Version of 15.8.08

Chapter 38

Automorphism groups

As with any mathematical structure, every measure algebra has an associated symmetry group, the group of all measure-preserving automorphisms. In this chapter I set out to describe some of the remarkable features of these groups. I begin with elementary results on automorphisms of general Boolean algebras (§381), introducing definitions for the rest of the chapter. In §382 I give a general theorem on the expression of an automorphism as the product of involutions (382M), with a description of the normal subgroups of certain groups of automorphisms (382R). Applications of these ideas to measure algebras are in §383. I continue with a discussion of circumstances under which these automorphism groups determine the underlying algebras and/or have few outer automorphisms (§384).

One of the outstanding open problems of the subject is the 'isomorphism problem', the classification of automorphisms of measure algebras up to conjugacy in the automorphism group. I offer two sections on 'entropy', the most important numerical invariant enabling us to distinguish some non-conjugate automorphisms (§§385-386). For Bernoulli shifts on the Lebesgue measure algebra (385Q-385S), the isomorphism problem is solved by Ornstein's theorem (387J, 387L); I present a complete proof of this theorem in §§386-387. Finally, in §388, I give Dye's theorem, describing the full subgroups generated by single automorphisms of measure algebras of countable Maharam type.

Version of 19.7.06

381 Automorphisms of Boolean algebras

I begin the chapter with a preparatory section of definitions (381B) and mostly elementary facts. A fundamental method of constructing automorphisms is in 381C-381D. The idea of 'support' of an endomorphism is explored in 381E-381G, a first look at 'periodic' and 'aperiodic' parts is in 381H, and basic facts about 'full subgroups' are in 381I-381J. We start to go deeper with the notion of 'recurrence', treated in 381L-381P. I describe how these phenomena appear when we represent an endomorphism as a map on the Stone space of an algebra (381Q). I end by introducing a 'cycle notation' for certain automorphisms.

381A The group $\operatorname{Aut}\mathfrak{A}$ For any Boolean algebra $\mathfrak{A}$, I write $\operatorname{Aut}\mathfrak{A}$ for the set of automorphisms of $\mathfrak{A}$, that is, the set of bijective Boolean homomorphisms $\pi:\mathfrak{A}\to\mathfrak{A}$. This is a group, being a subgroup of the group of all permutations of $\mathfrak{A}$ (use 312G). Note that every member of $\operatorname{Aut}\mathfrak{A}$ is order-continuous; this is because it must be an isomorphism of the order structure of $\mathfrak{A}$ (313Ld).

381B The primary aim of this chapter is to study automorphisms of probability algebras. In the context of the present section, this means that for a first reading you can take it that all algebras are Dedekind complete. The methods can however be used in many other contexts, at the price of complicating some of the statements of the lemmas. It is also interesting, and occasionally important, to apply some of the ideas to general Boolean homomorphisms. In the following definitions I try to set out a language to make this possible.

Definitions (a) If $\mathfrak{A}$ is a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean homomorphism, I say that $a\in\mathfrak{A}$ **supports** π if $\pi d=d$ for every $d\subseteq 1\setminus a$.

(b) Let $\mathfrak{A}$ be a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a Boolean homomorphism. If $\min\{a:a\in\mathfrak{A}\text{ supports }\pi\}$ is defined in $\mathfrak{A}$, I will call it the **support** $\operatorname{supp}\pi$ of π.

(c) If $\mathfrak{A}$ is a Boolean algebra, an automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is **periodic**, with **period** $n \geq 1$, if $\mathfrak{A} \neq \{0\}$, π^n is the identity operator and 1 is the support of π^i whenever $1 \leq i < n$.

(d) If $\mathfrak{A}$ is a Boolean algebra, a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is **aperiodic** if the support of π^n is 1 for every $n \geq 1$. I remark immediately that if π is aperiodic, so is π^n for every $n \geq 1$ (see 381Xc). Note that if $\mathfrak{A} = \{0\}$ then the trivial automorphism of $\mathfrak{A}$ is counted as aperiodic.

(e) If $\mathfrak{A}$ is a Boolean algebra, a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **full** if whenever $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, $\langle \pi_i\rangle_{i\in I}$ is a family in G, and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi d = \pi_i d$ whenever $i \in I$ and $d \subseteq a_i$, then $\pi \in G$.

(f) If $\mathfrak{A}$ is a Boolean algebra, a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **countably full** if whenever $\langle a_i\rangle_{i\in I}$ is a countable partition of unity in $\mathfrak{A}$, $\langle \pi_i\rangle_{i\in I}$ is a family in G, and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi d = \pi_i d$ whenever $i \in I$ and $d \subseteq a_i$, then $\pi \in G$.

(g) If $\mathfrak{A}$ is a Boolean algebra, $a \in \mathfrak{A}$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, I say that π is **recurrent on** a if for every non-zero $b \subseteq a$ there is a $k \geq 1$ such that $a \cap \pi^k b \neq 0$. If $\pi \in \operatorname{Aut}\mathfrak{A}$ and π and π^{-1} are both recurrent on a, I say that π is **doubly recurrent** on a.

381C Before setting out to explore the concepts just listed, I give a fundamental result on piecing automorphisms together from fragments.

Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Assume
either that I is finite
or that I is countable and $\mathfrak{A}$ is Dedekind σ-complete
or that $\mathfrak{A}$ is Dedekind complete.
Suppose that for each $i \in I$ we have an isomorphism $\pi_i : \mathfrak{A}_{a_i} \to \mathfrak{A}_{b_i}$ between the corresponding principal ideals. Then there is a unique $\pi \in \operatorname{Aut}\mathfrak{A}$ such that $\pi d = \pi_i d$ whenever $i \in I$ and $d \subseteq a_i$.

proof By 315F, we may identify $\mathfrak{A}$ with each of the products $\prod_{i\in I}\mathfrak{A}_{a_i}$, $\prod_{i\in I}\mathfrak{A}_{b_i}$; now π corresponds to the isomorphism between the two products induced by the π_i.

381D Corollary Let $\mathfrak{A}$ be a homogeneous Boolean algebra, and A, B two partitions of unity in $\mathfrak{A}$, neither containing 0. Let $\theta : A \to B$ be a bijection. Suppose
either that A, B are finite
or that A, B are countable and $\mathfrak{A}$ is Dedekind σ-complete
or that $\mathfrak{A}$ is Dedekind complete.
Then there is an automorphism of $\mathfrak{A}$ extending θ.

proof For every $a \in A$, the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_{\theta a}$ are isomorphic to the whole algebra $\mathfrak{A}$, and therefore to each other; let $\pi_a : \mathfrak{A}_a \to \mathfrak{A}_{\theta a}$ be an isomorphism. Now apply 381C.

381E Lemma Let $\mathfrak{A}$ be a Boolean algebra, and π, ϕ, $\psi : \mathfrak{A} \to \mathfrak{A}$ Boolean homomorphisms of which π is injective.

(a) If $a \in \mathfrak{A}$ supports ϕ then $\phi a = a$ and $\phi d \subseteq a$ for every $d \subseteq a$.

(b) If $a \in \mathfrak{A}$ supports both ϕ and ψ then it supports $\phi\psi$.

(c) Let A be the set of elements of $\mathfrak{A}$ supporting ϕ. Then A is non-empty and closed under $\cap$; also $b \in A$ whenever $b \supseteq a \in A$. If ϕ is order-continuous, then $\inf B \in A$ whenever $B \subseteq A$ has an infimum in $\mathfrak{A}$.

(d) If $a \in \mathfrak{A}$ supports $\pi\phi$, then ϕa supports $\pi\phi$.

(e) If π commutes with ϕ, and $a \in \mathfrak{A}$ is such that πa supports ϕ, then a supports ϕ.

(f) If ϕ is supported by a and ψ is supported by b, where $a \cap b = 0$, then $\phi\psi = \psi\phi$.

(g) For any $n \geq 1$ and $a \in \mathfrak{A}$, a supports π^n iff πa supports π^n. Consequently $\pi(\operatorname{supp}\pi^n) = \operatorname{supp}\pi^n$ if π^n has a support.

(h) If $\pi \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$ supports π, then a supports π^{-1}.

(i) If $\pi \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$, then

$$\begin{aligned} a \text{ supports } \pi &\iff d \triangle \pi d \subseteq a \text{ for every } d \in \mathfrak{A} \\ &\iff d \subseteq a \text{ whenever } d \cap \pi d = 0 \\ &\iff d \cap \pi d \ne 0 \text{ whenever } 0 \ne d \subseteq 1 \setminus a. \end{aligned}$$

(j) If $\pi \in \operatorname{Aut}\mathfrak{A}$ and $a \in \mathfrak{A}$ supports ϕ, then πa supports $\pi\phi\pi^{-1}$.
(k) If $a \in \mathfrak{A}$ supports ϕ, and π_1, $\pi_2 \in \operatorname{Aut}\mathfrak{A}$ agree on $\mathfrak{A}_a$, then $\pi_1\phi\pi_1^{-1} = \pi_2\phi\pi_2^{-1}$.

proof (a) $\phi(1 \setminus a) = 1 \setminus a$, so $\phi a = a$, and if $d \subseteq a$ then $\phi d \subseteq \phi a = a$.

(b) If $d \cap a = 0$ then $\phi d = d = \psi d$ so $\phi\psi d = d$.

(c) Of course $1 \in A$, because $\phi 0 = 0$; and it is also obvious that if $b \supseteq a \in A$ then $b \in A$. If a, $b \in A$ and $d \cap a \cap b = 0$, then $\phi d = \phi(d \setminus a) \cup \phi(d \setminus b) = d$. If ϕ is order-continuous, $B \subseteq A$ is non-empty and $c = \inf B$ is defined in $\mathfrak{A}$, then for any $d \subseteq 1 \setminus c$ we have

$$d = d \setminus c = \sup_{b\in B} d \setminus b,$$

and

$$\phi d = \sup_{b\in B} \phi(d \setminus b) = \sup_{b\in B} d \setminus b = d.$$

So in this case c supports ϕ.

(d) If $d \cap \phi a = 0$ then $\pi d \cap a = \pi d \cap \pi\phi a = 0$, so $\pi\phi\pi d = \pi d$ and (because π is injective) $\phi\pi d = d$.

(e) If $d \cap a = 0$ then $\pi d \cap \pi a = 0$, so $\pi\phi d = \phi\pi d = \pi d$ and $\phi d = d$.

(f) If $d \subseteq a$ then $\phi d \subseteq a$ and $\psi d = d$ so $\psi\phi d = \phi d = \phi\psi d$; if $d \subseteq b$ then $\psi\phi d = \phi\psi d = \psi d$; and if $d \subseteq 1 \setminus (a \cup b)$ then $\psi\phi d = \phi\psi d = d$.

(g) Because π is injective, so is π^{n-1}. So if a supports $\pi^n = \pi^{n-1}\pi$, so does πa, by (d). On the other hand, π commutes with π^n, so if πa supports π^n so does a, by (e).
If $c = \operatorname{supp}\pi^n$ then πc supports π^n so $c \subseteq \pi c$. Consequently $\pi^i c \subseteq \pi^{i+1} c$ for every $i \in \mathbb{N}$ and $c \subseteq \pi c \subseteq \pi^n c$. But as $\pi^n c = c$, by (a), $\pi c = c$.

(h) If $d \cap a = 0$ then $\pi d = d$ so $d = \pi^{-1} d$.

(i)(α) If a supports π and $d \in \mathfrak{A}$, then $\pi a = a$, by (a), so

$$(d \triangle \pi d) \setminus a = (d \setminus a) \triangle (\pi d \setminus \pi a) = (d \setminus a) \triangle \pi(d \setminus a) = (d \setminus a) \triangle (d \setminus a) = 0$$

and $d \triangle \pi d \subseteq a$.

(β) If $d \triangle \pi d \subseteq a$ and $d \cap \pi d = 0$, then $d \subseteq d \triangle \pi d \subseteq a$.

(γ) If $d \subseteq a$ whenever $d \cap \pi d = 0$, and $0 \ne d' \subseteq 1 \setminus a$, then of course $d' \cap \pi d' \ne 0$.

(δ) If a does not support π, there is a $c \subseteq 1 \setminus a$ such that $\pi c \ne c$. So one of $c \setminus \pi c$, $\pi c \setminus c$ is non-zero. If $c \setminus \pi c \ne 0$, take this for d; then $d \subseteq 1 \setminus a$ and $\pi d \cap d \subseteq \pi c \setminus \pi c = 0$. Otherwise, because π is an automorphism, we can take $d = \pi^{-1}(\pi c \setminus c)$; then $0 \ne d \subseteq c \subseteq 1 \setminus a$, while

$$d \cap \pi d = (c \setminus \pi^{-1} c) \cap (\pi c \setminus c) = 0.$$

(j) If $d \cap \pi a = 0$ then $\pi^{-1} d \cap a = 0$ so $\phi\pi^{-1} d = \pi^{-1} d$ and $\pi\phi\pi^{-1} d = d$.

(k) For $d \subseteq a$, $\pi_2^{-1}\pi_1 d = \pi_2^{-1}\pi_2 d = d$, so $\pi_2^{-1}\pi_1$ is supported by $1 \setminus a$. By (f), $\phi\pi_2^{-1}\pi_1 = \pi_2^{-1}\pi_1\phi$, so

$$\pi_1\phi\pi_1^{-1} = \pi_2\pi_2^{-1}\pi_1\phi\pi_1^{-1} = \pi_2\phi\pi_2^{-1}\pi_1\pi_1^{-1} = \pi_2\phi\pi_2^{-1}.$$

381F Corollary If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then every order-continuous Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ has a support.

proof By 381Ec, $\inf\{a : a \in \mathfrak{A} \text{ supports } \phi\}$ is the support of ϕ.

381G Corollary Let $\mathfrak{A}$ be a Boolean algebra, and suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ has a support e.

(a) $\pi e = e$.

(b) $e = \sup\{d \triangle \pi d : d \in \mathfrak{A}\} = \sup\{d : d \in \mathfrak{A},\ d \cap \pi d = 0\}$.

(c) e is the support of π^{-1}.

(d) For any $\phi \in \operatorname{Aut}\mathfrak{A}$, ϕe is the support of $\phi\pi\phi^{-1}$.

proof (a) 381Ea.

(b) 381Ei.

(c) 381Eh.

(d) By 381Ej, ϕe supports $\phi\pi\phi^{-1}$. At the same time, if $a \in \mathfrak{A}$ supports $\phi\pi\phi^{-1}$, then $\phi^{-1}a$ supports π, so $e \subseteq \phi^{-1}a$ and $a \supseteq \phi e$. Thus ϕe is the smallest element of $\mathfrak{A}$ supporting $\phi\pi\phi^{-1}$ and is the support of $\phi\pi\phi^{-1}$.

381H Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an injective Boolean homomorphism such that π^n has a support for every $n \in \mathbb{N}$. Then there is a partition of unity $\langle c_i\rangle_{1\le i\le\omega}$ in $\mathfrak{A}$ such that $\pi c_i \subseteq c_i$ for every i, $\pi\restriction\mathfrak{A}_{c_n}$ is periodic with period n whenever $n \in \mathbb{N}\setminus\{0\}$ and $c_n \ne 0$, and $\pi\restriction\mathfrak{A}_{c_\omega}$ is aperiodic.

proof Set

$$c_1 = 1 \setminus \operatorname{supp}\pi,$$

$$c_n = \inf_{i<n} \operatorname{supp}\pi^i \setminus \operatorname{supp}\pi^n \text{ for } n \ge 2,$$

$$c_\omega = \inf_{n\in\mathbb{N}} \operatorname{supp}\pi^n.$$

Then $\langle c_i\rangle_{1\le i\le\omega}$ is a partition of unity. By 381Eg, $\pi c_n = c_n$ for every n, so $\pi c_\omega \subseteq c_\omega$. If $d \subseteq c_n$, where $1 \le n \in \mathbb{N}$, then $d \cap \operatorname{supp}\pi^n = 0$ so $\pi^n d = d$. If $1 \le i < j \le \omega$ and $0 \ne a \subseteq c_j$, then $a \subseteq \operatorname{supp}\pi^i$ so there is a $d \subseteq a$ such that $(\pi\restriction\mathfrak{A}_{c_n})^i d = \pi^i d \ne d$; thus if $n \in \mathbb{N}\setminus\{0\}$ (and $c_n \ne 0$) $\pi\restriction\mathfrak{A}_{c_n}$ is periodic with period n, while $\pi\restriction\mathfrak{A}_{c_\omega}$ is aperiodic.

Remark The hypothesis 'every π^n has a support' will be satisfied if $\mathfrak{A}$ is Dedekind complete and π is order-continuous (381F). For other sufficient conditions see 382E.

381I Full and countably full subgroups If $\mathfrak{A}$ is a Boolean algebra, it is obvious that the intersection of any family of (countably) full subgroups of $\operatorname{Aut}\mathfrak{A}$ is again (countably) full. We may therefore speak of the (countably) full subgroup of $\mathfrak{A}$ generated by an element of $\operatorname{Aut}\mathfrak{A}$.

Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) Let G be a subgroup of $\operatorname{Aut}\mathfrak{A}$. Let H be the set of those $\pi \in \operatorname{Aut}\mathfrak{A}$ such that for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and a $\phi \in G$ such that $\pi c = \phi c$ for every $c \subseteq b$. Then H is a full subgroup of $\operatorname{Aut}\mathfrak{A}$, the smallest full subgroup of $\mathfrak{A}$ including G.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\phi c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a_n$.

(c) Suppose that $\mathfrak{A}$ is Dedekind complete, and π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and an $n \in \mathbb{Z}$ such that $\phi c = \pi^n c$ for every $c \subseteq b$;

(iii) ϕ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(iv) $\inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi^n\phi) = 0$.

proof (a)(i) $\pi_2\pi_1 \in H$ for all π_1, $\pi_2 \in H$. **P** Let $a \in \mathfrak{A}$ be non-zero; then there are a non-zero $b \subseteq a$ and a $\phi_1 \in G$ such that π_1 and ϕ_1 agree on the principal ideal $\mathfrak{A}_b$. Next, there are a non-zero $c \subseteq \pi_1 b$ and a $\phi_2 \in G$ such that π_2 and ϕ_2 agree on $\mathfrak{A}_c$. Set $d = \pi_1^{-1}c$; then $d \in \mathfrak{A}_a \setminus \{0\}$, and $\phi_2\phi_1$ is a member of G agreeing with $\pi_2\pi_1$ on $\mathfrak{A}_d$. As a is arbitrary, $\pi_2\pi_1 \in H$. **Q**

(ii) $\pi^{-1} \in H$ for every $\pi \in H$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, there are a non-zero $b \subseteq \pi^{-1}a$ and a $\phi \in G$ such that π and ϕ agree on $\mathfrak{A}_b$; now $0 \neq \pi b \subseteq a$ and π^{-1} and ϕ^{-1} agree on $\mathfrak{A}_{\pi b}$. As a is arbitrary, $\pi^{-1} \in H$. **Q** Of course $H \supseteq G$, so H is a subgroup of $\operatorname{Aut}\mathfrak{A}$.

(iii) Suppose now that $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, $\langle \pi_i\rangle_{i\in I}$ is a family in H, and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that π agrees with π_i on $\mathfrak{A}_{a_i}$ for every $i \in I$. Then $\pi \in H$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, there is an $i \in I$ such that $b = a \cap a_i$ is non-zero; now π agrees with π_i on b. **Q** So H is a full subgroup of $\operatorname{Aut}\mathfrak{A}$.

(iv) If H' is any full subgroup of $\operatorname{Aut}\mathfrak{A}$ including G, then $H' \supseteq H$. **P** If $\pi \in H$, then $B = \{b :$ there is a $\phi \in G$ agreeing with π on $\mathfrak{A}_b\}$ is an order-dense subset of $\mathfrak{A}$, so there is a partition $\langle a_i\rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that $a_i \in B$ for every i. For each $i \in I$, let $\pi_i \in G$ be such that π and π_i agree on $\mathfrak{A}_{a_i}$; then $\langle (a_i, \pi_i)\rangle_{i\in I}$ witnesses that $\pi \in H'$. As π is arbitrary, $H \subseteq H'$. **Q**

(b) (ii)$\Rightarrow$(i) is trivial. In the other direction, let G be the family of all those automorphisms ψ of $\mathfrak{A}$ such that there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\psi c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a_n$. Then G is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ containing π.

P Of course $\pi \in G$ (set $a_1 = 1$, $a_n = 0$ for $n \neq 1$).

Take ψ_1, $\psi_2 \in G$. Let $\langle a_n\rangle_{n\in\mathbb{Z}}$, $\langle a'_n\rangle_{n\in\mathbb{Z}}$ be partitions of unity in $\mathfrak{A}$ such that $\psi_1 c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a_n$, while $\psi_2 c = \pi^n c$ whenever $n \in \mathbb{Z}$ and $c \subseteq a'_n$. Then $\langle a'_n \cap \psi_2^{-1}a_m\rangle_{m,n\in\mathbb{Z}}$ is a partition of unity. If $c \subseteq a'_n \cap \psi_2^{-1}a_m$, then $\psi_2 c = \pi^n c \subseteq a_m$, so $\psi_1\psi_2 c = \pi^{m+n}c$. So if we set $b_n = \sup_{i\in\mathbb{Z}} a'_i \cap \psi_2^{-1}a_{n-i}$ for each $n \in \mathbb{Z}$, $\langle b_n\rangle_{n\in\mathbb{Z}}$ is a partition of unity in $\mathfrak{A}$ witnessing that $\psi_1\psi_2 \in G$. At the same time, $\langle \psi_1 a_{-n}\rangle_{n\in\mathbb{Z}}$ is a partition of unity witnessing that $\psi_1^{-1} \in G$. As ψ_1 and ψ_2 are arbitrary, G is a subgroup of $\operatorname{Aut}\mathfrak{A}$.

Now suppose that $\langle a_i\rangle_{i\in I}$ is a countable partition of unity in $\mathfrak{A}$ and that $\psi \in \operatorname{Aut}\mathfrak{A}$ is such that for every $i \in I$ there is a $\psi_i \in G$ such that $\psi c = \psi_i c$ for every $c \subseteq a_i$. For each $i \in I$ let $\langle a_{in}\rangle_{n\in\mathbb{Z}}$ be a partition of unity such that $\psi_i c = \pi^n c$ whenever $c \subseteq a_{in}$. Then $\langle a_i \cap a_{in}\rangle_{i\in I,n\in\mathbb{Z}}$ is a partition of unity such that $\psi c = \pi^n c$ whenever $c \subseteq c_i \cap a_{in}$. So setting $b_n = \sup_{i\in I} a_i \cap a_{in}$ for each $n \in \mathbb{Z}$, $\langle b_n\rangle_{n\in\mathbb{Z}}$ is a partition of unity witnessing that $\psi \in G$. As ψ is arbitrary, G is countably full. **Q**

Accordingly G must include (indeed, must coincide with) the countably full subgroup generated by π, and (i)$\Rightarrow$(ii).

(c)(i)$\Rightarrow$(ii) is a special case of (a).

(ii)$\Rightarrow$(iii) For $n \in \mathbb{Z}$, let B_n be the set of those $b \in \mathfrak{A}$ such that $\phi c = \pi^n c$ for every $c \subseteq b$. Set $b_n = \sup B_n$ for each n; then if $c \subseteq b_n$,

$$\phi c = \phi(\sup_{b\in B_n} b \cap c) = \sup_{b\in B_n} \phi(b \cap c) = \sup_{b\in B_n} \pi^n(b \cap c) = \pi^n c.$$

Set

$$\begin{aligned} a_n &= b_n \setminus \sup_{0\le i<n} b_i \text{ if } n \in \mathbb{N}, \\ &= b_n \setminus \sup_{i>n} b_i \text{ if } n \in \mathbb{Z} \setminus \mathbb{N}; \end{aligned}$$

then $\langle a_n\rangle_{n\in\mathbb{Z}}$ is disjoint,

$$\sup_{n\in\mathbb{Z}} a_n = \sup_{n\in\mathbb{Z}} b_n = \sup(\bigcup_{n\in\mathbb{Z}} B_n) = 1,$$

and $\phi c = \pi^n c$ for every $c \subseteq a_n$, $n \in \mathbb{Z}$. Thus ϕ satisfies condition (ii) of (a) and belongs to the countably full subgroup generated by π.

(iii)$\Rightarrow$(i) is trivial.

(ii)$\Leftrightarrow$(iv) The point is that, for $n \in \mathbb{Z}$ and $b \in \mathfrak{A}$,

$$\begin{aligned} \phi c = \pi^n c \text{ for every } c \subseteq b &\iff \pi^{-n}\phi c = c \text{ for every } c \subseteq b \\ &\iff b \cap \operatorname{supp}(\pi^{-n}\phi) = 0. \end{aligned}$$

So we have

$$\begin{aligned}\text{(ii)} &\iff \forall\, a \in \mathfrak{A} \setminus \{0\}\ \exists\, n \in \mathbb{Z},\ b \text{ such that } 0 \neq b \subseteq a \text{ and } b \cap \operatorname{supp}(\pi^{-n}\phi) = 0\\ &\iff \forall\, a \in \mathfrak{A} \setminus \{0\}\ \exists\, n \in \mathbb{Z},\ a \setminus \operatorname{supp}(\pi^{-n}\phi) \neq 0\\ &\iff \inf_{n\in\mathbb{Z}} \operatorname{supp}(\pi^{-n}\phi) = 0,\end{aligned}$$

as required.

381J Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\pi \in \operatorname{Aut}\mathfrak{A}$. Suppose that ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

(a) If $c \in \mathfrak{A}$ is such that $\pi c = c$, then $\phi c = c$.

(b) If $a \in \mathfrak{A}$ supports π then it supports ϕ.

proof (a) Let G be the set of all $\psi \in \operatorname{Aut}\mathfrak{A}$ such that $\psi c = c$. Then G is a subgroup of $\operatorname{Aut}\mathfrak{A}$ containing π. Also G is full. **P** If $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, $\langle \psi_i\rangle_{i\in I}$ is a family in G, and $\psi \in \operatorname{Aut}\mathfrak{A}$ is such that $\psi d = \psi_i d$ whenever $d \subseteq a_i$, then

$$\psi c = \sup_{i\in I} \psi(c \cap a_i) = \sup_{i\in I} \psi_i(c \cap a_i) = \sup_{i\in I} \psi_i c \cap \psi_i a_i = \sup_{i\in I} c \cap \psi_i a_i = c.$$

So $\psi \in G$; as ψ is arbitrary, G is full. **Q** So $\phi \in G$ and $\phi c = c$, as claimed.

(b) If $c \cap a = 0$ then $\pi c = c$ so $\phi c = c$.

381K Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism.

(a) If $a \in \mathfrak{A}$ and $a^* = \inf_{k\in\mathbb{N}} \sup_{i\geq k} \pi^i a$, then $\pi a^* = a^*$.

(b) If $a \in \mathfrak{A}$ is such that $a \subseteq \sup_{i\geq 1} \pi^i a$, then $\sup_{i\geq k} \pi^i a = \sup_{i\in\mathbb{N}} \pi^i a$ for every $k \in \mathbb{N}$.

proof (a) Because π is sequentially order-continuous,

$$\pi a^* = \inf_{k\in\mathbb{N}} \sup_{i\geq k} \pi^{i+1} a$$

(313Lc)

$$= \inf_{k\in\mathbb{N}} \sup_{i\geq k+1} \pi^i a = \inf_{k\geq 1} \sup_{i\geq k} \pi^i a = \inf_{k\in\mathbb{N}} \sup_{i\geq k} \pi^i a = a^*.$$

(b) Induce on k. For $k = 0$ the result is just the hypothesis. For the inductive step to $k + 1$, because π is sequentially order-continuous, so is π^k (313Ic), so

$$\begin{aligned}\sup_{i\geq k+1} \pi^i a &= \sup_{i\geq 1} \pi^k \pi^i a = \pi^k(\sup_{i\geq 1} \pi^i a)\\ &= \pi^k(\sup_{i\in\mathbb{N}} \pi^i a) = \sup_{i\geq k} \pi^i a = \sup_{i\in\mathbb{N}} \pi^i a,\end{aligned}$$

and the induction continues.

381L Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. Then for any $a \in \mathfrak{A}$, the following are equiveridical:

(i) π is recurrent on a;

(ii) $a \subseteq \sup_{n\geq 1} \pi^{-n} a$;

(iii) there is some $k \geq 1$ such that $a \subseteq \sup_{n\geq k} \pi^{-n} a$;

(iv) $a \subseteq \sup_{n\geq k} \pi^{-n} a$ for every $k \in \mathbb{N}$.

proof (i)⇒(ii) If (i) is true, set $b = a \setminus \sup_{n\geq 1} \pi^{-n} a$. Then $a \cap \pi^n b = 0$ for every $n \geq 1$, so $b = 0$, that is, $a \subseteq \sup_{n\geq 1} \pi^{-n} a$.

(ii)⇒(i) If (ii) is true and $0 \neq b \subseteq a$, then there is some $n \geq 1$ such that $b \cap \pi^{-n} a \neq 0$, that is, $\pi^n b \cap a \neq 0$; as b is arbitrary, π is recurrent on a.

(iv)⇒(ii)⇔(iii) are trivial.

(ii)⇒(iv) Apply 381Kb to π^{-1}.

381M It is with the idea of 'recurrence' that we start to get genuine surprises. The first fundamental construction is that of 'induced automorphism' in the following sense.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a \in \mathfrak{A}$. Suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ is doubly recurrent on a. Then we have a Boolean automorphism $\pi_a : \mathfrak{A}_a \to \mathfrak{A}_a$ defined by saying that $\pi_a d = \pi^n d$ whenever $n \ge 1$ and $d \subseteq a \cap \pi^{-n}a \setminus \sup_{1\le i<n} \pi^{-i}a$; I will call π_a the **induced automorphism** on $\mathfrak{A}_a$.

proof For $n \ge 1$ set

$$d_n = a \cap \pi^{-n}a \setminus \sup_{1\le i<n} \pi^{-i}a.$$

If $1 \le m < n$ then

$$d_n \subseteq \pi^{-n}a \setminus \pi^{-m}a, \quad d_m \subseteq \pi^{-m}a$$

so $d_m \cap d_n = 0$. Also

$$d_m \subseteq a, \quad \pi^{n-m}d_n \cap a = \pi^{n-m}(d_n \cap \pi^{-(n-m)}a) = 0$$

so

$$\pi^n d_n \cap \pi^m d_m = \pi^m(\pi^{n-m}d_n \cap d_m) = 0.$$

Finally, $\sup_{n\ge1} d_n = a \cap \sup_{n\ge1} \pi^{-n}a = a$, because π is recurrent on a (using (a)).

It follows that $\langle d_n\rangle_{n\ge1}$ is a partition of unity in $\mathfrak{A}_a$. Since $\langle \pi^n d_n\rangle_{n\ge1}$ also is a disjoint family in $\mathfrak{A}_a$, and

$$\begin{aligned}\sup_{n\ge1} \pi^n d_n &= \sup_{n\ge1}(\pi^n a \cap a \setminus \sup_{1\le i<n} \pi^{n-i}a)\\ &= a \cap \sup_{n\ge1}(\pi^n a \setminus \sup_{1\le i<n} \pi^i a) = a \cap \sup_{n\ge1} \pi^n a = a,\end{aligned}$$

(because π^{-1} is recurrent on a), $\langle \pi^n d_n\rangle_{n\ge1}$ is another partition of unity. So we have an automorphism $\pi_a : \mathfrak{A}_a \to \mathfrak{A}_a$ defined by setting $\pi_a d = \pi^n d$ if $d \subseteq d_n$ (381C).

381N Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a \in \mathfrak{A}$. Suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ is doubly recurrent on a. Let $\pi_a \in \operatorname{Aut}\mathfrak{A}_a$ be the induced automorphism.

(a) π^{-1} is doubly recurrent on a, and the induced automorphism $(\pi^{-1})_a$ is $(\pi_a)^{-1}$.

(b) For every $n \in \mathbb{N}$ there is a partition of unity $\langle b_i\rangle_{i\ge n}$ in $\mathfrak{A}_a$ such that $\pi_a^n b = \pi^i b$ whenever $i \ge n$ and $b \subseteq b_i$.

(c) If $n \ge 1$ and $0 \ne b \subseteq a \cap \pi^{-n}a$, there are a non-zero $b' \subseteq b$ and a j such that $1 \le j \le n$ and $\pi^n d = \pi_a^j d$ for every $d \subseteq b'$.

(d) Suppose that $m \ge 1$ is such that $a \cap \pi^i a = 0$ for $1 \le i < m$. Then for any $n \ge 1$ we have a disjoint family $\langle b_{ni}\rangle_{1\le i\le\lfloor n/m\rfloor}$, with supremum $a \cap \pi^{-n}a$, such that $\pi^n d = \pi_a^i d$ whenever $1 \le i \le \lfloor \frac{n}{m}\rfloor$ and $d \subseteq b_{ni}$.

(e) Suppose that $b \subseteq a$. Then π is doubly recurrent on b iff π_a is doubly recurrent on b, and in this case $\pi_b = (\pi_a)_b$, where $(\pi_a)_b$ is the automorphism of $\mathfrak{A}_b$ induced by π_a.

(f) Suppose that $c \in \mathfrak{A}$ is such that $\pi c = c$. Then π is doubly recurrent on $a \cap c$, and $\pi_{a\cap c} = \pi_a \restriction \mathfrak{A}_{a\cap c}$; in particular, $\pi_a(a \cap c) = a \cap c$.

(g) If π is aperiodic, so is π_a.

(h) Suppose that $a \cap \pi a = 0$, and that $b \subseteq a$ is such that $b \cap \pi_a b = 0$. Then b, πb and $\pi^2 b$ are all disjoint.

(i) There is an automorphism $\tilde\pi_a \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\tilde\pi_a d = \pi_a d$ for $d \subseteq a$, $\tilde\pi_a d = d$ for $d \subseteq 1 \setminus a$, and $\tilde\pi_a$ belongs to the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π.

proof Set $d_n = a \cap \pi^{-n}a \setminus \sup_{1\le i<n} \pi^{-i}a$ for $n \ge 1$, so that $\langle d_n\rangle_{n\ge1}$ and $\langle \pi^n d_n\rangle_{n\ge1}$ are partitions of unity in $\mathfrak{A}_a$, and $\pi_a b = \pi^n b$ for $b \subseteq d_n$.

(a) By the symmetry in the definition of 'doubly recurrent', π^{-1} is doubly recurrent on a iff π is. In this case,

$$\pi^n d_n = \pi^n a \cap a \setminus \sup_{1\le i<n} \pi^{n-i}a = a \cap \pi^n a \cap a \setminus \sup_{1\le i<n} \pi^i a$$

so $(\pi^{-1})_a b = \pi^{-n}b = (\pi_a)^{-1}b$ for every $b \subseteq \pi^n d_n$; as $\langle \pi_n d_n\rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}_a$, $(\pi^{-1})_a = (\pi_a)^{-1}$.

(b) Induce on n. For $n=0$ we can take $b_0=a$ and $b_i=0$ for $i>0$. For the inductive step to $n+1$, let $\langle b_i\rangle_{i\ge n}$ be a partition of unity in $\mathfrak{A}_a$ such that $\pi_a^n b=\pi^i b$ for $b\subseteq b_i$. Then $\langle\pi_a^{-1}b_i\rangle_{i\ge n}$ and $\langle d_k\cap\pi_a^{-1}b_i\rangle_{k\ge 1,i\ge n}$ are partitions of unity in $\mathfrak{A}_a$. If $b\subseteq d_k\cap\pi_a^{-1}b_i$, then $\pi_a b=\pi^k b\subseteq b_i$, so $\pi_a^{n+1}b=\pi^{k+i}b$. This means that if we set $b'_j=\sup_{k\ge 1,i\ge n,k+i=j}d_k\cap\pi_a^{-1}b_i$ for $j\ge n+1$, $\langle b'_j\rangle_{j\ge n+1}$ will be a partition of unity in $\mathfrak{A}_a$, and $\pi_a^{n+1}b=\pi^j b$ whenever $b\subseteq b_j$. So the induction continues.

(c) Induce on n. If $b\cap\pi^{-i}a=0$ for $1\le i<n$ then we can take $b'=b$ and $j=1$. Otherwise, take the first $i\ge 1$ such that $b_1=b\cap\pi^{-i}a\ne 0$. Then $\pi_a d=\pi^i d$ for every $d\subseteq b_1$. Also $\pi^{n-i}\pi^i b_1\subseteq a$, so, by the inductive hypothesis, there are a non-zero $c\subseteq\pi^i b_1$ and a j such that $1\le j\le n-i$ and $\pi^{n-i}d=\pi_a^j d$ for every $d\subseteq c$. Setting $b'=\pi^{-i}c\subseteq b_1$, we have $0\ne b'\subseteq b$ and

$$\pi^n d=\pi^{n-i}\pi^i d=\pi_a^j\pi_a d=\pi_a^{j+1}d$$

whenever $d\subseteq b'$. So the induction continues.

(d) Again induce on n. If $1\le n<m$ then $a\cap\pi^{-n}a=0$ and the result is trivial. If $n=m$, then $a\cap\pi^{-n}a=d_n$ and $\pi_a d=\pi^n d$ for every $d\subseteq d_n$, so we can set $b_{n1}=d_n$. For the inductive step to $n>m$, we have

$$\begin{aligned}a\cap\pi^{-n}a&=d_n\cup\sup_{m\le k<n}(d_k\cap\pi^{-n}a)=d_n\cup\sup_{m\le k<n}(d_k\cap\pi^{-k}(a\cap\pi^{-(n-k)}a))\\&=d_n\cup\sup_{\substack{m\le k\le n-m\\1\le j\le\lfloor(n-k)/m\rfloor}}(d_k\cap\pi^{-k}b_{n-k,j})\end{aligned}$$

by the inductive hypothesis, while $\langle d_k\cap\pi^{-k}b_{n-k,j}\rangle_{m\le k\le n-m,1\le j\le\lfloor(n-k)/m\rfloor}$ is disjoint. Now if $m\le k\le n-m$ and $1\le j\le\lfloor\frac{n-k}{m}\rfloor$ and $d\subseteq d_k\cap\pi^{-k}b_{n-k,j}$, we have $\pi_a d=\pi^k d\subseteq b_{n-k,j}$, so $\pi^n d=\pi^{n-k}\pi_a d=\pi_a^{j+1}d$; while if $d\subseteq d_n$ then $\pi^n d=\pi_a d$. So we can set

$$b_{n1}=d_n,\quad b_{ni}=\sup\nolimits_{m\le k\le n-m}d_k\cap b_{n-k,i-1}$$

for $2\le i\le\lfloor\frac{n}{m}\rfloor$, and the induction will continue.

(e) Applying (b) and (d) to π and π^{-1}, and using 381L and (a), we see that π is doubly recurrent on b iff π_a is doubly recurrent on b.

In this case, set $D=\{d:d\in\mathfrak{A}_b,\ \pi_b d=(\pi_a)_b d\}$. Then D is order-dense in $\mathfrak{A}_b$. **P** Take any non-zero $c\in\mathfrak{A}_b$. Since $b\subseteq\sup_{n\ge 1}\pi^{-n}b$, there is an $n\ge 1$ such that $c'=c\cap\pi^{-n}b\setminus\sup_{1\le i<n}\pi^{-i}b$ is non-zero. Next, there is a non-zero $d\subseteq c'$ such that for every $m\le n$ either $d\subseteq\pi^{-m}a$ or $d\cap\pi^{-m}a=0$. Enumerate $\{m:m\le n,\ d\subseteq\pi^{-m}a\}$ in ascending order as $(m_0,\dots,m_k)$ (note that as $c'\subseteq a\cap\pi^{-n}a$, we must have $m_0=0$ and $m_k=n$). Set $d_i=\pi^{m_i}d$ for $i\le k$, so that

$$d_0=d,\quad\pi^{m_{i+1}-m_i}d_i=d_{i+1}\subseteq a,$$

while

$$\pi^j d_i=\pi^{m_i+j}d\subseteq 1\setminus a$$

for $1\le j<m_{i+1}-m_i$; that is, $d_{i+1}=\pi_a d_i$ for $i<k$. Thus

$$\pi_a^k d=\pi^{m_k}d=\pi^n d\subseteq b,$$

while

$$\pi_a^i d=d_i=\pi^{m_i}d\subseteq\pi^{m_i}c'\subseteq 1\setminus b$$

for every $i<k$, and

$$(\pi_a)_b d=\pi_a^k d=\pi^n d=\pi_b d,$$

so that $d\in D$. As c is arbitrary, D is order-dense. **Q**

Because π_b and $(\pi_a)_b$ are both order-continuous Boolean homomorphisms on $\mathfrak{A}_b$, and every member of $\mathfrak{A}_b$ is a supremum of some subset of D (313K), $\pi_b=(\pi_a)_b$, as required.

(f) We have

$$a\cap c\subseteq\ \sup\nolimits_{n\ge 1}\pi^{-n}a\cap c=\sup\nolimits_{n\ge 1}\pi^{-n}a\cap\pi^{-n}c=\sup\nolimits_{n\ge 1}\pi^{-n}(a\cap c),$$

so π is recurrent on $a \cap c$; similarly, π^{-1} is recurrent on $a \cap c$. If $n \geq 1$ and

$$d \subseteq a \cap c \cap \pi^{-n}(a \cap c) \setminus \sup_{1 \leq i < n} \pi^{-i}(a \cap c) = c \cap a \cap \pi^{-n}a \setminus \sup_{1 \leq i < n} \pi^{-i}a,$$

then $\pi_{a \cap c}d = \pi^n d = \pi_a d$. So π_a extends $\pi_{a \cap c}$, as claimed.

(g) If $0 \neq b \subseteq a$, and $n \geq 1$, then (b) tells us that there are a non-zero $c \subseteq b$ and an $i \geq n$ such that $\pi_a^n d = \pi^i d$ for every $d \subseteq c$. Now we are supposing that $\operatorname{supp} \pi^i = 1$, so there is a $d \subseteq c$ such that $\pi^i d \neq d$, that is, $\pi_a^n d \neq d$. As b is arbitrary, $\operatorname{supp} \pi_a^n = a$; as n is arbitrary, π_a is aperiodic.

(h) Of course $\pi b \subseteq \pi a$ is disjoint from $b \subseteq a$; it follows that $\pi b \cap \pi^2 b = \pi(b \cap \pi b) = 0$. If $c = b \cap \pi^{-2}b$, then $c \subseteq a \cap \pi^{-2}a \setminus \pi^{-1}a$, so

$$\pi^2 b \cap b = \pi^2 c = \pi_a c \subseteq \pi_a b$$

is disjoint from b and must be 0. So b, πb and $\pi^2 b$ are all disjoint.

(i) By 381C, the formula defines an automorphism $\tilde{\pi}_a$. Setting $d_0 = 1 \setminus a$, $\langle d_n \rangle_{n \in \mathbb{N}}$ is a partition of unity in $\mathfrak{A}$ and $\tilde{\pi}_a d = \pi^n d$ for $d \subseteq d_n$, so $\tilde{\pi}_a$ belongs to the countably full subgroup of $\operatorname{Aut} \mathfrak{A}$ generated by π.

381O Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism. Then the following are equiveridical:

(i) π is recurrent on every $a \in \mathfrak{A}$;

(ii) for every non-zero $a \in \mathfrak{A}$ there is a $k \geq 1$ such that $a \cap \pi^k a \neq 0$;

(iii) $a = \sup_{k \geq 1} a \cap \pi^k a$ for every $a \in \mathfrak{A}$.

proof (i)$\Rightarrow$(ii) If (i) is true, and $a \in \mathfrak{A} \setminus \{0\}$, then taking $b = a$ in the definition 381Bg we see that there is a $k \geq 1$ such that $a \cap \pi^k a \neq 0$.

(ii)$\Rightarrow$(iii) Suppose (ii) is true. **?** If $a \in \mathfrak{A}$ is not the supremum of $\{a \cap \pi^k a : k \geq 1\}$, let $b \subseteq a$ be non-zero and disjoint from $\pi^k a$ for every $k \geq 1$. Then $b \cap \pi^k b = 0$ for every $k \geq 1$, which is impossible. **X**

(iii)$\Rightarrow$(i) Suppose (iii) is true. If $0 \neq b \subseteq a$ then $b = \sup_{k \geq 1} b \cap \pi^k b$, so there is certainly some $k \geq 1$ such that $b \cap \pi^k b \neq 0$, in which case $a \cap \pi^k b \neq 0$. As b is arbitrary, π is recurrent on a; as a is arbitrary, (i) is true.

Remark The condition 'recurrent on every $a \in \mathfrak{A}$' looks, and is, very restrictive; but it is satisfied by the homomorphisms we care about most (386A).

381P Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism which is recurrent on every $a \in \mathfrak{A}$. Then π is aperiodic iff $\mathfrak{A}$ is relatively atomless (definition: 331A) over the fixed-point algebra $\mathfrak{C} = \{c : c \in \mathfrak{A}, \pi c = c\}$. In particular, if π is ergodic, it is aperiodic iff $\mathfrak{A}$ is atomless.

proof It is elementary to check that $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$.

(a) Suppose that π is not aperiodic. Then there is a least $n \geq 1$ such that 1 is not the support of π^n; that is, there is a non-zero $a \in \mathfrak{A}$ such that $\pi^n d = d$ for every $d \subseteq a$. Now if $0 \neq b \subseteq a$ and $1 \leq i < n$ there is a non-zero $b' \subseteq b$ such that $b' \cap \pi^i b' = 0$. **P** We are supposing that the support of π^i is 1, so there is a $d \subseteq b$ such that $d \neq \pi^i d$. If $d \setminus \pi^i d \neq 0$, take $b' = d \setminus \pi^i d$. Otherwise, try $b' = d \setminus \pi^{n-i} d$; then

$$\pi^i b' = \pi^i d \setminus \pi^n d = \pi^i d \setminus d \neq 0,$$

so $b' \neq 0$, while $b' \cap \pi^i b' \subseteq d \setminus \pi^n d = 0$. **Q**

We can therefore find a non-zero $b \subseteq a$ such that $b \cap \pi^i b = 0$ whenever $1 \leq i < n$. Now b is a relative atom of $\mathfrak{A}$ over $\mathfrak{C}$. **P** If $d \subseteq b$, set $c = \sup_{0 \leq i < n} \pi^i d$. Then $\pi c = \sup_{1 \leq i \leq n} \pi^i d = c$, so $c \in \mathfrak{C}$, while $b \cap \pi^i d = 0$ for $1 \leq i < n$, so $d = b \cap c$. **Q** Thus b witnesses that $\mathfrak{A}$ is not relatively atomless over $\mathfrak{C}$.

(b)(i) Note that if $a \in \mathfrak{A}$ and $a \subseteq \pi a$ then $a = \pi a$. **P?** Otherwise, set $b = \pi a \setminus a$. Then $\pi^n b = \pi^{n+1} a \setminus \pi^n a$ for every n; also $a \subseteq \pi a \subseteq \pi^2 a \subseteq \ldots$, so $\langle \pi^n b \rangle_{n \in \mathbb{N}}$ is disjoint. But in this case π cannot be recurrent on b. **XQ**

(ii) Suppose that $\mathfrak{A}$ is not relatively atomless over $\mathfrak{C}$. Then there is a relative atom $a \in \mathfrak{A}$; as π is recurrent on a, there is a first $n \geq 1$ such that $a \cap \pi^n a \neq 0$. Then $\pi^n b = b$ for every $b \subseteq a \cap \pi^n a$. **P** Because a

is a relative atom over $\mathfrak{C}$, there is a $c \in \mathfrak{C}$ such that $b = a \cap c$. Now $\pi^n b = \pi^n a \cap c \supseteq b$. Set $b_1 = \sup_{0 \le i < n} \pi^i b$; then $\pi b_1 = \sup_{1 \le i \le n} \pi^i b \supseteq b_1$. So $b_1 = \pi b_1$, by (i), and $\pi^n b \subseteq \sup_{i<n} \pi^i b$. Next,

$$\pi^n b \cap \pi^i b = \pi^i(\pi^{n-i} b \cap b) \subseteq \pi^i(\pi^{n-i} a \cap a) = 0$$

for $0 < i < n$, so $\pi^n b \subseteq b$ and $\pi^n b = b$. **Q** Thus $a \cap \pi^n a$ witnesses that π is not aperiodic.

(c) Finally, if π is ergodic, then $\mathfrak{C} = \{0, 1\}$ (372Pa), so that 'relatively atomless over $\mathfrak{C}$' becomes 'atomless'.

381Q As far as possible I will express the ideas of this chapter in 'pure' Boolean algebra terms, without shifting to measure spaces or Stone spaces. However there is a crucial argument in §382 for which the Stone representation is an invaluable aid, and anyone studying the subject has to be able to use it.

Proposition Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. For $a \in \mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z; recall that $\widehat{a}$ can be identified with the Stone space of $\mathfrak{A}_a$ (312T). For a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ let $f_\pi : Z \to Z$ be the continuous function such that $\widehat{\pi a} = f_\pi^{-1}[\widehat{a}]$ for every $a \in \mathfrak{A}$ (312Q).

(a) If a, $b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ is a Boolean homomorphism represented by a continuous function $g : \widehat{b} \to \widehat{a}$, then $\pi \in \operatorname{Aut}\mathfrak{A}$ agrees with ϕ on $\mathfrak{A}_a$ iff f_π agrees with g on $\widehat{b}$.

(b) If $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, then $a \in \mathfrak{A}$ supports π iff $\widehat{a} \supseteq \{z : f_\pi(z) \ne z\}$. So a is the support of π iff $\widehat{a} = \overline{\{z : f_\pi(z) \ne z\}}$.

(c) Suppose that $\mathfrak{A}$ is Dedekind complete and π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Let G be the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. Then

$$\begin{aligned}\phi \in G &\iff \bigcup_{n\in\mathbb{Z}} \operatorname{int}\{x : f_\phi(z) = f_\pi^n(z)\} \text{ is dense in } Z \\ &\iff \{z : f_\phi(z) \in \{f_\pi^n(z) : n \in \mathbb{Z}\}\} \text{ is comeager in } Z.\end{aligned}$$

(d) A Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is recurrent on $a \in \mathfrak{A}$ iff $\widehat{a} \subseteq \overline{\bigcup_{n\ge1} f_\pi^n[\widehat{a}]}$.

(e) Suppose that $\mathfrak{A}$ is Dedekind σ-complete, $\pi \in \operatorname{Aut}\mathfrak{A}$ is recurrent on $a \in \mathfrak{A}$, and that $\pi_a \in \operatorname{Aut}\mathfrak{A}_a$ is the induced automorphism (381M). Let f_{π_a} be the corresponding autohomeomorphism of $\widehat{a}$. For $k \ge 1$, set $G_k = \{z : z \in \widehat{a},\ f^k(z) \in \widehat{a},\ f^i(z) \notin \widehat{a} \text{ for } 1 \le i < k\}$. Then $\bigcup_{k\ge1} G_k = \widehat{a} \cap \bigcup_{k\ge1} f^{-k}[\widehat{a}]$ is a dense open subset of $\widehat{a}$ and $f_{\pi_a}(z) = f_\pi^k(z)$ whenever $k \ge 1$ and $z \in G_k$.

proof Recall that $f_{\pi\phi} = f_\phi f_\pi$ for all Boolean homomorphisms π, $\phi : \mathfrak{A} \to \mathfrak{A}$ (312R).

(a) The point is that $\{\widehat{d} : d \subseteq a\}$ is a base for the Hausdorff topology of of $\widehat{a}$. So if $g \ne f_\pi{\restriction}\widehat{b}$, there are a $z \in \widehat{b}$ such that $f_\pi(z) \ne g(z)$ and a $d \subseteq a$ such that $g(z) \in \widehat{d}$ and $f_\pi(z) \notin \widehat{d}$. In this case,

$$z \in g^{-1}[\widehat{d}] \setminus f_\pi^{-1}[\widehat{d}] = \widehat{\phi d} \setminus \widehat{\pi d},$$

and $\phi \ne \pi{\restriction}\mathfrak{A}_a$. On the other hand, if $g = f_\pi{\restriction}\widehat{b}$, then

$$\widehat{\pi d} = f_\pi^{-1}[\widehat{d}] = g^{-1}[\widehat{d}] = \widehat{\phi d}$$

for every $d \subseteq a$, and $\phi = \pi{\restriction}\mathfrak{A}_a$.

(b)

$$\begin{aligned}a \in \mathfrak{A} \text{ supports } \pi &\iff \pi \text{ agrees with the identity on } 1 \setminus a \\ &\iff f_\pi(z) = z \text{ for every } z \in \widehat{\pi(1 \setminus a)} = Z \setminus \widehat{a} \\ &\iff \widehat{a} \supseteq \{z : f_\pi(z) \ne z\} \\ &\iff \widehat{a} \supseteq \overline{\{z : f_\pi(z) \ne z\}}.\end{aligned}$$

So the smallest such a, if there is one, must have $\widehat{a} = \overline{\{z : f_\pi(z) \ne z\}}$.

(c) If $\phi \in G$, let $\langle a_n\rangle_{n\in\mathbb{Z}}$ be a partition of unity in $\mathfrak{A}$ such that $\phi b = \pi^n b$ whenever $n \in \mathbb{Z}$ and $b \subseteq a_n$ (381I). Then $g(z) = f_\pi^n(z)$ whenever $z \in \widehat{\phi a_n}$ ((a) above). As $\sup_{n\in\mathbb{Z}} \phi a_n = 1$ in $\mathfrak{A}$,

$$\bigcup_{n\in\mathbb{Z}}\operatorname{int}\{x : f_\phi(z) = f_\pi^n(z)\} \supseteq \bigcup_{n\in\mathbb{Z}}\widehat{\phi a_n}$$

is dense (313Ca).

If $\bigcup_{n\in\mathbb{Z}}\operatorname{int}\{x : f_\phi(z) = f_\pi^n(z)\}$ is dense, it is a dense open subset of $\{z : f_\phi(z) \in \{f_\pi^n(z) : n \in \mathbb{Z}\}$, so the latter is comeager.

If $\{z : f_\phi(z) \in \{f_\pi^n(z) : n \in \mathbb{Z}\}\}$ is comeager, set $F_n = \{z : f_\phi(z) = f_\pi^n(z)\}$ for each n. Then $F_n \setminus \operatorname{int} F_n$ is nowhere dense for each n, and $Z \setminus \bigcup_{n\in\mathbb{Z}} F_n$ is meager, so $\bigcup_{n\in\mathbb{Z}}\operatorname{int} F_n$ is comeager, therefore dense (by Baire's theorem, 3A3G). If $a \in \mathfrak{A}$ is non-zero, there are an $n \in \mathbb{Z}$ such that $\widehat{\phi a} \cap \operatorname{int} F_n \ne \emptyset$ and a $b \in \mathfrak{A}$ such that $\emptyset \ne \widehat{b} \subseteq \widehat{\phi a} \cap F_n$, in which case $0 \ne \phi^{-1}b \subseteq a$ and $\phi c = \pi^n c$ for every $c \subseteq b$. By 381I(c-ii), $\phi \in G$. So the cycle is complete.

(d)

$$\begin{aligned}
\pi \text{ is recurrent on } a &\iff \text{whenever } 0 \ne b \subseteq a \text{ there is a } k \ge 1\\
&\qquad\qquad\qquad \text{such that } a \cap \pi^k b \ne 0\\
&\iff \text{whenever } 0 \ne b \subseteq a \text{ there is a } k \ge 1\\
&\qquad\qquad\qquad \text{such that } \widehat{a} \cap (f_\pi^k)^{-1}[\widehat{b}] \ne \emptyset\\
&\iff \text{whenever } 0 \ne b \subseteq a \text{ there is a } k \ge 1\\
&\qquad\qquad\qquad \text{such that } f_\pi^k[\widehat{a}] \cap \widehat{b} \ne \emptyset\\
&\iff \widehat{a} \cap \bigcup_{k\ge1} f_\pi^k[\widehat{a}] \text{ is dense in } \widehat{a}\\
&\iff \widehat{a} \subseteq \overline{\bigcup_{k\ge1} f_\pi^k[\widehat{a}]}.
\end{aligned}$$

(e) Set $d_k = a \cap \pi^{-k}a \setminus \sup_{1\le i<k}\pi^{-i}a$, so that $\pi^k d_k = a \cap \pi^k a \setminus \sup_{1\le i<k}\pi^i a$. Since π^k and π_a agree on $\mathfrak{A}_{d_k}$, (a) tells us that f_π^k and f_{π_a} agree on

$$\widehat{\pi^k d_k} = \widehat{\pi_a d_k} = f_{\pi_a}^{-1}[\widehat{d_k}] = G_k.$$

Because $\sup_{k\ge1}\pi^k d_k = a$, $\bigcup_{k\ge1} G_k$ is dense in $\widehat{a}$.

381R Cyclic automorphisms I end the section by describing a notation which is often useful.

Definition Let $\mathfrak{A}$ be a Boolean algebra.

(a) Suppose that a, b are disjoint members of $\mathfrak{A}$ and that $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi a = b$. I will write $(\overleftarrow{a\,{}_\pi\, b})$ for the member ψ of $\operatorname{Aut}\mathfrak{A}$ defined by setting

$$\begin{aligned}
\psi d &= \pi d \text{ if } d \subseteq a,\\
&= \pi^{-1}d \text{ if } d \subseteq b,\\
&= d \text{ if } d \subseteq 1 \setminus (a \cup b).
\end{aligned}$$

Observe that in this case (if $a \ne 0$) ψ is an involution, that is, has order 2 in the group $\operatorname{Aut}\mathfrak{A}$; I will call such a ψ an **exchanging involution**, and say that it **exchanges** a with b.

(b) More generally, if $a_1, \dots, a_n$ are disjoint elements of $\mathfrak{A}$ and $\pi_i \in \operatorname{Aut}\mathfrak{A}$ are such that $\pi_i a_i = a_{i+1}$ for each $i < n$, then I will write

$$(\overleftarrow{a_1\,{}_{\pi_1}\,a_2\,{}_{\pi_2}\,\dots\,{}_{\pi_{n-1}}\,a_n})$$

for that $\psi \in \operatorname{Aut}\mathfrak{A}$ such that

$$\begin{aligned}
\psi d &= \pi_i d \text{ if } 1 \le i < n,\ d \subseteq a_i,\\
&= \pi_1^{-1}\pi_2^{-1}\dots\pi_{n-1}^{-1}d \text{ if } d \subseteq a_n,\\
&= d \text{ if } d \subseteq 1 \setminus \sup_{i\le n} a_i.
\end{aligned}$$

(c) It will occasionally be convenient to use the same notation when each π_i is a Boolean isomorphism between the principal ideals $\mathfrak{A}_{a_i}$ and $\mathfrak{A}_{a_{i+1}}$, rather than an automorphism of the whole algebra $\mathfrak{A}$.

Remark The point of this notation is that we can expect to use the standard techniques for manipulating cycles that are (I suppose) familiar to you from elementary group theory; the principal change is that we have to keep track of the subscripted automorphisms ${}_{\pi}$. The following results are typical.

381S Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\psi = (\overleftarrow{a\,{}_{\pi}\,b})$ is an exchanging involution in $\operatorname{Aut}\mathfrak{A}$, then

$$\psi = (\overleftarrow{a\,{}_{\psi}\,b}) = (\overleftarrow{b\,{}_{\psi}\,a}) = (\overleftarrow{b\,{}_{\pi^{-1}}\,a})$$

has support $a \cup b$.

(b) If $\pi = (\overleftarrow{a\,{}_{\pi}\,b})$ is an exchanging involution in $\operatorname{Aut}\mathfrak{A}$, then for any $\phi \in \operatorname{Aut}\mathfrak{A}$,

$$\phi\pi\phi^{-1} = (\overleftarrow{\phi a\,{}_{\phi\pi\phi^{-1}}\,\phi b})$$

is another exchanging involution.

(c) If $\pi = (\overleftarrow{a\,{}_{\pi}\,b})$ and $\phi = (\overleftarrow{c\,{}_{\phi}\,d})$ are exchanging involutions, and a, b, c, d are all disjoint, then π and ϕ commute, and $\psi = \pi\phi = \phi\pi$ is another exchanging involution, being $(\overleftarrow{a \cup c\,{}_{\psi}\,b \cup d})$.

(d) If G is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$, $a_1, \dots , a_n \in \mathfrak{A}$ are disjoint, and $\pi_1, \dots , \pi_{n-1} \in G$, then

$$(\overleftarrow{a_1\,{}_{\pi_1}\,a_2\,{}_{\pi_2}\,\dots\,{}_{\pi_{n-1}}\,a_n}) \in G.$$

proof (a) Check the action of ψ on the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_b$, $\mathfrak{A}_{1\setminus(a\cup b)}$.

(b) $\phi a \cap \phi b = \phi(a \cap b) = 0$ and

$$\phi\pi\phi^{-1}\phi a = \phi\pi a = \phi b,$$

so $\psi = (\overleftarrow{\phi a\,{}_{\phi\pi\phi^{-1}}\,\phi b})$ is well-defined. Now check the action of ψ on the principal ideals $\mathfrak{A}_{\phi a}$, $\mathfrak{A}_{\phi b}$, $\mathfrak{A}_{1\setminus\phi(a\cup b)}$.

(c) Check the action of ψ on each of the principal ideals $\mathfrak{A}_a, \dots , \mathfrak{A}_e$, where $e = 1 \setminus (a \cup b \cup c \cup d)$.

(d) Immediate from the definitions in 381Rb and 381Be.

381T Remark I must emphasize that while, after a little practice, calculations of this kind become easy and safe, they are absolutely dependent on all the cycles present involving only members of one list of disjoint elements of $\mathfrak{A}$. If, for instance, a, b, c are disjoint, then

$$(\overleftarrow{a\,{}_{\pi}\,b})(\overleftarrow{b\,{}_{\phi}\,c}) = (\overleftarrow{a\,{}_{\pi}\,b\,{}_{\phi}\,c}).$$

But if $a \cap c \neq 0$ then there is no expression for the product in this language. Secondly, of course, we must be scrupulous in checking, at every use of the notation $(\overleftarrow{a_1\,{}_{\pi_1}\,\dots\,a_n})$, that $a_1, \dots , a_n$ are disjoint and that $\pi_i a_i = a_{i+1}$ for $i < n$. Thirdly, a significant problem can arise if the automorphisms involved don't match. Consider for instance the product

$$\psi = (\overleftarrow{a\,{}_{\pi}\,b})(\overleftarrow{a\,{}_{\phi}\,b}).$$

Then we have $\psi d = \pi^{-1}\phi d$ if $d \subseteq a$, $\pi\phi^{-1}d$ if $d \subseteq b$; ψ is not necessarily expressible as a product of 'disjoint' cycles. Clearly there are indefinitely complex variations possible on this theme. A possible formal expression of a sufficient condition to avoid these difficulties is the following. Restrict yourself to calculations involving a fixed list $a_1, \dots , a_n$ of disjoint elements of $\mathfrak{A}$ for which you can describe a family of isomorphisms $\phi_{ij} : \mathfrak{A}_{a_i} \to \mathfrak{A}_{a_j}$ such that ϕ_{ii} is always the identity on $\mathfrak{A}_{a_i}$, $\phi_{jk}\phi_{ij} = \phi_{ik}$ for all i, j, k, and whenever $a_i\,{}_{\pi}\,a_j$ appears in a cycle of the calculation, then π agrees with ϕ_{ij} on $\mathfrak{A}_{a_i}$. Of course this would be intolerably unwieldy if it were really necessary to exhibit all the ϕ_{ij} every time. I believe however that it is usually easy enough to form a mental picture of the actions of the isomorphisms involved sufficiently clear to offer confidence that such ϕ_{ij} are indeed present; and in cases of doubt, then *after* performing the formal operations it is always straightforward to check that the calculations are valid, by looking at the actions of the automorphisms on each relevant principal ideal.

381X Basic exercises (a) Let X be a set and Σ an algebra of subsets of X containing all singleton sets. Show that $\operatorname{Aut}\Sigma$ can be identified with the group of permutations $f : X \to X$ such that $f[E]$ and $f^{-1}[E]$ belong to Σ for every $E \in \Sigma$.

(b) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ partitions of unity in $\mathfrak{A}$, $\mathfrak{B}$ respectively. Assume *either* that I is finite *or* that I is countable and $\mathfrak{B}$ is Dedekind σ-complete *or* that $\mathfrak{B}$ is Dedekind complete. Suppose that for each $i \in I$ we have a Boolean homomorphism $\pi_i : \mathfrak{A}_{a_i} \to \mathfrak{B}_{b_i}$. (i) Show that there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ extending every π_i. (ii) Show that π is injective iff every π_i is. (iii) Show that if *either* I is finite *or* I is countable and $\mathfrak{A}$ is Dedekind σ-complete *or* $\mathfrak{A}$ is Dedekind complete, then π is surjective iff every π_i is. (iv) Show that π is order-continuous, or sequentially order-continuous, iff every π_i is.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Show that if $\pi \in \operatorname{Aut}\mathfrak{A}$ and $k \in \mathbb{Z} \setminus \{0\}$, then π is aperiodic iff π^k is.

(d) In 381H, show that the family $\langle c_i\rangle_{1\le i\le\omega}$ is uniquely determined.

>(e) Let (X, Σ, μ) be a countably separated measure space (definition: 343D), $\mathfrak{A}$ its measure algebra, $f : X \to X$ an inverse-measure-preserving function and $\pi : \mathfrak{A} \to \mathfrak{A}$ the induced homomorphism (343A). (i) Show that the support of π is $\{x : x \in X,\ f(x) \ne x\}^\bullet$. (ii) Show that π is periodic, with period $n \ge 1$, iff $\mu X > 0$, $f^n(x) = x$ for almost every x and $\{x : f^i(x) = x\}$ is negligible for $1 \le i < n$.

(f) Let (X, Σ, μ) be a localizable measure space, with measure algebra $(\mathfrak{A}, \bar\mu)$. Suppose that π and ϕ are automorphisms of $\mathfrak{A}$, and that π is represented by a measure space automorphism $f : X \to X$. Show that the following are equiveridical: (i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π; (ii) there is a function $g : X \to X$, representing ϕ, such that $g(x) \in \{f^n(x) : n \in \mathbb{Z}\}$ for every $x \in X$. (*Hint*: for (ii)$\Rightarrow$(i), consider measurable envelopes of sets $F \cap g[A_n]$, where $A_n = \{x : g(x) = f^n(x)\}$ and $\mu F < \infty$.)

(g) Let $\mathfrak{A}$ be a Boolean algebra, not $\{0\}$, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an automorphism with fixed-point subalgebra $\mathfrak{C}$. Show that π is periodic, with period $n \ge 1$, iff $\pi{\restriction}\mathfrak{A}_c$ has order n in the group $\operatorname{Aut}\mathfrak{A}_c$ whenever $c \in \mathfrak{C} \setminus \{0\}$. Show that π is aperiodic iff $\pi{\restriction}\mathfrak{A}_c$ has infinite order in the group $\operatorname{Aut}\mathfrak{A}_c$ whenever $c \in \mathfrak{C} \setminus \{0\}$.

(h) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$ and $\phi \in \operatorname{Aut}\mathfrak{A}$. Show that ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G iff $\inf_{\pi\in G}\operatorname{supp}(\pi\phi) = 0$.

(i) Let $\mathfrak{A}$ be a Boolean algebra. Let us say that a subgroup G of $\operatorname{Aut}\mathfrak{A}$ is **finitely full** if whenever $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$, $\langle \pi_i\rangle_{i\in I}$ is a family in G, and $\pi \in \operatorname{Aut}\mathfrak{A}$ is such that $\pi a = \pi_i a_i$ whenever $i \in I$ and $a \subseteq a_i$, then $\pi \in G$. Show that if π, $\phi \in \operatorname{Aut}\mathfrak{A}$ then ϕ belongs to the finitely full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π iff there are an $n \in \mathbb{N}$ and a partition of unity $\langle a_i\rangle_{-n\le i\le n}$ in $\mathfrak{A}$ such that $\phi d = \pi^i d$ whenever $|i| \le n$ and $d \subseteq a_i$.

(j) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism which is recurrent on $a \in \mathfrak{A}$. Show that for any non-zero $b \subseteq a$ and any $n \in \mathbb{N}$ there is a $k \ge n$ such that $a \cap \pi^k b \ne 0$.

(k) Let $\mathfrak{A}$ be a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism, and $a \in \mathfrak{A}$. Show that the following are equiveridical: (i) π is recurrent on every $b \subseteq a$; (ii) for every non-zero $b \subseteq a$ there is an $n \ge 1$ such that $b \cap \pi^n b \ne 0$; (iii) $b = \sup_{n\ge1} b \cap \pi^n b$ for every $b \subseteq a$.

>(l) Let (X, Σ, μ) be a measure space, $\mathfrak{A}$ its measure algebra, $f : X \to X$ a measure space automorphism, and π the corresponding automorphism of $\mathfrak{A}$. (i) Show that if $E \in \Sigma$ then π is doubly recurrent on $a = E^\bullet$ iff $E \setminus \bigcup_{n\ge1} f^{-n}[E]$ and $E \setminus \bigcup_{n\ge1} f^n[E]$ are negligible. (ii) Show that in this case there is a measurable $F \subseteq E$ such that $E \setminus F$ is negligible and $\{n : n \in \mathbb{Z},\ f^n(x) \in F\}$ is unbounded above and below in $\mathbb{Z}$ for every $x \in F$. (iii) For $x \in F$ let $k(x) = \min\{n : n \ge 1,\ f^n(x) \in F\}$. Show that $x \mapsto f^{k(x)}(x) : F \to F$ represents the induced automorphism π_a on the principal ideal $\mathfrak{A}_a$.

(m) For a Boolean algebra $\mathfrak{A}$, a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is **nowhere aperiodic** if $\inf\{a : a \in \mathfrak{A},\ a \text{ supports } \pi^n \text{ for some } n \ge 1\} = 0$. Show that if $\mathfrak{A}$ is Dedekind σ-complete and $\pi \in \operatorname{Aut}\mathfrak{A}$ is nowhere aperiodic and doubly recurrent on $a \in \mathfrak{A}$, then the induced automorphism π_a is nowhere aperiodic.

(n) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\pi \in \operatorname{Aut}\mathfrak{A}$ an automorphism and $\mathfrak{C}$ the fixed-point subalgebra of π. Suppose that π is doubly recurrent on $a \in \mathfrak{A}$ and that π_a is the induced automorphism on $\mathfrak{A}_a$. Show that the fixed-point subalgebra of π_a is $\{c \cap a : c \in \mathfrak{C}\}$, so that if π is ergodic, so is π_a.

(o) Let $\mathfrak{A}$ be a Boolean algebra with Stone space Z, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism corresponding to $f : Z \to Z$. (i) Show that π is periodic, with period $n \geq 1$, iff $Z \neq \emptyset$, $f^n(z) = z$ for every $z \in Z$ and $\{z : f^i(z) = z\}$ is nowhere dense whenever $1 \leq i < n$. (ii) Show that π is aperiodic iff $\{z : f^n(z) = z$, $f^n(w) \neq z$ for every $w \neq z\}$ is nowhere dense for every $n \geq 1$.

(p) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, G a subgroup of $\operatorname{Aut}\mathfrak{A}$ and G^* the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by G. Suppose that every member of G has a support. Show that every member of G^* has a support.

381Y Further exercises (a) (i) Give an example to show that the word 'injective' in the statement of 381H is essential. (ii) Give an example to show that, in 381H, we can have $\pi c_\omega \neq c_\omega$.

(b) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Let us say that G is **full** if whenever $\phi : \mathfrak{A} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism, and there is a partition of unity $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ such that for every $i \in I$ there is a $\pi_i \in G$ such that $\phi a = \pi_i a$ for every $a \subseteq a_i$, then $\phi \in G$. Show that if ϕ and π are order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself, then the following are equiveridical: (i) ϕ belongs to the full semigroup generated by π; (ii) for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and an $n \in \mathbb{N}$ such that $\phi d = \pi^n d$ for every $d \subseteq b$; (iii) there is a partition of unity $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$ such that $\phi a = \pi^n a$ whenever $n \in \mathbb{N}$ and $a \subseteq a_n$.

(c) Give an example of a Dedekind σ-complete Boolean algebra $\operatorname{Aut}\mathfrak{A}$ and an automorphism π of $\mathfrak{A}$ such that the countably full subgroup generated by π is not full.

(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and let G be the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by a subset A of $\operatorname{Aut}\mathfrak{A}$. Show that if *either* A is countable *or* $\mathfrak{A}$ is ccc, then G is full.

(e)(i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and a, b two elements of $\mathfrak{A}$. Suppose that $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$ is a Boolean isomorphism such that there is no disjoint sequence $\langle c_n \rangle_{n \in \mathbb{N}}$ of non-zero elements of $\mathfrak{A}_{a \cap b}$ such that $\pi c_n = c_{n+1}$ for every $n \in \mathbb{N}$. Show that there is a Boolean automorphism of $\mathfrak{A}$ extending π. (ii) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and a, $b \in \mathfrak{A}$ two elements of $\mathfrak{A}$ such that $\bar{\mu}(a \cap b) < \infty$. Show that any measure-preserving isomorphism from $\mathfrak{A}_a$ to $\mathfrak{A}_b$ extends to a measure-preserving automorphism of $\mathfrak{A}$. (Compare 332L.)

381 Notes and comments There are no long individual proofs in this section, and in so far as there is any delicacy in the arguments it is as often as not because (as in 381E) I am taking facts which are easy to prove for automorphisms of Dedekind complete algebras and separating out the parts which happen to be true in greater generality. However the parts are numerous enough for the sum to be not entirely predictable. The most important ideas are surely in 381M-381N.

In 381Q I give indications, including the minimum necessary for an application in the next section, of how to express the concepts here in terms of continuous functions on Stone spaces. When we come, in §383 and onwards, to look specifically at measure algebras, many of our homomorphisms will be derived from inverse-measure-preserving functions, and the results will be more effective if we can display them in terms of functions on measure spaces. Some appropriate translations are in 381Xe-381Xl. But these I will avoid in the proofs of the main theorems because not all automorphisms of measure algebras can be represented by automorphisms of the measure spaces we start from (343Jc). Of course Lebesgue measure is different, in ways explored in §344, and classical ergodic theory has not needed to make a clear distinction here. One of my purposes in this volume is to set out a framework in which transformations of measure *spaces* take their proper place as an inspiration for the theory rather than a foundation.

Version of 15.8.06

382 Factorization of automorphisms

My aim in this chapter is to investigate the automorphism groups of measure algebras, but as usual I prefer to begin with results which can be expressed in the language of general Boolean algebras. The principal theorems in this section are 382M, giving a sufficient condition for every member of a full group of automorphisms to be a product of involutions, and 382R, describing the normal subgroups of full groups. The former depends on Dedekind σ-completeness and the presence of 'separators' (382Aa); the latter needs a Dedekind complete algebra and a group with 'many involutions' (382O). Both concepts are chosen with a view to the next section, where the results will be applied to groups of measure-preserving automorphisms.

382A Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$.

(a) I say that $a \in \mathfrak{A}$ is a **separator** for π if $a \cap \pi a = 0$ and $\pi b = b$ whenever $b \in \mathfrak{A}$ and $b \cap \pi^n a = 0$ for every $n \in \mathbb{Z}$.

(b) I say that $a \in \mathfrak{A}$ is a **transversal** for π if $\sup_{n\in\mathbb{Z}} \pi^n a = 1$ and $\pi^n b = b$ whenever $n \in \mathbb{Z}$ and $b \subseteq a \cap \pi^n a$.

382B Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. If every power of π has a separator and π^n is the identity, where $n \ge 1$, then π has a transversal.

proof (a) For $0 \le j < n$ let $a_j \in \mathfrak{A}$ be a separator for π^j. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $A = \{\pi^i a_j : 0 \le i,\ j < n\}$. Because $\pi[A] = A$, $\pi[\mathfrak{B}] = \mathfrak{B}$. (The set $\{a : a \in \mathfrak{B},\ \pi a \in \mathfrak{B},\ \pi^{-1}a \in \mathfrak{B}\}$ is a subalgebra of $\mathfrak{A}$ including A, so must be $\mathfrak{B}$.) Because A is finite, so is $\mathfrak{B}$; let B be the set of atoms of $\mathfrak{B}$. Then $\pi{\restriction}B$ is a permutation of the finite set B.

(b) Let $\mathcal{C}$ be the set of orbits of $\pi{\restriction}B$, that is, the family of sets of the form $\{\pi^k b : k \in \mathbb{Z}\}$ for $b \in B$. If $b \in C \in \mathcal{C}$, set $m = \#(C)$; then $d = \pi^m d$ for every $d \subseteq b$. **P** If $m = n$ this is trivial. Otherwise, b is either disjoint from, or included in, $\pi^i a_m$ whenever $0 \le i < n$, and therefore for every $i \in \mathbb{Z}$. But we have $a_m \cap \pi^m a_m = 0$, so $\pi^i a_m \cap \pi^{i+m} a_m = 0$ for every i, and $b = \pi^m b$ must be disjoint from $\pi^i a_m$, for every i. By the other clause in the definition of 'separator', $\pi^m d = d$ for every $d \subseteq b$. **Q**

(c) For each $C \in \mathcal{C}$, choose $b_C \in C$. Set $c = \sup_{C\in\mathcal{C}} b_C$. Then c is a transversal for π. **P** If $C \in \mathcal{C}$, we have $\pi^n b_C = b_C$, so $k_C = \#(C)$ is a factor of n. Now

$$\sup_{0\le k<n} \pi^k c = \sup_{C\in\mathcal{C},0\le k<n} \pi^k b_C = \sup_{C\in\mathcal{C}} \sup C = \sup(\bigcup\mathcal{C}) = \sup B = 1.$$

So certainly $\sup_{k\in\mathbb{Z}} \pi^k c = 1$. Now suppose that $k \in \mathbb{Z}\setminus\{0\}$ and $d \subseteq c \cap \pi^k c$. Set $B_0 = \{b : b \in B,\ d \cap b \ne 0\}$. If $b \in B_0$, then $b \cap c \ne 0$, so $b = b_C$ where $C \in \mathcal{C}$ is the orbit of $\pi{\restriction}B$ containing b. Next, $d \cap b \cap \pi^k c \ne 0$, so $\pi^{-k}(d \cap b) \cap c \ne 0$ and there is a $b' \in B$ such that $\pi^{-k}(d\cap b) \cap b' \ne 0$; in this case we must have $b' = \pi^{-k} b \in C$. But as $b' \cap c \supseteq \pi^{-k}(d \cap b) \cap c$ is non-zero, $b' = b_C = b$. Thus $b = \pi^k b$ and k is a multiple of $\#(C)$. Since $\pi^{\#(C)}(d \cap b) = d \cap b$, by (b), $\pi^k(d \cap b) = d \cap b$.

This is true for every $b \in B$ meeting d; so

$$\pi^k d = \pi^k(\sup_{b\in B_0} d \cap b) = \sup_{b\in B_0} \pi^k(d \cap b) = \sup_{b\in B_0} d \cap b = d.$$

As k and d are arbitrary, c is a transversal for π. **Q**

382C Corollary If $\mathfrak{A}$ is a Boolean algebra and $\pi \in \mathfrak{A}$ is an involution, then π is an exchanging involution iff it has a separator iff it has a transversal.

proof If π exchanges a and πa then of course a is a separator for π. If π has a separator, then every power of π has a separator, so 382B tells us that π has a transversal. If a is a transversal for π then $a \cup \pi a = \sup_{n\in\mathbb{Z}} \pi^n a = 1$ and $\pi b = b$ whenever $b \subseteq a \cap \pi a$, so π exchanges $a \setminus \pi a$ and $\pi a \setminus a$.

382D Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) π has a separator;

(ii) there is an $a \in \mathfrak{A}$ such that $a \cap \pi a = 0$ and $a \cup \pi a \cup \pi^2 a$ supports π;

(iii) there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in\mathbb{N}} \pi a_n \setminus a_n$ supports π;

(iv) there is a partition of unity (a', a'', b', b'', c, e) in $\mathfrak{A}$ such that

$$\pi a' = b', \quad \pi a'' = b'', \quad \pi b'' = c, \quad \pi(b' \cup c) = a' \cup a'', \quad \pi d = d \text{ for every } d \subseteq e.$$

proof (i)⇒(ii) Suppose that a is a separator for π. Set $a^+ = \sup_{n\ge1} \pi^n a$, $a^- = \sup_{n\ge1} \pi^{-n} a$; we are supposing that $a \cap \pi a = 0$ and that $a \cup a^+ \cup a^-$ supports π. For $n \in \mathbb{N}$ set $a_n = \pi^n a \setminus \sup_{0\le i<n} \pi^i a$, so that $\langle a_n \rangle_{n\in\mathbb{N}}$ is disjoint and has supremum $a \cup a^+$. Set $b_1 = \sup_{n\in\mathbb{N}} a_{2n} \setminus \pi^{-1} a$. Since $a \cap \pi^{-1} a = \pi^{-1}(a \cap \pi a) = 0$, $a \subseteq b_1 \subseteq a \cup a^+$. For any $n \in \mathbb{N}$,

$$\pi(a_{2n} \setminus \pi^{-1} a) = \pi^{2n+1} a \setminus (a \cup \sup\nolimits_{1\le i\le 2n} \pi^i a) = a_{2n+1},$$

so $b_1 \cap \pi b_1 = 0$. Note that $\pi b_1 \subseteq a^+$, while $a^+ \setminus \pi^{-1} a \subseteq b_1 \cup \pi b_1$.

Set $c = a \setminus a^+$. Then

$$\pi^i c \cap \pi^j c = \pi^j(c \cap \pi^{i-j} c) \subseteq \pi^j(a \setminus \pi^{i-j} a^+) \subseteq \pi^j(a \setminus \pi^{i-j}\pi^{j-i} a) = 0$$

whenever $i < j$ in $\mathbb{Z}$, so $\langle \pi^k c \rangle_{k\in\mathbb{Z}}$ is disjoint. We have

$$\begin{aligned}\sup_{n\ge1} \pi^{-n} c &= \sup_{n\ge1}(\pi^{-n} a \setminus \sup_{i>-n} \pi^i a) = \sup_{n\ge1}(\pi^{-n} a \setminus \sup_{0\le i<n} \pi^{-i} a) \setminus (a \cup a^+)\\ &= \sup_{n\ge1} \pi^{-n} a \setminus (a \cup a^+) = a^- \setminus (a \cup a^+).\end{aligned}$$

If $k \ge 1$ and $i \ge 0$ then

$$\pi^{-k} c \cap \pi^i a = \pi^{-k}(c \cap \pi^{i+k} a) \subseteq \pi^{-k}(c \cap a^+) = 0;$$

as i is arbitrary, $\pi^{-k} c \cap b_1 = 0$. So if we set $b = b_1 \cup \sup_{k\ge1} \pi^{-2k} c$,

$$\begin{aligned}b \cap \pi b &= (b_1 \cap \pi b_1) \cup (\sup_{k\ge1} b_1 \cap \pi^{1-2k} c) \cup (\sup_{k\ge1} \pi^{-2k} c \cap \pi b_1) \cup (\sup_{j,k\ge1} \pi^{-2j} c \cap \pi^{1-2k} c)\\ &\subseteq 0 \cup 0 \cup \pi(\sup_{k\ge1} \pi^{-2k-1} c \cap b_1) \cup 0 = 0.\end{aligned}$$

Since

$$\begin{aligned}b \cup \pi b \cup \pi^{-1} b &\supseteq b_1 \cup \pi b_1 \cup \pi^{-1} a \cup \sup_{n\ge1} \pi^{-n} c\\ &\supseteq a \cup a^+ \cup (a^- \setminus (a \cup a^+)) = a \cup a^+ \cup a^-\end{aligned}$$

supports π, $\pi^{-1} b$ witnesses that (ii) is true.

(ii)⇒(iii) If $a \in \mathfrak{A}$ is such that $a \cap \pi a = 0$ and $a \cup \pi a \cup \pi^2 a$ supports π, then $\pi^{n+1} a = \pi^{n+1} a \setminus \pi^n a$ for every n, so we can set $a_n = \pi^{n-1} a$ for each n to obtain a sequence witnessing (iii).

(iii)⇒(i) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is such that $\sup_{n\in\mathbb{N}} \pi a_n \setminus a_n$ supports π, set $b_n = \sup_{k\in\mathbb{Z}} \pi^k(\pi a_n \setminus a_n)$, $c_n = b_n \setminus \sup_{0\le i<n} b_i$ for each $n \in \mathbb{N}$. Then $\pi b_n = b_n$ and $\pi c_n = c_n$ for every $n \in \mathbb{N}$, while $\langle c_n \rangle_{n\in\mathbb{N}}$ is disjoint. Set $a = \sup_{n\in\mathbb{N}} c_n \cap a_n \setminus \pi^{-1} a_n$. Then

$$\begin{aligned}a \cap \pi a &= \sup_{m,n\in\mathbb{N}} (c_m \cap a_m \setminus \pi^{-1} a_m) \cap (\pi c_n \cap \pi a_n \setminus a_n)\\ &= \sup_{m,n\in\mathbb{N}} (c_m \cap a_m \setminus \pi^{-1} a_m) \cap (c_n \cap \pi a_n \setminus a_n)\\ &= \sup_{n\in\mathbb{N}} c_n \cap (a_n \setminus \pi^{-1} a_n) \cap (\pi a_n \setminus a_n) = 0.\end{aligned}$$

Next,

$$\sup_{k\in\mathbb{Z}}\pi^k a=\sup_{n\in\mathbb{N},k\in\mathbb{Z}} c_n\cap\pi^k a_n\setminus\pi^{k-1}a_n$$
$$=\sup_{n\in\mathbb{N},k\in\mathbb{Z}} c_n\cap\pi^{k+1}a_n\setminus\pi^k a_n=\sup_{n\in\mathbb{N}}c_n\cap b_n$$
$$=\sup_{n\in\mathbb{N}}c_n=\sup_{n\in\mathbb{N}}b_n\supseteq\sup_{n\in\mathbb{N}}\pi a_n\setminus a_n$$

supports π. So a is a separator for π.

(ii)⇒(iv) Let a be such that $a\cap\pi a=0$ and $a\cup\pi a\cup\pi^2 a$ supports π. Set $c=\pi^2a\setminus(a\cup\pi a)$, $b''=\pi^{-1}c\subseteq\pi a$, $b'=\pi a\setminus b''$, $a''=\pi^{-1}b''\subseteq a$, $a'=a\setminus a''$ and $e=1\setminus(a\cup\pi a\cup\pi^2a)$. Then $(a,\pi a,c,e)$ and (a',a'',b',b'',c,e) are partitions of unity in $\mathfrak{A}$; $\pi a''=b''$; $\pi a'=\pi a\setminus b''=b'$; $\pi b''=c$; $\pi d=d$ for every $d\subseteq e$; so

$$\pi(b'\cup c)=\pi(1\setminus(a\cup b''\cup e))=1\setminus(\pi a\cup\pi b''\cup\pi e)=1\setminus(\pi a\cup c\cup e)=a=a'\cup a''.$$

(iv)⇒(ii) If a',a'',b',b'',c,e witness (iv), then $a=a'\cup a''$ witnesses (ii).

382E Corollary (a) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi\in\operatorname{Aut}\mathfrak{A}$ has a separator, then π has a support.

(b) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra then every $\pi\in\operatorname{Aut}\mathfrak{A}$ has a separator.

proof (a) Taking $a\in\mathfrak{A}$ such that $a\cap\pi a=0$ and $e=a\cup\pi a\cup\pi^2a$ supports π, we see that e must actually be the support of π (381Ei, 381Ea).

(b) If $\mathfrak{A}$ is Dedekind complete and $\pi\in\operatorname{Aut}\mathfrak{A}$, let P be the set $\{d:d\in\mathfrak{A},\ d\cap\pi d=0\}$. Then P has a maximal element. **P** Of course $P\ne\emptyset$, as $0\in P$. If $Q\subseteq P$ is non-empty and upwards-directed, set $a=\sup Q$, which is defined because $\mathfrak{A}$ is Dedekind complete; then $\pi a=\sup\pi[Q]$ (since π, being an automorphism, is surely order-continuous). If d_1, $d_2\in Q$, there is a $d\in Q$ such that $d_1\cup d_2\subseteq d$, so $d_1\cap\pi d_2\subseteq d\cap\pi d=0$. By 313Bc, $a\cap\pi a=0$. This means that $a\in P$ and is an upper bound for Q in P. As Q is arbitrary, Zorn's Lemma tells us that P has a maximal element. **Q**

Let $b\in P$ be maximal. Then $b\cap\pi b=0$. Set $e=b\cup\pi a\cup\pi^{-1}b$. **?** If e does not support π, let $d\subseteq 1\setminus e$ be such that $d\cap\pi d=0$ (381Ei). Then $d\cap\pi b\subseteq d\cap e=0$, while also $b\cap\pi d\subseteq\pi(\pi^{-1}b\cap d)\subseteq\pi(e\cap d)=0$; so $(b\cup d)\cap\pi(b\cup d)=0$, and $b\subset b\cup d\in P$, which is impossible. **X** So if we set $a=\pi^{-1}b$ we have a witness of 382D(ii), and π has a separator.

Remark 382Eb and 382D(i)⇔(ii) together amount to 'Frolík's theorem' (FROLÍK 68).

382F Corollary Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra.

(a) Every involution in $\operatorname{Aut}\mathfrak{A}$ is an exchanging involution.

(b) If $\pi\in\operatorname{Aut}\mathfrak{A}$ is periodic with period $n\ge 2$, there is an $a\in\mathfrak{A}$ such that $(a,\pi a,\pi^2a,\dots,\pi^{n-1}a)$ is a partition of unity in $\mathfrak{A}$; that is (in the language of 381R) π is of the form $(\overleftarrow{a_1\,{}_\pi\,a_2\,{}_\pi\,\dots\,{}_\pi\,a_n})$ where $(a_1,\dots,a_n)$ is a partition of unity in $\mathfrak{A}$.

proof (a) By 382Eb, every involution has a separator; now use 382C.

(b) Again because every automorphism has a separator, 382B tells us that π has a transversal a. In this case, $a\cap\pi^ka$ must be disjoint from the support of π^k for every $k\in\mathbb{Z}$; since $\operatorname{supp}\pi^k=1$ for $0<k<n$, $a\cap\pi^ka=0$ for $0<k<n$; of course it follows that $\pi^ia\cap\pi^ja=\pi^i(a\cap\pi^{j-i}a)=0$ if $0\le i<j<n$. So $a,\pi a,\dots,\pi^{n-1}a$ are disjoint; since $\sup_{0\le i<n}\pi^ia=\sup_{i\in\mathbb{Z}}\pi^ia=1$, they constitute a partition of unity.

382G Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi\in\operatorname{Aut}\mathfrak{A}$.

(a) Suppose that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a family in $\mathfrak{A}$ such that $\pi a_n=a_n$ and $\pi\restriction\mathfrak{A}_{a_n}$ has a transversal for every n. Set $a=\sup_{n\in\mathbb{N}}a_n$; then $\pi a=a$ and $\pi\restriction\mathfrak{A}_a$ has a transversal.

(b) If a is a transversal for π it is a transversal for π^{-1}.

(c) Suppose that $a \in \mathfrak{A}$. Set

$$a^* = \sup_{n\in\mathbb{Z}}(\pi^n a \setminus \sup_{i>n} \pi^i a), \quad a_* = \sup_{n\in\mathbb{Z}}(\pi^n a \setminus \sup_{i<n} \pi^i a).$$

Then $\pi a^* = a^*$, $\pi a_* = a_*$ and $\pi\restriction\mathfrak{A}_{a^*}$, $\pi\restriction\mathfrak{A}_{a_*}$ both have transversals.

proof (a) Of course $\pi a = \sup_{n\in\mathbb{N}} \pi a_n = a$, so we can speak of $\pi\restriction\mathfrak{A}_a$. For each $n \in \mathbb{N}$, let b_n be a transversal for $\pi\restriction\mathfrak{A}_{a_n}$. Set $b = \sup_{n\in\mathbb{N}}(b_n \setminus \sup_{i<n} a_i)$. Then b is a transversal for $\pi\restriction\mathfrak{A}_a$. **P** Of course $b \in \mathfrak{A}_a$. Now

$$\begin{aligned}\sup_{k\in\mathbb{Z}} \pi^k b &= \sup_{k\in\mathbb{Z}} \sup_{n\in\mathbb{N}} (\pi^k b_n \setminus \sup_{i<n} \pi^k a_i) = \sup_{n\in\mathbb{N}} \sup_{k\in\mathbb{Z}} (\pi^k b_n \setminus \sup_{i<n} a_i) \\ &= \sup_{n\in\mathbb{N}}\bigl((\sup_{k\in\mathbb{Z}} \pi^k b_n) \setminus \sup_{i<n} a_i\bigr) = \sup_{n\in\mathbb{N}}(a_n \setminus \sup_{i<n} a_i) = \sup_{n\in\mathbb{N}} a_n = a.\end{aligned}$$

Next, suppose that $k \in \mathbb{Z}$ and

$$\begin{aligned}d \subseteq b \cap \pi^k b &= \sup_{m,n\in\mathbb{N}} (b_m \setminus \sup_{i<m} a_i) \cap (\pi^k b_n \setminus \sup_{j<n} \pi^k a_j) \\ &= \sup_{m,n\in\mathbb{N}} (b_m \cap a_m \setminus \sup_{i<m} a_i) \cap (\pi^k b_n \cap a_n \setminus \sup_{j<n} a_j) = \sup_{n\in\mathbb{N}}(b_n \cap \pi^k b_n \setminus \sup_{i<n} a_i).\end{aligned}$$

Setting $d_n = d \cap b_n \cap \pi^k b_n$ for each n, we have

$$d = \sup_{n\in\mathbb{N}} d_n = \sup_{n\in\mathbb{N}} \pi^k d_n = \pi^k d.$$

As k and d are arbitrary, b is a transversal for $\pi\restriction\mathfrak{A}_a$. **Q**

(b) We have only to note that the definition in 382Ab is symmetric between π and π^{-1}.

(c)

$$\begin{aligned}\pi a^* &= \sup_{n\in\mathbb{Z}}(\pi^{n+1} a \setminus \sup_{i>n} \pi^{i+1} a) \\ &= \sup_{n\in\mathbb{Z}}(\pi^{n+1} a \setminus \sup_{i>n+1} \pi^i a) = \sup_{n\in\mathbb{Z}}(\pi^n a \setminus \sup_{i>n} \pi^i a) = a^*.\end{aligned}$$

Set $b_n = \pi^n a \setminus \sup_{i>n} \pi^i a$ for each n, $b = \sup_{n\in\mathbb{Z}} \pi^{-n} b_n \subseteq a$. Writing b^* for $\sup_{n\in\mathbb{Z}} \pi^n b$, we have $b^* \supseteq \sup_{n\in\mathbb{Z}} b_n = a^*$. Note that $\pi^{-n} b_n \cap \pi^i a = 0$ for every $i \ge 1$. So if $m < n$ in $\mathbb{Z}$,

$$\pi^m b \cap \pi^n b \subseteq \pi^m(\sup_{i\in\mathbb{Z}} \pi^{-i} b_i \cap \pi^{n-m} a) = 0.$$

Thus $\langle \pi^i b\rangle_{i\in\mathbb{Z}}$ is disjoint, and b is a transversal for $\pi\restriction\mathfrak{A}_{a^*}$.

Now

$$a_* = \sup_{n\in\mathbb{Z}}(\pi^n a \setminus \sup_{i<n} \pi^i a) = \sup_{n\in\mathbb{Z}}(\pi^{-n} a \setminus \sup_{i>n} \pi^{-i} a).$$

So $\pi^{-1} a_* = a_*$ and $\pi^{-1}\restriction\mathfrak{A}_{a_*}$ has a transversal. It follows at once that $\pi a_* = a_*$ and (using (b)) that $\pi\restriction\mathfrak{A}_{a_*}$ has a transversal.

382H Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. If π has a transversal, it is expressible as the product of at most two exchanging involutions both belonging to the countably full subgroup of $\mathfrak{A}$ generated by π.

proof Let a be a transversal for π. For $n \ge 1$, set $a_n = a \cap \pi^n a \setminus \sup_{1\le i<n} \pi^i a$; set $a_0 = a \setminus \sup_{i\ge 1} \pi^i a$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint and $\sup_{n\in\mathbb{N}} a_n = a$. We have $\pi^n b = b$ whenever $b \subseteq a_n$, while $\langle \pi^i a_0\rangle_{i\ge 1}$ is disjoint, so $\langle \pi^i a_0\rangle_{i\in\mathbb{Z}}$ is disjoint. For any $n \ge 1$, a_n is disjoint from $\pi^i a_n$ for $0 < i < n$, so $\langle \pi^i a_n\rangle_{i<n}$ is disjoint. If $0 \le i < m$ and $0 \le j < n$ and $i \le j$ and $\pi^i a_m \cap \pi^j a_n$ is non-zero, then $1 \le n - j + i \le n$ and

$$\begin{aligned}a_n \cap \pi^{n-j+i} a &= \pi^{n-j+i} a \cap \pi^n a_n = \pi^{n-j}(\pi^i a \cap \pi^j a_n) \\ &\supseteq \pi^{n-j}(\pi^i a_m \cap \pi^j a_n) \ne 0,\end{aligned}$$

so $i = j$; in this case $a_m \cap a_n \ne 0$ so $m = n$. If $0 \le i < n$ and $j \in \mathbb{Z}$ and $b = \pi^i a_n \cap \pi^j a_0$, then $\pi^n b = b$ and $\pi^n b$ is disjoint from b, so $b = 0$. This shows that all the $\pi^i a_n$ for $0 \le i < n$, and the $\pi^j a_0$ for $j \in \mathbb{Z}$, are disjoint. Also, because $\pi^n a_n = a_n$ for $n \ge 1$,

$$\sup_{0\le i<n}\pi^i a_n \cup \sup_{j\in\mathbb{Z}}\pi^j a_0 = \sup_{n\in\mathbb{N},j\in\mathbb{Z}}\pi^j a_n \cup \sup_{j\in\mathbb{Z}}\pi^j a_0 = \sup_{j\in\mathbb{Z}}\pi^j a = 1.$$

For any $n\ge 1$,

$$\langle\pi^{-2j}\pi^j a_n\rangle_{0\le j<n} = \langle\pi^{-j}a_n\rangle_{0\le j<n} = \langle\pi^{n-j}a_n\rangle_{0\le j<n},$$

$$\langle\pi^{1-2j}\pi^j a_n\rangle_{0\le j<n} = \langle\pi^{1-j}a_n\rangle_{0\le j<n} = \langle\pi^{n+1-j}a_n\rangle_{0\le j<n}$$

are disjoint and cover $\sup_{0\le j<n}\pi^j a_n$; while of course

$$\langle\pi^{-2j}\pi^j a_0\rangle_{j\in\mathbb{Z}} = \langle\pi^{-j}a_0\rangle_{j\in\mathbb{Z}},$$

$$\langle\pi^{1-2j}\pi^j a_0\rangle_{j\in\mathbb{Z}} = \langle\pi^{1-j}a_0\rangle_{j\in\mathbb{Z}}$$

are disjoint and cover $\sup_{j\in\mathbb{Z}}\pi^j a_0$. So we can define ϕ_1, $\phi_2\in\operatorname{Aut}\mathfrak{A}$ by setting

$$\begin{aligned}\phi_1 d = \pi^{-2j}d &\text{ if } j\in\mathbb{Z} \text{ and } d\subseteq\pi^j a_0\\ &\text{or if } 0\le j<n \text{ and } d\subseteq\pi^j a_n\\ \phi_2 d = \pi^{1-2j}d &\text{ if } j\in\mathbb{Z} \text{ and } d\subseteq\pi^j a_0\\ &\text{or if } 0\le j<n \text{ and } d\subseteq\pi^j a_n.\end{aligned}$$

Note that if $n\ge 1$ and $k\in\mathbb{Z}$ is arbitrary, then we have $\pi^k a_n = \pi^j a_n$ where $0\le j<n$ and $j\equiv k \bmod n$, so if $d\subseteq\pi^k a_n$ then

$$\phi_1 d = \pi^{-2j}d = \pi^{-2k}d,\quad \phi_2 d = \pi^{1-2j}d = \pi^{1-2k}d$$

because $\pi^n d = d$. So if $d\subseteq\pi^j a_n$ for any $n\in\mathbb{N}$ and $j\in\mathbb{Z}$, we have $\phi_1 d = \pi^{-2j}d\subseteq\pi^{-j}a_n$ and

$$\phi_2\phi_1 d = \pi^{1-2(-j)}\pi^{-2j}d = \pi d.$$

Because $\sup_{n\in\mathbb{N},j\in\mathbb{Z}}\pi^j a_n = 1$, $\phi_2\phi_1 = \pi$. Of course both ϕ_1 and ϕ_2 belong to the countably full subgroup generated by π. Next, ϕ_1 exchanges

$$\sup_{j\ge 1}\pi^j a_0 \cup \sup_{\substack{n\ge 2\\ 0<j\le\lfloor(n-1)/2\rfloor}}\pi^j a_n,$$

$$\sup_{j\le -1}\pi^j a_0 \cup \sup_{\substack{n\ge 2\\ -\lfloor(n-1)/2\rfloor\le j<0}}\pi^j a_n,$$

so is either the identity or an exchanging involution. In the same way, ϕ_2 exchanges

$$\sup_{j\ge 1}\pi^j a_0 \cup \sup_{\substack{n\ge 2\\ 1\le j\le\lfloor n/2\rfloor}}\pi^j a_n,$$

$$\sup_{j\le 0}\pi^j a_0 \cup \sup_{\substack{n\ge 2\\ -\lfloor n/2\rfloor<j\le 0}}\pi^j a_n,$$

so it too is either the identity or an exchanging involution. Thus we have a factorization of the desired type.

382I Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator.

(a) Every member of G has a support.

(b) Suppose $\pi\in G$ and $n\ge 1$ are such that π^n is the identity. Then π has a transversal.

(c) Let $\pi\in G$, and set $e^* = \inf_{n\ge 1}\operatorname{supp}(\pi^n)$. Then $\pi\restriction\mathfrak{A}_{1\setminus e^*}$ has a transversal.

(d) If $e\in\mathfrak{A}$ is such that $\pi e = e$ for every $\pi\in G$, then $\{\pi\restriction\mathfrak{A}_e : \pi\in G\}$ is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}_e$, and $\pi\restriction\mathfrak{A}_e$ has a separator for every $\pi\in G$.

proof (a) 382Ea.

(b) Induce on n. If $n=1$ then 1 is a transversal for π. For the inductive step to $n>1$, let $a\in\mathfrak{A}$ be such that $a\cap\pi a = 0$ and $\pi b = b$ whenever $b\cap\pi^i a = 0$ for every $i\in\mathbb{Z}$. Let $\mathfrak{B}$ be the (finite) subalgebra of $\mathfrak{A}$ generated by $\{\pi^i a : 0\le i<n\}$. Then $\pi^n a = a\in\mathfrak{B}$, so $\{b : \pi b\in\mathfrak{B}\}$ is a subalgebra of $\mathfrak{A}$ containing $\pi^i a$

whenever $i < n$, and includes $\mathfrak{B}$; thus $\pi b \in \mathfrak{B}$ for every $b \in \mathfrak{B}$. As π is injective, $\pi\restriction\mathfrak{B} \in \operatorname{Aut}\mathfrak{B}$. Let E be the set of atoms of $\mathfrak{B}$; then $\pi\restriction E$ is a permutation of E.

Let $C \subseteq E$ be an orbit of π. Then $\pi(\sup C) = \sup C$, and $\pi\restriction\mathfrak{A}_{\sup C}$ has a transversal. **P** Take $e \in C$, $k = \#(C)$. Then $\pi^i e \in C \setminus \{e\}$, so $e \cap \pi^i e = 0$, whenever $1 \le i < k$. As π^n is the identity, k is a factor of n. If $k = 1$, then e itself is a transversal for $\pi\restriction\mathfrak{A}_{\sup C} = \pi\restriction\mathfrak{A}_e$. If $k > 1$, define $\phi \in \operatorname{Aut}\mathfrak{A}$ by setting $\phi d = \pi^k(e \cap d) \cup (d \setminus e)$ for every $d \in \mathfrak{A}$. Then $\phi \in G$, because G is countably full, and $\phi^{n/k}$ is the identity. By the inductive hypothesis, ϕ has a transversal $c \in \mathfrak{A}$. There is some $m \in \mathbb{Z}$ such that $e' = e \cap \phi^m c \ne 0$. Now

$$\sup_{i\in\mathbb{Z}} \pi^{ki} e' = \sup_{i\in\mathbb{Z}} \phi^i e' = \sup_{i\in\mathbb{Z}} (e \cap \phi^{m+i} c) = e \cap \sup_{i\in\mathbb{Z}} \phi^i c = e,$$

so

$$\sup_{j\in\mathbb{Z}} \pi^j e' = \sup_{0\le j<k} \pi^j (\sup_{i\in\mathbb{Z}} \pi^{ki} e') = \sup_{0\le j<k} \pi^j e = \sup C.$$

Also, if $0 \le j < k$ and $i \in \mathbb{Z}$ and

$$0 \ne d \subseteq e' \cap \pi^{ki+j} e' \subseteq e \cap \pi^{ki+j} e = e \cap \pi^j e,$$

we must have $j = 0$ and $d \subseteq e' \cap \phi^i e'$, in which case $\pi^{ki+j} d = \phi^i d = d$. So e' is a transversal for $\pi\restriction\mathfrak{A}_{\sup C}$. **Q**

Let $\mathcal{C}$ be the set of orbits of $\pi\restriction E$, and for $C \in \mathcal{C}$ let c_C be a transversal for $\pi\restriction\mathfrak{A}_{\sup C}$. Then $\sup_{C\in\mathcal{C}} c_C$ is a transversal for π (382Ga). Thus the induction proceeds.

(c) Set $e_0 = 1 \setminus \operatorname{supp}\pi$, $e_n = \inf_{1\le i\le n} \operatorname{supp}(\pi^i) \setminus \operatorname{supp}(\pi^{n+1})$ for $n \ge 1$. Then $\langle e_n\rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}_{1\setminus e^*}$, and $\pi^{n+1} a = a$ whenever $a \subseteq e_n$. Also $\pi e_n = e_n$ for each n, by 381Eg. By (b), $\pi\restriction\mathfrak{A}_{e_n}$ has a transversal for every n; so $\pi\restriction\mathfrak{A}_{1\setminus e^*}$ has a transversal (382Ga again).

(d)(i) Write G_e for $\{\pi\restriction\mathfrak{A}_e : \pi \in G\}$. If $\langle a_i\rangle_{i\in I}$ is a countable partition of unity in $\mathfrak{A}_e$, $\langle \pi_i\rangle_{i\in I}$ a family in G, and $\phi \in \operatorname{Aut}\mathfrak{A}_e$ is such that $\phi d = \pi_i d$ whenever $i \in I$ and $d \subseteq a_i$, set $J = I \cup \{\infty\}$ for some object $\infty \notin I$, $a_\infty = 1 \setminus e$ and π_∞ the identity in $\operatorname{Aut}\mathfrak{A}$; then we have a $\tilde\phi \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\tilde\phi d = \phi(d \cap e) \cup (d \setminus e)$ for every $d \in \mathfrak{A}$, and $\langle a_i\rangle_{i\in J}$, $\langle \pi_i\rangle_{i\in J}$ witness that $\tilde\phi \in G$, so $\phi = \tilde\phi\restriction\mathfrak{A}_e$ belongs to G_e. As $\langle a_i\rangle_{i\in I}$ and $\langle \pi_i\rangle_{i\in I}$ are arbitrary, G_e is countably full.

(ii) If $\pi \in G$, let a be a separator for π, and consider $a' = a \cap e$. Then $a' \cap \pi a' = 0$ and $\sup_{k\in\mathbb{Z}} \pi^k a' = \sup_{k\in\mathbb{Z}} \pi^k a \cap e = e$, so a' is a separator for $\pi\restriction\mathfrak{A}_e$.

382J Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator, and $\pi \in G$ an aperiodic automorphism. Then there is a non-increasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $e_0 = 1$ and

(i) π is doubly recurrent on e_n, and in fact $\sup_{i\ge1} \pi^i e_n = \sup_{i\ge1} \pi^{-i} e_n = 1$,

(ii) e_{n+1}, $\pi_{e_n} e_{n+1}$ and $\pi_{e_n}^2 e_{n+1}$ are disjoint

for every $n \in \mathbb{N}$, where $\pi_{e_n} \in \operatorname{Aut}\mathfrak{A}_{e_n}$ is the automorphism induced by π (381M).

proof Construct $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with $a_0 = 1$. Given that $\sup_{i\ge1} \pi^i a_n = \sup_{i\ge1} \pi^{-i} a_n = 1$, then of course π is doubly recurrent on a_n (381L). Now there is an $a_{n+1} \subseteq a_n$ such that $a_{n+1} \cap \pi_{a_n} a_{n+1} = 0$ and $a_{n+1} \cup \pi_{a_n} a_{n+1} \cup \pi_{a_n}^2 a_{n+1} = a_n$. **P** We have a $\tilde\pi_{a_n} \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\tilde\pi_{a_n} d = \pi_{a_n} d$ for $d \subseteq a_n$, $\tilde\pi_{a_n} d = d$ for $d \subseteq 1 \setminus a_n$. Because π is aperiodic, so is π_{a_n} (381Ng); in particular, the support of π_{a_n} is a_n and this must also be the support of $\tilde\pi_{a_n}$. Because G is countably full, $\tilde\pi_{a_n} \in G$ (381Ni), so $\tilde\pi_{a_n}$ has a separator. By 382D, there is an $a_{n+1} \in \mathfrak{A}$ such that $a_{n+1} \cap \tilde\pi_{a_n} a_{n+1} = 0$ and $a_{n+1} \cup \tilde\pi_{a_n} a_{n+1} \cup \tilde\pi_{a_n}^2 a_{n+1}$ supports $\tilde\pi_{a_n}$, that is,

$$a_n = a_{n+1} \cup \tilde\pi_{a_n} a_{n+1} \cup \tilde\pi_{a_n}^2 a_{n+1} = a_{n+1} \cup \pi_{a_n} a_{n+1} \cup \pi_{a_n}^2 a_{n+1}. \quad \mathbf{Q}$$

Now

$$\sup_{i\ge1} \pi^i a_{n+1} = \sup_{i\ge1} \pi^i (\sup_{j\ge0} \pi^j a_{n+1}) \supseteq \sup_{i\ge1} \pi^i (\sup_{j\ge0} \pi_{a_n}^j a_{n+1})$$

(381Nb)

$$= \sup_{i\ge1} \pi^i a_n = 1.$$

Similarly, because we can identify $\pi_{a_n}^{-1}$ with $(\pi^{-1})_{a_n}$ (381Na), and

$$a_{n+1}\cup\pi_{a_n}^{-1}a_{n+1}\cup\pi_{a_n}^{-2}a_{n+1}=\pi_{a_n}^{-2}(a_{n+1}\cup\tilde\pi_{a_n}a_{n+1}\cup\tilde\pi_{a_n}^2a_{n+1})=a_n,$$

we have

$$\begin{aligned}\sup_{i\ge1}\pi^{-i}ia_{n+1}&=\sup_{i\ge1}\pi^{-i}(\sup_{j\ge0}\pi^{-j}a_{n+1})\\&\supseteq\sup_{i\ge1}\pi^{-i}(\sup_{j\ge0}\pi_{a_n}^{-j}a_{n+1})=\sup_{i\ge1}\pi^{-i}a_n=1,\end{aligned}$$

and the induction continues.

At the end of the induction, set $e_n=a_{2n}$ for every n. Then, for each n, we have

$$0=a_{2n+1}\cap\pi_{e_n}a_{2n+1}=e_{n+1}\cap\pi_{a_{2n+1}}e_{n+1}.$$

Since we can identify $\pi_{a_{2n+1}}$ with $(\pi_{e_n})_{a_{2n+1}}$ (381Ne), we can apply 381Nh to π_{e_n} to see that e_{n+1}, $\pi_{e_n}e_{n+1}$ and $\pi_{e_n}^2e_{n+1}$ are all disjoint.

382K Lemma Let $\mathfrak A$ be a Dedekind σ-complete Boolean algebra. Suppose that we have an aperiodic $\pi\in\operatorname{Aut}\mathfrak A$ and a non-increasing sequence $\langle e_n\rangle_{n\in\mathbb N}$ in $\mathfrak A$ such that $e_0=1$ and

$$\sup_{i\ge1}\pi^ie_n=\sup_{i\ge1}\pi^{-i}e_n=1,\quad e_{n+1},\ \pi_{e_n}(e_{n+1})\text{ and }\pi_{e_n}^2(e_{n+1})\text{ are disjoint}$$

for every $n\in\mathbb N$, writing $\pi_{e_n}\in\operatorname{Aut}\mathfrak A_{e_n}$ for the induced automorphism. Let G be the countably full subgroup of $\operatorname{Aut}\mathfrak A$ generated by π. Then there is a $\phi\in G$ such that ϕ is either the identity or an exchanging involution and $\inf_{n\ge1}\operatorname{supp}(\pi\phi)^n=0$.

proof (a) We need to check that every member of G has a support. **P** If $\phi\in G$, there is a partition $\langle a_n\rangle_{n\in\mathbb Z}$ of unity such that $\phi a=\pi^na$ whenever $n\in\mathbb Z$ and $a\subseteq a_n$ (381Ib). If $a\subseteq a_0$, then $\phi a=a$, so $1\setminus a_0$ supports ϕ. On the other hand, if $a\setminus a_0\ne0$, there is an $n\ne0$ such that $a\cap a_n\ne0$. As $\operatorname{supp}\pi^n=1$, there is a non-zero $d\subseteq a\cap a_n$ such that $0=d\cap\pi^nd=d\cap\phi d$. Thus $1\setminus a_0=\sup\{d:d\cap\phi d=0\}$ is the support of ϕ (381Ei). **Q**

(b) For each $n\in\mathbb N$, write π_n for π_{e_n} and $\tilde\pi_n\in G$ for the corresponding automorphism of $\mathfrak A$, as in 381Ni. Set

$$u_n'=\pi_n^{-1}e_{n+1},\quad u_n''=\pi_ne_{n+1}.$$

Then all the u_n', u_n'' are disjoint. **P**

$$u_n'\cap u_n''=\pi_n^{-1}(e_{n+1}\cap\pi_n^2e_{n+1})=0$$

for each n. And if $m<n$, then $u_n'\cup u_n''\subseteq e_n\subseteq e_{m+1}$ is disjoint from

$$u_m'\cup u_m''\subseteq\pi_m^{-1}(e_{m+1})\cup\pi_m(e_{m+1}).\ \mathbf{Q}$$

(c) By 381C, there is an automorphism $\phi_1\in\operatorname{Aut}\mathfrak A$ defined by setting

$$\begin{aligned}\phi_1d&=\pi_n\pi_{n+1}^{-1}\pi_nd=\tilde\pi_n\tilde\pi_{n+1}^{-1}\tilde\pi_nd\text{ if }n\in\mathbb N,\ d\subseteq u_n',\\&=\pi_n^{-1}\pi_{n+1}\pi_n^{-1}d=\tilde\pi_n^{-1}\tilde\pi_{n+1}\tilde\pi_n^{-1}d\text{ if }n\in\mathbb N,\ d\subseteq u_n'',\\&=d\text{ if }d\cap\sup_{n\in\mathbb N}(u_n'\cup u_n'')=0;\end{aligned}$$

$\phi_1\in G$ and ϕ_1^2 is the identity and ϕ_1 exchanges $\sup_{n\in\mathbb N}u_n'$ with $\sup_{n\in\mathbb N}u_n''$, so is either the identity or an exchanging involution. Set $c_0=\inf_{k\ge1}\operatorname{supp}(\pi\phi_1)^k$ and $c_1=\sup_{i\in\mathbb Z}\pi^ic_0$, so that $\pi c_1=c_1$ and $\phi_1c_1=c_1$ (381J).

(d) For $l\ge1$, set

$$v_l'=\pi^{-l}c_0\setminus\sup_{-l<i\le l}\pi^ic_0,\quad v_l''=\pi^lc_0\setminus\sup_{-l\le i<l}\pi^ic_0.$$

Then v_k', v_k'', v_l' and v_l'' are disjoint whenever $1\le k<l$. For j, $l\ge1$, set

$$d'_{lj} = v'_l \cap \pi^{-j} v''_l \setminus \sup_{1\le i<j} \pi^{-i} v''_l,$$

$$d''_{lj} = v''_l \cap \pi^{j} v'_l \setminus \sup_{1\le i<j} \pi^{i} v'_l,$$

$$d_{lj} = d'_{lj} \cap \pi^{-j} d''_{lj};$$

now define $\phi_2 \in \operatorname{Aut}\mathfrak{A}$ by setting

$$\begin{aligned} \phi_2 d &= \pi^j d \text{ if } d \subseteq d_{lj} \text{ for some } j,\, l \ge 1,\\ &= \pi^{-j} d \text{ if } d \subseteq \pi^j d_{lj} \text{ for some } j,\, l \ge 1,\\ &= d \text{ if } d \cap \sup_{j,l\ge 1}(d_{lj} \cup \pi^j d_{lj}) = 0, \end{aligned}$$

so that $\phi_2 \in G$, ϕ_2^2 is the identity and

$$\operatorname{supp}\phi_2 = \sup_{l,j\ge 1} d_{lj} \cup \pi^j d_{lj} \subseteq \sup_{l\ge 1} v'_l \cup v''_l \subseteq c_1.$$

As ϕ_2 exchanges $\sup_{j,l\ge1} d_{lj} \subseteq \sup_{j,l\ge1} d'_{lj}$ with $\sup_{j,l\ge1} \pi^j d_{lj} \subseteq \sup_{j,l\ge1} d''_{lj}$, it too is either trivial or an exchanging involution.

(e) Define $\phi \in \operatorname{Aut}\mathfrak{A}$ by setting

$$\begin{aligned} \phi d &= \phi_1 d \text{ if } d \subseteq 1 \setminus c_1,\\ &= \phi_2 d \text{ if } d \subseteq c_1. \end{aligned}$$

It is easy to check that ϕ is either the identity or an exchanging involution. Set $c_2 = \inf_{n\ge1} \operatorname{supp}(\pi\phi)^n$.

(f) I wish to show that $c_2 = 0$. The rest of the argument does not strictly speaking require the Stone representation (382Yb), but I think that most readers will find it easier to follow when expressed in terms of the Stone space Z of $\mathfrak{A}$. Let f, g_1, g_2 and g be the autohomeomorphisms of Z corresponding to π, ϕ_1, ϕ_2 and ϕ; write $\widehat{a} \subseteq Z$ for the open-and-closed set corresponding to $a \in \mathfrak{A}$. For each $n \in \mathbb{N}$, let $f_n : \widehat{e_n} \to \widehat{e_n}$ be the autohomeomorphism corresponding to π_{e_n}. Since

$\operatorname{supp}\pi^k = 1$ for every $k \ge 1$,
$\sup_{i\ge k} \pi^i e_n = \sup_{i\ge k} \pi^{-i} e_n = 1$ for every $n \in \mathbb{N}$, $k \in \mathbb{Z}$ (381L),
$c_0 = \inf_{k\ge1} \operatorname{supp}(\pi\phi_1)^k$,
$c_1 = \sup_{i\in\mathbb{Z}} \pi^i c_0$,
$\operatorname{supp}\phi_2 \setminus \sup_{l\ge1}(v'_l \cup v''_l) = 0$,
$c_2 = \inf_{k\ge1} \operatorname{supp}(\pi\phi)^k$,

the sets

$\{z : f^k(z) = z\}$, for $k \ge 1$,
$Z \setminus \bigcup_{i\ge k} f^{-i}[\widehat{e_n}]$, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,
$Z \setminus \bigcup_{i\le k} f^{-i}[\widehat{e_n}]$, for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,
$\widehat{c_0} \triangle \bigcap_{k\ge1}\{z : (g_1 f)^k(z) \ne z\}$,
$\widehat{c_1} \triangle \bigcup_{i\in\mathbb{Z}} f^{-i}[\widehat{c_0}]$,
$\widehat{\operatorname{supp}\phi_2} \setminus \bigcup_{l\ge1}(\widehat{v'_l} \cup \widehat{v''_l})$,
$\widehat{c_2} \triangle \{z : (gf)^k(x) \ne x \text{ for every } k \ge 1\}$,

as well as the sets

$\{z : g_1(z) \notin \{f^i(z) : i \in \mathbb{Z}\}\}$,
$\{z : g_2(z) \notin \{f^i(z) : i \in \mathbb{Z}\}\}$

are all meager (using 381Qb), and their union Y is meager. Set $Y' = \bigcup_{i\in\mathbb{Z}} f^{-i}[Y]$; then Y' also is meager, and $X = Z \setminus Y'$ is comeager, therefore dense, by Baire's theorem (3A3G). Of course $f^i(x) \in X$ whenever $x \in X$ and $i \in \mathbb{Z}$.

(g) Fix $x \in X \cap \widehat{c_1}$ for the time being. Because $f^k(x) \ne x$ for any $k \ge 1$, the map $i \mapsto f^i(x) : \mathbb{Z} \to X$ is injective. Because $g_k(f^i(z)) \in \{f^{i+j}(z) : j \in \mathbb{Z}\}$ for every $i \in \mathbb{Z}$ and both $k \in \{1, 2\}$, we can define g_1^x, $g_2^x : \mathbb{Z} \to \mathbb{Z}$ by saying that $g_k^x(i) = j$ if $g_k(f^i(x)) = f^j(x)$. Similarly, f is represented on $\{f^i(x) : i \in \mathbb{Z}\}$ by s, where $s(i) = i + 1$ for every $i \in \mathbb{Z}$.

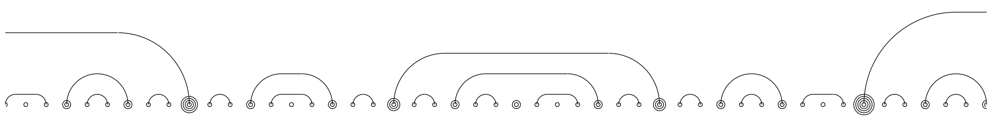
(i) For $n \in \mathbb{N}$, set

$$E_n = \{i : i \in \mathbb{Z},\ f^i(x) \in \widehat{e_n}\},$$

$$U'_n = \{i : f^i(x) \in \widehat{u'_n}\}, \quad U''_n = \{i : f^i(x) \in \widehat{u''_n}\}.$$

Because $x \in \bigcup_{i \geq k} f^{-i}[\widehat{e_n}] \cap \bigcup_{i \leq k} f^{-i}[\widehat{e_n}]$ for every k, E_n is unbounded above and below. If $i \in E_n$, then $f_n(f^i(x)) = f^{k+i}(x)$ where $k \geq 1$ is the first such that $f^{k+i}(x) \in \widehat{e_n}$ (381Qe), that is, such that $k + i \in E_n$. Turning this round, $f_n^{-1}(f^i(x)) = f^j(x)$ where j is the greatest member of E_n less than i. In particular, $i \in U'_n$ iff i is the next point of E_n above a point of E_{n+1}, and $i \in U''_n$ iff i is the next point of E_n below a point of E_{n+1}. If $i \in U'_n$, then $f_n^{-1} f^i(x) = f^j(x)$ where $j \in E_{n+1}$ is the next point of E_n below i, and $f_{n+1} f_n^{-1} f^i(x) = f^k(x)$ where k is the next point of E_{n+1} above j. Since g_1 must agree with $f_n^{-1} f_{n+1} f_n^{-1}$ on $\widehat{u'_n}$ (381Qa), $g_1 f^i(x) = f_n^{-1} f_{n+1} f_n^{-1} f_i(x) = f^l(x)$ where l is the next point of E_n below $f^k(x)$. This means that g_1^x exchanges pairs $i < l$ exactly when i, $l \in E_n$ are the first and last points in $E_n \cap \,]j, k[$ where j, k are successive points of E_{n+1}. In this case, there is no point of E_{n+1} in the interval $[i, l]$. Accordingly, if $i' < l'$ and g_1^x exchanges i' and l' and either i' or l' is in $]i, l[$, we must have i', $l' \in E_m$ for some $m < n$; and as the interval $[i', l']$ cannot meet $E_{m+1} \supseteq E_n$, it is included in $]i, l[$. Thus g_1^x fixes $]i, l[$ in the sense that if $i < i' < l$ then $g_1^x(i') = l'$ for some $l' \in \,]i, l[$. It follows that $g_1^x s$ fixes $[i, l[$. In this case, of course, every point of $[i, l[$ must be fixed by some power of $g_1^x s$.

The following diagram attempts to show how g_1^x links pairs of integers. The points of E_n, as n increases, are shown as progressively multiplied circles.



Pairs of points exchanged by g_1^x

Note that because e_{n+1}, $\phi_{e_n} e_{n+1}$ and $\pi_{e_n}^2 e_{n+1}$ are always disjoint, there are always at least two points of E_n between any two successive points of E_{n+1}.

(ii) Set $C_0 = \{i : f^i(x) \in \widehat{c_0}\}$. Then

$$C_0 = \mathbb{Z} \setminus \bigcup\{[i, l[\, : i < l = g_1^x(i)\}.$$

P Because X does not meet $\widehat{c_0} \triangle \bigcap_{k \geq 1}\{z : (g_1 f)^k(z) \neq z\}$,

$$C_0 = \{i : (g_1 f)^k f^i(x) \neq f^i(x) \text{ for every } k \geq 1\} = \{i : (g_1^x s)^k(i) \neq i \text{ for every } k \geq 1\}.$$

If $i < l = g_1^x(i)$ then (i) tells us that every point of $[i, l[$ is fixed by some power of $g_1^x s$ and cannot belong to C_0. Conversely, if $j \in \mathbb{Z}$ does not belong to any such interval $[i, l[$, then $g_1^x(i) > j$ for every $i > j$, so $g_1^x s(i) > j$ for every $i \geq j$ and $j \notin C_0$. **Q**

Because X does not meet $\widehat{c_1} \setminus \bigcup_{i \in \mathbb{Z}} f^{-i}[\widehat{c_0}]$, C_0 is not empty. Now C_0 has no greatest member. **P** Let $j_0 \in C_0$. Then $j_0 \notin [i, l[$ for any pair i, l exchanged by g_1^x. If $j_0 + 1 \in C_0$ we can stop. Otherwise, there are i_0, l_0 exchanged by g_1^x such that $i_0 \leq j_0 + 1 < l_0$. **?** If $l_0 \notin C_0$ there are i_1, l_1 exchanged by g_1^x such that $i_1 \leq l_0 < l_1$. But in this case $i_1 \leq j_0 < l_1$. **X** Thus $j_0 < l_0 \in C_0$ and j_0 cannot be the greatest member of C_0. **Q**

Similarly, C_0 has no least member. **P** If $j_0 \in C_0$ but $j_0 - 1 \notin C_0$, take i_0, l_0 exchanged by g_1^x such that $i_0 \leq j_0 - 1 < l_0$. **?** If $i_0 - 1 \notin C_0$, take i_1, l_1 exchanged by g_1^x such that $i_1 \leq i_0 - 1 < l_1$; then $i_1 \leq j_0 = l_0 < l_1$. **X** So $i_0 - 1$ is a member of C_0 less than j_0. **Q**

Thus C_0 is unbounded above and below.

(iii) For $l \geq 1$,

$$\widehat{v'_l} = f^l[\widehat{c_0}] \setminus \bigcup_{-l \leq j < l} f^j[\widehat{c_0}], \quad \widehat{v''_l} = f^{-l}[\widehat{c_0}] \setminus \bigcup_{-l < j \leq l} f^j[\widehat{c_0}];$$

so setting

$$V'_l = \{i : f^i(x) \in \widehat{v'_l}\}, \quad V''_l = \{i : f^i(x) \in \widehat{v''_l}\},$$

we see that

$$V'_l = \{i : i - l \in C_0,\, i + j \notin C_0 \text{ if } -l < j \le l\} = \{i + l : i \in C_0,\, C_0 \cap]i, i + 2l] = \emptyset\},$$

$$V''_l = \{i : i + l \in C_0,\, i + j \notin C_0 \text{ if } -l \le j < l\} = \{i - l : i \in C_0,\, C_0 \cap [i - 2l, i[= \emptyset\};$$

that is to say, if j, k are successive members of C_0, and $j + l < k - l$, then $j + l \in V'_l$ and $k - l \in V''_l$. Looking at this from the other direction, if j and k are successive members of C_0, and $l_0 = \lfloor\frac{k-j-1}{2}\rfloor$, then if $1 \le l \le l_0$ we have exactly one $i' \in V'_l \cap [j, k]$ and exactly one $i'' \in V''_l \cap [j, k]$ and $i' < i''$, while if $l > l_0$ then neither V'_l nor V''_l meets $[j, k]$.

(iv) Now the point is that every V'_l is unbounded above. **P** Because there are at least two points of E_n between any two points of E_{n+1}, successive points of E_n always differ by at least 3^n, for every n. Take n such that $3^n \ge 2l + 1$. For any $i_0 \in \mathbb{Z}$, there are an $i_1 \in C_0$ such that $i_1 \ge i_0$, and a $j \in E_{n+1}$ such that $j \ge i_1$; let k be the next point of E_{n+1} above j. Then we have points j', k' of $E_n \cap]j, k[$ such that C_0 is disjoint from $[j', k'[$. So if we take $i = \max(C_0 \cap]-\infty, j'[)$ and $i' = \min(C_0 \cap [j', \infty[)$, $i' - i \ge k' - j' \ge 2l + 1$ and $i + l \in V'_l$, while $i + l \ge i \ge i_1 \ge i_0$. As i_0 is arbitrary, V'_l is unbounded above. **Q** Similarly, turning the argument upside down, V''_l is unbounded below.

(v) Next consider

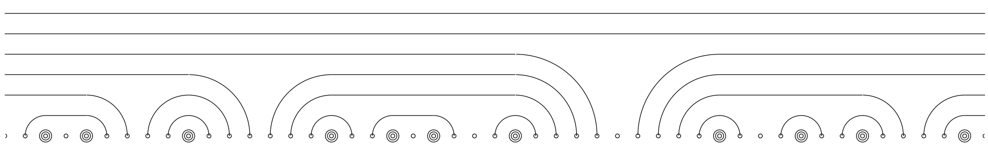
$$\begin{aligned} D'_{lj} = \{i : f^i(x) \in \widehat{d'_{lj}}\} &= V'_l \cap (V''_l + j) \setminus \bigcup_{1 \le i < j} V''_l + i \\ &= \{i : i \in V'_l,\, i - j = \max(V''_l \cap]-\infty, i[\}, \\ D''_{lj} = \{i : f^i(x) \in \widehat{d'_{lj}}\} &= V''_l \cap (V'_l + j) \setminus \bigcup_{1 \le i < j} V'_l + i \\ &= \{i : i \in V''_l,\, i + j = \min(V'_l \cap]i, \infty[\}, \\ D_{lj} = \{i : f^i(x) \in \widehat{d_{lj}}\} &= D'_{lj} \cap (D''_{lj} + j). \end{aligned}$$

Since ϕ_2 agrees with π^j on $\mathfrak{A}_{d_{lj}}$, g_2 agrees with f^j on $\widehat{\pi^j d_{lj}}$, and $g_2^x(i) = i + j$ whenever $f^i(x) \in f^{-j}[\widehat{d_{lj}}]$, that is, whenever $i + j \in D_{lj}$. This means that g_2^x exchanges pairs $i'' < i'$ exactly when, for some l, i'' is the greatest member of V''_l less than i' and i' is the least member of V'_l greater than i''. Since X does not meet $\widehat{\operatorname{supp}\phi_2} \setminus \bigcup_{l \ge 1}(\widehat{v'_l} \cup \widehat{v''_l})$, g_2^x does not move any other i.

But, starting from any $l \ge 1$ and $i' \in V'_l$, let i'' be the greatest element of V''_l less than i'. Then $i' - l$ and $i'' + l$ belong to C_0, and if k, k' are any successive members of C_0 such that $i'' < k < k' < i'$ then there is no member of V''_l in $[k, k']$ and therefore no member of V'_l in $[k, k']$. So i' is the least member of V'_l greater than i'', and $g_2^x(i') = i''$. Similarly, every member of every V''_l is moved by g_2^x.

At the same time we see that if $i'' \in V''_l$ and $i' \in V'_l$ are exchanged by g_2^x, and $m > l$, then there can be no interval of C_0 of length $2m + 1$ or greater between i'' and i', so there is no point of $V''_m \cup V'_m$ in $[i'', i']$. For the same reason, if $m < l$ then no pair of points in $V''_m \cup V'_m$ exchanged by g_2^x can bracket either i'' or i'. So g_2^x leaves the interval $[i'', i']$ invariant. Accordingly $g_2 s$ leaves $[i'', i'[$ invariant.

The next diagram attempts to illustrate g_2^x. Members of C_0 are shown as multiple circles[1].



Pairs of points exchanged by g_2^x

At this point observe that 0 belongs to some $g_2^x s$-invariant interval. **P** Let k, k' be successive members of C_0 such that $k \le 0 < k'$. Take l such that $k' - k \le 2l$. Let i' be the least member of V'_l greater than 0, and i'' the greatest member of V''_l less than 0; since neither V'_l nor V''_l meets $[k, k']$, i'' and i' are exchanged by g_2^x, while $0 \in [i'', i'[$. **Q** This means that there is a $k \ge 1$ such that $(g_2^x s)^k(0) = 0$, that is, $(g_2 f)^k(x) = x$.

[1] I have made no attempt to arrange these in a configuration compatible with the process by which C_0 was constructed; the diagram aims only to show how the links would be formed from a particular set.

(vi) We know that g agrees with g_2 on $\widehat{\phi_2 c_1} = \widehat{c_1}$. Since $x \in \widehat{c_1}$ and $f^{-1}[\widehat{c_1}] = \widehat{c_1}$, $(gf)^k(x) = x$. Because X does not meet $\widehat{c_2} \triangle \{z : (gf)^k(x) \neq x$ for every $k \geq 1\}$, $x \notin \widehat{c_2}$.

This is true for every $x \in X \cap \widehat{c_1}$. Since X is dense in Z, $\widehat{c_1} \cap \widehat{c_2}$ is empty, that is, $c_1 \cap c_2 = 0$.

(h) Since $\pi\phi$ agrees with $\pi\phi_1$ on $\mathfrak{A}_{1\setminus c_1}$, and $c_1 = \pi\phi c_1$, $\operatorname{supp}(\pi\phi)^k \setminus c_1 = \operatorname{supp}(\pi\phi_1)^k \setminus c_1$ for every k, and

$$c_2 \setminus c_1 = \inf_{k\geq 1} \operatorname{supp}(\pi\phi_1)^k \setminus c_1 \subseteq \inf_{k\geq 1} \operatorname{supp}(\pi\phi_1)^k \setminus c_0 = 0.$$

Putting this together with (g), we see that $c_2 = 0$, as required.

382L Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator. If $\pi \in G$, there is a $\phi \in G$ such that ϕ is either the identity or an exchanging involution and $\pi\phi$ has a transversal.

proof (a) We may suppose that G is the countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. π^n has a support for every $n \geq 1$ (382Ia); set $e = \inf_{n\geq 1} \operatorname{supp}\pi^n$, so that $\pi e = e$ and $\pi{\restriction}\mathfrak{A}_{1\setminus e}$ has a transversal (382Ic), while $\pi{\restriction}\mathfrak{A}_e$ is aperiodic (381H). By 381J, $\psi e = e$ for every $\psi \in G$; by 382Id, $G_e = \{\psi{\restriction}\mathfrak{A}_e : \psi \in G\}$ is a countably full subgroup of $\operatorname{Aut}\mathfrak{A}_e$ and every member of G_e has a separator.

(b) Applying 382J to $\pi{\restriction}\mathfrak{A}_e$, we can find $\langle e_n\rangle_{n\geq 1}$ such that $e_0 = e$, $\langle e_n\rangle_{n\in\mathbb{N}}$ is non-increasing, $\sup_{i\geq 1} \pi^i e_n = \sup_{i\geq 1} \pi^{-i} e_n = e$ for every n, and e_{n+1}, $\pi_{e_n} e_{n+1}$ and $\pi_{e_n}^2 e_{n+1}$ are disjoint for every n. (By 381Ne or otherwise, we can compute π_{e_n} either in $\operatorname{Aut}\mathfrak{A}$ or in $\operatorname{Aut}\mathfrak{A}_e$. Note that $\pi_e = \pi{\restriction}\mathfrak{A}_e$, by 381Nf or otherwise.) Now 382K tells us that there is a $\phi \in G_e$ such that ϕ is either the identity or an exchanging involution, and $\inf_{n\geq 1} \operatorname{supp}(\pi_e\phi)^n = 0$.

(c) Take $\tilde\phi \in \operatorname{Aut}\mathfrak{A}$ to agree with ϕ on $\mathfrak{A}_e$ and with the identity on $\mathfrak{A}_{1\setminus e}$, so that $\tilde\phi$ is either the identity or an exchanging involution. Now $\pi\tilde\phi{\restriction}\mathfrak{A}_{1\setminus e} = \pi{\restriction}\mathfrak{A}_{1\setminus e}$ and $\pi\tilde\phi{\restriction}\mathfrak{A}_e = \pi\phi{\restriction}\mathfrak{A}_e$ both have transversals (using 382I again). So $\pi\tilde\phi$ has a transversal (382Ga).

382M Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a countably full subgroup of $\operatorname{Aut}\mathfrak{A}$ such that every member of G has a separator. If $\pi \in G$, it can be expressed as the product of at most three exchanging involutions belonging to G.

proof By 382L, there is a $\phi \in G$, either the identity or an exchanging involution, such that $\pi\phi$ has a transversal. By 382H, $\pi\phi$ is the product of at most two exchanging involutions in G, so $\pi = \pi\phi\phi^{-1}$ is the product of at most three exchanging involutions.

382N Corollary If $\mathfrak{A}$ is a Dedekind complete Boolean algebra and G is a full subgroup of $\operatorname{Aut}\mathfrak{A}$, every $\pi \in G$ is expressible as the product of at most three involutions all belonging to G and all supported by $\operatorname{supp}\pi$.

proof We may suppose that G is the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π. By 382Eb, every member of G has a separator. By 382M, π is the product of at most three involutions all belonging to G; by 381Jb, they are all supported by $\operatorname{supp}\pi$.

382O Definition Let $\mathfrak{A}$ be a Boolean algebra, and G a subgroup of the automorphism group $\operatorname{Aut}\mathfrak{A}$. I will say that G **has many involutions** if for every non-zero $a \in \mathfrak{A}$ there is an involution $\pi \in G$ which is supported by a.

382P Lemma Let $\mathfrak{A}$ be an atomless homogeneous Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ has many involutions, and in fact every non-zero element of $\mathfrak{A}$ is the support of an exchanging involution.

proof If $a \in \mathfrak{A} \setminus \{0\}$, then there is a b such that $0 \neq b \subset a$. Let $\psi : \mathfrak{A}_b \to \mathfrak{A}_{a\setminus b}$ be an isomorphism; define $\pi \in \operatorname{Aut}\mathfrak{A}$ to agree with ψ on $\mathfrak{A}_b$, with ψ^{-1} on $\mathfrak{A}_{a\setminus b}$, and with the identity on $\mathfrak{A}_{1\setminus a}$. Then π is an exchanging involution with support a.

382Q Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Then every non-zero element of $\mathfrak{A}$ is the support of an exchanging involution belonging to G.

proof By the definition 382O,

$$C=\{\operatorname{supp}\pi:\pi\in G\text{ is an involution}\}$$

is order-dense in $\mathfrak{A}$. So if $a\in\mathfrak{A}\setminus\{0\}$ there is a disjoint $B\subseteq C$ such that $\sup B=a$ (313K). For each $b\in B$ let $\pi_b\in G$ be an involution with support b. Define $\pi\in G$ by setting $\pi d=\pi_b d$ for $d\subseteq b\in B$, $\pi d=d$ if $d\cap a=0$; then $\pi\in G$ is an involution with support a. By 382Fa it is an exchanging involution.

382R Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Then a subset H of G is a normal subgroup of G iff it is of the form

$$\{\pi:\pi\in G,\ \operatorname{supp}\pi\in I\}$$

for some ideal $I\triangleleft\mathfrak{A}$ which is G-**invariant**, that is, such that $\pi a\in I$ for every $a\in I$ and $\pi\in G$.

proof (a) I deal with the easy implication first. Let $I\triangleleft\mathfrak{A}$ be a G-invariant ideal and set $H=\{\pi:\pi\in G,\ \operatorname{supp}\pi\in I\}$. Because the support of the identity automorphism ι is $0\in I$, $\iota\in H$. If ϕ, $\psi\in H$ and $\pi\in G$, then

$$\operatorname{supp}(\phi\psi)\subseteq\operatorname{supp}\phi\cup\operatorname{supp}\psi\in I,$$

$$\operatorname{supp}(\psi^{-1})=\operatorname{supp}\psi\in I,$$

$$\operatorname{supp}(\pi\psi\pi^{-1})=\pi(\operatorname{supp}\psi)\in I$$

(381E), and $\phi\psi$, ψ^{-1}, $\pi\psi\pi^{-1}$ all belong to H; so $H\triangleleft G$.

(b) For the rest of the proof, therefore, I suppose that H is a normal subgroup of G and seek to express it in the given form. We can in fact describe the ideal I immediately, as follows. Set

$$J=\{a:a\in\mathfrak{A},\ \pi\in H\text{ whenever }\pi\in G\text{ is an involution and }\operatorname{supp}\pi\subseteq a\};$$

then $0\in J$ and $a\in J$ whenever $a\subseteq b\in J$. Also $\pi a\in J$ whenever $a\in J$ and $\pi\in G$. **P** If $\phi\in G$ is an involution and $\operatorname{supp}\phi\subseteq\pi a$ then $\phi_1=\pi^{-1}\phi\pi$ is an involution in G and

$$\operatorname{supp}\phi_1=\pi^{-1}(\operatorname{supp}\phi)\subseteq a,$$

so $\phi_1\in H$ and $\phi=\pi\phi_1\pi^{-1}\in H$. As ϕ is arbitrary, $\pi a\in J$. **Q**

I do not know how to prove directly that J is an ideal, so let us set

$$I=\{a_0\cup a_1\cup\ldots\cup a_n:a_0,\ldots,a_n\in J\};$$

then $I\triangleleft\mathfrak{A}$, and $\pi a\in I$ for every $a\in I$ and $\pi\in G$.

(c) If $a\in\mathfrak{A}$, $\psi\in H$ and $a\cap\psi a=0$ then $a\in J$. **P** If $a=0$, this is trivial. Otherwise, let $\pi\in G$ be an involution with $\operatorname{supp}\pi\subseteq a$; say $\pi=(\overleftarrow{b\,{}_{\pi}\,c})$ where $b\cup c\subseteq a$. By 382Q there is an involution $\pi_1\in G$ such that $\operatorname{supp}\pi_1=b$; say $\pi_1=(\overleftarrow{b'\,{}_{\pi_1}\,b''})$ where $b'\cup b''=b$. Set

$$c'=\pi b',\quad c''=\pi b''=c\setminus c',$$

$$\pi_2=\pi_1\pi\pi_1\pi^{-1}=(\overleftarrow{b'\,{}_{\pi_1}\,b''})(\overleftarrow{c'\,{}_{\pi\pi_1\pi^{-1}}\,c''}),\quad\pi_3=(\overleftarrow{b'\,{}_{\pi}\,c'}),$$

$$\phi=\pi_2^{-1}\psi\pi_2\psi^{-1}\in H,$$

$$\bar\pi=\pi_3^{-1}\phi\pi_3\phi^{-1}=\pi_3^{-1}\pi_2^{-1}\psi\pi_2\psi^{-1}\pi_3\psi\pi_2^{-1}\psi^{-1}\pi_2\in H.$$

Now

$$\operatorname{supp}(\psi\pi_2\psi^{-1})=\psi(\operatorname{supp}\pi_2)=\psi(b\cup c)\subseteq\psi a$$

is disjoint from

$$\operatorname{supp}\pi_3=b'\cup c'\subseteq a,$$

so π_3 commutes with $\psi\pi_2\psi^{-1}$, and

$$
\begin{aligned}
\bar\pi &= \pi_3^{-1}\pi_2^{-1}\pi_3\psi\pi_2\psi^{-1}\psi\pi_2^{-1}\psi^{-1}\pi_2\\
&= \pi_3^{-1}\pi_2^{-1}\pi_3\pi_2\\
&= (\overleftarrow{b'\,{}_{\pi}\,c'})(\overleftarrow{b'\,{}_{\pi_1}\,b''})(\overleftarrow{c'\,{}_{\pi\pi_1\pi^{-1}}\,c''})(\overleftarrow{b'\,{}_{\pi}\,c'})(\overleftarrow{b'\,{}_{\pi_1}\,b''})(\overleftarrow{c'\,{}_{\pi\pi_1\pi^{-1}}\,c''})\\
&= (\overleftarrow{b'\,{}_{\pi}\,c'})(\overleftarrow{b''\,{}_{\pi}\,c''})\\
&= \pi.
\end{aligned}
$$

So $\pi \in H$. As π is arbitrary, $a \in J$. **Q**

(d) If $\pi = (\overleftarrow{a\,{}_{\pi}\,b})$ is an involution in G and $a \in J$, then $\pi \in H$. **P** By 382Q again, there is an involution $\psi \in G$ such that $\operatorname{supp}\psi = a$; because $a \in J$, $\psi \in H$. Express ψ as $(\overleftarrow{a'\,{}_{\psi}\,a''})$ where $a' \cup a'' = a$. Set $b' = \pi a'$ and $b'' = \pi a''$, so that $\pi = (\overleftarrow{a'\,{}_{\pi}\,b'})(\overleftarrow{a''\,{}_{\pi}\,b''})$, and

$$\psi_1 = \psi\pi\psi\pi^{-1} = (\overleftarrow{a'\,{}_{\psi}\,a''})(\overleftarrow{b'\,{}_{\pi\psi\pi^{-1}}\,b''}) \in H.$$

As $\psi_1(a' \cup b') = a'' \cup b''$ is disjoint from $a' \cup b'$, $a' \cup b' \in J$, by (c), and $\pi_1 = (\overleftarrow{a'\,{}_{\pi}\,b'}) \in H$; similarly, $a'' \cup b'' \in J$, so $\pi_2 = (\overleftarrow{a''\,{}_{\pi}\,b''}) \in H$ and $\pi = \pi_1\pi_2$ belongs to H. **Q**

(e) If $\pi \in G$ is an involution and $\operatorname{supp}\pi \in I$, then $\pi \in H$. **P** Express π as $(\overleftarrow{a\,{}_{\pi}\,b})$. Let $a_0, \dots, a_n \in J$ be such that $a \cup b \subseteq a_0 \cup \ldots \cup a_n$. Set

$$c_j = a \cap a_j \setminus \sup_{i<j} a_i, \quad b_j = \pi c_j, \quad \pi_j = (\overleftarrow{c_j\,{}_{\pi}\,b_j})$$

for $j \le n$; then every c_j belongs to J, so every π_j belongs to H (by (d)) and $\pi = \pi_0 \ldots \pi_n \in H$. **Q**

(f) If $\pi \in G$ and $\operatorname{supp}\pi \in I$ then $\pi \in H$. **P** By 382N, π is a product of involutions in G all with supports included in $\operatorname{supp}\pi$; by (e), they all belong to H, so π also does. **Q**

(g) We are nearly home. So far we know that I is a G-invariant ideal and that $\pi \in H$ whenever $\pi \in G$ and $\operatorname{supp}\pi \in I$. On the other hand, $\operatorname{supp}\pi \in I$ for every $\pi \in H$. **P** By 382Eb, π has a separator; take a', a'', b', b'', c from 382D(iv). Then

$$a' \cap \pi a' = b' \cap \pi b' = \ldots = c \cap \pi c = 0,$$

so $a', \dots, c$ all belong to J, by (c), and $\operatorname{supp}\pi = a' \cup \ldots \cup c$ belongs to I. **Q**

So H is precisely the set of members of G with supports in I, as required.

382S Corollary Let $\mathfrak{A}$ be a homogeneous Dedekind complete Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ is simple.

proof If $\mathfrak{A}$ is $\{0\}$ or $\{0,1\}$ this is trivial. Otherwise, let H be a normal subgroup of $\operatorname{Aut}\mathfrak{A}$. Then by 382R and 382P there is an invariant ideal I of $\mathfrak{A}$ such that $H = \{\pi : \operatorname{supp}\pi \in I\}$. But if H is non-trivial so is I; say $a \in I \setminus \{0\}$. If $a = 1$ then certainly $1 \in I$ and $H = \operatorname{Aut}\mathfrak{A}$. Otherwise, there is a $\pi \in \operatorname{Aut}\mathfrak{A}$ such that $\pi a = 1 \setminus a$ (as in 381D), so $1 \setminus a \in I$, and again $1 \in I$ and $H = \operatorname{Aut}\mathfrak{A}$.

Remark I ought to remark that in fact $\operatorname{Aut}\mathfrak{A}$ is simple for any homogeneous Dedekind σ-complete Boolean algebra; see Štěpánek & Rubin 89, Theorem 5.9b.

382X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. Suppose that $\pi \in \operatorname{Aut}\mathfrak{A}$ is represented by $f_\pi : Z \to Z$. For $z \in Z$, write $\operatorname{Orb}_\pi(z) = \{f_\pi^n(z) : n \in \mathbb{Z}\}$. (i) Show that $a \in \mathfrak{A}$ is a separator for π iff $f_\pi^{-1}[\widehat{a}] \cap \widehat{a}$ is empty and $\{z : \operatorname{Orb}_\pi(z) \cap \widehat{a}\} \ne \emptyset\}$ is dense in $\{z : f_\pi(z) \ne z\}$. (ii) Show that $a \in \mathfrak{A}$ is a transversal for π iff $\{z : \operatorname{Orb}_\pi(z) \cap \widehat{a} \ne \emptyset\}$ is dense in Z and $\#(\operatorname{Orb}_\pi(z) \cap \widehat{a}) \le 1$ for every z.

>(b) Let X be a set. (i) Show that $\operatorname{Aut}(\mathcal{P}X)$ is isomorphic to the symmetric group on X, the group of all permutations of X. (ii) Show that any element of $\operatorname{Aut}(\mathcal{P}X)$ is expressible as a product of at most two involutions.

>(c) (Miller 04) Let X be a set and Σ a σ-algebra of subsets of X. Suppose that (X, Σ) is countably separated in the sense that there is a countable subset of Σ separating the points of X (cf. 343D). Let G be the group of permutations $f : X \to X$ such that $\Sigma = \{f^{-1}[E] : E \in \Sigma\}$. Show that every automorphism of the Boolean algebra Σ has a separator, so that every member of G is expressible as the product of at most three involutions belonging to G.

(d) Recall that in any group G, a **commutator** in G is an element of the form $ghg^{-1}h^{-1}$ where g, $h \in G$. Show that if $\mathfrak{A}$ is a Dedekind complete Boolean algebra and G is a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions then every involution in G is a commutator in G, so that every element of G is expressible as a product of three commutators, and any group homomorphism from G to an abelian group is constant.

(e) Give an example of a Dedekind complete Boolean algebra $\mathfrak{A}$ such that not every member of $\operatorname{Aut}\mathfrak{A}$ is a product of commutators in $\operatorname{Aut}\mathfrak{A}$.

(f) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and suppose that $\operatorname{Aut}\mathfrak{A}$ has many involutions. Show that if $H \lhd \operatorname{Aut}\mathfrak{A}$ then every member of H is expressible as the product of at most three involutions belonging to H.

(g) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show that the partially ordered set $\mathcal{H}$ of normal subgroups of G is a distributive lattice, that is, $H \cap K_1K_2 = (H \cap K_1)(H \cap K_2)$, $H(K_1 \cap K_2) = HK_1 \cap HK_2$ for all H, K_1, $K_2 \in \mathcal{H}$.

(h) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show that if H is the normal subgroup of G generated by a finite subset of G, then it is the normal subgroup generated by a single involution.

(i) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a full subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show (i) that there is an involution $\pi \in G$ such that every member of G is expressible as a product of conjugates of π in G (ii) any proper normal subgroup of G is included in a maximal proper normal subgroup of G.

(j) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra. Show that if $\pi : \mathfrak{A} \to \mathfrak{A}$ is an ergodic measure-preserving automorphism it has no transversal.

(k) Show that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra with countable Maharam type (definition: 331F), then every automorphism of $\mathfrak{A}$ has a separator. (*Hint*: show that if $b \in \mathfrak{A}$ then $\{a : a \triangle \pi a \subseteq b\}$ is an order-closed subalgebra.)

(l) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$. Show that π has a separator iff there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that π is supported by $\sup_{n\in\mathbb{N}} a_n \triangle \pi a_n$.

(m) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with many involutions. Show that for every $n \ge 2$ and every $a \in \mathfrak{A} \setminus \{0\}$ there is a $\pi \in G$ with period n and support a.

382Y Further exercises (a) Find a Dedekind σ-complete Boolean algebra with an involution which is not an exchanging involution.

(b) Devise an expression of the ideas of parts (f)-(h) of the proof of 382K which does not involve the Stone representation. (*Hint*: show that there is a non-increasing sequence in $\mathfrak{A}^+$ which makes enough decisions to play the role of the Boolean homomorphism $x : \mathfrak{A} \to \mathbb{Z}_2$.)

(c) Let $\mathcal{B}$ be the algebra of Borel subsets of $\mathbb{R}$. Show that $\operatorname{Aut}\mathcal{B}$ has exactly three proper normal subgroups. (*Hint*: re-work the proof of 382R, paying particular attention to calls on Lemma 382Q. You will need to know that if $E \in \mathcal{B}$ is uncountable then the subspace σ-algebra on E is isomorphic to $\mathcal{B}$; see §424 in Volume 4.)

(d) Find a Dedekind σ-complete Boolean algebra $\mathfrak{A}$ with an automorphism which cannot be expressed either as a product of finitely many involutions in $\operatorname{Aut}\mathfrak{A}$, or as a product of finitely many commutators in $\operatorname{Aut}\mathfrak{A}$. (This seems to require a certain amount of ingenuity.)

382 Notes and comments The ideas of 382A and 382G-382N are adapted from Miller 04, and (most conspicuously in part (g) of the proof of 382K) betray their origin in a study of Borel automorphisms of $\mathbb{R}$ (see 382Xc). The magic number of three involutions appears in Ryzhikov 93 and Truss 89. The idea of the method presented here is to shift from a 'separator' to a 'transversal'. Since there are many automorphisms without transversals (382Xj), something quite surprising has to happen. The diagrams in the proof of 382K are supposed to show the two steps involved in the argument. We are trying to draw non-overlapping links to build a function g^x such that every point of $\mathbb{Z}$ will belong to a finite orbit of $g^x s$. This must be done by some uniform, translation-invariant, process based on configurations already present; in particular, we are *not* permitted to single out any point of $\mathbb{Z}$ as a centre for the construction. The first attempt is based on the sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of sets corresponding to the decreasing sequence $\langle e_n\rangle_{n\in\mathbb{N}}$. The construction of such a sequence (382J) requires that there be many separators, which is why these results cannot be applied to all Boolean algebras, or even to all homogeneous ones. If this first attempt fails, however, the points not recurrent under $g_1^x s$ provide a set C_0 with arbitrarily large gaps both to left and to right, from which the second method can build an adequate family of links.

Of course the search for these factorizations was inspired by the well-known corresponding fact for algebras $\mathcal{P}X$ (382Xb). In those algebras we can use the axiom of choice unscrupulously to pick out a point of each orbit, thereby forming a transversal in one step without considering separators, and then apply 382H in its original simple form. Perhaps the principal psychological barrier we need to overcome in 382K is raised in the phrase 'fix $x \in X \cap \widehat{c_1}$'. What I could have said is 'fix an orbit of f meeting $\widehat{c_1}$, and order it by the transitive closure of the relation f'; because the whole point of the subsequent argument is that we do not have a marker to work from.

This volume is concerned with measure algebras, and all the most important measure algebras are Dedekind complete. I take the trouble to express the ideas down to Theorem 382M in terms of σ-complete algebras partly because this is the natural boundary of the arguments given and partly because in Volume 4 I will look at Borel automorphisms, as in 382Xc, and 382M as stated may then be illuminating. But note that in 382N σ-completeness is insufficient (382Yd). In 382S I allow myself for once to present a result with a stronger hypothesis than is required for the conclusion; the point being that homogeneous semi-finite measure algebras are necessarily Dedekind complete (383E), and the arguments for the more general case do not seem to tell us anything which we can use elsewhere in this treatise.

It is natural to ask whether the number 'three' in 382M is best possible (cf. 382Xb). It seems to be quite difficult to exhibit an automorphism requiring three involutions; examples may be found in Anzai 51 and Ornstein & Sheilds 73[2].

Just as well-known facts about symmetry groups lead us to the factorization theorem 382M, they suggest that automorphism groups of Boolean algebras may often have few normal subgroups; and once again we find that the form of the theorem changes significantly. However the root of the phenomenon remains the fact that our groups are multiply transitive. 382O-382S are derived from Štěpánek & Rubin 89 and Fathi 78. An obvious question arising from 382S is: does *every* homogeneous Boolean algebra have a simple automorphism group? This leads into deep water. As remarked after 382S, every homogeneous Dedekind σ-complete algebra has a simple automorphism group. Using the continuum hypothesis, it is possible to construct a homogeneous Boolean algebra which does not have a simple automorphism group; but as far as I am aware no such construction is known which does not rely on some special axiom outside ordinary set theory. See Štěpánek & Rubin 89, §5.

Version of 9.11.14

383 Automorphism groups of measure algebras

I turn now to the group of measure-preserving automorphisms of a measure algebra, seeking to apply the results of the last section. The principal theorems are 383D, which is a straightforward special case of 382N, and 383I, corresponding to 382S. I give another example of the use of 382R to describe the normal subgroups of $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ (383J), and conclude with an important fact about conjugacy in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\mathrm{Aut}\,\mathfrak{A}$ (383L).

[2] I am indebted to P.Biryukov and G.Hjorth for the references.

383A Definition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. I will write $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ for the set of all measure-preserving automorphisms of $\mathfrak{A}$. This is a group, being a subgroup of the group $\mathrm{Aut}\,\mathfrak{A}$ of all Boolean automorphisms of $\mathfrak{A}$.

383B Lemma Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Assume
either that I is countable
or that $(\mathfrak{A},\bar{\mu})$ is localizable.
Suppose that for each $i\in I$ we have a measure-preserving isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{A}_{b_i}$ between the corresponding principal ideals. Then there is a unique $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\pi c=\pi_i c$ whenever $i\in I$ and $c\subseteq a_i$.

proof (Compare 381C.) By 322Ld or 322Le, we may identify $\mathfrak{A}$ with each of the simple products $\prod_{i\in I}\mathfrak{A}_{a_i}$, $\prod_{i\in I}\mathfrak{A}_{b_i}$; now π corresponds to the isomorphism between the two products induced by the π_i.

383C Corollary If $(\mathfrak{A},\bar{\mu})$ is a localizable measure algebra, then, in the language of 381Be, $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is a full subgroup of $\mathrm{Aut}\,\mathfrak{A}$.

383D Theorem Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. Then every measure-preserving automorphism of $\mathfrak{A}$ is expressible as the product of at most three measure-preserving involutions.

proof This is immediate from 383C and 382N.

383E Lemma If $(\mathfrak{A},\bar{\mu})$ is a homogeneous semi-finite measure algebra, it is σ-finite, therefore localizable.

proof If $\mathfrak{A}=\{0\}$, this is trivial. Otherwise there is an $a\in\mathfrak{A}$ such that $0<\bar{\mu}a<\infty$. The principal ideal $\mathfrak{A}_a$ is ccc (322G(i)$\Rightarrow$(ii)), so $\mathfrak{A}$ also is, and $(\mathfrak{A},\bar{\mu})$ must be σ-finite, by 322G(ii)$\Rightarrow$(i).

383F Lemma Let $(\mathfrak{A},\bar{\mu})$ be a homogeneous semi-finite measure algebra.

(a) If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are partitions of unity in $\mathfrak{A}$ with $\bar{\mu}a_i=\bar{\mu}b_i$ for every i, there is a $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\pi a_i=b_i$ for each i.

(b) If $(\mathfrak{A},\bar{\mu})$ is totally finite, then whenever $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$ with $\bar{\mu}a_i=\bar{\mu}b_i$ for every i, there is a $\pi\in\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\pi a_i=b_i$ for each i.

proof (a) By 383E, $(\mathfrak{A},\bar{\mu})$ is σ-finite, therefore localizable. For each $i\in I$, the principal ideals $\mathfrak{A}_{a_i}$, $\mathfrak{A}_{b_i}$ are homogeneous, of the same measure and the same Maharam type (being $\tau(\mathfrak{A})$ if $a_i\neq 0$, 0 if $a_i=0$). Because they are ccc, they are of the same magnitude, as defined in 332Ga, and there is a measure-preserving isomorphism $\pi_i:\mathfrak{A}_{a_i}\to\mathfrak{A}_{b_i}$ (332J). By 383B there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\pi d=\pi_i d$ for every $i\in I$, $d\subseteq a_i$; and this π serves.

(b) Set $a^*=1\setminus\sup_{i\in I}a_i$, $b^*=1\setminus\sup_{i\in I}b_i$. We must have

$$\bar{\mu}a^*=\bar{\mu}1-\textstyle\sum_{i\in I}\bar{\mu}a_i=\bar{\mu}1-\sum_{i\in I}\bar{\mu}b_i=\bar{\mu}b^*,$$

so adding a^*, b^* to the families we obtain partitions of unity to which we can apply the result of (a).

383G Lemma (a) If $(\mathfrak{A},\bar{\mu})$ is an atomless semi-finite measure algebra, then $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ have many involutions.

(b) If $(\mathfrak{A},\bar{\mu})$ is an atomless localizable measure algebra, then every non-zero element of $\mathfrak{A}$ is the support of an involution in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$.

proof (a) If $a\in\mathfrak{A}\setminus\{0\}$, then by 332A there is a non-zero $b\subseteq a$, of finite measure, such that the principal ideal $\mathfrak{A}_b$ is (Maharam-type-)homogeneous. Now because $\mathfrak{A}$ is atomless, there is a $c\subseteq b$ such that $\bar{\mu}c=\frac{1}{2}\bar{\mu}b$ (331C), so that $\mathfrak{A}_c$ and $\mathfrak{A}_{b\setminus c}$ are isomorphic measure algebras. If $\theta:\mathfrak{A}_c\to\mathfrak{A}_{b\setminus c}$ is any measure-preserving isomorphism, then $\pi=\overleftarrow{(c\,{}_\theta\,b\setminus c)}$ is an involution in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ (and therefore in $\mathrm{Aut}\,\mathfrak{A}$) supported by a.

(b) Use 383C, (a) and 382Q.

383H Corollary Let $(\mathfrak{A},\bar{\mu})$ be an atomless localizable measure algebra. Then

(a) the lattice of normal subgroups of $\operatorname{Aut}\mathfrak{A}$ is isomorphic to the lattice of $\operatorname{Aut}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$;

(b) the lattice of normal subgroups of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is isomorphic to the lattice of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$.

proof Use 382R. Taking G to be either $\operatorname{Aut}\mathfrak{A}$ or $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, and $\mathcal{I}$ to be the family of G-invariant ideals in $\mathfrak{A}$, we have a map $I\mapsto H_I=\{\pi:\pi\in G,\ \operatorname{supp}\pi\in I\}$ from $\mathcal{I}$ to the family $\mathcal{H}$ of normal subgroups of G. Of course this map is order-preserving; 382R tells us that it is surjective; and 383Gb tells us that it is injective and its inverse is order-preserving, since if $a\in I\setminus J$ there is a $\pi\in G$ with $\operatorname{supp}\pi=a$, so that $\pi\in H_I\setminus H_J$. Thus we have an order-isomorphism between $\mathcal{H}$ and $\mathcal{I}$.

383I 382R provides the machinery for a full description of the normal subgroups of $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ when $(\mathfrak{A},\bar{\mu})$ is an atomless localizable measure algebra, as we know that they correspond exactly to the invariant ideals of $\mathfrak{A}$. The general case is complicated. But the following special cases are easy enough.

Theorem Let $(\mathfrak{A},\bar{\mu})$ be a homogeneous semi-finite measure algebra.

(a) $\operatorname{Aut}\mathfrak{A}$ is simple.

(b) If $(\mathfrak{A},\bar{\mu})$ is totally finite, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is simple.

(c) If $(\mathfrak{A},\bar{\mu})$ is not totally finite, $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ has exactly one non-trivial proper normal subgroup.

proof (a) $\mathfrak{A}$ is Dedekind complete (383E), so this is a special case of 382S.

(b)-(c) The point is that the only possible $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$ are $\{0\}$, $\mathfrak{A}^f$ and $\mathfrak{A}$. **P** If $\mathfrak{A}$ is $\{0\}$ or $\{0,1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless. Let $I\lhd\mathfrak{A}$ be an invariant ideal.

(i) If $I\not\subseteq\mathfrak{A}^f$, take $a\in I$ with $\bar{\mu}a=\infty$. By 383E, $\mathfrak{A}$ is σ-finite, so a has the same magnitude ω as 1. By 332I, there is a partition of unity $\langle e_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with $\bar{\mu}e_n=1$ for every n; setting $b=\sup_{n\in\mathbb{N}}e_{2n}$ and $b'=1\setminus b$, we see that both b and b' are of infinite measure. Similarly we can divide a into c and c', both of infinite measure. Now by 332J the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_{b'}$, $\mathfrak{A}_c$, $\mathfrak{A}_{1\setminus c}$ are all isomorphic as measure algebras, so that there are automorphisms π, $\phi\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that

$$\pi c=b,\quad \phi c=b'.$$

But this means that both b and b' belong to I, so that $1=b\cup b'\in I$ and $I=\mathfrak{A}$.

(ii) If $I\subseteq\mathfrak{A}^f$ and $I\ne\{0\}$, take any non-zero $a\in I$. If b is any member of $\mathfrak{A}$, then (because $\mathfrak{A}$ is atomless) b can be partitioned into $b_0,\dots,b_n$, all of measure at most $\bar{\mu}a$. Then for each i there is a $b_i'\subseteq a$ such that $\bar{\mu}b_i'=\bar{\mu}b_i$; since this common measure is finite, $\bar{\mu}(1\setminus b_i')=\bar{\mu}(1\setminus b_i)$. By 332J and 383Fa, there is a $\pi_i\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ such that $\pi_ib_i'=b_i$, so that b_i belongs to I. Accordingly $b\in I$. As b is arbitrary, $I=\mathfrak{A}^f$.

Thus the only invariant ideals of $\mathfrak{A}$ are $\{0\}$, $\mathfrak{A}^f$ and $\mathfrak{A}$. **Q**

By 383Hb we therefore have either one, two or three normal subgroups of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, according to whether $\bar{\mu}1$ is zero, finite and not zero, or infinite.

Remark For the Lebesgue probability algebra, (b) is due to Fathi 78. The extension to algebras of uncountable Maharam type is from Choksi & Prasad 82.

383J The language of §352 offers a way of describing another case.

Proposition Let $(\mathfrak{A},\bar{\mu})$ be an atomless totally finite measure algebra. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}$, and let K be $\{\kappa:e_\kappa\ne0\}$. Let $\mathcal{H}$ be the lattice of normal subgroups of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$. Then

(i) if K is finite, $\mathcal{H}$ is isomorphic, as partially ordered set, to $\mathcal{P}K$;

(ii) if K is infinite, then $\mathcal{H}$ is isomorphic, as partially ordered set, to the lattice of solid linear subspaces of ℓ^∞.

proof (a) Let $\mathcal{I}$ be the family of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant ideals of $\mathfrak{A}$, so that $\mathcal{H}\cong\mathcal{I}$, by 383Hb. For a, $b\in\mathfrak{A}$, say that $a\preccurlyeq b$ if there is some $k\in\mathbb{N}$ such that $\bar{\mu}(a\cap e_\kappa)\le k\bar{\mu}(b\cap e_\kappa)$ for every $\kappa\in K$. Then an ideal I of $\mathfrak{A}$ is $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$-invariant iff $a\in I$ whenever $a\preccurlyeq b\in I$. **P** (α) Suppose that I is $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ invariant, $a\in\mathfrak{A}$, $b\in I$ and $\bar{\mu}(a\cap e_\kappa)\le k\bar{\mu}(b\cap e_\kappa)$ for every $\kappa\in K$. Because $\mathfrak{A}$ is atomless, we can find, for each $\kappa\in K$, $a_{\kappa1},\dots,a_{\kappa k}$ such that $a\cap e_\kappa=\sup_{i\le k}a_{\kappa i}$ and $\bar{\mu}a_{\kappa i}\le\bar{\mu}(b\cap e_\kappa)$ for every i. Now for $\kappa\in K$ and $1\le i\le k$ there is a measure-preserving automorphisms $\pi_{\kappa i}$ of the principal ideal $\mathfrak{A}_{e_\kappa}$ such that $\pi_{\kappa i}a_{\kappa i}\subseteq b$. Setting $\pi_id=\sup_{\kappa\in K}\pi_{\kappa i}(d\cap e_\kappa)$ for every $d\in\mathfrak{A}$, and $a_i=\sup_{\kappa\in K}a_{\kappa i}$ for $1\le i\le k$, we have $\pi_i\in\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and

$\pi_i a_i \subseteq b$, so $a_i \in I$ for each i; also $a = \sup_{i\le k} a_i$, so $a \in I$. (β) On the other hand, if $a \in \mathfrak{A}$ and $\pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$, then

$$\bar\mu(\pi a \cap e_\kappa) = \bar\mu\pi(a \cap e_\kappa) = \bar\mu(a \cap e_\kappa)$$

for every $\kappa \in K$, because $\pi e_\kappa = e_\kappa$, so that $\pi a \preccurlyeq a$. So if I satisfies the condition, $\pi[I] \subseteq I$ for every $\pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and $I \in \mathcal{I}$. **Q**

(b) Consequently, for $I \in \mathcal{I}$ and $\kappa \in K$, $e_\kappa \in I$ iff there is some $a \in I$ such that $a \cap a_\kappa \neq 0$, since in this case $e_\kappa \preccurlyeq a$. (This is where I use the hypothesis that $(\mathfrak{A}, \bar\mu)$ is totally finite.) It follows that if K is finite, any $I \in \mathcal{I}$ is the principal ideal generated by $\sup\{e_\kappa : e_\kappa \in I\}$. Conversely, of course, all such ideals are $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$-invariant. Thus $\mathcal{I}$ is in a natural order-preserving correspondence with $\mathcal{P}K$, and $\mathcal{H} \cong \mathcal{P}K$.

(c) Now suppose that K is infinite; enumerate it as $\langle\kappa_n\rangle_{n\in\mathbb{N}}$. Define $\theta : \mathfrak{A} \to \ell^\infty$ by setting

$$\theta a = \langle\bar\mu(a \cap e_{\kappa_n})/\bar\mu(e_{\kappa_n})\rangle_{n\in\mathbb{N}}$$

for $a \in \mathfrak{A}$; so that

$$a \preccurlyeq b \text{ iff there is some } k \text{ such that } \theta a \le k\theta b,$$

$$\theta a \le \theta(a \cup b) \le \theta a + \theta b \le 2\theta(a \cup b)$$

for all a, $b \in \mathfrak{A}$, while $\theta(1_{\mathfrak{A}})$ is the standard order unit $\chi\mathbb{N}$ of ℓ^∞. Let $\mathcal{U}$ be the family of solid linear subspaces of ℓ^∞ and define functions $I \mapsto V_I : \mathcal{I} \to \mathcal{U}$, $U \mapsto J_U : \mathcal{U} \to \mathcal{I}$ by saying

$$V_I = \{f : f \in \ell^\infty,\ |f| \le k\theta a \text{ for some } a \in I,\ k \in \mathbb{N}\},$$

$$J_U = \{a : a \in \mathfrak{A},\ \theta a \in U\}.$$

The properties of θ just listed ensure that $V_I \in \mathcal{U}$ and $J_U \in \mathcal{I}$ for every $I \in \mathcal{I}$, $U \in \mathcal{U}$. Of course both $I \mapsto V_I$ and $U \mapsto J_U$ are order-preserving. If $I \in \mathcal{I}$, then

$$J_{V_I} = \{a : \exists\, b \in I,\ a \preccurlyeq b\} = I.$$

Finally, $V_{J_U} = U$ for every $U \in \mathcal{U}$. **P**

$$V_{J_U} = \{f : \exists\, a \in \mathfrak{A},\ k \in \mathbb{N},\ |f| \le k\theta a \in U\} \subseteq U$$

because U is a solid linear subspace. But also, given $g \in U$, there is an $a \in \mathfrak{A}$ such that $\bar\mu(a \cap e_{\kappa_n}) = \min(1, |g(n)|)\bar\mu(e_{\kappa_n})$ for every n (because $\mathfrak{A}$ is atomless); in which case

$$\theta a \le |g| \le \max(1, \|g\|_\infty)\theta a$$

so $a \in J_U$ and $g \in V_{J_U}$. Thus $U = V_{J_U}$. **Q** So the functions $I \mapsto V_I$ and $U \mapsto J_U$ are the two halves of an order-isomorphism between $\mathcal{I}$ and $\mathcal{U}$, and $\mathcal{H} \cong \mathcal{I} \cong \mathcal{U}$, as claimed.

383K Later in this chapter I will give a good deal of space to the question of when two automorphisms of a measure algebra are conjugate. Because, on any measure algebra $(\mathfrak{A}, \bar\mu)$, we have two groups $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ with claims on our attention, we have two different conjugacy relations to examine. To clear the ground, I give a result showing that in a significant number of cases the two coincide.

Proposition Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an ergodic measure-preserving Boolean homomorphism. If $\phi \in \operatorname{Aut}\mathfrak{A}$ is such that $\phi\pi\phi^{-1}$ is measure-preserving, then ϕ is measure-preserving.

proof Consider the functional $\nu : \mathfrak{A} \to \mathbb{R}$ defined by saying that $\nu a = \bar\mu(\phi a)$ for every $a \in \mathfrak{A}$. Because $\bar\mu$ is completely additive (321F) and strictly positive, so is ν. We therefore have a $c = [\![\nu > \bar\mu]\!]$ in $\mathfrak{A}$ such that $\nu a > \bar\mu a$ whenever $0 \neq a \subseteq c$ and $\nu a \le \bar\mu a$ whenever $a \cap c = 0$ (326T). Now $\pi c = c$. **P?** Otherwise, because π is measure-preserving,

$$\bar\mu(\pi c \setminus c) = \bar\mu(\pi c) - \bar\mu(c \cap \pi c) = \bar\mu c - \bar\mu(c \cap \pi c) = \bar\mu(c \setminus \pi c) = \frac{1}{2}\bar\mu(c \,\triangle\, \pi c) > 0.$$

Next,

$$\nu\pi c = \bar\mu(\phi\pi c) = \bar\mu(\phi\pi\phi^{-1}\phi c) = \nu c,$$

so we also have $\nu(\pi c\setminus c)=\nu(c\setminus\pi c)$. But now observe that

$$\nu(\pi c\setminus c)\le\bar\mu(\pi c\setminus c),\quad \nu(c\setminus\pi c)>\bar\mu(c\setminus\pi c)$$

by the choice of c, which is impossible. **XQ**

Because π is ergodic, c must be 0 or 1 (372Pa). But as $\nu\pi1=\nu1=\bar\mu1$, we cannot have $0\ne1\subseteq c$, so $c=0$. This means that $\nu a\le\bar\mu a$ for every $a\in\mathfrak{A}$; once again, $\nu1=\bar\mu1$, so in fact $\nu a=\bar\mu a$ for every a, that is, ϕ is measure-preserving.

383L Corollary Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and π_1, $\pi_2\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ two ergodic measure-preserving automorphisms. If they are conjugate in $\mathrm{Aut}\,\mathfrak{A}$ then they are conjugate in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

proof There is a $\phi\in\mathrm{Aut}\,\mathfrak{A}$ such that $\phi\pi_1\phi^{-1}=\pi_2$; now 383K tells us that $\phi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

383X Basic exercises (a) Let (X,Σ,μ) be a countably separated measure space (definition: 343D), and write $\mathrm{Aut}_\mu\Sigma$ for the group of automorphisms $\phi:\Sigma\to\Sigma$ such that $\mu\phi(E)=\mu E$ for every $E\in\Sigma$. Show that every member of $\mathrm{Aut}_\mu\Sigma$ is expressible as a product of at most three involutions belonging to $\mathrm{Aut}_\mu\Sigma$. (*Hint*: 382Xc.)

(b) Let (X,Σ,μ) be a measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Let S be the set of functions which are isomorphisms between conegligible measurable subsets of X with their subspace measures. (i) Show that the composition of two members of S belongs to S. (ii) Show that there is a map $f\mapsto\pi_f:S\to\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ defined by saying that $\pi_f(E^\bullet)=f^{-1}[E]^\bullet$ for every $E\in\Sigma$, and that $\pi_{fg}=\pi_g\pi_f$, $\pi_f^{-1}=\pi_{f^{-1}}$ for all f, $g\in S$. (iii) Show that $\{\pi_f:f\in S\}$ is a countably full subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$.

>(c) Let (X,Σ,μ) be a measure space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Let Φ be the group of measure space automorphisms of (X,Σ,μ). For $f\in\Phi$, let $\pi_f\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ be the corresponding automorphism, defined by setting $\pi_f(E^\bullet)=(f^{-1}[E])^\bullet$ for every $E\in\Sigma$. (i) Show that $f\mapsto\pi_f^{-1}$ is a group homomorphism from Φ to $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. (ii) Show that if $F\subseteq\Phi$ and the subgroup of Φ generated by F is Ψ, then the subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ generated by $\{\pi_f:f\in F\}$ is $\{\pi_f:f\in\Psi\}$. (iii) Show that if (X,Σ,μ) is countably separated and $F\subseteq\Phi$ is a countable subgroup, then the full subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ generated by $\{\pi_f:f\in F\}$ is $\{\pi_g:g\in F^*\}$, where

$$F^*=\{g:g\in\Phi,\,g(x)\in\{f(x):x\in F\}\text{ for every }x\in X\}.$$

>(d) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}$. (i) Show that $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is a simple group iff *either* there is just one infinite cardinal κ such that $e_\kappa\ne0$, that e_κ has finite measure and all the atoms of $\mathfrak{A}$ (if any) have different measures *or* $\mathfrak{A}$ is purely atomic and there is just one pair of atoms of the same measure *or* $\mathfrak{A}$ is purely atomic and all its atoms have different measures. (ii) Show that $\mathrm{Aut}\,\mathfrak{A}$ is a simple group iff *either* $(\mathfrak{A},\bar\mu)$ is σ-finite and there is just one infinite cardinal κ such that $e_\kappa\ne0$ and $\mathfrak{A}$ has at most one atom *or* $\mathfrak{A}$ is purely atomic and has at most two atoms.

(e) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. (i) Show that $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is simple iff it is isomorphic to one of the groups $\{\iota\}$, $\mathbb{Z}_2$ or $\mathrm{Aut}_{\bar\nu_\kappa}\mathfrak{B}_\kappa$ where κ is an infinite cardinal and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ is the measure algebra of the usual measure on $\{0,1\}^\kappa$. (ii) Show that $\mathrm{Aut}\,\mathfrak{A}$ is simple iff it is isomorphic to one of the groups $\{\iota\}$, $\mathbb{Z}_2$ or $\mathrm{Aut}\,\mathfrak{B}_\kappa$.

(f) Show that if $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra of magnitude greater than $\mathfrak{c}$, its automorphism group $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is not simple.

(g) Let $(\mathfrak{A},\bar\mu)$ be an atomless localizable measure algebra. For each infinite cardinal κ write e_κ for the Maharam-type-κ component of $\mathfrak{A}$. For π, $\psi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ show that π belongs to the normal subgroup of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ generated by ψ iff there is a $k\in\mathbb{N}$ such that

$$\mathrm{mag}(e_\kappa\cap\mathrm{supp}\,\pi)\le k\,\mathrm{mag}(e_\kappa\cap\mathrm{supp}\,\psi)\text{ for every infinite cardinal }\kappa,$$

writing $\mathrm{mag}\,a$ for the magnitude of a, and setting $k\zeta=\zeta$ if $k>0$ and ζ is an infinite cardinal.

>(h) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. For $n \in \mathbb{N}$ set $e_n = [-n, n]^\bullet \in \mathfrak{A}$. Let $G \le \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ be the group consisting of measure-preserving automorphisms π such that $\operatorname{supp} \pi \subseteq e_n$ for some n. Show that G is simple. (*Hint*: show that G is the union of an increasing sequence of simple subgroups.)

(i) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless totally finite measure algebra. Let $\mathcal{H}$ be the lattice of normal subgroups of $\operatorname{Aut} \mathfrak{A}$. Show that $\mathcal{H}$ is isomorphic, as partially ordered set, to $\mathcal{P}K$ for some countable set K.

(j) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless localizable measure algebra which is not σ-finite, and suppose that $\tau(\mathfrak{A}_a) = \tau(\mathfrak{A}_b)$ whenever a, $b \in \mathfrak{A}$ and $0 < \bar{\mu}a \le \bar{\mu}b < \infty$. Let κ be the magnitude of $\mathfrak{A}$. (i) Show that the lattice $\mathcal{H}$ of normal subgroups of $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ is well-ordered, with least member $\{\iota\}$ and one member H_ζ for each infinite cardinal ζ less than or equal to κ^+, setting

$$H_\zeta = \{\pi : \pi \in \operatorname{Aut}_{\bar{\mu}} \mathfrak{A},\ \operatorname{mag}(\operatorname{supp} \pi) < \zeta\},$$

where $\operatorname{mag} a$ is the magnitude of a. (ii) Show that the lattice $\mathcal{H}'$ of normal subgroups of $\operatorname{Aut} \mathfrak{A}$ is well-ordered, with least member $\{\iota\}$ and one member H'_ζ for each uncountable cardinal ζ less than or equal to κ^+, setting

$$H'_\zeta = \{\pi : \pi \in \operatorname{Aut} \mathfrak{A},\ \operatorname{mag}(\operatorname{supp} \pi) < \zeta\}.$$

>(k) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0, 1]$. Give an example of two measure-preserving automorphisms of $\mathfrak{A}$ which are conjugate in $\operatorname{Aut} \mathfrak{A}$ but not in $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$.

(l) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. For π, $\phi \in \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ set

$$\rho(\pi, \phi) = \sup_{a \in \mathfrak{A}} \bar{\mu}(\pi a \triangle \phi a), \quad \sigma(\pi, \phi) = \bar{\mu}(\operatorname{supp}(\pi^{-1}\phi)).$$

(i) Show that ρ and σ are metrics on $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$, and that $\rho \le \sigma \le \frac{3}{2}\rho$. (*Hint*: 381E, 381G, 382D-382E.) (ii) Show that $\rho(\psi\pi, \psi\phi) = \rho(\pi\psi, \phi\psi) = \rho(\pi, \phi)$, $\rho(\pi^{-1}, \phi^{-1}) = \rho(\pi, \phi)$, $\rho(\pi\psi, \phi\theta) \le \rho(\pi, \phi) + \rho(\psi, \theta)$, $\sigma(\psi\pi, \psi\phi) = \sigma(\pi\psi, \phi\psi) = \sigma(\pi, \phi)$, $\sigma(\pi^{-1}, \phi^{-1}) = \sigma(\pi, \phi)$, $\sigma(\pi\psi, \phi\theta) \le \sigma(\pi, \phi) + \sigma(\psi, \theta)$ for all π, ϕ, ψ, $\theta \in \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$. (iii) Show that $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ is complete under ρ and σ.

383Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless totally finite measure algebra. Show that $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ and $\operatorname{Aut} \mathfrak{A}$ have the same (cardinal) number of normal subgroups.

(b) Let X be a set. Show that $\operatorname{Aut} \mathcal{P}X$ has one normal subgroup if $\#(X) \le 1$, two if $\#(X) = 2$, three if $\#(X) = 3$ or $5 \le \#(X) < \omega$, four if $\#(X) = 4$ or $\#(X) = \omega$, five if $\#(X) = \omega_1$.

383 Notes and comments This section is short because there are no substantial new techniques to be developed. 383D is simply a matter of checking that the hypotheses of 382N are satisfied (and these hypotheses were of course chosen with 383D in mind), and 383I is similarly direct from 382R-382S. 383I-383J, 383Xg and 383Xj are variations on a theme. In a general Boolean algebra $\mathfrak{A}$ with a group G of automorphisms, we have a transitive, reflexive relation $\preccurlyeq_G$ defined by saying that $a \preccurlyeq_G b$ if there are $\pi_0, \dots, \pi_k \in G$ such that $a \subseteq \sup_{i \le k} \pi_i b$; the point about localizable measure algebras is that the functions 'Maharam type' and 'magnitude' enable us to describe this relation when $G = \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$, and the essence of 382R is that in that context π belongs to the normal subgroup of G generated by ψ iff $\operatorname{supp} \pi \preccurlyeq_G \operatorname{supp} \psi$.

Some of the most interesting questions concerning automorphism groups of measure algebras can be expressed in the form 'how can we determine when a given pair of automorphisms are conjugate?' Generally, people have concentrated on conjugacy in $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$. But the same question can be asked in $\operatorname{Aut} \mathfrak{A}$. In particular, it is possible for two members of $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ to be conjugate in $\operatorname{Aut} \mathfrak{A}$ but not in $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ (383Xk). However this phenomenon does not occur for *ergodic* automorphisms, or even for ergodic measure-preserving Boolean homomorphisms (383K-383L).

Most of the work of this chapter is focused on atomless measure algebras. There are various extra complications which appear if we allow atoms. The most striking are in the next section; here I mention only 383Xd and 383Yb.

Version of 5.11.14

384 Outer automorphisms

Continuing with the investigation of the abstract group-theoretic nature of the automorphism groups $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}_{\bar\mu}\,\mathfrak{A}$, I devote a section to some remarkable results concerning isomorphisms between them. Under any of a variety of conditions, any isomorphism between two groups $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Aut}\,\mathfrak{B}$ must correspond to an isomorphism between the underlying Boolean algebras (384E, 384F, 384J, 384M); consequently $\mathrm{Aut}\,\mathfrak{A}$ has few, or no, outer automorphisms (384G, 384K, 384O). I organise the section around a single general result (384D).

384A Lemma Let $\mathfrak{A}$ be a Boolean algebra and G a subgroup of $\mathrm{Aut}\,\mathfrak{A}$ which has many involutions (definition: 382O). Then for every non-zero $a\in\mathfrak{A}$ there is an automorphism $\psi\in G$, of order 4, which is supported by a.

proof Let $\pi\in G$ be an involution supported by a. Let $b\subseteq a$ be such that $\pi b\ne b$. Then at least one of $b\setminus\pi b$, $\pi b\setminus b=\pi(b\setminus\pi b)$ is non-zero, so in fact both are. Let ϕ be an involution supported by $b\setminus\pi b$. Then $\pi\phi\pi=\pi\phi\pi^{-1}$ is an involution supported by $\pi b\setminus b$, so commutes with ϕ, and $\phi\pi\phi\pi=\iota$. Also $\pi\phi b=\pi b\ne b$, so $\pi\phi$ and $\phi\pi$ are not the identity, and $\psi=\phi\pi$ has order 4. Of course ψ is supported by a because ϕ and π both are.

384B A note on supports Since in this section we shall be looking at more than one automorphism group at a time, I shall need to call on the following elementary extension of a fact in §381. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\theta:\mathfrak{A}\to\mathfrak{B}$ a Boolean isomorphism. If $\pi\in\mathrm{Aut}\,\mathfrak{A}$ is supported by $a\in\mathfrak{A}$, then $\theta\pi\theta^{-1}\in\mathrm{Aut}\,\mathfrak{B}$ is supported by θa. (Use the same argument as in 381Ej.) Accordingly, if a is the support of π then θa will be the support of $\theta\pi\theta^{-1}$, as in 381Gd.

384C Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Boolean algebras, and G a subgroup of $\mathrm{Aut}\,\mathfrak{A}$ with many involutions. If θ_1, $\theta_2:\mathfrak{A}\to\mathfrak{B}$ are distinct isomorphisms, then there is a $\phi\in G$ such that $\theta_1\phi\theta_1^{-1}\ne\theta_2\phi\theta_2^{-1}$.

proof Because $\theta_1\ne\theta_2$, $\theta=\theta_2^{-1}\theta_1$ is not the identity automorphism on $\mathfrak{A}$, and there is some non-zero $a\in\mathfrak{A}$ such that $\theta a\cap a=0$. Let $\pi\in G$ be an involution supported by a; then $\theta\pi\theta^{-1}$ is supported by θa, so cannot be equal to π, and $\theta_1\pi\theta_1^{-1}\ne\theta_2\pi\theta_2^{-1}$.

384D Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras and G and H subgroups of $\mathrm{Aut}\,\mathfrak{A}$, $\mathrm{Aut}\,\mathfrak{B}$ respectively, both having many involutions. Let $q:G\to H$ be an isomorphism. Then there is a unique Boolean isomorphism $\theta:\mathfrak{A}\to\mathfrak{B}$ such that $q(\phi)=\theta\phi\theta^{-1}$ for every $\phi\in G$.

proof (a) The first half of the proof is devoted to setting up some structures in the group G. Let $\pi\in G$ be any involution. Set

$$C_\pi=\{\phi:\phi\in G,\,\phi\pi=\pi\phi\},$$

the centralizer of π in G,

$$U_\pi=\{\phi:\phi\in C_\pi,\,\phi=\phi^{-1},\,\phi\psi\phi\psi^{-1}=\psi\phi\psi^{-1}\phi\text{ for every }\psi\in C_\pi\},$$

the set of involutions in C_π commuting with all their conjugates in C_π, together with the identity,

$$V_\pi=\{\phi:\phi\in G,\,\phi\psi=\psi\phi\text{ for every }\psi\in U_\pi\},$$

the centralizer of U_π in G,

$$S_\pi=\{\phi^2:\phi\in V_\pi\}$$

and

$$W_\pi=\{\phi:\phi\in G,\,\phi\psi=\psi\phi\text{ for every }\psi\in S_\pi\},$$

the centralizer of S_π in G.

(b) The point of this list is to provide a purely group-theoretic construction corresponding to the support of π in $\mathfrak{A}$. In the next few paragraphs of the proof (down to (f)), I set out to describe the objects just introduced in terms of their action on $\mathfrak{A}$. First, note that π is an exchanging involution (382Fa); express it as $(\overleftarrow{a' \,{}_{\pi}\, a''})$, so that the support of π is $a_\pi = a' \cup a''$.

(c) I start with two elementary properties of C_π:

(i) $\phi(a_\pi) = a_\pi$ for every $\phi \in C_\pi$. **P** As remarked in 381Gd, the support of $\pi = \phi\pi\phi^{-1}$ is $\phi(a_\pi)$, so this must be a_π. **Q**

(ii) If $\phi \in C_\pi$ and ϕ is not supported by a_π, there is a non-zero $d \subseteq 1 \setminus a_\pi$ such that $d \cap \phi d = 0$, by 381Ei.

(d) Now for the properties of U_π:

(i) If $\phi \in U_\pi$, then ϕ is supported by a_π. **P?** Otherwise, there is a non-zero $d \subseteq 1 \setminus a_\pi$ such that $\phi d \cap d = 0$. By 384A, there is a $\psi \in G$, of order 4, supported by d. Because $d \cap a_\pi = 0$, $\psi \in C_\pi$ (381Ef). Because $\psi \neq \psi^{-1}$, there is a $c \subseteq d$ such that $\psi c \neq \psi^{-1} c$; but now $\phi c \cap d = \phi\psi^{-1} c \cap d = 0$, so

$$\psi\phi\psi^{-1}\phi c = \psi\phi^2 c = \psi c \neq \psi^{-1} c = \phi^2\psi^{-1} c = \phi\psi\phi\psi^{-1} c,$$

and ϕ does not commute with its conjugate $\psi\phi\psi^{-1}$, contradicting the assumption that $\phi \in U_\pi$. **XQ**

(ii) If $u \in \mathfrak{A}$ and $\pi u = u$, then $\pi_u \in U_\pi$, where

$$\pi_u d = \pi d \text{ if } d \subseteq u, \quad \pi_u d = d \text{ if } d \cap u = 0,$$

that is, $\pi_u = (\overleftarrow{a' \cap u \,{}_{\pi}\, a'' \cap u})$. **P** (α) If $u = 0$ then $\pi_u = \iota \in U_\pi$. Otherwise, π_u is an involution. (β) For any $\psi \in \operatorname{Aut}\mathfrak{A}$,

$$\psi\pi_u\psi^{-1} = (\overleftarrow{\psi(a' \cap u) \,{}_{\psi\pi\psi^{-1}}\, \psi(a'' \cap u)})$$

(381Sb). Accordingly

$$\pi\pi_u\pi^{-1} = (\overleftarrow{a'' \cap u \,{}_{\pi}\, a' \cap u}) = \pi_u$$

and $\pi_u \in C_\pi$. (γ) If $\psi \in C_\pi$, then

$$\pi = \psi\pi\psi^{-1} = (\overleftarrow{\psi a' \,{}_{\psi\pi\psi^{-1}}\, \psi a''}) = (\overleftarrow{\psi a' \,{}_{\pi}\, \psi a''}).$$

So

$$\psi\pi_u\psi^{-1} = (\overleftarrow{\psi(a' \cap u) \,{}_{\psi\pi\psi^{-1}}\, \psi(a'' \cap u)}) = (\overleftarrow{\psi a' \cap \psi u \,{}_{\pi}\, \psi a'' \cap \psi u}) = \pi_{\psi u}.$$

Now if $\pi v = v$ then $\pi_u\pi_v = \pi_{u \triangle v} = \pi_v\pi_u$; in particular, $\pi_{\psi u}\pi_u = \pi_u\pi_{\psi u}$. As ψ is arbitrary, $\pi_u \in U_\pi$. **Q**

In particular, of course, $\pi = \pi_1$ belongs to U_π.

(e) The two parts of (d) lead directly to the properties we need of V_π.

(i) $V_\pi \subseteq C_\pi$, because $\pi \in U_\pi$. Consequently $\phi a_\pi = a_\pi$ for every $\phi \in V_\pi$.

(ii) If $\phi \in V_\pi$ then $\phi d \subseteq d \cup \pi d$ for every $d \subseteq a_\pi$. **P?** Suppose, if possible, otherwise. Set $u_0 = d \cup \pi d$, so that $\pi u_0 = u_0$, and $u = \phi u_0 \setminus u_0 \neq 0$; also $u \subseteq \phi a_\pi = a_\pi$. Since $\pi\phi u_0 = \phi\pi u_0 = \phi u_0$, $\pi u = u$. Set $v = u \cap a'$, so that $u = v \cup \pi v$ and $v \neq \pi v$. Because $u \cap \phi v \subseteq \phi(u_0 \cap u) = 0$,

$$\pi_u\phi v = \phi v \neq \phi\pi v = \phi\pi_u v,$$

which is impossible. **XQ**

(iii) It follows that $\phi^2 d = d$ whenever $\phi \in V_\pi$ and $d \subseteq a_\pi$. **P** Let e be the support of ϕ. Recall that $e = \sup\{c : c \cap \phi c = 0\}$ (381Gb), so that $d \cap e = \sup\{c : c \subseteq d,\ c \cap \phi c = 0\}$. Now if $c \subseteq a_\pi$ and $c \cap \phi c = 0$, we know that $\phi c \subseteq c \cup \pi c$, so in fact $\phi c \subseteq \pi c$. This shows that $\phi(d \cap e) \subseteq \pi(d \cap e)$. Also, because $\pi\phi = \phi\pi$, by (i), we have

$$\phi^2(d \cap e) \subseteq \phi\pi(d \cap e) = \pi\phi(d \cap e) \subseteq \pi^2(d \cap e) = d \cap e.$$

Of course $\phi^2(d \setminus e) = d \setminus e$, so $\phi^2 d \subseteq d$. This is true for every $d \subseteq a_\pi$. But as also $\phi^2 a_\pi = \phi a_\pi = a_\pi$, $\phi^2 d = d$ for every $d \subseteq a_\pi$. **Q**

(iv) The final thing we need to know about V_π is that $\phi\in V_\pi$ whenever $\phi\in G$ and $\operatorname{supp}\phi\cap a_\pi=0$; this is immediate from (d-i) above.

(f) From (e-iii), we see that if $\phi\in S_\pi$ then $\operatorname{supp}\phi\cap a_\pi=0$. But we also see from (e-iv) that if $0\ne c\subseteq 1\setminus a_\pi$ there is an involution in S_π supported by c; for there is a member ψ of G, of order 4, supported by c, and now $\psi\in V_\pi$ so $\psi^2\in S_\pi$, while ψ^2 is an involution.

(g) Consequently, W_π is just the set of members of G supported by a_π. **P** (i) If $\operatorname{supp}\phi\subseteq a_\pi$ and $\psi\in S_\pi$, then $\operatorname{supp}\psi\cap a_\pi=0$, as noted in (e), so $\phi\psi=\psi\phi$; as ψ is arbitrary, $\phi\in W_\pi$. (ii) If $\operatorname{supp}\phi\not\subseteq a_\pi$, then take a non-zero $d\subseteq 1\setminus a_\pi$ such that $\phi d\cap d=0$. Let $\psi\in S_\pi$ be an involution supported by d; then if $c\subseteq d$ is such that $\psi c\ne c$,

$$\phi\psi c\ne\phi c=\psi\phi c,$$

and $\phi\psi\ne\psi\phi$ so $\phi\notin W_\pi$. **Q**

(h) We can now return to consider the isomorphism $q:G\to H$. If $\pi\in G$ is an involution, then $q(\pi)\in H$ is an involution, and it is easy to check that

$$q[C_\pi]=C_{q(\pi)},$$

$$q[U_\pi]=U_{q(\pi)},$$

$$q[V_\pi]=V_{q(\pi)},$$

$$q[S_\pi]=S_{q(\pi)},$$

$$q[W_\pi]=W_{q(\pi)},$$

defining $C_{q(\pi)},\dots,W_{q(\pi)}\subseteq H$ as in (a) above. So we see that, for any $\phi\in G$,

$$\begin{aligned}\operatorname{supp}\phi\subseteq\operatorname{supp}\pi&\iff\phi\in W_\pi\iff q(\phi)\in W_{q(\pi)}\\&\iff\operatorname{supp}q(\phi)\subseteq\operatorname{supp}q(\pi).\end{aligned}$$

(i) Define $\theta:\mathfrak{A}\to\mathfrak{B}$ by writing

$$\theta a=\sup\{\operatorname{supp}q(\pi):\pi\in G\text{ is an involution and }\operatorname{supp}\pi\subseteq a\}$$

for every $a\in\mathfrak{A}$. Evidently θ is order-preserving. Now if $a\in\mathfrak{A}$, $\pi\in G$ is an involution and $\operatorname{supp}\pi\not\subseteq a$, $\operatorname{supp}q(\pi)\not\subseteq\theta a$. **P** There is a $\phi\in G$, of order 4, supported by $\operatorname{supp}\pi\setminus a$. Now ϕ^2 is an involution supported by $\operatorname{supp}\pi$, so $\operatorname{supp}q(\phi^2)\subseteq\operatorname{supp}q(\pi)$. On the other hand, if $\pi'\in G$ is an involution supported by a, then a supports every member of $U_{\pi'}$, by (d-i), so $\phi\in V_{\pi'}$, $q(\phi)\in V_{q(\pi')}$ and $\operatorname{supp}q(\phi^2)=\operatorname{supp}q(\phi)^2$ is disjoint from $\operatorname{supp}q(\pi')$, by (e-iii). As π' is arbitrary, $\operatorname{supp}q(\phi^2)\cap\theta a=0$; so

$$\operatorname{supp}q(\pi)\setminus\theta a\supseteq\operatorname{supp}q(\phi^2)\ne0.\ \mathbf{Q}$$

(j) In the same way, we can define $\theta^*:\mathfrak{B}\to\mathfrak{A}$ by setting

$$\theta^*b=\sup\{\operatorname{supp}q^{-1}(\tilde\pi):\tilde\pi\in H\text{ is an involution and }\operatorname{supp}\tilde\pi\subseteq b\}$$

for every $b\in\mathfrak{B}$. Now $\theta^*\theta a=a$ for every $a\in\mathfrak{A}$. **P** (α) If $0\ne u\subseteq a$, there is an involution $\pi\in G$ supported by u. Now $q(\pi)$ is an involution in H supported by θa, so

$$u\cap\theta^*\theta a\supseteq u\cap\operatorname{supp}q^{-1}q(\pi)=\operatorname{supp}\pi\ne0.$$

As u is arbitrary, $a\subseteq\theta^*\theta a$. ($\beta$) If $\tilde\pi\in H$ is an involution supported by θa, then $\phi=q^{-1}(\tilde\pi)$ is an involution in G with $\operatorname{supp}q(\phi)=\operatorname{supp}\tilde\pi\subseteq\theta a$, so $\operatorname{supp}\phi\subseteq a$, by (i) above; as $\tilde\pi$ is arbitrary, $\theta^*\theta a\subseteq a$. **Q**

Similarly, $\theta\theta^*b=b$ for every $b\in\mathfrak{B}$. But this means that θ and θ^* are the two halves of an order-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. By 312M, both are Boolean homomorphisms.

(k) If $\pi\in G$ is an involution, then $\theta(\operatorname{supp}\pi)=\operatorname{supp}q(\pi)$. **P** By the definition of θ, $\operatorname{supp}q(\pi)\subseteq\theta(\operatorname{supp}\pi)$. On the other hand,

$$\operatorname{supp}q(\pi)=\theta\theta^*(\operatorname{supp}q(\pi))\supseteq\theta(\operatorname{supp}q^{-1}q(\pi))=\theta(\operatorname{supp}\pi).\ \mathbf{Q}$$

Similarly, if $\tilde\pi \in H$ is an involution, $\theta^{-1}(\operatorname{supp}\tilde\pi) = \theta^*(\operatorname{supp}\tilde\pi) = \operatorname{supp} q^{-1}(\tilde\pi)$.

(l) We are nearly home. Let us confirm that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in G$. **P?** Otherwise, $\psi = q(\phi)^{-1}\theta\phi\theta^{-1}$ is not the identity automorphism on $\mathfrak{B}$, and there is a non-zero $b \in \mathfrak{B}$ such that $\psi b \cap b = 0$, that is, $\theta\phi\theta^{-1}b \cap q(\phi)b = 0$. Let $\tilde\pi \in H$ be an involution supported by b. Then $q^{-1}(\tilde\pi)$ is supported by $\theta^{-1}b$, by (j), so $\phi\theta^{-1}b$ supports $\phi q^{-1}(\tilde\pi)\phi^{-1}$ and $\theta\phi\theta^{-1}b$ supports $q(\phi q^{-1}(\tilde\pi)\phi^{-1}) = q(\phi)\tilde\pi q(\phi)^{-1}$. On the other hand, $q(\phi)b$ also supports $q(\phi)\tilde\pi q(\phi)^{-1}$, which is not the identity automorphism; so these two elements of $\mathfrak{B}$ cannot be disjoint. **XQ**

(m) Finally, θ is unique by 384C.

Remark The ideas of the proof here are taken from Eigen 82.

384E The rest of this section may be regarded as a series of corollaries of this theorem. But I think it will be apparent that they are very substantial results.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be atomless homogeneous Boolean algebras, and $q : \operatorname{Aut}\mathfrak{A} \to \operatorname{Aut}\mathfrak{B}$ an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof (a) Let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (314U). Then every $\phi \in \operatorname{Aut}\mathfrak{A}$ has a unique extension to a Boolean homomorphism $\hat\phi : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{A}}$ (314Tb). Because the extension is unique, we must have $(\phi\psi)\hat{} = \hat\phi\hat\psi$ for all ϕ, $\psi \in \operatorname{Aut}\mathfrak{A}$; consequently, $\hat\phi$ and $\widehat{\phi^{-1}}$ are inverses of each other, and $\hat\phi \in \operatorname{Aut}\widehat{\mathfrak{A}}$ for each $\phi \in \operatorname{Aut}\mathfrak{A}$; moreover, $\phi \mapsto \hat\phi$ is a group homomorphism. Of course it is injective, so we have a subgroup $G = \{\hat\phi : \phi \in \operatorname{Aut}\mathfrak{A}\}$ of $\operatorname{Aut}\widehat{\mathfrak{A}}$ which is isomorphic to $\operatorname{Aut}\mathfrak{A}$. Clearly

$$G = \{\phi : \phi \in \operatorname{Aut}\widehat{\mathfrak{A}},\ \phi u \in \mathfrak{A} \text{ for every } u \in \mathfrak{A}\}.$$

If $a \in \widehat{\mathfrak{A}}$ is non-zero, then there is a non-zero $u \subseteq a$ belonging to $\mathfrak{A}$. Because $\mathfrak{A}$ is atomless and homogeneous, there is an involution $\pi \in \operatorname{Aut}\mathfrak{A}$ supported by u (382P); now $\hat\pi \in G$ is an involution supported by a. As a is arbitrary, G has many involutions.

Similarly, writing $\widehat{\mathfrak{B}}$ for the Dedekind completion of $\mathfrak{B}$, we have a subgroup $H = \{\hat\psi : \psi \in \operatorname{Aut}\mathfrak{B}\}$ of $\operatorname{Aut}\widehat{\mathfrak{B}}$ isomorphic to $\operatorname{Aut}\mathfrak{B}$, and with many involutions. Let $\hat q : G \to H$ be the corresponding isomorphism, so that $\hat q(\hat\phi) = \widehat{q(\phi)}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

By 384D, there is a Boolean isomorphism $\hat\theta : \widehat{\mathfrak{A}} \to \widehat{\mathfrak{B}}$ such that $\hat q(\phi) = \hat\theta\phi\hat\theta^{-1}$ for every $\phi \in G$. Note that

$$\hat\theta(\operatorname{supp}\phi) = \operatorname{supp}(\hat\theta\phi\hat\theta^{-1}) = \operatorname{supp}\hat q(\phi)$$

for every $\phi \in G$, so that $\hat\theta(\operatorname{supp}\hat q^{-1}(\pi)) = \operatorname{supp}\pi$ for every $\pi \in H$.

(b) If $u \in \mathfrak{A}$, then $\hat\theta u \in \mathfrak{B}$. **P** It is enough to consider the case $u \notin \{0,1\}$, since surely $\hat\theta 0 = 0$ and $\hat\theta 1 = 1$. Take any $w \in \mathfrak{B}$ which is neither 0 nor 1; then there is an involution in $\operatorname{Aut}\mathfrak{B}$ with support w (382P again); the corresponding member π of H is still an involution with support w. Its image $\hat q^{-1}(\pi)$ in G is an involution with support $a = \hat\theta^{-1}w \in \widehat{\mathfrak{A}}$; of course $0 \ne a \ne 1$. Take non-zero u_1, $u_3 \in \mathfrak{A}$ such that $u_1 \subset a$ and $u_3 \subseteq 1 \setminus a$; set $u_2 = 1 \setminus (u_1 \cup u_3)$. Because $\mathfrak{A}$ is homogeneous, there are ϕ, $\psi \in G$ such that $\phi u_1 = u$, $\psi u_1 = u_1$, $\psi u_2 = u_3$; set $\phi_2 = \phi\psi$. Then we have

$$u = \phi u_1 \subseteq \phi(\operatorname{supp}\hat q^{-1}(\pi)) = \operatorname{supp}(\phi\hat q^{-1}(\pi)\phi^{-1}) \subseteq \phi(u_1 \cup u_2) = u \cup \phi u_2,$$

$$u = \phi_2 u_1 \subseteq \phi_2(\operatorname{supp}\hat q^{-1}(\pi)) = \operatorname{supp}(\phi_2\hat q^{-1}(\pi)\phi_2^{-1}) \subseteq u \cup \phi_2 u_2 = u \cup \phi u_3,$$

so

$$\phi(\operatorname{supp}\hat q^{-1}(\pi)) \cap \phi_2(\operatorname{supp}\hat q^{-1}(\pi)) = u,$$

and

$$\begin{aligned}\hat{\theta}u &= \hat{\theta}(\phi(\operatorname{supp}\hat{q}^{-1}(\pi))) \cap \hat{\theta}(\phi_2(\operatorname{supp}\hat{q}^{-1}(\pi)))\\ &= \hat{\theta}(\operatorname{supp}\phi\hat{q}^{-1}(\pi)\phi^{-1}) \cap \hat{\theta}(\operatorname{supp}\phi_2\hat{q}^{-1}(\pi)\phi_2^{-1})\\ &= \hat{\theta}(\operatorname{supp}\hat{q}^{-1}(\hat{q}(\phi)\pi\hat{q}(\phi)^{-1})) \cap \hat{\theta}(\operatorname{supp}\hat{q}^{-1}(\hat{q}(\phi_2)\pi\hat{q}(\phi_2)^{-1}))\\ &= \operatorname{supp}(\hat{q}(\phi)\pi\hat{q}(\phi)^{-1}) \cap \operatorname{supp}(\hat{q}(\phi_2)\pi\hat{q}(\phi_2)^{-1})\end{aligned}$$

(see the last sentence of (a) above)

$$= \hat{q}(\phi)(\operatorname{supp}\pi) \cap \hat{q}(\phi_2)(\operatorname{supp}\pi) = \hat{q}(\phi)w \cap \hat{q}(\phi_2)w \in \mathfrak{B}$$

because both $\hat{q}(\phi)$ and $\hat{q}(\phi_2)$ belong to H. **Q**

Similarly, $\hat{\theta}^{-1}v \in \mathfrak{A}$ for every $v \in \mathfrak{B}$, and $\theta = \hat{\theta}\restriction\mathfrak{A}$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

We now have

$$q(\phi) = \hat{q}(\hat{\phi})\restriction\mathfrak{B} = (\hat{\theta}\hat{\phi}\hat{\theta}^{-1})\restriction\mathfrak{B} = \theta\phi\theta^{-1}$$

for every $\phi \in \operatorname{Aut}\mathfrak{A}$. Finally, θ is unique by 384C, as before.

384F Corollary If $\mathfrak{A}$ and $\mathfrak{B}$ are atomless homogeneous Boolean algebras with isomorphic automorphism groups, they are isomorphic as Boolean algebras.

Remark Of course a one-element Boolean algebra $\{0\}$ and a two-element Boolean algebra $\{0,1\}$ have isomorphic automorphism groups without being isomorphic.

384G Corollary If $\mathfrak{A}$ is a homogeneous Boolean algebra, then $\operatorname{Aut}\mathfrak{A}$ has no outer automorphisms.

proof If $\mathfrak{A} = \{0,1\}$ this is trivial. Otherwise, $\mathfrak{A}$ is atomless, so if q is any automorphism of $\operatorname{Aut}\mathfrak{A}$, there is a Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{A}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$, and q is an inner automorphism.

384H Definitions Complementary to the notion of 'many involutions' is the following concept.

(a) A Boolean algebra $\mathfrak{A}$ is **rigid** if the only automorphism of $\mathfrak{A}$ is the identity automorphism.

(b) A Boolean algebra $\mathfrak{A}$ is **nowhere rigid** if no non-trivial principal ideal of $\mathfrak{A}$ is rigid.

384I Lemma Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is nowhere rigid;

(ii) for every $a \in \mathfrak{A} \setminus \{0\}$ there is a $\phi \in \operatorname{Aut}\mathfrak{A}$, not the identity, supported by a;

(iii) for every $a \in \mathfrak{A} \setminus \{0\}$ there are distinct b, $c \subseteq a$ such that the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_c$ they generate are isomorphic;

(iv) the automorphism group $\operatorname{Aut}\mathfrak{A}$ has many involutions.

proof (a)(ii)⇒(i) If $a \in \mathfrak{A} \setminus \{0\}$, let $\phi \in \operatorname{Aut}\mathfrak{A}$ be a non-trivial automophism supported by a; then $\phi\restriction\mathfrak{A}_a$ is a non-trivial automorphism of the principal ideal $\mathfrak{A}_a$, so $\mathfrak{A}_a$ is not rigid.

(b)(i)⇒(iii) There is a non-trivial automorphism ψ of $\mathfrak{A}_a$; now if $b \in \mathfrak{A}_a$ is such that $\psi b = c \ne b$, $\mathfrak{A}_b$ is isomorphic to $\psi[\mathfrak{A}_b] = \mathfrak{A}_c$.

(c)(iii)⇒(iv) Take any non-zero $a \in \mathfrak{A}$. By (iii), there are distinct b, $c \subseteq a$ such that $\mathfrak{A}_b$, $\mathfrak{A}_c$ are isomorphic. At least one of $b \setminus c$, $c \setminus b$ is non-zero; suppose the former. Let $\psi : \mathfrak{A}_b \to \mathfrak{A}_c$ be an isomorphism, and set $d = b \setminus c$, $d' = \psi(b \setminus c)$; then $d' \subseteq c$, so $d' \cap d = 0$, and $\phi = (\overleftarrow{d_{\psi} d'})$ is an involution supported by a.

(d)(iv)⇒(ii) is trivial.

384J Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be nowhere rigid Dedekind complete Boolean algebras and $q : \operatorname{Aut}\mathfrak{A} \to \operatorname{Aut}\mathfrak{B}$ an isomorphism. Then there is a unique Boolean isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \operatorname{Aut}\mathfrak{A}$.

proof Put 384I(i)⇒(iv) and 384D together.

384K Corollary Let $\mathfrak{A}$ be a nowhere rigid Dedekind complete Boolean algebra. Then $\operatorname{Aut}\mathfrak{A}$ has no outer automorphisms.

384L Examples I note the following examples of nowhere rigid algebras.

(a) A non-trivial homogeneous Boolean algebra is nowhere rigid.
(b) Any principal ideal of a nowhere rigid Boolean algebra is nowhere rigid.
(c) A simple product of nowhere rigid Boolean algebras is nowhere rigid.
(d) Any atomless semi-finite measure algebra is nowhere rigid.
(e) A free product of nowhere rigid Boolean algebras is nowhere rigid.
(f) The Dedekind completion of a nowhere rigid Boolean algebra is nowhere rigid.

Indeed, the difficulty is to find an atomless Boolean algebra which is not nowhere rigid; for a variety of constructions of rigid algebras, see Bekkali & Bonnet 89.

384M Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be atomless localizable measure algebras, and $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ the corresponding groups of measure-preserving automorphisms. Let $q:\operatorname{Aut}_{\bar\mu}\mathfrak{A}\to\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ be an isomorphism. Then there is a unique Boolean isomorphism $\theta:\mathfrak{A}\to\mathfrak{B}$ such that $q(\phi)=\theta\phi\theta^{-1}$ for every $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$.

proof The point is just that $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ has many involutions. **P** Let $a\in\mathfrak{A}\setminus\{0\}$. Then there is a non-zero $b\subseteq a$ such that the principal ideal $\mathfrak{A}_b$ is Maharam-type-homogeneous. Take $c\subseteq b$ and $d\subseteq b\setminus c$ such that $\bar\mu c=\bar\mu d=\min(1,\frac{1}{2}\bar\mu b)$ (331C). The principal ideals $\mathfrak{A}_c$, $\mathfrak{A}_d$ are now isomorphic as measure algebras (331I); let $\psi:\mathfrak{A}_c\to\mathfrak{A}_d$ be a measure-preserving isomorphism. Then $(\overleftarrow{c\,{}_\psi\,d})\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is an involution supported by a. **Q**

Similarly, $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ has many involutions, and the result follows at once from 384D.

384N To make proper use of the last theorem we need the following result.

Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras and $\theta:\mathfrak{A}\to\mathfrak{B}$ a Boolean isomorphism. For each infinite cardinal κ let e_κ be the Maharam-type-κ component of $\mathfrak{A}$ (332Gb) and for each $\gamma\in\,]0,\infty[$ let A_γ be the set of atoms of $\mathfrak{A}$ of measure γ. Then the following are equiveridical:

(i) for every $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, $\theta\phi\theta^{-1}\in\operatorname{Aut}_{\bar\nu}\mathfrak{B}$;

(ii)(α) for every infinite cardinal κ there is an $\alpha_\kappa>0$ such that $\bar\nu(\theta a)=\alpha_\kappa\bar\mu a$ for every $a\subseteq e_\kappa$,

(β) for every $\gamma\in\,]0,\infty[$ there is an $\alpha_\gamma>0$ such that $\bar\nu(\theta a)=\alpha_\gamma\bar\mu a$ for every $a\in A_\gamma$.

proof (a)(i)$\Rightarrow$(ii)(α) Let κ be an infinite cardinal. The point is that if a, $a'\subseteq e_\kappa$ and $\bar\mu a=\bar\mu a'<\infty$ then $\bar\nu(\theta a)=\bar\nu(\theta a')$. **P** The principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_{a'}$ are isomorphic as measure algebras; moreover, by 332J, the principal ideals $\mathfrak{A}_{e_\kappa\setminus a}$, $\mathfrak{A}_{e_\kappa\setminus a'}$ are isomorphic. We therefore have a $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ such that $\phi a=a'$. Consequently $\psi\theta a=\theta a'$, where $\psi=\theta\phi\theta^{-1}\in\operatorname{Aut}_{\bar\nu}\mathfrak{B}$, and $\bar\nu(\theta a)=\bar\nu(\theta a')$. **Q**

If $e_\kappa=0$ we can take $\alpha_\kappa=1$. Otherwise fix on some $c_0\subseteq e_\kappa$ such that $0<\bar\mu c_0<\infty$; take $b\subseteq\theta c_0$ such that $0<\bar\nu b<\infty$, and set $c=\theta^{-1}b$, $\alpha_\kappa=\bar\nu b/\bar\mu c$. Then we shall have $\bar\nu(\theta a)=\bar\nu(\theta c)=\alpha_\kappa\bar\mu a$ whenever $a\subseteq e_\kappa$ and $\bar\mu a=\bar\mu c$. But we can find for any $n\ge 1$ a partition $c_{n1},\dots,c_{nn}$ of c into elements of measure $\frac{1}{n}\bar\mu c$; since $\bar\nu(\theta c_{ni})=\bar\nu(\theta c_{nj})$ for all i, $j\le n$, we must have $\bar\nu(\theta c_{ni})=\frac{1}{n}\bar\nu(\theta c)=\alpha_\kappa\bar\mu c_{ni}$ for all i. So if $a\subseteq e_\kappa$ and $\bar\mu a=\frac{1}{n}\bar\mu c$, $\bar\nu(\theta a)=\bar\nu(\theta c_{n1})=\alpha_\kappa\bar\mu a$. Now suppose that $a\subseteq e_\kappa$ and $\bar\mu a=\frac{k}{n}\bar\mu c$ for some k, $n\ge 1$; then a can be partitioned into k elements of measure $\frac{1}{n}\bar\mu c$, so in this case also $\bar\nu(\theta a)=\alpha_\kappa\bar\mu a$. Finally, for any $a\subseteq e_\kappa$, set

$$D=\{d:d\subseteq a,\ \bar\mu d\text{ is a rational multiple of }\bar\mu c\},$$

and let $D'\subseteq D$ be a maximal upwards-directed set. Then $\sup D'=a$, so $\theta[D']$ is an upwards-directed set with supremum θa, and

$$\bar\nu(\theta a)=\sup_{d\in D'}\bar\nu(\theta d)=\sup_{d\in D'}\alpha_\kappa\bar\mu d=\alpha_\kappa\bar\mu a.$$

(β) Let $\gamma\in\,]0,\infty[$. If $A_\gamma=\emptyset$ take $\alpha_\gamma=1$. Otherwise, fix on any $c\in A_\gamma$ and set $\alpha_\gamma=\bar\nu(\theta c)/\gamma$. If $a\in A_\gamma$ then there is a $\phi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ exchanging the atoms a and c, so that $\theta\phi\theta^{-1}\in\operatorname{Aut}_{\bar\nu}\mathfrak{B}$ exchanges the atoms θa and θc, and

$$\bar{\nu}(\theta a) = \bar{\nu}(\theta c) = \alpha_\gamma \bar{\mu} a.$$

(b)(ii)⇒(i) Now suppose that the conditions (α) and (β) are satisfied, that $\phi \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ and that $a \in \mathfrak{A}$. For each infinite cardinal κ, we have $\phi e_\kappa = e_\kappa$, so

$$\bar{\nu}(\theta\phi(e_\kappa \cap a)) = \alpha_\kappa \bar{\mu}(\phi(e_\kappa \cap a)) = \alpha_\kappa \bar{\mu}(e_\kappa \cap a) = \bar{\nu}(\theta(e_\kappa \cap a)).$$

Similarly, if we write $a_\gamma = \sup A_\gamma$, then for each $\gamma \in \,]0, \infty[$ we have $\phi[A_\gamma] = A_\gamma$ and $\phi a_\gamma = a_\gamma$, and for $c \subseteq a_\gamma$ we have

$$\bar{\mu} c = \gamma \#(\{e : e \in A_\gamma,\, e \subseteq c\});$$

so

$$\begin{aligned} \bar{\nu}(\theta\phi(a_\gamma \cap a)) &= \alpha_\gamma \gamma \#(\{e : e \in A_\gamma,\, e \subseteq \phi a\}) \\ &= \alpha_\gamma \gamma \#(\{e : e \in A_\gamma,\, e \subseteq a\}) \\ &= \sum_{e \in A_\gamma, e \subseteq a} \bar{\nu}(\theta e) = \bar{\nu}(\theta(a_\gamma \cap a)). \end{aligned}$$

Putting these together,

$$\begin{aligned} \bar{\nu}(\theta\phi a) &= \sum_{\kappa \text{ is an infinite cardinal}} \bar{\nu}(\theta\phi(e_\kappa \cap a)) + \sum_{\gamma \in]0,\infty[} \bar{\nu}(\theta\phi(a_\gamma \cap a)) \\ &= \sum_{\kappa \text{ is an infinite cardinal}} \bar{\nu}(\theta(e_\kappa \cap a)) + \sum_{\gamma \in]0,\infty[} \bar{\nu}(\theta(a_\gamma \cap a)) = \bar{\nu}(\theta a). \end{aligned}$$

But this means that

$$\bar{\nu}(\theta\phi\theta^{-1} b) = \bar{\nu}(\theta\theta^{-1} b) = \bar{\nu} b$$

for every $b \in \mathfrak{B}$, and $\theta\phi\theta^{-1}$ is measure-preserving, as required by (i).

384O Corollary If $(\mathfrak{A}, \bar{\mu})$ is an atomless totally finite measure algebra, $\mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ has no outer automorphisms.

proof Let $q : \mathrm{Aut}_{\bar{\mu}} \mathfrak{A} \to \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ be any automorphism. By 384M, there is a corresponding $\theta \in \mathrm{Aut}\, \mathfrak{A}$ such that $q(\phi) = \theta\phi\theta^{-1}$ for every $\phi \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$. By 384N, there is for each infinite cardinal κ an $\alpha_\kappa > 0$ such that $\bar{\mu}(\theta a) = \alpha_\kappa \bar{\mu} a$ whenever $a \subseteq e_\kappa$, the Maharam-type-κ component of $\mathfrak{A}$. But since $\theta e_\kappa = e_\kappa$ and $\bar{\mu} e_\kappa < \infty$ for every κ, we must have $\alpha_\kappa = 1$ whenever $e_\kappa \neq 0$; as $\mathfrak{A}$ is atomless,

$$\begin{aligned} \bar{\mu}(\theta a) &= \sum_{\kappa \text{ is an infinite cardinal}} \bar{\mu}(\theta(a \cap e_\kappa)) \\ &= \sum_{\kappa \text{ is an infinite cardinal}} \alpha_\kappa \bar{\mu}(a \cap e_\kappa) \\ &= \sum_{\kappa \text{ is an infinite cardinal}} \bar{\mu}(a \cap e_\kappa) = \bar{\mu} a \end{aligned}$$

for every $a \in \mathfrak{A}$. Thus $\theta \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ and q is an inner automorphism.

384P The results above are satisfying and complete in their own terms, but leave open a number of obvious questions concerning whether some of the hypotheses can be relaxed. Atoms can produce a variety of complications (see 384Ya-384Yd below). To show that we really do need to assume that our algebras are Dedekind complete or localizable, I offer the following.

Examples (a) There are an atomless localizable measure algebra $(\mathfrak{A}, \bar{\mu})$ and an atomless semi-finite measure algebra $(\mathfrak{B}, \bar{\nu})$ such that $\mathrm{Aut}\, \mathfrak{A} \cong \mathrm{Aut}\, \mathfrak{B}$, $\mathrm{Aut}_{\bar{\mu}} \mathfrak{A} \cong \mathrm{Aut}_{\bar{\nu}} \mathfrak{B}$ but $\mathfrak{A}$ and $\mathfrak{B}$ are not isomorphic.

proof Let $(\mathfrak{A}_0,\bar\mu_0)$ be an atomless homogeneous probability algebra; for instance, the measure algebra of Lebesgue measure on the unit interval. Let $(\mathfrak{A},\bar\mu)$ be the simple product measure algebra $(\mathfrak{A}_0,\bar\mu_0)^{\omega_1}$ (322L); then $(\mathfrak{A},\bar\mu)$ is an atomless localizable measure algebra. In $\mathfrak{A}$ let I be the set

$$\{a : a\in\mathfrak{A} \text{ and the principal ideal } \mathfrak{A}_a \text{ is ccc}\};$$

then I is an ideal of $\mathfrak{A}$, the σ-ideal generated by the elements of finite measure (cf. 322G). Set

$$\mathfrak{B}=\{a : a\in\mathfrak{A}, \text{ either } a\in I \text{ or } 1\setminus a\in I\}.$$

Then $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, so if we set $\bar\nu=\bar\mu\restriction\mathfrak{B}$ then $(\mathfrak{B},\bar\nu)$ is a measure algebra in its own right.

The definition of I makes it plain that it is invariant under all Boolean automorphisms of $\mathfrak{A}$; so $\mathfrak{B}$ also is invariant under all automorphisms, and we have a homomorphism $\phi\mapsto q(\phi)=\phi\restriction\mathfrak{B}:\operatorname{Aut}\mathfrak{A}\to\operatorname{Aut}\mathfrak{B}$. On the other hand, because $\mathfrak{B}$ is order-dense in $\mathfrak{A}$, and $\mathfrak{A}$ is Dedekind complete, every automorphism of $\mathfrak{B}$ can be extended to an automorphism of $\mathfrak{A}$ (see part (a) of the proof of 384E). So q is actually an isomorphism between $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$. Moreover, still because $\mathfrak{B}$ is order-dense, $q(\phi)$ is measure-preserving iff ϕ is measure-preserving, so $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ is isomorphic to $\operatorname{Aut}_{\bar\nu}\mathfrak{B}$. But of course there is no Boolean isomorphism, let alone a measure algebra isomorphism, between $\mathfrak{A}$ and $\mathfrak{B}$, because $\mathfrak{A}$ is Dedekind complete while $\mathfrak{B}$ is not.

Remark Thus the hypothesis 'Dedekind complete' in 384D and 384J (and 'localizable' in 384M), and the hypothesis 'homogeneous' in 384E-384F, are essential.

(b) There is an atomless semi-finite measure algebra $(\mathfrak{C},\bar\lambda)$ such that $\operatorname{Aut}\mathfrak{C}$ has an outer automorphism.

proof In fact we can take $\mathfrak{C}$ to be the simple product of $\mathfrak{A}$ and $\mathfrak{B}$ above. I claim that the isomorphism between $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ gives rise to an outer automorphism of $\operatorname{Aut}\mathfrak{C}$; this seems very natural, but I think there is a fair bit to check, so I take the argument in easy stages.

(i) We may identify the Dedekind completion of $\mathfrak{C}=\mathfrak{A}\times\mathfrak{B}$ with $\mathfrak{A}\times\mathfrak{A}$. For $\phi\in\operatorname{Aut}\mathfrak{C}$, we have a corresponding $\hat\phi\in\operatorname{Aut}(\mathfrak{A}\times\mathfrak{A})$. Now $\mathfrak{B}\times\mathfrak{A}$ is invariant under $\hat\phi$. **P** Consider first $\phi(0,1)=(a_1,b_1)\in\mathfrak{C}$. The corresponding principal ideal $\mathfrak{C}_{(a_1,b_1)}\cong\mathfrak{A}_{a_1}\times\mathfrak{B}_{b_1}$ of $\mathfrak{C}$ must be isomorphic to the principal ideal $\mathfrak{C}_{(0,1)}\cong\mathfrak{B}$; so that if $(a,b)\in\mathfrak{C}$ and $(a,b)\subseteq(a_1,b_1)$, then just one of the principal ideals $\mathfrak{C}_{(a,b)}\cong\mathfrak{A}_a\times\mathfrak{B}_b$, $\mathfrak{C}_{(a_1\setminus a,b_1\setminus b)}\cong\mathfrak{A}_{a_1\setminus a}\times\mathfrak{B}_{b_1\setminus b}$ is ccc. But this can only happen if $\mathfrak{A}_{a_1}$ is ccc and $\mathfrak{B}_{b_1}$ is not; that is, if a_1 and $1\setminus b_1$ belong to I. Consequently $\hat\phi(0,a)\subseteq(a_1,b_1)$ belongs to $\mathfrak{B}\times\mathfrak{A}$ for every $a\in\mathfrak{A}$. We also find that

$$\phi(1,0)=(1,1)\setminus\phi(0,1)=(1\setminus a_1,1\setminus b_1)\in\mathfrak{B}\times\mathfrak{A}.$$

Now if $b\in I$, then

$$\mathfrak{C}_{\phi(b,0)}\cong\mathfrak{C}_{(b,0)}\cong\mathfrak{A}_b$$

is ccc and

$$\phi(b,0)\in I\times I\subseteq\mathfrak{B}\times\mathfrak{A};$$

while

$$\phi(1\setminus b,0)=(1\setminus a_1,1\setminus b_1)\setminus\phi(b,0)\in\mathfrak{B}\times\mathfrak{A}.$$

This shows that $\phi(b,0)\in\mathfrak{B}\times\mathfrak{A}$ for every $b\in\mathfrak{B}$. So

$$\hat\phi(b,a)=\hat\phi(b,0)\cup\hat\phi(0,a)\in\mathfrak{B}\times\mathfrak{A}$$

for every $b\in\mathfrak{B}$ and $a\in\mathfrak{A}$. **Q**

(ii) Let $\theta:\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}\times\mathfrak{A}$ be the involution defined by setting $\theta(a,b)=(b,a)$ for all a, $b\in\mathfrak{A}$. Take $\phi\in\operatorname{Aut}\mathfrak{C}$ and consider $\psi=\theta\hat\phi\theta^{-1}\in\operatorname{Aut}(\mathfrak{A}\times\mathfrak{A})$. If $c=(a,b)\in\mathfrak{C}$, then $\theta^{-1}c=(b,a)\in\mathfrak{B}\times\mathfrak{A}$, so $\hat\phi\theta^{-1}c\in\mathfrak{B}\times\mathfrak{A}$, by (i), and $\psi c\in\mathfrak{A}\times\mathfrak{B}=\mathfrak{C}$. This shows that $\psi\restriction\mathfrak{C}$ is a homomorphism from $\mathfrak{C}$ to itself. Of course $\psi^{-1}=\theta\hat\phi^{-1}\theta^{-1}$ has the same property. So we have a map $q:\operatorname{Aut}\mathfrak{C}\to\operatorname{Aut}\mathfrak{C}$ given by setting

$$q(\phi)=\theta\hat\phi\theta^{-1}\restriction\mathfrak{C}$$

for $\phi\in\operatorname{Aut}\mathfrak{C}$. Evidently q is an automorphism.

(iii) ? Suppose, if possible, that q were an inner automorphism. Let $\chi\in\operatorname{Aut}\mathfrak{C}$ be such that $q(\phi)=\chi\phi\chi^{-1}$ for every $\phi\in\operatorname{Aut}\mathfrak{C}$. Then

$$\hat{\chi}\hat{\phi}\hat{\chi}^{-1} = \widehat{q(\phi)} = \theta\hat{\phi}\theta^{-1}$$

for every $\phi \in \operatorname{Aut}\mathfrak{C}$. Since $G = \{\hat{\phi} : \phi \in \operatorname{Aut}\mathfrak{C}\}$ is a subgroup of $\operatorname{Aut}(\mathfrak{A} \times \mathfrak{A})$ with many involutions, the 'uniqueness' assertion of 384D tells us that $\hat{\chi} = \theta$. But

$$\theta[\mathfrak{C}] = \mathfrak{B} \times \mathfrak{A} \neq \mathfrak{C} = \chi[\mathfrak{C}] = \hat{\chi}[\mathfrak{C}],$$

so this cannot be. **X**

Thus q is the required outer automorphism of $\operatorname{Aut}\mathfrak{C}$.

Remark Thus the hypothesis 'homogeneous' in 384G, and the hypothesis 'Dedekind complete' in 384K, are necessary.

384Q Example Let μ be Lebesgue measure on $\mathbb{R}$, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Then $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ has an outer automorphism. **P** Set $f(x) = 2x$ for $x \in \mathbb{R}$. Then $E \mapsto f^{-1}[E] = \frac{1}{2}E$ is a Boolean automorphism of the domain Σ of μ, and $\mu(\frac{1}{2}E) = \frac{1}{2}\mu E$ for every $E \in \Sigma$ (263A, or otherwise). So we have a corresponding $\theta \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\theta E^\bullet = (\frac{1}{2}E)^\bullet$ for every $E \in \Sigma$, and $\bar{\mu}(\theta a) = \frac{1}{2}\bar{\mu}a$ for every $a \in \mathfrak{A}$. By 384N, we have an automorphism q of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ defined by setting $q(\phi) = \theta\phi\theta^{-1}$ for every measure-preserving automorphism ϕ. But q is now an outer automorphism of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, because (by 384D) the only possible automorphism of $\mathfrak{A}$ corresponding to q is θ, and θ is not measure-preserving. **Q**

Thus the hypothesis 'totally finite' in 384O cannot be omitted.

384X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is nowhere rigid; (ii) for every $a \in \mathfrak{A} \setminus \{0\}$ and $n \in \mathbb{N}$ there are disjoint non-zero $b_0, \dots, b_n \subseteq a$ such that the principal ideals $\mathfrak{A}_{b_i}$ they generate are all isomorphic; (iii) for every $a \in \mathfrak{A} \setminus \{0\}$ and $n \geq 1$ there is a $\phi \in \operatorname{Aut}\mathfrak{A}$, of order n, supported by a.

(b) Let $\mathfrak{A}$ be an atomless homogeneous Boolean algebra and $\mathfrak{B}$ a nowhere rigid Boolean algebra, and suppose that $\operatorname{Aut}\mathfrak{A}$ is isomorphic to $\operatorname{Aut}\mathfrak{B}$. Show that there is an invariant order-dense subalgebra of $\mathfrak{B}$ which is isomorphic to $\mathfrak{A}$.

(c) Let $\mathfrak{A}$ and $\mathfrak{B}$ be nowhere rigid Boolean algebras. Show that if $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ are isomorphic, then the Dedekind completions $\widehat{\mathfrak{A}}$ and $\widehat{\mathfrak{B}}$ are isomorphic.

(d) Find two non-isomorphic atomless totally finite measure algebras $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ such that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ are isomorphic. (This is easy.)

(e) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be semi-finite measure algebras and $\theta : \mathfrak{A} \to \mathfrak{B}$ a Boolean isomorphism. Show that the following are equiveridical: (i) for every $\phi \in \operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$, $\theta\phi\theta^{-1} \in \operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$; (ii)($\alpha$) for every infinite cardinal κ there is an $\alpha_\kappa > 0$ such that $\bar{\nu}(\theta a) = \alpha_\kappa\bar{\mu}a$ whenever $a \in \mathfrak{A}$ and the principal ideal $\mathfrak{A}_a$ is Maharam-type-homogeneous with Maharam type κ; (β) for every $\gamma \in \mathopen]0, \infty\mathclose[$ there is an $\alpha_\gamma > 0$ such that $\bar{\nu}(\theta a) = \alpha_\gamma\bar{\mu}a$ whenever $a \in \mathfrak{A}$ is an atom of measure γ.

(f) Show that if X is any set such that $\#(X) \neq 6$, the group G of all permutations of X has no outer automorphisms. (*Hint*: show that if $\tau \in G$ is an involution such that not every conjugate of τ commutes with τ, while $\tau\tau'$ and $\tau\tau''$ are conjugate whenever τ', τ'' are conjugates of τ which do not commute with τ, then τ is a transposition.)

(g) Let $q : \operatorname{Aut}\mathfrak{C} \to \operatorname{Aut}\mathfrak{C}$ be the automorphism of 384Pb. Show that $q(\phi)$ is measure-preserving whenever ϕ is measure-preserving, so that $q \restriction \operatorname{Aut}_{\bar{\lambda}}\mathfrak{C}$ is an outer automorphism of $\operatorname{Aut}_{\bar{\lambda}}\mathfrak{C}$.

384Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras such that $\operatorname{Aut}\mathfrak{A} \cong \operatorname{Aut}\mathfrak{B}$. Show that *either* $\mathfrak{A} \cong \mathfrak{B}$ *or* one of $\mathfrak{A}$, $\mathfrak{B}$ has just one atom and the other is atomless.

(b) Let $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras such that $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A} \cong \operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$. Show that *either* $(\mathfrak{A}, \bar{\mu}) \cong (\mathfrak{B}, \bar{\nu})$ *or* there is some $\gamma \in \mathopen]0, \infty\mathclose[$ such that one of $\mathfrak{A}$, $\mathfrak{B}$ has just one atom of measure γ and the other has none *or* there are γ, $\gamma' \in \mathopen]0, \infty\mathclose[$ such that the number of atoms of $\mathfrak{A}$ of measure γ is equal to the number of atoms of $\mathfrak{B}$ of measure γ', but not to the number of atoms of $\mathfrak{A}$ of measure γ'.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Show that there is an outer automorphism of $\operatorname{Aut}\mathfrak{A}$ iff $\mathfrak{A}$ has exactly six atoms.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. For each infinite cardinal κ let e_κ be the Maharam-type-κ component of $\mathfrak{A}$; for each $\gamma \in]0, \infty[$ let A_γ be the set of atoms of $\mathfrak{A}$ of measure γ. Show that there is an outer automorphism of $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ iff

either there is an infinite cardinal κ such that $\bar{\mu}e_\kappa = \infty$

or there are distinct γ, $\delta \in]0, \infty[$ such that $\#(A_\gamma) = \#(A_\delta) \geq 2$

or there is a $\gamma \in]0, \infty[$ such that $\#(A_\gamma) = 6$

or there are γ, $\delta \in]0, \infty[$ such that $\#(A_\gamma) = 2 < \#(A_\delta) < \omega$.

384 Notes and comments Let me recapitulate the results above. If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras, any isomorphism between $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Aut}\mathfrak{B}$ corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ if *either* $\mathfrak{A}$ and $\mathfrak{B}$ are atomless and homogeneous (384E) *or* they are Dedekind complete and nowhere rigid (384J). If $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are atomless localizable measure algebras, then any automorphism between $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ and $\operatorname{Aut}_{\bar{\nu}}\mathfrak{B}$ corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ (384M) which if $\bar{\mu} = \bar{\nu}$ is totally finite will be measure-preserving (384O).

These results may appear a little less surprising if I remark that the elementary Boolean algebras $\mathcal{P}X$ give rise to some of the same phenomena. The automorphism group of $\mathcal{P}X$ can be identified with the group of all permutations of X, and this has no outer automorphisms unless X has just six elements (384Xf). Some of the ideas of the fundamental theorem 384D can be traced through in the purely atomic case also, though of course there are significant changes to be made, and some serious complications arise, of which the most striking surround the remarkable fact that S_6 *does* have an outer automorphism (Burnside 1911, §162; Rotman 84, Theorem 7.8). I have not attempted to incorporate these into the main results. For localizable measure algebras, where the only rigid parts are atoms, the complications are superable, and I think I have listed them all (384Ya-384Yd).

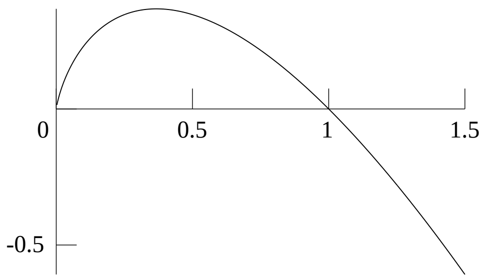
Version of 14.1.15

385 Entropy

Perhaps the most glaring problem associated with the theory of measure-preserving homomorphisms and automorphisms is the fact that we have no generally effective method of determining when two homomorphisms are the same, in the sense that two structures $(\mathfrak{A}, \bar{\mu}, \pi)$ and $(\mathfrak{B}, \bar{\nu}, \phi)$ are isomorphic, where $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are measure algebras and $\pi : \mathfrak{A} \to \mathfrak{A}$, $\phi : \mathfrak{B} \to \mathfrak{B}$ are Boolean homomorphisms. Of course the first part of the problem is to decide whether $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are isomorphic; but this is solved (at least for localizable algebras) by Maharam's theorem (see 332J). The difficulty lies in the homomorphisms. Even when we know that $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are both isomorphic to the Lebesgue measure algebra, the extraordinary variety of constructions of homomorphisms – corresponding in part to the variety of measure spaces with such measure algebras, each with its own natural inverse-measure-preserving functions – means that the question of which are isomorphic to each other is continually being raised. In this section I give the most elementary ideas associated with the concept of 'entropy', up to the Kolmogorov-Sinaĭ theorem (385P). This is an invariant which can be attached to any measure-preserving homomorphism on a probability algebra, and therefore provides a useful method for distinguishing non-isomorphic homomorphisms.

The main work of the section deals with homomorphisms on measure algebras, but as many of the most important ones arise from inverse-measure-preserving functions on measure spaces. I comment on the extra problems arising in the isomorphism problem for such functions (385T-385V). I should remark that some of the lemmas will be repeated in stronger forms in the next section.

385A Notation Throughout this section and the next two, I will use the letter q to denote the function from $[0, \infty[$ to $\mathbb{R}$ defined by saying that $q(t) = -t \ln t = t \ln \frac{1}{t}$ if $t > 0$, $q(0) = 0$.



The function q

We shall need the following straightforward facts concerning q.

(a) q is continuous on $[0,\infty[$ and differentiable on $]0,\infty[$; $q'(t) = -1 - \ln t$ and $q''(t) = -\frac{1}{t}$ for $t > 0$. Because q'' is negative, q is concave, that is, $-q$ is convex. q has a unique maximum at $(\frac{1}{e}, \frac{1}{e})$.

(b) If $s \ge 0$ and $t > 0$ then $q'(s+t) \le q'(t)$; consequently

$$q(s+t) = q(s) + \int_0^t q'(s+\tau)d\tau \le q(s) + q(t)$$

for s, $t \ge 0$. It follows that $q(\sum_{i=0}^n s_i) \le \sum_{i=0}^n q(s_i)$ for all $s_0, \dots, s_n \ge 0$ and (because q is continuous) $q(\sum_{i=0}^\infty s_i) \le \sum_{i=0}^\infty q(s_i)$ for every non-negative summable series $\langle s_i\rangle_{i\in\mathbb{N}}$.

(c) If s, $t \ge 0$ then $q(st) = sq(t) + tq(s)$; more generally, if $n \ge 1$ and $s_i \ge 0$ for $i \le n$ then

$$q(\prod_{i=0}^n s_i) = \sum_{j=0}^n q(s_j)\prod_{i\ne j} s_i.$$

(d) The function $t \mapsto q(t) + q(1-t)$ has a unique maximum at $(\frac{1}{2}, \ln 2)$. $(\frac{d}{dt}(q(t) + q(1-t)) = \ln\frac{1-t}{t}.)$ It follows that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|t - \frac{1}{2}| \le \epsilon$ whenever $q(t) + q(1-t) \ge \ln 2 - \delta$.

(e) If $0 \le t \le \frac{1}{2}$, then $q(1-t) \le q(t)$. **P** Set $f(t) = q(t) - q(1-t)$. Then

$$f''(t) = -\frac{1}{t} + \frac{1}{1-t} = \frac{2t-1}{t(1-t)} \le 0$$

for $0 < t \le \frac{1}{2}$, while $f(0) = f(\frac{1}{2}) = 0$, so $f(t) \ge 0$ for $0 \le t \le \frac{1}{2}$. **Q**

(f)(i) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, I will write $\bar q$ for the function from $L^0(\mathfrak{A})^+$ to $L^0(\mathfrak{A})$ defined from q (364H). Note that because $0 \le q(t) \le 1$ for $t \in [0,1]$, $0 \le \bar q(u) \le \chi 1$ if $0 \le u \le \chi 1$.

(ii) By (b), $\bar q(u+v) \le \bar q(u) + \bar q(v)$ for all u, $v \ge 0$ in $L^0(\mathfrak{A})$. (Represent $\mathfrak{A}$ as the measure algebra of a measure space, so that $\bar q(f^\bullet) = (qf)^\bullet$, as in 364Ib.)

(iii) Similarly, if u, $v \in L^0(\mathfrak{A})^+$, then $\bar q(u \times v) = u \times \bar q(v) + v \times \bar q(u)$.

385B Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$, and $P : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{A}, \bar\mu)$ the corresponding conditional expectation operator (365Q). Then $\int \bar q(u) \le q(\int u)$ and $P(\bar q(u)) \le \bar q(Pu)$ for every $u \in L^\infty(\mathfrak{A})^+$.

proof Apply the remarks in 365Qb to $-q$. ($\bar q(u) \in L^\infty \subseteq L^1$ for every $u \in (L^\infty)^+$ because q is bounded on every bounded interval in $[0,\infty[$.)

385C Definition Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra. If A is a partition of unity in $\mathfrak{A}$, its **entropy** is $H(A) = \sum_{a\in A} q(\bar\mu a)$, where q is the function defined in 385A.

Remarks (a) In the definition of 'partition of unity' (311Gc) I allowed 0 to belong to the family. In the present context this is a mild irritant, and when convenient I shall remove 0 from the partitions of unity considered here (as in 385F below). But because $q(0) = 0$, it makes no difference; $H(A) = H(A \setminus \{0\})$ whenever A is a partition of unity. So if you wish you can read 'partition of unity' in this section to mean 'partition of unity not containing 0', if you are willing to make an occasional amendment in a formula. In important cases, in fact, A is of the form $\{a_i : i \in I\}$ or $\{a_i : i \in I\} \setminus \{0\}$, where $\langle a_i\rangle_{i\in I}$ is an indexed partition of unity, with $a_i \cap a_j = 0$ for $i \neq j$, but no restriction in the number of i with $a_i = 0$; in this case, we still have $H(A) = \sum_{i \in I} q(\bar{\mu} a_i)$.

(b) Many authors prefer to use $\log_2$ in place of ln. This makes sense in terms of one of the intuitive approaches to entropy as the 'information' associated with a partition. See Petersen 83, §5.1.

385D Definition Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$. Let $P : L^1(\mathfrak{A}, \bar{\mu}) \to L^1(\mathfrak{A}, \bar{\mu})$ be the conditional expectation operator associated with $\mathfrak{B}$. Then the **conditional entropy of** A **on** $\mathfrak{B}$ is

$$H(A|\mathfrak{B}) = \sum_{a \in A} \int \bar{q}(P\chi a),$$

where $\bar{q}$ is defined as in 385Af.

385E Elementary remarks (a) In the formula

$$\sum_{a \in A} \int \bar{q}(P\chi a),$$

we have $0 \leq P(\chi a) \leq \chi 1$ for every a, so $\bar{q}(P\chi a) \geq 0$ and every term in the sum is non-negative; accordingly $H(A|\mathfrak{B})$ is well-defined in $[0, \infty]$.

(b) $H(A) = H(A|\{0, 1\})$, since if $\mathfrak{B} = \{0, 1\}$ then $P(\chi a) = \bar{\mu} a \chi 1$, so that $\int \bar{q}(P\chi a) = q(\bar{\mu} a)$. If $A \subseteq \mathfrak{B}$, $H(A|\mathfrak{B}) = 0$, since $P(\chi a) = \chi a$, $\bar{q}(P\chi a) = 0$ for every a.

385F Definition If $\mathfrak{A}$ is a Boolean algebra and A, $B \subseteq \mathfrak{A}$ are partitions of unity, I write $A \vee B$ for the partition of unity $\{a \cap b : a \in A,\ b \in B\} \setminus \{0\}$. (See 385Xf.)

385G Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra. Let $A \subseteq \mathfrak{A}$ be a partition of unity.

(a) If B is another partition of unity in $\mathfrak{A}$, then

$$H(A|\mathfrak{B}) \leq H(A \vee B|\mathfrak{B}) \leq H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

(b) If $\mathfrak{B}$ is purely atomic and D is the set of its atoms, then $H(A \vee D) = H(D) + H(A|\mathfrak{B})$.

(c) If $\mathfrak{C} \subseteq \mathfrak{B}$ is a smaller closed subalgebra of $\mathfrak{A}$, then $H(A|\mathfrak{C}) \geq H(A|\mathfrak{B})$. In particular, $H(A) \geq H(A|\mathfrak{B})$.

(d) Suppose that $\langle \mathfrak{B}_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\mathfrak{B} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{B}_n}$. If $H(A) < \infty$ then

$$H(A|\mathfrak{B}) = \lim_{n \to \infty} H(A|\mathfrak{B}_n).$$

In particular, if $A \subseteq \mathfrak{B}$ then $\lim_{n \to \infty} H(A|\mathfrak{B}_n) = 0$.

proof Write P for the conditional expectation operator on $L^1(\mathfrak{A}, \bar{\mu})$ associated with $\mathfrak{B}$.

(a)(i) If B is infinite, enumerate it as $\langle b_j \rangle_{j \in \mathbb{N}}$; if it is finite, enumerate it as $\langle b_j \rangle_{j \leq n}$ and set $b_j = 0$ for $j > n$. For any $a \in A$,

$$\chi a = \sum_{j=0}^{\infty} \chi(a \cap b_j), \quad P(\chi a) = \sum_{j=0}^{\infty} P\chi(a \cap b_j),$$

$$\begin{aligned}\bar{q}(P\chi a) &= \lim_{n \to \infty} \bar{q}(\sum_{j=0}^{n} P\chi(a \cap b_j)) \\ &\leq \lim_{n \to \infty} \sum_{j=0}^{n} \bar{q}(P\chi(a \cap b_j)) = \sum_{j=0}^{\infty} \bar{q}(P\chi(a \cap b_j))\end{aligned}$$

where all the infinite sums are to be regarded as order*-limits of the corresponding finite sums (see §367), and the middle inequality is a consequence of 385A(f-ii). Accordingly

$$H(A\vee B|\mathfrak{B})=\sum_{a\in A,b\in B,a\cap b\neq 0}\int\bar q(P\chi(a\cap b))$$

$$=\sum_{a\in A}\sum_{j=0}^{\infty}\int\bar q(P\chi(a\cap b_j))\ge\sum_{a\in A}\int\bar q(P\chi a)=H(A|\mathfrak{B}).$$

(ii) Suppose for the moment that A and B are both finite. For $a\in\mathfrak{A}$ set $u_a=P(\chi a)$. If a, $b\in\mathfrak{A}$ we have $0\le u_{a\cap b}\le u_b$ in $L^0(\mathfrak{B})$, so we may choose $v_{ab}\in L^0(\mathfrak{B})$ such that $0\le v_{ab}\le\chi 1$ and $u_{a\cap b}=v_{ab}\times u_b$.

For any $b\in B$, $\sum_{a\in A}u_{a\cap b}=u_b$ (because $\sum_{a\in A}\chi(a\cap b)=\chi b$), so $u_b\times\sum_{a\in A}v_{ab}=u_b$. Since $[\![|\bar q(u_b)|>0]\!]\subseteq[\![u_b>0]\!]$, $\bar q(u_b)\times\sum_{a\in A}v_{ab}=\bar q(u_b)$.

For any $a\in A$,

$$\bar q(u_a)=\bar q(\sum_{b\in B}u_{a\cap b})=\bar q(\sum_{b\in B}u_b\times v_{ab})=\bar q(P(\sum_{b\in B}\chi b\times v_{ab}))$$

(because $v_{ab}\in L^0(\mathfrak{B})$ for every b, so $P(\chi b\times v_{ab})=P(\chi b)\times v_{ab}$)

$$\ge P(\bar q(\sum_{b\in B}\chi b\times v_{ab}))$$

(385B)

$$=P(\sum_{b\in B}\chi b\times\bar q(v_{ab}))$$

(because B is disjoint)

$$=\sum_{b\in B}u_b\times\bar q(v_{ab})$$

(because $\bar q(v_{ab})\in L^0(\mathfrak{B})$ for every b).

Putting these together,

$$H(A\vee B|\mathfrak{B})=\sum_{a\in A,b\in B}\int\bar q(u_{a\cap b})=\sum_{a\in A,b\in B}\int\bar q(u_b\times v_{ab})$$

$$=\sum_{a\in A,b\in B}\int u_b\times\bar q(v_{ab})+\sum_{a\in A,b\in B}\int v_{ab}\times\bar q(u_b)$$

(385A(f-iii))

$$\le\sum_{a\in A}\int\bar q(u_a)+\sum_{b\in B}\int\bar q(u_b)=H(A|\mathfrak{B})+H(B|\mathfrak{B}).$$

(iii) For general partitions of unity A and B, take any finite set $C\subseteq A\vee B$. Then $C\subseteq\{a\cap b:a\in A_0,\,b\in B_0\}$ where $A_0\subseteq A$ and $B_0\subseteq B$ are finite. Set

$$A'=A_0\cup\{1\setminus\sup A_0\},\quad B'=B_0\cup\{1\setminus\sup B_0\},$$

so that A' and B' are finite partitions of unity and $C\subseteq A'\vee B'$. Now

$$\sum_{c\in C}\int\bar q(P\chi c)\le\sum_{c\in A'\vee B'}\int\bar q(P\chi c)=H(A'\vee B'|\mathfrak{B})\le H(A'|\mathfrak{B})+H(B'|\mathfrak{B})$$

(by (ii))

$$\le H(A'\vee A|\mathfrak{B})+H(B'\vee B|\mathfrak{B})$$

(by (i))

$$= H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

As C is arbitrary,

$$H(A \vee B|\mathfrak{B}) = \sum_{c \in A \vee B} \int \bar{q}(P\chi c) \le H(A|\mathfrak{B}) + H(B|\mathfrak{B}).$$

(b) Because $\mathfrak{B}$ is purely atomic and D is its set of atoms,

$$P(\chi a) = \sum_{d \in D} \frac{\bar{\mu}(a \cap d)}{\bar{\mu}d} \chi d, \quad \bar{q}(P(\chi a)) = \sum_{d \in D} q(\frac{\bar{\mu}(a \cap d)}{\bar{\mu}d}) \chi d$$

for every $a \in A$,

$$H(A|\mathfrak{B}) = \sum_{a \in A, d \in D} q(\frac{\bar{\mu}(a \cap d)}{\bar{\mu}d}) \bar{\mu}d.$$

Accordingly,

$$H(A \vee D) = \sum_{a \in A, d \in D} q(\bar{\mu}(a \cap d)) = \sum_{a \in A, d \in D} q(\frac{\bar{\mu}(a \cap d)}{\bar{\mu}d}) \bar{\mu}d + \frac{\bar{\mu}(a \cap d)}{\bar{\mu}d} q(\bar{\mu}d)$$

(385Ac)

$$= H(A|\mathfrak{B}) + \sum_{d \in D} q(\bar{\mu}d) = H(A|\mathfrak{B}) + H(D).$$

(c) Write $P_{\mathfrak{C}}$ for the conditional expectation operator corresponding to $\mathfrak{C}$. If $a \in \mathfrak{A}$,

$$\bar{q}(P_{\mathfrak{C}} \chi a) = \bar{q}(P_{\mathfrak{C}} P \chi a) \ge P_{\mathfrak{C}} \bar{q}(P \chi a)$$

by 385B. So

$$H(A|\mathfrak{C}) = \sum_{a \in A} \int \bar{q}(P_{\mathfrak{C}} \chi a) \ge \sum_{a \in A} \int P_{\mathfrak{C}} \bar{q}(P \chi a) = \sum_{a \in A} \int \bar{q}(P \chi a) = H(A|\mathfrak{B}).$$

Taking $\mathfrak{C} = \{0, 1\}$, we get $H(A) \ge H(A|\mathfrak{B})$.

(d) Let P_n be the conditional expectation operator corresponding to $\mathfrak{B}_n$, for each n. Fix $a \in A$. Then $P(\chi a)$ is the order*-limit of $\langle P_n(\chi a) \rangle_{n \in \mathbb{N}}$, by Lévy's martingale theorem (367Jb). Consequently (because q is continuous) $\langle \bar{q}(P_n \chi a) \rangle_{n \in \mathbb{N}}$ is order*-convergent to $\bar{q}(P \chi a)$ for every $a \in A$ (367H). Also, because $0 \le P_n \chi a \le \chi 1$ for every n, $0 \le \bar{q}(P_n \chi a) \le \frac{1}{e} \chi 1$ for every n. By the Dominated Convergence Theorem (367I), $\lim_{n \to \infty} \int \bar{q}(P_n \chi a) = \int \bar{q}(P \chi a)$.

By 385B, we also have

$$0 \le \int \bar{q}(P_n \chi a) \le q(\int P_n(\chi a)) = q(\int \chi a) = q(\bar{\mu}a)$$

for every $a \in A$ and $n \in \mathbb{N}$; since also

$$0 \le \int \bar{q}(P \chi a) \le q(\bar{\mu}a),$$

we have $|\int \bar{q}(P_n \chi a) - \int \bar{q}(P \chi a)| \le q(\bar{\mu}a)$ for every $a \in A$, $n \in \mathbb{N}$.

Now we are supposing that $H(A)$ is finite. Given $\epsilon > 0$, we can find a finite set $I \subseteq A$ such that $\sum_{a \in A \setminus I} q(\bar{\mu}a) \le \epsilon$, and an $n_0 \in \mathbb{N}$ such that

$$\sum_{a \in I} |\int \bar{q}(P_n \chi a) - \int \bar{q}(P \chi a)| \le \epsilon$$

for every $n \ge n_0$; in which case

$$\sum_{a \in A \setminus I} |\int \bar{q}(P_n \chi a) - \int \bar{q}(P \chi a)| \le \sum_{a \in A \setminus I} q(\bar{\mu}a) \le \epsilon$$

and $|H(A|\mathfrak{B}_n) - H(A|\mathfrak{B})| \le 2\epsilon$ for every $n \ge n_0$. As ϵ is arbitrary, $H(A|\mathfrak{B}) = \lim_{n \to \infty} H(A|\mathfrak{B}_n)$.

385H Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Then $H(A) \le H(A \vee B) \le H(A) + H(B)$.

proof Take $\mathfrak{B} = \{0, 1\}$ in 385Ga.

385I Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. If $A\subseteq\mathfrak{A}$ is a partition of unity, then $H(\pi[A])=H(A)$.

proof $\sum_{a\in A}q(\bar{\mu}\pi a)=\sum_{a\in A}q(\bar{\mu}a)$.

385J Lemma Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Let A be the set of its atoms. Then the following are equiveridical:
(i) *either* $\mathfrak{A}$ is not purely atomic *or* $\mathfrak{A}$ is purely atomic and $H(A)=\infty$;
(ii) there is a partition of unity $B\subseteq\mathfrak{A}$ such that $H(B)=\infty$;
(iii) for every $\gamma\in\mathbb{R}$ there is a finite partition of unity $C\subseteq\mathfrak{A}$ such that $H(C)\ge\gamma$.

proof (i)⇒(ii) We need examine only the case in which $\mathfrak{A}$ is not purely atomic. Let $a\in\mathfrak{A}$ be a non-zero element such that the principal ideal $\mathfrak{A}_a$ is atomless. By 331C we can choose inductively a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $a_n\subseteq a$ and $\bar{\mu}a_n=2^{-n-1}\bar{\mu}a$. Now, for each $n\in\mathbb{N}$, choose a disjoint set B_n such that

$$\#(B_n)=2^{2^n},\quad b\subseteq a_n\text{ and }\bar{\mu}b=2^{-2^n}\bar{\mu}a_n\text{ for each }b\in B_n.$$

Set

$$B=\bigcup_{n\in\mathbb{N}}B_n\cup\{1\setminus a\}.$$

Then B is a partition of unity in $\mathfrak{A}$ and

$$\begin{aligned}H(B)&\ge\sum_{n=0}^{\infty}\sum_{b\in B_n}q(\bar{\mu}B_n)=\sum_{n=0}^{\infty}2^{2^n}q\Bigl(\frac{\bar{\mu}a}{2^{n+1+2^n}}\Bigr)\\&=\sum_{n=0}^{\infty}\frac{\bar{\mu}a}{2^{n+1}}\ln\Bigl(\frac{2^{n+1+2^n}}{\bar{\mu}a}\Bigr)\ge\sum_{n=0}^{\infty}\frac{\bar{\mu}a}{2^{n+1}}2^n\ln 2=\infty.\end{aligned}$$

(ii)⇒(iii) Enumerate B as $\langle b_i\rangle_{i\in\mathbb{N}}$. For each $n\in\mathbb{N}$, $C_n=\{b_i:i\le n\}\cup\{1\setminus\sup_{i\le n}b_i\}$ is a finite partition of unity, and

$$\lim_{n\to\infty}H(C_n)\ge\lim_{n\to\infty}\sum_{i=0}^{n}q(\bar{\mu}b_i)=H(B)=\infty.$$

(iii)⇒(i) We need only consider the case in which $\mathfrak{A}$ is purely atomic. In this case, $A\vee C=A$ for every partition of unity $C\subseteq\mathfrak{A}$, so $H(C)\le H(A)$ for every C (385H), and $H(A)$ must be infinite.

385K Definition Let $\mathfrak{A}$ be a Boolean algebra. If $\pi:\mathfrak{A}\to\mathfrak{A}$ is an order-continuous Boolean homomorphism, $A\subseteq\mathfrak{A}$ is a partition of unity and $n\ge 1$, write $D_n(A,\pi)$ for the partition of unity generated by $\{\pi^i a:a\in A,\ 0\le i<n\}$, that is, $\{\inf_{i<n}\pi^i a_i:a_i\in A\text{ for every }i<n\}\setminus\{0\}$. It will occasionally be convenient to take $D_0(A,\pi)=\{1\}$ (or $\emptyset$ in the trivial case $\mathfrak{A}=\{0\}$). Observe that $D_1(A,\pi)=A\setminus\{0\}$ and

$$D_{n+1}(A,\pi)=D_n(A,\pi)\vee\pi^n[A]=A\vee\pi[D_n(A,\pi)]$$

for every $n\in\mathbb{N}$.

385L Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $A\subseteq\mathfrak{A}$ be a partition of unity. Then $\lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi))=\inf_{n\ge1}\frac{1}{n}H(D_n(A,\pi))$ is defined in $[0,\infty]$.

proof (a) Set $\alpha_0=0$, $\alpha_n=H(D_n(A,\pi))$ for $n\ge 1$. Then $\alpha_{m+n}\le\alpha_m+\alpha_n$ for all m, $n\ge 0$. **P** If m, $n\ge 1$, $D_{m+n}(A,\pi)=D_m(A,\pi)\vee\pi^m[D_n(A,\pi)]$. So 385Ga tells us that

$$H(D_{m+n}(A,\pi))\le H(D_m(A,\pi))+H(\pi^m[D_n(A,\pi)])=H(D_m(A,\pi))+H(D_n(A,\pi))$$

because π is measure-preserving. **Q**

(b) If $\alpha_1=\infty$ then of course $H(D_n(A,\pi))\ge H(A)=\infty$ for every $n\ge 1$, by 385H, so $\inf_{n\ge1}\frac{1}{n}H(D_n(A,\pi))=\infty=\lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi))$. Otherwise, $\alpha_n\le n\alpha_1$ is finite for every n. Set $\alpha=\inf_{n\ge1}\frac{1}{n}\alpha_n$. If $\epsilon>0$ there is an $m\ge1$ such that $\frac{1}{m}\alpha_m\le\alpha+\epsilon$. Set $M=\max_{j<m}\alpha_j$. Now, for any $n\ge m$, there are $k\ge 1$, $j<m$ such that $n=km+j$, so that

$$\alpha_n \le k\alpha_m + \alpha_j, \quad \frac{1}{n}\alpha_n \le \frac{k}{n}\alpha_m + \frac{M}{n} \le \frac{1}{m}\alpha_m + \frac{M}{n}.$$

Accordingly $\limsup_{n\to\infty} \frac{1}{n}\alpha_n \le \alpha + \epsilon$. As ϵ is arbitrary,

$$\alpha \le \liminf_{n\to\infty} \frac{1}{n}\alpha_n \le \limsup_{n\to\infty} \frac{1}{n}\alpha_n \le \alpha$$

and $\lim_{n\to\infty} \frac{1}{n}\alpha_n = \alpha$ is defined in $[0,\infty]$.

Remark See also 385Yc and 386Kc below.

385M Definition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. For any partition of unity $A \subseteq \mathfrak{A}$, set

$$h(\pi, A) = \inf_{n\ge1} \tfrac{1}{n} H(D_n(A,\pi)) = \lim_{n\to\infty} \tfrac{1}{n} H(D_n(A,\pi))$$

(385L). Now the **entropy** of π is

$$h(\pi) = \sup\{h(\pi, A) : A \subseteq \mathfrak{A} \text{ is a finite partition of unity}\}.$$

Remarks (a) For any partition A of unity,

$$h(\pi, A) \le H(D_1(A,\pi)) = H(A).$$

(b) Observe that if π is the identity automorphism then $D_n(A,\pi) = A \setminus \{0\}$ for every A and n, so that $h(\pi) = 0$.

385N Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be a measure-preserving Boolean homomorphism. Then $h(\pi, A) \le h(\pi, B) + H(A|\mathfrak{B})$, where $\mathfrak{B}$ is the closed subalgebra of $\mathfrak{A}$ generated by B.

proof We may suppose that $0 \notin B$, since removing 0 from B changes neither $D_n(B,\pi)$ nor $\mathfrak{B}$. For each $n \in \mathbb{N}$, set $A_n = \pi^n[A]$ and $B_n = \pi^n[B]$. Let $\mathfrak{B}_n = \pi^n[\mathfrak{B}]$ be the closed subalgebra of $\mathfrak{A}$ generated by B_n, and $\mathfrak{B}^*_n$ the closed subalgebra of $\mathfrak{A}$ generated by $D_n(B,\pi)$. Then $H(A_n|\mathfrak{B}_n) = H(A|\mathfrak{B})$. **P** The point is that, because $\mathfrak{B}$ is purely atomic and B is its set of atoms,

$$H(A|\mathfrak{B}) = \textstyle\sum_{a\in A, b\in B} q(\frac{\bar\mu(a\cap b)}{\bar\mu b})\bar\mu b$$

as in the proof of 385Gb. Similarly,

$$H(A_n|\mathfrak{B}_n) = \textstyle\sum_{a\in A, b\in B} q(\frac{\bar\mu(\pi^n a\cap\pi^n b)}{\bar\mu(\pi^n b)})\bar\mu(\pi^n b) = H(A|\mathfrak{B}). \quad \mathbf{Q}$$

Accordingly, for any $n \ge 1$,

$$H(D_n(A,\pi)|\mathfrak{B}^*_n) \le \sum_{i=0}^{n-1} H(A_i|\mathfrak{B}^*_n)$$

(by 385Ga, because $D_n(A,\pi) = A_0 \vee \ldots \vee A_n$)

$$\le \sum_{i=0}^{n-1} H(A_i|\mathfrak{B}_i)$$

(by 385Gc)

$$= nH(A|\mathfrak{B}).$$

Now

$$h(\pi, A) = \lim_{n\to\infty} \frac{1}{n} H(D_n(A,\pi)) \le \limsup_{n\to\infty} \frac{1}{n} H(D_n(A,\pi) \vee D_n(B,\pi))$$

(385Ga)

$$= \limsup_{n\to\infty} \frac{1}{n} H(D_n(B,\pi)) + \frac{1}{n} H(D_n(A,\pi)|\mathfrak{B}_n^*)$$

(385Gb)

$$\le \limsup_{n\to\infty} \frac{1}{n} H(D_n(B,\pi)) + \limsup_{n\to\infty} \frac{1}{n} H(D_n(A,\pi)|\mathfrak{B}_n^*) \le h(\pi,B) + H(A|\mathfrak{B}).$$

385O Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism, and $A\subseteq\mathfrak{A}$ a partition of unity such that $H(A)<\infty$. Then $h(\pi,A)\le h(\pi)$.

proof If A is finite, this is immediate from the definition of $h(\pi)$; so suppose that A is infinite. Enumerate A as $\langle a_i\rangle_{i\in\mathbb{N}}$. For each $n\in\mathbb{N}$ let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $a_0,\dots,a_n$, and B_n the set of its atoms; set $\mathfrak{B}=\overline{\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n}$. Then $A\subseteq\mathfrak{B}$, so

$$\lim_{n\to\infty} H(A|\mathfrak{B}_n) = H(A|\mathfrak{B}) = 0$$

by 385Eb and 385Gd. Accordingly, using 385N,

$$h(\pi,A)\le h(\pi,B_n)+H(A|\mathfrak{B}_n)\le h(\pi)+H(A|\mathfrak{B}_n)\to h(\pi)$$

as $n\to\infty$, and $h(\pi,A)\le h(\pi)$.

385P Theorem (Kolmogorov 58, Sinaĭ 59) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism.

(i) Suppose that $A\subseteq\mathfrak{A}$ is a partition of unity such that $H(A)<\infty$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}\pi^n[A]$ is $\mathfrak{A}$ itself. Then $h(\pi)=h(\pi,A)$.

(ii) Suppose that π is an automorphism, and that $A\subseteq\mathfrak{A}$ is a partition of unity such that $H(A)<\infty$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{Z}}\pi^n[A]$ is $\mathfrak{A}$ itself. Then $h(\pi)=h(\pi,A)$.

proof I take the two arguments together. In both cases, by 385O, we have $h(\pi,A)\le h(\pi)$, so I have to show that if $B\subseteq\mathfrak{A}$ is any finite partition of unity, then $h(\pi,B)\le h(\pi,A)$. For (i), let A_n be the partition of unity generated by $\bigcup_{0\le j<n}\pi^j[A]$; for (ii), let A_n be the partition of unity generated by $\bigcup_{-n\le j<n}\pi^j[A]$. Then $h(\pi,A_n)=h(\pi,A)$ for every n. **P** In case (i), we have $D_m(A_n,\pi)=D_{m+n}(A,\pi)$ for every m, so that

$$\lim_{m\to\infty}\frac{1}{m}H(D_m(A_n,\pi)) = \lim_{m\to\infty}\frac{1}{m}H(D_{m+n}(A,\pi)) = \lim_{m\to\infty}\frac{1}{m}H(D_m(A,\pi)).$$

In case (ii), we have $D_m(A_n,\pi)=\pi^{-n}[D_{m+2n}(A,\pi)]$ for every m, so that

$$\lim_{m\to\infty}\frac{1}{m}H(D_m(A_n,\pi)) = \lim_{m\to\infty}\frac{1}{m}H(D_{m+2n}(A,\pi)) = \lim_{m\to\infty}\frac{1}{m}H(D_m(A,\pi)).\ \mathbf{Q}$$

Let $\mathfrak{A}_n$ be the purely atomic closed subalgebra of $\mathfrak{A}$ generated by A_n; our hypothesis is that the closed subalgebra generated by $\bigcup_{n\in\mathbb{N}}A_n$ is $\mathfrak{A}$ itself, that is, that $\bigcup_{n\in\mathbb{N}}\mathfrak{A}_n$ is dense. But this means that $\lim_{n\to\infty}H(B|\mathfrak{A}_n)=0$ (385Gd). Since

$$h(\pi,B)\le h(\pi,A_n)+H(B|\mathfrak{A}_n)=h(\pi,A)+H(B|\mathfrak{A}_n)$$

for every n (385N), we have the result.

385Q Bernoulli shifts Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism.

(a) π is a **one-sided Bernoulli shift** if there is a closed subalgebra $\mathfrak{A}_0$ in $\mathfrak{A}$ such that (i) $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is stochastically independent (that is, $\bar\mu(\inf_{j\le k}\pi^j a_j)=\prod_{j=0}^k\bar\mu a_j$ for all $a_0,\dots,a_k\in\mathfrak{A}_0$; see 325L) (ii) the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}\pi^k[\mathfrak{A}_0]$ is $\mathfrak{A}$ itself. In this case $\mathfrak{A}_0$ is a **root algebra** for π.

(b) π is a **two-sided Bernoulli shift** if it is an automorphism and there is a closed subalgebra $\mathfrak{A}_0$ in $\mathfrak{A}$ such that (i) $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{Z}}$ is independent (ii) the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{Z}}\pi^k[\mathfrak{A}_0]$ is $\mathfrak{A}$ itself. In this case $\mathfrak{A}_0$ is a **root algebra** for π.

It is important to be aware that a Bernoulli shift can have many, and (in the case of a two-sided shift) very different, root algebras; this is the subject of §387 below.

385R Theorem Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Bernoulli shift, either one- or two-sided, with root algebra $\mathfrak{A}_0$.

(i) If $\mathfrak{A}_0$ is purely atomic, then $h(\pi)=H(A)$, where A is the set of atoms of $\mathfrak{A}_0$.

(ii) If $\mathfrak{A}_0$ is not purely atomic, then $h(\pi)=\infty$.

proof (a) The point is that for any partition of unity $C\subseteq\mathfrak{A}_0\setminus\{0\}$, $h(\pi,C)=H(C)$. **P** For any $n\ge 1$, $D_n(C,\pi)$ is the partition of unity consisting of elements of the form $\inf_{j<n}\pi^jc_j$, where $c_0,\dots,c_{n-1}\in C$. So

$$H(D_n(C,\pi))=\sum_{c_0,\dots,c_{n-1}\in C}q(\bar\mu(\inf_{j<n}\pi^jc_j))=\sum_{c_0,\dots,c_{n-1}\in C}q(\prod_{j=0}^{n-1}\bar\mu c_j))$$

$$=\sum_{c_0,\dots,c_{n-1}\in C}\sum_{j=0}^{n-1}q(\bar\mu c_j)\prod_{i\ne j}\bar\mu c_i$$

(385Ac)

$$=\sum_{j=0}^{n-1}\sum_{c\in C}q(\bar\mu c)=nH(C).$$

So

$$h(\pi,C)=\lim_{n\to\infty}\frac{1}{n}H(D_n(C,\pi))=H(C).\ \mathbf{Q}$$

(b) If $\mathfrak{A}_0$ is purely atomic and $H(A)<\infty$, the result can now be read off from 385P, because the closed subalgebra of $\mathfrak{A}$ generated by A is $\mathfrak{A}_0$ and the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}\pi^k[A]$ or $\bigcup_{k\in\mathbb{Z}}\pi^k[A]$ is $\mathfrak{A}$; so $h(\pi)=h(\pi,A)=H(A)$.

(c) Otherwise, 385J tells us that there are finite partitions of unity $C\subseteq\mathfrak{A}_0$ such that $H(C)$ is arbitrarily large. Since $h(\pi)\ge h(\pi,C)=H(C)$ for any such C, by (a) and the definition of $h(\pi)$, $h(\pi)$ must be infinite, as claimed.

385S Remarks (a) The standard construction of a Bernoulli shift is from a product space, as follows. If (X,Σ,μ_0) is any probability space, write μ for the product measure on $X^{\mathbb{N}}$; let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ, and $\mathfrak{A}_0\subseteq\mathfrak{A}$ the set of equivalence classes of sets of the form $\{x:x(0)\in E\}$ where $E\in\Sigma$, so that $(\mathfrak{A}_0,\bar\mu\restriction\mathfrak{A}_0)$ can be identified with the measure algebra of μ_0. We have an inverse-measure-preserving function $f:X^{\mathbb{N}}\to X^{\mathbb{N}}$ defined by setting

$$f(x)(n)=x(n+1)\text{ for every }x\in X^{\mathbb{N}},\ n\in\mathbb{N},$$

and f induces, as usual, a measure-preserving homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$. Now π is a one-sided Bernoulli shift with root algebra $\mathfrak{A}_0$. **P** (i) If $a_0,\dots,a_k\in\mathfrak{A}_0$, express each a_j as $\{x:x(0)\in E_j\}^\bullet$, where $E_j\in\Sigma$. Now

$$\pi^ja_j=\{x:(f^j(x))(0)\in E_j\}^\bullet=\{x:x(j)\in E_j\}^\bullet$$

for each j, so

$$\bar\mu(\inf_{j\le k}\pi^ja_j)=\mu(\bigcap_{j\le k}\{x:x(j)\in E_j\})=\prod_{j=0}^k\mu_0E_j=\prod_{j=0}^k\bar\mu a_j.$$

Thus $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ is independent. (ii) The closed subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}\pi^k[\mathfrak{A}_0]$ must contain $\{x:x(k)\in E\}^\bullet$ for every $k\in\mathbb{N}$ and $E\in\Sigma$, so must contain $W^\bullet$ for every W in the σ-algebra generated by sets of the form $\{x:x(k)\in E\}$; but every set measured by μ is equivalent to such a set W (254Ff). So $\mathfrak{A}'=\mathfrak{A}$. **Q**

(b) The same method gives us two-sided Bernoulli shifts. Again let (X,Σ,μ_0) be a probability space, and this time write μ for the product measure on $X^{\mathbb{Z}}$; again let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ, and $\mathfrak{A}_0\subseteq\mathfrak{A}$ the set of equivalence classes of sets of the form $\{x:x(0)\in E\}$ where $E\in\Sigma$, so that $(\mathfrak{A}_0,\bar\mu\restriction\mathfrak{A}_0)$ can once more be identified with the measure algebra of μ_0. This time, we have a measure space automorphism $f:X^{\mathbb{Z}}\to X^{\mathbb{Z}}$ defined by setting

$$f(x)(n)=x(n+1)\text{ for every }x\in X^{\mathbb{Z}},\,n\in\mathbb{Z},$$

and f induces a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$. The arguments used above show that π is a two-sided Bernoulli shift with root algebra $\mathfrak{A}_0$.

It follows that if $(\mathfrak{A},\bar\mu)$ is an atomless homogeneous probability algebra it has a two-sided Bernouilli shift. **P** We can identify $(\mathfrak{A},\bar\mu)$ with the measure algebra of the usual measure on $\{0,1\}^{\kappa\times\mathbb{Z}}\cong(\{0,1\}^\kappa)^{\mathbb{Z}}$, where κ is the Maharam type of $\mathfrak{A}$. **Q**

(c) I remarked above that a Bernoulli shift will normally have many root algebras. But it is important to know that, up to isomorphism, any probability algebra is the root algebra of just one Bernoulli shift of each type.

P(i) Given a probability algebra $(\mathfrak{A}_0,\bar\mu_0)$ then we can identify it with the measure algebra of a probability space (X,Σ,μ_0) (321J), and now the constructions of (a) and (b) provide Bernoulli shifts with root algebras isomorphic to $(\mathfrak{A}_0,\bar\mu_0)$.

(ii) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be probability algebras with one-sided Bernoulli shifts π, ϕ with root algebras $\mathfrak{A}_0$, $\mathfrak{B}_0$, and suppose that $\theta_0:\mathfrak{A}_0\to\mathfrak{B}_0$ is a measure-preserving isomorphism. Then $(\mathfrak{A},\bar\mu)$ can be identified with the probability algebra free product of $\langle\pi^k[\mathfrak{A}_0]\rangle_{k\in\mathbb{N}}$ (325L), while $(\mathfrak{B},\bar\nu)$ can be identified with the probability algebra free product of $\langle\pi^k[\mathfrak{B}_0]\rangle_{k\in\mathbb{N}}$. For each $k\in\mathbb{N}$, $\phi^k\theta_0(\pi^k)^{-1}$ is a measure-preserving isomorphism between $\pi^k[\mathfrak{A}_0]$ and $\phi^k[\mathfrak{B}_0]$. Assembling these, we have a measure-preserving Boolean homomorphism $\theta:\mathfrak{A}\to\mathfrak{B}$ such that $\theta a=\phi^k\theta_0(\pi^k)^{-1}a$ whenever $k\in\mathbb{N}$ and $a\in\pi^k[\mathfrak{A}_0]$ (325I), that is, $\theta\pi^ka=\phi^k\theta_0a$ for every $a\in\mathfrak{A}_0$, $k\in\mathbb{N}$. Of course θ extends θ_0. Also $\theta[\mathfrak{A}]$ is a closed subalgebra of $\mathfrak{B}$ (324Kb) including $\phi^k[\mathfrak{B}_0]$ for every k, so is the whole of $\mathfrak{B}$, and θ is a measure-preserving isomorphism.

If we set

$$\mathfrak{C}=\{a:a\in\mathfrak{A},\,\theta\pi a=\phi\theta a\},$$

then $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$. If $a\in\mathfrak{A}_0$ and $k\in\mathbb{N}$, then

$$\theta\pi(\pi^ka)=\theta\pi^{k+1}a=\phi^{k+1}\theta_0a=\phi(\phi^k\theta_0a)=\phi\theta(\pi^ka),$$

so $\pi^ka\in\mathfrak{C}$. Thus $\phi^k[\mathfrak{A}_0]\subseteq\mathfrak{C}$ for every $k\in\mathbb{N}$, and $\mathfrak{C}=\mathfrak{A}$.

This means that $\theta:\mathfrak{A}\to\mathfrak{B}$ is such that $\phi=\theta\pi\theta^{-1}$; θ is an isomorphism between the structures $(\mathfrak{A},\bar\mu,\pi)$ and $(\mathfrak{B},\bar\nu,\phi)$ extending the isomorphism θ_0 from $\mathfrak{A}_0$ to $\mathfrak{B}_0$.

(iii) Now suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are probability algebras with two-sided Bernoulli shifts π, ϕ with root algebras $\mathfrak{A}_0$, $\mathfrak{B}_0$, and suppose that $\theta_0:\mathfrak{A}_0\to\mathfrak{B}_0$ is a measure-preserving isomorphism. Repeating (ii) word for word, but changing each $\mathbb{N}$ into $\mathbb{Z}$, we find that θ_0 has an extension to a measure-preserving isomorphism $\theta:\mathfrak{A}\to\mathfrak{B}$ such that $\theta\pi=\phi\theta$, so that once more the structures $(\mathfrak{A},\bar\mu,\pi)$ and $(\mathfrak{B},\bar\nu,\phi)$ are isomorphic. **Q**

(d) The classic problem to which the theory of this section was directed was the following: suppose we have two two-sided Bernoulli shifts π and ϕ, one based on a root algebra with two atoms of measure $\frac{1}{2}$ and the other on a root algebra with three atoms of measure $\frac{1}{3}$; are they isomorphic? The Kolmogorov-Sinaĭ theorem tells us that they are not, because $h(\pi)=\ln 2$ and $h(\phi)=\ln 3$ are different. The question of which Bernoulli shifts *are* isomorphic is addressed, and (for countably-generated algebras) solved, in §387 below.

(e) We shall need to know that any Bernoulli shift (either one- or two-sided) is ergodic. In fact, it is mixing. **P** Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a Bernoulli shift with root algebra $\mathfrak{A}_0$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}}\pi^k[\mathfrak{A}_0]$ (if π is one-sided) or by $\bigcup_{k\in\mathbb{Z}}\pi^k[\mathfrak{A}_0]$ (if π is two-sided). If b, $c\in\mathfrak{B}$, there is some $n\in\mathbb{N}$ such that both belong to the algebra $\mathfrak{B}_n$ generated by $\bigcup_{j\le n}\pi^j[\mathfrak{A}_0]$ (if π is one-sided) or by $\bigcup_{|j|\le n}\pi^j[\mathfrak{A}_0]$ (if π is two-sided). If now $k>2n$, π^kb belongs to the algebra generated by $\bigcup_{j>n}\pi^j[\mathfrak{A}_0]$. But this is independent of $\mathfrak{B}_n$ (cf. 325Xg, 272K), so

$$\bar\mu(c\cap\pi^k b)=\bar\mu c\cdot\bar\mu(\pi^k b)=\bar\mu c\cdot\bar\mu b.$$

And this is true for every $k\ge n$. Generally, if b, $c\in\mathfrak A$ and $\epsilon>0$, there are b', $c'\in\mathfrak B$ such that $\bar\mu(b\mathbin{\triangle}b')\le\epsilon$ and $\bar\mu(c\mathbin{\triangle}c')\le\epsilon$, so that

$$\begin{aligned}\limsup_{k\to\infty}|\bar\mu(c\cap\pi^k b)-\bar\mu c\cdot\bar\mu b|&\le\limsup_{k\to\infty}|\bar\mu(c'\cap\pi^k b')-\bar\mu c'\cdot\bar\mu b'|\\&\qquad+\bar\mu(c\mathbin{\triangle}c')+\bar\mu(\pi^k b\mathbin{\triangle}\pi^k b')+|\bar\mu c\cdot\bar\mu b-\bar\mu c'\cdot\bar\mu b'|\\&\le 0+\epsilon+\epsilon+|\bar\mu c-\bar\mu c'|+|\bar\mu b-\bar\mu b'|\le 4\epsilon.\end{aligned}$$

As ϵ, b and c are arbitrary, π is mixing. By 372Qa, it is ergodic. **Q**

(f) The following elementary remark will be useful. If $(\mathfrak A,\bar\mu)$ is a probability algebra, $\pi:\mathfrak A\to\mathfrak A$ is a measure-preserving automorphism, and $\mathfrak A_0\subseteq\mathfrak A$ is a closed subalgebra such that $\langle\pi^k[\mathfrak A_0]\rangle_{k\in\mathbb N}$ is independent, then $\langle\pi^k[\mathfrak A_0]\rangle_{k\in\mathbb Z}$ is independent. **P** If $J\subseteq\mathbb Z$ is finite and $\langle a_j\rangle_{j\in J}$ is a family in $\mathfrak A_0$, take $n\in\mathbb N$ such that $-n\le j$ for every $j\in J$; then

$$\bar\mu(\inf_{j\in J}\pi^j a_j)=\bar\mu(\inf_{j\in J}\pi^{n+j}a_j)=\prod_{j\in J}\bar\mu a_j.\text{ **Q**}$$

(g) It is I hope obvious, but perhaps I should explicitly say: if $(\mathfrak A,\bar\mu)$ is a probability algebra, $\phi:\mathfrak A\to\mathfrak A$ is a measure-preserving automorphism, and $\pi:\mathfrak A\to\mathfrak A$ is a (one- or two-sided) Bernouilli shift with a root algebra $\mathfrak A_0$, then $\phi\pi\phi^{-1}$ is a Bernouilli shift and $\phi[\mathfrak A_0]$ is a root algebra for $\phi\pi\phi^{-1}$.

385T Isomorphic homomorphisms (a) In this section I have spoken of 'isomorphic homomorphisms' without offering a formal definition. I hope that my intention was indeed obvious, and that the next sentence will merely confirm what you have already assumed. If $(\mathfrak A_1,\bar\mu_1)$ and $(\mathfrak A_2,\bar\mu_2)$ are measure algebras, and $\pi_1:\mathfrak A_1\to\mathfrak A_2$, $\pi_2:\mathfrak A_2\to\mathfrak A_2$ are functions, then I say that $(\mathfrak A_1,\bar\mu_1,\pi_1)$ and $(\mathfrak A_2,\bar\mu_2,\pi_2)$ are isomorphic if there is a measure-preserving isomorphism $\phi:\mathfrak A_1\to\mathfrak A_2$ such that $\pi_2=\phi\pi_1\phi^{-1}$. In this context, using Maharam's theorem or otherwise, we can expect to be able to decide whether $(\mathfrak A_1,\bar\mu_1)$ and $(\mathfrak A_2,\bar\mu_2)$ are or are not isomorphic; and if they are, we have a good hope of being able to describe a measure-preserving isomorphism $\theta:\mathfrak A_1\to\mathfrak A_2$. In this case, of course, $(\mathfrak A_2,\bar\mu_2,\pi_2)$ will be isomorphic to $(\mathfrak A_1,\bar\mu_1,\pi_2')$ where $\pi_2'=\theta^{-1}\pi_2\theta$. So now we have to decide whether $(\mathfrak A_1,\bar\mu_1,\pi_1)$ is isomorphic to $(\mathfrak A_1,\bar\mu_1,\pi_2')$; and when π_1, π_2 are measure-preserving Boolean automorphisms, this is just the question of whether π_1, π_2' are conjugate in the group $\mathrm{Aut}_{\bar\mu_1}(\mathfrak A_1)$ of measure-preserving automorphisms of $\mathfrak A_1$. Thus the isomorphism problem, as stated here, is very close to the classical group-theoretic problem of identifying the conjugacy classes in $\mathrm{Aut}_{\bar\mu}(\mathfrak A)$ for a measure algebra $(\mathfrak A,\bar\mu)$. But we also want to look at measure-preserving homomorphisms which are not automorphisms, so there would be something left even if the conjugacy problem were solved. (In effect, we are studying conjugacy in the semigroup of all measure-preserving Boolean homomorphisms, not just in its group of invertible elements.)

The point of the calculation of the entropy of a homomorphism is that it is an invariant under this kind of isomorphism; so that if π_1, π_2 have different entropies then $(\mathfrak A_1,\bar\mu_1,\pi_1)$ and $(\mathfrak A_2,\bar\mu_2,\pi_2)$ cannot be isomorphic. Of course the properties of being 'ergodic' or 'mixing' (see 372O) are also invariant.

(b) All the main work of this section has been done in terms of measure algebras; part of my purpose in this volume has been to insist that this is often the right way to proceed, and to establish a language which makes the arguments smooth and natural. But of course a large proportion of the most important homomorphisms arise in the context of measure spaces, and I take a moment to discuss such applications. Suppose that we have two quadruples (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) where, for each i, (X_i,Σ_i,μ_i) is a measure space and $f_i:X_i\to X_i$ is an inverse-measure-preserving function. Then we have associated structures $(\mathfrak A_1,\bar\mu_1,\pi_1)$ and $(\mathfrak A_2,\bar\mu_2,\pi_2)$ where $(\mathfrak A_i,\bar\mu_i)$ is the measure algebra of (X_i,Σ_i,μ_i) and $\pi_i:\mathfrak A_i\to\mathfrak A_i$ is the measure-preserving homomorphism defined by the usual formula $\pi_i E^\bullet=f_i^{-1}[E]^\bullet$. Now we can call (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) isomorphic if there is a measure space isomorphism $g:X_1\to X_2$ such that $f_2=gf_1g^{-1}$. In this case $(\mathfrak A_1,\bar\mu_1,\pi_1)$ and $(\mathfrak A_2,\bar\mu_2,\pi_2)$ are isomorphic under the obvious isomorphism $\phi(E^\bullet)=g[E]^\bullet$ for every $E\in\Sigma_1$.

It is not the case that if the $(\mathfrak A_i,\bar\mu_i,\pi_i)$ are isomorphic, then the (X_i,Σ_i,μ_i,f_i) are; in fact we do not even need to have an isomorphism of the measure spaces (for instance, one could be Lebesgue measure, and the other the Stone space of the Lebesgue measure algebra). Even when $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ are actually identical, f_1 and f_2 need not be isomorphic. There are two examples in §343 of a probability space (X, Σ, μ) with a measure space automorphism $f : X \to X$ such that $f(x) \neq x$ for every $x \in X$ but the corresponding automorphism on the measure algebra is the identity (343I, 343J); writing ι for the identity map from X to itself, (X, Σ, μ, ι) and (X, Σ, μ, f) are non-isomorphic but give rise to the same $(\mathfrak{A}, \bar{\mu}, \pi)$.

(c) Even with Lebesgue measure, we can have a problem in a formal sense. Take (X, Σ, μ) to be $[0,1]$ with Lebesgue measure, and set $f(0) = 1$, $f(1) = 0$, $f(x) = x$ for $x \in]0,1[$; then f is not isomorphic to the identity function on X, but induces the identity automorphism on the measure algebra. But in this case we can sort things out just by discarding the negligible set $\{0,1\}$, and for Lebesgue measure such a procedure is effective in a wide variety of situations. To formalize it I offer the following definition.

385U Definition Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces, and $f_1 : X_1 \to X_1$, $f_2 : X_2 \to X_2$ two inverse-measure-preserving functions. I will say that the structures $(X_1, \Sigma_1, \mu_1, f_1)$ and $(X_2, \Sigma_2, \mu_2, f_2)$ are **almost isomorphic** if there are conegligible sets $X_i' \subseteq X_i$ such that $f_i[X_i'] \subseteq X_i'$ for both i and the structures $(X_i', \Sigma_i', \mu_i', f_i')$ are isomorphic in the sense of 385Tb, where Σ_i' is the algebra of relatively measurable subsets of X_i', μ_i' is the subspace measure on X_i' and $f_i' = f_i \restriction X_i'$.

385V I leave the elementary properties of this notion to the exercises (385Xq-385Xs), but I spell out the result for which the definition is devised. I phrase it in the language of §§342-343; if the terms are not immediately familiar, start by imagining that both (X_i, Σ_i, μ_i) are measurable subspaces of $\mathbb{R}$ endowed with some Radon measure (342J, 343H), or indeed that both are $[0,1]$ with Lebesgue measure.

Proposition Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be perfect, complete, strictly localizable and countably separated measure spaces, and $(\mathfrak{A}_1, \bar{\mu}_1)$, $(\mathfrak{A}_2, \bar{\mu}_2)$ their measure algebras. Suppose that $f_1 : X_1 \to X_1$, $f_2 : X_2 \to X_2$ are inverse-measure-preserving functions and that $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_1$, $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ are the measure-preserving Boolean homomorphisms they induce. If $(\mathfrak{A}_1, \bar{\mu}_1, \pi_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2, \pi_2)$ are isomorphic, then $(X_1, \Sigma_1, \mu_1, f_1)$ and $(X_2, \Sigma_2, \mu_2, f_2)$ are almost isomorphic.

proof Because $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ are isomorphic, we surely have $\mu_1 X_1 = \mu_2 X_2$. If both are zero, we can take $X_1' = X_2' = \emptyset$ and stop; so let us suppose that $\mu_1 X_1 > 0$. Let $\phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ be a measure-preserving automorphism such that $\pi_2 = \phi\pi_1\phi^{-1}$. Because both μ_1 and μ_2 are complete and strictly localizable and compact (343K), there are inverse-measure-preserving functions $g_1 : X_1 \to X_2$ and $g_2 : X_2 \to X_1$ representing ϕ^{-1}, ϕ respectively (343B). Now $g_1 g_2 : X_2 \to X_2$, $g_2 g_1 : X_1 \to X_1$, $f_2 g_1 : X_1 \to X_2$ and $g_1 f_1 : X_1 \to X_2$ represent, respectively, the identity automorphism on $\mathfrak{A}_2$, the identity automorphism on $\mathfrak{A}_1$, the homomorphism $\phi^{-1}\pi_2 = \pi_1\phi^{-1} : \mathfrak{A}_2 \to \mathfrak{A}_1$ and the homomorphism $\pi_1\phi^{-1}$ again. Next, because both μ_1 and μ_2 are countably separated, the sets $E_1 = \{x : g_2 g_1(x) = x\}$, $H = \{x : f_2 g_1(x) = g_1 f_1(x)\}$ and $E_2 = \{y : g_1 g_2(y) = y\}$ are all conegligible (343F). As in part (b) of the proof of 344I, $g_1 \restriction E_1$ and $g_2 \restriction E_2$ are the two halves of a bijection, a measure space isomorphism if E_1 and E_2 are given their subspace measures. Set $G_0 = E_1 \cap H$, and for $n \in \mathbb{N}$ set $G_{n+1} = G_n \cap f_1^{-1}[G_n]$. Then every G_n is conegligible, so $X_1' = \bigcap_{n \in \mathbb{N}} G_n$ is conegligible. Because X_1' is a conegligible subset of E_1, $h = g_1 \restriction X_1'$ is a measure space isomorphism between X_1' and $X_2' = g_1[X_1']$, which is conegligible in X_2. Because $f_1[G_{n+1}] \subseteq G_n$ for each n, $f_1[X_1'] \subseteq X_1'$. Because $X_1' \subseteq H$, $g_1 f_1(x) = f_2 g_1(x)$ for every $x \in X_1'$. Next, if $y \in X_2'$, $g_2(y) \in X_1'$, so

$$f_2(y) = f_2 g_1 g_2(y) = g_1 f_1 g_2(y) \in g_1[f_1[X_1']] \subseteq g_1[X_1'] = X_2'.$$

Accordingly we have $f_2' = h f_1' h^{-1}$, where $f_i' = f_i \restriction X_i'$ for both i.

Thus h is an isomorphism between (X_1', f_1') and (X_2', f_2'), and $(X_1, \Sigma_1, \mu_1, f_1)$ and $(X_2, \Sigma_2, \mu_2, f_2)$ are almost isomorphic.

385X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $A \subseteq \mathfrak{A}$ a partition of unity. Show that if $\#(A) = n$ then $H(A) \le \ln n$.

>(b) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$, enumerated as $\langle a_n \rangle_{n \in \mathbb{N}}$. Set $a_n^* = \sup_{i>n} a_i$, $A_n = \{a_0, \dots, a_n, a_n^*\}$ for each n. Show that $H(A_n|\mathfrak{B}) \le H(A_{n+1}|\mathfrak{B})$ for every n, and that $H(A|\mathfrak{B}) = \lim_{n \to \infty} H(A_n|\mathfrak{B})$.

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$. Show that $H(A|\mathfrak{B})=0$ iff $A\subseteq\mathfrak{B}$.

(d) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ and A a partition of unity in $\mathfrak{A}$. Show that $H(A|\mathfrak{B})=H(A)$ iff $\bar\mu(a\cap b)=\bar\mu a\cdot\bar\mu b$ for every $a\in A$, $b\in\mathfrak{B}$. (*Hint*: for 'only if', start with the case $\mathfrak{B}=\{0,b,1\setminus b,1\}$ and use 385Gc.)

(e) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and A, B two partitions of unity in $\mathfrak{A}$. Show that $H(A\vee B)=H(A)+H(B)$ iff $\bar\mu(a\cap b)=\bar\mu a\cdot\bar\mu b$ for all $a\in A$, $b\in B$. Show that $H(A\vee B)=H(A)$ iff every member of A is included in some member of B, that is, iff $A=A\vee B$.

(f) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and write $\mathcal{A}$ for the set of partitions of unity in $\mathfrak{A}$ not containing 0, ordered by saying that $A\le B$ if B refines A. (i) Show that $\mathcal{A}$ is a Dedekind complete lattice, and can be identified with the lattice of purely atomic closed subalgebras of $\mathfrak{A}$. Show that for A, $B\in\mathcal{A}$, $A\vee B$, as defined in 385F, is $\sup\{A,B\}$ in $\mathcal{A}$. (ii) Show that $H(A\vee B)+H(A\wedge B)\le H(A)+H(B)$ for all A, $B\in\mathcal{A}$, where $\vee$, $\wedge$ are the lattice operations on $\mathcal{A}$. (iii) Set $\mathcal{A}_1=\{A:A\in\mathcal{A},\,H(A)<\infty\}$. For A, $B\in\mathcal{A}_1$ set $\rho(A,B)=2H(A\vee B)-H(A)-H(B)$. Show that ρ is a metric on $\mathcal{A}_1$ (the **entropy metric**). (iv) Show that $H:\mathcal{A}_1\to[0,\infty[$ is 1-Lipschitz for ρ. (v) Show that the lattice operation $\vee$ is uniformly ρ-continuous on $\mathcal{A}_1$. (vi) Show that $H:\mathcal{A}_1\to[0,\infty[$ is order-continuous. (vii) Show that if $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$, then $A\mapsto H(A|\mathfrak{B})$ is order-continuous and 1-Lipschitz on $\mathcal{A}_1$. (viii) Show that if $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving Boolean homomorphism, $A\mapsto h(\pi,A):\mathcal{A}_1\to[0,\infty[$ is 1-Lipschitz for ρ.

(g) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C},\bar\lambda)$ (325K). Suppose that $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i\in I$, and that $\pi:\mathfrak{C}\to\mathfrak{C}$ is the measure-preserving Boolean homomorphism they induce. Show that $h(\pi)=\sum_{i\in I}h(\pi_i)$. (*Hint*: use 385Gb and 385Gd to show that $h(\pi)$ is the supremum of $h(\pi,A)$ as A runs over the finite partitions of unity in $\bigotimes_{i\in I}\mathfrak{A}_i$. Use this to reduce to the case $I=\{0,1\}$. Now show that if $A_i\subseteq\mathfrak{A}_i$ is a finite partition of unity for each i, and $A=\{a_0\otimes a_1:a_0\in A_0,\,a_1\in A_1\}$, then $H(A)=H(A_0)+H(A_1)$, so that $h(\pi,A)=h(\pi_0,A_0)+h(\pi_1,A_1)$.)

>(h) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving automorphism. Show that $h(\pi^{-1})=h(\pi)$.

(i) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Show that $h(\pi^k)=kh(\pi)$ for any $k\in\mathbb{N}$. (*Hint*: if $A\subseteq\mathfrak{A}$ is a partition of unity, $h(\pi^k,A)\le h(\pi^k,D_k(A,\pi))=kh(\pi,A)$.)

(j) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{B}$ a topologically dense subalgebra of $\mathfrak{A}$. (i) Show that if $\langle a_i\rangle_{i\le n}$ is a partition of unity in $\mathfrak{A}$ and $\epsilon>0$, there is a partition $\langle b_i\rangle_{i\le n}$ of unity in $\mathfrak{B}$ such that $\bar\mu(a_i\triangle b_i)\le\epsilon$ for every $i\le n$. (ii) Show that if A is a finite partition of unity in $\mathfrak{A}$ and $\epsilon>0$ then there is a finite partition of unity $D\subseteq\mathfrak{B}$ such that $H(A\vee D)\le H(A)+\epsilon$. (iii) Show that if $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving Boolean homomorphism, then $h(\pi)=\sup\{h(\pi,D):D\subseteq\mathfrak{B}\text{ is a finite partition of unity}\}$. (*Hint*: 385N, 385Gb.)

>(k) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. (i) Suppose there is a partition of unity $A\subseteq\mathfrak{A}$ such that (α) $\bar\mu(a\cap\pi b)=\bar\mu a\cdot\bar\mu b$ for every $a\in A$, $b\in\mathfrak{A}$ (β) $\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{n\in\mathbb{N}}\pi^n[A]$. Show that π is a one-sided Bernoulli shift, and that $h(\pi)=H(A)$. (ii) Suppose that π is a one-sided Bernoulli shift of finite entropy. Show that there is a partition of unity satisfying (α) and (β).

>(l) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1[$. Fix an integer $k\ge2$, and define $f:[0,1[\,\to[0,1[$ by setting $f(x)=\langle kx\rangle$, the fractional part of kx, for every $x\in[0,1[$; let $\pi:\mathfrak{A}\to\mathfrak{A}$ be the corresponding homomorphism. (Cf. 372Xt.) Show that π is a one-sided Bernoulli shift and that $h(\pi)=\ln k$. (*Hint*: in 385Xk, set $A=\{a_0,\dots,a_{k-1}\}$ where $a_i=\left[\frac{i}{k},\frac{i+1}{k}\right[^\bullet$ for $i<k$.)

>(m) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$. Set $f(x)=2\min(x,1-x)$ for $x\in[0,1]$ (see 372Xp). Show that the corresponding homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is a one-sided Bernoulli shift and that $h(\pi)=\ln 2$. (*Hint*: in 385Xk, set $A=\{a,1\setminus a\}$ where $a=[0,\frac{1}{2}]^\bullet$.)

(n) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a two-sided Bernoulli shift. (i) Show that π^{-1} is a two-sided Bernoulli shift. (ii) Show that π and π^{-1} are conjugate in $\mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$. (iii) Show that π is expressible as the product of at most two involutions. (*Hint*: 382Xb.)

(o) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, and $(\mathfrak{C},\bar{\lambda})$ their probability algebra free product. Suppose that for each $i\in I$ we have a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$, and that $\pi:\mathfrak{C}\to\mathfrak{C}$ is the measure-preserving homomorphism induced by $\langle\pi_i\rangle_{i\in I}$ (325Xe). (i) Show that if every π_i is a one-sided Bernoulli shift so is π. (ii) Show that if every π_i is a two-sided Bernoulli shift so is π.

(p) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an ergodic measure-preserving Boolean homomorphism. Show that if $h(\pi)>0$ then $\mathfrak{A}$ is atomless.

(q) Show that the relation 'almost isomorphic to' (385U) is an equivalence relation.

(r) Show that the concept of 'almost isomorphism' described in 385U is not changed if we amend the definition to require that the subspaces X_1', X_2' should be measurable.

(s) Show that if (X_1,Σ_1,μ_1,f_1) and (X_2,Σ_2,μ_2,f_2) are almost isomorphic quadruples as described in 385U, then $(\mathfrak{A}_1,\bar{\mu}_1,\pi_1)$ and $(\mathfrak{A}_2,\bar{\mu}_2,\pi_2)$ are isomorphic, where for each i $(\mathfrak{A}_i,\bar{\mu}_i)$ is the measure algebra of (X_i,Σ_i,μ_i) and $\pi_i:\mathfrak{A}_i\to\mathfrak{A}_i$ is the measure-preserving Boolean homomorphism derived from $f_i:X_i\to X_i$.

385Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and write $\mathbb{B}$ for the lattice of closed subalgebras of $\mathfrak{A}$. Show that if A is any partition of unity in $\mathfrak{A}$ of finite entropy, then the order-preserving function $\mathfrak{B}\mapsto -H(A|\mathfrak{B}):\mathbb{B}\to\,]-\infty,0]$ is order-continuous.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\mathcal{A}_1$ the set of partitions of unity of finite entropy not containing 0, as in 385Xf. Show that $\mathcal{A}_1$ is complete under the entropy metric. (*Hint*: show that if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}_1$ and $\sup_{n\in\mathbb{N}}H(A_n)<\infty$, then the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}A_n$ is purely atomic.)

(c) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, A a partition of unity in $\mathfrak{A}$ of finite entropy, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Show that $h(\pi,A)=\lim_{n\to\infty}H(A|\mathfrak{B}_n)$, where $\mathfrak{B}_n$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{1\le i\le n}\pi^i[A]$. (*Hint*: use 385Gb to show that $H(A|\mathfrak{B}_n)=H(D_{n+1}(A,\pi))-H(D_n(A,\pi))$ and observe that $\lim_{n\to\infty}H(A|\mathfrak{B}_n)$ is defined.)

(d) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Suppose that there is a partition of unity A of finite entropy such that the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\ge 1}\pi^i[A]$ is $\mathfrak{A}$. Show that $h(\pi)=0$. (*Hint*: use 385Yc and 385P.)

(e) Let μ be Lebesgue measure on $[0,1[$, and take any $\alpha\in\,]0,1[$. Let $f:[0,1[\,\to[0,1[$ be the measure space automorphism defined by saying that $f(x)$ is to be one of $x+\alpha$, $x+\alpha-1$. Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of $([0,1[\,,\mu)$ and $\pi:\mathfrak{A}\to\mathfrak{A}$ the measure-preserving automorphism corresponding to f. Show that $h(\pi)=0$. (*Hint*: if $\alpha\in\mathbb{Q}$, use 385Xi; otherwise use 385Yd with $A=\{a,1\setminus a\}$ where $a=\bigl[0,\frac{1}{2}\bigr[^\bullet$.)

(f) Set $X=[0,1]\setminus\mathbb{Q}$, let ν be the measure on X defined by setting $\nu E=\frac{1}{\ln 2}\int_E\frac{1}{1+x}dx$ for every Lebesgue measurable set $E\subseteq X$, and for $x\in X$ let $f(x)$ be the fractional part $<\frac{1}{x}>$ of $\frac{1}{x}$. Recall that f is inverse-measure-preserving for ν (see 372M). Let $(\mathfrak{A},\bar{\nu})$ be the measure algebra of (X,ν) and $\pi:\mathfrak{A}\to\mathfrak{A}$ the homomorphism corresponding to f. Show that $h(\pi)=\pi^2/6\ln 2$. (*Hint*: use the Kolmogorov-Sinaǐ theorem and 372Yh(v).)

(g) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\phi:\mathfrak{A}\to\mathfrak{A}$ a one-sided Bernoulli shift. Show that there are a probability algebra $(\mathfrak{C},\bar{\lambda})$, a two-sided Bernouilli shift $\tilde{\phi}:\mathfrak{C}\to\mathfrak{C}$, and a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{C}$ such that $\tilde{\phi}\pi=\pi\phi$. (*Hint*: 328J.)

(h) Consider the triplets $([0,1[\,,\mu_1,f_1)$ and $([0,1],\mu_2,f_2)$ where μ_1, μ_2 are Lebesgue measure on $[0,1[$, $[0,1]$ respectively, $f_1(x) = {<}2x{>}$ for each $x \in [0,1[$, and $f_2(x) = 2\min(x,1-x)$ for each $x \in [0,1]$. Show that these structures are almost isomorphic in the sense of 385U, and give a formula for an almost-isomorphism.

385 Notes and comments In preparing this section I have been heavily influenced by Petersen 83. I have taken almost the shortest possible route to Theorem 385P, the original application of the theory, ignoring both the many extensions of these ideas and their intuitive underpinning in the concept of the quantity of 'information' carried by a partition. For both of these I refer you to Petersen 83. The techniques described there are I think sufficiently powerful to make possible the calculation of the entropy of any of the measure-preserving homomorphisms which have yet appeared in this treatise.

Of course the idea of entropy of a partition, or of a homomorphism, can be translated into the language of probability spaces and inverse-measure-preserving functions; if (X,Σ,μ) is a probability space, with measure algebra $(\mathfrak{A},\bar\mu)$, then partitions of unity in $\mathfrak{A}$ correspond (subject to decisions on how to treat negligible sets) to countable partitions of X into measurable sets, and an inverse-measure-preserving function $f: X \to X$ gives rise to a measure-preserving homomorphism $\pi_f : \mathfrak{A} \to \mathfrak{A}$; so we can define the entropy of f to be $h(\pi_f)$. The whole point of the language I have sought to develop in this volume is that we can do this when and if we choose; in particular, we are not limited to those homomorphisms which are representable by inverse-measure-preserving functions. But of course a large proportion of the most important examples do arise in this way (see 385Xl, 385Xm). The same two examples are instructive from another point of view: the case $k = 2$ of 385Xl is (almost) isomorphic to the tent map of 385Xm. The similarity is obvious, but exhibiting an actual isomorphism is I think another matter (385Yh).

I must say 'almost' isomorphic here because the doubling map on $[0,1[$ is everywhere two-to-one, while the tent map is not, so they cannot be isomorphic in any exact sense. This is the problem grappled with in 385T-385V. In some moods I would say that a dislike of such contortions is a sign of civilized taste. Certainly it is part of my motivation for working with measure algebras whenever possible. But I have to say also that new ideas in this topic arise more often than not from actual measure spaces, and that it is absolutely necessary to be able to operate in the more concrete context.

Version of 20.8.15

386 More about entropy

In preparation for the next two sections, I present a number of basic facts concerning measure-preserving homomorphisms and entropy. Compared with the work to follow, they are mostly fairly elementary, but the Halmos-Rokhlin-Kakutani lemma (386C) and the Shannon-McMillan-Breiman theorem (386E), in their full strengths, go farther than one might expect.

As in §385, I write $q(0) = 0$, $q(t) = -t\ln t$ for $t > 0$.

386A I start by returning to the notion of 'recurrence' from 381L-381P, in its original home.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is recurrent on every $a \in \mathfrak{A}$.

proof If $a \in \mathfrak{A}$ is non-zero, then $\sum_{k=0}^{\infty}\bar\mu(\pi^k a) = \infty > \mu 1$, so there are $i < j$ such that $0 \ne \pi^i a \cap \pi^j a = \pi^i(a \cap \pi^{j-i}a)$ and $a \cap \pi^{j-i}a \ne 0$. Thus (ii) of 381O is satisfied; by 381O, π is recurrent on every $a \in \mathfrak{A}$.

386B Corollary Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{C}$ be its fixed-point subalgebra $\{c : c \in \mathfrak{A},\ \pi c = c\}$. Then

$$\sup_{k\ge n}\pi^k a = \mathrm{upr}(a,\mathfrak{C}) = \inf\{c : a \subseteq c \in \mathfrak{C}\} \in \mathfrak{C}$$

for any $a \in \mathfrak{A}$ and $n \in \mathbb{N}$.

proof By 386A and 381O, $a \subseteq \sup_{k\ge1}\pi^k a$. Set $a^* = \sup_{k\in\mathbb{N}}\pi^k a$; by 381Kb, $a^* = \sup_{k\ge n}\pi^k a$ for every n; by 381Ka, $a^* \in \mathfrak{C}$. Also, of course, $a^* \subseteq c$ whenever $a \subseteq c \in \mathfrak{C}$, so $a^* = \mathrm{upr}(a,\mathfrak{C})$.

386C The Halmos-Rokhlin-Kakutani lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism, with fixed-point subalgebra $\mathfrak{C}$. Then the following are equiveridical:

(i) π is aperiodic;

(ii) $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$ (definition: 331A);

(iii) whenever $n\ge 1$ and $0\le\gamma<\frac{1}{n}$ there is an $a\in\mathfrak{A}$ such that a, πa, $\pi^2a,\dots,\pi^{n-1}a$ are disjoint and $\bar\mu(a\cap c)=\gamma\bar\mu c$ for every $c\in\mathfrak{C}$;

(iv) whenever $n\ge 1$, $0\le\gamma<\frac{1}{n}$ and $B\subseteq\mathfrak{A}$ is finite, there is an $a\in\mathfrak{A}$ such that a, πa, $\pi^2a,\dots,\pi^{n-1}a$ are disjoint and $\bar\mu(a\cap b)=\gamma\bar\mu b$ for every $b\in B$.

proof Note that $\mathfrak{C}$ is (order-)closed because π is (order-)continuous (324Kb).

(i)⇒(ii) Put 386A and 381P together.

(ii)⇒(iii) Set $\delta=\frac{1}{n}(\frac{1}{n}-\gamma)>0$. By 331B, there is a $d\in\mathfrak{A}$ such that $\bar\mu(c\cap d)=\delta\bar\mu c$ for every $c\in\mathfrak{C}$. Set $d_k=\pi^kd\setminus\sup_{i<k}\pi^id$ for $k\in\mathbb{N}$. Note that

$$d_{j+k}=\pi^{j+k}d\setminus\sup_{i<j+k}\pi^id\subseteq\pi^{j+k}d\setminus\sup_{i<k}\pi^{j+i}d=\pi^jd_k$$

whenever j, $k\in\mathbb{N}$. Next, $\pi^id_j\cap d_k\subseteq\sup_{m\le i}d_m$ for any i, j, $k\in\mathbb{N}$ such that $i+j\ne k$. **P** (α) If $k\le i$ this is obvious. (β) If $i<k<i+j$ then

$$\pi^id_j\cap d_k\subseteq\pi^id_j\cap\pi^id_{k-i}=\pi^i(d_j\cap d_{k-i})=0.$$

(γ) If $i+j<k$, then

$$\pi^id_j\cap d_k\subseteq\pi^{i+j}d\cap d_k=0.\ \mathbf{Q}$$

Setting $c^*=\sup_{i\in\mathbb{N}}d_i=\sup_{i\in\mathbb{N}}\pi^id$, we have $c^*\in\mathfrak{C}$, by 386B, so that $\bar\mu(d\setminus c^*)=\delta\bar\mu(1\setminus c^*)$; but as $d\subseteq c^*$, $c^*=1$.

Set $a^*=\sup_{m\in\mathbb{N}}d_{mn}$ (the mn here is a product, not a double subscript!), $d^*=\sup_{i<n}d_i=\sup_{i<n}\pi^id$. Then

$$\bar\mu(c\cap d^*)\le\sum_{i=0}^{n-1}\bar\mu(c\cap\pi^id)=\sum_{i=0}^{n-1}\bar\mu\pi^i(c\cap d)=n\bar\mu(c\cap d)=n\delta\bar\mu c$$

for every $c\in\mathfrak{C}$. Next, $\pi^id_{mn}\supseteq d_{mn+i}$ for all m and i, so

$$\sup_{i<n}\pi^ia^*=\sup_{i\in\mathbb{N}}d_i=1.$$

Consequently

$$\bar\mu c\le\sum_{i=0}^{n-1}\bar\mu(c\cap\pi^ia^*)=n\bar\mu(c\cap a^*),$$

$$\bar\mu(c\cap a^*\setminus d^*)\ge\bar\mu(c\cap a^*)-\bar\mu(c\cap d^*)\ge(\tfrac{1}{n}-n\delta)\bar\mu c=\gamma\bar\mu c$$

for every $c\in\mathfrak{C}$.

By 331B again (applied to the principal ideal of $\mathfrak{A}$ generated by $a^*\setminus d^*$) there is an $a\subseteq a^*\setminus d^*$ such that $\bar\mu(a\cap c)=\gamma\bar\mu c$ for every $c\in\mathfrak{C}$. For $0<i<n$,

$$\pi^ia^*\cap a^*=\sup_{k,l\in\mathbb{N}}\pi^id_{kn}\cap d_{ln}\subseteq\sup_{m\le i}d_m\subseteq d^*,$$

so $\pi^ia\cap a=0$; accordingly a, $\pi a,\dots,\pi^{n-1}a$ are all disjoint and (iii) is satisfied.

(iii)⇒(iv) Note that $\mathfrak{A}$ is certainly atomless, since for every $k\ge 1$ we can find a $c\in\mathfrak{A}$ such that $c,\pi c,\dots,\pi^{k-1}c$ are disjoint and $\bar\mu c=\frac{\bar\mu 1}{k+1}$, so that we have a partition of unity consisting of sets of measure $\frac{\bar\mu 1}{k+1}$. Let B' be the set of atoms of the (finite) subalgebra of $\mathfrak{A}$ generated by B, and $m=\#(B')$. Let $\delta>0$ and $k\in\mathbb{N}$ be such that

$$3\delta\le(1-n\gamma)\bar\mu b\text{ for every }b\in B',\quad m(\bar\mu1)^2<k\delta^2,\quad k\delta\ge\bar\mu1.$$

By (iii), there is a $c\in\mathfrak{A}$ such that $c,\pi c,\dots,\pi^{nk^2-1}c$ are disjoint and $\bar\mu(\sup_{i<nk^2}\pi^ic)=1-\delta$. For $j<k$, set $e_j=\sup_{l<k,i<n}\pi^{nkj+nl+i}c$, $d_j=\sup_{l<k-1}\pi^{nkj+nl}c$. Observe that e_j, $\pi e_j,\dots,\pi^{n-1}e_j$ are disjoint, and that $\pi^id_j\subseteq e_j$ for $i<2n$. Set $e=\sup_{j<k}e_j=\sup_{i<nk^2}\pi^ic$, so that $\bar\mu e=1-\delta$.

Suppose we choose $d \in \mathfrak{A}$ by the following random process. Take $s(0),\dots,s(k-1)$ independently in $\{0,\dots,n-1\}$, so that $\Pr(s(j)=l)=\frac{1}{n}$ for each $l<n$, and set $d=\sup_{j<k}\pi^{s(j)}d_j$. Because we certainly have $\pi^i\pi^{s(j)}d_j \subseteq e_j$ whenever $i<n$, $d,\pi d,\dots,\pi^{n-1}d$ will be disjoint. Now for any $b\in\mathfrak{A}$,

$$\Pr\bigl(\bar\mu(d\cap b)\le\frac{1}{n}(\bar\mu b-3\delta)\bigr)<\frac{1}{m}.$$

P We can express the random variable $\bar\mu(d\cap b)$ as $X=\sum_{j=0}^{k-1}X_j$, where $X_j=\bar\mu(\pi^{s(j)}d_j\cap b)$. Then the X_j are independent random variables. For each j, X_j takes values between 0 and $\bar\mu d_j=(k-1)\bar\mu c\le\frac{\bar\mu 1}{nk}$, and has expectation $\frac{1}{n}\bar\mu(e'_j\cap b)$, where

$$e'_j=\sup_{i<n}\pi^i d_j=\sup_{l<k-1,i<n}\pi^{nkj+nl+i}c.$$

So X has expectation $\frac{1}{n}\bar\mu(e'\cap b)$ where $e'=\sup_{j<k}e'_j$. Now

$$e_j\setminus e'_j=\sup_{i<n}\pi^{nkj+n(k-1)+i}c$$

has measure $n\bar\mu c\le\frac{n\bar\mu 1}{nk^2}$ for each j, so $\bar\mu(e\setminus e')\le\frac{\bar\mu 1}{k}$ and $\bar\mu(1\setminus e')\le 2\delta$; thus $\mathbb{E}(X)\ge\frac{1}{n}(\bar\mu b-2\delta)$, while

$$\operatorname{Var}(X)=\sum_{j=0}^{k-1}\operatorname{Var}(X_j)\le k\bigl(\frac{\bar\mu 1}{nk}\bigr)^2=\frac{(\bar\mu 1)^2}{n^2k}.$$

But this means that

$$\frac{(\bar\mu 1)^2}{n^2k}\ge\bigl(\frac{\delta}{n}\bigr)^2\Pr\bigl(X\le\frac{1}{n}(\bar\mu b-3\delta)\bigr),$$

and

$$\Pr\bigl(X\le\frac{1}{n}(\bar\mu b-3\delta)\bigr)\le\frac{(\bar\mu 1)^2}{k\delta^2}<\frac{1}{m}$$

by the choice of k. **Q**

This is true for every $b\in B'$, while $\#(B')=m$. There must therefore be some choice of $s(0),\dots,s(k-1)$ such that, taking $d^*=\sup_{j<k}\pi^{s(j)}d_j$,

$$\bar\mu(d^*\cap b)\ge\frac{1}{n}(\bar\mu b-3\delta)\ge\gamma\bar\mu b$$

for every $b\in B'$, while $d^*,\pi d^*,\dots,\pi^{n-1}d^*$ are disjoint. Because $\mathfrak{A}$ is atomless, there is a $d\subseteq d^*$ such that $\bar\mu(d\cap b)=\gamma\bar\mu b$ for every $b\in B'$. Since every member of B is a disjoint union of members of B', $\bar\mu(d\cap b)=\gamma\bar\mu b$ for every $b\in B$.

(iv)⇒(i) If $a\in\mathfrak{A}\setminus\{0,1\}$ and $n\ge 1$ then (iv) tells us that there is a $b\in\mathfrak{A}$ such that $b,\pi b,\dots,\pi^n b$ are all disjoint and $\bar\mu(1\setminus\sup_{i\le n}\pi^i b)<\bar\mu a$. Now there must be some $i<n$ such that $d=\pi^i b\cap a\ne 0$, in which case

$$d\cap\pi^n d\subseteq\pi^i b\cap\pi^{i+n}b=\pi^i(b\cap\pi^n b)=0,$$

and $\pi^n d\ne d$. As n and a are arbitrary, π is aperiodic.

386D Corollary An ergodic measure-preserving Boolean homomorphism on an atomless totally finite measure algebra is aperiodic.

proof By 372Pa, this is (ii)⇒(i) of 386C in the case $\mathfrak{C}=\{0,1\}$ (compare 381P).

386E I turn now to a celebrated result which is a kind of strong law of large numbers.

The Shannon-McMillan-Breiman theorem Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism and $A\subseteq\mathfrak{A}$ a partition of unity of finite entropy. For each $n\ge 1$, set

$$w_n=\frac{1}{n}\sum_{d\in D_n(A,\pi)}\ln\bigl(\frac{1}{\bar\mu d}\bigr)\chi d,$$

where $D_n(A,\pi)$ is the partition of unity generated by $\{\pi^i a : a \in A,\ i < n\}$, as in 385K. Then $\langle w_n\rangle_{n\in\mathbb{N}}$ is norm-convergent in $L^1 = L^1(\mathfrak{A},\bar\mu)$ to w say; moreover, $\langle w_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to w (definition: 367A). If $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is the Riesz homomorphism defined by π, so that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$ (364P), then $Tw = w$.

proof (Petersen 83) We may suppose that $0 \notin A$.

(a) For each $n \in \mathbb{N}$, let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $\{\pi^i a : a \in A,\ 1 \le i \le n\}$, B_n the set of its atoms, and P_n the corresponding conditional expectation operator on L^1 (365Q). Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$, and P the corresponding conditional expectation operator. Observe that $B_n = \pi[D_n(A,\pi)]$ and that, in the language of 385F, $D_{n+1}(A,\pi) = A \vee B_n$. Let $\mathfrak{C}$ be the fixed-point subalgebra of π and Q the associated conditional expectation. Set $L^0 = L^0(\mathfrak{A})$, and let $\bar{\ln}$ be the function from $\{v : [\![v > 0]\!] = 1\}$ to L^0 corresponding to $\ln :]0,\infty[\to \mathbb{R}$ (364H).

(b) It will save a moment later if I note a simple fact here: if $v \in L^1$, then $\langle \frac{1}{n}T^n v\rangle_{n\ge1}$ is order*-convergent and $\| \|_1$-convergent to 0. **P** We know from the ergodic theorem (372G) that $\langle \tilde v_n\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to Qv, where $\tilde v_n = \frac{1}{n+1}\sum_{i=0}^n T^i v$. Now $\frac{1}{n}T^n v = \frac{n+1}{n}\tilde v_n - \tilde v_{n-1}$ is order*-convergent and $\| \|_1$-convergent to $Qv - Qv = 0$ (using 367C for 'order*-convergent'). **Q**

(c) Set

$$v_n = \sum_{a\in A} P_n(\chi a) \times \chi a = \sum_{a\in A, b\in B_n} \frac{\bar\mu(a\cap b)}{\bar\mu b}\chi(a \cap b).$$

By Lévy's martingale theorem (275I, 367Jb),

$$\langle v_n \times \chi a\rangle_{n\in\mathbb{N}} = \langle P_n(\chi a) \times \chi a\rangle_{n\in\mathbb{N}}$$

is order*-convergent to $P(\chi a) \times \chi a$ for every $a \in A$; consequently $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $v = \sum_{a\in A} P(\chi a) \times \chi a$. It follows that $\langle \bar{\ln}\, v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln}\, v$. **P** The point is that, for any $a \in A$ and $n \in \mathbb{N}$, $a \subseteq [\![P_n(\chi a) > 0]\!]$, so that $[\![v_n > 0]\!] = 1$ for every n, and $\bar{\ln}\, v_n$ is defined. Similarly, $\bar{\ln}\, v$ is defined, and $\langle \bar{\ln}\, v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln}\, v$ by 367H. **Q** As $0 \le v_n \le \chi 1$ for every n, $\langle v_n\rangle_{n\in\mathbb{N}} \to v$ for $\| \|_1$, by the Dominated Convergence Theorem (367I).

Next, $\langle \bar{\ln}\, v_n\rangle_{n\in\mathbb{N}}$ is order-bounded in L^1. **P** Of course $\bar{\ln}\, v_n \le 0$ for every n, because $P_n(\chi a) \le P_n(\chi 1) \le \chi 1$ for each a, so $v_n \le \chi 1$. To see that $\{\bar{\ln}\, v_n : n \in \mathbb{N}\}$ is bounded below in L^1, we use an idea from the fundamental martingale inequality 275D. Set $v_* = \inf_{n\in\mathbb{N}} v_n$. For $\alpha > 0$, $a \in A$ and $n \in \mathbb{N}$ set

$$b_{an}(\alpha) = [\![P_n(\chi a) < \alpha]\!] \cap \inf_{i<n} [\![P_i(\chi a) \ge \alpha]\!],$$

so that

$$[\![v_* < \alpha]\!] = \sup_{a\in A, n\in\mathbb{N}} a \cap b_{an}(\alpha).$$

Now $b_{an}(\alpha) \in \mathfrak{B}_n$, so

$$\bar\mu(a \cap b_{an}(\alpha)) = \int_{b_{an}(\alpha)} \chi a = \int_{b_{an}(\alpha)} P_n(\chi a) \le \alpha\bar\mu(b_{an}(\alpha)),$$

and

$$\begin{aligned}\bar\mu(a \cap [\![v_* < \alpha]\!]) &\le \min(\bar\mu a, \sum_{n=0}^\infty \bar\mu(a \cap b_{an}(\alpha)))\\ &\le \min(\bar\mu a, \alpha\sum_{n=0}^\infty \bar\mu b_{an}(\alpha)) \le \min(\bar\mu a, \alpha).\end{aligned}$$

Letting $\alpha \downarrow 0$, $\bar\mu(a \cap [\![v_* = 0]\!]) = 0$ for every $a \in A$, so $[\![v_* > 0]\!] = 1$, and $\bar{\ln}\, v_*$ is defined. Moreover,

$$\bar\mu(a \cap [\![-\bar{\ln}\, v_* > -\ln\alpha]\!]) = \bar\mu(a \cap [\![v_* < \alpha]\!]) \le \min(\bar\mu a, \alpha)$$

for every $a \in A$, $\alpha > 0$; that is,

$$\bar\mu(a \cap [\![-\bar{\ln}\, v_* > \beta]\!]) \le \min(\bar\mu a, e^{-\beta})$$

for every $a \in A$ and $\beta \in \mathbb{R}$. Accordingly

$$\begin{aligned}\int(-\bar{\ln} v_*) &= \int_0^\infty \bar\mu[\![-\bar{\ln} v_* > \beta]\!]d\beta = \sum_{a\in A}\int_0^\infty \bar\mu(a\cap[\![-\bar{\ln} v_* > \beta]\!])d\beta \\ &\le \sum_{a\in A}\int_0^\infty \min(\bar\mu a, e^{-\beta})d\beta \\ &= \sum_{a\in A}\Bigl(\int_0^{\ln(1/\bar\mu a)} \bar\mu a\,d\beta + \int_{\ln(1/\bar\mu a)}^\infty e^{-\beta}d\beta\Bigr) \\ &= \sum_{a\in A}\bigl(\ln(\frac{1}{\bar\mu a})\bar\mu a + e^{\ln\bar\mu a}\bigr) \\ &= \sum_{a\in A}\ln(\frac{1}{\bar\mu a})\bar\mu a + \sum_{a\in A}\bar\mu a = H(A)+1<\infty\end{aligned}$$

because A has finite entropy. But this means that $\bar{\ln} v_*$ belongs to L^1, and of course it is a lower bound for $\{\bar{\ln} v_n : n\in\mathbb{N}\}$. **Q**

By 367I again, $\bar{\ln} v \in L^1$ and $\langle\bar{\ln} v_n\rangle_{n\in\mathbb{N}} \to \bar{\ln} v$ for $\| \|_1$.

(d) Fix $n\in\mathbb{N}$ for the moment. For each $d\in D_{n+1}(A,\pi)$ let d' be the unique element of B_n such that $d\subseteq d'$. Then

$$\begin{aligned}(n+1)w_{n+1} &= \sum_{d\in D_{n+1}(A,\pi)}\ln(\frac{1}{\bar\mu d'})\chi d - \sum_{d\in D_{n+1}(A,\pi)}\ln(\frac{\bar\mu d}{\bar\mu d'})\chi d \\ &= \sum_{b\in B_n}\ln(\frac{1}{\bar\mu b})\chi b - \sum_{\substack{a\in A\\ b\in B_n\\ a\cap b\ne 0}}\ln(\frac{\bar\mu(a\cap b)}{\bar\mu b})\chi(a\cap b) \\ &= \sum_{d\in D_n(A,\pi)}\ln(\frac{1}{\bar\mu(\pi d)})\chi(\pi d) - \bar{\ln} v_n = T(nw_n) - \bar{\ln} v_n.\end{aligned}$$

Inducing on n, starting from

$$w_1 = \textstyle\sum_{a\in A}\ln(\frac{1}{\bar\mu a})\chi a = -\bar{\ln} v_0,$$

we get

$$nw_n = \textstyle\sum_{i=0}^{n-1} T^i(-\bar{\ln} v_{n-i-1}), \quad w_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i(-\bar{\ln} v_{n-i-1})$$

for every $n\ge 1$.

(e) Set $w'_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i(-\bar{\ln} v)$ for $n\ge 1$. By the Ergodic Theorem, $\langle w'_n\rangle_{n\ge 1}$ is order*-convergent and $\| \|_1$-convergent to $w = Q(-\bar{\ln} v)$, and $Tw = w$. To estimate $w_n - w'_n$, set $u^*_n = \sup_{k\ge n}|\bar{\ln} v_k - \bar{\ln} v|$ for each $n\in\mathbb{N}$. Then $\langle u^*_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence, $u^*_0\in L^1$ (by (c) above), and $\inf_{n\in\mathbb{N}} u^*_n = 0$ because $\langle\bar{\ln} v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{\ln} v$. Now, whenever $n > m\in\mathbb{N}$,

$$
\begin{aligned}
|w_n - w'_n| &\le \frac{1}{n}\sum_{i=0}^{n-1} T^i|\bar{\ln} v - \bar{\ln} v_{n-i-1}| \\
&= \frac{1}{n}\big(\sum_{i=0}^{n-m-1} T^i|\bar{\ln} v - \bar{\ln} v_{n-i-1}| + \sum_{i=n-m}^{n-1} T^i|\bar{\ln} v - \bar{\ln} v_{n-i-1}|\big) \\
&\le \frac{1}{n}\big(\sum_{i=0}^{n-m-1} T^i u_m^* + \sum_{j=0}^{m-1} T^{n-1-j}|\bar{\ln} v - \bar{\ln} v_j|\big) \\
&\le \frac{1}{n-m}\big(\sum_{i=0}^{n-m-1} T^i u_m^* + \sum_{j=0}^{m-1} T^{n-1-j} u_0^*\big) \\
&= \frac{1}{n-m}\sum_{i=0}^{n-m-1} T^i u_m^* + \frac{1}{n-m}T^{n-m}\sum_{j=0}^{m-1} T^{m-1-j} u_0^* \\
&= \frac{1}{n-m}\sum_{i=0}^{n-m-1} T^i u_m^* + \frac{1}{n-m}T^{n-m}\tilde{u}_m,
\end{aligned}
$$

setting $\tilde{u}_m = \sum_{j=0}^{m-1} T^{m-1-j}u_0^*$.

Holding m fixed and letting $n \to \infty$, we know that

$$\frac{1}{n-m}\sum_{i=0}^{n-m-1} T^i u_m^*$$

is order*-convergent and $\| \|_1$-convergent to Qu_m^*. As for the other term, $\frac{1}{n-m}T^{n-m}\tilde{u}_m$ is order*-convergent and $\| \|_1$-convergent to 0, by (b). What this means is that

$$\limsup_{n\to\infty} |w_n - w'_n| \le Qu_m^*,$$

$$\limsup_{n\to\infty} \|w_n - w'_n\|_1 \le \|Qu_m^*\|_1$$

for every $m \in \mathbb{N}$. Since $\langle Qu_m^*\rangle_{m\in\mathbb{N}}$ is surely a non-decreasing sequence with infimum 0,

$$\limsup_{n\to\infty} |w_n - w'_n| = 0, \quad \limsup_{n\to\infty} \|w_n - w'_n\|_1 = 0.$$

Since w'_n is order*-convergent and $\| \|_1$-convergent to w, so is w_n.

386F Corollary If, in 386E, π is ergodic, then $\langle w_n\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to $h(\pi, A)\chi 1$.

proof Because the limit w in 386E has $Tw = w$, it must be of the form $\gamma\chi 1$, because π is ergodic (372Q(a-iii)). Now $\gamma = \int w$ must be

$$\lim_{n\to\infty}\int w_n = \lim_{n\to\infty}\frac{1}{n}\sum_{d\in D_n(A,\pi)} \ln(\frac{1}{\bar{\mu}d})\bar{\mu}d = \lim_{n\to\infty}\frac{1}{n}\sum_{d\in D_n(A,\pi)} q(\bar{\mu}d)$$

(where q is the function of 385A)

$$= \lim_{n\to\infty}\frac{1}{n}H(D_n(A,\pi)) = h(\pi, A).$$

386G The Csisaár-Kullback inequality (Csiszár 67, Kullback 67) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and u a member of $L^1(\mathfrak{A}, \bar{\mu})^+$ such that $\int u = 1$. Then

$$\int \bar{q}(u) \le -\frac{1}{2}(\int |u - \chi 1|)^2.$$

proof Set $a = [\![u < 1]\!]$, $\alpha = \bar{\mu}a$, $\beta = \int_a u$, $b = 1 \setminus a$. Then $\bar{\mu}b = 1 - \alpha$ and $\int_b u = 1 - \beta$. Surely $\beta \le \alpha < 1$. If $\alpha = 0$ then $u = \chi 1$ and the result is trivial; so let us suppose that $0 < \alpha < 1$. Because the function q is concave,

$$\int_a \bar{q}(u) \le \bar{\mu}a \cdot q(\frac{1}{\bar{\mu}a}\int_a u) = \alpha q(\frac{\beta}{\alpha}) = q(\beta) + \beta \ln \alpha,$$

(using 233Ib/365Qb for the inequality), and similarly

$$\int_b \bar{q}(u) \le q(1-\beta) + (1-\beta)\ln(1-\alpha).$$

Also

$$\int |u - \chi 1| = \int_a (\chi 1 - u) + \int_b (u - \chi 1) = \alpha - \beta + (1-\beta) - (1-\alpha) = 2(\alpha - \beta),$$

so

$$\begin{aligned}\int \bar{q}(u) + \frac{1}{2}(\int |u - \chi 1|)^2 &\le q(\beta) + \beta \ln \alpha + q(1-\beta) + (1-\beta)\ln(1-\alpha) + 2(\alpha-\beta)^2 \\ &= \phi(\beta)\end{aligned}$$

say. Now ϕ is continuous on $[0,1]$ and arbitrarily often differentiable on $]0,1[$,

$$\phi(\alpha) = 0,$$

$$\phi'(t) = -\ln t + \ln \alpha + \ln(1-t) - \ln(1-\alpha) - 4(\alpha - t) \text{ for } t \in]0,1[,$$

$$\phi'(\alpha) = 0,$$

$$\phi''(t) = -\frac{1}{t} - \frac{1}{1-t} + 4 \le 0 \text{ for } t \in]0,1[.$$

So $\phi(t) \le 0$ for $t \in [0,1]$ and, in particular, $\phi(\beta) \le 0$; but this means that

$$\int \bar{q}(u) + \frac{1}{2}(\int |u - \chi 1|)^2 \ge 0,$$

that is, $\int \bar{q}(u) \le -\frac{1}{2}(\int |u - \chi 1|)^2$, as claimed.

386H Corollary Whenever $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and A, B are partitions of unity of finite entropy,

$$\sum_{a \in A, b \in B} |\bar{\mu}(a \cap b) - \bar{\mu}a \cdot \bar{\mu}b| \le \sqrt{2(H(A) + H(B) - H(A \vee B))}.$$

proof Replacing A, B by $A \setminus \{0\}$ and $B \setminus \{0\}$ if necessary, we may suppose that neither A nor B contains 0. Let $(\mathfrak{C}, \bar{\lambda})$ be the probability algebra free product of $(\mathfrak{A}, \bar{\mu})$ with itself (325E, 325K). Set

$$u = \sum_{a \in A, b \in B} \frac{\bar{\mu}(a \cap b)}{\bar{\mu}a \cdot \bar{\mu}b} \chi(a \otimes b) \in L^0(\mathfrak{C});$$

then u is non-negative and integrable and $\int u = \sum_{a \in A, b \in B} \bar{\mu}(a \cap b) = 1$. Now

$$\begin{aligned}\int \bar{q}(u) &= -\sum_{a \in A, b \in B} \bar{\mu}(a \cap b) \ln \frac{\bar{\mu}(a \cap b)}{\bar{\mu}a \cdot \bar{\mu}b} \\ &= H(A \vee B) + \sum_{a \in A, b \in B} \bar{\mu}(a \cap b) \ln \bar{\mu}a + \sum_{a \in A, b \in B} \bar{\mu}(a \cap b) \ln \bar{\mu}b \\ &= H(A \vee B) + \sum_{a \in A} \bar{\mu}a \ln \bar{\mu}a + \sum_{b \in B} \bar{\mu}b \ln \bar{\mu}b \\ &= H(A \vee B) - H(A) - H(B).\end{aligned}$$

On the other hand,

$$\int|u-\chi 1| = \sum_{a\in A,b\in B}\bar\mu a\cdot\bar\mu b|\frac{\bar\mu(a\cap b)}{\bar\mu a\cdot\bar\mu b}-1| = \sum_{a\in A,b\in B}|\bar\mu(a\cap b)-\bar\mu a\cdot\bar\mu b|.$$

Now 386G tells us that $(\int|u-\chi 1|)^2 \le -2\int\bar q(u)$, so

$$\sum_{a\in A,b\in B}|\bar\mu(a\cap b)-\bar\mu a\cdot\bar\mu b| \le \sqrt{-2\int\bar q} = \sqrt{2(H(A)+H(B)-H(A\vee B)},$$

as required.

386I The next six lemmas are notes on more or less elementary facts which will be used at various points in the next section. The first two are nearly trivial.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ two partitions of unity in $\mathfrak{A}$. Then

$$\bar\mu(\sup_{i\in I}a_i\cap b_i) = 1-\frac{1}{2}\sum_{i\in I}\bar\mu(a_i\triangle b_i).$$

proof

$$\begin{aligned}\bar\mu(\sup_{i\in I}a_i\cap b_i) &= \sum_{i\in I}\bar\mu(a_i\cap b_i) = \sum_{i\in I}\frac{1}{2}(\bar\mu a_i+\bar\mu b_i-\bar\mu(a_i\triangle b_i))\\ &= 1-\frac{1}{2}\sum_{i\in I}\bar\mu(a_i\triangle b_i).\end{aligned}$$

386J Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, $\langle B_k\rangle_{k\in\mathbb{N}}$ a non-decreasing sequence of subsets of $\mathfrak{A}$ such that $0\in B_0$ and $\langle c_i\rangle_{i\in I}$ a partition of unity in $\mathfrak{A}$. Set $B=\overline{\bigcup_{i<k}B_k}$. Then

$$\lim_{k\to\infty}\sup_{i\in I}\rho(c_i,B_k) = \sup_{i\in I}\rho(c_i,B),$$

writing $\rho(c,B)=\inf_{b\in B}\bar\mu(c\triangle b)$ for $c\in\mathfrak{A}$ and non-empty $B\subseteq\mathfrak{A}$, as in 3A4I, and counting $\sup\emptyset$ as 0.

proof Of course $\langle\sup_{i\in I}\rho(c_i,B_k)\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence and $\lim_{k\to\infty}\sup_{i\in I}\rho(c_i,B_k)\ge\sup_{i\in I}\rho(c_i,B)$. For the reverse inequality, let $\epsilon>0$. Then $J=\{j: j\in I, \bar\mu c_j\ge\epsilon\}$ is finite. For each $j\in J$, $\lim_{k\to\infty}\rho(c_j,B_k)=\rho(c_j,B)$, by 3A4I, while

$$\rho(c_i,B_k)\le\bar\mu(c_i\triangle 0)=\bar\mu c_i\le\epsilon$$

for every $i\in I\setminus J$. So

$$\lim_{k\to\infty}\sup_{i\in I}\rho(c_i,B_k)\le\max(\epsilon,\lim_{k\to\infty}\sup_{i\in J}\rho(c_i,B_k)) = \max(\epsilon,\sup_{i\in J}\lim_{k\to\infty}\rho(c_i,B_k))$$

(because J is finite)

$$= \max(\epsilon,\sup_{i\in J}\rho(c_i,B))$$

by 3A4J. As ϵ is arbitrary, $\lim_{k\to\infty}\sup_{i\in I}\rho(c_i,B_k)\le\sup_{i\in I}\rho(c_i,B)$ and we have the result.

386K Lemma Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let A, B and C be partitions of unity in $\mathfrak{A}$.

(a) $H(A\vee B\vee C)+H(C)\le H(B\vee C)+H(A\vee C)$.

(b) $h(\pi,A)\le h(\pi,A\vee B)\le h(\pi,A)+h(\pi,B)\le h(\pi,A)+H(B)$.

(c) If $H(A)<\infty$,

$$\begin{aligned}h(\pi,A) &= \inf_{n\in\mathbb{N}}H(D_{n+1}(A,\pi))-H(D_n(A,\pi))\\ &= \lim_{n\to\infty}H(D_{n+1}(A,\pi))-H(D_n(A,\pi)).\end{aligned}$$

(d) If $H(A)<\infty$ and $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]\subseteq\mathfrak{B}$, then $h(\pi,A)\le h(\pi\restriction\mathfrak{B})+H(A|\mathfrak{B})$.

proof (a) Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by C, so that $\mathfrak{C}$ is purely atomic and C is the set of its atoms. Then

$$\begin{aligned} H(A \vee B \vee C) + H(C) &= H(A \vee B|\mathfrak{C}) + 2H(C) \\ &\le H(A|\mathfrak{C}) + H(B|\mathfrak{C}) + 2H(C) = H(A \vee C) + H(B \vee C) \end{aligned}$$

by 385Gb and 385Ga.

(b) We need only observe that $D_n(A \vee B, \pi) = D_n(A, \pi) \vee D_n(B, \pi)$ for every $n \in \mathbb{N}$, being the partition of unity generated by $\{\pi^i a : i < n,\, a \in A\} \cup \{\pi^i b : i < n,\, b \in B\}$. Consequently

$$\begin{aligned} h(\pi, A) &= \lim_{n\to\infty} \frac{1}{n} H(D_n(A, \pi)) \le \lim_{n\to\infty} \frac{1}{n} H(D_n(A, \pi) \vee D_n(B, \pi)) \\ &= \lim_{n\to\infty} \frac{1}{n} H(D_n(A \vee B, \pi)) = h(\pi, A \vee B) \\ &\le \lim_{n\to\infty} \frac{1}{n} (H(D_n(A, \pi) + H(D_n(B, \pi))) = h(\pi, A) + h(\pi, B) \\ &\le h(\pi, A) + H(B) \end{aligned}$$

as remarked in 385M.

(c) Set $\gamma_n = H(D_{n+1}(A, \pi)) - H(D_n(A, \pi))$ for each $n \in \mathbb{N}$. By 385H, $\gamma_n \ge 0$. From (a) we see that

$$\begin{aligned} \gamma_{n+1} &= H(A \vee \pi[D_{n+1}(A, \pi)]) - H(A \vee \pi[D_n(A, \pi)]) \\ &\le H(\pi[D_{n+1}(A, \pi)]) - H(\pi[D_n(A, \pi]) = \gamma_n \end{aligned}$$

for every $n \in \mathbb{N}$. So $\lim_{n\to\infty} \gamma_n = \inf_{n\in\mathbb{N}} \gamma_n$; write γ for the common value. Now

$$h(\pi, A) = \lim_{n\to\infty} \frac{1}{n} H(D_n(A, \pi)) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \gamma_i = \gamma$$

(273Ca).

(d) Let $P : L^1_{\bar\mu} \to L^1_{\bar\mu}$ be the conditional expectation operator corresponding to $\mathfrak{B}$. Let $\langle b_k \rangle_{k\in\mathbb{N}}$ be a sequence running over $\{[\![P(\chi a) > p]\!] : a \in A,\, p \in \mathbb{Q}\}$, so that $b_k \in \mathfrak{B}$ for every k, and for each $k \in \mathbb{N}$ let $\mathfrak{B}_k \subseteq \mathfrak{B}$ be the subalgebra generated by $\{b_i : i \le k\}$; let P_k be the conditional expectation operator corresponding to $\mathfrak{B}_k$. Writing $\mathfrak{B}_\infty \subseteq \mathfrak{B}$ for $\overline{\bigcup_{k\in\mathbb{N}} \mathfrak{B}_k}$, and P_∞ for the corresponding conditional expectation operator, then $P(\chi a) \in L^0(\mathfrak{B}_\infty)$, so $P_\infty(\chi a) = P(\chi a)$, for every $a \in A$. So

$$H(A|\mathfrak{B}) = \sum_{a\in A} \int \bar q(P\chi a) = H(A|\mathfrak{B}_\infty) = \lim_{k\to\infty} H(A|\mathfrak{B}_k),$$

by 385Gd.

For each k, let B_k be the set of atoms of $\mathfrak{B}_k$. Then

$$h(\pi, A) \le h(\pi, B_k) + H(A|\mathfrak{B}_k) \le h(\pi{\restriction}\mathfrak{B}) + H(A|\mathfrak{B}_k)$$

by 385N and the definition of $h(\pi{\restriction}\mathfrak{B})$. So

$$h(\pi, A) \le h(\pi{\restriction}\mathfrak{B}) + \lim_{k\to\infty} H(A|\mathfrak{B}_k) = h(\pi{\restriction}\mathfrak{B}) + H(A|\mathfrak{B}).$$

386L Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra.

(a) There is a function $h : \mathfrak{A} \to \mathfrak{B}$ such that $\bar\mu(a \mathbin{\triangle} h(a)) = \rho(a, \mathfrak{B})$ for every $a \in \mathfrak{A}$ and $h(a) \cap h(a') = 0$ whenever $a \cap a' = 0$.

(b) If A is a partition of unity in $\mathfrak{A}$, then $H(A|\mathfrak{B}) \le \sum_{a\in A} q(\rho(a, \mathfrak{B}))$.

(c) If $\mathfrak{B}$ is atomless and $\langle a_i \rangle_{i\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$, then there is a partition of unity $\langle b_i \rangle_{i\in\mathbb{N}}$ in $\mathfrak{B}$ such that $\bar\mu b_i = \bar\mu a_i$ and $\bar\mu(b_i \mathbin{\triangle} a_i) \le 2\rho(a_i, \mathfrak{B})$ for every $i \in \mathbb{N}$.

proof (a) Let $P : L^1_{\bar\mu} \to L^1_{\bar\mu}$ be the conditional expectation operator associated with $\mathfrak{B}$. For any $b \in \mathfrak{B}$,

$$\begin{aligned} \int |P(\chi a) - \chi b| &= \int_{1\setminus b} P(\chi a) + \bar\mu b - \int_b P(\chi a) = \int_{1\setminus b} \chi a + \bar\mu b - \int_b \chi a \\ &= \bar\mu(a \setminus b) + \bar\mu b - \bar\mu(a \cap b) = \bar\mu(a \mathbin{\triangle} b). \end{aligned}$$

If $a \in \mathfrak{A}$ set $h(a) = [\![P(\chi a) > \frac{1}{2}]\!]$. Then $|P(\chi a) - \chi h(a)| \le |P(\chi a) - \chi b|$ for any $b \in \mathfrak{B}$, so

$$\rho(a, \mathfrak{B}) = \inf_{b \in \mathfrak{B}} \bar{\mu}(a \triangle b) = \inf_{b \in \mathfrak{B}} \int |P(\chi a) - \chi b|$$
$$= \int |P(\chi a) - \chi h(a)| = \bar{\mu}(a \triangle h(a)).$$

If $a \cap a' = 0$, then

$$P(\chi a) + P(\chi a') = P\chi(a \cup a') \le \chi 1,$$

so

$$h(a) \cap h(a') = [\![P(\chi a) > \tfrac{1}{2}]\!] \cap [\![P(\chi a') > \tfrac{1}{2}]\!] \subseteq [\![P(\chi a) + P(\chi a') > 1]\!] = 0,$$

by 364Ea.

(b) By 385Ae, $q(t) \le q(1-t)$ whenever $\frac{1}{2} \le t \le 1$. Consequently $q(t) \le q(\min(t, 1-t))$ for every $t \in [0,1]$, and $\bar{q}(u) \le \bar{q}(u \wedge (\chi 1 - u))$ whenever $u \in L^0(\mathfrak{A})$ and $0 \le u \le \chi 1$. Fix $a \in A$ for the moment. We have

$$\bar{q}(P(\chi a)) \le \bar{q}(P(\chi a) \wedge (\chi 1 - P(\chi a)) = \bar{q}(|P(\chi a) - \chi h(a)|).$$

Consequently

$$\int \bar{q}(P\chi a) \le \int \bar{q}(|P(\chi a) - \chi h(a)|) \le q\Bigl(\int |P(\chi a) - \chi h(a)|\Bigr)$$

(because q is concave)

$$= q(\rho(a, \mathfrak{B})).$$

Summing over a,

$$H(A|\mathfrak{B}) = \textstyle\sum_{a \in A} \int \bar{q}(P\chi a) \le \sum_{a \in A} q(\rho(a, \mathfrak{B})).$$

(c) Set $b'_i = h(a_i)$ for each $i \in \mathbb{N}$. Then $\langle b'_i \rangle_{i \in \mathbb{N}}$ is disjoint. Next, for each $i \in \mathbb{N}$, take $b''_i \in \mathfrak{B}$ such that $b''_i \subseteq b'_i$ and $\bar{\mu} b''_i = \min(\bar{\mu} a_i, \bar{\mu} b'_i)$; then $\langle b''_i \rangle_{i \in \mathbb{N}}$ is disjoint and $\bar{\mu} b''_i \le \bar{\mu} a_i$ for every i. We can therefore find a partition of unity $\langle b_i \rangle_{i \in \mathbb{N}}$ such that $b_i \supseteq b''_i$ and $\bar{\mu} b_i = \bar{\mu} a_i$ for every i. (Use 331C to choose $\langle d_i \rangle_{i \in \mathbb{N}}$ inductively so that $d_i \subseteq 1 \setminus (\sup_{j<i} d_j \cup \sup_{j \in \mathbb{N}} b''_j)$ and $\bar{\mu} d_i = \bar{\mu} a_i - \bar{\mu} b''_i$ for each i, and set $b_i = b''_i \cup d_i$.)

Take any $i \in \mathbb{N}$. If $\bar{\mu} b'_i > \bar{\mu} a_i$, then

$$\bar{\mu}(a_i \triangle b_i) = \bar{\mu}(a_i \triangle b''_i) \le \bar{\mu}(a_i \triangle b'_i) + \bar{\mu}(b'_i \triangle b''_i)$$
$$= \bar{\mu}(a_i \triangle b'_i) + \bar{\mu} b'_i - \bar{\mu} a_i \le 2\bar{\mu}(a_i \triangle b'_i) = 2\rho(a_i, \mathfrak{B}).$$

If $\bar{\mu} b'_i \le \bar{\mu} a_i$, then

$$\bar{\mu}(a_i \triangle b_i) \le \bar{\mu}(a_i \triangle b'_i) + \bar{\mu}(b'_i \triangle b_i)$$
$$= \bar{\mu}(a_i \triangle b'_i) + \bar{\mu} a_i - \bar{\mu} b'_i \le 2\bar{\mu}(a_i \triangle b'_i) = 2\rho(a_i, \mathfrak{B}).$$

386M Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism. Suppose that $B \subseteq \mathfrak{A}$. For $k \in \mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b : b \in B, |j| \le k\}$, and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b : b \in B, j \in \mathbb{Z}\}$.

(a) $\mathfrak{B}$ is the topological closure $\overline{\bigcup_{k \in \mathbb{N}} \mathfrak{B}_k}$.

(b) $\pi[\mathfrak{B}] = \mathfrak{B}$.

(c) If $\mathfrak{C}$ is any closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{C}] = \mathfrak{C}$, and $a \in \mathfrak{B}_k$, then

$$\rho(a, \mathfrak{C}) \le (2k+1) \textstyle\sum_{b \in B} \rho(b, \mathfrak{C}).$$

proof (a) Because $\langle \mathfrak{B}_k \rangle_{k \in \mathbb{N}}$ is non-decreasing, $\bigcup_{k \in \mathbb{N}} \mathfrak{B}_k$ is a subalgebra of $\mathfrak{A}$, so its closure also is (323J), and must be $\mathfrak{B}$.

(b) Of course $\pi^{-1}[\mathfrak{B}_{k+1}]$ is a closed subalgebra of $\mathfrak{A}$ containing $\pi^j b$ whenever $|j| \le k$ and $b \in B$, so includes $\mathfrak{B}_k$; thus $\pi[\mathfrak{B}_k] \subseteq \mathfrak{B}_{k+1} \subseteq \mathfrak{B}$ for every k, and

$$\pi[\mathfrak{B}] = \pi[\overline{\bigcup_{k\in\mathbb{N}} \mathfrak{B}_k}] \subseteq \overline{\bigcup_{k\in\mathbb{N}} \pi[\mathfrak{B}_k]} \subseteq \overline{\mathfrak{B}} \subseteq \mathfrak{B}$$

because π is continuous (324Kb again). Similarly, $\pi^{-1}[\mathfrak{B}] \subseteq \mathfrak{B}$ and $\pi[\mathfrak{B}] = \mathfrak{B}$.

(c) For each $b \in B$, choose $c_b \in \mathfrak{C}$ such that $\bar\mu(b \triangle c_b) = \rho(b, \mathfrak{C})$ (386La). Set

$$e = \sup_{|j|\le k} \sup_{b\in B} \pi^j(b \triangle c_b);$$

then

$$\bar\mu e \le (2k+1)\sum_{b\in B} \bar\mu(b \triangle c_b) = (2k+1)\sum_{b\in B} \rho(b, \mathfrak{C}).$$

Now

$$\mathfrak{B}' = \{d : d \in \mathfrak{A}, \exists\, c \in \mathfrak{C} \text{ such that } d \setminus e = c \setminus e\}$$

is a subalgebra of $\mathfrak{A}$. By 314F(a-i), applied to the order-continuous homomorphism $c \mapsto c \setminus e : \mathfrak{C} \to \mathfrak{A}_{1\setminus e}$, $\{c \setminus e : c \in \mathfrak{C}\}$ is an order-closed subalgebra of the principal ideal $\mathfrak{A}_{1\setminus e}$; by 313Id, applied to the order-continuous function $d \mapsto d \setminus e : \mathfrak{A} \to \mathfrak{A}_{1\setminus e}$, $\mathfrak{B}'$ is order-closed. If $b \in B$ and $|j| \le k$, then $\pi^j b \triangle \pi^j c_b \subseteq e$, so $\pi^j b \in \mathfrak{B}'$; accordingly $\mathfrak{B}' \supseteq \mathfrak{B}_k$. Now $a \in \mathfrak{B}_k$, so there is a $c \in \mathfrak{C}$ such that $a \triangle c \subseteq e$, and

$$\rho(a, \mathfrak{C}) \le \bar\mu(a \triangle c) \le \bar\mu e \le (2k+1)\sum_{b\in B} \rho(b, \mathfrak{C}),$$

as claimed.

386N Lemma Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and suppose *either* that $\mathfrak{A}$ is not purely atomic *or* that it is purely atomic and $H(D_0) = \infty$, where D_0 is the set of atoms of $\mathfrak{A}$. Then whenever $A \subseteq \mathfrak{A}$ is a partition of unity and $H(A) \le \gamma \le \infty$, there is a partition of unity B, refining A, such that $H(B) = \gamma$.

proof (a) By 385J, there is a partition of unity D_1 such that $H(D_1) = \infty$. Set $D = D_1 \vee A$; then we still have $H(D) = \infty$. Enumerate D as $\langle d_i \rangle_{i\in\mathbb{N}}$. Choose $\langle B_k \rangle_{k\in\mathbb{N}}$ inductively, as follows. $B_0 = A$. Given that B_k is a partition of unity, then if $H(B_k \vee \{d_k, 1 \setminus d_k\}) \le \gamma$, set $B_{k+1} = B_k \vee \{d_k, 1 \setminus d_k\}$; otherwise set $B_{k+1} = B_k$.

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{N}} B_k$. Note that, for each $d \in D$,

$$\{c : c \in \mathfrak{A},\ d \subseteq c \text{ or } d \cap c = 0\}$$

is a closed subalgebra of $\mathfrak{A}$ including every B_k, so includes $\mathfrak{B}$. If $b \in \mathfrak{B} \setminus \{0\}$, there is surely some $d \in D$ such that $b \cap d \ne 0$, so $b \supseteq d$; thus $\mathfrak{B}$ must be purely atomic. Let B be the set of atoms of $\mathfrak{B}$. Because $A = B_0 \subseteq \mathfrak{B}$, B refines A.

(b) $H(B) \le \gamma$. **P** For each $k \in \mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by B_k, so that $\mathfrak{B} = \overline{\bigcup_{k\in\mathbb{N}} \mathfrak{B}_k}$. Suppose that $b_0, \dots, b_n$ are distinct members of B. Then for each $k \in \mathbb{N}$ we can find disjoint $b_{0k}, \dots, b_{nk} \in \mathfrak{B}_k$ such that $\bar\mu(b_{ik} \triangle b_i) \le \rho(b_i, \mathfrak{B}_k)$ for every $i \le n$ (386La). Accordingly $\bar\mu b_i = \lim_{k\to\infty} \bar\mu b_{ik}$ for each i, and

$$\sum_{i=0}^n q(\bar\mu b_i) = \lim_{k\to\infty} \sum_{i=0}^n q(\bar\mu b_{ik}) \le \sup_{k\in\mathbb{N}} H(B_k) \le \gamma.$$

As $b_0, \dots, b_n$ are arbitrary, $H(B) \le \gamma$. **Q**

(c) $H(B) \ge \gamma$. **P?** Suppose otherwise. We know that

$$\lim_{k\to\infty} H(\{d_k, 1 \setminus d_k\}) = \lim_{k\to\infty} q(\bar\mu d_k) + q(1 - \bar\mu d_k) = 0.$$

Let $m \in \mathbb{N}$ be such that $H(B) + H(\{d_k, 1 \setminus d_k\}) \le \gamma$ for every $k \ge m$. Because B refines B_k, we must have

$$H(B_k \vee \{d_k, 1 \setminus d_k\}) \le H(B_k) + H(\{d_k, 1 \setminus d_k\}) \le \gamma,$$

so that $B_{k+1} = B_k \vee \{d_k, 1 \setminus d_k\}$ for every $k \ge m$. But this means that $d_k \in B$ for every $k \ge m$, so that

$$\gamma > H(B) \ge \sum_{k=m}^\infty q(\bar\mu d_k) = \infty,$$

which is impossible. **XQ**

Thus B has the required properties.

386X Basic exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism with fixed-point subalgebra $\mathfrak{C}$. Take any $a \in \mathfrak{A}$ and set $a_n = \pi^n a \setminus \sup_{1 \le i < n} \pi^i a$ for $n \ge 1$. Show that $\sum_{n=1}^{\infty} n\bar{\mu}(a \cap a_n) = \bar{\mu}(\mathrm{upr}(a, \mathfrak{C}))$. (*Hint*: for $0 \le j < k$ set $a_{jk} = \pi^j(a \cap a_{k-j})$. Show that if $r \in \mathbb{N}$, then $\langle a_{jk} \rangle_{j \le r < k}$ is disjoint.)

(b) Let (X, Σ, μ) be a totally finite measure space and $f : X \to X$ an inverse-measure-preserving function. Take $E \in \Sigma$ and set $F = \{x : \exists\, n \ge 1,\ f^n(x) \in E\}$. (i) Show that $E \setminus F$ is negligible. (ii) For $x \in E \cap F$ set $k_x = \min\{n : n \ge 1,\ f^n(x) \in E\}$. Show that $\int_E k_x \mu(dx) = \mu F$. (This is a simple form of the **Recurrence Theorem**.)

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $\langle B_k \rangle_{k \in \mathbb{N}}$ a non-decreasing sequence of subsets of $\mathfrak{A}$ such that $0 \in B_0$, and $\langle c_i \rangle_{i \in I}$ a partition of unity in $\mathfrak{A}$. Show that

$$\lim_{k \to \infty} \sum_{i \in I} \rho(c_i, B_k) = \sum_{i \in I} \rho(c_i, B)$$

where $B = \overline{\bigcup_{k \in \mathbb{N}} B_k}$.

>(d) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism and A a partition of unity in $\mathfrak{A}$. Show that $h(\pi, D_n(A, \pi)) = h(\pi, A) = h(\pi, \pi[A])$ for any $n \ge 1$.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Suppose that $B \subseteq \mathfrak{A}$. For $k \in \mathbb{N}$, let $\mathfrak{B}_k$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b : b \in B,\ j \le k\}$, and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b : b \in B,\ j \in \mathbb{N}\}$. Show that

$$\mathfrak{B} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{B}_k}, \quad \pi[\mathfrak{B}] \subseteq \mathfrak{B},$$

and that if $\mathfrak{C}$ is any subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{C}] \subseteq \mathfrak{C}$, and $a \in \mathfrak{B}_k$, then $\rho(a, \mathfrak{C}) \le (k+1) \sum_{b \in B} \rho(b, \mathfrak{C})$.

386Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving Boolean homomorphism. Set $\mathfrak{C} = \{c : \pi c = c\}$. Show that whenever $n \ge 1$, $0 \le \gamma < \frac{1}{n}$ and $B \subseteq \mathfrak{A}$ is finite, there is an $a \in \mathfrak{A}$ such that $a, \pi a, \pi^2 a, \dots, \pi^{n-1} a$ are disjoint and $\bar{\mu}(a \cap b \cap c) = \gamma \bar{\mu}(b \cap c)$ for every $b \in B$, $c \in \mathfrak{C}$.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{P}$ be the set of all closed subalgebras of $\mathfrak{A}$ which are invariant under π, ordered by inclusion. Show that $\mathfrak{B} \mapsto h(\pi \restriction \mathfrak{B}) : \mathfrak{P} \to [0, \infty]$ is order-preserving and **order-continuous on the left**, in the sense that if $\mathfrak{Q} \subseteq \mathfrak{P}$ is non-empty and upwards-directed then $h(\pi \restriction \sup \mathfrak{Q}) = \sup_{\mathfrak{B} \in \mathfrak{Q}} h(\pi \restriction \mathfrak{B})$.

386 Notes and comments I have taken the trouble to give sharp forms of the Halmos-Rokhlin-Kakutani lemma (386C) and the Cziszár-Kullback inequality (386G); while it is possible to get through the principal results of the next two sections with rather less, the formulae become better focused if we have the exact expressions available. Of course one can always go farther still (386Ya). Ornstein's theorem in §387 (though not Sinaĭ's, as stated there) can be deduced from the special case of the Shannon-McMillan-Breiman theorem (386E) in which the homomorphism π is a Bernoulli shift.

Version of 9.3.16

387 Ornstein's theorem

I come now to the most important of the handful of theorems known which enable us to describe automorphisms of measure algebras up to isomorphism: two two-sided Bernoulli shifts (on algebras of countable Maharam typre) of the same entropy are isomorphic (387J, 387L). This is hard work. It requires both delicate ϵ-δ analysis and substantial skill with the manipulation of measure-preserving homomorphisms. The proof is based on difficult lemmas (387C, 387G, 387K), and includes Sinaĭ's theorem (387E, 387M), describing the Bernoulli shifts which arise as factors of a given ergodic automorphism.

387A The following definitions offer a language in which to express the ideas of this section.

Definitions Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism.

(a) A **Bernoulli partition** for π is a partition of unity $\langle a_i\rangle_{i\in I}$ such that

$$\bar\mu(\inf_{j\le k}\pi^j a_{i(j)})=\prod_{j=0}^k\bar\mu a_{i(j)}$$

whenever $k\in\mathbb{N}$ and $i(0),\dots,i(k)\in I$.

(b) If $\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$, that is, π is a measure-preserving automorphism, a Bernoulli partition $\langle a_i\rangle_{i\in I}$ for π is **(two-sidedly) generating** if the closed subalgebra generated by $\{\pi^j a_i : i\in I,\ j\in\mathbb{Z}\}$ is $\mathfrak{A}$ itself.

(c) A **factor** of $(\mathfrak{A},\bar\mu,\pi)$ is a triple $(\mathfrak{B},\bar\mu\restriction\mathfrak{B},\pi\restriction\mathfrak{B})$ where $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]\subseteq\mathfrak{B}$.

(d) Let $\mathfrak{B}$, $\mathfrak{C}$ be closed subalgebras of $\mathfrak{A}$ such that $\pi[\mathfrak{B}]\subseteq\mathfrak{B}$ and $\pi[\mathfrak{C}]\subseteq\mathfrak{C}$. I will write $\operatorname{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$ for the set of Boolean homomorphisms $\phi:\mathfrak{B}\to\mathfrak{C}$ such that

$$\bar\mu\phi b=\bar\mu b,\quad \pi\phi b=\phi\pi b$$

for every $b\in\mathfrak{B}$. On $\operatorname{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$ the **weak uniformity** will be the uniformity generated by the pseudometrics

$$(\phi,\psi)\mapsto\bar\mu(\phi b\mathbin{\triangle}\psi b)$$

for $b\in\mathfrak{B}$ (3A4Ba); the **weak topology** on $\operatorname{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$ will be the associated topology (3A4Ab).

387B Elementary facts Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra, $\pi\in\operatorname{Aut}_{\bar\mu}\mathfrak{A}$ and that $\langle b_i\rangle_{i\in I}$ is a Bernoulli partition for π. Write $\mathfrak{B}_0$ for the closed subalgebra of $\mathfrak{A}$ generated by $\{b_i : i\in I\}$, $\mathfrak{B}$ for the closed subalgebra generated by $\{\pi^j b_i : i\in I,\ j\in\mathbb{Z}\}$, and B for $\{b_i : i\in I\}\setminus\{0\}$.

(a) $\pi\restriction\mathfrak{B}$ is a two-sided Bernoulli shift with root algebra $\mathfrak{B}_0$ and entropy $H(B)=h(\pi,B)\le h(\pi)$.

(b) If $H(B)>0$ then $\mathfrak{A}$ is atomless.

(c) Suppose now that $\langle c_i\rangle_{i\in I}$ is another Bernoulli partition for π with $\bar\mu c_i=\bar\mu b_i$ for every i; let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i : i\in I,\ j\in\mathbb{Z}\}$. Then we have a unique $\phi\in\operatorname{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$ such that $\phi b_i=c_i$ for every $i\in I$, and ϕ is an isomorphism between $(\mathfrak{B},\bar\mu\restriction\mathfrak{B},\pi\restriction\mathfrak{B})$ and $(\mathfrak{C},\bar\mu\restriction\mathfrak{C},\pi\restriction\mathfrak{C})$.

proof (a) I should begin by noting that $\pi[\mathfrak{B}]$ is the (order-)closed subalgebra generated by $\{\pi^{j+1}b_i : i\in I,\ j\in\mathbb{Z}\}$ (314H, 324L), so is equal to $\mathfrak{B}$; accordingly $\pi\restriction\mathfrak{B}\in\operatorname{Aut}_{\bar\mu\restriction\mathfrak{B}}\mathfrak{B}$.

Suppose that $d_j\in\pi^j[\mathfrak{B}_0]$ for $0\le j\le k$. Then each $\pi^{-j}d_j\in\mathfrak{B}_0$ is expressible as $\sup_{i\in I_j}b_i$ for some $I_j\subseteq I$. Now

$$\begin{aligned}\bar\mu(\inf_{j\le k}d_j)&=\bar\mu(\sup_{i_0\in I_0,\dots,i_k\in I_k}\inf_{j\le k}\pi^j b_{i_j})\\&=\sum_{i_0\in I_0,\dots,i_k\in I_k}\bar\mu(\inf_{j\le k}\pi^j b_{i_j})=\sum_{i_0\in I_0,\dots,i_k\in I_k}\prod_{j=0}^k\bar\mu b_{i_j}\\&=\prod_{j=0}^k\sum_{i\in I_j}\bar\mu b_i=\prod_{j=0}^k\bar\mu(\sup_{i\in I_j}b_i)=\prod_{j=0}^k\bar\mu d_j.\end{aligned}$$

As $d_0,\dots,d_k$ are arbitrary, $\langle\pi^k[\mathfrak{B}_0]\rangle_{k\in\mathbb{N}}$ is independent. By 385Sf, $\langle\pi^k[\mathfrak{B}_0]\rangle_{k\in\mathbb{Z}}$ is independent. Since $\mathfrak{B}$ is defined as the closed subalgebra generated by $\{\pi^j b_i : i\in I,\ j\in\mathbb{Z}\}$, $\pi\restriction\mathfrak{B}$ is a two-sided Bernoulli shift in which $\mathfrak{B}_0$ is a root algebra, as defined in 385Qb.

As in part (a) of the proof of 385R, $h(\pi\restriction\mathfrak{B})=H(B)=h(\pi,B)$, and of course $h(\pi,B)\le h(\pi)$.

(b) As B contains at least two elements of non-zero measure, $\gamma=\max_{b\in B}\bar\mu b<1$. Because $\langle b_i\rangle_{i\in I}$ is a Bernoulli partition, every member of $D_k(B,\pi)$ (definition: 385K) has measure at most γ^k, for any $k\in\mathbb{N}$. Thus any atom of $\mathfrak{A}$ could have measure at most $\inf_{k\in\mathbb{N}}\gamma^k=0$.

(c) If $i(0),\dots,i(n)\in I$ and $j(0),\dots,j(n)\in\mathbb{Z}$, set $L=\{j(k):k\le n\}$; then

$$\bar\mu(\inf_{k\le n}\pi^{j(k)}b_{i(k)})=\bar\mu(\inf_{l\in L}\pi^l(\inf_{\substack{k\le n\\ j(k)=l}}b_{i(k)}))=\prod_{l\in L}\bar\mu(\pi^l(\inf_{\substack{k\le n\\ j(k)=l}}b_{i(k)}))$$

(by (a) above)

$$=\prod_{l\in L}\bar\mu(\inf_{\substack{k\le n\\ j(k)=l}}b_{i(k)})$$

(because $\bar\mu\pi^l=\bar\mu$ for every l)

$$=\prod_{l\in L}\bar\mu(\inf_{\substack{k\le n\\ j(k)=l}}c_{i(k)})$$

(because if there are $k\ne k'$ such that $j(k)=j(k')$ but $i(k)\ne i(k')$ then both products are zero, and otherwise they are of the form $\prod_{l\in L}\bar\mu b_{r(l)}=\prod_{l\in L}\bar\mu c_{r(l)}$)

$$=\bar\mu(\inf_{k\le n}\pi^{j(k)}a_{i(k)}).$$

So we can apply 324P to see that there is a unique measure-preserving homomorphism $\phi:\mathfrak{B}\to\mathfrak{C}$ such that $\phi(\pi^j b_i)=\pi^j c_i$ for every $i\in I$ and $j\in\mathbb{Z}$. Now the set $\{b:b\in\mathfrak{B},\ \phi\pi b=\pi\phi b\}$ is a (metrically and order-) closed subset of $\mathfrak{B}$ including $\bigcup_{j\in\mathbb{Z}}\pi^j[B]$ and is therefore the whole of $\mathfrak{B}$. So $\phi\in\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$. Since $\phi[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{C}$ (324Kb) including $\{\pi^j c_i:i\in I,\ j\in\mathbb{Z}\}$, it is the whole of $\mathfrak{C}$, and $\phi:\mathfrak{B}\to\mathfrak{C}$ is an isomorphism.

387C Lemma Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra and $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ an ergodic measure-preserving automorphism. Let $\langle a_i\rangle_{i\in\mathbb{N}}$ be a partition of unity in $\mathfrak{A}$, of finite entropy, and $\langle\gamma_i\rangle_{i\in\mathbb{N}}$ a sequence of non-negative real numbers such that

$$\sum_{i=0}^\infty\gamma_i=1,\quad \sum_{i=0}^\infty q(\gamma_i)\le h(\pi),$$

where q is the function of 385A. Then for any $\epsilon>0$ we can find a partition $\langle a'_i\rangle_{i\in\mathbb{N}}$ of unity in $\mathfrak{A}$ such that

(i) $\{i:a'_i\ne 0\}$ is finite,

(ii) $\sum_{i=0}^\infty|\gamma_i-\bar\mu a'_i|\le\epsilon$,

(iii) $\sum_{i=0}^\infty\bar\mu(a'_i\mathbin{\triangle}a_i)\le\epsilon+6\sqrt{\sum_{i=0}^\infty|\bar\mu a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}}$

where $A=\{a_i:i\in\mathbb{N}\}\setminus\{0\}$,

(iv) $H(A')\le h(\pi,A')+\epsilon$

where $A'=\{a'_i:i\in\mathbb{N}\}\setminus\{0\}$.

proof (a) Of course $h(\pi,A)\le H(A)$, as remarked in 385M, so the square root $\sqrt{2(H(A)-h(\pi,A))}$ gives no difficulty. Set $\beta=\sqrt{\sum_{i=0}^\infty|\bar\mu a_i-\gamma_i|+\sqrt{2(H(A)-h(\pi,A))}}$, $\delta=\min(\frac14,\frac1{24}\epsilon)$.

There is a sequence $\langle\bar\gamma_i\rangle_{i\in\mathbb{N}}$ of non-negative real numbers such that $\{i:\bar\gamma_i>0\}$ is finite, $\sum_{i=0}^\infty\bar\gamma_i=1$, $\sum_{i=0}^\infty|\bar\gamma_i-\gamma_i|\le 2\delta^2$ and $\sum_{i=0}^\infty q(\bar\gamma_i)\le h(\pi)$. **P** Take $k\in\mathbb{N}$ such that $\sum_{i=k}^\infty\gamma_i\le\delta^2$, and set $\bar\gamma_i=\gamma_i$ for $i<k$, $\bar\gamma_k=\sum_{i=k}^\infty\gamma_i$ and $\bar\gamma_i=0$ for $i>k$; then $q(\bar\gamma_k)\le\sum_{i=k}^\infty q(\gamma_i)$ (385Ab), so

$$\sum_{i=0}^\infty q(\bar\gamma_i)\le\sum_{i=0}^\infty q(\gamma_i)\le h(\pi),$$

while

$$\sum_{i=0}^\infty|\bar\gamma_i-\gamma_i|\le\bar\gamma_k+\sum_{i=k}^\infty\gamma_i\le 2\delta^2.\ \mathbf{Q}$$

Because $\sum_{i=0}^\infty q(\bar\gamma_i)$ is finite, there is a partition of unity C in $\mathfrak{A}$, of finite entropy, such that $\sum_{i=0}^\infty q(\bar\gamma_i)\le h(\pi,C)+3\delta$; replacing C by $C\vee A$ if need be (note that $C\vee A$ still has finite entropy, by 385H), we may suppose that C refines A.

There is a sequence $\langle\gamma'_i\rangle_{i\in\mathbb{N}}$ of non-negative real numbers such that $\sum_{i=0}^\infty\gamma'_i=1$, $\{i:\gamma'_i>0\}$ is finite, $\sum_{i=0}^\infty|\gamma'_i-\gamma_i|\le 4\delta^2$ and

$$\sum_{i=0}^\infty q(\gamma'_i)=h(\pi,C)+3\delta.$$

P Take $k \in \mathbb{N}$ such that $\bar{\gamma}_i = 0$ for $i > k$. Take $r \geq 1$ such that $\delta^2 \ln(\frac{r}{\delta^2}) \geq h(\pi, C) + 3\delta$ and set

$$\begin{aligned}\tilde{\gamma}_i &= (1-\delta^2)\bar{\gamma}_i \text{ for } i \leq k\\ &= \frac{1}{r}\delta^2 \text{ for } k+1 \leq i \leq k+r\\ &= 0 \text{ for } i > k+r.\end{aligned}$$

Then

$$\sum_{i=0}^{\infty} |\tilde{\gamma}_i - \bar{\gamma}_i| = 2\delta^2, \quad \sum_{i=0}^{\infty} |\tilde{\gamma}_i - \gamma_i| \leq 4\delta^2,$$

$$\sum_{i=0}^{k+r} q(\bar{\gamma}_i) \leq h(\pi, C) + 3\delta \leq \delta^2 \ln(\frac{r}{\delta^2}) = rq(\frac{\delta^2}{r}) \leq \sum_{i=0}^{k+r} q(\tilde{\gamma}_i).$$

Now the function

$$\alpha \mapsto \sum_{i=0}^{k+r} q(\alpha\bar{\gamma}_i + (1-\alpha)\tilde{\gamma}_i) : [0,1] \to \mathbb{R}$$

is continuous, so there is some $\alpha \in [0,1]$ such that

$$\sum_{i=0}^{k+r} q(\alpha\bar{\gamma}_i + (1-\alpha)\tilde{\gamma}_i) = h(\pi, C) + 3\delta,$$

and we can set $\gamma_i' = \alpha\bar{\gamma}_i + (1-\alpha)\tilde{\gamma}_i$ for every i; of course

$$\sum_{i=0}^{\infty} |\gamma_i' - \gamma_i| \leq \alpha\sum_{i=0}^{\infty} |\bar{\gamma}_i - \gamma_i| + (1-\alpha)\sum_{i=0}^{\infty} |\tilde{\gamma}_i - \gamma_i| \leq 4\delta^2. \textbf{ Q}$$

Set $M = \{i : \gamma_i' \neq 0\}$, so that M is finite.

(b) Let $\eta \in \,]0, \delta]$ be so small that

(i) $|q(s) - q(t)| \leq \frac{\delta}{1+\#(M)}$ whenever $s, t \in [0,1]$ and $|s-t| \leq 3\eta$,

(ii) $\sum_{c \in C} q(\min(\bar{\mu}c, 2\eta)) \leq \delta$,

(iii) $\eta \leq \frac{1}{6}$.

(Actually, (iii) is a consequence of (i). For (ii) we must of course rely on the fact that $\sum_{c \in C} q(\bar{\mu}c)$ is finite.)

Let ν be the probability measure on M defined by saying that $\nu\{i\} = \gamma_i'$ for every $i \in M$, and λ the product measure on $M^{\mathbb{N}}$. Define $X_{ij} : M^{\mathbb{N}} \to \{0,1\}$, for $i \in M$ and $j \in \mathbb{N}$, and $Y_j : M^{\mathbb{N}} \to \mathbb{R}$, for $j \in \mathbb{N}$, by setting

$$\begin{aligned}X_{ij}(\omega) &= 1 \text{ if } \omega(j) = i,\\ &= 0 \text{ otherwise,}\\ Y_j(\omega) &= \ln(\gamma'_{\omega(j)}) \text{ for every } \omega \in M^{\mathbb{N}}.\end{aligned}$$

Then, for each $i \in M$, $\langle X_{ij}\rangle_{j\in\mathbb{N}}$ is an independent sequence of random variables, all with expectation γ_i', and $\langle Y_j\rangle_{j\in\mathbb{N}}$ also is an independent sequence of random variables, all with expectation

$$\sum_{i\in M} \gamma_i' \ln \gamma_i' = -\sum_{i=0}^{\infty} q(\gamma_i') = -h(\pi, C) - 3\delta.$$

Let $n \geq 1$ be so large that

(iv) $\bar{\mu}[\![w_n - h(\pi, C)\chi 1 \geq \delta]\!] < \eta$, where

$$w_n = \frac{1}{n}\sum_{d \in D_n(C,\pi)} \ln(\frac{1}{\bar{\mu}d})\chi d;$$

(v)

$$\Pr\bigl(\sum_{i\in M} |\frac{1}{n}\sum_{j=0}^{n-1} X_{ij} - \gamma_i'| \leq \eta\bigr) \geq 1 - \delta,$$

$$\Pr\bigl(|\frac{1}{n}\sum_{j=0}^{n-1} Y_j + h(\pi, C) + 3\delta| \leq \delta\bigr) \geq 1 - \delta;$$

(vi) $e^{n\delta} \geq 2, \quad \frac{1}{n+1} \leq \eta, \quad q(\frac{1}{n+1}) + q(\frac{n}{n+1}) \leq \delta;$

these will be true for all sufficiently large n, using the Shannon-McMillan-Breiman theorem (386E-386F; this is where we need to suppose that π is ergodic) for (iv) and the strong law of large numbers (in any of the forms 273D, 273H or 273I) for (v).

(c) There is a family $\langle b_{ji}\rangle_{j<n,i\in M}$ such that

(α) for each $j<n$, $\langle b_{ji}\rangle_{i\in M}$ is a partition of unity in $\mathfrak{A}$,

(β) $\bar\mu(\inf_{j<n} b_{j,i(j)}) = \prod_{j=0}^{n-1}\gamma'_{i(j)}$ for every $i(0),\dots,i(n-1)\in M$,

(γ) $\sum_{i\in M}\bar\mu(b_{ji}\cap\pi^j a_i)\ge 1-\beta^2-4\delta^2$ for every $j<n$.

P Construct $\langle b_{ji}\rangle_{i\in M}$ for $j=n-1,\,n-2,\dots,0$, as follows. Given b_{ji}, for $k<j<n$, such that

$$\bar\mu(\inf_{j\le k}\pi^j a_{i(j)}\cap\inf_{k<j<n} b_{j,i(j)}) = \bar\mu(\inf_{j\le k}\pi^j a_{i(j)})\cdot\prod_{j=k+1}^{n-1}\gamma'_{i(j)}$$

for every $i(0),\dots,i(n-1)\in M$ (of course this hypothesis is trivial for $k=n-1$), let B_k be the set of atoms of the (finite) subalgebra of $\mathfrak{A}$ generated by $\{b_{ji}: i\in M,\ k<j<n\}$. Then $\bar\mu(b\cap d)=\bar\mu b\cdot\bar\mu d$ for every $b\in B_k$ and $d\in D_{k+1}(A,\pi)$.

Now

$$\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\gamma'_i\bar\mu c|\\
&\quad\le\sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\bar\mu a_i\cdot\bar\mu c| + \sum_{i=0}^{\infty}|\bar\mu a_i-\gamma'_i|\sum_{c\in D_k(A,\pi)}\bar\mu c\\
&\quad\le\sum_{i=0}^{\infty}|\gamma_i-\gamma'_i| + \sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i| + \sum_{i=0}^{\infty}\sum_{c\in D_k(A,\pi)}|\bar\mu(\pi^k a_i\cap c)-\bar\mu a_i\cdot\bar\mu c|\\
&\quad\le 4\delta^2+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i| + \sqrt{2\bigl(H(\pi^k[A])+H(D_k(A,\pi))-H(D_{k+1}(A,\pi))\bigr)}
\end{aligned}$$

(by 386H, because $D_{k+1}(A,\pi)=\pi^k[A]\vee D_k(A,\pi)$)

$$\le 4\delta^2+\sum_{i=0}^{\infty}|\bar\mu a_i-\gamma_i| + \sqrt{2(H(A)-h(\pi,A))}$$

(because $h(\pi,A)\le H(D_{k+1}(A,\pi))-H(D_k(A,\pi))$, by 386Kc)

$$=\beta^2+4\delta^2.$$

Choose a partition of unity $\langle b_{ki}\rangle_{i\in M}$ such that, for each $c\in D_k(A,\pi)$, $b\in B_k$ and $i\in M$,

$$\bar\mu(b_{ki}\cap b\cap c)=\gamma'_i\bar\mu(b\cap c),$$

$$\text{if }\bar\mu(\pi^k a_i\cap b\cap c)\ge\gamma'_i\bar\mu(b\cap c)\text{ then }b_{ki}\cap b\cap c\subseteq\pi^k a_i,$$

$$\text{if }\bar\mu(\pi^k a_i\cap b\cap c)\le\gamma'_i\bar\mu(b\cap c)\text{ then }\pi^k a_i\cap b\cap c\subseteq b_{ki}.$$

(This is where I use the hypothesis that $\mathfrak{A}$ is atomless.) Note that in these formulae we always have

$$\pi^k a_i\cap c\in D_{k+1}(A,\pi),\quad \bar\mu(b\cap c)=\bar\mu b\cdot\bar\mu c,\quad \bar\mu(\pi^k a_i\cap b\cap c)=\bar\mu(\pi^k a_i\cap c)\cdot\bar\mu b.$$

Consequently

$$\begin{aligned}
\sum_{i\in M}\bar\mu(\pi^k a_i\cap b_{ki}) &= \sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i\in M}\bar\mu(b\cap c\cap(\pi^k a_i\cap b_{ki}))\\
&= \sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\min(\bar\mu(b\cap c\cap\pi^k a_i),\gamma'_i\bar\mu(b\cap c))\\
&\ge \sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\bar\mu(b\cap c\cap\pi^k a_i)-|\bar\mu(b\cap c\cap\pi^k a_i)-\gamma'_i\bar\mu(b\cap c)|\\
&= 1-\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}|\bar\mu(b\cap c\cap\pi^k a_i)-\gamma'_i\bar\mu(b\cap c)|\\
&= 1-\sum_{b\in B_k}\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}\bar\mu b\cdot|\bar\mu(c\cap\pi^k a_i)-\gamma'_i\bar\mu c|\\
&= 1-\sum_{c\in D_k(A,\pi)}\sum_{i=0}^{\infty}|\bar\mu(c\cap\pi^k a_i)-\gamma'_i\bar\mu c|\ge 1-\beta^2-4\delta^2.
\end{aligned}$$

Also we have

$$\bar\mu(b_{ki}\cap b\cap c)=\gamma'_i\bar\mu b\cdot\bar\mu c=\bar\mu(b_{ki}\cap b)\cdot\bar\mu c$$

for every $b\in B_k$, $c\in D_k(A,\pi)$ and $i\in M$, so the (downwards) induction proceeds. **Q**

(d) Let B be the set of atoms of the algebra generated by $\{b_{ji}:j<n,\,i\in M\}$. For $b\in B$ and $d\in D_n(C,\pi)$ set

$$I_{bd}=\{j:j<n,\,\exists\,i\in M,\,b\subseteq b_{ji},\,d\subseteq\pi^j a_i\}.$$

Then, for any $j<n$,

$$\sup\{b\cap d:b\in B,\,d\in D_n(C,\pi),\,j\in I_{bd}\}=\sup\nolimits_{i\in M}b_{ji}\cap\pi^j a_i,$$

because C refines A, so every $\pi^j a_i$ is a supremum of members of $D_n(C,\pi)$. Accordingly

$$\sum_{b\in B,d\in D_n(C,\pi)}\#(I_{bd})\bar\mu(b\cap d)=\sum_{j=0}^{n-1}\sum_{i\in M}\bar\mu(b_{ji}\cap\pi^j a_i)\ge n(1-\beta^2-4\delta^2).$$

Set

$$e_0=\sup\{b\cap d:b\in B,\,d\in D_n(C,\pi),\,\#(I_{bd})\ge n(1-\beta-4\delta)\};$$

then $\bar\mu e_0\ge 1-\beta-\delta$.

(e) Let $B'\subseteq B$ be the set of those $b\in B$ such that

$$\bar\mu b\le e^{-n(h(\pi,C)+2\delta)},\qquad\sum\nolimits_{i\in M}|\gamma'_i-\frac{1}{n}\#(\{j:j<n,\,b\subseteq b_{ji}\})|\le\eta.$$

Then $\bar\mu(\sup B')\ge 1-2\delta$. **P** Set

$$\begin{aligned}
B'_1&=\{b:b\in B,\,\bar\mu b\le e^{-n(h(\pi,C)+2\delta)}\}\\
&=\{b:b\in B,\,h(\pi,C)+2\delta+\frac{1}{n}\ln(\bar\mu b)\le 0\}\\
&=\{\inf_{j<n}b_{j,i(j)}:i(0),\dots,i(n-1)\in M,\,h(\pi,C)+2\delta+\frac{1}{n}\sum_{j=0}^{n-1}\ln\gamma'_{i(j)}\le 0\}.
\end{aligned}$$

Then

$$\begin{aligned}\bar\mu(\sup B_1') &= \Pr(h(\pi,C)+2\delta+\frac{1}{n}\sum_{j=0}^{n-1}Y_j\le 0)\\ &\ge \Pr(|h(\pi,C)+3\delta+\frac{1}{n}\sum_{j=0}^{n-1}Y_j|\le\delta)\ge 1-\delta\end{aligned}$$

by the choice of n. On the other hand, setting

$$\begin{aligned}B_2' &= \{b: b\in B,\ \sum_{i\in M}|\gamma_i'-\frac{1}{n}\#(\{j:j<n,\ b\subseteq b_{ji}\})|\le\eta\}\\ &= \{\inf_{j<n} b_{j,i(j)}: i(0),\dots,i(n-1)\in M,\ \sum_{i\in M}|\gamma_i'-\frac{1}{n}\#(\{j:i(j)=i\})|\le\eta\},\end{aligned}$$

we have

$$\bar\mu(\sup B_2') = \Pr(\textstyle\sum_{i\in M}|\gamma_i'-\frac{1}{n}\sum_{j=0}^{n-1}X_{ij}|\le\eta)\ge 1-\delta$$

by the other half of clause (b-v). Since $B'=B_1'\cap B_2'$, $\bar\mu(\sup B')\ge 1-2\delta$. **Q**

Let D' be the set of those $d\in D_n(C,\pi)$ such that

$$\frac{1}{n}\ln(\frac{1}{\bar\mu d})\le h(\pi,C)+\delta,\quad\text{i.e.,}\quad \bar\mu d\ge e^{-n(h(\pi,C)+\delta)};$$

by (b-iv), $\bar\mu(\sup D')>1-\eta$. Of course D' is finite. If $d\in D'$ and $b\in B'$ then

$$\bar\mu d\ge e^{-n(h(\pi,C)+\delta)}\ge e^{n\delta}\bar\mu b\ge 2\bar\mu b.$$

Since $\bar\mu(\sup D')\le 1\le 2\bar\mu(\sup B')$ (remember that $\delta\le\frac{1}{4}$), $\#(D')\le\#(B')$.

Set $e_1=e_0\cap\sup B'$, so that $\bar\mu e_1\ge 1-\beta-3\delta$, and

$$D''=\{d:d\in D',\ \bar\mu(d\cap e_1)\ge\frac{1}{2}\bar\mu d\};$$

then

$$\bar\mu(\sup(D'\setminus D''))\le 2\bar\mu(1\setminus e_1)\le 2\beta+6\delta,$$

so

$$\bar\mu(\sup D'')\ge 1-2\beta-6\delta-\eta\ge 1-2\beta-7\delta.$$

(f) If $d_1,\dots,d_k\in D''$ are distinct,

$$\bar\mu(\textstyle\sup_{1\le i\le k}d_i\cap e_1)\ge\frac{k}{2}\inf_{i\le k}\bar\mu d_i\ge k\sup_{b\in B'}\bar\mu b,$$

and

$$\#(\{b:b\in B',\ b\cap e_0\cap\textstyle\sup_{1\le i\le k}d_i\}\ne 0)\ge k.$$

By the Marriage Lemma (3A1K), there is an injective function $f_0:D''\to B'$ such that $d\cap f_0(d)\cap e_0\ne 0$ for every $d\in D''$. Because $\#(D')\le\#(B')$, we can extend f_0 to an injective function $f:D'\to B'$.

(g) By the Halmos-Rokhlin-Kakutani lemma, in the strong form 386C(iv), there is an $a\in\mathfrak{A}$ such that $a,\pi^{-1}a,\dots,\pi^{-n+1}a$ are disjoint and $\bar\mu(a\cap d)=\frac{1}{n+1}\bar\mu d$ for every $d\in D'\cup\{1\}$. Set $\tilde e=\sup\{\pi^{-j}(a\cap d):j<n,\ d\in D'\}$. Because $\langle\pi^{-j}(a\cap d)\rangle_{j<n,d\in D'}$ is disjoint,

$$\bar\mu\tilde e=\textstyle\sum_{j=0}^{n-1}\sum_{d\in D'}\bar\mu(a\cap d)=\frac{n}{n+1}\sum_{d\in D'}\bar\mu d\ge(1-\eta)^2\ge 1-2\eta.$$

(h) For $i\in M$, set

$$a_i'=\sup\{\pi^{-j}(a\cap d):j<n,\ d\in D',\ f(d)\subseteq b_{ji}\}.$$

Then the a'_i are disjoint. **P** Suppose that i, $i' \in M$ are distinct. If j, $j' < n$ and d, $d' \in D'$ and $f(d) \subseteq b_{ji}$, $f(d') \subseteq b_{j'i'}$, then either $j \ne j'$ or $j = j'$. In the former case,

$$\pi^{-j}(a \cap d) \cap \pi^{-j'}(a \cap d') \subseteq \pi^{-j}a \cap \pi^{-j'}a = 0.$$

In the latter case, $b_{ji} \cap b_{j'i'} = 0$, so $f(d) \ne f(d')$ and $d \ne d'$ and

$$\pi^{-j}(a \cap d) \cap \pi^{-j'}(a \cap d') \subseteq \pi^{-j}(d \cap d') = 0. \text{ **Q**}$$

Observe that

$$\sup_{i \in M} a'_i = \sup_{j<n, d \in D'} \pi^{-j}(a \cap d) = \tilde{e}$$

because if $j < n$ and $d \in D'$ then $f(d) \in B' \subseteq B$ and there must be some $i \in M$ such that $f(d) \subseteq b_{ji}$. Take any $m \in \mathbb{N} \setminus M$ and set $a'_m = 1 \setminus \tilde{e}$, $a'_i = 0$ for $i \in \mathbb{N} \setminus (M \cup \{m\})$; then $\langle a'_i \rangle_{i \in \mathbb{N}}$ is a partition of unity. Now

$$\begin{aligned}\sum_{i \in M} |\bar{\mu}a'_i - \gamma'_i| &\le \sum_{i \in M} \gamma'_i |1 - n\bar{\mu}(a \cap \sup D')| + \sum_{i \in M} |\bar{\mu}a'_i - n\gamma'_i\bar{\mu}(a \cap \sup D')| \\ &\le 1 - \tfrac{n}{n+1}\bar{\mu}(\sup D') \\ &\qquad + \sum_{i \in M} |\sum_{j=0}^{n-1} \sum_{\substack{d \in D' \\ f(d) \subseteq b_{ji}}} \bar{\mu}(\pi^{-j}(a \cap d)) - n\gamma'_i \sum_{d \in D'} \bar{\mu}(a \cap d)| \\ &\le 1 - (1-\eta)^2 \\ &\qquad + \sum_{d \in D'} \sum_{i \in M} |\bar{\mu}(a \cap d) \cdot \#(\{j : j < n,\, f(d) \subseteq b_{ji}\}) - n\gamma'_i\bar{\mu}(a \cap d)| \\ &\le 1 - (1-\eta)^2 + \sum_{d \in D'} \bar{\mu}(a \cap d)n\eta\end{aligned}$$

(see the definition of B'_2 in (e) above)

$$\le 2\eta + n\eta\bar{\mu}a \le 3\eta.$$

So

$$\begin{aligned}\sum_{i=0}^{\infty} |\bar{\mu}a'_i - \gamma_i| &\le \bar{\mu}a'_m + \sum_{i \in M} |\bar{\mu}a'_i - \gamma'_i| + \sum_{i=0}^{\infty} |\gamma'_i - \gamma_i| \\ &\le 2\eta + 3\eta + 4\delta^2 \le 6\delta \le \epsilon.\end{aligned}$$

We shall later want to know that $|\bar{\mu}a'_i - \gamma'_i| \le 3\eta$ for every i; for $i \in M$ this is covered by the formulae above, for $i = m$ it is true because $\bar{\mu}a'_m = 1 - \bar{\mu}\tilde{e} \le 2\eta$ (see (g)), and for other i it is trivial.

(i) The next step is to show that $\sum_{i=0}^{\infty} \bar{\mu}(a'_i \cap a_i) \ge 1 - 3\beta - 12\delta$. **P** It is enough to consider the case in which $3\beta + 12\delta < 1$. We know that

$$\begin{aligned}\sup_{i \in \mathbb{N}} a'_i \cap a_i &\supseteq \sup\{\pi^{-j}(a \cap d) : j < n,\, d \in D', \\ &\qquad \exists\, i \in M \text{ such that } f(d) \subseteq b_{ji} \text{ and } d \subseteq \pi^j a_i\} \\ &= \sup\{\pi^{-j}(a \cap d) : d \in D',\, j \in I_{f(d),d}\}\end{aligned}$$

(see (d) for the definition of I_{bd}) has measure at least $\sum_{d \in D'} \#(I_{f(d),d})\bar{\mu}(a \cap d)$.

For $d \in D''$, we arranged that $d \cap f(d) \cap e_0 \ne 0$. This means that there must be some $b \in B$ and $d' \in D_n(C, \pi)$ such that $d \cap f(d) \cap b \cap d' \ne 0$ and $\#(I_{bd'}) \ge n(1 - \beta - 4\delta)$; of course $d' = d$ and $b = f(d)$, so that $\#(I_{f(d),d})$ must be at least $n(1 - \beta - 4\delta)$. Accordingly

$$\begin{aligned}\sum_{i=0}^{\infty} \bar{\mu}(a'_i \cap a_i) &\ge \sum_{d \in D''} n(1 - \beta - 4\delta)\bar{\mu}(a \cap d) = n(1 - \beta - 4\delta)\tfrac{1}{n+1}\bar{\mu}(\sup D'') \\ &\ge (1 - \eta)(1 - \beta - 4\delta)(1 - 2\beta - 7\delta) \ge 1 - 3\beta - 12\delta. \text{ **Q**}\end{aligned}$$

But this means that

$$\sum_{i=0}^{\infty}\bar\mu(a'_i\triangle a_i)=2(1-\sum_{i=0}^{\infty}\bar\mu(a'_i\cap a_i))\le 6\beta+24\delta\le\epsilon+6\beta$$

(using 386I for the equality).

(j) Finally, we need to estimate $H(A')$ and $h(\pi,A')$, where $A'=\{a'_i:i\in\mathbb{N}\}\setminus\{0\}$. For the former, we have $H(A')\le h(\pi,C)+4\delta$. **P** $|\bar\mu a'_i-\gamma'_i|\le 3\eta$ for every i, by (h) above. So by (b-i),

$$H(A')=\sum_{i\in M\cup\{m\}}q(\bar\mu a'_i)\le\delta+\sum_{i=0}^{\infty}q(\gamma'_i)=h(\pi,C)+4\delta.\ \mathbf{Q}$$

(k) Consider the partition of unity

$$A''=A'\vee\{a,1\setminus a\}.$$

Let $\mathfrak{D}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^jc:j\in\mathbb{Z},\,c\in A''\}$.

(i) $a\cap d\in\mathfrak{D}$ for every $d\in D'$. **P** Of course $a\cap\tilde e\in\mathfrak{D}$, because $1\setminus\tilde e=a'_m$. If $d'\in D'$ and $d'\ne d$, then (because f is injective) $f(d)\ne f(d')$; there must therefore be some $k<n$ and distinct i, $i'\in M$ such that $f(d)\subseteq b_{ki}$ and $f(d')\subseteq b_{ki'}$. But this means that $\pi^{-k}(a\cap d)\subseteq a'_i$ and $\pi^{-k}(a\cap d')\subseteq a'_{i'}$, so that $a\cap d\subseteq\pi^ka'_i$ and $a\cap d'\cap\pi^ka'_i=0$.

What this means is that if we set

$$\tilde d=a\cap\tilde e\cap\inf\{\pi^ka'_i:k<n,\,i\in M,\,a\cap d\subseteq\pi^ka'_i\},$$

we get a member of $\mathfrak{D}$ (because every $a'_i\in\mathfrak{D}$, and $\pi[\mathfrak{D}]=\mathfrak{D}$) including $a\cap d$ and disjoint from $a\cap d'$ whenever $d'\in D'$ and $d'\ne d$. But as $a\cap\pi^{-j}a=0$ if $0<j<n$, $a\cap\tilde e$ must be $\sup\{a\cap d':d'\in D'\}$, and $a\cap d=\tilde d$ belongs to $\mathfrak{D}$. **Q**

(ii) Consequently $c\cap\tilde e\in\mathfrak{D}$ for every $c\in C$. **P** We have

$$\begin{aligned}c\cap\tilde e&=\sup\{c\cap\pi^{-j}(a\cap d):j<n,\,d\in D'\}\\&=\sup\{\pi^{-j}(\pi^jc\cap a\cap d):j<n,\,d\in D'\}\\&=\sup\{\pi^{-j}(a\cap d):j<n,\,d\in D',\,d\subseteq\pi^jc\}\end{aligned}$$

(because if $d\in D'$ and $j<n$ then either $d\subseteq\pi^jc$ or $d\cap\pi^jc=0$)

$$\in\mathfrak{D}$$

because $a\cap d\in\mathfrak{D}$ for every $d\in D'$ and $\pi^{-1}[\mathfrak{D}]=\mathfrak{D}$. **Q**

(iii) It follows that $h(\pi,A'')\ge h(\pi,C)-\delta$. **P** For any $c\in C$,

$$\rho(c,\mathfrak{D})\le\bar\mu(c\triangle(c\cap\tilde e))=\bar\mu(c\setminus\tilde e)\le\min(\bar\mu c,2\eta)\le\frac{1}{3}.$$

So

$$h(\pi,C)\le h(\pi\restriction\mathfrak{D})+H(C|\mathfrak{D})$$

(386Kd, because $\pi[\mathfrak{D}]=\mathfrak{D}$)

$$\le h(\pi,A'')+\sum_{c\in C}q(\rho(c,\mathfrak{D}))$$

(by the Kolmogorov-Sinaǐ theorem (385P) and 386Lb)

$$\le h(\pi,A'')+\sum_{c\in C}q(\min(\bar\mu c,2\eta))$$

(because q is monotonic on $[0,\frac{1}{3}]$)

$$\le h(\pi,A'')+\delta$$

by the choice of η. **Q**

(iv) Finally, $h(\pi,A')\ge h(\pi,C)-2\delta$. **P** Using 386Kb,

$$\begin{aligned} h(\pi, C) - \delta \le h(\pi, A'') &\le h(\pi, A') + H(\{a, 1 \setminus a\}) \\ &= h(\pi, A') + q(\bar{\mu}a) + q(1 - \bar{\mu}a) \\ &= h(\pi, A') + q(\tfrac{1}{n+1}) + q(\tfrac{n}{n+1}) \le h(\pi, A') + \delta \end{aligned}$$

by the choice of n. **Q**

(l) Putting these together,

$$H(A') \le h(\pi, C) + 4\delta \le h(\pi, A') + 6\delta \le h(\pi, A') + \epsilon,$$

and the proof is complete.

387D Corollary Suppose that $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra and $\pi \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$. Let $\langle a_i \rangle_{i \in \mathbb{N}}$ be a partition of unity in $\mathfrak{A}$, of finite entropy, and $\langle \gamma_i \rangle_{i \in \mathbb{N}}$ a sequence of non-negative real numbers such that

$$\sum_{i=0}^{\infty} \gamma_i = 1, \quad \sum_{i=0}^{\infty} q(\gamma_i) \le h(\pi).$$

Then for any $\epsilon > 0$ we can find a Bernoulli partition $\langle a_i^* \rangle_{i \in \mathbb{N}}$ for π such that $\bar{\mu}a_i^* = \gamma_i$ for every $i \in \mathbb{N}$ and

$$\sum_{i=0}^{\infty} \bar{\mu}(a_i^* \triangle a_i) \le \epsilon + 6\sqrt{\sum_{i=0}^{\infty} |\bar{\mu}a_i - \gamma_i| + \sqrt{2(H(A) - h(\pi, A))}},$$

writing $A = \{a_i : i \in \mathbb{N}\} \setminus \{0\}$.

proof (a) Set $\beta = \sqrt{\sum_{i=0}^{\infty} |\bar{\mu}a_i - \gamma_i| + \sqrt{2(H(A) - h(\pi, A))}}$. Let $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$ be a sequence of strictly positive real numbers such that

$$\sum_{n=0}^{\infty} \epsilon_n + 6\sqrt{\epsilon_n + \sqrt{2\epsilon_n}} \le \epsilon.$$

Using 387C, we can choose inductively, for $n \in \mathbb{N}$, partitions of unity $\langle a_{ni} \rangle_{i \in \mathbb{N}}$ such that, for each $n \in \mathbb{N}$,

$$\sum_{i=0}^{\infty} |\gamma_i - \bar{\mu}a_{ni}| \le \epsilon_n,$$

$$H(A_n) \le h(\pi, A_n) + \epsilon_n < \infty$$

(writing $A_n = \{a_{ni} : i \in \mathbb{N}\} \setminus \{0\}$),

$$\sum_{i=0}^{\infty} \bar{\mu}(a_{n+1,i} \triangle a_{ni}) \le \epsilon_{n+1} + 6\sqrt{\epsilon_n + \sqrt{2\epsilon_n}},$$

while

$$\sum_{i=0}^{\infty} \bar{\mu}(a_{0i} \triangle a_i) \le \epsilon_0 + 6\beta.$$

On completing the induction, we see that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \bar{\mu}(a_{n+1,i} \triangle a_{ni}) \le \sum_{n=1}^{\infty} \epsilon_n + \sum_{n=0}^{\infty} 6\sqrt{\epsilon_n + \sqrt{2\epsilon_n}} < \infty.$$

In particular, given $i \in \mathbb{N}$, $\sum_{n=0}^{\infty} \bar{\mu}(a_{n+1,i} \triangle a_{ni})$ is finite, so $\langle a_{ni} \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete metric space $\mathfrak{A}$ (323Gc), and has a limit a_i^*, with

$$\bar{\mu}a_i^* = \lim_{n \to \infty} \bar{\mu}a_{ni} = \gamma_i$$

(323C). If $i \ne j$,

$$a_i^* \cap a_j^* = \lim_{n \to \infty} a_{ni} \cap a_{nj} = 0$$

(using 323Ba), so $\langle a_i^* \rangle_{i \in \mathbb{N}}$ is disjoint; since

$$\sum_{i=0}^{\infty} \bar{\mu}a_i^* = \sum_{i=0}^{\infty} \gamma_i = 1,$$

$\langle a_i^* \rangle_{i \in \mathbb{N}}$ is a partition of unity. We also have

$$\begin{aligned} \sum_{i=0}^{\infty} \bar{\mu}(a_i^* \triangle a_i) &\le \sum_{i=0}^{\infty} \bar{\mu}(a_{0i} \triangle a_i) + \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \bar{\mu}(a_{n+1,i} \triangle a_{ni}) \\ &\le \epsilon_0 + 6\beta + \sum_{n=1}^{\infty} \epsilon_n + \sum_{n=0}^{\infty} 6\sqrt{\epsilon_n + \sqrt{2\epsilon_n}} \le \epsilon + 6\beta. \end{aligned}$$

(b) Now take any $i(0), \dots, i(k) \in \mathbb{N}$. For each $j < k$, $n \in \mathbb{N}$,

$$H(\pi^j[A_n]) + H(D_j(A_n, \pi)) - H(D_{j+1}(A_n, \pi)) \le H(A_n) - h(\pi, A_n) \le \epsilon_n$$

(using 386Kc). But this means that

$$\sum_{d \in D_j(A_n,\pi)} \sum_{i=0}^{\infty} |\bar{\mu}(d \cap \pi^j a_{ni}) - \bar{\mu}d \cdot \bar{\mu}a_{ni}| \le \sqrt{2\epsilon_n},$$

by 386H. *A fortiori*,

$$|\bar{\mu}(d \cap \pi^j a_{ni}) - \bar{\mu}d \cdot \bar{\mu}a_{ni}| \le \sqrt{2\epsilon_n}$$

for each $d \in D_j(A_n, \pi)$, $i \in \mathbb{N}$. Inducing on r, we see that

$$|\bar{\mu}(\inf_{j \le r} \pi^j a_{n,i(j)}) - \prod_{j=0}^{r} \bar{\mu}a_{n,i(j)}| \le r\sqrt{2\epsilon_n} \to 0$$

as $n \to \infty$, for any $r \le k$. Because $\bar{\mu}$, $\cap$ and π are all continuous (323C, 323Ba and the other part of 324Kb,

$$\begin{aligned}\bar{\mu}(\inf_{j \le k} \pi^j a^*_{i(j)}) &= \lim_{n\to\infty} \bar{\mu}(\inf_{j \le k} \pi^j a_{n,i(j)}) \\ &= \lim_{n\to\infty} \prod_{j=0}^{k} \bar{\mu}a_{n,i(j)} = \prod_{j=0}^{k} \gamma_{i(j)}.\end{aligned}$$

As $i(0), \dots, i(k)$ are arbitrary, $\langle a^*_i \rangle_{i\in\mathbb{N}}$ is a Bernoulli partition for π.

387E Sinaĭ's theorem (atomic case) (Sinaĭ 62) Suppose that $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra and that $\pi \in \mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$ is ergodic. Let $\langle \gamma_i \rangle_{i\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{i=0}^{\infty} \gamma_i = 1$ and $\sum_{i=0}^{\infty} q(\gamma_i) \le h(\pi)$. Then there is a Bernoulli partition $\langle a^*_i \rangle_{i\in\mathbb{N}}$ for π such that $\bar{\mu}a^*_i = \gamma_i$ for every $i \in \mathbb{N}$.

proof Apply 387D from any starting point, e.g., $a_0 = 1$, $a_i = 0$ for $i > 0$.

387F I devote a couple of pages to machinery concerning the spaces $\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$ of 387Ad. We do not really need to work at this level of abstraction, but it is easy, it fits naturally among the methods being developed in this volume, and it will simplify the language of some of the lemmas to follow.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, π a member of $\mathrm{Aut}_{\bar{\mu}}\,\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{C}$ closed subalgebras of $\mathfrak{A}$ such that $\pi[\mathfrak{B}] = \mathfrak{B}$ and $\pi[\mathfrak{C}] = \mathfrak{C}$.

(a) Suppose that $\phi \in \mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$.

(i) $\pi^j\phi = \phi\pi^j$ for every $j \in \mathbb{Z}$.

(ii) $\phi[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{C}$ and $\pi[\phi[\mathfrak{B}]] = \phi[\mathfrak{B}]$; ϕ is an isomorphism between $(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B}, \pi\restriction\mathfrak{B})$ and $(\phi[\mathfrak{B}], \bar{\mu}\restriction\phi[\mathfrak{B}], \pi\restriction\phi[\mathfrak{B}])$.

(iii) If $\psi \in \mathrm{Hom}_{\bar{\mu},\pi}(\phi[\mathfrak{B}]; \mathfrak{C})$ then $\psi\phi \in \mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$.

(iv) If $\langle b_i \rangle_{i\in I}$ is a Bernoulli partition for $\pi\restriction\mathfrak{B}$, then $\langle \phi b_i \rangle_{i\in I}$ is a Bernoulli partition for $\pi\restriction\mathfrak{C}$.

(b) $\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$ is complete under its weak uniformity.

(c) Let $B \subseteq \mathfrak{B}$ be such that $\mathfrak{B}$ is the closed subalgebra of itself generated by $\bigcup_{i\in\mathbb{Z}} \pi^i[B]$. Then the weak uniformity of $\mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$ is the uniformity defined by the pseudometrics $(\phi, \psi) \mapsto \bar{\mu}(\phi b \,\triangle\, \psi b)$ as b runs over B.

proof (a)(i) Since $\pi\phi = \phi\pi$, we can induce on j to get the result for $j \ge 0$. Now if $b \in \mathfrak{B}$ there is a $b' \in \mathfrak{B}$ such that $\pi b' = b$, in which case

$$\pi^{-1}\phi b = \pi^{-1}\phi\pi b' = \pi^{-1}\pi\phi b' = \phi b' = \phi\pi^{-1}b.$$

Thus $\pi^{-1}\phi = \phi\pi^{-1}$. Accordingly $\pi^{-j}\phi = \phi\pi^{-j}$ for every $j \ge 0$ and we have the result.

(ii) $\phi[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{C}$ by 324Kb again. Now $\pi[\phi[\mathfrak{B}]] = \phi[\pi[\mathfrak{B}]] = \phi[\mathfrak{B}]$. Because ϕ is injective, it is an isomorphism between the two structures.

(iii) By (ii), we can speak of $\mathrm{Hom}_{\bar{\mu},\pi}(\phi[\mathfrak{B}]; \mathfrak{C})$, and $\psi\phi : \mathfrak{B} \to \mathfrak{C}$ is a Boolean homomorphism. Now

$$\bar{\mu}\psi\phi = \bar{\mu}\phi = \bar{\mu}\restriction\mathfrak{B}, \quad \pi\psi\phi = \pi\phi = \pi\restriction\mathfrak{B}$$

so $\psi\phi \in \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$.

(iv) If $k \in \mathbb{N}$ and $i(0),\dots,i(k) \in I$,

$$\begin{aligned}\bar\mu(\inf_{j\le k}\pi^j\phi b_{i(j)}) = \bar\mu(\inf_{j\le k}\phi\pi^j b_{i(j)}) &= \bar\mu\phi(\inf_{j\le k}\pi^j b_{i(j)})\\ &= \bar\mu(\inf_{j\le k}\pi^j b_{i(j)}) = \prod_{j=0}^k \bar\mu b_{i(j)} = \prod_{j=0}^k \bar\mu\phi b_{i(j)}.\end{aligned}$$

(b) Let $\mathcal{F}$ be a Cauchy filter on $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$. Write ρ for the measure metric on $\mathfrak{C}$ (323Ad).

(i) For $b \in \mathfrak{B}$, let $\mathcal{F}_b$ be the image of $\mathcal{F}$ under the map $b \mapsto \phi b : \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C}) \to \mathfrak{C}$. Then $\mathcal{F}_b$ is ρ-Cauchy, since $(\phi,\psi) \mapsto \rho(\phi b,\psi b)$ is one of the pseudometrics defining the weak uniformity (see 3A4Fc). Since $\mathfrak{C}$ is complete (323Gc), we have an element $\lim\mathcal{F}_b = \lim_{\phi\to\mathcal{F}}\phi b$ defined in $\mathfrak{C}$; call it θb.

(ii)$(\boldsymbol{\alpha})$ Take b, $b' \in \mathfrak{B}$.

$$\theta(b\cap b') = \lim_{\phi\to\mathcal{F}}\phi(b\cap b') = \lim_{\phi\to\mathcal{F}}\phi b\cap\phi b' = \lim_{\phi\to\mathcal{F}}\phi b\cap\lim_{\phi\to\mathcal{F}}\phi b'$$

(because $\cap : \mathfrak{C}\times\mathfrak{C}\to\mathfrak{C}$ is continuous, by 323Ba)

$$= \theta b\cap\theta b',$$

and similarly $\theta(b\setminus b') = \theta b\setminus\theta b'$. Since of course

$$\theta 1_{\mathfrak{B}} = \lim_{\phi\to\mathcal{F}}\phi 1_{\mathfrak{B}} = \lim_{\phi\to\mathcal{F}} 1_{\mathfrak{C}} = 1_{\mathfrak{C}},$$

θ is a Boolean homomorphism.

$(\boldsymbol{\beta})$ Now for any $b \in \mathfrak{B}$, we have

$$\pi\theta b = \pi(\lim_{\phi\to\mathcal{F}}\phi b) = \lim_{\phi\to\mathcal{F}}\pi\phi b$$

(because $\pi\restriction\mathfrak{C}$ is continuous, by 324Kb once more)

$$= \lim_{\phi\to\mathcal{F}}\phi\pi b = \theta\pi b,$$

$$\bar\mu\theta b = \bar\mu(\lim_{\phi\to\mathcal{F}}\phi b) = \lim_{\phi\to\mathcal{F}}\bar\mu\phi b$$

(because $\bar\mu\restriction\mathfrak{C}$ is continuous, by 323Cb)

$$= \lim_{\phi\to\mathcal{F}}\bar\mu b = \bar\mu b.$$

So $\theta \in \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B},\mathfrak{C})$.

(iii) If $b \in \mathfrak{B}$ and $\epsilon > 0$, there is an $F \in \mathcal{F}$ such that $\rho(\phi b,\psi b) \le \epsilon$ for every ϕ, $\psi \in F$. So for $\phi \in F$,

$$\rho(\phi b,\theta b) = \lim_{\psi\to\mathcal{F}}\rho(\phi b,\psi b) \le \epsilon.$$

As b and ϵ are arbitrary, $\mathcal{F}\to\theta$ (2A3Sc). As $\mathcal{F}$ is arbitrary, $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B},\mathfrak{C})$ is complete.

(c)(i) Write $\mathcal{W}$ for the weak uniformity and $\mathcal{V}$ for the uniformity defined by the pseudometrics $(\phi,\psi) \mapsto \bar\mu(\phi b\triangle\psi b)$ as b runs over B. Since $\mathcal{W}$ is defined by a larger set of pseudometrics, we surely have $\mathcal{V}\subseteq\mathcal{W}$; I need to show that $\mathcal{W}\subseteq\mathcal{V}$, that is, that the identity map from $(\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C}),\mathcal{V})$ to $(\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C}),\mathcal{W})$ is uniformly continuous. Let D be the set of those $d \in \mathfrak{B}$ such that

for every $\epsilon > 0$ there are a finite subset $I \subseteq B$ and a $\delta > 0$ such that $\bar\mu(\phi d\triangle\psi d) \le \epsilon$ whenever ϕ, $\psi \in \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B},\mathfrak{C})$ and $\sup_{b\in I}\bar\mu(\phi b\triangle\psi b) \le \delta$.

Then $B\cup\{0,1\} \subseteq D$.

(i)(α) If d, $d'\in D$ then $d\cap d'\in D$. **P** Let $\epsilon>0$. Then there are I, $I'\in[B]^{\omega}$ and δ, $\delta'>0$ such that $\bar\mu(\phi d\,\triangle\,\psi d)\le\frac{1}{2}\epsilon$ whenever $\sup_{b\in I}\bar\mu(\phi b\,\triangle\,\psi b)\le\delta$ and $\bar\mu(\phi d'\,\triangle\,\psi d')\le\frac{1}{2}\epsilon$ whenever $\sup_{b\in I}\bar\mu(\phi b\,\triangle\,\psi b)\le\delta'$. If now $\sup_{b\in I\cup I'}\bar\mu(\phi b\,\triangle\,\psi b)\le\min(\delta,\delta')$ we shall have

$$\bar\mu(\phi(d\cap d')\,\triangle\,\psi(d\cap d'))=\bar\mu(\phi d\cap\phi d')\,\triangle\,(\psi d\cap\psi d'))\le\bar\mu(\phi d\,\triangle\,\psi d)+\bar\mu(\phi d'\,\triangle\,\psi d')$$

(see the proof of 323Ba)

$$\le\frac{1}{2}\epsilon+\frac{1}{2}\epsilon=\epsilon.$$

As ϵ is arbitrary, $d\cap d'\in D$. **Q**

(β) If $d\in D$ then $1\setminus d\in D$. **P** For any ϕ, $\psi\in\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$,

$$\bar\mu(\phi(1\setminus d)\,\triangle\,\psi(1\setminus d))=\bar\mu((\phi1\setminus\phi d)\,\triangle\,(\psi1\setminus\psi d))=\bar\mu((1\setminus\phi d)\,\triangle\,(1\setminus\psi d))=\bar\mu(\phi d\,\triangle\,\psi d).\ \mathbf{Q}$$

So D is a subalgebra of $\mathfrak{B}$ (312B).

(γ) D is a closed subalgebra of $\mathfrak{B}$. **P** Suppose that d belongs to the closure of D for the measure-algebra topology. Let $\epsilon>0$. Then there is a $d'\in D$ such that $\bar\mu(d\,\triangle\,d')\le\frac{1}{3}\epsilon$. Let $I\in[B]^{<\omega}$ and $\delta>0$ be such that $\bar\mu(\phi d'\,\triangle\,\psi d')\le\frac{1}{3}\epsilon$ whenever $\sup_{b\in I}\bar\mu(\phi b\,\triangle\,\psi b)\le\delta$. If now $\sup_{b\in I}\bar\mu(\phi b\,\triangle\,\psi b)\le\delta$,

$$\bar\mu(\phi d\,\triangle\,\psi d)\le\bar\mu(\phi d\,\triangle\,\phi d')+\bar\mu(\phi d'\,\triangle\,\psi d')+\bar\mu(\psi d'\,\triangle\,\psi d)\le\bar\mu\phi(d\,\triangle\,d')+\frac{\epsilon}{3}+\bar\mu\psi(d'\,\triangle\,d)$$

$$=\bar\mu(d\,\triangle\,d')+\frac{\epsilon}{3}+\bar\mu(d'\,\triangle\,d)\le\epsilon.$$

As ϵ is arbitrary, $d\in D$; as d is arbitrary, D is closed. **Q**

(δ) $\pi[D]\subseteq D$. **P** Suppose that $d\in D$ and $\epsilon>0$. Let $I\in[B]^{<\omega}$ and $\delta>0$ be such that $\bar\mu(\phi d\,\triangle\,\psi d)\le\epsilon$ whenever $\sup_{b\in I}\bar\mu(\phi b\,\triangle\,\psi b)\le\delta$. If now $\sup_{b\in I}\bar\mu(\phi b\,\triangle\,\psi b)\le\delta$,

$$\bar\mu(\phi\pi d\,\triangle\,\psi\pi d)=\bar\mu(\pi\phi d\,\triangle\,\pi\psi d)=\bar\mu\pi(\phi d\,\triangle\,\psi d)$$
$$=\bar\mu(\phi d\,\triangle\,\psi d)\le\epsilon.$$

As d and ϵ are arbitrary, $\pi[D]\subseteq D$. **Q**

Inducing on j, we see that $\pi^j d\in D$ whenever $j\in\mathbb{N}$ and $d\in D$.

(ε) $\pi^{-1}[D]\subseteq D$. **P** Since, as noted in (a-i) above, $\pi^{-1}\phi=\phi\pi^{-1}$ for every $\phi\in\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B},\mathfrak{C})$, we can repeat the argument of (δ) with π^{-1} in the place of π. **Q** So $\pi^{-j}d\in D$ whenever $j\in\mathbb{N}$ and $d\in D$, and $\pi^j d\in D$ whenever $j\in\mathbb{Z}$ and $d\in D$.

(iii) Thus D is a closed subalgebra of $\mathfrak{B}$ including $\bigcup_{i\in\mathbb{Z}}\pi^i[B]$ and must be the whole of $\mathfrak{B}$. But this means that the condition of 3A4Cc is satisfied by the defining families of pseudometrics for $\mathcal{V}$ and $\mathcal{W}$, so that the identity map from $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$ to itself is $(\mathcal{V}.\mathcal{W})$-uniformly continuous, $\mathcal{W}\subseteq\mathcal{V}$ and the two uniformities are the same.

387G Lemma Suppose that $(\mathfrak{A},\bar\mu)$ is an atomless probability algebra and $\pi\in\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Let $\langle b_i\rangle_{i\in\mathbb{N}}$ and $\langle c_i\rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras of $\mathfrak{A}$ generated by $\{\pi^j b_i:i\in\mathbb{N},\,j\in\mathbb{Z}\}$ and $\{\pi^j c_i:i\in\mathbb{N},\,j\in\mathbb{Z}\}$. Suppose that $\mathfrak{C}\subseteq\mathfrak{B}$. Then for any $\epsilon>0$ we can find a $\phi\in\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$ such that $\bar\mu(\phi c_i\,\triangle\,c_i)\le\epsilon$ for every $i\in\mathbb{N}$,

proof (a) Set $B=\{b_i:i\in\mathbb{N}\}\setminus\{0\}$, $C=\{c_i:i\in\mathbb{N}\}\setminus\{0\}$. If only one c_i is non-zero, then $H(C)=0$, so $H(B)=0$ and $\mathfrak{B}=\{0,1\}$, in which case $\mathfrak{B}=\mathfrak{C}$ and we take ϕ to be the identity homomorphism and stop. Otherwise, $\mathfrak{C}$ is atomless (387Bb).

For $k\in\mathbb{N}$, let $\mathfrak{B}_k\subseteq\mathfrak{B}$ be the finite subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_i:i\le k,\,|j|\le k\}$. Because $\mathfrak{C}\subseteq\mathfrak{B}$, there is an $m\in\mathbb{N}$ such that

$$\rho(c_i,\mathfrak{B}_m)\le\frac{1}{4}\epsilon\text{ for every }i\in\mathbb{N}$$

(386J). Let η, $\xi > 0$ be such that

$$\eta + 6\sqrt[4]{2\eta} \le \frac{\epsilon}{4(2m+1)}, \quad \xi \le \min(\frac{\epsilon}{4}, \frac{1}{6}), \quad \sum_{i=0}^{\infty} q(\min(2\xi, \bar\mu c_i)) \le \eta.$$

(The last is achievable because $\sum_{i=0}^{\infty} q(\bar\mu c_i) = H(C)$ is finite.) Let $r \ge m$ be such that

$$\rho(c_i, \mathfrak{B}_r) \le \xi \text{ for every } i \in \mathbb{N}.$$

Let $n \ge r$ be such that

$$\frac{2r+1}{2n+2} \le \xi, \quad \bar\mu c_i \le \xi \text{ for every } i > n.$$

(b) Let $\langle b_i' \rangle_{i \in \mathbb{N}}$ be a partition of unity in $\mathfrak{C}$ such that $\bar\mu b_i' = \bar\mu b_i$ for every $i \in \mathbb{N}$. Let U be the set of atoms of the subalgebra of $\mathfrak{B}$ generated by $\{\pi^j b_i : i \le n, |j| \le n\} \cup \{\pi^j c_i : i \le n, |j| \le n\}$, and V the set of atoms of the subalgebra of $\mathfrak{C}$ generated by $\{\pi^j b_i' : i \le n, |j| \le n\} \cup \{\pi^j c_i : i \le n, |j| \le n\}$. For each $v \in V$, choose a disjoint family $\langle d_{vu} \rangle_{u \in U}$ in $\mathfrak{C}$ such that $\sup_{u \in U} d_{vu} = v$ and $\bar\mu d_{vu} = \bar\mu(v \cap u)$ for every $u \in U$. By 386C(iv) again, there is an $a \in \mathfrak{C}$ such that $a, \pi a, \dots, \pi^{2n} a$ are disjoint and $\bar\mu(a \cap d_{vu}) = \frac{1}{2n+2}\bar\mu(d_{vu})$ for every $u \in U$ and $v \in V$. ($\pi \restriction \mathfrak{C}$ is a Bernoulli shift, therefore ergodic, by 385Se, therefore aperiodic, by 386D.) Set $e = \sup_{|j| \le n} \pi^j a$, $\tilde e = \sup_{|j| \le n-r} \pi^j a$; then

$$\bar\mu e = (2n+1)\bar\mu a = \frac{2n+1}{2n+2}, \quad \bar\mu \tilde e = (2(n-r)+1)\bar\mu a = 1 - \frac{2r+1}{2n+2}.$$

Let $\mathfrak{C}_{\tilde e}$ be the principal ideal of $\mathfrak{C}$ generated by $\tilde e$.

(c) The family $\langle \pi^{-j}(a \cap d_{vu}) \rangle_{|j| \le n, u \in U, v \in V}$ is disjoint. **P** All we have to note is that the families $\langle d_{vu} \rangle_{u \in U, v \in V}$ and

$$\langle \pi^{-j} a \rangle_{|j| \le n} = \langle \pi^{-n}(\pi^{n+j} a) \rangle_{|j| \le n}$$

are disjoint. **Q** Consequently, if we set

$$\hat b_i = \sup_{|j| \le n} \sup_{v \in V} \sup_{u \in U, u \subseteq \pi^j b_i} \pi^{-j}(a \cap d_{vu}) \in \mathfrak{C}$$

for $i \in \mathbb{N}$, $\langle \hat b_i \rangle_{i \in \mathbb{N}}$ is disjoint, since a given triple (j, u, v) can contribute to at most one $\hat b_i$.

Of course $\hat b_i \subseteq \sup_{|j| \le n} \pi^{-j} a = e$ for every i. If $i \le n$, we also have $\bar\mu \hat b_i = \bar\mu e \cdot \bar\mu b_i$. **P** For $|j| \le n$, $\pi^j b_i$ is a supremum of members of U, so

$$\bar\mu \hat b_i = \sum_{j=-n}^{n} \sum_{v \in V} \sum_{u \in U, u \subseteq \pi^j b_i} \bar\mu(\pi^{-j}(a \cap d_{vu}))$$

(because $\langle \pi^{-j}(a \cap d_{vu}) \rangle_{|j| \le n, u \in U, v \in V}$ is disjoint)

$$= \sum_{j=-n}^{n} \sum_{v \in V} \sum_{u \in U, u \subseteq \pi^j b_i} \bar\mu(a \cap d_{vu}) = \frac{1}{2n+2} \sum_{j=-n}^{n} \sum_{v \in V} \sum_{u \in U, u \subseteq \pi^j b_i} \bar\mu d_{vu}$$

(by the choice of a)

$$= \frac{1}{2n+2} \sum_{j=-n}^{n} \sum_{v \in V} \sum_{u \in U, u \subseteq \pi^j b_i} \bar\mu(v \cap u)$$

(by the choice of d_{vu})

$$= \frac{1}{2n+2} \sum_{j=-n}^{n} \sum_{u \in U, u \subseteq \pi^j b_i} \bar\mu u = \frac{1}{2n+2} \sum_{j=-n}^{n} \bar\mu(\pi^j b_i)$$

(because $\pi^j b_i$ is a disjoint union of members of U when $i \le n$, $|j| \le n$)

$$= \frac{2n+1}{2n+2}\bar\mu b_i = \bar\mu e \cdot \bar\mu b_i. \text{ } \mathbf{Q}$$

Again because $\mathfrak{C}$ is atomless, we can choose a partition of unity $\langle b_i^* \rangle_{i \in \mathbb{N}}$ in $\mathfrak{C}$ such that $\bar\mu b_i^* = \bar\mu b_i$ for every i, while $b_i^* \supseteq \hat b_i$ and $b_i^* \cap e = \hat b_i$ for $i \le n$.

(d) Let $\mathfrak{E}$ be the finite subalgebra of $\mathfrak{B}$ generated by $\{\pi^j b_i : i \le n,\ |j| \le r\} \cup \{\pi^j c_i : i \le n,\ |j| \le r\}$. Define $\theta : \mathfrak{E} \to \mathfrak{C}_{\tilde{e}}$ by setting

$$\theta b = \sup_{|j|\le n-r} \sup_{v\in V} \sup_{u\in U, u\subseteq \pi^j b} \pi^{-j}(a \cap d_{vu})$$

for $b \in \mathfrak{E}$.

(i) θ is a Boolean homomorphism. **P** The point is that if $|j| \le n-r$ and $b \in \mathfrak{E}$, then $\pi^j b$ belongs to the algebra generated by $\{\pi^k b_i : i \le n,\ |k| \le n\} \cup \{\pi^k c_i : i \le n,\ |k| \le n\}$, so is a union of members of U. Since each map

$$b \mapsto \pi^{-j}(a \cap d_{vu}) \text{ if } u \subseteq \pi^j b, \quad 0 \text{ otherwise}$$

is a Boolean homomorphism from $\mathfrak{E}$ to the principal ideal generated by $\pi^{-j}(a \cap d_{vu})$, and

$$\langle \pi^{-j}(a \cap d_{vu})\rangle_{|j|\le n-r, u\in U, v\in V}$$

is a partition of unity in $\mathfrak{C}_{\tilde{e}}$, θ also is a Boolean homomorphism. **Q**

(ii) $\bar{\mu}(\theta b) \le \bar{\mu} b$ for every $b \in \mathfrak{E}$. **P** (Compare (c) above.)

$$\begin{aligned}\bar{\mu}(\theta b) &= \sum_{j=-n+r}^{n-r} \sum_{v\in V} \sum_{u\in U, u\subseteq \pi^j b} \bar{\mu}\pi^{-j}(a \cap d_{vu}) \\ &= \frac{1}{2n+2} \sum_{j=-n+r}^{n-r} \sum_{v\in V} \sum_{u\in U, u\subseteq \pi^j b} \bar{\mu}(v \cap u) = \frac{2n-2r+1}{2n+2}\bar{\mu} b \le \bar{\mu} b. \text{ } \mathbf{Q}\end{aligned}$$

(iii) $\theta(\pi^k b_i) = \tilde{e} \cap \pi^k b_i^*$ for $i \le n$, $|k| \le r$. **P** Of course $\pi^k b_i \in \mathfrak{E}$. If $|j| \le n-r$, then $|j+k| \le n$, so

$$\begin{aligned}\pi^{-j}a \cap \theta(\pi^k b_i) &= \sup_{v\in V} \sup_{u\in U, u\subseteq \pi^{j+k} b_i} \pi^{-j}(a \cap d_{vu}) \\ &= \pi^k\bigl(\sup_{v\in V} \sup_{u\in U, u\subseteq \pi^{j+k} b_i} \pi^{-j-k}(a \cap d_{vu})\bigr) \\ &= \pi^k(\pi^{-j-k}a \cap \hat{b}_i) = \pi^{-j}a \cap \pi^k(e \cap b_i^*) = \pi^{-j}a \cap \pi^k b_i^*\end{aligned}$$

because $\pi^{-j}a \subseteq \pi^k e$. Taking the supremum of these pieces we have

$$\theta(\pi^k b_i) = \sup_{|j|\le n-r} \pi^{-j}a \cap \theta(\pi^k b_i) = \sup_{|j|\le n-r} \pi^{-j}a \cap \pi^k b_i^* = \tilde{e} \cap \pi^k b_i^*. \text{ } \mathbf{Q}$$

(iv) Finally, $\theta c_i = c_i \cap \tilde{e}$ for every $i \le n$. **P** If $|j| \le n-r$ and $v \in V$ then either $v \subseteq \pi^j c_i$ or $v \cap \pi^j c_i = 0$. In the former case,

$$d_{vu} = v \cap u = 0 \text{ whenever } u \in U \text{ and } u \not\subseteq \pi^j c_i,$$

so that

$$v = \sup_{u\in U} d_{vu} = \sup_{u\in U, u\subseteq \pi^j c_i} d_{vu};$$

in the latter case, $d_{vu} = v \cap u = 0$ whenever $u \subseteq \pi^j c_i$. So we have

$$v \cap \pi^j c_i = \sup_{u\in U, u\subseteq \pi^j c_i} d_{vu}$$

for every $v \in V$, and

$$\begin{aligned}\theta c_i &= \sup_{|j|\le n-r} \sup_{v\in V} \sup_{u\in U, u\subseteq \pi^j c_i} \pi^{-j}(a \cap d_{vu}) \\ &= \sup_{|j|\le n-r} \pi^{-j}(a \cap \sup_{v\in V} \sup_{u\in U, u\subseteq \pi^j c_i} d_{vu}) \\ &= \sup_{|j|\le n-r} \pi^{-j}(a \cap \sup_{v\in V}(v \cap \pi^j c_i)) \\ &= \sup_{|j|\le n-r} \pi^{-j}(a \cap \pi^j c_i) = c_i \cap \sup_{|j|\le n-r} \pi^{-j}a = c_i \cap \tilde{e}. \text{ } \mathbf{Q}\end{aligned}$$

(e) Let $\mathfrak{B}^*$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_i^* : i \in \mathbb{N},\ j \in \mathbb{Z}\}$. Then for every $b \in \mathfrak{B}_r$ there is a $b^* \in \mathfrak{B}^*$ such that $\theta b = b^* \cap \tilde{e}$. **P** The set of b for which this is true is a subalgebra of $\mathfrak{A}$ containing $\pi^k b_i$ for $i \le r$ and $|k| \le r$, by (d-iii). **Q** It follows that

$$\rho(c_i, \mathfrak{B}^*) \le 2\xi \text{ for } i \in \mathbb{N}.$$

P If $i > n$ this is trivial, because $\bar{\mu} c_i \le \xi$, by the choice of n. Otherwise, $c_i \in \mathfrak{C}$. Take $b \in \mathfrak{B}_r$ such that $\bar{\mu}(b \triangle c_i) = \rho(c_i, \mathfrak{B}_r) \le \xi$. Let $b^* \in \mathfrak{B}^*$ be such that $\theta b = b^* \cap \tilde{e}$. Then

$$\begin{aligned}\rho(c_i, \mathfrak{B}^*) &\le \bar{\mu}(c_i \triangle b^*) \le 1 - \bar{\mu}\tilde{e} + \bar{\mu}(\tilde{e} \cap (c_i \triangle b^*)) \\ &= \tfrac{2r+1}{2n+2} + \bar{\mu}((\tilde{e} \cap c_i) \triangle \theta b) = \tfrac{2r+1}{2n+2} + \bar{\mu}(\theta c_i \triangle \theta b)\end{aligned}$$

(by (d-iv))

$$= \tfrac{2r+1}{2n+2} + \bar{\mu}(\theta(c_i \triangle b)) \le \tfrac{2r+1}{2n+2} + \bar{\mu}(c_i \triangle b)$$

(by (d-ii))

$$\le 2\xi$$

by the choice of n. **Q**

(f) Set $B^* = \{b_i^* : i \in \mathbb{N}\} \setminus \{0\}$. Then $H(B^*) = h(\pi, C) \le h(\pi, B^*) + \eta$. **P**

$$H(B^*) = H(B) = H(C)$$

(because $\bar{\mu} b_i^* = \bar{\mu} b_i$ for every i, and we supposed from the beginning that $H(C) = H(B)$)

$$= h(\pi, C)$$

(because C is a Bernoulli partition, see 387Ba)

$$\le h(\pi \restriction \mathfrak{B}^*) + H(C|\mathfrak{B}^*)$$

(386Kd)

$$\le h(\pi \restriction \mathfrak{B}^*) + \sum_{i=0}^{\infty} q(\rho(c_i, \mathfrak{B}^*))$$

(386Lb)

$$\le h(\pi, B^*) + \sum_{i=0}^{\infty} q(\min(2\xi, \bar{\mu} c_i))$$

(by the Kolmogorov-Sinaĭ theorem, 385P(ii), and (e) above, recalling that $\xi \le \frac{1}{6}$, so that q is monotonic on $[0, 2\xi]$)

$$\le h(\pi, B^*) + \eta$$

by the choice of ξ. **Q**

Note also that $H(B^*) = h(\pi, C) \le h(\pi)$.

(g) By 387D, applied to $\pi \restriction \mathfrak{C}$ and the partition $\langle b_i^* \rangle_{i \in \mathbb{N}}$ of unity in $\mathfrak{C}$ and the sequence $\langle \gamma_i \rangle_{i \in \mathbb{N}} = \langle \bar{\mu} b_i^* \rangle_{i \in \mathbb{N}}$, we have a Bernoulli partition $\langle d_i \rangle_{i \in \mathbb{N}}$ in $\mathfrak{C}$ such that $\bar{\mu} d_i = \bar{\mu} b_i^* = \bar{\mu} b_i$ for every $i \in \mathbb{N}$ and

$$\textstyle\sum_{i=0}^{\infty} \bar{\mu}(d_i \triangle b_i^*) \le \eta + 6\sqrt[4]{2\eta} \le \frac{\epsilon}{4(2m+1)}.$$

Let $\mathfrak{D} \subseteq \mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j d_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$. Then we have a $\phi \in \mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{B}; \mathfrak{C})$ such that $\phi b_i = d_i$ for every $i \in \mathbb{N}$ (387Bc).

(h) Set

$$e^* = \tilde{e} \setminus \sup_{|j| \le m, i \in \mathbb{N}} \pi^j (d_i \triangle b_i^*).$$

Then $\phi(\pi^j b_i)\cap e^* = \theta(\pi^j b_i)\cap e^*$ whenever $i \le m$ and $|j| \le m$. **P**

$$\begin{aligned}\phi(\pi^j b_i)\cap e^* &= \pi^j(\phi b_i)\cap e^* = \pi^j d_i \cap e^*\\ &= \pi^j b_i^* \cap e^* = \pi^j b_i^* \cap \tilde e \cap e^* = \theta(\pi^j b_i)\cap e^*\end{aligned}$$

by (d-iii), because i and $|j|$ are both at most $m \le r \le n$. **Q** Since $b \mapsto \phi b \cap e^* : \mathfrak{A} \to \mathfrak{A}_{e^*}$, $b \mapsto \theta b \cap e^* : \mathfrak{C} \to \mathfrak{A}_{e^*}$ are Boolean homomorphisms, $\phi b \cap e^* = \theta b \cap e^*$ for every $b \in \mathfrak{B}_m$.

Now $\bar\mu(c_i \triangle \phi c_i) \le \epsilon$ for every $i \in \mathbb{N}$. **P** If $i > n$ then of course

$$\bar\mu(\phi c_i \triangle c_i) \le 2\bar\mu c_i \le 2\xi \le \epsilon.$$

If $i \le n$, then (by the choice of m) there is a $b \in \mathfrak{B}_m$ such that $\bar\mu(c_i, b) \le \frac14\epsilon$. So

$$\begin{aligned}\phi c_i \triangle c_i &\subseteq (\phi c_i \triangle \phi b)\cup(\phi b \triangle \theta b)\cup(\theta b \triangle \theta c_i)\cup(\theta c_i \triangle c_i)\\ &\subseteq \phi(c_i \triangle b)\cup(1\setminus e^*)\cup\theta(b \triangle c_i)\end{aligned}$$

(using the definition of e^* and (d-iv)) has measure at most

$$\bar\mu(c_i \triangle b) + \bar\mu(1\setminus e^*) + \bar\mu(b \triangle c_i)$$

(by (d-ii), since b and c_i both belong to $\mathfrak{C}$)

$$\begin{aligned}&\le 2\bar\mu(c_i \triangle b) + \bar\mu(1\setminus\tilde e) + (2m+1)\sum_{i=0}^\infty \bar\mu(d_i \triangle b_i^*)\\ &\le \frac{\epsilon}{2} + \frac{2r+1}{2n+2} + \frac{\epsilon}{4} \le \epsilon,\end{aligned}$$

as required. **Q**

Thus we have found a suitable ϕ.

387H Lemma Suppose that $(\mathfrak{A},\bar\mu)$ is an atomless probability algebra and $\pi \in \operatorname{Aut}_{\bar\mu}\mathfrak{A}$. Let $\langle b_i\rangle_{i\in\mathbb{N}}$ and $\langle c_i\rangle_{i\in\mathbb{N}}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras generated by $\{\pi^j b_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$ and $\{\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Suppose that $\mathfrak{C} \subseteq \mathfrak{B}$. Then for any $\epsilon > 0$ we can find a $\phi \in \operatorname{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$ such that $\bar\mu(\phi c_i \triangle c_i) \le \epsilon$ and $\rho(b_i, \phi[\mathfrak{C}]) \le \epsilon$ for every $i \in \mathbb{N}$.

proof (a) By 387G, there is a $\phi_0 \in \operatorname{Hom}_{\bar\mu,\pi}(\mathfrak{B};\mathfrak{C})$ such that $\bar\mu(\phi_0 c_i \triangle c_i) \le \frac14\epsilon$ for every $i \in \mathbb{N}$. Write $\mathfrak{B}^*$ for $\phi_0[\mathfrak{B}] \subseteq \mathfrak{C}$ and $b_i^* = \phi_0 b_i$ for $i \in \mathbb{N}$.

Let $m \in \mathbb{N}$ be such that

$$\rho(c_i, \mathfrak{B}_m) \le \tfrac14\epsilon \text{ for every } i \in \mathbb{N},$$

where $\mathfrak{B}_m$ is the subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_i : i \le m, |j| \le m\}$ (386J). Let $\eta \in \left]0,\epsilon\right]$ be such that

$$(2m+1)\sum_{i=0}^\infty \min(\eta, 2\bar\mu b_i) \le \tfrac14\epsilon.$$

We know that $\mathfrak{B}^*$ is a closed subalgebra of $\mathfrak{C}$ and $\pi[\mathfrak{B}^*] = \mathfrak{B}^*$ (387F(a-ii)), while $\langle b_i^*\rangle_{i\in\mathbb{N}}$ is a generating Bernoulli partition of $\mathfrak{B}^*$ because ϕ_0 is an isomorphism between $(\mathfrak{B},\bar\mu{\restriction}\mathfrak{B},\pi{\restriction}\mathfrak{B})$ and $(\mathfrak{B}^*,\bar\mu{\restriction}\mathfrak{B}^*,\pi{\restriction}\mathfrak{B}^*)$ (387Bc). By 387G again, there is a $\phi_1 \in \operatorname{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B}^*)$ such that $\bar\mu(\phi_1 b_i^* \triangle b_i^*) \le \eta$ for every $i \in \mathbb{N}$. Write $\mathfrak{C}^* = \phi_1[\mathfrak{C}]$, $c_i^* = \phi_1 c_i$ for $i \in \mathbb{N}$.

(b) Now $\bar\mu(c_i^* \triangle \phi_0 c_i) \le \epsilon$ for every $i \in \mathbb{N}$. **P** There is a $b \in \mathfrak{B}_m$ such that $\bar\mu(c_i \triangle b) \le \frac14\epsilon$. We know that $\phi_0[\mathfrak{B}_m]$ is the subalgebra of $\mathfrak{A}$ generated by $\{\phi_0\pi^j b_i : i \le m, |j| \le m\} = \{\pi^j b_i^* : i \le m, |j| \le m\}$, and contains $\phi_0 b$. Because

$$\phi_1(\phi_0 b)\triangle\phi_0 b \subseteq \sup_{i\in\mathbb{N},|j|\le m}\phi_1(\pi^j b_i^*)\triangle\pi^j b_i^* = \sup_{|j|\le m}\pi^j(\sup_{i\in\mathbb{N}}\phi_1 b_i^* \triangle b_i^*),$$

we have

$$\bar\mu(\phi_1\phi_0 b \mathbin{\triangle} \phi_0 b) \le (2m+1)\sum_{i=0}^{\infty}\bar\mu(\phi_1 b_i^* \mathbin{\triangle} b_i^*)$$
$$\le (2m+1)\sum_{i=0}^{\infty}\min(\eta, 2\bar\mu b_i) \le \frac{1}{4}\epsilon.$$

But this means that

$$\bar\mu(c_i^* \mathbin{\triangle} \phi_0 c_i) = \bar\mu(\phi_1 c_i \mathbin{\triangle} \phi_0 c_i) \le \bar\mu(\phi_1 c_i \mathbin{\triangle} \phi_1\phi_0 b) + \bar\mu(\phi_1\phi_0 b \mathbin{\triangle} \phi_0 b) + \bar\mu(\phi_0 b \mathbin{\triangle} \phi_0 c_i)$$
$$\le \bar\mu(c_i \mathbin{\triangle} \phi_0 b) + \frac{\epsilon}{4} + \bar\mu(b \mathbin{\triangle} c_i) \le \bar\mu(c_i \mathbin{\triangle} \phi_0 c_i) + \bar\mu(\phi_0 c_i \mathbin{\triangle} \phi_0 b) + \frac{\epsilon}{2}$$
$$\le \frac{\epsilon}{4} + \bar\mu(c_i \mathbin{\triangle} b) + \frac{\epsilon}{2} \le \epsilon. \quad \mathbf{Q}$$

(c) Set $\phi = \phi_0^{-1}\phi_1$; this is well-defined, with domain $\mathfrak{C}$, because ϕ_0 is injective and $\phi_1[\mathfrak{C}] \subseteq \phi_0[\mathfrak{B}]$. Because $\phi_1 : \mathfrak{C} \to \phi_0[\mathfrak{B}]$ and $\phi_0^{-1} : \phi_0[\mathfrak{B}] \to \mathfrak{B}$ are Boolean homomorphisms, $\phi : \mathfrak{C} \to \mathfrak{B}$ is a Boolean homomorphism. If $c \in \mathfrak{C}$, then

$$\bar\mu\phi c = \bar\mu\phi_1 c = \bar\mu c, \quad \pi\phi c = \pi\phi_1 c = \pi\phi c,$$

so $\phi \in \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$. Next,

$$\bar\mu(c_i \mathbin{\triangle} \phi c_i) = \bar\mu(\phi_0 c_i \mathbin{\triangle} \phi_0\phi c_i) = \bar\mu(\phi_0 c_i \mathbin{\triangle} c_i^*) \le \epsilon$$

for every i, by (b). Finally, if $i \in \mathbb{N}$, then $\phi_1 b_i^*$ belongs to $\phi_1[\mathfrak{C}]$, while $\mathfrak{D} = \phi_0^{-1}[\phi_1[\mathfrak{C}]]$, so

$$\rho(b_i, \phi[\mathfrak{C}]) = \rho(\phi_0 b_i, \phi_1[\mathfrak{C}]) \le \bar\mu(\phi_0 b_i \mathbin{\triangle} \phi_1 b_i^*) = \bar\mu(b_i^* \mathbin{\triangle} \phi_1 b_i^*) \le \eta \le \epsilon.$$

This completes the proof.

387I Lemma Suppose that $(\mathfrak{A}, \bar\mu)$ is an atomless probability algebra and $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Let $\langle b_i\rangle_{i\in I}$, $\langle c_i\rangle_{i\in I}$ be Bernoulli partitions for π, of the same finite entropy, and write $\mathfrak{B}$, $\mathfrak{C}$ for the closed subalgebras generated by $\{\pi^j b_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$ and $\{\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Suppose that $\mathfrak{C} \subseteq \mathfrak{B}$. Then for any $\epsilon > 0$ we can find $\phi \in \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$ such that $\phi[\mathfrak{C}] = \mathfrak{B}$ and $\bar\mu(\phi c_i \mathbin{\triangle} c_i) \le \epsilon$ for every $i \in \mathbb{N}$.

proof (a) To begin with (down to the end of (c) below) suppose that $I = \mathbb{N}$. Choose sequences $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ in $]0,\infty[$, $\langle\delta_n\rangle_{n\in\mathbb{N}}$ in $]0,\infty[$, $\langle r_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ and $\langle\phi_n\rangle_{n\in\mathbb{N}}$ in $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$ inductively, as follows. Start with $r_0 = 0$ and $\phi_0 : \mathfrak{C} \to \mathfrak{B}$ the identity. Given that $\phi_n \in \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$ is such that $\rho(b_i, \phi_n[\mathfrak{C}]) \le 2^{-n}$ for every $i \in \mathbb{N}$, let $r_n \in \mathbb{N}$ be such that $\rho(b_i, \mathfrak{D}_n) \le 2^{-n+1}$ for every $i \in \mathbb{N}$, where $\mathfrak{D}_n$ is the algebra generated by $\{\pi^j\phi_n c_i : i \le r_n, |j| \le r_n\}$ (386J).

Take ϵ_n, $\delta_n > 0$ such that

$$(2r_m + 1)\epsilon_n \le 2^{-n} \text{ for every } m \le n,$$

$$\delta_n \le 2^{-n-1}\epsilon, \quad \textstyle\sum_{i=0}^{\infty}\min(\delta_n, 2\bar\mu c_i) \le \epsilon_n,$$

and use 387H to find $\psi_n \in \mathrm{Hom}_{\bar\mu,\pi}(\phi_n[\mathfrak{C}];\mathfrak{B})$ such that

$$\bar\mu(\psi_n\phi_n c_i \mathbin{\triangle} \phi_n c_i) \le \delta_n, \quad \rho(b_i, \psi_n\phi_n[\mathfrak{C}]) \le 2^{-n-1}$$

for every $i \in \mathbb{N}$. Set $\phi_{n+1} = \psi_n\phi_n$, so that $\phi_{n+1} \in \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$ (387F(a-iii)) and $\rho(b_i, \phi_{n+1}[\mathfrak{C}]) \le 2^{-n-1}$ for every i. Continue.

(b) For any $i \in \mathbb{N}$,

$$\textstyle\sum_{n=0}^{\infty}\bar\mu(\phi_{n+1}c_i \mathbin{\triangle} \phi_n c_i) \le \sum_{n=0}^{\infty}\delta_n \le \epsilon,$$

so $\langle\phi_n c_i\rangle_{n\in\mathbb{N}}$ has a limit ϕc_i in $\mathfrak{A}$. This shows that $\langle\phi_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$ for the uniformity defined by the pseudometrics $(\psi,\psi') \mapsto \bar\mu(\psi c_i \mathbin{\triangle} \psi' c_i)$ as i runs over $\mathbb{N}$. But this is the weak uniformity of $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$, by 387Fc. Since $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$ is complete under this uniformity (387Fb), $\phi = \lim_{n\to\infty}\phi_n$ is defined in $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{C};\mathfrak{B})$. Of course

$$\bar{\mu}(\phi c_i \triangle c_i) \leq \sum_{n=0}^{\infty} \bar{\mu}(\phi_{n+1} c_i \triangle \phi_n c_i) \leq \epsilon$$

for every $i \in \mathbb{N}$.

(c) Now $b_j \in \phi[\mathfrak{C}]$ for every $j \in \mathbb{N}$. **P** Fix $m \in \mathbb{N}$. Then $\rho(b_j, \mathfrak{D}_m) \leq 2^{-m+1}$, so there is a $b \in \mathfrak{D}_m$ such that $\bar{\mu}(b_j \triangle b) \leq 2^{-m+1}$. Now

$$\sum_{i=0}^{\infty} \rho(\phi_m c_i, \phi[\mathfrak{C}]) \leq \sum_{i=0}^{\infty} \bar{\mu}(\phi_m c_i \triangle \phi c_i) \leq \sum_{i=0}^{\infty} \sum_{k=m}^{\infty} \bar{\mu}(\phi_{k+1} c_i \triangle \phi_k c_i)$$

$$\leq \sum_{k=m}^{\infty} \sum_{i=0}^{\infty} \min(\bar{\mu}\phi_{k+1} c_i + \bar{\mu}\phi_k c_i, \delta_k) = \sum_{k=m}^{\infty} \sum_{i=0}^{\infty} \min(2\bar{\mu} c_i, \delta_k) \leq \sum_{k=m}^{\infty} \epsilon_k.$$

So

$$\rho(b, \phi[\mathfrak{C}]) \leq (2r_m + 1) \sum_{i=0}^{\infty} \rho(\phi_m c_i, \phi[\mathfrak{C}])$$

(386Mc)

$$\leq \sum_{k=m}^{\infty} (2r_m + 1)\epsilon_k \leq \sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1},$$

and

$$\rho(b_j, \phi[\mathfrak{C}]) \leq \bar{\mu}(b_j \triangle b) + \rho(b, \phi[\mathfrak{C}]) \leq 2^{-m+1} + 2^{-m+1} = 2^{-m+2}.$$

As m is arbitrary, $\rho(b_j, \phi[\mathfrak{C}]) = 0$ and $b_j \in \phi[\mathfrak{C}]$. **Q**

(d) This completes the proof if $I = \mathbb{N}$. In general, if $I = \{0, \dots, n\}$, set $b_i = c_i = 0$ for $i > n$ and proceed as above; this shows that the result is true for any countable I. If we have been indulgent enough to allow an uncountable I to survive to this point, set $J = \{i : b_i \neq 0\} \cup \{i : c_i \neq 0\}$ and apply the result to $\langle b_i \rangle_{i \in J}$ and $\langle b_i \rangle_{i \in J}$.

387J Ornstein's theorem (finite entropy case) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be probability algebras, and $\pi : \mathfrak{A} \to \mathfrak{A}$, $\phi : \mathfrak{B} \to \mathfrak{B}$ two-sided Bernoulli shifts of the same finite entropy. Then $(\mathfrak{A}, \bar{\mu}, \pi)$ and $(\mathfrak{B}, \bar{\nu}, \phi)$ are isomorphic.

proof (a) By 385R, $\mathfrak{A}$ has a purely atomic root algebra $\mathfrak{A}_0$. If $\mathfrak{A}_0$ is infinite, enumerate its atoms as $\langle a_i \rangle_{i \in \mathbb{N}}$; if $\mathfrak{A}_0$ is finite, enumerate its atoms as $(a_0, \dots, a_n)$ and set $a_i = 0$ for $i > n$. In either case, $\langle a_i \rangle_{i \in \mathbb{N}}$ is a two-sided generating Bernoulli partition in $\mathfrak{A}$. Similarly, $\mathfrak{B}$ has a generating Bernoulli partition $\langle b_i \rangle_{i \in \mathbb{N}}$. By 385R, $\{a_i : i \in \mathbb{N}\}$ and $\{b_i : i \in \mathbb{N}\}$ both have entropy $h(\pi) = h(\phi)$. If this entropy is zero, then $\mathfrak{A}$ and $\mathfrak{B}$ are both $\{0, 1\}$, and the result is trivial; so let us assume that $h(\pi) > 0$, so that $\mathfrak{A}$ is atomless (387Bb).

(b) By Sinaĭ's theorem (387E), there is a Bernoulli partition $\langle c_i \rangle_{i \in \mathbb{N}}$ for π such that $\bar{\mu} c_i = \bar{\nu} b_i$ for every $i \in \mathbb{N}$. Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i : i \in \mathbb{N},\ j \in \mathbb{Z}\}$. By 387I, there is a $\psi \in \mathrm{Hom}_{\bar{\mu},\pi}(\mathfrak{C}, \mathfrak{A})$ such that $\psi[\mathfrak{C}] = \mathfrak{A}$. But now

$$(\mathfrak{A}, \bar{\mu}, \pi) \cong (\mathfrak{C}, \bar{\mu}\restriction\mathfrak{C}, \pi\restriction\mathfrak{C}) \cong (\mathfrak{B}, \bar{\nu}, \phi)$$

(387F(a-ii), 387Bc).

387K Using the same methods, we can extend the last result to the case of Bernoulli shifts of infinite entropy. The first step uses the ideas of 387C, as follows.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi \in \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ an ergodic measure-preserving automorphism. Suppose that $\langle a_i \rangle_{i \in I}$ is a finite Bernoulli partition for π, with $\#(I) = r \geq 1$ and $\bar{\mu} a_i = 1/r$ for every $i \in I$, and that $h(\pi) \geq \ln 2r$. Then for any $\epsilon > 0$ there is a Bernoulli partition $\langle b_{ij} \rangle_{i \in I, j \in \{0,1\}}$ for π such that

$$\bar{\mu}(a_i \triangle (b_{i0} \cup b_{i1})) \leq \epsilon, \quad \bar{\mu} b_{i0} = \bar{\mu} b_{i1} = \frac{1}{2r}$$

for every $i \in I$.

proof (a) Write A for $\{a_i : i \in I\}$. Let $\delta > 0$ be such that

$$\delta + 6\sqrt{4\delta} \le \epsilon.$$

Let $\eta > 0$ be such that

$$\eta < \ln 2, \quad \sqrt{8\eta} \le \delta$$

and

$$|t - \tfrac{1}{2}| \le \delta \text{ whenever } t \in [0,1] \text{ and } q(t) + q(1-t) \ge \ln 2 - 4\eta$$

(385Ad). We have

$$H(A) = rq(\frac{1}{r}) = \ln r,$$

and $\bar\mu d = r^{-n}$ whenever $n \in \mathbb{N}$ and $d \in D_n(A,\pi)$.

Note that $\mathfrak{A}$ is atomless. **P?** If $a \in \mathfrak{A}$ is an atom, then $\sup_{j\in\mathbb{Z}} \pi^j a = 1$ (because π is ergodic, 372Pb), and $\mathfrak{A}$ is purely atomic, with atoms all of the same size as a; but this means that $H(C) \le \ln(\frac{1}{\bar\mu a})$ for every partition of unity $C \subseteq \mathfrak{A}$, so that

$$h(\pi, C) = \lim_{n\to\infty} \frac{1}{n} H(D_n(C,\pi)) \le \lim_{n\to\infty} \frac{1}{n} \ln(\frac{1}{\bar\mu a}) = 0$$

for every partition of unity C, and

$$0 = h(\pi) \ge \ln 2r \ge \ln 2. \textbf{ XQ}$$

(b) There is a finite partition of unity $C \subseteq \mathfrak{A}$ such that

$$h(\pi, C) = \ln 2r - \eta,$$

and C refines A. **P** Because $h(\pi) \ge \ln 2r$, there is a finite partition of unity C' such that $h(\pi, C') \ge \ln 2r - \eta$; replacing C' by $C' \vee A$ if need be, we may suppose that C' refines A; take such a C' of minimal size. Because $H(C') \ge h(\pi, C') > H(A)$, there must be distinct c_0, $c_1 \in C'$ included in the same member of A. Because $\mathfrak{A}$ is atomless, the principal ideal generated by c_1 has a closed subalgebra isomorphic, as measure algebra, to the measure algebra of Lebesgue measure on $[0,1]$, up to a scalar multiple of the measure; and in particular there is a family $\langle d_t\rangle_{t\in[0,1]}$ such that $d_s \subseteq d_t$ whenever $s \le t$, $d_1 = c_1$ and $\bar\mu d_t = t\bar\mu c_1$ for every $t \in [0,1]$. Let D_t be the partition of unity

$$(C' \setminus \{c_0, c_1\}) \cup \{c_0 \cup d_t, c_1 \setminus d_t\}$$

for each $t \in [0,1]$. Then

$$h(\pi, D_1) = h(\pi, (C' \setminus \{c_0, c_1\}) \cup \{c_0 \cup c_1\}) < \ln 2r - \eta,$$

by the minimality of $\#(C')$, while

$$h(\pi, D_0) = h(\pi, C') \ge \ln 2r - \eta.$$

Using 385N, we also have, for any s, $t \in [0,1]$ such that $|s - t| \le \frac{1}{e}$,

$$h(\pi, D_s) - h(\pi, D_t) \le H(D_s|\mathfrak{D}_t)$$

(where $\mathfrak{D}_t$ is the closed subalgebra generated by D_t)

$$\le q(\rho(c_0 \cup d_s, \mathfrak{D}_t)) + q(\rho(c_1 \setminus d_s, \mathfrak{D}_t))$$

(by 386Lb, because $D_s \setminus \mathfrak{D}_t \subseteq \{c_0 \cup d_s, c_1 \setminus d_s\}$)

$$\le q(\bar\mu((c_0 \cup d_s) \triangle (c_0 \cup d_t))) + q(\bar\mu((c_1 \setminus d_s) \triangle (c_1 \setminus d_t)))$$
$$= 2q(\bar\mu(d_s \triangle d_t)) = 2q(|s-t|\bar\mu c_1)$$

because q is monotonic on $[0, |s-t|\bar\mu c_1]$. But this means that $t \mapsto h(\pi, D_t)$ is continuous and there must be some t such that $h(\pi, D_t) = \ln 2r - \eta$; take $C = D_t$. **Q**

(c) Let $\xi > 0$ be such that

$$\xi \le \eta, \quad \xi \le \frac{1}{6}, \quad q(2\xi) + q(1-2\xi) \le \eta, \quad \sum_{c\in C} q(\min(2\xi, \bar\mu c)) \le \eta.$$

Let $n \in \mathbb{N}$ be such that

$$\tfrac{1}{n+1} \le \xi, \quad q(\tfrac{1}{n+1}) + q(\tfrac{n}{n+1}) \le \eta, \quad \bar\mu[\![w_n - h(\pi, C)\chi 1 \ge \eta]\!] \le \xi,$$

where

$$w_n = \frac{1}{n}\sum_{d\in D_n(C,\pi)} \ln(\frac{1}{\bar\mu d})\chi d.$$

(The Shannon-McMillan-Breiman theorem, 386E-386F, assures us that any sufficiently large n has these properties.)

(d) Let D be the set of those $d \in D_n(C, \pi)$ such that

$$\bar\mu d \ge (2r)^{-n}, \quad \text{i.e.,} \quad \frac{1}{n}\ln(\frac{1}{\bar\mu d}) \le \ln 2r.$$

Then $\bar\mu(\sup D) \ge 1 - \xi$, by the choice of n, because $h(\pi, C) = \ln 2r - \eta$. Note that every member of D is included in some member of $D_n(A, \pi)$, because C refines A. If $b \in D_n(A, \pi)$, then $\bar\mu b = r^{-n}$, so $\#(\{d : d \in D, d \subseteq b\}) \le 2^n$; we can therefore find a function $f : D \to \{0, 1\}^n$ such that f is injective on $\{d : d \in D, d \subseteq b\}$ for every $b \in D_n(A, \pi)$.

(e) By 386C(iv), as usual, there is an $a \in \mathfrak{A}$ such that $a, \pi^{-1}a, \dots, \pi^{-n+1}a$ are disjoint and $\bar\mu(a \cap d) = \frac{1}{n+1}\bar\mu d$ for every $d \in D_n(C, \pi)$. Set

$$e = \sup_{d\in D, j<n} \pi^{-j}(a \cap d);$$

then

$$\bar\mu e = \sum_{j=0}^{n-1}\sum_{d\in D} \bar\mu(a \cap d) = \tfrac{n}{n+1}\bar\mu(\sup D) \ge (1 - \xi)^2 \ge 1 - 2\xi.$$

(f) Set

$$c^* = \sup\{\pi^{-j}(a \cap d) : j < n, d \in D, f(d)(j) = 1\}.$$

(I am identifying members of $\{0, 1\}^n$ with functions from $\{0, \dots, n-1\}$ to $\{0, 1\}$.) Set

$$A^* = A \vee \{c^*, 1 \setminus c^*\}, \quad A' = A^* \vee \{a, 1 \setminus a\} \vee \{e, 1 \setminus e\},$$

and let $\mathfrak{A}'$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a' : a' \in A', j \in \mathbb{Z}\}$. Then $a \cap d \in \mathfrak{A}'$ for every $d \in D$. **P** Set $\tilde d = \mathrm{upr}(a \cap d, \mathfrak{A}')$. Let b be the element of $D_n(A, \pi)$ including d. Because a, b, $e \in \mathfrak{A}'$,

$$\tilde d \subseteq a \cap b \cap e = \sup_{d'\in D} a \cap b \cap d' = \sup\{a \cap d' : d' \in D, d' \subseteq b\}.$$

Now if $d' \in D$, $d' \subseteq b$ and $d' \ne d$, then $f(d') \ne f(d)$. Let j be such that $f(d')(j) \ne f(d)(j)$; then $\pi^{-j}(a \cap d)$ is included in one of c^*, $1 \setminus c^*$ and $\pi^{-j}(a \cap d')$ in the other. This means that one of $\pi^j c^*$, $1 \setminus \pi^j c^*$ is a member of $\mathfrak{A}'$ including $a \cap d$ and disjoint from $a \cap d'$, so that $\tilde d \cap d' = 0$. Thus $\tilde d$ must be actually equal to $a \cap d$, and $a \cap d \in \mathfrak{A}'$. **Q**

Next, $c \cap e \in \mathfrak{A}'$ for every $c \in C$. **P** $\langle \pi^{-j}(a \cap d)\rangle_{j<n, d\in D}$ is a disjoint family in $\mathfrak{A}'$ with supremum e. But whenever $d \in D$ and $j < n$ we must have $d \subseteq \pi^j c'$ for some $c' \in C$, so either $d \subseteq \pi^j c$ or $d \cap \pi^j c = 0$; thus $\pi^{-j}(a \cap d)$ must be either included in c or disjoint from it. Accordingly

$$c \cap e = \sup\{\pi^{-j}(a \cap d) : j < n, d \in D, d \subseteq \pi^j c\} \in \mathfrak{A}'. \text{ **Q**}$$

Consequently $h(\pi, A') \ge \ln 2r - 2\eta$. **P** For any $c \in C$,

$$\rho(c, \mathfrak{A}') \le \bar\mu(c \triangle (c \cap e)) = \bar\mu(c \setminus e) \le \min(\bar\mu c, 2\xi) \le \frac{1}{3},$$

so

$$\ln 2r - \eta = h(\pi, C) \le h(\pi{\restriction}\mathfrak{A}') + H(C|\mathfrak{A}')$$

(386Kd)

$$\le h(\pi, A') + \sum_{c \in C} q(\rho(c, \mathfrak{A}'))$$

(by the Kolmogorov-Sinaĭ theorem and 386Lb)

$$\le h(\pi, A') + \sum_{c \in C} q(\min(\bar\mu c, 2\xi)) \le h(\pi, A') + \eta$$

by the choice of ξ. **Q**

Finally, $h(\pi, A^*) \ge \ln 2r - 4\eta$. **P**

$$\ln 2r - 2\eta \le h(\pi, A') \le h(\pi, A^*) + H(\{a, 1 \setminus a\}) + H(\{e, 1 \setminus e\})$$

(applying 386Kb twice)

$$= h(\pi, A^*) + q(\bar\mu a) + q(1 - \bar\mu a) + q(\bar\mu e) + q(1 - \bar\mu e)$$
$$\le h(\pi, A^*) + q(\frac{1}{n}) + q(\frac{n}{n+1}) + q(2\xi) + q(1 - 2\xi)$$
$$\le h(\pi, A^*) + \eta + \eta = h(\pi, A^*) + 2\eta.$$ **Q**

(g) We have

$$\ln 2r - 4\eta \le h(\pi, A^*) \le H(A^*)$$
$$\le H(A) + H(\{c^*, 1 \setminus c^*\}) = \ln r + H(\{c^*, 1 \setminus c^*\}) \le \ln 2r,$$

so

$$q(\bar\mu c^*) + q(1 - \bar\mu c^*) = H(\{c^*, 1 \setminus c^*\}) \ge \ln 2 - 4\eta.$$

By the choice of η, $|\bar\mu c^* - \frac{1}{2}| \le \delta$.

Next,

$$\textstyle\sum_{i \in I} |\bar\mu(a_i \cap c^*) - \frac{1}{2r}| + |\bar\mu(a_i \setminus c^*) - \frac{1}{2r}| \le 3\delta.$$

P By 386H,

$$\sum_{i \in I} |\bar\mu(a_i \cap c^*) - \tfrac{1}{r}\bar\mu c^*| + |\bar\mu(a_i \setminus c^*) - \tfrac{1}{r}\bar\mu(1 \setminus c^*)|$$
$$\le \sqrt{2(H(A) + H(\{c^*, 1 \setminus c^*\}) - H(A^*))}$$
$$\le \sqrt{2(\ln r + \ln 2 - \ln 2r + 4\eta)} = \sqrt{8\eta} \le \delta.$$

So

$$\sum_{i \in I} |\bar\mu(a_i \cap c^*) - \tfrac{1}{2r}| + |\bar\mu(a_i \setminus c^*) - \tfrac{1}{2r}|$$
$$\le \sum_{i \in I} \big(|\bar\mu(a_i \cap c^*) - \tfrac{1}{r}\bar\mu c^*| + \tfrac{1}{r}|\bar\mu c^* - \tfrac{1}{2}|$$
$$+ |\bar\mu(a_i \setminus c^*) - \tfrac{1}{r}\bar\mu(1 \setminus c^*)| + \tfrac{1}{r}|\bar\mu(1 \setminus c^*) - \tfrac{1}{2}|\big)$$
$$\le \delta + |\bar\mu c^* - \tfrac{1}{2}| + |\bar\mu(1 \setminus c^*) - \tfrac{1}{2}| \le 3\delta.$$ **Q**

(h) Now apply 387D to the partition of unity A^*, indexed as $\langle a^*_{ij}\rangle_{i\in I,j\in\{0,1\}}$, where $a^*_{i1}=a_i\cap c^*$ and $a^*_{i0}=a_i\setminus c^*$, and $\langle\gamma_{ij}\rangle_{i\in I,j\in\{0,1\}}$, where $\gamma_{ij}=\frac{1}{2r}$ for all i, j. We have

$$\sum_{i\in I,j\in\{0,1\}}|\bar\mu a^*_{ij}-\gamma_{ij}|\le 3\delta$$

by (g), while

$$H(A^*)-h(\pi,A^*)\le\ln 2r-\ln 2r+4\eta=4\eta,$$

so

$$\sum_{i\in I,j\in\{0,1\}}|\bar\mu a^*_{ij}-\gamma_{ij}|+\sqrt{2(H(A^*)-h(\pi,A^*))}\le 3\delta+\sqrt{8\eta}\le 4\delta.$$

Also

$$\sum_{i\in I,j\in\{0,1\}}q(\gamma_{ij})=\ln 2r\le h(\pi).$$

So 387D tells us that there is a Bernoulli partition $\langle b_{ij}\rangle_{i\in I,j\in\{0,1\}}$ for π such that $\bar\mu b_{ij}=\frac{1}{2r}$ for all i, j and

$$\sum_{i\in I,j\in\{0,1\}}\bar\mu(b_{ij}\triangle a^*_{ij})\le\delta+6\sqrt{4\delta}\le\epsilon.$$

Now of course

$$\begin{aligned}\sum_{i\in I}\bar\mu(a_i\triangle(b_{i0}\cup b_{i1}))&\le\sum_{i\in I}\bar\mu((a_i\cap c^*)\triangle b_{i1})+\bar\mu((a_i\setminus c^*)\triangle b_{i0})\\&=\sum_{i\in I,j\in\{0,1\}}\bar\mu(a^*_{ij}\triangle b_{ij})\le\epsilon,\end{aligned}$$

as required.

387L Ornstein's theorem (infinite entropy case) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra of countable Maharam type, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a two-sided Bernoulli shift of infinite entropy. Then $(\mathfrak{A},\bar\mu,\pi)$ is isomorphic to $(\mathfrak{B}_{\mathbb{Z}},\bar\nu_{\mathbb{Z}},\phi)$, where $(\mathfrak{B}_{\mathbb{Z}},\bar\nu_{\mathbb{Z}})$ is the measure algebra of the usual measure on $[0,1]^{\mathbb{Z}}$, and ϕ is the standard two-sided Bernoulli shift on $\mathfrak{B}_{\mathbb{Z}}$ (385Sb).

proof (a) We have to find a root algebra $\mathfrak{C}$ for π which is isomorphic to the measure algebra of Lebesgue measure on $[0,1]$. The materials we have to start with are a root algebra $\mathfrak{A}_0\subseteq\mathfrak{A}$ such that *either* $\mathfrak{A}_0$ is not purely atomic *or* $H(A_0)=\infty$, where A_0 is the set of atoms of $\mathfrak{A}_0$.

Because $\mathfrak{A}$ has countable Maharam type, there is a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_0$ such that $\{d_n:n\in\mathbb{N}\}$ is dense for the measure-algebra topology of $\mathfrak{A}_0$ (331O).

(b) There is a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of partitions of unity in $\mathfrak{A}_0$ such that C_{n+1} refines C_n, $H(C_n)=n\ln 2$ and d_n is a union of members of C_{n+1} for every n. **P** We have

$$\sup\{H(C):C\subseteq\mathfrak{A}_0\text{ is a partition of unity}\}=\infty$$

(385J). Choose the C_n inductively, as follows. Start with $C_0=\{0,1\}$. Given C_n with $H(C_n)=n\ln 2$, set $C'_n=C_n\vee\{d_n,1\setminus d_n\}$; then

$$H(C'_n)\le H(C_n)+H(\{d_n,1\setminus d_n\})\le(n+1)\ln 2$$

(385Ga, 385Ad). By 386N, there is a partition of unity C_{n+1}, refining C'_n, such that $H(C_{n+1})=(n+1)\ln 2$. Continue. **Q**

(c) For each $n\in\mathbb{N}$, let $\mathfrak{C}_n$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j a:a\in C_n,\ j\in\mathbb{Z}\}$. Then $\langle\mathfrak{C}_n\rangle_{n\in\mathbb{N}}$ is increasing. For each n, $\pi[\mathfrak{C}_n]=\mathfrak{C}_n$; because $C_n\subseteq\mathfrak{A}_0$, $\pi\restriction\mathfrak{C}_n$ is a Bernoulli shift with generating partition C_n. Accordingly

$$h(\pi\restriction\mathfrak{C}_n)=h(\pi,C_n)=H(C_n)=n\ln 2$$

(385R). Of course $d_n\in\mathfrak{C}_{n+1}$ for every n.

Choose inductively, for each $n\in\mathbb{N}$, $\epsilon_n>0$, $r_n\in\mathbb{N}$ and a Bernoulli partition $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^n}$ in $\mathfrak{C}_n$, as follows. Start with $b_{0\emptyset}=1$. (See 3A1H for the notation I am using here.) Given that $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^n}$ is a Bernoulli partition for π which generates $\mathfrak{C}_n$, in the sense that $\mathfrak{C}_n$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j b_{n\sigma}:\sigma\in\{0,1\}^n,\ j\in\mathbb{Z}\}$, and $\bar\mu b_{n\sigma}=2^{-n}$ for every σ, take $\epsilon_n>0$ such that

$$(2r_m+1)\epsilon_n \le 2^{-n} \text{ for every } m<n.$$

We know that

$$h(\pi\restriction\mathfrak{C}_{n+1})=(n+1)\ln 2=\ln(2\cdot 2^n).$$

So we can apply 387K to $(\mathfrak{C}_{n+1},\pi\restriction\mathfrak{C}_{n+1})$ to see that there is a Bernoulli partition $\langle b'_{n\tau}\rangle_{\tau\in\{0,1\}^{n+1}}$ for π such that

$$b'_{n\tau}\in\mathfrak{C}_{n+1},\quad \bar\mu b'_{n\tau}=2^{-n-1}$$

for every $\tau\in\{0,1\}^{n+1}$,

$$\bar\mu(b_{n\sigma}\triangle(b'_{n,\sigma^\frown<0>}\cup b'_{n,\sigma^\frown<1>}))\le 2^{-n}\epsilon_n$$

for every $\sigma\in\{0,1\}^n$. By 387I (with $\mathfrak{B}=\mathfrak{C}=\mathfrak{C}_{n+1}$), there is a Bernoulli partition $\langle b_{n+1,\tau}\rangle_{\tau\in\{0,1\}^{n+1}}$ for $\pi\restriction\mathfrak{C}_{n+1}$ such that the closed subalgebra generated by $\{\pi^j b_{n+1,\tau}:\tau\in\{0,1\}^{n+1},\ j\in\mathbb{Z}\}$ is $\mathfrak{C}_{n+1}$, $\bar\mu b_{n+1,\tau}=2^{-n-1}$ for every $\tau\in\{0,1\}^{n+1}$, and

$$\textstyle\sum_{\tau\in\{0,1\}^{n+1}}\bar\mu(b_{n+1,\tau}\triangle b'_{n\tau})\le\epsilon_n.$$

For each $k\in\mathbb{N}$, let $\mathfrak{B}^{(n+1)}_k$ be the finite subalgebra of $\mathfrak{C}_{n+1}$ generated by $\{\pi^j b_{n+1,\tau}:\tau\in\{0,1\}^{n+1},\ |j|\le k\}$. Since $d_m\in\mathfrak{C}_{m+1}\subseteq\mathfrak{C}_{n+1}$ for every $m\le n$, there is an $r_n\in\mathbb{N}$ such that

$$\rho(d_m,\mathfrak{B}^{(n+1)}_{r_n})\le 2^{-n} \text{ for every } m\le n.$$

Continue.

(d) Fix $m\le n\in\mathbb{N}$ for the moment. For $\sigma\in\{0,1\}^m$, set

$$b_{n\sigma}=\sup\{b_{n\tau}:\tau\in\{0,1\}^n,\ \tau\text{ extends }\sigma\}.$$

(If $n=m$, then of course σ is the unique member of $\{0,1\}^m$ extending itself, so this formula is safe.) Then

$$\bar\mu b_{n\sigma}=2^{-n}\#(\{\tau:\tau\in\{0,1\}^n,\ \tau\text{ extends }\sigma\})=2^{-n}2^{n-m}=2^{-m}.$$

Next, if σ, $\sigma'\in\{0,1\}^m$ are distinct, there is no member of $\{0,1\}^n$ extending both, so $b_{n\sigma}\cap b_{n\sigma'}=0$; thus $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^m}$ is a partition of unity. If $\sigma(0),\dots,\sigma(k)\in\{0,1\}^m$, then

$$\begin{aligned}\bar\mu(\inf_{j\le k}\pi^j b_{n,\sigma(j)})&=\bar\mu(\sup_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}\inf_{j\le k}\pi^j b_{n,\tau(j)})\\&=\sum_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}\bar\mu(\inf_{j\le k}\pi^j b_{n,\tau(j)})\\&=\sum_{\substack{\tau(0),\dots,\tau(k)\in\{0,1\}^n\\ \tau(j)\supseteq\sigma(j)\forall j\le k}}(2^{-n})^{k+1}\\&=(2^{n-m})^{k+1}(2^{-n})^{k+1}=(2^{-m})^{k+1}=\prod_{j=0}^k\bar\mu b_{n,\sigma(j)},\end{aligned}$$

so $\langle b_{n\sigma}\rangle_{\sigma\in\{0,1\}^m}$ is a Bernoulli partition.

(e) If $m\le n\in\mathbb{N}$, then

$$\textstyle\sum_{\sigma\in\{0,1\}^m}\bar\mu(b_{n\sigma}\triangle b_{n+1,\sigma})\le 2\epsilon_n.$$

P We have

$$\begin{aligned}b_{n\sigma}\triangle b_{n+1,\sigma}&=(\sup_{\sigma\subseteq\tau\in\{0,1\}^n}b_{n\tau})\triangle(\sup_{\sigma\subseteq\upsilon\in\{0,1\}^{n+1}}b_{n+1,\upsilon})\\&=(\sup_{\sigma\subseteq\tau\in\{0,1\}^n}b_{n\tau})\triangle(\sup_{\sigma\subseteq\tau\in\{0,1\}^n}b_{n+1,\tau^\frown<0>}\cup b_{n+1,\tau^\frown<1>})\\&\subseteq\sup_{\sigma\subseteq\tau\in\{0,1\}^n}b_{n\tau}\triangle(b_{n+1,\tau^\frown<0>}\cup b_{n+1,\tau^\frown<0>}),\end{aligned}$$

so

$$\sum_{\sigma\in\{0,1\}^m}\bar\mu(b_{n\sigma}\mathbin{\triangle}b_{n+1,\sigma})\le\sum_{\tau\in\{0,1\}^n}\bar\mu(b_{n\tau}\mathbin{\triangle}(b_{n+1,\tau^\frown<0>}\cup b_{n+1,\tau^\frown<1>}))$$

$$\le\sum_{\tau\in\{0,1\}^n}\bar\mu(b_{n\tau}\mathbin{\triangle}(b'_{n,\tau^\frown<0>}\cup b'_{n,\tau^\frown<1>}))$$

$$+\sum_{\upsilon\in\{0,1\}^{n+1}}\bar\mu(b'_{n\upsilon}\mathbin{\triangle}b_{n+1,\upsilon})$$

$$\le\sum_{\tau\in\{0,1\}^n}2^{-n}\epsilon_n+\epsilon_n=2\epsilon_n.\ \mathbf{Q}$$

(f) In particular, for any $m\in\mathbb{N}$ and $\sigma\in\{0,1\}^m$,

$$\textstyle\sum_{n=m}^\infty\bar\mu(b_{n\sigma}\mathbin{\triangle}b_{n+1,\sigma})\le\sum_{n=m}^\infty 2\epsilon_n<\infty.$$

So we can define $b_\sigma=\lim_{n\to\infty}b_{n\sigma}$ in $\mathfrak{A}$. We have

$$\bar\mu b_\sigma=\lim_{n\to\infty}\bar\mu b_{n\sigma}=2^{-m};$$

and if σ, $\sigma'\in\{0,1\}^m$ are distinct, then

$$b_\sigma\cap b_{\sigma'}=\lim_{n\to\infty}b_{n\sigma}\cap b_{n\sigma'}=0,$$

so $\langle b_\sigma\rangle_{\sigma\in\{0,1\}^m}$ is a partition of unity in $\mathfrak{A}$. If $\sigma(0),\dots,\sigma(k)\in\{0,1\}^m$, then

$$\bar\mu(\inf_{j\le k}\pi^jb_{\sigma(j)})=\lim_{n\to\infty}\bar\mu(\inf_{j\le k}\pi^jb_{n,\sigma(j)})$$

$$=\lim_{n\to\infty}\prod_{j=0}^k\bar\mu b_{n,\sigma(j)}=\prod_{j=0}^k\bar\mu b_{\sigma(j)},$$

so $\langle b_\sigma\rangle_{\sigma\in\{0,1\}^m}$ is a Bernoulli partition for π. If $\sigma\in\{0,1\}^m$, then $b_{n\sigma}=b_{n,\sigma^\frown<0>}\cup b_{n,\sigma^\frown<0>}$ for every $n\ge m+1$, so

$$b_{\sigma^\frown<0>}\cup b_{\sigma^\frown<1>}=\lim_{n\to\infty}b_{n,\sigma^\frown<0>}\cup b_{n,\sigma^\frown<1>}=\lim_{n\to\infty}b_{n,\sigma}=b_\sigma.$$

(g) Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{m\in\mathbb{N}}\{b_\sigma:\sigma\in\{0,1\}^m\}$. Then $\mathfrak{C}$ is atomless and countably τ-generated, so $(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ is isomorphic to the measure algebra of Lebesgue measure on $[0,1]$ (331P). Now $\bar\mu(\inf_{j\le k}\pi^je_j)=\prod_{j=0}^k\bar\mu e_j$ for all $e_0,\dots,e_k\in\mathfrak{C}$. $\mathbf{P}$ Let $\epsilon>0$. For $m\in\mathbb{N}$, let $\mathfrak{C}_m$ be the subalgebra of $\mathfrak{C}$ generated by $\{b_\sigma:\sigma\in\{0,1\}^m\}$. $\langle\mathfrak{C}_m\rangle_{m\in\mathbb{N}}$ is non-decreasing, so $\overline{\bigcup_{m\in\mathbb{N}}\mathfrak{C}_m}$ is a closed subalgebra of $\mathfrak{A}$, and must be $\mathfrak{C}$. Now the function

$$(a_0,\dots,a_k)\mapsto\bar\mu(\inf_{j\le k}\pi^ja_j)-\textstyle\prod_{j=0}^k\bar\mu a_j:\mathfrak{A}^{k+1}\to\mathbb{R}$$

is continuous and zero on $\mathfrak{C}_m^{k+1}$ for every m, by 387Ba, so is zero on $\mathfrak{C}^{k+1}$, and in particular is zero at $(e_0,\dots,e_k)$, as required. $\mathbf{Q}$

By 385Sf, $\langle\pi^j[\mathfrak{C}]\rangle_{j\in\mathbb{Z}}$ is independent.

(h) Let $\mathfrak{B}^*$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^jb_\sigma:\sigma\in\bigcup_{m\in\mathbb{N}}\{0,1\}^m,\ j\in\mathbb{Z}\}$; then $\mathfrak{B}^*$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{j\in\mathbb{Z}}\pi^j[\mathfrak{C}]$. It follows from (e) that, for any $m\in\mathbb{N}$,

$$\sum_{\sigma\in\{0,1\}^m}\rho(b_{m\sigma},\mathfrak{B}^*)\le\sum_{\sigma\in\{0,1\}^m}\bar\mu(b_{m\sigma}\mathbin{\triangle}b_\sigma)$$

$$\le\sum_{\sigma\in\{0,1\}^m}\sum_{n=m}^\infty\bar\mu(b_{n\sigma}\mathbin{\triangle}b_{n+1,\sigma})\le 2\sum_{n=m}^\infty\epsilon_n.$$

So if $b\in\mathfrak{B}_{r_m}^{(m+1)}$,

$$\rho(b, \mathfrak{B}^*) \le (2r_m + 1) \sum_{\sigma\in\{0,1\}^{m+1}} \rho(b_{m+1,\sigma}, \mathfrak{B}^*)$$

(386Mc, as $\pi[\mathfrak{B}^*] = \mathfrak{B}^*$)

$$\le 2(2r_m + 1) \sum_{n=m+1}^{\infty} \epsilon_n \le 2 \sum_{n=m+1}^{\infty} 2^{-n} = 2^{-m+1}.$$

It follows that, whenever $m \le n$ in $\mathbb{N}$,

$$\rho(d_m, \mathfrak{B}^*) \le \rho(d_m, \mathfrak{B}^{(n+1)}_{r_n}) + 2^{-n+1} \le 2^{-n} + 2^{-n+1}$$

by the choice of r_n. Letting $n \to \infty$, we see that $\rho(d_m, \mathfrak{B}^*) = 0$, that is, $d_m \in \mathfrak{B}^*$, for every $m \in \mathbb{N}$. But this means that $\mathfrak{A}_0 \subseteq \mathfrak{B}^*$, by the choice of $\langle d_m\rangle_{m\in\mathbb{N}}$. Accordingly $\pi^j[\mathfrak{A}_0] \subseteq \mathfrak{B}^*$ for every j and $\mathfrak{B}^*$ must be the whole of $\mathfrak{A}$.

(i) Thus π is a two-sided Bernoulli shift with root algebra $\mathfrak{C}$; by 385Sc, $(\mathfrak{A}, \bar\mu, \pi)$ is isomorphic to $(\mathfrak{B}_{\mathbb{Z}}, \bar\nu_{\mathbb{Z}}, \phi)$.

387M Corollary: Sinaǐ's theorem (general case) Suppose that $(\mathfrak{A}, \bar\mu)$ is an atomless probability algebra, and $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Let $(\mathfrak{B}, \bar\nu)$ be a probability algebra of countable Maharam type, and $\phi : \mathfrak{B} \to \mathfrak{B}$ a one- or two-sided Bernoulli shift with $h(\phi) \le h(\pi)$. Then $(\mathfrak{B}, \bar\nu, \phi)$ is isomorphic to a factor of $(\mathfrak{A}, \bar\mu, \pi)$.

proof (a) To begin with (down to the end of (b)) suppose that ϕ is two-sided. Let $\mathfrak{B}_0$ be a root algebra for ϕ. If $\mathfrak{B}_0$ is purely atomic, then there is a generating Bernoulli partition $\langle b_i\rangle_{i\in\mathbb{N}}$ for ϕ of entropy $h(\phi)$ (385R). By 387E, there is a Bernoulli partition $\langle c_i\rangle_{i\in\mathbb{N}}$ for π such that $\bar\mu c_i = \bar\nu b_i$ for every i. Let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$. Now $(\mathfrak{C}, \bar\mu{\restriction}\mathfrak{C}, \pi{\restriction}\mathfrak{C})$ is a factor of $(\mathfrak{A}, \bar\mu, \pi)$ isomorphic to $(\mathfrak{B}, \bar\nu, \phi)$.

(b) If $\mathfrak{B}_0$ is not purely atomic, then there is a partition of unity $\langle b_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{B}_0$ of infinite entropy (385J). Again, let $\mathfrak{C}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\pi^j c_i : i \in \mathbb{N}, j \in \mathbb{Z}\}$, where $\langle c_i\rangle_{i\in\mathbb{N}}$ is a Bernoulli partition for π such that $\bar\mu c_i = \bar\nu b_i$ for every i. Now $\pi{\restriction}\mathfrak{C}$ is a Bernoulli shift of infinite entropy and $\mathfrak{C}$ has countable Maharam type, so 387L tells us that there is a closed subalgebra $\mathfrak{C}_0 \subseteq \mathfrak{C}$ such that $\langle\pi^k[\mathfrak{C}_0]\rangle_{k\in\mathbb{Z}}$ is independent and $(\mathfrak{C}_0, \bar\mu{\restriction}\mathfrak{C}_0)$ is isomorphic to the measure algebra of Lebesgue measure on $[0, 1]$. But $(\mathfrak{B}_0, \bar\nu{\restriction}\mathfrak{B}_0)$ is a probability algebra of countable Maharam type, so is isomorphic to a closed subalgebra $\mathfrak{C}_1$ of $\mathfrak{C}_0$ (332N). Of course $\langle\pi^k[\mathfrak{C}_1]\rangle_{k\in\mathbb{Z}}$ is independent, so if we take $\mathfrak{C}_1^*$ to be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{k\in\mathbb{Z}} \pi^k[\mathfrak{C}_1]$, $\pi{\restriction}\mathfrak{C}_1^*$ will be a two-sided Bernoulli shift isomorphic to ϕ.

(c) If ϕ is a one-sided Bernoulli shift, then 385Sa shows that $(\mathfrak{B}, \bar\nu, \phi)$ can be represented in terms of a product measure on a space $X^{\mathbb{N}}$ and the standard shift operator on $X^{\mathbb{N}}$. Now this extends naturally to the standard two-sided Bernoulli shift represented by the product measure on $X^{\mathbb{Z}}$, as described in 385Sb (cf. 385Yg); so that $(\mathfrak{B}, \bar\nu, \phi)$ becomes represented as a factor of $(\mathfrak{B}', \bar\nu', \phi')$ where ϕ' is a two-sided Bernoulli shift with the same entropy as ϕ (since the entropy is determined by the root algebra, by 385R). By (a)-(b), $(\mathfrak{B}', \bar\nu', \phi')$ is isomorphic to a factor of $(\mathfrak{A}, \bar\mu, \pi)$, so $(\mathfrak{B}, \bar\nu, \phi)$ also is.

Remark Thus $(\mathfrak{A}, \bar\mu, \pi)$ has factors which are Bernoulli shifts based on root algebras of all countably-generated types permitted by the entropy of π.

387X Basic exercises (a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a one- or two-sided Bernoulli shift. Show that π^n is a Bernoulli shift for any $n \ge 1$. (*Hint*: if $\mathfrak{A}_0$ is a root algebra for π, the closed subalgebra generated by $\bigcup_{j<n} \pi^j[\mathfrak{A}_0]$ is a root algebra for π^n.)

(b) Suppose that $(\mathfrak{A}, \bar\mu)$ be a probability algebra, $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$ such that $\pi[\mathfrak{B}] = \mathfrak{B}$. Show that if π is ergodic or mixing, so is $\pi{\restriction}\mathfrak{B}$.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Show that $(\phi, \psi) \mapsto \psi\phi : \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{A}; \mathfrak{A}) \times \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{A}; \mathfrak{A}) \to \mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{A}; \mathfrak{A})$ is continuous for the weak topology on $\mathrm{Hom}_{\bar\mu,\pi}(\mathfrak{A}; \mathfrak{A})$.

(d) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and write ι for the identity map on $\mathfrak{A}$; regard $\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ as a subset of $\operatorname{Hom}_{\bar{\mu},\iota}(\mathfrak{A};\mathfrak{A})$ with its weak topology. Show that $\pi\mapsto\pi^{-1}:\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}\to\operatorname{Aut}_{\bar{\mu}}\mathfrak{A}$ is continuous.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra of countable Maharam type, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a two-sided Bernoulli shift. Show that for any $n\ge 1$ there is a Bernoulli shift $\phi:\mathfrak{A}\to\mathfrak{A}$ such that $\phi^n=\pi$. (*Hint*: construct a Bernoulli shift ψ such that $h(\psi)=\frac{1}{n}h(\pi)$, and use 385Xi and Ornstein's theorem to show that π is isomorphic to ψ^n.)

(f) Let $\langle\alpha_i\rangle_{i\in\mathbb{N}}$, $\langle\beta_i\rangle_{i\in\mathbb{N}}$ be non-negative real sequences such that $\sum_{i=0}^\infty\alpha_i=\sum_{i=0}^\infty\beta_i=1$ and $\sum_{i=0}^\infty q(\alpha_i)=\sum_{i=0}^\infty q(\beta_i)$. Let μ_0, ν_0 be the measures on $\mathbb{N}$ defined by the formulae

$$\mu_0E=\sum_{i\in E}\alpha_i,\quad \nu_0E=\sum_{i\in E}\beta_i$$

for $E\subseteq\mathbb{N}$. Set $X=\mathbb{N}^{\mathbb{Z}}$ and let μ, ν be the product measures on X derived from μ_0 and ν_0. Show that there is a permutation $f:X\to X$ such that ν is precisely the image measure μf^{-1} and f is translation-invariant, that is, $f(x\theta)=f(x)\theta$ for every $x\in X$, where $\theta(n)=n+1$ for every $n\in\mathbb{Z}$.

(g) Let $(\mathfrak{A},\bar{\mu},\pi)$ and $(\mathfrak{B},\bar{\nu},\phi)$ be probability algebras of countable Maharam type with two-sided Bernoulli shifts. Suppose that each is isomorphic to a factor of the other. Show that they are isomorphic.

387Y Further exercises (a) Suppose that $(\mathfrak{A},\bar{\mu},\pi)$ and $(\mathfrak{B},\bar{\nu},\phi)$ are probability algebras with one-sided Bernoulli shifts, and that they are isomorphic. Show that they have isomorphic root algebras. (*Hint*: apply the results of §333 to $(\mathfrak{A},\bar{\mu},\pi[\mathfrak{A}])$.)

387 Notes and comments The arguments here are expanded from Smorodinsky 71 and Ornstein 74. I have sought a reasonably direct path to 387J and 387L; of course there is a great deal more to be said (387Xe is a hint), and, in particular, extensions of the methods here provide powerful theorems enabling us to show that automorphisms are Bernoulli shifts. (See Ornstein 74.)

The ideas sketched in 387F can evidently be applied in many other ways; see 387Xc-387Xd here, or §494 in Volume 4.

Version of 6.6.16

388 Dye's theorem

I have repeatedly said that any satisfactory classification theorem for automorphisms of measure algebras remains elusive. There is however a classification, at least for the Lebesgue measure algebra, of the 'orbit structures' corresponding to measure-preserving automorphisms; in fact, they are defined by the fixed-point subalgebras, which I described in §333. We have to work hard for this result, but the ideas are instructive.

388A Orbit structures I said that this section was directed to a classification of 'orbit structures', without saying what these might be. In fact what I will do is to classify the full subgroups generated by measure-preserving automorphisms of the Lebesgue measure algebra. One aspect of the relation with 'orbits' is the following (cf. 381Qc).

Proposition Let (X,Σ,μ) be a localizable countably separated measure space (definition: 343D), with measure algebra $(\mathfrak{A},\bar{\mu})$. Suppose that f and g are measure space automorphisms from X to itself, inducing measure-preserving automorphisms π, ϕ of $\mathfrak{A}$. Then the following are equiveridical:

(i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π;

(ii) for almost every $x\in X$, there is an $n\in\mathbb{Z}$ such that $g(x)=f^n(x)$;

(iii) for almost every $x\in X$, $\{g^n(x):n\in\mathbb{Z}\}\subseteq\{f^n(x):n\in\mathbb{Z}\}$.

proof (i)$\Rightarrow$(ii) Let $\langle H_k\rangle_{k\in\mathbb{N}}$ be a sequence in Σ which separates the points of X; we may suppose that $H_0=X$. By 381Ib, there is a partition of unity $\langle a_n\rangle_{n\in\mathbb{Z}}$ in $\mathfrak{A}$ such that $\phi c=\pi^n c$ whenever $c\subseteq a_n$ and

$n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$ let $E_n \in \Sigma$ be such that $E_n^\bullet = a_n$; then $Y_0 = \bigcup_{n\in\mathbb{Z}} E_n$ is conegligible. The transformation f^n induces π^n, so for any $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ the set

$$F_{nk} = (f^n)^{-1}[E_n \cap H_k] \triangle g^{-1}[E_n \cap H_k]$$

is negligible, and $Y = g^{-1}[Y_0] \setminus \bigcup_{n\in\mathbb{Z},k\in\mathbb{N}} F_{nk}$ is conegligible. Now, for any $x \in Y$, there is some n such that $g(x)$ belongs to $E_n = E_n \cap H_0$, so that $f^n(x) \in E_n$, $\{k : g(x) \in H_k\} = \{k : f^n(x) \in H_k\}$ and $g(x) = f^n(x)$. As Y is conegligible, (ii) is satisfied.

(ii)⇒(iii) For $x \in X$, set $\Omega_x = \{f^n(x) : n \in \mathbb{Z}\}$; we are supposing that $A_0 = \{x : g(x) \notin \Omega_x\}$ is negligible. Set $A = \bigcup_{n\in\mathbb{Z}} g^{-n}[A_0]$, so that A is negligible and $g^n(x) \in X \setminus A$ for every $x \in X \setminus A$, $n \in \mathbb{Z}$.

Suppose that $x \in X \setminus A$ and $n \in \mathbb{N}$. Then $g^n(x) \in \Omega_x$. **P** Induce on n. Of course $g^0(x) = x \in \Omega_x$. For the inductive step to $n+1$, $g^n(x) \in \Omega_x \setminus A_0$, so there is a $k \in \mathbb{Z}$ such that $g^n(x) = f^k(x)$. At the same time, there is an $i \in \mathbb{Z}$ such that $g(g^n(x)) = f^i(g^n(x))$, so that $g^{n+1}(x) = f^{i+k}(x) \in \Omega_x$. Thus the induction continues. **Q**

Consequently $g^{-n}(x) \in \Omega_x$ whenever $x \in X \setminus A$ and $n \in \mathbb{N}$. **P** Since $g^{-n}(x) \in X \setminus A$, there is a $k \in \mathbb{Z}$ such that $x = g^n g^{-n}(x) = f^k g^{-n}(x)$ and $g^{-n}(x) = f^{-k}(x) \in \Omega_x$. **Q**

Thus $\{g^n(x) : n \in \mathbb{Z}\} \subseteq \Omega_x$ for every x in the conegligible set $X \setminus A$.

(iii)⇒(ii) is trivial.

(ii)⇒(i) Set

$$E_n = \{x : g(x) = f^n(x)\} = X \setminus \textstyle\bigcup_{k\in\mathbb{N}}(g^{-1}[H_k] \triangle f^{-n}[H_k]),$$

for $n \in \mathbb{Z}$. Then (ii) tells us that $\bigcup_{n\in\mathbb{Z}} E_n$ is conegligible, so $\bigcup_{n\in\mathbb{Z}} g[E_n]$ is conegligible. But also each E_n is measurable, so $g[E_n]$ also is, and we can set $a_n = g[E_n]^\bullet$. Now for $y \in g[E_n]$, $y = f^n(g^{-1}(y))$, that is, $g^{-1}(y) = f^{-n}(y)$; so $\phi a = \pi^n a$ for every $a \subseteq a_n$. Since $\sup_{n\in\mathbb{Z}} a_n = 1$ in $\mathfrak{A}$, ϕ belongs to the full subgroup generated by π.

Remark Of course the requirement 'countably separated' is essential here; for other measure spaces we can have ϕ and π actually equal without $g(x)$ and $f(x)$ being related for any particular x (see 343I and 343J).

388B Corollary Under the hypotheses of 388A, π and ϕ generate the same full subgroup of $\operatorname{Aut}\mathfrak{A}$ iff $\{f^n(x) : n \in \mathbb{Z}\} = \{g^n(x) : n \in \mathbb{Z}\}$ for almost every $x \in X$.

388C Extending some ideas from 381M-381N, we have the following fact.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism; let $\mathfrak{C}$ be its fixed-point subalgebra $\{c : \pi c = c\}$. Let $\langle d_i\rangle_{i\in I}$, $\langle e_i\rangle_{i\in I}$ be two disjoint families in $\mathfrak{A}$ such that $\bar\mu(c \cap d_i) = \bar\mu(c \cap e_i)$ for every $i \in I$ and $c \in \mathfrak{C}$. Then there is a $\phi \in G_\pi$, the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π, such that $\phi d_i = e_i$ for every $i \in I$.

proof Adding $d^* = 1 \setminus \sup_{i\in I} d_i$, $e^* = 1 \setminus \sup_{i\in I} e_i$ to the respective families, we may suppose that $\langle d_i\rangle_{i\in I}$, $\langle e_i\rangle_{i\in I}$ are partitions of unity. Define $\langle a_n\rangle_{n\in\mathbb{N}}$ inductively by the formula

$$a_n = \textstyle\sup_{i\in I}(d_i \setminus \sup_{m<n} a_m) \cap \pi^{-n}(e_i \setminus \sup_{m<n} \pi^m a_m).$$

Then $a_n \cap d_i \cap a_m = 0$ whenever $m < n$ and $i \in I$, so $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint. Also

$$\pi^n a_n \subseteq \textstyle\sup_{i\in I} e_i \setminus \sup_{m<n} \pi^m a_m$$

for each n, so $\langle \pi^n a_n\rangle_{n\in\mathbb{N}}$ is disjoint. Note that as $\pi^n(a_n \cap d_j) \subseteq e_j$ for each j,

$$\begin{aligned}\pi^n a_n \cap e_i &= \sup_{j\in I} \pi^n(a_n \cap d_j) \cap e_i = \sup_{j\in I} \pi^n(a_n \cap d_j) \cap e_j \cap e_i\\ &= \pi^n(a_n \cap d_i) \cap e_i = \pi^n(a_n \cap d_i)\end{aligned}$$

for every $i \in I$ and $n \in \mathbb{N}$.

? Suppose, if possible, that $a = 1 \setminus \sup_{n\in\mathbb{N}} a_n$ is non-zero. Then there is an $i \in I$ such that $a \cap d_i \neq 0$. Set $c = \sup_{n\in\mathbb{N}} \pi^n(a \cap d_i)$; then $\pi c \subseteq c$ so $c \in \mathfrak{C}$. Now

$$\sum_{n=0}^{\infty}\bar{\mu}(c\cap e_i\cap\pi^n a_n)=\sum_{n=0}^{\infty}\bar{\mu}(c\cap\pi^n(a_n\cap d_i))=\sum_{n=0}^{\infty}\bar{\mu}(\pi^n(c\cap a_n\cap d_i))$$
$$=\sum_{n=0}^{\infty}\bar{\mu}(c\cap a_n\cap d_i)=\bar{\mu}(c\cap d_i\setminus a)<\bar{\mu}(c\cap d_i)=\bar{\mu}(c\cap e_i).$$

So $b=c\cap e_i\setminus\sup_{n\in\mathbb{N}}\pi^n a_n$ is non-zero, and there is an $n\in\mathbb{N}$ such that $b\cap\pi^n(a\cap d_i)$ is non-zero. But look at $a'=\pi^{-n}(b\cap\pi^n(a\cap d_i))$. We have $0\ne a'\subseteq a\cap d_i$, so $a'\subseteq d_i\setminus\sup_{m<n}a_m$; while

$$\pi^n a'\subseteq b\subseteq e_i\setminus\sup_{m<n}\pi^m a_m.$$

But this means that $a'\subseteq a_n$, which is absurd. **X**

This shows that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. Since

$$\textstyle\sum_{n=0}^{\infty}\bar{\mu}(\pi^n a_n)=\sum_{n=0}^{\infty}\bar{\mu}a_n=\bar{\mu}1,$$

$\langle\pi^n a_n\rangle_{n\in\mathbb{N}}$ also is a partition of unity. We can therefore define $\phi\in G_\pi$ by setting $\phi d=\pi^n d$ whenever $n\in\mathbb{N}$ and $d\subseteq a_n$. Now, for any $i\in I$,

$$\phi d_i=\sup_{n\in\mathbb{N}}\phi(d_i\cap a_n)=\sup_{n\in\mathbb{N}}\pi^n(d_i\cap a_n)=\sup_{n\in\mathbb{N}}e_i\cap\pi^n a_n=e_i.$$

So we have found a suitable ϕ.

388D von Neumann automorphisms (a) Definitions Let $\mathfrak{A}$ be a Boolean algebra and $\pi\in\operatorname{Aut}\mathfrak{A}$ an automorphism. π is **weakly von Neumann** if there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $a_0=1$ and, for every n, $a_{n+1}\cap\pi^{2^n}a_{n+1}=0$, $a_{n+1}\cup\pi^{2^n}a_{n+1}=a_n$. In this case, π is **von Neumann** if $\langle a_n\rangle_{n\in\mathbb{N}}$ can be chosen in such a way that $\{\pi^m a_n: m,\,n\in\mathbb{N}\}$ τ-generates $\mathfrak{A}$, and **relatively von Neumann** if $\langle a_n\rangle_{n\in\mathbb{N}}$ can be chosen so that $\{\pi^m a_n: m,\,n\in\mathbb{N}\}\cup\{c:\pi c=c\}$ τ-generates $\mathfrak{A}$.

(b) There is another way of looking at automorphisms of this type which will be useful. If $\mathfrak{A}$ is a Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ an automorphism, then a **dyadic cycle system** for π is a finite or infinite family $\langle d_{mi}\rangle_{m\le n,i<2^m}$ or $\langle d_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ such that (α) for each m, $\langle d_{mi}\rangle_{i<2^m}$ is a partition of unity such that $\pi d_{mi}=d_{m,i+1}$ whenever $i<2^m-1$ (so that $\pi d_{m,2^m-1}$ must be d_{m0}) (β) $d_{m0}=d_{m+1,0}\cup d_{m+1,2^m}$ for every $m<n$ (in the finite case) or for every $m\in\mathbb{N}$ (in the infinite case). An easy induction on m shows that if $k\le m$ then

$$d_{ki}=\sup\{d_{mj}:j<2^m,\,j\equiv i\text{ mod }2^k\}$$

for every $i<2^k$.

Conversely, if d is such that $\langle\pi^j d\rangle_{j<2^n}$ is a partition of unity in $\mathfrak{A}$, then we can form a finite dyadic cycle system $\langle d_{mi}\rangle_{m\le n,i<2^m}$ by setting $d_{mi}=\sup\{\pi^j d:j<2^n,\,j\equiv i\text{ mod }2^m\}$ whenever $m\le n$ and $j<2^m$.

(c) Now an automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ is weakly von Neumann iff it has an infinite dyadic cycle system $\langle d_{mi}\rangle_{m\in\mathbb{N},i<2^m}$. (The a_m of (a) correspond to the d_{m0} of (b); starting from the definition in (a), you must check first, by induction on m, that $\langle\pi^i a_m\rangle_{i<2^m}$ is a partition of unity in $\mathfrak{A}$.) π is von Neumann iff it has a dyadic cycle system $\langle d_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ which τ-generates $\mathfrak{A}$.

388E Example The following is the basic example of a von Neumann transformation – in a sense, the only example of a measure-preserving von Neumann transformation. Let μ be the usual measure on $X=\{0,1\}^{\mathbb{N}}$, Σ its domain, and $(\mathfrak{A},\bar{\mu})$ its measure algebra. Define $f:X\to X$ by setting

$$f(x)(n)=1-x(n)\text{ if }x(i)=0\text{ for every }i<n,$$
$$=x(n)\text{ otherwise.}$$

Then f is a homeomorphism and a measure space automorphism. **P** (i) To see that f is a homeomorphism, perhaps the easiest way is to look at g, where

$$g(x)(n)=1-x(n)\text{ if }x(i)=1\text{ for every }i<n,$$
$$=x(n)\text{ otherwise,}$$

and check that f and g are both continuous and that fg and gf are both the identity function. (ii) To see that f is inverse-measure-preserving, it is enough to check that $\mu\{x : f(x)(i) = z(i) \text{ for every } i \le n\} = 2^{-n-1}$ for every $n \in \mathbb{N}$, $z \in X$ (254G). But

$$\{x : f(x)(i) = z(i) \text{ for every } i \le n\} = \{x : x(i) = g(z)(i) \text{ for every } i \le n\}.$$

(iii) Similarly, g is inverse-measure-preserving, so f is a measure space automorphism. **Q**

If $n \in \mathbb{N}$ and $x \in X$ then

$$\begin{aligned} f^{2^k}(x)(n) &= 1 - x(n) \text{ if } n \ge k \text{ and } x(i) = 0 \text{ whenever } k \le i < n, \\ &= x(n) \text{ otherwise.} \end{aligned}$$

(Induce on k. For the inductive step, observe that if we identify X with $\{0,1\} \times X$ then $f^2(\epsilon, y) = (\epsilon, f(y))$ for every $\epsilon \in \{0,1\}$ and $y \in X$.)

Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be the corresponding automorphism, setting $\pi E^\bullet = f^{-1}[E]^\bullet$ for $E \in \Sigma$. Then π is a measure-preserving von Neumann automorphism. **P** π is a measure-preserving automorphism because f is. Set $E_n = \{x : x \in X, x(i) = 1 \text{ for every } i < n\}$, $a_n = E_n^\bullet$. Then $f^{-2^n}[E_{n+1}] = \{x : x(i) = 1 \text{ for } i < n, x(n) = 0\}$, so a_{n+1} and $\pi^{2^n} a_{n+1}$ split a_n for each n, and $\langle a_n \rangle_{n\in\mathbb{N}}$ witnesses that π is weakly von Neumann. Next, inducing on n, we find that $\{f^{-i}[E_n] : i < 2^n\}$ runs over the basic cylinder sets of the form $\{x : x(i) = z(i) \text{ for every } i < n\}$ determined by coordinates less than n. Since the equivalence classes of such sets τ-generate $\mathfrak{A}$ (see part (a) of the proof of 331K), π is a von Neumann automorphism. **Q**

f is sometimes called the **odometer transformation**. For another way of looking at the functions f and g, see 445Xp in Volume 4.

388F We are now ready to approach the main results of this section.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism. Let $\mathfrak{C}$ be its fixed-point subalgebra. Then for any $a \in \mathfrak{A}$ there is a $b \subseteq a$ such that $\bar\mu(b \cap c) = \frac{1}{2}\bar\mu(a \cap c)$ for every $c \in \mathfrak{C}$ and π_b is a weakly von Neumann automorphism, writing π_b for the induced automorphism of the principal ideal $\mathfrak{A}_b$, as in 381M.

Remark On first reading, there is something to be said for supposing here that π is ergodic, that is, that $\mathfrak{C} = \{0,1\}$.

proof I should remark straight away that π is doubly recurrent on every $b \in \mathfrak{A}$ (386A), so we have an induced automorphism $\pi_b : \mathfrak{A}_b \to \mathfrak{A}_b$ for every $b \in \mathfrak{A}$ (381M).

(a) Set $\epsilon_n = \frac{1}{2}(1 + 2^{-n})$ for each $n \in \mathbb{N}$, so that $\langle \epsilon_n \rangle_{n\in\mathbb{N}}$ is strictly decreasing, with $\epsilon_0 = 1$ and $\lim_{n\to\infty} \epsilon_n = \frac{1}{2}$. Now there are $\langle b_n \rangle_{n\in\mathbb{N}}$, $\langle d_{ni} \rangle_{n\in\mathbb{N}, i<2^n}$ such that, for each $n \in \mathbb{N}$,

$$b_{n+1} \subseteq b_n \subseteq a, \quad \bar\mu(b_n \cap c) = \epsilon_n \bar\mu(a \cap c) \text{ for every } c \in \mathfrak{C},$$

$$\langle d_{ni} \rangle_{i<2^n} \text{ is disjoint}, \quad \sup_{i<2^n} d_{ni} = b_n,$$

$$\pi_{b_n} d_{ni} = d_{n,i+1} \text{ for every } i < 2^n - 1,$$

$$b_{n+1} \cap d_{ni} = d_{n+1,i} \cup d_{n+1,i+2^n} \text{ for every } i < 2^n.$$

P Start with $b_0 = d_{00} = a$. To construct b_{n+1} and $\langle d_{n+1,i} \rangle_{i<2^{n+1}}$, given $\langle d_{ni} \rangle_{i<2^n}$, note first that (because π_{b_n} is measure-preserving and $\pi_{b_n}(c \cap d) = c \cap \pi_{b_n} d$ for every $d \subseteq b_n$, see 381Nf) $\bar\mu(d_{n0} \cap c) = \bar\mu(d_{ni} \cap c)$ whenever $c \in \mathfrak{C}$ and $i < 2^n$, so

$$\bar\mu(d_{n0} \cap c) = 2^{-n}\bar\mu(b_n \cap c) = 2^{-n}\epsilon_n \bar\mu(a \cap c)$$

for every $c \in \mathfrak{C}$, and

$$d_{n0} = b_n \setminus \sup_{i<2^n-1} \pi_{b_n} d_{ni} = \pi_{b_n} d_{n,2^n-1} = \pi_{b_n}^{2^n} d_{n0}.$$

Now π_{b_n} is aperiodic (381Ng) so $\pi_{b_n}^{2^n}$ also is (381Bd), and there is a $d_{n+1,0} \subseteq d_{n0}$ such that

$$\pi_{b_n}^{2^n} d_{n+1,0} \cap d_{n+1,0} = 0, \quad \bar\mu(d_{n+1,0} \cap c) = 2^{-n-1}\epsilon_{n+1}\bar\mu(a \cap c) \text{ for every } c \in \mathfrak{C}$$

(applying 386C(iii) to $\pi_{b_n}^{2^n}\restriction\mathfrak{A}_{d_{n0}}$, with $\gamma=\epsilon_{n+1}/2\epsilon_n$). Set $d_{n+1,j}=\pi_{b_n}^j d_{n+1,0}$ for each $j<2^{n+1}$. Because $\pi_{b_n}^{2^n}d_{n+1,0}\subseteq d_{n0}\setminus d_{n+1,0}$, while $\langle\pi_{b_n}^j d_{n0}\rangle_{j<2^n}$ is disjoint, $\langle\pi_{b_n}^j d_{n+1,0}\rangle_{j<2^{n+1}}$ is disjoint. Set $b_{n+1}=\sup_{j<2^{n+1}}\pi_{b_n}^j d_{n+1,0}$; then $b_{n+1}\subseteq b_n$ and $\bar\mu(b_{n+1}\cap c)=\epsilon_{n+1}\bar\mu(a\cap c)$ for every $c\in\mathfrak{C}$. For $j<2^{n+1}$, $d_{n+1,j}\subseteq d_{ni}$ where i is either j or $j-2^n$, so $b_{n+1}\cap d_{ni}=d_{n+1,i}\cup d_{n+1,i+2^n}$ for every $i<2^n$.

For $j<2^{n+1}-1$,

$$\pi_{b_n}d_{n+1,j}=d_{n+1,j+1}\subseteq b_{n+1},$$

so we must also have

$$\pi_{b_{n+1}}d_{n+1,j}=(\pi_{b_n})_{b_{n+1}}d_{n+1,j}=d_{n+1,j+1}$$

(using 381Ne). Thus the induction continues. **Q**

(b) Set

$$b=\inf_{n\in\mathbb{N}}b_n,\quad e_{ni}=b\cap d_{ni}\text{ for }n\in\mathbb{N},\,i<2^n.$$

Because $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-increasing,

$$\bar\mu(b\cap c)=\lim_{n\to\infty}\bar\mu(b_n\cap c)=\frac{1}{2}\bar\mu(a\cap c)$$

for every $c\in\mathfrak{C}$. Next,

$$e_{ni}=b\cap b_{n+1}\cap d_{ni}=b\cap(d_{n+1,i}\cup d_{n+1,i+2^n})=e_{n+1,i}\cup e_{n+1,i+2^n}$$

whenever $i<2^n$.

If $m\le n$ and $j<2^m$ then

$$b_n\cap d_{mj}=\sup\{d_{ni}:i<2^n,\,i\equiv j\bmod 2^m\}$$

(induce on n). So

$$\bar\mu(b_n\cap d_{mj})=2^{n-m}\bar\mu d_{n0}=2^{-m}\epsilon_n;$$

taking the limit as $n\to\infty$, $\bar\mu e_{mj}=2^{-m}\bar\mu b$. Next,

$$\begin{aligned}\pi_{b_n}(b_n\cap d_{mj})&=\sup\{d_{n,i+1}:i<2^n,\,i\equiv j\bmod 2^m\}\\&=\sup\{d_{ni}:i<2^n,\,i\equiv j+1\bmod 2^m\}=b_n\cap d_{m,j+1},\end{aligned}$$

here interpreting $d_{n,2^n}$ as d_{n0}, $d_{m,2^m}$ as d_{m0}. Consequently $\pi_b e_{mj}\subseteq e_{m,j+1}$. **P?** Otherwise, there are a non-zero $e\subseteq d_{mj}\cap b$ and $k\ge1$ such that $\pi^i e\cap b=0$ for $1\le i<k$ and $\pi^k e\subseteq b\setminus d_{m,j+1}$. Take $n\ge m$ so large that $\bar\mu e>k\bar\mu(b_n\setminus b)$, so that

$$e'=e\setminus\sup_{1\le i<k}\pi^{-i}(b_n\setminus b)\ne0;$$

now $\pi^i e'\cap b_n=0$ for $1\le i<k$, while $\pi^k e'\subseteq b_n$, and

$$\pi_{b_n}e'=\pi^k e'\subseteq 1\setminus d_{m,j+1}.$$

But this means that $\pi_{b_n}(b_n\cap d_{mj})\not\subseteq d_{m,j+1}$, which is impossible. **XQ**

Since $\bar\mu(\pi_b e_{mj})=\bar\mu e_{m,j+1}$, we must have $\pi_b e_{mj}=e_{m,j+1}$. And this is true whenever $m\in\mathbb{N}$ and $j<2^m$, if we identify $e_{m,2^m}$ with e_{m0}. Thus $\langle e_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ is a dyadic cycle system for π_b and π_b is a weakly von Neumann automorphism.

388G Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and π, ψ two measure-preserving automorphisms of $\mathfrak{A}$. Suppose that ψ belongs to the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π and that there is a $b\in\mathfrak{A}$ such that $\sup_{n\in\mathbb{Z}}\psi^n b=1$ and the induced automorphisms ψ_b, π_b on $\mathfrak{A}_b$ are equal. Then $G_\psi=G_\pi$.

proof (a) The first fact to note is that if $0\ne b'\subseteq b$, $n\in\mathbb{Z}$ and $\pi^n b'\subseteq b$, then there are $m\in\mathbb{Z}$, $b''\subseteq b'$ such that $b''\ne0$ and $\pi^n d=\psi^m d$ for every $d\subseteq b''$. **P** (α) If $n=0$ take $b''=b'$, $m=0$. (β) Next, suppose that $n>0$. We have $0\ne b'\subseteq b\cap\pi^{-n}b$, so by 381Nc there are i, b'_1 such that $1\le i\le n$, $0\ne b'_1\subseteq b'$ and $\pi^n d=\pi_b^i d$ for every $d\subseteq b'_1$. Now by 381Nb there are a non-zero $b''\subseteq b'_1$ and an $m\ge i$ such that $\psi_b^i d=\psi^m d$ for every $d\subseteq b''$; so that $\pi^n d=\psi^m d$ for every $d\subseteq b''$. (γ) If $n<0$, then apply (β) to π^{-1} and ψ^{-1}, recalling that $(\pi^{-1})_b=\pi_b^{-1}=\psi_b^{-1}=(\psi^{-1})_b$ (381Na). **Q**

(b) Now take any non-zero $a \in \mathfrak{A}$. Then there are m, $n \in \mathbb{Z}$ such that $a_1 = a \cap \psi^m b \neq 0$, $a_2 = \pi a_1 \cap \psi^n b \neq 0$. Set $b_1 = \psi^{-m}\pi^{-1}a_2$. Because $\psi \in G_\pi$, there are a non-zero $b_2 \subseteq b_1$ and a $k \in \mathbb{Z}$ such that $\psi^{-n}\pi\psi^m d = \pi^k d$ for every $d \subseteq b_2$. Now

$$\pi^k b_2 = \psi^{-n}\pi\psi^m b_2 \subseteq \psi^{-n}\pi\psi^m b_1 = \psi^{-n}a_2 \subseteq b.$$

By (a), there are a non-zero $b_3 \subseteq b_2$ and an $r \in \mathbb{Z}$ such that $\pi^k d = \psi^r d$ for every $d \subseteq b_3$. Consider $a' = \psi^m b_3$. Then

$$0 \neq a' \subseteq \psi^m b_1 = \pi^{-1}a_2 \subseteq a_1 \subseteq a;$$

and, for $d \subseteq a'$, $\psi^{-m}d \subseteq b_3 \subseteq b_2$, so that

$$\pi d = \psi^n(\psi^{-n}\pi\psi^m)\psi^{-m}d = \psi^n\pi^k\psi^{-m}d = \psi^{n+r-m}d.$$

As a is arbitrary, this shows that $\pi \in G_\psi$, so that $G_\pi \subseteq G_\psi$ and the two are equal.

388H Lemma Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism, and ϕ any member of the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π. Suppose that $\langle d_{mi}\rangle_{m\le n,i<2^m}$ is a finite dyadic cycle system for ϕ. Then there is a weakly von Neumann automorphism ψ, with dyadic cycle system $\langle d'_{mi}\rangle_{m\in\mathbb{N},i<2^m}$, such that $G_\psi = G_\pi$, $\psi a = \phi a$ whenever $a \cap d_{n0} = 0$, and $d'_{mi} = d_{mi}$ whenever $m \le n$ and $i < 2^m$.

proof Write $\mathfrak{C}$ for the closed subalgebra $\{c : \pi c = c\}$. By 388F there is a $b \subseteq d_{n0}$ such that $\bar\mu(b \cap c) = \frac{1}{2}\bar\mu(d_{n0} \cap c)$ for every $c \in \mathfrak{C}$ and $\pi_b : \mathfrak{A}_b \to \mathfrak{A}_b$ is a weakly von Neumann automorphism. Let $\langle e_{ki}\rangle_{k\in\mathbb{N},i<2^k}$ be a dyadic cycle system for π_b.

If we define $\psi_1 \in \operatorname{Aut}\mathfrak{A}$ by setting

$$\psi_1 d = \pi_b d \text{ for } d \subseteq b, \quad \psi_1 d = \pi_{1\setminus b} d \text{ for } d \subseteq 1 \setminus b,$$

then $\psi_1 \in G_\pi$. Next, for any $c \in \mathfrak{C}$,

$$\bar\mu(\phi^{-2^n+1}b \cap c) = \bar\mu\phi^{-2^n+1}(b \cap c) = \bar\mu(b \cap c) = \frac{1}{2}\bar\mu(d_{n0} \cap c) = \bar\mu((d_{n0} \setminus b) \cap c)$$

because $\phi^{-2^n+1} \in G_\pi$, so $\phi^{-2^n+1}c = c$ (381Ja). By 388C, there is a $\psi_2 \in G_\pi$ such that $\psi_2(d_{n0} \setminus b) = \phi^{-2^n+1}b$. Set $\psi_3 = \phi^{-2^n+1}\psi_2^{-1}\phi^{-2^n+1}\psi_1$, so that $\psi_3 \in G_\pi$ and

$$\psi_3 b = \phi^{-2^n+1}\psi_2^{-1}\phi^{-2^n+1}b = \phi^{-2^n+1}(d_{n0} \setminus b).$$

Thus $\psi_3 b$ and $\psi_2(d_{n0} \setminus b)$ are disjoint and have union $\phi^{-2^n+1}d_{n0} = d_{n1}$ (if $n = 0$, we must read d_{01} as $d_{00} = 1$). Accordingly we can define $\psi \in G_\pi$ by setting

$$\begin{aligned}\psi d &= \psi_3 d \text{ if } d \subseteq b,\\ &= \psi_2 d \text{ if } d \subseteq d_{n0} \setminus b,\\ &= \phi d \text{ if } d \cap d_{n0} = 0.\end{aligned}$$

Since $\psi d_{n0} = d_{n1}$, we have $\psi d_{ni} = \phi d_{ni}$ for every $i < 2^n$, and therefore $\psi^i d_{m0} = d_{mi}$ whenever $m \le n$ and $i < 2^m$. Looking at ψ^{2^n}, we have

$$\psi^{2^n}d_{n0} = \phi^{2^n}d_{n0} = d_{n0}, \quad \psi^{2^n}b = \phi^{2^n-1}\psi_3 b = d_{n0} \setminus b,$$

so that $\psi^{2^n}(d_{n0} \setminus b) = b$ and $\psi^{2^{n+1}}b = b$. Accordingly

$$\psi^{2^{n+1}}d = \phi^{2^n-1}\psi_2\phi^{2^n-1}\psi_3 d = \psi_1 d = \pi_b d$$

for every $d \subseteq b$, and π_b is the automorphism of $\mathfrak{A}_b$ induced by ψ. Also $\sup_{i<2^{n+1}}\psi^i b = 1$, so 388G tells us that $G_\psi = G_\pi$.

Now define $\langle a_m\rangle_{m\in\mathbb{N}}$ as follows. For $m \le n$, $a_m = d_{m0}$; for $m > n$, $a_m = e_{m-n-1,0}$. Then for $m < n$ we have

$$\psi^{2^m}a_{m+1} = \psi^{2^m}d_{m+1,0} = \phi^{2^m}d_{m+1,0} = d_{m+1,2^m} = a_m \setminus a_{m+1},$$

for $m = n$ we have

$$\psi^{2^n}a_{n+1}=\psi^{2^n}e_{00}=\psi^{2^n}b=d_{n0}\setminus b=a_n\setminus a_{n+1},$$

and for $m>n$ we have

$$\psi^{2^m}a_{m+1}=(\psi^{2^{n+1}})^{2^{m-n-1}}e_{m-n,0}=(\pi_b)^{2^{m-n-1}}e_{m-n,0}$$
$$=e_{m-n,2^{m-n-1}}=e_{m-n-1,0}\setminus e_{m-n,0}=a_m\setminus a_{m+1}.$$

Thus $\langle a_m\rangle_{m\in\mathbb{N}}$ witnesses that ψ is a weakly von Neumann automorphism. If $d'_{mi}=\psi^i a_m$ for $m\in\mathbb{N}$, $i<2^m$ then $\langle d'_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ will be a dyadic cycle system for ψ and $d'_{mi}=d_{mi}$ for $m\le n$, as required.

388I Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$. For $a\in\mathfrak{A}$ write $\mathfrak{C}_a=\{a\cap c:c\in\mathfrak{C}\}$.

(a) Suppose that $b\in\mathfrak{A}$, $w\in\mathfrak{C}$ and $\delta>0$ are such that $\bar\mu(b\cap c)\ge\delta\bar\mu c$ whenever $c\in\mathfrak{C}$ and $c\subseteq w$. Then there is an $e\in\mathfrak{A}$ such that $e\subseteq b\cap w$ and $\bar\mu(e\cap c)=\delta\bar\mu c$ whenever $c\in\mathfrak{C}_w$.

(b) Suppose that $k\ge 1$ and that $(b_0,\dots,b_r)$ is a finite partition of unity in $\mathfrak{A}$. Then there is a partition E of unity in $\mathfrak{A}$ such that

$$\bar\mu(e\cap c)=\frac{1}{k}\bar\mu c \text{ for every } e\in E,\ c\in\mathfrak{C},$$

$$\#(\{e:e\in E,\exists\, i\le r,\ b_i\cap e\notin\mathfrak{C}_e\})\le r+1.$$

proof (a) Set $a=b\cap w$ and consider the principal ideal $\mathfrak{A}_a$ generated by $\mathfrak{A}$. We know that $(\mathfrak{A}_a,\bar\mu\restriction\mathfrak{A}_a)$ is a totally finite measure algebra (322H), and that $\mathfrak{C}_a$ is a closed subalgebra of $\mathfrak{A}_a$ (333Bc); and it is easy to see that $\mathfrak{A}_a$ is relatively atomless over $\mathfrak{C}_a$.

Let $\theta:\mathfrak{C}_w\to\mathfrak{C}_a$ be the Boolean homomorphism defined by setting $\theta c=c\cap b$ for $c\in\mathfrak{C}_w$. If $c\in\mathfrak{C}_w$ and $\theta c=0$, then $c\in\mathfrak{C}$ and $\delta\bar\mu c\le\bar\mu(c\cap b)=0$, so $c=0$; thus θ is injective; since it is certainly surjective, it is a Boolean isomorphism. We can therefore define a functional $\nu=\bar\mu\theta^{-1}:\mathfrak{C}_a\to[0,\infty[$, and we shall have

$$\delta\nu d=\delta\bar\mu(\theta^{-1}d)\le\bar\mu(b\cap\theta^{-1}d)=\bar\mu(\theta\theta^{-1}d)=\bar\mu d$$

for every $d\in\mathfrak{C}_a$. By 331B, there is an $e\in\mathfrak{A}_a$ such that $\delta\nu d=\bar\mu(d\cap e)$ for every $d\in\mathfrak{C}_a$, that is, $\delta\bar\mu c=\bar\mu(c\cap e)$ for every $c\in\mathfrak{C}_w$, as required.

(b)(i) Write D for the set of all those $e\in\mathfrak{A}$ such that $\bar\mu(c\cap e)=\frac{1}{k}\bar\mu c$ for every $c\in\mathfrak{C}$ and $b_i\cap e\in\mathfrak{C}_e$ for every $i\le r$. Then whenever $a\in\mathfrak{A}$ and $\gamma>\frac{r+1}{k}$ is such that $\mu(a\cap c)=\gamma\mu c$ for every $c\in\mathfrak{C}$, there is an $e\in D$ such that $e\subseteq a$. **P** For $d\in\mathfrak{A}$ and $c\in\mathfrak{C}$ set $\nu_d(c)=\bar\mu(d\cap c)$, so that $\nu_d:\mathfrak{C}\to[0,\infty[$ is a completely additive functional. For $i\le r$ set $v_i=[\![\bar\mu\restriction\mathfrak{C}>k\nu_{a\cap b_i}]\!]$, in the notation of 326T; so that $v_i\in\mathfrak{C}$ and $\bar\mu c\ge k\bar\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$ and $c\subseteq v_i$, while $\bar\mu c\le k\bar\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$ and $c\cap v_i=0$. Setting $v=\inf_{i\le r}v_i$, we have

$$k\gamma\bar\mu v=k\bar\mu(a\cap v)=\sum_{i=0}^r k\bar\mu(a\cap b_i\cap v)\le(r+1)\bar\mu v.$$

Since $k\gamma>r+1$, $v=0$. So if we now set $w_i=(\inf_{j<i}v_j)\setminus v_i$ for $i\le r$ (starting with $w_0=1\setminus v_0$), $(w_0,\dots,w_r)$ is a partition of unity in $\mathfrak{C}$, and $\bar\mu c\le k\bar\mu(a\cap b_i\cap c)$ whenever $c\in\mathfrak{C}$, $i\le r$ and $c\subseteq w_i$.

By (a), we can find for each $i\le r$ an $e_i\in\mathfrak{A}$ such that $e_i\subseteq a\cap b_i\cap w_i$ and $\bar\mu(c\cap e_i)=\frac{1}{k}\bar\mu c$ whenever $c\in\mathfrak{C}$ and $c\subseteq w_i$. Set $e=\sup_{i\le r}e_i$, so that $e\subseteq a$,

$$e\cap b_i=e\cap w_i\cap b_i=e_i=e\cap w_i\in\mathfrak{C}_e$$

for each i, and

$$\bar\mu(c\cap e)=\sum_{i=0}^r\bar\mu(c\cap e_i)=\sum_{i=0}^r\bar\mu(c\cap w_i\cap e_i)=\sum_{i=0}^r\frac{1}{k}\bar\mu(c\cap w_i)=\frac{1}{k}\bar\mu c$$

for every $c\in\mathfrak{C}$. So e has all the properties required. **Q**

(ii) Let $E_0\subseteq D$ be a maximal disjoint family, and set $m=\#(E_0)$, $a=1\setminus\sup E_0$. Then

$$\bar\mu(a\cap c)=\bar\mu c-\sum_{e\in E_0}\bar\mu(c\cap e)=(1-\frac{m}{k})\bar\mu c$$

for every $c\in\mathfrak{C}$, while a does not include any member of D. By (i), $1-\frac{m}{k}\le\frac{r+1}{k}$, that is, $k-m\le r+1$.

Applying (a) repeatedly, with $w = 1$ and $\delta = \frac{1}{k}$, we can find disjoint $d_0, \dots, d_{k-m-1} \subseteq a$ such that $\bar\mu(c \cap d_i) = \frac{1}{k}\bar\mu c$ for every $c \in \mathfrak{C}$ and $i < k-m$. So if we set $E = E_0 \cup \{d_i : i < k-m\}$ we shall have a partition of unity with the properties required.

388J Lemma Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism, with fixed-point subalgebra $\mathfrak{C}$. Suppose that ϕ is a member of the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π with a finite dyadic cycle system $\langle d_{mi}\rangle_{m\le n, i<2^m}$, and that $a \in \mathfrak{A}$ and $\epsilon > 0$. Then there is a $\psi \in G_\pi$ such that

(i) ψ has a dyadic cycle system $\langle d'_{mi}\rangle_{m\le k, i<2^m}$, with $k \ge n$ and $d'_{mi} = d_{mi}$ for $m \le n$, $i < 2^m$;

(ii) $\psi d = \phi d$ if $d \cap d_{n0} = 0$;

(iii) there is an a' in the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d'_{ki} : i < 2^k\}$ such that $\bar\mu(a \triangle a') \le \epsilon$.

proof (a) Take $k \ge n$ so large that $2^k\epsilon \ge 2^n 2^{2^n}\bar\mu 1$. Let $\mathfrak{D}$ be the subalgebra of the principal ideal $\mathfrak{A}_{d_{n1}}$ generated by $\{d_{n1} \cap \phi^{-j}a : j < 2^n\}$; then $\mathfrak{D}$ has atoms $b_0, \dots, b_r$ where $r < 2^{2^n}$. (If $n = 0$, take $d_{01} = d_{00} = 1$.) Applying 388Ib to the closed subalgebra $\mathfrak{C}_{d_{n1}}$ of $\mathfrak{A}_{d_{n1}}$, we can find a partition of unity E of $\mathfrak{A}_{d_{n1}}$ such that

$$\bar\mu(e \cap c) = 2^{n-k}\bar\mu(d_{n1} \cap c) = 2^{-k}\bar\mu c$$

for every $e \in E$ and $c \in \mathfrak{C}$, and

$$E_1 = \{e : e \in E, \text{ there is some } i \le r \text{ such that } b_i \cap e \notin \mathfrak{C}_e\}$$

has cardinal at most $r + 1 \le 2^{2^n}$. Of course $\bar\mu e = 2^{-k}\bar\mu 1$ for every $e \in E$, so $\#(E) = 2^{k-n}$ and $\bar\mu(\sup E_1) \le 2^{-k}2^{2^n}\bar\mu 1 \le 2^{-n}\epsilon$. Write e^* for $\sup E_1$.

(b) For $e \in E$ set $e' = \phi^{2^n-1}e$; then $\{e' : e \in E\}$ is a disjoint family, with cardinal 2^{k-n}; enumerate it as $\langle v_i\rangle_{i<2^{k-n}}$. Note that

$$\sup_{i<2^{k-n}} v_i = \phi^{2^n-1}(\sup E) = d_{n0},$$

$$\bar\mu(v_i \cap c) = \bar\mu(\phi^{-2^n+1}v_i \cap c) = 2^{-k}\bar\mu c$$

for every $c \in \mathfrak{C}$ and $i < 2^{k-n}$. There is therefore a $\psi_1 \in G_\pi$ such that

$$\psi_1 v_i = \phi^{-2^n+1}v_{i+1} \text{ for } i < 2^{k-n} - 1, \quad \psi_1 v_{2^{k-n}-1} = \phi^{-2^n+1}v_0$$

(388C). We have

$$\begin{aligned}\psi_1 d_{n0} = \psi_1(\sup_{i<2^{k-n}} v_i) &= \sup_{i<2^{k-n}} \psi_1 v_i = \sup_{i<2^{k-n}-1} \phi^{-2^n+1}v_{i+1} \cup \phi^{-2^n+1}v_0 \\ &= \sup_{i<2^{k-n}} \phi^{-2^n+1}v_i = \phi^{-2^n+1}d_{n0} = d_{n1} = \phi d_{n0}.\end{aligned}$$

So we may define $\psi \in G_\pi$ by setting

$$\begin{aligned}\psi d &= \psi_1 d \text{ if } d \subseteq d_{n0}, \\ &= \phi d \text{ if } d \cap d_{n0} = 0.\end{aligned}$$

(c) For each $i < 2^{k-n}$,

$$\psi^{2^n}v_i = \phi^{2^n-1}\psi_1 v_i = v_{i+1}$$

(identifying $v_{2^{k-n}}$ with v_0). Moreover, $\psi^j v_i \subseteq d_{nl}$ whenever $i < 2^{k-n}$ and $j \equiv l \bmod 2^n$. So $\langle\psi^j v_0\rangle_{j<2^k}$ is a partition of unity in $\mathfrak{A}$. What this means is that if we set

$$d'_{mj} = \sup\{\psi^i v_0 : i < 2^k,\, i \equiv j \bmod 2^m\}$$

for $m \le k$, then $\langle d'_{mj}\rangle_{m\le k, j<2^m}$ is a dyadic cycle system for ψ, with $d'_{mj} = d_{mj}$ if $m \le n$ and $j < 2^m$.

(d) Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d'_{kj} : j < 2^k\}$. Recall the definition of $\{v_i : i < 2^{k-n}\}$ as $\{\phi^{2^n-1}e : e \in E\}$; this implies that

$$\{\psi v_i : i < 2^{k-n}\} = \{\psi_1 v_i : i < 2^{k-n}\} = \{\phi^{-2^n+1} v_i : i < 2^{k-n}\} = E,$$

so that

$$\{\psi^{j+1} v_i : i < 2^{k-n}\} = \{\phi^j e : e \in E\}$$

for $j < 2^n$, and

$$\mathfrak{B} \supseteq \{d'_{kj} : j < 2^k\} = \{\psi^j v_i : i < 2^{k-n},\, j < 2^n\} = \{\phi^j e : e \in E,\, j < 2^n\}.$$

Set $E_0 = E \setminus E_1$. For $e \in E_0$ and $i \le r$ there is a $c_{ei} \in \mathfrak{C}$ such that $e \cap b_i = e \cap c_{ei}$. Set

$$K = \{(i,j) : 1 \le i \le r,\, j < 2^n,\, b_i \subseteq \phi^{-j} a\},$$

$$a' = \sup\{\phi^j e \cap c_{ei} : e \in E_0,\, (i,j) \in K\}.$$

Then a' is a supremum of (finitely many) members of $\mathfrak{B}$, so belongs to $\mathfrak{B}$. If $(i,j) \in K$ and $e \in E_0$, then

$$\phi^j e \cap c_{ei} = \phi^j (e \cap c_{ei}) = \phi^j (e \cap b_i) \subseteq a,$$

so $a' \subseteq a$. Next, $d_{n1} \cap \phi^{-j}(a \setminus a') \subseteq e^*$ for each $j < 2^n$. **P** Set

$$I = \{i : i \le r,\, (i,j) \in K\} = \{i : b_i \subseteq \phi^{-j} a\};$$

then $d_{n1} \cap \phi^{-j} a = \sup_{i \in I} b_i$. Now, for each $i \in I$,

$$b_i = \sup_{e \in E}(b_i \cap e) \subseteq \sup_{e \in E_0}(e \cap c_{ei}) \cup e^*,$$

so that

$$d_{n1} \cap \phi^{-j} a = \sup_{i \in I} b_i \subseteq \sup_{e \in E_0, i \in I}(e \cap c_{ei}) \cup e^* \subseteq (d_{n1} \cap \phi^{-j} a') \cup e^*. \textbf{ Q}$$

But this means that

$$\bar{\mu}(d_{n,j+1} \cap a \setminus a') = \bar{\mu}(d_{n1} \cap \phi^{-j}(a \setminus a')) \le \bar{\mu} e^* \le 2^{-n}\epsilon$$

for every $j < 2^n$ (interpreting $d_{n,2^n}$ as d_{n0}, as usual), and

$$\bar{\mu}(a \,\triangle\, a') = \sum_{j=1}^{2^n} \bar{\mu}(d_{nj} \cap a \setminus a') \le \epsilon,$$

so that the final condition of the lemma is satisfied.

388K Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, with Maharam type ω, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an aperiodic measure-preserving automorphism. Then there is a relatively von Neumann automorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ such that ϕ and π generate the same full subgroups of $\operatorname{Aut}\mathfrak{A}$.

proof (a) The idea is to construct ϕ as the limit of a sequence $\langle \phi_n \rangle_{n \in \mathbb{N}}$ of weakly von Neumann automorphisms such that $G_{\phi_n} = G_\pi$. Each ϕ_n will have a dyadic cycle system $\langle d_{nmi} \rangle_{m \in \mathbb{N}, i < 2^m}$; there will be a strictly increasing sequence $\langle k_n \rangle_{n \in \mathbb{N}}$ such that

$$d_{n+1,m,i} = d_{n,m,i} \text{ whenever } m \le k_n,\ i < 2^m,$$

$$\phi_{n+1} a = \phi_n a \text{ whenever } a \cap d_{n,k_n,0} = 0.$$

Interpolated between the ϕ_n will be a second sequence $\langle \psi_n \rangle_{n \in \mathbb{N}}$ in G_π, with associated (finite) dyadic cycle systems $\langle d'_{nmi} \rangle_{m \le k'_n, i < 2^m}$.

(b) Before starting on the inductive construction we must fix on a countable set $B \subseteq \mathfrak{A}$ which τ-generates $\mathfrak{A}$, and a sequence $\langle b_n \rangle_{n \in \mathbb{N}}$ in B such that every member of B recurs cofinally often in the sequence. (For instance, take the sequence of first members of an enumeration of $B \times \mathbb{N}$.) As usual, I write $\mathfrak{C}$ for the closed subalgebra $\{c : \pi c = c\}$. The induction begins with $\psi_0 = \pi$, $k'_0 = 0$, $d'_{000} = 1$. Given $\psi_n \in G_\pi$ and its dyadic cycle system $\langle d'_{nmi} \rangle_{m \le k'_n, i < 2^m}$, use 388H to find a weakly von Neumann automorphism ϕ_n, with dyadic cycle system $\langle d_{nmi} \rangle_{m \in \mathbb{N}, i < 2^m}$, such that $G_{\phi_n} = G_\pi$, $d_{nmi} = d'_{nmi}$ for $m \le k'_n$ and $i < 2^m$, and $\phi_n a = \psi_n a$ whenever $a \cap d'_{n,k'_n,0} = 0$.

(c) Given the weakly von Neumann automorphism ϕ_n, with its dyadic cycle system $\langle d_{nmi} \rangle_{m \in \mathbb{N}, i < 2^m}$, such that $G_{\phi_n} = G_\pi$, then we have a partition of unity $\langle e_{nj} \rangle_{j \in \mathbb{Z}}$ such that $\pi a = \phi_n^j a$ whenever $j \in \mathbb{Z}$

and $a \subseteq e_{nj}$ (381I). Take r_n such that $\bar{\mu}\tilde{e}_n \le 2^{-n}$, where $\tilde{e}_n = \sup_{|j|>r_n} e_{nj}$, and $k_n > k'_n$ such that $2^{-k_n}(2r_n+1)\bar{\mu}1 \le 2^{-n}$. Set

$$e^*_n = \sup_{|j|\le r_n} \phi_n^{-j} d_{n,k_n,0},$$

so that $\bar{\mu}e^*_n \le 2^{-n}$.

Now use 388J to find a $\psi_{n+1} \in G_\pi$, with a dyadic cycle system $\langle d'_{n+1,m,i}\rangle_{m\le k'_{n+1},i<2^m}$, such that $k'_{n+1} \ge k_n$, $d'_{n+1,m,i} = d_{nmi}$ if $m \le k_n$, $\psi_{n+1}a = \phi_n a$ if $a \cap d_{n,k_n,0} = 0$, and there is a b'_n in the algebra generated by $\mathfrak{C} \cup \{d'_{n+1,m,i} : m \le k'_{n+1},\ i < 2^m\}$ such that $\bar{\mu}(b_n \triangle b'_n) \le 2^{-n}$. Continue.

(d) The effect of this construction is to ensure that if $l < n$ in $\mathbb{N}$ then

$$d_{lmi} = d_{nmi} \text{ whenever } m \le k_l,\ i < 2^m,$$

$$\phi_n a = \phi_l a \text{ whenever } a \cap d_{l,k_l,0} = 0,$$

$$b'_l \text{ belongs to the subalgebra generated by } \mathfrak{C} \cup \{d_{nmi} : m \le k_n,\ i < 2^m\},$$

and, of course, $d_{n,k_n,0} \subseteq d_{l,k_l,0}$. Since $\langle k_n\rangle_{n\in\mathbb{N}}$ is strictly increasing, $\inf_{n\in\mathbb{N}} d_{n,k_n,0} = 0$. Now, for each $n \in \mathbb{N}$,

$$d_{n,k_n,1} = \phi_n d_{n,k_n,0} = \phi_{n+1} d_{n,k_n,0} \supseteq \phi_{n+1} d_{n+1,k_{n+1},0} = d_{n+1,k_{n+1},1},$$

so setting

$$a_0 = 1 \setminus d_{0,k_0,0}, \quad a_{n+1} = d_{n,k_n,0} \setminus d_{n+1,k_{n+1},0} \text{ for each } n,$$

we have

$$\phi_0 a_0 = 1 \setminus d_{0,k_0,1}, \quad \phi_{n+1} a_{n+1} = d_{n,k_n,1} \setminus d_{n+1,k_{n+1},1} \text{ for each } n,$$

and $\langle \phi_n a_n\rangle_{n\in\mathbb{N}}$ is a partition of unity. There is therefore a $\phi \in \operatorname{Aut}\mathfrak{A}$ defined by setting $\phi a = \phi_n a$ if $a \subseteq a_n$; because G_π is full, $\phi \in G_\pi$.

(e) If $m \le n$, then $a_m \cap d_{m,k_m,0} = 0$, so $\phi_n a = \phi_m a = \phi a$ for every $a \subseteq a_m$. Thus $\phi_n a = \phi a$ for every $a \subseteq \sup_{m\le n} a_m = 1 \setminus d_{n,k_n,0}$. In particular, $\phi d_{nmi} = d_{n,m,i+1}$ whenever $m \le k_n$ and $1 \le i < 2^m$ (counting $d_{n,m,2^m}$ as d_{nm0}, as usual); so that in fact $\phi d_{nmi} = d_{n,m,i+1}$ whenever $m \le k_n$ and $i < 2^m$.

For each n, we have $d_{nmi} = d'_{n+1,m,i} = d_{n+1,m,i}$ whenever $m \le k_n$ and $i < 2^m$. We therefore have a family $\langle d^*_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ defined by saying that $d^*_{mi} = d_{nmi}$ whenever $n \in \mathbb{N}$, $m \le k_n$ and $i < 2^m$. Now, for any $m \in \mathbb{N}$, there is a $k_n > m$, so that $\langle d^*_{mi}\rangle_{i<2^m} = \langle d_{nmi}\rangle_{i<2^m}$ is a partition of unity; and

$$d^*_{mi} = d_{nmi} = d_{n,m+1,i} \cup d_{n,m+1,i+2^m} = d^*_{m+1,i} \cup d^*_{m+1,i+2^m}$$

for each $i < 2^m$. Moreover,

$$\phi d^*_{m,i} = \phi_n d_{nmi} = d_{n,m,i+1} = d^*_{m,i+1}$$

at least for $1 \le i < 2^m$ (counting $d^*_{m,2^m}$ as $d^*_{m,0}$), so that in fact $\phi d^*_{mi} = d^*_{m,i+1}$ for every $i < 2^m$. Thus $\langle d^*_{mi}\rangle_{m\in\mathbb{N},i<2^m}$ is a dyadic cycle system for ϕ, and ϕ is a weakly von Neumann automorphism.

Writing $\mathfrak{B}$ for the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d^*_{mi} : m \in \mathbb{N},\ i < 2^m\}$, then

$$\begin{aligned}\mathfrak{C} \cup \{d'_{nmi} : m \le k'_n,\ i < 2^m\} &= \mathfrak{C} \cup \{d_{n+1,m,i} : m \le k'_n,\ i < 2^m\}\\ &= \mathfrak{C} \cup \{d^*_{mi} : m \le k'_n,\ i < 2^m\} \subseteq \mathfrak{B}\end{aligned}$$

for any $n \in \mathbb{N}$. So $b'_n \in \mathfrak{B}$ for every n. If $b \in B$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $2^{-n} \le \epsilon$ and $b_n = b$, so that $\bar{\mu}(b \triangle b'_n) \le \epsilon$; as every b'_n belongs to $\mathfrak{B}$, and $\mathfrak{B}$ is closed, $b \in \mathfrak{B}$; as b is arbitrary, and B τ-generates $\mathfrak{A}$, $\mathfrak{B} = \mathfrak{A}$. Thus ϕ is a relatively von Neumann automorphism.

(f) If $n \in \mathbb{N}$ and $d \cap e^*_n = 0$, then $\phi^j d = \phi_n^j d$ and $\phi^{-j} d = \phi_n^{-j} d$ whenever $0 \le j \le r_n$. **P** Induce on j. For $j = 0$ the result is trivial. For the inductive step to $j+1 \le r_n$, note that if $d' \cap d_{n,k_n,1} = 0$ then $\phi_n^{-1} d' \cap d_{n,k_n,0} = 0$, so

$$\phi^{-1} d' = \phi^{-1}\phi_n(\phi_n^{-1} d') = \phi^{-1}\phi(\phi_n^{-1} d') = \phi_n^{-1} d'.$$

Now we have

$$\phi^{j+1} d = \phi(\phi_n^j d) = \phi_n(\phi_n^j d) = \phi_n^{j+1} d$$

because

$$\phi_n^j d \cap d_{n,k_n,0} = \phi_n^j(d \cap \phi_n^{-j} d_{n,k_n,0}) = 0,$$

while

$$\phi^{-j-1}d = \phi^{-1}(\phi_n^{-j}d) = \phi_n^{-1}(\phi_n^{-j}d) = \phi_n^{-j-1}d$$

because

$$\phi_n^{-j}d \cap d_{n,k_n,1} = \phi_n^{-j}(d \cap \phi_n^{j+1}d_{n,k_n,0}) = 0. \textbf{ Q}$$

Thus $\phi^j d = \phi_n^j d$ whenever $|j| \le r_n$.

(g) Finally, $G_\phi = G_\pi$. **P** I remarked in (d) that $\phi \in G_\pi$, so that $G_\phi \subseteq G_\pi$. To see that $\pi \in G_\phi$, take any non-zero $a \in \mathfrak{A}$. Because $\bar\mu(e_n^* \cup \tilde e_n) \le 2^{-n+1}$ for each n, there is an n such that $a' = a \setminus (e_n^* \cup \tilde e_n) \ne 0$. Now there is some $j \in \mathbb{Z}$ such that $a'' = a' \cap e_{nj} \ne 0$; since $a' \cap \tilde e_n = 0$, $|j| \le r_n$. If $d \subseteq a''$, then $\pi d = \phi_n^j d$, by the definition of e_{nj}. But also $\phi_n^j d = \phi^j d$, by (f), because $d \cap e_n^* = 0$. So $\pi d = \phi^j d$ for every $d \subseteq a''$. As a is arbitrary, $\pi \in G_\phi$ and $G_\pi \subseteq G_\phi$. **Q**

This completes the proof.

388L Theorem Let $(\mathfrak{A}_1, \bar\mu_1)$ and $(\mathfrak{A}_2, \bar\mu_2)$ be totally finite measure algebras of countable Maharam type, and $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_1$, $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ measure-preserving automorphisms. For each i, let $\mathfrak{C}_i$ be the fixed-point subalgebra of π_i and G_{π_i} the full subgroup of $\operatorname{Aut}\mathfrak{A}_i$ generated by π_i. If $(\mathfrak{A}_1, \bar\mu_1, \mathfrak{C}_1)$ and $(\mathfrak{A}_2, \bar\mu_2, \mathfrak{C}_2)$ are isomorphic, so are $(\mathfrak{A}_1, \bar\mu_1, G_{\pi_1})$ and $(\mathfrak{A}_2, \bar\mu_2, G_{\pi_2})$.

proof (a) It is enough to consider the case in which $(\mathfrak{A}_1, \bar\mu_1, \mathfrak{C}_1)$ and $(\mathfrak{A}_2, \bar\mu_2, \mathfrak{C}_2)$ are actually equal; I therefore delete the subscripts and speak of a structure $(\mathfrak{A}, \bar\mu, \mathfrak{C})$, with two automorphisms π_1, π_2 of $\mathfrak{A}$ both with fixed-point subalgebra $\mathfrak{C}$.

(b) Suppose first that $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$, that is, that both the π_i are aperiodic (381P). In this case, 388K tells us that there are relatively von Neumann automorphisms ϕ_1 and ϕ_2 of $\mathfrak{A}$ such that $G_{\pi_1} = G_{\phi_1}$ and $G_{\pi_2} = G_{\phi_2}$. But $(\mathfrak{A}, \bar\mu, \phi_1)$ and $(\mathfrak{A}, \bar\mu, \phi_2)$ are isomorphic. **P** Let $\langle d_{mi}\rangle_{m\in\mathbb{N}, i<2^m}$ and $\langle d'_{mi}\rangle_{m\in\mathbb{N}, i<2^m}$ be dyadic cycle systems for ϕ_1, ϕ_2 respectively such that $\mathfrak{C} \cup \{d_{mi} : m \in \mathbb{N}, i < 2^m\}$ and $\mathfrak{C} \cup \{d'_{mi} : m \in \mathbb{N}, i < 2^m\}$ both τ-generate $\mathfrak{A}$.

Writing $\mathfrak{B}$, $\mathfrak{B}'$ for the subalgebras of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{d_{mi} : m \in \mathbb{N}, i < 2^m\}$ and $\mathfrak{C} \cup \{d'_{mi} : m \in \mathbb{N}, i < 2^m\}$ respectively, it is easy to see that these algebras are isomorphic: we just set $\theta_0 c = c$ for $c \in \mathfrak{C}$, $\theta_0 d_{mi} = d'_{mi}$ for $i < 2^m$ to obtain a measure-preserving isomorphism $\theta_0 : \mathfrak{B} \to \mathfrak{B}'$. Because these are topologically dense subalgebras of $\mathfrak{A}$, there is a unique extension of θ_0 to a measure-preserving automorphism $\theta : \mathfrak{A} \to \mathfrak{A}$ (324O). Next, we see that

$$\theta\phi_1\theta^{-1}c = c = \phi_2 c \text{ for every } c \in \mathfrak{C},$$

$$\theta\phi_1\theta^{-1}d'_{mi} = \theta\phi_1 d_{mi} = \theta d_{m,i+1} = d'_{m,i+1} = \phi_2 d'_{mi}$$

for $m \in \mathbb{N}$, $i < 2^m$ (as usual, taking $d_{m,2^m}$ to be d_{m0} and $d'_{m,2^m}$ to be d'_{m0}). But this means that $\theta\phi_1\theta^{-1}b = \phi_2 b$ for every $b \in \mathfrak{B}_2$, so (again because $\mathfrak{B}_2$ is dense in $\mathfrak{A}$) $\theta\phi_1\theta^{-1} = \phi_2$. Thus θ is an isomorphism between $(\mathfrak{A}, \bar\mu, \phi_1)$ and $(\mathfrak{A}, \bar\mu, \phi_2)$. **Q**

Of course θ is now also an isomorphism between $(\mathfrak{A}, \bar\mu, G_{\phi_1}) = (\mathfrak{A}, \bar\mu, G_{\pi_1})$ and $(\mathfrak{A}, \bar\mu, G_{\phi_2}) = (\mathfrak{A}, \bar\mu, G_{\pi_2})$.

(c) Next, consider the case in which π_1 is periodic, with period n, for some $n \ge 1$. In this case $\pi_2 \in G_{\pi_1}$. **P** Let $(d_0, \ldots, d_{n-1})$ be a partition of unity in $\mathfrak{A}$ such that $\pi_1 d_i = d_{i+1}$ for $i < n-1$ and $\pi_1 d_{n-1} = d_0$ (382Fb). If $d \subseteq d_j$, then $c = \sup_{i<n} \pi_1^i d \in \mathfrak{C}$ and $d = d_j \cap c$; so any member of $\mathfrak{A}$ is of the form $\sup_{j<n} d_j \cap c_j$ for some family $c_0, \ldots, c_{n-1}$ in $\mathfrak{C}$.

If $a \in \mathfrak{A} \setminus \{0\}$, take i, $j < n$ such that $a' = a \cap d_i \cap \pi_2^{-1}d_j \ne 0$. Then any $d \subseteq a'$ is of the form

$$d_i \cap c_1 = \pi_2^{-1}(d_j \cap c_2) = c_2 \cap \pi_2^{-1}d_j$$

for some c_1, $c_2 \in \mathfrak{C}$; setting $c = c_1 \cap c_2$, we have

$$d = d_i \cap c, \quad \pi_2 d = d_j \cap c = \pi_1^{j-i}d.$$

As a is arbitrary, this shows that $\pi_2 \in G_{\pi_1}$. **Q**

Now $\sup_{n\in\mathbb{Z}} \pi_2^n d_0$ belongs to $\mathfrak{C}$ and includes d_0, so must be 1. Finally, the two induced automorphisms $(\pi_1)_{d_0}$, $(\pi_2)_{d_0}$ on $\mathfrak{A}_{d_0}$ are both the identity. **P** If $0 \ne \tilde{d} \subseteq d_0$ there are a non-zero $d' \subseteq \tilde{d}$ and an $m \ge 1$ such that $(\pi_2)_{d_0} d = \pi_2^m d$ for every $d \subseteq d'$. As $\pi_2^m \in G_{\pi_1}$, there are a non-zero $d \subseteq d'$ and a $k \in \mathbb{Z}$ such that $\pi_2^m d = \pi_1^k d$. Now $\pi_1^k d \subseteq d_0$ so k is a multiple of n and $(\pi_2)_{d_0} d = \pi_1^k d = d$. This shows that $\{d : (\pi_2)_{d_0} d = d\}$ is order-dense in $\mathfrak{A}_{d_0}$ and must be the whole of $\mathfrak{A}_{d_0}$. As for π_1, we have $(\pi_1)_{d_0} d = \pi_1^n d = d$ for every $d \subseteq d_0$. **Q**

So 388G tells us that $G_{\pi_1} = G_{\pi_2}$.

(d) For the general case, we see from 381H that there is a partition of unity $\langle c_i \rangle_{1\le i\le\omega}$ in $\mathfrak{C}$ such that $\pi_1 \restriction \mathfrak{A}_{c_\omega}$ is aperiodic and if i is finite and $c_i \ne 0$ then $\pi_1 \restriction \mathfrak{A}_{c_i}$ is periodic with period i. For each i, let H_i be $\{\phi \restriction \mathfrak{A}_{c_i} : \phi \in G_{\pi_1}\}$; then H_i is a full subgroup of $\operatorname{Aut} \mathfrak{A}_{c_i}$, and

$$G_{\pi_1} = \{\phi : \phi \in \operatorname{Aut} \mathfrak{A},\ \phi \restriction \mathfrak{A}_{c_i} \in H_i \text{ whenever } 1 \le i \le \omega\}.$$

Similarly, writing $H_i' = \{\phi \restriction \mathfrak{A}_{c_i} : \phi \in G_{\pi_2}\}$,

$$G_{\pi_2} = \{\phi : \phi \in \operatorname{Aut} \mathfrak{A},\ \phi \restriction \mathfrak{A}_{c_i} \in H_i' \text{ whenever } 1 \le i \le \omega\}.$$

Note also that H_i, H_i' are the full subgroups of $\operatorname{Aut} \mathfrak{A}_{c_i}$ generated by $\pi_1 \restriction \mathfrak{A}_{c_i}$, $\pi_2 \restriction \mathfrak{A}_{c_i}$ respectively. By (b) and (c), $H_i = H_i'$ for finite i, while there is a measure-preserving automorphism $\theta : \mathfrak{A}_{c_\omega} \to \mathfrak{A}_{c_\omega}$ such that $\theta H_\omega \theta^{-1} = H_\omega'$. Now we can define a measure-preserving automorphism $\theta_1 : \mathfrak{A} \to \mathfrak{A}$ by setting $\theta_1 a = \theta a$ if $a \subseteq c_\omega$, $\theta_1 a = a$ if $a \cap c_\omega = 0$, and we shall have $\theta_1 G_{\pi_1} \theta_1^{-1} = G_{\pi_2}$. Thus $(\mathfrak{A}, \bar{\mu}, G_{\pi_1})$ and $(\mathfrak{A}, \bar{\mu}, G_{\pi_2})$ are isomorphic, as claimed.

388X Basic exercises >(a) Let $(\mathfrak{A}, \bar{\mu})$ be a Boolean algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ an automorphism. Let us say that a **pseudo-cycle** for π is a partition of unity $\langle a_i \rangle_{i<n}$, where $n \ge 1$, such that $\pi a_i = a_{i+1}$ for $i < n-1$ (so that $\pi a_{n-1} = a_0$). (i) Show that if we have pseudo-cycles $\langle a_i \rangle_{i<n}$ and $\langle b_j \rangle_{j<m}$, where m is a multiple of n, then we have a pseudo-cycle $\langle c_j \rangle_{j<m}$ with $c_0 \subseteq a_0$, so that $a_i = \sup\{c_j : j < m,\ j \equiv i \bmod n\}$ for every $i < n$. (ii) Show that π is weakly von Neumann iff it has a pseudo-cycle of length 2^n for any $n \in \mathbb{N}$.

(b) Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be probability algebras, and $\pi_1 : \mathfrak{A}_1 \to \mathfrak{A}_2$ and $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2$ measure-preserving von Neumann automorphisms. Show that there is a measure-preserving Boolean isomorphism $\theta : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that $\pi_2 = \theta \pi_2 \theta^{-1}$.

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a relatively von Neumann automorphism with fixed-point subalgebra $\mathfrak{C}$ and a dyadic cycle system $\langle d_{mi} \rangle_{m\in\mathbb{N}, i<2^m}$ such that $\{d_{mi} : m \in \mathbb{N},\ i < 2^m\} \cup \mathfrak{C}$ τ-generates $\mathfrak{A}$. Show that for any $n \in \mathbb{N}$ the fixed-point subalgebra of π^{2^n} is the subalgebra of $\mathfrak{A}$ generated by $\{d_{ni} : i < 2^n\} \cup \mathfrak{C}$.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving automorphism. (i) Show that π is weakly von Neumann iff it has a factor (definition: 387Ac) which is a von Neumann automorphism. (ii) Show that if π is a relatively von Neumann automorphism then no non-trivial factor of π can be weakly mixing.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra of countable Maharam type, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving von Neumann automorphism. (i) Show that for any ultrafilter $\mathcal{F}$ on $\mathbb{N}$ there is a $\phi_{\mathcal{F}} \in \operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ defined by the formula $\phi_{\mathcal{F}}(a) = \lim_{n\to\mathcal{F}} \pi^n a$ for every $a \in \mathfrak{A}$, the limit being taken in the measure-algebra topology. (ii) Show that $\{\phi_{\mathcal{F}} : \mathcal{F}$ is an ultrafilter on $\mathbb{N}\}$ is a subgroup of $\operatorname{Aut}_{\bar{\mu}} \mathfrak{A}$ homeomorphic to $\mathbb{Z}_2^{\mathbb{N}}$. (*Hint*: 388E.)

(f) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a weakly von Neumann automorphism. Show that π^n is a weakly von Neumann automorphism for every $n \in \mathbb{Z} \setminus \{0\}$. (*Hint*: consider $n = 2$, $n = -1$, odd $n \ge 3$ separately. The formula of 388E may be useful.)

(g) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a von Neumann automorphism. (i) Show that π^2 is not ergodic. (ii) Show that π^2 is relatively von Neumann. (iii) Show that π^n is von Neumann for every odd $n \in \mathbb{Z}$. (iv) Show that if $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and π is a measure-preserving von Neumann automorphism then π is ergodic.

388Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ such that the quotient algebra $\mathfrak{A} = \Sigma/\mathcal{I}$ is Dedekind complete and there is a countable subset of Σ separating the points of X. Suppose that f and g are automorphisms of the structure $(X, \Sigma, \mathcal{I})$ inducing π, $\phi \in \operatorname{Aut}\mathfrak{A}$. Show that the following are equiveridical: (i) ϕ belongs to the full subgroup of $\operatorname{Aut}\mathfrak{A}$ generated by π; (ii) $\{x : x \in X,\, f(x) \notin \{g^n(x) : n \in \mathbb{Z}\}\} \in \mathcal{I}$; (iii) $\{x : x \in X,\, \{f^n(x) : n \in \mathbb{Z}\} \not\subseteq \{g^n(x) : n \in \mathbb{Z}\}\} \in \mathcal{I}$.

(b)(i) Let π be a a weakly von Neumann automorphism of a Boolean algebra. Show that π is aperiodic. (ii) Let π be a relatively von Neumann measure-preserving automorphism of a probability algebra. Show that π has zero entropy.

(c) Give an example of an ergodic weakly von Neumann measure-preserving automorphism with zero entropy which is not a relatively von Neumann automorphism.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\pi \in \operatorname{Aut}\mathfrak{A}$ a relatively von Neumann automorphism; let $T = T_\pi : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ be the corresponding Riesz homomorphism (365N). (i) Show that $\bigcup_{n\ge1}\{u : T^n u = u\}$ is dense in $L^1_{\bar{\mu}}$. (ii) Show that $\{T^n : n \in \mathbb{Z}\}$ is relatively compact in $\mathsf{B}(L^1_{\bar{\mu}}; L^1_{\bar{\mu}})$ for the strong operator topology.

(e) Show that the odometer transformation on $\{0,1\}^{\mathbb{N}}$ is expressible as the product of two Borel measurable measure-preserving involutions.

(f) Give an example of a probability algebra $(\mathfrak{A}, \bar{\mu})$ and a von Neumann automorphism $\pi \in \operatorname{Aut}\mathfrak{A}$ which is not ergodic.

(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and π, ψ two doubly recurrent automorphisms of $\mathfrak{A}$. Suppose that ψ belongs to the full subgroup G_π of $\operatorname{Aut}\mathfrak{A}$ generated by π and that there is a $b \in \mathfrak{A}$ such that $\sup_{n\in\mathbb{Z}} \psi^n b = 1$ and the induced automorphisms ψ_b, π_b on $\mathfrak{A}_b$ are equal. Show that $G_\psi = G_\pi$.

(h) Let μ be Lebesgue measure on $[0,1]^2$, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra; let $\mathfrak{C}$ be the closed subalgebra of elements expressible as $(E \times [0,1])^\bullet$, where $E \subseteq [0,1]$ is measurable. Suppose that $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism such that $\mathfrak{C} = \{c : \pi c = c\}$. Show that there is a family $\langle f_x \rangle_{x\in[0,1]}$ of ergodic measure space automorphisms of $[0,1]$ such that $(x, y) \mapsto (x, f_x(y))$ is a measure space automorphism of $[0,1]^2$ representing π.

388 Notes and comments Dye's theorem (Dye 59) is actually Theorem 388L in the case in which π_1, π_2 are ergodic, that is, in which $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are both trivial. I take the trouble to give the generalized form here (a simplified version of that in Krieger 76) because it seems a natural target, once we have a classification of the relevant structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ (333R). The essential mathematical ideas are the same in both cases. You can find the special case worked out in Hajian Ito & Kakutani 75, from which I have taken the argument used here; and you may find it useful to go through the version above, to check what kind of simplifications arise if each $\mathfrak{C}$ is taken to be $\{0,1\}$. Essentially the difference will be that every 'aperiodic' turns into 'ergodic' (with an occasional 'atomless' thrown in) and '331B' turns into '331C'. As far as I know, there is no simplification available in the structure of the argument; of course the details become a bit easier, but with the possible exception of 388I-388J I think there is little difference.

Of course modifying a general argument to give a simpler proof of a special case is a standard exercise in this kind of mathematics. What is much more interesting is the reverse process. What kinds of theorem about ergodic automorphisms will in fact be true of all automorphisms? A variety of very powerful approaches to such questions have been developed in the last half-century, and I hope to describe some of the ideas in Volumes 4 and 5. The methods used in this section are relatively straightforward and do not require any deep theoretical underpinning beyond Maharam's lemma 331B. But an alternative approach can be found using 388Yh: in effect (at least for the Lebesgue measure algebra) any measure-preserving automorphism can be disintegrated into ergodic measure space automorphisms (the fibre maps f_x of 388Yh). It is sometimes possible to guess which theorems about ergodic transformations are 'uniformisable' in the sense that they can be applied to such a family $\langle f_x \rangle_{x\in[0,1]}$, in a systematic way, to provide a structure which can be interpreted on the product measure. The details tend to be complex, which is one of the reasons why I do not attempt to work through them here; but such disintegrations can be a most valuable aid to intuition.

In this section I use von Neumann automorphisms as an auxiliary tool: the point is, first, that two von Neumann automorphisms are isomorphic – that is, the von Neumann automorphisms on a given totally finite measure algebra $(\mathfrak{A}, \bar{\mu})$ (necessarily isomorphic to the Lebesgue measure algebra, since we must have $\mathfrak{A}$ atomless and $\tau(\mathfrak{A}) = \omega$) form a conjugacy class in the group $\mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ of measure-preserving automorphisms; and next, that for any ergodic measure-preserving automorphism π (on an atomless totally finite algebra of countable Maharam type) there is a von Neumann automorphism ϕ such that $G_\pi = G_\phi$ (388K). But I think they are remarkable in themselves. A (weakly) von Neumann automorphism has a 'pseudo-cycle' (388Xa) for every power of 2. For some purposes, existence is all we need to know; but in the arguments of 388H-388K we need to keep track of named pseudo-cycles in what I call 'dyadic cycle systems' (388D).

In this volume I have systematically preferred arguments which deal directly with measure algebras, rather than with measure spaces. I believe that such arguments can have a simplicity and clarity which repays the extra effort of dealing with more abstract structures. But undoubtedly it is necessary, if you are to have any hope of going farther in the subject, to develop methods of transferring intuitions and theorems between the two contexts. I offer 381Xl as an example. The description there of 'induced automorphism' requires a certain amount of manoeuvering around negligible sets, but gives a valuably graphic description. In the same way, 381Xf, 388A and 381Qc provide alternative ways of looking at full subgroups.

There are contexts in which it is useful to know whether an element of the full subgroup generated by π actually belongs to the full semigroup generated by π (381Yb); for instance, this happens in 388C.

Version of 14.1.15

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

385Xr Exercise 385Xr, referred to in the 2003, 2006 and 2013 editions of Volume 4, is now 385Xj.

Version of 25.8.17

References for Volume 3

Anderson I. [87] *Combinatorics of Finite Sets.* Oxford U.P., 1987. [332Xk, 3A1K.]

Anzai H. [51] 'On an example of a measure-preserving transformation which is not conjugate to its inverse', Proc. Japanese Acad. Sci. 27 (1951) 517-522. [§382 *notes*.]

Balcar B., Główczyński W. & Jech T. [98] 'The sequential topology on complete Boolean algebras', Fundamenta Math. 155 (1998) 59-78. [393L, 393Q.]

Balcar B., Jech T. & Pazák T. [05] 'Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann', Bull. London Math. Soc. 37 (2005) 885-898. [393L, 393Q.]

Becker H. & Kechris A.S. [96] *The descriptive set theory of Polish group actions.* Cambridge U.P., 1996 (London Math. Soc. Lecture Note Series 232). [§395 *notes*.]

Bekkali M. & Bonnet R. [89] 'Rigid Boolean algebras', pp. 637-678 in Monk 89. [384L.]

Bellow A. & Kölzow D. [76] *Measure Theory, Oberwolfach 1975.* Springer, 1976 (Lecture Notes in Mathematics 541).

Bhaskara Rao, K.P.S. & Bhaskara Rao, M. *Theory of Charges.* Academic, 1983.

Billingsley P. [65] *Ergodic Theory and Information.* Wiley, 1965. [§372 *notes*.]

Bollobás B. [79] *Graph Theory.* Springer, 1979. [332Xk, 3A1K.]

Bose R.C. & Manvel B. [84] *Introduction to Combinatorial Theory.* Wiley, 1984. [3A1K.]

Bourbaki N. [66] *General Topology.* Hermann/Addison-Wesley, 1968. [§3A3, §3A4.]

Bourbaki N. [68] *Theory of Sets.* Hermann/Addison-Wesley, 1968. [§315 *notes*.]

Bourbaki N. [87] *Topological Vector Spaces.* Springer, 1987. [§3A5.]

Bukhvalov A.V. [95] 'Optimization without compactness, and its applications', pp. 95-112 in Huijsmans Kaashoek Luxemburg & Pagter 95. [367U.]

Burke M.R. [93] 'Liftings for Lebesgue measure', pp. 119-150 in Judah 93. [341L, 345F.]

Burke M.R. [n95] 'Consistent liftings', privately circulated, 1995. [346Ya.]

Burnside W. [1911] *Theory of Groups of Finite Order.* Cambridge U.P., 1911 (reprinted by Dover, 1955). [§384 *notes*.]

Chacon R.V. [69] 'Weakly mixing transformations which are not strongly mixing', Proc. Amer. Math. Soc. 22 (1969) 559-562. [372R.]

Chacon R.V. & Krengel U. [64] 'Linear modulus of a linear operator', Proc. Amer. Math. Soc. 15 (1964) 553-559. [§371 *notes*.]

Choksi J.R. & Prasad V.S. [82] 'Ergodic theory of homogeneous measure algebras', pp. 367-408 of Kölzow & Maharam-Stone 82. [383I.]

Cohn H. [06] 'A short proof of the simple continued fraction expansion of e', Amer. Math. Monthly 113 (2006) 57-62; `arXiv:math.NT/0601660`. [372L.]

Coleman A.J. & Ribenboim P. [67] (eds.) *Proceedings of the Symposium in Analysis, Queen's University, June 1967.* Queen's University, Kingston, Ontario, 1967.

Comfort W.W. & Negrepontis S. [82] *Chain Conditions in Topology.* Cambridge U.P., 1982. [§391 *notes*.]

Cziszár I. [67] 'Information-type measures of difference of probability distributions and indirect observations', Studia Scientiarum Math. Hungarica 2 (1967) 299-318. [386G.]

Davey B.A. & Priestley H.A. [90] 'Introduction to Lattices and Order', Cambridge U.P., 1990. [3A6C.]

Dugundji J. [66] *Topology.* Allyn & Bacon, 1966. [§3A3, 3A4Bb.]

Dunford N. [1936] 'Integration and linear operators', Trans. Amer. Math. Soc. 40 (1936) 474-494. [§376 *notes*.]

Dunford N. & Schwartz J.T. [57] *Linear Operators I.* Wiley, 1957 (reprinted 1988). [§356 *notes*, §371 *notes*, §376 *notes*, §3A5.]

Dye H.A. [59] 'On groups of measure preserving transformations I', Amer. J. Math. 81 (1959) 119-159. [§388 *notes*.]

Eigen S.J. [82] 'The group of measure-preserving transformations of $[0,1]$ has no outer automorphisms', Math. Ann. 259 (1982) 259-270. [384D.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [§3A3, §3A4.]

Enderton H.B. [77] *Elements of Set Theory.* Academic, 1977. [§3A1.]

Erdős P. & Oxtoby J.C. [55] 'Partitions of the plane into sets having positive measure in every non-null product set', Trans. Amer. Math. Soc. 79 (1955) 91-102. [§325 *notes*.]

Fathi A. [78] 'Le groupe des transformations de $[0,1]$ qui préservent la mesure de Lebesgue est un groupe simple', Israel J. Math. 29 (1978) 302-308. [§382 *notes*, 383I.]

Fremlin D.H. [74a] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [Chap. 35 *intro.*, 354Yb, §355 *notes*, §356 *notes*, §363 *notes*, §365 *notes*, §371 *notes*, §376 *notes*.]

Fremlin D.H. [74b] 'A characterization of L-spaces', Indag. Math. 36 (1974) 270-275. [§371 *notes*.]

Fremlin D.H., de Pagter B. & Ricker W.J. [05] 'Sequential closedness of Boolean algebras of projections in Banach spaces', Studia Math. 167 (2005) 45-62. [§323 *notes*.]

Frolík Z. [68] 'Fixed points of maps of extremally disconnected spaces and complete Boolean algebras', Bull. Acad. Polon. Sci. 16 (1968) 269-275. [382E.]

Gaal S.A. [64] *Point Set Topology.* Academic, 1964. [§3A3, §3A4.]

Gaifman H. [64] 'Concerning measures on Boolean algebras', Pacific J. Math. 14 (1964) 61-73. [§391 *notes*.]

Gale D. [60] *The theory of linear economic models.* McGraw-Hill, 1960. [3A5D.]

Gnedenko B.V. & Kolmogorov A.N. [54] *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, 1954. [§342 *notes*.]

Graf S. & Weizsäcker H.von [76] 'On the existence of lower densities in non-complete measure spaces', pp. 155-158 in Bellow & Kölzow 76. [341L.]

Hajian A. & Ito Y. [69] 'Weakly wandering sets and invariant measures for a group of transformations', J. of Math. and Mech. 18 (1969) 1203-1216. [396B.]

Hajian A., Ito Y. & Kakutani S. [75] 'Full groups and a theorem of Dye', Advances in Math. 17 (1975) 48-59. [§388 *notes*.]

Halmos P.R. [1948] 'The range of a vector measure', Bull. Amer. Math. Soc. 54 (1948) 416-421. [326Yk.]

Halmos P.R. [60] *Naive Set Theory.* Van Nostrand, 1960. [3A1D.]

Huijsmans C.B., Kaashoek M.A., Luxemburg W.A.J. & de Pagter B. [95] (eds.) *Operator Theory in Function Spaces and Banach Lattices.* Birkhäuser, 1995.

Ionescu Tulcea C. & Ionescu Tulcea A. [69] *Topics in the Theory of Lifting.* Springer, 1969. [§341 *notes*.]

James I.M. [87] *Topological and Uniform Spaces.* Springer, 1987. [§3A3, §3A4.]

Jech T. [03] *Set Theory, Millennium Edition.* Springer, 2002. [§3A1.]

Jech T. [08] 'Algebraic characterizations of measure algebras', Proc. Amer. Math. Soc. 136 (2008) 1285-1294. [393Xj.]

Johnson R.A. [80] 'Strong liftings which are not Borel liftings', Proc. Amer. Math. Soc. 80 (1980) 234-236. [345F.]

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Kakutani S. [1941] 'Concrete representation of abstract L-spaces and the mean ergodic theorem', Annals of Math. 42 (1941) 523-537. [369E.]

Kalton N.J., Peck N.T. & Roberts J.W. [84] 'An F-space sampler', Cambridge U.P., 1984. [§375 *notes*.]

Kalton N.J. & Roberts J.W. [83] 'Uniformly exhaustive submeasures and nearly additive set functions', Trans. Amer. Math. Soc. 278 (1983) 803-816. [392D, §392 *notes*.]

Kantorovich L.V., Vulikh B.Z. & Pinsker A.G. [50] *Functional Analysis in Partially Ordered Spaces*, Gostekhizdat, 1950. [391D.]

Kawada Y. [1944] 'Über die Existenz der invarianten Integrale', Jap. J. Math. 19 (1944) 81-95. [§395 *notes*.]

Kelley J.L. [59] 'Measures on Boolean algebras', Pacific J. Math. 9 (1959) 1165-1177. [§391 *notes*.]

Kolmogorov A.N. [58] 'New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces', Dokl. Akad. Nauk SSSR 119 (1958) 861-864. [385P.]

Kölzow D. & Maharam-Stone D. [82] (eds.) *Measure Theory Oberwolfach 1981.* Springer, 1982 (Lecture Notes in Math. 945).

Koppelberg S. [89] *General Theory of Boolean Algebras*, vol. 1 of Monk 89. [Chap. 31 *intro.*, §332 *notes.*]

Köthe G. [69] *Topological Vector Spaces I.* Springer, 1969. [§356 *notes*, 3A5N.]

Kranz P. & Labuda I. [93] (eds.) *Proceedings of the Orlicz Memorial Conference, 1991*, unpublished manuscript, available from first editor (`mmkranz@olemiss.edu`).

Krengel, U. [63] '"Uber den Absolutbetrag stetiger linearer Operatoren und seine Anwendung auf ergodische Zerlegungen', Math. Scand. 13 (1963) 151-187. [371Xb.]

Krieger W. [76] 'On ergodic flows and the isomorphism of factors', Math. Ann. 223 (1976) 19-70. [§388 *notes.*]

Krivine J.-L. [71] *Introduction to Axiomatic Set Theory.* D.Reidel, 1971. [§3A1.]

Kullback S. [67] 'A lower bound for discrimination information in terms of variation', IEEE Trans. on Information Theory 13 (1967) 126-127. [386G.]

Kunen K. [80] *Set Theory.* North-Holland, 1980. [§3A1.]

Kwapien S. [73] 'On the form of a linear operator on the space of all measurable functions', Bull. Acad. Polon. Sci. 21 (1973) 951-954. [§375 *notes.*]

Lang S. [93] *Real and Functional Analysis.* Springer, 1993. [§3A5.]

Liapounoff A.A. [1940] 'Sur les fonctions-vecteurs complètement additives', Bull. Acad. Sci. URSS (Izvestia Akad. Nauk SSSR) 4 (1940) 465-478. [326H.]

Lindenstrauss J. & Tzafriri L. [79] *Classical Banach Spaces II.* Springer, 1979, reprinted in Lindenstrauss & Tzafriri 96. [§354 *notes*, 374Xj.]

Lindenstrauss J. & Tzafriri L. [96] *Classical Banach Spaces I & II.* Springer, 1996.

Lipschutz S. [64] *Set Theory and Related Topics.* McGraw-Hill, 1964 (Schaum's Outline Series). [3A1D.]

Luxemburg W.A.J. [67a] 'Is every integral normal?', Bull. Amer. Math. Soc. 73 (1967) 685-688. [363S.]

Luxemburg W.A.J. [67b] 'Rearrangement-invariant Banach function spaces', pp. 83-144 in Coleman & Ribenboim 67. [§374 *notes.*]

Luxemburg W.A.J. & Zaanen A.C. [71] *Riesz Spaces I.* North-Holland, 1971. [Chap. 35 *intro.*]

Macheras N.D., Musiał K. & Strauss W. [99] 'On products of admissible liftings and densities', J. for Analysis and its Applications 18 (1999) 651-668. [346G.]

Macheras N.D. & Strauss W. [95] 'Products of lower densities', J. for Analysis and its Applications 14 (1995) 25-32. [346Xf.]

Macheras N.D. & Strauss W. [96a] 'On products of almost strong liftings', J. Australian Math. Soc. (A) 60 (1996) 1-23. [346Yc.]

Macheras N.D. & Strauss W. [96b] 'The product lifting for arbitrary products of complete probability spaces', Atti Sem. Math. Fis. Univ. Modena 44 (1996) 485-496. [346H, 346Yd.]

Maharam D. [1942] 'On homogeneous measure algebras', Proc. Nat. Acad. Sci. U.S.A. 28 (1942) 108-111. [331F, 332B.]

Maharam D. [1947] 'An algebraic characterization of measure algebras', Ann. Math. 48 (1947) 154-167. [393J.]

Maharam D. [58] 'On a theorem of von Neumann', Proc. Amer. Math. Soc. 9 (1958) 987-994. [§341 *notes*, §346 *notes.*]

Marczewski E. [53] 'On compact measures', Fund. Math. 40 (1953) 113-124. [342A.]

McCune W. [97] 'Solution of the Robbins problem', J. Automated Reasoning 19 (1997) 263-276. [311Yc.]

Miller B.D. [04] PhD Thesis, University of California, Berkeley, 2004. [382Xc, §382 *notes.*]

Monk J.D. [89] (ed.) *Handbook of Boolean Algebra.* North-Holland, 1989.

Nadkarni M.G. [90] 'On the existence of a finite invariant measure', Proc. Indian Acad. Sci., Math. Sci. 100 (1990) 203-220. [§395 *notes.*]

Ornstein D.S. [74] *Ergodic Theory, Randomness and Dynamical Systems.* Yale U.P., 1974. [§387 *notes.*]

Ornstein D.S. & Shields P.C. [73] 'An uncountable family of K-automorphisms', Advances in Math. 10 (1973) 63-88. [§382 *notes.*]

Perović Ž. & Veličković B. [18] 'Ranks of Maharam algebras', Advances in Math. 330 (2018) 253-279. [394A.]

Petersen K. [83] *Ergodic Theory.* Cambridge U.P., 1983. [328Xa, 385C, §385 *notes*, 386E.]

Roberts J.W. [93] 'Maharam's problem', in Kranz & Labuda 93. [§394 *notes*.]

Rotman J.J. [84] 'An Introduction to the Theory of Groups', Allyn & Bacon, 1984. [§384 *notes*, 3A6B.]

Rudin W. [91] *Functional Analysis.* McGraw-Hill, 1991. [§3A5.]

Ryzhikov V.V. [93] 'Factorization of an automorphism of a complete Boolean algebra into a product of three involutions', Mat. Zametki (=Math. Notes of Russian Acad. Sci.) 54 (1993) 79-84. [§382 *notes*.]

Sazonov V.V. [66] 'On perfect measures', A.M.S. Translations (2) 48 (1966) 229-254. [§342 *notes*.]

Schaefer H.H. [66] *Topological Vector Spaces.* MacMillan, 1966; reprinted with corrections Springer, 1971. [3A4A, 3A5J, 3A5N.]

Schaefer H.H. [74] *Banach Lattices and Positive Operators.* Springer, 1974. [Chap. 35 *intro*., §354 *notes*.]

Schubert H. [68] *Topology.* Allyn & Bacon, 1968. [§3A3, §3A4.]

Shelah S. [98] 'The lifting problem with the full ideal', J. Applied Analysis 4 (1998) 1-17. [341L.]

Sikorski R. [64] *Boolean Algebras.* Springer, 1964. [Chap. 31 *intro*.]

Sinaĭ Ya.G. [59] 'The notion of entropy of a dynamical system', Dokl. Akad. Nauk SSSR 125 (1959) 768-771. [385P.]

Sinaĭ Ya.G. [62] 'Weak isomorphism of transformations with an invariant measure', Soviet Math. 3 (1962) 1725-1729. [387E.]

Smorodinsky M. [71] *Ergodic Theory, Entropy.* Springer, 1971 (Lecture Notes in Math., 214). [§387 *notes*.]

Štěpánek P. & Rubin M. [89] 'Homogeneous Boolean algebras', pp. 679-715 in Monk 89. [382S, §382 *notes*.]

Talagrand M. [82a] 'Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations', Ann. Institut Fourier (Grenoble) 32 (1982) 39-69. [§346 *notes*.]

Talagrand M. [82b] 'La pathologie des relèvements invariants', Proc. Amer. Math. Soc. 84 (1982) 379-382. [345F.]

Talagrand M. [84] *Pettis integral and measure theory.* Mem. Amer. Math. Soc. 307 (1984). [§346 *notes*.]

Talagrand M. [08] 'Maharam's problem', Annals of Math. 168 (2008) 981-1009. [§394.]

Taylor A.E. [64] *Introduction to Functional Analysis.* Wiley, 1964. [§3A5.]

Todorčević S. [04] 'A problem of von Neumann and Maharam about algebras supporting continuous submeasures', Fund. Math. 183 (2004) 169-183. [393S.]

Truss J.K. [89] 'Infinite permutation groups I: products of conjugacy classes', J. Algebra 120 (1989) 454-493. [§382 *notes*.]

Vladimirov D.A. [02] *Boolean algebras in analysis.* Kluwer, 2002 (Math. and its Appl. 540). [367L.]

Vulikh B.C. [67] *Introduction to the Theory of Partially Ordered Vector Spaces.* Wolters-Noordhoff, 1967. [§364 *notes*.]

Wagon S. [85] *The Banach-Tarski Paradox.* Cambridge U.P., 1985. [§395 *notes*.]

Wilansky A. [64] *Functional Analysis.* Blaisdell, 1964. [§3A5.]

Zaanen A.C. [83] *Riesz Spaces II.* North-Holland, 1983. [Chap. 35 *intro*., 376K, §376 *notes*.]