Chapter 36

Function Spaces

Chapter 24 of Volume 2 was devoted to the elementary theory of the 'function spaces' L^0, L^1, L^2 and L^∞ associated with a given measure space. In this chapter I return to these spaces to show how they can be related to the more abstract themes of the present volume. In particular, I develop constructions to demonstrate, as clearly as I can, the way in which the function spaces associated with a measure space in fact depend only on its measure algebra; and how many of their features can (in my view) best be understood in terms of constructions involving measure algebras.

The chapter is very long, not because there are many essentially new ideas, but because the intuitions I seek to develop depend, for their logical foundations, on technically complex arguments. This is perhaps best exemplified by §364. If two measure spaces (X,Σ,μ) and (Y,T,ν) have isomorphic measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ then the spaces $L^0(\mu)$, $L^0(\nu)$ are isomorphic as topological f-algebras; and more: for any isomorphism between $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ there is a unique corresponding isomorphism between the L^0 spaces. The intuition involved is in a way very simple. If f, g are measurable real-valued functions on X and Y respectively, then $f^\bullet\in L^0(\mu)$ will correspond to $g^\bullet\in L^0(\nu)$ if and only if $[\![f^\bullet>\alpha]\!]=\{x:f(x)>\alpha\}^\bullet\in\mathfrak{A}$ corresponds to $[\![g^\bullet>\alpha]\!]=\{y:g(y)>\alpha\}^\bullet\in\mathfrak{B}$ for every α. But the check that this formula is consistent, and defines an isomorphism of the required kind, involves a good deal of detailed work. It turns out, in fact, that the *measures* μ and ν do not enter this part of the argument at all, except through their ideals of negligible sets (used in the construction of $\mathfrak{A}$ and $\mathfrak{B}$). This is already evident, if you look for it, in the theory of $L^0(\mu)$; in §241, as written out, you will find that the measure of an individual set is not once mentioned, except in the exercises. Consequently there is an invitation to develop the theory with algebras $\mathfrak{A}$ which are not necessarily measure algebras. Here is another reason for the length of the chapter; substantial parts of the work are being done in greater generality than the corresponding sections of Chapter 24, necessitating a degree of repetition. Of course this is not 'measure theory' in the strict sense; but for thirty years now measure theory has been coloured by the existence of these generalizations, and I think it is useful to understand which parts of the theory apply only to measure algebras, and which can be extended to other σ-complete Boolean algebras, like the algebraic theory of L^0, or even to all Boolean algebras, like the theory of L^∞.

Here, then, are two of the objectives of this chapter: first, to express the ideas of Chapter 24 in ways making explicit their independence of particular measure spaces, by setting up constructions based exclusively on the measure algebras involved; second, to set out some natural generalizations to other algebras. But to justify the effort needed I ought to point to some mathematically significant idea which demands these constructions for its expression, and here I mention the categorical nature of the constructions. Between Boolean algebras we have a variety of natural and important classes of 'morphism'; for instance, the Boolean homomorphisms and the order-continuous Boolean homomorphisms; while between measure algebras we have in addition the measure-preserving Boolean homomorphisms. Now it turns out that if we construct the L^p spaces in the natural ways then morphisms between the underlying algebras give rise to morphisms between their L^p spaces. For instance, any Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces a multiplicative norm-contractive Riesz homomorphism from $L^\infty(\mathfrak{A})$ to $L^\infty(\mathfrak{B})$; if $\mathfrak{A}$ and $\mathfrak{B}$ are Dedekind σ-complete, then any sequentially order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces a sequentially order-continuous multiplicative Riesz homomorphism from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$; and if $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are measure algebras, then any measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces norm-preserving Riesz homomorphisms from $L^p(\mathfrak{A},\bar\mu)$ to $L^p(\mathfrak{B},\bar\nu)$ for every $p\in[1,\infty]$. All of these are 'functors', that is, a composition of homomorphisms between algebras gives rise to a composition of the corresponding operators between their function spaces, and are 'covariant', that is, a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ leads to an operator from $L^p(\mathfrak{A})$ to $L^p(\mathfrak{B})$. But the same constructions lead us to a functor which is 'contravariant': starting from an order-continuous Boolean homomorphism from a semi-finite measure algebra $(\mathfrak{A},\bar{\mu})$ to a measure algebra $(\mathfrak{B},\bar{\nu})$, we have an operator from $L^1(\mathfrak{B},\bar{\nu})$ to $L^1(\mathfrak{A},\bar{\mu})$. This last is in fact a kind of conditional expectation operator. In my view it is not possible to make sense of the theory of measure-preserving transformations without at least an intuitive grasp of these ideas.

Another theme is the characterization of each construction in terms of universal mapping theorems: for instance, each L^p space, for $1 \le p \le \infty$, can be characterized as Banach lattice in terms of factorizations of functions of an appropriate class from the underlying algebra to Banach lattices.

Now let me try to sketch a route-map for the journey ahead. I begin with two sections on the space $S(\mathfrak{A})$; this construction applies to any Boolean algebra (indeed, any Boolean ring), and corresponds to the space of 'simple functions' on a measure space. Just because it is especially close to the algebra (or ring) $\mathfrak{A}$, there is a particularly large number of universal mapping theorems corresponding to different aspects of its structure (§361). In §362 I seek to relate ideas on additive functionals on Boolean algebras from Chapter 23 and §§326-327 to the theory of Riesz space duals in §356. I then turn to a systematic discussion of the function spaces of Chapter 24: L^∞ (§363), L^0 (§364), L^1 (§365) and other L^p (§366), followed by an account of convergence in measure (§367). While all these sections are dominated by the objectives sketched in the paragraphs above, I do include a few major theorems not covered by the ideas of Volume 2, such as the Kelley-Nachbin characterization of the Banach spaces $L^\infty(\mathfrak{A})$ for Dedekind complete $\mathfrak{A}$ (363R). In the last two sections of the chapter I turn to the use of L^0 spaces in the representation of Archimedean Riesz spaces (§368) and of Banach lattices which are separated by their order-continuous duals (§369).

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This is the fundamental Riesz space associated with a Boolean ring $\mathfrak{A}$. When $\mathfrak{A}$ is a ring of sets, $S(\mathfrak{A})$ can be regarded as the linear space of 'simple functions' generated by the indicator functions of members of $\mathfrak{A}$ (361L). Its most important property is the universal mapping theorem 361F, which establishes a one-to-one correspondence between (finitely) additive functions on $\mathfrak{A}$ (361B-361C) and linear operators on $S(\mathfrak{A})$. Simple universal mapping theorems of this type can be interesting, but do not by themselves lead to new insights; what makes this one important is the fact that $S(\mathfrak{A})$ has a canonical Riesz space structure, norm and multiplication (361E). From this we can deduce universal mapping theorems for many other classes of function (361G, 361H, 361I, 361Xb). While the exact construction of $S(\mathfrak{A})$ can be varied (361D, 361L, 361M, 361Ya), its structure is uniquely defined, so homomorphisms between Boolean rings correspond to maps between their $S(\,)$-spaces (361J), and (when $\mathfrak{A}$ is a Boolean algebra) $\mathfrak{A}$ can be recovered from the Riesz space $S(\mathfrak{A})$ as the algebra of its projection bands (361K).

361A Boolean rings (b) If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a function, then the following are equiveridical: (i) π is a ring homomorphism; (ii) $\pi(a \setminus b) = \pi a \setminus \pi b$ for all a, $b \in \mathfrak{A}$; (iii) π is a lattice homomorphism and $\pi 0 = 0$.

(c) If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a ring homomorphism, then π is order-continuous iff $\inf \pi[A] = 0$ whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$ in $\mathfrak{A}$; while π is sequentially order-continuous iff $\inf_{n\in\mathbb{N}} \pi a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

(d) If $(\mathfrak{A},\bar{\mu})$ is a measure algebra, then the ideal $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\, \bar{\mu}a < \infty\}$ is a Boolean ring in its own right. Now suppose that $(\mathfrak{B},\bar{\nu})$ is another measure algebra and $\mathfrak{B}^f \subseteq \mathfrak{B}$ the corresponding ring of elements of finite measure. We can say that a ring homomorphism $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ is **measure-preserving** if $\bar{\nu}\pi a = \bar{\mu}a$ for every $a \in \mathfrak{A}^f$. In this case π is order-continuous.

361B Definition Let $\mathfrak{A}$ be a Boolean ring and U a linear space. A function $\nu : \mathfrak{A} \to U$ is **finitely additive**, or just **additive**, if $\nu(a \cup b) = \nu a + \nu b$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$.

361C Elementary facts Let $\mathfrak{A}$ be a Boolean ring, U a linear space and $\nu : \mathfrak{A} \to U$ an additive function.

(a) $\nu 0 = 0$.

(b) If $a_0, \dots , a_m$ are disjoint in $\mathfrak{A}$, then $\nu(\sup_{j\le m} a_j) = \sum_{j=0}^{m} \nu a_j$.

(c) If $\mathfrak{B}$ is another Boolean ring and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a ring homomorphism, then $\nu\pi : \mathfrak{B} \to U$ is additive. In particular, if $\mathfrak{B}$ is a subring of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B} : \mathfrak{B} \to U$ is additive.

(d) If V is another linear space and $T : U \to V$ is a linear operator, then $T\nu : \mathfrak{A} \to V$ is additive.

(e) If U is a partially ordered linear space, then ν is order-preserving iff it is non-negative, that is, $\nu a \ge 0$ for every $a \in \mathfrak{A}$.

(f) If U is a partially ordered linear space and ν is non-negative, then (i) ν is order-continuous iff $\inf \nu[A] = 0$ whenever $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0 (ii) ν is sequentially order-continuous iff $\inf_{n\in\mathbb{N}} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

361D Construction Let $\mathfrak{A}$ be a Boolean ring, and Z its Stone space. For $a \in \mathfrak{A}$ write χa for the indicator function of the open-and-compact subset $\widehat{a}$ of Z corresponding to a. Note that $\chi a = 0$ iff $a = 0$. Let $S(\mathfrak{A})$ be the linear subspace of $\mathbb{R}^Z$ generated by $\{\chi a : a \in \mathfrak{A}\}$. $S(\mathfrak{A})$ is a subspace of the M-space $\ell^\infty(Z)$ of all bounded real-valued functions on Z, and $\| \|_\infty$ is a norm on $S(\mathfrak{A})$. $S(\mathfrak{A})$ is closed under $\times$.

361E Proposition Let $\mathfrak{A}$ be a Boolean ring, with Stone space Z. Write S for $S(\mathfrak{A})$.

(a) If $a_0, \dots , a_n \in \mathfrak{A}$, there are disjoint $b_0, \dots , b_m$ such that each a_i is expressible as the supremum of some of the b_j.

(b) If $u \in S$, it is expressible in the form $\sum_{j=0}^{m} \beta_j\chi b_j$ where $b_0, \dots , b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j \in \mathbb{R}$ for each j. If all the b_j are non-zero then $\|u\|_\infty = \sup_{j\le m} |\beta_j|$.

(c) If $u \in S$ is non-negative, it is expressible in the form $\sum_{j=0}^{m} \beta_j\chi b_j$ where $b_0, \dots , b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j \ge 0$ for each j, and simultaneously in the form $\sum_{j=0}^{m} \gamma_j\chi c_j$ where $c_0 \supseteq c_1 \supseteq \dots \supseteq c_m$ and $\gamma_j \ge 0$ for every j.

(d) If $u = \sum_{j=0}^{m} \beta_j\chi b_j$ where $b_0, \dots , b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j \in \mathbb{R}$ for each j, then $|u| = \sum_{j=0}^{m} |\beta_j|\chi b_j \in S$.

(e) S is a Riesz subspace of $\mathbb{R}^Z$; in its own right, it is an Archimedean Riesz space. If $\mathfrak{A}$ is a Boolean algebra, then S has an order unit $\chi 1$ and $\|u\|_\infty = \min\{\alpha : \alpha \ge 0,\ |u| \le \alpha\chi 1\}$ for every $u \in S$.

(f) The map $\chi : \mathfrak{A} \to S$ is injective, additive, non-negative, a lattice homomorphism and order-continuous.

(g) Suppose that $u \ge 0$ in S and $\delta \ge 0$ in $\mathbb{R}$. Then

$$[\![u > \delta]\!] = \max\{a : a \in \mathfrak{A},\ (\delta + \eta)\chi a \le u \text{ for some } \eta > 0\}$$

is defined in $\mathfrak{A}$, and

$$\delta\chi[\![u > \delta]\!] \le u \le \delta\chi[\![u > 0]\!] \vee \|u\|_\infty[\![u > \delta]\!].$$

In particular, $u \le \|u\|_\infty\chi[\![u > 0]\!]$ and there is an $\eta > 0$ such that $\eta\chi[\![u > 0]\!] \le u$. If u, $v \ge 0$ in S then $u \wedge v = 0$ iff $[\![u > 0]\!] \cap [\![v > 0]\!] = 0$.

(h) Under $\times$, S is an f-algebra and a commutative normed algebra.

(i) For any $u \in S$, $u \ge 0$ iff $u = v \times v$ for some $v \in S$.

361F Theorem Let $\mathfrak{A}$ be a Boolean ring, and U any linear space. Then there is a one-to-one correspondence between additive functions $\nu : \mathfrak{A} \to U$ and linear operators $T : S(\mathfrak{A}) \to U$, given by the formula $\nu = T\chi$.

361G Theorem Let $\mathfrak{A}$ be a Boolean ring, and U a partially ordered linear space. Let $\nu : \mathfrak{A} \to U$ be an additive function, and $T : S(\mathfrak{A}) \to U$ the corresponding linear operator.

(a) ν is non-negative iff T is positive.

(b) In this case,

(i) if T is order-continuous or sequentially order-continuous, so is ν;

(ii) if U is Archimedean and ν is order-continuous or sequentially order-continuous, so is T.

(c) If U is a Riesz space, then the following are equiveridical:

(i) T is a Riesz homomorphism;

(ii) $\nu a \wedge \nu b = 0$ in U whenever $a \cap b = 0$ in $\mathfrak{A}$;

(iii) ν is a lattice homomorphism.

361H Theorem Let $\mathfrak{A}$ be a Boolean ring and U a Dedekind complete Riesz space. Suppose that $\nu : \mathfrak{A} \to U$ is an additive function and $T : S = S(\mathfrak{A}) \to U$ is the corresponding linear operator. Then $T \in \mathsf{L}^{\sim} = \mathsf{L}^{\sim}(S;U)$ iff $\{\nu b : b \subseteq a\}$ is order-bounded in U for every $a \in \mathfrak{A}$, and in this case $|T| \in \mathsf{L}^{\sim}$ corresponds to $|\nu| : \mathfrak{A} \to U$, defined by setting

$$\begin{aligned}|\nu|(a) &= \sup\{\sum_{j=0}^{n} |\nu a_i| : a_0, \dots , a_n \subseteq a \text{ are disjoint}\}\\ &= \sup\{\nu b - \nu(a \setminus b) : b \subseteq a\}\end{aligned}$$

for every $a \in \mathfrak{A}$.

361I Theorem Let $\mathfrak{A}$ be a Boolean ring, U a normed space and $\nu : \mathfrak{A} \to U$ an additive function. Give $S = S(\mathfrak{A})$ its norm $\| \|_\infty$, and let $T : S \to U$ be the linear operator corresponding to ν. Then T is a bounded linear operator iff $\{\nu a : a \in \mathfrak{A}\}$ is bounded, and in this case $\|T\| = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\|$.

361J Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ a ring homomorphism.

(a) We have a Riesz homomorphism $T_\pi : S(\mathfrak{A}) \to S(\mathfrak{B})$ given by the formula

$$T_\pi(\chi a) = \chi(\pi a) \text{ for every } a \in \mathfrak{A}.$$

For any $u \in S(\mathfrak{A})$, $\|T_\pi u\|_\infty = \min\{\|u'\|_\infty : u' \in S(\mathfrak{A}), T_\pi u' = T_\pi u\}$; $\|T_\pi u\|_\infty \le \|u\|_\infty$. $T_\pi(u \times u') = T_\pi u \times T_\pi u'$ for all u, $u' \in S(\mathfrak{A})$.

(b) T_π is surjective iff π is surjective, and in this case $\|v\|_\infty = \min\{\|u\|_\infty : u \in S(\mathfrak{A}), T_\pi u = v\}$ for every $v \in S(\mathfrak{B})$.

(c) The kernel of T_π is just the set of those $u \in S(\mathfrak{A})$ such that $\pi[\![|u| > 0]\!] = 0$, defining $[\![\ldots > \ldots]\!]$ as in 361Eg.

(d) T_π is injective iff π is injective, and in this case $\|T_\pi u\|_\infty = \|u\|_\infty$ for every $u \in S(\mathfrak{A})$.

(e) T_π is order-continuous iff π is order-continuous.

(f) T_π is sequentially order-continuous iff π is sequentially order-continuous.

(g) If $\mathfrak{C}$ is another Boolean ring and $\phi : \mathfrak{B} \to \mathfrak{C}$ is another ring homomorphism, then $T_{\phi\pi} = T_\phi T_\pi : S(\mathfrak{A}) \to S(\mathfrak{C})$.

361K Proposition Let $\mathfrak{A}$ be a Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the solid linear subspace of $S(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in $S(\mathfrak{A})$.

361L Proposition Let X be a set, and Σ a ring of subsets of X, that is, a subring of the Boolean ring $\mathcal{P}X$. Then $S(\Sigma)$ can be identified, as ordered linear space, with the linear subspace of $\ell^\infty(X)$ generated by the indicator functions of members of Σ, which is a Riesz subspace of $\ell^\infty(X)$. The norm of $S(\Sigma)$ corresponds to the uniform norm on $\ell^\infty(X)$, and its multiplication to pointwise multiplication of functions.

361M Proposition Let X be a set, Σ a ring of subsets of X, and $\mathcal{I}$ an ideal of Σ; write $\mathfrak{A}$ for the quotient ring $\Sigma/\mathcal{I}$. Let $\mathcal{S}$ be the linear span of $\{\chi E : E \in \Sigma\}$ in $\mathbb{R}^X$, and write

$$V = \{f : f \in \mathcal{S}, \{x : f(x) \ne 0\} \in \mathcal{I}\}.$$

V is a solid linear subspace of $\mathcal{S}$. $S(\mathfrak{A})$ becomes identified with the quotient Riesz space $\mathcal{S}/V$, if for every $E \in \Sigma$ we identify $\chi(E^\bullet) \in S(\mathfrak{A})$ with $(\chi E)^\bullet \in \mathcal{S}/V$. If we give $\mathcal{S}$ its uniform norm inherited from $\ell^\infty(X)$, V is a closed linear subspace of $\mathcal{S}$, and the quotient norm on $\mathcal{S}/V$ corresponds to the norm of $S(\mathfrak{A})$:

$$\|f^\bullet\| = \min\{\alpha : \{x : |f(x)| > \alpha\} \in \mathcal{I}\}.$$

If we write $\times$ for pointwise multiplication on $\mathcal{S}$, then V is an ideal of the ring $(\mathcal{S}, +, \times)$, and the multiplication induced on $\mathcal{S}/V$ corresponds to the multiplication of $S(\mathfrak{A})$.

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The next stage in our journey is the systematic investigation of linear functionals on spaces $S = S(\mathfrak{A})$. We already know that these correspond to additive real-valued functionals on the algebra $\mathfrak{A}$ (361F). My purpose here is to show how the structure of the Riesz space dual $S^\sim$ and its bands is related to the classes of additive functionals introduced in §§326-327. The first step is just to check the identification of the linear and order structures of $S^\sim$ and the space M of bounded finitely additive functionals (362A); all the ideas needed for this have already been set out, and the basic properties of $S^\sim$ are covered by the general results in §356. Next, we need to be able to describe the operations on M corresponding to the Riesz space operations $|\;|$, $\vee$, $\wedge$ on $S^\sim$, and the band projections from $S^\sim$ onto $S^\sim_c$ and $S^\times$; these are dealt with in 362B, with a supplementary remark in 362D. In the case of measure algebras, we have some further important bands which present themselves in M, rather than in $S^\sim$, and which are treated in 362C. Since all these spaces are L-spaces, it is worth taking a moment to identify their uniformly integrable subsets; I do this in 362E.

While some of the ideas here have interesting extensions to the case in which $\mathfrak{A}$ is a Boolean ring without identity, these can I think be left to one side; the work of this section will be done on the assumption that every $\mathfrak{A}$ is a Boolean algebra.

362A Theorem Let $\mathfrak{A}$ be a Boolean algebra. Write S for $S(\mathfrak{A})$.

(a) The partially ordered linear space of all finitely additive real-valued functionals on $\mathfrak{A}$ may be identified with the partially ordered linear space of all real-valued linear functionals on S.

(b) The linear space of bounded finitely additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\sim$ of order-bounded linear functionals on S. If $f \in S^\sim$ corresponds to $\nu : \mathfrak{A} \to \mathbb{R}$, then $f^+ \in S^\sim$ corresponds to ν^+, where

$$\nu^+ a = \sup_{b \subseteq a} \nu b$$

for every $a \in \mathfrak{A}$, and

$$\|f\| = \sup_{a \in \mathfrak{A}} \nu a - \nu(1 \setminus a).$$

(c) The linear space of bounded countably additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\sim_c$.

(d) The linear space of completely additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\times$.

362B Spaces of finitely additive functionals: Theorem Let $\mathfrak{A}$ be a Boolean algebra. Let M be the Riesz space of bounded finitely additive real-valued functionals on $\mathfrak{A}$, $M_\sigma \subseteq M$ the space of bounded countably additive functionals, and $M_\tau \subseteq M_\sigma$ the space of completely additive functionals.

(a) For any μ, $\nu \in M$, $\mu \vee \nu$, $\mu \wedge \nu$ and $|\nu|$ are defined by the formulae

$$(\mu \vee \nu)(a) = \sup_{b \subseteq a} \mu b + \nu(a \setminus b),$$

$$(\mu \wedge \nu)(a) = \inf_{b \subseteq a} \mu b + \nu(a \setminus b),$$

$$|\nu|(a) = \sup_{b \subseteq a} \nu b - \nu(a \setminus b) = \sup_{b,c \subseteq a} \nu b - \nu c$$

for every $a \in \mathfrak{A}$. Setting

$$\|\nu\| = |\nu|(1) = \sup_{a\in\mathfrak{A}} \nu a - \nu(1\setminus a),$$

M becomes an L-space.

(b) M_σ and M_τ are projection bands in M, therefore L-spaces in their own right. In particular, $|\nu| \in M_\sigma$ for every $\nu \in M_\sigma$, and $|\nu| \in M_\tau$ for every $\nu \in M_\tau$.

(c) The band projection $P_\sigma : M \to M_\sigma$ is defined by the formula

$$(P_\sigma\nu)(c) = \inf\{\sup_{n\in\mathbb{N}} \nu a_n : \langle a_n\rangle_{n\in\mathbb{N}} \text{ is a non-decreasing sequence with supremum } c\}$$

whenever $c \in \mathfrak{A}$ and $\nu \geq 0$ in M.

(d) The band projection $P_\tau : M \to M_\tau$ is defined by the formula

$$(P_\tau\nu)(c) = \inf\{\sup_{a\in A} \nu a : A \text{ is a non-empty upwards-directed set with supremum } c\}$$

whenever $c \in \mathfrak{A}$ and $\nu \geq 0$ in M.

(e) If $A \subseteq M$ is upwards-directed, then A is bounded above in M iff $\{\nu 1 : \nu \in A\}$ is bounded above in $\mathbb{R}$, and in this case (if $A \neq \emptyset$) $\sup A$ is defined by the formula

$$(\sup A)(a) = \sup_{\nu\in A} \nu a \text{ for every } a \in \mathfrak{A}.$$

(f) Suppose that μ, $\nu \in M$.

(i) The following are equiveridical:

(α) ν belongs to the band in M generated by μ;

(β) for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \leq \epsilon$ whenever $|\mu|a \leq \delta$;

(γ) $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ such that $\lim_{n\to\infty} |\mu|(a_n) = 0$.

(ii) Now suppose that μ, $\nu \geq 0$, and let ν_1, ν_2 be the components of ν in the band generated by μ and its complement. Then

$$\nu_1 c = \sup_{\delta>0} \inf_{\mu a\leq\delta} \nu(c\setminus a), \quad \nu_2 c = \inf_{\delta>0} \sup_{a\subseteq c, \mu a\leq\delta} \nu a$$

for every $c \in \mathfrak{A}$.

Remark The L-space norm $\| \|$ on M, described in (a) above, is the **total variation norm**.

362C Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and M be the Riesz space of bounded finitely additive real-valued functionals on $\mathfrak{A}$. Write

$$M_{ac} = \{\nu : \nu \in M \text{ is absolutely continuous with respect to } \bar\mu\},$$

$$M_{tc} = \{\nu : \nu \in M \text{ is continuous with respect to the measure-algebra topology on } \mathfrak{A}\},$$

$$M_t = \{\nu : \nu \in M,\ |\nu|1 = \sup_{\bar\mu a<\infty} |\nu|a\}.$$

Then M_{ac}, M_{tc} and M_t are bands in M.

362D Proposition Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive Boolean algebra. Let M be the space of bounded finitely additive functionals on $\mathfrak{A}$, $M_\tau \subseteq M$ the space of completely additive functionals, and $P_\tau : M \to M_\tau$ the band projection. Then for any $\nu \in M^+$ and $c \in \mathfrak{A}$ there is a non-empty upwards-directed set $A \subseteq \mathfrak{A}$ with supremum c such that $(P_\tau\nu)(c) = \sup_{a\in A} \nu a$.

362E Uniformly integrable sets: Theorem Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded finitely additive functionals on $\mathfrak{A}$. Then a norm-bounded set $C \subseteq M$ is uniformly integrable iff $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

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363 L^∞

In this section I set out to describe an abstract construction for L^∞ spaces on arbitrary Boolean algebras, corresponding to the $L^\infty(\mu)$ spaces of §243. I begin with the definition of $L^\infty(\mathfrak{A})$ (363A) and elementary facts concerning its own structure and the embedding $S(\mathfrak{A}) \subsetneq L^\infty(\mathfrak{A})$ (363B-363D). I give the basic universal mapping theorems which define the Banach lattice structure of L^∞ (363E) and a description of the action of Boolean homomorphisms on L^∞ spaces (363F-363G) before discussing the representation of $L^\infty(\Sigma)$ and $L^\infty(\Sigma/\mathcal{I})$ for σ-algebras Σ and ideals $\mathcal{I}$ of sets (363H). This leads at once to the identification of $L^\infty(\mu)$, as defined in Volume 2, with $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ (363I). Like $S(\mathfrak{A})$, $L^\infty(\mathfrak{A})$ determines the algebra $\mathfrak{A}$ (363J). I briefly discuss the dual spaces of L^∞; they correspond exactly to the duals of S described in §362 (363K). Linear functionals on L^∞ can for some purposes be treated as 'integrals' (363L).

In the second half of the section I present some of the theory of Dedekind complete and σ-complete algebras. First, $L^\infty(\mathfrak{A})$ is Dedekind (σ-)complete iff $\mathfrak{A}$ is (363M). The spaces $L^\infty(\mathfrak{A})$, for Dedekind σ-complete $\mathfrak{A}$, are precisely the Dedekind σ-complete Riesz spaces with order unit (363N-363P). The spaces $L^\infty(\mathfrak{A})$, for Dedekind complete $\mathfrak{A}$, are precisely the normed spaces which may be put in place of $\mathbb{R}$ in the Hahn-Banach theorem (363R). Finally, I mention some equivalent forms of the Banach-Ulam problem (363S).

363A Definition Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z. I will write $L^\infty(\mathfrak{A})$ for the space $C(Z) = C_b(Z)$ of continuous real-valued functions from Z to $\mathbb{R}$, endowed with the linear structure, order structure, norm and multiplication of $C(Z) = C_b(Z)$.

363B Theorem Let $\mathfrak{A}$ be any Boolean algebra; write L^∞ for $L^\infty(\mathfrak{A})$.

(a) L^∞ is an M-space; its standard order unit is the constant function taking the value 1 at each point; in particular, L^∞ is a Banach lattice with a Fatou norm and the Levi property.

(b) L^∞ is a commutative Banach algebra and an f-algebra.

(c) If $u \in L^\infty$ then $u \geq 0$ iff there is a $v \in L^\infty$ such that $u = v \times v$.

363C Proposition Let $\mathfrak{A}$ be any Boolean algebra. Then $S(\mathfrak{A})$ is a norm-dense, order-dense Riesz subspace of $L^\infty(\mathfrak{A})$, closed under multiplication.

363D Proposition Let $\mathfrak{A}$ be a Boolean algebra. If we regard $\chi a \in S(\mathfrak{A})$ as a member of $L^\infty(\mathfrak{A})$ for each $a \in \mathfrak{A}$, then $\chi : \mathfrak{A} \to L^\infty(\mathfrak{A})$ is additive, order-preserving, order-continuous and a lattice homomorphism.

363E Theorem Let $\mathfrak{A}$ be a Boolean algebra, and U a Banach space. Let $\nu : \mathfrak{A} \to U$ be a bounded additive function.

(a) There is a unique bounded linear operator $T : L^\infty(\mathfrak{A}) \to U$ such that $T\chi = \nu$; in this case $\|T\| = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\|$.

(b) If U is a Banach lattice then T is positive iff ν is non-negative; and in this case T is order-continuous iff ν is order-continuous, and sequentially order-continuous iff ν is sequentially order-continuous.

(c) If U is a Banach lattice then T is a Riesz homomorphism iff ν is a lattice homomorphism iff $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$.

363F Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) There is an associated multiplicative Riesz homomorphism $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$, of norm at most 1, defined by saying that $T_\pi(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$.

(b) For any $u \in L^\infty(\mathfrak{A})$, there is a $u' \in L^\infty(\mathfrak{A})$ such that $T_\pi u = T_\pi u'$ and $\|u'\|_\infty = \|T_\pi u\|_\infty \leq \|u\|_\infty$.

(c)(i) The kernel of T_π is the norm-closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : a \in \mathfrak{A}, \pi a = 0\}$.

(ii) The set of values of T_π is the norm-closed linear subspace of $L^\infty(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$.

(d) T_π is surjective iff π is surjective, and in this case $\|v\|_\infty = \min\{\|u\|_\infty : T_\pi u = v\}$ for every $v \in L^\infty(\mathfrak{B})$.

(e) T_π is injective iff π is injective, and in this case $\|T_\pi u\|_\infty = \|u\|_\infty$ for every $u \in L^\infty(\mathfrak{A})$.

(f) T_π is order-continuous, or sequentially order-continuous, iff π is.

(g) If $\mathfrak{C}$ is another Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ is another Boolean homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{C})$.

363G Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, then $L^\infty(\mathfrak{C})$ can be identified, as Banach lattice and as Banach algebra, with the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi c : c \in \mathfrak{C}\}$.

(b) If $\mathcal{I}$ is an ideal of $\mathfrak{A}$, then $L^\infty(\mathfrak{A}/\mathcal{I})$ can be identified, as Banach lattice and as Banach algebra, with the quotient space $L^\infty(\mathfrak{A})/V$, where V is the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : a \in \mathcal{I}\}$.

363H Representations of $L^\infty(\mathfrak{A})$: Proposition Let X be a set and Σ an algebra of subsets of X.

(a) Write $S(\Sigma)$ for the linear subspace of $\ell^\infty(X)$ generated by the indicator functions of members of Σ, and $\mathcal{L}^\infty$ for its $\| \|_\infty$-closure in $\ell^\infty(X)$.

(i) $L^\infty(\Sigma)$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty$; if $E \in \Sigma$, then χE, defined in $L^\infty(\Sigma)$ as in 361D, can be identified with the indicator function of E regarded as a subset of X.

(ii) A bounded function $f : X \to \mathbb{R}$ belongs to $\mathcal{L}^\infty$ iff whenever $\alpha < \beta$ in $\mathbb{R}$ there is an $E \in \Sigma$ such that $\{x : f(x) > \beta\} \subseteq E \subseteq \{x : f(x) > \alpha\}$.

(iii) In particular, $L^\infty(\mathcal{P}X)$ can be identified with $\ell^\infty(X)$.

(b) Now suppose that Σ is a σ-algebra of subsets of X.

(i) $\mathcal{L}^\infty$ is just the set of bounded Σ-measurable real-valued functions on X.

(ii) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ is a surjective sequentially order-continuous Boolean homomorphism with kernel $\mathcal{I}$, then $L^\infty(\mathfrak{A})$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty/\mathcal{W}$, where $\mathcal{W} = \{f : f \in \mathcal{L}^\infty, \{x : f(x) \neq 0\} \in \mathcal{I}\}$ is a solid linear subspace and closed ideal of $\mathcal{L}^\infty$. For $f \in \mathcal{L}^\infty$,

$$\|f^\bullet\|_\infty = \min\{\alpha : \alpha \geq 0, \{x : |f(x)| > \alpha\} \in \mathcal{I}\}.$$

(iii) In particular, if $\mathcal{I}$ is any σ-ideal of Σ and $E \mapsto E^\bullet$ is the canonical homomorphism from Σ onto $\mathfrak{A} = \Sigma/\mathcal{I}$, then we have an identification of $L^\infty(\mathfrak{A})$ with a quotient of $\mathcal{L}^\infty$, and for any $E \in \Sigma$ we can identify $\chi(E^\bullet) \in L^\infty(\mathfrak{A})$ with the equivalence class $(\chi E)^\bullet \in \mathcal{L}^\infty/\mathcal{W}$ of the indicator function χE.

363I Corollary Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$. Then $L^\infty(\mu)$ can be identified, as Banach lattice and Banach algebra, with $L^\infty(\mathfrak{A})$; the identification matches $(\chi E)^\bullet \in L^\infty(\mu)$ with $\chi(E^\bullet) \in L^\infty(\mathfrak{A})$, for every $E \in \Sigma$.

363J Recovering the algebra $\mathfrak{A}$: Proposition Let $\mathfrak{A}$ be a Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the solid linear subspace of $L^\infty(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in $L^\infty(\mathfrak{A})$.

363K Dual spaces of L^∞: Proposition Let $\mathfrak{A}$ be a Boolean algebra. Let M, M_σ and M_τ be the L-spaces of bounded finitely additive functionals, bounded countably additive functionals and completely additive functionals on $\mathfrak{A}$. Then the embedding $S(\mathfrak{A}) \subsetneqq L^\infty(\mathfrak{A})$ induces Riesz space isomorphisms between $S(\mathfrak{A})^\sim \cong M$ and $L^\infty(\mathfrak{A})^\sim = L^\infty(\mathfrak{A})^*$, $S(\mathfrak{A})^\sim_c \cong M_\sigma$ and $L^\infty(\mathfrak{A})^\sim_c$, and $S(\mathfrak{A})^\times \cong M_\tau$ and $L^\infty(\mathfrak{A})^\times$.

***363L Integration with respect to a finitely additive functional (a)** If $\mathfrak{A}$ is a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ is a bounded additive functional, then we have a corresponding functional $f_\nu \in L^\infty(\mathfrak{A})^*$ defined by saying that $f_\nu(\chi a) = \nu a$ for every $a \in \mathfrak{A}$. There are contexts in which it is convenient to use the formula $⨍ u\,d\nu$ in place of $f_\nu(u)$ for $u \in L^\infty = L^\infty(\mathfrak{A})$.

(b) Let M be the L-space of bounded finitely additive functionals on $\mathfrak{A}$. Then we have a function $(u, \nu) \mapsto ⨍ u\,d\nu : L^\infty \times M \to \mathbb{R}$. Now this map is bilinear.

(c) If ν is non-negative, we have $⨍ u\,d\nu \geq 0$ whenever $u \geq 0$. $(u, \nu) \mapsto ⨍ u\,d\nu$ has norm at most 1. If $\mathfrak{A} \neq 0$, the norm is exactly 1.

(e) If $\mathfrak{A}$ is any Boolean algebra, and $\nu : \mathfrak{A} \to [0, \infty[$ is a non-negative additive functional, and $u \in L^\infty(\mathfrak{A})^+$, then

$$⨍ u\,d\nu = \int_0^\infty \sup\{\nu a : t\chi a \leq u\} dt.$$

363M Theorem Let $\mathfrak{A}$ be a Boolean algebra.
(a) $\mathfrak{A}$ is Dedekind σ-complete iff $L^\infty(\mathfrak{A})$ is Dedekind σ-complete.
(b) $\mathfrak{A}$ is Dedekind complete iff $L^\infty(\mathfrak{A})$ is Dedekind complete.

363N Proposition Let U be a Dedekind σ-complete Riesz space with an order unit. Then U is isomorphic, as Riesz space, to $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of projection bands in U.

363O Corollary Let U be a Dedekind σ-complete M-space. Then U is isomorphic, as Banach lattice, to $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of projection bands of U.

363P Corollary Let U be any Dedekind σ-complete Riesz space and $e \in U^+$. Then the solid linear subspace U_e of U generated by e is isomorphic, as Riesz space, to $L^\infty(\mathfrak{A})$ for some Dedekind σ-complete Boolean algebra $\mathfrak{A}$; and if U is Dedekind complete, so is $\mathfrak{A}$.

363Q Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Then for any Banach lattice U, a linear operator $T : U \to L^\infty = L^\infty(\mathfrak{A})$ is continuous iff it is order-bounded, and in this case $\|T\| = \||T|\|$, where the modulus $|T|$ is taken in $\mathsf{L}^\sim(U; L^\infty)$.

363R Theorem Let U be a normed space over $\mathbb{R}$. Then the following are equiveridical:
(i) there is a Dedekind complete Boolean algebra $\mathfrak{A}$ such that U is isomorphic, as normed space, to $L^\infty(\mathfrak{A})$;
(ii) whenever V is a normed space, V_0 a linear subspace of V, and $T_0 : V_0 \to U$ is a bounded linear operator, there is an extension of T_0 to a bounded linear operator $T : V \to U$ with $\|T\| = \|T_0\|$.

363S The Banach-Ulam problem: Theorem The following statements are equiveridical.
(i) There are a set X and a probability measure ν, with domain $\mathcal{P}X$, such that $\nu\{x\} = 0$ for every $x \in X$.
(ii) There are a localizable measure space (X, Σ, μ) and an absolutely continuous countably additive functional $\nu : \Sigma \to \mathbb{R}$ which is not truly continuous, so has no Radon-Nikodým derivative.
(iii) There are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a countably additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ which is not completely additive.
(iv) There is a Dedekind complete Riesz space U such that $U_c^\sim \ne U^\times$.

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364 L^0

My next objective is to develop an abstract construction corresponding to the $L^0(\mu)$ spaces of §241. These generalized L^0 spaces will form the basis of the work of the rest of this chapter and also the next; partly because their own properties are remarkable, but even more because they form a framework for the study of Archimedean Riesz spaces in general (see §368). There seem to be significant new difficulties, and I take the space to describe an approach which can be made essentially independent of the route through Stone spaces used in the last three sections. I embark directly on a definition in the new language (364A), and relate it to the constructions of §241 (364B-364D, 364I) and §§361-363 (364J). The ideas of Chapter 27 can also be expressed in this language; I make a start on developing the machinery for this in 364F-364G, with the formula '$[\![u \in E]\!]$', 'the region in which u belongs to E', and some exercises (364Xe-364Xf). Following through the questions addressed in §363, I discuss Dedekind completeness in L^0 (364L-364M), properties of its multiplication (364N), the expression of the original algebra in terms of L^0 (364O), the action of Boolean homomorphisms on L^0 (364P) and product spaces (364R). In 364S-364V I describe representations of the L^0 space of a regular open algebra.

364A The set $L^0(\mathfrak{A})$ (a) Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. I will write $L^0(\mathfrak{A})$ for the set of all functions $\alpha \mapsto [\![u > \alpha]\!] : \mathbb{R} \to \mathfrak{A}$ such that

(α) $[\![u > \alpha]\!] = \sup_{\beta > \alpha} [\![u > \beta]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$,
(β) $\inf_{\alpha \in \mathbb{R}} [\![u > \alpha]\!] = 0$,
(γ) $\sup_{\alpha \in \mathbb{R}} [\![u > \alpha]\!] = 1$.

(e) In fact it will sometimes be convenient to note that the conditions of (a) can be replaced by

(α') $[\![u > \alpha]\!] = \sup_{q\in\mathbb{Q},q>\alpha} [\![u > q]\!]$ for every $\alpha \in \mathbb{R}$,

(β') $\inf_{n\in\mathbb{N}} [\![u > n]\!] = 0$,

(γ') $\sup_{n\in\mathbb{N}} [\![u > -n]\!] = 1$;

the point being that we need look only at suprema and infima of countable subsets of $\mathfrak{A}$.

***(f)** Indeed, we have the option of declaring $L^0(\mathfrak{A})$ to be the set of functions $\alpha \mapsto [\![u > \alpha]\!] : \mathbb{Q} \to \mathfrak{A}$ such that

(α'') $[\![u > q]\!] = \sup_{q'\in\mathbb{Q},q'>q} [\![u > q']\!]$ for every $q \in \mathbb{Q}$,

(β') $\inf_{n\in\mathbb{N}} [\![u > n]\!] = 0$,

(γ') $\sup_{n\in\mathbb{N}} [\![u > -n]\!] = 1$.

364B Proposition Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ.

(a) Write $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ for the space of all Σ-measurable functions from X to $\mathbb{R}$. Then $\mathcal{L}^0$, with its linear structure, ordering and multiplication inherited from $\mathbb{R}^X$, is a Dedekind σ-complete f-algebra with multiplicative identity.

(b) Set

$$\mathcal{W} = \mathcal{W}_\mathcal{I} = \{f : f \in \mathcal{L}^0,\ \{x : f(x) \neq 0\} \in \mathcal{I}\}.$$

Then

(i) $\mathcal{W}$ is a sequentially order-closed solid linear subspace and ideal of $\mathcal{L}^0$;

(ii) the quotient space $\mathcal{L}^0/\mathcal{W}$, with its inherited linear, order and multiplicative structures, is a Dedekind σ-complete Riesz space and an f-algebra with a multiplicative identity;

(iii) for f, $g \in \mathcal{L}^0$, $f^\bullet \le g^\bullet$ in $\mathcal{L}^0/\mathcal{W}$ iff $\{x : f(x) > g(x)\} \in \mathcal{I}$, and $f^\bullet = g^\bullet$ in $\mathcal{L}^0/\mathcal{W}$ iff $\{x : f(x) \neq g(x)\} \in \mathcal{I}$.

364C Theorem Let X be a set and Σ a σ-algebra of subsets of X. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ a surjective Boolean homomorphism, with kernel a σ-ideal $\mathcal{I}$; define $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ and $\mathcal{W} = \mathcal{W}_\mathcal{I}$ as in 364B, so that $U = \mathcal{L}^0/\mathcal{W}$ is a Dedekind σ-complete f-algebra with multiplicative identity.

(a) We have a canonical bijection $T : U \to L^0 = L^0(\mathfrak{A})$ defined by the formula

$$[\![Tf^\bullet > \alpha]\!] = \pi\{x : f(x) > \alpha\}$$

for every $f \in \mathcal{L}^0$ and $\alpha \in \mathbb{R}$.

(b)(i) For any u, $v \in U$,

$$[\![T(u+v) > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!]$$

for every $\alpha \in \mathbb{R}$.

(ii) For any $u \in U$ and $\gamma > 0$,

$$[\![T(\gamma u) > \alpha]\!] = [\![Tu > \tfrac{\alpha}{\gamma}]\!]$$

for every $\alpha \in \mathbb{R}$.

(iii) For any u, $v \in U$,

$$u \le v \iff [\![Tu > \alpha]\!] \subseteq [\![Tv > \alpha]\!] \text{ for every } \alpha \in \mathbb{R}.$$

(iv) For any u, $v \in U^+$,

$$[\![T(u \times v) > \alpha]\!] = \sup_{q\in\mathbb{Q},q>0} [\![Tu > q]\!] \cap [\![Tv > \tfrac{\alpha}{q}]\!]$$

for every $\alpha \ge 0$.

(v) Writing $e = (\chi X)^\bullet$ for the multiplicative identity of U, we have

$$[\![Te > \alpha]\!] = 1 \text{ if } \alpha < 1,\ 0 \text{ if } \alpha \ge 1.$$

364D Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then $L^0 = L^0(\mathfrak{A})$ has the structure of a Dedekind σ-complete f-algebra with multiplicative identity e, defined by saying

$$[\![u+v>\alpha]\!] = \sup_{q\in\mathbb{Q}} [\![u>q]\!] \cap [\![v>\alpha-q]\!],$$

whenever u, $v \in L^0$ and $\alpha \in \mathbb{R}$,

$$[\![\gamma u>\alpha]\!] = [\![u>\tfrac{\alpha}{\gamma}]\!]$$

whenever $u \in L^0$, $\gamma \in \,]0,\infty[$ and $\alpha \in \mathbb{R}$,

$$u \le v \iff [\![u>\alpha]\!] \subseteq [\![v>\alpha]\!] \text{ for every } \alpha \in \mathbb{R},$$

$$[\![u\times v>\alpha]\!] = \sup_{q\in\mathbb{Q},q>0} [\![u>q]\!] \cap [\![v>\tfrac{\alpha}{q}]\!]$$

whenever u, $v \ge 0$ in L^0 and $\alpha \ge 0$,

$$[\![e>\alpha]\!] = 1 \text{ if } \alpha<1,\ 0 \text{ if } \alpha \ge 1.$$

364E Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) If u, $v \in L^0 = L^0(\mathfrak{A})$ and α, $\beta \in \mathbb{R}$,

$$[\![u+v>\alpha+\beta]\!] \subseteq [\![u>\alpha]\!] \cup [\![v>\beta]\!].$$

(b) If u, $v \ge 0$ in L^0 and α, $\beta \ge 0$ in $\mathbb{R}$,

$$[\![u\times v>\alpha\beta]\!] \subseteq [\![u>\alpha]\!] \cup [\![v>\beta]\!].$$

364F Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there is a bijection between $L^0 = L^0(\mathfrak{A})$ and the set Φ of sequentially order-continuous Boolean homomorphisms from the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ to $\mathfrak{A}$, defined by saying that $u \in L^0$ corresponds to $\phi \in \Phi$ iff $[\![u>\alpha]\!] = \phi(]\alpha,\infty[)$ for every $\alpha \in \mathbb{R}$.

364G Definitions (a) In the context of 364F, I will write $[\![u \in E]\!]$, 'the region where u takes values in E', for $\phi(E)$, where $\phi : \mathcal{B} \to \mathfrak{A}$ is the homomorphism corresponding to $u \in L^0$. Thus $[\![u>\alpha]\!] = [\![u \in \,]\alpha,\infty[\,]\!]$. I write $[\![u \ge \alpha]\!]$ for $[\![u \in [\alpha,\infty[\,]\!] = \inf_{\beta<\alpha} [\![u>\beta]\!]$, $[\![u \ne 0]\!] = [\![|u|>0]\!] = [\![u>0]\!] \cup [\![u<0]\!]$ and so on, so that $[\![u=\alpha]\!] = [\![u \in \{\alpha\}]\!] = [\![u \ge \alpha]\!] \setminus [\![u>\alpha]\!]$ for $u \in L^0$ and $\alpha \in \mathbb{R}$.

(b) If $(\mathfrak{A},\bar{\mu})$ is a probability algebra, $\bar{\mu}\phi : \mathcal{B} \to [0,1]$ is a probability measure, so that its completion ν is a Radon probability measure on $\mathbb{R}$; I will call ν the **distribution** of u.

364H Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $E \subseteq \mathbb{R}$ a Borel set, and $h : E \to \mathbb{R}$ a Borel measurable function. Then whenever $u \in L^0 = L^0(\mathfrak{A})$ is such that $[\![u \in E]\!] = 1$, there is an element $\bar{h}(u)$ of L^0 defined by saying that $[\![\bar{h}(u) \in F]\!] = [\![u \in h^{-1}[F]]\!]$ for every Borel set $F \subseteq \mathbb{R}$.

364I Examples (a) Let X be a set and Σ a σ-algebra of subsets of X. Then we may identify $L^0(\Sigma)$ with the space $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ of Σ-measurable real-valued functions on X. For $f \in \mathcal{L}^0$, $[\![f \in E]\!]$ is just $f^{-1}[E]$, for any Borel set $E \subseteq \mathbb{R}$; and if h is a Borel measurable function, $\bar{h}(f)$ is just the composition hf, for any $f \in \mathcal{L}^0$.

(b) Now suppose that $\mathcal{I}$ is a σ-ideal of Σ and that $\mathfrak{A} = \Sigma/\mathcal{I}$. Then, as in 364C, we identify $L^0(\mathfrak{A})$ with a quotient $\mathcal{L}^0/\mathcal{W}_{\mathcal{I}}$. For $f \in \mathcal{L}^0$, $[\![f^\bullet \in E]\!] = f^{-1}[E]^\bullet$, and $\bar{h}(f^\bullet) = (hf)^\bullet$, for any Borel set E and any Borel measurable function $h : \mathbb{R} \to \mathbb{R}$.

(c) In particular, if (X,Σ,μ) is a measure space with measure algebra $\mathfrak{A}$, then $L^0(\mathfrak{A})$ becomes identified with $L^0(\mu)$ as defined in §241, and the distribution of $f \in \mathcal{L}^0(\mu)$, as defined in 271C, is the same as the distribution of $f^\bullet \in L^0(\mu) \cong L^0(\mathfrak{A})$, as defined in 364Gb.

364J Embedding S and L^∞ in L^0: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) We have a canonical embedding of $L^\infty = L^\infty(\mathfrak{A})$ as an order-dense solid linear subspace of $L^0 = L^0(\mathfrak{A})$; it is the solid linear subspace generated by the multiplicative identity e of L^0. Consequently $S = S(\mathfrak{A})$ also is embedded as an order-dense Riesz subspace and subalgebra of L^0.

(b) This embedding respects the linear, lattice and multiplicative structures of L^∞ and S, and the definition of $[\![u > \delta]\!]$, for $u \in S^+$ and $\delta \geq 0$, given in 361Eg.

(c) For $a \in \mathfrak{A}$, χa, when regarded as a member of L^0, can be described by the formula

$$\begin{aligned}[\![\chi a > \alpha]\!] &= 1 \text{ if } \alpha < 0,\\ &= a \text{ if } 0 \leq \alpha < 1,\\ &= 0 \text{ if } 1 \leq \alpha.\end{aligned}$$

The function $\chi : \mathfrak{A} \to L^0$ is additive, injective, order-continuous and a lattice homomorphism.

(d) For every $u \in (L^0)^+$ there is a non-decreasing sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in S such that $u_0 \geq 0$ and $\sup_{n\in\mathbb{N}} u_n = u$.

364K Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Then $S(\mathfrak{A}^f)$ can be embedded as a Riesz subspace of $L^0(\mathfrak{A})$, which is order-dense iff $(\mathfrak{A}, \bar{\mu})$ is semi-finite.

364L Suprema and infima in L^0: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $L^0 = L^0(\mathfrak{A})$.

(a) Let A be a subset of L^0.

(i) A is bounded above in L^0 iff there is a sequence $\langle c_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, with infimum 0, such that $[\![u > n]\!] \subseteq c_n$ for every $u \in A$.

(ii) If A is non-empty, then A has a supremum in L^0 iff $c_\alpha = \sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$ and $\inf_{n\in\mathbb{N}} c_n = 0$; and in this case $c_\alpha = [\![\sup A > \alpha]\!]$ for every α.

(iii) If A is non-empty and bounded above, then A has a supremum in L^0 iff $\sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$.

(b)(i) If u, $v \in L^0$, then $[\![u \wedge v > \alpha]\!] = [\![u > \alpha]\!] \cap [\![v > \alpha]\!]$ for every $\alpha \in \mathbb{R}$.

(ii) If A is a non-empty subset of $(L^0)^+$, then $\inf A = 0$ in L^0 iff $\inf_{u\in A} [\![u > \alpha]\!] = 0$ in $\mathfrak{A}$ for every $\alpha > 0$.

364M Theorem For a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, $L^0 = L^0(\mathfrak{A})$ is Dedekind complete iff $\mathfrak{A}$ is.

364N The multiplication of L^0: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then an element u of $L^0 = L^0(\mathfrak{A})$ has a multiplicative inverse in L^0 iff $|u|$ is a weak order unit in L^0 iff $[\![|u| > 0]\!] = 1$.

364O Recovering the algebra: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the band in $L^0 = L^0(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in L^0.

364P Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism.

(a) We have a multiplicative sequentially order-continuous Riesz homomorphism $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ defined by the formula

$$[\![T_\pi u > \alpha]\!] = \pi[\![u > \alpha]\!]$$

whenever $\alpha \in \mathbb{R}$ and $u \in L^0(\mathfrak{A})$.

(b) Defining $\chi a \in L^0(\mathfrak{A})$ as in 364J, $T_\pi(\chi a) = \chi(\pi a)$ in $L^0(\mathfrak{B})$ for every $a \in \mathfrak{A}$. If we regard $L^\infty(\mathfrak{A})$ and $L^\infty(\mathfrak{B})$ as embedded in $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, then T_π, as defined here, agrees on $L^\infty(\mathfrak{A})$ with T_π as defined in 363F.

(c) T_π is order-continuous iff π is order-continuous, injective iff π is injective, surjective iff π is surjective.

(d) $[\![T_\pi u \in E]\!] = \pi[\![u \in E]\!]$ for every $u \in L^0(\mathfrak{A})$ and every Borel set $E \subseteq \mathbb{R}$; consequently $\bar{h}T_\pi = T_\pi\bar{h}$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$, writing $\bar{h}$ indifferently for the associated maps from $L^0(\mathfrak{A})$ to itself and from $L^0(\mathfrak{B})$ to itself.

(e) If $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ another sequentially order-continuous Boolean homomorphism then $T_{\theta\pi} = T_\theta T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$.

364Q Proposition Let X and Y be sets, Σ, T σ-algebras of subsets of X, Y respectively, and $\mathcal{I}$, $\mathcal{J}$ σ-ideals of Σ, T. Set $\mathfrak{A} = \Sigma/\mathcal{I}$ and $\mathfrak{B} = \mathrm{T}/\mathcal{J}$. Suppose that $\phi : X \to Y$ is a function such that $\phi^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$ and $\phi^{-1}[F] \in \mathcal{I}$ for every $F \in \mathcal{J}$.

(a) There is a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by saying that $\pi F^\bullet = \phi^{-1}[F]^\bullet$ for every $F \in \mathrm{T}$.

(b) Let $T_\pi : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ be the Riesz homomorphism corresponding to π, as defined in 364P. If we identify $L^0(\mathfrak{B})$ with $\mathcal{L}^0_{\mathrm{T}}/\mathcal{W}_{\mathcal{J}}$ and $L^0(\mathfrak{A})$ with $\mathcal{L}^0_\Sigma/\mathcal{W}_{\mathcal{I}}$ in the manner of 364B-364C, then $T_\pi(g^\bullet) = (g\phi)^\bullet$ for every $g \in \mathcal{L}^0_{\mathrm{T}}$.

(c) Let Z be a third set, Υ a σ-algebra of subsets of Z, $\mathcal{K}$ a σ-ideal of Υ, and $\psi : Y \to Z$ a function such that $\psi^{-1}[G] \in \mathrm{T}$ for every $G \in \Upsilon$ and $\psi^{-1}[G] \in \mathcal{J}$ for every $G \in \mathcal{K}$. Let $\theta : \mathfrak{C} \to \mathfrak{B}$ and $T_\theta : L^0(\mathfrak{C}) \to L^0(\mathfrak{B})$ be the homomorphisms corresponding to ψ as in (a)-(b). Then $\pi\theta : \mathfrak{C} \to \mathfrak{A}$ and $T_\pi T_\theta : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ correspond to $\psi\phi : X \to Y$ in the same way.

(d) Now suppose that μ and ν are measures with domains Σ, T and null ideals $\mathcal{N}(\mu)$, $\mathcal{N}(\nu)$ respectively, and that $\mathcal{I} = \Sigma \cap \mathcal{N}(\mu)$ and $\mathcal{J} = \mathrm{T} \cap \mathcal{N}(\nu)$. In this case, identifying $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ with $L^0(\mu)$ and $L^0(\nu)$ as in 364Ic, we have $g\phi \in \mathcal{L}^0(\mu)$ and $T_\pi(g^\bullet) = (g\phi)^\bullet$ for every $g \in \mathcal{L}^0(\nu)$.

364R Products: Proposition Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a family of Dedekind σ-complete Boolean algebras, with simple product $\mathfrak{A}$. If $\pi_i : \mathfrak{A} \to \mathfrak{A}_i$ is the coordinate map for each i, and $T_i : L^0(\mathfrak{A}) \to L^0(\mathfrak{A}_i)$ the corresponding homomorphism, then $u \mapsto Tu = \langle T_i u \rangle_{i \in I} : L^0(\mathfrak{A}) \to \prod_{i \in I} L^0(\mathfrak{A}_i)$ is a multiplicative Riesz space isomorphism, so $L^0(\mathfrak{A})$ may be identified with the f-algebra product $\prod_{i \in I} L^0(\mathfrak{A}_i)$.

***364S Regular open algebras: Definition** Let $(X, \mathfrak{T})$ be a topological space and $f : X \to \mathbb{R}$ a function. For $x \in X$ write

$$\omega(f, x) = \inf_{G \in \mathfrak{T}, x \in G} \sup_{y,z \in G} |f(y) - f(z)|$$

(allowing ∞).

***364T Theorem** Let X be any topological space, and $\mathrm{RO}(X)$ its regular open algebra. Let U be the set of functions $f : X \to \mathbb{R}$ such that $\{x : \omega(f, x) < \epsilon\}$ is dense in X for every $\epsilon > 0$. Then U is a Riesz subspace of $\mathbb{R}^X$, closed under multiplication, and we have a surjective multiplicative Riesz homomorphism $T : U \to L^0(\mathrm{RO}(X))$ defined by writing

$$[\![Tf > \alpha]\!] = \sup_{\beta > \alpha} \operatorname{int} \overline{\{x : f(x) > \beta\}},$$

the supremum being taken in $\mathrm{RO}(X)$, for every $\alpha \in \mathbb{R}$ and $f \in U$. The kernel of T is the set W of functions $f : X \to \mathbb{R}$ such that $\operatorname{int}\{x : |f(x)| \le \epsilon\}$ is dense for every $\epsilon > 0$, so $L^0(\mathrm{RO}(X))$ can be identified, as f-algebra, with the quotient space U/W.

***364U Compact spaces** Suppose now that X is a compact Hausdorff topological space. In this case the space U of 364T is just the space of functions $f : X \to \mathbb{R}$ such that $\{x : f$ is continuous at $x\}$ is dense in X.

Now W, as defined in 364T, becomes $\{f : f \in U, \{x : f(x) = 0\}$ is dense$\}$.

***364V Theorem** Let X be a compact Hausdorff extremally disconnected space, and $\mathrm{RO}(X)$ its regular open algebra. Write $C^\infty = C^\infty(X)$ for the space of continuous functions $g : X \to [-\infty, \infty]$ such that $\{x : g(x) = \pm\infty\}$ is nowhere dense. Then we have a bijection $S : C^\infty \to L^0 = L^0(\mathrm{RO}(X))$ defined by saying that

$$[\![Sg > \alpha]\!] = \overline{\{x : g(x) > \alpha\}}$$

for every $\alpha \in \mathbb{R}$. Addition and multiplication in L^0 correspond to the operations $\dot{+}$, $\dot{\times}$ on C^∞ defined by saying that $g \dot{+} h$, $g \dot{\times} h$ are the unique elements of C^∞ agreeing with $g + h$, $g \times h$ on $\{x : g(x), h(x)$ are both finite$\}$. Scalar multiplication in L^0 corresponds to the operation

$$(\gamma g)(x) = \gamma g(x) \text{ for } x \in X,\ g \in C^\infty,\ \gamma \in \mathbb{R}$$

on C^∞ (counting $0 \cdot \infty$ as 0), while the ordering of L^0 corresponds to the relation

$$g \le h \iff g(x) \le h(x) \text{ for every } x \in X.$$

Version of 20.1.15

365 L^1

Continuing my programme of developing the ideas of Chapter 24 at a deeper level of abstraction, I arrive at last at L^1. As usual, the first step is to establish a definition which can be matched both with the constructions of the previous sections and with the definition of $L^1(\mu)$ in §242 (365A-365C, 365F). Next, I give what I regard as the most characteristic internal properties of L^1 spaces, including versions of the Radon-Nikodým theorem (365E), before turning to abstract versions of theorems in §235 (365H, 365S) and the duality between L^1 and L^∞ (365K-365M). As in §§361 and 363, the construction is associated with universal mapping theorems (365I-365J) which define the Banach lattice structure of L^1. As in §§361, 363 and 364, homomorphisms between measure algebras correspond to operators between their L^1 spaces; but now the duality theory gives us two types of operators (365N-365P), of which one class can be thought of as abstract conditional expectations (365Q). For localizable measure algebras, the underlying algebra can be recovered from its L^1 space (365R), but the measure cannot.

365A Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. For $u\in L^0(\mathfrak{A})$, write

$$\|u\|_1=\int_0^\infty \bar\mu[\![|u|>\alpha]\!]\,d\alpha,$$

the integral being with respect to Lebesgue measure on $\mathbb{R}$, and allowing ∞ as a value of the integral. Set $L^1_{\bar\mu}=L^1(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\ \|u\|_1<\infty\}$.

365B Theorem Let (X,Σ,μ) be a measure space with measure algebra $(\mathfrak{A},\bar\mu)$. Then the canonical isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ matches $L^1(\mu)\subseteq L^0(\mu)$ with $L^1(\mathfrak{A},\bar\mu)\subseteq L^0(\mathfrak{A})$, and the standard norm of $L^1(\mu)$ with $\|\,\|_1:L^1(\mathfrak{A},\bar\mu)\to[0,\infty[$.

365C Theorem For any measure algebra $(\mathfrak{A},\bar\mu)$, $L^1(\mathfrak{A},\bar\mu)$ is a solid linear subspace of $L^0(\mathfrak{A})$, and $\|\,\|_1$ is a norm on $L^1(\mathfrak{A},\bar\mu)$ under which $L^1(\mathfrak{A},\bar\mu)$ is an L-space. Consequently $L^1(\mathfrak{A},\bar\mu)$ is a perfect Riesz space with an order-continuous norm which has the Levi property, and if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing norm-bounded sequence in $L^1(\mathfrak{A},\bar\mu)$ then it converges for $\|\,\|_1$ to $\sup_{n\in\mathbb{N}}u_n$.

365D Integration Let $(\mathfrak{A},\bar\mu)$ be any measure algebra.

(a) If $u\in L^1=L^1(\mathfrak{A},\bar\mu)$, then u^+ and u^-, calculated in $L^0=L^0(\mathfrak{A})$, belong to L^1, and we may set

$$\int u=\|u^+\|_1-\|u^-\|_1=\int_0^\infty\bar\mu[\![u>\alpha]\!]\,d\alpha-\int_0^\infty\bar\mu[\![-u>\alpha]\!]\,d\alpha.$$

Now $\int:L^1\to\mathbb{R}$ is an order-continuous positive linear functional.

(b) $\|u\|_1=\int|u|\ge|\int u|$ for every $u\in L^1$.

(c) If $u\in L^1$ and $a\in\mathfrak{A}$ we may set $\int_a u=\int u\times\chi a$. If $\gamma>0$ and $0\ne a\subseteq[\![u>\gamma]\!]$ then

$$\int_a u>\gamma\bar\mu a.$$

In particular, $\bar\mu[\![u>\gamma]\!]$ must be finite.

(d)(i) If $u\in L^1$ then $u\ge0$ iff $\int_a u\ge0$ for every $a\in\mathfrak{A}^f$.

(ii) If u, $v\in L^1$ and $\int_a u=\int_a v$ for every $a\in\mathfrak{A}^f$ then $u=v$.

(iii) If $u\ge0$ in L^1 then $\int u=\sup\{\int_a u:a\in\mathfrak{A}^f\}$.

(e) If $u\in L^1$, $u\ge0$ and $\int u=0$ then $u=0$. If $u\in L^1$, $u\ge0$ and $\int_a u=0$ then $u\times\chi a=0$, that is, $a\cap[\![u>0]\!]=0$.

(f) If $C\subseteq L^1$ is non-empty and upwards-directed and $\sup_{v\in C}\int v$ is finite, then $\sup C$ is defined in L^1 and $\int\sup C=\sup_{v\in C}\int v$.

(g) I may write $\int u = \infty$ if $u \in L^0$, $u^- \in L^1$ and $u^+ \notin L^1$, while $\int u = -\infty$ if $u^+ \in L^1$ and $u^- \notin L^1$.

(h) On this convention, if $C \subseteq (L^0)^+$ is non-empty and upwards-directed and has a supremum u in L^0, then $\int u = \sup_{v \in C} \int v$ in $[0, \infty]$.

365E The Radon-Nikodým theorem again (a) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ an additive functional. Then the following are equiveridical:

(i) there is a $u \in L^1 = L^1(\mathfrak{A}, \bar{\mu})$ such that $\nu a = \int_a u$ for every $a \in \mathfrak{A}$;

(ii) ν is additive and continuous for the measure-algebra topology on $\mathfrak{A}$;

(iii) ν is completely additive.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be any measure algebra, and $\nu : \mathfrak{A}^f \to \mathbb{R}$ a function. Then the following are equiveridical:

(i) ν is additive and bounded and $\inf_{a \in A} |\nu a| = 0$ whenever $A \subseteq \mathfrak{A}^f$ is downwards-directed and has infimum 0;

(ii) there is a $u \in L^1$ such that $\nu a = \int_a u$ for every $a \in \mathfrak{A}^f$.

365F Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Write S^f for the intersection $S(\mathfrak{A}) \cap L^1(\mathfrak{A}, \bar{\mu})$. Then S^f is a norm-dense and order-dense Riesz subspace of $L^1(\mathfrak{A}, \bar{\mu})$, and can be identified with $S(\mathfrak{A}^f)$. The function $\chi : \mathfrak{A}^f \to S^f \subseteq L^1(\mathfrak{A}, \bar{\mu})$ is an injective order-continuous additive lattice homomorphism. If $u \geq 0$ in $L^1(\mathfrak{A}, \bar{\mu})$, there is a non-decreasing sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $(S^f)^+$ such that $u = \sup_{n \in \mathbb{N}} u_n = \lim_{n \to \infty} u_n$.

365G Semi-finite algebras: Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ is order-dense in $L^0 = L^0(\mathfrak{A})$.

(b) In this case, writing $S^f = S(\mathfrak{A}) \cap L^1$, $\int u = \sup\{\int v : v \in S^f,\ 0 \leq v \leq u\}$ in $[0, \infty]$ for every $u \in (L^0)^+$.

365H Measurable transformations: Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be the sequentially order-continuous Riesz homomorphism associated with π.

(a) Suppose that $w \geq 0$ in $L^0(\mathfrak{B})$ is such that $\int_{\pi a} w\, d\bar{\nu} = \bar{\mu} a$ whenever $a \in \mathfrak{A}$ and $\bar{\mu} a < \infty$. Then for any $u \in L^1(\mathfrak{A}, \bar{\mu})$ and $a \in \mathfrak{A}$, $\int_{\pi a} Tu \times w\, d\bar{\nu}$ is defined and equal to $\int_a u\, d\bar{\mu}$.

(b) Suppose that $w' \geq 0$ in $L^0(\mathfrak{A})$ is such that $\int_a w' d\bar{\mu} = \bar{\nu}(\pi a)$ for every $a \in \mathfrak{A}$. Then $\int Tu\, d\bar{\nu} = \int u \times w'\, d\bar{\mu}$ whenever $u \in L^0(\mathfrak{A})$ and either integral is defined in $[-\infty, \infty]$.

365I Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and U a Banach space. Let $\nu : \mathfrak{A}^f \to U$ be a function. Then the following are equiveridical:

(i) there is a continuous linear operator T from $L^1(\mathfrak{A}, \bar{\mu})$ to U such that $\nu a = T(\chi a)$ for every $a \in \mathfrak{A}^f$;

(ii)(α) ν is additive

(β) there is an $M \geq 0$ such that $\|\nu a\| \leq M \bar{\mu} a$ for every $a \in \mathfrak{A}^f$.

Moreover, in this case, T is unique and $\|T\|$ is the smallest number M satisfying the condition in (ii-β).

365J Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, U a Banach lattice, and T a bounded linear operator from $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ to U. Let $\nu : \mathfrak{A}^f \to U$ be the corresponding additive function, as in 365I.

(a) T is a positive linear operator iff $\nu a \geq 0$ in U for every $a \in \mathfrak{A}^f$; in this case, T is order-continuous.

(b) If U is Dedekind complete and $T \in \mathrm{L}^\sim(L^1; U)$, then $|T| : L^1 \to U$ corresponds to $|\nu| : \mathfrak{A}^f \to U$, where

$$|\nu|(a) = \sup\{\textstyle\sum_{i=0}^n |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}$$

for every $a \in \mathfrak{A}^f$.

(c) T is a Riesz homomorphism iff ν is a lattice homomorphism.

365K The duality between L^1 and L^∞ Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and set $L^1 = L^1(\mathfrak{A}, \bar{\mu})$, $L^\infty = L^\infty(\mathfrak{A})$. If we identify L^∞ with the solid linear subspace of $L^0 = L^0(\mathfrak{A})$ generated by $e = \chi 1_{\mathfrak{A}}$, then we have a bilinear operator $(u, v) \mapsto u \times v : L^1 \times L^\infty \to L^1$. $\|u \times v\|_1 \leq \|u\|_1 \|v\|_\infty$, so the bilinear operator $(u, v) \mapsto u \times v$ has norm at most 1. Consequently we have a bilinear functional $(u, v) \mapsto \int u \times v : L^1 \times L^\infty \to \mathbb{R}$, which also has norm at most 1, corresponding to linear operators $S : L^1 \to (L^\infty)^*$ and $T : L^\infty \to (L^1)^*$, both of norm at most 1, defined by the formula

$$(Su)(v) = (Tv)(u) = \int u \times v \text{ for } u \in L^1,\ v \in L^\infty.$$

$(L^1)^* = (L^1)^\sim$ and $(L^\infty)^* = (L^\infty)^\sim$. $(L^1)^* = (L^1)^\times$.

365L Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and set $L^1 = L^1(\mathfrak{A}, \bar\mu)$, $L^\infty = L^\infty(\mathfrak{A})$. Let $S : L^1 \to (L^\infty)^* = (L^\infty)^\sim$, $T : L^\infty \to (L^1)^* = (L^1)^\sim = (L^1)^\times$ be the canonical maps defined by the duality between L^1 and L^∞. Then

(a) S and T are order-continuous Riesz homomorphisms, $S[L^1] \subseteq (L^\infty)^\times$, S is norm-preserving and $T[L^\infty]$ is order-dense in $(L^1)^\sim$.

(b) $(\mathfrak{A}, \bar\mu)$ is semi-finite iff T is injective, and in this case T is norm-preserving, while S is a normed Riesz space isomorphism between L^1 and $(L^\infty)^\times$.

(c) $(\mathfrak{A}, \bar\mu)$ is localizable iff T is bijective, and in this case T is a normed Riesz space isomorphism between L^∞ and $(L^1)^* = (L^1)^\sim = (L^1)^\times$.

365M Corollary If $(\mathfrak{A}, \bar\mu)$ is a localizable measure algebra, $L^\infty(\mathfrak{A})$ is a perfect Riesz space.

365N Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras. Let $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ be a measure-preserving ring homomorphism.

(a) There is a unique order-continuous norm-preserving Riesz homomorphism $T_\pi : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{B}, \bar\nu)$ such that $T_\pi(\chi a) = \chi(\pi a)$ whenever $a \in \mathfrak{A}^f$. We have $T_\pi(u \times \chi a) = T_\pi u \times \chi(\pi a)$ whenever $a \in \mathfrak{A}^f$ and $u \in L^1(\mathfrak{A}, \bar\mu)$.

(b) $\int T_\pi u = \int u$ and $\int_{\pi a} T_\pi u = \int_a u$ for every $u \in L^1(\mathfrak{A}, \bar\mu)$ and $a \in \mathfrak{A}^f$.

(c) $[\![T_\pi u > \alpha]\!] = \pi[\![u > \alpha]\!]$ for every $u \in L^1(\mathfrak{A}, \bar\mu)$ and $\alpha > 0$.

(d) T_π is surjective iff π is.

(e) If $(\mathfrak{C}, \bar\lambda)$ is another measure algebra and $\theta : \mathfrak{B}^f \to \mathfrak{C}^f$ another measure-preserving ring homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{C}, \bar\lambda)$.

365O Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and $\pi : \mathfrak{A}^f \to \mathfrak{B}$ an order-continuous ring homomorphism.

(a) There is a unique positive linear operator $P_\pi : L^1(\mathfrak{B}, \bar\nu) \to L^1(\mathfrak{A}, \bar\mu)$ such that $\int_a P_\pi v = \int_{\pi a} v$ for every $v \in L^1(\mathfrak{B}, \bar\nu)$ and $a \in \mathfrak{A}^f$.

(b) P_π is order-continuous and norm-continuous, and $\|P_\pi\| \le 1$.

(c) If $a \in \mathfrak{A}^f$ and $v \in L^1(\mathfrak{B}, \bar\nu)$ then $P_\pi(v \times \chi\pi a) = P_\pi v \times \chi a$.

(d) If $\pi[\mathfrak{A}^f]$ is order-dense in $\mathfrak{B}$ then P_π is a norm-preserving Riesz homomorphism; in particular, P_π is injective.

(e) If $(\mathfrak{B}, \bar\nu)$ is semi-finite and π is injective, then P_π is surjective, and there is for every $u \in L^1(\mathfrak{A}, \bar\mu)$ a $v \in L^1(\mathfrak{B}, \bar\nu)$ such that $P_\pi v = u$ and $\|v\|_1 = \|u\|_1$.

(f) Suppose again that $(\mathfrak{B}, \bar\nu)$ is semi-finite. If $(\mathfrak{C}, \bar\lambda)$ is another measure algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ an order-continuous Boolean homomorphism, then $P_{\theta\pi} = P_\pi P_{\theta'} : L^1(\mathfrak{C}, \bar\lambda) \to L^1(\mathfrak{A}, \bar\mu)$, where I write θ' for the restriction of θ to $\mathfrak{B}^f$.

365P Proposition Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\mu)$ be measure algebras and $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ a measure-preserving ring homomorphism.

(a) In the language of 365N-365O above, $P_\pi T_\pi$ is the identity operator on $L^1(\mathfrak{A}, \bar\mu)$.

(b) If π is surjective then $P_\pi = T_\pi^{-1} = T_{\pi^{-1}}$ and $T_\pi = P_\pi^{-1} = P_{\pi^{-1}}$.

365Q Conditional expectations (a) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra; write $\bar\nu$ for the restriction $\bar\mu\restriction\mathfrak{B}$. The identity map from $\mathfrak{B}$ to $\mathfrak{A}$ induces operators $T : L^1(\mathfrak{B}, \bar\nu) \to L^1(\mathfrak{A}, \bar\mu)$ and $P : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{B}, \bar\nu)$. If we take $L^0(\mathfrak{A})$ to be defined as the set of functions from $\mathbb{R}$ to $\mathfrak{A}$ described in 364Aa, then $L^0(\mathfrak{B})$ becomes a subset of $L^0(\mathfrak{A})$ in the literal sense, and T is actually the identity operator associated with the subset $L^1(\mathfrak{B}, \bar\nu) \subseteq L^1(\mathfrak{A}, \bar\mu)$; $L^1(\mathfrak{B}, \bar\nu)$ is a norm-closed and order-closed Riesz subspace of $L^1(\mathfrak{A},\bar\mu)$. P is a positive linear operator, while PT is the identity, so P is a projection from $L^1(\mathfrak{A},\bar\mu)$ onto $L^1(\mathfrak{B},\bar\nu)$. P is defined by the formula

$$\int_b Pu = \int_b u \text{ for every } u \in L^1(\mathfrak{A},\bar\mu),\ b \in \mathfrak{B},$$

so is the conditional expectation operator in the sense of 242J. $L^1(\mathfrak{B},\bar\nu)$ is just $L^1(\mathfrak{A},\bar\mu)\cap L^0(\mathfrak{B})$. $P(u\times v) = Pu \times v$ whenever $u \in L^1(\mathfrak{A},\bar\mu)$, $v \in L^0(\mathfrak{B})$ and $u \times v \in L^1(\mathfrak{A},\bar\mu)$.

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be a convex function and $\bar h : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ the corresponding map. If $u \in L^1(\mathfrak{A},\bar\mu)$, then $h(\int u) \le \int \bar h(u)$; and if $\bar h(u) \in L^1(\mathfrak{A},\bar\mu)$, then $\bar h(Pu) \le P(\bar h(u))$.

(c) If $u \in L^1(\mathfrak{A},\bar\mu)$ is non-negative, then $[\![Pu > 0]\!] = \operatorname{upr}([\![u > 0]\!],\mathfrak{B})$, the upper envelope of $[\![u > 0]\!]$ in $\mathfrak{B}$.

(d) Suppose now that $(\mathfrak{C},\bar\lambda)$ is another probability algebra and $\pi : \mathfrak{A} \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism. Then $\mathfrak{D} = \pi[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{C}$. Let $Q : L^1(\mathfrak{C},\bar\lambda) \to L^1(\mathfrak{D},\bar\lambda\restriction\mathfrak{D}) \subseteq L^1(\mathfrak{C},\bar\lambda)$ be the conditional expectation associated with $\mathfrak{D}$, and $T_\pi : L^1(\mathfrak{A},\bar\mu) \to L^1(\mathfrak{C},\bar\lambda)$ the norm-preserving Riesz homomorphism defined by π. Then $T_\pi P = QT_\pi$.

365R Recovering the algebra: Proposition (a) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. Then $\mathfrak{A}$ is isomorphic to the band algebra of $L^1(\mathfrak{A},\bar\mu)$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\bar\mu$, $\bar\nu$ two measures on $\mathfrak{A}$ such that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A},\bar\nu)$ are both semi-finite measure algebras. Then $L^1(\mathfrak{A},\bar\mu)$ is isomorphic, as Banach lattice, to $L^1(\mathfrak{A},\bar\nu)$.

365S Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\bar\mu : \mathfrak{A} \to [0,\infty]$, $\bar\nu : \mathfrak{A} \to [0,\infty]$ two functions such that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A},\bar\nu)$ are both semi-finite measure algebras.

(a) There is a unique $u \in L^0 = L^0(\mathfrak{A})$ such that $\int_a u\,d\bar\mu = \bar\nu a$ for every $a \in \mathfrak{A}$.

(b) For $v \in L^0(\mathfrak{A})$, $\int v\,d\bar\nu = \int u \times v\,d\bar\mu$ if either is defined in $[-\infty,\infty]$.

(c) u is strictly positive and, writing $\frac{1}{u}$ for the multiplicative inverse of u, $\int_a \frac{1}{u} d\bar\nu = \bar\mu a$ for every $a \in \mathfrak{A}$.

365T Uniform integrability: Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Set $L^1 = L^1(\mathfrak{A},\bar\mu)$.

(a) For a non-empty subset A of L^1, the following are equiveridical:

(i) A is uniformly integrable in the sense of 354P;

(ii) for every $\epsilon > 0$ there are an $a \in \mathfrak{A}^f$ and an $M \ge 0$ such that $\int(|u| - M\chi a)^+ \le \epsilon$ for every $u \in \mathfrak{A}$;

(iii)(α) $\sup_{u\in A} |\int_a u|$ is finite for every atom $a \in \mathfrak{A}$,

(β) for every $\epsilon > 0$ there are $c \in \mathfrak{A}^f$ and $\delta > 0$ such that $|\int_a u| \le \epsilon$ whenever $u \in A$, $a \in \mathfrak{A}$ and $\bar\mu(a \cap c) \le \delta$;

(iv)(α) $\sup_{u\in A} |\int_a u|$ is finite for every atom $a \in \mathfrak{A}$,

(β) $\lim_{n\to\infty} \sup_{u\in A} |\int_{a_n} u| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$;

(v) A is relatively weakly compact in L^1.

(b) If $(\mathfrak{A},\bar\mu)$ is a probability algebra and $A \subseteq L^1$ is uniformly integrable, then there is a solid convex norm-closed uniformly integrable set $C \supseteq A$ such that $P[C] \subseteq C$ whenever $P : L^1 \to L^1$ is the conditional expectation operator associated with a closed subalgebra of $\mathfrak{A}$.

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366 L^p

In this section I apply the methods of this chapter to L^p spaces, where $1 < p < \infty$. The constructions proceed without surprises up to 366E, translating the ideas of §244 by the methods used in §365. Turning to the action of Boolean homomorphisms on L^p spaces, I introduce a space M^0, which can be regarded as the part of L^0 that can be determined from the ring $\mathfrak{A}^f$ of elements of $\mathfrak{A}$ of finite measure (366F), and which includes L^p whenever $1 \le p < \infty$. Now a measure-preserving ring homomorphism from $\mathfrak{A}^f$ to $\mathfrak{B}^f$ acts on the M^0 spaces in a way which includes injective Riesz homomorphisms from $L^p(\mathfrak{A},\bar\mu)$ to $L^p(\mathfrak{B},\bar\nu)$ and surjective positive linear operators from $L^p(\mathfrak{B},\bar\nu)$ to $L^p(\mathfrak{A},\bar\mu)$ (366H). The latter may be regarded as conditional expectation operators (366J). The case $p=2$ (366K-366L) is of course by far the most important. As with the familiar spaces $L^p(\mu)$ of Chapter 24, we have complex versions $L^p_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ with the expected properties (366M).

366A Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and suppose that $1<p<\infty$. For $u\in L^0(\mathfrak{A})$, define $|u|^p\in L^0(\mathfrak{A})$ by setting

$$\begin{aligned}[\![|u|^p>\alpha]\!] &= [\![|u|>\alpha^{1/p}]\!] \text{ if } \alpha\ge 0,\\ &= 1 \text{ if } \alpha<0.\end{aligned}$$

Set

$$L^p_{\bar\mu}=L^p(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\ |u|^p\in L^1(\mathfrak{A},\bar\mu)\},$$

and for $u\in L^0(\mathfrak{A})$ set

$$\|u\|_p=(\int|u|^p)^{1/p}=\||u|^p\|_1^{1/p},$$

counting $\infty^{1/p}$ as ∞

366B Theorem Let (X,Σ,μ) be a measure space, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Then the canonical isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ makes $L^p(\mu)$ correspond to $L^p(\mathfrak{A},\bar\mu)$.

366C Corollary For any measure algebra $(\mathfrak{A},\bar\mu)$ and $p\in\,]1,\infty[$, $L^p=L^p(\mathfrak{A},\bar\mu)$ is a solid linear subspace of $L^0(\mathfrak{A})$. It is a Dedekind complete Banach lattice under its uniformly convex norm $\|\,\|_p$. Setting $q=p/(p-1)$, $(L^p)^*$ is identified with $L^q(\mathfrak{A},\bar\mu)$ by the duality $(u,v)\mapsto\int u\times v$. Writing $\mathfrak{A}^f$ for the ring $\{a:a\in\mathfrak{A},\ \bar\mu a<\infty\}$, $S(\mathfrak{A}^f)$ is norm-dense in L^p.

366D Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $p\in\,]1,\infty[$.

(a) The norm $\|\,\|_p$ on $L^p=L^p(\mathfrak{A},\bar\mu)$ is order-continuous.

(b) L^p has the Levi property.

(c) Setting $q=p/(p-1)$, the canonical identification of $L^q=L^q(\mathfrak{A},\bar\mu)$ with $(L^p)^*$ is a Riesz space isomorphism between L^q and $(L^p)^\sim=(L^p)^\times$.

(d) L^p is a perfect Riesz space.

366E Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $p\in[1,\infty]$. Set $q=p/(p-1)$ if $1<p<\infty$, $q=\infty$ if $p=1$ and $q=1$ if $p=\infty$. Then

$$L^q(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\ u\times v\in L^1(\mathfrak{A},\bar\mu)\text{ for every }v\in L^p(\mathfrak{A},\bar\mu)\}.$$

366F Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Write

$$M^0_{\bar\mu}=M^0(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\ \bar\mu[\![|u|>\alpha]\!]<\infty\text{ for every }\alpha>0\},$$

$$M^{1,0}_{\bar\mu}=M^{1,0}(\mathfrak{A},\bar\mu)=\{u:u\in M^0_{\bar\mu},\ u\times\chi a\in L^1(\mathfrak{A},\bar\mu)\text{ whenever }\bar\mu a<\infty\}.$$

366G Lemma Let $(\mathfrak{A},\bar\mu)$ be any measure algebra. Write $M^0=M^0(\mathfrak{A},\bar\mu)$, etc.

(a) M^0 and $M^{1,0}$ are Dedekind complete solid linear subspaces of L^0 which include L^p for every $p\in[1,\infty[$; moreover, M^0 is closed under multiplication.

(b) If $u\in M^0$ and $u\ge 0$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)$ such that $u=\sup_{n\in\mathbb{N}}u_n$.

(c) $M^{1,0}=\{u:u\in L^0,\ (|u|-\epsilon\chi 1)^+\in L^1\text{ for every }\epsilon>0\}=L^1+(L^\infty\cap M^0)$.

(d) If $u,\,v\in M^{1,0}$ and $\int_a u\le\int_a v$ whenever $\bar\mu a<\infty$, then $u\le v$; so if $\int_a u=\int_a v$ whenever $\bar\mu a<\infty$, $u=v$.

366H Theorem Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras. Let $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ be a measure-preserving ring homomorphism.

(a)(i) We have a unique order-continuous Riesz homomorphism $T=T_\pi:M^0(\mathfrak{A},\bar{\mu})\to M^0(\mathfrak{B},\bar{\nu})$ such that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}^f$.

(ii) $[\![Tu>\alpha]\!]=\pi[\![u>\alpha]\!]$ for every $u\in M^0(\mathfrak{A},\bar{\mu})$ and $\alpha>0$.

(iii) T is injective and multiplicative.

(iv) For $p\in[1,\infty]$ and $u\in M^0(\mathfrak{A},\bar{\mu})$, $\|Tu\|_p=\|u\|_p$; in particular, $Tu\in L^p(\mathfrak{B},\bar{\nu})$ iff $u\in L^p(\mathfrak{A},\bar{\mu})$. $\int Tu=\int u$ whenever $u\in L^1(\mathfrak{A},\bar{\mu})$.

(v) For $u\in M^0(\mathfrak{A},\bar{\mu})$, $Tu\in M^{1,0}(\mathfrak{B},\bar{\nu})$ iff $u\in M^{1,0}(\mathfrak{A},\bar{\mu})$.

(b)(i) We have a unique order-continuous positive linear operator $P=P_\pi:M^{1,0}(\mathfrak{B},\bar{\nu})\to M^{1,0}(\mathfrak{A},\bar{\mu})$ such that $\int_a Pv=\int_{\pi a}v$ whenever $v\in M^{1,0}(\mathfrak{B},\bar{\nu})$ and $a\in\mathfrak{A}^f$.

(ii) If $u\in M^0(\mathfrak{A},\bar{\mu})$, $v\in M^{1,0}(\mathfrak{B},\bar{\nu})$ and $v\times Tu\in M^{1,0}(\mathfrak{B},\bar{\nu})$, then $P(v\times Tu)=u\times Pv$.

(iii) If $q\in[1,\infty[$ and $v\in L^q(\mathfrak{B},\bar{\nu})$, then $Pv\in L^q(\mathfrak{A},\bar{\mu})$ and $\|Pv\|_q\le\|v\|_q$; if $v\in L^\infty(\mathfrak{B})\cap M^0(\mathfrak{B},\bar{\nu})$, then $Pv\in L^\infty(\mathfrak{A})$ and $\|Pv\|_\infty\le\|v\|_\infty$.

(iv) $PTu=u$ for every $u\in M^{1,0}(\mathfrak{A},\bar{\mu})$; in particular, $P[L^p(\mathfrak{B},\bar{\nu})]=L^p(\mathfrak{A},\bar{\mu})$ for every $p\in[1,\infty[$.

(c) If $(\mathfrak{C},\lambda)$ is another measure algebra and $\theta:\mathfrak{B}^f\to\mathfrak{C}^f$ another measure-preserving ring homomorphism, then $T_{\theta\pi}=T_\theta T_\pi:M^0(\mathfrak{A},\bar{\mu})\to M^0(\mathfrak{C},\bar{\lambda})$ and $P_{\theta\pi}=P_\pi P_\theta:M^{1,0}(\mathfrak{C},\bar{\lambda})\to M^{1,0}(\mathfrak{A},\bar{\mu})$.

(d) Now suppose that $\pi[\mathfrak{A}^f]=\mathfrak{B}^f$, so that π is a measure-preserving isomorphism between the rings $\mathfrak{A}^f$ and $\mathfrak{B}^f$.

(i) T is a Riesz space isomorphism between $M^0(\mathfrak{A},\bar{\mu})$ and $M^0(\mathfrak{B},\bar{\nu})$, and its inverse is $T_{\pi^{-1}}$.

(ii) P is a Riesz space isomorphism between $M^{1,0}(\mathfrak{B},\bar{\nu})$ and $M^{1,0}(\mathfrak{A},\bar{\mu})$, and its inverse is $P_{\pi^{-1}}$.

(iii) The restriction of T to $M^{1,0}(\mathfrak{A},\bar{\mu})$ is $P^{-1}=P_{\pi^{-1}}$; the restriction of $T^{-1}=T_{\pi^{-1}}$ to $M^{1,0}(\mathfrak{B},\bar{\nu})$ is P.

(iv) For any $p\in[1,\infty[$, $T{\restriction}L^p(\mathfrak{A},\bar{\mu})=P_{\pi^{-1}}{\restriction}L^p(\mathfrak{A},\bar{\mu})$ and $P{\restriction}L^p(\mathfrak{B},\bar{\nu})=T_{\pi^{-1}}{\restriction}L^p(\mathfrak{B},\bar{\nu})$ are the two halves of a Banach lattice isomorphism between $L^p(\mathfrak{A},\bar{\mu})$ and $L^p(\mathfrak{B},\bar{\nu})$.

366I Corollary Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Then, for any $p\in[1,\infty[$, $L^p(\mathfrak{B},\bar{\mu}{\restriction}\mathfrak{B})$ can be identified, as Banach lattice, with the closed linear subspace of $L^p(\mathfrak{A},\bar{\mu})$ generated by $\{\chi b:b\in\mathfrak{B},\ \bar{\mu}b<\infty\}$.

366J Corollary If $(\mathfrak{A},\bar{\mu})$ is a probability algebra, $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, and $P:L^1(\mathfrak{A},\bar{\mu})\to L^1(\mathfrak{B},\bar{\mu}{\restriction}\mathfrak{B})$ is the conditional expectation operator, then $\|Pu\|_p\le\|u\|_p$ whenever $p\in[1,\infty]$ and $u\in L^p(\mathfrak{A},\bar{\mu})$.

366K Corollary Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ a measure-preserving ring homomorphism. Let $T:L^2(\mathfrak{A},\bar{\mu})\to L^2(\mathfrak{B},\bar{\nu})$ and $P:L^2(\mathfrak{B},\bar{\nu})\to L^2(\mathfrak{A},\bar{\mu})$ be the corresponding operators. Then $TP:L^2(\mathfrak{B},\bar{\nu})\to L^2(\mathfrak{B},\bar{\nu})$ is an orthogonal projection, its range $TP[L^2(\mathfrak{B},\bar{\nu})]$ being isomorphic, as Banach lattice, to $L^2(\mathfrak{A},\bar{\mu})$. The kernel of TP is just

$$\{v:v\in L^2(\mathfrak{B},\bar{\nu}),\ \int_{\pi a}v=0\text{ for every }a\in\mathfrak{A}^f\}.$$

366L Corollary Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\pi:\mathfrak{A}^f\to\mathfrak{A}^f$ a measure-preserving ring automorphism. Then there is a corresponding Banach lattice isomorphism T of $L^2=L^2(\mathfrak{A},\bar{\mu})$ defined by writing $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}^f$. Its inverse is defined by the formula

$$\int_a T^{-1}u=\int_{\pi a}u\text{ for every }u\in L^2,\ a\in\mathfrak{A}^f.$$

***366M Complex L^p spaces (a)** Just as in §§241-244, we have 'complex' versions of all the spaces considered in this chapter. Thus for any Boolean algebra $\mathfrak{A}$ with Stone space Z, we can identify $L^\infty_{\mathbb{C}}(\mathfrak{A})$ with the space $C(Z;\mathbb{C})$ of continuous functions from Z to $\mathbb{C}$; inside this, we have a $\|\,\|_\infty$-dense subspace $S_{\mathbb{C}}(\mathfrak{A})$ consisting of complex linear combinations of indicator functions of open-and-closed sets. If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, identified with a quotient $\Sigma/\mathcal{M}$ where Σ is a σ-algebra of subsets of a set Z and $\mathcal{M}$ is a σ-ideal of Σ, then we can write $\mathcal{L}^0_{\mathbb{C}}$ for the set of functions from Z to $\mathbb{C}$ such that their real and imaginary parts are both Σ-measurable, $\mathcal{W}_{\mathbb{C}}$ for the set of those $f \in \mathcal{L}^0_{\mathbb{C}}$ such that $\{z : f(z) \neq 0\}$ belongs to $\mathcal{M}$, and $L^0_{\mathbb{C}} = L^0_{\mathbb{C}}(\mathfrak{A})$ for the linear space quotient $\mathcal{L}^0_{\mathbb{C}}/\mathcal{W}_{\mathbb{C}}$. As in 241J, we have a natural embedding of $L^0 = L^0(\mathfrak{A})$ in $L^0_{\mathbb{C}}$ and functions

$$\mathcal{R}\mathrm{e} : L^0_{\mathbb{C}} \to L^0, \quad \mathcal{I}\mathrm{m} : L^0_{\mathbb{C}} \to L^0, \quad |\ | : L^0_{\mathbb{C}} \to L^0, \quad {}^- : L^0_{\mathbb{C}} \to L^0_{\mathbb{C}}$$

such that

$$u = \mathcal{R}\mathrm{e}(u) + i\,\mathcal{I}\mathrm{m}(u), \quad \mathcal{R}\mathrm{e}(u+v) = \mathcal{R}\mathrm{e}(u) + \mathcal{R}\mathrm{e}(v), \quad \mathcal{I}\mathrm{m}(u+v) = \mathcal{I}\mathrm{m}(u) + \mathcal{I}\mathrm{m}(v),$$

$$\mathcal{R}\mathrm{e}(\alpha u) = \mathcal{R}\mathrm{e}(\alpha)\,\mathcal{R}\mathrm{e}(u) - \mathcal{I}\mathrm{m}(\alpha)\,\mathcal{I}\mathrm{m}(u), \quad \mathcal{I}\mathrm{m}(\alpha u) = \mathcal{R}\mathrm{e}(\alpha)\,\mathcal{I}\mathrm{m}(u) + \mathcal{I}\mathrm{m}(\alpha)\,\mathcal{R}\mathrm{e}(u),$$

$$|\alpha u| = |\alpha||u|, \quad |u+v| \le |u| + |v|, \quad |u| = \sup_{|\gamma|=1} \mathcal{R}\mathrm{e}(\gamma u),$$

$$\bar{u} = \mathcal{R}\mathrm{e}(u) - i\,\mathcal{I}\mathrm{m}(u), \quad \overline{u+v} = \bar{u} + \bar{v}, \quad \overline{\alpha u} = \bar{\alpha}\bar{u}$$

for all u, $v \in L^0_{\mathbb{C}}$ and $\alpha \in \mathbb{C}$.

I seem to have omitted to mention it in 241J, but of course we also have a multiplication

$$u \times v = (\mathcal{R}\mathrm{e}(u) \times \mathcal{R}\mathrm{e}(v) - \mathcal{I}\mathrm{m}(u) \times \mathcal{I}\mathrm{m}(v)) + i(\mathcal{R}\mathrm{e}(u) \times \mathcal{I}\mathrm{m}(v) + \mathcal{I}\mathrm{m}(u) \times \mathcal{R}\mathrm{e}(v)),$$

for which we have

$$u \times v = v \times u, \quad u \times (v \times w) = (u \times v) \times w, \quad u \times (v + w) = (u \times v) + (u \times w),$$

$$(\alpha u) \times v = u \times (\alpha v) = \alpha(u \times v),$$

$$\overline{u \times v} = \bar{u} \times \bar{v}, \quad |u \times v| = |u| \times |v|, \quad u \times \bar{u} = |u|^2 = (\mathcal{R}\mathrm{e}(u))^2 + (\mathcal{I}\mathrm{m}(u))^2$$

for u, $v \in L^0_{\mathbb{C}}$ and $\alpha \in \mathbb{C}$.

(b) If $(\mathfrak{A}, \bar{\mu})$ is a measure algebra and $1 \le p < \infty$, we can think of $L^p_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$ as the set of those $u \in L^0_{\mathbb{C}}$ such that $|u| \in L^p(\mathfrak{A}, \bar{\mu})$, with its norm defined by the formula $\|u\|_p = \||u|\|_p$; this will make $L^p_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$ a Banach space, with dual $L^q(\mathfrak{A}, \bar{\mu})$ where $\frac{1}{p} + \frac{1}{q} = 1$ if $p > 1$. (Similarly, if $(\mathfrak{A}, \bar{\mu})$ is localizable, the dual of $L^1_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$ can be identified with $L^\infty_{\mathbb{C}}$.)

Writing $S_{\mathbb{C}}(\mathfrak{A}^f)$ for the space of linear combinations of indicator functions of elements of $\mathfrak{A}$ of finite measure, $S_{\mathbb{C}}(\mathfrak{A}^f)$ is dense in $L^p_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$ whenever $1 \le p < \infty$.

(c) L^1- and L^2-spaces have integrals and inner products. Here we set

$$\int u = \int \mathcal{R}\mathrm{e}(u) + i \int \mathcal{I}\mathrm{m}(u)$$

for $u \in L^1_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$, and $\int : L^1_{\mathbb{C}}(\mathfrak{A}, \bar{\mu}) \to \mathbb{C}$ becomes a $\mathbb{C}$-linear functional. As for L^2,

$$|u \times v| = |u| \times |v| \in L^1(\mathfrak{A}, \bar{\mu}), \quad u \times v \in L^1_{\mathbb{C}}(\mathfrak{A}, \bar{\mu}), \quad \int u \times \bar{u} = \|u\|_2^2$$

for u, $v \in L^2_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$. So if we set

$$(u|v) = \int u \times \bar{v}$$

for u, $v \in L^2_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$, $L^2_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$ becomes a complex Hilbert space.

(d) If $\mathfrak{A}$, $\mathfrak{B}$ are Dedekind σ-complete Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, then we have a linear operator $T_\pi : L^0_{\mathbb{C}}(\mathfrak{A}) \to L^0_{\mathbb{C}}(\mathfrak{B})$ defined by setting $T_\pi u = T_\pi^{\mathrm{real}}(\mathcal{R}\mathrm{e}(u)) + iT_\pi^{\mathrm{real}}(\mathcal{I}\mathrm{m}(u))$, where $T_\pi^{\mathrm{real}} : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ is the Riesz homomorphism described in 364P. T_π, like T_π^{real}, will be multiplicative; $T_\pi|u| = |T_\pi u|$ for every $u \in L^0_{\mathbb{C}}(\mathfrak{A})$. $T_\pi\bar{u} = \overline{T_\pi u}$ for every $u \in L^0_{\mathbb{C}}(\mathfrak{A})$. Also, if $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\phi : \mathfrak{B} \to \mathfrak{C}$ are sequentially order-continuous Boolean homomorphisms, $T_{\phi\pi} = T_\phi T_\pi$. So if $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean automorphism, T_π will be a bijection with inverse $T_{\pi^{-1}}$.

(e) Similarly, if $(\mathfrak{A}, \bar{\mu})$ is a measure algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean homomorphism, $\int T_\pi u = \int u$ for every $u \in L^1_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$. If u, $v \in L^2_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$, then

$$(T_\pi u | T_\pi v) = \int T_\pi u \times \overline{T_\pi v} = \int T_\pi u \times T_\pi \bar{v} = \int T_\pi(u \times \bar{v}) = \int u \times \bar{v} = (u|v).$$

If π is a measure-preserving Boolean automorphism, we shall have

$$(T_\pi u|v) = (T_{\pi^{-1}} T_\pi u|T_{\pi^{-1}} v) = (u|T_\pi^{-1} v)$$

for all u, $v \in L^2_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$.

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367 Convergence in measure

Continuing through the ideas of Chapter 24, I come to 'convergence in measure'. The basic results of §245 all translate easily into the new language (367L-367M, 367P). The associated concept of (sequential) order-convergence can also be expressed in abstract terms (367A), and I take the trouble to do this in the context of general lattices (367A-367B), since the concept can be applied in many ways (367C-367E, 367K). In the particular case of L^0 spaces, which are the first aim of this section, the idea is most naturally expressed by 367F. It enables us to express some of the basic theorems in Volumes 1 and 2 in the language of this chapter (367I-367J).

In 367N and 367O I give two of the most characteristic properties of the topology of convergence in measure on L^0; it is one of the fundamental types of topological Riesz space. Another striking fact is the way it is determined by the Riesz space structure (367T). In 367U I set out a theorem which is the basis of many remarkable applications of the concept; for the sake of a result in §369 I give one such application (367V).

367A Order*-convergence: Definition Let P be a lattice, p an element of P and $\langle p_n\rangle_{n\in\mathbb{N}}$ a sequence in P. I will say that $\langle p_n\rangle_{n\in\mathbb{N}}$ **order*-converges** to p if

$$\begin{aligned} p &= \inf\{q : \exists\, n \in \mathbb{N},\, q \ge (p' \vee p_i) \wedge p'' \;\forall\, i \ge n\} \\ &= \sup\{q : \exists\, n \in \mathbb{N},\, q \le p' \vee (p_i \wedge p'') \;\forall\, i \ge n\} \end{aligned}$$

whenever $p' \le p \le p''$ in P.

367B Lemma Let P be a lattice.

(a) A sequence in P can order*-converge to at most one point.

(b) A constant sequence order*-converges to its constant value.

(c) Any subsequence of an order*-convergent sequence is order*-convergent, with the same limit.

(d) If $\langle p_n\rangle_{n\in\mathbb{N}}$ and $\langle p'_n\rangle_{n\in\mathbb{N}}$ both order*-converge to p, and $p_n \le q_n \le p'_n$ for every n, then $\langle q_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(e) If $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in P, then it order*-converges to $p \in P$ iff

$$\begin{aligned} p &= \inf\{q : \exists\, n \in \mathbb{N},\, q \ge p_i \;\forall\, i \ge n\} \\ &= \sup\{q : \exists\, n \in \mathbb{N},\, q \le p_i \;\forall\, i \ge n\}. \end{aligned}$$

(f) If P is a Dedekind σ-complete lattice and $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in P, then it order*-converges to $p \in P$ iff

$$p = \sup_{n\in\mathbb{N}} \inf_{i\ge n} p_i = \inf_{n\in\mathbb{N}} \sup_{i\ge n} p_i.$$

367C Proposition Let U be a Riesz space.

(a) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ are two sequences in U order*-converging to u, v respectively.

(i) $\langle u_n + w\rangle_{n\in\mathbb{N}}$ order*-converges to $u + w$ for every $w \in U$, and αu_n order*-converges to αu for every $\alpha \in \mathbb{R}$.

(ii) $\langle u_n \vee v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u \vee v$ and $\langle u_n \wedge v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u \wedge v$.

(iii) If $\langle w_n\rangle_{n\in\mathbb{N}}$ is any sequence in U, then it order*-converges to $w \in U$ iff $\langle |w_n - w|\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

(iv) $\langle u_n + v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u + v$.

(v) If $\langle w_n\rangle_{n\in\mathbb{N}}$ and $\langle z_n\rangle_{n\in\mathbb{N}}$ are sequences in U, $\langle w_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 and $|z_n| \le |w_n|$ for every n, then $\langle z_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

(b) Now suppose that U is Archimedean.

(i) If $\langle \alpha_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to $\alpha \in \mathbb{R}$, and $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U order*-converging to $u \in U$, then $\langle \alpha_n u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu.

(ii) A sequence $\langle w_n\rangle_{n\in\mathbb{N}}$ in U^+ is *not* order*-convergent to 0 iff there is a $\tilde{w} > 0$ such that $\tilde{w} = \sup_{i\ge n} \tilde{w} \wedge w_i$ for every $n \in \mathbb{N}$.

(iii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U^+ such that $\{\sum_{i=0}^n u_i : n \in \mathbb{N}\}$ is bounded above, then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

367D Proposition Let U be a Riesz space with a Riesz norm $\| \,\|$.

(a) If a sequence in U is both order*-convergent and norm-convergent, the two limits are the same.

(b) $\| \,\|$ is order-continuous iff every order-bounded order*-convergent sequence in U is norm-convergent.

367E Proposition Let U be an Archimedean Riesz space and V a regularly embedded Riesz subspace. If $\langle v_n\rangle_{n\in\mathbb{N}}$ is a sequence in V and $v \in V$, then $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to v when regarded as a sequence in V, iff it order*-converges to v when regarded as a sequence in U.

367F Proposition Let X be a set, Σ a σ-algebra of subsets of X, $\mathfrak{A}$ a Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ a sequentially order-continuous surjective Boolean homomorphism; let $\mathcal{I}$ be its kernel. Write $\mathcal{L}^0$ for the space of Σ-measurable functions from X to $\mathbb{R}$, and let $T = T_\pi : \mathcal{L}^0 \to L^0 = L^0(\mathfrak{A})$ be the canonical Riesz homomorphism. Then for any $\langle f_n\rangle_{n\in\mathbb{N}}$ and f in $\mathcal{L}^0$, $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf in L^0 iff $X \setminus \{x : f(x) = \lim_{n\to\infty} f_n(x)\} \in \mathcal{I}$.

367G Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) Any order*-convergent sequence in $L^0 = L^0(\mathfrak{A})$ is order-bounded.

(b) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0, then it is order*-convergent to $u \in L^0$ iff

$$u = \inf_{n\in\mathbb{N}} \sup_{i\ge n} u_i = \sup_{n\in\mathbb{N}} \inf_{i\ge n} u_i.$$

367H Proposition Suppose that $E \subseteq \mathbb{R}$ is a Borel set and $h : E \to \mathbb{R}$ is a continuous function. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and set $Q_E = \{u : u \in L^0, [\![u \in E]\!] = 1\}$, where $L^0 = L^0(\mathfrak{A})$. Let $\bar{h} : Q_E \to L^0$ be the function defined by h. Then $\langle \bar{h}(u_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{h}(u)$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in Q_E order*-converging to $u \in Q_E$.

367I Dominated convergence: Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^1 = L^1_{\bar{\mu}}$ which is order-bounded and order*-convergent in L^1, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to some $u \in L^1$; in particular, $\int u = \lim_{n\to\infty} \int u_n$.

367J The Martingale Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\langle \mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$. For each $n \in \mathbb{N}$ let $P_n : L^1 = L^1_{\bar{\mu}} \to L^1 \cap L^0(\mathfrak{B}_n)$ be the conditional expectation operator; let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$, and P the conditional expectation operator onto $L^1 \cap L^0(\mathfrak{B})$.

(a) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a norm-bounded sequence in L^1 such that $P_n(u_{n+1}) = u_n$ for every $n \in \mathbb{N}$, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent in L^1.

(b) If $u \in L^1$ then $\langle P_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \,\|_1$-convergent to Pu.

367K Proposition Let X be a locally compact Hausdorff space, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in $C(X)$. Then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$ iff $\{x : x \in X, \limsup_{n\to\infty} |u_n(x)| > 0\}$ is meager. In particular, $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 if $\lim_{n\to\infty} u_n(x) = 0$ for every x.

367L Convergence in measure Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. For $a\in\mathfrak{A}^f=\{a:\bar{\mu}a<\infty\}$, $u\in L^0=L^0(\mathfrak{A})$ and $\epsilon>0$ set $\tau_a(u)=\int|u|\wedge\chi a$ and $\tau_{a\epsilon}(u)=\bar{\mu}(a\cap[\![|u|>\epsilon]\!])$. Then the **topology of convergence in measure** on L^0 is defined *either* as the topology generated by the F-seminorms τ_a *or* by saying that $G\subseteq L^0$ is open iff for every $u\in G$ there are $a\in\mathfrak{A}^f$ and $\epsilon>0$ such that $v\in G$ whenever $\tau_{a\epsilon}(u-v)\le\epsilon$.

367M Theorem (a) For any measure algebra $(\mathfrak{A},\bar{\mu})$, the topology $\mathfrak{T}$ of convergence in measure on $L^0=L^0(\mathfrak{A})$ is a linear space topology, and any order*-convergent sequence in L^0 is $\mathfrak{T}$-convergent to the same limit.

(b) $u\mapsto|u|:L^0\to L^0$ and $(u,v)\mapsto u\vee v$, $(u,v)\mapsto u\times v:L^0\times L^0\to L^0$ are continuous.

(c) $(\mathfrak{A},\bar{\mu})$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(d) $(\mathfrak{A},\bar{\mu})$ is localizable iff $\mathfrak{T}$ is Hausdorff and L^0 is complete under the uniformity corresponding to $\mathfrak{T}$.

(e) $(\mathfrak{A},\bar{\mu})$ is σ-finite iff $\mathfrak{T}$ is metrizable.

367N Proposition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and give $L^0=L^0(\mathfrak{A})$ its topology of convergence in measure.

(a) If $A\subseteq L^0$ is a non-empty, downwards-directed set with infimum 0, then for every neighbourhood G of 0 in L^0 there is a $u\in A$ such that $v\in G$ whenever $|v|\le u$.

(b) If $U\subseteq L^0$ is an order-dense Riesz subspace, it is topologically dense.

(c) In particular, $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ are topologically dense.

367O Theorem Let U be a Banach lattice and $(\mathfrak{A},\bar{\mu})$ a measure algebra. Give $L^0=L^0(\mathfrak{A})$ its topology of convergence in measure. If $T:U\to L^0$ is a positive linear operator, then it is continuous.

367P Proposition Let $(\mathfrak{A},\bar{\mu})$ be a σ-finite measure algebra.

(a) A sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $L^0=L^0(\mathfrak{A})$ converges in measure to $u\in L^0$ iff every subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence which order*-converges to u.

(b) A set $F\subseteq L^0$ is closed for the topology of convergence in measure iff $u\in F$ whenever there is a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in F order*-converging to $u\in L^0$.

367Q Theorem Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra; for each closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, let $P_{\mathfrak{B}}$ be the corresponding conditional expectation operator from $L^1=L^1_{\bar{\mu}}$ to $L^1\cap L^0(\mathfrak{B})=L^1_{\bar{\mu}\restriction\mathfrak{B}}$.

(a) If $\mathbb{B}$ is a non-empty downwards-directed family of closed subalgebras of $\mathfrak{A}$ with intersection $\mathfrak{C}$, and $u\in L^1=L^1_{\bar{\mu}}$, then $P_{\mathfrak{C}}u$ is the $\|\,\|_1$-limit $\lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\downarrow)}P_{\mathfrak{B}}u$, where $\mathcal{F}(\mathbb{B}\downarrow)$ is the filter on $\mathbb{B}$ generated by $\{\{\mathfrak{B}:\mathfrak{B}_0\supseteq\mathfrak{B}\in\mathbb{B}\}:\mathfrak{B}_0\in\mathbb{B}\}$.

(b) If $\mathbb{B}$ is a non-empty upwards-directed family of closed subalgebras of $\mathfrak{A}$ and $\mathfrak{C}$ is the closed subalgebra generated by $\bigcup\mathbb{B}$, then for every $u\in L^1$, $P_{\mathfrak{C}}u$ is the $\|\,\|_1$-limit $\lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)}P_{\mathfrak{B}}u$, where $\mathcal{F}(\mathbb{B}\uparrow)$ is the filter on $\mathbb{B}$ generated by $\{\{\mathfrak{B}:\mathfrak{B}_0\subseteq\mathfrak{B}\in\mathbb{B}\}:\mathfrak{B}_0\in\mathbb{B}\}$. as $\mathfrak{B}$ decreases through $\mathbb{B}$.

(c) Suppose that $\mathbb{B}$ is a non-empty upwards-directed family of closed subalgebras of $\mathfrak{A}$, and $\langle u_{\mathfrak{B}}\rangle_{\mathfrak{B}\in\mathbb{B}}$ is a $\|\,\|_1$-bounded family in L^1 such that $u_{\mathfrak{B}}=P_{\mathfrak{B}}u_{\mathfrak{C}}$ whenever $\mathfrak{B}$, $\mathfrak{C}\in\mathbb{B}$ and $\mathfrak{B}\subseteq\mathfrak{C}$. Then $\lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)}u_{\mathfrak{B}}$ is defined for the topology of convergence in measure and belongs to L^1.

367R Proposition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Give $\mathfrak{A}$ its measure-algebra topology and $L^0=L^0(\mathfrak{A})$ the topology of convergence in measure.

(a) The map $\chi:\mathfrak{A}\to L^0$ is a homeomorphism between $\mathfrak{A}$ and its image in L^0.

(b) If $\mathfrak{A}$ has countable Maharam type, then L^0 is separable.

(c) Suppose that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ which is closed for the measure-algebra topology. Then $L^0(\mathfrak{B})$ is closed in $L^0(\mathfrak{A})$.

(d) A non-empty set $A\subseteq L^0$ is bounded in the linear topological space sense iff $\inf_{k\in\mathbb{N}}\sup_{u\in A}\bar{\mu}(a\cap[\![|u|>k]\!])=0$ for every $a\in\mathfrak{A}^f$.

367S Proposition Let $E\subseteq\mathbb{R}$ be a Borel set, and $h:E\to\mathbb{R}$ a continuous function. Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\bar{h}:Q_E\to L^0=L^0(\mathfrak{A})$ the associated function, where $Q_E=\{u:u\in L^0,\,[\![u\in E]\!]=1\}$. Then $\bar{h}$ is continuous for the topology of convergence in measure.

367T Intrinsic description of convergence in measure: Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and U an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$. Suppose that $A \subseteq U$ and $u^* \in U$. Then u^* belongs to the closure of A for the topology of convergence in measure iff

there is an order-dense Riesz subspace V of U such that

for every $v \in V^+$ there is a non-empty downwards-directed $B \subseteq U$, with infimum 0, such that

for every $w \in B$ there is a $u \in A$ such that

$$|u-u^*| \wedge v \le w.$$

***367U Theorem** Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra; write L^1 for $L^1_{\bar{\mu}}$. Let $P:(L^1)^{**} \to L^1$ be the linear operator corresponding to the band projection from $(L^1)^{**} = (L^1)^{\times\sim}$ onto $(L^1)^{\times\times}$ and the canonical isomorphism between L^1 and $(L^1)^{\times\times}$. For $A \subseteq L^1$ write A^* for the weak* closure of the image of A in $(L^1)^{**}$. Then for every $A \subseteq L^1$

$$P[A^*] \subseteq \overline{\Gamma(A)},$$

where $\Gamma(A)$ is the convex hull of A and $\overline{\Gamma(A)}$ is the closure of $\Gamma(A)$ in $L^0 = L^0(\mathfrak{A})$ for the topology of convergence in measure.

***367V Corollary** Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. Let $\mathcal{C}$ be a family of convex subsets of $L^0 = L^0(\mathfrak{A})$, all closed for the topology of convergence in measure, with the finite intersection property, and suppose that for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and a $C \in \mathcal{C}$ such that $\sup_{u\in C}\int_b |u| < \infty$. Then $\bigcap\mathcal{C} \ne \emptyset$.

***367W Independence:** Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra. Then a family $\langle u_i\rangle_{i\in I}$ in $L^0(\mathfrak{A})$ is **stochastically independent** if $\bar{\mu}(\inf_{i\in J}[\![u_i > \alpha_i]\!]) = \prod_{i\in J}\bar{\mu}[\![u_i > \alpha_i]\!]$ whenever $J \subseteq I$ is a non-empty finite set and $\alpha_i \in \mathbb{R}$ for every $i \in I$.

Proposition Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra, and I any set. Give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. Then the collection of independent families $\langle u_i\rangle_{i\in I}$ is a closed set in $(L^0)^I$.

Version of 16.9.09

368 Embedding Riesz spaces in L^0

In this section I turn to the representation of Archimedean Riesz spaces as function spaces. Any Archimedean Riesz space U can be represented as an order-dense subspace of $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is its band algebra (368E). Consequently we get representations of Archimedean Riesz spaces as quotients of subspaces of $\mathbb{R}^X$ (368F) and as subspaces of $C^\infty(X)$ (368G), and a notion of 'Dedekind completion' (368I-368J). Closely associated with these is the fact that we have a very general extension theorem for order-continuous Riesz homomorphisms into L^0 spaces (368B). I give a characterization of L^0 spaces in terms of lateral completeness (368M), and I discuss weakly (σ,∞)-distributive Riesz spaces (368N-368S).

368A Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $A \subseteq (L^0)^+$ a set with no upper bound in L^0, where $L^0 = L^0(\mathfrak{A})$. If *either* A is countable *or* $\mathfrak{A}$ is Dedekind complete, there is a $v > 0$ in L^0 such that $nv = \sup_{u\in A} u \wedge nv$ for every $n \in \mathbb{N}$.

368B Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, U an Archimedean Riesz space, V an order-dense Riesz subspace of U and $T: V \to L^0 = L^0(\mathfrak{A})$ an order-continuous Riesz homomorphism. Then T has a unique extension to an order-continuous Riesz homomorphism $\tilde{T}: U \to L^0$.

368C Corollary Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras and U, V order-dense Riesz subspaces of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively. Then any Riesz space isomorphism between U and V extends uniquely to a Riesz space isomorphism between $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$; and in this case $\mathfrak{A}$ and $\mathfrak{B}$ must be isomorphic as Boolean algebras.

368D Corollary Suppose that $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, and that U is an order-dense Riesz subspace of $L^0(\mathfrak{A})$ which is isomorphic, as Riesz space, to $L^0(\mathfrak{B})$ for some Dedekind complete Boolean algebra $\mathfrak{B}$. Then $U = L^0(\mathfrak{A})$ and $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ (so, in particular, is Dedekind complete).

368E Theorem Let U be any Archimedean Riesz space, and $\mathfrak{A}$ its band algebra. Then U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{A})$.

368F Corollary A Riesz space U is Archimedean iff it is isomorphic to a Riesz subspace of some reduced power $\mathbb{R}^X|\mathcal{F}$, where X is a set and $\mathcal{F}$ is a filter on X such that $\bigcap_{n\in\mathbb{N}} F_n \in \mathcal{F}$ whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$.

368G Corollary Every Archimedean Riesz space U is isomorphic to an order-dense Riesz subspace of some space $C^\infty(X)$, where X is an extremally disconnected compact Hausdorff space.

368H Corollary Any Dedekind complete Riesz space U is isomorphic to an order-dense solid linear subspace of $L^0(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$.

368I Corollary Let U be an Archimedean Riesz space. Then U can be embedded as an order-dense Riesz subspace of a Dedekind complete Riesz space V in such a way that the solid linear subspace of V generated by U is V itself, and this can be done in essentially only one way. If W is any other Dedekind complete Riesz space and $T : U \to W$ is an order-continuous positive linear operator, there is a unique positive linear operator $\tilde{T} : V \to W$ extending T.

368J Definition If U is an Archimedean Riesz space, a **Dedekind completion** of U is a Dedekind complete Riesz space V together with an embedding of U in V as an order-dense Riesz subspace of V such that the solid linear subspace of V generated by U is V itself.

368K Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Suppose that $A \subseteq L^0(\mathfrak{A})^+$ is disjoint. If *either* A is countable *or* $\mathfrak{A}$ is Dedekind complete, A is bounded above in $L^0(\mathfrak{A})$.

368L Definition A Riesz space U is called **laterally complete** or **universally complete** if A is bounded above whenever $A \subseteq U^+$ is disjoint.

368M Theorem Let U be an Archimedean Riesz space. Then the following are equiveridical:

(i) there is a Dedekind complete Boolean algebra $\mathfrak{A}$ such that U is isomorphic to $L^0(\mathfrak{A})$;

(ii) U is Dedekind σ-complete and laterally complete;

(iii) whenever V is an Archimedean Riesz space, V_0 is an order-dense Riesz subspace of V and $T : V_0 \to U$ is an order-continuous Riesz homomorphism, there is a positive linear operator $\tilde{T} : V \to U$ extending T.

368N Weakly (σ,∞)-distributive Riesz spaces: Definition Let U be a Riesz space. Then U is **weakly (σ,∞)-distributive** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty downwards-directed subsets of U^+, each with infimum 0, and $\bigcup_{n\in\mathbb{N}} A_n$ has an upper bound in U, then

$$\{u : u \in U, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } v \le u\}$$

has infimum 0 in U.

368O Lemma Let U be an Archimedean Riesz space. Then the following are equiveridical:

(i) U is not weakly (σ,∞)-distributive;

(ii) there are a $u > 0$ in U and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\sup_{n\in\mathbb{N}} u_n = u$ whenever $u_n \in A_n$ for every $n \in \mathbb{N}$.

368P Proposition (a) A regularly embedded Riesz subspace of an Archimedean weakly (σ,∞)-distributive Riesz space is weakly (σ,∞)-distributive.

(b) An Archimedean Riesz space with a weakly (σ,∞)-distributive order-dense Riesz subspace is weakly (σ,∞)-distributive.

(c) If U is a Riesz space such that $U^\times$ separates the points of U, then U is weakly (σ,∞)-distributive; in particular, $U^\sim$ and $U^\times$ are weakly (σ,∞)-distributive for every Riesz space U.

368Q Theorem (a) For any Boolean algebra $\mathfrak{A}$, $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $S(\mathfrak{A})$ is weakly (σ,∞)-distributive iff $L^\infty(\mathfrak{A})$ is weakly (σ,∞)-distributive.

(b) For a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is weakly (σ,∞)-distributive.

368R Corollary An Archimedean Riesz space is weakly (σ,∞)-distributive iff its band algebra is weakly (σ,∞)-distributive.

368S Corollary If $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra, any regularly embedded Riesz subspace (in particular, any solid linear subspace and any order-dense Riesz subspace) of $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive.

Version of 23.11.16

369 Banach function spaces

In this section I continue the work of §368 with results which involve measure algebras. The first step is a modification of the basic representation theorem for Archimedean Riesz spaces. If U is any Archimedean Riesz space, it can be represented as a subspace of $L^0 = L^0(\mathfrak{A})$, where $\mathfrak{A}$ is its band algebra (368E); now if $U^\times$ separates the points of U, there is a measure rendering $\mathfrak{A}$ a localizable measure algebra (369A). Moreover, we get a simultaneous representation of $U^\times$ as a subspace of L^0 (369C-369D), the duality between U and $U^\times$ corresponding exactly to the familiar duality between L^p and L^q. In particular, every L-space can be represented as an L^1-space (369E).

Still drawing inspiration from the classical L^p spaces, we have a general theory of 'associated Fatou norms' (369F-369M, 369R). I include notes on the spaces $M^{1,\infty}$, $M^{\infty,1}$ and $M^{1,0}$ (369N-369Q), which will be particularly useful in the next chapter.

369A Theorem Let U be a Riesz space such that $U^\times$ separates the points of U. Then U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{A})$ for some localizable measure algebra $(\mathfrak{A},\bar\mu)$.

369B Corollary Let U be a Banach lattice with order-continuous norm. Then U can be embedded as an order-dense solid linear subspace of $L^0(\mathfrak{A})$ for some localizable measure algebra $(\mathfrak{A},\bar\mu)$.

369C Theorem Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $U \subseteq L^0 = L^0(\mathfrak{A})$ an order-dense Riesz subspace. Set

$$V = \{v : v \in L^0,\ v \times u \in L^1 \text{ for every } u \in U\},$$

writing L^1 for $L^1(\mathfrak{A},\bar\mu) \subseteq L^0$. Then V is a solid linear subspace of L^0, and we have an order-continuous injective Riesz homomorphism $T : V \to U^\times$ defined by setting

$$(Tv)(u) = \int u \times v \text{ for all } u \in U,\ v \in V.$$

The image of V is order-dense in $U^\times$. If $(\mathfrak{A},\bar\mu)$ is localizable, then T is surjective, so is a Riesz space isomorphism between V and $U^\times$.

369D Corollary Let U be any Riesz space such that $U^\times$ separates the points of U. Then there is a localizable measure algebra $(\mathfrak{A},\bar\mu)$ such that the pair $(U,U^\times)$ can be represented by a pair (V,W) of order-dense Riesz subspaces of $L^0 = L^0(\mathfrak{A})$ such that $W = \{w : w \in L^0,\ v \times w \in L^1 \text{ for every } v \in V\}$, writing L^1 for $L^1(\mathfrak{A},\bar\mu)$. In this case, $U^{\times\times}$ becomes represented by $\tilde V = \{v : v \in L^0,\ v \times w \in L^1 \text{ for every } w \in W\} \supseteq V$.

369E Kakutani's theorem If U is any L-space, there is a localizable measure algebra $(\mathfrak{A},\bar\mu)$ such that U is isomorphic, as Banach lattice, to $L^1(\mathfrak{A},\bar\mu)$.

369F Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. An **extended Fatou norm** on $L^0 = L^0(\mathfrak{A})$ is a function $\tau : L^0 \to [0, \infty]$ such that

(i) $\tau(u + v) \le \tau(u) + \tau(v)$ for all u, $v \in L^0$;

(ii) $\tau(\alpha u) = |\alpha|\tau(u)$ whenever $u \in L^0$ and $\alpha \in \mathbb{R}$;

(iii) $\tau(u) \le \tau(v)$ whenever $|u| \le |v|$ in L^0;

(iv) $\sup_{u \in A} \tau(u) = \tau(v)$ whenever $A \subseteq (L^0)^+$ is a non-empty upwards-directed set with supremum v in L^0;

(v) $\tau(u) > 0$ for every non-zero $u \in L^0$;

(vi) whenever $u > 0$ in L^0 there is a $v \in L^0$ such that $0 < v \le u$ and $\tau(v) < \infty$.

369G Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then $L^\tau = \{u : u \in L^0, \tau(u) < \infty\}$ is an order-dense solid linear subspace of L^0, and τ, restricted to L^τ, is a Fatou norm under which L^τ is a Banach lattice. If $\langle u_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing norm-bounded sequence in $(L^\tau)^+$, then it has a supremum in L^τ; if $\mathfrak{A}$ is Dedekind complete, then L^τ has the Levi property.

369H Associate norms: Definition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Define $\tau' : L^0 \to [0, \infty]$ by setting

$$\tau'(u) = \sup\{\|u \times v\|_1 : v \in L^0, \tau(v) \le 1\}$$

for every $u \in L^0$; then τ' is the **associate** of τ.

369I Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then

(i) its associate τ' is also an extended Fatou norm on L^0;

(ii) τ is the associate of τ';

(iii) $\|u \times v\|_1 \le \tau(u)\tau'(v)$ for all u, $v \in L^0$.

369J Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$, with associate θ. Then

$$L^\theta = \{v : v \in L^0, u \times v \in L^1(\mathfrak{A}, \bar{\mu}) \text{ for every } u \in L^\tau\}.$$

369K Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra, and τ an extended Fatou norm on $L^0(\mathfrak{A})$, with associate θ. Then L^θ may be identified, as normed Riesz space, with $(L^\tau)^\times \subseteq (L^\tau)^*$, and L^τ is a perfect Riesz space.

369L L^p Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra and $p \in [1, \infty]$. Then $\| \|_p$ is an extended Fatou norm.

As usual, set $q = p/(p-1)$ if $1 < p < \infty$, ∞ if $p = 1$, and 1 if $p = \infty$. Then $\| \|_q$ is the associate extended Fatou norm of $\| \|_p$.

369M Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then

(a) the embedding $L^\tau \subsetneqq L^0$ is continuous for the norm topology of L^τ and the topology of convergence in measure on L^0;

(b) $\tau : L^0 \to [0, \infty]$ is lower semi-continuous;

(c) if $\langle u_n \rangle_{n \in \mathbb{N}}$ is a sequence in L^0 which is order*-convergent to $u \in L^0$, then $\tau(u)$ is at most $\liminf_{n \to \infty} \tau(u_n)$.

369N Definition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Set

$$M^{\infty,1}_{\bar{\mu}} = M^{\infty,1}(\mathfrak{A}, \bar{\mu}) = L^1(\mathfrak{A}, \bar{\mu}) \cap L^\infty(\mathfrak{A}),$$

$$M^{1,\infty}_{\bar{\mu}} = M^{1,\infty}(\mathfrak{A}, \bar{\mu}) = L^1(\mathfrak{A}, \bar{\mu}) + L^\infty(\mathfrak{A}),$$

and

$$\|u\|_{\infty,1} = \max(\|u\|_1, \|u\|_\infty)$$

for $u \in L^0(\mathfrak{A})$.

369O Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra.

(a) $\| \|_{\infty,1}$ is an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$, and the corresponding Banach lattice is $M^{\infty,1}(\mathfrak{A}, \bar{\mu})$.

(b) The associate of $\| \|_{\infty,1}$ is $\| \|_{1,\infty}$, which may be defined by any of the formulae

$$\begin{aligned}\|u\|_{1,\infty} &= \sup\{\|u \times v\|_1 : v \in L^0,\ \|v\|_{\infty,1} \le 1\}\\ &= \min\{\|v\|_1 + \|w\|_\infty : v \in L^1,\ w \in L^\infty,\ v + w = u\}\\ &= \min\{\alpha + \int (|u| - \alpha\chi 1)^+ : \alpha \ge 0\}\\ &= \int_0^\infty \min(1, \bar{\mu}[\![|u| > \alpha]\!])d\alpha\end{aligned}$$

for every $u \in L^0$, writing $L^1 = L^1(\mathfrak{A}, \bar{\mu})$, $L^\infty = L^\infty(\mathfrak{A})$.

(c)

$$\{u : u \in L^0,\ \|u\|_{1,\infty} < \infty\} = M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar{\mu}),$$

$$\{u : u \in L^0,\ \|u\|_{\infty,1} < \infty\} = M^{\infty,1} = M^{\infty,1}(\mathfrak{A}, \bar{\mu}).$$

(d) Writing $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$, $S(\mathfrak{A}^f)$ is norm-dense in $M^{\infty,1}$ and $S(\mathfrak{A})$ is norm-dense in $M^{1,\infty}$.

(e) For any $p \in [1, \infty]$,

$$\|u\|_{1,\infty} \le \|u\|_p \le \|u\|_{\infty,1}$$

for every $u \in L^0$.

369P Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) $M^{1,0} = M^{1,0}(\mathfrak{A}, \bar{\mu})$ is a norm-closed solid linear subspace of $M^{1,\infty}(\mathfrak{A}, \bar{\mu})$.

(b) The norm $\| \|_{1,\infty}$ is order-continuous on $M^{1,0}$.

(c) $S(\mathfrak{A}^f)$ and $L^1(\mathfrak{A}, \bar{\mu})$ are norm-dense and order-dense in $M^{1,0}$.

369Q Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Set $M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar{\mu})$, etc.

(a) $(M^{1,\infty})^\times$ and $(M^{1,0})^\times$ can both be identified with $M^{\infty,1}$.

(b) $(M^{\infty,1})^\times$ can be identified with $M^{1,\infty}$; $M^{1,\infty}$ and $M^{\infty,1}$ are perfect Riesz spaces.

369R Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra, and τ_1, τ_2 two extended Fatou norms on $L^0 = L^0(\mathfrak{A})$ with associates τ_1', τ_2'. Then we have an extended Fatou norm τ defined by the formula

$$\tau(u) = \min\{\tau_1(v) + \tau_2(w) : v,\ w \in L^0,\ v + w = u\}$$

for every $u \in L^0$, and its associate τ' is given by the formula

$$\tau'(u) = \max(\tau_1'(u), \tau_2'(u))$$

for every $u \in L^0$. Moreover, the corresponding function spaces are

$$L^\tau = L^{\tau_1} + L^{\tau_2}, \quad L^{\tau'} = L^{\tau_1'} \cap L^{\tau_2'}.$$

Version of 20.7.11

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

364Be $L^0(\mathfrak{A})$ This re-phrasing of the definition of $L^0(\mathfrak{A})$, referred to in the 2008 edition of Volume 5, is now 364Af.

364D L^0 **as** f**-algebra** This paragraph, referred to in the 2008 edition of Volume 5, is now 364C.

364E Algebraic operations on L^0 This paragraph, referred to in the 2008 edition of Volume 5, is now 364D.

364G The identification of $L^0(\mathfrak{A})$ with the set of sequentially order-continuous Boolean homomorphisms from $\mathcal{B}(\mathbb{R})$ to $\mathfrak{A}$, referred to in the 2008 edition of Volume 5, is now 364F.

364I Action of Borel functions on L^0 This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 364H.

364J $L^0(\Sigma/\mathcal{I})$ The identification of $L^0(\Sigma/\mathcal{I})$ as a space of equivalence classes of functions, referred to in the 2003 and 2206 editions of Volume 4 and the 2008 edition of Volume 5, is now 364I.

364K Embedding S **and** L^∞ **in** L^0 This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 364J.

364M-364N Suprema and infima in $L^0(\mathfrak{A})$ These paragraphs, referred to in the 2003 and 2006 editions of Volume 4 and the 2008 edition of Volume 5, have now been amalgamated as 364L.

364O Dedekind completeness of L^0 This paragraph, referred to in the 2008 edition of Volume 5, is now 364M.

364P Multiplicative inverses in L^0 This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 364J.

364R Action of Boolean homomorphisms on L^0 This paragraph, referred to the 2003 and 2006 editions of Volume 4 and in the 2008 edition of Volume 5, is now 364P.

364Xw Extension of $\int$ This exercise, referred to in the 2008 edition of Volume 5, is now 364Xj.

364Yn $L^0_{\mathbb{C}}(\mathfrak{A})$ This exercise on complex L^0 spaces, referred to in the 2003 and 2006 editions of Volume 4, has been moved to 366M.

365K Additive functions on $\mathfrak{A}^f$ **and linear operators on** L^1 This theorem, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 365J.

365M L^1 **and** L^∞ This theorem, referred to in the 2008 printing of Volume 5, is now 365L.

365O Ring homomorphisms on $\mathfrak{A}^f$ **and Riesz homomorphisms on** L^1 This theorem, referred to in the 2013 printing of Volume 4 and the 2008 printing of Volume 5, is now 365N.

365P Order-continuous ring homomorphisms on $\mathfrak{A}^f$ and conditional expectations This theorem, referred to in the 2008 printing of Volume 5, is now 365O.

365R Conditional expectations These notes, referred to in the 2006 and 2013 printings of Volume 4 and the 2008 printing of Volume 5, is now 365Q.

365T Change of measure This proposition, referred to in the 2008 printing of Volume 5, is now 365S.