$, so

$$c_1 \subseteq [\![v > \delta]\!] \subseteq [\![\theta(a \setminus b_\zeta) > 0]\!] = \operatorname{upr}(a \setminus b_\zeta, \mathfrak{C}),$$

and $c_1 \cap (a \setminus b_\zeta) \ne 0$; there is therefore an $\eta' \ge \zeta$ such that $d = c_1 \cap a_{\eta'} \ne 0$. Since $\theta d \le u - \theta b_\zeta$ and c_1 is included in $[\![u - \theta b_\zeta \le \delta]\!] \cap [\![v > \delta]\!]$, $[\![v - \theta d > 0]\!] \supseteq c_1$.

Choose $\langle d_\xi\rangle_{\xi < \kappa}$ inductively, as follows. Given that $\langle d_\eta\rangle_{\eta < \xi}$ is a disjoint family in $\mathfrak{A}_{a \setminus d}$ such that d_η is G-τ-equidecomposable with $a_\eta \cap c_1$ for every $\eta < \xi$, then $e_\xi = \sup_{\eta < \xi} d_\eta$ is G-τ-equidecomposable with $b_\xi \cap c_1$, so that $\theta e_\xi \le \theta b_\xi$, and

$$\begin{aligned}[\![\theta(a \setminus (d \cup e_\xi)) - \theta a_\xi > 0]\!] &= [\![\theta a - \theta d - \theta e_\xi - \theta a_\xi > 0]\!] \supseteq [\![\theta a - \theta d - \theta b_\xi - \theta a_\xi > 0]\!] \\ &= [\![\theta a - \theta d - \theta b_{\xi+1} > 0]\!] \supseteq [\![v - \theta d > 0]\!] \supseteq c_1.\end{aligned}$$

By (g), $a_\xi \cap c_1 \preccurlyeq^\tau_G a \setminus (d \cup e_\xi)$; take $d_\xi \subseteq a \setminus (d \cup e_\xi)$ G-τ-equidecomposable with $a_\xi \cap c_1$, and continue.

At the end of this induction, we have a disjoint family $\langle d_\xi\rangle_{\xi < \kappa}$ in $\mathfrak{A}_{a \setminus d}$ such that d_ξ is G-τ-equidecomposable with $a_\xi \cap c_1$ for every ξ. But this means that $a' = \sup_{\xi < \kappa} d_\xi$ is G-τ-equidecomposable with $a \cap c_1$, while $a' \subseteq (a \setminus d) \cap c_1$; since $d \cap a \cap c_1 \ne 0$, G cannot be fully non-paradoxical. **X**

Thus $\theta a = u = \sum_{\xi < \kappa} \theta a_\xi$ and the induction continues. **Q**

(i) It follows that θ is order-continuous. **P** (α) If $B \subseteq \mathfrak{A}$ is non-empty and upwards-directed and has supremum e, then $\bigcup_{b \in B} \mathfrak{A}_b$ is order-dense in $\mathfrak{A}_e$, so includes a partition of unity A of $\mathfrak{A}_e$; now (h) tells us that

$$\theta e = \textstyle\sum_{a \in A} \theta a \le \sup_{b \in B} \theta b.$$

Since of course $\theta b \le \theta e$ for every $b \in B$, $\theta e = \sup_{b \in B} \theta b$. (β) If $B \subseteq \mathfrak{A}$ is non-empty and downwards-directed and has infimum e, then, using (α), we see that

$$\theta 1-\theta e=\theta(1\setminus e)=\sup_{b\in B}\theta(1\setminus b)=\sup_{b\in B}\theta 1-\theta b,$$

so that $\theta e=\inf_{b\in B}\theta b$. **Q**

(j) I still have to show that θ is unique. Let $\theta':\mathfrak{A}\to L^\infty$ be any non-negative order-continuous G-invariant additive function such that $\theta' c=\chi c$ for every $c\in\mathfrak{C}$.

(i) Just as in (e) of this proof, but more easily, we see that $\theta'(b\cap c)=\theta' b\times\chi c$ whenever $b\in\mathfrak{A}$ and $c\in\mathfrak{C}$.

(ii) If $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a, then $\langle\sup_{i\in J}a_i\rangle_{J\subseteq I\text{ is finite}}$ is an upwards-directed family with supremum a, so that

$$\theta' a=\sup_{J\subseteq I\text{ is finite}}\theta'(\sup_{i\in J}a_i)=\sup_{J\subseteq I\text{ is finite}}\sum_{i\in J}\theta' a_i=\sum_{i\in I}\theta' a_i.$$

(iii) $\theta' a=\theta' b$ whenever a and b are G-τ-equidecomposable. **P** Take a partition $\langle a_i\rangle_{i\in I}$ of a and a family $\langle\pi_i\rangle_{i\in I}$ in G such that $\langle\pi_i a_i\rangle_{i\in I}$ is a partition of b. Then

$$\theta' a=\sum_{i\in I}\theta' a_i=\sum_{i\in I}\theta'\pi_i a_i=\theta' b.\ \mathbf{Q}$$

Consequently $\theta' a\le\theta' b$ whenever $a\preccurlyeq^\tau_G b$.

(iv) Take $a\in A^*$, $b\in\mathfrak{A}$ and for j, $k\in\mathbb{N}$ set $c_{jk}=[\![\lfloor 1:a\rfloor=j]\!]\cap[\![\lceil b:a\rceil=k]\!]$. Then

$$\lceil b:a\rceil\times\chi c_{jk}\ge\theta' b\times\lfloor 1:a\rfloor\times\chi c_{jk}.$$

P If $c_{jk}=0$ this is trivial; suppose $c_{jk}\ne 0$. Now we have sets I, J such that $\#(I)=j$, $\#(J)\le k$, $a\cap c_{jk}\preccurlyeq^\tau_G d$ for every $d\in I$, $e\preccurlyeq^\tau_G a$ for every $e\in J$, I is disjoint, and $b\cap c_{jk}\subseteq\sup J$. So

$$\begin{aligned}\theta' b\times\lfloor 1:a\rfloor\times\chi c_{jk}&=j\theta' b\times\chi c_{jk}=j\theta'(b\cap c_{jk})\le j\sum_{e\in J}\theta'(e\cap c_{jk})\\&\le jk\theta'(a\cap c_{jk})\le k\sum_{d\in I}\theta'(d\cap c_{jk})\le k\theta' c_{jk}\\&=k\chi c_{jk}=\lceil b:a\rceil\times\chi c_{jk}.\ \mathbf{Q}\end{aligned}$$

Summing over j and k, $\lceil b:a\rceil\ge\theta' b\times\lfloor 1:a\rfloor$, that is, $\theta_a b\ge\theta' b$. Taking the infimum over a, $\theta b\ge\theta' b$. But also

$$\theta b=\chi 1-\theta(1\setminus b)\le\chi 1-\theta'(1\setminus b)=\theta' b,$$

so $\theta b=\theta' b$. As b is arbitrary, $\theta=\theta'$. This completes the proof.

395O We have reached the summit. The rest of the section is a list of easy corollaries.

Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, not $\{0\}$, and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Then there is a G-invariant additive functional $\nu:\mathfrak{A}\to[0,1]$ such that $\nu 1=1$.

proof Let $\mathfrak{C}$ be the fixed-point subalgebra of G, and $\theta:\mathfrak{A}\to L^\infty(\mathfrak{C})$ the function of 395N. By 311D, there is a ring homomorphism $\nu_0:\mathfrak{C}\to\{0,1\}$ such that $\nu_0 1=1$; now ν_0 can also be regarded as an additive functional from $\mathfrak{C}$ to $\mathbb{R}$. Let $f_0:L^\infty(\mathfrak{C})\to\mathbb{R}$ be the corresponding positive linear functional (363K). Set $\nu=f_0\theta$. Then ν is order-preserving and additive because f_0 and θ are, $\nu 1=f_0(\chi 1)=\nu_0 1>0$, and ν is G-invariant because θ is.

395P Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Then the following are equiveridical:
(i) $\mathfrak{A}$ is a measurable algebra;
(ii) $\mathfrak{C}$ is a measurable algebra;
(iii) there is a strictly positive G-invariant countably additive real-valued functional on $\mathfrak{A}$.

proof (iii)⇒(i)⇒(ii) are trivial. For (ii)⇒(iii), let $\theta:\mathfrak{A}\to L^\infty(\mathfrak{C})$ be the function of 395N, and $\bar\nu:\mathfrak{C}\to\mathbb{R}$ a strictly positive countably additive functional. Let $f:L^\infty(\mathfrak{C})\to\mathbb{R}$ be the corresponding linear operator; then f is sequentially order-continuous (363K again). Set $\bar\mu=f\theta$. Then $\bar\mu$ is additive and order-preserving and sequentially order-continuous because f and θ are. It is also strictly positive, because if $a \in \mathfrak{A} \setminus \{0\}$ then $\theta a > 0$ (395N(ii)), that is, there is some $\delta > 0$ such that $[\![\theta a > \delta]\!] \neq 0$, so that

$$\bar{\mu}a \geq \delta\bar{\nu}[\![\theta a > \delta]\!] > 0.$$

Finally, $\bar{\mu}$ is G-invariant because θ is.

395Q Corollary: Kawada's theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that $\operatorname{Aut}\mathfrak{A}$ has a subgroup which is ergodic and fully non-paradoxical. Then $\mathfrak{A}$ is measurable.

proof By 395Gf, this is the case $\mathfrak{C} = \{0, 1\}$ of 395P.

395R Thus the existence of an ergodic fully non-paradoxical subgroup is a sufficient condition for a Dedekind complete Boolean algebra to be measurable. It is not quite necessary, because if a measure algebra $\mathfrak{A}$ is not homogeneous then its automorphism group is not ergodic. But for homogeneous algebras the condition is necessary as well as sufficient, by the following result.

Proposition If $(\mathfrak{A}, \bar{\mu})$ is a homogeneous totally finite measure algebra, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is ergodic.

proof If $\mathfrak{A} = \{0, 1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless. If a, $b \in \mathfrak{A} \setminus \{0, 1\}$, set $\gamma = \min(\bar{\mu}a, \bar{\mu}b)$; then there are $a' \subseteq a$ and $b' \subseteq b$ such that $\bar{\mu}a' = \bar{\mu}b' = \gamma$. By 383Fb, there is a $\pi \in G$ such that $\pi a' = b'$, so that $\pi a \cap a \neq 0$. As b is arbitrary, $\sup_{\pi \in G} \pi a = 1$; as a is arbitrary, G is ergodic.

395X Basic exercises (a) Re-write the section on the assumption that every group G is ergodic, so that $L^0(\mathfrak{C})$ may be identified with $\mathbb{R}$, the functions $\lceil \ldots \rceil$ and $\lfloor \ldots \rfloor$ become real-valued, the functionals θ_a (395N) become submeasures and θ becomes a measure.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that $\langle c_i \rangle_{i \in I}$ is a partition of unity in $\mathfrak{C}$ and that a, $b \in \mathfrak{A}$ are such that $a \cap c_i \preccurlyeq_G^\tau b$ for every $i \in I$. Show that $a \preccurlyeq_G^\tau b$.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$ iff the full subgroup generated by G has many involutions (definition: 382O).

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that the following are equiveridical: (i) $\mathfrak{A}$ is chargeable (definition: 391Bb); (ii) $\mathfrak{C}$ is chargeable; (iii) there is a strictly positive G-invariant real-valued additive functional on $\mathfrak{A}$.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Show that the following are equiveridical: (i) there is a non-zero completely additive functional on $\mathfrak{A}$; (ii) there is a non-zero completely additive functional on $\mathfrak{C}$; (iii) there is a non-zero G-invariant completely additive functional on $\mathfrak{A}$.

(f) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra. Show that it is a measurable algebra iff there is a fully non-paradoxical subgroup G of $\operatorname{Aut}\mathfrak{A}$ such that the fixed-point subalgebra of G is purely atomic.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Show that the following are equiveridical: (i) $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is ergodic; (ii) $\mathfrak{A}$ is quasi-homogeneous in the sense of 374G.

(h) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Show that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is fully non-paradoxical iff (i) for every infinite cardinal κ, the Maharam-type-κ component of $\mathfrak{A}$ (definition: 332Gb) has finite measure (ii) for every $\gamma \in \left]0, \infty\right[$ there are only finitely many atoms of measure γ.

(i) Let $\mathfrak{A}$ be a Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$, and G^* the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G. Show that G^* is ergodic iff G is ergodic.

395Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$, and G^*_τ the full local semigroup generated by G. For ϕ, $\psi \in G^*_\tau$, say that $\phi \le \psi$ if ψ extends ϕ. (i) Show that every member of G^*_τ can be extended to a maximal member of G^*_τ. (ii) Show that G is fully non-paradoxical iff every maximal member of G^*_τ is actually a Boolean automorphism of $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Show that G is fully non-paradoxical iff $\langle \pi_n a_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and $\langle \pi_n\rangle_{n\in\mathbb{N}}$ is a sequence in G.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a fully non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. (i) Show that $\mathfrak{A}$ is ccc iff $\mathfrak{C}$ is ccc. (*Hint*: if $\mathfrak{C}$ is ccc, $L^\infty(\mathfrak{C})$ has the countable sup property.) (ii) Show that $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $\mathfrak{C}$ is. (iii) Show that $\mathfrak{A}$ is a Maharam algebra iff $\mathfrak{C}$ is.

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, G an ergodic subgroup of $\operatorname{Aut}\mathfrak{A}$, and G^*_τ the full local semigroup generated by G. Suppose that there is a non-zero $a \in \mathfrak{A}$ for which there is no $\phi \in G^*_\tau$ such that $\phi a \subset a$. Show that there is a measure $\bar\mu$ such that $(\mathfrak{A}, \bar\mu)$ is a localizable measure algebra. (*Hint*: show that $\mathfrak{A}_a$ is a measurable algebra.)

(e) Show that there are a semi-finite measure algebra $(\mathfrak{A}, \bar\mu)$ and a subgroup G of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ such that G is not ergodic but has fixed-point algebra $\{0, 1\}$.

395Z Problem Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, not $\{0\}$, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ such that whenever $\langle a_i\rangle_{i\le n}$ is a finite partition of unity in $\mathfrak{A}$ and we are given π_i, $\pi'_i \in G$ for every $i \le n$, then the elements $\pi_0 a_0$, $\pi'_0 a_0$, $\pi_1 a_1$, $\pi'_1 a_1, \dots, \pi'_n a_n$ are not all disjoint. Must there be a non-zero non-negative G-invariant finitely additive functional θ on $\mathfrak{A}$?

(See 'Tarski's theorem' in the notes below.)

395 Notes and comments Regarded as a sufficient condition for measurability, Kawada's theorem suffers from the obvious defect that it is going to be rather rarely that we can verify the existence of an ergodic fully non-paradoxical group of automorphisms without having some quite different reason for supposing that our algebra is measurable. If we think of it as a criterion for the existence of a G-invariant measure, rather than as a criterion for measurability in the abstract, it seems to make better sense. But if we know from the start that the algebra $\mathfrak{A}$ is measurable, the argument short-circuits, as we shall see in §396.

I take the trouble to include the 'τ' in every 'G-τ-equidecomposable', 'G^*_τ' and '$\preccurlyeq^\tau_G$' because there are important variations on the concept, in which the partitions $\langle a_i\rangle_{i\in I}$ of 395A are required to be finite or countable. Indeed **Tarski's theorem** relies on one of these. I spell it out because it is close to Kawada's in spirit, though there are significant differences in the ideas needed in the proof:

> Let X be a set and G a subgroup of $\operatorname{Aut}\mathcal{P}X$. Then the following are equiveridical: (i) there is a G-invariant additive functional $\theta : \mathcal{P}X \to [0, 1]$ such that $\theta A = 1$; (ii) there are no $A_0, \dots, A_n$, $\pi_0, \dots, \pi_n$, $\pi'_0, \dots, \pi'_n$ such that $A_0, \dots, A_n$ are subsets of X covering X, $\pi_0, \dots, \pi'_n$ belong to G, and $\pi_0[A_0]$, $\pi'_0[A_0]$, $\pi_1[A_1]$, $\pi'_1[A_1], \dots, \pi'_n[A_n]$ are all disjoint.

For a proof, see 449L in Volume 4; for an illuminating discussion of this theorem, see Wagon 85, Chapter 9. But it seems to be unknown whether the natural translation of this result is valid in all Dedekind complete Boolean algebras (395Z). Note that we are looking for theorems which do not depend on any special properties of the group G or the Boolean algebra $\mathfrak{A}$. For abelian or 'amenable' groups, or weakly (σ,∞)-distributive algebras, for instance, much more can be done, as described in 396Ya and §449.

The methods of this section can, however, be used to prove similar results for *countable* groups of automorphisms on Dedekind σ-complete Boolean algebras; I will return to such questions in §448. The presentation here owes a good deal to Nadkarni 90 and something to Becker & Kechris 96.

As noted, Kawada (Kawada 1944) treated the case in which the group G of automorphisms is ergodic, that is, the fixed-point subalgebra $\mathfrak{C}$ is trivial. Under this hypothesis the proof is of course very much simpler. (You may find it useful to reconstruct the original version, as suggested in 395Xa.) I give the more general argument partly for the sake of 395O, partly to separate out the steps which really need ergodicity from those which depend only on non-paradoxicality, partly to prepare the ground for the countable version in the next volume, partly to show off the power of the construction in §364, and partly to get you used to 'Boolean-valued' arguments. A bolder use of language could indeed simplify some formulae slightly by writing (for instance) $[\![k\lceil a_0 : a\rceil < \lfloor 1 : a\rfloor]\!]$ in place of $[\![\lfloor 1 : a\rfloor - k\lceil a_0 : a\rceil > 0]\!]$ (see part (f) of the proof of 395N). As in §388, the differences involved in the extension to non-ergodic groups are, in a sense, just a matter of technique; but this time the technique is more obtrusive. In §556 of Volume 5 I will try to explain a general approach to questions of this kind, using metamathematical ideas.

Version of 15.8.08

396 The Hajian-Ito theorem

In the notes to the last section, I said that the argument there short-circuits if we are told that we are dealing with a measurable algebra. The point is that in this case there is a much simpler criterion for the existence of a G-invariant measure (396B(ii)), with a proof which is independent of §395 in all its non-trivial parts, which makes it easy to prove that non-paradoxicality is sufficient as well as necessary.

396A Lemma Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra.

(a) Let $\pi \in \operatorname{Aut}\mathfrak{A}$ be a Boolean automorphism (not necessarily measure-preserving), and T_π the corresponding Riesz homomorphism from $L^0 = L^0(\mathfrak{A})$ to itself (364P). Then there is a unique $w_\pi \in (L^0)^+$ such that $\int w_\pi \times v = \int T_\pi v$ for every $v \in (L^0)^+$.

(b) If ϕ, $\pi \in \operatorname{Aut}\mathfrak{A}$ then $w_{\pi\phi} = w_\phi \times T_{\phi^{-1}} w_\pi$.

(c) For each $\pi \in \operatorname{Aut}\mathfrak{A}$ we have a norm-preserving isomorphism U_π from $L^2 = L^2(\mathfrak{A}, \bar\mu)$ to itself defined by setting

$$U_\pi v = T_\pi v \times \sqrt{w_{\pi^{-1}}}$$

for every $v \in L^2$, and $U_{\pi\phi} = U_\pi U_\phi$ for all π, $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof (a) Set $\bar\nu a = \bar\mu(\pi a)$ for $a \in \mathfrak{A}$. Then $(\mathfrak{A}, \bar\nu)$ is a semi-finite measure algebra. **P** $\bar\nu 0 = \bar\mu 0 = 0$; if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, then $\langle \pi a_n\rangle_{n\in\mathbb{N}}$ is disjoint and (because π is sequentially order-continuous) $a = \sup_{n\in\mathbb{N}} \pi a_n$, so $\bar\nu a = \sum_{n=0}^\infty \bar\nu a_n$; if $a \ne 0$ then $\pi a \ne 0$ so $\bar\nu a > 0$. Thus $(\mathfrak{A}, \bar\nu)$ is a measure algebra. If $a \ne 0$ there is a $b \subseteq \pi a$ such that $0 < \bar\mu b < \infty$, and now $\pi^{-1}b \subseteq a$ and $0 < \bar\nu(\pi^{-1}b) < \infty$; thus $\bar\nu$ is semi-finite. **Q**

By 365S, there is a unique $w_\pi \in (L^0)^+$ such that $\int_a w_\pi = \bar\mu(\pi a)$ for every $a \in \mathfrak{A}$. If we look at

$$W = \{v : v \in (L^0)^+, \int v \times w_\pi = \int T_\pi v\},$$

we see that W contains χa for every $a \in \mathfrak{A}$, that $v + v' \in W$ and $\alpha v \in W$ whenever v, $v' \in W$ and $\alpha \ge 0$, and that $\sup_{n\in\mathbb{N}} v_n \in W$ whenever $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in W which is bounded above in L^0. By 364Jd, $W = (L^0)^+$, as required.

(b) For any $v \in (L^0)^+$,

$$\int w_{\pi\phi} \times v = \int T_{\pi\phi} v = \int T_\pi T_\phi v$$

(364Pe)

$$= \int w_\pi \times T_\phi v = \int T_\phi(T_{\phi^{-1}} w_\pi \times v)$$

(recalling that T_ϕ is multiplicative)

$$= \int w_\phi \times T_{\phi^{-1}} w_\pi \times v.$$

As v is arbitrary (and $(\mathfrak{A}, \bar\mu)$ is semi-finite), $w_{\pi\phi} = w_\phi \times T_{\phi^{-1}} w_\pi$.

(c)(i) For any $v \in L^0$,

$$\int (T_\pi v \times \sqrt{w_{\pi^{-1}}})^2 = \int T_\pi v^2 \times w_{\pi^{-1}} = \int T_{\pi^{-1}} T_\pi v^2 = \int v^2.$$

So $U_\pi v \in L^2$ and $\|U_\pi v\|_2 = \|v\|_2$ whenever $v \in L^2$, and U_π is a norm-preserving operator on L^2.

(ii) Now consider $U_{\pi\phi}$. For any $v \in L^2$, we have

$$U_\pi U_\phi v = T_\pi(T_\phi v \times \sqrt{w_{\phi^{-1}}}) \times \sqrt{w_{\pi^{-1}}}$$
$$= T_\pi T_\phi v \times \sqrt{T_\pi w_{\phi^{-1}} \times w_{\pi^{-1}}}$$

(using 364Pd)

$$= T_{\pi\phi} v \times \sqrt{w_{\phi^{-1}\pi^{-1}}}$$

(by (b) above)

$$= U_{\pi\phi} v.$$

So $U_{\pi\phi} = U_\pi U_\phi$.

(iii) Writing ι for the identity operator on $\mathfrak{A}$, we see that T_ι is the identity operator on L^0, $w_\iota = \chi 1$ and U_ι is the identity operator on L^2. Since $U_{\pi^{-1}} U_\pi = U_\pi U_{\pi^{-1}} = U_\iota$, $U_\pi : L^2 \to L^2$ is an isomorphism, with inverse $U_{\pi^{-1}}$, for every $\pi \in \operatorname{Aut}\mathfrak{A}$.

396B Theorem (Hajian & Ito 69) Let $\mathfrak{A}$ be a measurable algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is a G-invariant functional $\bar\nu$ such that $(\mathfrak{A}, \bar\nu)$ is a totally finite measure algebra;

(ii) whenever $a \in \mathfrak{A} \setminus \{0\}$ and $\langle \pi_n \rangle_{n\in\mathbb{N}}$ is a sequence in G, $\langle \pi_n a \rangle_{n\in\mathbb{N}}$ is not disjoint;

(iii) G is fully non-paradoxical (definition: 395E).

proof (a) (i)$\Rightarrow$(iii) by the argument of 395F, and (iii)$\Rightarrow$(ii) by the criterion (ii) of 395E. So for the rest of the proof I assume that (ii) is true and seek to prove (i).

(b) Let $\bar\mu$ be such that $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra. If $a \in \mathfrak{A} \setminus \{0\}$, then $\inf_{\pi\in G} \bar\mu(\pi a) > 0$. **P?** Otherwise, let $\langle \pi_n \rangle_{n\in\mathbb{N}}$ be a sequence in G such that $\bar\mu \pi_n a \le 2^{-n}$ for each $n \in \mathbb{N}$. Set $b_n = \sup_{k\ge n} \pi_k a$ for each n; then $\inf_{n\in\mathbb{N}} b_n = 0$, so that

$$\inf_{n\in\mathbb{N}} \pi b_n = 0, \quad \lim_{n\to\infty} \bar\mu(\pi\pi_n a) = 0$$

for every $\pi \in \operatorname{Aut}\mathfrak{A}$. Choose $\langle n_i \rangle_{i\in\mathbb{N}}$ inductively so that

$$\bar\mu(\pi_{n_i}^{-1}\pi_{n_j} a) \le 2^{-j-2}\bar\mu a$$

whenever $i < j$. Set

$$c = a \setminus \sup_{i<j} \pi_{n_i}^{-1}\pi_{n_j} a.$$

Because

$$\sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \bar\mu(\pi_{n_i}^{-1}\pi_{n_j} a) < \bar\mu a,$$

$c \ne 0$, while $\pi_{n_i} c \cap \pi_{n_j} c = 0$ whenever $i < j$, contrary to the hypothesis (ii). **XQ**

(c) For each $\pi \in G$, define $w_\pi \in L^0 = L^0(\mathfrak{A})$ and $U_\pi : L^2 \to L^2$ as in 396A, where $L^2 = L^2(\mathfrak{A}, \bar\mu)$. If $a \in \mathfrak{A} \setminus \{0\}$, then $\inf_{\pi\in G} \int_a \sqrt{w_\pi} > 0$. **P?** Otherwise, there is a sequence $\langle \pi_n \rangle_{n\in\mathbb{N}}$ in G such that $\int_a v_n \le 4^{-n-2}\bar\mu a$ for every n, where $v_n = \sqrt{w_{\pi_n}}$. In this case, $\bar\mu(a \cap [\![v_n \ge 2^{-n}]\!]) \le 2^{-n-2}\bar\mu a$ for every n, so that $b = a \setminus \sup_{n\in\mathbb{N}} [\![v_n \ge 2^{-n}]\!]$ is non-zero. But now

$$\bar\mu(\pi_n b) = \int_b w_{\pi_n} = \int_b v_n^2 \le 4^{-n}\bar\mu b \to 0$$

as $n \to \infty$, contradicting (b) above. **XQ**

(d) Write $e = \chi 1$ for the standard weak order unit of L^0 or L^2. Let $C \subseteq L^2$ be the convex hull of $\{U_\pi e : \pi \in G\}$. Then C and its norm closure $\overline{C}$ are G-invariant in the sense that $U_\pi v \in C$, $U_\pi v' \in \overline{C}$

whenever $v \in C$, $v' \in \overline{C}$ and $\pi \in G$. By 3A5Md, there is a unique $u_0 \in \overline{C}$ such that $\|u_0\|_2 \le \|u\|_2$ for every $u \in \overline{C}$. Now if $\pi \in G$, $U_\pi u_0 \in \overline{C}$, while $\|U_\pi u_0\|_2 = \|u_0\|_2$; so $U_\pi u_0 = u_0$. Also, if $a \in \mathfrak{A} \setminus \{0\}$,

$$\int_a u_0 \ge \inf_{u\in\overline{C}} \int_a u = \inf_{u\in C} \int_a u$$

(because $u \mapsto \int_a u$ is $\| \|_2$-continuous)

$$= \inf_{\pi\in G} \int_a U_\pi e = \inf_{\pi\in G} \int_a T_\pi e \times \sqrt{w_{\pi^{-1}}}$$

$$= \inf_{\pi\in G} \int_a \sqrt{w_{\pi^{-1}}} = \inf_{\pi\in G} \int_a \sqrt{w_\pi} > 0$$

by (c). So $[\![u_0 > 0]\!] = 1$.

(e) For $a \in \mathfrak{A}$, set $\bar\nu a = \int_a u_0^2$. Because $u_0 \in L^2$, $\bar\nu$ is a non-negative countably additive functional on $\mathfrak{A}$; because $[\![u_0^2 > 0]\!] = [\![u_0 > 0]\!] = 1$, $\bar\nu$ is strictly positive, and $(\mathfrak{A}, \bar\nu)$ is a totally finite measure algebra. Finally, $\bar\nu$ is G-invariant. **P** If $a \in \mathfrak{A}$ and $\pi \in G$, then

$$\bar\nu(\pi a) = \int_{\pi a} u_0^2 = \int u_0^2 \times \chi(\pi a) = \int T_\pi(T_{\pi^{-1}} u_0^2 \times \chi a)$$

$$= \int w_\pi \times T_{\pi^{-1}} u_0^2 \times \chi a = \int_a (T_{\pi^{-1}} u_0 \times \sqrt{w_\pi})^2$$

$$= \int_a (U_{\pi^{-1}} u_0)^2 = \int_a u_0^2 = \bar\nu a. \ \mathbf{Q}$$

So (i) is true.

396C Remark If $\mathfrak{A}$ is a Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$, a non-zero element a of $\mathfrak{A}$ is called **weakly wandering** if there is a sequence $\langle \pi_n \rangle_{n\in\mathbb{N}}$ in G such that $\langle \pi_n a \rangle_{n\in\mathbb{N}}$ is disjoint. Thus condition (ii) of 396B may be read as 'there is no weakly wandering element of $\mathfrak{A}$'.

396X Basic exercises (a) Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an order-continuous Boolean homomorphism. Let $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ be the corresponding Riesz homomorphism. Show that there is a unique $w_\pi \in L^1(\mathfrak{A}, \bar\mu)$ such that $\int T_\pi v = \int v \times w_\pi$ for every $v \in L^0(\mathfrak{A})^+$.

(b) In 396A, show that the map $\pi \mapsto U_\pi : \operatorname{Aut}\mathfrak{A} \to \mathsf{B}(L^2; L^2)$ is injective.

(c) Let $\mathfrak{A}$ be a measurable algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Suppose that there is a strictly positive G-invariant finitely additive functional on $\mathfrak{A}$. Show that there is a G-invariant $\bar\mu$ such that $(\mathfrak{A}, \bar\mu)$ is a totally finite measure algebra.

396Y Further exercises (a) Let $\mathfrak{A}$ be a weakly (σ, ∞)-distributive Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For a, $b \in \mathfrak{A}$, say that a and b are G**-equidecomposable** if there are *finite* partitions of unity $\langle a_i \rangle_{i\in I}$ in $\mathfrak{A}_a$ and $\langle b_i \rangle_{i\in I}$ in $\mathfrak{A}_b$, and a family $\langle \pi_i \rangle_{i\in I}$ in G, such that $\pi_i a_i = b_i$ for every $i \in I$. Show that the following are equiveridical: (i) G is fully non-paradoxical in the sense of 395E; (ii) if $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence of mutually G-equidecomposable elements of $\mathfrak{A}$, they must all be 0.

396 Notes and comments I have separated these few pages from §395 partly because §395 was already up to full weight and partly in order that the ideas here should not be entirely overshadowed by those of the earlier section. It will be evident that the construction of the U_π in 396A, providing us with a faithful representation, acting on a Hilbert space, of the whole group $\operatorname{Aut}\mathfrak{A}$, is a basic tool for the study of that group.

Version of 21.10.07

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

393B The association of a metric with a strictly positive submeasure, used in the 2003 and 2006 editions of Volume 4, is now in 392H and 393H.

393C The result that a non-negative additive functional on a Boolean algebra can be factored through a measure algebra, used in the 2003 and 2006 editions of Volume 4, is now in 392I.

393O The note on control measures for vector measures, referred to in the 2003 and 2006 editions of Volume 4, is now in 394Q.

§**394** Kawada's theorem, referred to in the 2003 and 2006 editions of Volume 4, is now in §395.

Version of 25.8.17

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