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Chapter 37

Linear operators between function spaces

As everywhere in functional analysis, the function spaces of measure theory cannot be properly understood without investigating linear operators between them. In this chapter I have collected a number of results which rely on, or illuminate, the measure-theoretic aspects of the theory. §371 is devoted to a fundamental property of linear operators on L-spaces, if considered abstractly, that is, of L^1-spaces, if considered in the languages of Chapters 24 and 36, and to an introduction to the class $\mathcal{T}$ of operators which are norm-decreasing for both $\| \|_1$ and $\| \|_\infty$. This makes it possible to prove a version of Birkhoff's Ergodic Theorem for operators which need not be positive (372D). In §372 I give various forms of this theorem, for linear operators between function spaces, for measure-preserving Boolean homomorphisms between measure algebras, and for inverse-measure-preserving functions between measure spaces, with an excursion into the theory of continued fractions. In §373 I make a fuller analysis of the class $\mathcal{T}$, with a complete characterization of those u, v such that $v = Tu$ for some $T \in \mathcal{T}$. Using this we can describe 'rearrangement-invariant' function spaces and extended Fatou norms (§374). Returning to ideas left on one side in §§364 and 368, I investigate positive linear operators defined on L^0 spaces (§375). In the penultimate section of the chapter (§376), I look at operators which can be defined in terms of kernels on product spaces. Finally, in §377, I examine the function spaces of reduced products, projective limits and inductive limits of probability algebras.

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371 The Chacon-Krengel theorem

The first topic I wish to treat is a remarkable property of L-spaces: if U and V are L-spaces, then every continuous linear operator $T : U \to V$ is order-bounded, and $\||T|\| = \|T\|$ (371D). This generalizes in various ways to other V (371B, 371C). I apply the result to a special type of operator between $M^{1,0}$ spaces which will be conspicuous in the next section (371F-371H).

371A Lemma Let U be an L-space, V a Banach lattice and $T : U \to V$ a bounded linear operator. Take $u \ge 0$ in U and set

$$B = \{\sum_{i=0}^n |Tu_i| : u_0, \dots , u_n \in U^+, \sum_{i=0}^n u_i = u\} \subseteq V^+.$$

Then B is upwards-directed and $\sup_{v\in B} \|v\| \le \|T\|\|u\|$.

proof (a) Suppose that v, $v' \in B$. Then we have $u_0, \dots , u_m, u'_0, \dots , u'_n \in U^+$ such that $\sum_{i=0}^m u_i = \sum_{j=0}^n u'_j = u$, $v = \sum_{i=0}^m |Tu_i|$ and $v' = \sum_{j=0}^n |Tu'_j|$. Now there are $v_{ij} \ge 0$ in U, for $i \le m$ and $j \le n$, such that $u_i = \sum_{j=0}^n v_{ij}$ for $i \le m$ and $u'_j = \sum_{i=0}^m v_{ij}$ for $j \le n$ (352Fd). We have $u = \sum_{i=0}^m \sum_{j=0}^n v_{ij}$, so that $v'' = \sum_{i=0}^m \sum_{j=0}^n |Tv_{ij}| \in B$. But

$$v = \sum_{i=0}^m |Tu_i| = \sum_{i=0}^m |T(\sum_{j=0}^n v_{ij})| \le \sum_{i=0}^m \sum_{j=0}^m |Tv_{ij}| = v'',$$

and similarly $v' \le v''$. As v and v' are arbitrary, B is upwards-directed.

(b) The other part is easy. If $v \in B$ is expressed as $\sum_{i=0}^n |Tu_i|$ where $u_i \ge 0$ for every i and $\sum_{i=0}^n u_i = u$, then

$$\|v\| \le \sum_{i=0}^n \|Tu_i\| \le \|T\|\sum_{i=0}^n \|u_i\| = \|T\|\|u\|$$

because U is an L-space.

371B Theorem Let U be an L-space and V a Dedekind complete Banach lattice U with a Fatou norm. Then the Riesz space $\mathsf{L}^{\sim}(U;V)=\mathsf{L}^{\times}(U;V)$ is a closed linear subspace of the Banach space $\mathsf{B}(U;V)$ and is in itself a Banach lattice with a Fatou norm.

proof (a) I start by noting that $\mathsf{L}^{\sim}(U;V)=\mathsf{L}^{\times}(U;V)\subseteq\mathsf{B}(U;V)$ just because V has a Riesz norm and U is a Banach lattice with an order-continuous norm (355Kb, 355C).

(b) The first new step is to check that $\||T|\|\le\|T\|$ for any $T\in\mathsf{L}^{\sim}(U;V)$. **P** Start with any $u\in U^+$. Set

$$B=\{\sum_{i=0}^n|Tu_i|:u_0,\ldots,u_n\in U^+,\,\sum_{i=0}^n u_i=u\}\subseteq V^+,$$

as in 371A. If $u_0,\ldots,u_n\ge 0$ are such that $\sum_{i=0}^n u_i=u$, then $|Tu_i|\le|T|u_i$ for each i, so that $\sum_{i=0}^n|Tu_i|\le\sum_{i=0}^n|T|u_i=|T|u$; thus B is bounded above by $|T|u$ and $\sup B\le|T|u$. On the other hand, if $|v|\le u$ in U, then $v^++v^-+(u-|v|)=u$, so $|Tv^+|+|Tv^-|+|T(u-|v|)|\in B$ and

$$|Tv|=|Tv^++Tv^-|\le|Tv^+|+|Tv^-|\le\sup B.$$

As v is arbitrary, $|T|u\le\sup B$ and $|T|u=\sup B$. Consequently

$$\||T|u\|\le\|\sup B\|=\sup_{w\in B}\|w\|\le\|T\|\|u\|$$

because V has a Fatou norm and B is upwards-directed.

For general $u\in U$,

$$\||T|u\|\le\||T||u|\|\le\|T\|\||u|\|=\|T\|\|u\|.$$

This shows that $\||T|\|\le\|T\|$. **Q**

(c) Now if $|S|\le|T|$ in $\mathsf{L}^{\sim}(U;V)$, and $u\in U$, we must have

$$\|Su\|\le\||S||u|\|\le\||T||u|\|\le\||T|\|\||u|\|\le\|T\|\|u\|;$$

as u is arbitrary, $\|S\|\le\|T\|$. This shows that the norm of $\mathsf{L}^{\sim}(U;V)$, inherited from $\mathsf{B}(U;V)$, is a Riesz norm.

(d) Suppose next that $T\in\mathsf{B}(U;V)$ belongs to the norm-closure of $\mathsf{L}^{\sim}(U;V)$. For each $n\in\mathbb{N}$ choose $T_n\in\mathsf{L}^{\sim}(U;V)$ such that $\|T-T_n\|\le 2^{-n}$. Set $S_n=|T_{n+1}-T_n|\in\mathsf{L}^{\sim}(U;V)$ for each n. Then

$$\|S_n\|=\|T_{n+1}-T_n\|\le 3\cdot 2^{-n-1}$$

for each n, so $S=\sum_{n=0}^\infty S_n$ is defined in the Banach space $\mathsf{B}(U;V)$. But if $u\in U^+$, we surely have

$$Su=\sum_{n=0}^\infty S_nu\ge 0$$

in V. Moreover, if $u\in U^+$ and $|v|\le u$, then for any $n\in\mathbb{N}$

$$|T_{n+1}v-T_0v|=|\sum_{i=0}^n(T_{i+1}-T_i)v|\le\sum_{i=0}^n S_iu\le Su,$$

and $T_0v-Su\le T_{n+1}v\le T_0v+Su$; letting $n\to\infty$, we see that

$$-|T_0|u-Su\le T_0v-Su\le Tv\le T_0v+Su\le|T_0|u+Su.$$

So $|Tv|\le|T_0|u+Su$ whenever $|v|\le u$. As u is arbitrary, $T\in\mathsf{L}^{\sim}(U;V)$.

This shows that $\mathsf{L}^{\sim}(U;V)$ is closed in $\mathsf{B}(U;V)$ and is therefore a Banach space in its own right; putting this together with (b), we see that it is a Banach lattice.

(e) Finally, the norm of $\mathsf{L}^{\sim}(U;V)$ is a Fatou norm. **P** Let $A\subseteq\mathsf{L}^{\sim}(U;V)^+$ be a non-empty, upwards-directed set with supremum $T_0\in\mathsf{L}^{\sim}(U;V)$. For any $u\in U$,

$$\|T_0u\|=\||T_0u|\|\le\|T_0|u|\|=\|\sup_{T\in A}T|u|\|$$

by 355Ed. But $\{T|u|:T\in A\}$ is upwards-directed and the norm of V is a Fatou norm, so

$$\|T_0u\|\le\sup_{T\in A}\|T|u|\|\le\sup_{T\in A}\|T\|\|u\|.$$

As u is arbitrary, $\|T_0\|\le\sup_{T\in A}\|T\|$. As A is arbitrary, the norm of $\mathsf{L}^{\sim}(U;V)$ is Fatou. **Q**

371C Theorem Let U be an L-space and V a Dedekind complete Banach lattice with a Fatou norm and the Levi property. Then $\mathsf{B}(U;V)=\mathsf{L}^{\sim}(U;V)=\mathsf{L}^{\times}(U;V)$ is a Dedekind complete Banach lattice with a Fatou norm and the Levi property. In particular, $|T|$ is defined and $\||T|\|=\|T\|$ for every $T\in\mathsf{B}(U;V)$.

proof (a) Let $T:U\to V$ be any bounded linear operator. Then $T\in\mathsf{L}^{\sim}(U;V)$. **P** Take any $u\ge 0$ in U. Set

$$B=\{\sum_{i=0}^{n}|Tu_i|:u_0,\dots,u_n\in U^+,\ \sum_{i=0}^{n}u_i=u\}\subseteq V^+$$

as in 371A. Then 371A tells us that B is upwards-directed and norm-bounded. Because V has the Levi property, B is bounded above. But just as in part (b) of the proof of 371B, any upper bound of B is also an upper bound of $\{Tv:|v|\le u\}$. As u is arbitrary, $T\in\mathsf{L}^{\sim}(U;V)$. **Q**

(b) Accordingly $\mathsf{L}^{\sim}(U;V)=\mathsf{B}(U;V)$. By 371B, this is a Banach lattice with a Fatou norm, and equal to $\mathsf{L}^{\times}(U;V)$. To see that it also has the Levi property, let $A\subseteq\mathsf{L}^{\sim}(U;V)$ be any non-empty norm-bounded upwards-directed set. For $u\in U^+$, $\{Tu:T\in A\}$ is non-empty, norm-bounded and upwards-directed in V, so is bounded above in V. By 355Ed, A is bounded above in $\mathsf{L}^{\sim}(U;V)$.

371D Corollary Let U and V be L-spaces. Then $\mathsf{L}^{\sim}(U;V)=\mathsf{L}^{\times}(U;V)=\mathsf{B}(U;V)$ is a Dedekind complete Banach lattice with a Fatou norm and the Levi property.

371E Remarks Note that both these theorems show that $\mathsf{L}^{\sim}(U;V)$ is a Banach lattice with properties similar to those of V whenever U is an L-space. They can therefore be applied repeatedly, to give facts about $\mathsf{L}^{\sim}(U_1;\mathsf{L}^{\sim}(U_2;V))$ where U_1, U_2 are L-spaces and V is a Banach lattice, for instance. I hope that this formula will recall some of those in the theory of bilinear operators and tensor products (see 253Xa-253Xb).

371F The class $\mathcal{T}^{(0)}$ For the sake of applications in the next section, I introduce now a class of operators of great intrinsic interest.

Definition Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be measure algebras. Recall that $M^{1,0}(\mathfrak{A},\bar\mu)$ is the space of those $u\in L^1(\mathfrak{A},\bar\mu)+L^\infty(\mathfrak{A})$ such that $\bar\mu[\![|u|>\alpha]\!]<\infty$ for every $\alpha>0$ (366F-366G, 369P). Write $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ for the set of all linear operators $T:M^{1,0}(\mathfrak{A},\bar\mu)\to M^{1,0}(\mathfrak{B},\bar\nu)$ such that $Tu\in L^1(\mathfrak{B},\bar\nu)$ and $\|Tu\|_1\le\|u\|_1$ for every $u\in L^1(\mathfrak{A},\bar\mu)$, $Tu\in L^\infty(\mathfrak{B})$ and $\|Tu\|_\infty\le\|u\|_\infty$ for every $u\in L^\infty(\mathfrak{A})\cap M^{1,0}(\mathfrak{A},\bar\mu)$.

371G Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras.

(a) $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ is a convex set in the unit ball of $\mathsf{B}(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu))$. If $T_0:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{B},\bar\nu)$ is a linear operator of norm at most 1, and $T_0u\in L^\infty(\mathfrak{B})$ and $\|T_0u\|_\infty\le\|u\|_\infty$ for every $u\in L^1(\mathfrak{A},\bar\mu)\cap L^\infty(\mathfrak{A})$, then T_0 has a unique extension to a member of $\mathcal{T}^{(0)}$.

(b) If $T\in\mathcal{T}^{(0)}$ then T is order-bounded and $|T|$, taken in

$$\mathsf{L}^{\sim}(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu))=\mathsf{L}^{\times}(M^{1,0}(\mathfrak{A},\bar\mu);M^{1,0}(\mathfrak{B},\bar\nu)),$$

also belongs to $\mathcal{T}^{(0)}$.

(c) If $T\in\mathcal{T}^{(0)}$ then $\|Tu\|_{1,\infty}\le\|u\|_{1,\infty}$ for every $u\in M^{1,0}(\mathfrak{A},\bar\mu)$.

(d) If $T\in\mathcal{T}^{(0)}$, $p\in[1,\infty[$ and $w\in L^p(\mathfrak{A},\bar\mu)$ then $Tw\in L^p(\mathfrak{B},\bar\nu)$ and $\|Tw\|_p\le\|w\|_p$.

(e) If $(\mathfrak{C},\bar\lambda)$ is another measure algebra then $ST\in\mathcal{T}^{(0)}_{\bar\mu,\bar\lambda}$ whenever $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ and $S\in\mathcal{T}^{(0)}_{\bar\nu,\bar\lambda}$.

proof I write $M^{1,0}_{\bar\mu}$, $L^p_{\bar\nu}$ for $M^{1,0}_{\bar\mu}$, $L^p(\mathfrak{B},\bar\nu)$, etc.

(a)(i) If $T\in\mathcal{T}^{(0)}$ and $u\in M^{1,0}_{\bar\mu}$ then there are $v\in L^1_{\bar\mu}$, $w\in L^\infty_{\bar\mu}$ such that $u=v+w$ and $\|v\|_1+\|w\|_\infty=\|u\|_{1,\infty}$ (369Ob); so that

$$\|Tu\|_{1,\infty}\le\|Tv\|_1+\|Tw\|_\infty\le\|v\|_1+\|w\|_\infty\le\|u\|_{1,\infty}.$$

As u is arbitrary, T is in the unit ball of $\mathsf{B}(M^{1,0}_{\bar\mu};M^{1,0}_{\bar\nu})$.

(ii) Because the unit balls of $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})$ and $\mathsf{B}(L^\infty_{\bar\mu};L^\infty_{\bar\nu})$ are convex, so is $\mathcal{T}^{(0)}$.

(iii) Now suppose that $T_0:L^1_{\bar\mu}\to L^1_{\bar\nu}$ is a linear operator of norm at most 1 such that $\|T_0u\|_\infty\le\|u\|_\infty$ for every $u\in L^1_{\bar\mu}\cap L^\infty_{\bar\mu}$. By the argument of (i), T_0 is a bounded operator for the $\|\ \|_{1,\infty}$ norms; since $L^1_{\bar\mu}$ is dense in $M^{1,0}_{\bar\mu}$ (369Pc), T_0 has a unique extension to a bounded linear operator $T : M^{1,0}_{\bar\mu} \to M^{1,0}_{\bar\nu}$. Of course $\|Tu\|_1 = \|T_0u\|_1 \le \|u\|_1$ for every $u \in L^1_{\bar\mu}$.

Now suppose that $u \in L^\infty_{\bar\mu} \cap M^{1,0}_{\bar\mu}$; set $\gamma = \|u\|_\infty$. Let $\epsilon > 0$, and set

$$v = (u^+ - \epsilon\chi1)^+ - (u^- - \epsilon\chi1)^+;$$

then $|v| \le |u|$ and $\|u - v\|_\infty \le \epsilon$ and $v \in L^1_{\bar\mu} \cap L^\infty_{\bar\mu}$. Accordingly

$$\|Tu - Tv\|_{1,\infty} \le \|u - v\|_{1,\infty} \le \epsilon, \quad \|Tv\|_\infty = \|T_0v\|_\infty \le \|v\|_\infty \le \gamma.$$

So if we set $w = (|Tu - Tv| - \epsilon\chi1)^+ \in L^1_{\bar\nu}$, $\|w\|_1 \le \epsilon$; while

$$|Tu| \le |Tv| + w + \epsilon\chi1 \le (\gamma + \epsilon)\chi1 + w,$$

so

$$\|(|Tu| - (\gamma + \epsilon)\chi1)^+\|_1 \le \|w\|_1 \le \epsilon.$$

As ϵ is arbitrary, $|Tu| \le \gamma\chi1$, that is, $\|Tu\|_\infty \le \|u\|_\infty$. As u is arbitrary, $T \in \mathcal{T}^{(0)}$.

(b) Because $M^{1,0}_{\bar\mu}$ has an order-continuous norm (369Pb), $\mathsf{L}^\sim(M^{1,0}_{\bar\mu}; M^{1,0}_{\bar\nu}) = \mathsf{L}^\times(M^{1,0}_{\bar\mu}; M^{1,0}_{\bar\nu})$ (355Kb). Take any $T \in \mathcal{T}^{(0)}$ and consider $T_0 = T\restriction L^1_{\bar\mu} : L^1_{\bar\mu} \to L^1_{\bar\nu}$. This is an operator of norm at most 1. By 371D, T_0 is order-bounded, and $\||T_0|\| \le 1$, where $|T_0|$ is taken in $\mathsf{L}^\sim(L^1_{\bar\mu}; L^1_{\bar\mu}) = \mathsf{B}(L^1_{\bar\mu}; L^1_{\bar\nu})$. Now if $u \in L^1_{\bar\mu} \cap L^\infty_{\bar\nu}$,

$$||T_0|u| \le |T_0||u| = \sup_{|u'| \le |u|} |T_0u'| \le \|u\|_\infty\chi1,$$

so $\||T_0|u\|_\infty \le \|u\|_\infty$. By (a), there is a unique $S \in \mathcal{T}^{(0)}$ extending $|T_0|$. Now $Su^+ \ge 0$ for every $u \in L^1_{\bar\mu}$, so $Su^+ \ge 0$ for every $u \in M^{1,0}_{\bar\mu}$ (since the function $u \mapsto (Su^+)^+ - Su^+ : M^{1,0}_{\bar\mu} \to M^{1,0}_{\bar\nu}$ is continuous and zero on the dense set $L^1_{\bar\mu}$), that is, S is a positive operator; also $S|u| \ge |Tu|$ for every $u \in L^1_{\bar\mu}$, so $Sv \ge S|u| \ge |Tu|$ whenever u, $v \in M^{1,0}_{\bar\mu}$ and $|u| \le v$. This means that $T : M^{1,0}_{\bar\mu} \to M^{1,0}_{\bar\nu}$ is order-bounded. Because $M^{1,0}_{\bar\nu}$ is Dedekind complete (366Ga), $|T|$ is defined in $\mathsf{L}^\sim(M^{1,0}_{\bar\mu}; M^{1,0}_{\bar\mu})$.

If $v \ge 0$ in $L^1_{\bar\mu}$, then

$$|T|v = \sup_{|u| \le v} Tu = \sup_{|u| \le v} T_0u = |T_0|v = Sv.$$

Thus $|T|$ agrees with S on $L^1_{\bar\mu}$. Because $M^{1,0}_{\bar\mu}$ is a Banach lattice (or otherwise), $|T|$ is a bounded operator, therefore continuous (2A4Fc), so $|T| = S \in \mathcal{T}^{(0)}$, which is what we needed to know.

(c) We can express u as $v + w$ where $\|v\|_1 + \|w\|_\infty = \|u\|_{1,\infty}$; now $w = u - v \in M^{1,0}_{\bar\mu}$, so we can speak of Tw, and

$$\|Tu\|_{1,\infty} = \|Tv + Tw\|_{1,\infty} \le \|Tv\|_1 + \|Tw\|_\infty \le \|v\|_1 + \|w\|_\infty = \|u\|_{1,\infty},$$

as required.

(d) This can be thought of as a generalization of 244M. We need to revisit the proof of Jensen's inequality in 233H-233J.

(i) Suppose that T, p, w are as described, and that in addition T is positive. As in the proof of 244M, the function $t \mapsto |t|^p$ is convex, so we can find families $\langle\beta_q\rangle_{q\in\mathbb{Q}}$, $\langle\gamma_q\rangle_{q\in\mathbb{Q}}$ of real numbers such that $|t|^p = \sup_{q\in\mathbb{Q}} \beta_q + \gamma_q(t - q)$ for every $t \in \mathbb{R}$ (233Hb). Then $|u|^p = \sup_{q\in\mathbb{Q}} \beta_q\chi1 + \gamma_q(u - q\chi1)$ for every $u \in L^0$. (The easiest way to check this is perhaps to think of L^0 as a quotient of a space of functions, as in 364C; it is also a consequence of 364Xg(iii).) We know that $|w|^p \in L^1_{\bar\mu}$, so we may speak of $T(|w|^p)$; while $w \in M^{1,0}_{\bar\mu}$ (366Ga), so we may speak of Tw.

For any $q \in \mathbb{Q}$, $0^p \ge \beta_q - q\gamma_q$, that is, $q\gamma_q - \beta_q \ge 0$, while $\gamma_qw - |w|^p \le (q\gamma_q - \beta_q)\chi1$ and $\|(\gamma_qw - |w|^p)^+\|_\infty \le q\gamma_q - \beta_q$. Now this means that

$$\begin{aligned} T(\gamma_qw - |w|^p) &\le T(\gamma_qw - |w|^p)^+ \le \|T(\gamma_qw - |w|^p)^+\|_\infty\chi1 \\ &\le \|(\gamma_qw - |w|^p)^+\|_\infty\chi1 \le (q\gamma_q - \beta_q)\chi1. \end{aligned}$$

Turning this round again,

$$\beta_q\chi1 + \gamma_q(Tw - q\chi1) \le T(|w|^p).$$

Taking the supremum over q, $|Tw|^p \le T(|w|^p)$, so that $\int |Tw|^p \le \int |w|^p$ (because $\|Tv\|_1 \le \|v\|_1$ for every $v \in L^1$). Thus $Tw \in L^p$ and $\|Tw\|_p \le \|w\|_p$.

(ii) For a general $T \in \mathcal{T}^{(0)}$, we have $|T| \in \mathcal{T}^{(0)}$, by (b), and $|Tw| \le |T||w|$, so that $\|Tw\|_p \le \||T||w|\|_p \le \|w\|_p$, as required.

(e) This is elementary, because

$$\|STu\|_1 \le \|Tu\|_1 \le \|u\|_1, \quad \|STv\|_\infty \le \|Tu\|_\infty \le \|u\|_\infty$$

whenever $u \in L^1_{\bar\mu}$ and $v \in L^\infty_{\bar\mu} \cap M^{1,0}_{\bar\mu}$.

371H Remark In the context of 366H, $T_\pi \restriction M^{1,0}_{\bar\mu} \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, while $P_\pi \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$. Thus 366H(a-iv) and 366H(b-iii) are special cases of 371Gd.

371X Basic exercises >(a) Let U be an L-space, V a Banach lattice with an order-continuous norm and $T : U \to V$ a bounded linear operator. Let B be the unit ball of U. Show that $|T|[B] \subseteq \overline{T[B]}$.

(b) Let U and V be Banach spaces. (i) Show that the space $\mathsf{K}(U;V)$ of compact linear operators from U to V (definition: 3A5La) is a closed linear subspace of $\mathsf{B}(U;V)$. (ii) Show that if U is an L-space and V is a Banach lattice with an order-continuous norm, then $\mathsf{K}(U;V)$ is a norm-closed Riesz subspace of $\mathsf{L}^\sim(U;V)$. (See Krengel 63.)

(c) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and set $U = L^1(\mathfrak{A}, \bar\mu)$. Show that $\mathsf{L}^\sim(U;U) = \mathsf{B}(U;U)$ is a Banach lattice with a Fatou norm and the Levi property. Show that its norm is order-continuous iff $\mathfrak{A}$ is finite. (*Hint*: consider operators $u \mapsto u \times \chi a$, where $a \in \mathfrak{A}$.)

>(d) Let U be a Banach lattice, and V a Dedekind complete M-space. Show that $\mathsf{L}^\sim(U;V) = \mathsf{B}(U;V)$ is a Banach lattice with a Fatou norm and the Levi property.

(e) Let U and V be Riesz spaces, of which V is Dedekind complete, and let $T \in \mathsf{L}^\sim(U;V)$. Define $T' \in \mathsf{L}^\sim(V^\sim;U^\sim)$ by writing $T'(h) = hT$ for $h \in V^\sim$. (i) Show that $|T|' \ge |T'|$ in $\mathsf{L}^\sim(V^\sim;U^\sim)$. (ii) Show that $|T|'h = |T'|h$ for every $h \in V^\times$. (*Hint*: show that if $u \in U^+$ and $h \in (V^\times)^+$ then $(|T'|h)(u)$ and $h(|T|u)$ are both equal to $\sup\{\sum_{i=0}^n g_i(Tu_i) : |g_i| \le h,\ u_i \ge 0,\ \sum_{i=0}^n u_i = u\}$.)

>(f) Using 371D, but nothing about uniformly integrable sets beyond the definition (354P), show that if U and V are L-spaces, $A \subseteq U$ is uniformly integrable in U, and $T : U \to V$ is a bounded linear operator, then $T[A]$ is uniformly integrable in V.

371Y Further exercises (a) Let U and V be Banach spaces. (i) Show that the space $\mathsf{K}_w(U;V)$ of weakly compact linear operators from U to V (definition: 3A5Lb) is a closed linear subspace of $\mathsf{B}(U;V)$. (ii) Show that if U is an L-space and V is a Banach lattice with an order-continuous norm, then $\mathsf{K}_w(U;V)$ is a norm-closed Riesz subspace of $\mathsf{L}^\sim(U;V)$.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, U a Banach space, and $T : L^1(\mathfrak{A}, \bar\mu) \to U$ a bounded linear operator. Show that T is a compact linear operator iff $\{\frac{1}{\bar\mu a} T(\chi a) : a \in \mathfrak{A},\ 0 < \bar\mu a < \infty\}$ is relatively compact in U.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and set $L^1 = L^1(\mathfrak{A}, \bar\mu)$. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a stochastically independent sequence of elements of $\mathfrak{A}$ of measure $\frac{1}{2}$, and define $T : L^1 \to \mathbb{R}^\mathbb{N}$ by setting $Tu(n) = \int u - 2\int_{a_n} u$ for each n. Show that $T \in \mathsf{B}(L^1;\boldsymbol{c}_0) \setminus \mathsf{L}^\sim(L^1;\boldsymbol{c}_0)$, where $\boldsymbol{c}_0$ is the Banach lattice of sequences converging to 0. (See 272Ye[1].)

(d) Regarding T of 371Yc as a map from L^1 to ℓ^∞, show that $|T'| \ne |T|'$ in $\mathsf{L}^\sim((\ell^\infty)^*, L^\infty(\mathfrak{A}))$.

[1] Formerly 272Yd.

(e)(i) In ℓ^2 define e_i by setting $e_i(i)=1$, $e_i(j)=0$ if $j\ne i$. Show that if $T\in\mathsf{L}^\sim(\ell^2;\ell^2)$ then $(|T|e_i|e_j)=|(Te_i|e_j)|$ for all i, $j\in\mathbb{N}$. (ii) Show that for each $n\in\mathbb{N}$ there is an orthogonal $(2^n\times 2^n)$-matrix $\mathbf{A}_n$ such that every coefficient of $\mathbf{A}_n$ has modulus $2^{-n/2}$. (*Hint*: $\mathbf{A}_{n+1}=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{A}_n & \mathbf{A}_n\\ -\mathbf{A}_n & \mathbf{A}_n\end{pmatrix}$.) (iii) Show that there is a linear isometry $S:\ell^2\to\ell^2$ such that $|(Se_i|e_j)|=2^{-n/2}$ if $2^n\le i$, $j<2^{n+1}$. (iv) Show that $S\notin\mathsf{L}^\sim(\ell^2;\ell^2)$.

371 Notes and comments The 'Chacon-Krengel theorem', properly speaking (Chacon & Krengel 64), is 371D in the case in which $U=L^1(\mu)$, $V=L^1(\nu)$; of course no new ideas are required in the generalizations here, which I have copied from Fremlin 74a.

Anyone with a training in functional analysis will automatically seek to investigate properties of operators $T:U\to V$ in terms of properties of their adjoints $T':V^*\to U^*$, as in 371Xe and 371Yd. When U is an L-space, then U^* is a Dedekind complete M-space, and it is easy to see that this forces T' to be order-bounded, for any Banach lattice V (371Xd). But since no important L-space is reflexive, this approach cannot reach 371B-371D without a new idea of some kind. It can however be adapted to the special case in 371Gb (Dunford & Schwartz 57, VIII.6.4).

In fact the results of 371B-371C are characteristic of L-spaces (Fremlin 74b). To see that they fail in the simplest cases in which U is not an L-space and V is not an M-space, see 371Yc-371Ye.

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372 The ergodic theorem

I come now to one of the most remarkable topics in measure theory. I cannot do it justice in the space I have allowed for it here, but I can give the basic theorem (372D, 372F) and a variety of the corollaries through which it is regularly used (372E, 372G-372J), together with brief notes on one of its most famous and characteristic applications (to continued fractions, 372L-372N) and on 'ergodic' and 'mixing' transformations (372O-372S). In the first half of the section (down to 372G) I express the arguments in the abstract language of measure algebras and their associated function spaces, as developed in Chapter 36; the second half, from 372H onwards, contains translations of the results into the language of measure spaces and measurable functions, the more traditional, and more readily applicable, forms.

372A Lemma Let U be a reflexive Banach space and $T:U\to U$ a bounded linear operator of norm at most 1. Then

$$V=\{u+v-Tu: u,\,v\in U,\,Tv=v\}$$

is dense in U.

proof Of course V is a linear subspace of U. **?** Suppose, if possible, that it is not dense. Then there is a non-zero $h\in U^*$ such that $h(v)=0$ for every $v\in V$ (3A5Ad). Take $u\in U$ such that $h(u)\ne 0$. Set

$$u_n=\frac{1}{n+1}\sum_{i=0}^n T^i u$$

for each $n\in\mathbb{N}$, taking T^0 to be the identity operator; because

$$\|T^iu\|\le\|T^i\|\|u\|\le\|T\|^i\|u\|\le\|u\|$$

for each i, $\|u_n\|\le\|u\|$ for every n. Note also that $T^{i+1}u-T^iu\in V$ for every i, so that $h(T^{i+1}u-T^iu)=0$; accordingly $h(T^iu)=h(u)$ for every i, and $h(u_n)=h(u)$ for every n.

Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Because U is reflexive, $v=\lim_{n\to\mathcal{F}}u_n$ is defined in U for the weak topology on U (3A5Gc). Now $Tv=v$. **P** For each $n\in\mathbb{N}$,

$$Tu_n-u_n=\frac{1}{n+1}\sum_{i=0}^n(T^{i+1}u-T^iu)=\frac{1}{n+1}(T^{n+1}u-u)$$

has norm at most $\frac{2}{n+1}\|u\|$. So $\langle Tu_n-u_n\rangle_{n\in\mathbb{N}}\to 0$ for the norm topology U and therefore for the weak topology, and surely $\lim_{n\to\mathcal{F}}Tu_n-u_n=0$. On the other hand (because T is continuous for the weak topology, 2A5If)

$$Tv = \lim_{n\to\mathcal{F}} Tu_n = \lim_{n\to\mathcal{F}}(Tu_n - u_n) + \lim_{n\to\mathcal{F}} u_n = 0 + v = v,$$

where all the limits are taken for the weak topology. **Q**

But this means that $v \in V$, while

$$h(v) = \lim_{n\to\mathcal{F}} h(u_n) = h(u) \neq 0,$$

contradicting the assumption that $h \in V^\circ$. **X**

372B Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $T : L^1 \to L^1$ a positive linear operator of norm at most 1, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$. Take any $u \in L^1$ and $m \in \mathbb{N}$, and set

$$a = [\![u > 0]\!] \cup [\![u + Tu > 0]\!] \cup [\![u + Tu + T^2u > 0]\!] \cup \ldots \cup [\![u + Tu + \ldots + T^m u > 0]\!].$$

Then $\int_a u \geq 0$.

proof Set $u_0 = u$, $u_1 = u + Tu, \ldots, u_m = u + Tu + \ldots + T^m u$, $v = \sup_{i\leq m} u_i$, so that $a = [\![v > 0]\!]$. Consider $u + T(v^+)$. We have $T(v^+) \geq Tv \geq Tu_i$ for every $i \leq m$ (because T is positive), so that $u + T(v^+) \geq u + Tu_i = u_{i+1}$ for $i < m$, and $u + T(v^+) \geq \sup_{1\leq i\leq m} u_i$. Also $u + T(v^+) \geq u$ because $T(v^+) \geq 0$, so $u + T(v^+) \geq v$. Accordingly

$$\int_a u \geq \int_a v - \int_a T(v^+) = \int v^+ - \int_a T(v^+) \geq \|v^+\|_1 - \|Tv^+\|_1 \geq 0$$

because $\|T\| \leq 1$.

372C Maximal Ergodic Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $T : L^1 \to L^1$ a linear operator, where $L^1 = L^1(\mathfrak{A}, \bar\mu)$, such that $\|Tu\|_1 \leq \|u\|_1$ for every $u \in L^1$ and $\|Tu\|_\infty \leq \|u\|_\infty$ for every $u \in L^1 \cap L^\infty(\mathfrak{A})$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for each $n \in \mathbb{N}$. Then for any $u \in L^1$, $u^* = \sup_{n\in\mathbb{N}} A_n u$ is defined in $L^0(\mathfrak{A})$, and $\alpha\bar\mu[\![u^* > \alpha]\!] \leq \|u\|_1$ for every $\alpha > 0$.

proof (a) To begin with, suppose that T is positive and that $u \geq 0$ in L^1. Note that if $v \in L^1 \cap L^\infty$, then $\|T^i v\|_\infty \leq \|v\|_\infty$ for every $i \in \mathbb{N}$, so $\|A_n v\|_\infty \leq \|v\|_\infty$ for every n; in particular, $A_n(\chi a) \leq \chi 1$ for every n and every a of finite measure.

For $m \in \mathbb{N}$ and $\alpha > 0$, set

$$a_{m\alpha} = [\![\sup_{i\leq m} A_i u > \alpha]\!].$$

Then $\alpha\bar\mu a_{m\alpha} \leq \|u\|_1$. **P** Set $a = a_{m\alpha}$, $w = u - \alpha\chi a$. Of course $\sup_{i\leq m} A_i u$ belongs to L^1, so $\bar\mu a$ is finite and $w \in L^1$. For any $i \leq m$,

$$A_i w = A_i u - \alpha A_i(\chi a) \geq A_i u - \alpha\chi 1,$$

so $[\![A_i w > 0]\!] \supseteq [\![A_i u > \alpha]\!]$. Accordingly $a \subseteq b$, where

$$b = \sup_{i\leq m} [\![A_i w > 0]\!] = \sup_{i\leq m} [\![w + Tw + \ldots + T^i w > 0]\!].$$

By 372B, $\int_b w \geq 0$. But this means that

$$\alpha\bar\mu a = \alpha\int_b \chi a = \int_b u - \int_b w \leq \int_b u \leq \|u\|_1,$$

as claimed. **Q**

It follows that if we set $c_\alpha = \sup_{n\in\mathbb{N}} a_{n\alpha}$, $\bar\mu c_\alpha \leq \alpha^{-1}\|u\|_1$ for every $\alpha > 0$ and $\inf_{\alpha>0} c_\alpha = 0$. But this is exactly the criterion in 364L(a-ii) for $u^* = \sup_{n\in\mathbb{N}} A_n u$ to be defined in L^0. And $[\![u^* > \alpha]\!] = c_\alpha$, so $\alpha\bar\mu[\![u^* > \alpha]\!] \leq \|u\|_1$ for every $\alpha > 0$, as required.

(b) Now consider the case of general T, u. In this case T is order-bounded and $\||T|\| \leq 1$, where $|T|$ is the modulus of T in $\mathsf{L}^\sim(L^1; L^1) = \mathsf{B}(L^1; L^1)$ (371D). If $w \in L^1 \cap L^\infty$, then

$$||T|w| \leq |T||w| = \sup_{|w'|\leq|w|} |Tw'| \leq \|w\|_\infty\chi 1,$$

so $\||T|w\|_\infty \leq \|w\|_\infty$. Thus $|T|$ also satisfies the conditions of the theorem. Setting $B_n = \frac{1}{n+1}\sum_{i=0}^n |T|^i$, $B_n \geq A_n$ in $\mathsf{L}^\sim(L^1; L^1)$ and $B_n|u| \geq A_n u$ for every n. But by (a), $v = \sup_{n\in\mathbb{N}} B_n|u|$ is defined in L^0 and $\alpha\bar\mu[\![v > \alpha]\!] \leq \||u|\|_1 = \|u\|_1$ for every $\alpha > 0$. Consequently $u^* = \sup_{n\in\mathbb{N}} A_n u$ is defined in L^0 and $u^* \leq v$, so that $\alpha\bar\mu[\![u^* > \alpha]\!] \leq \|u\|_1$ for every $\alpha > 0$.

372D We are now ready for a very general form of the Ergodic Theorem. I express it in terms of the space $M^{1,0}$ from 366F and the class $\mathcal{T}^{(0)}$ of operators from 371F. If these formulae are unfamiliar, you may like to glance at the statement of 372F before looking them up.

The Ergodic Theorem: first form Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and set $M^{1,0}=M^{1,0}(\mathfrak{A},\bar{\mu})$, $\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar{\mu},\bar{\mu}}\subseteq \mathsf{B}(M^{1,0};M^{1,0})$ as in 371F-371G. Take any $T\in\mathcal{T}^{(0)}$, and set $A_n=\frac{1}{n+1}\sum_{i=0}^{n}T^i:M^{1,0}\to M^{1,0}$ for every n. Then for any $u\in M^{1,0}$, $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent (definition: 367A) and $\| \|_{1,\infty}$-convergent to a member Pu of $M^{1,0}$. The operator $P:M^{1,0}\to M^{1,0}$ is a projection onto the linear subspace $\{u:u\in M^{1,0},\,Tu=u\}$, and $P\in\mathcal{T}^{(0)}$.

proof (a) It will be convenient to start with some elementary remarks. First, every A_n belongs to $\mathcal{T}^{(0)}$, by 371Ge and 371Ga. Next, $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order-bounded in $L^0=L^0(\mathfrak{A})$ for any $u\in M^{1,0}$; this is because if $u=v+w$, where $v\in L^1=L^1(\mathfrak{A},\bar{\mu})$ and $w\in L^\infty=L^\infty(\mathfrak{A})$, then $\langle A_nv\rangle_{n\in\mathbb{N}}$ and $\langle A_n(-v)\rangle_{n\in\mathbb{N}}$ are bounded above, by 372C, while $\langle A_nw\rangle_{n\in\mathbb{N}}$ is norm- and order-bounded in L^∞. Accordingly I can uninhibitedly speak of $P^*(u)=\inf_{n\in\mathbb{N}}\sup_{i\ge n}A_iu$ and $P_*(u)=\sup_{n\in\mathbb{N}}\inf_{i\ge n}A_iu$ for any $u\in M^{1,0}$, these both being defined in L^0.

(b) Write V_1 for the set of those $u\in M^{1,0}$ such that $\langle A_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent in L^0; that is, $P^*(u)=P_*(u)$ (367Be). It is easy to see that V_1 is a linear subspace of $M^{1,0}$ (use 367Ca). Also it is closed for $\| \|_{1,\infty}$.

P We know that $|T|$, taken in $\mathsf{L}^\sim(M^{1,0};M^{1,0})$, belongs to $\mathcal{T}^{(0)}$ (371Gb); set $B_n=\frac{1}{n+1}\sum_{i=0}^{n}|T|^i$ for each i.

Suppose that $u_0\in\overline{V}_1$. Then for any $\epsilon>0$ there is a $u\in V_1$ such that $\|u_0-u\|_{1,\infty}\le\epsilon^2$. Write $Pu=P^*(u)=P_*(u)$ for the order*-limit of $\langle A_nu\rangle_{n\in\mathbb{N}}$. Express u_0-u as $v+w$ where $v\in L^1$, $w\in L^\infty$ and $\|v\|_1+\|w\|_\infty\le 2\epsilon^2$.

Set $v^*=\sup_{n\in\mathbb{N}}B_n|v|$. Then $\bar{\mu}[\![v^*>\epsilon]\!]\le 2\epsilon$, by 372C. Next, if $w^*=\sup_{n\in\mathbb{N}}B_n|w|$, we surely have $w^*\le 2\epsilon^2\chi 1$. Now

$$|A_nu_0-A_nu|=|A_nv+A_nw|\le B_n|v|+B_n|w|\le v^*+w^*$$

for every $n\in\mathbb{N}$, that is,

$$A_nu-v^*-w^*\le A_nu_0\le A_nu+v^*+w^*$$

for every n. Because $\langle A_nu\rangle_{n\in\mathbb{N}}$ order*-converges to Pu,

$$Pu-v^*-w^*\le P_*(u_0)\le P^*(u_0)\le Pu+v^*+w^*,$$

and $P^*(u_0)-P_*(u_0)\le 2(v^*+w^*)$. On the other hand,

$$\bar{\mu}[\![2(v^*+w^*)>2\epsilon+4\epsilon^2]\!]\le\bar{\mu}[\![v^*>\epsilon]\!]+\bar{\mu}[\![w^*>2\epsilon^2]\!]=\bar{\mu}[\![v^*>\epsilon]\!]\le 2\epsilon$$

(using 364Ea for the first inequality). So

$$\bar{\mu}[\![P^*(u_0)-P_*(u_0)>2\epsilon(1+2\epsilon)]\!]\le 2\epsilon.$$

Since ϵ is arbitrary, $\langle A_nu_0\rangle_{n\in\mathbb{N}}$ order*-converges to $P^*(u_0)=P_*(u_0)$, and $u_0\in V_1$. As u_0 is arbitrary, V_1 is closed. **Q**

(c) Similarly, the set V_2 of those $u\in M^{1,0}$ for which $\langle A_nu\rangle_{n\in\mathbb{N}}$ is norm-convergent is a linear subspace of $M^{1,0}$, and it also is closed. **P** This is a standard argument. If $u_0\in\overline{V}_2$ and $\epsilon>0$, there is a $u\in V_2$ such that $\|u_0-u\|_{1,\infty}\le\epsilon$. There is an $n\in\mathbb{N}$ such that $\|A_iu-A_ju\|_{1,\infty}\le\epsilon$ for all i, $j\ge n$, and now $\|A_iu_0-A_ju_0\|_{1,\infty}\le 3\epsilon$ for all i, $j\ge n$, because every A_i has norm at most 1 in $\mathsf{B}(M^{1,0};M^{1,0})$ (371Gc). As ϵ is arbitrary, $\langle A_iu_0\rangle_{n\in\mathbb{N}}$ is Cauchy; because $M^{1,0}$ is complete, it is convergent, and $u_0\in V_2$. As u_0 is arbitrary, V_2 is closed. **Q**

(d) Now let V be $\{u+v-Tu:u\in M^{1,0}\cap L^\infty,\,v\in M^{1,0},\,Tv=v\}$. Then $V\subseteq V_1\cap V_2$. **P** If $u\in M^{1,0}\cap L^\infty$, then for any $n\in\mathbb{N}$

$$A_n(u-Tu)=\tfrac{1}{n+1}(u-T^{n+1}u)\to 0$$

for $\| \|_\infty$, and therefore is both order*-convergent and convergent for $\| \|_{1,\infty}$; so $u-Tu\in V_1\cap V_2$. On the other hand, if $Tv=v$, then of course $A_nv=v$ for every n, so again $v\in V_1\cap V_2$. **Q**

(e) Consequently $L^2 = L^2(\mathfrak{A},\bar\mu) \subseteq V_1 \cap V_2$. **P** $L^2 \cap V_1 \cap V_2$ is a linear subspace; but also it is closed for the norm topology of L^2, because the identity map from L^2 to $M^{1,0}$ is continuous (369Oe). We know also that $T\restriction L^2$ is an operator of norm at most 1 from L^2 to itself (371Gd). Consequently $W = \{u + v - Tu : u, v \in L^2, Tv = v\}$ is dense in L^2 (372A). On the other hand, given $u \in L^2$ and $\epsilon > 0$, there is a $u' \in L^2 \cap L^\infty$ such that $\|u-u'\|_2 \le \epsilon$ (take $u' = (u \wedge \gamma\chi 1) \vee (-\gamma\chi 1)$ for any γ large enough), and now $\|(u-Tu)-(u'-Tu')\|_2 \le 2\epsilon$. Thus $W' = \{u' + v - Tu' : u' \in L^2 \cap L^\infty, v \in L^2, Tv = v\}$ is dense in L^2. But $W' \subseteq V_1 \cap V_2$, by (d) above. Thus $L^2 \cap V_1 \cap V_2$ is dense in L^2, and is therefore the whole of L^2. **Q**

(f) $L^2 \supseteq S(\mathfrak{A}^f)$ is dense in $M^{1,0}$, by 369Pc, so $V_1 = V_2 = M^{1,0}$. This shows that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is norm-convergent and order*-convergent for every $u \in M^{1,0}$. By 367Da, the limits are the same. Write Pu for the common value of the limits.

(g) Of course we now have

$$\|Pu\|_\infty \le \sup_{n\in\mathbb{N}} \|A_n u\|_\infty \le \|u\|_\infty$$

for every $u \in L^\infty \cap M^{1,0}$, while

$$\|Pu\|_1 \le \liminf_{n\to\infty} \|A_n u\|_1 \le \|u\|_1$$

for every $u \in L^1$, by Fatou's Lemma. So $P \in \mathcal{T}^{(0)}$. If $u \in M^{1,0}$ and $Tu = u$, then surely $Pu = u$, because $A_n u = u$ for every u. On the other hand, for any $u \in M^{1,0}$, $TPu = Pu$. **P** Because $\langle A_n u\rangle_{n\in\mathbb{N}}$ is norm-convergent to Pu,

$$\begin{aligned}\|TPu - Pu\|_{1,\infty} &= \lim_{n\to\infty} \|TA_n u - A_n u\|_{1,\infty} \\ &= \lim_{n\to\infty} \frac{1}{n+1}\|T^{n+1}u - u\|_{1,\infty} = 0. \ \mathbf{Q}\end{aligned}$$

Thus, writing $U = \{u : Tu = u\}$, $P[M^{1,0}] = U$ and $Pu = u$ for every $u \in U$.

372E Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\pi : \mathfrak{A}^f \to \mathfrak{A}^f$ a measure-preserving ring homomorphism, where $\mathfrak{A}^f = \{a : \bar\mu a < \infty\}$. Let $T : M^{1,0} \to M^{1,0}$ be the corresponding Riesz homomorphism, where $M^{1,0} = M^{1,0}(\mathfrak{A},\bar\mu)$ (366H, in particular part (a-v)). Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for $n \in \mathbb{N}$. Then for every $u \in M^{1,0}$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_{1,\infty}$-convergent to some v such that $Tv = v$.

proof By 366H(a-iv), $T \in \mathcal{T}^{(0)}$, as defined in 371F. So the result follows at once from 372D.

372F The Ergodic Theorem: second form Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and let $T : L^1 \to L^1$, where $L^1 = L^1(\mathfrak{A},\bar\mu)$, be a linear operator of norm at most 1 such that $Tu \in L^\infty = L^\infty(\mathfrak{A})$ and $\|Tu\|_\infty \le \|u\|_\infty$ whenever $u \in L^1 \cap L^\infty$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i : L^1 \to L^1$ for every n. Then for any $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to an element Pu of L^1. The operator $P : L^1 \to L^1$ is a projection of norm at most 1 onto the linear subspace $\{u : u \in L^1, Tu = u\}$.

proof By 371Ga, there is an extension of T to a member $\tilde T$ of $\mathcal{T}^{(0)}$. So 372D tells us that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to some $Pu \in L^1$ for every $u \in L^1$, and $P : L^1 \to L^1$ is a projection of norm at most 1, because P is the restriction of a projection $\tilde P \in \mathcal{T}^{(0)}$. Also we still have $TPu = Pu$ for every $u \in L^1$, and $Pu = u$ whenever $Tu = u$, so the set of values $P[L^1]$ of P must be exactly $\{u : u \in L^1, Tu = u\}$.

Remark In 372D and 372F I have used the phrase 'order*-convergent' from §367 without always being specific about the partially ordered set in which it is to be interpreted. But, as remarked in 367E, the notion is robust enough for the omission to be immaterial here. Since both $M^{1,0}$ and L^1 are solid linear subspaces of L^0, a sequence in $M^{1,0}$ is order*-convergent to a member of $M^{1,0}$ (when order*-convergence is interpreted in the partially ordered set $M^{1,0}$) iff it is order*-convergent to the same point (when convergence is interpreted in the set L^0); and the same applies to L^1 in place of $M^{1,0}$.

372G Corollary Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $T : L^1 \to L^1$ be the corresponding Riesz homomorphism, where $L^1 = L^1(\mathfrak{A},\bar\mu)$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$ for $n \in \mathbb{N}$. Then for every $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent.

If we set $Pu = \lim_{n\to\infty} A_n u$ for each u, P is the conditional expectation operator corresponding to the fixed-point subalgebra $\mathfrak{C} = \{a : \pi a = a\}$ of $\mathfrak{A}$.

proof (a) The first part is just a special case of 372E; the point is that because $(\mathfrak{A}, \bar\mu)$ is totally finite, $L^\infty(\mathfrak{A}) \subseteq L^1$, so $M^{1,0}(\mathfrak{A}, \bar\mu) = L^1$. Also (because $\bar\mu 1 = 1$) $\|u\|_\infty \le \|u\|_1$ for every $u \in L^\infty$, so the norm $\| \|_{1,\infty}$ is actually equal to $\| \|_1$.

(b) For the last sentence, recall that $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (cf. 333R). By 372D or 372F, P is a projection operator onto the subspace $\{u : Tu = u\}$. Now $[\![Tu > \alpha]\!] = \pi[\![u > \alpha]\!]$ (365Nc), so $Tu = u$ iff $[\![u > \alpha]\!] \in \mathfrak{C}$ for every $\alpha \in \mathbb{R}$, that is, iff u belongs to the canonical image of $L^1(\mathfrak{C}, \bar\mu\restriction\mathfrak{C})$ in L^1 (365Q). To identify Pu further, observe that if $u \in L^1$ and $a \in \mathfrak{C}$ then

$$\int_a Tu = \int_{\pi a} Tu = \int_a u$$

(365Nb). Consequently $\int_a T^i u = \int_a u$ for every $i \in \mathbb{N}$, $\int_a A_n u = \int_a u$ for every $n \in \mathbb{N}$, and $\int_a Pu = \int_a u$ (because Pu is the limit of $\langle A_n u\rangle_{n\in\mathbb{N}}$ for $\| \|_1$). But this is enough to define Pu as the conditional expectation of u on $\mathfrak{C}$ (365Q).

372H The Ergodic Theorem is most often expressed in terms of transformations of measure spaces. In the next few corollaries I will formulate such expressions. The translation is straightforward.

Corollary Let (X, Σ, μ) be a measure space and $\phi : X \to X$ an inverse-measure-preserving function. Let f be a real-valued function which is integrable over X. Then

$$g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$$

is defined for almost every $x \in X$, and $g\phi(x) = g(x)$ for almost every x.

proof Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of (X, Σ, μ), and $\pi : \mathfrak{A} \to \mathfrak{A}$, $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ the homomorphisms corresponding to ϕ, as in 364Qd. Set $u = f^\bullet$ in $L^1(\mathfrak{A}, \bar\mu)$. Then for any $i \in \mathbb{N}$, $T^i u = (f\phi^i)^\bullet$ (364Q(c)-(d)), so setting $A_n = \frac{1}{n+1}\sum_{i=0}^n T^i$, $A_n u = g_n^\bullet$, where $g_n(x) = \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$ whenever this is defined. Now we know from 372F or 372E that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent to some v such that $Tv = v$, so $\langle g_n\rangle_{n\in\mathbb{N}}$ must be convergent almost everywhere (367F), and taking $g = \lim_{n\to\infty} g_n$ where this is defined, $g^\bullet = v$. Accordingly $(g\phi)^\bullet = Tv = v = g^\bullet$ and $g\phi =_{\text{a.e.}} g$, as claimed.

372I The following facts will be useful in the next version of the theorem, and elsewhere.

Lemma Let (X, Σ, μ) be a measure space with measure algebra $(\mathfrak{A}, \bar\mu)$. Let $\phi : X \to X$ be an inverse-measure-preserving function and $\pi : \mathfrak{A} \to \mathfrak{A}$ the associated homomorphism, as in 343A and 364Qd. Set $\mathfrak{C} = \{c : c \in \mathfrak{A}, \pi c = c\}$, $\mathrm{T} = \{E : E \in \Sigma, \phi^{-1}[E]\triangle E \text{ is negligible}\}$ and $\mathrm{T}_0 = \{E : E \in \Sigma, \phi^{-1}[E] = E\}$. Then T and T_0 are σ-subalgebras of Σ; $\mathrm{T}_0 \subseteq \mathrm{T}$, $\mathrm{T} = \{E : E \in \Sigma, E^\bullet \in \mathfrak{C}\}$, and $\mathfrak{C} = \{E^\bullet : E \in \mathrm{T}_0\}$.

proof It is easy to see that T and T_0 are σ-subalgebras of Σ and that $\mathrm{T}_0 \subseteq \mathrm{T} = \{E : E^\bullet \in \mathfrak{C}\}$. So we have only to check that if $c \in \mathfrak{C}$ there is an $E \in \mathrm{T}_0$ such that $E^\bullet = c$. **P** Start with any $F \in \Sigma$ such that $F^\bullet = c$. Now $F\triangle\phi^{-i}[F]$ is negligible for every $i \in \mathbb{N}$, because $(\phi^{-i}[F])^\bullet = \pi^i c = c$. So if we set

$$E = \bigcup_{n\in\mathbb{N}}\bigcap_{i\ge n}\phi^{-i}[F]$$
$$= \{x : \text{ there is an } n \in \mathbb{N} \text{ such that } \phi^i(x) \in F \text{ for every } i \ge n\},$$

$E^\bullet = c$. On the other hand, it is easy to check that $E \in \mathrm{T}_0$. **Q**

372J The Ergodic Theorem: third form Let (X, Σ, μ) be a probability space and $\phi : X \to X$ an inverse-measure-preserving function. Let f be a real-valued function which is integrable over X. Then

$$g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$$

is defined for almost every $x \in X$; $g\phi =_{\text{a.e.}} g$, and g is a conditional expectation of f on the σ-algebra $\mathrm{T} = \{E : E \in \Sigma, \phi^{-1}[E]\triangle E \text{ is negligible}\}$. If *either* f is Σ-measurable and defined everywhere in X

or $\phi[E]$ is negligible for every negligible set E, then g is a conditional expectation of f on the σ-algebra $\mathrm{T}_0 = \{E : E \in \Sigma,\, \phi^{-1}[E] = E\}$.

proof (a) We know by 372H that g is defined almost everywhere and that $g\phi =_{\text{a.e.}} g$. In the language of the proof of 372H, $g^\bullet = v$ is the conditional expectation of $u = f^\bullet$ on the closed subalgebra

$$\mathfrak{C} = \{a : a \in \mathfrak{A},\, \pi a = a\} = \{F^\bullet : F \in \mathrm{T}\} = \{F^\bullet : F \in \mathrm{T}_0\},$$

by 372G and 372I. So v must be expressible as $h^\bullet$ where $h : X \to \mathbb{R}$ is T_0-measurable and is a conditional expectation of f on T_0 (and also on T). Since every set of measure zero belongs to T, $g = h$ $\mu\restriction\mathrm{T}$-a.e., and g also is a conditional expectation of f on T.

(b) Suppose now that f is defined everywhere and Σ-measurable. Here I come to a technical obstruction. The definition of 'conditional expectation' in 233D asks for g to be $\mu\restriction\mathrm{T}_0$-integrable, and since μ-negligible sets do not need to be $\mu\restriction\mathrm{T}_0$-negligible we have some more checking to do, to confirm that $\{x : x \in \operatorname{dom} g,\, g(x) = h(x)\}$ is $\mu\restriction\mathrm{T}_0$-conegligible as well as μ-conegligible.

(i) For $n \in \mathbb{N}$, set $\Sigma_n = \{\phi^{-n}[E] : E \in \Sigma\}$; then Σ_n is a σ-subalgebra of Σ, including T_0. Set $\Sigma_\infty = \bigcap_{n\in\mathbb{N}} \Sigma_n$, still a σ-algebra including T_0. Now any negligible set $E \in \Sigma_\infty$ is $\mu\restriction\mathrm{T}_0$-negligible. **P** For each $n \in \mathbb{N}$ choose $F_n \in \Sigma$ such that $E = \phi^{-n}[F_n]$. Because ϕ is inverse-measure-preserving, every F_n is negligible, so that

$$E^* = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}, j\ge m} \phi^{-j}[F_n]$$

is negligible. Of course $E = \bigcap_{m\in\mathbb{N}} \phi^{-m}[F_m]$ is included in E^*. Now

$$\phi^{-1}[E^*] = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}, j\ge m} \phi^{-j-1}[F_n] = \bigcap_{m\ge 1} \bigcup_{n\in\mathbb{N}, j\ge m} \phi^{-j}[F_n] = E^*$$

because

$$\bigcup_{n\in\mathbb{N}, j\ge 1} \phi^{-j}[F_n] \subseteq \bigcup_{n\in\mathbb{N}, j\ge 0} \phi^{-j}[F_n].$$

So $E^* \in \mathrm{T}_0$ and E is included in a negligible member of T_0, which is what we needed to know. **Q**

(ii) We are assuming that f is Σ-measurable and defined everywhere, so that $g_n = \frac{1}{n+1}\sum_{i=0}^n f\circ\phi^i$ is Σ-measurable and defined everywhere. If we set $g^* = \limsup_{n\to\infty} g_n$, then $g^* : X \to [-\infty, \infty]$ is Σ_∞-measurable. **P** For any $m \in \mathbb{N}$, $f\circ\phi^i$ is Σ_m-measurable for every $i \ge m$, since $\{x : f(\phi^i(x)) > \alpha\} = \phi^{-m}[\{x : f(\phi^{i-m}(x)) > \alpha\}]$ for every α. Accordingly

$$g^* = \limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=m}^n f\circ\phi^i$$

is Σ_m-measurable. As m is arbitrary, g^* is Σ_∞-measurable. **Q**

Since h is surely Σ_∞-measurable, and $h = g^*$ μ-a.e., (i) tells us that $h = g^*$ $\mu\restriction\mathrm{T}_0$-a.e. But similarly $h = \liminf_{n\to\infty} g_n$ $\mu\restriction\mathrm{T}_0$-a.e., so we must have $h = g$ $\mu\restriction\mathrm{T}_0$-a.e.; and g, like h, is a conditional expectation of f on T_0.

(c) Finally, suppose that $\phi[E]$ is negligible for every negligible set E. Then every μ-negligible set is $\mu\restriction\mathrm{T}_0$-negligible. **P** If E is μ-negligible, then $\phi[E]$, $\phi^2[E] = \phi[\phi[E]], \dots$ are all negligible, so $E^* = \bigcup_{n\in\mathbb{N}} \phi^n[E]$ is negligible, and there is a measurable negligible set $F \supseteq E^*$. Now $F_* = \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \phi^{-n}[F]$ is a negligible set in T_0 including E, so E is $\mu\restriction\mathrm{T}_0$-negligible. **Q** Consequently $g = h$ $\mu\restriction\mathrm{T}_0$-a.e., and in this case also g is a conditional expectation of f on T_0.

372K Remark Parts (b)-(c) of the proof above are dominated by the technical question of the exact definition of 'conditional expectation of f on T_0', and it is natural to be impatient with such details. The kind of example I am concerned about is the following. Let $C \subseteq [0,1]$ be the Cantor set (134G), and $\phi : [0,1] \to [0,1]$ a Borel measurable function such that $\phi[C] = [0,1]$ and $\phi(x) = x$ for $x \in [0,1] \setminus C$. (For instance, we could take ϕ agreeing with the Cantor function on C (134H).) Because C is negligible, ϕ is inverse-measure-preserving for Lebesgue measure μ, and if f is any Lebesgue integrable function then $g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(\phi^i(x))$ is defined and equal to $f(x)$ for every $x \in \operatorname{dom} f \setminus C$. But for $x \in C$ we can, by manipulating ϕ, arrange for $g(x)$ to be almost anything; and if f is undefined on C then g will also be undefined on C. On the other hand, C is not $\mu\restriction\mathrm{T}_0$-negligible, because the only member of T_0 including C is $[0,1]$. So we cannot be sure of being able to form $\int g\,d(\mu\restriction\mathrm{T}_0)$.

If instead of Lebesgue measure itself we took its restriction $\mu_{\mathcal{B}}$ to the algebra of Borel subsets of $[0,1]$, then ϕ would still be inverse-measure-preserving for $\mu_{\mathcal{B}}$, but we should now have to worry about the possibility that $f\restriction C$ was non-measurable, so that $g\restriction C$ came out to be non-measurable, even if everywhere defined, and g was not $\mu_{\mathcal{B}}\restriction\mathrm{T}_0$-virtually measurable.

In the statement of 372J I have offered two ways of being sure that the problem does not arise: check that $\phi[E]$ is negligible whenever E is negligible (so that all negligible sets are $\mu\restriction\mathrm{T}_0$-negligible), or check that f is defined everywhere and Σ-measurable. Even if these conditions are not immediately satisfied in a particular application, it may be possible to modify the problem so that they are. For instance, completing the measure will leave ϕ inverse-measure-preserving (234Ba[2]), will not change the integrable functions but will make them all measurable (212F, 212Bc), and may enlarge T_0 enough to make a difference. If our function f is measurable (because the measure is complete, or otherwise) we can extend it to a measurable function defined everywhere (121I) and the corresponding extension of g will be $\mu\restriction\mathrm{T}_0$-integrable. Alternatively, if the difficulty seems to lie in the behaviour of ϕ rather than in the behaviour of f (as in the example above), it may help to modify ϕ on a negligible set.

372L Continued fractions A particularly delightful application of the results above is to a question which belongs as much to number theory as to analysis. It takes a bit of space to describe, but I hope you will agree with me that it is well worth knowing in itself, and that it also illuminates some of the ideas above.

(a) Set $X=[0,1]\setminus\mathbb{Q}$. For $x\in X$, set $\phi(x)=<\frac{1}{x}>$, the fractional part of $\frac{1}{x}$, and $k_1(x)=\frac{1}{x}-\phi(x)=\lfloor\frac{1}{x}\rfloor$, the integer part of $\frac{1}{x}$; then $\phi(x)\in X$ for each $x\in X$, so we may define $k_n(x)=k_1(\phi^{n-1}(x))$ for every $n\ge 1$. The strictly positive integers $k_1(x)$, $k_2(x)$, $k_3(x),\dots$ are the **continued fraction coefficients** of x. Of course $k_{n+1}(x)=k_n(\phi(x))$ for every $n\ge 1$. Now define $\langle p_n(x)\rangle_{n\in\mathbb{N}}$, $\langle q_n(x)\rangle_{n\in\mathbb{N}}$ inductively by setting

$$p_0(x)=0,\quad p_1(x)=1,\quad p_n(x)=p_{n-2}(x)+k_n(x)p_{n-1}(x)\text{ for }n\ge 1,$$

$$q_0(x)=1,\quad q_1(x)=k_1(x),\quad q_n(x)=q_{n-2}(x)+k_n(x)q_{n-1}(x)\text{ for }n\ge 1.$$

The **continued fraction approximations** or **convergents** to x are the quotients $p_n(x)/q_n(x)$.

(I do not discuss rational x, because for my purposes here these are merely distracting. But if we set $k_1(0)=\infty$, $\phi(0)=0$ then the formulae above produce the conventional values for $k_n(x)$ for rational $x\in[0,1[$. As for the p_n and q_n, use the formulae above until you get to $x=p_n(x)/q_n(x)$, $\phi^n(x)=0$, $k_{n+1}(x)=\infty$, and then set $p_m(x)=p_n(x)$, $q_m(x)=q_n(x)$ for $m\ge n$.)

(b) The point is that the quotients $r_n(x)=p_n(x)/q_n(x)$ are, in a strong sense, good rational approximations to x. (See 372Xl(v).) We have $r_n(x)<x<r_{n+1}(x)$ for every even n (372Xl). If $x=\pi-3$, then the first few coefficients are

$$k_1=7,\quad k_2=15,\quad k_3=1,$$

$$r_1=\frac{1}{7},\quad r_2=\frac{15}{106},\quad r_3=\frac{16}{113};$$

the first and third of these corresponding to the classical approximations $\pi\simeq\dfrac{22}{7}$, $\pi\simeq\dfrac{355}{113}$. Or if we take $x=e-2$, we get

$$k_1=1,\quad k_2=2,\quad k_3=1,\quad k_4=1,\quad k_5=4,\quad k_6=1,\quad k_7=1,$$

$$r_1=1,\quad r_2=\frac{2}{3},\quad r_3=\frac{3}{4},\quad r_4=\frac{5}{7},\quad r_5=\frac{23}{32},\quad r_6=\frac{28}{39},\quad r_7=\frac{51}{71};$$

note that the obvious approximations $\frac{17}{24}$, $\frac{86}{120}$ derived from the series for e are not in fact as close as the even terms $\frac{5}{7}$, $\frac{28}{39}$ above, and involve larger numbers[3].

[2] Formerly 235Hc.

[3] There is a remarkable expression for the continued fraction expansion of e, due essentially to Euler; $k_{3m-1}=2m$, $k_{3m}=$

(c) Now we need a variety of miscellaneous facts about these coefficients, which I list here.

(i) For any $x \in X$, $n \ge 1$ we have

$$p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x) = (-1)^n, \quad \phi^n(x) = \frac{p_n(x)-xq_n(x)}{xq_{n-1}(x)-p_{n-1}(x)}$$

(induce on n), so

$$x = \frac{p_n(x)+p_{n-1}(x)\phi^n(x)}{q_n(x)+q_{n-1}(x)\phi^n(x)}.$$

(ii) Another easy induction on n shows that for any finite string $\mathbf{m} = (m_1, \dots, m_n)$ of strictly positive integers the set $D_{\mathbf{m}} = \{x : x \in X,\ k_i(x) = m_i \text{ for } 1 \le i \le n\}$ is an interval in X on which ϕ^n is monotonic, being strictly increasing if n is even and strictly decreasing if n is odd. (For the inductive step, note just that

$$D_{(m_1,\dots,m_n)} = [\tfrac{1}{m_1+1}, \tfrac{1}{m_1}] \cap \phi^{-1}[D_{(m_2,\dots,m_n)}].)$$

(iii) We also need to know that the intervals $D_{\mathbf{m}}$ of (ii) are small; specifically, that if $\mathbf{m} = (m_1, \dots, m_n)$, the length of $D_{\mathbf{m}}$ is at most 2^{-n+1}. **P** All the coefficients p_i, q_i, for $i \le n$, take constant values p_i^*, q_i^* on $D_{\mathbf{m}}$, since they are determined from the coefficients k_i which are constant on $D_{\mathbf{m}}$ by definition. Now every $x \in D_{\mathbf{m}}$ is of the form $(p_n^* + tp_{n-1}^*)/(q_n^* + tq_{n-1}^*)$ for some $t \in X$ (see (i) above) and therefore lies between p_{n-1}^*/q_{n-1}^* and p_n^*/q_n^*. But the distance between these is

$$\left|\frac{p_n^*q_{n-1}^* - p_{n-1}^*q_n^*}{q_n^*q_{n-1}^*}\right| = \frac{1}{q_n^*q_{n-1}^*},$$

by the first formula in (i). Next, noting that $q_i^* \ge q_{i-1}^* + q_{i-2}^*$ for each $i \ge 2$, we see that $q_i^*q_{i-1}^* \ge 2q_{i-1}^*q_{i-2}^*$ for $i \ge 2$, and therefore that $q_n^*q_{n-1}^* \ge 2^{n-1}$, so that the length of $D_{\mathbf{m}}$ is at most 2^{-n+1}. **Q**

372M Theorem Set $X = [0,1]\setminus\mathbb{Q}$, and define $\phi : X \to X$ as in 372L. Then for every Lebesgue integrable function f on X,

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x)) = \frac{1}{\ln 2}\int_0^1 \frac{f(t)}{1+t}dt$$

for almost every $x \in X$.

proof (a) The integral just written, and the phrase 'almost every', refer of course to Lebesgue measure; but the first step is to introduce another measure, so I had better give a name μ_L to Lebesgue measure on X. Let ν be the indefinite-integral measure on X defined by saying that $\nu E = \frac{1}{\ln 2}\int_E \frac{1}{1+x}\mu_L(dx)$ whenever this is defined. The coefficient $\frac{1}{\ln 2}$ is of course chosen to make $\nu X = 1$. Because $\frac{1}{1+x} > 0$ for every $x \in X$, $\operatorname{dom}\nu = \operatorname{dom}\mu_L$ and ν has just the same negligible sets as μ_L (234Lc[4]); I can therefore safely use the terms 'measurable set', 'almost everywhere' and 'negligible' without declaring which measure I have in mind each time.

(b) Now ϕ is inverse-measure-preserving when regarded as a function from (X,ν) to itself. **P** For each $k \ge 1$, set $I_k = \left[\frac{1}{k+1}, \frac{1}{k}\right[$. On $X \cap I_k$, $\phi(x) = \frac{1}{x} - k$. Observe that $\phi\restriction I_k : X \cap I_k \to X$ is bijective and differentiable relative to its domain in the sense of 262Fb. Consider, for any measurable $E \subseteq X$,

$$\begin{aligned}\int_E \frac{1}{(y+k)(y+k+1)}\mu_L(dy) &= \int_{I_k\cap\phi^{-1}[E]} \frac{1}{(\phi(x)+k)(\phi(x)+k+1)}|\phi'(x)|\mu_L(dx)\\ &= \int_{I_k\cap\phi^{-1}[E]} \frac{x^2}{x+1}\frac{1}{x^2}\mu_L(dx) = \ln 2\cdot\nu(I_k \cap \phi^{-1}[E]),\end{aligned}$$

using 263D (or more primitive results, of course). But

$k_{3m+1} = 1$ for $m \ge 2$. See Cohn 06.

[4]Formerly 234D.

$$\sum_{k=1}^{\infty} \frac{1}{(y+k)(y+k+1)} = \sum_{k=1}^{\infty} \frac{1}{y+k} - \frac{1}{y+k+1} = \frac{1}{y+1}$$

for every $y \in [0,1]$, so

$$\nu E = \frac{1}{\ln 2} \sum_{k=1}^{\infty} \int_E \frac{1}{(y+k)(y+k+1)} \mu_L(dy) = \sum_{k=1}^{\infty} \nu(I_k \cap \phi^{-1}[E]) = \nu\phi^{-1}[E].$$

As E is arbitrary, ϕ is inverse-measure-preserving. **Q**

(c) The next thing we need to know is that if $E \subseteq X$ and $\phi^{-1}[E] = E$ then E is either negligible or conegligible. **P** I use the sets $D_{\mathbf{m}}$ of 372L(c-ii).

(i) For any string $\mathbf{m} = (m_1, \dots, m_n)$ of strictly positive integers, we have

$$x = \frac{p_n^* + p_{n-1}^* \phi^n(x)}{q_n^* + q_{n-1}^* \phi^n(x)}$$

for every $x \in D_{\mathbf{m}}$, where p_n^*, etc., are defined from $\mathbf{m}$ as in 372L(c-iii). Recall also that ϕ^n is strictly monotonic on $D_{\mathbf{m}}$. So for any interval $I \subseteq [0,1]$ (open, closed or half-open) with endpoints $\alpha < \beta$, $\phi^{-n}[I] \cap D_{\mathbf{m}}$ will be of the form $X \cap J$, where J is an interval with endpoints $(p_n^* + p_{n-1}^*\alpha)/(q_n^* + q_{n-1}^*\alpha)$, $(p_n^*+p_{n-1}^*\beta)/(q_n^*+q_{n-1}^*\beta)$ in some order. This means that we can estimate $\mu_L(\phi^{-n}[I] \cap D_{\mathbf{m}})/\mu_L D_{\mathbf{m}}$, because it is

$$\frac{\left|\frac{p_n^*+p_{n-1}^*\alpha}{q_n^*+q_{n-1}^*\alpha} - \frac{p_n^*+p_{n-1}^*\beta}{q_n^*+q_{n-1}^*\beta}\right|}{\left|\frac{p_n^*}{q_n^*} - \frac{p_n^*+p_{n-1}^*}{q_n^*+q_{n-1}^*}\right|} = \frac{(\beta-\alpha)q_n^*(q_n^*+q_{n-1}^*)}{(q_n^*+q_{n-1}^*\alpha)(q_n^*+q_{n-1}^*\beta)} \ge \frac{(\beta-\alpha)q_n^*}{q_n^*+q_{n-1}^*} \ge \frac{1}{2}(\beta-\alpha).$$

Now look at

$$\mathcal{A} = \{E : E \subseteq [0,1] \text{ is Lebesgue measurable, } \mu_L(\phi^{-n}[E] \cap D_{\mathbf{m}}) \ge \frac{1}{2}\mu_L E \cdot \mu_L D_{\mathbf{m}}\}.$$

Clearly the union of two disjoint members of $\mathcal{A}$ belongs to $\mathcal{A}$. Because $\mathcal{A}$ contains every subinterval of $[0,1]$ it includes the algebra $\mathcal{E}$ of subsets of $[0,1]$ consisting of finite unions of intervals. Next, the union of any non-decreasing sequence in $\mathcal{A}$ belongs to $\mathcal{A}$, and the intersection of a non-increasing sequence likewise. But this means that $\mathcal{A}$ must include the σ-algebra generated by $\mathcal{E}$ (136G), that is, the Borel σ-algebra. But also, if $E \in \mathcal{A}$ and $H \subseteq [0,1]$ is negligible, then

$$\mu_L(\phi^{-n}[E \triangle H] \cap D_{\mathbf{m}}) = \mu_L(\phi^{-n}[E] \cap D_{\mathbf{m}}) \ge \frac{1}{2}\mu_L E \cdot \mu_L D_{\mathbf{m}} = \frac{1}{2}\mu_L(E \triangle H) \cdot \mu_L D_{\mathbf{m}}$$

and $E \triangle H \in \mathcal{A}$. And this means that every Lebesgue measurable subset of $[0,1]$ belongs to $\mathcal{A}$ (134Fb).

(ii) ? Now suppose, if possible, that E is a measurable subset of X and that $\phi^{-1}[E] = E$ and E is neither negligible nor conegligible in X. Set $\gamma = \frac{1}{2}\mu_L E > 0$. By Lebesgue's density theorem (223B) there is some $x \in X \setminus E$ such that $\lim_{\delta\downarrow 0} \psi(\delta) = 0$, where $\psi(\delta) = \frac{1}{2\delta}\mu_L(E \cap [x-\delta, x+\delta])$ for $\delta > 0$. Take n so large that $\psi(\delta) < \frac{1}{2}\gamma$ whenever $0 < \delta \le 2^{-n+1}$, and set $m_i = k_i(x)$ for $i \le n$, so that $x \in D_{\mathbf{m}}$. Taking the least δ such that $D_{\mathbf{m}} \subseteq [x-\delta, x+\delta]$, we must have $\delta \le 2^{-n+1}$, because the length of $D_{\mathbf{m}}$ is at most 2^{-n+1} (372L(c-iii)), while $\mu_L D_{\mathbf{m}} \ge \delta$, because $D_{\mathbf{m}}$ is an interval. Accordingly

$$\mu_L(E \cap D_{\mathbf{m}}) \le \mu_L(E \cap [x-\delta, x+\delta]) = 2\delta\psi(\delta) < \gamma\delta \le \gamma\mu_L D_{\mathbf{m}}.$$

But we also have

$$\mu_L(E \cap D_{\mathbf{m}}) = \mu_L(\phi^{-n}[E] \cap D_{\mathbf{m}}) \ge \gamma\mu_L D_{\mathbf{m}},$$

by (i). **X**

This proves the result. **Q**

(d) The final fact we need in preparation is that $\phi[E]$ is negligible for every negligible $E \subseteq X$. This is because ϕ is differentiable relative to its domain (see 263D(ii)).

(e) Now let f be any μ_L-integrable function. Because $\frac{1}{1+x} \le 1$ for every x, f is also ν-integrable (235K[5]);

[5]Formerly 235M.

consequently, using (b) above and 372J,

$$g(x) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x))$$

is defined for almost every $x \in X$, and is a conditional expectation of f (with respect to the measure ν) on the σ-algebra $\mathrm{T}_0 = \{E : E \text{ is measurable}, \phi^{-1}[E] = E\}$. But we have just seen that T_0 consists only of negligible and conegligible sets, so g must be essentially constant; since $\int g\,d\nu = \int f d\nu$, we must have

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x)) = \int f d\nu = \frac{1}{\ln 2}\int_0^1 \frac{f(t)}{1+t}\mu_L(dt)$$

for almost every x (using 235K to calculate $\int f d\nu$).

372N Corollary For almost every $x \in [0,1]\setminus\mathbb{Q}$,

$$\lim_{n\to\infty} \frac{1}{n}\#(\{i : 1 \le i \le n,\ k_i(x) = k\}) = \frac{1}{\ln 2}(2\ln(k+1) - \ln k - \ln(k+2))$$

for every $k \ge 1$, where $k_1(x), \dots$ are the continued fraction coefficients of x.

proof In 372M, set $f = \chi(X \cap [\frac{1}{k+1}, \frac{1}{k}])$. Then (for $i \ge 1$) $f(\phi^i(x)) = 1$ if $k_i(x) = k$ and zero otherwise. So

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n}\#(\{i : 1 \le i \le n,\ k_i(x) = k\})\\
&\quad = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(\phi^i(x)) = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x))\\
&\quad = \frac{1}{\ln 2}\int_0^1 \frac{f(t)}{1+t}dt = \frac{1}{\ln 2}\int_{1/k+1}^{1/k} \frac{1}{1+t}dt\\
&\quad = \frac{1}{\ln 2}(\ln(1+\tfrac{1}{k}) - \ln(1+\tfrac{1}{k+1})) = \frac{1}{\ln 2}(2\ln(k+1) - \ln k - \ln(k+2)),
\end{aligned}$$

for almost every $x \in X$.

372O Mixing and ergodic transformations This seems an appropriate moment for some brief notes on three special types of measure-preserving homomorphism or inverse-measure-preserving function.

Definitions (a)(i) Let $\mathfrak{A}$ be a Boolean algebra. Then a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ is **ergodic** if whenever $a, b \in \mathfrak{A}\setminus\{0\}$ there are $m, n \in \mathbb{N}$ such that $\pi^m a \cap \pi^n b \ne 0$.

(ii) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is **mixing** (sometimes called **strongly mixing**) if $\lim_{n\to\infty} \bar\mu(\pi^n a \cap b) = \bar\mu a \cdot \bar\mu b$ for all $a, b \in \mathfrak{A}$.

(iii) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. Then π is **weakly mixing** if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} |\bar\mu(\pi^n a \cap b) - \bar\mu a \cdot \bar\mu b| = 0$ for all $a, b \in \mathfrak{A}$.

(b) Let (X, Σ, μ) be a probability space and $\phi : X \to X$ an inverse-measure-preserving function.

(i) ϕ is **ergodic** (also called **metrically transitive**, **indecomposable**) if every measurable set E such that $\phi^{-1}[E] = E$ is either negligible or conegligible.

(ii) ϕ is **mixing** if $\lim_{n\to\infty} \mu(F \cap \phi^{-n}[E]) = \mu E \cdot \mu F$ for all $E, F \in \Sigma$.

(iii) ϕ is **weakly mixing** if if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} |\mu(F \cap \phi^{-n}[E]) - \mu E \cdot \mu F| = 0$ for all $E, F \in \Sigma$.

372P For the principal applications of the idea in 372O(a-i), we have an alternative definition in terms of fixed-point subalgebras.

Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism, with fixed-point subalgebra $\mathfrak{C}$.

(a) If π is ergodic, then $\mathfrak{C} = \{0, 1\}$.

(b) If π is an automorphism, then π is ergodic iff $\sup_{n\in\mathbb{Z}} \pi^n a = 1$ for every $a \in \mathfrak{A} \setminus \{0\}$.

(c) If π is an automorphism and $\mathfrak{A}$ is Dedekind σ-complete, then π is ergodic iff $\mathfrak{C} = \{0, 1\}$.

proof (a) If $c \in \mathfrak{C}$, then $\pi^m c = c$ is disjoint from $\pi^n(1 \setminus c) = 1 \setminus c$ for all m, $n \in \mathbb{N}$, so one of c, $1 \setminus c$ must be zero.

(b)(i) If π is ergodic and $a \neq 0$ and $b \cap \pi^n a = 0$ for every $n \in \mathbb{Z}$, then $\pi^m b \cap \pi^n a = \pi^m(b \cap \pi^{n-m} a) = 0$ for all m, $n \in \mathbb{N}$, so $b = 0$. As b is arbitrary, $\sup_{n\in\mathbb{Z}} \pi^n a = 1$; as a is arbitrary, π satisfies the condition.

(ii) If π satisfies the condition, and a, $b \in \mathfrak{A} \setminus \{0\}$, then there is an $m \in \mathbb{Z}$ such that $\pi^m a \cap b \neq 0$; setting $n = \max(-m, 0)$, $\pi^{m+n} a \cap \pi^n b \neq 0$, while $m + n$ and n both belong to $\mathbb{N}$. As a and b are arbitrary, π is ergodic.

(c) If π is ergodic then $\mathfrak{C} = \{0, 1\}$, by (a). If $\mathfrak{C} = \{0, 1\}$ and $a \in \mathfrak{A} \setminus \{0\}$, consider $c = \sup_{n\in\mathbb{Z}} \pi^n a$, which is defined because $\mathfrak{A}$ is Dedekind σ-complete. Being an automorphism, π is order-continuous (313Ld), so $\pi c = \sup_{n\in\mathbb{Z}} \pi^{n+1} a = c$ and $c \in \mathfrak{C}$. Since $c \supseteq a$ is non-zero, $c = 1$. As a is arbitrary, π is ergodic, by (b).

372Q The following facts are equally straightforward.

Proposition (a) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism, and $T : L^0 = L^0(\mathfrak{A}) \to L^0$ the Riesz homomorphism such that $T(\chi a) = \chi\pi a$ for every $a \in \mathfrak{A}$.

(i) If π is mixing, it is weakly mixing.

(ii) If π is weakly mixing, it is ergodic.

(iii) The following are equiveridical: (α) π is ergodic; (β) the only $u \in L^0$ such that $Tu = u$ are the multiples of $\chi 1$; (γ) for every $u \in L^1 = L^1(\mathfrak{A}, \bar{\mu})$, $\langle \frac{1}{n+1} \sum_{i=0}^n T^i u\rangle_{n\in\mathbb{N}}$ order*-converges to $(\int u)\chi 1$.

(iv) The following are equiveridical: (α) π is mixing; (β) $\lim_{n\to\infty}(T^n u|v) = \int u \int v$ for all u, $v \in L^2(\mathfrak{A}, \bar{\mu})$.

(v) The following are equiveridical: (α) π is weakly mixing; (β) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |(T^k u|v) - \int u \int v| = 0$ for all u, $v \in L^2(\mathfrak{A}, \bar{\mu})$.

(b) Let (X, Σ, μ) be a probability space, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Let $\phi : X \to X$ be an inverse-measure-preserving function and $\pi : \mathfrak{A} \to \mathfrak{A}$ the associated homomorphism such that $\pi E^\bullet = (\phi^{-1}[E])^\bullet$ for every $E \in \Sigma$.

(i) The following are equiveridical: (α) ϕ is ergodic; (β) π is ergodic; (γ) for every μ-integrable real-valued function f, $\langle \frac{1}{n+1} \sum_{i=0}^n f(\phi^i(x))\rangle_{n\in\mathbb{N}}$ converges to $\int f$ for almost every $x \in X$.

(ii) ϕ is mixing iff π is, and in this case ϕ is weakly mixing.

(iii) ϕ is weakly mixing iff π is, and in this case ϕ is ergodic.

proof (a)(i)-(ii) Immediate from the definitions.

(iii)(α)⇒(β) $Tu = u$ iff $\pi[\![u > \alpha]\!] = [\![u > \alpha]\!]$ for every α; if π is ergodic, this means that $[\![u > \alpha]\!] \in \{0, 1\}$ for every α, by 372Pa, and u must be of the form $\gamma\chi 1$, where $\gamma = \inf\{\alpha : [\![u > \alpha]\!] = 0\}$.

(β)⇒(γ) If (β) is true and $u \in L^1$, then we know from 372G that $\langle \frac{1}{n+1} \sum_{i=0}^n T^i u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to some v such that $Tv = v$; by (β), v is of the form $\gamma\chi 1$; and

$$\gamma = \int v = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n \int T^i u = \int u.$$

(γ)⇒(α) Assuming (γ), take any $a \in \mathfrak{A}$ such that $\pi a = a$, and consider $u = \chi a$. Then $T^i u = \chi a$ for every i, so

$$\chi a = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^n T^i u = (\int u)\chi 1 = \bar{\mu} a \cdot \chi 1,$$

and a must be either 0 or 1. By 372Pc, π is ergodic.

(iv)(α)⇒(β) Since π is mixing,

$$\lim_{n\to\infty}(T^n\chi a|\chi b) = \lim_{n\to\infty}(\chi\pi^n a|\chi b) = \lim_{n\to\infty}\bar\mu(\pi^n a\cap b)$$
$$= \bar\mu a\cdot\bar\mu b = \int\chi a\int\chi b$$

for all a, $b\in\mathfrak{A}$. Because $(u,v)\mapsto(T^n u|v)$ and $(u,v)\mapsto\int u\int v$ are both bilinear,

$$\lim_{n\to\infty}(T^n u|v) = \int u\int v$$

for all u, $v\in S(\mathfrak{A})$. For general u, $v\in L^2(\mathfrak{A},\bar\mu)$, take any $\epsilon>0$. Then there are u', $v'\in S(\mathfrak{A})$ such that

$$(\|u-u'\|_2+\|v-v'\|_2)\max(\|u\|_2,\|v\|_2+\|v-v'\|_2)\le\epsilon$$

(366C), so that

$$|(T^n u|v)-(T^n u'|v')| \le |(T^n u|v-v')|+|(T^n u-T^n u'|v')|$$
$$\le \|T^n u\|_2\|v-v'\|_2+\|T^n u-T^n u'\|_2\|v'\|_2$$
$$\le \|u\|_2\|v-v'\|_2+\|u-u'\|_2(\|v\|_2+\|v-v'\|_2)$$

(366H(a-iv))

$$\le\epsilon,$$
$$|\int u\int v-\int u'\int v'| \le |\int u||\int v-v'|+|\int u-u'||\int v'|$$
$$\le \|u\|_2\|v-v'\|_2+\|u-u'\|_2\|v'\|_2\le\epsilon$$

for every n, and

$$\limsup_{n\to\infty}|(T^n u|v)-\int u\int v|\le 2\epsilon+\lim_{n\to\infty}|(T^n u'|v')-\int u'\int v'| = 2\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty}(T^n u|v)=\int u\int v$, as required.

$(\beta)\Rightarrow(\alpha)$ This is elementary, as (α) is just the case $u=\chi a$, $v=\chi b$ of (β).

(v) The argument is essentially the same as in (iv); (α) is a special case of (β); if (α) is true, then by linearity (β) is true when u, $v\in S(\mathfrak{A})$, and the functional $(u,v)\mapsto\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|(T^k u|v)-\int u\int v|$ is continuous.

(b)(i)$(\alpha)\Rightarrow(\beta)$ If $\pi a=a$ there is an E such that $\phi^{-1}[E]=E$ and $E^\bullet=a$, by 372I; now $\bar\mu a=\mu E\in\{0,1\}$, so $a\in\{0,1\}$. Thus the fixed-point subalgebra of π is $\{0,1\}$; by 372Pc again, π is ergodic.

$(\beta)\Rightarrow(\gamma)$ Set $u=f^\bullet\in L^1$. In the language of (a), $T^i u=(f\phi^i)^\bullet$ for each i, as in the proof of 372H, so that

$$(\tfrac{1}{n+1}\sum_{i=0}^n f\phi^i)^\bullet = \tfrac{1}{n+1}\sum_{i=0}^n T^i u$$

is order*-convergent to $(\int u)\chi 1=(\int f)\chi 1$, and $\frac{1}{n+1}\sum_{i=0}^n f\phi^i\to\int f$ a.e.

$(\gamma)\Rightarrow(\alpha)$ If $\phi^{-1}[E]=E$ then, applying (γ) to $f=\chi E$, we see that $\chi E=_{\text{a.e.}}\mu E\cdot\chi X$, so that E is either negligible or conegligible.

(ii)-(iii) Simply translating the definitions, we see that π is mixing, or weakly mixing, iff ϕ is. So the results here are reformulations of (a-i) and (a-ii).

372R Remarks (a) The reason for introducing 'ergodic' homomorphisms in this section is of course 372G/372J; if π in 372G, or ϕ in 372J, is ergodic, then the limit Pu or g must be (essentially) constant, being a conditional expectation on a trivial subalgebra.

(b) In the definition 372O(b-i) I should perhaps emphasize that we look only at *measurable* sets E. We certainly expect that there will generally be many sets E for which $\phi^{-1}[E]=E$, since any union of orbits of ϕ will have this property.

(c) Part (c) of the proof of 372M was devoted to showing that the function ϕ there was ergodic; see also 372Xm. For another ergodic transformation see 372Xr. For examples of mixing transformations see 333P, 372Xp, 372Xq, 372Xt, 372Xw and 372Xx.

(d) It seems to be difficult to display explicitly a weakly mixing transformation which is not mixing. There is an example in CHACON 69, and I give another in 494F in Volume 4. In a certain sense, however, 'most' measure-preserving automorphisms of the Lebesgue probability algebra are weakly mixing but not mixing; I will return to this in 494E.

372S There is a useful sufficient condition for a homomorphism or function to be mixing.

Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. If $\bigcap_{n\in\mathbb{N}}\pi^n[\mathfrak{A}]=\{0,1\}$, then π is mixing.

(b) Let (X,Σ,μ) be a probability space, and $\phi:X\to X$ an inverse-measure-preserving function. Set

$$\mathrm{T}=\{E:\text{for every }n\in\mathbb{N}\text{ there is an }F\in\Sigma\text{ such that }E=\phi^{-n}[F]\}.$$

If every member of T is either negligible or conegligible, ϕ is mixing.

proof (a) Let $T:L^0=L^0(\mathfrak{A})\to L^0$ be the Riesz homomorphism associated with π. Take any a, $b\in\mathfrak{A}$ and any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Then $\langle T^n(\chi a)\rangle_{n\in\mathbb{N}}$ is a bounded sequence in the reflexive space $L^2_{\bar\mu}=L^2(\mathfrak{A},\bar\mu)$, so $v=\lim_{n\to\mathcal{F}}T^n(\chi a)$ is defined for the weak topology of $L^2_{\bar\mu}$. Now for each $n\in\mathbb{N}$ set $\mathfrak{B}_n=\pi^n[\mathfrak{A}]$. This is a closed subalgebra of $\mathfrak{A}$ (314F(a-i)), and contains $\pi^i a$ for every $i\ge n$. So if we identify $L^2(\mathfrak{B}_n,\bar\mu\restriction\mathfrak{B}_n)$ with the corresponding subspace of $L^2_{\bar\mu}$ (366I), it contains $T^i(\chi a)$ for every $i\ge n$; but also it is norm-closed, therefore weakly closed (3A5Ee), so contains v. This means that $[\![v>\alpha]\!]$ must belong to $\mathfrak{B}_n$ for every α and every n. But in this case $[\![v>\alpha]\!]\in\bigcap_{n\in\mathbb{N}}\mathfrak{B}_n=\{0,1\}$ for every α, and v is of the form $\gamma\chi1$. Also

$$\gamma=\int v=\lim_{n\to\mathcal{F}}\int T^n(\chi a)=\bar\mu a.$$

So

$$\lim_{n\to\mathcal{F}}\bar\mu(\pi^n a\cap b)=\lim_{n\to\mathcal{F}}\int T^n(\chi a)\times\chi b=\int v\times\chi b=\gamma\bar\mu b=\bar\mu a\cdot\bar\mu b.$$

But this is true of every non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$, so we must have $\lim_{n\to\infty}\bar\mu(\pi^n a\cap b)=\bar\mu a\cdot\bar\mu b$ (3A3Lc). As a and b are arbitrary, π is mixing.

(b) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of (X,Σ,μ), and $\pi:\mathfrak{A}\to\mathfrak{A}$ the measure-preserving homomorphism corresponding to ϕ. The point is that if $a\in\bigcap_{n\in\mathbb{N}}\pi^n[\mathfrak{A}]$, there is an $E\in\mathrm{T}$ such that $E^\bullet=a$. **P** For each $n\in\mathbb{N}$ there is an $a_n\in\mathfrak{A}$ such that $\pi^n a_n=a$; say $a_n=F_n^\bullet$ where $F_n\in\Sigma$. Then $\phi^{-n}[F_n]^\bullet=a$. Set

$$E=\bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m}\phi^{-n}[F_n],\quad E_k=\bigcup_{m\ge k}\bigcap_{n\ge m}\phi^{-(n-k)}[F_n]$$

for each k; then $E^\bullet=a$ and

$$\phi^{-k}[E_k]=\bigcup_{m\ge k}\bigcap_{n\ge m}\phi^{-n}[F_n]=\bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m}\phi^{-n}[F_n]=E$$

for every k, so $E\in\mathrm{T}$. **Q**

So $\bigcap_{n\in\mathbb{N}}\mathfrak{A}_n=\{0,1\}$ and π and ϕ are mixing.

372X Basic exercises (a) Let U be any reflexive Banach space, and $T:U\to U$ an operator of norm at most 1. Set $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$ for each $n\in\mathbb{N}$. Show that $Pu=\lim_{n\to\infty}A_n u$ is defined (as a limit for the norm topology) for every $u\in U$, and that $P:U\to U$ is a projection onto $\{u:Tu=u\}$. (*Hint*: show that $\{u:Pu\text{ is defined}\}$ is a closed linear subspace of U containing $Tu-u$ for every $u\in U$.)

(This is a version of the **mean ergodic theorem**.)

>(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$; set $A_n=\frac{1}{n+1}\sum_{i=0}^n T^i$ for $n\in\mathbb{N}$. Take any $p\in[1,\infty[$ and $u\in L^p=L^p(\mathfrak{A},\bar\mu)$. Show that $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_p$-convergent to some $v\in L^p$. (*Hint*: put 372Xa together with 372D.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. Let $P:L^1\to L^1$ be the operator defined as in 365O/366Hb, where $L^1=L^1_{\bar\mu}$, so that $\int_a Pu=\int_{\pi a}u$ for

$u \in L^1$ and $a \in \mathfrak{A}$. Set $A_n = \frac{1}{n+1}\sum_{i=0}^n P^i : L^1 \to L^1$ for each i. Show that for any $u \in L^1$, $\langle A_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to the conditional expectation of u on the subalgebra $\{a : \pi a = a\}$.

(d) Show that if f is any Lebesgue integrable function on $\mathbb{R}$, and $y \in \mathbb{R}\setminus\{0\}$, then

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^n f(x+ky) = 0$$

for almost every $x \in \mathbb{R}$.

(e) Let (X,Σ,μ) be a measure space and $\phi : X \to X$ an inverse-measure-preserving function. Set $\mathrm{T} = \{E : E \in \Sigma,\ \mu(\phi^{-1}[E]\triangle E) = 0\}$, $\mathrm{T}_0 = \{E : E \in \Sigma,\ \phi^{-1}[E] = E\}$. (i) Show that $\mathrm{T} = \{E\triangle F : E \in \mathrm{T}_0,\ F \in \Sigma,\ \mu F = 0\}$. (ii) Show that a set $A \subseteq X$ is $\mu\restriction\mathrm{T}_0$-negligible iff $\phi^n[A]$ is μ-negligible for every $n \in \mathbb{N}$.

>(f) Let ν be a Radon probability measure on $\mathbb{R}$ such that $\int |t|\nu(dt)$ is finite (cf. 271F). On $X = \mathbb{R}^{\mathbb{N}}$ let λ be the product measure obtained when each factor is given the measure ν. Define $\phi : X \to X$ by setting $\phi(x)(n) = x(n+1)$ for $x \in X$, $n \in \mathbb{N}$. (i) Show that ϕ is inverse-measure-preserving. (*Hint*: 254G. See also 372Xw below.) (iii) Set $\gamma = \int t\nu(dt)$, the expectation of the distribution ν. By considering $\frac{1}{n+1}\sum_{i=0}^n f\circ\phi^i$, where $f(x) = x(0)$ for $x \in X$, show that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n x(i) = \gamma$ for λ-almost every $x \in X$.

>(g) Use the Ergodic Theorem to prove Kolmogorov's Strong Law of Large Numbers (273I), as follows. Given a complete probability space (Ω,Σ,μ) and an independent identically distributed sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of measurable functions from Ω to $\mathbb{R}$, set $X = \mathbb{R}^{\mathbb{N}}$ and $f(\omega) = \langle f_n(\omega)\rangle_{n\in\mathbb{N}}$ for $\omega \in \Omega$. Show that if we give each copy of $\mathbb{R}$ the distribution of f_0 then f is inverse-measure-preserving for μ and the product measure λ on X. Now use 372Xf.

>(h) Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of real-valued random variables with finite expectation such that $(f_0, f_1,\dots,f_n)$ has the same joint distribution as $(f_1, f_2,\dots,f_{n+1})$ for every $n \in \mathbb{N}$. Show that $\langle\frac{1}{n+1}\sum_{i=0}^n f_i\rangle_{n\in\mathbb{N}}$ converges a.e. (*Hint*: Let (X,Σ,μ) be the underlying probability space. Reduce to the case in which every f_i is measurable and defined everywhere in X. Define $\theta : X \to \mathbb{R}^{\mathbb{N}}$ by setting $\theta(x)(n) = f_n(x)$ for $x \in X$, $n \in \mathbb{N}$. Let λ be the image measure $\mu\theta^{-1}$. Set $\phi(z)(n) = z(n+1)$ for $z \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Show that ϕ is inverse-measure-preserving for λ, and apply 372J.)

(i) Show that the continued fraction coefficients of $\frac{1}{\sqrt{2}}$ are 1, 2, 2, 2, $\dots$.

>(j) For $x \in X = [0,1]\setminus\mathbb{Q}$ let $k_1(x), k_2(x),\dots$ be its continued-fraction coefficients. Show that $x \mapsto \langle k_{n+1}(x) - 1\rangle_{n\in\mathbb{N}}$ is a bijection between X and $\mathbb{N}^{\mathbb{N}}$ which is a homeomorphism if X is given its usual topology (as a subset of $\mathbb{R}$) and $\mathbb{N}^{\mathbb{N}}$ is given its usual product topology (each copy of $\mathbb{N}$ being given its discrete topology).

(k) Set $x = \frac{1}{2}(\sqrt{5}-1)$. Show that, in the notation of 372L, $k_n(x) = 1$ and $q_n(x) = p_{n-1}(x)$ for every $n \ge 1$ and that $\langle p_n(x)\rangle_{n\in\mathbb{N}}$ is the Fibonacci sequence.

(l) For any irrational $x \in [0,1]$ let $k_1(x), k_2(x),\dots$ be its continued-fraction coefficients and $p_n(x)$, $q_n(x)$ the numerators and denominators of its continued-fraction approximations, as described in 372L. Write $r_n(x) = p_n(x)/q_n(x)$. (i) Show that x lies between $r_n(x)$ and $r_{n+1}(x)$ for every $n \in \mathbb{N}$. (ii) Show that $r_{n+1}(x) - r_n(x) = (-1)^n/q_n(x)q_{n+1}(x)$ for every $n \in \mathbb{N}$. (iii) Show that $|x - r_n(x)| \le 1/q_n(x)^2 k_{n+1}(x)$ for every $n \ge 1$. (iv) Hence show that for almost every $\gamma \in \mathbb{R}$, the set $\{(p,q) : p \in \mathbb{Z},\ q \ge 1,\ |\gamma - \frac{p}{q}| \le \epsilon/q^2\}$ is infinite for every $\epsilon > 0$. (v) Show that if $n \ge 1$, p, $q \in \mathbb{N}$ and $0 < q \le q_n(x)$, then $|x - \frac{p}{q}| \ge |x - r_n(x)|$, with equality only when $p = p_n(x)$ and $q = q_n(x)$.

(m) In 372M, let T_1 be the family $\{E$: for every $n \in \mathbb{N}$ there is a measurable set $F \subseteq X$ such that $\phi^{-n}[F] = E\}$. Show that every member of T_1 is either negligible or conegligible. (*Hint*: the argument of part (c) of the proof of 372M still works.) Hence show that ϕ is mixing for the measure ν.

(n) Let $(\mathfrak{A},\bar\mu)$ be an atomless probability algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is homogeneous; (ii) there is an ergodic measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$; (iii) there is a mixing measure-preserving automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$. (*Hint*: 333P.)

(o) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism. (i) Show that if $n\ge 1$ then π is mixing iff π^n is mixing. (ii) Show that if $n\ge 1$ then π is weakly mixing iff π^n is weakly mixing. (iii) Show that if $n\ge 1$ and π^n is ergodic then π is ergodic. (iv) Show that if π is an automorphism then it is ergodic, or mixing, or weakly mixing, iff π^{-1} is.

>(p) Consider the **tent map** $\phi_\alpha(x)=\alpha\min(x,1-x)$ for $x\in[0,1]$, $\alpha\in[0,2]$. Show that ϕ_2 is inverse-measure-preserving and mixing for Lebesgue measure on $[0,1]$. (*Hint*: show that $\phi_2^{n+1}(x)=\phi_2(<2^nx>)$ for $n\ge 1$, and hence that $\mu(I\cap\phi_2^{-n}[J])=\mu I\cdot\mu J$ whenever I is of the form $[2^{-n}k,2^{-n}(k+1)]$ and J is an interval.)

(q) Consider the **logistic map** $\psi_\beta(x)=\beta x(1-x)$ for $x\in[0,1]$, $\beta\in[0,4]$. Show that ψ_4 is inverse-measure-preserving and mixing for the Radon measure on $[0,1]$ with density function $t\mapsto\frac{1}{\pi\sqrt{t(1-t)}}$. (*Hint*: show that the transformation $t\mapsto\sin^2\frac{\pi t}{2}$ matches it with the tent map.) Show that for almost every x,

$$\lim_{n\to\infty}\frac{1}{n+1}\#(\{i:i\le n,\ \psi_4^i(x)\le\alpha\})=\frac{2}{\pi}\arcsin\sqrt{\alpha}$$

for every $\alpha\in[0,1]$.

(r) Let μ be Lebesgue measure on $[0,1[$, and fix an irrational number $\alpha\in[0,1[$. (i) Set $\phi(x)=x+_1\alpha$ for every $x\in[0,1[$, where $x+_1\alpha$ is whichever of $x+\alpha$, $x+\alpha-1$ belongs to $[0,1[$. Show that ϕ is inverse-measure-preserving. (ii) Show that if $I\subseteq[0,1[$ is an interval then $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n\chi I(\phi^i(x))=\mu I$ for almost every $x\in[0,1[$. (*Hint*: this is Weyl's Equidistribution Theorem (281N).) (iii) Show that ϕ is ergodic. (*Hint*: take the conditional expectation operator P of 372G, and look at $P(\chi I^\bullet)$ for intervals I.) (iv) Show that ϕ^n is ergodic for any $n\in\mathbb{Z}\setminus\{0\}$. (v) Show that ϕ is not weakly mixing.

(s) Let p, $q\in[1,\infty]$ be such that $\frac{1}{p}+\frac{1}{q}=1$. (i) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a mixing measure-preserving homomorphism, and $T:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ the corresponding homomorphism. Show that $\lim_{n\to\infty}\int T^nu\times v=\int u\int v$ whenever $u\in L^p(\mathfrak{A},\bar{\mu})$ and $v\in L^q(\mathfrak{A},\bar{\mu})$. (*Hint*: start with u, $v\in S(\mathfrak{A})$.) (ii) Let (X,Σ,μ) be a probability space and $\phi:X\to X$ a mixing inverse-measure-preserving function. Show that $\lim_{n\to\infty}\int f(\phi^n(x))g(x)dx=\int f\int g$ whenever $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$.

(t) Give $[0,1[$ Lebesgue measure μ, and let $k\ge 2$ be an integer. Define $\phi:[0,1[\,\to[0,1[$ by setting $\phi(x)=<kx>$, the fractional part of kx. Show that ϕ is inverse-measure-preserving. Show that ϕ is mixing. (*Hint*: if $I=[k^{-n}i,k^{-n}(i+1)[$, $J=[k^{-n}j,k^{-n}(j+1)[$ then $\mu(I\cap\phi^{-m}[J])=\mu I\cdot\mu J$ for all $m\ge n$.)

(u) Let (X,Σ,μ) be a probability space and $\phi:X\to X$ an ergodic inverse-measure-preserving function. Let f be a μ-virtually measurable function defined almost everywhere in X such that $\int fd\mu=\infty$. Show that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f\phi^i$ is infinite a.e. (*Hint*: look at the corresponding limits for $f_m=f\wedge m\chi X$.)

(v) For irrational $x\in[0,1]$, write $k_1(x),k_2(x),\dots$ for the continued-fraction coefficients of x. Show that the limit $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n k_i(x)$ is infinite for almost every x. (*Hint*: take ϕ, ν as in 372M, and show that $\int k_1d\nu=\infty$.)

(w) Let (X,Σ,μ) be any probability space, and let λ be the product measure on $X^{\mathbb{N}}$. Define $\phi:X^{\mathbb{N}}\to X^{\mathbb{N}}$ by setting $\phi(x)(n)=x(n{+}1)$. Show that ϕ is inverse-measure-preserving. Show that ϕ satisfies the conditions of 372S, so is mixing.

(x) Let (X,Σ,μ) be any probability space, and λ the product measure on $X^{\mathbb{Z}}$. Define $\phi:X^{\mathbb{Z}}\to X^{\mathbb{Z}}$ by setting $\phi(x)(n)=x(n+1)$. Show that ϕ is inverse-measure-preserving. Show that ϕ is mixing. (*Hint*: show that if C, C' are basic cylinder sets then $\mu(C\cap\phi^{-n}[C'])=\mu C\cdot\mu C'$ for all n large enough.) Show that ϕ does not ordinarily satisfy the conditions of 372S. (Compare 333P.)

(y)(i) Let $\mathfrak{A}$ be a Boolean algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a Boolean homomorphism, and $\phi:\mathfrak{A}\to\mathfrak{A}$ an automorphism. Show that if π is ergodic then $\phi\pi\phi^{-1}$ is ergodic. (ii) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean homomorphism, and $\phi:\mathfrak{A}\to\mathfrak{A}$ a measure-preserving Boolean automorphism. Show that if π is mixing, or weakly mixing, then so is $\phi\pi\phi^{-1}$.

372Y Further exercises (a) In 372D, show that the null space of the limit operator P is precisely the closure in $M^{1,0}$ of the subspace $\{Tu-u:u\in M^{1,0}\}$.

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$, $p\in\,]1,\infty[$ and $u\in L^p(\mathfrak{A},\bar\mu)$. Set $u^*=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{i=0}^n|T^iu|$. (i) Show that for any $\gamma>0$,

$$\bar\mu[\![u^*>\gamma]\!]\le\frac{2}{\gamma}\int_{[\![|u|>\gamma/2]\!]}|u|.$$

(*Hint*: apply 372C to $(|u|-\frac{1}{2}\gamma\chi 1)^+$.) (ii) Show that $\|u^*\|_p\le 2(\frac{p}{p-1})^{1/p}\|u\|_p$. (*Hint*: show that $\int_{[\![|u|>\alpha]\!]}|u|=\alpha\bar\mu[\![|u|>\alpha]\!]+\int_\alpha^\infty\bar\mu[\![|u|>\beta]\!]d\beta$; see 365A. Use 366Xa to show that

$$\|u^*\|_p^p\le 2p\int_0^\infty\gamma^{p-2}\int_{\gamma/2}^\infty\bar\mu[\![|u|>\beta]\!]d\beta d\gamma+2^p\|u\|_p^p,$$

and reverse the order of integration. Compare 275Yd.) (This is **Wiener's Dominated Ergodic Theorem**.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and T an operator in $\mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$. Take $u\in L^1=L^1(\mathfrak{A},\bar\mu)$ such that $\bar h(|u|)\in L^1$, where $h(t)=t\ln t$ for $t\ge 1$, 0 for $t\le 1$, and $\bar h$ is the corresponding function from $L^0(\mathfrak{A})$ to itself. Set $u^*=\sup_{n\in\mathbb{N}}\frac{1}{n+1}\sum_{i=0}^n|T^iu|$. Show that $u^*\in L^1$. (*Hint*: use the method of 372Yb to show that $\int_2^\infty\bar\mu[\![u^*>\gamma]\!]d\gamma\le 2\int\bar h(u)$.)

(d) Let U be a Banach space, $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra and $\langle T_n\rangle_{n\in\mathbb{N}}$ a sequence of continuous linear operators from U to $L^0=L^0(\mathfrak{A})$ with its topology of convergence in measure. Suppose that $\sup_{n\in\mathbb{N}}T_nu$ is defined in L^0 for every $u\in U$. Show that $\{u:u\in U,\ \langle T_nu\rangle_{n\in\mathbb{N}}$ is order*-convergent in $L^0\}$ is a norm-closed linear subspace of U.

(e) In 372G, suppose that $\mathfrak{A}$ is atomless. Show that there is always an $a\in\mathfrak{A}$ such that $\bar\mu a\le\frac{1}{2}$ and $\inf_{i\le n}\pi^ia\ne 0$ for every n, so that (except in trivial cases) $\langle A_n(\chi a)\rangle_{n\in\mathbb{N}}$ will not be $\|\,\|_\infty$-convergent.

(f) Let (X,Σ,μ) be a measure space with measure algebra $(\mathfrak{A},\bar\mu)$. Let Φ be a family of inverse-measure-preserving functions from X to itself, and for $\phi\in\Phi$ let $\pi_\phi:\mathfrak{A}\to\mathfrak{A}$ be the associated homomorphism. Set $\mathfrak{C}=\{c:c\in\mathfrak{A},\ \pi_\phi c=c$ for every $\phi\in\Phi\}$, $\mathrm{T}=\{E:E\in\Sigma,\ \phi^{-1}[E]\triangle E$ is negligible for every $\phi\in\Phi\}$ and $\mathrm{T}_0=\{E:E\in\Sigma,\ \phi^{-1}[E]=E$ for every $\phi\in\Phi\}$. Show that (i) T and T_0 are σ-subalgebras of Σ (ii) $\mathrm{T}_0\subseteq\mathrm{T}$ (iii) $\mathrm{T}=\{E:E\in\Sigma,\ E^\bullet\in\mathfrak{C}\}$ (iv) if Φ is countable and $\phi\psi=\psi\phi$ for all $\phi,\ \psi\in\Phi$, then $\mathfrak{C}=\{E^\bullet:E\in\mathrm{T}_0\}$.

(g) Show that an irrational $x\in\,]0,1[$ has an eventually periodic sequence of continued fraction coefficients iff it is a solution of a quadratic equation with integral coefficients.

(h) In the language of 372L-372N and 372Xl, show the following. (i) For any $x\in X$ and $n\ge 2$, $q_n(x)q_{n-1}(x)\ge 2^{n-1}$ and $p_n(x)p_{n+1}(x)\ge 2^{n-1}$, so that $q_{n+1}(x)p_n(x)\ge 2^{n-1}$, $|1-x/r_n(x)|\le 2^{-n+1}$ and $|\ln x-\ln r_n(x)|\le 2^{-n+2}$. Also $|x-r_n(x)|\ge 1/q_n(x)q_{n+2}(x)$. (ii) For any $x\in X$, $n\ge 1$, $p_{n+1}(x)=q_n(\phi(x))$ and $q_n(x)\prod_{i=0}^{n-1}r_{n-i}(\phi^i(x))=1$. (iii) For any $x\in X$, $n\ge 1$, $|\ln q_n(x)+\sum_{i=0}^{n-1}\ln\phi^i(x)|\le 4$. (iv) For almost every $x\in X$,

$$\lim_{n\to\infty}\frac{1}{n}\ln q_n(x)=-\frac{1}{\ln 2}\int_0^1\frac{\ln t}{1+t}dt=\frac{\pi^2}{12\ln 2}.$$

(*Hint*: 225Xg, 282Xo.) (v) For almost every $x\in X$, $\lim_{n\to\infty}\frac{1}{n}\ln|x-r_n(x)|=-\frac{\pi^2}{6\ln 2}$. (vi) For almost every $x\in X$, $11^{-n}\le|x-r_n(x)|\le 10^{-n}$ and $3^n\le q_n(x)\le 4^n$ for all but finitely many n.

(i)(i) Let (X,Σ,μ) and (Y,T,ν) be probability spaces, with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Suppose that $\phi:X\to X$ is a weakly mixing inverse-measure-preserving function and $\psi:Y\to Y$ is an ergodic inverse-measure-preserving function. Define $\theta:X\times Y\to X\times Y$ by setting $\theta(x,y)=(\phi(x),\psi(y))$ for all x, y. Show that θ is an ergodic inverse-measure-preserving function. (ii) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be probability algebras, with probability algebra free product $(\mathfrak{C},\bar\lambda)$. Suppose that $\phi:\mathfrak{A}\to\mathfrak{A}$ is a weakly mixing measure-preserving Boolean homomorphism and $\psi:\mathfrak{B}\to\mathfrak{B}$ is an ergodic measure-preserving Boolean homomorphism. Let

$\theta : \mathfrak{C} \to \mathfrak{C}$ be the measure-preserving Boolean homomorphism such that $\theta(a \otimes b) = \phi a \otimes \psi b$ for all $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ (325Xe). (α) Show that θ is ergodic. (β) Show that if ψ is weakly mixing then θ is weakly mixing. (γ) Show that if ϕ and ψ are mixing then θ is mixing.

(j) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be any family of probability spaces, with product (X, Λ, λ). Suppose that for each $i \in I$ we are given an inverse-measure-preserving function $\phi_i : X_i \to X_i$. (i) Show that there is a corresponding inverse-measure-preserving function $\phi : X \to X$ given by setting $\phi(x)(i) = \phi_i(x(i))$ for $x \in X$, $i \in I$. (ii) Show that if every ϕ_i is mixing so is ϕ. (iii) Show that if every ϕ_i is weakly mixing so is ϕ.

(k) Give an example of an ergodic measure-preserving automorphism $\phi : [0, 1[\to [0, 1[$ such that ϕ^2 is not ergodic. (*Hint*: set $\phi(x) = \frac{1}{2}(1 + \phi_0(2x))$ for $x < \frac{1}{2}$, $x - \frac{1}{2}$ for $x \ge \frac{1}{2}$. See also 388Xg.)

(l) Show that there is an ergodic $\phi : [0, 1] \to [0, 1]$ such that $(\xi_1, \xi_2) \mapsto (\phi(\xi_1), \phi(\xi_2)) : [0, 1]^2 \to [0, 1]^2$ is not ergodic. (*Hint*: 372Xr.)

(m) Let M be an $r \times r$ matrix with integer coefficients and non-zero determinant, where $r \ge 1$. Let $\phi : [0, 1[^r \to [0, 1[^r$ be the function such that $\phi(x) - Mx \in \mathbb{Z}^r$ for every $x \in [0, 1[^r$. Show that ϕ is inverse-measure-preserving for Lebesgue measure on $[0, 1[^r$.

(n)(i) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ a weakly mixing measure-preserving Boolean homomorphism, and $T = T_\pi : L^1_{\bar\mu} \to L^1_{\bar\mu}$ the corresponding linear operator (365N). Show that if $u \in L^1_{\bar\mu}$ is such that $\{T^n u : n \in \mathbb{N}\}$ is relatively compact for the norm topology, then $u = \alpha\chi 1$ for some α. (ii) Let μ be Lebesgue measure on $[0, 1[$, $(\mathfrak{A}, \bar\mu)$ its measure algebra, $\alpha \in [0, 1[$ an irrational number, $\phi(x) = x +_1 \alpha$ for $x \in [0, 1[$ (as in 372Xr), and $T : L^1(\mu) \to L^1(\mu)$ the linear operator defined by setting $Tg^\bullet = (g\phi)^\bullet$ for $g \in \mathcal{L}^1(\mu)$. Show that $\{T^n : n \in \mathbb{Z}\}$ is relatively compact for the strong operator topology on $\mathsf{B}(L^1(\mu); L^1(\mu))$.

(o) In 372M, show that for any measurable set $E \subseteq X$, $\lim_{n\to\infty} \mu_L \phi^{-n}[E] = \nu E$. (*Hint*: recall that ϕ is mixing for ν (372Xm). Hence show that $\lim_{n\to\infty} \int_{\phi^{-n}[E]} g\,d\nu = \nu E \cdot \int g\,d\nu$ for any integrable g. Apply this to a Radon-Nikodým derivative of μ_L with respect to ν.) (I understand that this result is due to Gauss.)

(p) (i) Show that there are a Boolean algebra $\mathfrak{A}$ and an automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ which is not ergodic, but has fixed-point algebra $\{0, 1\}$. (ii) Show that there are a σ-finite measure algebra $(\mathfrak{A}, \bar\mu)$ and a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ which is not ergodic, but has fixed-point algebra $\{0, 1\}$.

(q) For a Boolean algebra $\mathfrak{A}$ and a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$, write T_π for the corresponding operator from $L^\infty(\mathfrak{A})$ to itself, as defined in 363F. (i) Suppose that $\mathfrak{A}$ is a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, $u \in L^\infty(\mathfrak{A})$ and $T_\pi u = u$. Show that if either π is ergodic or $\mathfrak{A}$ is Dedekind σ-complete and the fixed-point subalgebra of π is $\{0, 1\}$, then u must be a multiple of $\chi 1$. (ii) Find a Boolean algebra $\mathfrak{A}$, an automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ with fixed-point algebra $\{0, 1\}$, and a $u \in L^\infty(\mathfrak{A})$, not a multiple of $\chi 1$, such that $T_\pi u = u$.

(r) Set $\mathcal{F}_d = \{I : I \subseteq \mathbb{N}, \lim_{n\to\infty} \frac{1}{n}\#(I \cap n) = 1\}$. (i) Show that $\mathcal{F}_d$ is a filter on $\mathbb{N}$. (ii) Show that for a bounded sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$, the following are equiveridical: (α) $\lim_{n\to\mathcal{F}_d} \alpha_n = 0$; ($\beta$) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^n |\alpha_k| = 0$; ($\gamma$) $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^n \alpha_k^2 = 0$. ($\mathcal{F}_d$ is called the **(asymptotic) density filter**.)

(s) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\phi : \mathfrak{A} \to \mathfrak{A}$ a measure-preserving Boolean homomorphism. (i) Show that there are a probability algebra $(\mathfrak{C}, \bar\lambda)$, a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$, and a measure-preserving automorphism $\tilde\phi : \mathfrak{C} \to \mathfrak{C}$ such that $\tilde\phi\pi = \pi\phi$ and $\mathfrak{C}$ is the closure of $\bigcup_{n\in\mathbb{N}} \tilde\phi^{-n}[\pi[\mathfrak{A}]]$ for the measure-algebra topology. (*Hint*: 328J.) (ii) Show that $\tilde\phi$ is ergodic, or mixing, or weakly mixing iff ϕ is.

372 Notes and comments I have chosen an entirely conventional route to the Ergodic Theorem here, through the Mean Ergodic Theorem (372Xa) or, rather, the fundamental lemma underlying it (372A), and the Maximal Ergodic Theorem (372B-372C). What is not to be found in every presentation is the generality here. I speak of arbitrary $T \in \mathcal{T}^{(0)}$, the operators which are contractions both for $\| \|_1$ and for $\| \|_\infty$, not requiring T to be positive, let alone correspond to a measure-preserving homomorphism. (I do not mention $\mathcal{T}^{(0)}$ in the statement of 372C, but of course it is present in spirit.) The work we have done up to this point puts this extra generality within easy reach, but as the rest of the section shows, it is not needed for the principal examples. Only in 372Xc do I offer an application not associated in the usual way with a measure-preserving homomorphism or an inverse-measure-preserving function.

The Ergodic Theorem is an 'almost-everywhere pointwise convergence theorem', like the strong law(s) of large numbers and the martingale theorem(s) (§273, §275). Indeed Kolmogorov's form of the strong law can be derived from the Ergodic Theorem (372Xg). There are some very strong family resemblances. For instance, the Maximal Ergodic Theorem corresponds to the most basic of all the martingale inequalities (275D). Consequently we have similar results, obtained by similar methods, concerning the domination of sequences starting from members of L^p (372Yb, 275Yd), though the inequalities are not identical. (Compare also 372Yc with 275Ye.) There are some tantalising reflections of these traits in results surrounding Carleson's theorem on the pointwise convergence of square-integrable Fourier series (see §286 *notes*), but Carleson's theorem seems to be much harder than the others. Other forms of the strong law (273D, 273H) do not appear to fit into quite the same pattern, but I note that here, as with the Ergodic Theorem, we begin with a study of square-integrable functions (see part (e) of the proof of 372D).

After 372D, there is a contraction and concentration in the scope of the results, starting with a simple replacement of $M^{1,0}$ with L^1 (372F). Of course it is almost as easy to prove 372D from 372F as the other way about; I give precedence to 372D only because $M^{1,0}$ is the space naturally associated with the class $\mathcal{T}^{(0)}$ of operators to which these methods apply. Following this I turn to the special family of operators to which the rest of the section is devoted, those associated with measure-preserving homomorphisms (372E), generally on probability spaces (372G). This is the point at which we can begin to identify the limit as a conditional expectation as well as an invariant element.

Next comes the translation into the language of measure spaces and inverse-measure-preserving functions, all perfectly straightforward in view of 372I. These turn 372E into 372H and 372G into the main part of 372J.

In 372J-372K I find myself writing at some length about a technical problem. The root of the difficulty is in the definition of 'conditional expectation'. Now it is generally accepted that any pure mathematician has 'Humpty Dumpty's privilege': 'When *I* use a word, it means just what I choose it to mean – neither more nor less'. With any privilege come duties and responsibilities; here, the duty to be self-consistent, and the responsibility to try to use terms in ways which will not mystify or mislead the unprepared reader. Having written down a definition of 'conditional expectation' in Volume 2, I must either stick to it, or go back and change it, or very carefully explain exactly what modification I wish to make here. I don't wish to suggest that absolute consistency – in terminology or anything else – is supreme among mathematical virtues. Surely it is better to give local meanings to words, or tolerate ambiguities, than to suppress ideas which cannot be formulated effectively otherwise, and among 'ideas' I wish to include the analogies and resonances which a suitable language can suggest. But I do say that it is always best to be conscious of what one is doing – I go farther: one of the things which mathematics is for, is to raise our consciousness of what our thoughts really are. So I believe it is right to pause occasionally over such questions.

In 372L-372N (see also 372Xl, 372Xv, 372Xm, 372Xk, 372Yh, 372Yo) I make an excursion into number theory. This is a remarkable example of the power of advanced measure theory to give striking results in other branches of mathematics. Everything here is derived from Billingsley 65, who goes farther than I have space for, and gives references to more. Here let me point to 372Xj; almost accidentally, the construction offers a useful formula for a homeomorphism between two of the most important spaces of descriptive set theory, which will be important to us in Volume 4.

I end the section by introducing three terms, 'ergodic', 'mixing' and 'weakly mixing' transformations, not because I wish to use them for any new ideas (apart from the elementary 372P-372S, these must wait for §§385-387 below and §494 in Volume 4), but because it may help if I immediately classify some of the inverse-measure-preserving functions we have seen (372Xp-372Xr, 372Xt, 372Xw, 372Xx). Of course in any application of any ergodic theorem it is of great importance to be able to identify the limits promised by the theorem, and the point about an ergodic transformation is just that our averages converge to constant limits (372Q). Actually proving that a given inverse-measure-preserving function is ergodic is rarely quite trivial (see 372M, 372Xq, 372Xr), though a handful of standard techniques cover a large number of cases, and it is usually obvious when a map is *not* ergodic, so that if an invariant region does not leap to the eye one has a good hope of ergodicity. The extra concept of 'weakly mixing' transformation is hardly relevant to anything in this volume (though see 372Yi-372Yj), but is associated with a remarkable topological fact about automorphism groups of probability algebras, to come in 494E.

I ought to remark on the odd shift between the definitions of 'ergodic Boolean homomorphism' and 'ergodic inverse-measure-preserving function' in 372O. The point is that the version in 372O(b-i) is the standard formulation in this context, but that its natural translation into the version 'a Boolean homomorphism from a probability algebra to itself is ergodic if its fixed-point subalgebra is trivial', although perfectly satisfactory in that context, allows unwelcome phenomena if applied to general Boolean algebras (372Yp, 372Yq). The definition in 372O(a-i) is rather closer to the essential idea of ergodicity of a dynamical system, which asks that the system should always evolve along a path which approximates all possible states. In practice, however, we shall nearly always be dealing with automorphisms of Dedekind σ-complete algebras, for which we can use the fixed-point criterion of 372Pc.

I take the opportunity to mention two famous functions from $[0,1]$ to itself, the 'tent' and 'logistic' maps (372Xp, 372Xq). In the formulae ϕ_α, ψ_β I include redundant parameters; this is because the real importance of these functions lies in the way their behaviour depends, in bewildering complexity, on these parameters. It is only at the extreme values $\alpha=2$, $\beta=4$ that the methods of this volume can tell us anything interesting.

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373 Decreasing rearrangements

I take a section to discuss operators in the class $\mathcal{T}^{(0)}$ of 371F-371H and two associated classes $\mathcal{T}$, $\mathcal{T}^\times$ (373A). These turn out to be intimately related to the idea of 'decreasing rearrangement' (373C). In 373D-373F I give elementary properties of decreasing rearrangements; then in 373G-373O I show how they may be used to characterize the set $\{Tu:T\in\mathcal{T}\}$ for a given u. The argument uses a natural topology on the set $\mathcal{T}$ (373K). I conclude with remarks on the possible values of $\int Tu\times v$ for $T\in\mathcal{T}$ (373P-373Q) and identifications between $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ and $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ (373R-373T).

373A Definition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Recall that $M^{1,\infty}(\mathfrak{A},\bar\mu)=L^1(\mathfrak{A},\bar\mu)+L^\infty(\mathfrak{A})$ is the set of those $u\in L^0(\mathfrak{A})$ such that $(|u|-\alpha\chi1)^+$ is integrable for some α, its norm $\|\,\|_{1,\infty}$ being defined by the formulae

$$\begin{aligned}\|u\|_{1,\infty}&=\min\{\|v\|_1+\|w\|_\infty:v\in L^1,\,w\in L^\infty,\,v+w=u\}\\&=\min\{\alpha+\int(|u|-\alpha\chi1)^+:\alpha\ge0\}\end{aligned}$$

(369Ob).

(a) $\mathcal{T}_{\bar\mu,\bar\nu}$ will be the space of linear operators $T:M^{1,\infty}(\mathfrak{A},\bar\mu)\to M^{1,\infty}(\mathfrak{B},\bar\nu)$ such that $\|Tu\|_1\le\|u\|_1$ for every $u\in L^1(\mathfrak{A},\bar\mu)$ and $\|Tu\|_\infty\le\|u\|_\infty$ for every $u\in L^\infty(\mathfrak{A})$. (Compare the definition of $\mathcal{T}^{(0)}$ in 371F.)

(b) If $\mathfrak{B}$ is Dedekind complete, so that $M^{1,\infty}(\mathfrak{A},\bar\mu)$, being a solid linear subspace of the Dedekind complete space $L^0(\mathfrak{B})$, is Dedekind complete, $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ will be $\mathcal{T}_{\bar\mu,\bar\nu}\cap\mathsf{L}^\times(M^{1,\infty}(\mathfrak{A},\bar\mu);M^{1,\infty}(\mathfrak{A},\bar\mu))$.

373B Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras.

(a) $\mathcal{T}=\mathcal{T}_{\bar\mu,\bar\nu}$ is a convex subset of the unit ball of $\mathsf{B}(M^{1,\infty}(\mathfrak{A},\bar\mu);M^{1,\infty}(\mathfrak{B},\bar\nu))$.

(b) If $T\in\mathcal{T}$ then $T\restriction M^{1,0}(\mathfrak{A},\bar\mu)$ belongs to $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$. So if $T\in\mathcal{T}$, $p\in[1,\infty[$ and $u\in L^p(\mathfrak{A},\bar\mu)$ then $Tu\in L^p(\mathfrak{B},\bar\nu)$ and $\|Tu\|_p\le\|u\|_p$.

(c) If $\mathfrak{B}$ is Dedekind complete, then $\mathcal{T}$ is a solid subset of $\mathsf{L}^\sim(M^{1,\infty}(\mathfrak{A},\bar\mu);M^{1,\infty}(\mathfrak{B},\bar\nu))$.

(d) If $\pi:\mathfrak{A}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism, then we have a corresponding operator $T\in\mathcal{T}$ defined by saying that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$. If π is order-continuous, then so is T.

(e) If $(\mathfrak{C},\bar\lambda)$ is another measure algebra, $T\in\mathcal{T}$ and $S\in\mathcal{T}_{\bar\nu,\bar\lambda}$ then $ST\in\mathcal{T}_{\bar\mu,\bar\lambda}$.

proof (a) As 371G, parts (a-i) and (a-ii) of the proof.

(b) If $u\in M^{1,0}_{\bar\mu}$ and $\epsilon>0$, then u is expressible as $u'+u''$ where $\|u''\|_\infty\le\epsilon$ and $u'\in L^1_{\bar\mu}$. (Set

$$u''=(u^+\wedge\epsilon\chi1)-(u^-\wedge\epsilon\chi1).)$$

So

$$(|Tu|-\epsilon\chi1)^+\le(|Tu|-|Tu''|)^+\le||Tu|-|Tu''||\le|Tu-Tu''|=|Tu'|\in L^1_{\bar\nu}.$$

As ϵ is arbitrary, $Tu\in M^{1,0}_{\bar\nu}$; as u is arbitrary, $T\restriction M^{1,0}_{\bar\mu}\in\mathcal{T}^{(0)}$. Now the rest is a consequence of 371Gd.

(c)(i) Because $M^{1,\infty}_{\bar\nu}$ is a solid linear subspace of $L^0(\mathfrak{B})$, which is Dedekind complete because $\mathfrak{B}$ is, $\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu};M^{1,\infty}_{\bar\nu})$ is a Riesz space (355Ea).

(ii) $\mathcal{T}\subseteq\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu};M^{1,\infty}_{\bar\nu})$. **P** Suppose that $T\in\mathcal{T}$. Take any $u\ge0$ in $M^{1,\infty}_{\bar\mu}$. Let $\alpha\ge0$ be such that $(u-\alpha\chi1)^+\in L^1_{\bar\mu}$. Because $T\restriction L^1_{\bar\mu}$ belongs to $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})=\mathsf{L}^\sim(L^1_{\bar\mu};L^1_{\bar\nu})$ (371D), $w_0=\sup\{Tv:v\in L^1_{\bar\mu},\ 0\le v\le(u-\alpha\chi1)^+\}$ is defined in $L^1_{\bar\nu}$. Now if $v\in M^{1,\infty}_{\bar\mu}$ and $0\le v\le u$, we must have

$$Tv=T(v-\alpha\chi1)^++T(v\wedge\alpha\chi1)\le w_0+\alpha\chi1\in M^{1,\infty}_{\bar\nu}.$$

Thus $\{Tv:0\le v\le u\}$ is bounded above in $M^{1,\infty}_{\bar\nu}$. As u is arbitrary, $T\in\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu};M^{1,\infty}_{\bar\nu})$ (355Ba). **Q**

(iii) $\mathcal{T}$ is solid in $\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu};M^{1,\infty}_{\bar\nu})$. **P** Suppose that $T\in\mathcal{T}$, $T_1\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu};M^{1,\infty}_{\bar\nu})$ and $|T_1|\le|T|$. Then

$$\|T_1u\|_1\le\||T_1||u|\|_1$$

(355Eb)

$$\le\||T||u|\|_1\le\||T|\restriction L^1_{\bar\mu}\|\||u|\|_1=\|T\restriction L^1_{\bar\mu}\|\|u\|_1$$

(371D)

$$\le\|u\|_1$$

for every $u\in L^1_{\bar\mu}$. At the same time, if $u\in L^\infty(\mathfrak{A})$, then

$$|T_1u|\le|T_1||u|\le|T||u|=\sup_{|v|\le|u|}Tv$$
$$\le\sup_{|v|\le|u|}\|Tv\|_\infty\chi1\le\sup_{|v|\le|u|}\|v\|_\infty\chi1=\|u\|_\infty\chi1,$$

so $\|T_1u\|_\infty\le\|u\|_\infty$. Thus $T_1\in\mathcal{T}$. By 352Ja, this is enough to show that $\mathcal{T}$ is solid. **Q**

(d) By 365N and 363F, we have norm-preserving positive linear operators $T_1:L^1_{\bar\mu}\to L^1_{\bar\nu}$ and $T_\infty:L^\infty(\mathfrak{A})\to L^\infty(\mathfrak{B})$ defined by saying that $T_1(\chi a)=\chi(\pi a)$ whenever $\bar\mu a<\infty$ and $T_\infty(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$. If $u\in S(\mathfrak{A}^f)=L^1_{\bar\mu}\cap S(\mathfrak{A})$ (365F), then $T_1u=T_\infty u$, because both T_1 and T_∞ are linear and they agree on $\{\chi a:\bar\mu a<\infty\}$. If $u\ge0$ in $M^{\infty,1}_{\bar\mu}=L^1_{\bar\mu}\cap L^\infty(\mathfrak{A})$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)$ such that $u=\sup_{n\in\mathbb{N}}u_n$ and

$$\lim_{n\to\infty}\|u-u_n\|_1=\lim_{n\to\infty}\|u-u_n\|_\infty=0$$

(see the proof of 369Od), so that

$$T_1u=\sup_{n\in\mathbb{N}}T_1u_n=\sup_{n\in\mathbb{N}}T_\infty u_n=T_\infty u.$$

Accordingly T_1 and T_∞ agree on $L^1_{\bar\mu}\cap L^\infty(\mathfrak{A})$. But this means that if $u\in M^{1,\infty}_{\bar\mu}$ is expressed as $v+w=v'+w'$, where v, $v'\in L^1_{\bar\mu}$ and w, $w'\in L^\infty(\mathfrak{A})$, we shall have

$$T_1v'+T_\infty w'=T_1v+T_\infty w+T_1(v'-v)-T_\infty(w-w')=T_1v+T_\infty w,$$

because $v'-v=w-w'\in M^{\infty,1}_{\bar\mu}$. Accordingly we have an operator $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ defined by setting

$$T(v+w)=T_1v+T_\infty w \text{ whenever } v\in L^1_{\bar\mu},\ w\in L^\infty(\mathfrak{A}).$$

This formula makes it easy to check that T is linear and positive, and it clearly belongs to $\mathcal{T}$.

To see that T is uniquely defined, observe that $T\restriction L^1_{\bar\mu}$ and $T\restriction L^\infty(\mathfrak{A})$ are uniquely defined by the values T takes on $S(\mathfrak{A}^f)$, $S(\mathfrak{A})$ respectively, because these spaces are dense for the appropriate norms.

Now suppose that π is order-continuous. Then T_1 and T_∞ are also order-continuous (365Na, 363Ff). If $A\subseteq M^{1,\infty}_{\bar\mu}$ is non-empty and downwards-directed and has infimum 0, take $u_0\in A$ and $\gamma>0$ such that $(u_0-\gamma\chi1)^+\in L^1_{\bar\mu}$. Set

$$A_1=\{(u-\gamma\chi1)^+:u\in A,\ u\le u_0\},\quad A_\infty=\{u\wedge\gamma\chi1:u\in A\}.$$

Then $A_1\subseteq L^1_{\bar\mu}$ and $A_\infty\subseteq L^\infty(\mathfrak{A})$ are both downwards-directed and have infimum 0, so $\inf T_1[A_1]=\inf T_\infty[A_\infty]=0$ in $L^0(\mathfrak{B})$. But this means that $\inf(T_1[A_1]+T_\infty[A_\infty])=0$ (351Dc). Now any $w\in T_1[A_1]+T_\infty[A_\infty]$ is expressible as $T(u-\gamma\chi1)^++T(u'\wedge\gamma\chi1)$ where u, $u'\in A$; because A is downwards-directed, there is a $v\in A$ such that $v\le u\wedge u'$, in which case $Tv\le w$. Accordingly $T[A]$ must also have infimum 0. As A is arbitrary, T is order-continuous.

(e) is obvious, as usual.

373C Decreasing rearrangements The following concept is fundamental to any understanding of the class $\mathcal{T}$. Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Write $M^{0,\infty}_{\bar\mu}=M^{0,\infty}(\mathfrak{A},\bar\mu)$ for the set of those $u\in L^0(\mathfrak{A})$ such that $\bar\mu[\![|u|>\alpha]\!]$ is finite for some $\alpha\in\mathbb{R}$. (See 369N for the ideology of this notation.) It is easy to see that $M^{0,\infty}(\mathfrak{A},\bar\mu)$ is a solid linear subspace of $L^0(\mathfrak{A})$. Let $(\mathfrak{A}_L,\bar\mu_L)$ be the measure algebra of Lebesgue measure on $[0,\infty[$. For $u\in M^{0,\infty}(\mathfrak{A},\bar\mu)$ its **decreasing rearrangement** is $u^*\in M^{0,\infty}_{\bar\mu_L}=M^{0,\infty}(\mathfrak{A}_L,\bar\mu_L)$, defined by setting $u^*=g^\bullet$, where

$$g(t)=\min\{\alpha:\alpha\ge0,\ \bar\mu[\![|u|>\alpha]\!]\le t\}$$

for every $t>0$. (The infimum is always finite because $\inf_{\alpha\in\mathbb{R}}\bar\mu[\![|u|>\alpha]\!]=0$, by 364Aa($\beta$) and 321F, and by 364Aa(α) the infimum is attained.)

I will maintain this usage of the symbols $\mathfrak{A}_L$, $\bar\mu_L$, u^* for the rest of this section.

373D Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) For any $u\in M^{0,\infty}(\mathfrak{A},\bar\mu)$, its decreasing rearrangement u^* may be defined by the formula

$$[\![u^*>\alpha]\!]=[0,\bar\mu[\![|u|>\alpha]\!][^\bullet \text{ for every } \alpha\ge0,$$

that is,

$$\bar\mu_L[\![u^*>\alpha]\!]=\bar\mu[\![|u|>\alpha]\!] \text{ for every } \alpha\ge0.$$

(b) If $|u|\le|v|$ in $M^{0,\infty}(\mathfrak{A},\bar\mu)$, then $u^*\le v^*$; in particular, $|u|^*=u^*$.

(c)(i) If $u=\sum_{i=0}^n\alpha_i\chi a_i$, where $a_0\supseteq a_1\supseteq\ldots\supseteq a_n$ and $\alpha_i\ge0$ for each i, then $u^*=\sum_{i=0}^n\alpha_i\chi\,[0,\bar\mu a_i[^\bullet$.

(ii) If $u=\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\dots,a_n$ are disjoint and $|\alpha_0|\ge|\alpha_1|\ge\ldots\ge|\alpha_n|$, then $u^*=\sum_{i=0}^n|\alpha_i|\chi\,[\beta_i,\beta_{i+1}[^\bullet$, where $\beta_i=\sum_{j<i}\bar\mu a_i$ for $i\le n+1$.

(d) If $E\subseteq\,]0,\infty[$ is any Borel set, and $u\in M^0(\mathfrak{A},\bar\mu)$, then $\bar\mu_L[\![u^*\in E]\!]=\bar\mu[\![|u|\in E]\!]$.

(e) Let $h:[0,\infty[\,\to[0,\infty[$ be a non-decreasing function such that $h(0)=0$, and write $\bar h$ for the corresponding functions on $L^0(\mathfrak{A})^+$ and $L^0(\mathfrak{A}_L)^+$ (364H). Then $(\bar h(u))^*=\bar h(u^*)$ whenever $u\ge0$ in $M^0(\mathfrak{A},\bar\mu)$. If h is continuous on the left, $(\bar h(u))^*=\bar h(u^*)$ whenever $u\ge0$ in $M^{0,\infty}(\mathfrak{A},\bar\mu)$.

(f) If $u\in M^{0,\infty}(\mathfrak{A},\bar\mu)$ and $\alpha\ge0$, then

$$(u^*-\alpha\chi1)^+=((|u|-\alpha\chi1)^+)^*.$$

(g) If $u\in M^{0,\infty}(\mathfrak{A},\bar\mu)$, then for any $t>0$

$$\int_0^t u^* = \inf_{\alpha\ge 0} \alpha t + \int (|u| - \alpha\chi 1)^+.$$

(h) If $A \subseteq (M^{0,\infty}(\mathfrak{A},\bar\mu))^+$ is non-empty and upwards-directed and has supremum $u_0 \in M^{0,\infty}(\mathfrak{A},\bar\mu)$, then $u_0^* = \sup_{u\in A} u^*$.

proof (a) Set

$$g(t) = \inf\{\alpha : \bar\mu[\![|u| > \alpha]\!] \le t\}$$

as in 373C. If $\alpha \ge 0$,

$$g(t) > \alpha \iff \bar\mu[\![|u| > \beta]\!] > t \text{ for some } \beta > \alpha \iff \bar\mu[\![|u| > \alpha]\!] > t$$

(because $[\![|u| > \alpha]\!] = \sup_{\beta>\alpha} [\![|u| > \beta]\!]$), so

$$[\![u^* > \alpha]\!] = \{t : g(t) > \alpha\}^\bullet = [0, \bar\mu[\![|u| > \alpha]\!][^\bullet.$$

Of course this formula defines u^*.

(b) This is obvious, either from the definition in 373C or from (a) just above.

(c)(i) Setting $v = \sum_{i=0}^n \alpha_i \chi\,[0, \bar\mu a_i[^\bullet$, we have

$$\begin{aligned}[\![v > \alpha]\!] &= 0 \text{ if } \sum_{i=0}^n \alpha_i \le \alpha,\\ &= [0, \bar\mu a_j[^\bullet \text{ if } \sum_{i=0}^{j-1} \alpha_i \le \alpha < \sum_{i=0}^{j} \alpha_i,\\ &= [0, \bar\mu a_0[^\bullet \text{ if } 0 \le \alpha < \alpha_0,\end{aligned}$$

and in all cases is equal to $[0, \bar\mu[\![|u| > \alpha]\!][^\bullet$.

(ii) A similar argument applies. (If any a_j has infinite measure, then a_i is irrelevant for $i > j$.)

(d) Fix $\gamma > 0$ for the moment, and consider

$$\mathcal{A} = \{E : E \subseteq\,]\gamma, \infty[\text{ is a Borel set}, \bar\mu_L[\![u^* \in E]\!] = \bar\mu[\![|u| \in E]\!]\},$$

$$\mathcal{I} = \{]\alpha, \infty[: \alpha \ge \gamma\}.$$

Then $\mathcal{I} \subseteq \mathcal{A}$ (by (a)), $I \cap J \in \mathcal{I}$ for all I, $J \in \mathcal{I}$, $E \setminus F \in \mathcal{A}$ whenever E, $F \in \mathcal{A}$ and $F \subseteq E$ (because $u \in M^0_{\bar\mu}$, so $\bar\mu[\![|u| \in E]\!] < \infty$), and $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$. So, by the Monotone Class Theorem (136B), $\mathcal{A}$ includes the σ-algebra of subsets of $]\gamma, \infty[$ generated by $\mathcal{I}$; but this must contain $E \cap\,]\gamma, \infty[$ for every Borel set $E \subseteq \mathbb{R}$.

Accordingly, for any Borel set $E \subseteq\,]0, \infty[$,

$$\bar\mu_L[\![u^* \in E]\!] = \sup_{n\in\mathbb{N}} \bar\mu_L[\![u^* \in E \cap\,]2^{-n}, \infty[\,]\!] = \bar\mu[\![|u| \in E]\!].$$

(e) For any $\alpha > 0$, $E_\alpha = \{t : h(t) > \alpha\}$ is a Borel subset of $]0, \infty[$. If $u \in M^0_{\bar\mu}$ then, using (d) above,

$$\bar\mu_L[\![\bar h(u^*) > \alpha]\!] = \bar\mu_L[\![u^* \in E_\alpha]\!] = \bar\mu[\![u \in E_\alpha]\!] = \bar\mu[\![\bar h(u) > \alpha]\!] = \bar\mu_L[\![(\bar h(u))^* > \alpha]\!].$$

As both $(\bar h(u))^*$ and $\bar h(u^*)$ are equivalence classes of non-increasing functions, they must be equal.

If h is continuous on the left, then $E_\alpha =]\gamma, \infty[$ for some γ, so we no longer need to use (d), and the argument works for any $u \in (M^{0,\infty}_{\bar\mu})^+$.

(f) Apply (e) with $h(\beta) = \max(0, \beta - \alpha)$.

(g) Express u^* as $g^\bullet$, where

$$g(s) = \inf\{\alpha : \bar\mu[\![|u| > \alpha]\!] \le s\}$$

for every $s > 0$. Because g is non-increasing, it is easy to check that, for $t > 0$,

$$\int_0^t g = tg(t) + \int_0^\infty \max(0, g(s) - g(t))ds \le \alpha t + \int_0^\infty \max(0, g(s) - \alpha)ds$$

for every $\alpha \ge 0$; so that

$$\int_0^t u^* = \min_{\alpha\ge 0} \alpha t + \int (u^* - \alpha\chi 1)^+.$$

Now

$$\begin{aligned}\int (u^* - \alpha\chi 1)^+ &= \int_0^\infty \bar\mu_L [\![(u^* - \alpha\chi 1)^+ > \beta]\!] d\beta \\ &= \int_0^\infty \bar\mu [\![(|u| - \alpha\chi 1)^+ > \beta]\!] d\beta = \int (|u| - \alpha\chi 1)^+\end{aligned}$$

for every $\alpha \ge 0$, using (f) and 365A, and

$$\int_0^t u^* = \min_{\alpha\ge 0} \alpha t + \int (|u| - \alpha\chi 1)^+.$$

(h)

$$\bar\mu [\![u_0 > \alpha]\!] = \bar\mu(\sup_{u\in A} [\![u > \alpha]\!]) = \sup_{u\in A} \bar\mu [\![u > \alpha]\!]$$

for any $\alpha > 0$, using 364L(a-ii) and 321D. So

$$[\![u_0^* > \alpha]\!] = [0, \bar\mu [\![u_0 > \alpha]\!] [^\bullet = \sup_{u\in A} [0, \bar\mu [\![u > \alpha]\!] [^\bullet = \sup_{u\in A} [\![u^* > \alpha]\!]$$

for every α, and $u_0^* = \sup_{u\in A} u^*$.

373E Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Then $\int |u \times v| \le \int u^* \times v^*$ for all u, $v \in M^{0,\infty}(\mathfrak{A}, \bar\mu)$.

proof (a) Consider first the case u, $v \ge 0$ in $S(\mathfrak{A})$. Then we may express u, v as $\sum_{i=0}^m \alpha_i \chi a_i$, $\sum_{j=0}^n \beta_j \chi b_j$ where $a_0 \supseteq a_1 \supseteq \ldots \supseteq a_m$, $b_0 \supseteq \ldots \supseteq b_n$ in $\mathfrak{A}$ and α_i, $\beta_j \ge 0$ for all i, j (361Ec). Now u^*, v^* are given by

$$u^* = \sum_{i=0}^m \alpha_i \chi [0, \bar\mu a_i[^\bullet, \quad v^* = \sum_{j=0}^n \beta_j \chi [0, \bar\mu b_j[^\bullet$$

(373Dc). So

$$\begin{aligned}\int u \times v &= \sum_{i=0}^m \sum_{j=0}^n \alpha_i \beta_j \bar\mu(a_i \cap b_j) \le \sum_{i=0}^m \sum_{j=0}^n \alpha_i \beta_j \min(\bar\mu a_i, \bar\mu b_j) \\ &= \sum_{i=0}^m \sum_{j=0}^n \alpha_i \beta_j \mu_L([0, \bar\mu a_i[\cap [0, \bar\mu b_j[) = \int u^* \times v^*.\end{aligned}$$

(b) For the general case, we have non-decreasing sequences $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A})^+$ with suprema $|u|$, $|v|$ respectively (364Jd), so that

$$|u \times v| = |u| \times |v| = \sup_{n\in\mathbb{N}} |u| \times v_n = \sup_{m,n\in\mathbb{N}} u_m \times v_n = \sup_{n\in\mathbb{N}} u_n \times v_n$$

and

$$\int |u \times v| = \int \sup_{n\in\mathbb{N}} u_n \times v_n = \sup_{n\in\mathbb{N}} \int u_n \times v_n \le \sup_{n\in\mathbb{N}} \int u_n^* \times v_n^* \le \int u^* \times v^*,$$

using 373Db.

373F Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and u any member of $M^{0,\infty}(\mathfrak{A}, \bar\mu)$.

(a) For any $p \in [1, \infty]$, $u \in L^p(\mathfrak{A}, \bar\mu)$ iff $u^* \in L^p(\mathfrak{A}_L, \bar\mu_L)$, and in this case $\|u\|_p = \|u^*\|_p$.

(b)(i) $u \in M^0(\mathfrak{A}, \bar\mu)$ iff $u^* \in M^0(\mathfrak{A}_L, \bar\mu_L)$;

(ii) $u \in M^{1,\infty}(\mathfrak{A}, \bar\mu)$ iff $u^* \in M^{1,\infty}(\mathfrak{A}_L, \bar\mu_L)$, and in this case $\|u\|_{1,\infty} = \|u^*\|_{1,\infty}$;

(iii) $u \in M^{1,0}(\mathfrak{A}, \bar\mu)$ iff $u^* \in M^{1,0}(\mathfrak{A}_L, \bar\mu_L)$;

(iv) $u \in M^{\infty,1}(\mathfrak{A}, \bar\mu)$ iff $u^* \in M^{\infty,1}(\mathfrak{A}_L, \bar\mu_L)$, and in this case $\|u\|_{\infty,1} = \|u^*\|_{\infty,1}$.

proof (a)(i) Consider first the case $p = 1$. In this case

$$\int |u| = \int_0^\infty \bar\mu [\![|u| > \alpha]\!] d\alpha = \int_0^\infty \bar\mu_L [\![u^* > \alpha]\!] d\alpha = \int u^*.$$

(ii) If $1 < p < \infty$, then by 373De we have $(|u|^p)^* = (u^*)^p$, so that

$$\|u\|_p^p = \int |u|^p = \int (|u|^p)^* = \int (u^*)^p = \|u^*\|_p^p$$

if either $\|u\|_p$ or $\|u^*\|_p$ is finite. (iii) As for $p=\infty$,

$$\|u\|_\infty \le \gamma \iff [\![|u|>\gamma]\!]=0 \iff [\![u^*>\gamma]\!]=0 \iff \|u^*\|_\infty \le \gamma.$$

(b)(i)

$$\begin{aligned} u \in M^0_{\bar\mu} &\iff \bar\mu[\![|u|>\alpha]\!]<\infty \text{ for every } \alpha>0 \\ &\iff \bar\mu_L[\![u^*>\alpha]\!]<\infty \text{ for every } \alpha>0 \iff u^* \in M^0_{\bar\mu_L}. \end{aligned}$$

(ii) For any $\alpha\ge0$,

$$\int(|u|-\alpha\chi1)^+=\int(u^*-\alpha\chi1)^+$$

as in the proof of 373Dg. So $\|u\|_{1,\infty}=\|u^*\|_{1,\infty}$ if either is finite, by the formula in 369Ob.

(iii) This follows from (i) and (ii), because $M^{1,0}=M^0\cap M^{1,\infty}$.

(iv) Allowing ∞ as a value of an integral, we have

$$\begin{aligned} \|u\|_{1,\infty} &= \min\{\alpha+\int(|u|-\alpha\chi1)^+ : \alpha\ge0\} \\ &= \min\{\alpha+\int(u^*-\alpha\chi1)^+ : \alpha\ge0\} = \|u^*\|_{1,\infty} \end{aligned}$$

by 369Ob; in particular, $u\in M^{1,\infty}_{\bar\mu}$ iff $u^*\in M^{1,\infty}_{\bar\mu_L}$.

373G Lemma Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. If

either $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $T\in\mathcal{T}_{\bar\mu,\bar\nu}$

or $u\in M^{1,0}(\mathfrak{A},\bar\mu)$ and $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$,

then $\int_0^t(Tu)^*\le\int_0^t u^*$ for every $t\ge0$.

proof Set $T_1=T\restriction L^1_{\bar\mu}$, so that $\|T_1\|\le1$ in $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})$, and $|T_1|$ is defined in $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\nu})$, also with norm at most 1. If $\alpha\ge0$, then we can express u as u_1+u_2 where $|u_1|\le(|u|-\alpha\chi1)^+$ and $|u_2|\le\alpha\chi1$. (Let $w\in L^\infty(\mathfrak{A})$ be such that $\|w\|_\infty\le1$, $u=|u|\times w$; set $u_2=w\times(|u|\wedge\alpha\chi1)$.) So if $\int(|u|-\alpha\chi1)^+<\infty$,

$$|Tu|\le|Tu_1|+|Tu_2|\le|T_1||u_1|+\alpha\chi1$$

and

$$\int(|Tu|-\alpha\chi1)^+\le\int|T_1||u_1|\le\int|u_1|\le\int(|u|-\alpha\chi1)^+.$$

The formula of 373Dg now tells us that $\int_0^t(Tu)^*\le\int_0^t u^*$ for every t.

373H Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\theta:\mathfrak{A}^f\to\mathbb{R}$ an additive functional, where $\mathfrak{A}^f=\{a:\bar\mu a<\infty\}$.

(a) The following are equiveridical:

(α) $\lim_{t\downarrow0}\sup_{\bar\mu a\le t}|\theta a|=\lim_{t\to\infty}\frac{1}{t}\sup_{\bar\mu a\le t}|\theta a|=0$,

(β) there is some $u\in M^{1,0}(\mathfrak{A},\bar\mu)$ such that $\theta a=\int_a u$ for every $a\in\mathfrak{A}^f$,

and in this case u is uniquely defined.

(b) Now suppose that $(\mathfrak{A},\bar\mu)$ is localizable. Then the following are equiveridical:

(α) $\lim_{t\downarrow0}\sup_{\bar\mu a\le t}|\theta a|=0$, $\limsup_{t\to\infty}\frac{1}{t}\sup_{\bar\mu a\le t}|\theta a|<\infty$,

(β) there is some $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$ such that $\theta a=\int_a u$ for every $a\in\mathfrak{A}^f$,

and again this u is uniquely defined.

proof (a)(i) Assume (α). For a, $c\in\mathfrak{A}^f$, set $\theta_c(a)=\theta(a\cap c)$. Then for each $c\in\mathfrak{A}^f$, there is a unique $u_c\in L^1_{\bar\mu}$ such that $\theta_c a=\int_a u_c$ for every $a\in\mathfrak{A}^f$ (365Eb). Because u_c is unique we must have $u_c=u_d\times\chi c$

whenever $c \subseteq d \in \mathfrak{A}^f$. Next, given $\alpha > 0$, there is a $t_0 \ge 0$ such that $|\theta a| \le \alpha\bar\mu a$ whenever $a \in \mathfrak{A}^f$ and $\bar\mu a \ge t_0$; so that $\bar\mu[\![u_c > \alpha]\!] \le t_0$ for every $c \in \mathfrak{A}^f$, and $e(\alpha) = \sup_{c\in\mathfrak{A}^f}[\![u_c^+ > \alpha]\!]$ is defined in $\mathfrak{A}^f$. Of course $e(\alpha) = [\![u_{e(1)}^+ > \alpha]\!]$ for every $\alpha \ge 1$, so $\inf_{\alpha\in\mathbb{R}} e(\alpha) = 0$, and $v_1 = \sup_{c\in\mathfrak{A}^f} u_c^+$ is defined in $L^0 = L^0(\mathfrak{A})$ (364L(a-ii) again). Because $[\![v_1 > \alpha]\!] = e(\alpha) \in \mathfrak{A}^f$ for each $\alpha > 0$, $v_1 \in M^0_{\bar\mu}$. For any $a \in \mathfrak{A}^f$,

$$v_1 \times \chi a = \sup_{c\in\mathfrak{A}^f} u_c^+ \times \chi a = u_a^+,$$

so $v_1 \in M^{1,0}_{\bar\mu}$ and $\int_a v_1 = \int_a u_a^+$ for every $a \in \mathfrak{A}^f$.

Similarly, $v_2 = \sup_{c\in\mathfrak{A}^f} u_c^-$ is defined in $M^{1,0}_{\bar\mu}$ and $\int_a v_2 = \int_a u_a^-$ for every $a \in \mathfrak{A}^f$. So we can set $u = v_1 - v_2 \in M^{1,0}_{\bar\mu}$ and get

$$\int_a u = \int_a u_a = \theta a$$

for every $a \in \mathfrak{A}^f$. Thus (β) is true.

(ii) Assume (β). If $\epsilon > 0$, there is a $\delta > 0$ such that $\int_a(|u| - \epsilon\chi 1)^+ \le \epsilon$ whenever $\bar\mu a \le \delta$ (365Ea), so that $|\int_a u| \le \epsilon(1 + \bar\mu a)$ whenever $\bar\mu a \le \delta$. As ϵ is arbitrary, $\lim_{t\downarrow 0}\sup_{\bar\mu a\le t}|\int_a u| = 0$. Moreover, whenever $t > 0$ and $\bar\mu a \le t$, $\frac{1}{t}|\int_a u| \le \epsilon + \frac{1}{t}\int(|u| - \epsilon\chi 1)^+$. Thus

$$\limsup_{t\to\infty}\tfrac{1}{t}\sup_{\bar\mu a\le t}|\int_a u| \le \epsilon.$$

As ϵ is arbitrary, θ satisfies the conditions in (α).

(iii) The uniqueness of u is a consequence of 366Gd.

(b) The argument for (b) uses the same ideas.

(i) Assume (α). Again, for each $c \in \mathfrak{A}^f$, we have a unique $u_c \in L^1_{\bar\mu}$ such that $\theta_c a = \int_a u_c$ for every $a \in \mathfrak{A}^f$; again, set $e(\alpha) = \sup_{c\in\mathfrak{A}^f}[\![u_c^+ > \alpha]\!]$, which is still defined because $\mathfrak{A}$ is supposed to be Dedekind complete. This time, there are t_0, $\gamma \ge 0$ such that $|\theta a| \le \gamma\bar\mu a$ whenever $a \in \mathfrak{A}^f$ and $\bar\mu a \ge t_0$; so that $\bar\mu[\![u_c > \gamma]\!] \le t_0$ for every $c \in \mathfrak{A}^f$, and $\bar\mu e(\gamma) < \infty$. Accordingly

$$\inf_{\alpha\ge\gamma} e(\alpha) = \inf_{\alpha\ge\gamma}[\![u_{e(\gamma)}^+ > \alpha]\!] = 0,$$

and once more $v_1 = \sup_{c\in\mathfrak{A}^f} u_c^+$ is defined in $L^0 = L^0(\mathfrak{A})$. As before, $v_1 \times \chi a = u_a^+ \in L^1_{\bar\mu}$ for any $a \in \mathfrak{A}^f$, Because $[\![v_1 > \gamma]\!] = e(\gamma) \in \mathfrak{A}^f$, $v_1 \in M^{1,\infty}_{\bar\mu}$. Similarly, $v_2 = \sup_{c\in\mathfrak{A}^f} u_c^-$ is defined in $M^{1,\infty}_{\bar\mu}$, with $v_2 \times \chi a = u_a^-$ for every $a \in \mathfrak{A}^f$. So $u = v_1 - v_2 \in M^{1,\infty}_{\bar\mu}$, and

$$\int_a u = \int_a u_a = \theta a$$

for every $a \in \mathfrak{A}^f$.

(ii) Assume (β). Take $\gamma \ge 0$ such that $\beta = \int(|u| - \gamma\chi 1)^+$ is finite. If $\epsilon > 0$, there is a $\delta > 0$ such that $\int_a(|u| - \gamma\chi 1)^+ \le \epsilon$ whenever $\bar\mu a \le \delta$, so that $|\int_a u| \le \epsilon + \gamma\bar\mu a$ whenever $\bar\mu a \le \delta$. As ϵ is arbitrary, $\lim_{t\downarrow 0}\sup_{\bar\mu a\le t}|\int_a u| = 0$. Moreover, whenever $t > 0$ and $\bar\mu a \le t$, then $\frac{1}{t}|\int_a u| \le \gamma + \frac{1}{t}\int(|u| - \epsilon\chi 1)^+$. Thus

$$\limsup_{t\to\infty}\tfrac{1}{t}\sup_{\bar\mu a\le t}|\int_a u| \le \gamma < \infty,$$

and the function $a \mapsto \int_a u$ satisfies the conditions in (β).

(iii) u is uniquely defined because $u \times \chi a$ must be u_a, as defined in (i), for every $a \in \mathfrak{A}^f$, and $(\mathfrak{A}, \bar\mu)$ is semi-finite.

373I Lemma Suppose that u, v, $w \in M^{0,\infty}(\mathfrak{A}_L, \bar\mu_L)$ are all equivalence classes of non-negative non-increasing functions. If $\int_0^t u \le \int_0^t v$ for every $t \ge 0$, then $\int u \times w \le \int v \times w$.

proof For $n \in \mathbb{N}$, $i \le 4^n$ set $a_{ni} = [\![w > 2^{-n}i]\!]$; set $w_n = \sum_{i=1}^{4^n} 2^{-n}\chi a_{ni}$. Then each a_{ni} is of the form $[0,t]^\bullet$, so

$$\int u \times w_n = \sum_{i=1}^{4^n} 2^{-n}\int_{a_{ni}} u \le \sum_{i=1}^{4^n} 2^{-n}\int_{a_{ni}} v = \int v \times w_n.$$

Also $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum w, so

$$\int u \times w = \sup_{n\in\mathbb{N}} \int u \times w_n \le \sup_{n\in\mathbb{N}} \int v \times w_n = \int v \times w.$$

373J Corollary Suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are measure algebras and $v \in M^{0,\infty}(\mathfrak{B},\bar\nu)$. If

either $u \in M^{1,0}(\mathfrak{A},\bar\mu)$ and $T \in \mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$

or $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $T \in \mathcal{T}_{\bar\mu,\bar\nu}$

then $\int |Tu \times v| \le \int u^* \times v^*$.

proof Put 373E, 373G and 373I together.

373K The very weak operator topology Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be two measure algebras. For $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $w \in M^{\infty,1}(\mathfrak{B},\bar\nu)$ set

$$\tau_{uw}(T) = |\int Tu \times w| \text{ for } T \in \mathsf{B} = \mathsf{B}(M^{1,\infty}(\mathfrak{A},\bar\mu); M^{1,\infty}(\mathfrak{B},\bar\nu)).$$

Then τ_{uw} is a seminorm on B. I will call the topology generated by $\{\tau_{uw} : u \in M^{1,\infty}(\mathfrak{A},\bar\mu),\ w \in M^{\infty,1}(\mathfrak{B},\bar\nu)\}$ (2A5B) the **very weak operator topology** on B.

373L Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a localizable measure algebra. Then $\mathcal{T} = \mathcal{T}_{\bar\mu,\bar\nu}$ is compact in the very weak operator topology.

proof Let $\mathcal{F}$ be an ultrafilter on $\mathcal{T}$. If $u \in M^{1,\infty}_{\bar\mu}$ and $w \in M^{\infty,1}_{\bar\nu}$ then

$$|\int Tu \times w| \le \int u^* \times w^* < \infty$$

for every $T \in \mathcal{T}$ (373J); $\int u^* \times w^*$ is finite because $u^* \in M^{1,\infty}_{\bar\mu_L}$ and $w^* \in M^{\infty,1}_{\bar\mu_L}$ (373F).

In particular, $\{\int Tu \times w : T \in \mathcal{T}\}$ is bounded. Consequently $h_u(w) = \lim_{T\to\mathcal{F}} \int Tu \times w$ is defined in $\mathbb{R}$ (2A3Se). Because $w \mapsto \int Tu \times w$ is additive for every $T \in \mathcal{T}$, so is h_u. Also

$$|h_u(w)| \le \int u^* \times w^* \le \|u^*\|_{1,\infty}\|w^*\|_{\infty,1} = \|u\|_{1,\infty}\|w\|_{\infty,1}$$

for every $w \in M^{\infty,1}_{\bar\nu}$.

$|h_u(\chi b)| \le \int_0^t u^*$ whenever $b \in \mathfrak{B}^f$ and $\bar\nu b \le t$. So

$$\lim_{t\downarrow 0} \sup_{\bar\nu b\le t} |h_u(\chi b)| \le \lim_{t\downarrow 0} \int_0^t u^* = 0,$$

$$\limsup_{t\to\infty} \tfrac{1}{t} \sup_{\bar\nu b\le t} |h_u(\chi b)| \le \limsup_{t\to\infty} \tfrac{1}{t}\int_0^t u^* < \infty.$$

Of course $b \mapsto h_u(\chi b)$ is additive, so by 373Hb there is a unique $Su \in M^{1,\infty}_{\bar\nu}$ such that $h_u(\chi b) = \int_b Su$ for every $b \in \mathfrak{B}^f$. Since both h_u and $w \mapsto \int Su \times w$ are linear and continuous on $M^{\infty,1}_{\bar\nu}$, and $S(\mathfrak{B}^f)$ is dense in $M^{\infty,1}_{\bar\nu}$ (369Od),

$$\int Su \times w = h_u(w) = \lim_{T\to\mathcal{F}} \int Tu \times w$$

for every $w \in M^{\infty,1}_{\bar\nu}$. And this is true for every $u \in M^{1,\infty}_{\bar\mu}$.

For any particular $w \in M^{\infty,1}_{\bar\nu}$, all the maps $u \mapsto \int Tu \times w$ are linear, so $u \mapsto \int Su \times w$ also is; that is, $S : M^{1,\infty}_{\bar\mu} \to M^{1,\infty}_{\bar\nu}$ is linear.

Now $S \in \mathcal{T}$. $\mathbf{P}$ (α) If $u \in L^1_{\bar\mu}$ and $b, c \in \mathfrak{B}^f$, then

$$\begin{aligned}\int_b Su - \int_c Su &= \lim_{T\to\mathcal{F}} \int Tu \times (\chi b - \chi c) \le \sup_{T\in\mathcal{T}} \int Tu \times (\chi b - \chi c)\\ &\le \sup_{T\in\mathcal{T}} \|Tu\|_1 \|\chi b - \chi c\|_\infty \le \|u\|_1.\end{aligned}$$

But, setting $e = [\![Su > 0]\!]$, we have

$$\begin{aligned}\int |Su| &= \int_e Su - \int_{1\setminus e} Su\\ &= \sup_{b\in\mathfrak{B}^f, b\subseteq e} \int_b Su + \sup_{c\in\mathfrak{B}^f, c\subseteq 1\setminus e} \int_c (-Su) \le \|u\|_1.\end{aligned}$$

(β) If $u \in L^\infty(\mathfrak{A})$, then

$$|\int_b Su| \le \sup_{T\in\mathcal{T}} |\int Tu \times \chi b| \le \sup_{T\in\mathcal{T}} \|Tu\|_\infty \bar\nu b \le \|u\|_\infty \bar\nu b$$

for every $b \in \mathfrak{B}^f$. So $[\![Su > \|u\|_\infty]\!] = [\![-Su > \|u\|_\infty]\!] = 0$ and $\|Su\|_\infty \le \|u\|_\infty$. (Note that both parts of this argument depend on knowing that $(\mathfrak{B},\bar\nu)$ is semi-finite, so that we cannot be troubled by purely infinite elements of $\mathfrak{B}$.) **Q**

Of course we now have $\lim_{T\to\mathcal{F}} \tau_{uw}(T - S) = 0$ for all $u \in M^{1,\infty}_{\bar\mu}$, $w \in M^{\infty,1}_{\bar\nu}$, so that $S = \lim \mathcal{F}$ in $\mathcal{T}$. As $\mathcal{F}$ is arbitrary, $\mathcal{T}$ is compact (2A3R).

373M Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a localizable measure algebra, and u any member of $M^{1,\infty}(\mathfrak{A},\bar\mu)$. Then $B = \{Tu : T \in \mathcal{T}_{\bar\mu,\bar\nu}\}$ is compact in $M^{1,\infty}(\mathfrak{B},\bar\nu)$ for the topology $\mathfrak{T}_s(M^{1,\infty}(\mathfrak{B},\bar\nu), M^{\infty,1}(\mathfrak{B},\bar\nu))$.

proof The point is just that the map $T \mapsto Tu : \mathcal{T}_{\bar\mu,\bar\nu} \to M^{1,\infty}_{\bar\nu}$ is continuous for the very weak operator topology on $\mathcal{T}_{\bar\mu,\bar\nu}$ and $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu}, M^{\infty,1}_{\bar\nu})$. So B is a continuous image of a compact set, therefore compact (2A3N(b-ii)).

373N Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $(\mathfrak{B},\bar\nu)$ a localizable measure algebra and u any member of $M^{1,\infty}(\mathfrak{A},\bar\mu)$; set $B = \{Tu : T \in \mathcal{T}_{\bar\mu,\bar\nu}\}$. If $\langle v_n\rangle_{n\in\mathbb{N}}$ is any non-decreasing sequence in B, then $\sup_{n\in\mathbb{N}} v_n$ is defined in $M^{1,\infty}(\mathfrak{B},\bar\nu)$ and belongs to B.

proof (a) The point is that $(M^{1,\infty}_{\bar\nu})^+$ is a closed set for $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu}, M^{\infty,1}_{\bar\nu})$. **P** If $w \in M^{1,\infty}_{\bar\nu}$ and $w \not\ge 0$, then $b = [\![-w > 0]\!] \ne 0$. As $(\mathfrak{B},\bar\nu)$ is semi-finite, there is a non-zero $c \in \mathfrak{B}^f$ with $c \subseteq b$, and $\int_c(-w) > 0$, that is, $\int w \times \chi c < 0$. Now $\chi c \in M^{\infty,1}_{\bar\nu}$ so $\{w' : \int_c w' < 0\}$ is a neighbourhood of w disjoint from $(M^{1,\infty}_{\bar\nu})^+$. Thus $M^{1,\infty}_{\bar\nu} \setminus (M^{1,\infty}_{\bar\nu})^+$ is open and $(M^{1,\infty}_{\bar\nu})^+$ is closed. **Q**

(b) Because $\mathfrak{T}_s(M^{1,\infty}_{\bar\nu}, M^{\infty,1}_{\bar\nu})$ is a linear space topology, the sets $\{w : v \le w\} = \{w : w - v \in (M^{1,\infty}_{\bar\nu})^+\}$ and $\{w : w \le v\} = \{w : v - w \in (M^{1,\infty}_{\bar\nu})^+\}$ are closed for every $w \in M^{1,\infty}_{\bar\nu}$. Now consider the given sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in B. By 373M, it has a cluster point $v \in B$. Since $\{w : v_n \le w\}$ is a closed set containing v_i whenever $i \ge n$, $v_n \le v$, for every $n \in \mathbb{N}$. On the other hand, if v' is any upper bound of $\{v_n : n \in \mathbb{N}\}$ in $M^{1,\infty}_{\bar\nu}$ then $v \le v'$ because $\{w : w \le v'\}$. Accordingly $\sup_{n\in\mathbb{N}} v_n = v$ is defined and belongs to B.

373O Theorem Suppose that $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ are measure algebras, $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$ and $v \in M^{1,\infty}(\mathfrak{B},\bar\nu)$. Then the following are equiveridical:

(i) there is a $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu = v$,

(ii) $\int_0^t v^* \le \int_0^t u^*$ for every $t \ge 0$.

In particular, given $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$, there are $S \in \mathcal{T}_{\bar\mu,\bar\mu_L}$, $T \in \mathcal{T}_{\bar\mu_L,\bar\mu}$ such that $Su = u^*$ and $Tu^* = u$.

proof (i)$\Rightarrow$(ii) is covered by Lemma 373G. Accordingly I shall devote the rest of the proof to showing that (ii)$\Rightarrow$(i).

(a) If $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ are measure algebras, $u \in M^{1,\infty}_{\bar\mu}$ and $v \in M^{1,\infty}_{\bar\nu}$, I will say that $v \preccurlyeq u$ if there is a $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu = v$, and that $v \sim u$ if $v \preccurlyeq u$ and $u \preccurlyeq v$. (Properly speaking, I ought to write $(u,\bar\mu) \preccurlyeq (v,\bar\nu)$, because we could in principle have two different measures on the same algebra. But I do not think any confusion is likely to arise in the argument which follows.) By 373Be, $\preccurlyeq$ is transitive and $\sim$ is an equivalence relation. Now we have the following facts.

(b) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and u_1, $u_2 \in M^{1,\infty}_{\bar\mu}$ are such that $|u_1| \le |u_2|$, then $u_1 \preccurlyeq u_2$. **P** There is a $w \in L^\infty(\mathfrak{A})$ such that $u_1 = w \times u_2$ and $\|w\|_\infty \le 1$. Set $Tv = w \times v$ for for $v \in M^{1,\infty}_{\bar\mu}$; then $T \in \mathcal{T}_{\bar\mu,\bar\mu}$ and $Tu_2 = u_1$. **Q** So $u \sim |u|$ for every $u \in M^{1,\infty}_{\bar\mu}$.

(c)(i) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u \ge 0$ in $S(\mathfrak{A})$, then $u^* \preccurlyeq u$. **P** If $u = 0$ this is trivial. Otherwise, express u as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0,\dots,a_n$ are disjoint and non-zero and $\alpha_0 > \alpha_1 \ldots > \alpha_n > 0 \in \mathbb{R}$. If $\bar\mu a_i = \infty$ for any i, take m to be minimal subject to $\bar\mu a_m = \infty$; otherwise, set $m = n$. Then $u^* = \sum_{i=0}^m \alpha_i \chi\,[\beta_i,\beta_{i+1}[^\bullet$, where $\beta_j = \sum_{i=0}^{j-1} \bar\mu a_i$ for $j \le m+1$.

For $i < m$, and for $i = m$ if $\bar\mu a_m < \infty$, define $h_i : M^{1,\infty}_{\bar\mu} \to \mathbb{R}$ by setting

$$h_i(v) = \frac{1}{\bar\mu a_i}\int_{a_i} v$$

for every $v \in M^{1,\infty}_{\bar\mu}$. If $\bar\mu a_m = \infty$, then we need a different idea to define h_m, as follows. Let I be $\{a : a \in \mathfrak{A},\ \bar\mu(a \cap a_m) < \infty\}$. Then I is an ideal of $\mathfrak{A}$ not containing a_m, so there is a Boolean homomorphism $\pi : \mathfrak{A} \to \{0,1\}$ such that $\pi a = 0$ for $a \in I$ and $\pi a_m = 1$ (311D). This induces a corresponding $\| \|_\infty$-continuous linear operator $h : L^\infty(\mathfrak{A}) \to L^\infty(\{0,1\}) \cong \mathbb{R}$, as in 363F. Now $h(\chi a) = 0$ whenever $\bar\mu a < \infty$, and accordingly $h(v) = 0$ whenever $v \in M^{\infty,1}_{\bar\mu}$, since $S(\mathfrak{A}^f)$ is dense in $M^{\infty,1}_{\bar\mu}$ for $\| \|_{\infty,1}$ and therefore also for $\| \|_\infty$. But this means that h has a unique extension to a linear functional $h_m : M^{1,\infty}_{\bar\mu} \to \mathbb{R}$ such that $h_m(v) = 0$ for every $v \in L^1_{\bar\mu}$, while $h_m(\chi a_m) = 1$ and $|h_m(v)| \le \|v\|_\infty$ for every $v \in L^\infty(\mathfrak{A})$.

Having defined h_i for every $i \le m$, define $T : M^{1,\infty}_{\bar\mu} \to M^{1,\infty}_{\bar\mu_L}$ by setting

$$Tv = \sum_{i=0}^m h_i(v)\chi\,[\beta_i,\beta_{i+1}[^\bullet$$

for every $v \in M^{1,\infty}_{\bar\mu}$.

For any $i \le m$ and $v \in L^1_{\bar\mu}$,

$$\int_{\beta_i}^{\beta_{i+1}} |Tv| = |h_i(v)|\bar\mu a_i \le \int_{a_i} |v|;$$

summing over i, $\|Tv\|_1 \le \|v\|_1$. Similarly, for any $i \le m$ and $v \in L^\infty(\mathfrak{B})$, $|h_i(v)| \le \|v\|_\infty$, so $\|Tv\|_\infty \le \|v\|_\infty$.

Thus $T \in \mathcal{T}_{\bar\mu,\bar\mu_L}$. Since $u^* = Tu$, we conclude that $u^* \preccurlyeq u$, as claimed. **Q**

(ii) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$, then $u^* \preccurlyeq u$. **P** Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A})$ with $u_0 \ge 0$ and $\sup_{n\in\mathbb{N}} u_n = u$. Then $\langle u_n^*\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $M^{1,\infty}_{\bar\mu_L}$ with supremum u^*, by 373Db and 373Dh. Also $u_n^* \preccurlyeq u_n \preccurlyeq u$ for every n, by (b) above and (i) here. By 373N, $u^* \preccurlyeq u$. **Q**

(d)(i) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u \ge 0$ in $S(\mathfrak{A})$, then $u \preccurlyeq u^*$. **P** The argument is very similar to that of (c-i). Again, the result is trivial if $u = 0$; suppose that $u > 0$ and define α_i, a_i, m, β_i as before. This time, set $a'_i = a_i$ for $i < m$, $a'_m = \sup_{m\le j\le n} a_j$, $\tilde u = \sum_{i=0}^m \alpha_i\chi a'_i$; then $u \le \tilde u$ and $\tilde u^* = u^*$. Set

$$h_i(v) = \frac{1}{\beta_{i+1}-\beta_i}\int_{\beta_i}^{\beta_{i+1}} v$$

if $i \le m$, $\beta_{i+1} < \infty$ (that is, $\bar\mu a_i < \infty$) and $v \in M^{1,\infty}_{\bar\mu_L}$; and if $\bar\mu a_m = \infty$, set

$$h_m(v) = \lim_{k\to\mathcal{F}} \frac{1}{k}\int_0^k v$$

for some non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. As before, we have

$$|h_i(v)|\bar\mu a'_i \le \int_{\beta_i}^{\beta_{i+1}} |v|,$$

whenever $v \in L^1_{\bar\mu_L}$ and $i \le m$, while $|h_i(v)| \le \|v\|_\infty$ whenever $v \in L^\infty(\mathfrak{A}_L)$ and $i \le m$. So we can define $T \in \mathcal{T}_{\bar\mu_L,\bar\mu}$ by setting $Tv = \sum_{i=0}^m h_i(v)\chi a'_i$ for every $v \in M^{1,\infty}_{\bar\mu_L}$, and get

$$u \preccurlyeq \tilde u = Tu^* \preccurlyeq u^*. \text{ **Q**}$$

(ii) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$, then $u \preccurlyeq u^*$. **P** This time I seek to copy the ideas of (c-ii); there is a new obstacle to circumvent, since $(\mathfrak{A},\bar\mu)$ might not be localizable. Set

$$\alpha_0 = \inf\{\alpha : \alpha \ge 0,\ \bar\mu[\![u > \alpha]\!] < \infty\}, \quad e = [\![u > \alpha_0]\!].$$

Then $e = \sup_{n\in\mathbb{N}} [\![u > \alpha_0 + 2^{-n}]\!]$ is a countable supremum of elements of finite measure, so the principal ideal $\mathfrak{A}_e$, with its induced measure $\bar\mu_e$, is σ-finite. Now let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A})$ with $u_0 \ge 0$ and $\sup_{n\in\mathbb{N}} u_n = u$; set $\tilde u = u \times \chi e$ and $\tilde u_n = u_n \times \chi e$, regarded as members of $S(\mathfrak{A}_e)$, for each n. In this case

$$\tilde u_n \preccurlyeq \tilde u_n^* \preccurlyeq u^*$$

for every n. Because $(\mathfrak{A}_e,\bar\mu_e)$ is σ-finite, therefore localizable, 373N tells us that $\tilde u \preccurlyeq u^*$.

Let $S \in \mathcal{T}_{\bar\mu_L,\bar\mu_e}$ be such that $Su^* = \tilde u$. As in (i), choose a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ and set

$$h(v) = \lim_{k\to\mathcal{F}} \frac{1}{k}\int_0^k v$$

for $v \in M^{1,\infty}_{\bar\mu_L}$. Now define $T : M^{1,\infty}_{\bar\mu_L} \to M^{1,\infty}_{\bar\mu}$ by setting

$$Tv = Sv + h(v)\chi(1 \setminus e),$$

here regarding Sv as a member of $M^{1,\infty}_{\bar\mu}$. (I am taking it to be obvious that $M^{1,\infty}_{\bar\mu_e}$ can be identified with $\{w \times \chi e : w \in M^{1,\infty}_{\bar\mu}\}$.) Then it is easy to see that $T \in \mathcal{T}_{\bar\mu_L,\bar\mu}$. Also $u \le Tu^*$, because

$$h(u^*) = \inf\{\alpha : \bar\mu_L[\![u^* > \alpha]\!] < \infty\} = \alpha_0,$$

while $u \times \chi(1 \setminus e) \le \alpha_0\chi(1 \setminus e)$. So we get $u \preccurlyeq Tu^* \preccurlyeq u^*$. **Q**

(e)(i) Now suppose that u, $v \ge 0$ in $M^{1,\infty}_{\bar\mu_L}$, that $\int_0^t u^* \ge \int_0^t v^*$ for every $t \ge 0$, and that v is of the form $\sum_{i=1}^n \alpha_i\chi a_i$ where $\alpha_1 > \ldots > \alpha_n > 0$, $a_1, \ldots, a_n \in \mathfrak{A}_L$ are disjoint and $0 < \bar\mu_L a_i < \infty$ for each i. Then $v \preccurlyeq u$. **P** Induce on n. If $n = 0$ then $v = 0$ and the result is trivial. For the inductive step to $n \ge 1$, if $v^* \le u^*$ we have

$$v \sim v^* \preccurlyeq u^* \sim u,$$

using (b)-(d) above. Otherwise, look at $\phi(t) = \frac{1}{t}\int_0^t u^*$ for $t > 0$. We have

$$\phi(t) \ge \frac{1}{t}\int_0^t v^* = \alpha_1$$

for $t \le \beta = \bar\mu a_1$, while $\lim_{t\to\infty} \phi(t) < \alpha_1$, because $(\lim_{t\to\infty} \phi(t))\chi 1 \le u^*$ and $v^* \le \alpha_1\chi 1$ and $v^* \not\le u^*$. Because ϕ is continuous, there is a $\gamma \ge \beta$ such that $\phi(\gamma) = \alpha_1$. Define $T_0 \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ by setting

$$T_0 w = (\frac{1}{\gamma}\int_0^\gamma w)\chi\,[0,\gamma[^\bullet + (w \times \chi\,[\gamma,\infty[^\bullet)$$

for every $w \in M^{1,\infty}_{\bar\mu_L}$. Then $T_0 u^* \preccurlyeq u^* \sim u$, and

$$T_0 u^* \times \chi\,[0,\gamma[^\bullet = (\frac{1}{\gamma}\int_0^\gamma u^*)\chi\,[0,\gamma[^\bullet = \alpha_1\chi\,[0,\gamma[^\bullet.$$

We need to know that $\int_0^t T_0 u^* \ge \int_0^t v^*$ for every t; this is because

$$\begin{aligned}\int_0^t T_0 u^* &= \alpha_1 t \ge \int_0^t v^* \text{ whenever } t \le \gamma,\\ &= \int_0^\gamma T_0 u^* + \int_\gamma^t T_0 u^* = \int_0^t u^* \ge \int_0^t v^* \text{ whenever } t \ge \gamma.\end{aligned}$$

Set

$$u_1 = T_0 u^* \times \chi\,[\beta,\infty[^\bullet, \quad v_1 = v^* \times \chi\,[\beta,\infty[^\bullet.$$

Then u_1^*, v_1^* are just translations of $T_0 u^*$, v^* to the left, so that

$$\int_0^t u_1^* = \int_\beta^{\beta+t} T_0 u^* = \int_0^{\beta+t} T_0 u^* - \alpha_1\beta \ge \int_0^{\beta+t} v^* - \alpha_1\beta = \int_\beta^{\beta+t} v^* = \int_0^t v_1^*$$

for every $t \ge 0$. Also $v_1 = \sum_{i=2}^n \alpha_i\chi\,[\beta_{i-1},\beta_i[^\bullet$ where $\beta_i = \sum_{j=1}^i \bar\mu a_j$ for each j. So by the inductive hypothesis, $v_1 \preccurlyeq u_1$.

Let $S \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ be such that $Su_1 = v_1$, and define $T \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ by setting

$$Tw = w \times \chi\,[0,\beta[^\bullet + S(w \times \chi\,[\beta,\infty[^\bullet) \times \chi\,[\beta,\infty[^\bullet$$

for every $w \in M^{1,\infty}_{\bar\mu_L,\bar\mu_L}$. Then $TT_0 u^* = v^*$, so $v \sim v^* \preccurlyeq u^* \sim u$, as required. **Q**

(ii) We are nearly home. If u, $v \ge 0$ in $M^{1,\infty}_{\bar\mu_L}$ and $\int_0^t v^* \le \int_0^t u^*$ for every $t \ge 0$, then $v \preccurlyeq u$. **P** Let $\langle v_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A}_L^f)^+$ with supremum v. Then $v_n^* \le v^*$ for each n, so (i) tells us that $v_n \preccurlyeq u$ for every n. By 373N, for the last time, $v \preccurlyeq u$. **Q**

(f) Finally, suppose that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are arbitrary measure algebras and that $u \in M^{1,\infty}_{\bar\mu}$, $v \in M^{1,\infty}_{\bar\nu}$ are such that $\int_0^t v^* \le \int_0^t u^*$ for every $t \ge 0$. Then

$$v \sim |v| \preccurlyeq |v|^* = v^*$$

(by (b), (d) and 373Db)

$$\preccurlyeq u^*$$

(by (e))

$$= |u|^* \preccurlyeq |u|$$

(by (c))

$$\sim u$$

and $v \preccurlyeq u$, as claimed

373P Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $(\mathfrak{B}, \bar{\nu})$ a semi-finite measure algebra. Then for any $u \in M^{1,\infty}(\mathfrak{A}, \bar{\mu})$ and $v \in M^0(\mathfrak{B}, \bar{\nu})$, there is a $T \in \mathcal{T} = \mathcal{T}_{\bar{\mu},\bar{\nu}}$ such that $\int Tu \times v = \int u^* \times v^*$.

proof (a) It is convenient to dispose immediately of some elementary questions.

(i) We need only find a $T \in \mathcal{T}$ such that $\int |Tu \times v| \ge \int u^* \times v^*$. **P** Take $v_0 \in L^\infty(\mathfrak{B})$ such that $|Tu \times v| = v_0 \times Tu \times v$ and $\|v_0\|_\infty \le 1$, and set $T_1 w = v_0 \times Tw$ for $w \in M^{1,\infty}_{\bar{\mu}}$; then $T_1 \in \mathcal{T}$ and

$$\int T_1 u \times v = \int |Tu \times v| \ge \int u^* \times v^* \ge \int T_1 u \times v$$

by 373J. **Q**

(ii) Consequently it will be enough to consider $v \ge 0$, since of course $\int |Tu \times v| = \int |Tu \times |v||$, while $|v|^* = v^*$.

(iii) It will be enough to consider $u = u^*$. **P** If we can find $T \in \mathcal{T}_{\bar{\mu}_L,\bar{\nu}}$ such that $\int Tu^* \times v = \int (u^*)^* \times v^*$, then we know from 373O that there is an $S \in \mathcal{T}_{\bar{\mu},\bar{\mu}_L}$ such that $Su = u^*$, so that $TS \in \mathcal{T}$ and

$$\int TSu \times v = \int (u^*)^* \times v^* = \int u^* \times v^*. \text{ Q}$$

(iv) It will be enough to consider localizable $(\mathfrak{B}, \bar{\nu})$. **P** Assuming that $v \ge 0$, following (ii) above, set $e = [\![v > 0]\!] = \sup_{n\in\mathbb{N}} [\![v > 2^{-n}]\!]$, and let $\bar{\nu}_e$ be the restriction of $\bar{\nu}$ to the principal ideal $\mathfrak{B}_e$ generated by e. Then if we write $\tilde{v}$ for the member of $L^0(\mathfrak{B}_e)$ corresponding to v (so that $[\![\tilde{v} > \alpha]\!] = [\![v > \alpha]\!]$ for every $\alpha > 0$), $\tilde{v}^* = v^*$. Also $(\mathfrak{B}_e, \bar{\nu}_e)$ is σ-finite, therefore localizable. Now if we can find $T \in \mathcal{T}_{\bar{\mu},\bar{\nu}_e}$ such that $\int Tu \times \tilde{v} = \int u^* \times \tilde{v}^*$, then ST will belong to $\mathcal{T}_{\bar{\mu},\bar{\nu}}$, where $S : L^0(\mathfrak{B}_e) \to L^0(\mathfrak{B})$ is the canonical embedding defined by the formula

$$[\![Sw > \alpha]\!] = [\![w > \alpha]\!] \text{ if } \alpha \ge 0,$$
$$= [\![w > \alpha]\!] \cup (1 \setminus e) \text{ if } \alpha < 0,$$

and

$$\int STu \times v = \int Tu \times \tilde{v} = \int u^* \times \tilde{v}^* = \int u^* \times v^*. \text{ Q}$$

(b) So let us suppose henceforth that $\bar{\mu} = \bar{\mu}_L$, $u = u^*$ is the equivalence class of a non-increasing non-negative function, $v \ge 0$ and $(\mathfrak{B}, \bar{\nu})$ is localizable.

For n, $i \in \mathbb{N}$ set

$$b_{ni} = [\![v > 2^{-n} i]\!], \quad \beta_{ni} = \bar{\nu} b_{ni}, \quad c_{ni} = b_{ni} \setminus b_{n,i+1}, \quad \gamma_{ni} = \bar{\nu} c_{ni} = \beta_{ni} - \beta_{n,i+1}$$

(because $\beta_{ni} < \infty$ if $i > 0$; this is really where I use the hypothesis that $v \in M^0$). For $n \in \mathbb{N}$ set

$$K_n = \{i : i \ge 1,\, \gamma_{ni} > 0\},$$

$$T_n w = \sum_{i\in K_n} \Big(\frac{1}{\gamma_{ni}} \int_{\beta_{n,i+1}}^{\beta_{ni}} w\Big) \chi c_{ni}$$

for $w \in M^{1,\infty}_{\bar\mu_L}$; this is defined in $L^0(\mathfrak{B})$ because K_n is countable and $\langle c_{ni}\rangle_{i\in\mathbb{N}}$ is disjoint. Of course $T_n : M^{1,\infty}_{\bar\mu_L} \to L^0(\mathfrak{B})$ is linear. If $w \in L^\infty(\mathfrak{A}_L)$ then

$$\|T_n w\|_\infty = \sup_{i\in K_n}\Bigl|\frac{1}{\gamma_{ni}}\int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr| \le \|w\|_\infty,$$

and if $w \in L^1_{\bar\mu_L}$ then

$$\|T_n w\|_1 = \sum_{i\in K_n}\Bigl|\frac{1}{\gamma_{ni}}\int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr|\bar\nu c_{ni} = \sum_{i\in K_n}\Bigl|\int_{\beta_{n,i+1}}^{\beta_{ni}} w\Bigr| \le \|w\|_1;$$

so $T_n w \in M^{1,\infty}_{\bar\nu}$ for every $w \in M^{1,\infty}_{\bar\mu_L}$, and $T_n \in \mathcal{T}$. It will be helpful to observe that

$$\int_{c_{ni}} T_n w = \int_{\beta_{n,i+1}}^{\beta_{ni}} w$$

whenever $i \ge 1$, since if $i \notin K_n$ then both sides are 0.

Note next that for every n, $i \in \mathbb{N}$,

$$b_{ni} = b_{n+1,2i}, \quad \beta_{ni} = \beta_{n+1,2i}, \quad c_{ni} = c_{n+1,2i} \cup c_{n+1,2i+1}, \quad \gamma_{ni} = \gamma_{n+1,2i} + \gamma_{n+1,2i+1},$$

so that, for $i \ge 1$,

$$\int_{c_{ni}} T_n u = \int_{\beta_{n,i+1}}^{\beta_{ni}} u = \int_{c_{ni}} T_{n+1} u.$$

This means that if T is any cluster point of $\langle T_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{T}$ for the very weak operator topology (and such a cluster point exists, by 373L), $\int_{c_{mi}} Tu$ must be a cluster point of $\langle\int_{c_{mi}} T_n u\rangle_{n\in\mathbb{N}}$, and therefore equal to $\int_{c_{mi}} T_m u$, whenever $m \in \mathbb{N}$ and $i \ge 1$.

Consequently, if $m \in \mathbb{N}$,

$$\int |Tu \times v| \ge \sum_{i=0}^{\infty}\int_{c_{mi}} |Tu| \times v \ge \sum_{i=0}^{\infty} 2^{-m} i\int_{c_{mi}} |Tu|$$

(because $c_{mi} \subseteq [\![v > 2^{-m}i]\!]$)

$$\ge \sum_{i=1}^{\infty} 2^{-m} i\Bigl|\int_{c_{mi}} Tu\Bigr| = \sum_{i=1}^{\infty} 2^{-m} i\int_{c_{mi}} T_m u$$

$$= \sum_{i=0}^{\infty} 2^{-m} i\int_{\beta_{m,i+1}}^{\beta_{mi}} u \ge \int u \times (v^* - 2^{-m}\chi 1)^+$$

because

$$[\beta_{m,i+1}, \beta_{mi}]^\bullet \subseteq [\![v^* \le 2^{-m}(i+1)]\!] = [\![(v^* - 2^{-m}\chi 1)^+ \le 2^{-m} i]\!]$$

for each $i \in \mathbb{N}$. But letting $m \to \infty$, we have

$$\int |Tu \times v| \ge \lim_{m\to\infty}\int u \times (v^* - 2^{-m}\chi 1)^+ = \int u \times v^*$$

because $\langle u \times (v^* - 2^{-m}\chi 1)^+\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence with supremum $u \times v^*$. In view of the reductions in (a) above, this is enough to complete the proof.

373Q Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, $(\mathfrak{B}, \bar\nu)$ a semi-finite measure algebra, $u \in M^{1,\infty}(\mathfrak{A}, \bar\mu)$ and $v \in M^{0,\infty}(\mathfrak{B}, \bar\nu)$. Then

$$\int u^* \times v^* = \sup\{\int |Tu \times v| : T \in \mathcal{T}_{\bar\mu,\bar\nu}\} = \sup\{\int Tu \times v : T \in \mathcal{T}_{\bar\mu,\bar\nu}\}.$$

proof There is a non-decreasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}^f$ such that $v^* = \sup_{n\in\mathbb{N}} v_n^*$, where $v_n = v \times \chi c_n$ for each n. **P** For each rational $q > 0$, we can find a countable non-empty set $B_q \subseteq \mathfrak{B}$ such that

$$b \subseteq [\![|v| > q]\!],\ \bar\nu b < \infty \text{ for every } b \in B_q,$$

$$\sup_{b\in B_q} \bar\nu b = \bar\nu[\![|v| > q]\!]$$

(because $(\mathfrak{B},\bar{\nu})$ is semi-finite). Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\bigcup_{q\in\mathbb{Q},q>0}B_q$ and set $c_n=\sup_{i\le n}b_i$, $v_n=v\times\chi c_n$ for each n. Then $\langle|v_n|\rangle_{n\in\mathbb{N}}$ and $\langle v_n^*\rangle_{n\in\mathbb{N}}$ are non-decreasing and $\sup_{n\in\mathbb{N}}v_n^*\le v^*$ in $L^0(\mathfrak{A}_L)$. But in fact $\sup_{n\in\mathbb{N}}v_n^*=v^*$, because

$$\bar{\mu}_L[\![v^*>q]\!]=\bar{\mu}[\![|v|>q]\!]=\sup_{n\in\mathbb{N}}\bar{\mu}[\![v_n>q]\!]=\sup_{n\in\mathbb{N}}\bar{\mu}_L[\![v_n^*>q]\!]=\bar{\mu}_L[\![\sup_{n\in\mathbb{N}}v_n^*>q]\!]$$

for every rational $q>0$, by 373Da. **Q**

For each $n\in\mathbb{N}$ we have a $T_n\in\mathcal{T}_{\bar{\mu},\bar{\nu}}$ such that $\int T_nu\times v_n=\int u^*\times v_n^*$ (373P). Set $S_nw=T_nw\times\chi c_n$ for $n\in\mathbb{N}$, $w\in M_{\bar{\mu}}^{1,\infty}$; then every S_n belongs to $\mathcal{T}_{\bar{\mu},\bar{\nu}}$, so

$$\begin{aligned}\sup\{\int Tu\times v:T\in\mathcal{T}_{\bar{\mu},\bar{\nu}}\}&\ge\sup_{n\in\mathbb{N}}\int S_nu\times v=\sup_{n\in\mathbb{N}}\int T_nu\times v_n\\&=\sup_{n\in\mathbb{N}}\int u^*\times v_n^*=\int u^*\times v^*\\&\ge\sup\{\int|Tu\times v|:T\in\mathcal{T}_{\bar{\mu},\bar{\nu}}\}\ge\sup\{\int Tu\times v:T\in\mathcal{T}_{\bar{\mu},\bar{\nu}}\}\end{aligned}$$

by 373J, as usual.

373R Order-continuous operators: Proposition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, $(\mathfrak{B},\bar{\nu})$ a localizable measure algebra, and $T_0\in\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar{\mu},\bar{\nu}}$. Then there is a $T\in\mathcal{T}^\times=\mathcal{T}^\times_{\bar{\mu},\bar{\nu}}$ extending T_0. If $(\mathfrak{A},\bar{\mu})$ is semi-finite, T is uniquely defined.

proof (a) Suppose first that $T_0\in\mathcal{T}^{(0)}$ is non-negative, regarded as a member of $\mathrm{L}^\sim(M_{\bar{\mu}}^{1,0};M_{\bar{\nu}}^{1,0})$. In this case T_0 has an extension to an order-continuous positive linear operator $T:M_{\bar{\mu}}^{1,\infty}\to L^0(\mathfrak{B})$ defined by saying that $Tw=\sup\{T_0u:u\in M_{\bar{\mu}}^{1,0},\,0\le u\le w\}$ for every $w\ge0$ in $M_{\bar{\mu}}^{1,\infty}$. **P** I use 355F. $M_{\bar{\mu}}^{1,0}$ is a solid linear subspace of $M_{\bar{\mu}}^{1,\infty}$. T_0 is order-continuous when its codomain is taken to be $M_{\bar{\nu}}^{1,0}$, as noted in 371Gb, and therefore if its codomain is taken to be $L^0(\mathfrak{B})$, because $M^{1,0}$ is a solid linear subspace in L^0, so the embedding is order-continuous. If $w\ge0$ in $M_{\bar{\mu}}^{1,\infty}$, let $\gamma\ge0$ be such that $u_1=(w-\gamma\chi1)^+$ is integrable. If $u\in M_{\bar{\mu}}^{1,0}$ and $0\le u\le w$, then $(u-\gamma\chi1)^+\le u_1$, so

$$T_0u=T_0(u-\gamma\chi1)^++T_0(u\wedge\gamma\chi1)\le T_0u_1+\gamma\chi1\in L^0(\mathfrak{B}).$$

Thus $\{T_0u:u\in M_{\bar{\nu}}^{1,0},\,0\le u\le w\}$ is bounded above in $L^0(\mathfrak{B})$, for any $w\ge0$ in $M_{\bar{\mu}}^{1,\infty}$. $L^0(\mathfrak{B})$ is Dedekind complete, because $(\mathfrak{B},\bar{\nu})$ is localizable, so $\sup\{T_0u:0\le u\le w\}$ is defined in $L^0(\mathfrak{B})$; and this is true for every $w\in(M_{\bar{\mu}}^{1,\infty})^+$. Thus the conditions of 355F are satisfied and we have the result. **Q**

(b) Now suppose that T_0 is any member of $\mathcal{T}^{(0)}$. Then T_0 has an extension to a member of $\mathcal{T}^\times$. **P** $|T_0|$, $T_0^+=\frac{1}{2}(|T_0|+T_0)$ and $T_0^-=\frac{1}{2}(|T_0|-T_0)$, taken in $\mathrm{L}^\sim(M_{\bar{\mu}}^{1,0};M_{\bar{\nu}}^{1,0})$, all belong to $\mathcal{T}^{(0)}$ (371G), so have extensions S, S_1 and S_2 to order-continuous positive linear operators from $M_{\bar{\mu}}^{1,\infty}$ to $L^0(\mathfrak{B})$ as defined in (a). Now for any $w\in L^1_{\bar{\mu}}$,

$$\|Sw\|_1=\||T_0|w\|_1\le\|w\|_1,$$

and for any $w\in L^\infty(\mathfrak{A})$,

$$|Sw|\le S|w|=\sup\{|T_0|u:u\in M_{\bar{\mu}}^{1,0},\,0\le u\le w\}\le\|w\|_\infty\chi1,$$

so $\|Sw\|_\infty\le\|w\|_\infty$. Thus $S\in\mathcal{T}$; similarly, S_1 and S_2 can be regarded as operators from $M_{\bar{\mu}}^{1,\infty}$ to $M_{\bar{\nu}}^{1,\infty}$, and as such belong to $\mathcal{T}$. Next, for $w\ge0$ in $M_{\bar{\mu}}^{1,\infty}$,

$$\begin{aligned}S_1w+S_2w&=\sup\{T_0^+u:u\in M_{\bar{\mu}}^{1,0},\,0\le u\le w\}+\sup\{T_0^-u:u\in M_{\bar{\mu}}^{1,0},\,0\le u\le w\}\\&=\sup\{T_0^+u+T_0^-u:u\in M_{\bar{\mu}}^{1,0},\,0\le u\le w\}=Sw.\end{aligned}$$

But this means that

$$S=S_1+S_2\ge|S_1-S_2|$$

and $T=S_1-S_2\in\mathcal{T}$, by 373Bc; while of course T extends $T_0^+-T_0^-=T_0$. Finally, because S_1 and S_2 are order-continuous, $T\in\mathsf{L}^\times(M^{1,\infty}_{\bar\mu};M^{1,\infty}_{\bar\nu})$, so $T\in\mathcal{T}^\times$. **Q**

(c) If $(\mathfrak{A},\bar\mu)$ is semi-finite, then $M^{1,0}_{\bar\mu}$ is order-dense in $M^{1,\infty}_{\bar\mu}$ (because it includes $L^1_{\bar\mu}$, which is order-dense in $L^0(\mathfrak{A})$); so that the extension T is unique, by 355Fe.

373S Adjoints in $\mathcal{T}^{(0)}$: Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and T any member of $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$. Then there is a unique operator $T'\in\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ such that $\int_a T'(\chi b)=\int_b T(\chi a)$ whenever $a\in\mathfrak{A}^f$ and $b\in\mathfrak{B}^f$, and now $\int u\times T'v=\int Tu\times v$ whenever $u\in M^{1,0}(\mathfrak{A},\bar\mu)$, $v\in M^{1,0}(\mathfrak{B},\bar\nu)$ are such that $\int u^*\times v^*<\infty$.

proof (a) For each $v\in M^{1,0}_{\bar\nu}$ we can define $T'v\in M^{1,0}_{\bar\mu}$ by the formula

$$\int_a T'v=\int T(\chi a)\times v$$

for every $a\in\mathfrak{A}^f$. **P** Set $\theta a=\int T(\chi a)\times v$ for each $a\in\mathfrak{A}^f$; because $\int(\chi a)^*\times v^*<\infty$, the integral is defined and finite (373J). Of course $\theta:\mathfrak{A}^f\to\mathbb{R}$ is additive because χ is additive and T, $\times$ and $\int$ are linear. Also

$$\lim_{t\downarrow 0}\sup_{\bar\mu a\le t}|\theta a|\le\lim_{t\downarrow 0}\int_0^t v^*=0,$$

$$\lim_{t\to\infty}\frac{1}{t}\sup_{\bar\mu a\le t}|\theta a|\le\lim_{t\to\infty}\tfrac{1}{t}\int_0^t v^*=0$$

because $v\in M^{1,0}_{\bar\nu}$, so $v^*\in M^{1,0}_{\bar\mu_L}$. By 373Ha, there is a unique $T'v\in M^{1,0}_{\bar\mu}$ such that $\int_a T'v=\theta a$ for every $a\in\mathfrak{A}^f$. **Q**

(b) Because the formula uniquely determines $T'v$, we see that $T':M^{1,0}_{\bar\nu}\to M^{1,0}_{\bar\mu}$ is linear. Now $T'\in\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$. **P** (i) If $v\in L^1_{\bar\nu}$, then (because $T'v\in M^{1,0}_{\bar\mu}$) $|T'v|=\sup_{a\in\mathfrak{A}^f}|T'v|\times\chi a$, and

$$\begin{aligned}\|T'v\|_1&=\int|T'v|=\sup_{a\in\mathfrak{A}^f}\int_a|T'v|=\sup_{b,c\in\mathfrak{A}^f}\Big(\int_b T'v-\int_c T'v\Big)\\&=\sup_{b,c\in\mathfrak{A}^f}\int T(\chi b-\chi c)\times v\le\sup_{b,c\in\mathfrak{A}^f}\int(\chi b-\chi c)^*\times v^*\\&=\int v^*=\|v\|_1.\end{aligned}$$

(ii) Now suppose that $v\in L^\infty(\mathfrak{B})\cap M^{1,0}_{\bar\nu}$, and set $\gamma=\|v\|_\infty$. **?** If $a=[\![|T'v|>\gamma]\!]\ne 0$, then $T'v\ne 0$ so $v\ne 0$ and $\gamma>0$ and $\bar\mu a<\infty$, because $T'v\in M^{1,0}_{\bar\mu}$. Set $b=[\![(T'v)^+>\gamma]\!]$, $c=[\![(T'v)^->\gamma]\!]$; then

$$\begin{aligned}\gamma\bar\mu a&<\int_a|T'v|=\int_b T'v-\int_c T'v=\int T(\chi b-\chi c)\times v\\&\le\gamma\|T(\chi b-\chi c)\|_1\le\gamma\|\chi b-\chi c\|_1=\gamma\bar\mu a,\end{aligned}$$

which is impossible. **X** Thus $[\![|T'v|>\gamma]\!]=0$ and $\|T'v\|_\infty\le\gamma=\|v\|_\infty$.

Putting this together with (i), we see that $T'\in\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$. **Q**

(c) Let $|T|$ be the modulus of T in $\mathsf{L}^\sim(M^{1,0}_{\bar\mu};M^{1,0}_{\bar\nu})$, so that $|T|\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, by 371Gb. If $u\ge 0$ in $M^{1,0}_{\bar\mu}$, $v\ge 0$ in $M^{1,0}_{\bar\nu}$ are such that $\int u^*\times v^*<\infty$, let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $S(\mathfrak{A}^f)^+$ with supremum u. In this case $|T|u=\sup_{n\in\mathbb{N}}|T|u_n$, so $\int|T|u\times v=\sup_{n\in\mathbb{N}}\int|T|u_n\times v$ and

$$|\int Tu\times v-\int Tu_n\times v|\le\int|T|(u-u_n)\times v\to 0$$

as $n\to\infty$, because

$$\int|T|u\times v\le\int u^*\times v^*<\infty.$$

At the same time,

$$|\int u\times T'v-\int u_n\times T'v|\le\int(u-u_n)\times|T'v|\to 0$$

because $\int u\times|T'v|\le\int u^*\times v^*<\infty$. So

$$\int Tu \times v = \lim_{n\to\infty} \int Tu_n \times v = \lim_{n\to\infty} \int u_n \times T'v = \int u \times T'v,$$

the middle equality being valid because each u_n is a linear combination of indicator functions.

Because T and T' are linear, it follows at once that $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,0}_{\bar\mu}$, $v \in M^{1,0}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$.

(d) Finally, to see that T' is uniquely defined by the formula in the statement of the theorem, observe that this surely defines $T'(\chi b)$ for every $b \in \mathfrak{B}^f$, by the remarks in (a). Consequently it defines T' on $S(\mathfrak{B}^f)$. Since $S(\mathfrak{B}^f)$ is order-dense in $M^{1,0}_{\bar\nu}$, and any member of $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ must belong to $\mathsf{L}^\times(M^{1,0}_{\bar\nu}; M^{1,0}_{\bar\mu})$ (371Gb), the restriction of T' to $S(\mathfrak{B}^f)$ determines T' (355J).

373T Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras. Then for any $T \in \mathcal{T}^\times_{\bar\mu,\bar\nu}$ there is a unique $T' \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ such that $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,\infty}(\mathfrak{A},\bar\mu)$, $v \in M^{1,\infty}(\mathfrak{B},\bar\nu)$ are such that $\int u^* \times v^* < \infty$.

proof The restriction $T{\restriction}M^{1,0}_{\bar\mu}$ belongs to $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ (373Bb), so there is a unique $S \in \mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ such that $\int u \times Sv = \int Tu \times v$ whenever $u \in M^{1,0}_{\bar\mu}$, $v \in M^{1,0}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$ (373S). Now there is a unique $T' \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ extending S (373R). If $u \ge 0$ in $M^{1,\infty}_{\bar\mu}$, $v \ge 0$ in $M^{1,\infty}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$, then $\int u \times T'v = \int Tu \times v$. **P** If $T \ge 0$, then both are

$$\sup\{\int u_0 \times T'v_0 : u_0 \in M^{1,0}_{\bar\mu},\ v \in M^{1,0}_{\bar\nu},\ 0 \le u_0 \le u,\ 0 \le v_0 \le v\}$$

because both T and T' are (order-)continuous. In general, we can apply the same argument to T^+ and T^-, taken in $\mathsf{L}^\sim(M^{1,\infty}_{\bar\mu}; M^{1,\infty}_{\bar\nu})$, since these belong to $\mathcal{T}^\times_{\bar\mu,\bar\nu}$, by 373B and 355H, and we shall surely have $T' = (T^+)' - (T^-)'$. **Q** As in 373S, it follows that $\int u \times T'v = \int Tu \times v$ whenever $u \in M^{1,\infty}_{\bar\mu}$, $v \in M^{1,\infty}_{\bar\nu}$ are such that $\int u^* \times v^* < \infty$.

373U Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be localizable measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous measure-preserving Boolean homomorphism. Then the associated map $T \in \mathcal{T}^\times_{\bar\mu,\bar\nu}$ (373Bd) has an adjoint $P \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ defined by the formula $\int_a P(\chi b) = \bar\nu(b \cap \pi a)$ for $a \in \mathfrak{A}^f$, $b \in \mathfrak{B}^f$.

proof By 373T, T has an adjoint $P = T'$ such that

$$\int_a P(\chi b) = \int \chi a \times P(\chi b) = \int T(\chi a) \times \chi b = \int \chi(\pi a) \times \chi b = \bar\nu(\pi a \cap b)$$

whenever $a \in \mathfrak{A}^f$ and $b \in \mathfrak{B}^f$. To see that this defines P uniquely, let $S \in \mathcal{T}^\times_{\bar\nu,\bar\mu}$ be any other operator with the same property. By 373Hb, $S(\chi b) = P(\chi b)$ for every $b \in \mathfrak{B}^f$, so S and P agree on $S(\mathfrak{B}^f)$. Because both P and S are supposed to belong to $\mathsf{L}^\times(M^{1,\infty}_{\bar\nu}; M^{1,\infty}_{\bar\mu})$, and $S(\mathfrak{B}^f)$ is order-dense in $M^{1,\infty}_{\bar\nu}$, $S = P$, by 355J.

373X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a *ring* homomorphism such that $\bar\nu\pi a \le \bar\mu a$ for every $a \in \mathfrak{A}$. (i) Show that there is a unique $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$, and that T is a Riesz homomorphism. (ii) Show that T is (sequentially) order-continuous iff π is.

>(b) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\mu)$ be measure algebras, and $\phi : \mathbb{R} \to \mathbb{R}$ a convex function such that $\phi(0) \le 0$. Show that if $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ and $T \ge 0$, then $\bar\phi(Tu) \le T(\bar\phi(u))$ whenever $u \in M^{1,\infty}_{\bar\mu}$ is such that $\bar\phi(u) \in M^{1,\infty}_{\bar\mu}$. (*Hint*: 371Gd.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Show that if $w \in L^\infty(\mathfrak{A})$ and $\|w\|_\infty \le 1$ then $u \mapsto u \times w : M^{1,\infty}_{\bar\mu} \to M^{1,\infty}_{\bar\mu}$ belongs to $\mathcal{T}^\times_{\bar\mu,\bar\mu}$.

(d) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Show that if $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}$ are disjoint families in $\mathfrak{A}$, $\mathfrak{B}$ respectively, and $\langle T_i\rangle_{i\in I}$ is any family in $\mathcal{T}_{\bar\mu,\bar\nu}$, and *either* I is countable *or* $\mathfrak{B}$ is Dedekind complete, then we have an operator $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ such that $Tu \times \chi b_i = T_i(u \times \chi a_i) \times \chi b_i$ for every $u \in M^{1,\infty}_{\bar\mu,\bar\nu}$, $i \in I$.

>(e) Let I, J be sets and write $\mu = \bar\mu$, $\nu = \bar\nu$ for counting measure on I, J respectively. Show that there is a natural one-to-one correspondence between $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ and the set of matrices $\langle a_{ij}\rangle_{i\in I,j\in J}$ such that $\sum_{i\in I} |a_{ij}| \le 1$ for every $j \in J$, $\sum_{j\in J} |a_{ij}| \le 1$ for every $i \in I$.

>(f) Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$, and product measure λ on $X\times Y$. Let $h:X\times Y\to\mathbb{R}$ be a measurable function such that $\int|h(x,y)|dx\le 1$ for ν-almost every $y\in Y$ and $\int|h(x,y)|dy\le 1$ for μ-almost every $x\in X$. Show that there is a corresponding $T\in\mathcal{T}^{\times}_{\bar\mu,\bar\nu}$ defined by writing $T(f^\bullet)=g^\bullet$ whenever $f\in\mathcal{L}^1(\mu)+\mathcal{L}^\infty(\mu)$ and $g(y)=\int h(x,y)f(x)dx$ for almost every y.

>(g) Let μ be Lebesgue measure on $\mathbb{R}$, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Show that for any μ-integrable function h with $\int|h|d\mu\le 1$ we have a corresponding $T\in\mathcal{T}^{\times}_{\bar\mu,\bar\mu}$ defined by setting $T(f^\bullet)=(h*f)^\bullet$ whenever $g\in\mathcal{L}^1(\mu)+\mathcal{L}^\infty(\mu)$, writing $h*f$ for the convolution of h and f (255E). Explain how this may be regarded as a special case of 373Xf.

>(h) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $u\in L^0(\mathfrak{A})^+$; let ν_u be its distribution (364GB). Show that each of u^*, ν_u is uniquely determined by the other.

(i) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism; let $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ be the corresponding operator (373Bd). Show that $(Tu)^*=u^*$ for every $u\in M^{1,\infty}_{\bar\mu}$.

(j) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and A a subset of $L^1_{\bar\mu}$. Show that the following are equiveridical: (i) A is uniformly integrable; (ii) $\{u^*:u\in A\}$ is uniformly integrable in $L^1_{\bar\mu_L}$; (iii) $\lim_{t\downarrow 0}\sup_{u\in A}\int_0^t u^*=0$.

(k) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $A\subseteq(M^0_{\bar\mu})^+$ a non-empty downwards-directed set. Show that $(\inf A)^*=\inf_{u\in A}u^*$ in $L^0(\mathfrak{A}_L)$.

(l) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Show that $\|u\|_{1,\infty}=\int_0^1 u^*$ for every $u\in M^{1,\infty}(\mathfrak{A},\bar\mu)$.

(m) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and ϕ a Young's function (369Xc). Write $U_{\phi,\bar\mu}\subseteq L^0(\mathfrak{A})$, $U_{\phi,\bar\nu}\subseteq L^0(\mathfrak{B})$ for the corresponding Orlicz spaces. (i) Show that if $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ and $u\in U_{\phi,\bar\mu}$, then $Tu\in U_{\phi,\bar\nu}$ and $\|Tu\|_\phi\le\|u\|_\phi$. (ii) Show that $u\in U_{\phi,\bar\mu}$ iff $u^*\in U_{\phi,\bar\mu_L}$, and in this case $\|u\|_\phi=\|u^*\|_\phi$.

>(n) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $(\mathfrak{B},\bar\nu)$ a totally finite measure algebra. Show that if $A\subseteq L^1_{\bar\mu}$ is uniformly integrable, then $\{Tu:u\in A,\,T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$ is uniformly integrable in $L^1_{\bar\nu}$.

(o)(i) Give examples of u, $v\in L^1(\mathfrak{A}_L)$ such that $(u+v)^*\not\le u^*+v^*$. (ii) Show that if $(\mathfrak{A},\bar\mu)$ is any measure algebra and u, $v\in M^{0,\infty}_{\bar\mu}$, then $\int_0^t(u+v)^*\le\int_0^t u^*+v^*$ for every $t\ge 0$.

(p) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be two measure algebras. For $u\in M^{1,0}_{\bar\mu}$, $w\in M^{\infty,1}_{\bar\nu}$ set

$$\rho_{uw}(S,T)=|\int Su\times w-\int Tu\times w|\text{ for }S,\,T\in\mathcal{T}^{(0)}=\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}.$$

The topology generated by the pseudometrics ρ_{uw} is the **very weak operator topology** on $\mathcal{T}^{(0)}$. Show that $\mathcal{T}^{(0)}$ is compact in this topology.

(q) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras and let $u\in M^{1,0}_{\bar\mu}$. (i) Show that $B=\{Tu:T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}\}$ is compact for the topology $\mathfrak{T}_s(M^{1,0}_{\bar\nu},M^{\infty,1}_{\bar\nu})$. (ii) Show that any non-decreasing sequence in B has a supremum in $L^0(\mathfrak{B})$ which belongs to B.

(r) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $u\in M^{1,0}_{\bar\mu}$, $v\in M^{1,0}_{\bar\nu}$. Show that the following are equiveridical: (i) there is a $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ such that $Tu=v$; (ii) $\int_0^t u^*\le\int_0^t v^*$ for every $t\ge 0$.

(s) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Suppose that u_1, $u_2\in M^{1,\infty}_{\bar\mu}$ and $v\in M^{1,\infty}_{\bar\nu}$ are such that $\int_0^t v^*\le\int_0^t(u_1+u_2)^*$ for every $t\ge 0$. Show that there are v_1, $v_2\in M^{1,\infty}_{\bar\nu}$ such that $v_1+v_2=v$ and $\int_0^t v_i^*\le\int_0^t u_i^*$ for both i, every $t\ge 0$.

>(t) Set $g(t)=t/(t+1)$ for $t\ge 0$, and set $v=g^\bullet$, $u=\chi[0,1]^\bullet\in L^\infty(\mathfrak{A}_L)$. Show that $\int u^*\times v^*=1>\int Tu\times v$ for every $T\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$.

(u) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and for $T\in\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ define $T'\in\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ as in 373S. Show that $T''=T$.

(v) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and give $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$, $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ their very weak operator topologies (373Xp). Show that the map $T\mapsto T':\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}\to\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$ is an isomorphism for the convex, order and topological structures of the two spaces. (By the 'convex structure' of a convex set C in a linear space I mean the operation $(x,y,t)\mapsto tx+(1-t)y:C\times C\times[0,1]\to C$.)

373Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. Set $u=f^\bullet$ and $v=g^\bullet$ in $L^0(\mathfrak{A})$, where $f(t)=t$, $g(t)=1-2|t-\frac{1}{2}|$ for $t\in[0,1]$. Show that $u^*=v^*$, but that there is no measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $T_\pi v=u$, writing $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ for the operator induced by π, as in 364P. (*Hint*: show that $\{[\![v>\alpha]\!]:\alpha\in\mathbb{R}\}$ does not τ-generate $\mathfrak{A}$.)

(b) Let $(\mathfrak{A},\bar\mu)$ be a totally finite homogeneous measure algebra of uncountable Maharam type. Let u, $v\in(M^{1,\infty}_{\bar\mu})^+$ be such that $u^*=v^*$. Show that there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $T_\pi u=v$.

(c) Let u, $v\in M^{1,\infty}_{\bar\mu_L}$ be such that $u=u^*$, $v=v^*$ and $\int_0^t v\le\int_0^t u$ for every $t\ge 0$. (i) Show that there is a non-negative $T\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ such that $Tu=v$ and $\int_0^t Tw\le\int_0^t w$ for every $w\in(M^{1,\infty}_{\bar\mu_L})^+$. (ii) Show that any such T must belong to $\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$.

(d) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $u\in M^{1,\infty}_{\bar\mu}$. (i) Suppose that $w\in S(\mathfrak{B}^f)$. Show directly (without quoting the result of 373O, but possibly using some of the ideas of the proof) that for every $\gamma<\int u^*\times w^*$ there is a $T\in\mathcal{T}_{\bar\mu,\bar\nu}$ such that $\int Tu\times w\ge\gamma$. (ii) Suppose that $(\mathfrak{B},\bar\nu)$ is localizable and that $v\in M^{1,\infty}_{\bar\nu}\setminus\{Tu:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$. Show that there is a $w\in S(\mathfrak{B}^f)$ such that $\int v\times w>\sup\{\int Tu\times w:T\in\mathcal{T}_{\bar\mu,\bar\nu}\}$. (*Hint*: use 373M and the Hahn-Banach theorem in the following form: if U is a linear space with the topology $\mathfrak{T}_s(U,V)$ defined by a linear subspace V of $\mathsf{L}(U;\mathbb{R})$, $C\subseteq U$ is a non-empty closed convex set, and $v\in U\setminus C$, then there is an $f\in V$ such that $f(v)>\sup_{u\in C}f(u)$.) (iii) Hence prove 373O for localizable $(\mathfrak{B},\bar\nu)$. (iv) Now prove 373O for general $(\mathfrak{B},\bar\nu)$.

(e)(i) Define $v\in L^\infty(\mathfrak{A}_L)$ as in 373Xt. Show that there is no $T\in\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$ such that $Tv=v^*$. (ii) Set $h(t)=1+\max(0,\frac{\sin t}{t})$ for $t>0$, $w=h^\bullet\in L^\infty(\mathfrak{A}_L)$. Show that there is no $T\in\mathcal{T}^\times_{\bar\mu_L,\bar\mu_L}$ such that $Tw^*=w$.

(f) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. Show that $\mathcal{T}_{\bar\mu,\bar\mu_L}=\mathcal{T}^\times_{\bar\mu,\bar\mu_L}$ can be identified, as convex ordered space, with $\mathcal{T}^\times_{\bar\mu_L,\bar\mu}$, and that this is a proper subset of $\mathcal{T}_{\bar\mu_L,\bar\mu}$.

(g) Show that the adjoint operation of 373T is not as a rule continuous for the very weak operator topologies of $\mathcal{T}^\times_{\bar\mu,\bar\nu}$, $\mathcal{T}^\times_{\bar\nu,\bar\mu}$.

373 Notes and comments 373A-373B are just alternative expressions of concepts already treated in 371F-371H. My use of the simpler formula $\mathcal{T}_{\bar\mu,\bar\nu}$ symbolizes my view that $\mathcal{T}$, rather than $\mathcal{T}^{(0)}$ or $\mathcal{T}^\times$, is the most natural vehicle for these ideas; I used $\mathcal{T}^{(0)}$ in §§371 only because that made it possible to give theorems which applied to all measure algebras, without demanding localizability (compare 371Gb with 373Bc).

The obvious examples of operators in $\mathcal{T}$ are those derived from measure-preserving Boolean homomorphisms, as in 373Bd, and their adjoints (373U). Note that the latter include conditional expectation operators. In return, we find that operators in $\mathcal{T}$ share some of the characteristic properties of the operators derived from Boolean homomorphisms (373Bb, 373Xb, 373Xm). Other examples are multiplication operators (373Xc), operators obtained by piecing others together (373Xd) and kernel operators of the type described in 373Xe-373Xf, including convolution operators (373Xg). (For a general theory of kernel operators, see §376 below.)

Most of the section is devoted to the relationships between the classes $\mathcal{T}$ of operators and the 'decreasing rearrangements' of 373C. If you like, the decreasing rearrangement u^* of u describes the 'distribution' of $|u|$ (373Xh); but for $u \notin M^0$ it loses some information (373Xt, 373Ye). It is important to be conscious that even when $u \in L^0(\mathfrak{A}_L)$, u^* is not necessarily obtained by 'rearranging' the elements of the algebra $\mathfrak{A}_L$ by a measure-preserving automorphism (which would, of course, correspond to an automorphism of the measure space $([0,\infty[\,,\mu_L)$, by 344C). I will treat 'rearrangements' of this narrower type in the next section; for the moment, see 373Ya. Apart from this, the basic properties of decreasing rearrangements are straightforward enough (373D-373F). The only obscure area concerns the relationship between $(u+v)^*$ and u^*, v^* (see 373Xo).

In 373G I embark on results involving both decreasing rearrangements and operators in $\mathcal{T}$, leading to the characterization of the sets $\{Tu : T \in \mathcal{T}\}$ in 373O. In one direction this is easy, and is the content of 373G. In the other direction it depends on a deeper analysis, and the easiest method seems to be through studying the 'very weak operator topology' on $\mathcal{T}$ (373K-373L), even though this is an effective tool only when one of the algebras involved is localizable (373L). A functional analyst is likely to feel that the method is both natural and illuminating; but from the point of view of a measure theorist it is not perfectly satisfactory, because it is essentially non-constructive. While it tells us that there are operators $T \in \mathcal{T}$ acting in the required ways, it gives only the vaguest of hints concerning what they actually look like.

Of course the very weak operator topology is interesting in its own right; and see also 373Xp-373Xq.

The proof of 373O can be thought of as consisting of three steps. Given that $\int_0^t v^* \le \int_0^t u^*$ for every t, then I set out to show that v is expressible as $T_1 v^*$ (parts (c)-(d) of the proof), that v^* is expressible as $T_2 u^*$ (part (g)) and that u^* is expressible as $T_3 u$ (parts (e)-(f)), each T_i belonging to an appropriate $\mathcal{T}$. In all three steps the general case follows easily from the case in which $u \in S(\mathfrak{A})$ and $v \in S(\mathfrak{B})$. If we are willing to use a more sophisticated version of the Hahn-Banach theorem than those given in 3A5A and 363R, there is an alternative route (373Yd). I note that the central step above, from u^* to v^*, can be performed with an order-continuous T_2 (373Yc), but that in general neither of the other steps can (373Ye), so that we cannot use $\mathcal{T}^\times$ in place of $\mathcal{T}$ here.

A companion result to 373O, in that it also shows that $\{Tu : T \in \mathcal{T}\}$ is large enough to reach natural bounds, is 373P; given u and v, we can find T such that $\int Tu \times v$ is as large as possible. In this form the result is valid only for $v \in M^{(0)}$ (373Xt). But if we do not demand that the supremum should be attained, we can deal with other v (373Q).

We already know that every operator in $\mathcal{T}^{(0)}$ is a difference of order-continuous operators, just because $M^{1,0}$ has an order-continuous norm (371Gb). It is therefore not surprising that members of $\mathcal{T}^{(0)}$ can be extended to members of $\mathcal{T}^\times$, at least when the codomain $M^{1,\infty}_{\bar\nu}$ is Dedekind complete (373R). It is also very natural to look for a correspondence between $\mathcal{T}_{\bar\mu,\bar\nu}$ and $\mathcal{T}_{\bar\nu,\bar\mu}$, because if $T \in \mathcal{T}_{\bar\mu,\bar\nu}$ we shall surely have an adjoint operator $(T\restriction L^1_{\bar\mu})'$ from $(L^1_{\bar\nu})^*$ to $(L^1_{\bar\mu})^*$, and we can hope that this will correspond to some member of $\mathcal{T}_{\bar\nu,\bar\mu}$. But when we come to the details, the normed-space properties of a general member of $\mathcal{T}$ are not enough (373Yf), and we need some kind of order-continuity. For members of $\mathcal{T}^{(0)}$ this is automatically present (373S), and now the canonical isomorphism between $\mathcal{T}^{(0)}$ and $\mathcal{T}^\times$ gives us an isomorphism between $\mathcal{T}^\times_{\bar\mu,\bar\nu}$ and $\mathcal{T}^\times_{\bar\nu,\bar\mu}$ when $\bar\mu$ and $\bar\nu$ are localizable (373T).

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374 Rearrangement-invariant spaces

As is to be expected, many of the most important function spaces of analysis are symmetric in various ways; in particular, they share the symmetries of the underlying measure algebras. The natural expression of this is to say that they are 'rearrangement-invariant' (374E). In fact it turns out that in many cases they have the stronger property of '$\mathcal{T}$-invariance' (374A). In this section I give a brief account of the most important properties of these two kinds of invariance. In particular, $\mathcal{T}$-invariance is related to a kind of transfer mechanism, enabling us to associate function spaces on different measure algebras (374C-374D). As for rearrangement-invariance, the salient fact is that on the most important measure algebras many rearrangement-invariant spaces are $\mathcal{T}$-invariant (374K, 374M).

374A $\mathcal{T}$-invariance: Definitions Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Recall that I write

$$M^{1,\infty}_{\bar\mu} = L^1_{\bar\mu} + L^\infty(\mathfrak{A}) \subseteq L^0(\mathfrak{A}),$$

$$M^{\infty,1}_{\bar\mu} = L^1_{\bar\mu} \cap L^\infty(\mathfrak{A}),$$

$$M^{0,\infty}_{\bar\mu} = \{u : u \in L^0(\mathfrak{A}),\ \inf_{\alpha>0} \bar\mu[\![|u| > \alpha]\!] < \infty\},$$

(369N, 373C).

(a) I will say that a subset A of $M^{1,\infty}_{\bar\mu}$ is **$\mathcal{T}$-invariant** if $Tu \in A$ whenever $u \in A$ and $T \in \mathcal{T} = \mathcal{T}_{\bar\mu,\bar\mu}$ (definition: 373Aa).

(b) An extended Fatou norm τ on L^0 is **$\mathcal{T}$-invariant** or **fully symmetric** if $\tau(Tu) \le \tau(u)$ whenever $u \in M^{1,\infty}_{\bar\mu}$ and $T \in \mathcal{T}$.

(c) As in §373, I will write $(\mathfrak{A}_L, \bar\mu_L)$ for the measure algebra of Lebesgue measure on $[0,\infty[$, and $u^* \in M^{0,\infty}_{\bar\mu_L}$ for the decreasing rearrangement of any u belonging to any $M^{0,\infty}_{\bar\mu}$ (373C).

374B The first step is to show that the associate of a $\mathcal{T}$-invariant norm is $\mathcal{T}$-invariant.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. Let L^τ be the Banach lattice defined from τ (369G), and τ' the associate extended Fatou norm (369H-369I). Then

(i) $M^{\infty,1}_{\bar\mu} \subseteq L^\tau \subseteq M^{1,\infty}_{\bar\mu}$;

(ii) τ' is also $\mathcal{T}$-invariant, and $\int u^* \times v^* \le \tau(u)\tau'(v)$ for all u, $v \in M^{0,\infty}_{\bar\mu}$.

proof (a) I check first that $L^\tau \subseteq M^{0,\infty}_{\bar\mu}$. **P** Take any $u \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar\mu}$. There is surely some $w > 0$ in L^τ, and we can suppose that $w = \chi a$ for some a of finite measure. Now, for any $n \in \mathbb{N}$,

$$(|u| \wedge n\chi 1)^* = n\chi 1 \ge nw^*$$

in $L^0(\mathfrak{A}_L)$, because $\bar\mu[\![|u| > n]\!] = \infty$. So there is a $T \in \mathcal{T}_{\bar\mu,\bar\mu}$ such that $T(|u| \wedge n\chi 1) = nw$, by 373O, and

$$\tau(u) \ge \tau(|u| \wedge n\chi 1) \ge \tau(T(|u| \wedge n\chi 1)) = \tau(nw) = n\tau(w).$$

As n is arbitrary, $\tau(u) = \infty$. As u is arbitrary, $L^\tau \subseteq M^{0,\infty}_{\bar\mu}$. **Q**

(b) Next, $\int u^* \times v^* \le \tau(u)\tau'(v)$ for all u, $v \in M^{0,\infty}_{\bar\mu}$. **P** If $u \in M^{1,\infty}_{\bar\mu}$, then

$$\int u^* \times v^* = \sup\{\int |Tu \times v| : T \in \mathcal{T}_{\bar\mu,\bar\mu}\}$$

(373Q)

$$\le \sup\{\tau(Tu)\tau'(v) : T \in \mathcal{T}_{\bar\mu,\bar\mu}\} = \tau(u)\tau'(v).$$

Generally, setting $u_n = |u| \wedge n\chi 1$, $\langle u_n^*\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u^* (373Db, 373Dh), so

$$\int u^* \times v^* = \sup_{n\in\mathbb{N}} \int u_n^* \times v^* \le \sup_{n\in\mathbb{N}} \tau(u_n)\tau'(v) = \tau(u)\tau'(v). \quad \textbf{Q}$$

(c) Consequently, $L^\tau \subseteq M^{1,\infty}_{\bar\mu}$. **P** If $\mathfrak{A} = \{0\}$, this is trivial. Otherwise, take $u \in L^\tau$. There is surely some non-zero a such that $\tau'(\chi a) < \infty$; now, setting $v = \chi a$,

$$\int_0^{\bar\mu a} u^* = \int u^* \times v^* \le \tau(u)\tau'(v) < \infty$$

by (b) above. But this means that $u^* \in M^{1,\infty}_{\bar\mu}$, so that $u \in M^{1,\infty}_{\bar\mu}$ (373F(b-ii)). **Q**

(d) Next, τ' is $\mathcal{T}$-invariant. **P** Suppose that $v \in M^{1,\infty}_{\bar\mu}$, $T \in \mathcal{T}_{\bar\mu,\bar\mu}$, $u \in L^0(\mathfrak{A})$ and $\tau(u) \le 1$. Then $u \in M^{1,\infty}_{\bar\mu}$, by (c), so

$$\int |u \times Tv| \le \int u^* \times v^* \le \tau(u)\tau'(v) \le \tau'(v),$$

using 373J for the first inequality. Taking the supremum over u, we see that $\tau'(Tv) \le \tau'(v)$; as T and v are arbitrary, τ' is $\mathcal{T}$-invariant. **Q**

(e) Finally, putting (d) and (c) together, $L^{\tau'} \subseteq M^{1,\infty}_{\bar\mu}$, so that $L^\tau \supseteq M^{\infty,1}_{\bar\mu}$, using 369J and 369O.

374C For any $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$ there are corresponding norms on $L^0(\mathfrak{A})$ for any semi-finite measure algebra, as follows.

Theorem Let θ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$, and $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra.
(a) There is a $\mathcal{T}$-invariant extended Fatou norm τ on $L^0(\mathfrak{A})$ defined by setting

$$\begin{aligned}\tau(u) &= \theta(u^*) \text{ if } u \in M^{0,\infty}_{\bar\mu},\\ &= \infty \text{ if } u \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar\mu}.\end{aligned}$$

(b) Writing θ', τ' for the associates of θ and τ, we now have

$$\begin{aligned}\tau'(v) &= \theta'(v^*) \text{ if } v \in M^{0,\infty}_{\bar\mu},\\ &= \infty \text{ if } v \in L^0(\mathfrak{A}) \setminus M^{0,\infty}_{\bar\mu}.\end{aligned}$$

(c) If θ is an order-continuous norm on the Banach lattice L^θ, then τ is an order-continuous norm on L^τ.

proof (a)(i) The argument seems to run better if I use a different formula to define τ: set

$$\tau(u) = \sup\{\int |u \times Tw| : T \in \mathcal{T}_{\bar\mu_L,\bar\mu},\, w \in L^0(\mathfrak{A}_L),\, \theta'(w) \le 1\}$$

for $u \in L^0(\mathfrak{A})$. (By 374B(i), $w \in M^{1,\infty}_{\bar\mu_L}$ whenever $\theta'(w) \le 1$, so there is no difficulty in defining Tw.) Now $\tau(u) = \theta(u^*)$ for every $u \in M^{0,\infty}_{\bar\mu}$. **P** ($\alpha$) If $w \in L^0(\mathfrak{A}_L)$ and $\theta'(w) \le 1$, then $w \in M^{1,\infty}_{\bar\mu_L}$, so there is an $S \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ such that $Sw = w^*$ (373O). Accordingly $\theta'(w^*) \le \theta'(w)$ (because θ' is $\mathcal{T}$-invariant, by 374B); now

$$\int |u \times Tw| \le \int u^* \times w^* \le \theta(u^*)\theta'(w^*) \le \theta(u^*)\theta'(w) \le \theta(u^*);$$

as w is arbitrary, $\tau(u) \le \theta(u^*)$. (β) If $w \in L^0(\mathfrak{A}_L)$ and $\theta'(w) \le 1$, then

$$\int |u^* \times w| \le \int (u^*)^* \times w^*$$

(373E)

$$= \int u^* \times w^* = \sup\{\int |u \times Tw| : T \in \mathcal{T}_{\bar\mu_L,\bar\mu}\}$$

(373Q)

$$\le \tau(u).$$

But because θ is the associate of θ' (369I(ii)), this means that $\theta(u^*) \le \tau(u)$. **Q**

(ii) Now τ is an extended Fatou norm on $L^0(\mathfrak{A})$. **P** Of the conditions in 369F, (i)-(iv) are satisfied just because $\tau(u) = \sup_{v\in B} \int |u \times v|$ for some set $B \subseteq L^0$. As for (v) and (vi), observe that if $u \in M^{\infty,1}_{\bar\mu}$ then $u^* \in M^{\infty,1}_{\bar\mu_L}$ (373F(b-iv)), so that $\tau(u) = \theta(u^*) < \infty$, by 374B(i), while also

$$u \ne 0 \Longrightarrow u^* \ne 0 \Longrightarrow \tau(u) = \theta(u^*) > 0.$$

As $M^{\infty,1}_{\bar\mu}$ is order-dense in $L^0(\mathfrak{A})$ (this is where I use the hypothesis that $(\mathfrak{A},\bar\mu)$ is semi-finite), 369F(v)-(vi) are satisfied, and τ is an extended Fatou norm. **Q**

(iii) τ is $\mathcal{T}$-invariant. **P** Take $u \in M^{1,\infty}_{\bar\mu}$ and $T \in \mathcal{T}_{\bar\mu,\bar\mu}$. There are $S_0 \in \mathcal{T}_{\bar\mu_L,\bar\mu}$ and $S_1 \in \mathcal{T}_{\bar\mu,\bar\mu_L}$ such that $S_0u^* = u$, $S_1Tu = (Tu)^*$ (373O); now $S_1TS_0 \in \mathcal{T}_{\bar\mu_L,\bar\mu_L}$ (373Be), so

$$\tau(Tu) = \theta((Tu)^*) = \theta(S_1TS_0u^*) \le \theta(u^*) = \tau(u)$$

because θ is $\mathcal{T}$-invariant. **Q**

(iv) We can now return to the definition of τ. I have already remarked that $\tau(u)=\theta(u^*)$ if $u\in M^{0,\infty}_{\bar\mu}$. For other u, we must have $\tau(u)=\infty$ just because τ is a $\mathcal{T}$-invariant extended Fatou norm (374B(i)). So the definitions in the statement of the theorem and (i) above coincide.

(b) We surely have $\tau'(v)=\infty$ if $v\in L^0(\mathfrak{A})\setminus M^{0,\infty}_{\bar\mu}$, by 374B, because τ', like τ, is a $\mathcal{T}$-invariant extended Fatou norm. So take $v\in M^{0,\infty}_{\bar\mu}$.

(i) If $u\in L^0(\mathfrak{A})$ and $\tau(u)\le 1$, then

$$\int|v\times u|\le\int v^*\times u^*\le\theta'(v^*)\theta(u^*)=\theta'(v^*)\tau(u)\le\theta'(v^*);$$

as u is arbitrary, $\tau'(v)\le\theta'(v^*)$.

(ii) If $w\in L^0(\mathfrak{A}_L)$ and $\theta(w)\le 1$, then

$$\int|v^*\times w|\le\int v^*\times w^*=\sup\{\int|v\times Tw|:T\in\mathcal{T}_{\bar\mu_L,\bar\mu}\}$$

(373Q)

$$\le\sup\{\tau'(v)\tau(Tw):T\in\mathcal{T}_{\bar\mu_L,\bar\mu}\}=\sup\{\tau'(v)\theta((Tw)^*):T\in\mathcal{T}_{\bar\mu_L,\bar\mu}\}$$
$$\le\sup\{\tau'(v)\theta(STw):T\in\mathcal{T}_{\bar\mu_L,\bar\mu},\,S\in\mathcal{T}_{\bar\mu,\bar\mu_L}\}$$

(because, given T, we can find an S such that $STw=(Tw)^*$, by 373O)

$$\le\sup\{\tau'(v)\theta(Tw):T\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}\}\le\tau'(v).$$

As w is arbitrary, $\theta'(v^*)\le\tau'(v)$ and the two are equal. This completes the proof of (b).

(c)(i) The first step is to note that $L^\tau\subseteq M^0_{\bar\mu}$. **P?** Suppose that $u\in L^\tau\setminus M^0_{\bar\mu}$, that is, that $\bar\mu[\![|u|>\alpha]\!]=\infty$ for some $\alpha>0$. Then $u^*\ge\alpha\chi 1$ in $L^0(\mathfrak{A}_L)$, so $L^\infty(\mathfrak{A}_L)\subseteq L^\theta$. For each $n\in\mathbb{N}$, set $v_n=\chi\,[n,\infty[^\bullet$. Then $v_n^*=v_0$, so we can find a $T_n\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ such that $T_nv_n=v_0$ (373O), and $\theta(v_n)\ge\theta(v_0)$ for every n. But as $\langle v_n\rangle_{n\in\mathbb{N}}$ is a decreasing sequence with infimum 0, this means that θ is not an order-continuous norm. **XQ**

(ii) Now suppose that $A\subseteq L^\tau$ is non-empty and downwards-directed and has infimum 0. Then $\inf_{u\in A}\bar\mu[\![u>\alpha]\!]=0$ for every $\alpha>0$ (put 364L(b-ii) and 321F together). But this means that $B=\{u^*:u\in A\}$ must have infimum 0; since B is surely downwards-directed, $\inf_{v\in B}\theta(v)=0$, that is, $\inf_{u\in A}\tau(u)=0$. As A is arbitrary, τ is an order-continuous norm.

374D What is more, every $\mathcal{T}$-invariant extended Fatou norm can be represented in this way.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. Then there is a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{A}_L)$ such that $\tau(u)=\theta(u^*)$ for every $u\in M^{0,\infty}_{\bar\mu}$.

proof I use the method of 374C. If $\mathfrak{A}=\{0\}$ the result is trivial; assume that $\mathfrak{A}\ne\{0\}$.

(a) Set

$$\theta(w)=\sup\{\int|w\times Tv|:T\in\mathcal{T}_{\bar\mu,\bar\mu_L},\,v\in L^0(\mathfrak{A}),\,\tau'(v)\le 1\}$$

for $w\in L^0(\mathfrak{A}_L)$. Note that

$$\theta(w)=\sup\{\int w^*\times v^*:v\in L^0(\mathfrak{A}),\,\tau'(v)\le 1\}$$

for every $w\in M^{0,\infty}_{\bar\mu_L}$, by 373Q again.

θ is an extended Fatou norm on $L^0(\mathfrak{A}_L)$. **P** As in 374C, the conditions 369F(i)-(iv) are elementary. If $w>0$ in $L^0(\mathfrak{A}_L)$, take any $v\in L^0(\mathfrak{A})$ such that $0<\tau'(v)\le 1$; then $w^*\times v^*\ne 0$ so $\theta(w)\ge\int w^*\times v^*>0$. So 369F(v) is satisfied. As for 369F(vi), if $w>0$ in $L^0(\mathfrak{A}_L)$, take a non-zero $a\in\mathfrak{A}$ of finite measure such that $\alpha=\tau(\chi a)<\infty$. Let $\beta>0$, $b\in\mathfrak{A}_L$ be such that $0<\bar\mu_Lb\le\bar\mu a$ and $\beta\chi b\le w$; then

$$\theta(\chi b)=\sup_{\tau'(v)\le 1}\int(\chi b)^*\times v^*\le\sup_{\tau'(v)\le 1}\int(\chi a)^*\times v^*\le\tau(\chi a)<\infty$$

by 374B(ii). So $\theta(\beta\chi b)<\infty$ and 369F(vi) is satisfied. Thus θ is an extended Fatou norm. **Q**

(b) θ is $\mathcal{T}$-invariant. **P** If $T\in\mathcal{T}_{\bar\mu_L,\bar\mu_L}$ and $w\in M^{1,\infty}_{\bar\mu_L}$, then

$$\theta(Tw) = \sup_{\tau'(v)\le 1} \int (Tw)^* \times v^* \le \sup_{\tau'(v)\le 1} \int w^* \times v^* = \theta(w)$$

by 373G and 373I. **Q**

(c) $\theta(u^*) = \tau(u)$ for every $u \in M^{0,\infty}_{\bar\mu}$. **P** We have

$$\tau(u) = \sup_{\tau'(v)\le 1} \int |u \times v| \le \sup_{\tau'(v)\le 1} \int u^* \times v^* \le \tau(u),$$

using 369I, 373E and 374B. So

$$\theta(u^*) = \sup_{\tau'(v)\le 1} \int u^* \times v^* = \tau(u)$$

by the remark in (a) above. **Q**

374E I turn now to rearrangement-invariance. Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) I will say that a subset A of $L^0 = L^0(\mathfrak{A})$ is **rearrangement-invariant** if $T_\pi u \in A$ whenever $u \in A$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean automorphism, writing $T_\pi : L^0 \to L^0$ for the isomorphism corresponding to π (364P).

(b) I will say that an extended Fatou norm τ on L^0 is **rearrangement-invariant** if $\tau(T_\pi u) = \tau(u)$ whenever $u \in L^0$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism.

374F Remarks (a) If $(\mathfrak{A}, \bar\mu)$ is a semi-finite measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a sequentially order-continuous measure-preserving Boolean homomorphism, then $T_\pi{\restriction}M^{1,\infty}_{\bar\mu}$ belongs to $\mathcal{T}_{\bar\mu,\bar\mu}$; this is obvious from the definition of $M^{1,\infty} = L^1 + L^\infty$ and the basic properties of T_π (364P). Accordingly, any $\mathcal{T}$-invariant extended Fatou norm τ on $L^0(\mathfrak{A})$ must be rearrangement-invariant, since (by 374B) we shall have $\tau(u) = \tau(T_\pi(u)) = \infty$ when $u \notin M^{1,\infty}_{\bar\mu}$. Similarly, any $\mathcal{T}$-invariant subset of $M^{1,\infty}_{\bar\mu}$ will be rearrangement-invariant.

(b) I seek to describe cases in which rearrangement-invariance implies $\mathcal{T}$-invariance. This happens only for certain measure algebras; in order to shorten the statements of the main theorems I introduce a special phrase.

374G Definition I say that a measure algebra $(\mathfrak{A}, \bar\mu)$ is **quasi-homogeneous** if for any non-zero a, $b \in \mathfrak{A}$ there is a measure-preserving Boolean automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi a \cap b \ne 0$.

374H Proposition Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Then the following are equiveridical:

(i) $(\mathfrak{A}, \bar\mu)$ is quasi-homogeneous;

(ii) *either* $\mathfrak{A}$ is purely atomic and every atom of $\mathfrak{A}$ has the same measure *or* there is a $\kappa \ge \omega$ such that the principal ideal $\mathfrak{A}_a$ is homogeneous, with Maharam type κ, for every $a \in \mathfrak{A}$ of non-zero finite measure.

proof (i)⇒(ii) Suppose that $(\mathfrak{A}, \bar\mu)$ is quasi-homogeneous.

(α) Suppose that $\mathfrak{A}$ has an atom a. In this case, for any $b \in \mathfrak{A} \setminus \{0\}$ there is an automorphism π of $(\mathfrak{A}, \bar\mu)$ such that $\pi a \cap b \ne 0$; now πa must be an atom, so $\pi a = \pi a \cap b$ and πa is an atom included in b. As b is arbitrary, $\mathfrak{A}$ is purely atomic; moreover, if b is an atom, then it must be equal to πa and therefore of the same measure as a, so all atoms of $\mathfrak{A}$ have the same measure.

(β) Now suppose that $\mathfrak{A}$ is atomless. In this case, if $a \in \mathfrak{A}$ has finite non-zero measure, $\mathfrak{A}_a$ is homogeneous. **P?** Otherwise, there are non-zero b, $c \subseteq a$ such that the principal ideals $\mathfrak{A}_b$, $\mathfrak{A}_c$ are homogeneous and of different Maharam types, by Maharam's theorem (332B, 332H). But now there is supposed to be an automorphism π such that $\pi b \cap c \ne 0$, in which case $\mathfrak{A}_b$, $\mathfrak{A}_{\pi b}$, $\mathfrak{A}_{\pi b \cap c}$ and $\mathfrak{A}_c$ must all have the same Maharam type. **XQ**

Consequently, if a, $b \in \mathfrak{A}$ are both of non-zero finite measure, the Maharam types of $\mathfrak{A}_a$, $\mathfrak{A}_{a\cup b}$ and $\mathfrak{A}_b$ must all be the same infinite cardinal κ.

(ii)⇒(i) Assume (ii), and take a, $b \in \mathfrak{A} \setminus \{0\}$. If $a \cap b \ne 0$ we can take π to be the identity automorphism and stop. So let us suppose that $a \cap b = 0$.

(α) If $\mathfrak{A}$ is purely atomic and every atom has the same measure, then there are atoms $a_0 \subseteq a$, $b_0 \subseteq b$. Set

$$\pi c = c \text{ if } c \supseteq a_0 \cup b_0 \text{ or } c \cap (a_0 \cup b_0) = 0,$$
$$= c \triangle (a_0 \cup b_0) \text{ otherwise.}$$

Then it is easy to check that π is a measure-preserving automorphism of $\mathfrak{A}$ such that $\pi a_0 = b_0$, so that $\pi a \cap b \neq 0$.

(β) If $\mathfrak{A}_c$ is Maharam-type-homogeneous with the same infinite Maharam type κ for every non-zero c of finite measure, set $\gamma = \min(1, \bar\mu a, \bar\mu b) > 0$. Because $\mathfrak{A}$ is atomless, there are $a_0 \subseteq a$, $b_0 \subseteq b$ with $\bar\mu a_0 = \bar\mu b_0 = \gamma$ (331C). Now $\mathfrak{A}_{a_0}$ and $\mathfrak{A}_{b_0}$ are homogeneous with the same Maharam type and the same magnitude, so by Maharam's theorem (331I) there is a measure-preserving isomorphism $\pi_0 : \mathfrak{A}_{a_0} \to \mathfrak{A}_{b_0}$. Define $\pi : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\pi c = (c \setminus (a_0 \cup b_0)) \cup \pi_0(c \cap a_0) \cup \pi_0^{-1}(c \cap b_0)$$

for $c \in \mathfrak{A}$; then it is easy to see that π is a measure-preserving automorphism of $\mathfrak{A}$ and that $\pi a \cap b \neq 0$.

Remark We shall return to these ideas in Chapter 38. In particular, the construction of π from π_0 in the last part of the proof will be of great importance; in the language of 381R, $\pi = \overleftarrow{(a_0 \,{}_{\pi_0}\, b_0)}$.

374I Corollary Let $(\mathfrak{A}, \bar\mu)$ be a quasi-homogeneous semi-finite measure algebra. Then

(a) whenever a, $b \in \mathfrak{A}$ have the same finite measure, the principal ideals $\mathfrak{A}_a$, $\mathfrak{A}_b$ are isomorphic as measure algebras;

(b) there is a subgroup Γ of the additive group $\mathbb{R}$ such that (α) $\bar\mu a \in \Gamma$ whenever $a \in \mathfrak{A}$ and $\bar\mu a < \infty$ (β) whenever $a \in \mathfrak{A}$, $\gamma \in \Gamma$ and $0 \le \gamma \le \bar\mu a$ then there is a $c \subseteq a$ such that $\bar\mu c = \gamma$.

proof If $\mathfrak{A}$ is purely atomic, with all its atoms of measure γ_0, set $\Gamma = \gamma_0\mathbb{Z}$, and the results are elementary. If $\mathfrak{A}$ is atomless, set $\Gamma = \mathbb{R}$; then (a) is a consequence of Maharam's theorem, and (b) is a consequence of 331C, already used in the proof of 374H.

374J Lemma Let $(\mathfrak{A}, \bar\mu)$ be a quasi-homogeneous semi-finite measure algebra and u, $v \in M^{0,\infty}_{\bar\mu}$. Let $\mathrm{Aut}_{\bar\mu}$ be the group of measure-preserving automorphisms of $\mathfrak{A}$. Then

$$\int u^* \times v^* = \sup_{\pi \in \mathrm{Aut}_{\bar\mu}} \int |u \times T_\pi v|,$$

where $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is the isomorphism corresponding to π.

proof (a) Suppose first that u, v are non-negative and belong to $S(\mathfrak{A}^f)$, where $\mathfrak{A}^f$ is the ring $\{a : \bar\mu a < \infty\}$, as usual. Then they can be expressed as $u = \sum_{i=0}^m \alpha_i \chi a_i$, $v = \sum_{j=0}^n \beta_j \chi b_j$ where $\alpha_0 \ge \dots \alpha_m \ge 0$, $\beta_0 \ge \dots \ge \beta_n \ge 0$, $a_0, \dots, a_m$ are disjoint and of finite measure, and $b_0, \dots, b_n$ are disjoint and of finite measure. Extending each list by a final term having a coefficient of 0, if need be, we may suppose that $\sup_{i \le m} a_i = \sup_{j \le n} b_j$.

Let $(t_0, \dots, t_s)$ enumerate in ascending order the set

$$\{0\} \cup \{\textstyle\sum_{i=0}^k \bar\mu a_i : k \le m\} \cup \{\textstyle\sum_{j=0}^k \bar\mu b_j : k \le n\}.$$

Then every t_r belongs to the subgroup Γ of 374Ib, and $t_s = \sum_{i=0}^m \bar\mu a_i = \sum_{j=0}^n \bar\mu b_j$. For $1 \le r \le s$ let $k(r)$, $l(r)$ be minimal subject to the requirements $t_r \le \sum_{i=0}^{k(r)} \bar\mu a_i$, $t_r \le \sum_{j=0}^{l(r)} \bar\mu b_j$. Then $\bar\mu a_i = \sum_{k(r)=i} t_r - t_{r-1}$, so (using 374Ib) we can find a disjoint family $\langle c_r \rangle_{1 \le r \le s}$ such that $c_r \subseteq a_{k(r)}$ and $\bar\mu c_r = t_r - t_{r-1}$ for each r. Similarly, there is a disjoint family $\langle d_r \rangle_{1 \le r \le s}$ such that $d_r \subseteq b_{l(r)}$ and $\bar\mu d_r = t_r - t_{r-1}$ for each r. Now the principal ideals $\mathfrak{A}_{c_r}$, $\mathfrak{A}_{d_r}$ are isomorphic for every r, by 374Ia; let $\pi_r : \mathfrak{A}_{d_r} \to \mathfrak{A}_{c_r}$ be measure-preserving isomorphisms. Define $\pi : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\pi a = (a \setminus \sup_{1 \le r \le s} d_r) \cup \sup_{1 \le r \le s} \pi_r(a \cap d_r);$$

because

$$\sup_{r \le s} c_r = \sup_{i \le m} a_i = \sup_{j \le n} b_j = \sup_{r \le s} d_r,$$

$\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism.

Now

$$u = \sum_{r=1}^{s} \alpha_{k(r)} \chi c_r, \quad v = \sum_{r=1}^{s} \beta_{l(r)} \chi d_r,$$

$$u^* = \sum_{r=1}^{s} \alpha_{k(r)} \chi\, [t_{r-1}, t_r[^\bullet, \quad v^* = \sum_{r=1}^{s} \beta_{l(r)} \chi\, [t_{r-1}, t_r[^\bullet,$$

so

$$\int u \times T_\pi v = \sum_{r=1}^{s} \alpha_{k(r)} \beta_{l(r)} \bar\mu c_r = \sum_{r=1}^{s} \alpha_{k(r)} \beta_{l(r)} (t_r - t_{r-1}) = \int u^* \times v^*.$$

(b) Now take any u_0, $v_0 \in M^{0,\infty}_{\bar\mu}$. Set

$$A = \{u : u \in S(\mathfrak{A}^f),\ 0 \le u \le |u_0|\}, \quad B = \{v : v \in S(\mathfrak{A}^f),\ 0 \le v \le |v_0|\}.$$

Then A is an upwards-directed set with supremum $|u_0|$, because $(\mathfrak{A}, \bar\mu)$ is semi-finite, so $\{u^* : u \in A\}$ is an upwards-directed set with supremum $|u_0|^* = u_0^*$ (373Db, 373Dh). Similarly $\{v^* : v \in B\}$ is upwards-directed and has supremum v_0^*, so $\{u^* \times v^* : u \in A,\ v \in B\}$ is upwards-directed and has supremum $u_0^* \times v_0^*$.

Consequently, if $\gamma < \int u_0^* \times v_0^*$, there are $u \in A$, $v \in B$ such that $\gamma \le \int u^* \times v^*$. Now, by (a), there is a $\pi \in \mathrm{Aut}_{\bar\mu}$ such that

$$\gamma \le \int u \times T_\pi v \le \int |u_0| \times T_\pi |v_0| = \int |u_0 \times T_\pi v_0|$$

because T_π is a Riesz homomorphism. As γ is arbitrary,

$$\int u_0^* \times v_0^* \le \sup_{\pi \in \mathrm{Aut}_{\bar\mu}} \int |u_0 \times T_\pi v_0|.$$

But the reverse inequality is immediate from 373J.

374K Theorem Let $(\mathfrak{A}, \bar\mu)$ be a quasi-homogeneous semi-finite measure algebra, and τ a rearrangement -invariant extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then τ is $\mathcal{T}$-invariant.

proof Write τ' for the associate of τ. Then 374J tells us that for any u, $v \in M^{0,\infty}_{\bar\mu}$,

$$\int u^* \times v^* = \sup_{\pi \in \mathrm{Aut}_{\bar\mu}} \int |T_\pi u \times v| \le \sup_{\pi \in \mathrm{Aut}_{\bar\mu}} \tau(T_\pi u)\tau'(v) = \tau(u)\tau'(v),$$

writing u^*, v^* for the decreasing rearrangements of u and v, and $\mathrm{Aut}_{\bar\mu}$ for the group of measure-preserving automorphisms of $(\mathfrak{A}, \bar\mu)$. But now, if $u \in M^{1,\infty}_{\bar\mu}$ and $T \in \mathcal{T}_{\bar\mu,\bar\mu}$,

$$\tau(Tu) = \sup\{\int |Tu \times v| : \tau'(v) \le 1\}$$

(369I)

$$\le \sup\{\int u^* \times v^* : \tau'(v) \le 1\}$$

(373J)

$$\le \tau(u).$$

As T, u are arbitrary, τ is $\mathcal{T}$-invariant.

374L Lemma Let $(\mathfrak{A}, \bar\mu)$ be a quasi-homogeneous semi-finite measure algebra. Suppose that u, $v \in (M^{0,\infty}_{\bar\mu})^+$ are such that $\int u^* \times v^* = \infty$. Then there is a measure-preserving automorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\int u \times T_\pi v = \infty$.

proof I take three cases separately.

(a) Suppose that $\mathfrak{A}$ is purely atomic; then u, $v \in L^\infty(\mathfrak{A})$ and u^*, $v^* \in L^\infty(\mathfrak{A}_L)$, so neither u^* nor v^* can belong to $L^1_{\bar\mu_L}$ and neither u nor v can belong to $L^1_{\bar\mu}$. Let γ be the common measure of the atoms of $\mathfrak{A}$. For each $n \in \mathbb{N}$, set

$$\alpha_n = \inf\{\alpha : \alpha \ge 0,\ \bar\mu[\![u > \alpha]\!] \le 3^n\gamma\}, \quad \tilde a_n = [\![u > \tfrac{1}{2}\alpha_n]\!].$$

Then $\bar{\mu}[\![u > \alpha_n]\!] \le 3^n\gamma$; also $\alpha_n > 0$, since otherwise u would belong to $L^1_{\bar{\mu}}$, so $\bar{\mu}\tilde{a}_n \ge 3^n\gamma$. We can therefore choose $\langle a'_n\rangle_{n\in\mathbb{N}}$ inductively such that $a'_n \subseteq \tilde{a}_n$ and $\bar{\mu}a'_n = 3^n\gamma$ for each n (using 374Ib). For each $n \ge 1$, set $a''_n = a'_n \setminus \sup_{i<n} a'_i$; then $\bar{\mu}a''_n \ge \frac{1}{2}\cdot 3^{-n}\gamma$, so we can choose an $a_n \subseteq a''_n$ such that $\bar{\mu}a_n = 3^{n-1}\gamma$.

Also, of course, $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is non-increasing. We now see that

$$\langle a_n\rangle_{n\ge 1} \text{ is disjoint}, \quad u \ge \frac{1}{2}\alpha_n\chi a_n \text{ for every } n \ge 1,$$

$$u^* \le \|u\|_\infty \chi\, [0,\gamma[^\bullet \vee \sup_{n\in\mathbb{N}} \alpha_n\chi\, [3^n\gamma, 3^{n+1}\gamma[^\bullet.$$

Similarly, there are a non-increasing sequence $\langle\beta_n\rangle_{n\in\mathbb{N}}$ in $[0,\infty[$ and a disjoint sequence $\langle b_n\rangle_{n\ge 1}$ in $\mathfrak{A}$ such that

$$\bar{\mu}b_n = 3^{n-1}\gamma, \quad v \ge \frac{1}{2}\beta_n\chi b_n \text{ for every } n \ge 1,$$

$$v^* \le \|v\|_\infty \chi\, [0,\gamma[^\bullet \vee \sup_{n\in\mathbb{N}} \beta_n\chi\, [3^n\gamma, 3^{n+1}\gamma[^\bullet.$$

We are supposing that

$$\begin{aligned}\infty = \int u^* \times v^* &= \gamma\|u\|_\infty\|v\|_\infty + \sum_{n=0}^{\infty} 2\cdot 3^n\gamma\alpha_n\beta_n \\ &= \gamma\|u\|_\infty\|v\|_\infty + 2\gamma\alpha_0\beta_0 + 2\gamma\sum_{n=0}^{\infty} 3^{2n+1}(\alpha_{2n+1}\beta_{2n+1} + 3\alpha_{2n+2}\beta_{2n+2}) \\ &\le \gamma\|u\|_\infty\|v\|_\infty + 2\gamma\alpha_0\beta_0 + 24\sum_{n=0}^{\infty} 3^{2n}\gamma\alpha_{2n+1}\beta_{2n+1},\end{aligned}$$

so $\sum_{n=0}^\infty 3^{2n}\alpha_{2n+1}\beta_{2n+1} = \infty$.

At this point, recall that we are dealing with a purely atomic algebra in which every atom has measure γ. Let A_n, B_n be the sets of atoms included in a_n, b_n for each $n \ge 1$, and $A = \bigcup_{n\ge 1} A_n \cup B_n$. Then $\#(A_n) = \#(B_n) = 3^{n-1}$ for each $n \ge 1$. We therefore have a permutation $\phi : A \to A$ such that $\phi[B_{2n+1}] = A_{2n+1}$ for every n. (The point is that $A \setminus \bigcup_{n\in\mathbb{N}} A_{2n+1}$ and $A \setminus \bigcup_{n\in\mathbb{N}} B_{2n+1}$ are both countably infinite.) Define $\pi : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\pi c = (c \setminus \sup A) \cup \sup_{a\in A, a\subseteq c} \phi a$$

for $c \in \mathfrak{A}$. Then π is well-defined (because A is countable), and it is easy to check that it is a measure-preserving Boolean automorphism (because it is just a permutation of the atoms); and $\pi b_{2n+1} = a_{2n+1}$ for every n. Consequently

$$\int u \times T_\pi v \ge \sum_{n=0}^\infty \frac{1}{4}\alpha_{2n+1}\beta_{2n+1}\bar{\mu}a_{2n+1} = \frac{1}{4}\gamma\sum_{n=0}^\infty 3^{2n}\alpha_{2n+1}\beta_{2n+1} = \infty.$$

So we have found a suitable automorphism.

(b) Next, consider the case in which $(\mathfrak{A},\bar{\mu})$ is atomless and of finite magnitude γ. Of course $\gamma > 0$. For each $n \in \mathbb{N}$ set

$$\alpha_n = \inf\{\alpha : \alpha \ge 0,\ \bar{\mu}[\![u > \alpha]\!] \le 3^{-n}\gamma\}, \quad \tilde{a}_n = [\![u > \frac{1}{2}\alpha_n]\!].$$

Then $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and

$$u^* \le \sup_{n\in\mathbb{N}} \alpha_{n+1}\chi\, [3^{-n-1}\gamma, 3^{-n}\gamma[^\bullet.$$

This time, $\bar{\mu}\tilde{a}_n \ge 3^{-n}\gamma$, and we are in an atomless measure algebra, so we can choose $a'_n \subseteq \tilde{a}_n$ such that $\bar{\mu}a'_n = 3^{-n}\gamma$; taking $a''_n = a'_n \setminus \sup_{i>n} a'_i$, $\bar{\mu}a''_n \ge \frac{1}{2}\cdot 3^{-n}\gamma$, and we can choose $a_n \subseteq a''_n$ such that $\bar{\mu}a_n = 3^{-n-1}\gamma$ for every n. As before, $u \ge \frac{1}{2}\alpha_n\chi a_n$ for every n, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint.

In the same way, we can find $\langle\beta_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ such that $\langle b_n\rangle_{n\in\mathbb{N}}$ is disjoint,

$$v^* \le \sup_{n\in\mathbb{N}} \beta_{n+1}\chi\,[3^{-n-1}\gamma, 3^{-n}\gamma[^\bullet, \quad v \ge \sup_{n\in\mathbb{N}} \frac{1}{2}\beta_n\chi b_n$$

and $\bar\mu b_n = 3^{-n-1}\gamma$ for each n. In this case, we have

$$\infty = \int u^* \times v^* \le \sum_{n=0}^{\infty} 2\cdot 3^{-n-1}\gamma\alpha_{n+1}\beta_{n+1},$$

and $\sum_{n=0}^{\infty} 3^{-n}\alpha_n\beta_n$ is infinite.

Now all the principal ideals $\mathfrak{A}_{a_n}$, $\mathfrak{A}_{b_n}$ are homogeneous and of the same Maharam type, so there are measure-preserving isomorphisms $\pi_n : \mathfrak{A}_{b_n} \to \mathfrak{A}_{a_n}$; similarly, setting $\tilde a = 1\setminus \sup_{n\in\mathbb{N}} a_n$ and $\tilde b = 1\setminus \sup_{n\in\mathbb{N}} b_n$, there is a measure-preserving isomorphism $\tilde\pi : \mathfrak{A}_{\tilde b} \to \mathfrak{A}_{\tilde a}$. Define $\pi : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\pi c = \tilde\pi(c\cap\tilde b) \cup \sup_{n\in\mathbb{N}} \pi_n(c\cap a_n)$$

for every $c\in\mathfrak{A}$; then π is a measure-preserving automorphism of $\mathfrak{A}$, and $\pi b_n = a_n$ for each n. In this case,

$$\int u\times T_\pi v \ge \frac{1}{4}\sum_{n=0}^{\infty} 3^{-n-1}\gamma\alpha_n\beta_n = \infty,$$

and again we have a suitable automorphism.

(c) Thirdly, consider the case in which $\mathfrak{A}$ is atomless and not totally finite; take κ to be the common Maharam type of all the principal ideals $\mathfrak{A}_a$ where $0<\bar\mu a<\infty$. In this case, set

$$\alpha_n = \inf\{\alpha : \bar\mu[\![u>\alpha]\!] \le 3^n\}, \quad \beta_n = \inf\{\alpha : \bar\mu[\![v>\alpha]\!] \le 3^n\}$$

for each $n\in\mathbb{Z}$. This time

$$u^* \le \sup_{n\in\mathbb{Z}} \alpha_n\chi\,[3^n, 3^{n+1}[^\bullet, \quad v^* \le \sup_{n\in\mathbb{Z}} \beta_n\chi\,[3^n, 3^{n+1}[^\bullet,$$

so

$$\infty = \int u^*\times v^* = 2\sum_{n=-\infty}^{\infty} 3^n\alpha_n\beta_n \le 8\sum_{n=-\infty}^{\infty} 3^{2n}\alpha_{2n}\beta_{2n}.$$

For each $n\in\mathbb{Z}$, $3^n \le \bar\mu[\![u>\frac{1}{2}\alpha_n]\!]$, so there is an a''_n such that

$$a''_n \subseteq [\![u>\tfrac{1}{2}\alpha_n]\!], \quad \bar\mu a''_n = 3^n.$$

Set $a'_n = a''_n \setminus \sup_{-\infty<i<n} a''_i$; then $\bar\mu a'_n \ge \frac{1}{2}\cdot 3^n$ for each n; choose $a_n \subseteq a'_n$ such that $\bar\mu a_n = 3^{n-1}$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is disjoint and $u \ge \frac{1}{2}\alpha_n\chi a_n$ for each n.

Similarly, there is a disjoint sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ such that

$$\bar\mu b_n = 3^{n-1}, \quad v \ge \frac{1}{2}\beta_n\chi b_n$$

for each $n\in\mathbb{N}$.

Set $d^* = \sup_{n\in\mathbb{Z}} a_n \cup \sup_{n\in\mathbb{Z}} b_n$. Then

$$\tilde a = d^*\setminus \sup_{n\in\mathbb{Z}} a_{2n}, \quad \tilde b = d^*\setminus \sup_{n\in\mathbb{Z}} b_{2n}$$

both have magnitude ω and Maharam type κ. So there is a measure-preserving isomorphism $\tilde\pi : \mathfrak{A}_{\tilde b} \to \mathfrak{A}_{\tilde a}$ (332J). At the same time, for each $n\in\mathbb{Z}$ there is a measure-preserving isomorphism $\pi_n : \mathfrak{A}_{b_{2n}} \to \mathfrak{A}_{a_{2n}}$. So once again we can assemble these to form a measure-preserving automorphism $\pi : \mathfrak{A}\to\mathfrak{A}$, defined by the formula

$$\pi c = (c\setminus d^*) \cup \tilde\pi(c\cap\tilde b) \cup \sup_{n\in\mathbb{Z}} \pi_n(c\cap b_{2n}).$$

Just as in (a) and (b) above,

$$\int u\times T_\pi v \ge \sum_{n=-\infty}^{\infty} \frac{1}{4}\cdot 3^{2n-1}\alpha_{2n}\beta_{2n} = \infty.$$

Thus we have a suitable π in any of the cases allowed by 374H.

374M Proposition Let $(\mathfrak{A},\bar\mu)$ be a quasi-homogeneous localizable measure algebra, and $U\subseteq L^0 = L^0(\mathfrak{A})$ a solid linear subspace which, regarded as a Riesz space, is perfect. If U is rearrangement-invariant and $M^{\infty,1}_{\bar\mu} \subseteq U \subseteq M^{1,\infty}_{\bar\mu}$, then U is $\mathcal{T}$-invariant.

proof Set $V = \{v : u \times v \in L^1$ for every $u \in U\}$, so that V is a solid linear subspace of L^0 which can be identified with $U^\times$ (369C), and U becomes $\{u : u \times v \in L^1$ for every $v \in V\}$; note that $M^{\infty,1}_{\bar\mu} \subseteq V \subseteq M^{1,\infty}_{\bar\mu}$ (using 369Q).

If $u \in U^+$, $v \in V^+$ and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving automorphism, then $T_\pi u \in U$, so $\int v \times T_\pi u < \infty$; by 374L, $\int u^* \times v^*$ is finite. But this means that if $u \in U$, $v \in V$ and $T \in \mathcal{T}_{\bar\mu,\bar\mu}$,

$$\int |Tu \times v| \le \int u^* \times v^* < \infty.$$

As v is arbitrary, $Tu \in U$; as T and u are arbitrary, U is $\mathcal{T}$-invariant.

374X Basic exercises >(a) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $A \subseteq M^{1,\infty}_{\bar\mu}$ a $\mathcal{T}$-invariant set. (i) Show that A is solid. (ii) Show that if A is a linear subspace and not $\{0\}$, then it includes $M^{\infty,1}_{\bar\mu}$. (iii) Show that if $u \in A$, $v \in M^{0,\infty}_{\bar\mu}$ and $\int_0^t v^* \le \int_0^t u^*$ for every $t > 0$, then $v \in A$. (iv) Show that if $(\mathfrak{B}, \bar\nu)$ is any other measure algebra, then $B = \{Tu : u \in A, T \in \mathcal{T}_{\bar\mu,\bar\nu}\}$ and $C = \{v : v \in M^{1,\infty}_{\bar\nu}, Tv \in A$ for every $T \in \mathcal{T}_{\bar\nu,\bar\mu}\}$ are $\mathcal{T}$-invariant subsets of $M^{1,\infty}_{\bar\nu}$, and that $B \subseteq C$. Give two examples in which $B \subset C$. Show that if $(\mathfrak{A}, \bar\mu) = (\mathfrak{A}_L, \bar\mu_L)$ then $B = C$.

>(b) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that the extended Fatou norm $\| \|_p$ on $L^0(\mathfrak{A})$ is $\mathcal{T}$-invariant for every $p \in [1, \infty]$. (*Hint*: 371Gd.)

(c) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be semi-finite measure algebras, and ϕ a Young's function (369Xc). Let τ_ϕ, $\tilde\tau_\phi$ be the corresponding Orlicz norms on $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$. Show that $\tilde\tau_\phi(Tu) \le \tau_\phi(u)$ for every $u \in L^0(\mathfrak{A})$, $T \in \mathcal{T}_{\bar\mu,\bar\nu}$. (*Hint*: 369Xn, 373Xm.) In particular, τ_ϕ is $\mathcal{T}$-invariant.

(d) Show that if $(\mathfrak{A}, \bar\mu)$ is a semi-finite measure algebra and τ is a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$, then the Banach lattice L^τ defined from τ is $\mathcal{T}$-invariant.

(e) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$ which is an order-continuous norm on L^τ. Show that $L^\tau \subseteq M^{1,0}_{\bar\mu}$.

(f) Let θ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$ and $(\mathfrak{A}, \bar\mu)$, $(\mathfrak{B}, \bar\nu)$ two semi-finite measure algebras. Let τ_1, τ_2 be the extended Fatou norms on $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ defined from θ by the method of 374C. Show that $\tau_2(Tu) \le \tau_1(u)$ whenever $u \in M^{1,\infty}_{\bar\mu}$ and $T \in \mathcal{T}_{\bar\mu,\bar\nu}$.

>(g) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, not $\{0\}$, and set $\tau(u) = \sup_{0<\bar\mu a<\infty} \frac{1}{\sqrt{\bar\mu a}} \int_a |u|$ for $u \in L^0(\mathfrak{A})$. Show that τ is a $\mathcal{T}$-invariant extended Fatou norm. Find examples of $(\mathfrak{A}, \bar\mu)$ for which τ is, and is not, order-continuous on L^τ.

(h) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be semi-finite measure algebras and τ a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A})$. (i) Show that there is a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{B})$ defined by setting $\theta(v) = \sup\{\tau(Tv) : T \in \mathcal{T}_{\bar\nu,\bar\mu}\}$ for $v \in M^{1,\infty}_{\bar\nu}$. (ii) Show that when $(\mathfrak{A}, \bar\mu) = (\mathfrak{A}_L, \bar\mu_L)$ then $\theta(v) = \tau(v^*)$ for every $v \in M^{0,\infty}_{\bar\nu}$. (iii) Show that when $(\mathfrak{B}, \bar\nu) = (\mathfrak{A}_L, \bar\mu_L)$ then $\tau(u) = \theta(u^*)$ for every $u \in M^{0,\infty}_{\bar\mu}$.

(i) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Suppose that L^τ is a $\mathcal{T}$-invariant subset of L^0. Show that there is a $\mathcal{T}$-invariant extended Fatou norm $\tilde\tau$ which is equivalent to τ in the sense that, for some $M > 0$, $\tilde\tau(u) \le M\tau(u) \le M^2\tilde\tau(u)$ for every $u \in L^0$. (*Hint*: show first that $\int u^* \times v^* < \infty$ for every $u \in L^\tau$ and $v \in L^{\tau'}$, then that $\sup_{\tau(u)\le 1, \tau'(v)\le 1} \int u^* \times v^* < \infty$.)

(j) Suppose that τ is a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mathfrak{A}_L)$, and that $0 < w = w^* \in M^{1,\infty}_{\bar\mu_L}$. Let $(\mathfrak{A}, \bar\mu)$ be any semi-finite measure algebra. Show that the function $u \mapsto \tau(w \times u^*)$ extends to a $\mathcal{T}$-invariant extended Fatou norm θ on $L^0(\mathfrak{A})$. (*Hint*: $\tau(w \times u^*) = \sup\{\tau(w \times Tu) : T \in \mathcal{T}_{\bar\mu,\bar\mu_L}\}$ for $u \in M^{1,\infty}_{\bar\mu_L}$.) (When $\tau = \| \|_p$ these norms are called **Lorentz norms**; see Lindenstrauss & Tzafriri 79, p. 121.)

(k) Let $(\mathfrak{A}, \bar\mu)$ be $\mathcal{P}\mathbb{N}$ with counting measure. Identify $L^0(\mathfrak{A})$ with $\mathbb{R}^{\mathbb{N}}$. Let U be $\{u : u \in \mathbb{R}^{\mathbb{N}}$, $\{n : u(n) \ne 0\}$ is finite$\}$. Show that U is a perfect Riesz space, and is rearrangement-invariant but not $\mathcal{T}$-invariant.

(l) Let $(\mathfrak{A},\bar{\mu})$ be an atomless quasi-homogeneous localizable measure algebra, and $U\subseteq L^0(\mathfrak{A})$ a rearrangement-invariant solid linear subspace which is a perfect Riesz space. Show that $U\subseteq M^{1,\infty}_{\bar{\mu}}$ and that U is $\mathcal{T}$-invariant. (*Hint*: assume $U\neq\{0\}$. Show that (i) $\chi a\in U$ whenever $\bar{\mu}a<\infty$ (ii) $V=\{v:v\times u\in L^1\ \forall\ u\in U\}$ is rearrangement-invariant (iii) U, $V\subseteq M^{1,\infty}$.)

374Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra and $U\subseteq M^{1,\infty}_{\bar{\mu}}$ a non-zero $\mathcal{T}$-invariant Riesz subspace which, regarded as a Riesz space, is perfect. (i) Show that U includes $M^{\infty,1}_{\bar{\mu}}$. (ii) Show that its dual $\{v:v\in L^0(\mathfrak{A}),\ v\times u\in L^1_{\bar{\mu}}\ \forall\ u\in U\}$ (which in this exercise I will denote by $U^\times$) is also $\mathcal{T}$-invariant, and is $\{v:v\in M^{0,\infty}_{\bar{\mu}},\ \int u^*\times v^*<\infty\ \forall\ u\in U\}$. (iii) Show that for any localizable measure algebra $(\mathfrak{B},\bar{\nu})$ the set $V=\{v:v\in M^{1,\infty}_{\bar{\nu}},\ Tv\in U\ \forall\ T\in\mathcal{T}_{\bar{\nu},\bar{\mu}}\}$ is a perfect Riesz subspace of $L^0(\mathfrak{B})$, and that $V^\times=\{v:v\in M^{1,\infty}_{\bar{\nu}},\ Tv\in U^\times\ \forall\ T\in\mathcal{T}_{\bar{\nu},\bar{\mu}}\}$. (iv) Show that if, in (i)-(iii), $(\mathfrak{A},\bar{\mu})=(\mathfrak{A}_L,\bar{\mu}_L)$, then $V=\{v:v\in M^{0,\infty},\ v^*\in U\}$. (v) Show that if, in (iii), $(\mathfrak{B},\bar{\nu})=(\mathfrak{A}_L,\bar{\mu}_L)$, then $U=\{u:u\in M^{0,\infty}_{\bar{\mu}},\ u^*\in V\}$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and suppose that $1\le q\le p<\infty$. Let $w_{pq}\in L^0(\mathfrak{A}_L)$ be the equivalence class of the function $t\mapsto t^{(q-p)/p}$. (i) Show that for any $u\in L^0(\mathfrak{A})$,

$$\int w_{pq}\times(u^*)^q=p\int_0^\infty t^{q-1}(\bar{\mu}[\![|u|>t]\!])^{q/p}dt.$$

(ii) Show that we have an extended Fatou norm $\|\ \|_{p,q}$ on $L^0(\mathfrak{A})$ defined by setting

$$\|u\|_{p,q}=\big(p\int_0^\infty t^{q-1}(\bar{\mu}[\![|u|>t]\!])^{q/p}dt\big)^{1/q}$$

for every $u\in L^0(\mathfrak{A})$. (*Hint*: use 374Xj with $w=w_{pq}^{1/q}$, $\|\ \|=\|\ \|_q$.) (iii) Show that if $(\mathfrak{B},\bar{\nu})$ is another semi-finite measure algebra and $T\in\mathcal{T}_{\bar{\mu},\bar{\nu}}$, then $\|Tu\|_{p,q}\le\|u\|_{p,q}$ for every $u\in M^{1,\infty}_{\bar{\mu}}$. (iv) Show that $\|\ \|_{p,q}$ is an order-continuous norm on $L^{\|\ \|_{p,q}}$.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a homogeneous measure algebra of uncountable Maharam type, and u, $v\ge 0$ in $M^0_{\bar{\mu}}$ such that $u^*=v^*$. Show that there is a measure-preserving automorphism π of $\mathfrak{A}$ such that $T_\pi u=v$, where $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ is the isomorphism corresponding to π.

(d) In $L^0(\mathfrak{A}_L)$ let u be the equivalence class of the function $f(t)=te^{-t}$. Show that there is no Boolean automorphism π of $\mathfrak{A}_L$ such that $T_\pi u=u^*$. (*Hint*: show that $\mathfrak{A}_L$ is τ-generated by $\{[\![u^*>\alpha]\!]:\alpha>0\}$.)

(e) Let $(\mathfrak{A},\bar{\mu})$ be a quasi-homogeneous semi-finite measure algebra and $C\subseteq L^0(\mathfrak{A})$ a solid convex order-closed rearrangement-invariant set. Show that $C\cap M^{1,\infty}_{\bar{\mu}}$ is $\mathcal{T}$-invariant.

374 Notes and comments I gave this section the title 'rearrangement-invariant spaces' because it looks good on the Contents page, and it follows what has been common practice since Luxemburg 67b; but actually I think that it's $\mathcal{T}$-invariance which matters, and that rearrangement-invariant spaces are significant largely because the important ones are $\mathcal{T}$-invariant. The particular quality of $\mathcal{T}$-invariance which I have tried to bring out here is its transferability from one measure algebra (or measure space, of course) to another. This is what I take at a relatively leisurely pace in 374B-374D and 374Xf, and then encapsulate in 374Xh and 374Ya. The special place of the Lebesgue algebra $(\mathfrak{A}_L,\bar{\mu}_L)$ arises from its being more or less the simplest algebra over which every $\mathcal{T}$-invariant set can be described; see 374Xa.

I don't think this work is particularly easy, and (as in §373) there are rather a lot of unattractive names in it; but once one has achieved a reasonable familiarity with the concepts, the techniques used can be seen to amount to half a dozen ideas – non-trivial ideas, to be sure – from §§369 and 373. From §369 I take concepts of duality: the symmetric relationship between a perfect Riesz space $U\subseteq L^0$ and the representation of its dual (369C-369D), and the notion of associate extended Fatou norms (369H-369K). From §373 I take the idea of 'decreasing rearrangement' and theorems guaranteeing the existence of useful members of $\mathcal{T}_{\bar{\mu},\bar{\nu}}$ (373O-373Q). The results of the present section all depend on repeated use of these facts, assembled in a variety of patterns.

There is one new method here, but an easy one: the construction of measure-preserving automorphisms by joining isomorphisms together, as in the proofs of 374H and 374J. I shall return to this idea, in greater generality and more systematically investigated, in §381. I hope that the special cases here will give no difficulty.

While $\mathcal{T}$-invariance is a similar phenomenon for both extended Fatou norms and perfect Riesz spaces (see 374Xh, 374Ya), the former seem easier to deal with. The essential difference is I think in 374B(i); with a $\mathcal{T}$-invariant extended Fatou norm, we are necessarily confined to $M^{1,\infty}$, the natural domain of the methods used here. For perfect Riesz spaces we have examples like $\mathbb{R}^{\mathbb{N}} \cong L^0(\mathcal{P}\mathbb{N})$ and its dual, the space of eventually-zero sequences (374Xk); these are rearrangement-invariant but not $\mathcal{T}$-invariant, as I have defined it. This problem does not arise over atomless algebras (374Xl).

I think it is obvious that for algebras which are not quasi-homogeneous (374G) rearrangement-invariance is going to be of limited interest; there will be regions between which there is no communication by means of measure-preserving automorphisms, and the best we can hope for is a discussion of quasi-homogeneous components, if they exist, corresponding to the partition of unity used in the proof of 332J. There is a special difficulty concerning rearrangement-invariance in $L^0(\mathfrak{A}_L)$: two elements can have the same decreasing rearrangement without being rearrangements of each other in the strict sense (373Ya, 374Yd). The phenomenon of 373Ya is specific to algebras of countable Maharam type (374Yc). You will see that some of the labour of 374L is because we have to make room for the pieces to move in. 374J is easier just because in that context we can settle for a supremum, rather than an actual infinity, so the rearrangement needed (part (a) of the proof) can be based on a region of finite measure.

Version of 30.1.10

375 Kwapien's theorem

In §368 and the first part of §369 I examined maps from various types of Riesz space into L^0 spaces. There are equally striking results about maps out of L^0 spaces. I start with some relatively elementary facts about positive linear operators from L^0 spaces to Archimedean Riesz spaces in general (375A-375D), and then turn to a remarkable analysis, due essentially to S.Kwapien, of the positive linear operators from a general L^0 space to the L^0 space of a semi-finite measure algebra (375J), with a couple of simple corollaries.

375A Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and W an Archimedean Riesz space. If $T : L^0(\mathfrak{A}) \to W$ is a positive linear operator, it is sequentially order-continuous.

proof (a) The first step is to observe that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is any non-increasing sequence in $L^0 = L^0(\mathfrak{A})$ with infimum 0, and $\epsilon > 0$, then $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ is bounded above in L^0. **P** For $k \in \mathbb{N}$ set $a_k = \sup_{n\in\mathbb{N}} [\![n(u_n - \epsilon u_0) > k]\!]$; set $a = \inf_{k\in\mathbb{N}} a_k$. **?** Suppose, if possible, that $a \neq 0$. Because $u_n \le u_0$, $n(u_n - \epsilon u_0) \le nu_0$ for every n and

$$a \subseteq a_0 \subseteq [\![u_0 > 0]\!] = [\![\epsilon u_0 > 0]\!] = \sup_{n\in\mathbb{N}} [\![\epsilon u_0 - u_n > 0]\!].$$

So there is some $m \in \mathbb{N}$ such that $a' = a \cap [\![\epsilon u_0 - u_m > 0]\!] \neq 0$. Now, for any $n \ge m$, any $k \in \mathbb{N}$,

$$a' \cap [\![n(u_n - \epsilon u_0) > k]\!] \subseteq [\![\epsilon u_0 - u_m > 0]\!] \cap [\![u_m - \epsilon u_0 > 0]\!] = 0.$$

But $a' \subseteq \sup_{n\in\mathbb{N}} [\![n(u_n - \epsilon u_0) > k]\!]$, so in fact

$$a' \subseteq \sup_{n\le m} [\![n(u_n - \epsilon u_0) > k]\!] = [\![v > k]\!],$$

where $v = \sup_{n\le m} n(u_n - \epsilon u_0)$. And this means that $\inf_{k\in\mathbb{N}} [\![v > k]\!] \supseteq a' \neq 0$, which is impossible. **X** Accordingly $a = 0$; by 364L(a-i), $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ is bounded above. **Q**

(b) Now suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in L^0 with infimum 0, and that $w \in W$ is a lower bound for $\{Tu_n : n \in \mathbb{N}\}$. Take any $\epsilon > 0$. By (a), $\{n(u_n - \epsilon u_0) : n \in \mathbb{N}\}$ has an upper bound v in L^0. Because T is positive,

$$w \le Tu_n = T(u_n - \epsilon u_0) + T(\epsilon u_0) \le T(\frac{1}{n}v) + T(\epsilon u_0) = \frac{1}{n}Tv + \epsilon Tu_0$$

for every $n \ge 1$. Because W is Archimedean, $w \le \epsilon Tu_0$. But this is true for every $\epsilon > 0$, so (again because W is Archimedean) $w \le 0$. As w is arbitrary, $\inf_{n\in\mathbb{N}} Tu_n = 0$. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous (351Gb).

375B Proposition Let $\mathfrak{A}$ be an atomless Dedekind σ-complete Boolean algebra. Then $L^0(\mathfrak{A})^\times = \{0\}$.

proof ? Suppose, if possible, that $h : L^0(\mathfrak{A}) \to \mathbb{R}$ is a non-zero order-continuous positive linear functional. Then there is a $u > 0$ in L^0 such that $h(v) > 0$ whenever $0 < v \le u$ (356H). Because $\mathfrak{A}$ is atomless, there is a disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ such that $a_n \subseteq [\![u > 0]\!]$ for each n, so that $u_n = u \times \chi a_n > 0$, while $u_m \wedge u_n = 0$ if $m \ne n$. Now however

$$v = \sup_{n\in\mathbb{N}} \frac{n}{h(u_n)} u_n$$

is defined in L^0, by 368K, and $h(v) \ge n$ for every n, which is impossible. **X**

375C Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, W an Archimedean Riesz space, and $T : L^0(\mathfrak{A}) \to W$ an order-continuous Riesz homomorphism. Then $V = T[L^0(\mathfrak{A})]$ is an order-closed Riesz subspace of W.

proof The kernel U of T is a band in $L^0 = L^0(\mathfrak{A})$ (352Oe), and must be a projection band (353J), because L^0 is Dedekind complete (364M). Since $U + U^\perp = L^0$, $T[U] + T[U^\perp] = V$, that is, $T[U^\perp] = V$; since $U \cap U^\perp = \{0\}$, T is an isomorphism between $U^\perp$ and V. Now suppose that $A \subseteq V$ is upwards-directed and has a least upper bound $w \in W$. Then $B = \{u : u \in U^\perp, Tu \in A\}$ is upwards-directed and $T[B] = A$. The point is that B is bounded above in L^0. **P?** If not, then $\{u^+ : u \in B\}$ cannot be bounded above, so there is a $u_0 > 0$ in L^0 such that $nu_0 = \sup_{u\in B} nu_0 \wedge u^+$ for every $n \in \mathbb{N}$ (368A). Since $B \subseteq U^\perp$, $u_0 \in U^\perp$ and $Tu_0 > 0$. But now, because T is an order-continuous Riesz homomorphism,

$$nTu_0 = \sup_{u\in B} T(nu_0 \wedge u^+) = \sup_{v\in A} nTu_0 \wedge v^+ \le w^+$$

for every $n \in \mathbb{N}$, which is impossible. **XQ**

Set $u^* = \sup B$; then $Tu^* = \sup A = w$ and $w \in V$. As A is arbitrary, V is order-closed.

375D Corollary Let W be a Riesz space and V an order-dense Riesz subspace which is isomorphic to $L^0(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$. Then $V = W$.

proof By 353G, W is Archimedean. So we can apply 375C to an isomorphism $T : L^0(\mathfrak{A}) \to V$ to see that V is order-closed in W.

375E Theorem Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, $(\mathfrak{B}, \bar\nu)$ any measure algebra, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ an order-continuous positive linear operator. Then T is continuous for the topologies of convergence in measure.

proof ? Otherwise, we can find $w \in L^0(\mathfrak{A})$, $b \in \mathfrak{B}^f$ and $\epsilon > 0$ such that whenever $a \in \mathfrak{A}^f$ and $\delta > 0$ there is a $u \in L^0(\mathfrak{A})$ such that $\bar\mu(a \cap [\![|u - w| > \delta]\!]) \le \delta$ and $\bar\nu(b \cap [\![|Tu - Tw| > \epsilon]\!]) \ge \epsilon$ (367L, 2A3H). Of course it follows that whenever $a \in \mathfrak{A}^f$ and $\delta > 0$ there is a $u \in L^0(\mathfrak{A})$ such that $\bar\mu(a \cap [\![|u| > \delta]\!]) \le \delta$ and $\bar\nu(b \cap [\![|Tu| > \epsilon]\!]) \ge \epsilon$. Choose $\langle a_n \rangle_{n\in\mathbb{N}}$ and $\langle u_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $a_0 = 0$. Given that $a_n \in \mathfrak{A}^f$, let $u_n \in L^0(\mathfrak{A})$ be such that $\bar\mu(a_n \cap [\![|u_n| > 2^{-n}]\!]) \le 2^{-n}$ and $\bar\nu(b \cap [\![|Tu_n| > \epsilon]\!]) \ge \epsilon$. Of course it follows that $\bar\nu(b \cap [\![T|u_n| > \epsilon]\!]) \ge \epsilon$. Because $(\mathfrak{A}, \bar\mu)$ is semi-finite, $|u_n| = \sup_{a\in\mathfrak{A}^f} |u_n| \times \chi a)$; because T is order-continuous, $T|u_n| = \sup_{a\in\mathfrak{A}^f} T(|u_n| \times \chi a)$, and we can find $a_{n+1} \in \mathfrak{A}^f$ such that $\bar\nu(b \cap [\![T(|u_n| \times \chi a_{n+1}) > \epsilon]\!]) \ge \frac{1}{2}\epsilon$. Enlarging a_{n+1} if necessary, arrange that $a_{n+1} \supseteq a_n$. Continue.

At the end of the induction, set $v_n = 2^n |u_n| \times \chi a_{n+1}$; then $\bar\mu(a_n \cap [\![v_n > 1]\!]) \le 2^{-n}$, for each $n \in \mathbb{N}$. It follows that $\{v_n : n \in \mathbb{N}\}$ is bounded above. **P** For $k \in \mathbb{N}$, set $c_k = \sup_{n\in\mathbb{N}} [\![v_n > k]\!]$. Then $c_k \subseteq \sup_{n\in\mathbb{N}} a_n$. If $n \in \mathbb{N}$ and $\delta > 0$, let $m \ge n$ be such that $2^{-m+1} \le \delta$, and $k \ge 1$ such that $\bar\mu(a_n \cap [\![\sup_{m<n} v_m > k]\!]) \le \delta$. Then

$$\bar\mu(a_n \cap c_k) \le \bar\mu(a_n \cap [\![\sup_{m<n} v_m > k]\!]) + \sum_{i=m}^\infty \bar\mu(a_i \cap [\![v_i > 1]\!]) \le 2\delta.$$

As δ is arbitrary, $a_n \cap \inf_{k\in\mathbb{N}} c_k = 0$; as n is arbitrary, $\inf_{k\in\mathbb{N}} c_k = 0$; by 364L(a-i) again, $\{v_n : n \in \mathbb{N}\}$ is bounded above. **Q**

Set $v = \sup_{n\in\mathbb{N}} v_n$. Then $2^{-n}v \ge |u_n| \times \chi a_{n+1}$, so $2^{-n}Tv \ge T(|u_n| \times \chi a_{n+1})$ and $\bar\nu(b \cap [\![2^{-n}Tv > \epsilon]\!]) \ge \frac{1}{2}\epsilon$, for each $n \in \mathbb{N}$. But $\inf_{n\in\mathbb{N}} 2^{-n}Tv = 0$, so $\inf_{n\in\mathbb{N}} [\![2^{-n}Tv > \epsilon]\!] = 0$ (364L(b-ii)) and $\inf_{n\in\mathbb{N}} \bar\nu(b \cap [\![2^{-n}Tv > \epsilon]\!]) = 0$. **X**

So we have the result.

375F I come now to the deepest result of this section, concerning positive linear operators from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$ where $\mathfrak{B}$ is a measure algebra. I approach through a couple of lemmas which are striking enough in their own right.

The following temporary definition will be useful.

Definition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. I will say that a function $\phi:\mathfrak{A}\to\mathfrak{B}$ is a **σ-subhomomorphism** if

$\phi(a\cup a')=\phi(a)\cup\phi(a')$ for all a, $a'\in\mathfrak{A}$,

$\inf_{n\in\mathbb{N}}\phi(a_n)=0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

Now we have the following easy facts.

375G Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\phi:\mathfrak{A}\to\mathfrak{B}$ a σ-subhomomorphism.

(a) $\phi(0)=0$, $\phi(a)\subseteq\phi(a')$ whenever $a\subseteq a'$, and $\phi(a)\setminus\phi(a')\subseteq\phi(a\setminus a')$ for every a, $a'\in\mathfrak{A}$.

(b) If $\bar\mu$, $\bar\nu$ are measures such that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are totally finite measure algebras, then for every $\epsilon>0$ there is a $\delta>0$ such that $\bar\nu\phi(a)\le\epsilon$ whenever $\bar\mu a\le\delta$.

proof (a) This is elementary. Set every $a_n=0$ in the second clause of the definition 375F to see that $\phi(0)=0$. The other two parts are immediate consequences of the first clause.

(b) (Compare 232Ba, 327Bb.) **?** Suppose, if possible, otherwise. Then for every $n\in\mathbb{N}$ there is an $a_n\in\mathfrak{A}$ such that $\bar\mu a_n\le 2^{-n}$ and $\bar\nu\phi(a_n)\ge\epsilon$. Set $c_n=\sup_{i\ge n}a_i$ for each n; then $\langle c_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0 (since $\bar\mu c_n\le 2^{-n+1}$ for each n), but $\bar\nu\phi(c_n)\ge\epsilon$ for every n, so $\inf_{n\in\mathbb{N}}\phi c_n$ cannot be 0. **X**

375H Lemma Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be totally finite measure algebras and $\phi:\mathfrak{A}\to\mathfrak{B}$ a σ-subhomomorphism. Then for every non-zero $b_0\in\mathfrak{B}$ there are a non-zero $b\subseteq b_0$ and an $m\in\mathbb{N}$ such that $b\cap\inf_{j\le m}\phi(a_j)=0$ whenever $a_0,\dots,a_m\in\mathfrak{A}$ are disjoint.

proof (a) Suppose first that $\mathfrak{A}$ is atomless and that $\bar\mu 1=1$.

Set $\epsilon=\frac{1}{5}\bar\nu b_0$ and let $m\ge 1$ be such that $\bar\nu\phi(a)\le\epsilon$ whenever $\bar\mu a\le\frac{1}{m}$ (375Gb). We need to know that $(1-\frac{1}{m})^m\le\frac{1}{2}$; this is because (if $m\ge 2$) $\ln m-\ln(m-1)\ge\frac{1}{m}$, so $m\ln(1-\frac{1}{m})\le -1\le -\ln 2$.

Set

$$C=\{\inf_{j\le m}\phi(a_j):a_0,\dots,a_m\in\mathfrak{A}\text{ are disjoint}\}.$$

? Suppose, if possible, that $b_0\subseteq\sup C$. Then there are $c_0,\dots,c_k\in C$ such that $\bar\nu(b_0\cap\sup_{i\le k}c_i)\ge 4\epsilon$. For each $i\le k$ choose disjoint $a_{i0},\dots,a_{im}\in\mathfrak{A}$ such that $c_i=\inf_{j\le m}\phi(a_{ij})$. Let D be the set of atoms of the finite subalgebra of $\mathfrak{A}$ generated by $\{a_{ij}:i\le k,\,j\le m\}$, so that D is a finite partition of unity in $\mathfrak{A}$, and every a_{ij} is the join of the members of D it includes. Set $p=\#(D)$, and for each $d\in D$ take a maximal disjoint set $E_d\subseteq\{e:e\subseteq d,\,\bar\mu e=\frac{1}{pm}\}$, so that $\bar\mu(d\setminus\sup E_d)<\frac{1}{pm}$; set

$$d^*=1\setminus\sup(\bigcup_{d\in D}E_d)=\sup_{d\in D}(d\setminus\sup E_d),$$

so that $\bar\mu d^*$ is a multiple of $\frac{1}{pm}$ and is less than $\frac{1}{m}$. Let E^* be a disjoint set of elements of measure $\frac{1}{pm}$ with union d^*, and take $E=E^*\cup\bigcup_{d\in D}E_d$, so that E is a partition of unity in $\mathfrak{A}$, $\bar\mu e=\frac{1}{pm}$ for every $e\in E$, and $a_{ij}\setminus d^*$ is the join of the members of E it includes for every $i\le k$ and $j\le m$.

Set

$$\mathcal{K}=\{K:K\subseteq E,\,\#(K)=p\},\qquad M=\#(\mathcal{K})=\frac{(mp)!}{p!(mp-p)!}.$$

For every $K\in\mathcal{K}$, $\bar\mu(\sup K)=\frac{1}{m}$ so $\bar\nu\phi(\sup K)\le\epsilon$. So if we set

$$v=\sum_{K\in\mathcal{K}}\chi\phi(\sup K),$$

$\int v\le\epsilon M$. On the other hand,

$$\bar\nu(b_0\cap\sup_{i\le k}c_i)\ge 4\epsilon,\quad\bar\nu\phi(d^*)\le\epsilon,$$

so $\bar\nu b_1\ge 3\epsilon$, where

$$b_1=b_0\cap\sup_{i\le k}c_i\setminus\phi(d^*).$$

Accordingly $\int v \le \frac{1}{3}M\bar{\nu}b_1$ and

$$b_2 = b_1 \cap [\![v < \tfrac{1}{2}M]\!]$$

is non-zero.

Because $b_2 \subseteq b_1$, there is an $i \le k$ such that $b_2 \cap c_i \ne 0$. Now

$$b_2 \cap c_i \subseteq c_i \setminus \phi(d^*) = \inf_{j\le m} \phi(a_{ij}) \setminus \phi(d^*) \subseteq \inf_{j\le m} \phi(a_{ij} \setminus d^*).$$

But every $a_{ij} \setminus d^*$ is the join of the members of E it includes, so

$$\begin{aligned} b_2 \cap c_i &\subseteq \inf_{j\le m} \phi(a_{ij} \setminus d^*) \subseteq \inf_{j\le m} \phi(\sup\{e : e \in E,\, e \subseteq a_{ij}\}) \\ &= \inf_{j\le m} \sup\{\phi(e) : e \in E,\, e \subseteq a_{ij}\} \\ &= \sup\{\inf_{j\le m} \phi(e_j) : e_0, \dots, e_m \in E \text{ and } e_j \subseteq a_{ij} \text{ for every } j\}. \end{aligned}$$

So there are $e_0, \dots, e_m \in E$ such that $e_j \subseteq a_{ij}$ for each j and $b_3 = b_2 \cap \inf_{j\le m} \phi(e_j) \ne 0$. Because $a_{i0}, \dots, a_{im}$ are disjoint, $e_0, \dots, e_m$ are distinct; set $J = \{e_0, \dots, e_m\}$. Then whenever $K \in \mathcal{K}$ and $K \cap J \ne \emptyset$, $b_3 \subseteq \phi(\sup K)$.

So let us calculate the size of $\mathcal{K}_1 = \{K : K \in \mathcal{K},\, K \cap J \ne \emptyset\}$. This is

$$\begin{aligned} M - \tfrac{(mp-m-1)!}{p!(mp-p-m-1)!} &= M\big(1 - \tfrac{(mp-p)(mp-p-1)\dots(mp-p-m)}{mp(mp-1)\dots(mp-m)}\big) \\ &\ge M\big(1 - (\tfrac{mp-p}{mp})^{m+1}\big) \ge \frac{1}{2}M. \end{aligned}$$

But this means that $b_3 \subseteq [\![v \ge \frac{1}{2}M]\!]$, while also $b_3 \subseteq [\![v < \frac{1}{2}M]\!]$; which is surely impossible. **X**

Accordingly $b_0 \not\subseteq \sup C$, and we can take $b = b_0 \setminus \sup C$.

(b) Now for the general case. Let A be the set of atoms of $\mathfrak{A}$, and set $d = 1 \setminus \sup A$. Then the principal ideal $\mathfrak{A}_d$ is atomless, so there are a non-zero $b_1 \subseteq b_0$ and an $n \in \mathbb{N}$ such that $b_1 \cap \inf_{j\le n} \phi(a_j) = 0$ whenever $a_0, \dots, a_n \in \mathfrak{A}_d$ are disjoint. **P** If $\bar{\mu}d > 0$ this follows from (a), if we apply it to $\phi\restriction\mathfrak{A}_d$ and $(\bar{\mu}d)^{-1}\bar{\mu}\restriction\mathfrak{A}_d$. If $\bar{\mu}d = 0$ then we can just take $b_1 = b_0$ and $n = 0$. **Q**

Let $\delta > 0$ be such that $\bar{\nu}\phi(a) < \bar{\nu}b_1$ whenever $\bar{\mu}a \le \delta$. Let $A_1 \subseteq A$ be a finite set such that $\bar{\mu}(\sup A_1) \ge \bar{\mu}(\sup A) - \delta$, and set $r = \#(A)$, $d^* = \sup(A \setminus A_1)$. Then $\bar{\mu}d^* \le \delta$ so $b = b_1 \setminus \phi(d^*) \ne 0$. Try $m = n + r$. If $a_0, \dots, a_m$ are disjoint, then at most r of them can meet $\sup A_1$, so (re-ordering if necessary) we can suppose that $a_0, \dots, a_n$ are disjoint from $\sup A_1$, in which case $a_j \setminus d^* \subseteq d$ for each $j \le m$. But in this case (because $b \cap \phi(d^*) = 0$)

$$b \cap \inf_{j\le m} \phi(a_j) \subseteq b \cap \inf_{j\le n} \phi(a_j) = b \cap \inf_{j\le n} \phi(a_j \cap d) = 0$$

by the choice of n and b_1.

Thus in the general case also we can find appropriate b and m.

375I Lemma Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism. Then for every non-zero $b_0 \in \mathfrak{B}$ there are a non-zero $b \subseteq b_0$ and a finite partition of unity $C \subseteq \mathfrak{A}$ such that $a \mapsto b \cap \phi(a \cap c)$ is a ring homomorphism for every $c \in C$.

proof By 375H, we can find b_1, m such that $0 \ne b_1 \subseteq b_0$ and $b_1 \cap \inf_{j\le m} \phi(a_j) = 0$ whenever $a_0, \dots, a_m \in \mathfrak{A}$ are disjoint. Do this with the smallest possible m. If $m = 0$ then $b_1 \cap \phi(1) = 0$, so we can take $b = b_1$, $C = \{1\}$. Otherwise, because m is minimal, there must be disjoint $c_1, \dots, c_m \in \mathfrak{A}$ such that $b = b_1 \cap \inf_{1\le j\le m} \phi(c_j) \ne 0$. Set $c_0 = 1 \setminus \sup_{1\le j\le m} c_j$, $C = \{c_0, c_1, \dots, c_m\}$; then C is a partition of unity in $\mathfrak{A}$. Set $\pi_j(a) = b \cap \phi(a \cap c_j)$ for each $a \in \mathfrak{A}$ and $j \le m$. Then we always have $\pi_j(a \cup a') = \pi_j(a) \cup \pi_j(a')$ for all a, $a' \in \mathfrak{A}$, because ϕ is a subhomomorphism.

To see that every π_j is a ring homomorphism, we need only check that $\pi_j(a \cap a') = 0$ whenever $a \cap a' = 0$. (Compare 312H(iv).) In the case $j = 0$, we actually have $\pi_0(a) = 0$ for every a, because $b \cap \phi(c_0) = b_1 \cap \inf_{0\le j\le m} \phi(c_j) = 0$ by the choice of b_1 and m. When $1 \le j \le m$, if $a \cap a' = 0$, then

$$\pi_j(a) \cap \pi_j(a') = b_1 \cap \inf_{1\le i\le m, i\ne j} \phi(c_j) \cap \phi(a) \cap \phi(a')$$

is again 0, because a, a', $c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_m$ are disjoint. So we have a suitable pair b, C.

375J Theorem Let $\mathfrak{A}$ be any Dedekind σ-complete Boolean algebra and $(\mathfrak{B},\bar\nu)$ a semi-finite measure algebra. Let $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ be a positive linear operator. Then we can find B, $\langle A_b\rangle_{b\in B}$ such that B is a partition of unity in $\mathfrak{B}$, each A_b is a finite partition of unity in $\mathfrak{A}$, and $u\mapsto T(u\times\chi a)\times\chi b$ is a Riesz homomorphism whenever $b\in B$ and $a\in A_b$.

proof (a) Write B^* for the set of potential members of B; that is, the set of those $b\in\mathfrak{B}$ such that there is a finite partition of unity $A\subseteq\mathfrak{A}$ such that T_{ab} is a Riesz homomorphism for every $a\in A$, writing $T_{ab}(u)=T(u\times\chi a)\times\chi b$. If I can show that B^* is order-dense in $\mathfrak{B}$, this will suffice, since there will then be a partition of unity $B\subseteq B^*$.

(b) So let b_0 be any non-zero member of $\mathfrak{B}$; I seek a non-zero member of B^* included in b_0. Of course there is a non-zero $b_1\subseteq b_0$ with $\bar\nu b_1<\infty$. Let $\gamma>0$ be such that $b_2=b_1\cap[\![T(\chi1)\le\gamma]\!]$ is non-zero. Define $\mu:\mathfrak{A}\to[0,\infty[$ by setting $\mu a=\int_{b_2}T(\chi a)$ for every $a\in\mathfrak{A}$. Then μ is countably additive, because χ, T and $\int$ are all additive and sequentially order-continuous (using 375A). Set $\mathcal{N}=\{a:\mu a=0\}$; then $\mathcal{N}$ is a σ-ideal of $\mathfrak{A}$, and $(\mathfrak{C},\bar\mu)$ is a totally finite measure algebra, where $\mathfrak{C}=\mathfrak{A}/\mathcal{N}$ and $\bar\mu a^\bullet=\mu a$ for every $a\in\mathfrak{A}$ (just as in 321H).

(c) We have a function ϕ from $\mathfrak{C}$ to the principal ideal $\mathfrak{B}_{b_2}$ defined by saying that $\phi a^\bullet=b_2\cap[\![T(\chi a)>0]\!]$ for every $a\in\mathfrak{A}$. **P** If a_1, $a_2\in\mathfrak{A}$ are such that $a_1^\bullet=a_2^\bullet$ in $\mathfrak{C}$, this means that $a_1\triangle a_2\in\mathcal{N}$; now

$$\begin{aligned}[\![T(\chi a_1)>0]\!]\triangle[\![T(\chi a_2)>0]\!]&\subseteq[\![|T(\chi a_1)-T(\chi a_2)|>0]\!]\\&\subseteq[\![T(|\chi a_1-\chi a_2|)>0]\!]=[\![T\chi(a_1\triangle a_2)>0]\!]\end{aligned}$$

is disjoint from b_2 because $\int_{b_2}T\chi(a_1\triangle a_2)=0$. Accordingly $b_2\cap[\![T(\chi a_1)>0]\!]=b_2\cap[\![T(\chi a_2)>0]\!]$ and we can take this common value for $\phi(a_1^\bullet)=\phi(a_2^\bullet)$. **Q**

(d) Now ϕ is a σ-subhomomorphism. **P** (i) For any a_1, $a_2\in\mathfrak{A}$ we have

$$[\![T\chi(a_1\cup a_2)>0]\!]=[\![T(\chi a_1)>0]\!]\cup[\![T(\chi a_2)>0]\!]$$

because

$$T(\chi a_1)\vee T(\chi a_2)\le T\chi(a_1\cup a_2)\le T(\chi a_1)+T(\chi a_2).$$

So $\phi(c_1\cup c_2)=\phi(c_1)\cup\phi(c_2)$ for all c_1, $c_2\in\mathfrak{C}$. (ii) If $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{C}$ with infimum 0, choose $a_n\in\mathfrak{A}$ such that $a_n^\bullet=c_n$ for each n, and set $\tilde a_n=\inf_{i\le n}a_i\setminus\inf_{i\in\mathbb{N}}a_i$ for each n; then $\tilde a_n^\bullet=c_n$ so $\phi(c_n)=[\![T(\chi\tilde a_n)>0]\!]$ for each n, while $\langle\tilde a_n\rangle_{n\in\mathbb{N}}$ is non-increasing and $\inf_{n\in\mathbb{N}}\tilde a_n=0$. **?** Suppose, if possible, that $b'=\inf_{n\in\mathbb{N}}\phi(c_n)\ne0$; set $\epsilon=\frac12\bar\nu b'$. Then $\bar\nu(b_2\cap[\![T(\chi\tilde a_n)>0]\!])\ge2\epsilon$ for every $n\in\mathbb{N}$. For each n, take $\alpha_n>0$ such that $\bar\nu(b_2\cap[\![T(\chi\tilde a_n)>\alpha_n]\!])\ge\epsilon$. Then $u=\sup_{n\in\mathbb{N}}n\alpha_n^{-1}\chi\tilde a_n$ is defined in $L^0(\mathfrak{A})$ (because $\sup_{n\in\mathbb{N}}[\![n\alpha_n^{-1}\chi\tilde a_n>k]\!]\subseteq\tilde a_m$ if $k\ge\max_{i\le m}i\alpha_i^{-1}$, so $\inf_{k\in\mathbb{N}}\sup_{n\in\mathbb{N}}[\![n\alpha_n^{-1}\chi\tilde a_n>k]\!]=0$). But now

$$\bar\nu(b_2\cap[\![Tu>n]\!])\ge\bar\nu(b_2\cap[\![T(\chi\tilde a_n)>\alpha_n]\!])\ge\epsilon$$

for every n, so $\inf_{n\in\mathbb{N}}[\![Tu>n]\!]\ne0$, which is impossible. **X** Thus $\inf_{n\in\mathbb{N}}\phi(c_n)=0$; as $\langle c_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ϕ is a σ-subhomomorphism. **Q**

(e) By 375I, there are a non-zero $b\in\mathfrak{B}_{b_2}$ and a finite partition of unity $C\subseteq\mathfrak{C}$ such that $d\mapsto b\cap\phi(d\cap c)$ is a ring homomorphism for every $c\in C$. There is a partition of unity $A\subseteq\mathfrak{A}$, of the same size as C, such that $C=\{a^\bullet:a\in A\}$. Now T_{ab} is a Riesz homomorphism for every $a\in A$. **P** It is surely a positive linear operator. If u_1, $u_2\in L^0(\mathfrak{A})$ and $u_1\wedge u_2=0$, set $e_i=[\![u_i>0]\!]$ for each i, so that $e_1\cap e_2=0$. Observe that $u_i=\sup_{n\in\mathbb{N}}u_i\wedge n\chi e_i$, so that

$$[\![T_{ab}u_i>0]\!]=\sup_{n\in\mathbb{N}}[\![T_{ab}(u_i\wedge n\chi e_i)>0]\!]\subseteq[\![T_{ab}(\chi e_i)>0]\!]=b\cap[\![T\chi(e_i\cap a)>0]\!]$$

for both i (of course T_{ab}, like T, is sequentially order-continuous). But this means that

$$\begin{aligned}[\![T_{ab}u_1>0]\!]\cap[\![T_{ab}u_2>0]\!]&\subseteq b\cap[\![T\chi(e_1\cap a)>0]\!]\cap[\![T\chi(e_2\cap a)>0]\!]\\&=b\cap\phi(e_1^\bullet\cap a^\bullet)\cap\phi(e_2^\bullet\cap a^\bullet)=0\end{aligned}$$

because $a^\bullet\in C$, so $d\mapsto b\cap\phi(d\cap a^\bullet)$ is a ring homomorphism, while $e_1^\bullet\cap e_2^\bullet=0$. So $T_{ab}u_1\wedge T_{ab}u_2=0$. As u_1 and u_2 are arbitrary, T_{ab} is a Riesz homomorphism (352G(iv)). **Q**

(f) Thus $b\in B^*$. As b_0 is arbitrary, B^* is order-dense, and we're home.

375K Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and U a Dedekind complete Riesz space such that $U^\times$ separates the points of U. If $T : L^0(\mathfrak{A}) \to U$ is a positive linear operator, there is a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ of Riesz homomorphisms from $L^0(\mathfrak{A})$ to U such that $T = \sum_{n=0}^\infty T_n$, in the sense that $Tu = \sup_{n\in\mathbb{N}} \sum_{i=0}^n T_i u$ for every $u \ge 0$ in $L^0(\mathfrak{A})$.

proof By 369A, U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{B})$ for some localizable measure algebra $(\mathfrak{B}, \bar\nu)$; being Dedekind complete, it is solid in $L^0(\mathfrak{B})$ (353L). Regard T as an operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$, and take B, $\langle A_b\rangle_{b\in B}$ as in 375J. Note that $L^0(\mathfrak{B})$ can be identified with $\prod_{b\in B} L^0(\mathfrak{B}_b)$ (364R, 322L). For each $b \in B$ let $f_b : A_b \to \mathbb{N}$ be an injection. If $b \in B$ and $n \in f_b[A_b]$, set $T_{nb}(u) = \chi b \times T(u \times \chi f_b^{-1}(n))$; otherwise set $T_{nb} = 0$. Then $T_{nb} : L^0(\mathfrak{A}) \to L^0(\mathfrak{B}_b)$ is a Riesz homomorphism; because A_b is a finite partition of unity, $\sum_{n=0}^\infty T_{nb} u = \chi b \times Tu$ for every $u \in L^0(\mathfrak{A})$. But this means that if we set $T_n u = \langle T_{nb} u\rangle_{b\in B}$,

$$T_n : L^0(\mathfrak{A}) \to \prod_{b\in B} L^0(\mathfrak{B}_b) \cong L^0(\mathfrak{B})$$

is a Riesz homomorphism for each n; and $T = \sum_{n=0}^\infty T_n$. Of course every T_n is an operator from $L^0(\mathfrak{A})$ to U because $|T_n u| \le T|u| \in U$ for every $u \in L^0(\mathfrak{A})$.

375L Corollary (a) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, $(\mathfrak{B}, \bar\nu)$ is a semi-finite measure algebra, and there is any non-zero positive linear operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$, then there is a non-trivial sequentially order-continuous ring homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(b) If $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are homogeneous probability algebras and $\tau(\mathfrak{A}) > \tau(\mathfrak{B})$, then $\mathrm{L}^\sim(L^0(\mathfrak{A}); L^0(\mathfrak{B})) = \{0\}$.

proof (a) It is probably quickest to look at the proof of 375J: starting from a non-zero positive linear operator $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$, we move to a non-zero σ-subhomomorphism $\phi : \mathfrak{A}/\mathcal{N} \to \mathfrak{B}$ and thence to a non-zero ring homomorphism from $\mathfrak{A}/\mathcal{N}$ to $\mathfrak{B}$, corresponding to a non-zero ring homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, which is sequentially order-continuous because it is dominated by ϕ. Alternatively, quoting 375J, we have a non-zero Riesz homomorphism $T_1 : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$, and it is easy to check that $a \mapsto [\![T(\chi a) > 0]\!]$ is a non-zero sequentially order-continuous ring homomorphism.

(b) Use (a) and 331J.

375X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and W an Archimedean Riesz space. Let $T : L^0(\mathfrak{A}) \to W$ be a positive linear operator. Show that T is order-continuous iff $T\chi : \mathfrak{A} \to W$ is order-continuous.

(b) Let $\mathfrak{A}$ be an atomless Dedekind σ-complete Boolean algebra and W a Banach lattice. Show that the only order-continuous positive linear operator from $L^0(\mathfrak{A})$ to W is the zero operator.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and W a Riesz space. Let $T : L^0(\mathfrak{A}) \to W$ be an order-continuous Riesz homomorphism such that $T[L^0(\mathfrak{A})]$ is order-dense in W. Show that T is surjective.

>(d) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism as defined in 375F. Show that ϕ is sequentially order-continuous.

>(e) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $[0,1]$ and $\mathfrak{G}$ the regular open algebra of $\mathbb{R}$. (i) Show that there is no non-zero positive linear operator from $L^0(\mathfrak{G})$ to $L^0(\mathfrak{A})$. (*Hint*: suppose $T : L^0(\mathfrak{G}) \to L^0(\mathfrak{A})$ were such an operator. Reduce to the case $T(\chi 1) \le \chi 1$. Let $\langle b_n\rangle_{n\in\mathbb{N}}$ enumerate an order-dense subset of $\mathfrak{G}$ (316Yo). For each $n \in \mathbb{N}$ take non-zero $b'_n \subseteq b_n$ such that $\int T(\chi b'_n) \le 2^{-n-2} \int T(\chi 1)$ and consider $T\chi(\sup_{n\in\mathbb{N}} b'_n)$. See also 375Yf-375Ye.) (ii) Show that there is no non-zero positive linear operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{G})$. (*Hint*: suppose $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{G})$ were such an operator. For each $n \in \mathbb{N}$ choose $a_n \in \mathfrak{A}$, $\alpha_n > 0$ such that $\bar\mu a_n \le 2^{-n}$ and if $b_n \subseteq [\![T(\chi 1) > 0]\!]$ then $b_n \cap [\![T(\chi a_n) > \alpha_n]\!] \ne 0$. Consider Tu where $u = \sum_{n=0}^\infty n\alpha_n^{-1} \chi a_n$.)

(f) In 375K, show that for any $u \in L^0(\mathfrak{A})$

$$\inf_{n\in\mathbb{N}} \sup_{m\ge n} [\![|Tu - \sum_{i=0}^m T_i u| > 0]\!] = 0.$$

>(g) Prove directly, without quoting 375F-375L, that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra then every positive linear functional from $L^0(\mathfrak{A})$ to $\mathbb{R}$ is a finite sum of Riesz homomorphisms.

(h) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ a Riesz homomorphism. Show that there are a sequentially order-continuous ring homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ and a $w \in L^0(\mathfrak{B})^+$ such that $Tu = w \times T_\pi u$ for every $u \in L^0(\mathfrak{A})$, where $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ is defined as in 364Yg.

375Y Further exercises (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ a linear operator. (i) Show that if T is order-bounded, then $\langle Tu_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{B})$ (definition: 367A) whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{A})$. (ii) Show that if $\mathfrak{B}$ is ccc and weakly (σ, ∞)-distributive and $\langle Tu_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{B})$ whenever $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $L^0(\mathfrak{A})$, then T is order-bounded.

(b) Show that the following are equiveridical: (i) there is a purely atomic probability space (X, Σ, μ) such that $\Sigma = \mathcal{P}X$ and $\mu\{x\} = 0$ for every $x \in X$; (ii) there are a set X and a Riesz homomorphism $f : \mathbb{R}^X \to \mathbb{R}$ which is not order-continuous; (iii) there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a positive linear operator $f : L^0(\mathfrak{A}) \to \mathbb{R}$ which is not order-continuous; (iv) there are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \{0, 1\}$ which is not order-continuous; (v) there are a Dedekind complete Riesz space U and a sequentially order-continuous Riesz homomorphism $f : U \to \mathbb{R}$ which is not order-continuous; *(vi) there are an atomless Dedekind complete Boolean algebra $\mathfrak{A}$ and a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \{0, 1\}$ which is not order-continuous. (Compare 363S.)

(c) Give an example of an atomless Dedekind σ-complete Boolean algebra $\mathfrak{A}$ such that $L^0(\mathfrak{A})^\sim \neq \{0\}$.

(d) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $[0, 1]$, and set $L^0 = L^0(\mathfrak{A})$. Show that there is a positive linear operator $T : L^0 \to L^0$ such that $T[L^0]$ is not order-closed in L^0.

(e) Show that the following are equiveridical: (i) there is a probability space (X, Σ, μ) such that $\Sigma = \mathcal{P}X$ and $\mu\{x\} = 0$ for every $x \in X$; (ii) there are localizable measure algebras $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ and a positive linear operator $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ which is not order-continuous.

(f) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras of which $\mathfrak{B}$ is weakly σ-distributive. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be a positive linear operator. Show that $a \mapsto [\![T(\chi a) > 0]\!] : \mathfrak{A} \to \mathfrak{B}$ is a σ-subhomomorphism.

(g) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras of which $\mathfrak{B}$ is weakly σ-distributive. Let $\phi : \mathfrak{A} \to \mathfrak{B}$ be a σ-subhomomorphism such that $\pi a \neq 0$ whenever $a \in \mathfrak{A} \setminus \{0\}$. Show that $\mathfrak{A}$ is weakly σ-distributive.

(h) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras, and $\phi : \mathfrak{A} \to \mathfrak{B}$ a σ-subhomomorphism such that $\phi 1_{\mathfrak{A}} = 1_{\mathfrak{B}}$. Show that there is a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\pi a \subseteq \phi a$ for every $a \in \mathfrak{A}$.

(i) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, and $L^0 = L^0(\mathfrak{G})$. Give an example of a non-zero positive linear operator $T : L^0 \to L^0$ such that there is no non-zero Riesz homomorphism $S : L^0 \to L^0$ with $S \leq T$.

375Z Problem Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$, and $L^0 = L^0(\mathfrak{G})$. If $T : L^0 \to L^0$ is a positive linear operator, must $T[L^0]$ be order-closed?

375 Notes and comments Both this section, and the earlier work on linear operators into L^0 spaces, can be regarded as describing different aspects of a single fact: L^0 spaces are very large. The most explicit statements of this principle are 368E and 375D: every Archimedean Riesz space can be embedded into a Dedekind complete L^0 space, but no such L^0 space can be properly embedded as an order-dense Riesz subspace of any other Archimedean Riesz space. Consequently there are many maps into L^0 spaces (368B).

But by the same token there are few maps out of them (375B, 375Lb), and those which do exist have a variety of special properties (375A, 375J).

The original version of Kwapien's theorem (Kwapien 73) was the special case of 375J in which $\mathfrak{A}$ is the Lebesgue measure algebra. The ideas of the proof here are mostly taken from Kalton Peck & Roberts 84. I have based my account on the concept of 'subhomomorphism' (375F); this seems to be an effective tool when $\mathfrak{B}$ is weakly (σ,∞)-distributive (375Yf), but less useful in other cases. The case $\mathfrak{B}=\{0,1\}$, $L^0(\mathfrak{B})\cong\mathbb{R}$ is not entirely trivial and is worth working through on its own (375Xg).

Version of 8.4.10

376 Kernel operators

The theory of linear integral equations is in large part the theory of operators T defined from formulae of the type

$$(Tf)(y)=\int k(x,y)f(x)dx$$

for some function k of two variables. I make no attempt to study the general theory here. However, the concepts developed in this book make it easy to discuss certain aspects of such operators defined between the 'function spaces' of measure theory, meaning spaces of equivalence classes of functions, and indeed allow us to do some of the work in the abstract theory of Riesz spaces, omitting all formal mention of measures (376D, 376H, 376P). I give a very brief account of two theorems characterizing kernel operators in the abstract (376E, 376H), with corollaries to show the form these theorems can take in the ordinary language of integral kernels (376J, 376N). To give an idea of the kind of results we can hope for in this area, I go a bit farther with operators with domain L^1 (376Mb, 376P, 376S).

I take the opportunity to spell out versions of results from §253 in the language of this volume (376B-376C).

376A Kernel operators To give an idea of where this section is going, I will try to describe the central idea in a relatively concrete special case. Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces; you can take them both to be $[0,1]$ with Lebesgue measure if you like. Let λ be the product measure on $X\times Y$. If $k\in\mathcal{L}^1(\lambda)$, then $\int k(x,y)dx$ is defined for almost every y, by Fubini's theorem; so if $f\in\mathcal{L}^\infty(\mu)$ then $g(y)=\int k(x,y)f(x)dx$ is defined for almost every y. Also

$$\int g(y)dy=\int k(x,y)f(x)dxdy$$

is defined, because $(x,y)\mapsto k(x,y)f(x)$ is λ-virtually measurable, defined λ-a.e. and is dominated by a multiple of the integrable function k. Thus k defines a function from $\mathcal{L}^\infty(\mu)$ to $\mathcal{L}^1(\nu)$. Changing f on a set of measure 0 will not change g, so we can think of this as an operator from $L^\infty(\mu)$ to $\mathcal{L}^1(\nu)$; and of course we can move immediately to the equivalence class of g in $L^1(\nu)$, so getting an operator T_k from $L^\infty(\mu)$ to $L^1(\nu)$. This operator is plainly linear; also it is easy to check that $\pm T_k\le T_{|k|}$, so that $T_k\in\mathsf{L}^\sim(L^\infty(\mu);L^1(\nu))$, and that $\|T_k\|\le\int|k|$. Moreover, changing k on a λ-negligible set does not change T_k, so that in fact we can speak of T_w for any $w\in L^1(\lambda)$.

I think it is obvious, even before investigating them, that operators representable in this way will be important. We can immediately ask what their properties will be and whether there is any straightforward way of recognising them. We can look at the properties of the map $w\mapsto T_w:L^1(\lambda)\to\mathsf{L}^\sim(L^\infty(\mu);L^1(\nu))$. And we can ask what happens when $L^\infty(\mu)$ and $L^1(\nu)$ are replaced by other function spaces, defined by extended Fatou norms or otherwise. Theorems 376E and 376H are answers to questions of this kind.

It turns out that the formula $g(y)=\int k(x,y)f(x)dx$ gives rise to a variety of technical problems, and it is much easier to characterize Tu in terms of its action on the dual. In the language of the special case above, if $h\in\mathcal{L}^\infty(\nu)$, then we shall have

$$\int k(x,y)f(x)h(y)d(x,y)=\int g(y)h(y)dy;$$

since $g^\bullet\in L^1(\nu)$ is entirely determined by the integrals $\int g(y)h(y)dy$ as h runs over $\mathcal{L}^\infty(\nu)$, we can define the operator T in terms of the functional $(f,h)\mapsto\int k(x,y)f(x)h(y)d(x,y)$. This enables us to extend the results from the case of σ-finite spaces to general strictly localizable spaces; perhaps more to the point in the present context, it gives them natural expressions in terms of function spaces defined from measure algebras rather than measure spaces, as in 376E.

Before going farther along this road, however, I give a couple of results relating the theorems of §253 to the methods of this volume.

376B The canonical map $L^0 \times L^0 \to L^0$: Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be semi-finite measure algebras, and $(\mathfrak{C}, \bar{\lambda})$ their localizable measure algebra free product (325E). Then we have a bilinear operator $(u, v) \mapsto u \otimes v : L^0(\mathfrak{A}) \times L^0(\mathfrak{B}) \to L^0(\mathfrak{C})$ with the following properties.

(a) For any $u \in L^0(\mathfrak{A})$, $v \in L^0(\mathfrak{B})$ and $\alpha \in \mathbb{R}$,

$$[\![u \otimes \chi 1_{\mathfrak{B}} > \alpha]\!] = [\![u > \alpha]\!] \otimes 1_{\mathfrak{B}}, \quad [\![\chi 1_{\mathfrak{A}} \otimes v > \alpha]\!] = 1_{\mathfrak{A}} \otimes [\![v > \alpha]\!]$$

where for $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ I write $a \otimes b$ for the corresponding member of $\mathfrak{A} \otimes \mathfrak{B}$ (315N), identified with a subalgebra of $\mathfrak{C}$ (325Dc).

(b)(i) For any $u \in L^0(\mathfrak{A})^+$, the map $v \mapsto u \otimes v : L^0(\mathfrak{B}) \to L^0(\mathfrak{C})$ is an order-continuous multiplicative Riesz homomorphism.

(ii) For any $v \in L^0(\mathfrak{B})^+$, the map $u \mapsto u \otimes v : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$ is an order-continuous multiplicative Riesz homomorphism.

(c) In particular, $|u \otimes v| = |u| \otimes |v|$ for all $u \in L^0(\mathfrak{A})$ and $v \in L^0(\mathfrak{B})$.

(d) For any $u \in L^0(\mathfrak{A})^+$ and $v \in L^0(\mathfrak{B})^+$, $[\![u \otimes v > 0]\!] = [\![u > 0]\!] \otimes [\![v > 0]\!]$.

proof The canonical maps $a \mapsto a \otimes 1_{\mathfrak{B}}$, $b \mapsto 1_{\mathfrak{A}} \otimes b$ from $\mathfrak{A}$, $\mathfrak{B}$ to $\mathfrak{C}$ are order-continuous Boolean homomorphisms (325Da), so induce order-continuous multiplicative Riesz homomorphisms from $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ to $L^0(\mathfrak{C})$ (364P); write $\tilde{u}$, $\tilde{v}$ for the images of $u \in L^0(\mathfrak{A})$, $v \in L^0(\mathfrak{B})$. Observe that $|\tilde{u}| = |u|^\sim$, $|\tilde{v}| = |v|^\sim$ and $(\chi 1_{\mathfrak{A}})^\sim = (\chi 1_{\mathfrak{B}})^\sim = \chi 1_{\mathfrak{C}}$. Now set $u \otimes v = \tilde{u} \times \tilde{v}$. The properties listed in (a)-(c) are just a matter of putting the definition in 364Pa together with the fact that $L^0(\mathfrak{C})$ is an f-algebra (364D). As for $[\![u \otimes v > 0]\!] = [\![\tilde{u} \times \tilde{v} > 0]\!]$, this is (for non-negative u, v) just

$$[\![\tilde{u} > 0]\!] \cap [\![\tilde{v} > 0]\!] = ([\![u > 0]\!] \otimes 1_{\mathfrak{B}}) \cap (1_{\mathfrak{A}} \otimes [\![v > 0]\!]) = [\![u > 0]\!] \otimes [\![v > 0]\!].$$

376C For L^1 spaces we have a similar result, with additions corresponding to the Banach lattice structures of the three spaces.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be semi-finite measure algebras with localizable measure algebra free product $(\mathfrak{C}, \bar{\lambda})$.

(a) If $u \in L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$ and $v \in L^1_{\bar{\nu}} = L^1(\mathfrak{B}, \bar{\nu})$ then $u \otimes v \in L^1_{\bar{\lambda}} = L^1(\mathfrak{C}, \bar{\lambda})$ and

$$\int u \otimes v = \int u \int v, \quad \|u \otimes v\|_1 = \|u\|_1 \|v\|_1.$$

(b) Let W be a Banach space and $\phi : L^1_{\bar{\mu}} \times L^1_{\bar{\nu}} \to W$ a bounded bilinear operator. Then there is a unique bounded linear operator $T : L^1_{\bar{\lambda}} \to W$ such that $T(u \otimes v) = \phi(u, v)$ for all $u \in L^1_{\bar{\mu}}$ and $v \in L^1_{\bar{\nu}}$, and $\|T\| = \|\phi\|$.

(c) Suppose, in (b), that W is a Banach lattice. Then

(i) T is positive iff $\phi(u, v) \ge 0$ for all u, $v \ge 0$;

(ii) T is a Riesz homomorphism iff $u \mapsto \phi(u, v_0) : L^1_{\bar{\mu}} \to W$ and $v \mapsto \phi(u_0, v) : L^1_{\bar{\nu}} \to W$ are Riesz homomorphisms for all $v_0 \ge 0$ in $L^1_{\bar{\nu}}$ and $u_0 \ge 0$ in $L^1_{\bar{\mu}}$.

proof (a) I refer to the proof of 325D. Let (X, Σ, μ) and (Y, T, ν) be the Stone spaces of $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ (321K), so that $(\mathfrak{C}, \bar{\lambda})$ can be identified with the measure algebra of the c.l.d. product measure λ on $X \times Y$ (part (a) of the proof of 325D), and $L^1_{\bar{\mu}}$, $L^1_{\bar{\nu}}$, $L^1_{\bar{\lambda}}$ can be identified with $L^1(\mu)$, $L^1(\nu)$ and $L^1(\lambda)$ (365B). Now if $f \in \mathcal{L}^0(\mu)$ and $g \in \mathcal{L}^0(\nu)$ then $f \otimes g \in \mathcal{L}^0(\lambda)$ (253Cb), and it is easy to check that $(f \otimes g)^\bullet \in L^0(\bar{\lambda})$ corresponds to $f^\bullet \otimes g^\bullet$ as defined in 376B. (Look first at the cases in which one of f, g is a constant function with value 1.) By 253E, we have a canonical map $(f^\bullet, g^\bullet) \mapsto (f \otimes g)^\bullet$ from $L^1(\mu) \times L^1(\nu)$ to $L^1(\lambda)$, with $\int f \otimes g = \int f \int g$ (253D); so that if $u \in L^1_{\bar{\mu}}$ and $v \in L^1_{\bar{\nu}}$ we must have $u \otimes v \in L^1_{\bar{\lambda}}$, with $\int u \otimes v = \int u \int v$. As in 253E, it follows that $\|u \otimes v\|_1 = \|u\|_1 \|v\|_1$.

(b) In view of the situation described in (a) above, this is now just a translation of the same result about $L^1(\mu)$, $L^1(\nu)$ and $L^1(\lambda)$, which is Theorem 253F.

(c) Identifying the algebraic free product $\mathfrak{A}\otimes\mathfrak{B}$ with its canonical image in $\mathfrak{C}$ (325Dc), I write $(\mathfrak{A}\otimes\mathfrak{B})^f$ for $\{c : c\in\mathfrak{A}\otimes\mathfrak{B},\ \bar\lambda c<\infty\}$, so that $(\mathfrak{A}\otimes\mathfrak{B})^f$ is a subring of $\mathfrak{C}$. Recall that any member of $\mathfrak{A}\otimes\mathfrak{B}$ is expressible as $\sup_{i\le n} a_i\otimes b_i$ where $a_0,\dots,a_n$ are disjoint (315Oa); evidently this will belong to $(\mathfrak{A}\otimes\mathfrak{B})^f$ iff $\bar\mu a_i\cdot\bar\nu b_i$ is finite for every i.

The next fact to lift from previous theorems is in part (e) of the proof of 253F: the linear span M of $\{\chi(a\otimes b) : a\in\mathfrak{A}^f,\ b\in\mathfrak{B}^f\}$ is norm-dense in $L^1_{\bar\lambda}$. Of course M can also be regarded as the linear span of $\{\chi c : c\in(\mathfrak{A}\otimes\mathfrak{B})^f\}$, or $S(\mathfrak{A}\otimes\mathfrak{B})^f$. (Strictly speaking, this last remark relies on 361J; the identity map from $(\mathfrak{A}\otimes\mathfrak{B})^f$ to $\mathfrak{C}$ induces an injective Riesz homomorphism from $S(\mathfrak{A}\otimes\mathfrak{B})^f$ into $S(\mathfrak{C})\subseteq L^0(\mathfrak{C})$. To see that $\chi c\in M$ for every $c\in(\mathfrak{A}\otimes\mathfrak{B})^f$, we need to know that c can be expressed as a disjoint union of members of $\mathfrak{A}\otimes\mathfrak{B}$, as noted above.)

(i) If T is positive then of course $\phi(u,v)=T(u\otimes v)\ge 0$ whenever $u,\ v\ge 0$, since $u\otimes v\ge 0$. On the other hand, if ϕ is non-negative on $U^+\times V^+$, then, in particular, $T\chi(a\otimes b)=\phi(\chi a,\chi b)\ge 0$ whenever $\bar\mu a\cdot\bar\nu b<\infty$. Consequently $T(\chi c)\ge 0$ for every $c\in(\mathfrak{A}\otimes\mathfrak{B})^f$ and $Tw\ge 0$ whenever $w\ge 0$ in $M\cong S(\mathfrak{A}\otimes\mathfrak{B})^f$, as in 361Ga.

Now this means that $T|w|\ge 0$ whenever $w\in M$. But as M is norm-dense in $L^1_{\bar\lambda}$, $w\mapsto T|w|$ is continuous and W^+ is closed, it follows that $T|w|\ge 0$ for every $w\in L^1_{\bar\lambda}$, that is, that T is positive.

(ii) If T is a Riesz homomorphism then of course $u\mapsto\phi(u,v_0)=T(u\otimes v_0)$ and $v\mapsto\phi(u_0,v)=T(u_0\otimes v)$ are Riesz homomorphisms for $v_0,\ u_0\ge 0$. On the other hand, if all these maps are Riesz homomorphisms, then, in particular,

$$
\begin{aligned}
T\chi(a\otimes b)\wedge T\chi(a'\otimes b') &= \phi(\chi a,\chi b)\wedge\phi(\chi a',\chi b')\\
&\le \phi(\chi a,\chi b+\chi b')\wedge\phi(\chi a',\chi b+\chi b')\\
&= \phi(\chi a\wedge\chi a',\chi b+\chi b')=0
\end{aligned}
$$

whenever $a,\ a'\in\mathfrak{A}^f$, $b,\ b'\in\mathfrak{B}^f$ and $a\cap a'=0$. Similarly, $T\chi(a\otimes b)\wedge T\chi(a'\otimes b')=0$ if $b\cap b'=0$. But this means that $T\chi c\wedge T\chi c'=0$ whenever $c,\ c'\in(\mathfrak{A}\otimes\mathfrak{B})^f$ and $c\cap c'=0$. **P** Express $c,\ c'$ as $\sup_{i\le m} a_i\otimes b_i$, $\sup_{j\le n} a'_j\otimes b'_j$ where $a_i,\ a'_j,\ b_i,\ b'_j$ all have finite measure. Now if $i\le m,\ j\le n$, $(a_i\cap a'_j)\otimes(b_i\cap b'_j)=(a_i\otimes b_i)\cap(a'_j\otimes b'_j)=0$, so one of $a_i\cap a'_j$, $b_i\cap b'_j$ must be zero, and in either case $T\chi(a_i\otimes b_i)\wedge T\chi(a'_j\otimes b'_j)=0$. Accordingly

$$
\begin{aligned}
T\chi c\wedge T\chi c' &\le \Bigl(\sum_{i=0}^m T\chi(a_i\otimes b_i)\Bigr)\wedge\Bigl(\sum_{j=0}^n T\chi(a'_j\times b'_j)\Bigr)\\
&\le \sum_{i=0}^m\sum_{j=0}^n T\chi(a_i\otimes b_i)\wedge T\chi(a'_j\otimes b'_j)=0,
\end{aligned}
$$

using 352F(a-ii) for the second inequality. **Q**

This implies that $T\restriction M$ must be a Riesz homomorphism (361Gc), that is, $T|w|=|Tw|$ for all $w\in M$. Again because M is dense in $L^1_{\bar\lambda}$, $T|w|=|Tw|$ for every $w\in L^1_{\bar\lambda}$, and T is a Riesz homomorphism.

376D Abstract integral operators: Definition The following concept will be used repeatedly in the theorems below; it is perhaps worth giving it a name. Let U be a Riesz space and V a Dedekind complete Riesz space, so that $\mathsf{L}^\times(U;V)$ is a Dedekind complete Riesz space (355H). If $f\in U^\times$ and $v\in V$ write $P_{fv}u=f(u)v$ for each $u\in U$; then $P_{fv}\in\mathsf{L}^\times(U;V)$. **P** If $f\ge 0$ in $U^\times$ and $v\ge 0$ in $V^\times$ then P_{fv} is a positive linear operator from U to V which is order-continuous because if $A\subseteq U$ is non-empty, downwards-directed and has infimum 0, then (as V is Archimedean)

$$\inf_{u\in A} P_{fv}(u)=\inf_{u\in A} f(u)v=0.$$

Of course $(f,g)\mapsto P_{fg}$ is bilinear, so $P_{fv}\in\mathsf{L}^\times(U;V)$ for every $f\in U^\times$, $v\in V$. **Q** Now I call a linear operator from U to V an **abstract integral operator** if it is in the band in $\mathsf{L}^\times(U;V)$ generated by $\{P_{fv} : f\in U^\times,\ v\in V\}$.

The first result describes these operators when U, V are expressed as subspaces of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ for measure algebras $\mathfrak{A}$, $\mathfrak{B}$ and V is perfect.

376E Theorem Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras, with localizable measure algebra free product $(\mathfrak{C},\bar{\lambda})$, and $U\subseteq L^0(\mathfrak{A})$, $V\subseteq L^0(\mathfrak{B})$ order-dense Riesz subspaces. Write W for the set of those $w\in L^0(\mathfrak{C})$ such that $w\times(u\otimes v)$ is integrable for every $u\in U$ and $v\in V$. Then we have an operator $w\mapsto T_w:W\to\mathsf{L}^\times(U;V^\times)$ defined by setting

$$T_w(u)(v)=\int w\times(u\otimes v)$$

for every $w\in W$, $u\in U$ and $v\in V$. The map $w\mapsto T_w$ is a Riesz space isomorphism between W and the band of abstract integral operators in $\mathsf{L}^\times(U;V^\times)$.

proof (a) The first thing to check is that the formula offered does define a member $T_w(u)$ of $V^\times$ for any $w\in W$ and $u\in U$. **P** Of course $T_w(u)$ is a linear operator because $\int$ is linear and $\otimes$ and $\times$ are bilinear. It belongs to $V^\sim$ because, writing $g(v)=\int|w|\times(|u|\otimes v)$, g is a positive linear operator and $|T_w(u)(v)|\le g(|v|)$ for every v. (I am here using 376Bc to see that $|w\times(u\otimes Fv)|=|w|\times(|u|\otimes|v|)$.) Also $g\in V^\times$ because $v\mapsto|u|\otimes v$, $w'\mapsto|w|\times w'$ and $\int$ are all order-continuous; so $T_w(u)$ also belongs to $V^\times$. **Q**

(b) Next, for any given $w\in W$, the map $T_w:U\to V^\times$ is linear (again because $\otimes$ and $\times$ are bilinear). It is helpful to note that W is a solid linear subspace of $L^0(\mathfrak{C})$. Now if $w\ge0$ in W, then $T_w\in\mathsf{L}^\times(U;V^\times)$. **P** If u, $v\ge0$ then $u\otimes v\ge0$, $w\times(u\otimes v)\ge0$ and $T_w(u)(v)\ge0$; as v is arbitrary, $T_w(u)\ge0$ whenever $u\ge0$; as u is arbitrary, T_w is positive. If $A\subseteq U$ is non-empty, downwards-directed and has infimum 0, then $T_w[A]$ is downwards-directed, and for any $v\in V^+$

$$(\inf T_w[A])(v)=\inf_{u\in A}T_w(u)(v)=\inf_{u\in A}\int w\times(u\otimes v)=0$$

because $u\mapsto u\otimes v$ is order-continuous. So $\inf T_w[A]=0$; as A is arbitrary, T_w is order-continuous. **Q**

For general $w\in W$, we now have $T_w=T_{w^+}-T_{w^-}\in L^\times(U;V^\times)$.

(c) Ths shows that $w\mapsto T_w$ is a map from W to $L^\times(U;V^\times)$. Running through the formulae once again, it is linear, positive and order-continuous; this last because, given a non-empty downwards-directed $C\subseteq W$ with infimum 0, then for any $u\in U^+$, $v\in V^+$

$$(\inf_{w\in C}T_w)(u)(v)\le\inf_{w\in C}\int w\times(u\otimes v)=0$$

(because $\int$ and $\times$ are order-continuous); as v is arbitrary, $(\inf_{w\in C}T_w)(u)=0$; as u is arbitrary, $\inf_{w\in C}T_w=0$.

(d) All this is easy, being nothing but a string of applications of the elementary properties of $\otimes$, $\times$ and $\int$. But I think a new idea is needed for the next fact: the map $w\mapsto T_w:W\to\mathsf{L}^\times(U;V^\times)$ is a Riesz homomorphism. **P** Write $\mathfrak{D}$ for the set of those $d\in\mathfrak{C}$ such that $T_w\wedge T_{w'}=0$ whenever w, $w'\in W^+$, $[\![w>0]\!]\subseteq d$ and $[\![w'>0]\!]\subseteq1_{\mathfrak{C}}\setminus d$. (i) If d_1, $d_2\in\mathfrak{D}$, w, $w'\in W^+$, $[\![w>0]\!]\subseteq d_1\cup d_2$ and $[\![w'>0]\!]\cap(d_1\cup d_2)=0$, then set $w_1=w\times\chi d_1$, $w_2=w-w_1$. In this case

$$[\![w_1>0]\!]\subseteq d_1,\quad[\![w_2>0]\!]\subseteq d_2,$$

so

$$T_{w_1}\wedge T_{w'}=T_{w_2}\wedge T_{w'}=0,\quad T_w\wedge T_{w'}\le(T_{w_1}\wedge T_{w'})+(T_{w_2}\wedge T_{w'})=0.$$

As w, w' are arbitrary, $d_1\cup d_2\in\mathfrak{D}$. Thus $\mathfrak{D}$ is closed under $\cup$. (ii) The symmetry of the definition of $\mathfrak{D}$ means that $1_{\mathfrak{C}}\setminus d\in\mathfrak{D}$ whenever $d\in\mathfrak{D}$. (iii) Of course $0\in\mathfrak{D}$, just because $T_w=0$ if $w\in W^+$ and $[\![w>0]\!]=0$; so $\mathfrak{D}$ is a subalgebra of $\mathfrak{C}$. (iv) If $D\subseteq\mathfrak{D}$ is non-empty and upwards-directed, with supremum c in $\mathfrak{C}$, and if w, $w'\in W^+$ are such that $[\![w>0]\!]\subseteq c$, $[\![w'>0]\!]\cap c=0$, then consider $\{w\times\chi d:d\in D\}$. This is upwards-directed, with supremum w; so $T_w=\sup_{d\in D}T_{w\times\chi d}$, because the map $q\mapsto T_q$ is order-continuous. Also $T_{w\times\chi d}\wedge T_{w'}=0$ for every $d\in D$, so $T_w\wedge T_{w'}=0$. As w, w' are arbitrary, $c\in\mathfrak{D}$; as D is arbitrary, $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{C}$. (v) If $a\in\mathfrak{A}$ and w, $w'\in W^+$ are such that $[\![w>0]\!]\subseteq a\otimes1_{\mathfrak{B}}$ and $[\![w'>0]\!]\cap(a\otimes1_{\mathfrak{B}})=0$, then any $u\in U^+$ is expressible as u_1+u_2 where $u_1=u\times\chi a$, $u_2=u\times\chi(1_{\mathfrak{A}}\setminus a)$. Now

$$T_w(u_2)(v)=\int w\times(u_2\otimes v)=\int w\times\chi(a\otimes1_{\mathfrak{B}})\times(u\otimes v)\times\chi((1_{\mathfrak{A}}\setminus a)\otimes1_{\mathfrak{B}})=0$$

for every $v\in V$, so $T_w(u_2)=0$. Similarly, $T_{w'}(u_1)=0$. But this means that

$$(T_w\wedge T_{w'})(u)\le T_w(u_2)+T_{w'}(u_1)=0.$$

As u is arbitrary, $T_w \wedge T_{w'} = 0$; as w and w' are arbitrary, $a \otimes 1_{\mathfrak{B}} \in \mathfrak{D}$. (vi) Now suppose that $b \in \mathfrak{B}$ and that w, $w' \in W^+$ are such that $[\![w > 0]\!] \subseteq 1_{\mathfrak{A}} \otimes b$ and $[\![w' > 0]\!] \cap (1_{\mathfrak{A}} \otimes b) = 0$. If $u \in U^+$ and $v \in V^+$ then

$$(T_w \wedge T_{w'})(u)(v) \le \int w \times (u \otimes (v \times \chi(1_{\mathfrak{B}} \setminus b))) + \int w' \times (u \otimes (v \times \chi b)) = 0.$$

As u, v are arbitrary, $T_w \wedge T_{w'} = 0$; as w and w' are arbitrary, $1_{\mathfrak{A}} \otimes b \in \mathfrak{D}$. (vii) This means that $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{C}$ including $\mathfrak{A} \otimes \mathfrak{B}$, and is therefore the whole of $\mathfrak{C}$ (325D(c-ii)). (viii) Now take any w, $w' \in W$ such that $w \wedge w' = 0$, and consider $c = [\![w > 0]\!]$. Then $[\![w' > 0]\!] \subseteq 1_{\mathfrak{C}} \setminus c$ and $c \in \mathfrak{D}$, so $T_w \wedge T_{w'} = 0$. This is what we need to be sure that $w \mapsto T_w$ is a Riesz homomorphism (352G). **Q**

(e) The map $w \mapsto T_w$ is injective. **P** (i) If $w > 0$ in W, then consider

$$A = \{a : a \in \mathfrak{A}, \exists\, u \in U, \chi a \le u\}, \quad B = \{b : b \in \mathfrak{B}, \exists\, v \in V, \chi b \le v\}.$$

Because U and V are order-dense in $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, A and B are order-dense in $\mathfrak{A}$ and $\mathfrak{B}$. Also both are upwards-directed. So $\sup_{a\in A, b\in B} a \otimes b = 1_{\mathfrak{C}}$ and $0 < \int w = \sup_{a\in A, b\in B} \int_{a\otimes b} w$. Take $a \in A$, $b \in B$ such that $\int_{a\otimes b} w > 0$; then there are $u \in U$, $v \in V$ such that $\chi a \le u$ and $\chi b \le v$, so that

$$T_w(u)(v) \ge \int_{a\otimes b} w > 0$$

and $T_w > 0$. (ii) For general non-zero $w \in W$, we now have $|T_w| = T_{|w|} > 0$ so $T_w \ne 0$. **Q**

Thus $w \mapsto T_w$ is an order-continuous injective Riesz homomorphism.

(f) Write $\tilde{W}$ for $\{T_w : w \in W\}$, so that $\tilde{W}$ is a Riesz subspace of $\mathsf{L}^\times(U; V^\times)$ isomorphic to W, and $\widehat{W}$ for the band it generates in $\mathsf{L}^\times(U; V^\times)$. Then $\tilde{W}$ is order-dense in $\widehat{W}$. **P** Suppose that $S > 0$ in $\widehat{W} = \tilde{W}^{\perp\perp}$ (353Ba). Then $S \notin \tilde{W}^\perp$, so there is a $w \in W$ such that $S \wedge T_w > 0$. Set $w_1 = w \wedge \chi 1_{\mathfrak{C}}$. Then $w = \sup_{n\in\mathbb{N}} w \wedge n w_1$, so $T_w = \sup_{n\in\mathbb{N}} T_w \wedge n T_{w_1}$ and $R = S \wedge T_{w_1} > 0$.

Set $U_1 = U \cap L^1(\mathfrak{A}, \bar{\mu})$. Because U is an order-dense Riesz subspace of $L^0(\mathfrak{A})$, U_1 is an order-dense Riesz subspace of $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$, therefore also norm-dense. Similarly $V_1 = V \cap L^1(\mathfrak{B}, \bar{\nu})$ is a norm-dense Riesz subspace of $L^1_{\bar{\nu}} = L^1(\mathfrak{B}, \bar{\nu})$. Define $\phi_0 : U_1 \times V_1 \to \mathbb{R}$ by setting $\phi_0(u, v) = R(u)(v)$ for $u \in U_1$ and $v \in V_1$. Then ϕ_0 is bilinear, and

$$\begin{aligned}|\phi_0(u, v)| &= |R(u)(v)| \le |R(u)|(|v|) \le R(|u|)(|v|) \le T_{w_1}(|u|)(|v|)\\ &= \int w_1 \times (|u| \otimes |v|) \le \int |u| \otimes |v| = \|u\|_1 \|v\|_1\end{aligned}$$

for all $u \in U_1$, $v \in V_1$, because $0 \le R \le T_{w_1}$ in $\mathsf{L}^\times(U; V^\times)$. Because U_1, V_1 are norm-dense in $L^1_{\bar{\mu}}$, $L^1_{\bar{\nu}}$ respectively, ϕ_0 has a unique extension to a continuous bilinear operator $\phi : L^1_{\bar{\mu}} \times L^1_{\bar{\nu}} \to \mathbb{R}$. (To reduce this to standard results on linear operators, think of R as a function from U_1 to V_1^*; since every member of V_1^* has a unique extension to a member of $(L^1_{\bar{\nu}})^*$, we get a corresponding function $R_1 : U_1 \to (L^1_{\bar{\nu}})^*$ which is continuous and linear, so has a unique extension to a continuous linear operator $R_2 : L^1_{\bar{\mu}} \to (L^1_{\bar{\nu}})^*$, and we set $\phi(u, v) = R_2(u)(v)$.)

By 376C, there is a unique $h \in (L^1_{\bar{\lambda}})^* = L^1(\mathfrak{C}, \bar{\lambda})^*$ such that $h(u \otimes v) = \phi(u, v)$ for every $u \in L^1_{\bar{\mu}}$ and $v \in L^1_{\bar{\nu}}$. Because $(\mathfrak{C}, \bar{\lambda})$ is localizable, this h corresponds to a $w' \in L^\infty(\mathfrak{C})$ (365Lc), and

$$\int w' \times (u \otimes v) = h(u \otimes v) = \phi_0(u, v) = R(u)(v)$$

for every $u \in U_1$, $v \in V_1$.

Because U_1 is norm-dense in $L^1_{\bar{\mu}}$, U_1^+ is dense in $(L^1_{\bar{\mu}})^+$, and similarly V_1^+ is dense in $(L^1_{\bar{\nu}})^+$, so $U_1^+ \times V_1^+$ is dense in $(L^1_{\bar{\mu}})^+ \times (L^1_{\bar{\nu}})^+$; now ϕ_0 is non-negative on $U_1^+ \times V_1^+$, so ϕ (being continuous) is non-negative on $(L^1_{\bar{\mu}})^+ \times (L^1_{\bar{\nu}})^+$. By 376Cc, $h \ge 0$ in $(L^1_{\bar{\lambda}})^*$ and $w' \ge 0$ in $L^\infty(\mathfrak{C})$. In the same way, because $\phi_0(u, v) \le T_w(u)(v)$ for $u \in U_1^+$ and $v \in V_1^+$, $w' \le w_1 \le w$ in $L^0(\mathfrak{C})$, so $w' \in W$. We have

$$T_{w'}(u)(v) = \int w' \times (u \otimes v) = R(u)(v)$$

for all $u \in U_1$, $v \in V_1$. If $u \in U_1^+$, then $T_{w'}(u)$ and $R(u)$ are both order-continuous, so must be identical, since V_1 is order-dense in V. This means that $T_{w'}$ and R agree on U_1. But as both are themselves order-continuous linear operators, and U_1 is order-dense in U, they must be equal.

Thus $0 < T_{w'} \le S$ in $\mathsf{L}^\times(U; V^\times)$. As S is arbitrary, $\tilde{W}$ is quasi-order-dense in $\widehat{W}$, therefore order-dense (353A). **Q**

(g) Because $w \mapsto T_w : W \mapsto \tilde{W}$ is an injective Riesz homomorphism, we have an inverse map $Q : \tilde{W} \to L^0(\mathfrak{C})$, setting $Q(T_w) = w$; this is a Riesz homomorphism, and it is order-continuous because W is solid in $L^0(\mathfrak{C})$, so that the embedding $W \subsetneq L^0(\mathfrak{C})$ is order-continuous. By 368B, Q has an extension to an order-continuous Riesz homomorphism $\tilde{Q} : \widehat{W} \to L^0(\mathfrak{C})$. Because $Q(S) > 0$ whenever $S > 0$ in $\tilde{W}$, $\tilde{Q}(S) > 0$ whenever $S > 0$ in $\widehat{W}$, so $\tilde{Q}$ is injective. Now $\tilde{Q}(S) \in W$ for every $S \in \widehat{W}$. **P** It is enough to look at non-negative S. In this case, $\tilde{Q}(S)$ must be $\sup\{\tilde{Q}(T_w) : w \in W, T_w \le S\} = \sup C$, where $C = \{w : T_w \le S\} \subseteq W$. Take $u \in U^+$ and $v \in V^+$. Then $\{w \times (u \otimes v) : w \in C\}$ is upwards-directed, because C is, and

$$\sup_{w\in C} \int w \times (u \otimes v) = \sup_{w\in C} T_w(u)(v) \le S(u)(v) < \infty.$$

So $\tilde{Q}(S) \times (u \otimes v) = \sup_{w\in C} w \times (u \otimes v)$ belongs to L^1_{λ} (365Df). As u and v are arbitrary, $\tilde{Q}(S) \in W$. **Q**

(h) Of course this means that $\tilde{W} = \widehat{W}$ and $\tilde{Q} = Q$, that is, that $w \mapsto T_w : W \mapsto \widehat{W}$ is a Riesz space isomorphism.

(i) I have still to check on the identification of $\widehat{W}$ as the band Z of abstract integral operators in $\mathrm{L}^{\times}(U; V^{\times})$. Write $P_{fg}(u) = f(u)g$ for $f \in U^{\times}$, $g \in V^{\times}$ and $u \in U$.

Set

$$U^{\#} = \{u : u \in L^0(\mathfrak{A}),\, u \times u' \in L^1_{\bar{\mu}} \text{ for every } u' \in U\},$$

$$V^{\#} = \{v : v \in L^0(\mathfrak{B}),\, v \times v' \in L^1_{\bar{\nu}} \text{ for every } v' \in V\}.$$

From 369C we know that if we set $f_u(u') = \int u \times u'$ for $u \in U^{\#}$ and $u' \in U$, then $f_u \in U^{\times}$ for every $u \in U^{\#}$, and $u \mapsto f_u$ is an isomorphism between $U^{\#}$ and an order-dense Riesz subspace of $U^{\times}$. Similarly, setting $g_v(v') = \int v \times v'$ for $v \in V^{\#}$ and $v' \in V$, $v \mapsto g_v$ is an isomorphism between $V^{\#}$ and an order-dense Riesz subspace of $V^{\times}$.

If $u \in U^{\#}$ and $v \in V^{\#}$ then

$$\int (u \otimes v) \times (u' \otimes v') = \int (u \times u') \otimes (v \times v') = (\int u \times u')(\int v \times v') = f_u(u')g_v(v')$$

for every $u' \in U$, $v' \in V$, so $u \otimes v \in W$ and $T_{u\otimes v} = P_{f_u g_v}$.

Now take $f \in (U^{\times})^+$ and $g \in (V^{\times})^+$. Set $A = \{u : u \in U^{\#}, u \ge 0, f_u \le f\}$ and $B = \{v : v \in V^{\#}, v \ge 0, g_v \le g\}$. These are upwards-directed, so $C = \{u \otimes v : u \in A, v \in B\}$ is upwards-directed in $L^0(\mathfrak{C})$. Because $\{f_u : u \in U^{\#}\}$ is order-dense in $U^{\times}$, $f = \sup_{u\in A} f_u$; by 355Ed, $f(u') = \sup_{u\in A} f_u(u')$ for every $u' \in U^+$. Similarly, $g(v') = \sup_{v\in B} f_v(v')$ for every $v' \in V^+$.

? Suppose, if possible, that C is not bounded above in $L^0(\mathfrak{C})$. Because $\mathfrak{C}$ and $L^0(\mathfrak{C})$ are Dedekind complete,

$$c = \inf_{n\in\mathbb{N}} \sup_{u\in A, v\in B} [\![u \otimes v \ge n]\!]$$

must be non-zero (364L(a-i)). Because U and V are order-dense in $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively,

$$1_{\mathfrak{A}} = \sup\{[\![u' > 0]\!] : u' \in U\}, \quad 1_{\mathfrak{B}} = \sup\{[\![v' > 0]\!] : v' \in V\},$$

and there are $u' \in U^+$, $v' \in V^+$ such that $c \cap [\![u' > 0]\!] \otimes [\![v' > 0]\!] \ne 0$, so that $\int_c u' \otimes v' > 0$. But now, for any $n \in \mathbb{N}$,

$$\begin{aligned}
f(u')g(v') &\ge \sup_{u\in A, v\in B} f_u(u')g_v(v') \\
&= \sup_{u\in A, v\in B} \int (u \otimes v) \times (u' \otimes v') \\
&\ge \sup_{u\in A, v\in B} \int ((u \otimes v) \wedge n\chi c) \times (u' \otimes v') \\
&= \int \sup_{u\in A, v\in B} ((u \otimes v) \wedge n\chi c) \times (u' \otimes v')
\end{aligned}$$

(because $w \mapsto \int w \times (u' \otimes v')$ is order-continuous)

$$= \int (n\chi c) \times (u' \otimes v') = n \int_c u' \otimes v',$$

which is impossible. **X**

Thus C is bounded above in $L^0(\mathfrak{C})$, and has a supremum $w \in L^0(\mathfrak{C})$. If $u' \in U^+$, $v' \in V^+$ then

$$\int w \times (u' \otimes v') = \sup_{u \in A, v \in B} \int (u \otimes v) \times (u' \otimes v')$$
$$= \sup_{u \in A, v \in B} f_u(u') g_v(v') = f(u')g(v') = P_{fg}(u')(v').$$

Thus $w \in W$ and

$$P_{fg} = T_w \in \tilde{W} \subseteq \widehat{W}.$$

And this is true for any non-negative $f \in U^\times$ and $g \in V^\times$. Of course it follows that $P_{fg} \in \widehat{W}$ for every $f \in U^\times$, $g \in V^\times$; as $\widehat{W}$ is a band, it must include Z.

(j) Finally, $\widehat{W} \subseteq Z$. **P** Since $Z = Z^{\perp\perp}$, it is enough to show that $\widehat{W} \cap Z^\perp = \{0\}$. Take any $T > 0$ in $\widehat{W}$. There are $u_0' \in U^+$, $v_0' \in V^+$ such that $T(u_0')(v_0') > 0$. So there is a $v \in V^\#$ such that $0 \le g_v \le T(u_0')$ and $g_v(v_0') > 0$, that is, $\int v \times v_0' > 0$. Because V is order-dense in $L^0(\mathfrak{B})$, there is a $v_1' \in V$ such that $0 < v_1' \le v_0' \times \chi[\![v > 0]\!]$, so that

$$0 < \int v \times v_1' = g_v(v_1') \le T(u_0')(v_1')$$

and $[\![v_1' > 0]\!] \subseteq [\![v > 0]\!]$.

Now consider the functional $u' \mapsto h(u') = T(u')(v_1') : U \to \mathbb{R}$. This belongs to $(U^\times)^+$ and $h(u_0') > 0$, so there is a $u \in U^\#$ such that $0 \le f_u \le h$ and $f_u(u_0') > 0$. This time, $\int u \times u_0' > 0$ so (because U is order-dense in $L^0(\mathfrak{A})$) there is a $u_1' \in U$ such that $h(u_1') > 0$ and $[\![u_1' > 0]\!] \subseteq [\![u > 0]\!]$.

We can express T as T_w where $w \in W^+$. In this case, we have

$$\int w \times (u_1' \otimes v_1') = T(u_1')(v_1') = h(u_1') > 0,$$

so

$$0 \ne [\![w > 0]\!] \cap [\![u_1' \otimes v_1' > 0]\!] = [\![w > 0]\!] \cap ([\![u_1' > 0]\!] \otimes [\![v_1' > 0]\!])$$
$$\subseteq [\![w > 0]\!] \cap ([\![u > 0]\!] \otimes [\![v > 0]\!]) = [\![w > 0]\!] \cap [\![u \otimes v > 0]\!],$$

and $w \wedge (u \otimes v) > 0$, so

$$T_w \wedge P_{f_u g_v} = T_w \wedge T_{u \otimes v} = T_{w \wedge (u \otimes v)} > 0.$$

Thus $T \notin Z^\perp$. Accordingly $\widehat{W} \cap Z^\perp = \{0\}$ and $\widehat{W} \subseteq Z^{\perp\perp} = Z$. **Q**

Since we already know that $Z \subseteq \widehat{W}$, this completes the proof.

376F Corollary Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras, with localizable measure algebra free product $(\mathfrak{C}, \bar{\lambda})$. Let $U \subseteq L^0(\mathfrak{A})$, $V \subseteq L^0(\mathfrak{B})$ be perfect order-dense solid linear subspaces, and $T : U \to V$ a linear operator. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) there is a $w \in L^0(\mathfrak{C})$ such that $\int w \times (u \otimes v')$ is defined and equal to $\int Tu \times v'$ whenever $u \in U$ and $v' \in L^0(\mathfrak{B})$ is such that $v' \times v$ is integrable for every $v \in V$.

proof Setting $V^\# = \{v' : v' \in L^0(\mathfrak{B}),\ v \times v' \in L^1 \text{ for every } v \in V\}$, we know that we can identify $V^\#$ with $V^\times$ and V with $(V^\#)^\times$ (369C). So the equivalence of (i) and (ii) is just 376E applied to $V^\#$ in place of V.

376G Lemma Let U be a Riesz space, V an Archimedean Riesz space, $T : U \to V$ a linear operator, $f \in (U^\sim)^+$ and $e \in V^+$. Suppose that $0 \le Tu \le f(u)e$ for every $u \in U^+$. Then if $\langle u_n \rangle_{n \in \mathbb{N}}$ is a sequence in U such that $\lim_{n\to\infty} g(u_n) = 0$ whenever $g \in U^\sim$ and $|g| \le f$, $\langle Tu_n \rangle_{n \in \mathbb{N}}$ order*-converges to 0 in V (definition: 367A).

proof Let V_e be the solid linear subspace of V generated by e; then $Tu \in V_e$ for every $u \in U$. We can identify V_e with an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, with e corresponding to χX (353N). For $x \in X$, set $g_x(u) = (Tu)(x)$ for every $u \in U$; then $0 \le g_x(u) \le f(u)$ for $u \ge 0$, so $|g_x| \le f$ and $\lim_{n\to\infty}(Tu_n)(x) = 0$. As x is arbitrary, $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$, by 367K, and therefore in V_e, because V_e is order-dense in $C(X)$ (367E). But V_e, regarded as a subspace of V, is solid, so 367E tells us also that $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

376H Theorem Let U be a Riesz space and V a weakly (σ,∞)-distributive Dedekind complete Riesz space (definition: 368N). Suppose that $T \in \mathsf{L}^\times(U;V)$. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ and $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V;

(iii) whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U and $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

proof For $f \in U^\times$, $v \in V$ and $u \in U$ set $P_{fv}(u) = f(u)v$. Write $Z \subseteq \mathsf{L}^\times(U;V)$ for the band of abstract integral operators.

(a)(i)⇒(iii) Suppose that $T \in Z^+$, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U such that $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$. Note that $\{P_{fv} : f \in U^{\times +},\ v \in V^+\}$ is upwards-directed, so that $T = \sup\{T \wedge P_{fv} : f \in U^{\times +},\ v \in V^+\}$ (352Va).

Take $u^* \in U^+$ such that $|u_n| \le u^*$ for every n, and set $w = \inf_{n\in\mathbb{N}}\sup_{m\ge n} Tu_m$, which is defined because $|Tu_n| \le Tu^*$ for every n. Now $w \le (T-P_{fv})^+(u^*)$ for every $f \in U^{\times +}$ and $v \in V^+$. **P** Setting $T_1 = T \wedge P_{fv}$, $w_0 = (T-P_{fv})^+(u^*)$ we have

$$Tu_n - T_1u_n \le |T-T_1|(u^*) = (T-P_{fv})^+(u^*) = w_0$$

for every $n \in \mathbb{N}$, so $Tu_n \le w_0 + T_1u_n$. On the other hand, $0 \le T_1u \le f(u)v$ for every $u \in U^+$, so by 376G we must have $\inf_{n\in\mathbb{N}}\sup_{m\ge n} T_1u_m = 0$. Accordingly

$$w \le w_0 + \inf_{n\in\mathbb{N}}\sup_{m\ge n} T_1u_m = w_0. \text{ } \mathbf{Q}$$

But as $\inf\{(T-P_{fv})^+ : f \in U^{\times +},\ v \in V^+\} = 0$, $w \le 0$. Similarly (or applying the same argument to $\langle -u_n\rangle_{n\in\mathbb{N}}$), $\sup_{n\in\mathbb{N}}\inf_{n\in\mathbb{N}} Tu_n \ge 0$ and $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to zero.

For general $T \in Z$, this shows that $\langle T^+u_n\rangle_{n\in\mathbb{N}}$ and $\langle T^-u_n\rangle_{n\in\mathbb{N}}$ both order*-converge to 0, so $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367C(a-iv). As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, (iii) is satisfied.

(b)(iii)⇒(ii) is trivial.

(c)(ii)⇒(i) ? Now suppose, if possible, that (ii) is satisfied, but that $T \notin Z$. Because $\mathsf{L}^\times(U;V)$ is Dedekind complete (355H), Z is a projection band (353J), so T is expressible as $T_1 + T_2$ where $T_1 \in Z$, $T_2 \in Z^\perp$ and $T_2 \ne 0$. At least one of T_2^+, T_2^- is non-zero; replacing T by $-T$ if need be, we may suppose that $T_2^+ > 0$.

Because T_2^+, like T, belongs to $\mathsf{L}^\times(U;V)$, its kernel U_0 is a band in U, which cannot be the whole of U, and there is a $u_0 > 0$ in $U_0^\perp$. In this case $T_2^+u_0 > 0$; because $T_2^+ \wedge (T_2^- + |T_1|) = 0$, there is a $u_1 \in [0,u_0]$ such that $T_2^+(u_0-u_1) + (T_2^- + |T_1|)(u_1) \not\ge T_2^+u_0$, so that

$$Tu_1 \ge T_2u_1 - |T_1|(u_1) \not\le 0$$

and $Tu_1 \ne 0$. Now this means that the sequence $(Tu_1, Tu_1, \dots)$ is not order*-convergent to zero, so there must be some $f \in U^\times$ such that $(f(u_1), f(u_1), \dots)$ does not converge to 0, that is, $f(u_1) \ne 0$; replacing f by $|f|$ if necessary, we may suppose that $f \ge 0$ and that $f(u_1) > 0$.

By 356H, there is a u_2 such that $0 < u_2 \le u_1$ and $g(u_2) = 0$ whenever $g \in U^\times$ and $g \wedge f = 0$. Because $0 < u_2 \le u_0$, $u_2 \in U_0^\perp$ and $v_0 = T_2^+u_2 > 0$. Consider $P_{fv_0} \in Z$. Because $T_2 \in Z^\perp$, $T_2^+ \wedge P_{fv_0} = 0$; set $S = P_{fv_0} + T_2^-$, so that $T_2^+ \wedge S = 0$. Then

$$\inf_{u\in[0,u_2]} T_2^+(u_2-u) + Su = 0, \quad \sup_{u\in[0,u_2]} T_2^+u - Su = v_0$$

(use 355Ec for the first equality, and then subtract both sides from v_0). Now $Su \ge f(u)v_0$ for every $u \ge 0$, so that for any $\epsilon > 0$

$$\sup_{u\in[0,u_2],f(u)\ge\epsilon} T_2^+u - Su \le (1-\epsilon)v_0$$

and accordingly

$$\sup_{u\in[0,u_2],f(u)\le\epsilon} T_2^+u = v_0,$$

since the join of these two suprema is surely at least v_0, while the second is at most v_0. Note also that

$$v_0 = \sup_{u\in[0,u_2],f(u)\le\epsilon} T_2^+u = \sup_{0\le u'\le u\le u_2,f(u)\le\epsilon} T_2u' = \sup_{0\le u'\le u_2,f(u')\le\epsilon} T_2u'.$$

For $k\in\mathbb{N}$ set $A_k = \{u : 0\le u\le u_2,\ f(u)\le 2^{-k}\}$. We know that

$$B_k = \{\sup_{u\in I} T_2u : I\subseteq A_k \text{ is finite}\}$$

is an upwards-directed set with supremum v_0 for each k. Because V is weakly (σ,∞)-distributive, we can find a sequence $\langle v'_k\rangle_{k\in\mathbb{N}}$ such that $v'_k\in B_k$ for every k and $v_1 = \inf_{k\in\mathbb{N}} v'_k > 0$. For each k let $I_k\subseteq A_k$ be a finite set such that $v'_k = \sup_{u\in I_k} T_2u$.

Because each I_k is finite, we can build a sequence $\langle u'_n\rangle_{n\in\mathbb{N}}$ in $[0,u_2]$ enumerating each in turn, so that $\lim_{n\to\infty} f(u'_n) = 0$ (since $f(u)\le 2^{-k}$ if $u\in I_k$) while $\sup_{m\ge n} T_2u'_m \ge v_1$ for every n (since $\{u'_m : m\ge n\}$ always includes some I_k). Now $\langle T_2u'_n\rangle_{n\in\mathbb{N}}$ does not order*-converge to 0.

However, $\lim_{n\to\infty} g(u'_n) = 0$ for every $g\in U^\times$. **P** Express $|g|$ as g_1+g_2 where g_1 belongs to the band of $U^\times$ generated by f and $g_2\wedge f = 0$ (353Ic). Then $g_2(u'_n) = g_2(u_2) = 0$ for every n, by the choice of u_2. Also $g_1 = \sup_{n\in\mathbb{N}} g_1\wedge nf$ (352Vb); so, given $\epsilon>0$, there is an $m\in\mathbb{N}$ such that $(g_1-mf)^+(u_2)\le\epsilon$ and $(g_1-mf)^+(u'_n)\le\epsilon$ for every $n\in\mathbb{N}$. But this means that

$$|g(u'_n)| \le |g|(u'_n) \le \epsilon + mf(u'_n)$$

for every n, and $\limsup_{n\to\infty} |g(u'_n)|\le\epsilon$; as ϵ is arbitrary, $\lim_{n\to\infty} g(u'_n) = 0$. **Q**

Now, however, part (a) of this proof tells us that $\langle T_1u'_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0, because $T_1\in Z$, while $\langle Tu'_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0, by hypothesis; so $\langle T_2u'_n\rangle_{n\in\mathbb{N}} = \langle Tu'_n - T_1u'_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0. **X**

This contradiction shows that every operator satisfying the condition (ii) must be in Z.

376I The following elementary remark will be useful for the next corollary and also for Theorem 376S.

Lemma Let (X,Σ,μ) be a σ-finite measure space and U an order-dense solid linear subspace of $L^0(\mu)$. Then there is a non-decreasing sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of measurable subsets of X, with union X, such that $\chi X_n^\bullet\in U$ for every $n\in\mathbb{N}$.

proof Write $\mathfrak{A}$ for the measure algebra of μ, so that $L^0(\mu)$ can be identified with $L^0(\mathfrak{A})$ (364Ic). $A = \{a : a\in\mathfrak{A}\setminus\{0\},\ \chi a\in U\}$ is order-dense in $\mathfrak{A}$, so includes a partition of unity $\langle a_i\rangle_{i\in I}$. Because μ is σ-finite, $\mathfrak{A}$ is ccc (322G) and I is countable, so we can take I to be a subset of $\mathbb{N}$. Choose $E_i\in\Sigma$ such that $E_i^\bullet = a_i$ for $i\in I$; set $E = X\setminus\bigcup_{i\in I}E_i$, $X_n = E\cup\bigcup_{i\in I,i\le n}E_i$ for $n\in\mathbb{N}$.

376J Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with product measure λ on $X\times Y$. Let $U\subseteq L^0(\mu)$, $V\subseteq L^0(\nu)$ be perfect order-dense solid linear subspaces, and $T:U\to V$ a linear operator. Write $\mathcal{U} = \{f : f\in\mathcal{L}^0(\mu),\ f^\bullet\in U\}$, $\mathcal{V}^\# = \{h : h\in\mathcal{L}^0(\nu),\ h^\bullet\times v\in L^1$ for every $v\in V\}$. Then the following are equiveridical:

(i) T is an abstract integral operator;

(ii) there is a $k\in\mathcal{L}^0(\lambda)$ such that

(α) $\int|k(x,y)f(x)h(y)|d(x,y)<\infty$ for every $f\in\mathcal{U}$, $h\in\mathcal{V}^\#$,

(β) if $f\in\mathcal{U}$ and we set $g(y) = \int k(x,y)f(x)dx$ wherever this is defined, then $g\in\mathcal{L}^0(\nu)$ and $Tf^\bullet = g^\bullet$;

(iii) $T\in\mathrm{L}^\sim(U;V)$ and whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ and $\lim_{n\to\infty} h(u_n) = 0$ for every $h\in U^\times$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

Remark I write '$d(x,y)$' above to indicate integration with respect to the product measure λ. Recall that in the terminology of §251, λ can be taken to be either the 'primitive' or 'c.l.d.' product measure (251K).

proof The idea is of course to identify $L^0(\mu)$ and $L^0(\nu)$ with $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$, where $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are the measure algebras of μ and ν, so that their localizable measure algebra free product can be identified with the measure algebra of λ (325E), while $V^\# = \{h^\bullet : h \in \mathcal{V}^\#\}$ can be identified with $V^\times$, because (T, T, ν) is localizable (see the last sentence in 369C).

(a)(i)⇒(ii) By 376F, there is a $w \in L^0(\lambda)$ such that $\int w \times (u \otimes v')$ is defined and equal to $\int Tu \times v'$ whenever $u \in U$ and $v' \in V^\#$. Express w as $k^\bullet$ where $k \in \mathcal{L}^0(\lambda)$. If $f \in \mathcal{U}$ and $h \in \mathcal{V}^\#$ then $\int |k(x,y)f(x)h(y)|d(x,y) = \int |w \times (f^\bullet \otimes h^\bullet|$ is finite, so (ii-α) is satisfied.

Now take any $f \in \mathcal{U}$, and set $g(y) = \int k(x,y)f(x)dx$ whenever this is defined in $\mathbb{R}$. Write $\mathcal{F}$ for the set of those $F \in \mathrm{T}$ such that $\chi F \in \mathcal{V}^\#$. Then for any $F \in \mathcal{F}$, g is defined almost everywhere in F and $g\restriction F$ is ν-virtually measurable. **P** $\int k(x,y)f(x)\chi F(y)d(x,y)$ is defined in $\mathbb{R}$, so by Fubini's theorem (252B, 252C) $g_F(y) = \int k(x,y)f(x)\chi F(y)dx$ is defined for almost every y, and is ν-virtually measurable; now $g\restriction F = g_F\restriction F$. **Q** Next, there is a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ with union Y, by 376I, because V is perfect and order-dense, so $V^\#$ must also be order-dense in $L^0(\nu)$.

For each $n \in \mathbb{N}$, there is a measurable set $F'_n \subseteq F_n \cap \operatorname{dom} g$ such that $g\restriction F_n$ is measurable and $F_n \setminus F'_n$ is negligible. Setting $G = \bigcup_{n\in\mathbb{N}} F'_n$, G is conegligible and $g\restriction G$ is measurable, so $g \in \mathcal{L}^0(\nu)$.

If $\tilde g \in L^0(\nu)$ represents $Tu \in L^0(\nu)$, then for any $F \in \mathcal{F}$

$$\int_F \tilde g = \int Tu \times (\chi F)^\bullet = \int_F g.$$

In particular, this is true whenever $F \in \mathrm{T}$ and $F \subseteq F_n$. So g and $\tilde g$ agree almost everywhere in F_n, for each n, and $g =_{\text{a.e.}} \tilde g$. Thus g also represents Tu, as required in (ii-β).

(b)(ii)⇒(i) Set $w = k^\bullet$ in $L^0(\lambda)$. If $f \in \mathcal{U}$ and $h \in \mathcal{V}^\#$ the hypothesis (α) tells us that $(x,y) \mapsto k(x,y)f(x)h(y)$ is integrable (because it surely belongs to $\mathcal{L}^0(\lambda)$). By Fubini's theorem,

$$\int k(x,y)f(x)h(y)d(x,y) = \int g(y)h(y)dy$$

where $g(y) = \int k(x,y)f(x)dx$ for almost every y, so that $Tf^\bullet = g^\bullet$, by (β). But this means that, setting $u = f^\bullet$ and $v' = h^\bullet$,

$$\int w \times (u \otimes v') = \int Tu \times v';$$

and this is true for every $u \in U$, $v' \in V^\#$.

Thus T satisfies the condition 376F(ii), and is an abstract integral operator.

(b)(i)⇒(iii) Because V is weakly (σ,∞)-distributive (368S), this is covered by 376H(i)⇒(iii).

(c)(iii)⇒(i) Suppose that T satisfies (iii). The point is that T^+ is order-continuous. **P?** Otherwise, let $A \subseteq U$ be a non-empty downwards-directed set, with infimum 0, such that $v_0 = \inf_{u\in A} T^+(u) > 0$. Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering X, and set $a_n = X_n^\bullet$ for each n. For each n, $\inf_{u\in A} [\![u > 2^{-n}]\!] = 0$, so we can find $\tilde u_n \in A$ such that $\bar\mu(a_n \cap [\![\tilde u_n > 2^{-n}]\!]) \le 2^{-n}$. Set $u_n = \inf_{i\le n} \tilde u_i$ for each n; then $\langle u_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0; also, $[0, u_n]$ meets A for each n, so that $v_0 \le \sup\{Tu : 0 \le u \le u_n\}$ for each n. Because V is weakly (σ,∞)-distributive, we can find a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of finite sets such that $I_n \subseteq [0, u_n]$ for each n and $v_1 = \inf_{n\in\mathbb{N}} \sup_{u\in I_n} (Tu)^+ > 0$. Enumerating $\bigcup_{n\in\mathbb{N}} I_n$ as $\langle u'_n\rangle_{n\in\mathbb{N}}$, as in part (c) of the proof of 376H, we see that $\langle u'_n\rangle_{n\in\mathbb{N}}$ is order-bounded and $\lim_{n\to\infty} f(u'_n) = 0$ for every $f \in U^\times$ (indeed, $\langle u'_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in U), while $\langle Tu'_n\rangle_{n\in\mathbb{N}} \not\to^* 0$ in V. **XQ**

Similarly, T^- is order-continuous, so $T \in \mathsf{L}^\times(U;V)$. Accordingly T is an abstract integral operator by condition (ii) of 376H.

376K As an application of the ideas above, I give a result due to N.Dunford (376N) which was one of the inspirations underlying the theory. Following the method of Zaanen 83, I begin with a couple of elementary lemmas.

Lemma Let U and V be Riesz spaces. Then there is a Riesz space isomorphism $T \mapsto T' : \mathsf{L}^\times(U;V^\times) \to \mathsf{L}^\times(V;U^\times)$ defined by the formula

$$(T'v)(u) = (Tu)(v) \text{ for every } u \in U,\ v \in V.$$

If we write $P_{fg}(u) = f(u)g$ for $f \in U^\times$, $g \in V^\times$ and $u \in U$, then $P_{fg} \in \mathsf{L}^\times(U;V^\times)$ and $P'_{fg} = P_{gf}$ in $\mathsf{L}^\times(V;U^\times)$. Consequently T is an abstract integral operator iff T' is.

proof All the ideas involved have already appeared. For positive $T \in \mathsf{L}^\times(U;V^\times)$ the functional $(u,v) \mapsto (Tu)(v)$ is bilinear and order-continuous in each variable separately; so (just as in the first part of the proof of 376E) corresponds to a $T' \in \mathsf{L}^\times(V;U^\times)$. The map $T \mapsto T' : \mathsf{L}^\times(U;V^\times)^+ \to \mathsf{L}^\times(V;U^\times)^+$ is evidently an additive, order-preserving bijection, so extends to an isomorphism between $\mathsf{L}^\times(U;V^\times)$ and $\mathsf{L}^\times(V;U^\times)$ given by the same formula. I remarked in part (i) of the proof of 376E that every P_{fg} belongs to $\mathsf{L}^\times(U;V^\times)$, and the identification $P'_{fg} = P_{gf}$ is just a matter of checking the formulae. Of course it follows at once that the bands of abstract integral operators must also be matched by the map $T \mapsto T'$.

376L Lemma Let U be a Banach lattice with an order-continuous norm. If $w \in U^+$ there is a $g \in (U^\times)^+$ such that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|u\| \le \epsilon$ whenever $0 \le u \le w$ and $g(u) \le \delta$.

proof (a) As remarked in 356D, $U^* = U^\sim = U^\times$. Set

$$A = \{v : v \in U \text{ and there is an } f \in (U^\times)^+ \text{ such that } f(u) > 0 \text{ whenever } 0 < u \le |v|\}.$$

Then $v' \in A$ whenever $|v'| \le |v| \in A$ and $v + v' \in A$ for all v, $v' \in A$ (if $f(u) > 0$ whenever $0 < u \le |v|$ and $f'(u) > 0$ whenever $0 < u \le |v'|$, then $(f + f')(u) > 0$ whenever $0 < u \le |v + v'|$); moreover, if $v_0 > 0$ in U, there is a $v \in A$ such that $0 < v \le v_0$. **P** Because $U^\times = U^*$ separates the points of U, there is a $g > 0$ in $U^\times$ such that $g(v_0) > 0$; now by 356H there is a $v \in \,]0, v_0]$ such that g is strictly positive on $]0, v]$, so that $v \in A$. **Q** But this means that A is an order-dense solid linear subspace of U.

(b) In fact $w \in A$. **P** $w = \sup B$, where $B = A \cap [0, w]$. Because B is upwards-directed, $w \in \overline{B}$ (354Ea), and there is a sequence $\langle u'_n\rangle_{n\in\mathbb{N}}$ in B converging to w for the norm. For each n, choose $f_n \in (U^\times)^+$ such that $f_n(u) > 0$ whenever $0 < u \le u'_n$. Set

$$f = \sum_{n=0}^\infty \frac{1}{2^n(1+\|f_n\|)} f_n$$

in $U^* = U^\times$. Then whenever $0 < u \le w$ there is some $n \in \mathbb{N}$ such that $u \wedge u'_n > 0$, so that $f_n(u) > 0$ and $f(u) > 0$. So f witnesses that $w \in A$. **Q**

(c) Take $g \in (U^\times)^+$ such that $g(u) > 0$ whenever $0 < u \le w$. This g serves. **P?** Otherwise, there is some $\epsilon > 0$ such that for every $n \in \mathbb{N}$ we can find a $u_n \in [0, w]$ with $g(u_n) \le 2^{-n}$ and $\|u_n\| \ge \epsilon$. Set $v_n = \sup_{i\ge n} u_i$; then $0 \le v_n \le w$, $g(v_n) \le 2^{-n+1}$ and $\|v_n\| \ge \epsilon$ for every $n \in \mathbb{N}$. But $\langle v_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so $v = \inf_{n\in\mathbb{N}} v_n$ must be non-zero, while $0 \le v \le w$ and $g(v) = 0$; which is impossible. **X** **Q**

Thus we have found an appropriate g.

376M Theorem (a) Let U be a Banach lattice with an order-continuous norm and V a Dedekind complete M-space. Then every bounded linear operator from U to V is an abstract integral operator.

(b) Let U be an L-space and V a Banach lattice with order-continuous norm. Then every bounded linear operator from U to $V^\times$ is an abstract integral operator.

proof (a) By 355Kb and 355C, $\mathsf{L}^\times(U;V) = \mathsf{L}^\sim(U;V) \subseteq \mathsf{B}(U;V)$; but since norm-bounded sets in V are also order-bounded, $\{Tu : |u| \le u_0\}$ is bounded above in V for every $T \in \mathsf{B}(U;V)$ and $u_0 \in U^+$, and $\mathsf{B}(U;V) = \mathsf{L}^\times(U;V)$.

I repeat ideas from the proof of 376H. (I cannot quote 376H directly as I am not assuming that V is weakly (σ,∞)-distributive.) **?** Suppose, if possible, that $\mathsf{B}(U;V)$ is not the band Z of abstract integral operators. In this case there is a $T > 0$ in $Z^\perp$. Take $u_1 \ge 0$ such that $v_0 = Tu_1$ is non-zero. Let $f \ge 0$ in $U^\times$ be such that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|u\| \le \epsilon$ whenever $0 \le u \le u_1$ and $f(u) \le \delta$ (376L). Then, just as in part (c) of the proof of 376H,

$$\sup_{u\in[0,u_1], f(u)\le\delta} Tu = v_0$$

for every $\delta > 0$. But there is a $\delta > 0$ such that $\|T\|\|u\| \le \frac{1}{2}\|v_0\|$ whenever $0 \le u \le u_1$ and $f(u) \le \delta$; in which case $\|\sup_{u\in[0,u_1], f(u)\le\delta} Tu\| \le \frac{1}{2}\|v_0\|$, which is impossible. **X**

Thus $Z = \mathsf{B}(U;V)$, as required.

(b) Because V has an order-continuous norm, $V^* = V^\times = V^\sim$; and the norm of V^* is a Fatou norm with the Levi property (356Da). So $\mathsf{B}(U;V^*) = \mathsf{L}^\times(U;V^\times)$, by 371C. By 376K, this is canonically isomorphic to $\mathsf{L}^\times(V;U^\times)$. Now $U^\times = U^*$ is an M-space (356Pb). By (a), every member of $\mathsf{L}^\times(V;U^\times)$ is an abstract integral operator; but the isomorphism between $\mathsf{L}^\times(V;U^\times)$ and $\mathsf{L}^\times(U;V^\times)$ matches the abstract integral operators in each space (376K), so every member of $\mathsf{B}(U;V^*)$ is also an abstract integral operator, as claimed.

376N Corollary: Dunford's theorem Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces and $T: L^1(\mu) \to L^p(\nu)$ a bounded linear operator, where $1 < p \le \infty$. Then there is a measurable function $k: X \times Y \to \mathbb{R}$ such that $Tf^\bullet = g_f^\bullet$, where $g_f(y) = \int k(x,y)f(x)dx$ almost everywhere, for every $f \in \mathcal{L}^1(\mu)$.

proof Set $q = \frac{p}{p-1}$ if p is finite, 1 if $p = \infty$. We can identify $L^p(\nu)$ with $V^\times$, where $V = L^q(\nu) \cong L^p(\nu)^\times$ (366Dc, 365Lc) has an order-continuous norm because $1 \le q < \infty$. By 376Mb, T is an abstract integral operator. By 376F/376J, T is represented by a kernel, as claimed.

376O Under the right conditions, weakly compact operators are abstract integral operators.

Lemma Let U be a Riesz space, and W a solid linear subspace of $U^\sim$. If $C \subseteq U$ is relatively compact for the weak topology $\mathfrak{T}_s(U,W)$ (3A5E), then for every $g \in W^+$ and $\epsilon > 0$ there is a $u^* \in U^+$ such that $g(|u| - u^*)^+ \le \epsilon$ for every $u \in C$.

proof Let W_g be the solid linear subspace of W generated by g. Then W_g is an Archimedean Riesz space with order unit, so $W_g^\times$ is a band in the L-space $W_g^* = W_g^\sim$ (356Na), and is therefore an L-space in its own right (354O). For $u \in U$, $h \in W_g^\times$ set $(Tu)(h) = h(u)$; then T is an order-continuous Riesz homomomorphism from U to $W_g^\times$ (356F).

Now W_g is perfect. **P** I use 356K. W_g is Dedekind complete because it is a solid linear subspace of the Dedekind complete space $U^\sim$. $W_g^\times$ separates the points of W because $T[U]$ does. If $A \subseteq W_g$ is upwards-directed and $\sup_{h\in A}\phi(h)$ is finite for every $\phi \in W_g^\times$, then A acts on $W_g^\times$ as a set of bounded linear functionals which, by the Uniform Boundedness Theorem (3A5Ha), is uniformly bounded; that is, there is some $M \ge 0$ such that $\sup_{h\in A}|\phi(h)| \le M\|\phi\|$ for every $\phi \in W_g^\times$. Because g is the standard order unit of W_g, we have $\|\phi\| = |\phi|(g)$ and $|\phi(h)| \le M|\phi|(g)$ for every $\phi \in W_g^\times$ and $h \in A$. In particular,

$$h(u) \le |h(u)| = |(Tu)(h)| \le M|Tu|(g) = M(Tu)(g) = Mg(u)$$

for every $h \in A$ and $u \in U^+$. But this means that $h \le Mg$ for every $h \in A$ and A is bounded above in W_g. Thus all the conditions of 356K are satisfied and W_g is perfect. **Q**

Accordingly T is continuous for the topologies $\mathfrak{T}_s(U,W)$ and $\mathfrak{T}_s(W_g^\times, W_g^{\times\times})$, because every element ϕ of $W_g^{\times\times}$ corresponds to a member of $W_g \subseteq W$, so 3A5Ec applies.

Now we are supposing that C is relatively compact for $\mathfrak{T}_s(U,W)$, that is, is included in some compact set C'; accordingly $T[C']$ is compact and $T[C]$ is relatively compact for $\mathfrak{T}_s(W_g^\times, W_g^{\times\times})$. Since $W_g^\times$ is an L-space, $T[C]$ is uniformly integrable (356Q); consequently (ignoring the trivial case $C = \emptyset$) there are $\phi_0,\dots,\phi_n \in T[C]$ such that $\|(|\phi| - \sup_{i\le n}|\phi_i|)^+\| \le \epsilon$ for every $\phi \in T[C]$ (354Rb), so that $(|\phi| - \sup_{i\le n}|\phi_i|)^+(g) \le \epsilon$ for every $\phi \in T[C]$.

Translating this back into terms of C itself, and recalling that T is a Riesz homomorphism, we see that there are $u_0,\dots,u_n \in C$ such that $g(|u| - \sup_{i\le n}|u_i|)^+ \le \epsilon$ for every $u \in C$. Setting $u^* = \sup_{i\le n}|u_i|$ we have the result.

376P Theorem Let U be an L-space and V a perfect Riesz space. If $T: U \to V$ is a linear operator such that $\{Tu : u \in U,\ \|u\| \le 1\}$ is relatively compact for the weak topology $\mathfrak{T}_s(V,V^\times)$, then T is an abstract integral operator.

proof (a) For any $g \ge 0$ in $V^\times$, $M_g = \sup_{\|u\|\le 1} g(|Tu|)$ is finite. **P** By 376O, there is a $v^* \in V^+$ such that $g(|Tu| - v^*)^+ \le 1$ whenever $\|u\| \le 1$; now $M_g \le g(v^*) + 1$. **Q** Considering $\|u\|^{-1}u$, we see that $g(|Tu|) \le M_g\|u\|$ for every $u \in U$.

Next, we find that $T \in \mathsf{L}^\sim(U;V)$. **P** Take $u \in U^+$. Set

$$B = \{\sum_{i=0}^n |Tu_i| : u_0,\dots,u_n \in U^+,\ \sum_{i=0}^n u_i = u\} \subseteq V^+.$$

Then B is upwards-directed. (Cf. 371A.) If $g \ge 0$ in $V^\times$,

$$\sup_{v\in B} g(v) = \sup\{\sum_{i=0}^{n} g(|Tu_i|) : \sum_{i=0}^{n} u_i = u\}$$
$$\le \sup\{\sum_{i=0}^{n} M_g\|u_i\| : \sum_{i=0}^{n} u_i = u\} = M_g\|u\|$$

is finite. By 356K, B is bounded above in V; and of course any upper bound for B is also an upper bound for $\{Tu' : 0 \le u' \le u\}$. As u is arbitrary, T is order-bounded. **Q**

Because U is a Banach lattice with an order-continuous norm, $T \in \mathsf{L}^\times(U;V)$ (355Kb).

(b) Since we can identify $\mathsf{L}^\times(U;V)$ with $\mathsf{L}^\times(U;V^{\times\times})$, we have an adjoint operator $T' \in \mathsf{L}^\times(V^\times;U^\times)$, as in 376K. Now if $g \ge 0$ in $V^\times$ and $\langle g_n\rangle_{n\in\mathbb{N}}$ is a sequence in $[0,g]$ such that $\lim_{n\to\infty} g_n(v) = 0$ for every $v \in V$, $\langle T'g_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $U^\times$. **P** For any $\epsilon > 0$, there is a $v^* \in V^+$ such that $g(|Tu| - v^*)^+ \le \epsilon$ whenever $\|u\| \le 1$; consequently

$$\|T'g_n\| = \sup_{\|u\|\le 1} (T'g_n)(u) = \sup_{\|u\|\le 1} g_n(Tu)$$
$$\le g_n(v^*) + \sup_{\|u\|\le 1} g_n(|Tu| - v^*)^+$$
$$\le g_n(v^*) + \sup_{\|u\|\le 1} g(|Tu| - v^*)^+ \le g_n(v^*) + \epsilon$$

for every $n \in \mathbb{N}$. As $\lim_{n\to\infty} g_n(v^*) = 0$, $\limsup_{n\to\infty} \|T'g_n\| \le \epsilon$; as ϵ is arbitrary, $\langle\|T'g_n\|\rangle_{n\in\mathbb{N}} \to 0$. But as $U^\times$ is an M-space (356Pb), it follows that $\langle T'g_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0. **Q**

By 368Pc, $U^\times$ is weakly (σ,∞)-distributive. By 376H, T' is an abstract integral operator, so T also is, by 376K.

376Q Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces and $T : L^1(\mu) \to L^1(\nu)$ a weakly compact linear operator. Then there is a function $k : X \times Y \to \mathbb{R}$ such that $Tf^\bullet = g_f^\bullet$, where $g_f(y) = \int k(x,y)f(x)dx$ almost everywhere, for every $f \in \mathcal{L}^1(\mu)$.

proof This follows from 376P and 376J, just as in 376N.

376R So far I have mentioned actual kernel functions $k(x,y)$ only as a way of giving slightly more concrete form to the abstract kernels of 376E. But of course they can provide new structures and insights. I give one result as an example. The following lemma is useful.

Lemma Let (X,Σ,μ) be a measure space, (Y,T,ν) a σ-finite measure space, and λ the c.l.d. product measure on $X \times Y$. Suppose that k is a λ-integrable real-valued function. Then for any $\epsilon > 0$ there is a finite partition $E_0,\dots,E_n$ of X into measurable sets such that $\|k - k_1\|_1 \le \epsilon$, where

$$k_1(x,y) = \frac{1}{\mu E_i}\int_{E_i} k(t,y)dt \text{ whenever } x \in E_i,\ 0 < \mu E_i < \infty$$
$$\text{and the integral is defined in } \mathbb{R},$$
$$= 0 \text{ in all other cases.}$$

proof Once again I refer to the proof of 253F: there are sets $H_0,\dots,H_r$ of finite measure in X, sets $F_0,\dots,F_r$ of finite measure in Y, and $\alpha_0,\dots,\alpha_r$ such that $\|k - k_2\|_1 \le \frac{1}{2}\epsilon$, where $k_2 = \sum_{j=0}^{r} \alpha_i\chi(H_j \times F_j)$. Let $E_0,\dots,E_n$ be the partition of X generated by $\{H_i : i \le r\}$. Then for any $i \le n$, $\int_{E_i\times Y} |k - k_1|$ is defined and is at most $2\int_{E_i\times Y} |k - k_2|$. **P** If $\mu E_i = 0$, this is trivial, as both are zero. If $\mu E_i = \infty$, then again the result is elementary, since both k_1 and k_2 are zero on $E_i \times Y$. So let us suppose that $0 < \mu E_i < \infty$. In this case $\int_{E_i} k(t,y)dt$ must be defined for almost every y, by Fubini's theorem. So k_1 is defined almost everywhere in $E_i \times Y$, and

$$\int_{E_i\times Y} |k - k_1| = \int_Y \int_{E_i} |k(x,y) - k_1(x,y)|dxdy.$$

Now take some fixed $y \in Y$ such that

$$\beta = \frac{1}{\mu E_i} \int_{E_i} k(t,y)dt$$

is defined. Then $\beta = k_1(x,y)$ for every $x \in E_i$. For every $x \in E_i$, we must have $k_2(x,y) = \alpha$ where $\alpha = \sum\{\alpha_j : E_i \subseteq H_j,\, y \in F_j\}$. But in this case, because $\int_{E_i} k(x,y) - \beta\, dx = 0$, we have

$$\int_{E_i} \max(0, k(x,y) - \beta)dx = \int_{E_i} \max(0, \beta - k(x,y))dx = \frac{1}{2}\int_{E_i} |k(x,y) - k_1(x,y)|dx.$$

If $\beta \ge \alpha$,

$$\int_{E_i} \max(0, k(x,y) - \beta)dx \le \int_{E_i} \max(0, k(x,y) - \alpha)dx \le \int_{E_i} |k(x,y) - k_2(x,y)|dx;$$

if $\beta \le \alpha$,

$$\int_{E_i} \max(0, \beta - k(x,y))dx \le \int_{E_i} \max(0, \alpha - k(x,y))dx \le \int_{E_i} |k(x,y) - k_2(x,y)|dx;$$

in either case,

$$\frac{1}{2}\int_{E_i} |k(x,y) - k_1(x,y)|dx \le \int_{E_i} |k(x,y) - k_2(x,y)|dx.$$

This is true for almost every y, so integrating with respect to y we get the result. **Q**

Now, summing over i, we get

$$\int |k - k_1| \le 2\int |k - k_2| \le \epsilon,$$

as required.

376S Theorem Let (X, Σ, μ) be a complete locally determined measure space, (Y, T, ν) a σ-finite measure space, and λ the c.l.d. product measure on $X \times Y$. Let τ be an extended Fatou norm on $L^0(\nu)$ and write $\mathcal{L}^{\tau'}$ for $\{g : g \in \mathcal{L}^0(\nu),\, \tau'(g^\bullet) < \infty\}$, where τ' is the associate extended Fatou norm of τ (369H-369I). Suppose that $k \in \mathcal{L}^0(\lambda)$ is such that $k \times (f \otimes g)$ is integrable whenever $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^{\tau'}$. Then we have a corresponding linear operator $T : L^1(\mu) \to L^\tau$ defined by saying that $\int (Tf^\bullet) \times g^\bullet = \int k \times (f \otimes g)$ whenever $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^{\tau'}$.

For $x \in X$ set $k_x(y) = k(x,y)$ whenever this is defined. Then $k_x \in \mathcal{L}^0(\nu)$ for almost every x; set $v_x = k_x^\bullet \in L^0(\nu)$ for such x. In this case $x \mapsto \tau(v_x)$ is measurable and defined and finite almost everywhere, and $\|T\| = \operatorname{ess\,sup}_x \tau(v_x)$.

Remarks The discussion of extended Fatou norms in §369 regarded them as functionals on spaces of the form $L^0(\mathfrak{A})$. I trust that no-one will be offended if I now speak of an extended Fatou norm on $L^0(\nu)$, with the associated function spaces L^τ, $L^{\tau'} \subseteq L^0$, taking for granted the identification in 364Ic.

Recall that $(f \otimes g)(x,y) = f(x)g(y)$ for $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} g$ (253B).

By '$\operatorname{ess\,sup}_x \tau(v_x)$' I mean

$$\inf\{M : M \ge 0,\, \{x : v_x \text{ is defined and } \tau(v_x) \le M\} \text{ is conegligible}\}$$

(see 243D).

proof (a) To see that the formula $(f,g) \mapsto \int k \times (f \otimes g)$ gives rise to an operator in $\mathrm{L}^\times(U; (L^{\tau'})^\times)$, it is perhaps quickest to repeat the argument of parts (a) and (b) of the proof of 376E. (We are not quite in a position to quote 376E, as stated, because the localizable measure algebra free product there might be strictly larger than the measure algebra of λ; see 325B.) The first step, of course, is to note that changing f or g on a negligible set does not affect the integral $\int k \times (f \otimes g)$, so that we have a bilinear functional on $L^1 \times L^{\tau'}$; and the other essential element is the fact that the maps $f^\bullet \mapsto (f \otimes \chi Y)^\bullet$, $g^\bullet \mapsto (\chi X \otimes g)^\bullet$ are order-continuous (put 325A and 364Pc together).

By 369K, we can identify $(L^{\tau'})^\times$ with L^τ, so that T becomes an operator in $\mathrm{L}^\times(U; L^\tau)$. Note that it must be norm-bounded (355C).

(b) By 376I, there is a non-decreasing sequence $\langle Y_n \rangle_{n \in \mathbb{N}}$ of measurable sets in Y, covering Y, such that $\chi Y_n \in \mathcal{L}^{\tau'}$ for every n. Set $X_0 = \{x : x \in X,\, k_x \in \mathcal{L}^0(\nu)\}$. Then X_0 is conegligible in X. **P** Let $E \in \Sigma$ be any set of finite measure. Then for any $n \in \mathbb{N}$, $k \times (\chi E \otimes \chi Y_n)$ is integrable, that is, $\int_{E \times Y_n} k$ is defined and finite; so by Fubini's theorem $\int_{Y_n} k_x$ is defined and finite for almost every $x \in E$. Consequently, for almost every $x \in E$, $k_x \times \chi Y_n \in \mathcal{L}^0(\nu)$ for every $n \in \mathbb{N}$, that is, $k_x \in \mathcal{L}^0(\nu)$, that is, $x \in X_0$.

Thus $E \setminus X_0$ is negligible for every set E of finite measure. Because μ is complete and locally determined, X_0 is conegligible. **Q**

This means that v_x and $\tau(v_x)$ are defined for almost every x.

(c) $\tau(v_x) \le \|T\|$ for almost every x. **P** Take any $E \in \Sigma$ of finite measure, and $n \in \mathbb{N}$. Then $k \times \chi(E \times Y_n)$ is integrable. For each $r \in \mathbb{N}$, there is a finite partition $E_{r0}, \dots , E_{r,m(r)}$ of E into measurable sets such that $\int_{E \times Y_n} |k - k^{(r)}| \le 2^{-r}$, where

$$\begin{aligned} k^{(r)}(x, y) &= \frac{1}{\mu E_{ri}} \int_{E_{ri}} k(t, y)dt \text{ whenever } y \in Y_n,\ x \in E_{ri},\ \mu E_{ri} > 0 \\ &\qquad\qquad\qquad\qquad\qquad \text{and the integral is defined in } \mathbb{R} \\ &= 0 \text{ otherwise} \end{aligned}$$

(376R). Now $k^{(r)}$ also is integrable over $E \times Y_n$, so $k^{(r)}_x \in \mathcal{L}^0(\nu)$ for almost every $x \in E$, writing $k^{(r)}_x(y) = k^{(r)}(x, y)$, and we can speak of $v^{(r)}_x = (k^{(r)}_x)^\bullet$ for almost every x. Note that $k^{(r)}_x = k^{(r)}_{x'}$ whenever x, x' belong to the same E_{ri}.

If $\mu E_{ri} > 0$, then $v^{(r)}_x$ must be defined for every $x \in E_{ri}$. If $v' \in L^{\tau'}$ is represented by $g \in \mathcal{L}^{\tau'}$ then

$$\begin{aligned} \int k \times (\chi E_{ri} \otimes (g \times \chi Y_n)) &= \int_{E_{ri} \times Y_n} k(t, y)g(y)d(t, y) \\ &= \mu E_{ri} \int k^{(r)}(x, y)g(y)dy = \mu E_{ri} \int v^{(r)}_x \times v' \end{aligned}$$

for any $x \in E_{ri}$. But this means that

$$\mu E_{ri} \int v^{(r)}_x \times v' = \int T(\chi E^\bullet_{ri}) \times v' \times \chi Y^\bullet_n$$

for every $v' \in L^{\tau'}$, so

$$v^{(r)}_x = \frac{1}{\mu E_{ri}} T(\chi E^\bullet_{ri}) \times \chi Y^\bullet_n, \quad \tau(v^{(r)}_x) \le \frac{1}{\mu E_{ri}} \|T\| \|\chi E^\bullet_{ri}\|_1 = \|T\|$$

for every $x \in E_{ri}$. This is true whenever $\mu E_{ri} > 0$, so in fact $\tau(v^{(r)}_x) \le \|T\|$ for almost every $x \in E$.

Because $\sum_{r\in\mathbb{N}} \int_{E\times Y_n} |k - k^{(r)}| < \infty$, we must have $k(x, y) = \lim_{r\to\infty} k^{(r)}(x, y)$ for almost every $(x, y) \in E \times Y_n$. Consequently, for almost every $x \in E$, $k(x, y) = \lim_{r\to\infty} k^{(r)}(x, y)$ for almost every $y \in Y_n$, that is, $\langle v^{(r)}_x \rangle_{r\in\mathbb{N}}$ order*-converges to $v_x \times \chi Y^\bullet_n$ (in $L^0(\nu)$) for almost every $x \in E$. But this means that, for almost every $x \in E$,

$$\tau(v_x \times \chi Y^\bullet_n) \le \liminf_{r\to\infty} \tau(v^{(r)}_x) \le \|T\|$$

(369Mc). Now

$$\tau(v_x) = \lim_{n\to\infty} \tau(v_x \times \chi Y^\bullet_n) \le \|T\|$$

for almost every $x \in E$.

As in (b), this implies (since E is arbitrary) that $\tau(v_x) \le \|T\|$ for almost every $x \in X$. **Q**

(d) I now show that $x \mapsto \tau(v_x)$ is measurable. **P** Take $\gamma \in [0, \infty[$ and set $A = \{x : x \in X_0,\ \tau(v_x) \le \gamma\}$. Suppose that $\mu E < \infty$. Let G be a measurable envelope of $A \cap E$ (132Ee). Set $\tilde{k}(x, y) = k(x, y)$ when $x \in G$ and $(x, y) \in \operatorname{dom} k$, 0 otherwise. If $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^{\tau'}$, then

$$\int \tilde{k}(x, y)f(x)g(y)d(x, y) = \int_{G\times Y} k(x, y)f(x)g(y)d(x, y) = \int_G f(x)\int_Y k(x, y)g(y)dydx$$

is defined.

Take any $g \in \mathcal{L}^{\tau'}$. For $x \in X_0$, set $h(x) = \int |\tilde{k}(x, y)g(y)|dy$. Then h is finite almost everywhere and measurable. For $x \in A \cap E$,

$$\int|\tilde{k}(x,y)g(y)|dy=\int|v_x\times g^\bullet|\le\gamma\tau'(g^\bullet).$$

So the measurable set $G'=\{x:h(x)\le\gamma\tau'(g^\bullet)\}$ includes $A\cap E$, and $\mu(G\setminus G')=0$. Consequently

$$|\int\tilde{k}(x,y)f(x)g(y)d(x,y)|\le\int_G|f(x)|h(x)dx\le\gamma\|f\|_1\tau'(g^\bullet),$$

and this is true whenever $f\in\mathcal{L}^1(\mu)$.

Now we have an operator $\tilde{T}:L^1(\mu)\to L^\tau$ defined by the formula

$$\int(\tilde{T}f^\bullet)\times g^\bullet=\int\tilde{k}\times(f\otimes g)\text{ when }f\in\mathcal{L}^1(\nu)\text{ and }g\in\mathcal{L}^{\tau'},$$

and the formula just above tells us that $|\int\tilde{T}u\times v'|\le\gamma\|u\|_1\tau'(v')$ for every $u\in L^1(\nu)$ and $v'\in L^{\tau'}$; that is, $\tau(\tilde{T}u)\le\gamma\|u\|_1$ for every $u\in L^1(\mu)$; that is, $\|\tilde{T}\|\le\gamma$. But now (c) tells us that $\tau(\tilde{v}_x)\le\gamma$ for almost every $x\in X$, where $\tilde{v}_x$ is the equivalence class of $y\mapsto\tilde{k}(x,y)$, that is, $\tilde{v}_x=v_x$ for $x\in G\cap X_0$, 0 for $x\in X\setminus G$. So $\tau(v_x)\le\gamma$ for almost every $x\in G$, and $G\setminus A$ is negligible. But this means that $A\cap E$ is measurable. As E is arbitrary, A is measurable; as γ is arbitrary, $x\mapsto\tau(v_x)$ is measurable. **Q**

(e) Finally, the ideas in (d) show that $\|T\|\le\operatorname{ess\,sup}_x\tau(v_x)$. **P** Set $M=\operatorname{ess\,sup}_x\tau(v_x)$. If $f\in\mathcal{L}^1(\mu)$ and $g\in\mathcal{L}^{\tau'}$, then

$$\int|k(x,y)f(x)g(y)|d(x,y)\le\int|f(x)|\tau(v_x)\tau'(g^\bullet)dx\le M\|f\|_1\tau'(g^\bullet);$$

as g is arbitrary, $\tau(Tf^\bullet)\le M\|f\|_1$; as f is arbitrary, $\|T\|\le M$. **Q**

376X Basic exercises >(a) Let μ be Lebesgue measure on $\mathbb{R}$. Let h be a μ-integrable real-valued function with $\|h\|_1\le1$, and set $k(x,y)=h(y-x)$ whenever this is defined. Show that if f is in either $\mathcal{L}^1(\mu)$ or $\mathcal{L}^\infty(\mu)$ then $g(y)=\int k(x,y)f(x)dx$ is defined for almost every $y\in\mathbb{R}$, and that this formula gives rise to an operator $T\in\mathcal{T}^\times_{\bar{\mu},\bar{\mu}}$ as defined in 373Ab. (*Hint*: 255H.)

(b) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras with localizable measure algebra free product $(\mathfrak{C},\bar{\lambda})$, and take $p\in[1,\infty]$. Show that if $u\in L^p(\mathfrak{A},\bar{\mu})$ and $v\in L^p(\mathfrak{B},\bar{\nu})$ then $u\otimes v\in L^p(\mathfrak{C},\bar{\lambda})$ and $\|u\otimes v\|_p=\|u\|_p\|v\|_p$.

>(c) Let U, V, W be Riesz spaces, of which V and W are Dedekind complete, and suppose that $T\in\mathsf{L}^\times(U;V)$ and $S\in\mathsf{L}^\times(V;W)$. Show that if either S or T is an abstract integral operator, so is ST.

(d) Let h be a Lebesgue integrable function on $\mathbb{R}$, and f a square-integrable function. Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable functions such that (α) $|f_n|\le f$ for every n (β) $\lim_{n\to\infty}\int_E f_n=0$ for every measurable set E of finite measure. Show that $\lim_{n\to\infty}(h*f_n)(y)=0$ for almost every $y\in\mathbb{R}$, where $h*f_n$ is the convolution of h and f_n. (*Hint*: 376Xa, 376H.)

(e) Let U and V be Riesz spaces, of which V is Dedekind complete. Suppose that $W\subseteq U^\sim$ is a solid linear subspace, and that T belongs to the band in $\mathsf{L}^\sim(U;V)$ generated by operators of the form $u\mapsto f(u)v$, where $f\in W$ and $v\in V$. Show that whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U such that $\lim_{n\to\infty}f(u_n)=0$ for every $f\in W$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V.

(f) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and $U\subseteq L^0=L^0(\mathfrak{A})$ an order-dense Riesz subspace such that $U^\times$ separates the points of U. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be an order-bounded sequence in U. Show that the following are equiveridical: (i) $\lim_{n\to\infty}f(|u_n|)=0$ for every $f\in U^\times$; (ii) $\langle u_n\rangle_{n\in\mathbb{N}}\to0$ for the topology of convergence in measure on L^0. (*Hint*: by 367T, condition (ii) is intrinsic to U, so we can replace $(\mathfrak{A},\bar{\mu})$ by a localizable algebra and use the representation in 369D.)

(g) Let U be a Banach lattice with an order-continuous norm, and V a weakly (σ,∞)-distributive Riesz space. Show that for $T\in\mathsf{L}^\sim(U;V)$ the following are equiveridical: (i) T belongs to the band in $\mathsf{L}^\sim(U;V)$ generated by operators of the form $u\mapsto f(u)v$ where $f\in U^\sim$, $v\in V$; (ii) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U^+ which is norm-convergent to 0; (iii) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in U which is weakly convergent to 0.

(h) Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, with product measure λ on $X\times Y$, and measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$. Suppose that $k\in\mathcal{L}^0(\lambda)$. Show that the following are equiveridical: (i)(α) if $f\in\mathcal{L}^1(\mu)$ then $g_f(y)=\int k(x,y)f(x)dx$ is defined for almost every y and $g_f\in\mathcal{L}^1(\nu)$ (β) there is an operator $T\in\mathcal{T}^{\times}_{\bar\mu,\bar\nu}$ defined by setting $Tf^\bullet=g_f^\bullet$ for every $f\in\mathcal{L}^1(\mu)$; (ii) $\int|k(x,y)|dy\le 1$ for almost every $x\in X$, $\int|k(x,y)|dx\le 1$ for almost every $y\in Y$.

>(i)(i) Show that there is a compact linear operator from ℓ^2 to itself which is not in $\mathsf{L}^\sim(\ell^2;\ell^2)$. (*Hint*: start from the operator S of 371Ye.) (ii) Show that the identity operator on ℓ^2 is an abstract integral operator.

>(j) Let μ be Lebesgue measure on $[0,1]$. (i) Give an example of a measurable function $k:[0,1]^2\to\mathbb{R}$ such that, for any $f\in\mathcal{L}^2(\mu)$, $g_f(y)=\int k(x,y)f(x)dx$ is defined for every y and $\|g_f\|_2=\|f\|_2$, but k is not integrable, so the linear isometry on $L^2=L^2(\mu)$ defined by k does not belong to $\mathsf{L}^\sim(L^2;L^2)$. (ii) Show that the identity operator on L^2 is not an abstract integral operator.

(k) Let (X,Σ,μ) be a σ-finite measure space and (Y,T,ν) a complete locally determined measure space. Let $U\subseteq L^0(\mu)$, $V\subseteq L^0(\nu)$ be solid linear subspaces, of which V is order-dense; write $V^\#=\{v:v\in L^0(\nu),\ v\times v'\text{ is integrable for every }v'\in V\}$, $\mathcal{U}=\{f:f\in\mathcal{L}^0(\nu),\ f^\bullet\in U\}$, $\mathcal{V}=\{g:g\in\mathcal{L}^0(\nu),\ g^\bullet\in V\}$, $\mathcal{V}^\#=\{h:h\in\mathcal{L}^0(\nu),\ h^\bullet\in V^\#\}$. Let λ be the c.l.d. product measure on $X\times Y$, and $k\in\mathcal{L}^0(\lambda)$ a function such that $k\times(f\otimes g)$ is integrable for whenever $f\in\mathcal{U}$ and $g\in\mathcal{V}$. (i) Show that for any $f\in\mathcal{U}$, $h_f(y)=\int k(x,y)f(x)dx$ is defined for almost every $y\in Y$, and that $h_f\in\mathcal{V}^\#$. (ii) Show that we have a map $T\in\mathsf{L}^\times(U;V^\#)$ defined *either* by writing $Tf^\bullet=h_f^\bullet$ for every $f\in\mathcal{U}$ *or* by writing $\int(Tf^\bullet)\times g^\bullet=\int k\times(f\otimes g)$ for every $f\in\mathcal{U}$ and $g\in\mathcal{V}$.

(l) Let (X,Σ,μ), (Y,T,ν) and (Z,Λ,λ) be σ-finite measure spaces, and U, V, W perfect order-dense solid linear subspaces of $L^0(\mu)$, $L^0(\nu)$ and $L^0(\lambda)$ respectively. Suppose that $T:U\to V$ and $S:V\to W$ are abstract integral operators corresponding to kernels $k_1\in\mathcal{L}^0(\mu\times\nu)$, $k_2\in\mathcal{L}^0(\nu\times\lambda)$, writing $\mu\times\nu$ for the (c.l.d. or primitive) product measure on $X\times Y$. Show that $ST:U\to W$ is represented by the kernel $k\in\mathcal{L}^0(\mu\times\lambda)$ defined by setting $k(x,z)=\int k_1(x,y)k_2(y,z)dy$ whenever this integral is defined.

(m) Let U be a perfect Riesz space. Show that a set $C\subseteq U$ is relatively compact for $\mathfrak{T}_s(U,U^\times)$ iff for every $g\in(U^\times)^+$, $\epsilon>0$ there is a $u^*\in U$ such that $g(|u|-u^*)^+\le\epsilon$ for every $u\in C$. (*Hint*: 376O and the proof of 356Q.)

>(n) Let μ be Lebesgue measure on $[0,1]$, and ν counting measure on $[0,1]$. Set $k(x,y)=1$ if $x=y$, 0 otherwise. Show that 376S fails in this context (with, e.g., $\tau=\|\;\|_\infty$).

(o) Suppose, in 376Xk, that $U=L^\tau$ for some extended Fatou norm on $L^0(\mu)$ and that $V=L^1(\nu)$, so that $V^\#=L^\infty(\nu)$. Set $k_y(x)=k(x,y)$ whenever this is defined, $w_y=k_y^\bullet$ whenever $k_y\in\mathcal{L}^0(\mu)$. Show that $w_y\in L^{\tau'}$ for almost every $y\in Y$, and that the norm of T in $\mathsf{B}(L^\tau;L^\infty)$ is $\operatorname{ess\,sup}_y\tau'(w_y)$. (*Hint*: do the case of totally finite Y first.)

376Y Further exercises (a) Let U, V and W be linear spaces (over any field F) and $\phi:U\times V\to W$ a bilinear operator. Let W_0 be the linear subspace of W generated by $\phi[U\times V]$. Show that the following are equiveridical: (i) for every linear space Z over F and every bilinear $\psi:U\times V\to Z$, there is a (unique) linear operator $T:W_0\to Z$ such that $T\phi=\psi$ (ii) whenever $u_0,\dots,u_n\in U$ are linearly independent and $v_0,\dots,v_n\in V$ are non-zero, $\sum_{i=0}^n\phi(u_i,v_i)\ne 0$ (iii) whenever $u_0,\dots,u_n\in U$ are non-zero and $v_0,\dots,v_n\in V$ are linearly independent, $\sum_{i=0}^n\phi(u_i,v_i)\ne 0$ (iv) for any Hamel bases $\langle u_i\rangle_{i\in I}$, $\langle v_j\rangle_{j\in J}$ of U and V, $\langle\phi(u_i,v_j)\rangle_{i\in I,j\in J}$ is a Hamel basis of W_0 (v) for some pair $\langle u_i\rangle_{i\in I}$, $\langle v_j\rangle_{j\in J}$ of Hamel bases of U and V, $\langle\phi(u_i,v_j)\rangle_{i\in I,j\in J}$ is a Hamel basis of W_0.

(b) Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $(\mathfrak{C},\bar\lambda)$ their localizable measure algebra free product. Show that $\otimes:L^0(\mathfrak{A})\times L^0(\mathfrak{B})\to L^0(\mathfrak{C})$ satisfies the equivalent conditions of 376Ya.

(c) Let (X,Σ,μ) and (Y,T,ν) be semi-finite measure spaces and λ the c.l.d. product measure on $X\times Y$. Show that the map $(f,g)\mapsto f\otimes g:\mathcal{L}^0(\mu)\times\mathcal{L}^0(\nu)\to\mathcal{L}^0(\lambda)$ induces a map $(u,v)\mapsto u\otimes v:L^0(\mu)\times L^0(\nu)\to L^0(\lambda)$ possessing all the properties described in 376B and 376Ya, subject to a suitable interpretation of the formula $\otimes:\mathfrak{A}\times\mathfrak{B}\to\mathfrak{C}$.

(d) Let $(\mathfrak{B}_{\omega_1},\bar\nu_{\omega_1})$ be the measure algebra of $\{0,1\}^{\omega_1}$ with its usual measure, and $\langle a_\xi\rangle_{\xi<\omega_1}$ a stochastically independent (definition: 325Xf) family of elements of measure $\frac12$ in $\mathfrak{B}_{\omega_1}$. Set $U=L^2(\mathfrak{B}_{\omega_1},\bar\nu_{\omega_1})$ and $V=\{v:v\in\mathbb{R}^{\omega_1},\ \{\xi:v(\xi)\ne0\}$ is countable$\}$. Define $T:U\to\mathbb{R}^{\omega_1}$ by setting $Tu(\xi)=2\int_{a_\xi}u-\int u$ for $\xi<\omega_1$, $u\in U$. Show that (i) $Tu\in V$ for every $u\in U$ (ii) $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\lim_{n\to\infty}f(u_n)=0$ for every $f\in U^\times$ (iii) $T\notin\mathsf{L}^\sim(U;V)$.

(e) Let U be a Riesz space with the countable sup property (definition: 241Ye) such that $U^\times$ separates the points of U, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U. Show that the following are equiveridical: (i) $\lim_{n\to\infty}f(v\wedge|u_n|)=0$ for every $f\in U^\times$, $v\in U^+$; (ii) every subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence which is order*-convergent to 0.

(f) Let U be an Archimedean Riesz space and $\mathfrak{A}$ a weakly (σ,∞)-distributive Dedekind complete Boolean algebra. Suppose that $T:U\to L^0=L^0(\mathfrak{A})$ is a linear operator such that $\langle|Tu_n|\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in L^0 whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-bounded and order*-convergent to 0 in U. Show that $T\in\mathsf{L}^\sim_c(U;L^0)$ (definition: 355G), so that if U has the countable sup property then $T\in\mathsf{L}^\times(U;L^0)$.

(g) Suppose that (Y,T,ν) is a probability space in which $\mathrm{T}=\mathcal{P}Y$ and $\nu\{y\}=0$ for every $y\in Y$. (See 363S.) Take $X=Y$ and let μ be counting measure on X; let λ be the c.l.d. product measure on $X\times Y$, and set $k(x,y)=1$ if $x=y$, 0 otherwise. Show that we have an operator $T:L^\infty(\mu)\to L^\infty(\nu)$ defined by setting $Tf=g^\bullet$ whenever $f\in L^\infty(\mu)\cong\ell^\infty(X)$ and $g(y)=\int k(x,y)f(x)dx=f(y)$ for every $y\in Y$. Show that T satisfies the conditions (ii) and (iii) of 376J but does not belong to $\mathsf{L}^\times(L^\infty(\mu);L^\infty(\nu))$.

(h) Give an example of an abstract integral operator $T:\ell^2\to L^1(\mu)$, where μ is Lebesgue measure on $[0,1]$, such that $\langle Te_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent in $L^1(\mu)$, where $\langle e_n\rangle_{n\in\mathbb{N}}$ is the standard orthonormal sequence in ℓ^2.

(i) Set $k(m,n)=1/\pi(n-m+\frac12)$ for m, $n\in\mathbb{Z}$. (i) Show that $\sum_{n=-\infty}^\infty k(m,n)^2=1$ and $\sum_{n=-\infty}^\infty k(m,n)k(m',n)=0$ for all distinct m, $m'\in\mathbb{Z}$. (*Hint*: find the Fourier series of $x\mapsto e^{i(m+\frac12)x}$ and use 282K.) (ii) Show that there is a norm-preserving linear operator T from $\ell^2=\ell^2(\mathbb{Z})$ to itself given by the formula $(Tu)(n)=\sum_{m=-\infty}^\infty k(m,n)u(m)$. (iii) Show that T^2 is the identity operator on ℓ^2. (iv) Show that $T\notin\mathsf{L}^\sim(\ell^2;\ell^2)$. (*Hint*: consider $\sum_{m,n=-\infty}^\infty|k(m,n)|x(m)x(n)$ where $x(n)=1/\sqrt{|n|}\ln|n|$ for $|n|\ge2$.) (T is a form of the **Hilbert transform.**)

(j) Let U be an L-space and V a Banach lattice with an order-continuous norm. Let $T\in\mathsf{L}^\sim(U;V)$. Show that the following are equiveridical: (i) T is an abstract integral operator; (ii) $T[C]$ is norm-compact in V whenever C is weakly compact in U. (*Hint*: start with the case in which C is order-bounded, and remember that it is weakly sequentially compact.)

(k) Let (X,Σ,μ) be a complete locally determined measure space and (Y,T,ν), (Z,Λ,λ) two σ-finite measure spaces. Suppose that τ, θ are extended Fatou norms on $L^0(\nu)$, $L^0(\lambda)$ respectively, and that $T:L^1(\mu)\to L^\tau$ is an abstract integral operator, with corresponding kernel $k\in\mathcal{L}^0(\mu\times\nu)$, while $S\in\mathsf{L}^\times(L^\tau;L^\theta)$, so that $ST:L^1(\mu)\to L^\theta$ is an abstract integral operator (376Xc); let $\tilde k\in\mathcal{L}^0(\mu\times\lambda)$ be the corresponding kernel. For $x\in X$ set $v_x=k_x^\bullet$ when this is defined in L^τ, as in 376S, and similarly take $w_x=\tilde k_x^\bullet\in L^\theta$. Show that $Sv_x=w_x$ for almost every $x\in X$.

376 Notes and comments I leave 376Yb to the exercises because I do not rely on it for any of the work here, but of course it is an essential aspect of the map $\otimes:L^0(\mathfrak{A})\times L^0(\mathfrak{B})\to L^0(\mathfrak{C})$ I discuss in this section. The conditions in 376Ya are characterizations of the 'tensor product' of two linear spaces, a construction of great importance in abstract linear algebra (and, indeed, in modern applied linear algebra; it is by no means trivial even in the finite-dimensional case). In particular, note that conditions (ii), (iii) of 376Ya apply to arbitrary subspaces of U and V if they apply to U and V themselves.

The principal ideas used in 376B-376C have already been set out in §§253 and 325. Here I do little more than list the references. I remark however that it is quite striking that $L^1(\mathfrak{C},\bar\lambda)$ should have no fewer than three universal mapping theorems attached to it (376Cb, 376C(c-i) and 376C(c-ii)).

The real work of this section begins in 376E. As usual, much of the proof is taken up with relatively straightforward verifications, as in parts (a) and (b), while part (i) is just a manoeuvre to show that it doesn't matter if $\mathfrak{A}$ and $\mathfrak{B}$ aren't Dedekind complete, because $\mathfrak{C}$ is. But I think that parts (d), (f) and (j) have ideas in them. In particular, part (f) is a kind of application of the Radon-Nikodým theorem (through the identification of $L^1(\mathfrak{C},\bar{\lambda})^*$ with $L^\infty(\mathfrak{C})$).

I have split 376E from 376H because the former demands the language of measure algebras, while the latter can be put into the language of pure Riesz space theory. Asking for a weakly (σ,∞)-distributive space V in 376H is a way of applying the ideas to $V = L^0$ as well as to Banach function spaces. (When $V = L^0$, indeed, variations on the hypotheses are possible, using 376Yf.) But it is a reminder of one of the directions in which it is often possible to find generalizations of ideas beginning in measure theory.

The condition '$\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$' (376H(ii)) seems natural in this context, and gives marginally greater generality than some alternatives (because it does the right thing when $U^\times$ does not separate the points of U), but it is not the only way of expressing the idea; see 376Xf and 376Ye. Note that the conditions (ii) and (iii) of 376H are significantly different. In 376H(iii) we could easily have $|u_n| = u^*$ for every n; for instance, if $u_n = 2\chi a_n - \chi 1$ for some stochastically independent sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of elements of measure $\frac{1}{2}$ in a probability algebra (272Ye).

If you have studied compact linear operators between Banach spaces (definition: 3A5La), you will have encountered the condition '$Tu_n \to 0$ strongly whenever $u_n \to 0$ weakly'. The conditions in 376H and 376J are of this type. If a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in a Riesz space U is order-bounded and order*-convergent to 0, then $\lim_{n\to\infty} f(u_n) = 0$ for every $f \in U^\times$ (367Xg). Visibly this latter condition is associated with weak convergence, and order*-convergence is (in Banach lattices) closely related to norm convergence (367D). In the context of 376H, an abstract integral operator is one which transforms convergent sequences of a weak type into convergent sequences of a stronger type. The relationship between the classes of (weakly) compact operators and abstract integral operators is interesting, but outside the scope of this book; I leave you with 376P-376Q and 376Y, and a pair of elementary examples to guard against extravagant conjecture (376Xi).

376O belongs to an extensive general theory of weak compactness in perfect Riesz spaces, based on adaptations of the concept of 'uniform integrability'. I give the next step in 376Xm. For more information see Fremlin 74a, chap. 8.

Note that 376Mb and 376P overlap when $V^\times$ in 376Mb is reflexive – for instance, when V is an L^p space for some $p \in]1,\infty[$ – since then every bounded linear operator from L^1 to $V^\times$ must be weakly compact. For more information on the representation of operators see Dunford & Schwartz 57, particularly Table VI in the notes to Chapter VI.

As soon as we leave formulations in terms of the spaces $L^0(\mathfrak{A})$ and their subspaces, and return to the original conception of a kernel operator in terms of integrating functions against sections of a kernel, we are necessarily involved in the pathology of Fubini's theorem for general measure spaces. In general, the repeated integrals $\iint k(x,y)dxdy$, $\iint k(x,y)dydx$ need not be equal, and something has to give (376Xn). Of course this particular worry disappears if the spaces are σ-finite, as in 376J. In 376S I take the trouble to offer a more general condition, mostly as a reminder that the techniques developed in Volume 2 do enable us sometimes to go beyond the σ-finite case. Note that this is one of the many contexts in which anything we can prove about probability spaces will be true of all σ-finite spaces; but that we cannot make the next step, to all strictly localizable spaces.

376S verges on the theory of integration of vector-valued functions, which I don't wish to enter here; but it also seems to have a natural place in the context of this chapter. It is of course a special property of L^1 spaces. The formula $\|T_k\| = \operatorname{ess\,sup}_x \tau(k_x^\bullet)$ shows that $\|T_{|k|}\| = \|T_k\|$; now we know from 376E that $T_{|k|} = |T_k|$, so we get a special case of the Chacon-Krengel theorem (371D). Reversing the roles of X and Y, we find ourselves with an operator from L^τ to L^∞ (376Xo), which is the other standard context in which $\|T\| = \||T|\|$ (371Xd). I include two exercises on L^2 spaces (376Xj, 376Yi) designed to emphasize the fact that $\mathsf{B}(U;V)$ is included in $\mathsf{L}^\sim(U;V)$ only in very special cases.

The history of the theory here is even more confusing than that of mathematics in general, because so many of the ideas were developed in national schools in very imperfect contact with each other. My own account gives no hint of how this material arose; I ought in particular to note that 376N is one of the oldest results, coming (essentially) from Dunford 1936. For further references, see Zaanen 83, chap. 13.

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*377 Function spaces of reduced products

In §328 I introduced 'reduced products' of probability algebras. In this section I seek to describe the function spaces of reduced products as images of subspaces of products of function spaces of the original algebras. I add a group of universal mapping theorems associated with the constructions of projective and inductive limits of directed families of probability algebras (377G-377H).

377A Proposition If $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a non-empty family of Boolean algebras with simple product $\mathfrak{A}$, then $L^\infty(\mathfrak{A})$ can be identified, as normed space and f-algebra, with the subspace W_∞ of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$ consisting of families $u=\langle u_i\rangle_{i\in I}$ such that $\|u\|_\infty=\sup_{i\in I}\|u_i\|_\infty$ is finite.

proof (a) I begin by noting that W_∞ is, in itself, an Archimedean f-algebra and $\|\,\|_\infty$ is a Riesz norm on W_∞. **P** W_∞ is a solid linear subspace of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$, so inherits a Riesz space structure (352K, 352Ia). Now it is easy to check that $e=\langle\chi 1_{\mathfrak{A}_i}\rangle_{i\in I}$ is an order unit in W_∞ and that $\|\,\|_\infty$ is the corresponding order-unit norm (354F-354G). Finally, because W_∞ is the solid linear subspace of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$ generated by e, and e is the multiplicative identity of $\prod_{i\in I}L^\infty(\mathfrak{A}_i)$, W_∞ is closed under multiplication, and is an f-algebra. **Q**

(b) We have a natural function $\theta:\mathfrak{A}\to W_\infty$ defined by saying that $\theta a=\langle\chi a_i\rangle_{i\in I}$ whenever $a=\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Clearly θ is additive and $\|\theta a\|_\infty\le 1$ for every $a\in\mathfrak{A}$; moreover, $\theta a\wedge\theta b=0$ when a, $b\in\mathfrak{A}$ are disjoint. By 363E, we have a corresponding Riesz homomorphism $T:L^\infty(\mathfrak{A})\to W_\infty$ of norm at most 1.

(c) In fact $\|Tw\|_\infty=\|w\|_\infty$ for every $w\in L^\infty(\mathfrak{A})$. **P** If $w=0$, this is trivial. If $w\in S(\mathfrak{A})\setminus\{0\}$, express it as $\sum_{k=0}^n\alpha_k\chi a^{(k)}$ where $\langle a^{(k)}\rangle_{k\le n}$ is a disjoint family of non-zero elements. Expressing each $a^{(k)}$ as $\langle a_{ki}\rangle_{i\in I}$,

$$Tw=\langle\textstyle\sum_{k=0}^n\alpha_k\chi a_{ki}\rangle_{i\in I}.$$

There must be a j such that $|\alpha_j|=\|w\|_\infty$; now there is an i such that $a_{ji}\ne 0$; as $\langle a_{ki}\rangle_{k\le n}$ is disjoint,

$$\|Tw\|_\infty\ge\|\textstyle\sum_{k=0}^n\alpha_k\chi a_{ki}\|_\infty\ge|\alpha_j|=\|w\|_\infty.$$

If now w is any member of $L^\infty(\mathfrak{A})$,

$$\begin{aligned}\|w\|_\infty&=\sup\{\|w'\|_\infty:w'\in S(\mathfrak{A}),\,|w'|\le|w|\}\\&=\sup\{\|Tw'\|_\infty:w'\in S(\mathfrak{A}),\,|w'|\le|w|\}\le\|Tw\|_\infty\end{aligned}$$

because T is a Riesz homomorphism. **Q**

Thus T is norm-preserving, therefore injective.

(d) Next, T is surjective. **P** Suppose that $\langle u_i\rangle_{i\in I}\in W_\infty^+$ is non-negative, and that $\epsilon>0$. Let $n\in\mathbb{N}$ be such that $n\epsilon\ge\sup_{i\in I}\|u_i\|_\infty$, and for $k\le n$, $i\in I$ set $a_{ki}=[\![u_i>k\epsilon]\!]$. Set $w=\epsilon\sum_{k=1}^n\chi(\langle a_{ki}\rangle_{i\in I})$. Then $w\in L^\infty(\mathfrak{A})$ and $Tw=\langle v_i\rangle_{i\in I}$, where $v_i=\epsilon\sum_{k=1}^n\chi a_{ki}$, so that $v_i\le u_i$ and $\|u_i-v_i\|_\infty\le\epsilon$, for every $i\in I$. Thus $\|Tw-\langle u_i\rangle_{i\in I}\|_\infty\le\epsilon$.

As $\langle u_i\rangle_{i\in I}$ and ϵ are arbitrary, $T[L^\infty(\mathfrak{A})]\cap W_\infty^+$ is norm-dense in W_∞^+. But T is an isometry and $L^\infty(\mathfrak{A})$ is norm-complete, so $T[L^\infty(\mathfrak{A})]$ is closed in W_∞ and includes W_∞^+ and therefore W_∞; that is, T is surjective. **Q**

So T is a norm-preserving bijective Riesz homomorphism, that is, a normed Riesz space isomorphism. Finally, by 353Qd or otherwise, T is multiplicative, so is an f-algebra isomorphism.

377B Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $(\mathfrak{B},\bar\nu)$ a probability algebra. Let $\mathfrak{A}$ be the simple product of $\langle\mathfrak{A}_i\rangle_{i\in I}$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})\le\sup_{i\in I}\bar\mu_i a_i$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Let W_0 be the subspace of $\prod_{i\in I}L^0(\mathfrak{A}_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\inf_{k\in\mathbb{N}}\sup_{i\in I}\bar\mu_i[\![|u_i|>k]\!]=0$.

(a) W_0 is a solid linear subspace and a subalgebra of $\prod_{i\in I}L^0(\mathfrak{A}_i)$, and there is a unique Riesz homomorphism $T:W_0\to L^0(\mathfrak{B})$ such that $T(\langle\chi a_i\rangle_{i\in I})=\chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$. Moreover, T is multiplicative, and $[\![Tu>0]\!]\subseteq\pi(\langle[\![u_i>0]\!]\rangle_{i\in I})$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

(b) If $h:\mathbb{R}\to\mathbb{R}$ is a continuous function, and we write $\bar{h}$ for the corresponding maps from L^0 to itself for any of the spaces $L^0=L^0(\mathfrak{A}_i)$, $L^0=L^0(\mathfrak{B})$ (364H), then $\langle\bar{h}(u_i)\rangle_{i\in I}\in W_0$ and $T(\langle\bar{h}(u_i)\rangle_{i\in I})=\bar{h}(Tu)$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_0.

proof (a) For $u=\langle u_i\rangle_{i\in I}\in\prod_{i\in I}L^0(\mathfrak{A}_i)$ and $k\in\mathbb{N}$, set $\gamma_k(u)=\sup_{i\in I}\bar{\mu}_i[\![|u_i|>k]\!]$.

(i) W_0 is a solid linear subspace and subalgebra of the f-algebra $\prod_{i\in I}L^0(\mathfrak{A}_i)$. **P** For $k\in\mathbb{N}$ and u, $v\in\prod_{i\in I}L^0(\mathfrak{A}_i)$,

$$\gamma_k(u)\le\gamma_k(v) \text{ whenever } |u|\le|v|,$$

$$\gamma_{2k}(u+v)\le\gamma_k(u)+\gamma_k(v),$$

$$\gamma_{k^2}(u\times v)\le\gamma_k(u)+\gamma_k(v)$$

for all u, $v\in\prod_{i\in I}L^0(\mathfrak{A}_i)$ and $k\in\mathbb{N}$. So W_0 is solid, is closed under addition, and is closed under multiplication. **Q**

(ii) Let $W_\infty\subseteq W_0$ be the set of families $\langle u_i\rangle_{i\in I}\in\prod_{i\in I}L^\infty(\mathfrak{A}_i)$ such that $\sup_{i\in I}\|u_i\|_\infty$ is finite; by 377A, we can identify W_∞ with $L^\infty(\mathfrak{A})$. We therefore have a corresponding multiplicative Riesz homomorphism $S:W_\infty\to L^\infty(\mathfrak{B})$ such that $S(\langle\chi a_i\rangle_{i\in I})=\chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}$ (363F); note that $S(\langle\chi 1_{\mathfrak{A}_i}\rangle_{i\in I})=\chi 1_{\mathfrak{B}}$.

(iii) If $u=\langle u_i\rangle_{i\in I}\in W_\infty$ and $k\in\mathbb{N}$, then $[\![Su>k]\!]\subseteq\pi(\langle[\![u_i>k]\!]\rangle_{i\in I})$. **P** Setting $a_i=[\![u_i>k]\!]$, we have $u_i\times\chi(1_{\mathfrak{A}_i}\setminus a_i)\le k\chi 1_{\mathfrak{A}_i}$ for every i. Set $a=\langle a_i\rangle_{i\in I}$. Since S is a multiplicative Riesz homomorphism,

$$\begin{aligned}Su\times\chi(1_{\mathfrak{B}}\setminus\pi a)&=Su\times\chi\pi(\langle 1_{\mathfrak{A}_i}\setminus a_i\rangle_{i\in I})=S(\langle u_i\rangle_{i\in I})\times S(\langle\chi(1_{\mathfrak{A}_i}\setminus a_i)\rangle_{i\in I})\\&=S(\langle u_i\rangle_{i\in I}\times\langle\chi(1_{\mathfrak{A}_i}\setminus a_i\rangle_{i\in I}))=S(\langle u_i\times\chi(1_{\mathfrak{A}_i}\setminus a_i)\rangle_{i\in I})\\&\le S(\langle k\chi 1_{\mathfrak{A}_i}\rangle_{i\in I})=k\chi 1_{\mathfrak{B}}\end{aligned}$$

and $[\![Su>k]\!]\subseteq\pi a$, as claimed. **Q**

(iv) If $u=\langle u_i\rangle_{i\in I}\in W_0^+$, then $\sup\{Sv:v\in W_\infty,\ 0\le v\le u\}$ is defined in $L^0(\mathfrak{B})$. **P** Set $A_u=S[W_\infty\cap[0,u]\,]$. Because $W_\infty\cap[0,u]$ is upwards-directed, so is A. If $v=\langle v_i\rangle_{i\in I}\in W_\infty\cap[0,u]$, then $[\![Sv>k]\!]\subseteq\pi(\langle[\![v_i>k]\!]\rangle_{i\in I})$, by (iii), so

$$\bar{\nu}[\![Sv>k]\!]\le\sup\nolimits_{i\in I}\bar{\mu}_i[\![v_i>k]\!]\le\gamma_k(u).$$

Thus $\bar{\nu}[\![w>k]\!]\le\gamma_k(u)$ for every $w\in A$. Since $u\in W_0$, $\lim_{k\to\infty}\gamma_k(u)=0$; so 364L(a-ii) tells us that $\sup A_u$ is defined in $L^0(\mathfrak{B})$. **Q**

By 355F, there is a Riesz homomorphism $T:W_0\to L^0(\mathfrak{B})$ extending S and such that $Tu=A_u$ for every $u\in W_0^+$. By 353Qd, T is multiplicative.

(v) Because T is multiplicative, we can repeat the calculations of (iii), with T in place of S, to see that

$$[\![Tu>k]\!]\subseteq\pi(\langle[\![u_i>k]\!]\rangle_{i\in I})$$

whenever $u=\langle u_i\rangle_{i\in I}\in W_0$; in particular, $[\![Tu>0]\!]\subseteq\pi(\langle[\![u_i>0]\!]\rangle_{i\in I})$.

(vi) To see that T is uniquely defined, let $T':W_0\to L^0(\mathfrak{B})$ be another Riesz homomorphism agreeing with T on families of the form $\langle\chi a_i\rangle_{i\in I}$. Then T and T' agree on $W_\infty\cong L^\infty(\mathfrak{A})$, by the uniqueness guaranteed in 363Fa, and T' also is multiplicative, by 353Qd once more. As in (v), we therefore have

$$[\![Tu>k]\!]\cup[\![T'u>k]\!]\subseteq\pi(\langle[\![u_i>k]\!]\rangle_{i\in I}),\quad\bar{\nu}([\![Tu>k]\!]\cup[\![T'u>k]\!])\le\gamma_k(u)$$

whenever $u\in W_0$ and $k\in\mathbb{N}$.

Suppose that $u\in W_0^+$ and $\epsilon>0$. Then there is a $k\in\mathbb{N}$ such that $\gamma_k(u)\le\epsilon$. Set $v_i=u_i\wedge k\chi 1_{\mathfrak{A}_i}$ for each i, and $v=\langle v_i\rangle_{i\in I}$. Then $Tv=T'v$, so

$$\bar{\nu}[\![|Tu-T'u|>0]\!]\le\bar{\nu}([\![Tu-Tv>0]\!]\cup[\![T'u-T'v>0]\!])\le\gamma_0(u-v)=\gamma_k(u)\le\epsilon.$$

As ϵ is arbitrary, $Tu=T'u$; as u is arbitrary, $T=T'$.

(b)(i) If $\epsilon > 0$, there is a $k \in \mathbb{N}$ such that $\bar{\mu}_i[\![|u_i| > k]\!] \le \epsilon$ for every $i \in I$. Now there is an $l \in \mathbb{N}$ such that $|h(t)| \le l$ whenever $|t| \le k$. So $[\![|\bar{h}(u_i)| > l]\!] \subseteq [\![|u_i| > k]\!]$ and $\bar{\mu}_i[\![|\bar{h}(u_i)| > l]\!] \le \epsilon$ for every $i \in I$. As ϵ is arbitrary, $\langle \bar{h}(u_i)\rangle_{i\in I} \in W_0$.

(ii) Again take any $\epsilon > 0$. Let $k \in \mathbb{N}$ be such that $\bar{\mu}_i a_i \le \epsilon$ for every $i \in I$, where $a_i = [\![|u_i| > k]\!]$. By the Stone-Weierstrass theorem in the form 281E, there is a polynomial $g : \mathbb{R} \to \mathbb{R}$ such that $|g(t) - h(t)| \le \epsilon$ whenever $|t| \le k$. Setting $v_i = \bar{h}(u_i)$, $v'_i = \bar{g}(u_i)$, $v = \langle u_i\rangle_{i\in I}$ and $v' = \langle v'_i\rangle_{i\in I}$, we have $[\![|v_i - v'_i| > \epsilon]\!] \subseteq a_i$ for every i (use 364Ib for a quick check of the calculation). Because T is multiplicative (and $T(\langle \chi 1_{\mathfrak{A}_i}\rangle_{i\in I}) = \chi 1_{\mathfrak{B}}$), $Tv' = \bar{g}(Tu)$. So

$$\begin{aligned}[\![|Tv - \bar{h}(Tu)| > 2\epsilon]\!] &\subseteq [\![T|v - v'| > \epsilon]\!] \cup [\![|\bar{g}(Tu) - \bar{h}(Tu)| > \epsilon]\!]\\ &\subseteq \pi(\langle [\![|v_i - v'_i| > \epsilon]\!]\rangle_{i\in I}) \cup [\![|Tu| > k]\!]\end{aligned}$$

(using (b))

$$\subseteq \pi(\langle a_i\rangle_{i\in I})$$

(see (a-v) above), which has measure at most ϵ. As ϵ is arbitrary, $Tv = \bar{h}(Tu)$, as claimed.

377C Theorem Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, $(\mathfrak{B}, \bar{\nu})$ a probability algebra, and $\pi : \prod_{i\in I}\mathfrak{A}_i \to \mathfrak{B}$ a Boolean homomorphism such that $\bar{\nu}\pi(\langle a_i\rangle_{i\in I}) \le \sup_{i\in I}\bar{\mu}_i a_i$ whenever $\langle a_i\rangle_{i\in I} \in \prod_{i\in I}\mathfrak{A}_i$. Let $W_0 \subseteq \prod_{i\in I} L^0(\mathfrak{A}_i)$ and $T : W_0 \to L^0(\mathfrak{B})$ be as in 377B. Suppose *either* that every $\mathfrak{A}_i$ is atomless *or* that there is an ultrafilter $\mathcal{F}$ on I such that $\bar{\nu}\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}}\bar{\mu}_i a_i$ whenever $\langle a_i\rangle_{i\in I}$ in $\prod_{i\in I}\mathfrak{A}_i$. For $1 \le p \le \infty$ let W_p be the subspace of $\prod_{i\in I} L^0(\mathfrak{A}_i)$ consisting of families $\langle u_i\rangle_{i\in I}$ such that $\sup_{i\in I}\|u_i\|_p$ is finite. Then $T[W_p] \subseteq L^p(\mathfrak{B}, \bar{\nu})$, and $\|Tu\|_p \le \sup_{i\in I}\|u_i\|_p$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_p.

proof (a) I should begin by explaining why $W_1 \subseteq W_0$. All we need to observe is that if $u = \langle u_i\rangle_{i\in I}$ belongs to W_1, so that $\gamma = \sup_{i\in I}\|u_i\|_1$ is finite, then

$$\inf_{k\ge1}\sup_{i\in I}\bar{\mu}_i[\![u_i > k]\!] \le \inf_{k\ge1}\frac{\gamma}{k} = 0,$$

so $u \in W_0$. Of course we now have $W_p \subseteq W_1$ for $p \ge 1$, because every $(\mathfrak{A}_i, \bar{\mu}_i)$ is a probability algebra.

(b) I start real work on the proof with a note on the case in which every $\mathfrak{A}_i$ is atomless. Suppose that this is so, and that we are given a family $\langle a_i\rangle_{i\in I} \in \prod_{i\in I}\mathfrak{A}_i$ and $\gamma \in \mathbb{Q} \cap [0, 1]$. Then there is a family $\langle a'_i\rangle_{i\in I}$ such that $a'_i \subseteq a_i$ and $\bar{\mu}_i a'_i = \gamma\mu_i a_i$ for every $i \in I$, and

$$\gamma\bar{\nu}\pi(\langle a_i\rangle_{i\in I}) \le \bar{\nu}\pi(\langle a'_i\rangle_{i\in I}).$$

P For each $i \in I$, we can find a non-decreasing family $\langle a_{it}\rangle_{t\in[0,1]}$ in $\mathfrak{A}_i$ such that $a_{i1} = a_i$ and $\bar{\mu}_i a_{it} = t\bar{\mu}a_i$ for every $t \in [0, 1]$. Set $b(t) = \pi(\langle a_{it}\rangle_{i\in I})$ and $\beta(t) = \bar{\nu}b(t)$ for $t \in [0, 1]$; then $\beta(s) \le \beta(t) \le \beta(s) + t - s$ for $0 \le s \le t \le 1$, because

$$\beta(t) - \beta(s) = \bar{\nu}\pi(\langle a_{it} \setminus a_{is}\rangle_{i\in I}) \le \sup_{i\in I}\bar{\mu}_i(a_{it} \setminus a_{is}) = (t - s)\sup_{i\in I}\bar{\mu}_i a_i \le t - s.$$

Let $n \ge 1$ be such that $\frac{1}{n} \le \epsilon$ and $m = n\gamma$ is an integer, and set $\alpha_i = \beta(\frac{i+1}{n}) - \beta(\frac{i}{n})$ for $i < n$; then

$$\sum_{i=0}^{n-1}\alpha_i = \beta(1) = \bar{\nu}b(1).$$

Consider the possible values of $\gamma_K = \sum_{k\in K}\alpha_k$ for sets $K \in [n]^m$. (I am thinking of n as the set $\{0, 1, \dots, n-1\}$.) The average value of γ_K over all m-element subsets of n is just $\frac{m}{n}\beta(1) = \gamma\beta(1)$, so there is some K such that $\gamma_K \ge \gamma\beta(1)$.

Set

$$a'_i = \sup_{k\in K} a_{i,(k+1)/n} \setminus a_{i,k/n}$$

for $i \in I$. Then $\bar{\mu}_i a'_i = \gamma\bar{\mu}_i a_i$ for every i, while

$$\bar{\nu}\pi(\langle a'_i\rangle_{i\in I}) = \sup_{k\in K}\bar{\nu}(b(\tfrac{k+1}{n}) \setminus b(\tfrac{k}{n})) = \sum_{k\in K}\alpha_k$$

is at least $\gamma\beta(1)$, as required. **Q**

(c) We find now that under either of the hypotheses proposed,

$$\sum_{k=0}^{n}\gamma_k\bar\nu\pi(\langle a_{ki}\rangle_{i\in I})\le\sup_{i\in I}\sum_{k=0}^{n}\gamma_k\mu_i a_{ki}$$

whenever $\gamma_0,\dots,\gamma_n\ge 0$ are rational and $\langle a_{ki}\rangle_{k\le n}$ is a disjoint family in $\mathfrak{A}_i$ for each $i\in I$.

P(i) Consider first the case in which every $\mathfrak{A}_i$ is atomless and every γ_k is between 0 and 1. In this case, given $\epsilon>0$, (b) above tells us that we can find $a'_{ki}\subseteq a_{ki}$, for $i\in I$ and $k\le n$, such that $\bar\mu_i a'_{ki}=\gamma_k\bar\mu_i a_{ki}$ and

$$\gamma_k\bar\nu\pi(\langle a_{ki}\rangle_{i\in I})\le\bar\nu\pi(\langle a'_{ki}\rangle_{i\in I}).$$

Set $c_i=\sup_{k\le n}a'_{ki}$ for $i\in I$; then

$$\begin{aligned}\sum_{k=0}^{n}\gamma_k\bar\nu\pi(\langle a_{ki}\rangle_{i\in I})&\le\sum_{k=0}^{n}\bar\nu\pi(\langle a'_{ki}\rangle_{i\in I})=\bar\nu\pi(\sup_{k\le n}\langle a'_{ki}\rangle_{i\in I})=\bar\nu\pi(\langle c_i\rangle_{i\in I})\\&\le\sup_{i\in I}\bar\mu_i c_i=\sup_{i\in I}\sum_{k=0}^{n}\bar\mu_i a'_{ki}=\sup_{i\in I}\sum_{k=0}^{n}\gamma_k\bar\mu_i a_{ki},\end{aligned}$$

as required.

(ii) Because T is linear, it follows at once that the result is true for any rational $\gamma_0,\dots,\gamma_n\ge 0$, if every $\mathfrak{A}_i$ is atomless.

(iii) Now consider the case in which there is an ultrafilter $\mathcal{F}$ on I such that $\bar\nu\pi(\langle a_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu_i a_i$ for every $\langle a_i\rangle_{i\in I}$. In this case, given $\epsilon>0$, the set

$$J=\{j:j\in I,\ \bar\nu(\langle a_{ki}\rangle_{i\in I})\le\bar\mu_j a_{kj}+\epsilon\text{ for every }k\le n\}$$

belongs to $\mathcal{F}$ and is not empty. Take any $j\in J$; then

$$\sum_{k=0}^{n}\gamma_k\bar\nu\pi(\langle a_{ki}\rangle_{i\in I})\le\sum_{k=0}^{n}\gamma_k(\bar\mu_j a_{kj}+\epsilon)\le\epsilon\sum_{k=0}^{n}\gamma_k+\sup_{i\in I}\sum_{k=0}^{n}\gamma_k\bar\mu_i a_{ki}.$$

As ϵ is arbitrary, we again have the result. **Q**

(d) Next, $\int Tu\le\sup_{i\in I}\int u_i$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_∞^+. **P** Let $\epsilon>0$ and let $n\in\mathbb{N}$ be such that $\|u_i\|_\infty\le n\epsilon$ for every $i\in I$. For $i\in I$ and $k\le n$, set $a_{ki}=[\![u_i>k\epsilon]\!]\setminus[\![u_i>(k+1)\epsilon]\!]$; for $i\in I$, set $u'_i=\sum_{k=0}^{n}k\epsilon\chi a_{ki}$; then $u'_i\le u_i\le u'_i+\epsilon\chi 1_{\mathfrak{A}_i}$. Setting $u'=\langle u'_i\rangle_{i\in I}$, $Tu\le Tu'+\epsilon\chi 1_{\mathfrak{B}}$, so

$$\begin{aligned}\int Tu-\epsilon\le\int Tu'&=\int\sum_{k=0}^{n}k\epsilon\chi\pi(\langle a_{ki}\rangle_{i\in I})\\&=\sum_{k=0}^{n}k\epsilon\bar\nu\pi(\langle a_{ki}\rangle_{i\in I})\le\sup_{i\in I}\sum_{k=0}^{n}k\epsilon\bar\mu_i a_{ki}\end{aligned}$$

(by (c))

$$=\sup_{i\in I}\int u'_i\le\sup_{i\in I}\int u_i.$$

As ϵ is arbitrary, we have the result. **Q**

(d) It follows that $Tu\in L^1(\mathfrak{B},\bar\nu)$ and $\int Tu\le\sup_{i\in I}\int u_i$ whenever $u=\langle u_i\rangle_{i\in I}$ belongs to W_1^+ and $u\ge 0$. **P** Set $\gamma=\sup_{i\in I}\int u_i$. Let $\epsilon>0$. Set $\gamma'=\gamma/\epsilon$. For $i\in I$ set $v_i=u_i\wedge\gamma'\chi 1_{\mathfrak{A}_i}$; set $v=\langle v_i\rangle_{i\in I}$. Then $v\in W_\infty$ and

$$\int Tv\le\sup_{i\in I}\int v_i\le\sup_{i\in I}\int u_i=\gamma$$

by (c) above. Also $[\![Tu-Tv>0]\!]\subseteq\pi(\langle[\![u_i>\gamma']\!]\rangle_{i\in I})$, by 377Ba, so

$$\bar\nu[\![Tu-Tv>0]\!]\le\sup_{i\in I}\bar\mu_i[\![u_i>\gamma']\!]\le\epsilon.$$

Thus for each $n \in \mathbb{N}$ we can find a $w_n \in L^\infty(\mathfrak{B})$ such that $0 \le w_n \le Tu$, $\int w_n \le \gamma$ and $\bar\nu[\![Tu - w_n > 0]\!] \le 2^{-n}$. Set $w'_n = \inf_{i \ge n} w_i$ for each n; then $\langle w'_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum Tu in $L^0(\mathfrak{B})$, while $\int w'_n \le \gamma$ for every n. Consequently $Tu \in L^1(\mathfrak{B}, \bar\nu)$ and $\int Tu \le \gamma$, as claimed. **Q**

(e) Because T is a Riesz homomorphism, $Tu \in L^1(\mathfrak{B}, \bar\nu)$ and $\|Tu\|_1 = \int T|u|$ is at most $\sup_{i\in I} \int |u_i| = \sup_{i\in I} \|u_i\|_1$ for every $u \in W_1$.

(f) Now suppose that $p \in]1, \infty[$ and that $u = \langle u_i \rangle_{i\in I}$ belongs to W_p. In this case, $\langle |u_i|^p \rangle_{i\in I}$ belongs to W_1, so $T(\langle |u_i|^p \rangle_{i\in I}) \in L^1(\mathfrak{B}, \bar\nu)$ and $\int T(\langle |u_i|^p \rangle_{i\in I}) \le \sup_{i\in I} \int |u_i|^p$. By 377Bb, with $h(t) = |t|^p$, $T(\langle |u_i|^p \rangle_{i\in I}) = |Tu|^p$. So $Tu \in L^p(\mathfrak{B}, \bar\nu)$ and

$$\|Tu\|_p = (\int |Tu|^p)^{1/p} \le \sup_{i\in I} (\int |u_i|^p)^{1/p} = \sup_{i\in I} \|u_i\|_p$$

as claimed.

377D The original motivation for the work of this section was to understand the function spaces associated with the reduced products of §328. For these we have various simplifications in addition to that observed in 377C.

Theorem Let $\langle (\mathfrak{A}_i, \bar\mu_i) \rangle_{i\in I}$ be a family of probability algebras, $\mathcal{F}$ an ultrafilter on I, and $(\mathfrak{B}, \bar\nu)$ a probability algebra. Let $\mathfrak{A}$ be the simple product $\prod_{i\in I} \mathfrak{A}_i$ and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism such that $\bar\nu\pi(\langle a_i \rangle_{i\in I}) = \lim_{i\to\mathcal{F}} \bar\mu_i a_i$ whenever $\langle a_i \rangle_{i\in I} \in \mathfrak{A}$. Let $W_0 \subseteq \prod_{i\in I} L^0(\mathfrak{A}_i)$ and $T : W_0 \to L^0(\mathfrak{B})$ be as in 377B-377C.

(a) If $u = \langle u_i \rangle_{i\in I}$ belongs to W_0 and $\{i : i \in I,\ u_i = 0\} \in \mathcal{F}$, then $Tu = 0$.

(b) For $1 \le p \le \infty$, write W_p for the set of those families $\langle u_i \rangle_{i\in I} \in \prod_{i\in I} L^p(\mathfrak{A}_i, \bar\mu_i)$ such that $\sup_{i\in I} \|u_i\|_p$ is finite. Then $Tu \in L^p(\mathfrak{B}, \bar\nu)$ and $\|Tu\|_p \le \lim_{i\to\mathcal{F}} \|u_i\|_p$ whenever $u = \langle u_i \rangle_{i\in I}$ belongs to W_p.

(c) Let W_{ui} be the subspace of $\prod_{i\in I} L^1(\mathfrak{A}_i, \bar\mu_i)$ consisting of families $\langle u_i \rangle_{i\in I}$ such that $\inf_{k\in\mathbb{N}} \sup_{i\in I} \int (|u_i| - k\chi 1_{\mathfrak{A}_i})^+ = 0$. Then $\int Tu = \lim_{i\to\mathcal{F}} \int u_i$ and $\|Tu\|_1 = \lim_{i\to\mathcal{F}} \|u_i\|_1$ whenever $u = \langle u_i \rangle_{i\in I}$ belongs to W_{ui}.

(d) Suppose now that $\pi[\mathfrak{A}] = \mathfrak{B}$.

(i) $T[W_0] = L^0(\mathfrak{B})$.

(ii) $T[W_{ui}] = L^1(\mathfrak{B}, \bar\nu)$.

(iii) If $p \in [1, \infty]$, then $T[W_p] = L^p(\mathfrak{B}, \bar\nu)$ and for every $w \in L^p(\mathfrak{B}, \bar\nu)$ there is a $u = \langle u_i \rangle_{i\in I}$ in W_p such that $Tu = w$ and $\sup_{i\in I} \|u_i\|_p = \|w\|_p$.

proof (a) Setting

$$a_i = 1_{\mathfrak{A}_i} \text{ if } u_i \ne 0,$$
$$= 0 \text{ if } u_i = 0,$$

$\langle a_i \rangle_{i\in I} \in \mathfrak{A}$ and $\bar\nu\pi(\langle a_i \rangle_{i\in I}) = \lim_{i\to\mathcal{F}} \bar\mu_i a_i = 0$, so $\pi(\langle a_i \rangle_{i\in I}) = 0$. Accordingly

$$Tu = T(\langle u_i \times \chi a_i \rangle_{i\in I}) = Tu \times T(\langle \chi a_i \rangle_{i\in I}) = Tu \times \chi\pi(\langle a_i \rangle_{i\in I}) = 0.$$

(b) Suppose that $u = \langle u_i \rangle_{i\in I} \in W_p$ and that $J \in \mathcal{F}$. Set

$$v_i = u_i \text{ if } i \in J,$$
$$= 0 \text{ if } i \in I \setminus J;$$

then, putting (a) and 377C together,

$$\|Tu\|_p = \|Tv\|_p \le \sup_{i\in I} \|v_i\|_p = \sup_{i\in J} \|u_i\|_p.$$

As J is arbitrary, $\|Tu\|_p \le \lim_{i\to\mathcal{F}} \|u_i\|_p$.

(c)(i) Clearly W_{ui} is a solid linear subspace of W_1. Suppose that $u = \langle u_i \rangle_{i\in I} \in W_{ui}^+$ and $\epsilon > 0$. Let $n \ge 1$ be such that $\int (u_i - n\epsilon\chi 1_{\mathfrak{A}_i})^+ \le \epsilon$ for every $i \in I$. For $i \in I$ and $k \le n$, set $a_{ki} = [\![u_i > k\epsilon]\!]$; set $v_i = \sum_{k=1}^n k\epsilon\chi a_{ki}$, so that

$$v_i \le u_i \le v_i + \epsilon\chi 1_{\mathfrak{A}_i} + (u_i - n\epsilon\chi 1_{\mathfrak{A}_i})^+, \quad \int u_i \le \int v_i + 2\epsilon.$$

If $v = \langle v_i \rangle_{i\in I}$, then

$$\begin{aligned}\int Tu &= \|Tu\|_1 \le \lim_{i\to\mathcal{F}} \|u_i\|_1 = \lim_{i\to\mathcal{F}} \int u_i \\ &\le 2\epsilon + \lim_{i\to\mathcal{F}} \int v_i = 2\epsilon + \sum_{k=1}^{n} k\epsilon \lim_{i\to\mathcal{F}} \bar\mu_i a_{ki} \\ &= 2\epsilon + \sum_{k=1}^{n} k\epsilon\bar\nu\pi(\langle a_{ki}\rangle_{i\in I}) = 2\epsilon + \int \sum_{k=1}^{n} k\epsilon\chi\pi(\langle a_{ki}\rangle_{i\in I}) \\ &= 2\epsilon + \int Tv \le 2\epsilon + \int Tu.\end{aligned}$$

As ϵ is arbitrary, $\int Tu = \lim_{i\to\mathcal{F}} \int u_i$.

(ii) It follows at once that $\int Tu = \lim_{i\to\mathcal{F}} \int u_i$ and that

$$\|Tu\|_1 = \int |Tu| = \int T|u| = \lim_{i\to\mathcal{F}} \int |u_i| = \lim_{i\to\mathcal{F}} \|u_i\|_1$$

whenever $u = \langle u_i\rangle_{i\in I} \in W_{ui}$.

(d)(i)($\boldsymbol{\alpha}$) Let $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$ be the Riesz homomorphism associated with the Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$. Since π is surjective, 363Fd tells us that T_π is surjective. Identifying W_∞ with $L^\infty(\mathfrak{A})$, and $T\restriction W_\infty$ with T_π, as in part (a) of the the proof of 377B, we see that $T[W_\infty] = L^\infty(\mathfrak{B})$. Moreover, 363Fd tells us also that if $w \in L^\infty$ there is a $v \in L^\infty(\mathfrak{A})$ such that $T_\pi v = w$ and $\|v\|_\infty = \|w\|_\infty$; translating this into terms of W_∞, we have a $u = \langle u_i\rangle_{i\in I} \in W_\infty$ such that $Tu = w$ and $\sup_{i\in I} \|u_i\|_\infty = \|w\|_\infty$.

It will be useful to know that if $b \in \mathfrak{B}$ and $\epsilon > 0$ there is a family $\langle a_i\rangle_{i\in I} \in \mathfrak{A}$ such that $\pi(\langle a_i\rangle_{i\in I}) = b$ and $\sup_{i\in I} \bar\mu_i a_i \le \bar\nu b + \epsilon$. **P** By hypothesis, there is a family $\langle a'_i\rangle_{i\in I} \in \mathfrak{A}$ such that $\pi(\langle a'_i\rangle_{i\in I}) = b$, and $\bar\nu b = \lim_{i\to\mathcal{F}} \bar\mu_i a_i$. Set

$$\begin{aligned} a_i &= a'_i \text{ if } \bar\mu_i a_i \le \bar\nu b + \epsilon, \\ &= 0 \text{ for other } i \in I. \end{aligned}$$

Then $\lim_{i\to\mathcal{F}} \bar\mu_i(a'_i \triangle a_i) = 0$ so $\pi(\langle a'_i \triangle a_i\rangle_{i\in I}) = 0$ and $\pi(\langle a_i\rangle_{i\in I}) = b$, while $\bar\mu_i a_i \le \bar\nu b + \epsilon$ for every $i \in I$. **Q**

($\boldsymbol{\beta}$) Now suppose that $w \in L^0(\mathfrak{B})^+$. For each $n \in \mathbb{N}$, set $w_n = w \wedge n\chi 1_{\mathfrak{B}}$ and let $u^{(n)} = \langle u_{ni}\rangle_{i\in I} \in W_\infty$ be such that $Tu^{(n)} = w_{n+1} - w_n$ and $\|u_{ni}\|_\infty \le 1$ for every $i \in I$. Next, for each n, set $b_n = [\![w_{n+1} - w_n > 0]\!]$, and let $\langle a_{ni}\rangle_{i\in I} \in \mathfrak{A}$ be such that $\pi(\langle a_{ni}\rangle_{i\in I}) = b_n$ and $\sup_{i\in I} \bar\mu_i a_{ni} \le \bar\nu b_n + 2^{-n}$. If we set $a'_{ni} = \inf_{m\le n} a_{mi}$ and $u'_{ni} = u_{ni} \times \chi a'_{ni}$, we shall have

$$\begin{aligned} T(\langle u'_{ni}\rangle_{i\in I}) &= T(\langle u_{ni}\rangle_{i\in I}) \times \chi\pi(\langle a'_{ni}\rangle_{i\in I}) \\ &= (w_{n+1} - w_n) \times \inf_{m\le n} \chi b_m = w_{n+1} - w_n \end{aligned}$$

for every n. Also, for each $i \in I$, $\langle a'_{ni}\rangle_{n\in\mathbb{N}}$ is non-increasing and

$$\lim_{n\to\infty} \bar\mu_i a'_{ni} \le \lim_{n\to\infty} \bar\nu b_n + 2^{-n} = 0.$$

So $v_i = \sup_{n\in\mathbb{N}} \sum_{m=0}^{n} u'_{ni}$ is defined in $L^0(\mathfrak{A}_i)$, and

$$\inf_{k\in\mathbb{N}} \sup_{i\in I} \bar\mu_i [\![v_i > k]\!] \le \inf_{k\in\mathbb{N}} \sup_{i\in I} \bar\mu_i a'_{ki} = 0.$$

Thus $v = \langle v_i\rangle_{i\in I}$ belongs to W_0 and we can speak of Tv. Of course

$$Tv \ge \sum_{m=0}^{n} T(\langle u'_{ni}\rangle_{i\in I}) = w_{n+1}$$

for every n, so $Tv \ge w$. On the other hand, for any $n \in \mathbb{N}$,

$$[\![v_i - \sum_{m=0}^{n} u'_{ni} > 0]\!] \subseteq a'_{ni}$$

for every i, so $[\![Tv - w_{n+1} > 0]\!] \subseteq b_n$, by 377B; as $\inf_{n\in\mathbb{N}} b_n = 0$, $Tv = \sup_{n\in\mathbb{N}} w_n = w$.

($\boldsymbol{\gamma}$) Thus $T[W_0] \supseteq L^0(\mathfrak{B})^+$; as T is linear, $T[W_0] = L^0(\mathfrak{B})$.

(ii) Now suppose that $w \in L^1(\mathfrak{B},\bar\nu)^+$. In this case, repeat the process of (i-β) above. This time, observe that as $\chi b_{n+1} \le w_{n+1} - w_n$ for every n, $\sum_{n=0}^\infty \bar\nu b_n \le 1 + \int w$ is finite. Consequently, in the first place,

$$\sum_{n=0}^\infty \int u'_{ni} \le \sum_{n=0}^\infty \bar\mu_i a_{ni} \le \sum_{n=0}^\infty \bar\nu b_n + 2^{-n}$$

is finite, and $v_i \in L^1(\mathfrak{A}_i,\bar\mu_i)$, for every $i \in I$. But also, for any $k \in \mathbb{N}$ and $i \in I$,

$$\int (v_i - k\chi 1_{\mathfrak{A}_i})^+ \le \sum_{n=k}^\infty \int u'_{ni} \le \sum_{n=k}^\infty \bar\nu b_n + 2^{-n} \to 0$$

$k \to \infty$. So $v \in W_{ui}$ and $w \in T[W_{ui}]$. Because W_{ui} is a linear subspace of W_0, $T[W_{ui}] = L^1(\mathfrak{B},\bar\nu)$.

(iii)$(\boldsymbol{\alpha})$ If $p = \infty$ the result has already been dealt with in (i-α) above.

$(\boldsymbol{\beta})$ For the case $p = 1$, take $w \in L^1(\mathfrak{B},\bar\nu)$. Let $v = \langle v_i\rangle_{i\in I} \in W_{ui}$ be such that $Tv = w$. For $i \in I$ set

$$\begin{aligned} u_i &= \tfrac{\|w\|_1}{\|v_i\|_1} v_i \text{ if } \|v_i\|_1 > \|w\|_1, \\ &= v_i \text{ otherwise.} \end{aligned}$$

Then

$$\bar\mu[\![(|u_i| - k > 0]\!] \le \bar\mu[\![|v_i| - k > 0]\!]$$

for all $k \in \mathbb{N}$ and $i \in I$, so $u = \langle u_i\rangle_{i\in I} \in W_{ui}$. Since $\lim_{i\to\mathcal{F}} \|v_i\|_1 = \|w\|_1$, by (c) above, $\lim_{i\to\mathcal{F}} \|u_i - v_i\|_1 = 0$ and $Tu = Tv = w$, by (b). And of course $\|u_i\|_1 \le \|w\|_1$ for every i.

$(\boldsymbol{\gamma})$ Now suppose that $1 < p < \infty$ and that $w \in L^p(\mathfrak{B},\bar\nu)$. By ($\beta$), there is a $v = \langle v_i\rangle_{i\in I} \in W_1$ such that $Tv = |w|^p$ and $\sup_{i\in I} \|v_i\|_1 = \|w\|_p^p$. Set $v'_i = |v_i|^{1/p}$ for each i; then $v' = \langle v'_i\rangle_{i\in I} \in W_p$ and $Tv' = |w|$, by 377Bb. Next, w is expressible as $|w| \times \tilde w$, where $\tilde w \in L^\infty(\mathfrak{B})$ and $\|\tilde w\|_\infty \le 1$. There is a $\tilde v = \langle \tilde v_i\rangle_{i\in I} \in W_\infty$ such that $T\tilde v = \tilde w$ and $\sup_{i\in I} \|\tilde v_i\|_\infty = 1$. Set $u_i = v'_i \times \tilde v_i$ for each i; then $u = \langle u_i\rangle_{i\in I}$ belongs to W_p, $\|u_i\|_p \le \|w\|_p$ for every i, and $Tu = w$.

377E In the case of a reduced power of a probability algebra we can express these ideas in a slightly different way.

Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be probability algebras, I a set and $\mathcal{F}$ an ultrafilter on I. Let $\pi : \mathfrak{A}^I \to \mathfrak{B}$ be a Boolean homomorphism such that $\bar\nu\pi(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}} \bar\mu a_i$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}^I$. Let W_0 be the set of families in $L^0(\mathfrak{A})^I$ which are bounded for the topology of convergence in measure on $L^0(\mathfrak{A})$.

(a)(i) W_0 is a solid linear subspace and a subalgebra of $L^0(\mathfrak{A})^I$, and there is a unique multiplicative Riesz homomorphism $T : W_0 \to L^0(\mathfrak{B})$ such that $T(\langle \chi a_i\rangle_{i\in I}) = \chi\pi(\langle a_i\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{A}^I$.

(ii) $[\![Tu > 0]\!] \subseteq \pi(\langle [\![u_i > 0]\!]\rangle_{i\in I})$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_0.

(iii) If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, and we write $\bar h$ for the corresponding maps from L^0 to itself for either of the spaces $L^0 = L^0(\mathfrak{A})$, $L^0 = L^0(\mathfrak{B})$, then $\langle \bar h(u_i)\rangle_{i\in I} \in W_0$ and $T(\langle \bar h(u_i)\rangle_{i\in I}) = \bar h(Tu)$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_0.

(b)(i) For $1 \le p \le \infty$ let W_p be the subspace of $L^p(\mathfrak{A},\bar\mu)^I$ consisting of $\| \|_p$-bounded families. Then $T[W_p] \subseteq L^p(\mathfrak{B},\bar\nu)$, and $\|Tu\|_p \le \lim_{i\to\mathcal{F}} \|u_i\|_p$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_p.

(ii) Let W_{ui} be the subspace of $L^1(\mathfrak{A}_i,\bar\mu_i)^I$ consisting of uniformly integrable families. Then $\int Tu = \lim_{i\to\mathcal{F}} \int u_i$ and $\|Tu\|_1 = \lim_{i\to\mathcal{F}} \|u_i\|_1$ whenever $u = \langle u_i\rangle_{i\in I}$ belongs to W_{ui}.

(c)(i) We have a measure-preserving Boolean homomorphism $\tilde\pi : \mathfrak{A} \to \mathfrak{B}$ defined by setting $\tilde\pi a = \pi(\langle a\rangle_{i\in I})$ for each $a \in \mathfrak{A}$.

(ii) Let $P_{\tilde\pi} : L^1(\mathfrak{B},\bar\nu) \to L^1(\mathfrak{A},\bar\mu)$ be the conditional-expectation operator corresponding to $\tilde\pi : \mathfrak{A} \to \mathfrak{B}$ (365O). If $\langle u_i\rangle_{i\in I}$ is a uniformly integrable family in $L^1(\mathfrak{A})$, then $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})$ is the limit $\lim_{i\to\mathcal{F}} u_i$ for the weak topology of $L^1(\mathfrak{A},\bar\mu)$.

(iii) Suppose that $1 < p < \infty$ and that $\langle u_i\rangle_{i\in I}$ is a bounded family in $L^p(\mathfrak{A},\bar\mu)$. Then $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})$ is the limit $\lim_{i\to\mathcal{F}} u_i$ for the weak topology of $L^p(\mathfrak{A},\bar\mu)$.

proof (a) By 367Rd, a family $\langle u_i\rangle_{i\in I}$ in $L^0(\mathfrak{A})$ is bounded for the topology of convergence in measure iff $\inf_{k\in\mathbb{N}} \sup_{i\in I} \bar\mu[\![|u_i| > k]\!] = 0$. So we just have a special case of 377B.

(b) Similarly, the condition '$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i|-k\chi 1_{\mathfrak{A}_i})^+=0$' translates into '$\{u_i:i\in I\}$ is uniformly integrable' (cf. 246Bd), so we are looking at a special case of 377Db-377Dc.

(c)(i) $\tilde\pi$ is a Boolean homomorphism just because the function taking $a\in\mathfrak{A}$ into the constant family with value a is a Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{A}^I$. The formula '$\bar\nu\pi(\langle a_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu a_i$' now ensures that $\tilde\pi$ is measure-preserving.

(ii) By the defining formula for $P_{\tilde\pi}$ (365Oa),

$$\begin{aligned}\int_a P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})&=\int T(\langle u_i\rangle_{i\in I})\times\chi\tilde\pi(a)=\int T(\langle u_i\rangle_{i\in I})\times\chi\pi(\langle a\rangle_{i\in I})\\&=\int T(\langle u_i\rangle_{i\in I})\times T(\langle\chi a\rangle_{i\in I})\\&=\int T(\langle u_i\times\chi a\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\int u_i\times\chi a\end{aligned}$$

(because $\{u_i\times\chi a:i\in I\}$ is uniformly integrable)

$$=\lim_{i\to\mathcal{F}}\int_a u_i$$

for every $a\in\mathfrak{A}$. It follows that $P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}u_i$. **P** We have

$$\int P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})\times v=\lim_{i\to\mathcal{F}}\int u_i\times v$$

whenever $v=\chi a$, for any $a\in\mathfrak{A}$; by linearity, whenever $v\in S(\mathfrak{A})$, the space of $\mathfrak{A}$-simple functions; and by continuity, whenever $v\in L^\infty(\mathfrak{A})$ (because $\{u_i:i\in I\}$ is $\|\,\|_1$-bounded, and $S(\mathfrak{A})$ is $\|\,\|_\infty$-dense in $L^\infty(\mathfrak{A})$). Since $L^\infty(\mathfrak{A})$ can be identified with the dual of $L^1(\mathfrak{A},\bar\mu)$ (365Lc), we have the required weak convergence. **Q**

(iii) If $\{u_i:i\in I\}$ is $\|\,\|_p$-bounded, where $1<p<\infty$, then it is uniformly integrable. **P** Set $q=\frac{p}{p-1}$. If $k\ge 1$,

$$\inf_{k\ge1}\sup_{i\in I}\int(|u_i|-k\chi 1_{\mathfrak{A}})^+\le\inf_{k\ge1}\frac{1}{k^{p-1}}\sup_{i\in I}\|u_i\|_p^p=0.\ \mathbf{Q}$$

So

$$\int P_{\tilde\pi}T(\langle u_i\rangle_{i\in I})\times v=\lim_{i\to\mathcal{F}}\int u_i\times v$$

for every $v\in S(\mathfrak{A})$, and therefore for every $v\in L^q(\mathfrak{A},\bar\mu)$, since v can be $\|\,\|_q$-approximated by members of $S(\mathfrak{A})$ (366C). Since $L^q(\mathfrak{A},\bar\mu)$ can be identified with $L^p(\mathfrak{A},\bar\mu)^*$, we again have weak convergence.

377F Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A}',\bar\mu')$ be probability algebras, I a set and $\mathcal{F}$ an ultrafilter on I; let $(\mathfrak{B},\bar\nu)$ and $(\mathfrak{B}',\bar\nu')$ be the reduced powers $(\mathfrak{A},\bar\mu)^I|\mathcal{F}$, $(\mathfrak{A}',\bar\mu')^I|\mathcal{F}$ as described in 328A-328C, with corresponding homomorphisms $\pi:\mathfrak{A}^I\to\mathfrak{B}$ and $\pi':\mathfrak{A}'^I\to\mathfrak{B}'$.

(a) Writing W_0, W_0' for the spaces of topologically bounded families in $L^0(\mathfrak{A})^I$, $L^0(\mathfrak{A}')^I$ respectively, we have unique Riesz homomorphisms $T:W_0\to L^0(\mathfrak{B})$ and $T':W_0'\to L^0(\mathfrak{B}')$ such that $T(\langle\chi a_i\rangle_{i\in I})=\chi\pi(\langle a_i\rangle_{i\in I})$, $T'(\langle\chi a_i'\rangle_{i\in I})=\chi\pi'(\langle a_i'\rangle_{i\in I})$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{A}^I$ and $\langle a_i'\rangle_{i\in I}\in(\mathfrak{A}')^I$.

(b) Suppose that $S:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{A}',\bar\mu')$ is a bounded linear operator. Then we have a unique bounded linear operator $\hat S:L^1(\mathfrak{B},\bar\nu)\to L^1(\mathfrak{B}',\bar\nu')$ such that $\hat ST(\langle u_i\rangle_{i\in I})=T'(\langle Su_i\rangle_{i\in I})$ whenever $\langle u_i\rangle_{i\in I}$ is a uniformly integrable family in $L^1(\mathfrak{A},\bar\mu)$.

(c) The map $S\mapsto\hat S$ is a norm-preserving Riesz homomorphism from $\mathsf{B}(L^1(\mathfrak{A},\bar\mu);L^1(\mathfrak{A}',\bar\mu'))$ to $\mathsf{B}(L^1(\mathfrak{B},\bar\nu);L^1(\mathfrak{B}',\bar\nu'))$.

proof (a) Once again, this is nothing but a specialization of the corresponding fragments of 377Ba and 377Ea.

(b) Write W_{ui} for the space of uniformly integrable families in $L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)$. If $\langle u_i\rangle_{i\in I}\in W_{ui}$, then $\langle Su_i\rangle_{i\in I}$ is uniformly integrable in $L^1_{\bar\mu'}=L^1(\mathfrak{A}',\bar\mu')$ (because $\{u_i:i\in I\}$ and $\{Su_i:i\in I\}$ are relatively weakly compact, as in 247D), so belongs to W_0', and we can speak of $T'(\langle Su_i\rangle_{i\in I})$. If moreover $T(\langle u_i\rangle_{i\in I})=0$, then $\lim_{i\to\mathcal{F}}\|u_i\|_1=0$ (377E(b-ii)), so $\lim_{i\to\mathcal{F}}\|Su_i\|_1=0$ and $T'(\langle Su_i\rangle_{i\in I})=0$. Finally, $T[W_{ui}]=L^1_{\bar\nu}=L^1(\mathfrak{B},\bar\nu)$ by 377D(d-ii). So the given formula defines a linear operator $\hat S:L^1_{\bar\nu}\to L^1_{\bar\nu'}=L^1(\mathfrak{B}',\bar\nu')$. Next, if $w\in L^1_{\bar\nu}$, we can take any family $\langle u_i\rangle_{i\in I}\in W_{ui}$ such that $T(\langle u_i\rangle_{i\in I})=w$, and

$$\|\hat Sw\|_1=\|T'(\langle Su_i\rangle_{i\in I})\|_1=\lim_{i\to\mathcal{F}}\|Su_i\|_1$$

(377E(b-ii))

$$\le\|S\|\lim_{i\to\mathcal{F}}\|u_i\|_1=\|S\|\|w\|_1.$$

As w is arbitrary, $\hat S$ is a bounded linear operator, and $\|\hat S\|\le\|S\|$. On the other hand, if $u\in L^1_{\bar\mu}$ and $\|u\|_1\le 1$, $\|T(\langle u\rangle_{i\in I})\|_1\le 1$ so

$$\|\hat S\|\ge\|\hat ST(\langle u\rangle_{i\in I})\|=\|T'(\langle Su\rangle_{i\in I})\|_1=\|Su\|_1;$$

as u is arbitrary, $\|\hat S\|\ge\|S\|$.

(c)(i) Recall from 371D that the Banach space $\mathsf{B}(L^1_{\bar\mu};L^1_{\bar\mu'})$ of continuous linear operators is also the Dedekind complete Riesz space $L^\sim(L^1_{\bar\mu};L^1_{\bar\mu'})$ of order-bounded linear operators, and its norm is a Riesz norm; similarly, $\mathsf{B}(L^1_{\bar\nu};L^1_{\bar\nu'})=L^\sim(L^1_{\bar\nu};L^1_{\bar\nu'})$. We have already seen that $S\mapsto\hat S$ is norm-preserving, and it is clearly linear. If $w\in(L^1_{\bar\nu})^+$, then, by 377D(d-ii), $w=T(\langle u_i\rangle_{i\in I})$ for a family $\langle u_i\rangle_{i\in I}\in W_{ui}$; since T is a Riesz homomorphism, $w=w^+=T(\langle u_i^+\rangle_{i\in I}$; so that if $S\ge 0$ we shall have $\hat Sw=T'(\langle Su_i^+\rangle_{i\in I})\ge 0$. This shows that $\hat S\ge 0$ whenever $S\ge 0$, so that $S\mapsto\hat S$ is a positive linear operator.

(ii) To show that $S\mapsto\hat S$ is a Riesz homomorphism, I argue as follows. Take any bounded linear operator $S:L^1_{\bar\mu}\to L^1_{\bar\mu'}$ and $\epsilon>0$. Then

$$B=\{\sum_{k=0}^n|Sv_k|:v_0,\dots,v_k\in(L^1_{\bar\mu})^+,\ \sum_{k=0}^n v_k=\chi 1_{\mathfrak{A}}\}$$

is an upwards-directed set in $L^1_{\bar\mu'}$ with supremum $|S|(\chi 1_{\mathfrak{A}})$ (371A, part (b) of the proof of 371B). So we can find $v_0,\dots,v_n\in(L^1_{\bar\mu})^+$ such that $\sum_{k=0}^n v_k=\chi 1_{\mathfrak{A}}$ and $\|v'\|_1\le\epsilon$, where $v'=|S|(\chi 1_{\mathfrak{A}})-\sum_{k=0}^n|Sv_k|\ge 0$.

Next, if $0\le u\le\chi 1_{\mathfrak{A}}$ in $L^1_{\bar\mu}$, set $u'=\chi 1_{\mathfrak{A}}-u$; we have

$$\begin{aligned}|S|(\chi 1_{\mathfrak{A}})-v'=\sum_{k=0}^n|Sv_k|&\le\sum_{k=0}^n|S(u\times v_k)|+\sum_{k=0}^n|S(u'\times v_k)|\\&\le|S|(u)+|S|(u')=|S|(\chi 1_{\mathfrak{A}}).\end{aligned}$$

So $|S|(u)-\sum_{k=0}^n|S(u\times v_k)|\le v'$ and $\||S|(u)-\sum_{k=0}^n|S(u\times v_k)|\|_1\le\epsilon$.

Now take any $w\in L^1_{\bar\nu}$ such that $0\le w\le\chi 1_{\mathfrak{B}}$. Again because T is a Riesz homomorphism and $T(\langle\chi 1_{\mathfrak{A}}\rangle_{i\in I})=\chi 1_{\mathfrak{B}}$, we can express w as $T(\langle u_i\rangle_{i\in I})$ where $0\le u_i\le\chi 1_{\mathfrak{A}}$ for every i. Consequently, setting $v_i'=|S|u_i-\sum_{k=0}^n|S(u_i\times v_k)|$ for each i, and $w'=T'(\langle v_i'\rangle_{i\in I})$,

$$\begin{aligned}|S|^\frown(w)=T'(\langle|S|u_i\rangle_{i\in I})&=T'(\langle\sum_{k=0}^n|S(u_i\times v_k)|+v_i'\rangle_{i\in I})\\&=\sum_{k=0}^n|T'(\langle S(u_i\times v_k)\rangle_{i\in I})|+T'(\langle v_i'\rangle_{i\in I})\\&=\sum_{k=0}^n|\hat ST(\langle u_i\times v_k\rangle_{i\in I})|+w'=\sum_{k=0}^n|\hat S\bigl(T(\langle u_i\rangle_{i\in I})\times T(\langle v_k\rangle_{i\in I})\bigr)|+w'\\&\le\sum_{k=0}^n|\hat S|\bigl(T(\langle u_i\rangle_{i\in I})\times T(\langle v_k\rangle_{i\in I})\bigr)+w'=|\hat S|(w)+w'\end{aligned}$$

because

$$\sum_{k=0}^n T(\langle v_k\rangle_{i\in I})=T(\langle\chi 1_{\mathfrak{A}}\rangle_{i\in I})=\chi 1_{\mathfrak{B}}.$$

But we also have $\|w'\|_1 = \lim_{i\to\mathcal{F}} \|v_i'\|_1 \le \epsilon$, while $|\hat{S}| \le |S|\hat{}$. So we conclude that $\||S|\hat{}(w) - |\hat{S}|(w)\|_1 \le \epsilon$; as ϵ is arbitrary, $|S|\hat{}(w) = |\hat{S}|(w)$.

This is true whenever $0 \le w \le \chi 1_{\mathfrak{B}}$. But as both $|S|\hat{}$ and $|\hat{S}|$ are continuous linear operators, and $L^\infty(\mathfrak{B})$ is dense in $L^1_{\bar{\nu}}$, $|S|\hat{} = |\hat{S}|$. As S is arbitrary, we have a Riesz homomorphism (352G).

377G Projective limits: Proposition Let $(I, \le)$, $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ and $\langle\pi_{ij}\rangle_{i\le j}$ be such that $(I, \le)$ is a non-empty upwards-directed partially ordered set, every $(\mathfrak{A}_i, \bar{\mu}_i)$ is a probability algebra, $\pi_{ij} : \mathfrak{A}_j \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism whenever $i \le j$ in I, and $\pi_{ik} = \pi_{ij}\pi_{jk}$ whenever $i \le j \le k$. Let $(\mathfrak{C}, \bar{\lambda}, \langle\pi_i\rangle_{i\in I})$ be the corresponding projective limit (328I). Write $L^1_{\bar{\mu}_i}$ for $L^1(\mathfrak{A}_i, \bar{\mu}_i)$ and $L^1_{\bar{\lambda}}$ for $L^1(\mathfrak{C}, \bar{\lambda})$. For $i \le j$ in I, let $T_{ij} : L^1_{\bar{\mu}_j} \to L^1_{\bar{\mu}_i}$ and $P_{ij} : L^1_{\bar{\mu}_i} \to L^1_{\bar{\mu}_j}$ be the norm-preserving Riesz homomorphism and the positive linear operator corresponding to $\pi_{ij} : \mathfrak{A}_j \to \mathfrak{A}_i$ (365N, 365O), and $T_i : L^1_{\bar{\lambda}} \to L^1_{\bar{\mu}_i}$, $P_i : L^1_{\bar{\mu}_i} \to L^1_{\bar{\lambda}}$ the operators corresponding to $\pi_i : \mathfrak{C} \to \mathfrak{A}_i$. Let X be any set.

(a) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar{\mu}_i} \to X$ such that $S_i T_{ij} = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar{\lambda}} \to X$ such that $S = S_i T_i$ for every $i \in I$.

(b) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar{\mu}_i}$ such that $T_{ij} S_j = S_i$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar{\lambda}}$ such that $T_i S = S_i$ for every $i \in I$.

(c) Suppose that X is a topological space, and for each $i \in I$ we are given a norm-continuous function $S_i : L^1_{\bar{\mu}_i} \to X$ such that $S_j P_{ij} = S_i$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar{\lambda}} \to X$ such that $SP_i = S_i$ for every $i \in I$.

(d) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar{\mu}_i}$ such that $P_{ij} S_i = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar{\lambda}}$ such that $S = P_i S_i$ for every $i \in I$.

proof: preliminary remarks (i) It will be helpful to recall some basic facts from §§328 and 365. If $i \le j$ in I, then by the definition of 'projective limit' we have $\pi_{ij}\pi_j = \pi_i$ so $T_{ij}T_j = T_i$ and $P_j P_{ij} = P_i$. Also $P_{ij}T_{ij}$ is the identity operator on $L^1_{\bar{\mu}_j}$, and $P_i T_i$ is the identity operator on $L^1_{\bar{\lambda}}$.

(ii) At a deeper level, we have useful concretizations of $(\mathfrak{C}, \bar{\lambda})$, as follows. Fix $i \in I$ for the moment. For $j \ge i$, set $\mathfrak{B}_j = \pi_{ij}[\mathfrak{A}_j]$, $\bar{\nu}_j = \bar{\mu}_i \restriction \mathfrak{B}_j$; then $\mathfrak{B}_j$ is a closed subalgebra of $\mathfrak{A}_i$, isomorphic (as probability algebra) to $\mathfrak{A}_j$. If $u \in L^1_{\bar{\mu}_i}$ and $b \in \mathfrak{B}_j$, set $b' = \pi_{ij}^{-1} b \in \mathfrak{A}_j$; then

$$\int_b u = \int_{\pi_{ij}b'} u = \int_{b'} P_{ij}u = \int_b T_{ij}P_{ij}u;$$

thus $T_{ij}P_{ij}$ is the conditional expectation $P_{\mathfrak{B}_j} : L^1_{\bar{\mu}_i} \to L^1(\mathfrak{B}_j, \bar{\nu}_j)$, identifying $L^1(\mathfrak{B}_j, \bar{\nu}_j)$ with $L^1_{\bar{\mu}_i} \cap L^0(\mathfrak{B}_j)$ as in 365Qa.

If $k \ge j$, then $\pi_{ik} = \pi_{ij}\pi_{jk}$ so $\mathfrak{B}_k \subseteq \mathfrak{B}_j$; thus $\mathbb{B} = \{\mathfrak{B}_j : j \ge i\}$ is downwards-directed. Set $\mathfrak{D} = \bigcap \mathbb{B}$, $\bar{\nu} = \bar{\mu} \restriction \mathfrak{D}$.

For $k \ge i$, set $\phi_k = \pi_{ik}^{-1} \restriction \mathfrak{D} : \mathfrak{D} \to \mathfrak{A}_k$; then ϕ_k is a measure-preserving Boolean homomorphism, and $\phi_j = \pi_{jk}\phi_k$ whenever $i \le j \le k$. We can therefore define $\phi_j : \mathfrak{D} \to \mathfrak{A}_j$, for any $j \in I$, by saying that $\phi_j = \pi_{jk}\phi_k$ whenever $k \in I$ is greater than or equal to both i and j, and we shall have $\phi_j = \pi_{jk}\phi_k$ whenever $j \le k$ in I. **P** If $j \in I$ and k_0, k_1 are two upper bounds of $\{i, j\}$ in I, take an upper bound k of $\{k_0, k_1\}$; then

$$\pi_{jk_0}\phi_{k_0} = \pi_{jk_0}\pi_{k_0k}\phi_k = \pi_{jk}\phi_k = \pi_{jk_1}\pi_{k_1k}\phi_k = \pi_{jk_1}\phi_{k_1},$$

so ϕ_j is well-defined. If j, $k \in I$ and $j \le k$, let k' be an upper bound of $\{i, k\}$; then

$$\pi_{jk}\phi_k = \pi_{jk}\pi_{kk'}\phi_{k'} = \pi_{jk'}\phi_{k'} = \phi_j. \quad \mathbf{Q}$$

Of course every ϕ_j is a measure-preserving Boolean homomorphism.

By the definition of $(\mathfrak{C}, \bar{\lambda})$, there is a measure-preserving Boolean homomorphism $\phi : \mathfrak{D} \to \mathfrak{C}$ such that $\pi_j\phi = \phi_j$ for every $j \in I$. In this case, $\pi_i\phi = \phi_i$ is the identity embedding of $\mathfrak{D}$ in $\mathfrak{A}_i$, and $\pi_i[\mathfrak{C}] = \mathfrak{D}$. Accordingly $P_{\mathfrak{D}} = T_i P_i$. By the generalized reverse martingale theorem 367Qa, $T_i P_i$ is the limit of $P_{\mathfrak{B}}$ as $\mathfrak{B}$ decreases in $\mathbb{B}$, in the sense that for every $u \in L^1(\mathfrak{A}_i)$ and $\epsilon > 0$ there is a $j \ge i$ in I such that

$$\|T_i P_i u - T_{ik}P_{ik}u\|_1 = \|P_{\mathfrak{D}}u - P_{\mathfrak{B}_k}u\|_1 \le \epsilon$$

whenever $k \ge j$ in I. If we write $\mathcal{F}(I\uparrow)$ for the filter on I generated by $\{\{k : k \ge j\} : j \in I\}$, we have

$$T_i P_i u = \lim_{j\to\mathcal{F}(I\uparrow)} T_{ij}P_{ij}u,$$

for the norm in $L^1_{\bar\mu_i}$, for every $u \in L^1_{\bar\mu_i}$.

Now let us turn to (a)-(d) as listed above.

(a) All we have to know is that

$$S_iT_i = S_iT_{ij}T_j = S_jT_j$$

whenever $i \le j$ in I; because I is upwards-directed, $S_iT_i = S_jT_j$ for all i, $j \in I$, and we have a sound definition for S.

(b) The point is that $T_iP_iS_i = S_i$ for every $i \in I$. **P** For $j \ge i$,

$$T_{ij}P_{ij}S_i = T_{ij}P_{ij}T_{ij}S_j = T_{ij}S_j = S_i.$$

If $x \in X$,

$$T_iP_iS_ix = \lim_{j\to\mathcal{F}(I\uparrow)} T_{ij}P_{ij}S_ix = S_ix. \textbf{ Q}$$

If now $i \le j$ in I,

$$P_iS_i = P_jP_{ij}T_{ij}S_j = P_jS_j.$$

As I is upwards-directed, $P_iS_i = P_jS_j$ for all i, $j \in I$; write S for this common value. Then

$$T_iS = T_iP_iS_i = S_i$$

for every $i \in I$. As T_i is injective for every $i \in I$, the formula uniquely defines the function S.

(c) This time, we have $S_iT_iP_i = S_i$ for every $i \in I$. **P** For any $u \in L^1_{\bar\lambda}$,

$$S_iT_iP_iu = \lim_{j\to\mathcal{F}(I\uparrow)} S_iT_{ij}P_{ij}u$$

(because S_i is continuous)

$$= \lim_{j\to\mathcal{F}(I\uparrow)} S_jP_{ij}T_{ij}P_{ij}u = \lim_{j\to\mathcal{F}(I\uparrow)} S_jP_{ij}u = S_iu. \textbf{ Q}$$

If $i \le j$ in I,

$$S_iT_i = S_jP_{ij}T_{ij}T_j = S_jT_j;$$

consequently $S_iT_i = S_jT_j$ for all i, $j \in I$, and we can call this common function S. In this case, $SP_i = S_iT_iP_i = S_i$ for every $i \in I$. Since $P_i[L^1_{\bar\mu_i}] = L^1_{\bar\lambda}$, this defines S uniquely.

(d) As in (a), all we have to check is that if $i \le j$ in I then

$$P_jS_j = P_jP_{ij}S_i = P_iS_i.$$

377H Inductive limits: Proposition Let $(I, \le)$, $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ and $\langle\pi_{ji}\rangle_{i\le j}$ be such that $(I, \le)$ is a non-empty upwards-directed partially ordered set, every $(\mathfrak{A}_i, \bar\mu_i)$ is a probability algebra, $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$ is a measure-preserving Boolean homomorphism whenever $i \le j$ in I, and $\pi_{ki} = \pi_{kj}\pi_{ji}$ whenever $i \le j \le k$. Let $(\mathfrak{C}, \bar\lambda, \langle\pi_i\rangle_{i\in I})$ be the corresponding inductive limit (328H). Write $L^1_{\bar\mu_i}$ for $L^1(\mathfrak{A}_i, \bar\mu_i)$ and $L^1_{\bar\lambda}$ for $L^1(\mathfrak{C}, \bar\lambda)$. For $i \le j$ in I, let $T_{ji} : L^1_{\bar\mu_i} \to L^1_{\bar\mu_j}$ and $P_{ji} : L^1_{\bar\mu_j} \to L^1_{\bar\mu_i}$ be the Riesz homomorphism and the positive linear operator corresponding to $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$, and $T_i : L^1_{\bar\mu_i} \to L^1_{\bar\lambda}$, $P_i : L^1_{\bar\lambda} \to L^1_{\bar\mu_i}$ the operators corresponding to $\pi_i : \mathfrak{A}_i \to \mathfrak{C}$. Let X be a set.

(a) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_jT_{ji} = S_i$ whenever $i \le j$ in I. Then there is a function $S : L^1_{\bar\lambda} \to X$ such that $S_i = ST_i$ for every $i \in I$.

(b) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $T_{ji}S_i = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : X \to L^1_{\bar\lambda}$ such that $T_iS_i = S$ for every $i \in I$.

(c) Suppose that for each $i \in I$ we are given a function $S_i : L^1_{\bar\mu_i} \to X$ such that $S_iP_{ji} = S_j$ whenever $i \le j$ in I. Then there is a unique function $S : L^1_{\bar\lambda} \to X$ such that $S = S_iP_i$ for every $i \in I$.

(d) Suppose that for each $i \in I$ we are given a function $S_i : X \to L^1_{\bar\mu_i}$ such that $P_{ji}S_j = S_i$ whenever $i \le j$ in I, and that

$$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|S_ix|-k\chi 1_{\mathfrak{A}_i})^+=0$$

for every $x\in X$. Then there is a unique function $S:X\to L^1_{\bar\lambda}$ such that $S_i=P_iS$ for every $i\in I$.

proof We can follow the same programme as in the proof of 377G, but with a couple of new twists.

preliminary remarks (i) If $i\le j$ in I, then by the definition of 'inductive limit' we have $\pi_j\pi_{ji}=\pi_i$ so $T_jT_{ji}=T_i$ and $P_{ji}P_j=P_i$. $P_{ji}T_{ji}$ and P_iT_i are the identity operator on $L^1_{\bar\mu_i}$.

(ii) Let $\mathcal{F}(I\uparrow)$ be the filter on I generated by $\{\{k:k\ge j\}:j\in I\}$. Then $\lim_{i\to\mathcal{F}(I\uparrow)}T_iP_iu=u$ for every $u\in L^1_{\bar\lambda}$. **P** Setting $\mathfrak{B}_i=T_i[\mathfrak{A}_i]$ for each $i\in I$, $\mathbb{B}=\{\mathfrak{B}_i:i\in I\}$ is an upwards-directed family of closed subalgebras of $\mathfrak{C}$; set $\mathfrak{D}=\overline{\bigcup\mathbb{B}}$ and $\bar\nu=\bar\lambda\restriction\mathfrak{D}$, so that $(\mathfrak{D},\bar\nu)$ is a probability algebra. Since $\pi_i:\mathfrak{A}_i\to\mathfrak{D}$ is a measure-preserving Boolean homomorphism and $\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$ in I, there is a measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{D}$ such that $\phi\pi_i=\pi_i$ for every i. But this means that $\mathfrak{C}=\mathfrak{D}$.

As in 377G, we can identify each $T_iP_i:L^1_{\bar\lambda}\to L^1_{\bar\lambda}$ with the conditional expectation $P_{\mathfrak{B}_i}$. This time, 367Qb tells us that $P_{\mathfrak{B}}u\to P_{\mathfrak{D}}u=u$ as $\mathfrak{B}$ increases through $\mathbb{B}$, that is, $u=\lim_{i\to\mathcal{F}(I\uparrow)}T_iP_iu$, for every $u\in L^1_{\bar\lambda}$. **Q**

(a) The point is that if i, $j\in I$, $u\in L^1_{\bar\mu_i}$, $v\in L^1_{\bar\mu_j}$ and $T_iu=T_jv$, then $S_iu=S_jv$. **P** Let $k\in I$ be such that $i\le k$ and $j\le k$. Then

$$T_kT_{ki}u=T_iu=T_jv=T_kT_{kj}v;$$

since T_k is injective, $T_{ki}u=T_{kj}v$. Accordingly

$$S_iu=S_kT_{ki}u=S_kT_{kj}v=S_jv.\ \mathbf{Q}$$

There is therefore a function $S':\bigcup_{i\in I}S_i[L^1_{\bar\mu_i}]\to X$ defined by saying that $S(T_iu)=S_iu$ whenever $i\in I$ and $u\in L^1_{\bar\mu_i}$; extending S' arbitrarily to a function $S:L^1_{\bar\lambda}\to X$, we get the result.

(b) All we have to do is to check that if $i\le j$ in I, then

$$T_iS_i=T_jT_{ji}S_i=T_jS_j.$$

(c) In this case, we have

$$S_jP_j=S_iP_{ji}P_j=S_iP_i$$

whenever $i\le j$ in I.

(d)(i) For each $x\in X$, $\{T_iS_ix:i\in I\}\subseteq L^1_{\bar\lambda}$ is uniformly integrable. **P** If $k\in\mathbb{N}$ and $i\in I$,

$$|T_iS_ix|\le T_i(|S_ix|-k\chi 1_{\mathfrak{A}_i})^++T_i(k\chi 1_{\mathfrak{A}_i})\le T_i(|S_ix|-k\chi 1_{\mathfrak{A}_i})^++k\chi 1_{\mathfrak{C}},$$

so

$$\int(|T_iS_ix|-k\chi 1_{\mathfrak{C}})^+\le\int T_i(|S_ix|-k\chi 1_{\mathfrak{A}_i})^+=\int(|S_ix|-k\chi 1_{\mathfrak{A}_i})^+.$$

Accordingly

$$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|T_iS_ix|-k\chi 1_{\mathfrak{C}})^+\le\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|S_ix|-k\chi 1_{\mathfrak{A}_i})^+=0.\ \mathbf{Q}$$

(ii) Fix an ultrafilter $\mathcal{G}$ on I including $\mathcal{F}(I\uparrow)$. For each $x\in X$, $\{T_iS_ix:i\in I\}$ is relatively weakly compact in $L^1_{\bar\lambda}$, so $Sx=\lim_{i\to\mathcal{G}}T_iS_ix$ is defined for the weak topology on $L^1_{\bar\lambda}$. Now for any $i\in I$,

$$P_iSx=\lim_{j\to\mathcal{G}}P_iT_jS_jx$$

(for the weak topology on $L^1_{\bar\mu_i}$)

$$=\lim_{j\to\mathcal{G}}P_{ji}P_jT_jS_jx$$

(because $\{j:j\ge i\}\in\mathcal{G}$)

$$=\lim_{j\to\mathcal{G}}P_{ji}S_jx=S_ix.$$

(iii) To see that S is uniquely defined, it is enough to recall that

$$Sx = \lim_{i\to\mathcal{F}(I\uparrow)} T_iP_iSx = \lim_{i\to\mathcal{F}(I\uparrow)} T_iS_ix$$

is uniquely defined by the family $\langle S_ix\rangle_{i\in I}$, for every $x \in X$.

377X Basic exercises (a) In 377B, show that $\langle u_i\rangle_{i\in I} \in \prod_{i\in I} L^0(\mathfrak{A}_i)$ belongs to W_0 iff $\{u_i^* : i \in I\}$ is bounded above in $L^0(\mathfrak{A}_L)$, where $\mathfrak{A}_L$ is the measure algebra of Lebesgue measure on $[0,\infty[$, and u_i^* is the decreasing rearrangement of u_i for each i (373C).

(b) In 377D, suppose that $u = \langle u_i\rangle_{i\in I}$ and $v = \langle v_i\rangle_{i\in I}$ belong to W_2, and that at least one of $|u|^2$, $|v|^2$ belongs to W_{ui}. Show that $(Tu|Tv) = \lim_{i\to\mathcal{F}}(u_i|v_i)$.

(c) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and suppose that we have $u_i \in L^1(\mathfrak{A}_i,\bar\mu_i)$ for each i. Show that the following are equiveridical: (i) $\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i| - k\chi 1_{\mathfrak{A}_i})^+ = 0$; (ii) $\{u_i^* : i \in I\}$ is uniformly integrable in $L^1(\mu_L)$, where μ_L is Lebesgue measure on $[0,\infty[$, and u_i^* is the decreasing rearrangement of u_i for each $i \in I$.

(d) Take any $p \in]1,\infty[$. Show that 377G remains true if we replace every 'L^1' by 'L^p'.

(e) Take any $p \in]1,\infty[$. Show that 377H remains true if we replace every 'L^1' by 'L^p' and in part (d) we replace '$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|S_ix| - k\chi 1_{\mathfrak{A}_i})^+ = 0$' by '$\sup_{i\in I}\|S_ix\|_p < \infty$'.

(f) In 377Ha, suppose that X has a metric ρ under which it is complete, and that $\langle S_i\rangle_{i\in I}$ is uniformly equicontinuous in the sense that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\rho(S_iu, S_iv) \le \epsilon$ whenever $i \in I$, u, $v \in L^1_{\bar\mu_i}$ and $\|u - v\|_1 \le \delta$. Show that there is a unique continuous function $S : L^1_{\bar\lambda} \to X$ such that $S_i = ST_i$ for every $i \in I$.

377Y Further exercises (a) Find a non-empty family $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ of probability algebras, a probability algebra $(\mathfrak{B},\bar\nu)$, a Boolean homomorphism $\pi : \prod_{i\in I}\mathfrak{A}_i \to \mathfrak{B}$ such that $\bar\nu\pi(\langle a_i\rangle_{i\in I}) \le \sup_{i\in I}\bar\mu_ia_i$ whenever $\langle a_i\rangle_{i\in I} \in \prod_{i\in I}\mathfrak{A}_i$, and an element $u = \langle u_i\rangle_{i\in I}$ of W_0^+, as described in 377B, such that $\|Tu\|_1 > \sup_{i\in I}\|u_i\|_1$, where $T : W_0 \to L^0(\mathfrak{B})$ is the Riesz homomorphism of 377B-377C. (*Hint*: $\#(I) = 2$.)

(b) Show that if, in 377Gc, we omit the hypothesis that the S_i are to be continuous, then the result can fail.

(c) Let $\langle U_i\rangle_{i\in I}$ be a non-empty family of L-spaces and $\mathcal{F}$ an ultrafilter on I. (i) Show that $\prod_{i\in I} U_i$ is a Dedekind complete Riesz space (see 352K) in which $W_\infty = \{\langle u_i\rangle_{i\in I} : \sup_{i\in I}\|u_i\| < \infty\}$ is a solid linear subspace. (ii) Let $W_0 \subseteq W_\infty$ be $\{\{\langle u_i\rangle_{i\in I} : \sup_{i\in I}\|u_i\| < \infty, \lim_{i\to\mathcal{F}}\|u_i\| = 0\}$; show that W_0 is a solid linear subspace of W_∞. (iii) Let U be the quotient Riesz space W_∞/W_0 (352U). Show that U is an L-space under the norm $\|\langle u_i\rangle^\bullet_{i\in I}\| = \lim_{i\to\mathcal{F}}\|u_i\|$ for $\langle u_i\rangle_{i\in I} \in W_\infty$.

(d) Let V be a normed space, and suppose that for every finite-dimensional subspace V_0 of V there are an L-space U and a norm-preserving linear map $T : V_0 \to U$. Show that there are an L-space U and a norm-preserving linear map $T : V \to U$.

377 Notes and comments Although my main target in this section has been to understand the function spaces of reduced products of probability algebras, I have as usual felt that the ideas are clearer if each is developed in a context closer to the most general case in which it is applicable. Only in part (b) of the proof of 377C, I think, does this involve us in extra work.

The new techniques of this section are forced on us by the fact that we are looking at Boolean homomorphisms $\pi : \prod_{i\in I}\mathfrak{A}_i \to \mathfrak{B}$ which are not normally sequentially order-continuous. While we have a natural Riesz homomorphism from $L^\infty(\prod_{i\in I}\mathfrak{A}_i)$ to $L^\infty(\mathfrak{B})$, as in 363F, we cannot expect a similar operator from the whole of $L^0(\prod_{i\in I}\mathfrak{A}_i) \cong \prod_{i\in I} L^0(\mathfrak{A}_i)$ to $L^0(\mathfrak{B})$. However the condition '$\bar\nu\pi(\langle a_i\rangle_{i\in I}) \le \sup_{i\in I}\bar\mu_ia_i$' ensures that there is a space $W_0 \subseteq \prod_{i\in I} L^0(\mathfrak{A}_i)$ on which an operator to $L^0(\mathfrak{B})$ can be defined, and which is large enough to give us a method of investigating the spaces $L^p(\mathfrak{B},\bar{\nu})$ as images of subspaces W_p of products $\prod_{i\in I} L^p(\mathfrak{A}_i,\bar{\mu}_i)$.

In 377E, the case $p=1$ is special because we can identify W_{ui} as the space of relatively weakly compact families in $L^1(\mathfrak{A},\bar{\mu})$, and for such a family $u=\langle u_i\rangle_{i\in I}$ we have $\|Tu\|_1=\lim_{i\to\mathcal{F}}\|u_i\|_1$. So the Banach space $L^1(\mathfrak{B},\bar{\nu})$ is a kind of reduced power, describable in terms of the normed space $L^1(\mathfrak{A},\bar{\nu})$. For other L^p spaces we need to know something more, e.g., the lattice structure, if we are to identify those $u\in W_p$ such that $Tu=0$. The difference becomes significant when we come to look at morphisms of $L^p(\mathfrak{B},\bar{\nu})$ corresponding to morphisms of $L^p(\mathfrak{A},\bar{\mu})$, as in 377F.

In 377G-377H I give a string of results which are visibly mass-produced. What is striking is that in eight cases out of eight we have a straightforward formula corresponding to the idea that $(\mathfrak{C},\bar{\lambda})$ is a limit of $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$. What is curious is that in two of the eight cases (377Gc, 377Hd) we have to impose different special conditions on the functions S_i which the target S is supposed to approximate, and in just one case (377Ha) the target S is not uniquely defined in the absence of further constraints (377Xf). I think the ideas take up enough room when given only in their application to L^1 spaces, but of course there are versions, only slightly modified, which apply to other L^p spaces (377Xd-377Xe).

The repeated conditions of the form

$$\inf_{k\in\mathbb{N}}\sup_{i\in I}\bar{\mu}_i[\![|u_i|>k]\!]=0,$$

$$\inf_{k\in\mathbb{N}}\sup_{i\in I}\int(|u_i|-k\chi 1_{\mathfrak{A}_i})^+=0,$$

(377B, 377Dc, 377Hd) both have expressions in terms of decreasing rearrangements (377Xa, 377Xc). The latter is clearly associated with uniform integrability and weak compactness, and unsurprisingly we use it to show that a weak limit will be defined. The former is there to ensure that a set appearing in an L^0 space will be bounded above, so that we can apply 355F to extend a Riesz homomorphism.

Version of 7.12.08

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

372I The version of the Ergodic Theorem in 372I, referred to in the 2003 and 2006 editions of Volume 4, is now 372H.

372K The version of the Ergodic Theorem in 372K, referred to in the 2003 and 2006 editions of Volume 4, is now 372J.

372P Mixing and ergodic transformations The definitions in 372P are now in 372O.

372Xm The tent map, referred to in the 2003 and 2006 editions of Volume 4, is now in 372Xp.

Version of 25.8.17

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