Version of 27.6.16

Chapter 37

Linear operators between function spaces

As everywhere in functional analysis, the function spaces of measure theory cannot be properly understood without investigating linear operators between them. In this chapter I have collected a number of results which rely on, or illuminate, the measure-theoretic aspects of the theory. §371 is devoted to a fundamental property of linear operators on L-spaces, if considered abstractly, that is, of L^1-spaces, if considered in the languages of Chapters 24 and 36, and to an introduction to the class $\mathcal{T}$ of operators which are norm-decreasing for both $\| \|_1$ and $\| \|_\infty$. This makes it possible to prove a version of Birkhoff's Ergodic Theorem for operators which need not be positive (372D). In §372 I give various forms of this theorem, for linear operators between function spaces, for measure-preserving Boolean homomorphisms between measure algebras, and for inverse-measure-preserving functions between measure spaces, with an excursion into the theory of continued fractions. In §373 I make a fuller analysis of the class $\mathcal{T}$, with a complete characterization of those u, v such that $v = Tu$ for some $T \in \mathcal{T}$. Using this we can describe 'rearrangement-invariant' function spaces and extended Fatou norms (§374). Returning to ideas left on one side in §§364 and 368, I investigate positive linear operators defined on L^0 spaces (§375). In the penultimate section of the chapter (§376), I look at operators which can be defined in terms of kernels on product spaces. Finally, in §377, I examine the function spaces of reduced products, projective limits and inductive limits of probability algebras.

Version of 13.12.06

371 The Chacon-Krengel theorem

The first topic I wish to treat is a remarkable property of L-spaces: if U and V are L-spaces, then every continuous linear operator $T : U \to V$ is order-bounded, and $\||T|\| = \|T\|$ (371D). This generalizes in various ways to other V (371B, 371C). I apply the result to a special type of operator between $M^{1,0}$ spaces which will be conspicuous in the next section (371F-371H).

371A Lemma Let U be an L-space, V a Banach lattice and $T : U \to V$ a bounded linear operator. Take $u \ge 0$ in U and set

$$B = \{\sum_{i=0}^n |Tu_i| : u_0, \dots , u_n \in U^+, \sum_{i=0}^n u_i = u\} \subseteq V^+.$$

Then B is upwards-directed and $\sup_{v\in B} \|v\| \le \|T\|\|u\|$.

371B Theorem Let U be an L-space and V a Dedekind complete Banach lattice U with a Fatou norm. Then the Riesz space $\mathsf{L}^\sim(U; V) = \mathsf{L}^\times(U; V)$ is a closed linear subspace of the Banach space $\mathsf{B}(U; V)$ and is in itself a Banach lattice with a Fatou norm.

371C Theorem Let U be an L-space and V a Dedekind complete Banach lattice with a Fatou norm and the Levi property. Then $\mathsf{B}(U; V) = \mathsf{L}^\sim(U; V) = \mathsf{L}^\times(U; V)$ is a Dedekind complete Banach lattice with a Fatou norm and the Levi property. In particular, $|T|$ is defined and $\||T|\| = \|T\|$ for every $T \in \mathsf{B}(U; V)$.

371D Corollary Let U and V be L-spaces. Then $\mathsf{L}^\sim(U; V) = \mathsf{L}^\times(U; V) = \mathsf{B}(U; V)$ is a Dedekind complete Banach lattice with a Fatou norm and the Levi property.

371F The class $\mathcal{T}^{(0)}$: Definition Let $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{B}, \bar\nu)$ be measure algebras. Write $\mathcal{T}^{(0)} = \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ for the set of all linear operators $T : M^{1,0}(\mathfrak{A}, \bar\mu) \to M^{1,0}(\mathfrak{B}, \bar\nu)$ such that $Tu \in L^1(\mathfrak{B}, \bar\nu)$ and $\|Tu\|_1 \le \|u\|_1$ for every $u \in L^1(\mathfrak{A}, \bar\mu)$, $Tu \in L^\infty(\mathfrak{B})$ and $\|Tu\|_\infty \le \|u\|_\infty$ for every $u \in L^\infty(\mathfrak{A}) \cap M^{1,0}(\mathfrak{A}, \bar\mu)$.

371G Proposition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras.

(a) $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$ is a convex set in the unit ball of $\mathsf{B}(M^{1,0}(\mathfrak{A},\bar{\mu});M^{1,0}(\mathfrak{B},\bar{\nu}))$. If $T_0:L^1(\mathfrak{A},\bar{\mu})\to L^1(\mathfrak{B},\bar{\nu})$ is a linear operator of norm at most 1, and $T_0u\in L^\infty(\mathfrak{B})$ and $\|T_0u\|_\infty\le\|u\|_\infty$ for every $u\in L^1(\mathfrak{A},\bar{\mu})\cap L^\infty(\mathfrak{A})$, then T_0 has a unique extension to a member of $\mathcal{T}^{(0)}$.

(b) If $T\in\mathcal{T}^{(0)}$ then T is order-bounded and $|T|$, taken in

$$\mathsf{L}^\sim(M^{1,0}(\mathfrak{A},\bar{\mu});M^{1,0}(\mathfrak{B},\bar{\nu}))=\mathsf{L}^\times(M^{1,0}(\mathfrak{A},\bar{\mu});M^{1,0}(\mathfrak{B},\bar{\nu})),$$

also belongs to $\mathcal{T}^{(0)}$.

(c) If $T\in\mathcal{T}^{(0)}$ then $\|Tu\|_{1,\infty}\le\|u\|_{1,\infty}$ for every $u\in M^{1,0}(\mathfrak{A},\bar{\mu})$.

(d) If $T\in\mathcal{T}^{(0)}$, $p\in[1,\infty[$ and $w\in L^p(\mathfrak{A},\bar{\mu})$ then $Tw\in L^p(\mathfrak{B},\bar{\nu})$ and $\|Tw\|_p\le\|w\|_p$.

(e) If $(\mathfrak{C},\bar{\lambda})$ is another measure algebra then $ST\in\mathcal{T}^{(0)}_{\bar{\mu},\bar{\lambda}}$ whenever $T\in\mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$ and $S\in\mathcal{T}^{(0)}_{\bar{\nu},\bar{\lambda}}$.

Version of 7.12.08/17.7.11

372 The ergodic theorem

I come now to one of the most remarkable topics in measure theory. I cannot do it justice in the space I have allowed for it here, but I can give the basic theorem (372D, 372F) and a variety of the corollaries through which it is regularly used (372E, 372G-372J), together with brief notes on one of its most famous and characteristic applications (to continued fractions, 372L-372N) and on 'ergodic' and 'mixing' transformations (372O-372S). In the first half of the section (down to 372G) I express the arguments in the abstract language of measure algebras and their associated function spaces, as developed in Chapter 36; the second half, from 372H onwards, contains translations of the results into the language of measure spaces and measurable functions, the more traditional, and more readily applicable, forms.

372A Lemma Let U be a reflexive Banach space and $T:U\to U$ a bounded linear operator of norm at most 1. Then

$$V=\{u+v-Tu:u,\,v\in U,\,Tv=v\}$$

is dense in U.

372B Lemma Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $T:L^1\to L^1$ a positive linear operator of norm at most 1, where $L^1=L^1(\mathfrak{A},\bar{\mu})$. Take any $u\in L^1$ and $m\in\mathbb{N}$, and set

$$a=[\![u>0]\!]\cup[\![u+Tu>0]\!]\cup[\![u+Tu+T^2u>0]\!]\cup\ldots\cup[\![u+Tu+\ldots+T^mu>0]\!].$$

Then $\int_a u\ge 0$.

372C Maximal Ergodic Theorem Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $T:L^1\to L^1$ a linear operator, where $L^1=L^1(\mathfrak{A},\bar{\mu})$, such that $\|Tu\|_1\le\|u\|_1$ for every $u\in L^1$ and $\|Tu\|_\infty\le\|u\|_\infty$ for every $u\in L^1\cap L^\infty(\mathfrak{A})$. Set $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$ for each $n\in\mathbb{N}$. Then for any $u\in L^1$, $u^*=\sup_{n\in\mathbb{N}}A_nu$ is defined in $L^0(\mathfrak{A})$, and $\alpha\bar{\mu}[\![u^*>\alpha]\!]\le\|u\|_1$ for every $\alpha>0$.

372D The Ergodic Theorem: first form Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and set $M^{1,0}=M^{1,0}(\mathfrak{A},\bar{\mu})$, $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar{\mu},\bar{\mu}}\subseteq\mathsf{B}(M^{1,0};M^{1,0})$. Take any $T\in\mathcal{T}^{(0)}$, and set $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i:M^{1,0}\to M^{1,0}$ for every n. Then for any $u\in M^{1,0}$, $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\|\,\|_{1,\infty}$-convergent to a member Pu of $M^{1,0}$. The operator $P:M^{1,0}\to M^{1,0}$ is a projection onto the linear subspace $\{u:u\in M^{1,0},\,Tu=u\}$, and $P\in\mathcal{T}^{(0)}$.

372E Corollary Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\pi:\mathfrak{A}^f\to\mathfrak{A}^f$ a measure-preserving ring homomorphism, where $\mathfrak{A}^f=\{a:\bar{\mu}a<\infty\}$. Let $T:M^{1,0}\to M^{1,0}$ be the corresponding Riesz homomorphism, where $M^{1,0}=M^{1,0}(\mathfrak{A},\bar{\mu})$. Set $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$ for $n\in\mathbb{N}$. Then for every $u\in M^{1,0}$, $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\|\,\|_{1,\infty}$-convergent to some v such that $Tv=v$.

372F The Ergodic Theorem: second form Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and let $T:L^1\to L^1$, where $L^1=L^1(\mathfrak{A},\bar{\mu})$, be a linear operator of norm at most 1 such that $Tu\in L^\infty=L^\infty(\mathfrak{A})$ and $\|Tu\|_\infty\le$

$\|u\|_\infty$ whenever $u \in L^1 \cap L^\infty$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i : L^1 \to L^1$ for every n. Then for any $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to an element Pu of L^1. The operator $P : L^1 \to L^1$ is a projection of norm at most 1 onto the linear subspace $\{u : u \in L^1,\, Tu = u\}$.

372G Corollary Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $T : L^1 \to L^1$ be the corresponding Riesz homomorphism, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for $n \in \mathbb{N}$. Then for every $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent. If we set $Pu = \lim_{n\to\infty} A_n u$ for each u, P is the conditional expectation operator corresponding to the fixed-point subalgebra $\mathfrak{C} = \{a : \pi a = a\}$ of $\mathfrak{A}$.

372H Corollary Let (X, Σ, μ) be a measure space and $\phi : X \to X$ an inverse-measure-preserving function. Let f be a real-valued function which is integrable over X. Then

$$g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$$

is defined for almost every $x \in X$, and $g\phi(x) = g(x)$ for almost every x.

372I Lemma Let (X, Σ, μ) be a measure space with measure algebra $(\mathfrak{A}, \bar\mu)$. Let $\phi : X \to X$ be an inverse-measure-preserving function and $\pi : \mathfrak{A} \to \mathfrak{A}$ the associated homomorphism. Set $\mathfrak{C} = \{c : c \in \mathfrak{A},\, \pi c = c\}$, $\mathrm{T} = \{E : E \in \Sigma,\, \phi^{-1}[E] \triangle E$ is negligible$\}$ and $\mathrm{T}_0 = \{E : E \in \Sigma,\, \phi^{-1}[E] = E\}$. Then T and T_0 are σ-subalgebras of Σ; $\mathrm{T}_0 \subseteq \mathrm{T}$, $\mathrm{T} = \{E : E \in \Sigma,\, E^\bullet \in \mathfrak{C}\}$, and $\mathfrak{C} = \{E^\bullet : E \in \mathrm{T}_0\}$.

372J The Ergodic Theorem: third form Let (X, Σ, μ) be a probability space and $\phi : X \to X$ an inverse-measure-preserving function. Let f be a real-valued function which is integrable over X. Then

$$g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$$

is defined for almost every $x \in X$; $g\phi =_{\text{a.e.}} g$, and g is a conditional expectation of f on the σ-algebra $\mathrm{T} = \{E : E \in \Sigma,\, \phi^{-1}[E] \triangle E$ is negligible$\}$. If *either* f is Σ-measurable and defined everywhere in X *or* $\phi[E]$ is negligible for every negligible set E, then g is a conditional expectation of f on the σ-algebra $\mathrm{T}_0 = \{E : E \in \Sigma,\, \phi^{-1}[E] = E\}$.

372L Continued fractions (a) Set $X = [0, 1] \setminus \mathbb{Q}$. For $x \in X$, set $\phi(x) = <\frac{1}{x}>$, the fractional part of $\frac{1}{x}$, and $k_1(x) = \frac{1}{x} - \phi(x) = \lfloor \frac{1}{x} \rfloor$, the integer part of $\frac{1}{x}$; then $\phi(x) \in X$ for each $x \in X$, so we may define $k_n(x) = k_1(\phi^{n-1}(x))$ for every $n \ge 1$. The strictly positive integers $k_1(x), k_2(x), k_3(x), \ldots$ are the **continued fraction coefficients** of x. $k_{n+1}(x) = k_n(\phi(x))$ for every $n \ge 1$. Now define $\langle p_n(x)\rangle_{n\in\mathbb{N}}$, $\langle q_n(x)\rangle_{n\in\mathbb{N}}$ inductively by setting

$$p_0(x) = 0, \quad p_1(x) = 1, \quad p_n(x) = p_{n-2}(x) + k_n(x)p_{n-1}(x) \text{ for } n \ge 1,$$

$$q_0(x) = 1, \quad q_1(x) = k_1(x), \quad q_n(x) = q_{n-2}(x) + k_n(x)q_{n-1}(x) \text{ for } n \ge 1.$$

The **continued fraction approximations** to x are the quotients $p_n(x)/q_n(x)$.

(c)(i) For any $x \in X$, $n \ge 1$ we have

$$p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x) = (-1)^n, \quad \phi^n(x) = \frac{p_n(x) - xq_n(x)}{xq_{n-1}(x) - p_{n-1}(x)},$$

$$x = \frac{p_n(x) + p_{n-1}(x)\phi^n(x)}{q_n(x) + q_{n-1}(x)\phi^n(x)}.$$

(ii) For any finite string $\mathbf{m} = (m_1, \ldots, m_n)$ of strictly positive integers the set $D_{\mathbf{m}} = \{x : x \in X,\, k_i(x) = m_i \text{ for } 1 \le i \le n\}$ is an interval in X on which ϕ^n is monotonic, being strictly increasing if n is even and strictly decreasing if n is odd.

(iii) We also need to know that if $\mathbf{m} = (m_1, \ldots, m_n)$, the length of $D_{\mathbf{m}}$ is at most 2^{-n+1}.

372M Theorem Set $X = [0,1]\setminus\mathbb{Q}$, and define $\phi : X \to X$ as in 372L. Then for every Lebesgue integrable function f on X,

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x)) = \frac{1}{\ln 2}\int_0^1 \frac{f(t)}{1+t}dt$$

for almost every $x \in X$.

372N Corollary For almost every $x \in [0,1] \setminus \mathbb{Q}$,

$$\lim_{n\to\infty} \frac{1}{n}\#(\{i : 1 \le i \le n,\ k_i(x) = k\}) = \frac{1}{\ln 2}(2\ln(k+1) - \ln k - \ln(k+2))$$

for every $k \ge 1$, where $k_1(x), \dots$ are the continued fraction coefficients of x.

372O Mixing and ergodic transformations: Definitions (a)(i) Let $\mathfrak{A}$ be a Boolean algebra. Then a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is **ergodic** if whenever $a, b \in \mathfrak{A} \setminus \{0\}$ there are $m, n \in \mathbb{N}$ such that $\pi^m a \cap \pi^n b \ne 0$.

(ii) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is **mixing** if $\lim_{n\to\infty} \bar\mu(\pi^n a \cap b) = \bar\mu a \cdot \bar\mu b$ for all $a, b \in \mathfrak{A}$.

(iii) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is **weakly mixing** if $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\bar\mu(\pi^n a \cap b) - \bar\mu a \cdot \bar\mu b| = 0$ for all $a, b \in \mathfrak{A}$.

(b) Let (X, Σ, μ) be a probability space and $\phi : X \to X$ an inverse-measure-preserving function.

(i) ϕ is **ergodic** if every measurable set E such that $\phi^{-1}[E] = E$ is either negligible or conegligible.

(ii) ϕ is **mixing** if $\lim_{n\to\infty} \mu(F \cap \phi^{-n}[E]) = \mu E \cdot \mu F$ for all $E, F \in \Sigma$.

(iii) ϕ is **weakly mixing** if if $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(F \cap \phi^{-n}[E]) - \mu E \cdot \mu F| = 0$ for all $E, F \in \Sigma$.

372P Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism, with fixed-point subalgebra $\mathfrak{C}$.
(a) If π is ergodic, then $\mathfrak{C} = \{0,1\}$.
(b) If π is an automorphism, then π is ergodic iff $\sup_{n\in\mathbb{Z}} \pi^n a = 1$ for every $a \in \mathfrak{A} \setminus \{0\}$.
(c) If π is an automorphism and $\mathfrak{A}$ is Dedekind σ-complete, then π is ergodic iff $\mathfrak{C} = \{0,1\}$.

372Q Proposition (a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, and $T : L^0 = L^0(\mathfrak{A}) \to L^0$ the Riesz homomorphism such that $T(\chi a) = \chi\pi a$ for every $a \in \mathfrak{A}$.

(i) If π is mixing, it is weakly mixing.

(ii) If π is weakly mixing, it is ergodic.

(iii) The following are equiveridical: (α) π is ergodic; (β) the only $u \in L^0$ such that $Tu = u$ are the multiples of $\chi 1$; (γ) for every $u \in L^1 = L^1(\mathfrak{A}, \bar\mu)$, $\langle \frac{1}{n+1} \sum_{i=0}^{n} T^i u\rangle_{n\in\mathbb{N}}$ order*-converges to $(\int u)\chi 1$.

(iv) The following are equiveridical: (α) π is mixing; (β) $\lim_{n\to\infty}(T^n u|v) = \int u \int v$ for all $u, v \in L^2(\mathfrak{A}, \bar\mu)$.

(v) The following are equiveridical: (α) π is weakly mixing; (β) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |(T^k u|v) - \int u \int v| = 0$ for all $u, v \in L^2(\mathfrak{A}, \bar\mu)$.

(b) Let (X, Σ, μ) be a probability space, with measure algebra $(\mathfrak{A}, \bar\mu)$. Let $\phi : X \to X$ be an inverse-measure-preserving function and $\pi : \mathfrak{A} \to \mathfrak{A}$ the associated homomorphism such that $\pi E^\bullet = (\phi^{-1}[E])^\bullet$ for every $E \in \Sigma$.

(i) The following are equiveridical: (α) ϕ is ergodic; (β) π is ergodic; (γ) for every μ-integrable real-valued function f, $\langle \frac{1}{n+1} \sum_{i=0}^{n} f(\phi^i(x))\rangle_{n\in\mathbb{N}}$ converges to $\int f$ for almost every $x \in X$.

(ii) ϕ is mixing iff π is, and in this case ϕ is weakly mixing.

(iii) ϕ is weakly mixing iff π is, and in this case ϕ is ergodic.

372S Proposition (a) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. If $\bigcap_{n\in\mathbb{N}}\pi^n[\mathfrak{A}]=\{0,1\}$, then π is mixing.

(b) Let (X,Σ,μ) be a probability space, and $\phi:X\to X$ an inverse-measure-preserving function. Set

$$\mathrm{T}=\{E:\text{ for every } n\in\mathbb{N} \text{ there is an } F\in\Sigma \text{ such that } E=\phi^{-n}[F]\}.$$

If every member of T is either negligible or conegligible, ϕ is mixing.

Version of 25.5.16

373 Decreasing rearrangements

I take a section to discuss operators in the class $\mathcal{T}^{(0)}$ of 371F-371H and two associated classes $\mathcal{T}$, $\mathcal{T}^\times$ (373A). These turn out to be intimately related to the idea of 'decreasing rearrangement' (373C). In 373D-373F I give elementary properties of decreasing rearrangements; then in 373G-373O I show how they may be used to characterize the set $\{Tu:T\in\mathcal{T}\}$ for a given u. The argument uses a natural topology on the set $\mathcal{T}$ (373K). I conclude with remarks on the possible values of $\int Tu\times v$ for $T\in\mathcal{T}$ (373P-373Q) and identifications between $\mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$, $\mathcal{T}^{(0)}_{\bar{\nu},\bar{\mu}}$ and $\mathcal{T}^\times_{\bar{\mu},\bar{\nu}}$ (373R-373T).

373A Definition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras.

(a) $\mathcal{T}_{\bar{\mu},\bar{\nu}}$ will be the space of linear operators $T:M^{1,\infty}(\mathfrak{A},\bar{\mu})\to M^{1,\infty}(\mathfrak{B},\bar{\nu})$ such that $\|Tu\|_1\le\|u\|_1$ for every $u\in L^1(\mathfrak{A},\bar{\mu})$ and $\|Tu\|_\infty\le\|u\|_\infty$ for every $u\in L^\infty(\mathfrak{A})$.

(b) If $\mathfrak{B}$ is Dedekind complete, $\mathcal{T}^\times_{\bar{\mu},\bar{\nu}}$ will be $\mathcal{T}_{\bar{\mu},\bar{\nu}}\cap\mathsf{L}^\times(M^{1,\infty}(\mathfrak{A},\bar{\mu});M^{1,\infty}(\mathfrak{A},\bar{\mu}))$.

373B Proposition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras.

(a) $\mathcal{T}=\mathcal{T}_{\bar{\mu},\bar{\nu}}$ is a convex subset of the unit ball of $\mathsf{B}(M^{1,\infty}(\mathfrak{A},\bar{\mu});M^{1,\infty}(\mathfrak{B},\bar{\nu}))$.

(b) If $T\in\mathcal{T}$ then $T\restriction M^{1,0}(\mathfrak{A},\bar{\mu})$ belongs to $\mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$. So if $T\in\mathcal{T}$, $p\in[1,\infty[$ and $u\in L^p(\mathfrak{A},\bar{\mu})$ then $Tu\in L^p(\mathfrak{B},\bar{\nu})$ and $\|Tu\|_p\le\|u\|_p$.

(c) If $\mathfrak{B}$ is Dedekind complete, then $\mathcal{T}$ is a solid subset of $\mathsf{L}^\sim(M^{1,\infty}(\mathfrak{A},\bar{\mu});M^{1,\infty}(\mathfrak{B},\bar{\nu}))$.

(d) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism, then we have a corresponding operator $T\in\mathcal{T}$ defined by saying that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$. If π is order-continuous, then so is T.

(e) If $(\mathfrak{C},\bar{\lambda})$ is another measure algebra, $T\in\mathcal{T}$ and $S\in\mathcal{T}_{\bar{\nu},\bar{\lambda}}$ then $ST\in\mathcal{T}_{\bar{\mu},\bar{\lambda}}$.

373C Decreasing rearrangements Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Write $M^{0,\infty}(\mathfrak{A},\bar{\mu})$ for the set of those $u\in L^0(\mathfrak{A})$ such that $\bar{\mu}[\![|u|>\alpha]\!]$ is finite for some $\alpha\in\mathbb{R}$. $M^{0,\infty}(\mathfrak{A},\bar{\mu})$ is a solid linear subspace of $L^0(\mathfrak{A})$. Let $(\mathfrak{A}_L,\bar{\mu}_L)$ be the measure algebra of Lebesgue measure on $[0,\infty[$. For $u\in M^{0,\infty}(\mathfrak{A},\bar{\mu})$ its **decreasing rearrangement** is $u^*\in M^{0,\infty}(\mathfrak{A}_L,\bar{\mu}_L)$, defined by setting $u^*=g^\bullet$, where

$$g(t)=\min\{\alpha:\alpha\ge 0,\ \bar{\mu}[\![|u|>\alpha]\!]\le t\}$$

for every $t>0$.

373D Lemma Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra.

(a) For any $u\in M^{0,\infty}(\mathfrak{A},\bar{\mu})$, its decreasing rearrangement u^* may be defined by the formula

$$[\![u^*>\alpha]\!]=[0,\bar{\mu}[\![|u|>\alpha]\!][^\bullet \text{ for every } \alpha\ge 0,$$

that is,

$$\bar{\mu}_L[\![u^*>\alpha]\!]=\bar{\mu}[\![|u|>\alpha]\!] \text{ for every } \alpha\ge 0.$$

(b) If $|u|\le|v|$ in $M^{0,\infty}(\mathfrak{A},\bar{\mu})$, then $u^*\le v^*$; $|u|^*=u^*$.

(c)(i) If $u=\sum_{i=0}^n\alpha_i\chi a_i$, where $a_0\supseteq a_1\supseteq\ldots\supseteq a_n$ and $\alpha_i\ge 0$ for each i, then $u^*=\sum_{i=0}^n\alpha_i\chi\,[0,\bar{\mu}a_i[^\bullet$.

(ii) If $u=\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\ldots,a_n$ are disjoint and $|\alpha_0|\ge|\alpha_1|\ge\ldots\ge|\alpha_n|$, then $u^*=\sum_{i=0}^n|\alpha_i|\chi\,[\beta_i,\beta_{i+1}[^\bullet$, where $\beta_i=\sum_{j<i}\bar{\mu}a_i$ for $i\le n+1$.

(d) If $E \subseteq]0,\infty[$ is any Borel set, and $u \in M^0(\mathfrak{A},\bar\mu)$, then $\bar\mu_L[\![u^* \in E]\!] = \bar\mu[\![|u| \in E]\!]$.

(e) Let $h : [0,\infty[\to [0,\infty[$ be a non-decreasing function such that $h(0) = 0$, and write $\bar h$ for the corresponding functions on $L^0(\mathfrak{A})^+$ and $L^0(\mathfrak{A}_L)^+$. Then $(\bar h(u))^* = \bar h(u^*)$ whenever $u \ge 0$ in $M^0(\mathfrak{A},\bar\mu)$. If h is continuous on the left, $(\bar h(u))^* = \bar h(u^*)$ whenever $u \ge 0$ in $M^{0,\infty}(\mathfrak{A},\bar\mu)$.

(f) If $u \in M^{0,\infty}(\mathfrak{A},\bar\mu)$ and $\alpha \ge 0$, then

$$(u^* - \alpha\chi 1)^+ = ((|u| - \alpha\chi 1)^+)^*.$$

(g) If $u \in M^{0,\infty}(\mathfrak{A},\bar\mu)$, then for any $t > 0$

$$\int_0^t u^* = \inf_{\alpha\ge 0} \alpha t + \int (|u| - \alpha\chi 1)^+.$$

(h) If $A \subseteq (M^{0,\infty}(\mathfrak{A},\bar\mu))^+$ is non-empty and upwards-directed and has supremum $u_0 \in M^{0,\infty}(\mathfrak{A},\bar\mu)$, then $u_0^* = \sup_{u\in A} u^*$.

373E Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Then $\int |u \times v| \le \int u^* \times v^*$ for all u, $v \in M^{0,\infty}(\mathfrak{A},\bar\mu)$.

373F Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and u any member of $M^{0,\infty}(\mathfrak{A},\bar\mu)$.

(a) For any $p \in [1,\infty]$, $u \in L^p(\mathfrak{A},\bar\mu)$ iff $u^* \in L^p(\mathfrak{A}_L,\bar\mu_L)$, and in this case $\|u\|_p = \|u^*\|_p$.

(b)(i) $u \in M^0(\mathfrak{A},\bar\mu)$ iff $u^* \in M^0(\mathfrak{A}_L,\bar\mu_L)$;

(ii) $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ iff $u^* \in M^{1,\infty}(\mathfrak{A}_L,\bar\mu_L)$, and in this case $\|u\|_{1,\infty} = \|u^*\|_{1,\infty}$;

(iii) $u \in M^{1,0}(\mathfrak{A},\bar\mu)$ iff $u^* \in M^{1,0}(\mathfrak{A}_L,\bar\mu_L)$;

(iv) $u \in M^{\infty,1}(\mathfrak{A},\bar\mu)$ iff $u^* \in M^{\infty,1}(\mathfrak{A}_L,\bar\mu_L)$, and in this case $\|u\|_{\infty,1} = \|u^*\|_{\infty,1}$.

373G Lemma Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. If

either $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $T \in \mathcal{T}_{\bar\mu,\bar\nu}$

or $u \in M^{1,0}(\mathfrak{A},\bar\mu)$ and $T \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$,

then $\int_0^t (Tu)^* \le \int_0^t u^*$ for every $t \ge 0$.

373H Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\theta : \mathfrak{A}^f \to \mathbb{R}$ an additive functional, where $\mathfrak{A}^f = \{a : \bar\mu a < \infty\}$.

(a) The following are equiveridical:

(α) $\lim_{t\downarrow 0} \sup_{\bar\mu a\le t} |\theta a| = \lim_{t\to\infty} \frac{1}{t} \sup_{\bar\mu a \le t} |\theta a| = 0$,

(β) there is some $u \in M^{1,0}(\mathfrak{A},\bar\mu)$ such that $\theta a = \int_a u$ for every $a \in \mathfrak{A}^f$,

and in this case u is uniquely defined.

(b) Now suppose that $(\mathfrak{A},\bar\mu)$ is localizable. Then the following are equiveridical:

(α) $\lim_{t\downarrow 0} \sup_{\bar\mu a\le t} |\theta a| = 0$, $\quad \limsup_{t\to\infty} \frac{1}{t} \sup_{\bar\mu a\le t} |\theta a| < \infty$,

(β) there is some $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ such that $\theta a = \int_a u$ for every $a \in \mathfrak{A}^f$,

and again this u is uniquely defined.

373I Lemma Suppose that u, v, $w \in M^{0,\infty}(\mathfrak{A}_L,\bar\mu_L)$ are all equivalence classes of non-negative non-increasing functions. If $\int_0^t u \le \int_0^t v$ for every $t \ge 0$, then $\int u \times w \le \int v \times w$.

373J Corollary Suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are measure algebras and $v \in M^{0,\infty}(\mathfrak{B},\bar\nu)$. If

either $u \in M^{1,0}(\mathfrak{A},\bar\mu)$ and $T \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$

or $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $T \in \mathcal{T}_{\bar\mu,\bar\nu}$

then $\int |Tu \times v| \le \int u^* \times v^*$.

373K The very weak operator topology Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be two measure algebras. For $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $w \in M^{\infty,1}(\mathfrak{B},\bar\nu)$ set

$$\tau_{uw}(T) = |\int Tu \times w| \text{ for } T \in \mathsf{B} = \mathsf{B}(M^{1,\infty}(\mathfrak{A},\bar\mu); M^{1,\infty}(\mathfrak{B},\bar\nu)).$$

Then τ_{uw} is a seminorm on B. I will call the topology generated by $\{\tau_{uw} : u \in M^{1,\infty}(\mathfrak{A},\bar\mu),\ w \in M^{\infty,1}(\mathfrak{B},\bar\nu)\}$ the **very weak operator topology** on B.

373L Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a localizable measure algebra. Then $\mathcal{T}_{\bar\mu,\bar\nu}$ is compact in the very weak operator topology.

373M Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a localizable measure algebra, and u any member of $M^{1,\infty}(\mathfrak{A},\bar\mu)$. Then $B=\{Tu:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$ is compact in $M^{1,\infty}(\mathfrak{B},\bar\nu)$ for the topology $\mathfrak{T}_s(M^{1,\infty}(\mathfrak{B},\bar\nu),M^{\infty,1}(\mathfrak{B},\bar\nu))$.

373N Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $(\mathfrak{B},\bar\nu)$ a localizable measure algebra and u any member of $M^{1,\infty}(\mathfrak{A},\bar\mu)$; set $B=\{Tu:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$. If $\langle v_n\rangle_{n\in\mathbb{N}}$ is any non-decreasing sequence in B, then $\sup_{n\in\mathbb{N}}v_n$ is defined in $M^{1,\infty}(\mathfrak{B},\bar\nu)$ and belongs to B.

373O Theorem Suppose that $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ are measure algebras, $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $v\in M^{1,\infty}(\mathfrak{B},\bar\nu)$. Then the following are equiveridical:

(i) there is a $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu=v$,

(ii) $\int_0^t v^*\le\int_0^t u^*$ for every $t\ge 0$.

In particular, given $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$, there are $S\in\mathcal{T}_{\bar\mu,\bar\mu_L}$, $T\in\mathcal{T}_{\bar\mu_L,\bar\mu}$ such that $Su=u^*$ and $Tu^*=u$.

373P Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a semi-finite measure algebra. Then for any $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $v\in M^0(\mathfrak{B},\bar\nu)$, there is a $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $\int Tu\times v=\int u^*\times v^*$.

373Q Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $(\mathfrak{B},\bar\nu)$ a semi-finite measure algebra, $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $v\in M^{0,\infty}(\mathfrak{B},\bar\nu)$. Then

$$\int u^*\times v^*=\sup\{\int|Tu\times v|:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}=\sup\{\int Tu\times v:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}.$$

373R Order-continuous operators: Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $(\mathfrak{B},\bar\nu)$ a localizable measure algebra, and $T_0\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$. Then there is a $T\in\mathcal{T}^\times=\mathcal{T}^\times_{\bar\mu,\bar\nu}$ extending T_0. If $(\mathfrak{A},\bar\mu)$ is semi-finite, T is uniquely defined.

373S Adjoints in $\mathcal{T}^{(0)}$: Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and T any member of $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$. Then there is a unique operator $T'\in\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ such that $\int_a T'(\chi b)=\int_b T(\chi a)$ whenever $a\in\mathfrak{A}^f$ and $b\in\mathfrak{B}^f$, and now $\int u\times T'v=\int Tu\times v$ whenever $u\in M^{1,0}(\mathfrak{A},\bar\mu)$, $v\in M^{1,0}(\mathfrak{B},\bar\nu)$ are such that $\int u^*\times v^*<\infty$.

373T Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras. Then for any $T\in\mathcal{T}^\times_{\bar\mu,\bar\nu}$ there is a unique $T'\in\mathcal{T}^\times_{\bar\nu,\bar\mu}$ such that $\int u\times T'v=\int Tu\times v$ whenever $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$, $v\in M^{1,\infty}(\mathfrak{B},\bar\nu)$ are such that $\int u^*\times v^*<\infty$.

373U Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ an order-continuous measure-preserving Boolean homomorphism. Then the associated map $T\in\mathcal{T}^\times_{\bar\mu,\bar\nu}$ has an adjoint $P\in\mathcal{T}^\times_{\bar\nu,\bar\mu}$ defined by the formula $\int_a P(\chi b)=\bar\nu(b\cap\pi a)$ for $a\in\mathfrak{A}^f$, $b\in\mathfrak{B}^f$.

Version of 15.6.09

374 Rearrangement-invariant spaces

As is to be expected, many of the most important function spaces of analysis are symmetric in various ways; in particular, they share the symmetries of the underlying measure algebras. The natural expression of this is to say that they are 'rearrangement-invariant' (374E). In fact it turns out that in many cases they have the stronger property of '$\mathcal{T}$-invariance' (374A). In this section I give a brief account of the most important properties of these two kinds of invariance. In particular, $\mathcal{T}$-invariance is related to a kind of transfer mechanism, enabling us to associate function spaces on different measure algebras (374C-374D). As for rearrangement-invariance, the salient fact is that on the most important measure algebras many rearrangement-invariant spaces are $\mathcal{T}$-invariant (374K, 374M).

374A $\mathcal{T}$-invariance: Definitions Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) I will say that a subset A of $M^{1,\infty}_{\bar\mu}$ is **$\mathcal{T}$-invariant** if $Tu\in A$ whenever $u\in A$ and $T\in\mathcal{T}=\mathcal{T}_{\bar\mu,\bar\mu}$.

(b) An extended Fatou norm τ on L^0 is **$\mathcal{T}$-invariant** if $\tau(Tu)\le\tau(u)$ whenever $u\in M^{1,\infty}_{\bar\mu}$ and $T\in\mathcal{T}$.

(c) I will write $(\mathfrak{A}_L,\bar\mu_L)$ for the measure algebra of Lebesgue measure on $[0,\infty[$, and $u^*\in M^{0,\infty}_{\bar\mu_L}$ for the decreasing rearrangement of any u belonging to any $M^{0,\infty}_{\bar\mu}$.

374B Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. Let L^τ be the Banach lattice defined from τ, and τ' the associate extended Fatou norm. Then
(i) $M^{\infty,1}_{\bar\mu}\subseteq L^\tau\subseteq M^{1,\infty}_{\bar\mu}$;
(ii) τ' is also $\mathcal{T}$-invariant, and $\int u^*\times v^*\le\tau(u)\tau'(v)$ for all u, $v\in M^{0,\infty}_{\bar\mu}$.

374C Theorem Let θ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$, and $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra.
(a) There is a $\mathcal{T}$-invariant extended Fatou norm τ on $L^0(\mathfrak{A})$ defined by setting

$$\begin{aligned}\tau(u)&=\theta(u^*)\text{ if }u\in M^{0,\infty}_{\bar\mu},\\&=\infty\text{ if }u\in L^0(\mathfrak{A})\setminus M^{0,\infty}_{\bar\mu}.\end{aligned}$$

(b) Writing θ', τ' for the associates of θ and τ, we now have

$$\begin{aligned}\tau'(v)&=\theta'(v^*)\text{ if }v\in M^{0,\infty}_{\bar\mu},\\&=\infty\text{ if }v\in L^0(\mathfrak{A})\setminus M^{0,\infty}_{\bar\mu}.\end{aligned}$$

(c) If θ is an order-continuous norm on the Banach lattice L^θ, then τ is an order-continuous norm on L^τ.

374D Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. Then there is a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{A}_L)$ such that $\tau(u)=\theta(u^*)$ for every $u\in M^{0,\infty}_{\bar\mu}$.

374E Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) I will say that a subset A of $L^0=L^0(\mathfrak{A})$ is **rearrangement-invariant** if $T_\pi u\in A$ whenever $u\in A$ and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving Boolean automorphism, writing $T_\pi:L^0\to L^0$ for the isomorphism corresponding to π.

(b) I will say that an extended Fatou norm τ on L^0 is **rearrangement-invariant** if $\tau(T_\pi u)=\tau(u)$ whenever $u\in L^0$ and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving automorphism.

374F Remarks If $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a sequentially order-continuous measure-preserving Boolean homomorphism, then $T_\pi\restriction M^{1,\infty}_{\bar\mu}$ belongs to $\mathcal{T}_{\bar\mu,\bar\mu}$. Accordingly, any $\mathcal{T}$-invariant extended Fatou norm τ on $L^0(\mathfrak{A})$ must be rearrangement-invariant. Similarly, any $\mathcal{T}$-invariant subset of $M^{1,\infty}_{\bar\mu}$ will be rearrangement-invariant.

374G Definition I say that a measure algebra $(\mathfrak{A},\bar\mu)$ is **quasi-homogeneous** if for any non-zero a, $b\in\mathfrak{A}$ there is a measure-preserving Boolean automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\pi a\cap b\ne 0$.

374H Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Then the following are equiveridical:
(i) $(\mathfrak{A},\bar\mu)$ is quasi-homogeneous;
(ii) *either* $\mathfrak{A}$ is purely atomic and every atom of $\mathfrak{A}$ has the same measure *or* there is a $\kappa\ge\omega$ such that the principal ideal $\mathfrak{A}_a$ is homogeneous, with Maharam type κ, for every $a\in\mathfrak{A}$ of non-zero finite measure.

374I Corollary Let $(\mathfrak{A},\bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra. Then

(a) whenever a, $b\in\mathfrak{A}$ have the same finite measure, the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_b$ are isomorphic as measure algebras;

(b) there is a subgroup Γ of the additive group $\mathbb{R}$ such that (α) $\bar{\mu}a\in\Gamma$ whenever $a\in\mathfrak{A}$ and $\bar{\mu}a<\infty$ (β) whenever $a\in\mathfrak{A}$, $\gamma\in\Gamma$ and $0\le\gamma\le\bar{\mu}a$ then there is a $c\subseteq a$ such that $\bar{\mu}c=\gamma$.

374J Lemma Let $(\mathfrak{A},\bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra and u, $v\in M^{0,\infty}_{\bar{\mu}}$. Let $\mathrm{Aut}_{\bar{\mu}}$ be the group of measure-preserving automorphisms of $\mathfrak{A}$. Then

$$\int u^*\times v^*=\sup_{\pi\in\mathrm{Aut}_{\bar{\mu}}}\int|u\times T_\pi v|,$$

where $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ is the isomorphism corresponding to π.

374K Theorem Let $(\mathfrak{A},\bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra, and τ a rearrangement -invariant extended Fatou norm on $L^0=L^0(\mathfrak{A})$. Then τ is $\mathcal{T}$-invariant.

374L Lemma Let $(\mathfrak{A},\bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra. Suppose that u, $v\in(M^{0,\infty}_{\bar{\mu}})^+$ are such that $\int u^*\times v^*=\infty$. Then there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $\int u\times T_\pi v=\infty$.

374M Proposition Let $(\mathfrak{A},\bar{\mu})$ be a quasi-homogeneous localizable measure algebra, and $U\subseteq L^0=L^0(\mathfrak{A})$ a solid linear subspace which, regarded as a Riesz space, is perfect. If U is rearrangement-invariant and $M^{\infty,1}_{\bar{\mu}}\subseteq U\subseteq M^{1,\infty}_{\bar{\mu}}$, then U is $\mathcal{T}$-invariant.

Version of 30.1.10

375 Kwapien's theorem

In §368 and the first part of §369 I examined maps from various types of Riesz space into L^0 spaces. There are equally striking results about maps out of L^0 spaces. I start with some relatively elementary facts about positive linear operators from L^0 spaces to Archimedean Riesz spaces in general (375A-375D), and then turn to a remarkable analysis, due essentially to S.Kwapien, of the positive linear operators from a general L^0 space to the L^0 space of a semi-finite measure algebra (375J), with a couple of simple corollaries.

375A Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and W an Archimedean Riesz space. If $T:L^0(\mathfrak{A})\to W$ is a positive linear operator, it is sequentially order-continuous.

375B Proposition Let $\mathfrak{A}$ be an atomless Dedekind σ-complete Boolean algebra. Then $L^0(\mathfrak{A})^\times=\{0\}$.

375C Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, W an Archimedean Riesz space, and $T:L^0(\mathfrak{A})\to W$ an order-continuous Riesz homomorphism. Then $V=T[L^0(\mathfrak{A})]$ is an order-closed Riesz subspace of W.

375D Corollary Let W be a Riesz space and V an order-dense Riesz subspace which is isomorphic to $L^0(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$. Then $V=W$.

375E Theorem Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, $(\mathfrak{B},\bar{\nu})$ any measure algebra, and $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ an order-continuous positive linear operator. Then T is continuous for the topologies of convergence in measure.

375F Definition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. I will say that a function $\phi:\mathfrak{A}\to\mathfrak{B}$ is a **σ-subhomomorphism** if

$\phi(a\cup a')=\phi(a)\cup\phi(a')$ for all a, $a'\in\mathfrak{A}$,

$\inf_{n\in\mathbb{N}}\phi(a_n)=0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

375G Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism.

(a) $\phi(0) = 0$, $\phi(a) \subseteq \phi(a')$ whenever $a \subseteq a'$, and $\phi(a) \setminus \phi(a') \subseteq \phi(a \setminus a')$ for every a, $a' \in \mathfrak{A}$.

(b) If $\bar{\mu}$, $\bar{\nu}$ are measures such that $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are totally finite measure algebras, then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\bar{\nu}\phi(a) \leq \epsilon$ whenever $\bar{\mu}a \leq \delta$.

375H Lemma Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism. Then for every non-zero $b_0 \in \mathfrak{B}$ there are a non-zero $b \subseteq b_0$ and an $m \in \mathbb{N}$ such that $b \cap \inf_{j \leq m} \phi(a_j) = 0$ whenever $a_0, \ldots , a_m \in \mathfrak{A}$ are disjoint.

375I Lemma Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism. Then for every non-zero $b_0 \in \mathfrak{B}$ there are a non-zero $b \subseteq b_0$ and a finite partition of unity $C \subseteq \mathfrak{A}$ such that $a \mapsto b \cap \phi(a \cap c)$ is a ring homomorphism for every $c \in C$.

375J Theorem Let $\mathfrak{A}$ be any Dedekind σ-complete Boolean algebra and $(\mathfrak{B}, \bar{\nu})$ a semi-finite measure algebra. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be a positive linear operator. Then we can find B, $\langle A_b \rangle_{b \in B}$ such that B is a partition of unity in $\mathfrak{B}$, each A_b is a finite partition of unity in $\mathfrak{A}$, and $u \mapsto T(u \times \chi a) \times \chi b$ is a Riesz homomorphism whenever $b \in B$ and $a \in A_b$.

375K Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and U a Dedekind complete Riesz space such that $U^\times$ separates the points of U. If $T : L^0(\mathfrak{A}) \to U$ is a positive linear operator, there is a sequence $\langle T_n \rangle_{n \in \mathbb{N}}$ of Riesz homomorphisms from $L^0(\mathfrak{A})$ to U such that $T = \sum_{n=0}^{\infty} T_n$, in the sense that $Tu = \sup_{n \in \mathbb{N}} \sum_{i=0}^{n} T_i u$ for every $u \geq 0$ in $L^0(\mathfrak{A})$.

375L Corollary (a) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, $(\mathfrak{B}, \bar{\nu})$ is a semi-finite measure algebra, and there is any non-zero positive linear operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$, then there is a non-trivial sequentially order-continuous ring homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(b) If $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are homogeneous probability algebras and $\tau(\mathfrak{A}) > \tau(\mathfrak{B})$, then $\mathsf{L}^\sim(L^0(\mathfrak{A}); L^0(\mathfrak{B})) = \{0\}$.

375Z Problem Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, and $L^0 = L^0(\mathfrak{G})$. If $T : L^0 \to L^0$ is a positive linear operator, must $T[L^0]$ be order-closed?

Version of 8.4.10

376 Kernel operators

The theory of linear integral equations is in large part the theory of operators T defined from formulae of the type

$$(Tf)(y) = \int k(x, y) f(x) dx$$

for some function k of two variables. I make no attempt to study the general theory here. However, the concepts developed in this book make it easy to discuss certain aspects of such operators defined between the 'function spaces' of measure theory, meaning spaces of equivalence classes of functions, and indeed allow us to do some of the work in the abstract theory of Riesz spaces, omitting all formal mention of measures (376D, 376H, 376P). I give a very brief account of two theorems characterizing kernel operators in the abstract (376E, 376H), with corollaries to show the form these theorems can take in the ordinary language of integral kernels (376J, 376N). To give an idea of the kind of results we can hope for in this area, I go a bit farther with operators with domain L^1 (376Mb, 376P, 376S).

I take the opportunity to spell out versions of results from §253 in the language of this volume (376B-376C).

376B The canonical map $L^0 \times L^0 \to L^0$: Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be semi-finite measure algebras, and $(\mathfrak{C}, \bar{\lambda})$ their localizable measure algebra free product. Then we have a bilinear operator $(u, v) \mapsto u \otimes v : L^0(\mathfrak{A}) \times L^0(\mathfrak{B}) \to L^0(\mathfrak{C})$ with the following properties.

(a) For any $u\in L^0(\mathfrak{A})$, $v\in L^0(\mathfrak{B})$ and $\alpha\in\mathbb{R}$,

$$[\![u\otimes\chi 1_{\mathfrak{B}}>\alpha]\!]=[\![u>\alpha]\!]\otimes 1_{\mathfrak{B}},\quad [\![\chi 1_{\mathfrak{A}}\otimes v>\alpha]\!]=1_{\mathfrak{A}}\otimes[\![v>\alpha]\!]$$

where for $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ I write $a\otimes b$ for the corresponding member of $\mathfrak{A}\otimes\mathfrak{B}$, identified with a subalgebra of $\mathfrak{C}$.

(b)(i) For any $u\in L^0(\mathfrak{A})^+$, the map $v\mapsto u\otimes v:L^0(\mathfrak{B})\to L^0(\mathfrak{C})$ is an order-continuous multiplicative Riesz homomorphism.

(ii) For any $v\in L^0(\mathfrak{B})^+$, the map $u\mapsto u\otimes v:L^0(\mathfrak{A})\to L^0(\mathfrak{C})$ is an order-continuous multiplicative Riesz homomorphism.

(c) In particular, $|u\otimes v|=|u|\otimes|v|$ for all $u\in L^0(\mathfrak{A})$ and $v\in L^0(\mathfrak{B})$.

(d) For any $u\in L^0(\mathfrak{A})^+$ and $v\in L^0(\mathfrak{B})^+$, $[\![u\otimes v>0]\!]=[\![u>0]\!]\otimes[\![v>0]\!]$.

376C Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$.

(a) If $u\in L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$ and $v\in L^1_{\bar\nu}=L^1(\mathfrak{B},\bar\nu)$ then $u\otimes v\in L^1_{\bar\lambda}=L^1(\mathfrak{C},\bar\lambda)$ and

$$\int u\otimes v=\int u\int v,\quad \|u\otimes v\|_1=\|u\|_1\|v\|_1.$$

(b) Let W be a Banach space and $\phi:L^1_{\bar\mu}\times L^1_{\bar\nu}\to W$ a bounded bilinear operator. Then there is a unique bounded linear operator $T:L^1_{\bar\lambda}\to W$ such that $T(u\otimes v)=\phi(u,v)$ for all $u\in L^1_{\bar\mu}$ and $v\in L^1_{\bar\nu}$, and $\|T\|=\|\phi\|$.

(c) Suppose, in (b), that W is a Banach lattice. Then

(i) T is positive iff $\phi(u,v)\ge 0$ for all u, $v\ge 0$;

(ii) T is a Riesz homomorphism iff $u\mapsto\phi(u,v_0):L^1_{\bar\mu}\to W$ and $v\mapsto\phi(u_0,v):L^1_{\bar\nu}\to W$ are Riesz homomorphisms for all $v_0\ge 0$ in $L^1_{\bar\nu}$ and $u_0\ge 0$ in $L^1_{\bar\mu}$.

376D Abstract integral operators: Definition Let U be a Riesz space and V a Dedekind complete Riesz space. If $f\in U^\times$ and $v\in V$ write $P_{fv}u=f(u)v$ for each $u\in U$; then $P_{fv}\in\mathsf{L}^\times(U;V)$. I call a linear operator from U to V an **abstract integral operator** if it is in the band in $\mathsf{L}^\times(U;V)$ generated by $\{P_{fv}:f\in U^\times,\,v\in V\}$.

376E Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$, and $U\subseteq L^0(\mathfrak{A})$, $V\subseteq L^0(\mathfrak{B})$ order-dense Riesz subspaces. Write W for the set of those $w\in L^0(\mathfrak{C})$ such that $w\times(u\otimes v)$ is integrable for every $u\in U$ and $v\in V$. Then we have an operator $w\mapsto T_w:W\to\mathsf{L}^\times(U;V^\times)$ defined by setting

$$T_w(u)(v)=\int w\times(u\otimes v)$$

for every $w\in W$, $u\in U$ and $v\in V$. The map $w\mapsto T_w$ is a Riesz space isomorphism between W and the band of abstract integral operators in $\mathsf{L}^\times(U;V^\times)$.

376F Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras, with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$. Let $U\subseteq L^0(\mathfrak{A})$, $V\subseteq L^0(\mathfrak{B})$ be perfect order-dense solid linear subspaces, and $T:U\to V$ a linear operator. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) there is a $w\in L^0(\mathfrak{C})$ such that $\int w\times(u\otimes v')$ is defined and equal to $\int Tu\times v'$ whenever $u\in U$ and $v'\in L^0(\mathfrak{B})$ is such that $v'\times v$ is integrable for every $v\in V$.

376G Lemma Let U be a Riesz space, V an Archimedean Riesz space, $T:U\to V$ a linear operator, $f\in(U^\sim)^+$ and $e\in V^+$. Suppose that $0\le Tu\le f(u)e$ for every $u\in U^+$. Then if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\lim_{n\to\infty}g(u_n)=0$ whenever $g\in U^\sim$ and $|g|\le f$, $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

376H Theorem Let U be a Riesz space and V a weakly (σ,∞)-distributive Dedekind complete Riesz space. Suppose that $T\in\mathsf{L}^\times(U;V)$. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ and $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V;

(iii) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U and $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

376I Lemma Let (X,Σ,μ) be a σ-finite measure space and U an order-dense solid linear subspace of $L^0(\mu)$. Then there is a non-decreasing sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of measurable subsets of X, with union X, such that $\chi X_n^\bullet \in U$ for every $n \in \mathbb{N}$.

376J Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with product measure λ on $X\times Y$. Let $U \subseteq L^0(\mu)$, $V \subseteq L^0(\nu)$ be perfect order-dense solid linear subspaces, and $T : U \to V$ a linear operator. Write $\mathcal{U} = \{f : f \in \mathcal{L}^0(\mu),\ f^\bullet \in U\}$, $\mathcal{V}^\# = \{h : h \in \mathcal{L}^0(\nu),\ h^\bullet \times v \in L^1$ for every $v \in V\}$. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) there is a $k \in \mathcal{L}^0(\lambda)$ such that

(α) $\int |k(x,y)f(x)h(y)|d(x,y) < \infty$ for every $f \in \mathcal{U}$, $h \in \mathcal{V}^\#$,

(β) if $f \in \mathcal{U}$ and we set $g(y) = \int k(x,y)f(x)dx$ wherever this is defined, then $g \in \mathcal{L}^0(\nu)$ and $Tf^\bullet = g^\bullet$;

(iii) $T \in \mathsf{L}^\sim(U;V)$ and whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ and $\lim_{n\to\infty} h(u_n) = 0$ for every $h \in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

376K Lemma Let U and V be Riesz spaces. Then there is a Riesz space isomorphism $T \mapsto T' : \mathsf{L}^\times(U;V^\times) \to \mathsf{L}^\times(V;U^\times)$ defined by the formula

$$(T'v)(u) = (Tu)(v) \text{ for every } u \in U,\ v \in V.$$

If we write $P_{fg}(u) = f(u)g$ for $f \in U^\times$, $g \in V^\times$ and $u \in U$, then $P_{fg} \in \mathsf{L}^\times(U;V^\times)$ and $P'_{fg} = P_{gf}$ in $\mathsf{L}^\times(V;U^\times)$. Consequently T is an abstract integral operator iff T' is.

376L Lemma Let U be a Banach lattice with an order-continuous norm. If $w \in U^+$ there is a $g \in (U^\times)^+$ such that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|u\| \le \epsilon$ whenever $0 \le u \le w$ and $g(u) \le \delta$.

376M Theorem (a) Let U be a Banach lattice with an order-continuous norm and V a Dedekind complete M-space. Then every bounded linear operator from U to V is an abstract integral operator.

(b) Let U be an L-space and V a Banach lattice with order-continuous norm. Then every bounded linear operator from U to $V^\times$ is an abstract integral operator.

376N Corollary: Dunford's theorem Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces and $T : L^1(\mu) \to L^p(\nu)$ a bounded linear operator, where $1 < p \le \infty$. Then there is a measurable function $k : X\times Y \to \mathbb{R}$ such that $Tf^\bullet = g_f^\bullet$, where $g_f(y) = \int k(x,y)f(x)dx$ almost everywhere, for every $f \in \mathcal{L}^1(\mu)$.

376O Lemma Let U be a Riesz space, and W a solid linear subspace of $U^\sim$. If $C \subseteq U$ is relatively compact for the weak topology $\mathfrak{T}_s(U,W)$, then for every $g \in W^+$ and $\epsilon > 0$ there is a $u^* \in U^+$ such that $g(|u| - u^*)^+ \le \epsilon$ for every $u \in C$.

376P Theorem Let U be an L-space and V a perfect Riesz space. If $T : U \to V$ is a linear operator such that $\{Tu : u \in U,\ \|u\| \le 1\}$ is relatively compact for the weak topology $\mathfrak{T}_s(V,V^\times)$, then T is an abstract integral operator.

376Q Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces and $T : L^1(\mu) \to L^1(\nu)$ a weakly compact linear operator. Then there is a function $k : X\times Y \to \mathbb{R}$ such that $Tf^\bullet = g_f^\bullet$, where $g_f(y) = \int k(x,y)f(x)dx$ almost everywhere, for every $f \in \mathcal{L}^1(\mu)$.

376R Lemma Let (X,Σ,μ) be a measure space, (Y,T,ν) a σ-finite measure space, and λ the c.l.d. product measure on $X\times Y$. Suppose that k is a λ-integrable real-valued function. Then for any $\epsilon > 0$ there is a finite partition $E_0,\dots,E_n$ of X into measurable sets such that $\|k - k_1\|_1 \le \epsilon$, where

$$k_1(x,y) = \frac{1}{\mu E_i}\int_{E_i} k(t,y)dt \text{ whenever } x \in E_i,\ 0 < \mu E_i < \infty$$

and the integral is defined in $\mathbb{R}$,

$= 0$ in all other cases.

376S Theorem Let (X,Σ,μ) be a complete locally determined measure space, (Y,T,ν) a σ-finite measure space, and λ the c.l.d. product measure on $X\times Y$. Let τ be an extended Fatou norm on $L^0(\nu)$ and write $\mathcal{L}^{\tau'}$ for $\{g : g\in\mathcal{L}^0(\nu),\ \tau'(g^\bullet)<\infty\}$, where τ' is the associate extended Fatou norm of τ. Suppose that $k\in\mathcal{L}^0(\lambda)$ is such that $k\times(f\otimes g)$ is integrable whenever $f\in\mathcal{L}^1(\mu)$ and $g\in\mathcal{L}^{\tau'}$. Then we have a corresponding linear operator $T : L^1(\mu)\to L^\tau$ defined by saying that $\int(Tf^\bullet)\times g^\bullet = \int k\times(f\otimes g)$ whenever $f\in\mathcal{L}^1(\mu)$ and $g\in\mathcal{L}^{\tau'}$.

For $x\in X$ set $k_x(y) = k(x,y)$ whenever this is defined. Then $k_x\in\mathcal{L}^0(\nu)$ for almost every x; set $v_x = k_x^\bullet\in L^0(\nu)$ for such x. In this case $x\mapsto\tau(v_x)$ is measurable and defined and finite almost everywhere, and $\|T\| = \text{ess sup}_x\,\tau(v_x)$.

Version of 30.12.09

*377 Function spaces of reduced products

In §328 I introduced 'reduced products' of probability algebras. In this section I seek to describe the function spaces of reduced products as images of subspaces of products of function spaces of the original algebras. I add a group of universal mapping theorems associated with the constructions of projective and inductive limits of directed families of probability algebras (377G-377H).

377A Proposition If $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a non-empty family of Boolean algebras with simple product $\mathfrak{A}$, then $L^\infty(\mathfrak{A})$ can be identified, as normed space and f-algebra, with the subspace W_∞ of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$ consisting of families $u=\langle u_i\rangle_{i\in I}$ such that $\|u\|_\infty = \sup_{i\in I}\|u_i\|_\infty$ is finite.

377B Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $(\mathfrak{B},\bar\nu)$ a probability algebra. Let $\mathfrak{A}$ be the simple product of $\langle\mathfrak{A}_i\rangle_{i\in I}$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})\le\sup_{i\in I}\bar\mu_i a_i$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Let W_0 be the subspace of $\prod_{i\in I}L^0(\mathfrak{A}_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\inf_{k\in\mathbb{N}}\sup_{i\in I}\bar\mu_i[\![|u_i|>k]\!]=0$.

(a) W_0 is a solid linear subspace and a subalgebra of $\prod_{i\in I}L^0(\mathfrak{A}_i)$, and there is a unique Riesz homomorphism $T:W_0\to L^0(\mathfrak{B})$ such that $T(\langle\chi a_i\rangle_{i\in I}) = \chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Moreover, T is multiplicative, and $[\![Tu>0]\!]\subseteq\pi(\langle[\![u_i>0]\!]\rangle_{i\in I})$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

(b) If $h:\mathbb{R}\to\mathbb{R}$ is a continuous function, and we write $\bar h$ for the corresponding maps from L^0 to itself for any of the spaces $L^0=L^0(\mathfrak{A}_i)$, $L^0=L^0(\mathfrak{B})$, then $\langle\bar h(u_i)\rangle_{i\in I}\in W_0$ and $T(\langle\bar h(u_i)\rangle_{i\in I}) = \bar h(Tu)$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

377C Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, $(\mathfrak{B},\bar\nu)$ a probability algebra, and $\pi:\prod_{i\in I}\mathfrak{A}_i\to\mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})\le\sup_{i\in I}\bar\mu_i a_i$ whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$. Let $W_0\subseteq\prod_{i\in I}L^0(\mathfrak{A}_i)$ and $T:W_0\to L^0(\mathfrak{B})$ be as in 377B. Suppose *either* that every $\mathfrak{A}_i$ is atomless *or* that there is an ultrafilter $\mathcal{F}$ on I such that $\bar\nu\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}}\bar\mu_i a_i$ whenever $\langle a_i\rangle_{i\in I}$ in $\prod_{i\in I}\mathfrak{A}_i$. For $1\le p\le\infty$ let W_p be the subspace of $\prod_{i\in I}L^0(\mathfrak{A}_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\sup_{i\in I}\|u_i\|_p$ is finite. Then $T[W_p]\subseteq L^p(\mathfrak{B},\bar\nu)$, and $\|Tu\|_p\le\sup_{i\in I}\|u_i\|_p$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_p.

377D Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, $\mathcal{F}$ an ultrafilter on I, and $(\mathfrak{B},\bar\nu)$ a probability algebra. Let $\mathfrak{A}$ be the simple product $\prod_{i\in I}\mathfrak{A}_i$ and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}}\bar\mu_i a_i$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Let $W_0\subseteq\prod_{i\in I}L^0(\mathfrak{A}_i)$ and $T:W_0\to L^0(\mathfrak{B})$ be as in 377B-377C.

(a) If $u=\langle u_i\rangle_{i\in I}$ belongs to W_0 and $\{i:i\in I,\ u_i=0\}\in\mathcal{F}$, then $Tu=0$.

(b) For $1\le p\le\infty$, write W_p for the set of those families $\langle u_i\rangle_{i\in I}\in\prod_{i\in I}L^p(\mathfrak{A}_i,\bar\mu_i)$ such that $\sup_{i\in I}\|u_i\|_p$ is finite. Then $Tu\in L^p(\mathfrak{B},\bar\nu)$ and $\|Tu\|_p\le\lim_{i\to\mathcal{F}}\|u_i\|_p$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_p.

(c) Let W_{ui} be the subspace of $\prod_{i\in I}L^1(\mathfrak{A}_i,\bar\mu_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i|-k\chi 1_{\mathfrak{A}_i})^+=0$. Then $\int Tu=\lim_{i\to\mathcal{F}}\int u_i$ and $\|Tu\|_1=\lim_{i\to\mathcal{F}}\|u_i\|_1$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_{ui}.

(d) Suppose now that $\pi[\mathfrak{A}]=\mathfrak{B}$.

(i) $T[W_0]=L^0(\mathfrak{B})$.

(ii) $T[W_{ui}]=L^1(\mathfrak{B},\bar\nu)$.

(iii) If $p\in[1,\infty]$, then $T[W_p]=L^p(\mathfrak{B},\bar\nu)$ and for every $w\in L^p(\mathfrak{B},\bar\nu)$ there is a $u=\langle u_i\rangle_{i\in I}$ in W_p such that $Tu=w$ and $\sup_{i\in I}\|u_i\|_p=\|w\|_p$.

377E Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be probability algebras, I a set and $\mathcal{F}$ an ultrafilter on I. Let $\pi:\mathfrak{A}^I\to\mathfrak{B}$ be a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu a_i$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}^I$. Let W_0 be the set of families in $L^0(\mathfrak{A})^I$ which are bounded for the topology of convergence in measure on $L^0(\mathfrak{A})$.

(a)(i) W_0 is a solid linear subspace and a subalgebra of $L^0(\mathfrak{A})^I$, and there is a unique multiplicative Riesz homomorphism $T:W_0\to L^0(\mathfrak{B})$ such that $T(\langle\chi a_i\rangle_{i\in I})=\chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}^I$.

(ii) $[\![Tu>0]\!]\subseteq\pi(\langle[\![u_i>0]\!]\rangle_{i\in I})$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

(iii) If $h:\mathbb{R}\to\mathbb{R}$ is a continuous function, and we write $\bar h$ for the corresponding maps from L^0 to itself for either of the spaces $L^0=L^0(\mathfrak{A})$, $L^0=L^0(\mathfrak{B})$, then $\langle\bar h(u_i)\rangle_{i\in I}\in W_0$ and $T(\langle\bar h(u_i)\rangle_{i\in I})=\bar h(Tu)$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

(b)(i) For $1\le p\le\infty$ let W_p be the subspace of $L^p(\mathfrak{A},\bar\mu)^I$ consisting of $\|\ \|_p$-bounded families. Then $T[W_p]\subseteq L^p(\mathfrak{B},\bar\nu)$, and $\|Tu\|_p\le\lim_{i\to\mathcal{F}}\|u_i\|_p$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_p.

(ii) Let W_{ui} be the subspace of $L^1(\mathfrak{A}_i,\bar\mu_i)^I$ consisting of uniformly integrable families. Then $\int Tu=\lim_{i\to\mathcal{F}}\int u_i$ and $\|Tu\|_1=\lim_{i\to\mathcal{F}}\|u_i\|_1$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_{ui}.

(c)(i) We have a measure-preserving Boolean homomorphism $\tilde\pi:\mathfrak{A}\to\mathfrak{B}$ defined by setting $\tilde\pi a=\pi(\langle a\rangle_{i\in I})$ for each $a\in\mathfrak{A}$.

(ii) Let $P_{\tilde\pi}:L^1(\mathfrak{B},\bar\nu)\to L^1(\mathfrak{A},\bar\mu)$ be the conditional-expectation operator corresponding to $\tilde\pi:\mathfrak{A}\to\mathfrak{B}$. If $\langle u_i\rangle_{i\in I}$ is a uniformly integrable family in $L^1(\mathfrak{A})$, then $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})$ is the limit $\lim_{i\to\mathcal{F}}u_i$ for the weak topology of $L^1(\mathfrak{A},\bar\mu)$.

(iii) Suppose that $1<p<\infty$ and that $\langle u_i\rangle_{i\in I}$ is a bounded family in $L^p(\mathfrak{A},\bar\mu)$. Then $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})$ is the limit $\lim_{i\to\mathcal{F}}u_i$ for the weak topology of $L^p(\mathfrak{A},\bar\mu)$.

377F Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A}',\bar\mu')$ be probability algebras, I a set and $\mathcal{F}$ an ultrafilter on I; let $(\mathfrak{B},\bar\nu)$ and $(\mathfrak{B}',\bar\nu')$ be the reduced powers $(\mathfrak{A},\bar\mu)^I|\mathcal{F}$, $(\mathfrak{A}',\bar\mu')^I|\mathcal{F}$, with corresponding homomorphisms $\pi:\mathfrak{A}^I\to\mathfrak{B}$ and $\pi':\mathfrak{A}'^I\to\mathfrak{B}'$.

(a) Writing W_0, W_0' for the spaces of topologically bounded families in $L^0(\mathfrak{A})^I$, $L^0(\mathfrak{A}')^I$ respectively, we have unique Riesz homomorphisms $T:W_0\to L^0(\mathfrak{B})$ and $T':W_0'\to L^0(\mathfrak{B}')$ such that $T(\langle\chi a_i\rangle_{i\in I})=\chi\pi(\langle a_i\rangle_{i\in I})$, $T'(\langle\chi a_i'\rangle_{i\in I})=\chi\pi'(\langle a_i'\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}^I$ and $\langle a_i'\rangle_{i\in I}\in(\mathfrak{A}')^I$.

(b) Suppose that $S:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{A}',\bar\mu')$ is a bounded linear operator. Then we have a unique bounded linear operator $\hat S:L^1(\mathfrak{B},\bar\nu)\to L^1(\mathfrak{B}',\bar\nu')$ such that $\hat ST(\langle u_i\rangle_{i\in I})=T'(\langle Su_i\rangle_{i\in I})$ whenever $\langle u_i\rangle_{i\in I}$ is a uniformly integrable family in $L^1(\mathfrak{A},\bar\mu)$.

(c) The map $S\mapsto\hat S$ is a norm-preserving Riesz homomorphism from $\mathsf{B}(L^1(\mathfrak{A},\bar\mu);L^1(\mathfrak{A}',\bar\mu'))$ to $\mathsf{B}(L^1(\mathfrak{B},\bar\nu);L^1(\mathfrak{B}',\bar\nu'))$.

377G Projective limits: Proposition Let $(I,\le)$, $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ and $\langle\pi_{ij}\rangle_{i\le j}$ be such that $(I,\le)$ is a non-empty upwards-directed partially ordered set, every $(\mathfrak{A}_i,\bar\mu_i)$ is a probability algebra, $\pi_{ij}:\mathfrak{A}_j\to\mathfrak{A}_i$ is a measure-preserving Boolean homomorphism whenever $i\le j$ in I, and $\pi_{ik}=\pi_{ij}\pi_{jk}$ whenever $i\le j\le k$. Let $(\mathfrak{C},\bar\lambda,\langle\pi_i\rangle_{i\in I})$ be the corresponding projective limit. Write $L^1_{\bar\mu_i}$ for $L^1(\mathfrak{A}_i,\bar\mu_i)$ and $L^1_{\bar\lambda}$ for $L^1(\mathfrak{C},\bar\lambda)$. For $i\le j$ in I, let $T_{ij}:L^1_{\bar\mu_j}\to L^1_{\bar\mu_i}$ and $P_{ij}:L^1_{\bar\mu_i}\to L^1_{\bar\mu_j}$ be the norm-preserving Riesz homomorphism and the positive linear operator corresponding to $\pi_{ij}:\mathfrak{A}_j\to\mathfrak{A}_i$, and $T_i:L^1_{\bar\lambda}\to L^1_{\bar\mu_i}$, $P_i:L^1_{\bar\mu_i}\to L^1_{\bar\lambda}$ the operators corresponding to $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$. Let X be any set.

(a) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_i T_{ij} = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar\lambda} \to X$ such that $S = S_i T_i$ for every $i \in I$.

(b) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $T_{ij} S_j = S_i$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $T_i S = S_i$ for every $i \in I$.

(c) Suppose that X is a topological space, and for each $i \in I$ we are given a norm-continuous function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_j P_{ij} = S_i$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar\lambda} \to X$ such that $SP_i = S_i$ for every $i \in I$.

(d) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $P_{ij} S_i = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $S = P_i S_i$ for every $i \in I$.

377H Inductive limits: Proposition Let $(I, \le)$, $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ and $\langle\pi_{ji}\rangle_{i\le j}$ be such that $(I, \le)$ is a non-empty upwards-directed partially ordered set, every $(\mathfrak{A}_i, \bar\mu_i)$ is a probability algebra, $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$ is a measure-preserving Boolean homomorphism whenever $i \le j$ in I, and $\pi_{ki} = \pi_{kj}\pi_{ji}$ whenever $i \le j \le k$. Let $(\mathfrak{C}, \bar\lambda, \langle\pi_i\rangle_{i\in I})$ be the corresponding inductive limit. Write $L^1_{\bar\mu_i}$ for $L^1(\mathfrak{A}_i, \bar\mu_i)$ and $L^1_{\bar\lambda}$ for $L^1(\mathfrak{C}, \bar\lambda)$. For $i \le j$ in I, let $T_{ji} : L^1_{\bar\mu_i} \to L^1_{\bar\mu_j}$ and $P_{ji} : L^1_{\bar\mu_j} \to L^1_{\bar\mu_i}$ be the Riesz homomorphism and the positive linear operator corresponding to $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$, and $T_i : L^1_{\bar\mu_i} \to L^1_{\bar\lambda}$, $P_i : L^1_{\bar\lambda} \to L^1_{\bar\mu_i}$ the operators corresponding to $\pi_i : \mathfrak{A}_i \to \mathfrak{C}$. Let X be a set.

(a) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_j T_{ji} = S_i$ whenever $i \le j$ in I. Then there is a function $S : L^1_{\bar\lambda} \to X$ such that $S_i = ST_i$ for every $i \in I$.

(b) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $T_{ji} S_i = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $T_i S_i = S$ for every $i \in I$.

(c) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_i P_{ji} = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar\lambda} \to X$ such that $S = S_i P_i$ for every $i \in I$.

(d) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $P_{ji} S_j = S_i$ whenever $i \le j$ in I, and that

$$\inf_{k\in\mathbb{N}} \sup_{i\in I} \int (|S_i x| - k\chi 1_{\mathfrak{A}_i})^+ = 0$$

for every $x \in X$. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $S_i = P_i S$ for every $i \in I$.

Version of 7.12.08

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

372I The version of the Ergodic Theorem in 372I, referred to in the 2003 and 2006 editions of Volume 4, is now 372H.

372K The version of the Ergodic Theorem in 372K, referred to in the 2003 and 2006 editions of Volume 4, is now 372J.

372P Mixing and ergodic transformations The definitions in 372P are now in 372O.

372Xm The tent map, referred to in the 2003 and 2006 editions of Volume 4, is now in 372Xp.