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Chapter 38

Automorphism groups

As with any mathematical structure, every measure algebra has an associated symmetry group, the group of all measure-preserving automorphisms. In this chapter I set out to describe some of the remarkable features of these groups. I begin with elementary results on automorphisms of general Boolean algebras (§381), introducing definitions for the rest of the chapter. In §382 I give a general theorem on the expression of an automorphism as the product of involutions (382M), with a description of the normal subgroups of certain groups of automorphisms (382R). Applications of these ideas to measure algebras are in §383. I continue with a discussion of circumstances under which these automorphism groups determine the underlying algebras and/or have few outer automorphisms (§384).

One of the outstanding open problems of the subject is the 'isomorphism problem', the classification of automorphisms of measure algebras up to conjugacy in the automorphism group. I offer two sections on 'entropy', the most important numerical invariant enabling us to distinguish some non-conjugate automorphisms (§§385-386). For Bernoulli shifts on the Lebesgue measure algebra (385Q-385S), the isomorphism problem is solved by Ornstein's theorem (387J, 387L); I present a complete proof of this theorem in §§386-387. Finally, in §388, I give Dye's theorem, describing the full subgroups generated by single automorphisms of measure algebras of countable Maharam type.

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381 Automorphisms of Boolean algebras

I begin the chapter with a preparatory section of definitions (381B) and mostly elementary facts. A fundamental method of constructing automorphisms is in 381C-381D. The idea of 'support' of an endomorphism is explored in 381E-381G, a first look at 'periodic' and 'aperiodic' parts is in 381H, and basic facts about 'full subgroups' are in 381I-381J. We start to go deeper with the notion of 'recurrence', treated in 381L-381P. I describe how these phenomena appear when we represent an endomorphism as a map on the Stone space of an algebra (381Q). I end by introducing a 'cycle notation' for certain automorphisms.

381A The group Aut $\mathfrak{A}$ For any Boolean algebra $\mathfrak{A}$, I write $\operatorname{Aut}\mathfrak{A}$ for the set of automorphisms of $\mathfrak{A}$, that is, the set of bijective Boolean homomorphisms $\pi:\mathfrak{A}\to\mathfrak{A}$. This is a group. Note that every member of $\operatorname{Aut}\mathfrak{A}$ is order-continuous.

381B Definitions (a) If $\mathfrak{A}$ is a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean homomorphism, $a\in\mathfrak{A}$ **supports** π if $\pi d=d$ for every $d\subseteq 1\setminus a$.

(b) Let $\mathfrak{A}$ be a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a Boolean homomorphism. If $\min\{a:a\in\mathfrak{A}\text{ supports }\pi\}$ is defined in $\mathfrak{A}$, I will call it the **support** $\operatorname{supp}\pi$ of π.

(c) If $\mathfrak{A}$ is a Boolean algebra, an automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is **periodic**, with **period** $n\ge 1$, if $\mathfrak{A}\ne\{0\}$, π^n is the identity operator and 1 is the support of π^i whenever $1\le i<n$.

(d) If $\mathfrak{A}$ is a Boolean algebra, a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is **aperiodic** if the support of π^n is 1 for every $n\ge 1$. I remark immediately that if π is aperiodic, so is π^n for every $n\ge 1$.

(e) If $\mathfrak{A}$ is a Boolean algebra, a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **full** if whenever $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, $\langle\pi_i\rangle_{i\in I}$ is a family in G, and $\pi\in\operatorname{Aut}\mathfrak{A}$ is such that $\pi d=\pi_i d$ whenever $i\in I$ and $d\subseteq a_i$, then $\pi\in G$.

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(f) If $\mathfrak{A}$ is a Boolean algebra, a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **countably full** if whenever $\langle a_i\rangle_{i\in I}$ is a countable partition of unity in $\mathfrak{A}$, $\langle\pi_i\rangle_{i\in I}$ is a family in G, and $\pi\in\operatorname{Aut}\mathfrak{A}$ is such that $\pi d=\pi_i d$ whenever $i\in I$ and $d\subseteq a_i$, then $\pi\in G$.

(g) If $\mathfrak{A}$ is a Boolean algebra, $a\in\mathfrak{A}$ and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean homomorphism, I say that π is **recurrent on** a if for every non-zero $b\subseteq a$ there is a $k\ge 1$ such that $a\cap\pi^k b\ne 0$. If $\pi\in\operatorname{Aut}\mathfrak{A}$ and π and π^{-1} are both recurrent on a, I say that π is **doubly recurrent** on a.

381C Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Assume
either that I is finite
or that I is countable and $\mathfrak{A}$ is Dedekind σ-complete
or that $\mathfrak{A}$ is Dedekind complete.
Suppose that for each $i\in I$ we have an isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{A}_{b_i}$ between the corresponding principal ideals. Then there is a unique $\pi\in\operatorname{Aut}\mathfrak{A}$ such that $\pi d=\pi_i d$ whenever $i\in I$ and $d\subseteq a_i$.

381D Corollary Let $\mathfrak{A}$ be a homogeneous Boolean algebra, and A, B two partitions of unity in $\mathfrak{A}$, neither containing 0. Let $\theta:A\to B$ be a bijection. Suppose
either that A, B are finite
or that A, B are countable and $\mathfrak{A}$ is Dedekind σ-complete
or that $\mathfrak{A}$ is Dedekind complete.
Then there is an automorphism of $\mathfrak{A}$ extending θ.

381E Lemma Let $\mathfrak{A}$ be a Boolean algebra, and π, ϕ, $\psi:\mathfrak{A}\to\mathfrak{A}$ Boolean homomorphisms of which π is injective.

(a) If $a\in\mathfrak{A}$ supports ϕ then $\phi a=a$ and $\phi d\subseteq a$ for every $d\subseteq a$.

(b) If $a\in\mathfrak{A}$ supports both ϕ and ψ then it supports $\phi\psi$.

(c) Let A be the set of elements of $\mathfrak{A}$ supporting ϕ. Then A is non-empty and closed under $\cap$; also $b\in A$ whenever $b\supseteq a\in A$. If ϕ is order-continuous, then $\inf B\in A$ whenever $B\subseteq A$ has an infimum in $\mathfrak{A}$.

(d) If $a\in\mathfrak{A}$ supports $\pi\phi$, then ϕa supports $\pi\phi$.

(e) If π commutes with ϕ, and $a\in\mathfrak{A}$ is such that πa supports ϕ, then a supports ϕ.

(f) If ϕ is supported by a and ψ is supported by b, where $a\cap b=0$, then $\phi\psi=\psi\phi$.

(g) For any $n\ge 1$ and $a\in\mathfrak{A}$, a supports π^n iff πa supports π^n. $\pi(\operatorname{supp}\pi^n)=\operatorname{supp}\pi^n$ if π^n has a support.

(h) If $\pi\in\operatorname{Aut}\mathfrak{A}$ and $a\in\mathfrak{A}$ supports π, then a supports π^{-1}.

(i) If $\pi\in\operatorname{Aut}\mathfrak{A}$ and $a\in\mathfrak{A}$, then

$$\begin{aligned} a\text{ supports }\pi &\iff d\mathbin{\triangle}\pi d\subseteq a\text{ for every }d\in\mathfrak{A}\\ &\iff d\subseteq a\text{ whenever }d\cap\pi d=0\\ &\iff d\cap\pi d\ne 0\text{ whenever }0\ne d\subseteq 1\setminus a.\end{aligned}$$

(j) If $\pi\in\operatorname{Aut}\mathfrak{A}$ and $a\in\mathfrak{A}$ supports ϕ, then πa supports $\pi\phi\pi^{-1}$.

(k) If $a\in\mathfrak{A}$ supports ϕ, and π_1, $\pi_2\in\operatorname{Aut}\mathfrak{A}$ agree on $\mathfrak{A}_a$, then $\pi_1\phi\pi_1^{-1}=\pi_2\phi\pi_2^{-1}$.

381F Corollary If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then every order-continuous Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{A}$ has a support.

381G Corollary Let $\mathfrak{A}$ be a Boolean algebra, and suppose that $\pi\in\operatorname{Aut}\mathfrak{A}$ has a support e.

(a) $\pi e=e$.

(b) $e=\sup\{d\mathbin{\triangle}\pi d:d\in\mathfrak{A}\}=\sup\{d:d\in\mathfrak{A},\ d\cap\pi d=0\}$.

(c) e is the support of π^{-1}.

(d) For any $\phi\in\operatorname{Aut}\mathfrak{A}$, ϕe is the support of $\phi\pi\phi^{-1}$.

381H Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an injective Boolean homomorphism such that π^n has a support for every $n\in\mathbb{N}$. Then there is a partition of unity $\langle c_i\rangle_{1\le i\le\omega}$ in $\mathfrak{A}$ such that $\pi c_i\subseteq c_i$ for every i, $\pi\restriction\mathfrak{A}_{c_n}$ is periodic with period n whenever $n\in\mathbb{N}\setminus\{0\}$ and $c_n\ne 0$, and $\pi\restriction\mathfrak{A}_{c_\omega}$ is aperiodic.

381I Full and countably full subgroups If $\mathfrak{A}$ is a Boolean algebra, the intersection of any family of (countably) full subgroups of $\operatorname{Aut}\mathfrak{A}$ is again (countably) full. We may therefore speak of the (countably) full subgroup of $\mathfrak{A}$ generated by an element of $\operatorname{Aut}\mathfrak{A}$.

Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) Let G be a subgroup of $\operatorname{Aut}\mathfrak{A}$. Let H be the set of those $\pi \in \operatorname{Aut}\mathfrak{A}$ such that for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and a $\phi \in G$ such that $\pi c = \phi c$ for every $c \subseteq b$. Then H is a full subgroup of $\operatorname{Aut}\mathfrak{A}$, the smallest full subgroup of $\mathfrak{A}$ including G.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\phi c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a_n$.

(c) Suppose that $\mathfrak{A}$ is Dedekind complete, and π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and an $n \in \mathbb{Z}$ such that $\phi c = \pi^n c$ for every $c \subseteq b$;

(iii) ϕ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(iv) $\inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi^n\phi) = 0$.

381J Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\pi \in \operatorname{Aut}\mathfrak{A}$. Suppose that ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

(a) If $c \in \mathfrak{A}$ is such that $\pi c = c$, then $\phi c = c$.

(b) If $a \in \mathfrak{A}$ supports π then it supports ϕ.

381K Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism.

(a) If $a \in \mathfrak{A}$ and $a^* = \inf_{k\in\mathbb{N}} \sup_{i\ge k} \pi^i a$, then $\pi a^* = a^*$.

(b) If $a \in \mathfrak{A}$ is such that $a \subseteq \sup_{i\ge 1} \pi^i a$, then $\sup_{i\ge k} \pi^i a = \sup_{i\in\mathbb{N}} \pi^i a$ for every $k \in \mathbb{N}$.

381L Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. Then for any $a \in \mathfrak{A}$, the following are equiveridical:

(i) π is recurrent on a;

(ii) $a \subseteq \sup_{n\ge 1} \pi^{-n} a$;

(iii) there is some $k \ge 1$ such that $a \subseteq \sup_{n\ge k} \pi^{-n} a$;

(iv) $a \subseteq \sup_{n\ge k} \pi^{-n} a$ for every $k \in \mathbb{N}$.

381M Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a \in \mathfrak{A}$. Suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ is doubly recurrent on a. Then we have a Boolean automorphism $\pi_a : \mathfrak{A}_a \to \mathfrak{A}_a$ defined by saying that $\pi_a d = \pi^n d$ whenever $n \ge 1$ and $d \subseteq a \cap \pi^{-n} a \setminus \sup_{1\le i<n} \pi^{-i} a$; I will call π_a the **induced automorphism** on $\mathfrak{A}_a$.

381N Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a \in \mathfrak{A}$. Suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ is doubly recurrent on a. Let $\pi_a \in \operatorname{Aut}\mathfrak{A}_a$ be the induced automorphism.

(a) π^{-1} is doubly recurrent on a, and the induced automorphism $(\pi^{-1})_a$ is $(\pi_a)^{-1}$.

(b) For every $n \in \mathbb{N}$ there is a partition of unity $\langle b_i\rangle_{i\ge n}$ in $\mathfrak{A}_a$ such that $\pi_a^n b = \pi^i b$ whenever $i \ge n$ and $b \subseteq b_i$.

(c) If $n \ge 1$ and $0 \ne b \subseteq a \cap \pi^{-n} a$, there are a non-zero $b' \subseteq b$ and a j such that $1 \le j \le n$ and $\pi^n d = \pi_a^j d$ for every $d \subseteq b'$.

(d) Suppose that $m \ge 1$ is such that $a \cap \pi^i a = 0$ for $1 \le i < m$. Then for any $n \ge 1$ we have a disjoint family $\langle b_{ni}\rangle_{1\le i\le\lfloor n/m\rfloor}$, with supremum $a \cap \pi^{-n} a$, such that $\pi^n d = \pi_a^i d$ whenever $1 \le i \le \lfloor \frac{n}{m} \rfloor$ and $d \subseteq b_{ni}$.

(e) Suppose that $b \subseteq a$. Then π is doubly recurrent on b iff π_a is doubly recurrent on b, and in this case $\pi_b = (\pi_a)_b$, where $(\pi_a)_b$ is the automorphism of $\mathfrak{A}_b$ induced by π_a.

(f) Suppose that $c \in \mathfrak{A}$ is such that $\pi c = c$. Then π is doubly recurrent on $a \cap c$, and $\pi_{a\cap c} = \pi_a \restriction \mathfrak{A}_{a\cap c}$; in particular, $\pi_a(a \cap c) = a \cap c$.

(g) If π is aperiodic, so is π_a.

(h) Suppose that $a\cap\pi a=0$, and that $b\subseteq a$ is such that $b\cap\pi_a b=0$. Then b, πb and $\pi^2 b$ are all disjoint.

(i) There is an automorphism $\tilde\pi_a\in\operatorname{Aut}\mathfrak{A}$ defined by setting $\tilde\pi_a d=\pi_a d$ for $d\subseteq a$, $\tilde\pi_a d=d$ for $d\subseteq 1\setminus a$, and $\tilde\pi_a$ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

381O Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a Boolean homomorphism. Then the following are equiveridical:

(i) π is recurrent on every $a\in\mathfrak{A}$;

(ii) for every non-zero $a\in\mathfrak{A}$ there is a $k\ge 1$ such that $a\cap\pi^k a\ne 0$;

(iii) $a=\sup_{k\ge1}a\cap\pi^k a$ for every $a\in\mathfrak{A}$.

381P Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a Boolean homomorphism which is recurrent on every $a\in\mathfrak{A}$. Then π is aperiodic iff $\mathfrak{A}$ is relatively atomless over the fixed-point algebra $\mathfrak{C}=\{c:c\in\mathfrak{A},\ \pi c=c\}$. In particular, if π is ergodic, it is aperiodic iff $\mathfrak{A}$ is atomless.

381Q Proposition Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. For $a\in\mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z. For a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ let $f_\pi:Z\to Z$ be the continuous function such that $\widehat{\pi a}=f_\pi^{-1}[\widehat{a}]$ for every $a\in\mathfrak{A}$.

(a) If a, $b\in\mathfrak{A}$ and $\phi:\mathfrak{A}_a\to\mathfrak{A}_b$ is a Boolean homomorphism represented by a continuous function $g:\widehat{b}\to\widehat{a}$, then $\pi\in\operatorname{Aut}\mathfrak{A}$ agrees with ϕ on $\mathfrak{A}_a$ iff f_π agrees with g on $\widehat{b}$.

(b) If $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean homomorphism, then $a\in\mathfrak{A}$ supports π iff $\widehat{a}\supseteq\{z:f_\pi(z)\ne z\}$. So a is the support of π iff $\widehat{a}=\overline{\{z:f_\pi(z)\ne z\}}$.

(c) Suppose that $\mathfrak{A}$ is Dedekind complete and π, $\phi\in\operatorname{Aut}\mathfrak{A}$. Let G be the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. Then

$$\begin{aligned}\phi\in G&\iff\bigcup_{n\in\mathbb{Z}}\operatorname{int}\{x:f_\phi(z)=f_\pi^n(z)\}\text{ is dense in }Z\\&\iff\{z:f_\phi(z)\in\{f_\pi^n(z):n\in\mathbb{Z}\}\}\text{ is comeager in }Z.\end{aligned}$$

(d) A Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is recurrent on $a\in\mathfrak{A}$ iff $\widehat{a}\subseteq\overline{\bigcup_{n\ge1}f_\pi^n[\widehat{a}]}$.

(e) Suppose that $\mathfrak{A}$ is Dedekind σ-complete, $\pi\in\operatorname{Aut}\mathfrak{A}$ is recurrent on $a\in\mathfrak{A}$, and that $\pi_a\in\operatorname{Aut}\mathfrak{A}_a$ is the induced automorphism. Let f_{π_a} be the corresponding autohomeomorphism of $\widehat{a}$. For $k\ge1$, set $G_k=\{z:z\in\widehat{a},\ f^k(z)\in\widehat{a},\ f^i(z)\notin\widehat{a}\text{ for }1\le i<k\}$. Then $\bigcup_{k\ge1}G_k=\widehat{a}\cap\bigcup_{k\ge1}f^{-k}[\widehat{a}]$ is a dense open subset of $\widehat{a}$ and $f_{\pi_a}(z)=f_\pi^k(z)$ whenever $k\ge1$ and $z\in G_k$.

381R Cyclic automorphisms: Definition Let $\mathfrak{A}$ be a Boolean algebra.

(a) Suppose that a, b are disjoint members of $\mathfrak{A}$ and that $\pi\in\operatorname{Aut}\mathfrak{A}$ is such that $\pi a=b$. I will write $(\overleftarrow{a\,{}_{\pi}\,b})$ for the member ψ of $\operatorname{Aut}\mathfrak{A}$ defined by setting

$$\begin{aligned}\psi d&=\pi d\text{ if }d\subseteq a,\\&=\pi^{-1}d\text{ if }d\subseteq b,\\&=d\text{ if }d\subseteq 1\setminus(a\cup b).\end{aligned}$$

Observe that in this case (if $a\ne0$) ψ is an involution, that is, has order 2 in the group $\operatorname{Aut}\mathfrak{A}$; I will call such a ψ an **exchanging involution**, and say that it **exchanges** a with b.

(b) More generally, if $a_1,\dots,a_n$ are disjoint elements of $\mathfrak{A}$ and $\pi_i\in\operatorname{Aut}\mathfrak{A}$ are such that $\pi_i a_i=a_{i+1}$ for each $i<n$, then I will write

$$(\overleftarrow{a_1\,{}_{\pi_1}\,a_2\,{}_{\pi_2}\,\dots\,{}_{\pi_{n-1}}\,a_n})$$

for that $\psi\in\operatorname{Aut}\mathfrak{A}$ such that

$$
\begin{aligned}
\psi d &= \pi_i d \text{ if } 1 \le i < n,\ d \subseteq a_i,\\
&= \pi_1^{-1}\pi_2^{-1}\ldots\pi_{n-1}^{-1} d \text{ if } d \subseteq a_n,\\
&= d \text{ if } d \subseteq 1 \setminus \sup_{i \le n} a_i.
\end{aligned}
$$

(c) It will occasionally be convenient to use the same notation when each π_i is a Boolean isomorphism between the principal ideals $\mathfrak{A}_{a_i}$ and $\mathfrak{A}_{a_{i+1}}$, rather than an automorphism of the whole algebra $\mathfrak{A}$.

381S Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\psi = (\overleftarrow{a \,{}_{\pi}\, b})$ is an exchanging involution in $\operatorname{Aut}\mathfrak{A}$, then

$$\psi = (\overleftarrow{a \,{}_{\psi}\, b}) = (\overleftarrow{b \,{}_{\psi}\, a}) = (\overleftarrow{b \,{}_{\pi^{-1}}\, a})$$

has support $a \cup b$.

(b) If $\pi = (\overleftarrow{a \,{}_{\pi}\, b})$ is an exchanging involution in $\operatorname{Aut}\mathfrak{A}$, then for any $\phi \in \operatorname{Aut}\mathfrak{A}$,

$$\phi\pi\phi^{-1} = (\overleftarrow{\phi a \,{}_{\phi\pi\phi^{-1}}\, \phi b})$$

is another exchanging involution.

(c) If $\pi = (\overleftarrow{a \,{}_{\pi}\, b})$ and $\phi = (\overleftarrow{c \,{}_{\phi}\, d})$ are exchanging involutions, and a, b, c, d are all disjoint, then π and ϕ commute, and $\psi = \pi\phi = \phi\pi$ is another exchanging involution, being $(\overleftarrow{a \cup c \,{}_{\psi}\, b \cup d})$.

(d) If G is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$, $a_1, \ldots, a_n \in \mathfrak{A}$ are disjoint, and $\pi_1, \ldots, \pi_{n-1} \in G$, then

$$(\overleftarrow{a_1 \,{}_{\pi_1}\, a_2 \,{}_{\pi_2}\, \ldots \,{}_{\pi_{n-1}}\, a_n}) \in G.$$

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382 Factorization of automorphisms

My aim in this chapter is to investigate the automorphism groups of measure algebras, but as usual I prefer to begin with results which can be expressed in the language of general Boolean algebras. The principal theorems in this section are 382M, giving a sufficient condition for every member of a full group of automorphisms to be a product of involutions, and 382R, describing the normal subgroups of full groups. The former depends on Dedekind σ-completeness and the presence of 'separators' (382Aa); the latter needs a Dedekind complete algebra and a group with 'many involutions' (382O). Both concepts are chosen with a view to the next section, where the results will be applied to groups of measure-preserving automorphisms.

382A Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$.

(a) I say that $a \in \mathfrak{A}$ is a **separator** for π if $a \cap \pi a = 0$ and $\pi b = b$ whenever $b \in \mathfrak{A}$ and $b \cap \pi^n a = 0$ for every $n \in \mathbb{Z}$.

(b) I say that $a \in \mathfrak{A}$ is a **transversal** for π if $\sup_{n\in\mathbb{Z}} \pi^n a = 1$ and $\pi^n b = b$ whenever $n \in \mathbb{Z}$ and $b \subseteq a \cap \pi^n a$.

382B Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. If every power of π has a separator and π^n is the identity, where $n \ge 1$, then π has a transversal.

382C Corollary If $\mathfrak{A}$ is a Boolean algebra and $\pi \in \mathfrak{A}$ is an involution, then π is an exchanging involution iff it has a separator iff it has a transversal.

382D Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) π has a separator;

(ii) there is an $a \in \mathfrak{A}$ such that $a \cap \pi a = 0$ and $a \cup \pi a \cup \pi^2 a$ supports π;

(iii) there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in\mathbb{N}}\pi a_n\setminus a_n$ supports π;

(iv) there is a partition of unity (a',a'',b',b'',c,e) in $\mathfrak{A}$ such that

$$\pi a'=b',\quad \pi a''=b'',\quad \pi b''=c,\quad \pi(b'\cup c)=a'\cup a'',\quad \pi d=d \text{ for every } d\subseteq e.$$

382E Corollary (a) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi\in\operatorname{Aut}\mathfrak{A}$ has a separator, then π has a support.

(b) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra then every $\pi\in\operatorname{Aut}\mathfrak{A}$ has a separator.

382F Corollary Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra.

(a) Every involution in $\operatorname{Aut}\mathfrak{A}$ is an exchanging involution.

(b) If $\pi\in\operatorname{Aut}\mathfrak{A}$ is periodic with period $n\ge 2$, there is an $a\in\mathfrak{A}$ such that $(a,\pi a,\pi^2a,\dots,\pi^{n-1}a)$ is a partition of unity in $\mathfrak{A}$.

382G Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi\in\operatorname{Aut}\mathfrak{A}$.

(a) Suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a family in $\mathfrak{A}$ such that $\pi a_n=a_n$ and $\pi{\restriction}\mathfrak{A}_{a_n}$ has a transversal for every n. Set $a=\sup_{n\in\mathbb{N}}a_n$; then $\pi a=a$ and $\pi{\restriction}\mathfrak{A}_a$ has a transversal.

(b) If a is a transversal for π it is a transversal for π^{-1}.

(c) Suppose that $a\in\mathfrak{A}$. Set

$$a^*=\sup_{n\in\mathbb{Z}}(\pi^na\setminus\sup_{i>n}\pi^ia),\quad a_*=\sup_{n\in\mathbb{Z}}(\pi^na\setminus\sup_{i<n}\pi^ia).$$

Then $\pi a^*=a^*$, $\pi a_*=a_*$ and $\pi{\restriction}\mathfrak{A}_{a^*}$, $\pi{\restriction}\mathfrak{A}_{a_*}$ both have transversals.

382H Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi\in\operatorname{Aut}\mathfrak{A}$. If π has a transversal, it is expressible as the product of at most two exchanging involutions both belonging to the countably full subgroup of $\mathfrak{A}$ generated by π.

382I Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator.

(a) Every member of G has a support.

(b) Suppose $\pi\in G$ and $n\ge1$ are such that π^n is the identity. Then π has a transversal.

(c) Let $\pi\in G$, and set $e^*=\inf_{n\ge1}\operatorname{supp}(\pi^n)$. Then $\pi{\restriction}\mathfrak{A}_{1\setminus e^*}$ has a transversal.

(d) If $e\in\mathfrak{A}$ is such that $\pi e=e$ for every $\pi\in G$, then $\{\pi{\restriction}\mathfrak{A}_e:\pi\in G\}$ is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}_e$, and $\pi{\restriction}\mathfrak{A}_e$ has a separator for every $\pi\in G$.

382J Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator, and $\pi\in G$ an aperiodic automorphism. Then there is a non-increasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $e_0=1$ and

(i) π is doubly recurrent on e_n, and in fact $\sup_{i\ge1}\pi^ie_n=\sup_{i\ge1}\pi^{-i}e_n=1$,

(ii) e_{n+1}, $\pi_{e_n}e_{n+1}$ and $\pi_{e_n}^2e_{n+1}$ are disjoint

for every $n\in\mathbb{N}$, where $\pi_{e_n}\in\operatorname{Aut}\mathfrak{A}_{e_n}$ is the automorphism induced by π.

382K Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Suppose that we have an aperiodic $\pi\in\operatorname{Aut}\mathfrak{A}$ and a non-increasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $e_0=1$ and

$$\sup_{i\ge1}\pi^ie_n=\sup_{i\ge1}\pi^{-i}e_n=1,\quad e_{n+1},\ \pi_{e_n}(e_{n+1}) \text{ and } \pi_{e_n}^2(e_{n+1}) \text{ are disjoint}$$

for every $n\in\mathbb{N}$, writing $\pi_{e_n}\in\operatorname{Aut}\mathfrak{A}_{e_n}$ for the induced automorphism. Let G be the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. Then there is a $\phi\in G$ such that ϕ is either the identity or an exchanging involution and $\inf_{n\ge1}\operatorname{supp}(\pi\phi)^n=0$.

382L Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator. If $\pi\in G$, there is a $\phi\in G$ such that ϕ is either the identity or an exchanging involution and $\pi\phi$ has a transversal.

382M Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator. If $\pi \in G$, it can be expressed as the product of at most three exchanging involutions belonging to G.

382N Corollary If $\mathfrak{A}$ is a Dedekind complete Boolean algebra and G is a full subgroup of $\operatorname{Aut}\mathfrak{A}$, every $\pi \in G$ is expressible as the product of at most three involutions all belonging to G and all supported by $\operatorname{supp}\pi$.

382O Definition Let $\mathfrak{A}$ be a Boolean algebra, and G a subgroup of the automorphism group $\operatorname{Aut}\mathfrak{A}$. I will say that G **has many involutions** if for every non-zero $a \in \mathfrak{A}$ there is an involution $\pi \in G$ which is supported by a.

382P Lemma Let $\mathfrak{A}$ be an atomless homogeneous Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ has many involutions, and in fact every non-zero element of $\mathfrak{A}$ is the support of an exchanging involution.

382Q Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Then every non-zero element of $\mathfrak{A}$ is the support of an exchanging involution belonging to G.

382R Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Then a subset H of G is a normal subgroup of G iff it is of the form

$$\{\pi : \pi \in G,\ \operatorname{supp}\pi \in I\}$$

for some ideal $I \triangleleft \mathfrak{A}$ which is G**-invariant**, that is, such that $\pi a \in I$ for every $a \in I$ and $\pi \in G$.

382S Corollary Let $\mathfrak{A}$ be a homogeneous Dedekind complete Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ is simple.

Version of 9.11.14

383 Automorphism groups of measure algebras

I turn now to the group of measure-preserving automorphisms of a measure algebra, seeking to apply the results of the last section. The principal theorems are 383D, which is a straightforward special case of 382N, and 383I, corresponding to 382S. I give another example of the use of 382R to describe the normal subgroups of $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ (383J), and conclude with an important fact about conjugacy in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{A}$ (383L).

383A Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. I will write $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ for the set of all measure-preserving automorphisms of $\mathfrak{A}$. This is a group.

383B Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Assume
either that I is countable
or that $(\mathfrak{A},\bar\mu)$ is localizable.
Suppose that for each $i \in I$ we have a measure-preserving isomorphism $\pi_i : \mathfrak{A}_{a_i} \to \mathfrak{A}_{b_i}$ between the corresponding principal ideals. Then there is a unique $\pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ such that $\pi c = \pi_i c$ whenever $i \in I$ and $c \subseteq a_i$.

383C Corollary If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra, then $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is a full subgroup of $\operatorname{Aut}\mathfrak{A}$.

383D Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Then every measure-preserving automorphism of $\mathfrak{A}$ is expressible as the product of at most three measure-preserving involutions.

383E Lemma If $(\mathfrak{A},\bar\mu)$ is a homogeneous semi-finite measure algebra, it is σ-finite, therefore localizable.

383F Lemma Let $(\mathfrak{A},\bar{\mu})$ be a homogeneous semi-finite measure algebra.

(a) If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are partitions of unity in $\mathfrak{A}$ with $\bar{\mu}a_i=\bar{\mu}b_i$ for every i, there is a $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\pi a_i=b_i$ for each i.

(b) If $(\mathfrak{A},\bar{\mu})$ is totally finite, then whenever $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ with $\bar{\mu}a_i=\bar{\mu}b_i$ for every i, there is a $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\pi a_i=b_i$ for each i.

383G Lemma (a) If $(\mathfrak{A},\bar{\mu})$ is an atomless semi-finite measure algebra, then $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ have many involutions.

(b) If $(\mathfrak{A},\bar{\mu})$ is an atomless localizable measure algebra, then every non-zero element of $\mathfrak{A}$ is the support of an involution in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$.

383H Corollary Let $(\mathfrak{A},\bar{\mu})$ be an atomless localizable measure algebra. Then

(a) the lattice of normal subgroups of $\mathrm{Aut}\,\mathfrak{A}$ is isomorphic to the lattice of $\mathrm{Aut}\,\mathfrak{A}$-invariant ideals of $\mathfrak{A}$;

(b) the lattice of normal subgroups of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is isomorphic to the lattice of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$.

383I Theorem Let $(\mathfrak{A},\bar{\mu})$ be a homogeneous semi-finite measure algebra.

(a) $\mathrm{Aut}\,\mathfrak{A}$ is simple.

(b) If $(\mathfrak{A},\bar{\mu})$ is totally finite, $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is simple.

(c) If $(\mathfrak{A},\bar{\mu})$ is not totally finite, $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ has exactly one non-trivial proper normal subgroup.

383J Proposition Let $(\mathfrak{A},\bar{\mu})$ be an atomless totally finite measure algebra. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}$, and let K be $\{\kappa:e_\kappa\ne0\}$. Let $\mathcal{H}$ be the lattice of normal subgroups of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Then

(i) if K is finite, $\mathcal{H}$ is isomorphic, as partially ordered set, to $\mathcal{P}K$;

(ii) if K is infinite, then $\mathcal{H}$ is isomorphic, as partially ordered set, to the lattice of solid linear subspaces of ℓ^∞.

383K Proposition Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an ergodic measure-preserving Boolean homomorphism. If $\phi\in\mathrm{Aut}\,\mathfrak{A}$ is such that $\phi\pi\phi^{-1}$ is measure-preserving, then ϕ is measure-preserving.

383L Corollary Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra, and π_1, $\pi_2\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ two ergodic measure-preserving automorphisms. If they are conjugate in $\mathrm{Aut}\,\mathfrak{A}$ then they are conjugate in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$.

Version of 5.11.14

384 Outer automorphisms

Continuing with the investigation of the abstract group-theoretic nature of the automorphism groups $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$, I devote a section to some remarkable results concerning isomorphisms between them. Under any of a variety of conditions, any isomorphism between two groups $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}\,\mathfrak{B}$ must correspond to an isomorphism between the underlying Boolean algebras (384E, 384F, 384J, 384M); consequently $\mathrm{Aut}\,\mathfrak{A}$ has few, or no, outer automorphisms (384G, 384K, 384O). I organise the section around a single general result (384D).

384A Lemma Let $\mathfrak{A}$ be a Boolean algebra and G a subgroup of $\mathrm{Aut}\,\mathfrak{A}$ which has many involutions. Then for every non-zero $a\in\mathfrak{A}$ there is an automorphism $\psi\in G$, of order 4, which is supported by a.

384B A note on supports Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\theta:\mathfrak{A}\to\mathfrak{B}$ a Boolean isomorphism. If $\pi\in\mathrm{Aut}\,\mathfrak{A}$ is supported by $a\in\mathfrak{A}$, then $\theta\pi\theta^{-1}\in\mathrm{Aut}\,\mathfrak{B}$ is supported by θa. Accordingly, if a is the support of π then θa will be the support of $\theta\pi\theta^{-1}$.

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384C Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Boolean algebras, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. If θ_1, $\theta_2 : \mathfrak{A} \to \mathfrak{B}$ are distinct isomorphisms, then there is a $\phi \in G$ such that $\theta_1\phi\theta_1^{-1} \neq \theta_2\phi\theta_2^{-1}$.

384D Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras and G and H subgroups of $\operatorname{Aut}\mathfrak{A}$, $\operatorname{Aut}\mathfrak{B}$ respectively, both having many involutions. Let $q : G \to H$ be an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in G$.

384E Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be atomless homogeneous Boolean algebras, and $q : \operatorname{Aut}\mathfrak{A} \to \operatorname{Aut}\mathfrak{B}$ an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

384F Corollary If $\mathfrak{A}$ and $\mathfrak{B}$ are atomless homogeneous Boolean algebras with isomorphic automorphism groups, they are isomorphic as Boolean algebras.

384G Corollary If $\mathfrak{A}$ is a homogeneous Boolean algebra, then $\operatorname{Aut}\mathfrak{A}$ has no outer automorphisms.

384H Definitions (a) A Boolean algebra $\mathfrak{A}$ is **rigid** if the only automorphism of $\mathfrak{A}$ is the identity automorphism.

(b) A Boolean algebra $\mathfrak{A}$ is **nowhere rigid** if no non-trivial principal ideal of $\mathfrak{A}$ is rigid.

384I Lemma Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is nowhere rigid;

(ii) for every $a \in \mathfrak{A} \setminus \{0\}$ there is a $\phi \in \operatorname{Aut}\mathfrak{A}$, not the identity, supported by a;

(iii) for every $a \in \mathfrak{A} \setminus \{0\}$ there are distinct b, $c \subseteq a$ such that the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_c$ they generate are isomorphic;

(iv) the automorphism group $\operatorname{Aut}\mathfrak{A}$ has many involutions.

384J Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be nowhere rigid Dedekind complete Boolean algebras and $q : \operatorname{Aut}\mathfrak{A} \to \operatorname{Aut}\mathfrak{B}$ an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

384K Corollary Let $\mathfrak{A}$ be a nowhere rigid Dedekind complete Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ has no outer automorphisms.

384L Examples (a) A non-trivial homogeneous Boolean algebra is nowhere rigid.

(b) Any principal ideal of a nowhere rigid Boolean algebra is nowhere rigid.

(c) A simple product of nowhere rigid Boolean algebras is nowhere rigid.

(d) Any atomless semi-finite measure algebra is nowhere rigid.

(e) A free product of nowhere rigid Boolean algebras is nowhere rigid.

(f) The Dedekind completion of a nowhere rigid Boolean algebra is nowhere rigid.

384M Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be atomless localizable measure algebras, and $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ the corresponding groups of measure-preserving automorphisms. Let $q : \operatorname{Aut}_{\bar{\mu}}\mathfrak{A} \to \operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ be an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$.

384N Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras and $\theta : \mathfrak{A} \to \mathfrak{B}$ a Boolean isomorphism. For each infinite cardinal κ let e_κ be the Maharam-type-κ component of $\mathfrak{A}$ and for each $\gamma \in \,]0, \infty[$ let A_γ be the set of atoms of $\mathfrak{A}$ of measure γ. Then the following are equiveridical:

(i) for every $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $\theta\phi\theta^{-1} \in \operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$;

(ii)(α) for every infinite cardinal κ there is an $\alpha_\kappa > 0$ such that $\bar{\nu}(\theta a) = \alpha_\kappa\bar{\mu}a$ for every $a \subseteq e_\kappa$,

(β) for every $\gamma \in \,]0, \infty[$ there is an $\alpha_\gamma > 0$ such that $\bar{\nu}(\theta a) = \alpha_\gamma\bar{\mu}a$ for every $a \in A_\gamma$.

384O Corollary If $(\mathfrak{A},\bar{\mu})$ is an atomless totally finite measure algebra, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ has no outer automorphisms.

384P Examples (a) There are an atomless localizable measure algebra $(\mathfrak{A},\bar{\mu})$ and an atomless semi-finite measure algebra $(\mathfrak{B},\bar{\nu})$ such that $\operatorname{Aut}\mathfrak{A}\cong\operatorname{Aut}\mathfrak{B}$, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}\cong\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ but $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic.

(b) There is an atomless semi-finite measure algebra $(\mathfrak{C},\bar{\lambda})$ such that $\operatorname{Aut}\mathfrak{C}$ has an outer automorphism.

384Q Example Let μ be Lebesgue measure on $\mathbb{R}$, and $(\mathfrak{A},\bar{\mu})$ its measure algebra. Then $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ has an outer automorphism.

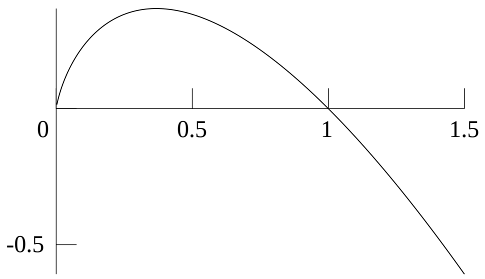
Version of 14.1.15

385 Entropy

Perhaps the most glaring problem associated with the theory of measure-preserving homomorphisms and automorphisms is the fact that we have no generally effective method of determining when two homomorphisms are the same, in the sense that two structures $(\mathfrak{A},\bar{\mu},\pi)$ and $(\mathfrak{B},\bar{\nu},\phi)$ are isomorphic, where $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are measure algebras and $\pi:\mathfrak{A}\to\mathfrak{A}$, $\phi:\mathfrak{B}\to\mathfrak{B}$ are Boolean homomorphisms. Of course the first part of the problem is to decide whether $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are isomorphic; but this is solved (at least for localizable algebras) by Maharam's theorem. The difficulty lies in the homomorphisms. Even when we know that $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are both isomorphic to the Lebesgue measure algebra, the extraordinary variety of constructions of homomorphisms – corresponding in part to the variety of measure spaces with such measure algebras, each with its own natural inverse-measure-preserving functions – means that the question of which are isomorphic to each other is continually being raised. In this section I give the most elementary ideas associated with the concept of 'entropy', up to the Kolmogorov-Sinaǐ theorem (385P). This is an invariant which can be attached to any measure-preserving homomorphism on a probability algebra, and therefore provides a useful method for distinguishing non-isomorphic homomorphisms.

The main work of the section deals with homomorphisms on measure algebras, but as many of the most important ones arise from inverse-measure-preserving functions on measure spaces. I comment on the extra problems arising in the isomorphism problem for such functions (385T-385V).

385A Notation Throughout this section and the next two, I will use the letter q to denote the function from $[0,\infty[$ to $\mathbb{R}$ defined by saying that $q(t)=-t\ln t=t\ln\frac{1}{t}$ if $t>0$, $q(0)=0$.



The function q

(a) q is continuous on $[0,\infty[$ and differentiable on $]0,\infty[$; $q'(t)=-1-\ln t$ and $q''(t)=-\frac{1}{t}$ for $t>0$. q is concave. q has a unique maximum at $(\frac{1}{e},\frac{1}{e})$.

(b)

$$q(s+t)\le q(s)+q(t)$$

for s, $t\ge 0$. $q(\sum_{i=0}^{\infty}s_i)\le\sum_{i=0}^{\infty}q(s_i)$ for every non-negative summable series $\langle s_i\rangle_{i\in\mathbb{N}}$.

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(c) If $s,\,t\ge 0$ then $q(st)=sq(t)+tq(s)$; more generally, if $n\ge 1$ and $s_i\ge 0$ for $i\le n$ then

$$q(\prod_{i=0}^n s_i)=\sum_{j=0}^n q(s_j)\prod_{i\ne j}s_i.$$

(d) The function $t\mapsto q(t)+q(1-t)$ has a unique maximum at $(\frac12,\ln 2)$.

(e) If $0\le t\le\frac12$, then $q(1-t)\le q(t)$.

(f)(i) If $\mathfrak A$ is a Dedekind σ-complete Boolean algebra, I will write $\bar q$ for the function from $L^0(\mathfrak A)^+$ to $L^0(\mathfrak A)$ defined from q. $0\le\bar q(u)\le\chi 1$ if $0\le u\le\chi 1$.

(ii) $\bar q(u+v)\le\bar q(u)+\bar q(v)$ for all $u,\,v\ge 0$ in $L^0(\mathfrak A)$.

(iii) Similarly, if $u,\,v\in L^0(\mathfrak A)^+$, then $\bar q(u\times v)=u\times\bar q(v)+v\times\bar q(u)$.

385B Lemma Let $(\mathfrak A,\bar\mu)$ be a probability algebra, $\mathfrak B$ a closed subalgebra of $\mathfrak A$, and $P:L^1(\mathfrak A,\bar\mu)\to L^1(\mathfrak A,\bar\mu)$ the corresponding conditional expectation operator. Then $\int\bar q(u)\le q(\int u)$ and $P(\bar q(u))\le\bar q(Pu)$ for every $u\in L^\infty(\mathfrak A)^+$.

385C Definition Let $(\mathfrak A,\bar\mu)$ be a probability algebra. If A is a partition of unity in $\mathfrak A$, its **entropy** is $H(A)=\sum_{a\in A}q(\bar\mu a)$.

385D Definition Let $(\mathfrak A,\bar\mu)$ be a probability algebra, $\mathfrak B$ a closed subalgebra of $\mathfrak A$ and A a partition of unity in $\mathfrak A$. Let $P:L^1(\mathfrak A,\bar\mu)\to L^1(\mathfrak A,\bar\mu)$ be the conditional expectation operator associated with $\mathfrak B$. Then the **conditional entropy of** A **on** $\mathfrak B$ is

$$H(A|\mathfrak B)=\sum_{a\in A}\int\bar q(P\chi a).$$

385E Elementary remarks (a) In the formula

$$\sum_{a\in A}\int\bar q(P\chi a),$$

every term in the sum is non-negative; accordingly $H(A|\mathfrak B)$ is well-defined in $[0,\infty]$.

(b) $H(A)=H(A|\{0,1\})$. If $A\subseteq\mathfrak B$, $H(A|\mathfrak B)=0$.

385F Definition If $\mathfrak A$ is a Boolean algebra and $A,\,B\subseteq\mathfrak A$ are partitions of unity, I write $A\vee B$ for the partition of unity $\{a\cap b:a\in A,\,b\in B\}\setminus\{0\}$.

385G Lemma Let $(\mathfrak A,\bar\mu)$ be a probability algebra and $\mathfrak B$ a closed subalgebra. Let $A\subseteq\mathfrak A$ be a partition of unity.

(a) If B is another partition of unity in $\mathfrak A$, then

$$H(A|\mathfrak B)\le H(A\vee B|\mathfrak B)\le H(A|\mathfrak B)+H(B|\mathfrak B).$$

(b) If $\mathfrak B$ is purely atomic and D is the set of its atoms, then $H(A\vee D)=H(D)+H(A|\mathfrak B)$.

(c) If $\mathfrak C\subseteq\mathfrak B$ is a smaller closed subalgebra of $\mathfrak A$, then $H(A|\mathfrak C)\ge H(A|\mathfrak B)$. In particular, $H(A)\ge H(A|\mathfrak B)$.

(d) Suppose that $\langle\mathfrak B_n\rangle_{n\in\mathbb N}$ is a non-decreasing sequence of closed subalgebras of $\mathfrak A$ such that $\mathfrak B=\overline{\bigcup_{n\in\mathbb N}\mathfrak B_n}$. If $H(A)<\infty$ then

$$H(A|\mathfrak B)=\lim_{n\to\infty}H(A|\mathfrak B_n).$$

In particular, if $A\subseteq\mathfrak B$ then $\lim_{n\to\infty}H(A|\mathfrak B_n)=0$.

385H Corollary Let $(\mathfrak A,\bar\mu)$ be a probability algebra and A, B two partitions of unity in $\mathfrak A$. Then $H(A)\le H(A\vee B)\le H(A)+H(B)$.

385I Lemma Let $(\mathfrak A,\bar\mu)$ be a probability algebra, and $\pi:\mathfrak A\to\mathfrak A$ a measure-preserving Boolean homomorphism. If $A\subseteq\mathfrak A$ is a partition of unity, then $H(\pi[A])=H(A)$.

385J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Let A be the set of its atoms. Then the following are equiveridical:

(i) *either* $\mathfrak{A}$ is not purely atomic *or* $\mathfrak{A}$ is purely atomic and $H(A) = \infty$;

(ii) there is a partition of unity $B \subseteq \mathfrak{A}$ such that $H(B) = \infty$;

(iii) for every $\gamma \in \mathbb{R}$ there is a finite partition of unity $C \subseteq \mathfrak{A}$ such that $H(C) \geq \gamma$.

385K Definition Let $\mathfrak{A}$ be a Boolean algebra. If $\pi : \mathfrak{A} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism, $A \subseteq \mathfrak{A}$ is a partition of unity and $n \geq 1$, write $D_n(A, \pi)$ for $\{\inf_{i<n} \pi^i a_i : a_i \in A$ for every $i < n\} \setminus \{0\}$.

385L Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $A \subseteq \mathfrak{A}$ be a partition of unity. Then $\lim_{n\to\infty} \frac{1}{n} H(D_n(A, \pi)) = \inf_{n\geq 1} \frac{1}{n} H(D_n(A, \pi))$ is defined in $[0, \infty]$.

385M Definition Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. For any partition of unity $A \subseteq \mathfrak{A}$, set

$$h(\pi, A) = \inf_{n\geq 1} \tfrac{1}{n} H(D_n(A, \pi)) = \lim_{n\to\infty} \tfrac{1}{n} H(D_n(A, \pi)).$$

Now the **entropy** of π is

$$h(\pi) = \sup\{h(\pi, A) : A \subseteq \mathfrak{A} \text{ is a finite partition of unity}\}.$$

Remarks (a) For any partition A of unity,

$$h(\pi, A) \leq H(A).$$

(b) Observe that if π is the identity automorphism then $h(\pi) = 0$.

385N Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be a measure-preserving Boolean homomorphism. Then $h(\pi, A) \leq h(\pi, B) + H(A|\mathfrak{B})$, where $\mathfrak{B}$ is the closed subalgebra of $\mathfrak{A}$ generated by B.

385O Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, and $A \subseteq \mathfrak{A}$ a partition of unity such that $H(A) < \infty$. Then $h(\pi, A) \leq h(\pi)$.

385P Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism.

(i) Suppose that $A \subseteq \mathfrak{A}$ is a partition of unity such that $H(A) < \infty$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \pi^n[A]$ is $\mathfrak{A}$ itself. Then $h(\pi) = h(\pi, A)$.

(ii) Suppose that π is an automorphism, and that $A \subseteq \mathfrak{A}$ is a partition of unity such that $H(A) < \infty$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{Z}} \pi^n[A]$ is $\mathfrak{A}$ itself. Then $h(\pi) = h(\pi, A)$.

385Q Bernoulli shifts Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism.

(a) π is a **one-sided Bernoulli shift** if there is a closed subalgebra $\mathfrak{A}_0$ in $\mathfrak{A}$ such that (i) $\langle \pi^k[\mathfrak{A}_0] \rangle_{k\in\mathbb{N}}$ is stochastically independent (ii) the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}} \pi^k[\mathfrak{A}_0]$ is $\mathfrak{A}$ itself. In this case $\mathfrak{A}_0$ is a **root algebra** for π.

(b) π is a **two-sided Bernoulli shift** if it is an automorphism and there is a closed subalgebra $\mathfrak{A}_0$ in $\mathfrak{A}$ such that (i) $\langle \pi^k[\mathfrak{A}_0] \rangle_{k\in\mathbb{Z}}$ is independent (ii) the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{Z}} \pi^k[\mathfrak{A}_0]$ is $\mathfrak{A}$ itself. In this case $\mathfrak{A}_0$ is a **root algebra** for π.

385R Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Bernoulli shift, either one- or two-sided, with root algebra $\mathfrak{A}_0$.

(i) If $\mathfrak{A}_0$ is purely atomic, then $h(\pi) = H(A)$, where A is the set of atoms of $\mathfrak{A}_0$.

(ii) If $\mathfrak{A}_0$ is not purely atomic, then $h(\pi) = \infty$.

385S Remarks (a) If (X, Σ, μ_0) is any probability space, write μ for the product measure on $X^{\mathbb{N}}$; let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of μ, and $\mathfrak{A}_0 \subseteq \mathfrak{A}$ the set of equivalence classes of sets of the form $\{x : x(0) \in E\}$ where $E \in \Sigma$. We have an inverse-measure-preserving function $f : X^{\mathbb{N}} \to X^{\mathbb{N}}$ defined by setting

$$f(x)(n) = x(n+1) \text{ for every } x \in X^{\mathbb{N}},\ n \in \mathbb{N},$$

and f induces a measure-preserving homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$. Now π is a one-sided Bernoulli shift with root algebra $\mathfrak{A}_0$.

(b) Again let (X, Σ, μ_0) be a probability space, and write μ for the product measure on $X^{\mathbb{Z}}$; let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of μ, and $\mathfrak{A}_0 \subseteq \mathfrak{A}$ the set of equivalence classes of sets of the form $\{x : x(0) \in E\}$ where $E \in \Sigma$. This time, we have a measure space automorphism $f : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ defined by setting

$$f(x)(n) = x(n+1) \text{ for every } x \in X^{\mathbb{Z}},\ n \in \mathbb{Z},$$

and f induces a measure-preserving automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$. π is a two-sided Bernoulli shift with root algebra $\mathfrak{A}_0$.

It follows that if $(\mathfrak{A}, \bar{\mu})$ is an atomless homogeneous probability algebra it has a two-sided Bernouilli shift.

(c) I remarked above that a Bernoulli shift will normally have many root algebras. But, up to isomorphism, any probability algebra is the root algebra of just one Bernoulli shift of each type.

(d) The classic problem to which the theory of this section was directed was the following: suppose we have two two-sided Bernoulli shifts π and ϕ, one based on a root algebra with two atoms of measure $\frac{1}{2}$ and the other on a root algebra with three atoms of measure $\frac{1}{3}$; are they isomorphic? The Kolmogorov-Sinaǐ theorem tells us that they are not, because $h(\pi) = \ln 2$ and $h(\phi) = \ln 3$.

(e) We shall need to know that any Bernoulli shift (either one- or two-sided) is ergodic. In fact, it is mixing.

(f) If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism, and $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is a closed subalgebra such that $\langle \pi^k[\mathfrak{A}_0] \rangle_{k \in \mathbb{N}}$ is independent, then $\langle \pi^k[\mathfrak{A}_0] \rangle_{k \in \mathbb{Z}}$ is independent.

(g) It is I hope obvious, but perhaps I should say: if $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $\phi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism, and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a (one- or two-sided) Bernouilli shift with a root algebra $\mathfrak{A}_0$, then $\phi\pi\phi^{-1}$ is a Bernouilli shift and $\phi[\mathfrak{A}_0]$ is a root algebra for $\phi\pi\phi^{-1}$.

385U Definition Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces, and $f_1 : X_1 \to X_1$, $f_2 : X_2 \to X_2$ two inverse-measure-preserving functions. I will say that the structures $(X_1, \Sigma_1, \mu_1, f_1)$ and $(X_2, \Sigma_2, \mu_2, f_2)$ are **almost isomorphic** if there are conegligible sets $X_i' \subseteq X_i$ such that $f_i[X_i'] \subseteq X_i'$ for both i and the structures $(X_i', \Sigma_i', \mu_i', f_i')$ are isomorphic, where Σ_i' is the algebra of relatively measurable subsets of X_i', μ_i' is the subspace measure on X_i' and $f_i' = f_i{\restriction}X_i'$.

385V Proposition Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be perfect, complete, strictly localizable and countably separated measure spaces, and $(\mathfrak{A}_1, \bar{\mu}_1)$, $(\mathfrak{A}_2, \bar{\mu}_2)$ their measure algebras. Suppose that $f_1 : X_1 \to X_1$, $f_2 : X_2 \to X_2$ are inverse-measure-preserving functions and that $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_1$, $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ are the measure-preserving Boolean homomorphisms they induce. If $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ are isomorphic, then $(X_1, \Sigma_1, \mu_1, f_1)$ and $(X_2, \Sigma_2, \mu_2, f_2)$ are almost isomorphic.

Version of 20.8.15

386 More about entropy

In preparation for the next two sections, I present a number of basic facts concerning measure-preserving homomorphisms and entropy. Compared with the work to follow, they are mostly fairly elementary, but the Halmos-Rokhlin-Kakutani lemma (386C) and the Shannon-McMillan-Breiman theorem (386E), in their full strengths, go farther than one might expect.

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386A Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is recurrent on every $a \in \mathfrak{A}$.

386B Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{C}$ be its fixed-point subalgebra $\{c : c \in \mathfrak{A},\ \pi c = c\}$. Then

$$\sup_{k \ge n} \pi^k a = \operatorname{upr}(a, \mathfrak{C})$$

for any $a \in \mathfrak{A}$ and $n \in \mathbb{N}$.

386C The Halmos-Rokhlin-Kakutani lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, with fixed-point subalgebra $\mathfrak{C}$. Then the following are equiveridical:

(i) π is aperiodic;

(ii) $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$;

(iii) whenever $n \ge 1$ and $0 \le \gamma < \frac{1}{n}$ there is an $a \in \mathfrak{A}$ such that a, πa, $\pi^2 a, \dots, \pi^{n-1} a$ are disjoint and $\bar{\mu}(a \cap c) = \gamma \bar{\mu} c$ for every $c \in \mathfrak{C}$;

(iv) whenever $n \ge 1$, $0 \le \gamma < \frac{1}{n}$ and $B \subseteq \mathfrak{A}$ is finite, there is an $a \in \mathfrak{A}$ such that a, πa, $\pi^2 a, \dots, \pi^{n-1} a$ are disjoint and $\bar{\mu}(a \cap b) = \gamma \bar{\mu} b$ for every $b \in B$.

386D Corollary An ergodic measure-preserving Boolean homomorphism on an atomless totally finite measure algebra is aperiodic.

386E The Shannon-McMillan-Breiman theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism and $A \subseteq \mathfrak{A}$ a partition of unity of finite entropy. For each $n \ge 1$, set

$$w_n = \frac{1}{n} \sum_{d \in D_n(A,\pi)} \ln\left(\frac{1}{\bar{\mu} d}\right) \chi d,$$

where $D_n(A, \pi)$ is the partition of unity generated by $\{\pi^i a : a \in A,\ i < n\}$. Then $\langle w_n \rangle_{n \in \mathbb{N}}$ is norm-convergent in $L^1(\mathfrak{A}, \bar{\mu})$ to w say; moreover, $\langle w_n \rangle_{n \in \mathbb{N}}$ is order*-convergent to w. If $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is the Riesz homomorphism defined by π, then $Tw = w$.

386F Corollary If, in 386E, π is ergodic, then $\langle w_n \rangle_{n \in \mathbb{N}}$ is order*-convergent and $\| \ \|_1$-convergent to $h(\pi, A)\chi 1$.

386G The Csisaár-Kullback inequality Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and u a member of $L^1(\mathfrak{A}, \bar{\mu})^+$ such that $\int u = 1$. Then

$$\int \bar{q}(u) \le -\frac{1}{2}\left(\int |u - \chi 1|\right)^2.$$

386H Corollary Whenever $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and A, B are partitions of unity of finite entropy,

$$\sum_{a \in A, b \in B} |\bar{\mu}(a \cap b) - \bar{\mu} a \cdot \bar{\mu} b| \le \sqrt{2(H(A) + H(B) - H(A \vee B))}.$$

386I Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\langle a_i \rangle_{i \in I}$, $\langle b_i \rangle_{i \in I}$ two partitions of unity in $\mathfrak{A}$. Then

$$\bar{\mu}(\sup_{i \in I} a_i \cap b_i) = 1 - \frac{1}{2} \sum_{i \in I} \bar{\mu}(a_i \triangle b_i).$$

386J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $\langle B_k \rangle_{k \in \mathbb{N}}$ a non-decreasing sequence of subsets of $\mathfrak{A}$ such that $0 \in B_0$ and $\langle c_i \rangle_{i \in I}$ a partition of unity in $\mathfrak{A}$. Set $B = \overline{\bigcup_{i<k} B_k}$. Then

$$\lim_{k \to \infty} \sup_{i \in I} \rho(c_i, B_k) = \sup_{i \in I} \rho(c_i, B).$$

386K Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let A, B and C be partitions of unity in $\mathfrak{A}$.

(a) $H(A\vee B\vee C)+H(C)\le H(B\vee C)+H(A\vee C)$.

(b) $h(\pi,A)\le h(\pi,A\vee B)\le h(\pi,A)+h(\pi,B)\le h(\pi,A)+H(B)$.

(c) If $H(A)<\infty$,

$$\begin{aligned} h(\pi,A) &= \inf_{n\in\mathbb{N}} H(D_{n+1}(A,\pi))-H(D_n(A,\pi)) \\ &= \lim_{n\to\infty} H(D_{n+1}(A,\pi))-H(D_n(A,\pi)). \end{aligned}$$

(d) If $H(A)<\infty$ and $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]\subseteq\mathfrak{B}$, then $h(\pi,A)\le h(\pi\restriction\mathfrak{B})+H(A|\mathfrak{B})$.

386L Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra.

(a) There is a function $h:\mathfrak{A}\to\mathfrak{B}$ such that $\bar{\mu}(a\triangle h(a))=\rho(a,\mathfrak{B})$ for every $a\in\mathfrak{A}$ and $h(a)\cap h(a')=0$ whenever $a\cap a'=0$.

(b) If A is a partition of unity in $\mathfrak{A}$, then $H(A|\mathfrak{B})\le\sum_{a\in A}q(\rho(a,\mathfrak{B}))$.

(c) If $\mathfrak{B}$ is atomless and $\langle a_i\rangle_{i\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$, then there is a partition of unity $\langle b_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\bar{\mu}b_i=\bar{\mu}a_i$ and $\bar{\mu}(b_i\triangle a_i)\le 2\rho(a_i,\mathfrak{B})$ for every $i\in\mathbb{N}$.

386M Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving automorphism. Suppose that $B\subseteq\mathfrak{A}$. For $k\in\mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b: b\in B,\ |j|\le k\}$, and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b: b\in B,\ j\in\mathbb{Z}\}$.

(a) $\mathfrak{B}$ is the topological closure $\overline{\bigcup_{k\in\mathbb{N}}\mathfrak{B}_k}$.

(b) $\pi[\mathfrak{B}]=\mathfrak{B}$.

(c) If $\mathfrak{C}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{C}]=\mathfrak{C}$, and $a\in\mathfrak{B}_k$, then

$$\rho(a,\mathfrak{C})\le(2k+1)\textstyle\sum_{b\in B}\rho(b,\mathfrak{C}).$$

386N Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and suppose *either* that $\mathfrak{A}$ is not purely atomic *or* that it is purely atomic and $H(D_0)=\infty$, where D_0 is the set of atoms of $\mathfrak{A}$. Then whenever $A\subseteq\mathfrak{A}$ is a partition of unity and $H(A)\le\gamma\le\infty$, there is a partition of unity B, refining A, such that $H(B)=\gamma$.

Version of 9.3.16

387 Ornstein's theorem

I come now to the most important of the handful of theorems known which enable us to describe automorphisms of measure algebras up to isomorphism: two two-sided Bernoulli shifts (on algebras of countable Maharam typre) of the same entropy are isomorphic (387J, 387L). This is hard work. It requires both delicate ϵ-δ analysis and substantial skill with the manipulation of measure-preserving homomorphisms. The proof is based on difficult lemmas (387C, 387G, 387K), and includes Sinaĭ's theorem (387E, 387M), describing the Bernoulli shifts which arise as factors of a given ergodic automorphism.

387A Definitions Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism.

(a) A **Bernoulli partition** for π is a partition of unity $\langle a_i\rangle_{i\in I}$ such that

$$\bar{\mu}(\inf\nolimits_{j\le k}\pi^j a_{i(j)})=\textstyle\prod_{j=0}^k\bar{\mu}a_{i(j)}$$

whenever $k\in\mathbb{N}$ and $i(0),\dots,i(k)\in I$.

(b) If $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$, a Bernoulli partition $\langle a_i\rangle_{i\in I}$ for π is **(two-sidedly) generating** if the closed subalgebra generated by $\{\pi^j a_i: i\in I,\ j\in\mathbb{Z}\}$ is $\mathfrak{A}$ itself.

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(c) A **factor** of $(\mathfrak{A},\bar{\mu},\pi)$ is a triple $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B},\pi\restriction\mathfrak{B})$ where $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]\subseteq\mathfrak{B}$.

(d) Let $\mathfrak{B}$, $\mathfrak{C}$ be closed subalgebras of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]\subseteq\mathfrak{B}$ and $\pi[\mathfrak{C}]\subseteq\mathfrak{C}$. I will write $\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B};\mathfrak{C})$ for the set of Boolean homomorphisms $\phi:\mathfrak{B}\to\mathfrak{C}$ such that

$$\bar{\mu}\phi b=\bar{\mu}b,\quad \pi\phi b=\phi\pi b$$

for every $b\in\mathfrak{B}$. On $\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B};\mathfrak{C})$ the **weak uniformity** will be the uniformity generated by the pseudometrics

$$(\phi,\psi)\mapsto\bar{\mu}(\phi b\,\triangle\,\psi b)$$

for $b\in\mathfrak{B}$; the **weak topology** on $\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B};\mathfrak{C})$ will be the associated topology.

387B Elementary facts Suppose that $(\mathfrak{A},\bar{\mu})$ is a probability algebra, $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ and that $\langle b_i\rangle_{i\in I}$ is a Bernoulli partition for π. Write $\mathfrak{B}_0$ for the closed subalgebra of $\mathfrak{A}$ generated by $\{b_i:i\in I\}$, $\mathfrak{B}$ for the closed subalgebra generated by $\{\pi^j b_i:i\in I,\,j\in\mathbb{Z}\}$, and B for $\{b_i:i\in I\}\setminus\{0\}$.

(a) $\pi\restriction\mathfrak{B}$ is a two-sided Bernoulli shift with root algebra $\mathfrak{B}_0$ and entropy $H(B)=h(\pi,B)\le h(\pi)$.

(b) If $H(B)>0$ then $\mathfrak{A}$ is atomless.

(c) Suppose now that $\langle c_i\rangle_{i\in I}$ is another Bernoulli partition for π with $\bar{\mu}c_i=\bar{\mu}b_i$ for every i; let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i:i\in I,\,j\in\mathbb{Z}\}$. Then we have a unique $\phi\in\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B};\mathfrak{C})$ such that $\phi b_i=c_i$ for every $i\in I$, and ϕ is an isomorphism between $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B},\pi\restriction\mathfrak{B})$ and $(\mathfrak{C},\bar{\mu}\restriction\mathfrak{C},\pi\restriction\mathfrak{C})$.

387C Lemma Let $(\mathfrak{A},\bar{\mu})$ be an atomless probability algebra and $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ an ergodic measure-preserving automorphism. Let $\langle a_i\rangle_{i\in\mathbb{N}}$ be a partition of unity in $\mathfrak{A}$, of finite entropy, and $\langle\gamma_i\rangle_{i\in\mathbb{N}}$ a sequence of non-negative real numbers such that

$$\sum_{i=0}^{\infty}\gamma_i=1,\quad \sum_{i=0}^{\infty}q(\gamma_i)\le h(\pi),$$

where q is the function of 385A. Then for any $\epsilon>0$ we can find a partition $\langle a'_i\rangle_{i\in\mathbb{N}}$ of unity in $\mathfrak{A}$ such that

(i) $\{i:a'_i\ne 0\}$ is finite,

(ii) $\sum_{i=0}^{\infty}|\gamma_i-\bar{\mu}a'_i|\le\epsilon$,

(iii) $\sum_{i=0}^{\infty}\bar{\mu}(a'_i\triangle a_i)\le\epsilon+6\sqrt{\sum_{i=0}^{\infty}|\bar{\mu}a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}}$

where $A=\{a_i:i\in\mathbb{N}\}\setminus\{0\}$,

(iv) $H(A')\le h(\pi,A')+\epsilon$

where $A'=\{a'_i:i\in\mathbb{N}\}\setminus\{0\}$.

387D Corollary Suppose that $(\mathfrak{A},\bar{\mu})$ is an atomless probability algebra and $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. Let $\langle a_i\rangle_{i\in\mathbb{N}}$ be a partition of unity in $\mathfrak{A}$, of finite entropy, and $\langle\gamma_i\rangle_{i\in\mathbb{N}}$ a sequence of non-negative real numbers such that

$$\sum_{i=0}^{\infty}\gamma_i=1,\quad \sum_{i=0}^{\infty}q(\gamma_i)\le h(\pi).$$

Then for any $\epsilon>0$ we can find a Bernoulli partition $\langle a^*_i\rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}a^*_i=\gamma_i$ for every $i\in\mathbb{N}$ and

$$\sum_{i=0}^{\infty}\bar{\mu}(a^*_i\triangle a_i)\le\epsilon+6\sqrt{\sum_{i=0}^{\infty}|\bar{\mu}a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}},$$

writing $A=\{a_i:i\in\mathbb{N}\}\setminus\{0\}$.

387E Sinaǐ's theorem (atomic case) Suppose that $(\mathfrak{A},\bar{\mu})$ is an atomless probability algebra and that $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is ergodic. Let $\langle\gamma_i\rangle_{i\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{i=0}^{\infty}\gamma_i=1$ and $\sum_{i=0}^{\infty}q(\gamma_i)\le h(\pi)$. Then there is a Bernoulli partition $\langle a^*_i\rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}a^*_i=\gamma_i$ for every $i\in\mathbb{N}$.

387F Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, π a member of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{C}$ closed subalgebras of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]=\mathfrak{B}$ and $\pi[\mathfrak{C}]=\mathfrak{C}$.

(a) Suppose that $\phi\in\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B};\mathfrak{C})$.

(i) $\pi^j\phi=\phi\pi^j$ for every $j\in\mathbb{Z}$.

(ii) $\phi[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{C}$ and $\pi[\phi[\mathfrak{B}]] = \phi[\mathfrak{B}]$; ϕ is an isomorphism between $(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B}, \pi\restriction\mathfrak{B})$ and $(\phi[\mathfrak{B}], \bar{\mu}\restriction\phi[\mathfrak{B}], \pi\restriction\phi[\mathfrak{B}])$.

(iii) If $\psi \in \mathrm{Hom}_{\bar{\mu},\pi}(\phi[\mathfrak{B}]; \mathfrak{C})$ then $\psi\phi \in \mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$.

(iv) If $\langle b_i\rangle_{i\in I}$ is a Bernoulli partition for $\pi\restriction\mathfrak{B}$, then $\langle \phi b_i\rangle_{i\in I}$ is a Bernoulli partition for $\pi\restriction\mathfrak{C}$.

(b) $\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$ is complete under its weak uniformity.

(c) Let $B \subseteq \mathfrak{B}$ be such that $\mathfrak{B}$ is the closed subalgebra of itself generated by $\bigcup_{i\in\mathbb{Z}} \pi^i[B]$. Then the weak uniformity of $\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$ is the uniformity defined by the pseudometrics $(\phi, \psi) \mapsto \bar{\mu}(\phi b \triangle \psi b)$ as b runs over B.

387G Lemma Suppose that $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra and $\pi \in \mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$. Let $\langle b_i\rangle_{i\in\mathbb{N}}$ and $\langle c_i\rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras of $\mathfrak{A}$ generated by $\{\pi^j b_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$ and $\{\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Suppose that $\mathfrak{C} \subseteq \mathfrak{B}$. Then for any $\epsilon > 0$ we can find a $\phi \in \mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$ such that $\bar{\mu}(\phi c_i \triangle c_i) \le \epsilon$ for every $i \in \mathbb{N}$,

387H Lemma Suppose that $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra and $\pi \in \mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$. Let $\langle b_i\rangle_{i\in\mathbb{N}}$ and $\langle c_i\rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras generated by $\{\pi^j b_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$ and $\{\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Suppose that $\mathfrak{C} \subseteq \mathfrak{B}$. Then for any $\epsilon > 0$ we can find a $\phi \in \mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{C}; \mathfrak{B})$ such that $\bar{\mu}(\phi c_i \triangle c_i) \le \epsilon$ and $\rho(b_i, \phi[\mathfrak{C}]) \le \epsilon$ for every $i \in \mathbb{N}$.

387I Lemma Suppose that $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra and $\pi \in \mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$. Let $\langle b_i\rangle_{i\in I}$, $\langle c_i\rangle_{i\in I}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras generated by $\{\pi^j b_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$ and $\{\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Suppose that $\mathfrak{C} \subseteq \mathfrak{B}$. Then for any $\epsilon > 0$ we can find $\phi \in \mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{C}; \mathfrak{B})$ such that $\phi[\mathfrak{C}] = \mathfrak{B}$ and $\bar{\mu}(\phi c_i \triangle c_i) \le \epsilon$ for every $i \in \mathbb{N}$.

387J Ornstein's theorem (finite entropy case) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be probability algebras, and $\pi : \mathfrak{A} \to \mathfrak{A}$, $\phi : \mathfrak{B} \to \mathfrak{B}$ two-sided Bernoulli shifts of the same finite entropy. Then $(\mathfrak{A}, \bar{\mu}, \pi)$ and $(\mathfrak{B}, \bar{\nu}, \phi)$ are isomorphic.

387K Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi \in \mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$ an ergodic measure-preserving automorphism. Suppose that $\langle a_i\rangle_{i\in I}$ is a finite Bernoulli partition for π, with $\#(I) = r \ge 1$ and $\bar{\mu}a_i = 1/r$ for every $i \in I$, and that $h(\pi) \ge \ln 2r$. Then for any $\epsilon > 0$ there is a Bernoulli partition $\langle b_{ij}\rangle_{i\in I, j\in\{0,1\}}$ for π such that

$$\bar{\mu}(a_i \triangle (b_{i0} \cup b_{i1})) \le \epsilon, \quad \bar{\mu}b_{i0} = \bar{\mu}b_{i1} = \frac{1}{2r}$$

for every $i \in I$.

387L Ornstein's theorem (infinite entropy case) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra of countable Maharam type, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a two-sided Bernoulli shift of infinite entropy. Then $(\mathfrak{A}, \bar{\mu}, \pi)$ is isomorphic to $(\mathfrak{B}_{\mathbb{Z}}, \bar{\nu}_{\mathbb{Z}}, \phi)$, where $(\mathfrak{B}_{\mathbb{Z}}, \bar{\nu}_{\mathbb{Z}})$ is the measure algebra of the usual measure on $[0,1]^{\mathbb{Z}}$, and ϕ is the standard two-sided Bernoulli shift on $\mathfrak{B}_{\mathbb{Z}}$.

387M Corollary: Sinaĭ's theorem (general case) Suppose that $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra, and $\pi \in \mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$. Let $(\mathfrak{B}, \bar{\nu})$ be a probability algebra of countable Maharam type, and $\phi : \mathfrak{B} \to \mathfrak{B}$ a one- or two-sided Bernoulli shift with $h(\phi) \le h(\pi)$. Then $(\mathfrak{B}, \bar{\nu}, \phi)$ is isomorphic to a factor of $(\mathfrak{A}, \bar{\mu}, \pi)$.

Version of 6.6.16

388 Dye's theorem

I have repeatedly said that any satisfactory classification theorem for automorphisms of measure algebras remains elusive. There is however a classification, at least for the Lebesgue measure algebra, of the 'orbit structures' corresponding to measure-preserving automorphisms; in fact, they are defined by the fixed-point subalgebras, which I described in §333. We have to work hard for this result, but the ideas are instructive.

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388A Orbit structures: Proposition Let (X, Σ, μ) be a localizable countably separated measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Suppose that f and g are measure space automorphisms from X to itself, inducing measure-preserving automorphisms π, ϕ of $\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) for almost every $x \in X$, there is an $n \in \mathbb{Z}$ such that $g(x) = f^n(x)$;

(iii) for almost every $x \in X$, $\{g^n(x) : n \in \mathbb{Z}\} \subseteq \{f^n(x) : n \in \mathbb{Z}\}$.

388B Corollary Under the hypotheses of 388A, π and ϕ generate the same full subgroup of $\operatorname{Aut}\mathfrak{A}$ iff $\{f^n(x) : n \in \mathbb{Z}\} = \{g^n(x) : n \in \mathbb{Z}\}$ for almost every $x \in X$.

388C Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism; let $\mathfrak{C}$ be its fixed-point subalgebra. Let $\langle d_i \rangle_{i \in I}$, $\langle e_i \rangle_{i \in I}$ be two disjoint families in $\mathfrak{A}$ such that $\bar{\mu}(c \cap d_i) = \bar{\mu}(c \cap e_i)$ for every $i \in I$ and $c \in \mathfrak{C}$. Then there is a $\phi \in G_\pi$, the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π, such that $\phi d_i = e_i$ for every $i \in I$.

388D von Neumann automorphisms (a) Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$ an automorphism. π is **weakly von Neumann** if there is a sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$ such that $a_0 = 1$ and, for every n, $a_{n+1} \cap \pi^{2^n} a_{n+1} = 0$, $a_{n+1} \cup \pi^{2^n} a_{n+1} = a_n$. In this case, π is **von Neumann** if $\langle a_n \rangle_{n \in \mathbb{N}}$ can be chosen in such a way that $\{\pi^m a_n : m, n \in \mathbb{N}\}$ τ-generates $\mathfrak{A}$, and **relatively von Neumann** if $\langle a_n \rangle_{n \in \mathbb{N}}$ can be chosen so that $\{\pi^m a_n : m, n \in \mathbb{N}\} \cup \{c : \pi c = c\}$ τ-generates $\mathfrak{A}$.

(b) If $\mathfrak{A}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an automorphism, then a **dyadic cycle system** for π is a finite or infinite family $\langle d_{mi} \rangle_{m \le n, i < 2^m}$ or $\langle d_{mi} \rangle_{m \in \mathbb{N}, i < 2^m}$ such that (α) for each m, $\langle d_{mi} \rangle_{i < 2^m}$ is a partition of unity such that $\pi d_{mi} = d_{m,i+1}$ whenever $i < 2^m - 1$ (β) $d_{m0} = d_{m+1,0} \cup d_{m+1,2^m}$ for every $m < n$ (in the finite case) or for every $m \in \mathbb{N}$ (in the infinite case). An easy induction on m shows that if $k \le m$ then

$$d_{ki} = \sup\{d_{mj} : j < 2^m,\ j \equiv i \bmod 2^k\}$$

for every $i < 2^k$.

Conversely, if d is such that $\langle \pi^j d \rangle_{j < 2^n}$ is a partition of unity in $\mathfrak{A}$, then we can form a finite dyadic cycle system $\langle d_{mi} \rangle_{m \le n, i < 2^m}$ by setting $d_{mi} = \sup\{\pi^j d : j < 2^n,\ j \equiv i \bmod 2^m\}$ whenever $m \le n$ and $j < 2^m$.

(c) Now an automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is weakly von Neumann iff it has an infinite dyadic cycle system $\langle d_{mi} \rangle_{m \in \mathbb{N}, i < 2^m}$. π is von Neumann iff it has a dyadic cycle system $\langle d_{mi} \rangle_{m \in \mathbb{N}, i < 2^m}$ which τ-generates $\mathfrak{A}$.

388E Example Let μ be the usual measure on $X = \{0,1\}^{\mathbb{N}}$, Σ its domain, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Define $f : X \to X$ by setting

$$\begin{aligned} f(x)(n) &= 1 - x(n) \text{ if } x(i) = 0 \text{ for every } i < n, \\ &= x(n) \text{ otherwise.} \end{aligned}$$

Then f is a homeomorphism and a measure space automorphism.

Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be the corresponding automorphism. Then π is a measure-preserving von Neumann automorphism.

f is sometimes called the **odometer transformation**.

388F Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism. Let $\mathfrak{C}$ be its fixed-point subalgebra. Then for any $a \in \mathfrak{A}$ there is a $b \subseteq a$ such that $\bar{\mu}(b \cap c) = \frac{1}{2}\bar{\mu}(a \cap c)$ for every $c \in \mathfrak{C}$ and π_b is a weakly von Neumann automorphism, writing π_b for the induced automorphism of the principal ideal $\mathfrak{A}_b$.

388G Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and π, ψ two measure-preserving automorphisms of $\mathfrak{A}$. Suppose that ψ belongs to the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π and that there is a $b \in \mathfrak{A}$ such that $\sup_{n \in \mathbb{Z}} \psi^n b = 1$ and the induced automorphisms ψ_b, π_b on $\mathfrak{A}_b$ are equal. Then $G_\psi = G_\pi$.

388H Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism, and ϕ any member of the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π. Suppose that $\langle d_{mi}\rangle_{m\le n,i<2^m}$ is a finite dyadic cycle system for ϕ. Then there is a weakly von Neumann automorphism ψ, with dyadic cycle system $\langle d'_{mi}\rangle_{m\in\mathbb{N},i<2^m}$, such that $G_\psi = G_\pi$, $\psi a = \phi a$ whenever $a \cap d_{n0} = 0$, and $d'_{mi} = d_{mi}$ whenever $m \le n$ and $i < 2^m$.

388I Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$. For $a \in \mathfrak{A}$ write $\mathfrak{C}_a = \{a \cap c : c \in \mathfrak{C}\}$.

(a) Suppose that $b \in \mathfrak{A}$, $w \in \mathfrak{C}$ and $\delta > 0$ are such that $\bar{\mu}(b \cap c) \ge \delta\bar{\mu}c$ whenever $c \in \mathfrak{C}$ and $c \subseteq w$. Then there is an $e \in \mathfrak{A}$ such that $e \subseteq b \cap w$ and $\bar{\mu}(e \cap c) = \delta\bar{\mu}c$ whenever $c \in \mathfrak{C}_w$.

(b) Suppose that $k \ge 1$ and that $(b_0, \dots , b_r)$ is a finite partition of unity in $\mathfrak{A}$. Then there is a partition E of unity in $\mathfrak{A}$ such that

$$\bar{\mu}(e \cap c) = \frac{1}{k}\bar{\mu}c \text{ for every } e \in E,\ c \in \mathfrak{C},$$

$$\#(\{e : e \in E, \exists\, i \le r,\ b_i \cap e \notin \mathfrak{C}_e\}) \le r + 1.$$

388J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism, with fixed-point subalgebra $\mathfrak{C}$. Suppose that ϕ is a member of the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π with a finite dyadic cycle system $\langle d_{mi}\rangle_{m\le n,i<2^m}$, and that $a \in \mathfrak{A}$ and $\epsilon > 0$. Then there is a $\psi \in G_\pi$ such that

(i) ψ has a dyadic cycle system $\langle d'_{mi}\rangle_{m\le k,i<2^m}$, with $k \ge n$ and $d'_{mi} = d_{mi}$ for $m \le n$, $i < 2^m$;

(ii) $\psi d = \phi d$ if $d \cap d_{n0} = 0$;

(iii) there is an a' in the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d'_{ki} : i < 2^k\}$ such that $\bar{\mu}(a \,\triangle\, a') \le \epsilon$.

388K Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, with Maharam type ω, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism. Then there is a relatively von Neumann automorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ such that ϕ and π generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

388L Theorem Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be totally finite measure algebras of countable Maharam type, and $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_1$, $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ measure-preserving automorphisms. For each i, let $\mathfrak{C}_i$ be the fixed-point subalgebra of π_i and G_{π_i} the full subgroup of $\operatorname{Aut}\mathfrak{A}_i$ generated by π_i. If $(\mathfrak{A}_1, \bar{\mu}_1, \mathfrak{C}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \mathfrak{C}_2)$ are isomorphic, so are $(\mathfrak{A}_1, \bar{\mu}_1, G_{\pi_1})$ and $(\mathfrak{A}_2, \bar{\mu}_2, G_{\pi_2})$.

Version of 14.1.15

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

385Xr Exercise 385Xr, referred to in the 2003, 2006 and 2013 editions of Volume 4, is now 385Xj.

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