Version of 13.8.98/27.7.17

Chapter 36

Function Spaces

Chapter 24 of Volume 2 was devoted to the elementary theory of the 'function spaces' L^0, L^1, L^2 and L^∞ associated with a given measure space. In this chapter I return to these spaces to show how they can be related to the more abstract themes of the present volume. In particular, I develop constructions to demonstrate, as clearly as I can, the way in which the function spaces associated with a measure space in fact depend only on its measure algebra; and how many of their features can (in my view) best be understood in terms of constructions involving measure algebras.

The chapter is very long, not because there are many essentially new ideas, but because the intuitions I seek to develop depend, for their logical foundations, on technically complex arguments. This is perhaps best exemplified by §364. If two measure spaces (X,Σ,μ) and (Y,T,ν) have isomorphic measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ then the spaces $L^0(\mu)$, $L^0(\nu)$ are isomorphic as topological f-algebras; and more: for any isomorphism between $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ there is a unique corresponding isomorphism between the L^0 spaces. The intuition involved is in a way very simple. If f, g are measurable real-valued functions on X and Y respectively, then $f^\bullet\in L^0(\mu)$ will correspond to $g^\bullet\in L^0(\nu)$ if and only if $[\![f^\bullet>\alpha]\!]=\{x:f(x)>\alpha\}^\bullet\in\mathfrak{A}$ corresponds to $[\![g^\bullet>\alpha]\!]=\{y:g(y)>\alpha\}^\bullet\in\mathfrak{B}$ for every α. But the check that this formula is consistent, and defines an isomorphism of the required kind, involves a good deal of detailed work. It turns out, in fact, that the *measures* μ and ν do not enter this part of the argument at all, except through their ideals of negligible sets (used in the construction of $\mathfrak{A}$ and $\mathfrak{B}$). This is already evident, if you look for it, in the theory of $L^0(\mu)$; in §241, as written out, you will find that the measure of an individual set is not once mentioned, except in the exercises. Consequently there is an invitation to develop the theory with algebras $\mathfrak{A}$ which are not necessarily measure algebras. Here is another reason for the length of the chapter; substantial parts of the work are being done in greater generality than the corresponding sections of Chapter 24, necessitating a degree of repetition. Of course this is not 'measure theory' in the strict sense; but for thirty years now measure theory has been coloured by the existence of these generalizations, and I think it is useful to understand which parts of the theory apply only to measure algebras, and which can be extended to other σ-complete Boolean algebras, like the algebraic theory of L^0, or even to all Boolean algebras, like the theory of L^∞.

Here, then, are two of the objectives of this chapter: first, to express the ideas of Chapter 24 in ways making explicit their independence of particular measure spaces, by setting up constructions based exclusively on the measure algebras involved; second, to set out some natural generalizations to other algebras. But to justify the effort needed I ought to point to some mathematically significant idea which demands these constructions for its expression, and here I mention the categorical nature of the constructions. Between Boolean algebras we have a variety of natural and important classes of 'morphism'; for instance, the Boolean homomorphisms and the order-continuous Boolean homomorphisms; while between measure algebras we have in addition the measure-preserving Boolean homomorphisms. Now it turns out that if we construct the L^p spaces in the natural ways then morphisms between the underlying algebras give rise to morphisms between their L^p spaces. For instance, any Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces a multiplicative norm-contractive Riesz homomorphism from $L^\infty(\mathfrak{A})$ to $L^\infty(\mathfrak{B})$; if $\mathfrak{A}$ and $\mathfrak{B}$ are Dedekind σ-complete, then any sequentially order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces a sequentially order-continuous multiplicative Riesz homomorphism from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$; and if $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are measure algebras, then any measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ produces norm-preserving Riesz homomorphisms from $L^p(\mathfrak{A},\bar\mu)$ to $L^p(\mathfrak{B},\bar\nu)$ for every $p\in[1,\infty]$. All of these are 'functors', that is, a composition of homomorphisms between algebras gives rise to a composition of the corresponding operators between their function spaces, and are 'covariant', that is, a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ leads to an operator from $L^p(\mathfrak{A})$ to $L^p(\mathfrak{B})$. But the same constructions lead us to a functor which is 'contravariant': starting from an order-continuous Boolean homomorphism from a semi-finite measure algebra $(\mathfrak{A},\bar{\mu})$ to a measure algebra $(\mathfrak{B},\bar{\nu})$, we have an operator from $L^1(\mathfrak{B},\bar{\nu})$ to $L^1(\mathfrak{A},\bar{\mu})$. This last is in fact a kind of conditional expectation operator. In my view it is not possible to make sense of the theory of measure-preserving transformations without at least an intuitive grasp of these ideas.

Another theme is the characterization of each construction in terms of universal mapping theorems: for instance, each L^p space, for $1 \le p \le \infty$, can be characterized as Banach lattice in terms of factorizations of functions of an appropriate class from the underlying algebra to Banach lattices.

Now let me try to sketch a route-map for the journey ahead. I begin with two sections on the space $S(\mathfrak{A})$; this construction applies to any Boolean algebra (indeed, any Boolean ring), and corresponds to the space of 'simple functions' on a measure space. Just because it is especially close to the algebra (or ring) $\mathfrak{A}$, there is a particularly large number of universal mapping theorems corresponding to different aspects of its structure (§361). In §362 I seek to relate ideas on additive functionals on Boolean algebras from Chapter 23 and §§326-327 to the theory of Riesz space duals in §356. I then turn to a systematic discussion of the function spaces of Chapter 24: L^∞ (§363), L^0 (§364), L^1 (§365) and other L^p (§366), followed by an account of convergence in measure (§367). While all these sections are dominated by the objectives sketched in the paragraphs above, I do include a few major theorems not covered by the ideas of Volume 2, such as the Kelley-Nachbin characterization of the Banach spaces $L^\infty(\mathfrak{A})$ for Dedekind complete $\mathfrak{A}$ (363R). In the last two sections of the chapter I turn to the use of L^0 spaces in the representation of Archimedean Riesz spaces (§368) and of Banach lattices which are separated by their order-continuous duals (§369).

Version of 6.2.08

361 *S*

This is the fundamental Riesz space associated with a Boolean ring $\mathfrak{A}$. When $\mathfrak{A}$ is a ring of sets, $S(\mathfrak{A})$ can be regarded as the linear space of 'simple functions' generated by the indicator functions of members of $\mathfrak{A}$ (361L). Its most important property is the universal mapping theorem 361F, which establishes a one-to-one correspondence between (finitely) additive functions on $\mathfrak{A}$ (361B-361C) and linear operators on $S(\mathfrak{A})$. Simple universal mapping theorems of this type can be interesting, but do not by themselves lead to new insights; what makes this one important is the fact that $S(\mathfrak{A})$ has a canonical Riesz space structure, norm and multiplication (361E). From this we can deduce universal mapping theorems for many other classes of function (361G, 361H, 361I, 361Xb). (Particularly important are countably additive and completely additive real-valued functionals, which will be dealt with in the next section.) While the exact construction of $S(\mathfrak{A})$ (and the associated map from $\mathfrak{A}$ to $S(\mathfrak{A})$) can be varied (361D, 361L, 361M, 361Ya), its structure is uniquely defined, so homomorphisms between Boolean rings correspond to maps between their $S(\,)$-spaces (361J), and (when $\mathfrak{A}$ is a Boolean algebra) $\mathfrak{A}$ can be recovered from the Riesz space $S(\mathfrak{A})$ as the algebra of its projection bands (361K).

361A Boolean rings In this section I speak of Boolean *rings* rather than *algebras*; there are ideas in §365 below which are more naturally expressed in terms of the ring of elements of finite measure in a measure algebra than in terms of the whole algebra. I should perhaps therefore recall some of the ideas of §311, which is the last time when Boolean rings without identity were mentioned, and set out some simple facts.

(a) Any Boolean ring $\mathfrak{A}$ can be represented as the ring of compact open subsets of its Stone space Z, which is a zero-dimensional locally compact Hausdorff space (311I); Z is just the set of surjective ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$ (311E).

(b) If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a function, then the following are equiveridical: (i) π is a ring homomorphism; (ii) $\pi(a \setminus b) = \pi a \setminus \pi b$ for all a, $b \in \mathfrak{A}$; (iii) π is a lattice homomorphism and $\pi 0 = 0$. **P** See 312H. To prove (ii)⇒(iii), observe that if a, $b \in \mathfrak{A}$ then

$$\pi(a \cap b) = \pi a \setminus \pi(a \setminus b) = \pi a \setminus (\pi a \setminus \pi b) = \pi a \cap \pi b,$$

$$\pi a = \pi((a \cup b) \cap a) = \pi(a \cup b) \cap \pi a \subseteq \pi(a \cup b),$$

$$\pi(b \setminus a) = \pi((a \cup b) \setminus a) = \pi(a \cup b) \setminus \pi a,$$

$$\pi(a \cup b) = \pi a \cup \pi(b \setminus a) = \pi a \cup (\pi b \setminus \pi a) = \pi a \cup \pi b. \textbf{ Q}$$

(c) If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a ring homomorphism, then π is order-continuous iff $\inf \pi[A] = 0$ whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$ in $\mathfrak{A}$; while π is sequentially order-continuous iff $\inf_{n\in\mathbb{N}} \pi a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. (See 313L.)

(d) The following will be a particularly important type of Boolean ring for us. If $(\mathfrak{A}, \bar\mu)$ is a measure algebra, then the ideal $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar\mu a < \infty\}$ is a Boolean ring in its own right. Now suppose that $(\mathfrak{B}, \bar\nu)$ is another measure algebra and $\mathfrak{B}^f \subseteq \mathfrak{B}$ the corresponding ring of elements of finite measure. We can say that a ring homomorphism $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ is **measure-preserving** if $\bar\nu\pi a = \bar\mu a$ for every $a \in \mathfrak{A}^f$. In this case π is order-continuous. **P** If $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, then $\inf_{a\in A} \bar\mu a = 0$, by 321F; but this means that $\inf_{a\in A} \bar\nu\pi a = 0$, and $\inf \pi[A] = 0$ in $\mathfrak{B}^f$. **Q**

361B Definition Let $\mathfrak{A}$ be a Boolean ring and U a linear space. A function $\nu : \mathfrak{A} \to U$ is **finitely additive**, or just **additive**, if $\nu(a \cup b) = \nu a + \nu b$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$.

361C Elementary facts We have the following immediate consequences of this definition, corresponding to 326B and 313L. Let $\mathfrak{A}$ be a Boolean ring, U a linear space and $\nu : \mathfrak{A} \to U$ an additive function.

(a) $\nu 0 = 0$ (because $\nu 0 = \nu 0 + \nu 0$).

(b) If $a_0, \dots , a_m$ are disjoint in $\mathfrak{A}$, then $\nu(\sup_{j\le m} a_j) = \sum_{j=0}^m \nu a_j$. (Induce on m.)

(c) If $\mathfrak{B}$ is another Boolean ring and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a ring homomorphism, then $\nu\pi : \mathfrak{B} \to U$ is additive. In particular, if $\mathfrak{B}$ is a subring of $\mathfrak{A}$, then $\nu{\restriction}\mathfrak{B} : \mathfrak{B} \to U$ is additive.

(d) If V is another linear space and $T : U \to V$ is a linear operator, then $T\nu : \mathfrak{A} \to V$ is additive.

(e) If U is a partially ordered linear space, then ν is order-preserving iff it is non-negative, that is, $\nu a \ge 0$ for every $a \in \mathfrak{A}$. **P** (α) If ν is order-preserving, then of course $0 = \nu 0 \le \nu a$ for every $a \in \mathfrak{A}$. (β) If ν is non-negative, and $a \subseteq b$ in $\mathfrak{A}$, then

$$\nu a \le \nu a + \nu(b \setminus a) = \nu b. \textbf{ Q}$$

(f) If U is a partially ordered linear space and ν is non-negative, then (i) ν is order-continuous iff $\inf \nu[A] = 0$ whenever $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0 (ii) ν is sequentially order-continuous iff $\inf_{n\in\mathbb{N}} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0.

P(i) If ν is order-continuous, then of course $\inf \nu[A] = \nu 0 = 0$ whenever $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0. If ν satisfies the condition, and $A \subseteq \mathfrak{A}$ is a non-empty upwards-directed set with supremum c, then $\{c \setminus a : a \in \mathfrak{A}\}$ is downwards-directed with infimum 0 (313Aa), so that

$$\sup_{a\in A} \nu a = \sup_{a\in A} \nu c - \nu(c \setminus a) = \nu c - \inf_{a\in A} \nu(c \setminus a)$$

(by 351Db)

$$= \nu c.$$

Similarly, if $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum c, then

$$\inf_{a\in A} \nu a = \inf_{a\in A} \nu c + \nu(a \setminus c) = \nu c + \inf_{a\in A} \nu(a \setminus c) = \nu c.$$

Putting these together, ν is order-continuous.

(ii) If ν is sequentially order-continuous, then of course $\inf_{n\in\mathbb{N}} \nu a_n = \nu 0 = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. If ν satisfies the condition, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum c, then $\langle c\setminus a_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so that

$$\sup_{n\in\mathbb{N}} \nu a_n = \sup_{n\in\mathbb{N}} \nu c - \nu(c\setminus a_n) = \nu c - \inf_{n\in\mathbb{N}} \nu(c\setminus a_n) = \nu c.$$

Similarly, if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum c, then $\langle a_n\setminus c\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so that

$$\inf_{n\in\mathbb{N}} \nu a_n = \inf_{n\in\mathbb{N}} \nu c + \nu(c\setminus a_n) = \nu c + \inf_{n\in\mathbb{N}} \nu(c\setminus a_n) = \nu c.$$

Thus ν is sequentially order-continuous. **Q**

361D Construction Let $\mathfrak{A}$ be a Boolean ring, and Z its Stone space. For $a\in\mathfrak{A}$ write χa for the indicator function of the open-and-compact subset $\widehat{a}$ of Z corresponding to a. Note that $\chi a = 0$ iff $a = 0$. Let $S(\mathfrak{A})$ be the linear subspace of $\mathbb{R}^Z$ generated by $\{\chi a : a \in \mathfrak{A}\}$. Because χa is a bounded function for every a, $S(\mathfrak{A})$ is a subspace of the M-space $\ell^\infty(Z)$ of all bounded real-valued functions on Z (354Ha), and $\| \|_\infty$ is a norm on $S(\mathfrak{A})$. Because $\chi a\times\chi b = \chi(a\cap b)$ for all a, $b\in\mathfrak{A}$ (writing $\times$ for pointwise multiplication of functions, as in 281B), $S(\mathfrak{A})$ is closed under $\times$.

361E I give a portmanteau proposition running through the elementary, mostly algebraic, properties of $S(\mathfrak{A})$.

Proposition Let $\mathfrak{A}$ be a Boolean ring, with Stone space Z. Write S for $S(\mathfrak{A})$.

(a) If $a_0,\dots,a_n\in\mathfrak{A}$, there are disjoint $b_0,\dots,b_m$ such that each a_i is expressible as the supremum of some of the b_j.

(b) If $u\in S$, it is expressible in the form $\sum_{j=0}^m \beta_j\chi b_j$ where $b_0,\dots,b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j\in\mathbb{R}$ for each j. If all the b_j are non-zero then $\|u\|_\infty = \sup_{j\le m}|\beta_j|$.

(c) If $u\in S$ is non-negative, it is expressible in the form $\sum_{j=0}^m \beta_j\chi b_j$ where $b_0,\dots,b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j\ge 0$ for each j, and simultaneously in the form $\sum_{j=0}^m \gamma_j\chi c_j$ where $c_0\supseteq c_1\supseteq\ldots\supseteq c_m$ and $\gamma_j\ge 0$ for every j.

(d) If $u = \sum_{j=0}^m \beta_j\chi b_j$ where $b_0,\dots,b_m$ are disjoint members of $\mathfrak{A}$ and $\beta_j\in\mathbb{R}$ for each j, then $|u| = \sum_{j=0}^m |\beta_j|\chi b_j\in S$.

(e) S is a Riesz subspace of $\mathbb{R}^Z$; in its own right, it is an Archimedean Riesz space. If $\mathfrak{A}$ is a Boolean algebra, then S has an order unit $\chi 1$ and $\|u\|_\infty = \min\{\alpha : \alpha\ge 0,\ |u|\le\alpha\chi 1\}$ for every $u\in S$.

(f) The map $\chi:\mathfrak{A}\to S$ is injective, additive, non-negative, a lattice homomorphism and order-continuous.

(g) Suppose that $u\ge 0$ in S and $\delta\ge 0$ in $\mathbb{R}$. Then

$$[\![u>\delta]\!] = \max\{a : a\in\mathfrak{A},\ (\delta+\eta)\chi a\le u \text{ for some } \eta>0\}$$

is defined in $\mathfrak{A}$, and

$$\delta\chi[\![u>\delta]\!] \le u \le \delta\chi[\![u>0]\!]\vee\|u\|_\infty[\![u>\delta]\!].$$

In particular, $u\le\|u\|_\infty\chi[\![u>0]\!]$ and there is an $\eta>0$ such that $\eta\chi[\![u>0]\!]\le u$. If u, $v\ge 0$ in S then $u\wedge v = 0$ iff $[\![u>0]\!]\cap[\![v>0]\!] = 0$.

(h) Under $\times$, S is an f-algebra (352W) and a commutative normed algebra (2A4J).

(i) For any $u\in S$, $u\ge 0$ iff $u = v\times v$ for some $v\in S$.

proof Write $\widehat{a}$ for the open-and-compact subset of Z corresponding to $a\in\mathfrak{A}$.

(a) Induce on n. If $n=0$ take $m=0$, $b_0=a_0$. For the inductive step to $n\ge 1$, take disjoint $b_0,\dots,b_m$ such that a_i is the supremum of some of the b_j for each $i<n$; now replace $b_0,\dots,b_m$ with $b_0\cap a_n,\dots,b_m\cap a_n, b_0\setminus a_n,\dots,b_m\setminus a_n, a_n\setminus\sup_{j\le m} b_j$ to obtain a suitable string for $a_0,\dots,a_n$.

(b) If $u=0$ set $m=0$, $b_0=0$, $\beta_0=0$. Otherwise, express u as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0,\dots,a_n\in\mathfrak{A}$ and $\alpha_0,\dots,\alpha_n$ are real numbers. Let $b_0,\dots,b_m$ be disjoint and such that every a_i is expressible as the supremum of some of the b_j. Set $\gamma_{ij}=1$ if $b_j\subseteq a_i$, 0 otherwise, so that, because the b_j are disjoint, $\chi a_i = \sum_{j=0}^m \gamma_{ij}\chi b_j$ for each i. Then

$$u = \sum_{i=0}^n \alpha_i\chi a_i = \sum_{i=0}^n\sum_{j=0}^m \alpha_i\gamma_{ij}\chi b_j = \sum_{j=0}^m \beta_j\chi b_j,$$

setting $\beta_j=\sum_{i=0}^n\alpha_i\gamma_{ij}$ for each $j\le m$.

The expression for $\|u\|_\infty$ is now obvious.

(c)(i) If $u\ge 0$ in (b), we must have $\beta_j=u(z)\ge 0$ whenever $z\in\widehat{b}_j$, so that $\beta_j\ge 0$ whenever $b_j\ne 0$; consequently $u=\sum_{j=0}^m|\beta_j|\chi b_j$ is in the required form.

(ii) If we suppose that every β_j is non-negative, and rearrange the terms of the sum so that $\beta_0\le\ldots\le\beta_m$, then we may set $\gamma_0=\beta_0$, $\gamma_j=\beta_j-\beta_{j-1}$ for $1\le j\le m$, $c_j=\sup_{i\ge j}b_i$ to get

$$\textstyle\sum_{j=0}^m\gamma_j\chi c_j=\sum_{j=0}^m\sum_{i=j}^m\gamma_j\chi b_i=\sum_{i=0}^m\sum_{j=0}^i\gamma_j\chi b_i=\sum_{i=0}^m\beta_i\chi b_i=u.$$

(d) is trivial, because $\widehat{b}_0,\ldots,\widehat{b}_n$ are disjoint.

(e) By (d), $|u|\in S$ for every $u\in S$, so S is a Riesz subspace of $\mathbb{R}^Z$, and in itself is an Archimedean Riesz space. If $\mathfrak{A}$ is a Boolean algebra, then $\chi 1$, the constant function with value 1, belongs to S, and is an order unit of S; while

$$\|u\|_\infty=\min\{\alpha:\alpha\ge 0,\,|u(z)|\le\alpha\;\forall\,z\in Z\}=\min\{\alpha:\alpha\ge 0,\,|u|\le\alpha\chi 1\}$$

for every $u\in S$.

(f) χ is injective because $\widehat{a}\ne\widehat{b}$ whenever $a\ne b$. χ is additive because $\widehat{a}\cap\widehat{b}=\emptyset$ whenever $a\cap b=0$. Of course χ is non-negative. It is a lattice homomorphism because $a\mapsto\widehat{a}:\mathfrak{A}\to\mathcal{P}Z$ and $E\mapsto\chi E:\mathcal{P}Z\to\mathbb{R}^Z$ are. To see that ν is order-continuous, take a non-empty downwards-directed $A\subseteq\mathfrak{A}$ with infimum 0. **?** Suppose, if possible, that $\{\chi a:a\in A\}$ does not have infimum 0 in S. Then there is a $u>0$ in S such that $u\le\chi a$ for every $a\in\mathfrak{A}$. Now u can be expressed as $\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\ldots,b_m$ are disjoint. There must be some $z_0\in Z$ such that $u(z_0)>0$; take j such that $z_0\in\widehat{b}_j$, so that $b_j\ne 0$ and $\beta_j=u(z_0)>0$. But now, for any $z\in\widehat{b}_j$, $a\in A$,

$$(\chi a)(z)\ge u(z)=\beta_j>0$$

and $z\in\widehat{a}$. As z is arbitrary, $\widehat{b}_j\subseteq\widehat{a}$ and $b_j\subseteq a$; as a is arbitrary, b_j is a non-zero lower bound for A in $\mathfrak{A}$. **X** So $\inf\chi[A]=0$ in S. As A is arbitrary, χ is order-continuous, by the criterion of 361C(f-i).

(g) Express u as $\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\ldots,b_m$ are disjoint and every $\beta_j\ge 0$. Then given $\delta\ge 0$, $\eta>0$ and $a\in\mathfrak{A}$ we have $(\delta+\eta)\chi a\le u$ iff $a\subseteq\sup\{b_j:j\le m,\,\beta_j\ge\delta+\eta\}$. So $[\![u>\delta]\!]=\sup\{b_j:j\le m,\,\beta_j>\delta\}$. Writing $c=[\![u>\delta]\!]$, $d=[\![u>0]\!]=\sup\{b_j:\beta_j>0\}$, we have

$$\begin{aligned}u(z)&\le\|u\|_\infty\text{ if }z\in\widehat{c},\\&\le\delta\text{ if }z\in\widehat{d}\setminus\widehat{c},\\&=0\text{ if }z\in Z\setminus\widehat{d}.\end{aligned}$$

So

$$\delta\chi c\le u\le\|u\|_\infty\chi c\vee\delta\chi d,$$

as claimed. Taking $\delta=0$ we get $u\le\|u\|_\infty\chi d$. Set

$$\eta=\min(\{1\}\cup\{\beta_j:j\le m,\,\beta_j>0\});$$

then $\eta>0$ and $\eta\chi d\le u$.

If u, $v\in S^+$ take η, $\eta'>0$ such that

$$\eta\chi[\![u>0]\!]\le u,\quad\eta'\chi[\![v>0]\!]\le v.$$

Then

$$\min(\eta,\eta')\chi([\![u>0]\!]\cap[\![v>0]\!])\le u\wedge v\le\max(\|u\|_\infty,\|v\|_\infty)\chi([\![u>0]\!]\cap[\![v>0]\!]).$$

So

$$u\wedge v=0\Longrightarrow[\![u>0]\!]\cap[\![v>0]\!]=0\Longrightarrow u\wedge v=0.$$

(h) S is a commutative f-algebra and normed algebra just because it is a Riesz subspace of the f-algebra and commutative normed algebra $\ell^\infty(Z)$ and is closed under multiplication.

(i) If $u=\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\dots,b_m$ are disjoint and $\beta_j\ge 0$ for every j, then $u=v\times v$ where $v=\sum_{j=0}^m\sqrt{\beta_j}\chi b_j$.

361F I now turn to the universal mapping theorems which really define the construction.

Theorem Let $\mathfrak{A}$ be a Boolean ring, and U any linear space. Then there is a one-to-one correspondence between additive functions $\nu:\mathfrak{A}\to U$ and linear operators $T:S(\mathfrak{A})\to U$, given by the formula $\nu=T\chi$.

proof (a) The core of the proof is the following observation. Let $\nu:\mathfrak{A}\to U$ be additive. If $a_0,\dots,a_n\in\mathfrak{A}$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$ are such that $\sum_{i=0}^n\alpha_i\chi a_i=0$ in $S=S(\mathfrak{A})$, then $\sum_{i=0}^n\alpha_i\nu a_i=0$ in U. **P** By 361Ea, we can find disjoint $b_0,\dots,b_m$ such that each a_i is the supremum of some of the b_j; set $\gamma_{ij}=1$ if $b_j\subseteq a_i$, 0 otherwise, so that $\chi a_i=\sum_{j=0}^m\gamma_{ij}\chi b_j$ and $\nu a_i=\sum_{j=0}^m\gamma_{ij}\nu b_j$ for each i. Set $\beta_j=\sum_{i=0}^n\alpha_i\gamma_{ij}$ for each j, so that

$$0=\sum_{i=0}^n\alpha_i\chi a_i=\sum_{j=0}^m\beta_j\chi b_j.$$

Now $\beta_j\nu b_j=0$ in U for each j, because either $b_j=0$ and $\nu b_j=0$, or there is some $z\in\widehat{b}_j$ so that β_j must be 0. Accordingly

$$0=\sum_{j=0}^m\beta_j\nu b_j=\sum_{j=0}^m\sum_{i=0}^n\alpha_i\gamma_{ij}\nu b_j=\sum_{i=0}^n\alpha_i\nu a_i.\ \mathbf{Q}$$

(b) It follows that if $u\in S$ is expressible simultaneously as $\sum_{i=0}^n\alpha_i\chi a_i=\sum_{j=0}^m\beta_j\chi b_j$, then

$$\sum_{i=0}^n\alpha_i\chi a_i+\sum_{j=0}^m(-\beta_j)\chi b_j=0\text{ in }S,$$

so that

$$\sum_{i=0}^n\alpha_i\nu a_i+\sum_{j=0}^m(-\beta_j)\nu b_j=0\text{ in }U,$$

and

$$\sum_{i=0}^n\alpha_i\nu a_i=\sum_{j=0}^m\beta_j\nu b_j.$$

We can therefore define $T:S\to U$ by setting

$$T(\sum_{i=0}^n\alpha_i\chi a_i)=\sum_{i=0}^n\alpha_i\nu a_i$$

whenever $a_0,\dots,a_n\in\mathfrak{A}$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$.

(c) It is now elementary to check that T is linear, and that $T\chi a=\nu a$ for every $a\in\mathfrak{A}$. Of course this last condition uniquely defines T, because $\{\chi a:a\in\mathfrak{A}\}$ spans the linear space S.

361G Theorem Let $\mathfrak{A}$ be a Boolean ring, and U a partially ordered linear space. Let $\nu:\mathfrak{A}\to U$ be an additive function, and $T:S(\mathfrak{A})\to U$ the corresponding linear operator.

(a) ν is non-negative iff T is positive.

(b) In this case,

(i) if T is order-continuous or sequentially order-continuous, so is ν;

(ii) if U is Archimedean and ν is order-continuous or sequentially order-continuous, so is T.

(c) If U is a Riesz space, then the following are equiveridical:

(i) T is a Riesz homomorphism;

(ii) $\nu a\wedge\nu b=0$ in U whenever $a\cap b=0$ in $\mathfrak{A}$;

(iii) ν is a lattice homomorphism.

proof Write S for $S(\mathfrak{A})$.

(a) If T is positive, then surely $\nu a=T\chi a\ge 0$ for every $a\in\mathfrak{A}$, so $\nu=T\chi$ is non-negative. If ν is non-negative, and $u\ge 0$ in S, then u is expressible as $\sum_{j=0}^m\beta_j\chi b_j$ where $b_0,\dots,b_m\in\mathfrak{A}$ and $\beta_j\ge 0$ for every j (361Ec), so that

$$Tu=\sum_{j=0}^m\beta_j\nu b_j\ge 0.$$

Thus T is positive.

(b)(i) If T is order-continuous (resp. sequentially order-continuous) then $\nu = T\chi$ is the composition of two order-continuous (resp. sequentially order-continuous) functions (361Ef), so must be order-continuous (resp. sequentially order-continuous).

(ii) Assume now that U is Archimedean.

(α) Suppose that ν is order-continuous and that $A \subseteq S$ is non-empty, downwards-directed and has infimum 0. Fix $u_0 \in A$, set $\alpha = \|u\|_\infty$ and $a_0 = [\![u > 0]\!]$ (in the language of 361Eg). If $\alpha = 0$ then of course $\inf_{u\in A} Tu = Tu_0 = 0$. Otherwise, take any $w \in U$ such that $w \not\le 0$. Then there is some $\delta > 0$ such that $w \not\le \delta\nu a_0$, because U is Archimedean. Set $A' = \{u : u \in A,\, u \le u_0\}$; because A is downwards-directed, A' has the same lower bounds as A, and $\inf A' = 0$, while A' is still downwards-directed. For $u \in A'$ set $c_u = [\![u > \delta]\!]$, so that

$$\delta\chi c_u \le u \le \alpha\chi c_u + \delta\chi[\![u > 0]\!] \le \alpha\chi c_u + \delta\chi a_0$$

(361Eg). If u, $v \in A'$ and $u \le v$, then $c_u \subseteq c_v$, so $C = \{c_u : u \in A'\}$ is downwards-directed; but if c is any lower bound for C in $\mathfrak{A}$, $\delta\chi c$ is a lower bound for A' in S, so is zero, and $c = 0$ in $\mathfrak{A}$. Thus $\inf C = 0$ in $\mathfrak{A}$, and $\inf_{u\in A'} \nu c_u = 0$ in U. But this means, in particular, that $\frac{1}{\alpha}(w - \delta\nu a_0)$ is not a lower bound for $\nu[C]$, and there is some $u \in A'$ such that $\frac{1}{\alpha}(w - \delta\nu a_0) \not\le \nu c_u$, that is, $w - \delta\nu a_0 \not\le \alpha\nu c_u$, that is, $w \not\le \delta\nu a_0 + \alpha\nu c_u$. As $u \le \alpha\chi c_u + \delta\chi a_0$,

$$Tu \le T(\alpha\chi c_u + \delta\chi a_0) = \alpha\nu c_u + \delta\nu a_0,$$

and $w \not\le Tu$. Since w is arbitrary, this means that $0 = \inf T[A]$; as A is arbitrary, T is order-continuous.

(β) The argument for sequential order-continuity is essentially the same. Suppose that ν is sequentially order-continuous and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S with infimum 0. Again set $\alpha = \|u_0\|$, $a_0 = [\![u_0 > 0]\!]$; again we may suppose that $\alpha > 0$; again take any $w \in U$ such that $w \not\le 0$. As before, there is some $\delta > 0$ such that $w \not\le \delta\nu a_0$. For $n \in \mathbb{N}$ set $c_n = [\![u_n > \delta]\!]$, so that

$$\delta\chi c_n \le u_n \le \alpha\chi c_n + \delta\chi a_0.$$

The sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ is non-increasing because $\langle u_n\rangle_{n\in\mathbb{N}}$ is, and if $c \subseteq c_n$ for every n, then $\delta\chi c \le u_n$ for every n, so is zero, and $c = 0$ in $\mathfrak{A}$. Thus $\inf_{n\in\mathbb{N}} c_n = 0$ in $\mathfrak{A}$, and $\inf_{n\in\mathbb{N}} \nu c_n = 0$ in U, because ν is sequentially order-continuous. Replacing A', C in the argument above by $\{u_n : n \in \mathbb{N}\}$, $\{c_n : n \in \mathbb{N}\}$ we find an n such that $w \not\le Tu_n$. Since w is arbitrary, this means that $0 = \inf_{n\in\mathbb{N}} Tu_n$; as $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous.

(c)(i)$\Rightarrow$(iii) If $T : S(\mathfrak{A}) \to U$ is a Riesz homomorphism, and $\nu = T\chi$, then surely ν is a lattice homomorphism because T and χ are.

(iii)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i) If $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$, then for any $u \in S(\mathfrak{A})$ we have an expression of u as $\sum_{j=0}^m \beta_j\chi b_j$, where $b_0, \dots, b_m \in \mathfrak{A}$ are disjoint. Now

$$|Tu| = |\textstyle\sum_{j=0}^m \beta_j\nu b_j| = \sum_{j=0}^m |\beta_j|\nu b_j = T(\sum_{j=0}^m |\beta_j|\chi b_j) = T(|u|)$$

by 352Fb and 361Ed. As u is arbitrary, T is a Riesz homomorphism (352G).

361H Theorem Let $\mathfrak{A}$ be a Boolean ring and U a Dedekind complete Riesz space. Suppose that $\nu : \mathfrak{A} \to U$ is an additive function and $T : S = S(\mathfrak{A}) \to U$ is the corresponding linear operator. Then $T \in \mathrm{L}^\sim = \mathrm{L}^\sim(S; U)$ iff $\{\nu b : b \subseteq a\}$ is order-bounded in U for every $a \in \mathfrak{A}$, and in this case $|T| \in \mathrm{L}^\sim$ corresponds to $|\nu| : \mathfrak{A} \to U$, defined by setting

$$\begin{aligned}|\nu|(a) &= \sup\{\sum_{j=0}^n |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}\\ &= \sup\{\nu b - \nu(a \setminus b) : b \subseteq a\}\end{aligned}$$

for every $a \in \mathfrak{A}$.

proof (a) Suppose that $T \in \mathsf{L}^\sim$ and $a \in \mathfrak{A}$. Then for any $b \subseteq a$, we have $\chi b \le \chi a$ so

$$|\nu b| = |T\chi b| \le |T|(\chi a).$$

Accordingly $\{\nu b : b \subseteq a\}$ is order-bounded in U.

(b) Now suppose that $\{\nu b : b \subseteq a\}$ is order-bounded in U for every $a \in \mathfrak{A}$. Then for any $a \in \mathfrak{A}$ we can define $w_a = \sup\{|\nu b| : b \subseteq a\}$; in this case, $\nu b - \nu(a \setminus b) \le 2w_a$ whenever $b \subseteq a$, so $\theta a = \sup_{b \subseteq a} \nu b - \nu(a \setminus b)$ is defined in U. Considering $b = a$, $b = 0$ we see that $\theta a \ge |\nu a|$. Next, $\theta : \mathfrak{A} \to U$ is additive. **P** Take a_1, $a_2 \in \mathfrak{A}$ such that $a_1 \cap a_2 = 0$; set $a_0 = a_1 \cup a_2$. For each $j \le 2$ set

$$A_j = \{\nu(a_j \cap b) - \nu(a_j \setminus b) : b \in \mathfrak{A}\} \subseteq U.$$

Then $A_0 \subseteq A_1 + A_2$, because

$$\nu(a_0 \cap b) - \nu(a_0 \setminus b) = \nu(a_1 \cap b) - \nu(a_1 \setminus b) + \nu(a_2 \cap b) - \nu(a_2 \setminus b)$$

for every $b \in \mathfrak{A}$. But also $A_1 + A_2 \subseteq A_0$, because if b_1, $b_2 \in \mathfrak{A}$ then

$$\nu(a_1 \cap b_1) - \nu(a_1 \setminus b_1) + \nu(a_2 \cap b_2) - \nu(a_2 \setminus b_2) = \nu(a_0 \cap b) - \nu(a_0 \setminus b)$$

where $b = (a_1 \cap b_1) \cup (a_2 \cap b_2)$. So $A_0 = A_1 + A_2$, and

$$\theta a_0 = \sup A_0 = \sup A_1 + \sup A_2 = \theta a_1 + \theta a_2$$

(351Dc). **Q**

We therefore have a corresponding positive operator $T_1 : S \to U$ such that $\theta = T_1\chi$. But we also see that $\theta a = \sup\{\sum_{i=0}^n |\nu a_i| : a_0, \dots, a_n \subseteq a$ are disjoint$\}$ for every $a \in \mathfrak{A}$. **P** If $a_0, \dots, a_n$ are disjoint and included in a, then

$$\textstyle\sum_{i=0}^n |\nu a_i| \le \sum_{i=0}^n \theta a_i = \theta(\sup_{i \le n} a_i) \le \theta a.$$

On the other hand,

$$\theta a \le \sup_{b \subseteq a} |\nu b| + |\nu(a \setminus b)| \le \sup\{\textstyle\sum_{i=0}^n |\nu a_i| : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}. \quad \mathbf{Q}$$

It follows that $T \in \mathsf{L}^\sim$. **P** Take any $u \ge 0$ in S. Set $a = [\![u > 0]\!]$ (361Eg) and $\alpha = \|u\|_\infty$. If $0 < |v| \le u$, then v is expressible as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint and no α_i nor a_i is zero. Since $|v| \le \alpha\chi a$, we must have $|\alpha_i| \le \alpha$, $a_i \subseteq a$ for each i. So

$$|Tv| = |\textstyle\sum_{i=0}^n \alpha_i \nu a_i| \le \sum_{i=0}^n |\alpha_i||\nu a_i| \le \alpha \sum_{i=0}^n |\nu a_i| \le \alpha\theta a.$$

Thus $\{|Tv| : |v| \le u\}$ is bounded above by $\alpha\theta a$. As u is arbitrary, $T \in \mathsf{L}^\sim$. **Q**

(c) Thus $T \in \mathsf{L}^\sim$ iff ν is order-bounded on the sets $\{b : b \subseteq a\}$, and in this case the two formulae offered for $|\nu|$ are consistent and make $|\nu| = \theta$. Finally, $\theta = |T|\chi$. **P** Take $a \in \mathfrak{A}$. If $a_0, \dots, a_n \subseteq a$ are disjoint, then

$$\textstyle\sum_{i=0}^n |\nu a_i| = \sum_{i=0}^n |T\chi a_i| \le \sum_{i=0}^n |T|(\chi a_i) \le |T|(\chi a);$$

so $\theta a \le |T|(\chi a)$. On the other hand, the argument at the end of (b) above shows that $|T|(\chi a) \le \theta a$ for every a. Thus $|T|(\chi a) = \theta a$ for every $a \in \mathfrak{A}$, as required. **Q**

361I Theorem Let $\mathfrak{A}$ be a Boolean ring, U a normed space and $\nu : \mathfrak{A} \to U$ an additive function. Give $S = S(\mathfrak{A})$ its norm $\| \, \|_\infty$, and let $T : S \to U$ be the linear operator corresponding to ν. Then T is a bounded linear operator iff $\{\nu a : a \in \mathfrak{A}\}$ is bounded, and in this case $\|T\| = \sup_{a,b \in \mathfrak{A}} \|\nu a - \nu b\|$.

proof (a) If T is bounded, then

$$\|\nu a - \nu b\| = \|T(\chi a - \chi b)\| \le \|T\|\|\chi a - \chi b\|_\infty \le \|T\|$$

for every $a \in \mathfrak{A}$, so ν is bounded and $\sup_{a,b \in \mathfrak{A}} \|\nu a - \nu b\| \le \|T\|$.

(b)(i) For the converse, we need a refinement of an idea in 361Ec. If $u \in S$ and $u \ge 0$ and $\|u\|_\infty \le 1$, then u is expressible as $\sum_{i=0}^m \gamma_i \chi c_i$ where $\gamma_i \ge 0$ and $\sum_{i=0}^m \gamma_i = 1$. **P** If $u = 0$, take $n = 0$, $c_0 = 0$, $\gamma_0 = 1$. Otherwise, start from an expression $u = \sum_{j=0}^n \gamma_j \chi c_j$ where $c_0 \supseteq \dots \supseteq c_n$ and every γ_j is non-negative, as in 361Ec. We may suppose that $c_n \ne 0$, in which case

$$\sum_{j=0}^{n}\gamma_j = u(z) \le 1$$

for every $z \in \widehat{c}_n \subseteq Z$, the Stone space of $\mathfrak{A}$. Set $m = n+1$, $c_m = 0$ and $\gamma_m = 1-\sum_{j=0}^{n}\gamma_j$ to get the required form. **Q**

(ii) The next fact we need is an elementary property of real numbers: if $\gamma_0,\dots,\gamma_m,\gamma'_0,\dots,\gamma'_n \ge 0$ and $\sum_{i=0}^{m}\gamma_i = \sum_{j=0}^{n}\gamma'_j$, then there are $\delta_{ij} \ge 0$ such that $\gamma_i = \sum_{j=0}^{n}\delta_{ij}$ for every $i \le m$ and $\gamma'_j = \sum_{i=0}^{m}\delta_{ij}$ for every $j \le n$. **P** This is just the case $U = \mathbb{R}$ of 352Fd. **Q**

(iii) Now suppose that ν is bounded; set $\alpha_0 = \sup_{a\in\mathfrak{A}}\|\nu a\| < \infty$. Then

$$\alpha = \sup_{a,b\in\mathfrak{A}}\|\nu a - \nu b\| \le 2\alpha_0$$

is also finite. If $u \in S$ and $\|u\|_\infty \le 1$, then we can express u as $u^+ - u^-$ where u^+, u^- are non-negative and also of norm at most 1. By (i), we can express these as

$$u^+ = \sum_{i=0}^{m}\gamma_i\chi c_i, \quad u^- = \sum_{j=0}^{n}\gamma'_j\chi c'_j$$

where all the γ_i, γ'_j are non-negative and $\sum_{i=0}^{m}\gamma_i = \sum_{j=0}^{n}\gamma'_j = 1$. Take $\langle\delta_{ij}\rangle_{i\le m,j\le n}$ from (ii). Set $c_{ij} = c_i$, $c'_{ij} = c'_j$ for all i, j, so that

$$u^+ = \sum_{i=0}^{m}\sum_{j=0}^{n}\delta_{ij}\chi c_{ij}, \quad u^- = \sum_{i=0}^{m}\sum_{j=0}^{n}\delta_{ij}\chi c'_{ij},$$

$$u = \sum_{i=0}^{m}\sum_{j=0}^{n}\delta_{ij}(\chi c_{ij} - \chi c'_{ij}),$$

$$Tu = \sum_{i=0}^{m}\sum_{j=0}^{n}\delta_{ij}(\nu c_{ij} - \nu c'_{ij}),$$

$$\|Tu\| \le \sum_{i=0}^{m}\sum_{j=0}^{n}\delta_{ij}\|\nu c_{ij} - \nu c'_{ij}\| \le \sum_{i=0}^{m}\sum_{j=0}^{n}\delta_{ij}\alpha = \alpha.$$

As u is arbitrary, T is a bounded linear operator and $\|T\| \le \alpha$, as required.

361J The last few paragraphs describe the properties of $S(\mathfrak{A})$ in terms of universal mapping theorems. The next theorem looks at the construction as a functor which converts Boolean algebras into Riesz spaces and ring homomorphisms into Riesz homomorphisms.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings and $\pi : \mathfrak{A} \to \mathfrak{B}$ a ring homomorphism.

(a) We have a Riesz homomorphism $T_\pi : S(\mathfrak{A}) \to S(\mathfrak{B})$ given by the formula

$$T_\pi(\chi a) = \chi(\pi a) \text{ for every } a \in \mathfrak{A}.$$

For any $u \in S(\mathfrak{A})$, $\|T_\pi u\|_\infty = \min\{\|u'\|_\infty : u' \in S(\mathfrak{A}), T_\pi u' = T_\pi u\}$; in particular, $\|T_\pi u\|_\infty \le \|u\|_\infty$. Moreover, $T_\pi(u \times u') = T_\pi u \times T_\pi u'$ for all u, $u' \in S(\mathfrak{A})$.

(b) T_π is surjective iff π is surjective, and in this case $\|v\|_\infty = \min\{\|u\|_\infty : u \in S(\mathfrak{A}), T_\pi u = v\}$ for every $v \in S(\mathfrak{B})$.

(c) The kernel of T_π is just the set of those $u \in S(\mathfrak{A})$ such that $\pi[\![|u| > 0]\!] = 0$, defining $[\![\ldots > \ldots]\!]$ as in 361Eg.

(d) T_π is injective iff π is injective, and in this case $\|T_\pi u\|_\infty = \|u\|_\infty$ for every $u \in S(\mathfrak{A})$.

(e) T_π is order-continuous iff π is order-continuous.

(f) T_π is sequentially order-continuous iff π is sequentially order-continuous.

(g) If $\mathfrak{C}$ is another Boolean ring and $\phi : \mathfrak{B} \to \mathfrak{C}$ is another ring homomorphism, then $T_{\phi\pi} = T_\phi T_\pi : S(\mathfrak{A}) \to S(\mathfrak{C})$.

proof (a) The map $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$ is additive (361Cc), so corresponds to a linear operator $T = T_\pi : S(\mathfrak{A}) \to S(\mathfrak{B})$, by 361F. χ and π are both lattice homomorphisms, so $\chi\pi$ also is, and T is a Riesz homomorphism (361Gc). If $u = \sum_{i=0}^{n}\alpha_i\chi a_i$, where $a_0,\dots,a_n$ are disjoint, then look at $I = \{i : i \le n, \pi a_i \ne 0\}$. We have

$$Tu = \sum_{i=0}^{n}\alpha_i\chi(\pi a_i) = \sum_{i\in I}\alpha_i\chi(\pi a_i)$$

and $\pi a_0,\dots,\pi a_n$ are disjoint, so that

$$\|Tu\|_\infty = \sup_{i\in I}|\alpha_i| = \|u'\|_\infty \le \sup_{a_i\ne 0}|\alpha_i| \le \|u\|_\infty,$$

where $u' = \sum_{i\in I}\alpha_i\chi a_i$, so that $Tu' = Tu$. If a, $a' \in \mathfrak{A}$, then

$$T(\chi a \times \chi a') = T\chi(a \cap a') = \chi\pi(a \cap a') = \chi\pi a \times \chi\pi a' = T\chi a \times T\chi a',$$

so T is multiplicative.

(b) If π is surjective, then $T[S(\mathfrak{A})]$ must be the linear span of

$$\{T(\chi a) : a \in \mathfrak{A}\} = \{\chi(\pi a) : a \in \mathfrak{A}\} = \{\chi b : b \in \mathfrak{B}\},$$

so is the whole of $S(\mathfrak{B})$. If T is surjective, and $b \in \mathfrak{B}$, then there must be a $u \in \mathfrak{A}$ such that $Tu = \chi b$. We can express u as $\sum_{i=0}^{n} \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint; now

$$\chi b = Tu = \sum_{i=0}^{n} \alpha_i \chi(\pi a_i),$$

and $\pi a_0, \dots, \pi a_n$ are disjoint in $\mathfrak{B}$, so we must have

$$b = \sup_{i \in I} \pi a_i = \pi(\sup_{i \in I} a_i) \in \pi[\mathfrak{A}],$$

where $I = \{i : \alpha_i = 1\}$. As b is arbitrary, π is surjective. Of course the formula for $\|v\|_\infty$ is a consequence of the formula for $\|Tu\|_\infty$ in (a).

(c)(i) If $\pi[\![|u| > 0]\!] = 0$ then $|u| \le \alpha\chi a$, where $\alpha = \|u\|_\infty$, and $a = [\![|u| > 0]\!]$, so

$$|Tu| = T|u| \le \alpha T(\chi a) = \alpha\chi(\pi a) = 0,$$

and $Tu = 0$.

(ii) If $u \in S(\mathfrak{A})$ and $Tu = 0$, express u as $\sum_{i=0}^{n} \alpha_i \chi a_i$ where $a_0, \dots, a_n$ are disjoint and every α_i is non-zero (361Eb). In this case

$$0 = |Tu| = T|u| = \sum_{i=0}^{n} |\alpha_i| \chi(\pi a_i),$$

so $\pi a_i = 0$ for every i, and

$$\pi[\![|u| > 0]\!] = \pi(\sup_{i \le n} a_i) = \sup_{i \le n} \pi a_i = 0.$$

(d) If T is injective and $a \in \mathfrak{A} \setminus \{0\}$, then $\chi(\pi a) = T(\chi a) \ne 0$, so $\pi a \ne 0$; as a is arbitrary, π is injective. If π is injective then $\pi[\![|u| > 0]\!] \ne 0$ for every non-zero $u \in S(\mathfrak{A})$, so T is injective, by (c). In this case the formula in (a) shows that T is norm-preserving.

(e)(i) If T is order-continuous and $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{A}$, let b be any lower bound for $\pi[A]$ in $\mathfrak{B}$. Then

$$\chi b \le \chi(\pi a) = T(\chi a)$$

for any $a \in A$. But $T\chi$ is order-continuous, by 361Ef, so $\inf_{a \in A} T(\chi a) = 0$, and b must be 0. As b is arbitrary, $\inf_{a \in A} \pi a = 0$; as A is arbitrary, π is order-continuous.

(ii) If π is order-continuous, so is $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$, using 361Ef again; but now by 361G(b-ii) T must be order-continuous.

(f)(i) If T is sequentially order-continuous, and $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, let b be any lower bound for $\{\pi a_n : n \in \mathbb{N}\}$ in $\mathfrak{B}$. Then

$$\chi b \le \chi(\pi a_n) = T(\chi a_n)$$

for any $a \in A$. But $T\chi$ is sequentially order-continuous so $\inf_{n \in \mathbb{N}} T(\chi a_n) = 0$, and b must be 0. As b is arbitrary, $\inf_{n \in \mathbb{N}} \pi a_n = 0$; as A is arbitrary, π is sequentially order-continuous.

(ii) If π is sequentially order-continuous, so is $\chi\pi : \mathfrak{A} \to S(\mathfrak{B})$; but now T must be sequentially order-continuous.

(g) We need only check that

$$T_{\phi\pi}(\chi a) = \chi(\phi(\pi a)) = T_\phi(\chi(\pi a)) = T_\phi T_\pi(\chi a)$$

for every $a \in \mathfrak{A}$.

361K Proposition Let $\mathfrak{A}$ be a Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the solid linear subspace of $S(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in $S(\mathfrak{A})$.

proof Write S for $S(\mathfrak{A})$.

(a) The point is that, for any $a \in \mathfrak{A}$,

(i) $|u| \wedge |v| = 0$ whenever $u \in V_a$, $v \in V_{1\setminus a}$,

(ii) $V_a + V_{1\setminus a} = S$.

P (i) is just because $\chi a \wedge \chi(1 \setminus a) = 0$. As for (ii), if $w \in S$ then

$$w = (w \times \chi a) + (w \times \chi(1 \setminus a)) \in V_a + V_{1\setminus a}. \text{ **Q**}$$

(b) Accordingly $V_a + V_a^\perp \supseteq V_a + V_{1\setminus a} = S$ and V_a is a projection band (352R). Next, any projection band $U \subseteq S$ is of the form V_a. **P** We know that $\chi 1 = u + v$ where $u \in U$, $v \in U^\perp$. Since $|u| \wedge |v| = 0$, u and v must be the indicator functions of complementary subsets of Z, the Stone space of $\mathfrak{A}$. But $\{z : u(z) \neq 0\} = \{z : u(z) \geq 1\}$ must be of the form $\widehat{a}$, where $a = [\![u > 0]\!]$, in which case $u = \chi a$ and $v = \chi(1 \setminus a)$. Accordingly $U \supseteq V_a$ and $U^\perp \supseteq V_{1\setminus a}$. But this means that U must be V_a precisely. **Q**

(c) Thus $a \mapsto V_a$ is a surjective function from $\mathfrak{A}$ onto the algebra of projection bands in S. Now

$$a \subseteq b \iff \chi a \in V_b \iff V_a \subseteq V_b,$$

so $a \mapsto V_a$ is order-preserving and bijective. By 312M it is a Boolean isomorphism.

361L Proposition Let X be a set, and Σ a ring of subsets of X, that is, a subring of the Boolean ring $\mathcal{P}X$. Then $S(\Sigma)$ can be identified, as ordered linear space, with the linear subspace of $\ell^\infty(X)$ generated by the indicator functions of members of Σ, which is a Riesz subspace of $\ell^\infty(X)$. The norm of $S(\Sigma)$ corresponds to the uniform norm on $\ell^\infty(X)$, and its multiplication to pointwise multiplication of functions.

proof Let Z be the Stone space of Σ, and for $E \in \Sigma$ write χE for the indicator function of E as a subset of X, $\hat\chi E$ for the indicator function of the open-and-compact subset of Z corresponding to E. Of course $\chi : \Sigma \to \ell^\infty(X)$ is additive, so by 361F there is a linear operator $T : S \to \ell^\infty(X)$, writing S for $S(\Sigma)$, such that $T(\hat\chi E) = \chi E$ for every $E \in \Sigma$.

If $u \in S$, $Tu \geq 0$ iff $u \geq 0$. **P** Express u as $\sum_{j=0}^m \beta_j \hat\chi E_j$ where $E_0, \dots, E_m$ are disjoint. Then $Tu = \sum_{j=0}^m \beta_j \chi E_j$, so

$$u \geq 0 \iff \beta_j \geq 0 \text{ whenever } E_j \neq \emptyset \iff Tu \geq 0. \text{ **Q**}$$

But this means (α) that

$$Tu = 0 \iff Tu \geq 0 \;\&\; T(-u) \geq 0 \iff u \geq 0 \;\&\; -u \geq 0 \iff u = 0,$$

so that T is injective and is a linear space isomorphism between S and its image $\mathcal{S}$, which must be the linear space spanned by $\{\chi E : E \in \Sigma\}$ (β) that T is an order-isomorphism between S and $\mathcal{S}$.

Because $\chi E \wedge \chi F = 0$ whenever E, $F \in \Sigma$ and $E \cap F = \emptyset$, T is a Riesz homomorphism and $\mathcal{S}$ is a Riesz subspace of $\ell^\infty(X)$ (361Gc). Now

$$\|u\|_\infty = \inf\{\alpha : |u| \leq \alpha\hat\chi X\} = \inf\{\alpha : |Tu| \leq \alpha\chi X\} = \|Tu\|_\infty$$

for every $u \in S$. Finally,

$$T(\hat\chi E \times \hat\chi F) = T(\hat\chi(E \cap F)) = \chi(E \cap F) = T(\hat\chi E) \times T(\hat\chi F)$$

for all E, $F \in \Sigma$, so $\mathcal{S}$ is closed under pointwise multiplication and the multiplications of S, $\mathcal{S}$ are identified by T.

361M Proposition Let X be a set, Σ a ring of subsets of X, and $\mathcal{I}$ an ideal of Σ; write $\mathfrak{A}$ for the quotient ring $\Sigma/\mathcal{I}$. Let $\mathcal{S}$ be the linear span of $\{\chi E : E \in \Sigma\}$ in $\mathbb{R}^X$, and write

$$V = \{f : f \in \mathcal{S}, \{x : f(x) \neq 0\} \in \mathcal{I}\}.$$

Then V is a solid linear subspace of $\mathcal{S}$. Now $S(\mathfrak{A})$ becomes identified with the quotient Riesz space $\mathcal{S}/V$, if for every $E \in \Sigma$ we identify $\chi(E^\bullet) \in S(\mathfrak{A})$ with $(\chi E)^\bullet \in \mathcal{S}/V$. If we give $\mathcal{S}$ its uniform norm inherited from $\ell^\infty(X)$, V is a closed linear subspace of $\mathcal{S}$, and the quotient norm on $\mathcal{S}/V$ corresponds to the norm of $S(\mathfrak{A})$:

$$\|f^\bullet\| = \min\{\alpha : \{x : |f(x)| > \alpha\} \in \mathcal{I}\}.$$

If we write $\times$ for pointwise multiplication on $\mathcal{S}$, then V is an ideal of the ring $(\mathcal{S},+,\times)$, and the multiplication induced on $\mathcal{S}/V$ corresponds to the multiplication of $S(\mathfrak{A})$.

proof Use 361J and 361L. We can identify $\mathcal{S}$ with $S(\Sigma)$. Now the canonical ring homomorphism $E\mapsto E^\bullet$ corresponds to a surjective Riesz homomorphism T from $S(\Sigma)$ to $S(\mathfrak{A})$ which takes χE to $\chi(E^\bullet)$. For $f\in\mathcal{S}$, $[\![|f|>0]\!]$ is just $\{x:f(x)\ne 0\}$, so the kernel of T is just the set of those $f\in\mathcal{S}$ such that $\{x:f(x)\ne 0\}\in\mathcal{I}$, which is V. So

$$S(\mathfrak{A})=T[\mathcal{S}]\cong\mathcal{S}/V.$$

As noted in 361Ja, $T(f\times g)=Tf\times Tg$ for all f, $g\in\mathcal{S}$, so the multiplications of $\mathcal{S}/V$ and $S(\mathfrak{A})$ match. As for the norms, the norm of $S(\mathfrak{A})$ corresponds to the norm of $\mathcal{S}/V$ by the formulae in 361Ja or 361Jb. To see that V is closed in $\mathcal{S}$, we need note only that if $f\in\overline{V}$ then

$$\|Tf\|_\infty=\inf_{g\in V}\|f+g\|_\infty=\inf_{g\in V}\|f-g\|_\infty=0,$$

so that $Tf=0$ and $f\in V$. To check the formula for $\|f^\bullet\|$, take any $f\in\mathcal{S}$. Express it as $\sum_{i=0}^n\alpha_i\chi E_i$ where $E_0,\dots,E_n\in\Sigma$ are disjoint. Set $I=\{i:E_i\notin\mathcal{I}\}$; then

$$\|Tf\|_\infty=\max_{i\in I}|\alpha_i|=\min\{\alpha:\{x:|f(x)|>\alpha\}\in\mathcal{I}\}.$$

361X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean ring and U a linear space. Show that a function $\nu:\mathfrak{A}\to U$ is additive iff $\nu 0=0$ and $\nu(a\cup b)+\nu(a\cap b)=\nu a+\nu b$ for all a, $b\in\mathfrak{A}$.

>(b) Let U be an **algebra over** $\mathbb{R}$, that is, a real linear space endowed with a multiplication $\times$ such that $(U,+,\times)$ is a ring and $\alpha(w\times z)=(\alpha w)\times z=w\times(\alpha z)$ for all w, $z\in U$ and all $\alpha\in\mathbb{R}$. Let $\mathfrak{A}$ be a Boolean ring, $\nu:\mathfrak{A}\to U$ an additive function and $T:S(\mathfrak{A})\to U$ the corresponding linear operator. Show that T is multiplicative iff $\nu(a\cap b)=\nu a\times\nu b$ for all a, $b\in\mathfrak{A}$.

>(c) Let $\mathfrak{A}$ be a Boolean ring, and U a Dedekind complete Riesz space. Suppose that $\nu:\mathfrak{A}\to U$ is an additive function such that the corresponding linear operator $T:S(\mathfrak{A})\to U$ belongs to $\mathsf{L}^\sim=\mathsf{L}^\sim(S(\mathfrak{A});U)$. Show that $T^+\in\mathsf{L}^\sim$ corresponds to $\nu^+:\mathfrak{A}\to U$, where $\nu^+a=\sup_{b\subseteq a}\nu b$ for every $a\in\mathfrak{A}$.

(d) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. Show that there is a natural one-to-one correspondence between Boolean homomorphisms $\pi:\mathfrak{A}\to\mathfrak{B}$ and Riesz homomorphisms $T:S(\mathfrak{A})\to S(\mathfrak{B})$ such that $T(\chi 1_{\mathfrak{A}})=\chi 1_{\mathfrak{B}}$, given by setting $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$.

(e) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean rings and $T:S(\mathfrak{A})\to S(\mathfrak{B})$ a linear operator such that $T(u\times v)=Tu\times Tv$ for all u, $v\in S(\mathfrak{A})$. Show that there is a ring homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ such that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings. Show that any isomorphism of the algebras $S(\mathfrak{A})$ and $S(\mathfrak{B})$ (using the word 'algebra' in the sense of 361Xb) must be a Riesz space isomorphism, and therefore corresponds to an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

(g) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T:S(\mathfrak{A})\to S(\mathfrak{B})$ a Riesz homomorphism. Show that there are a ring homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ and a non-negative $v\in S(\mathfrak{B})$ such that $T(\chi a)=v\times\chi(\pi a)$ for every $a\in\mathfrak{A}$.

(h) Let $\mathfrak{A}$ be a Boolean algebra, $\pi:\mathfrak{A}\to\mathfrak{A}$ a Boolean homomorphism and $T_\pi:S(\mathfrak{A})\to S(\mathfrak{A})$ the associated Riesz homomorphism. Let $\mathfrak{C}$ be the fixed-point subalgebra of π (312K). Show that $S(\mathfrak{C})$ may be identified with the linear subspace of $S(\mathfrak{A})$ generated by $\{\chi c:c\in\mathfrak{C}\}$, and that this is $\{u:u\in S(\mathfrak{A}),\ T_\pi u=u\}$.

(i) Let $\mathfrak{A}$ be a Boolean ring. Show that for any $u\in S(\mathfrak{A})$ the solid linear subspace of $S(\mathfrak{A})$ generated by u is a projection band in $S(\mathfrak{A})$. Show that the set of such bands is an ideal in the algebra of all projection bands, and is isomorphic to $\mathfrak{A}$.

>(j) Let X be a set and Σ a σ-algebra of subsets of X. Show that the linear span $\mathcal{S}$ in $\mathbb{R}^X$ of $\{\chi E:E\in\Sigma\}$ is just the set of Σ-measurable functions $f:X\to\mathbb{R}$ which take only finitely many values.

(k) For any Boolean ring $\mathfrak{A}$, we may define its 'complex S-space' $S_{\mathbb{C}}(\mathfrak{A})$ as the linear span in $\mathbb{C}^Z$ of the indicator functions of open-and-compact subsets of the Stone space Z of $\mathfrak{A}$. State and prove results corresponding to 361Eb, 361Ed, 361Eh, 361F, 361L and 361M.

(l) Let $\mathfrak{A}$ be a Boolean algebra, U a partially ordered linear space and $\nu : \mathfrak{A} \to U$ a non-negative additive function. (i) Show that ν is order-continuous iff $\nu 1 = \sup_{J \subseteq I \text{ is finite}} \sum_{i \in J} \nu a_i$ whenever $\langle a_i \rangle_{i \in I}$ is a partition of unity in U. (i) Show that ν is order-continuous iff $\nu 1 = \sup_{n \in \mathbb{N}} \sum_{i=0}^{n} \nu a_i$ whenever $\langle a_i \rangle_{i \in \mathbb{N}}$ is a partition of unity in U.

361Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean ring. For $a \in \mathfrak{A}$ let $e_a \in \mathbb{R}^{\mathfrak{A}}$ be the function such that $e_a(a) = 1$, $e_a(b) = 0$ for $b \in \mathfrak{A} \setminus \{a\}$; let V be the linear subspace of $\mathbb{R}^{\mathfrak{A}}$ generated by $\{e_a : a \in \mathfrak{A}\}$. Let $W \subseteq V$ be the linear subspace spanned by members of V of the form $e_{a \cup b} - e_a - e_b$ where a, $b \in \mathfrak{A}$ are disjoint. Define $\chi' : \mathfrak{A} \to V/W$ by taking $\chi' a = e_a^\bullet$ to be the image in V/W of $e_a \in V$. Show, without using the axiom of choice, that the pair $(V/W, \chi')$ has the universal mapping property of $(S(\mathfrak{A}), \chi)$ as described in 361F and that V/W has a Riesz space structure, a norm and a multiplicative structure as described in 361D-361E. Prove results corresponding to 361E-361M.

(b) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras, with free product $\mathfrak{A}$; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical maps, and

$$C = \{\inf_{j \in J} \varepsilon_j(a_j) : J \subseteq I \text{ is finite, } a_j \in \mathfrak{A}_j \text{ for every } j \in J\}.$$

Suppose that U is a linear space and $\theta : C \to U$ is such that

$$\theta c = \theta(c \cap \varepsilon_i(a)) + \theta(c \cap \varepsilon_i(1 \setminus a))$$

whenever $c \in C$, $i \in I$ and $a \in \mathfrak{A}_i$. Show that there is a unique additive function $\nu : \mathfrak{A} \to U$ extending θ. (*Hint*: 326E.)

(c) Let $\mathfrak{A}$ be a Boolean ring and U a Dedekind complete Riesz space. Let $A \subseteq \mathsf{L}^\sim = \mathsf{L}^\sim(S(\mathfrak{A}); U)$ be a non-empty set. Suppose that $\tilde{T} = \sup A$ is defined in $\mathsf{L}^\sim$, and that $\tilde{\nu} = \tilde{T}\chi$. Show that for any $a \in \mathfrak{A}$,

$$\tilde{\nu} a = \sup\{\sum_{i=0}^{n} T_i(\chi a_i) : T_0, \dots, T_n \in A,\ a_0, \dots, a_n \subseteq a \text{ are disjoint, } \sup_{i \le n} a_i = a\}.$$

(d) Let $\mathfrak{A}$ be a Boolean algebra. Show that the algebra of all bands of $S(\mathfrak{A})$ can be identified with the Dedekind completion of $\mathfrak{A}$ (314U).

(e) Let $\mathfrak{A}$ be a Boolean ring, and U a complex normed space. Let $\nu : \mathfrak{A} \to U$ be an additive function and $T : S_{\mathbb{C}}(\mathfrak{A}) \to U$ the corresponding linear operator (cf. 361Xk). Show that (giving $S_{\mathbb{C}}(\mathfrak{A})$ its usual norm $\| \|_\infty$)

$$\|T\| = \sup\{\|\sum_{j=0}^{n} \zeta_j \nu a_j\| : a_0, \dots, a_n \in \mathfrak{A} \text{ are disjoint, } |\zeta_j| = 1 \text{ for every } j\}$$

if either is finite.

(f) Let U be a Riesz space. Show that it is isomorphic to $S(\mathfrak{A})$, for some Boolean algebra $\mathfrak{A}$, iff it has an order unit and every solid linear subspace of U is a projection band.

361 Notes and comments The space $S(\mathfrak{A})$ corresponds of course to the idea of 'simple function' which belongs to the very beginnings of the theory of integration (122A). All that 361D is trying to do is to set up a logically sound description of this obvious concept which can be derived from the Boolean ring $\mathfrak{A}$ itself. To my eye, there is a defect in the construction there. It relies on the axiom of choice, since it uses the Stone space; but none of the elementary properties of $S(\mathfrak{A})$ have anything to do with the axiom of choice. In 361Ya I offer an alternative construction which is in a formal sense more 'elementary'. If you work through the suggestion there you will find, however, that the technical details become significantly more complicated, and would be intolerable were it not for the intuition provided by the Stone space construction. Of course this intuition is chiefly valuable in the finitistic arguments used in 361E, 361F and 361I; and for these arguments we really need the Stone representation only for finite Boolean rings, which does not depend on the axiom of choice.

It is quite true that in most of this volume (and in most of this chapter) I use the axiom of choice without scruple and without comment. I mention it here only because I find myself using arguments dependent on choice to prove theorems of a type to which the axiom cannot be relevant.

The linear space structure of $S(\mathfrak{A})$, together with the map χ, are uniquely determined by the first universal mapping theorem here, 361F. This result says nothing about the order structure, which needs the further refinement in 361Ga. What is striking is that the partial order defined by 361Ga is actually a lattice ordering, so that we can have a universal mapping theorem for functions to Riesz spaces, as in 361Gc and 361Ja. Moreover, the same ordering provides a happy abundance of results concerning order-continuous functions (361Gb, 361Je-361Jf). When the codomain is a Dedekind complete Riesz space, so that we have a Riesz space $\mathsf{L}^\sim(S;U)$, and a corresponding modulus function $T \mapsto |T|$ for linear operators, there are reasonably natural formulae for $|T|\chi$ in terms of $T\chi$ (361H); see also 361Xc and 361Yc. The multiplicative structure of $S(\mathfrak{A})$ is defined by 361Xb, and its norm by 361I.

The Boolean ring $\mathfrak{A}$ cannot be recovered from the linear space structure of $S(\mathfrak{A})$ alone (since this tells us only the cardinality of $\mathfrak{A}$), but if we add either the ordering or the multiplication of $S(\mathfrak{A})$ then $\mathfrak{A}$ is easy to identify (361K, 361Xf).

The most important Boolean algebras of measure theory arise either as algebras of sets or as their quotients, so it is a welcome fact that in such cases the spaces $S(\mathfrak{A})$ have straightforward representations in terms of the construction of $\mathfrak{A}$ (361L-361M).

In Chapter 24 I offered a paragraph in each section to sketch a version of the theory based on the field of complex numbers rather than the field of real numbers. This was because so many of the most important applications of these ideas involve complex numbers, even though (in my view) the ideas themselves are most clearly and characteristically expressed in terms of real numbers. In the present chapter we are one step farther away from these applications, and I therefore relegate complex numbers to the exercises, as in 361Xk and 361Ye.

Version of 31.12.10

362 $S^\sim$

The next stage in our journey is the systematic investigation of linear functionals on spaces $S = S(\mathfrak{A})$. We already know that these correspond to additive real-valued functionals on the algebra $\mathfrak{A}$ (361F). My purpose here is to show how the structure of the Riesz space dual $S^\sim$ and its bands is related to the classes of additive functionals introduced in §§326-327. The first step is just to check the identification of the linear and order structures of $S^\sim$ and the space M of bounded finitely additive functionals (362A); all the ideas needed for this have already been set out, and the basic properties of $S^\sim$ are covered by the general results in §356. Next, we need to be able to describe the operations on M corresponding to the Riesz space operations $|\;|$, $\vee$, $\wedge$ on $S^\sim$, and the band projections from $S^\sim$ onto $S^\sim_c$ and $S^\times$; these are dealt with in 362B, with a supplementary remark in 362D. In the case of measure algebras, we have some further important bands which present themselves in M, rather than in $S^\sim$, and which are treated in 362C. Since all these spaces are L-spaces, it is worth taking a moment to identify their uniformly integrable subsets; I do this in 362E.

While some of the ideas here have interesting extensions to the case in which $\mathfrak{A}$ is a Boolean ring without identity, these can I think be left to one side; the work of this section will be done on the assumption that every $\mathfrak{A}$ is a Boolean algebra.

362A Theorem Let $\mathfrak{A}$ be a Boolean algebra. Write S for $S(\mathfrak{A})$.

(a) The partially ordered linear space of all finitely additive real-valued functionals on $\mathfrak{A}$ may be identified with the partially ordered linear space of all real-valued linear functionals on S.

(b) The linear space of bounded finitely additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\sim$ of order-bounded linear functionals on S. If $f \in S^\sim$ corresponds to $\nu : \mathfrak{A} \to \mathbb{R}$, then $f^+ \in S^\sim$ corresponds to ν^+, where

$$\nu^+ a = \sup_{b \subseteq a} \nu b$$

for every $a \in \mathfrak{A}$, and

$$\|f\| = \sup_{a\in\mathfrak{A}} \nu a - \nu(1\setminus a).$$

(c) The linear space of bounded countably additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S_c^\sim$.

(d) The linear space of completely additive real-valued functionals on $\mathfrak{A}$ may be identified with the L-space $S^\times$.

proof By 361F, we have a canonical one-to-one correspondence between linear functionals $f : S \to \mathbb{R}$ and additive functionals $\nu_f : \mathfrak{A} \to \mathbb{R}$, given by setting $\nu_f = f\chi$.

(a) Now it is clear that $\nu_{f+g} = \nu_f + \nu_g$, $\nu_{\alpha f} = \alpha\nu_f$ for all f, g and α, so this one-to-one correspondence is a linear space isomorphism. To see that it is also an order-isomorphism, we need note only that ν_f is non-negative iff f is, by 361Ga.

(b) Recall from 356N that, because S is a Riesz space with order unit (361Ee), $S^\sim$ has a corresponding norm under which it is an L-space.

(i) If $f \in S^\sim$, then

$$\sup_{b\in\mathfrak{A}} |\nu_f b| = \sup_{b\in\mathfrak{A}} |f(\chi b)| \le \sup\{|f(u)| : u \in S,\ |u| \le \chi 1\}$$

is finite, and ν_f is bounded.

(ii) Now suppose that ν_f is bounded and that $v \in S^+$. Then there is an $\alpha \ge 0$ such that $v \le \alpha\chi 1$ (361Ee). If $u \in S$ and $|u| \le v$, then we can express u as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \dots, a_n$ are disjoint (361Eb); now $|\alpha_i| \le \alpha$ whenever $a_i \ne 0$, so

$$|f(u)| = |\sum_{i=0}^n \alpha_i\nu_f a_i| \le \alpha\sum_{i=0}^n |\nu_f a_i| = \alpha(\nu_f c_1 - \nu_f c_2) \le 2\alpha \sup_{b\in\mathfrak{A}} |\nu_f b|,$$

setting $c_1 = \sup\{a_i : i \le n,\ \nu_f a_i \ge 0\}$, $c_2 = \sup\{a_i : i \le n,\ \nu_f a_i < 0\}$. This shows that $\{f(u) : |u| \le v\}$ is bounded. As v is arbitrary, $f \in S^\sim$ (356Aa).

(iii) To check the correspondence between f^+ and ν_f^+, refine the arguments of (i) and (ii) as follows. Take any $f \in S^\sim$. If $a \in \mathfrak{A}$,

$$\nu_f^+ a = \sup_{b\subseteq a} \nu_f b = \sup_{b\subseteq a} f(\chi b) \le \sup\{f(u) : u \in S,\ 0 \le u \le \chi a\} = f^+(\chi a).$$

On the other hand, if $u \in S$ and $0 \le u \le \chi a$, then we can express u as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \dots, a_n$ are disjoint; now $0 \le \alpha_i \le 1$ whenever $a_i \ne 0$, so

$$f(u) = \sum_{i=0}^n \alpha_i\nu_f a_i \le \nu_f c \le \nu_f^+ a,$$

where $c = \sup\{a_i : i \le n,\ \nu_f a_i \ge 0\}$. As u is arbitrary, $f^+(\chi a) \le \nu_f^+ a$. This shows that $\nu_f^+ = f^+\chi$ is finitely additive, and that $\nu_f^+ = \nu_{f^+}$, as claimed.

(iv) Now, for any $f \in S^\sim$,

$$\|f\| = |f|(\chi 1)$$

(356N)

$$= (2f^+ - f)(\chi 1) = 2\nu_f^+ 1 - \nu_f 1$$

(by (iii) just above)

$$= \sup_{a\in\mathfrak{A}} 2\nu_f a - \nu_f 1 = \sup_{a\in\mathfrak{A}} \nu_f a - \nu_f(1\setminus a).$$

(c) If $f \ge 0$ in $S^\sim$, then f is sequentially order-continuous iff ν_f is sequentially order-continuous (361Gb), that is, iff ν_f is countably additive (326Kc). Generally, an order-bounded linear functional belongs to $S_c^\sim$ iff it is expressible as the difference of two sequentially order-continuous positive linear functionals (356Ab), while a bounded finitely additive functional is countably additive iff it is expressible as the difference of two non-negative countably additive functionals (326L); so in the present context $f \in S_c^\sim$ iff ν_f is bounded and countably additive.

(d) If $f \geq 0$ in $S^\sim$, then f is order-continuous iff ν_f is order-continuous (361Gb), that is, iff ν_f is completely additive (326Oc). Generally, an order-bounded linear functional belongs to $S^\times$ iff it is expressible as the difference of two order-continuous positive linear functionals (356Ac), while a finitely additive functional is completely additive iff it is expressible as the difference of two non-negative completely additive functionals (326Q); so in the present context $f \in S^\times$ iff ν_f is completely additive.

362B Spaces of finitely additive functionals The identifications in the last theorem mean that we can relate the Riesz space structure of $S(\mathfrak{A})^\sim$ to constructions involving finitely additive functionals. I have already set out the most useful facts as exercises (326Yd, 326Ym, 326Yn, 326Yp, 326Yq); it is now time to repeat them more formally.

Theorem Let $\mathfrak{A}$ be a Boolean algebra. Let M be the Riesz space of bounded finitely additive real-valued functionals on $\mathfrak{A}$, $M_\sigma \subseteq M$ the space of bounded countably additive functionals, and $M_\tau \subseteq M_\sigma$ the space of completely additive functionals.

(a) For any μ, $\nu \in M$, $\mu \vee \nu$, $\mu \wedge \nu$ and $|\nu|$ are defined by the formulae

$$(\mu \vee \nu)(a) = \sup_{b \subseteq a} \mu b + \nu(a \setminus b),$$

$$(\mu \wedge \nu)(a) = \inf_{b \subseteq a} \mu b + \nu(a \setminus b),$$

$$|\nu|(a) = \sup_{b \subseteq a} \nu b - \nu(a \setminus b) = \sup_{b,c \subseteq a} \nu b - \nu c$$

for every $a \in \mathfrak{A}$. Setting

$$\|\nu\| = |\nu|(1) = \sup_{a \in \mathfrak{A}} \nu a - \nu(1 \setminus a),$$

M becomes an L-space.

(b) M_σ and M_τ are projection bands in M, therefore L-spaces in their own right. In particular, $|\nu| \in M_\sigma$ for every $\nu \in M_\sigma$, and $|\nu| \in M_\tau$ for every $\nu \in M_\tau$.

(c) The band projection $P_\sigma : M \to M_\sigma$ is defined by the formula

$$(P_\sigma \nu)(c) = \inf\{\sup_{n \in \mathbb{N}} \nu a_n : \langle a_n \rangle_{n \in \mathbb{N}} \text{ is a non-decreasing sequence with supremum } c\}$$

whenever $c \in \mathfrak{A}$ and $\nu \geq 0$ in M.

(d) The band projection $P_\tau : M \to M_\tau$ is defined by the formula

$$(P_\tau \nu)(c) = \inf\{\sup_{a \in A} \nu a : A \text{ is a non-empty upwards-directed set with supremum } c\}$$

whenever $c \in \mathfrak{A}$ and $\nu \geq 0$ in M.

(e) If $A \subseteq M$ is upwards-directed, then A is bounded above in M iff $\{\nu 1 : \nu \in A\}$ is bounded above in $\mathbb{R}$, and in this case (if $A \neq \emptyset$) $\sup A$ is defined by the formula

$$(\sup A)(a) = \sup_{\nu \in A} \nu a \text{ for every } a \in \mathfrak{A}.$$

(f) Suppose that μ, $\nu \in M$.

(i) The following are equiveridical:

(α) ν belongs to the band in M generated by μ;

(β) for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \leq \epsilon$ whenever $|\mu| a \leq \delta$;

(γ) $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ such that $\lim_{n\to\infty} |\mu|(a_n) = 0$.

(ii) Now suppose that μ, $\nu \geq 0$, and let ν_1, ν_2 be the components of ν in the band generated by μ and its complement. Then

$$\nu_1 c = \sup_{\delta > 0} \inf_{\mu a \leq \delta} \nu(c \setminus a), \quad \nu_2 c = \inf_{\delta > 0} \sup_{a \subseteq c, \mu a \leq \delta} \nu a$$

for every $c \in \mathfrak{A}$.

proof (a) Of course $\mu \vee \nu = \nu + (\mu - \nu)^+$, $\mu \wedge \nu = \nu - (\nu - \mu)^+$, $|\nu| = \nu \vee (-\nu)$ (352D), so the formula of 362Ab gives

$$\begin{aligned}(\mu\vee\nu)(a)&=\nu a+\sup_{b\subseteq a}\mu b-\nu b=\sup_{b\subseteq a}\mu b+\nu(a\setminus b),\\(\mu\wedge\nu)(a)&=\nu a-\sup_{b\subseteq a}\nu b-\mu b=\inf_{b\subseteq a}\mu b+\nu(a\setminus b),\\|\nu|(a)&=\sup_{b\subseteq a}\nu b-\nu(a\setminus b)\le\sup_{b,c\subseteq a}\nu b-\nu c=\sup_{b,c\subseteq a}\nu(b\setminus c)-\nu(c\setminus b)\\&\le\sup_{b,c\subseteq a}|\nu|(b\setminus c)+|\nu|(c\setminus b)=\sup_{b,c\subseteq a}|\nu|(b\triangle c)\le|\nu|(a).\end{aligned}$$

The formula offered for $\|\nu\|$ corresponds exactly to the formula in 362Ab for the norm of the associated member of $S(\mathfrak{A})^\sim$; because $S(\mathfrak{A})^\sim$ is an L-space under its norm, so is M.

(b) By 362Ac-362Ad, M_σ and M_τ may be identified with $S(\mathfrak{A})^\sim_c$ and $S(\mathfrak{A})^\times$, which are bands in $S(\mathfrak{A})^\sim$ (356B), therefore projection bands (353J); so that M_σ and M_τ are projection bands in M, and are L-spaces in their own right (354O).

(c) Take any $\nu\ge 0$ in M. Set

$$\nu_\sigma c=\inf\{\sup_{n\in\mathbb{N}}\nu a_n:\langle a_n\rangle_{n\in\mathbb{N}}\text{ is a non-decreasing sequence with supremum }c\}$$

for every $c\in\mathfrak{A}$. Then of course $0\le\nu_\sigma c\le\nu c$ for every c. The point is that ν_σ is countably additive. **P** Let $\langle c_i\rangle_{i\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$, with supremum c. Then for any $\epsilon>0$ we have non-decreasing sequences $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle a_{in}\rangle_{n\in\mathbb{N}}$, for $i\in\mathbb{N}$, such that

$$\sup_{n\in\mathbb{N}}a_n=c,\quad\sup_{n\in\mathbb{N}}a_{in}=c_i\text{ for }i\in\mathbb{N},$$

$$\sup_{n\in\mathbb{N}}\nu a_n\le\nu_\sigma c+\epsilon,$$

$$\sup_{n\in\mathbb{N}}\nu a_{in}\le\nu_\sigma c_i+2^{-i}\epsilon\text{ for every }i\in\mathbb{N}.$$

Set $b_n=\sup_{i\le n}a_{in}$ for each n; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, and

$$\sup_{n\in\mathbb{N}}b_n=\sup_{i,n\in\mathbb{N}}a_{in}=\sup_{i\in\mathbb{N}}c_i=c,$$

so

$$\begin{aligned}\nu_\sigma c&\le\sup_{n\in\mathbb{N}}\nu b_n=\sup_{n\in\mathbb{N}}\sum_{i=0}^n\nu a_{in}\\&=\sum_{i=0}^\infty\sup_{n\in\mathbb{N}}\nu a_{in}\le\sum_{i=0}^\infty\nu_\sigma c_i+2^{-i}\epsilon=\sum_{i=0}^\infty\nu_\sigma c_i+2\epsilon.\end{aligned}$$

On the other hand, $\langle a_n\cap c_i\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum $c\cap c_i=c_i$ for each i, so $\nu_\sigma c_i\le\sup_{n\in\mathbb{N}}\nu(a_n\cap c_i)$, and

$$\sum_{i=0}^\infty\nu_\sigma c_i\le\sum_{i=0}^\infty\sup_{n\in\mathbb{N}}\nu(a_n\cap c_i)=\sup_{n\in\mathbb{N}}\sum_{i=0}^\infty\nu(a_n\cap c_i)$$

(because $\langle a_n\rangle_{n\in\mathbb{N}}$ is non-decreasing)

$$\le\sup_{n\in\mathbb{N}}\nu a_n$$

(because $\langle c_i\rangle_{i\in\mathbb{N}}$ is disjoint)

$$\le\nu_\sigma c+\epsilon.$$

As ϵ is arbitrary, $\nu_\sigma c=\sum_{i=0}^\infty\nu_\sigma c_i$; as $\langle c_i\rangle_{i\in\mathbb{N}}$ is arbitrary, ν_σ is countably additive. **Q**

Thus $\nu_\sigma\in M_\sigma$. On the other hand, if $\nu'\in M_\sigma$ and $0\le\nu'\le\nu$, then whenever $c\in\mathfrak{A}$ and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum c,

$$\nu'c=\sup_{n\in\mathbb{N}}\nu'a_n\le\sup_{n\in\mathbb{N}}\nu a_n.$$

So we must have $\nu' c \le \nu_\sigma c$. This means that

$$\nu_\sigma = \sup\{\nu' : \nu' \in M_\sigma,\, \nu' \le \nu\} = P_\sigma \nu,$$

as claimed.

(d) The same ideas, with essentially elementary modifications, deal with the completely additive part. Take any $\nu \ge 0$ in M. Set

$$\nu_\tau c = \inf\{\sup_{a\in A} \nu a : A \text{ is a non-empty upwards-directed set with supremum } c\}$$

for every $c \in \mathfrak{A}$. Then of course $0 \le \nu_\tau c \le \nu c$ for every c. The point is that ν_τ is completely additive. **P** Note first that if $c \in \mathfrak{A}$, $\epsilon > 0$ there is a non-empty upwards-directed A, with supremum c, such that $\sup_{a\in A} \nu a \le \nu_\tau c + \epsilon\nu c$; for if $\nu c = 0$ we can take $A = \{c\}$. Now let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$. Then for any $\epsilon > 0$ we have non-empty upwards-directed sets A, A_i, for $i \in I$, such that

$$\sup A = 1, \quad \sup A_i = c_i \text{ for } i \in I, \quad \sup_{a\in A} \nu a \le \nu_\tau 1 + \epsilon\nu 1,$$

$$\sup_{a\in A_i} \nu a \le \nu_\tau c_i + \epsilon\nu c_i \text{ for every } i \in I.$$

Set

$$B = \{\sup_{i\in J} a_i : J \subseteq I \text{ is finite, } a_i \in A_i \text{ for every } i \in J\};$$

then B is non-empty and upwards-directed, and

$$\sup B = \sup(\bigcup_{i\in I} A_i) = 1,$$

so

$$\begin{aligned}\nu_\tau 1 \le \sup_{b\in B} \nu b &= \sup\{\sum_{i\in J} \nu a_i : J \subseteq I \text{ is finite, } a_i \in A_i\ \forall\, i \in J\}\\ &\le \sum_{i\in I} \nu_\tau c_i + \epsilon\nu c_i \le \epsilon\nu 1 + \sum_{i\in I} \nu_\tau c_i.\end{aligned}$$

On the other hand, $A_i' = \{a \cap c_i : a \in A\}$ is a non-empty upwards-directed set with supremum c_i for each i, so $\nu_\tau c_i \le \sup_{a\in A_i'} \nu a$, and

$$\begin{aligned}\sum_{i\in I} \nu_\tau c_i \le \sum_{i\in I} \sup_{a\in A} \nu(a \cap c_i) &= \sup_{a\in A} \sum_{i\in I} \nu(a \cap c_i)\\ &\le \sup_{a\in A} \nu a \le \nu_\tau 1 + \epsilon\nu 1.\end{aligned}$$

As ϵ is arbitrary, $\nu_\tau c = \sum_{i\in I} \nu_\tau c_i$; as $\langle c_i\rangle_{i\in I}$ is arbitrary, ν_τ is completely additive, by 326R. **Q**

Thus $\nu_\tau \in M_\tau$. On the other hand, if $\nu' \in M_\tau$ and $0 \le \nu' \le \nu$, then whenever $c \in \mathfrak{A}$ and A is a non-empty upwards-directed set with supremum c,

$$\nu' c = \sup_{a\in A} \nu' a \le \sup_{a\in A} \nu a$$

(using 326Oc). So we must have $\nu' c \le \nu_\tau c$. This means that

$$\nu_\tau = \sup\{\nu' : \nu' \in M_\tau,\, \nu' \le \nu\} = P_\tau \nu,$$

as claimed.

(e) If A is empty, of course it is bounded above in M, and $\{\nu 1 : \nu \in A\} = \emptyset$ is bounded above in $\mathbb{R}$; so let us suppose that A is not empty. In this case, if $\lambda_0 \in M$ is an upper bound for A, then $\lambda_0 1$ is an upper bound for $\{\nu 1 : \nu \in A\}$. On the other hand, if $\sup_{\nu\in A} \nu 1 = \gamma$ is finite, $\gamma^* = \sup\{\nu a : \nu \in A,\, a \in \mathfrak{A}\}$ is finite. **P** Fix $\nu_0 \in A$. Set $\gamma_1 = \sup_{a\in\mathfrak{A}} |\nu_0 a| < \infty$. Then for any $\nu \in A$ and $a \in \mathfrak{A}$ there is a $\nu' \in A$ such that $\nu_0 \vee \nu \le \nu'$, so that

$$\nu a \le \nu' a = \nu' 1 - \nu'(1 \setminus a) \le \gamma - \nu_0(1 \setminus a) \le \gamma + \gamma_1.$$

So

$$\gamma^* \le \gamma + \gamma_1 < \infty. \text{ } \mathbf{Q}$$

Set $\lambda a = \sup_{\nu\in A} \nu a$ for every $a \in \mathfrak{A}$. Then $\lambda : \mathfrak{A} \to \mathbb{R}$ is additive. **P** If a, $b \in \mathfrak{A}$ are disjoint, then

$$\lambda(a \cup b) = \sup_{\nu\in A} \nu(a \cup b) = \sup_{\nu\in A} \nu a + \nu b = \sup_{\nu\in A} \nu a + \sup_{\nu\in A} \nu b$$

(because A is upwards-directed)

$$= \lambda a + \lambda b. \text{ } \mathbf{Q}$$

Also $\lambda a \le \gamma^*$ for every a, so

$$|\lambda a| = \max(\lambda a, -\lambda a) = \max(\lambda a, \lambda(1 \setminus a) - \lambda 1) \le \gamma^* + |\lambda 1|$$

for every $a \in \mathfrak{A}$, and λ is bounded.

This shows that $\lambda \in M$, so that A is bounded above in M. Of course λ must be actually the least upper bound of A in M.

(f)(i)(α)$\Rightarrow$(β) Suppose that ν belongs to the band in M generated by μ, that is, $|\nu| = \sup_{n\in\mathbb{N}} |\nu| \wedge n|\mu|$ (352Vb). Let $\epsilon > 0$. Then there is an $n \in \mathbb{N}$ such that $|\nu|(1) \le \frac{1}{2}\epsilon + (|\nu| \wedge n|\mu|)(1)$ ((e) above). Set $\delta = \frac{1}{2n+1}\epsilon > 0$. If $|\mu|(a) \le \delta$, then

$$|\nu a| \le |\nu|(a) = (|\nu| \wedge n|\mu|)(a) + (|\nu| - |\nu| \wedge n|\mu|)(a)$$
$$\le n|\mu|(a) + (|\nu| - |\nu| \wedge n|\mu|)(1) \le n\delta + \frac{1}{2}\epsilon \le \epsilon.$$

So (β) is satisfied.

not-(α)$\Rightarrow$not-(β) Suppose that ν does not belong to the band in M generated by $|\mu|$. Then there is a $\nu_1 > 0$ such that $\nu_1 \le |\nu|$ and $\nu_1 \wedge |\mu| = 0$ (353C). For any $\delta > 0$, there is an $a \in \mathfrak{A}$ such that $\nu_1(1 \setminus a) + |\mu|(a) \le \min(\delta, \frac{1}{2}\nu_1 1)$ ((a) above); now $|\mu|(a) \le \delta$ but

$$|\nu|(a) \ge \nu_1 a = \nu_1 1 - \nu_1(1 \setminus a) \ge \nu_1 1 - \tfrac{1}{2}\nu_1 1 = \tfrac{1}{2}\nu_1 1.$$

Thus μ, ν do not satisfy (β) (with $\epsilon = \frac{1}{2}\nu_1 1$).

(β)$\Rightarrow$(γ) is trivial.

(γ)$\Rightarrow$(α) Observe first that if $\langle c_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ such that $\lim_{k\to\infty} |\mu|c_k = 0$, then $\lim_{k\to\infty} \nu^+ c_k = 0$. **P** Let $\epsilon > 0$. Because $\nu^+ \wedge \nu^- = 0$, there is a $b \in \mathfrak{A}$ such that $\nu^+ b + \nu^-(1 \setminus b) \le \epsilon$, by part (a). Now $\langle c_k \setminus b\rangle_{k\in\mathbb{N}}$ is non-increasing and $\lim_{k\to\infty} |\mu|(c_k \setminus b) = 0$, so $\lim_{k\to\infty} \nu(c_k \setminus b) = 0$ and

$$\limsup_{k\to\infty} \nu^+ c_k = \limsup_{k\to\infty} \nu^+(c_k \cap b) + \nu(c_k \setminus b) + \nu^-(c_k \setminus b)$$
$$\le \nu^+ b + \nu^-(1 \setminus b) \le \epsilon.$$

As ϵ is arbitrary, $\lim_{k\to\infty} \nu^+ c_k = 0$. **Q**

? Now suppose, if possible, that ν^+ does not belong to the band generated by μ. Then there is a $\nu_1 > 0$ such that $\nu_1 \le \nu^+$ and $\nu_1 \wedge |\mu| = 0$. Set $\epsilon = \frac{1}{4}\nu_1 1 > 0$. For each $n \in \mathbb{N}$, we can choose $a_n \in \mathfrak{A}$ such that $|\mu|a_n + \nu_1(1 \setminus a_n) \le 2^{-n}\epsilon$, by part (a) again. For $n \ge k$, set $b_{kn} = \sup_{k\le i\le n} a_i$; then

$$|\mu|b_{kn} \le \textstyle\sum_{i=k}^n |\mu|a_i \le 2^{-k+1}\epsilon,$$

and $\langle b_{kn}\rangle_{n\ge k}$ is non-decreasing. Set $\gamma_k = \sup_{n\ge k} \nu_1 b_{kn}$ and choose $m(k) \ge k$ such that $\nu_1 b_{k,m(k)} \ge \gamma_k - 2^{-k}\epsilon$. Setting $b_k = b_{k,m(k)}$, we see that $b_k \cup b_{k+1} = b_{kn}$ where $n = \max(m(k), m(k+1))$, so that

$$\nu_1(b_k \cup b_{k+1}) \le \gamma_k \le \nu_1 b_k + 2^{-k}\epsilon$$

and $\nu_1(b_{k+1} \setminus b_k) \le 2^{-k}\epsilon$. Set $c_k = \inf_{i\le k} b_i$ for each k; then

$$\nu_1(b_{k+1} \setminus c_{k+1}) = \nu_1(b_{k+1} \setminus c_k) \le \nu_1(b_{k+1} \setminus b_k) + \nu_1(b_k \setminus c_k) \le 2^{-k}\epsilon + \nu_1(b_k \setminus c_k)$$

for each k; inducing on k, we see that

$$\nu_1(b_k \setminus c_k) \le \sum_{i=0}^{k-1} 2^{-i}\epsilon \le 2\epsilon$$

for every k. This means that

$$\nu^+ c_k \ge \nu_1 c_k \ge \nu_1 b_k - 2\epsilon \ge \nu_1 a_k - 2\epsilon = \nu_1 1 - \nu_1(1 \setminus a_k) - 2\epsilon \ge 4\epsilon - \epsilon - 2\epsilon = \epsilon$$

for every $k \in \mathbb{N}$. On the other hand, $\langle c_k \rangle_{k\in\mathbb{N}}$ is a non-increasing sequence and

$$|\mu| c_k \le |\mu| b_k \le 2^{-k+1}\epsilon$$

for every k, which contradicts the paragraph just above. **X**

This means that ν^+ must belong to the band generated by μ. Similarly $\nu^- = (-\nu)^+$ belongs to the band generated by μ and $\nu = \nu^+ + \nu^-$ also does.

(ii) Take $c \in \mathfrak{A}$. Set

$$\beta_1 = \sup_{\delta>0} \inf_{\mu a \le \delta} \nu(c \setminus a), \quad \beta_2 = \inf_{\delta>0} \sup_{a \subseteq c, \mu a \le \delta} \nu a.$$

Then

$$\beta_1 = \sup_{\delta>0} \inf_{a \subseteq c, \mu a \le \delta} \nu(c \setminus a) = \nu c - \beta_2.$$

Take any $\epsilon > 0$. Because ν_1 belongs to the band generated by μ, part (i) tells us that there is a $\delta > 0$ such that $\nu_1 a \le \epsilon$ whenever $\mu a \le \delta$. In this case, if $\mu a \le \delta$,

$$\nu(c \setminus a) = \nu c - \nu(c \cap a) \ge \nu c - \epsilon \ge \nu_1 c - \epsilon;$$

thus

$$\beta_1 \ge \inf_{\mu a \le \delta} \nu(c \setminus a) \ge \nu_1 c - \epsilon.$$

As ϵ is arbitrary, $\beta_1 \ge \nu_1 c$. On the other hand, given ϵ, $\delta > 0$, there is an $a \subseteq c$ such that $\mu a + \nu_2(c \setminus a) \le \min(\delta, \epsilon)$, because $\mu \wedge \nu_2 = 0$ (using (a) again). In this case, of course, $\mu a \le \delta$, while

$$\nu a \ge \nu_2 a = \nu_2 c - \nu_2(c \setminus a) \ge \nu_2 c - \epsilon.$$

Thus $\sup_{a \subseteq c, \mu a \le \delta} \nu a \ge \nu_2 c - \epsilon$. As δ is arbitrary, $\beta_2 \ge \nu_2 c - \epsilon$. As ϵ is arbitrary, $\beta_2 \ge \nu_2 c$; but as

$$\beta_1 + \beta_2 = \nu c = \nu_1 c + \nu_2 c,$$

$\beta_i = \nu_i c$ for both i, as claimed.

Remark The L-space norm $\| \ \|$ on M, described in (a) above, is the **total variation norm**.

362C The formula in 362B(f-i) has, I hope, already reminded you of the concept of 'absolutely continuous' additive functional from the Radon-Nikodým theorem (Chapter 23, §327). The expressions in 362Bf are limited by the assumption that μ, like ν, is finite-valued. If we relax this we get an alternative version of some of the same ideas.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and M be the Riesz space of bounded finitely additive real-valued functionals on $\mathfrak{A}$. Write

$$M_{ac} = \{\nu : \nu \in M \text{ is absolutely continuous with respect to } \bar{\mu}\}$$

(see 327A),

$$M_{tc} = \{\nu : \nu \in M \text{ is continuous with respect to the measure-algebra topology on } \mathfrak{A}\},$$

$$M_t = \{\nu : \nu \in M, \, |\nu| 1 = \sup_{\bar{\mu} a < \infty} |\nu| a\}.$$

Then M_{ac}, M_{tc} and M_t are bands in M.

proof (a)(i) It is easy to check that M_{ac} is a linear subspace of M.

(ii) If $\nu \in M_{ac}$, $\nu' \in M$ and $|\nu'| \le |\nu|$ then $\nu' \in M_{ac}$. **P** Given $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \le \frac{1}{2}\epsilon$ whenever $\bar{\mu} a \le \delta$. Now

$$|\nu' a| \le |\nu'|(a) \le |\nu|(a) \le 2 \sup_{c \subseteq a} |\nu c| \le \epsilon$$

(using the formula for $|\nu|$ in 362Ba) whenever $\bar{\mu} a \le \delta$. As ϵ is arbitrary, ν' is absolutely continuous. **Q**

(iii) If $A \subseteq M_{ac}$ is non-empty and upwards-directed and $\nu = \sup A$ in M, then $\nu \in M_{ac}$. **P** Let $\epsilon > 0$. Then there is a $\nu' \in A$ such that $\nu 1 \le \nu' 1 + \frac{1}{2}\epsilon$ (362Be). Now there is a $\delta > 0$ such that $|\nu a| \le \frac{1}{2}\epsilon$ whenever $\bar{\mu} a \le \delta$. If now $\bar{\mu} a \le \delta$,

$$|\nu a| \le |\nu' a| + (\nu - \nu')(a) \le \tfrac{1}{2}\epsilon + (\nu - \nu')(1) \le \epsilon.$$

As ϵ is arbitrary, ν is absolutely continuous with respect to $\bar{\mu}$. **Q**

Putting these together, we see that M_{ac} is a band.

(b)(i) We know that M_{tc} consists just of those $\nu \in M$ which are continuous at 0 (327Bc). Of course this is a linear subspace of M.

(ii) If $\nu \in M_{tc}$, $\nu' \in M$ and $|\nu'| \le |\nu|$ then $|\nu| \in M_{tc}$. **P** Write $\mathfrak{A}^f = \{d : d \in \mathfrak{A}, \bar{\mu} d < \infty\}$. Given $\epsilon > 0$ there are $d \in \mathfrak{A}^f$, $\delta > 0$ such that $|\nu a| \le \frac{1}{2}\epsilon$ whenever $\bar{\mu}(a \cap d) \le \delta$. Now

$$|\nu' a| \le |\nu'|(a) \le |\nu|(a) \le 2 \sup_{c \subseteq a} |\nu c| \le \epsilon$$

whenever $\bar{\mu}(a \cap d) \le \delta$. As ϵ is arbitrary, ν' is continuous at 0 and belongs to M_{tc}. **Q**

(iii) If $A \subseteq M_{tc}$ is non-empty and upwards-directed and $\nu = \sup A$ in M, then $\nu \in M_{tc}$. **P** Let $\epsilon > 0$. Then there is a $\nu' \in A$ such that $\nu 1 \le \nu' 1 + \frac{1}{2}\epsilon$. There are $d \in \mathfrak{A}^f$, $\delta > 0$ such that $|\nu a| \le \frac{1}{2}\epsilon$ whenever $\bar{\mu}(a \cap d) \le \delta$. If now $\bar{\mu}(a \cap d) \le \delta$,

$$|\nu a| \le |\nu' a| + (\nu - \nu')(a) \le \tfrac{1}{2}\epsilon + (\nu - \nu')(1) \le \epsilon.$$

As ϵ is arbitrary, ν is continuous at 0, therefore belongs to M_{tc}. **Q**

Putting these together, we see that M_{tc} is a band.

(c)(i) M_t is a linear subspace of M. **P** Suppose that ν_1, $\nu_2 \in M_t$ and $\alpha \in \mathbb{R}$. Given $\epsilon > 0$, there are a_1, $a_2 \in \mathfrak{A}^f$ such that $|\nu_1|(1 \setminus a_1) \le \frac{\epsilon}{1+|\alpha|}$ and $|\nu_2|(1 \setminus a_2) \le \epsilon$. Set $a = a_1 \cup a_2$; then $\bar{\mu} a < \infty$ and

$$|\nu_1 + \nu_2|(1 \setminus a) \le 2\epsilon, \quad |\alpha \nu_1|(1 \setminus a) \le \epsilon.$$

As ϵ is arbitrary, $\nu_1 + \nu_2$ and $\alpha\nu_1$ belong to M_t; as ν_1, ν_2 and α are arbitrary, M_t is a linear subspace of M. **Q**

(ii) If $\nu \in M_t$, $\nu' \in M$ and $|\nu'| \le |\nu|$ then

$$\inf_{\bar{\mu} a < \infty} |\nu'|(1 \setminus a) \le \inf_{\bar{\mu} a < \infty} |\nu|(1 \setminus a) = 0,$$

so $\nu' \in M_t$. Thus M_t is a solid linear subspace of M.

(iii) If $A \subseteq M_t^+$ is non-empty and upwards-directed and $\nu = \sup A$ is defined in M, then $\nu \in M_t$. **P**

$$|\nu| 1 = \nu 1 = \sup_{\nu' \in A} \nu' 1 = \sup_{\nu' \in A, \bar{\mu} a < \infty} \nu' a = \sup_{\bar{\mu} a < \infty} \nu a.$$

As A is arbitrary, $\nu \in M_t$. **Q** Thus M_t is a band in M.

362D For semi-finite measure algebras, among others, the formula of 362Bd takes a special form.

Proposition Let $\mathfrak{A}$ be a weakly (σ, ∞)-distributive Boolean algebra. Let M be the space of bounded finitely additive functionals on $\mathfrak{A}$, $M_\tau \subseteq M$ the space of completely additive functionals, and $P_\tau : M \to M_\tau$ the band projection, as in 362B. Then for any $\nu \in M^+$ and $c \in \mathfrak{A}$ there is a non-empty upwards-directed set $A \subseteq \mathfrak{A}$ with supremum c such that $(P_\tau \nu)(c) = \sup_{a \in A} \nu a$; that is, the 'inf' in 362Bd can be read as 'min'.

proof By 362Bd, we can find for each n a non-empty upwards-directed A_n, with supremum c, such that $\sup_{a \in A_n} \nu a \le (P_\tau \nu)(c) + 2^{-n}$. Set $B_n = \{c \setminus a : a \in A_n\}$ for each n, so that B_n is downwards-directed and has infimum 0. Because $\mathfrak{A}$ is weakly (σ, ∞)-distributive,

$$B = \{b : \text{for every } n \in \mathbb{N} \text{ there is a } b' \in B_n \text{ such that } b \supseteq b'\}$$

is also a downwards-directed set with infimum 0. Consequently $A = \{c \setminus b : b \in B\}$ is upwards-directed and has supremum c. Moreover, for any $n \in \mathbb{N}$ and $a \in A$, there is an $a' \in A_n$ such that $a \subseteq a'$; so, using 362Bd again and referring to the choice of A_n,

$$(P_\tau \nu)(c) \le \sup_{a \in A} \nu a \le \sup_{a' \in A_n} \nu a' \le (P_\tau \nu)(c) + 2^{-n}.$$

As n is arbitrary, A has the required property.

362E Uniformly integrable sets The spaces $S^\sim$, $S_c^\sim$ and $S^\times$ of 362A, or, if you prefer, the spaces M, M_σ, M_τ, M_{ac}, M_{tc}, M_t of 362B-362C, are all L-spaces, and any serious study of them must involve a discussion of their uniformly integrable (= relatively weakly compact) subsets. The basic work has been done in 356O; I spell out its application in this context.

Theorem Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded finitely additive functionals on $\mathfrak{A}$. Then a norm-bounded set $C \subseteq M$ is uniformly integrable iff $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$.

proof Write S for $S(\mathfrak{A})$ and $\tilde{C}$ for the set $\{f : f \in S^\sim,\ f\chi \in C\}$. Because the map $f \mapsto f\chi$ is a normed Riesz space isomorphism between $S^\sim$ and M, $\tilde{C}$ is uniformly integrable in M iff C is uniformly integrable in $S^\sim$.

(a) Suppose that C is uniformly integrable and that $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$. Then $\langle \chi a_n\rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $S^\sim$, while $\tilde{C}$ is uniformly integrable, so $\lim_{n\to\infty} \sup_{f\in\tilde{C}} |f(\chi a_n)| = 0$, by 356O; but this means that $\lim_{n\to\infty} \sup_{\nu\in C} |\nu a_n| = 0$. Thus the condition is satisfied.

(b) Now suppose that C is not uniformly integrable. By 356O, in the other direction, there is a disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S such that $0 \le u_n \le \chi 1$ for each n and $\limsup_{n\to\infty} \sup_{f\in\tilde{C}} |f(u_n)| > 0$. For each n, take $c_n = [\![u_n > 0]\!]$ (361Eg); then $0 \le u_n \le \chi c_n$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ is disjoint. Now

$$\begin{aligned}\limsup_{n\to\infty} \sup_{\nu\in C} |\nu|(c_n) &= \limsup_{n\to\infty} \sup_{f\in\tilde{C}} |f|(\chi c_n)\\ &\ge \limsup_{n\to\infty} \sup_{f\in\tilde{C}} |f(u_n)| > 0.\end{aligned}$$

So if we choose $\nu_n \in C$ such that $|\nu_n|(c_n) \ge \frac{1}{2}\sup_{\nu\in C} |\nu|(c_n)$, we shall have $\limsup_{n\to\infty} |\nu_n|(c_n) > 0$. Next, for each n, we can find $a_n \subseteq c_n$ such that $|\nu_n a_n| \ge \frac{1}{2}|\nu_n|(c_n)$, so that

$$\limsup_{n\in\mathbb{N}} \sup_{\nu\in C} |\nu a_n| \ge \limsup_{n\to\infty} |\nu_n a_n| > 0.$$

Since $\langle a_n\rangle_{n\in\mathbb{N}}$, like $\langle c_n\rangle_{n\in\mathbb{N}}$, is disjoint, the condition is not satisfied. This completes the proof.

362X Basic exercises >(a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν_1, ν_2 two countably additive functionals on $\mathfrak{A}$. Show that $|\nu_1| \wedge |\nu_2| = 0$ in the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$ iff there is a $c \in \mathfrak{A}$ such that $\nu_1 a = \nu_1(a \cap c)$ and $\nu_2 a = \nu_2(a \setminus c)$ for every $a \in \mathfrak{A}$.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and take M, M_{ac} as in 362C. Show that for any non-negative $\nu \in M$, the component ν_{ac} of ν in M_{ac} is given by the formula

$$\nu_{ac} c = \sup_{\delta>0} \inf_{\bar\mu a \le \delta} \nu(c \setminus a).$$

(c) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and take M, M_t as in 362C. (i) Show that M_t is just the set of those $\nu \in M$ such that $\nu a = \lim_{b\to\mathcal{F}} \nu(a \cap b)$ for every $a \in \mathfrak{A}$, where $\mathcal{F}$ is the filter on $\mathfrak{A}$ generated by the sets $\{b : b \in \mathfrak{A}^f,\ b \supseteq b_0\}$ as b_0 runs over the set $\mathfrak{A}^f$ of elements of $\mathfrak{A}$ of finite measure. (ii) Show that the complementary band $M_t^\perp$ of M_t in M is just the set of those $\nu \in M$ such that $\nu a = 0$ for every $a \in \mathfrak{A}^f$. (iii) Show that for any $\nu \in M$, its component ν_t in M_t is given by the formula $\nu_t a = \lim_{b\to\mathcal{F}} \nu(a \cap b)$ for every $a \in \mathfrak{A}$.

(d) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Write M, M_σ, M_τ, M_{ac}, M_{tc} and M_t as in 362B-362C. Show that (i) $M_\sigma \subseteq M_{ac}$ (ii) $M_{ac} \cap M_t = M_{tc} \subseteq M_\tau$ (iii) if $(\mathfrak{A}, \bar\mu)$ is σ-finite, then $M_\sigma = M_{tc}$.

(e) Let $\mathfrak{A}$ be a Boolean algebra, and M the space of bounded additive functionals on $\mathfrak{A}$. Let us say that a non-zero finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **atomic** if whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$ then at least one of νa, νb is zero. (i) Show that for a non-zero finitely additive functional ν on $\mathfrak{A}$ the following are equiveridical: (α) ν is atomic; (β) $\nu \in M$ and $|\nu|$ is atomic; (γ) $\nu \in M$ and the corresponding linear functional $f_{|\nu|} = |f_\nu| \in S(\mathfrak{A})^\sim$ is a Riesz homomorphism; (δ) there are a multiplicative linear functional $f : S(\mathfrak{A}) \to \mathbb{R}$ and an $\alpha \in \mathbb{R}$ such that $\nu a = \alpha f(\chi a)$ for every $a \in \mathfrak{A}$; (ϵ) $\nu \in M$ and the band in M generated by ν is the set of multiples of ν. (ii) Show that a completely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is atomic iff there are $a \in \mathfrak{A}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ such that a is an atom in $\mathfrak{A}$ and $\nu b = \alpha$ when $a \subseteq b$, 0 when $a \cap b = 0$.

(f) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the properly atomless functionals (definition: 326F) form a band M_c in the Riesz space M of all bounded finitely additive functionals on $\mathfrak{A}$. (ii) Show that the complementary band $M_c^\perp$ consists of just those $\nu \in M$ expressible as a sum $\sum_{i\in I}\nu_i$ of countably many atomic functionals $\nu_i \in M$. (iii) Show that if $\mathfrak{A}$ is purely atomic then a properly atomless completely additive functional on $\mathfrak{A}$ must be 0.

(g) Let X be a set and Σ an algebra of subsets of X. Let M be the Riesz space of bounded finitely additive functionals on Σ, M_τ the space of completely additive functionals and M_p the space of functionals expressible in the form $\nu E = \sum_{x\in E}\alpha_x$ for some absolutely summable family $\langle\alpha_x\rangle_{x\in X}$ of real numbers. (i) Show that M_p is a band in M. (ii) Show that if all singleton subsets of X belong to Σ then $M_p = M_\tau$. (iii) Show that if Σ is a σ-algebra then every member of M_p is countably additive. (iv) Show that if X is a compact zero-dimensional Hausdorff space and Σ is the algebra of open-and-closed subsets of X then the complementary band $M_p^\perp$ of M_p in M is the band M_c of properly atomless functionals described in 362Xf.

(h) Let (X,Σ,μ) be a measure space. Let M be the Riesz space of bounded finitely additive functionals on Σ and M_σ the space of bounded countably additive functionals. Let M_{tc}, M_{ac} be the spaces of truly continuous and bounded absolutely continuous additive functionals as defined in 232A. Show that M_{tc} and M_{ac} are bands in M and that $M_{tc} \subseteq M_\sigma \cap M_{ac}$. Show that if μ is σ-finite then $M_{tc} = M_\sigma \cap M_{ac}$.

(i) Let $\mathfrak{A}$ be a Boolean algebra and M the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$. (i) For any non-empty downwards-directed set $A \subseteq \mathfrak{A}$ set $N_A = \{\nu : \nu \in M, \inf_{a\in A}|\nu|a = 0\}$. Show that N_A is a band in M. (ii) For any non-empty set $\mathcal{A}$ of non-empty downwards-directed sets in $\mathfrak{A}$ set $M_{\mathcal{A}} = \{\nu : \nu \in M, \inf_{a\in A}|\nu|a = 0 \ \forall\ A \in \mathcal{A}\}$. Show that $M_{\mathcal{A}}$ is a band in M. (iii) Explain how to represent as such $M_{\mathcal{A}}$ the bands M_σ, M_τ, M_t, M_{ac}, M_{tc} described in 362B-362C, and also any band generated by a single element of M. (iv) Suppose, in (ii), that $\mathcal{A}$ has the property that for any A, $A' \in \mathcal{A}$ there is a $B \in \mathcal{A}$ such that for every $b \in B$ there are $a \in A$, $a' \in A'$ such that $a \cup a' \subseteq b$. Show that for any non-negative $\nu \in M$, the component ν_1 of ν in $M_{\mathcal{A}}$ is given by the formula $\nu_1 c = \inf_{A\in\mathcal{A}}\sup_{a\in A}\nu(c\setminus a)$, so that the component ν_2 of ν in $M_{\mathcal{A}}^\perp$ is given by the formula $\nu_2 c = \sup_{A\in\mathcal{A}}\inf_{a\in A}\nu(c\cap a)$. (Cf. 356Yb.)

362Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Let $\mathfrak{C}$ be the band algebra of the Riesz space M of bounded finitely additive functionals on $\mathfrak{A}$ (353B). Show that the bands M_σ, M_τ, M_c (362B, 362Xf) generate a subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ with at most six atoms. Give an example in which $\mathfrak{C}_0$ has six atoms. How many atoms can it have if (i) $\mathfrak{A}$ is atomless (ii) $\mathfrak{A}$ is purely atomic (iii) $\mathfrak{A}$ is Dedekind σ-complete?

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Let $\mathfrak{C}$ be the band algebra of the Riesz space M of bounded finitely additive functionals on $\mathfrak{A}$. Show that the bands M_σ, M_τ, M_c, M_{ac}, M_{tc}, M_t (362B, 362C, 362Xf) generate a subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ with at most twelve atoms. Give an example in which $\mathfrak{C}_0$ has twelve atoms. How many atoms can it have if (i) $\mathfrak{A}$ is atomless (ii) $\mathfrak{A}$ is purely atomic (iii) $(\mathfrak{A},\bar\mu)$ is semi-finite (iv) $(\mathfrak{A},\bar\mu)$ is localizable (v) $(\mathfrak{A},\bar\mu)$ is σ-finite (vi) $(\mathfrak{A},\bar\mu)$ is totally finite?

(c) Give an example of a set X, a σ-algebra Σ of subsets of X, and a functional in M_p (as defined in 362Xg) which is not completely additive.

(d) Let U be a Riesz space and f, $g \in U^\sim$. Show that the following are equiveridical: (α) g is in the band in $U^\sim$ generated by f; (β) for every $u \in U^+$, $\epsilon > 0$ there is a $\delta > 0$ such that $|g(v)| \le \epsilon$ whenever $0 \le v \le u$ and $|f|(v) \le \delta$; (γ) $\lim_{n\to\infty} g(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U^+ and $\lim_{n\to\infty}|f|(u_n) = 0$.

(e) Let $\mathfrak{A}$ be a weakly σ-distributive Boolean algebra (316Ye). Show that the 'inf' in the formula for $P_\sigma\nu$ in 362Bc can be replaced by 'min'.

(f) Let $\mathfrak{A}$ be any Boolean algebra and M the space of bounded finitely additive functionals on $\mathfrak{A}$. Let $C \subseteq M$ be such that $\sup_{\nu\in C}|\nu a| < \infty$ for every $a \in \mathfrak{A}$. (i) Suppose that $\sup_{n\in\mathbb{N}}\sup_{\nu\in C}|\nu a_n|$ is finite for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is norm-bounded. (ii) Suppose that $\lim_{n\to\infty}\sup_{\nu\in C}|\nu a_n| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is uniformly integrable.

(g) Let $\mathfrak{A}$ be a Boolean algebra and M_τ the space of completely additive functionals on $\mathfrak{A}$. Let $C \subseteq M_\tau$ be such that $\sup_{\nu\in C}|\nu a| < \infty$ for every atom $a \in \mathfrak{A}$. (i) Suppose that $\sup_{n\in\mathbb{N}}\sup_{\nu\in C}|\nu a_n|$ is finite for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is norm-bounded. (ii) Suppose that $\lim_{n\to\infty}\sup_{\nu\in C}|\nu a_n| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. Show that C is uniformly integrable.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle \nu_n\rangle_{n\in\mathbb{N}}$ a sequence of countably additive real-valued functionals on $\mathfrak{A}$. Suppose that $\nu a = \lim_{n\to\infty}\nu_n a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$. Show that ν is countably additive and that $\{\nu_n : n \in \mathbb{N}\}$ is uniformly integrable. (*Hint*: 246Yg.) Show that if every ν_n is completely additive, so is ν.

(i) Let $\mathfrak{A}$ be a Boolean algebra, M the Riesz space of bounded finitely additive functionals on $\mathfrak{A}$, and $M_c \subseteq M$ the band of properly atomless functionals (362Xf). Show that for a non-negative $\nu \in M$ the component ν_c of ν in M_c is given by the formula

$$\nu_c a = \inf_{\delta>0}\sup\{\sum_{i=0}^n \nu a_i : a_0,\dots,a_n \subseteq a \text{ are disjoint, } \nu a_i \le \delta \text{ for every } i\}$$

for each $a \in \mathfrak{A}$.

(j) Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded additive real-valued functionals on $\mathfrak{A}$. Show that the complexification of M, as defined in 354Yl, can be identified with the Banach space of bounded additive functionals $\nu : \mathfrak{A} \to \mathbb{C}$, writing

$$\|\nu\| = \sup\{\sum_{i=0}^n |\nu a_i| : a_0,\dots,a_n \text{ are disjoint elements of } \mathfrak{A}\}$$

for such ν.

(k) Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded additive real-valued functionals on $\mathfrak{A}$. Suppose that M_0 is a norm-closed linear subspace of M and that $a \mapsto \nu(a\cap c) : \mathfrak{A} \to \mathbb{R}$ belongs to M_0 whenever $\nu \in M_0$ and $c \in \mathfrak{A}$. Show that M_0 is a band in M. (*Hint*: 436L.)

362 Notes and comments The Boolean algebras most immediately important in measure theory are of course σ-algebras of measurable sets and their quotient measure algebras. It is therefore natural to begin any investigation by concentrating on Dedekind σ-complete algebras. Nevertheless, in this section and the last (and in §326), I have gone to some trouble not to specialize to σ-complete algebras except when necessary. Partly this is just force of habit, but partly it is because I wish to lay a foundation for a further step forward: the investigation of the ways in which additive functionals on general Boolean algebras reflect the concepts of measure theory, and indeed can generate them. Some of the results in this direction can be surprising. I do not think it obvious that the condition (γ) in 362B(f-i), for instance, is sufficient in the absence of any hypothesis of Dedekind σ-completeness or countable additivity.

Given a Boolean algebra $\mathfrak{A}$ with the associated Riesz space $M \cong S(\mathfrak{A})^\sim$ of bounded additive functionals on $\mathfrak{A}$, we now have a substantial list of bands in M: M_σ, M_τ, M_c (362Xf), and for a measure algebra the further bands M_{ac}, M_{tc} and M_t; for an algebra of sets we also have M_p (362Xg). These bands can be used to generate finite subalgebras of the band algebra of M (362Ya-362Yb), and for any such finite subalgebra we have a corresponding decomposition of M as a direct sum of the bands which are the atoms of the subalgebra (352Tb). This decomposition of M can be regarded as a recipe for decomposing its members into finite sums of functionals with special properties. What I called the 'Lebesgue decomposition' in 232I is just such a recipe. In that context I had a measure space (X,Σ,μ) and was looking at the countably additive functionals from Σ to $\mathbb{R}$, that is, at M_σ in the language of this section, and the bands involved in the decomposition were M_p, M_{ac} and M_{tc}. But I hope that it will be plain that these ideas can be refined indefinitely, as we refine the classification of additive functionals. At each stage, of course, the exact enumeration of the subalgebra of bands generated by the classification (as in 362Ya-362Yb) is a necessary check that we have understood the relationships between the classes we have described.

These decompositions are of such importance that it is worth examining the corresponding band projections. I give formulae for the action of band projections on (non-negative) functionals in 362Bc, 362Bd, 362B(f-ii), 362Xb, 362Xc(iii), 362Xi(iv) and 362Yi. Of course these are readily adapted to give formulae for the projections onto the complementary bands, as in 362Bf and 362Xi.

If we have an algebra of sets, the completely additive functionals are (usually) of relatively minor importance; in the standard examples, they correspond to functionals defined as weighted sums of point masses (362Xg(ii)). The point is that measure algebras $\mathfrak{A}$ appear as quotients of σ-algebras Σ of sets by σ-ideals $\mathcal{I}$; consequently the countably additive functionals on $\mathfrak{A}$ correspond exactly to the countably additive functionals on Σ which are zero on $\mathcal{I}$; but the canonical homomorphism from Σ to $\mathfrak{A}$ is hardly ever order-continuous, so completely additive functionals on $\mathfrak{A}$ rarely correspond to completely additive functionals on Σ. On the other hand, when we are looking at countably additive functionals on Σ, we have to consider the possibility that they are singular in the sense that they are carried on some member of $\mathcal{I}$; in the measure algebra context this possibility disappears, and we can often be sure that every countably additive functional is absolutely continuous, as in 327Bb.

For any Boolean algebra $\mathfrak{A}$, we can regard it as the algebra of open-and-closed subsets of its Stone space Z; the points of Z correspond to Boolean homomorphisms from $\mathfrak{A}$ to $\{0,1\}$, which are the normalised 'atomic elements' in the space of additive functionals on $\mathfrak{A}$ (362Xe, 362Xg(iv)). It is the case that all non-negative additive functionals on a Boolean algebra $\mathfrak{A}$ can be represented by appropriate measures on its Stone space (see 416Q in Volume 4), but I prefer to hold this result back until it can take its place among other theorems on representing functionals by measures and integrals.

It is one of the leitmotivs of this chapter, that Boolean algebras and Riesz spaces are Siamese twins; again and again, matching results are proved by the application of identical ideas. A typical example is the pair 362B(f-i) and 362Yd. Many of us have been tempted to try to describe something which would provide a common generalization of Boolean algebras and Riesz spaces (and lattice-ordered groups). I have not yet seen any such structure which was worth the trouble. Most of the time, in this chapter, I shall be using ideas from the general theory of Riesz spaces to suggest and illuminate questions in measure theory; but if you pursue this subject you will surely find that intuitions often come to you first in the context of Boolean algebras, and the applications to Riesz spaces are secondary.

In 362E I give a condition for uniform integrability in terms of disjoint sequences, following the pattern established in 246G and repeated in 354R and 356O. The condition of 362E assumes that the set is norm-bounded; but if you have 246G to hand, you will see that it can be done with weaker assumptions involving atoms, as in 362Yf-362Yg.

I mention once again the Banach-Ulam problem: if $\mathfrak{A}$ is Dedekind complete, can $S(\mathfrak{A})^\sim_c$ be different from $S(\mathfrak{A})^\times$? This is obviously equivalent to the form given in the notes to §326 above. See 363S below.

Version of 4.3.08

363 L^∞

In this section I set out to describe an abstract construction for L^∞ spaces on arbitrary Boolean algebras, corresponding to the $L^\infty(\mu)$ spaces of §243. I begin with the definition of $L^\infty(\mathfrak{A})$ (363A) and elementary facts concerning its own structure and the embedding $S(\mathfrak{A}) \subsetneq L^\infty(\mathfrak{A})$ (363B-363D). I give the basic universal mapping theorems which define the Banach lattice structure of L^∞ (363E) and a description of the action of Boolean homomorphisms on L^∞ spaces (363F-363G) before discussing the representation of $L^\infty(\Sigma)$ and $L^\infty(\Sigma/\mathcal{I})$ for σ-algebras Σ and ideals $\mathcal{I}$ of sets (363H). This leads at once to the identification of $L^\infty(\mu)$, as defined in Volume 2, with $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ (363I). Like $S(\mathfrak{A})$, $L^\infty(\mathfrak{A})$ determines the algebra $\mathfrak{A}$ (363J). I briefly discuss the dual spaces of L^∞; they correspond exactly to the duals of S described in §362 (363K). Linear functionals on L^∞ can for some purposes be treated as 'integrals' (363L).

In the second half of the section I present some of the theory of Dedekind complete and σ-complete algebras. First, $L^\infty(\mathfrak{A})$ is Dedekind (σ-)complete iff $\mathfrak{A}$ is (363M). The spaces $L^\infty(\mathfrak{A})$, for Dedekind σ-complete $\mathfrak{A}$, are precisely the Dedekind σ-complete Riesz spaces with order unit (363N-363P). The spaces $L^\infty(\mathfrak{A})$, for Dedekind complete $\mathfrak{A}$, are precisely the normed spaces which may be put in place of $\mathbb{R}$ in the Hahn-Banach theorem (363R). Finally, I mention some equivalent forms of the Banach-Ulam problem (363S).

363A Definition Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z. I will write $L^\infty(\mathfrak{A})$ for the space $C(Z) = C_b(Z)$ of continuous real-valued functions from Z to $\mathbb{R}$, endowed with the linear structure, order structure, norm and multiplication of $C(Z) = C_b(Z)$. (Recall that because Z is compact (311I), $\{u(z) : z \in Z\}$ is bounded for every $u \in L^\infty(\mathfrak{A}) = C(Z)$ (2A3N(b-iii)), that is, $C(Z) = C_b(Z)$. Of course if $\mathfrak{A} = \{0\}$, so that $Z = \emptyset$, then $C(Z)$ has just one member, the empty function.)

363B Theorem Let $\mathfrak{A}$ be any Boolean algebra; write L^∞ for $L^\infty(\mathfrak{A})$.

(a) L^∞ is an M-space; its standard order unit is the constant function taking the value 1 at each point; in particular, L^∞ is a Banach lattice with a Fatou norm and the Levi property.

(b) L^∞ is a commutative Banach algebra and an f-algebra.

(c) If $u \in L^\infty$ then $u \geq 0$ iff there is a $v \in L^\infty$ such that $u = v \times v$.

proof (a) See 354Hb and 354J.

(b)-(c) are obvious from the definitions of Banach algebra (2A4J) and f-algebra (352W) and the ordering of $L^\infty = C(Z)$.

363C Proposition Let $\mathfrak{A}$ be any Boolean algebra. Then $S(\mathfrak{A})$ is a norm-dense, order-dense Riesz subspace of $L^\infty(\mathfrak{A})$, closed under multiplication.

proof Let Z be the Stone space of $\mathfrak{A}$. Using the definition of $S = S(\mathfrak{A})$ set out in 361D, it is obvious that S is a linear subspace of $L^\infty = L^\infty(\mathfrak{A}) = C(Z)$ closed under multiplication. Because S, like L^∞, is a Riesz subspace of $\mathbb{R}^Z$ (361Ee), S is a Riesz subspace of L^∞. By the Stone-Weierstrass theorem (in either of the forms given in 281A and 281E), S is norm-dense in L^∞. Consequently it is order-dense (354I).

363D Proposition Let $\mathfrak{A}$ be a Boolean algebra. If we regard $\chi a \in S(\mathfrak{A})$ (361D) as a member of $L^\infty(\mathfrak{A})$ for each $a \in \mathfrak{A}$, then $\chi : \mathfrak{A} \to L^\infty(\mathfrak{A})$ is additive, order-preserving, order-continuous and a lattice homomorphism.

proof Because the embedding $S = S(\mathfrak{A}) \subsetneqq L^\infty(\mathfrak{A}) = L^\infty$ is a Riesz homomorphism, $\chi : \mathfrak{A} \to L^\infty$ is additive and a lattice homomorphism (361F-361G). Because S is order-dense in L^∞ (363C), the embedding $S \subsetneqq L^\infty$ is order-continuous (352Nb), so $\chi : \mathfrak{A} \to L^\infty$ is order-continuous (361Gb).

363E Theorem Let $\mathfrak{A}$ be a Boolean algebra, and U a Banach space. Let $\nu : \mathfrak{A} \to U$ be a bounded additive function.

(a) There is a unique bounded linear operator $T : L^\infty(\mathfrak{A}) \to U$ such that $T\chi = \nu$; in this case $\|T\| = \sup_{a,b\in\mathfrak{A}} \|\nu a - \nu b\|$.

(b) If U is a Banach lattice then T is positive iff ν is non-negative; and in this case T is order-continuous iff ν is order-continuous, and sequentially order-continuous iff ν is sequentially order-continuous.

(c) If U is a Banach lattice then T is a Riesz homomorphism iff ν is a lattice homomorphism iff $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$.

proof Write $S = S(\mathfrak{A})$, $L^\infty = L^\infty(\mathfrak{A})$.

(a) By 361I there is a unique bounded linear operator $T_0 : S \to U$ such that $T_0\chi = \nu$, and $\|T_0\| = \sup\{\|\nu a - \nu b\| : a,\ b \in \mathfrak{A}\}$. But because U is a Banach space and S is dense in L^∞, T_0 has a unique extension to a bounded linear operator $T : L^\infty \to U$ with the same norm (2A4I).

(b)(i) If T is positive then T_0 is positive so ν is non-negative, by 361Ga.

(ii) If ν is non-negative then T_0 is positive, by 361Ga in the other direction. But if $u \in L^{\infty +}$ and $\epsilon > 0$, then by 354I there is a $v \in S^+$ such that $\|u - v\|_\infty \leq \epsilon$; now $\|Tu - Tv\| \leq \epsilon\|T\|$. But $Tv = T_0 v$ belongs to the positive cone U^+ of U. As ϵ is arbitrary, Tu belongs to the closure of U^+, which is U^+ (354Bc). As u is arbitrary, T is positive.

(iii) Now suppose that ν is order-continuous as well as non-negative, and that $A \subseteq L^\infty$ is a non-empty downwards-directed set with infimum 0. Set

$$B = \{v : v \in S, \text{ there is some } u \in A \text{ such that } v \geq u\}.$$

Then B is downwards-directed (indeed, $v_1 \wedge v_2 \in B$ for every $v_1,\ v_2 \in B$), and $u = \inf\{v : v \in B,\ u \leq v\}$ for every $u \in A$ (354I again), so B has the same lower bounds as A and $\inf B = 0$ in L^∞ and in S. But we know from 361Gb that T_0 is order-continuous, while any lower bound for $\{Tu : u \in A\}$ in U must also be a lower bound for $\{Tv : v \in B\} = \{T_0v : v \in B\}$, so $\inf_{u\in A} Tu = \inf_{v\in B} T_0v = 0$ in U. As A is arbitrary, T is order-continuous (351Ga).

(iv) Suppose next that ν is only sequentially order-continuous, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in L^∞ with infimum 0. For each n, k choose $w_{nk} \in S$ such that $u_n \le w_{nk}$ and $\|w_{nk} - u_n\|_\infty \le 2^{-k}$ (354I once more), and set $w'_n = \inf_{j,k\le n} w_{jk}$ for each n. Then $\langle w'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S, and any lower bound of $\{w'_n : n \in \mathbb{N}\}$ is also a lower bound of $\{u_n : n \in \mathbb{N}\}$, so $0 = \inf_{n\in\mathbb{N}} w'_n$ in S and L^∞. Since $T_0 : S \to U$ is sequentially order-continuous (361Gb),

$$\inf_{n\in\mathbb{N}} Tu_n \le \inf_{n\in\mathbb{N}} Tw'_n = \inf_{n\in\mathbb{N}} T_0w'_n = 0$$

in U. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, T is sequentially order-continuous.

(v) On the other hand, if T is order-continuous or sequentially order-continuous, so is $\nu = T\chi$, because χ is order-continuous (363D).

(c) We know that $T_0 : S \to U$ is a Riesz homomorphism iff ν is a lattice homomorphism iff $\nu a \wedge \nu b = 0$ whenever $a \cap b = 0$, by 361Gc. But T_0 is a Riesz homomorphism iff T is. **P** If T is a Riesz homomorphism so is T_0, because the embedding $S \subsetneqq L^\infty$ is a Riesz homomorphism. On the other hand, if T_0 is a Riesz homomorphism, then the functions $u \mapsto u^+ \mapsto T(u^+)$, $u \mapsto Tu \mapsto (Tu)^+$ are continuous (by 354Bb) and agree on S, so agree on L^∞, and T is a Riesz homomorphism, by 352G. **Q**

363F Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) There is an associated multiplicative Riesz homomorphism $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$, of norm at most 1, defined by saying that $T_\pi(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}$.

(b) For any $u \in L^\infty(\mathfrak{A})$, there is a $u' \in L^\infty(\mathfrak{A})$ such that $T_\pi u = T_\pi u'$ and $\|u'\|_\infty = \|T_\pi u\|_\infty \le \|u\|_\infty$.

(c)(i) The kernel of T_π is the norm-closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : a \in \mathfrak{A},\ \pi a = 0\}$.

(ii) The set of values of T_π is the norm-closed linear subspace of $L^\infty(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$.

(d) T_π is surjective iff π is surjective, and in this case $\|v\|_\infty = \min\{\|u\|_\infty : T_\pi u = v\}$ for every $v \in L^\infty(\mathfrak{B})$.

(e) T_π is injective iff π is injective, and in this case $\|T_\pi u\|_\infty = \|u\|_\infty$ for every $u \in L^\infty(\mathfrak{A})$.

(f) T_π is order-continuous, or sequentially order-continuous, iff π is.

(g) If $\mathfrak{C}$ is another Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ is another Boolean homomorphism, then $T_{\theta\pi} = T_\theta T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{C})$.

proof Let Z and W be the Stone spaces of $\mathfrak{A}$ and $\mathfrak{B}$. By 312Q there is a continuous function $\phi : W \to Z$ such that $\widehat{\pi a} = \phi^{-1}[\widehat{a}]$ for every $a \in \mathfrak{A}$, where $\widehat{a}$ is the open-and-closed subset of Z corresponding to $a \in \mathfrak{A}$. Write T for T_π.

(a) For $u \in L^\infty(\mathfrak{A}) = C(Z)$, set $Tu = u\phi : W \to \mathbb{R}$. Then $Tu \in C(W) = L^\infty(\mathfrak{B})$. It is obvious, or at any rate very easy to check, that $T : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$ is linear, multiplicative, a Riesz homomorphism and of norm 1 unless $\mathfrak{B} = \{0\}$, $W = \emptyset$. If $a \in \mathfrak{A}$, then

$$T(\chi a) = (\chi a)\phi = (\chi\widehat{a})\phi = \chi(\phi^{-1}[\widehat{a}]) = \chi(\pi a),$$

identifying $\chi a \in L^\infty(\mathfrak{A})$ with the indicator function $\chi\widehat{a} : Z \to \{0,1\}$ of the set $\widehat{a}$. Of course $T_\pi = T$ is the only continuous linear operator with these properties, by 363Ea.

(b) Set $\alpha = \|Tu\|_\infty$, $u'(z) = \mathrm{med}(-\alpha, u(z), \alpha)$ for $z \in Z$; that is, $u' = \mathrm{med}(-\alpha e, u, \alpha e)$ in $L^\infty(\mathfrak{A})$, where e is the standard order unit of $L^\infty(\mathfrak{A})$. Then Te is the standard order unit of $L^\infty(\mathfrak{B})$, so

$$Tu' = \mathrm{med}(-\alpha Te, Tu, \alpha Te) = Tu$$

(because T is a lattice homomorphism, see 3A1Ic), while

$$\|u'\|_\infty \le \alpha = \|Tu\|_\infty = \|Tu'\|_\infty \le \|u'\|_\infty \le \|u\|_\infty.$$

(c)(i) Let U be the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : \pi a = 0\}$, and U_0 the kernel of T. Because T is continuous and linear, U_0 is a closed linear subspace, and $T(\chi a) = \chi 0 = 0$ whenever $\pi a = 0$; so $U \subseteq U_0$. Now take any $u \in U_0$ and $\epsilon > 0$. Then $T(u^+) = (Tu)^+ = 0$, so $u^+ \in U_0$. By 354I there is a $u' \in S(\mathfrak{A})$ such that $0 \le u' \le u^+$ and $\|u^+ - u'\|_\infty \le \epsilon$. Now $0 \le Tu' \le Tu^+ = 0$, so $Tu' = 0$. Express

u' as $\sum_{i=0}^{n} \alpha_i \chi a_i$ where $\alpha_i \geq 0$ for each i. For each i, $\alpha_i \chi(\pi a_i) = T(\alpha_i \chi a_i) = 0$, so $\pi a_i = 0$ or $\alpha_i = 0$; in either case $\alpha_i \chi a_i \in U$. Consequently $u' \in U$. As ϵ is arbitrary and U is closed, $u^+ \in U$. Similarly, $u^- = (-u)^+ \in U$ and $u = u^+ - u^- \in U$. As u is arbitrary, $U_0 \subseteq U$ and $U_0 = U$.

(ii) Let V be the closed linear subspace of $L^\infty(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$, and $V_0 = T[L^\infty(\mathfrak{A})]$. Then $T[S(\mathfrak{A})] \subseteq V$, so

$$V_0 = T[\overline{S(\mathfrak{A})}] \subseteq \overline{T[S(\mathfrak{A})]} \subseteq \overline{V} = V.$$

On the other hand, V_0 is a closed linear subspace in $L^\infty(\mathfrak{B})$. **P** It is a linear subspace because T is a linear operator. To see that it is closed, take any $v \in \overline{V}_0$. Then there is a sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in V_0 such that $\|v - v_n\|_\infty \leq 2^{-n}$ for every $n \in \mathbb{N}$. Choose $u_n \in L^\infty(\mathfrak{A})$ such that $Tu_0 = v_0$, while $Tu_n = v_n - v_{n-1}$ and $\|u_n\|_\infty = \|v_n - v_{n-1}\|_\infty$ for $n \geq 1$ (using (b) above). Then

$$\sum_{n=1}^{\infty} \|u_n\|_\infty \leq \sum_{n=1}^{\infty} \|v - v_n\|_\infty + \|v - v_{n-1}\|_\infty$$

is finite, so $u = \lim_{n\to\infty} \sum_{i=0}^{n} u_i$ is defined in the Banach space $L^\infty(\mathfrak{A})$, and

$$Tu = \lim_{n\to\infty} \sum_{i=0}^{n} Tu_i = \lim_{n\to\infty} v_n = v.$$

As v is arbitrary, V_0 is closed. **Q** Since $\chi(\pi a) = T(\chi a) \in V_0$ for every $a \in \mathfrak{A}$, $V \subseteq V_0$ and $V = V_0$, as required.

(d) If π is surjective, then T is surjective, by (c-ii). If T is surjective and $b \in \mathfrak{B}$, then there is a $u \in L^\infty(\mathfrak{A})$ such that $Tu = \chi b$. Now there is a $u' \in S(\mathfrak{A})$ such that $\|u - u'\|_\infty \leq \frac{1}{3}$, so that $\|Tu' - \chi b\|_\infty \leq \frac{1}{3}$. Taking $a \in \mathfrak{A}$ such that $\{z : u'(z) \geq \frac{1}{2}\} = \widehat{a}$, we must have $\pi a = b$, since

$$\widehat{b} = \{w : (Tu')(w) \geq \tfrac{1}{2}\} = \phi^{-1}[\widehat{a}] = \widehat{\pi a}.$$

As b is arbitrary, π is surjective.

Now (b) tells us that in this case $\|v\|_\infty = \min\{\|u\|_\infty : Tu = v\}$ for every $v \in L^\infty(\mathfrak{B})$.

(e) By (c-i), T is injective iff π is injective. In this case, for any $u \in L^\infty(\mathfrak{A})$,

$$\|Tu\|_\infty = \|T|u|\|_\infty$$

(because T is a Riesz homomorphism)

$$\geq \sup\{\|Tu'\|_\infty : u' \in S(\mathfrak{A}),\, u' \leq |u|\}$$
$$= \sup\{\|u'\|_\infty : u' \in S(\mathfrak{A}),\, u' \leq |u|\}$$

(by 361Jd)

$$= \|u\|_\infty$$

(by 354I)

$$\geq \|Tu\|_\infty,$$

and $\|Tu\|_\infty = \|u\|_\infty$.

(f) If T is (sequentially) order-continuous then $\pi = T\chi$ is (sequentially) order-continuous, by 363D. If π is (sequentially) order-continuous then $\chi\pi : \mathfrak{A} \to L^\infty(\mathfrak{B})$ is (sequentially) order-continuous, so T is (sequentially) order-continuous, by 363Eb.

(g) This is elementary, in view of the uniqueness of $T_{\theta\pi}$.

363G Corollary Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, then $L^\infty(\mathfrak{C})$ can be identified, as Banach lattice and as Banach algebra, with the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi c : c \in \mathfrak{C}\}$.

(b) If $\mathcal{I}$ is an ideal of $\mathfrak{A}$, then $L^\infty(\mathfrak{A}/\mathcal{I})$ can be identified, as Banach lattice and as Banach algebra, with the quotient space $L^\infty(\mathfrak{A})/V$, where V is the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi a : a \in \mathcal{I}\}$.

proof Apply 363Fc-363Fd to the identity map from $\mathfrak{C}$ to $\mathfrak{A}$ and the canonical map from $\mathfrak{A}$ onto $\mathfrak{A}/\mathcal{I}$.

363H Representations of $L^\infty(\mathfrak{A})$ Much of the importance of the concept of $L^\infty(\mathfrak{A})$ arises from the way it is naturally represented in the contexts in which the most familiar Boolean algebras appear.

Proposition Let X be a set and Σ an algebra of subsets of X.

(a) Write $S(\Sigma)$ for the linear subspace of $\ell^\infty(X)$ generated by the indicator functions of members of Σ, and $\mathcal{L}^\infty$ for its $\|\,\|_\infty$-closure in $\ell^\infty(X)$.

(i) $L^\infty(\Sigma)$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty$; if $E \in \Sigma$, then χE, defined in $L^\infty(\Sigma)$ as in 361D, can be identified with the indicator function of E regarded as a subset of X.

(ii) A bounded function $f : X \to \mathbb{R}$ belongs to $\mathcal{L}^\infty$ iff whenever $\alpha < \beta$ in $\mathbb{R}$ there is an $E \in \Sigma$ such that $\{x : f(x) > \beta\} \subseteq E \subseteq \{x : f(x) > \alpha\}$.

(iii) In particular, $L^\infty(\mathcal{P}X)$ can be identified with $\ell^\infty(X)$.

(b) Now suppose that Σ is a σ-algebra of subsets of X.

(i) $\mathcal{L}^\infty$ is just the set of bounded Σ-measurable real-valued functions on X.

(ii) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ is a surjective sequentially order-continuous Boolean homomorphism with kernel $\mathcal{I}$, then $L^\infty(\mathfrak{A})$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty/\mathcal{W}$, where $\mathcal{W} = \{f : f \in \mathcal{L}^\infty, \{x : f(x) \neq 0\} \in \mathcal{I}\}$ is a solid linear subspace and closed ideal of $\mathcal{L}^\infty$. For $f \in \mathcal{L}^\infty$,

$$\|f^\bullet\|_\infty = \min\{\alpha : \alpha \geq 0, \{x : |f(x)| > \alpha\} \in \mathcal{I}\}.$$

(iii) In particular, if $\mathcal{I}$ is any σ-ideal of Σ and $E \mapsto E^\bullet$ is the canonical homomorphism from Σ onto $\mathfrak{A} = \Sigma/\mathcal{I}$, then we have an identification of $L^\infty(\mathfrak{A})$ with a quotient of $\mathcal{L}^\infty$, and for any $E \in \Sigma$ we can identify $\chi(E^\bullet) \in L^\infty(\mathfrak{A})$ with the equivalence class $(\chi E)^\bullet \in \mathcal{L}^\infty/\mathcal{W}$ of the indicator function χE.

proof (a)(i) By 361L, $S(\Sigma)$, as described here, can be identified with $S(\Sigma)$ as defined in 361D. Because the normed space $\ell^\infty(X)$ is complete, $\mathcal{L}^\infty$ can be identified with the normed space completion of $S(\Sigma)$ for $\|\,\|_\infty$; but 363C shows that the same is true of $L^\infty(\Sigma)$. Thus we have a canonical Banach space isomorphism between $\mathcal{L}^\infty$ and $L^\infty(\Sigma)$. Because multiplication and the lattice operations are $\|\,\|_\infty$-continuous, both in $\mathcal{L}^\infty$ and in $L^\infty(\Sigma)$, this isomorphism is multiplicative and order-preserving, that is, identifies $\mathcal{L}^\infty$ with $L^\infty(\Sigma)$ as Banach algebra and Banach lattice. In the language of 363E, $\mathcal{L}^\infty$ is the image of $L^\infty(\Sigma)$ in $\ell^\infty(X)$ under the operator associated with the additive function $E \mapsto \chi E : \Sigma \to \ell^\infty(X)$.

(ii)($\boldsymbol{\alpha}$) If $f \in \mathcal{L}^\infty$ and $\alpha < \beta$ in $\mathbb{R}$, let $g \in S(\Sigma)$ be such that $\|f - g\|_\infty \leq \frac{1}{2}(\beta - \alpha)$. Set $E = \{x : g(x) > \frac{1}{2}(\alpha + \beta)\}$; by 361G or otherwise, $E \in \Sigma$, and $\{x : f(x) > \beta\} \subseteq E \subseteq \{x : f(x) > \alpha\}$.

($\boldsymbol{\beta}$) If f satisfies the condition, take any $\epsilon > 0$. Let $n \in \mathbb{N}$ be such that $\|f\|_\infty < n\epsilon$. For $-n \leq i \leq n$, let $E_i \in \Sigma$ be such that $\{x : f(x) > (i+1)\epsilon\} \subseteq E_i \subseteq \{x : f(x) > i\epsilon\}$. Set $g(x) = \epsilon \sum_{i=-n}^{n} \chi E_i - \epsilon n$ for $x \in X$; then $g \in S(\Sigma)$ and $\|f - g\|_\infty \leq \epsilon$. As ϵ is arbitrary, $f \in \mathcal{L}^\infty$.

(iii) Now (ii) shows that if $\Sigma = \mathcal{P}X$ we shall have $\mathcal{L}^\infty = \ell^\infty(X)$ and $L^\infty(\mathcal{P}X)$ becomes identified with $\ell^\infty(X)$.

(b)(i) If Σ is a σ-algebra and $f : X \to \mathbb{R}$ is bounded then

$$\begin{aligned} f \text{ is } \Sigma\text{-measurable} &\iff \{x : f(x) > \alpha\} \in \Sigma \text{ for every } \alpha \in \mathbb{R} \\ &\iff \text{ whenever } \alpha \in \mathbb{R},\, n \in \mathbb{N} \text{ there is an } E \in \Sigma \\ &\qquad \text{such that } \{x : f(x) > \alpha + 2^{-n}\} \subseteq E \subseteq \{x : f(x) > \alpha\} \\ &\iff \text{ whenever } \beta > \alpha \text{ there is an } E \in \Sigma \\ &\qquad \text{such that } \{x : f(x) > \beta\} \subseteq E \subseteq \{x : f(x) > \alpha\} \\ &\iff f \in \mathcal{L}^\infty \end{aligned}$$

by (a-ii) above.

(ii)($\boldsymbol{\alpha}$) By 363F, we have a multiplicative Riesz homomorphism $T = T_\pi$ from $L^\infty(\Sigma)$ to $L^\infty(\mathfrak{A})$ which is surjective (363Fd) and has kernel the closed linear subspace W of $L^\infty(\Sigma)$ generated by $\{\chi E : E \in \mathcal{I}\}$. Now under the identification described in (a), W corresponds to $\mathcal{W}$. **P** $\mathcal{W}$ is a linear subspace of $\mathcal{L}^\infty$ because

$$\{x : (f+g)(x) \neq 0\} \subseteq \{x : f(x) \neq 0\} \cup \{x : g(x) \neq 0\} \in \mathcal{I},$$

$$\{x : (\alpha f)(x) \neq 0\} \subseteq \{x : f(x) \neq 0\} \in \mathcal{I}$$

whenever f, $g \in \mathcal{W}$ and $\alpha \in \mathbb{R}$. If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}$ converging to $f \in \mathcal{L}^\infty$, then

$$\{x : f(x) \neq 0\} \subseteq \bigcup_{n\in\mathbb{N}}\{x : f_n(x) \neq 0\} \in \mathcal{I},$$

so $f \in \mathcal{W}$. Thus $\mathcal{W}$ is a closed linear subspace of $\mathcal{L}^\infty$. If $E \in \mathcal{I}$, then χE, taken in $S(\Sigma)$ or $L^\infty(\Sigma)$, corresponds to the function $\chi E : X \to \{0, 1\}$, which belongs to $\mathcal{W}$; so that W must correspond to the closed linear span in $\mathcal{L}^\infty$ of such indicator functions, which is a subspace of $\mathcal{W}$. On the other hand, if $f \in \mathcal{W}$ and $\epsilon > 0$, set

$$E_n = \{x : n\epsilon < f(x) \leq (n + 1)\epsilon\}, \quad E'_n = \{x : -(n + 1)\epsilon \leq f(x) < -n\epsilon\}$$

for $n \in \mathbb{N}$; all these belong to $\mathcal{I}$, so $g = \epsilon\sum_{n=0}^{\infty}(\chi E_n - \chi E'_n) \in \mathcal{W}$ corresponds to a member of W, while $\|f - g\|_\infty \leq \epsilon$. As W is closed, f also must correspond to some member of W. As f is arbitrary, W and $\mathcal{W}$ match exactly. **Q**

(β) Because T is a multiplicative Riesz homomorphism, $L^\infty(\mathfrak{A}) \cong L^\infty(\Sigma)/W$ is matched canonically, in its linear, order and multiplicative structures, with $\mathcal{L}^\infty/\mathcal{W}$. We know also that

$$\|v\|_\infty = \min\{\|u\|_\infty : u \in L^\infty(\Sigma),\, Tu = v\}$$

for every $v \in L^\infty(\mathfrak{A})$ (363Fd), that is, that the norm of $L^\infty(\mathfrak{A})$ corresponds to the quotient norm on $L^\infty(\Sigma)/W$.

As for the given formula for the norm, take any $f \in \mathcal{L}^\infty$. There is a $g \in \mathcal{L}^\infty$ such that $Tf = Tg$ and $\|Tf\|_\infty = \|g\|_\infty$. (Here I am treating T as an operator from $\mathcal{L}^\infty$ onto $L^\infty(\mathfrak{A})$.) In this case

$$\{x : |f(x)| > \|Tf\|_\infty\} \subseteq \{x : f(x) \neq g(x)\} \in \mathcal{I}.$$

On the other hand, if $\alpha \geq 0$ and $\{x : |f(x)| > \alpha\} \in \mathcal{I}$, and we set $h = \mathrm{med}(-\alpha\chi X, f, \alpha\chi X)$, then $Th = Tf$, so $\|Tf\|_\infty \leq \|h\|_\infty \leq \alpha$.

(iii) Put (a-i) and (ii) just above together.

363I Corollary Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$. Then $L^\infty(\mu)$ can be identified, as Banach lattice and Banach algebra, with $L^\infty(\mathfrak{A})$; the identification matches $(\chi E)^\bullet \in L^\infty(\mu)$ with $\chi(E^\bullet) \in L^\infty(\mathfrak{A})$, for every $E \in \Sigma$.

Remark The space I called $\mathcal{L}^\infty(\mu)$ in Chapter 24 is not strictly speaking the space $\mathcal{L}^\infty \cong L^\infty(\Sigma)$ of 363H; I took $\mathcal{L}^\infty(\mu) \subseteq \mathcal{L}^0(\mu)$ to be the set of essentially bounded, virtually measurable functions defined almost everywhere in X, and in general this is larger. But, as remarked in the notes to §243, $L^\infty(\mu)$ can equally well be regarded as a quotient of what I there called $\mathcal{L}^\infty_\Sigma$, which is the $\mathcal{L}^\infty$ above, because every function in $\mathcal{L}^\infty(\mu)$ is equal almost everywhere to some member of $\mathcal{L}^\infty_\Sigma$.

363J Recovering the algebra $\mathfrak{A}$: Proposition Let $\mathfrak{A}$ be a Boolean algebra. For $a \in \mathfrak{A}$ write V_a for the solid linear subspace of $L^\infty(\mathfrak{A})$ generated by χa. Then $a \mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in $L^\infty(\mathfrak{A})$.

proof The proof is nearly identical to that of 361K. If $a \in \mathfrak{A}$, $u \in V_a$ and $v \in V_{1\setminus a}$, then $|u| \wedge |v| = 0$ because $\chi a \wedge \chi(1 \setminus a) = 0$; and if $w \in L^\infty(\mathfrak{A})$ then

$$w = (w \times \chi a) + (w \times \chi(1 \setminus a)) \in V_a + V_{1\setminus a}$$

because $|w\times\chi a| \leq \|w\|_\infty\chi a$ and $|w\times\chi(1 \setminus a)| \leq \|w\|_\infty\chi(1 \setminus a)$. So V_a and $V_{1\setminus a}$ are complementary projection bands in $L^\infty = L^\infty(\mathfrak{A})$. Next, if $U \subseteq L^\infty$ is a projection band, then $\chi 1$ is expressible as $u + v$ where $u \in U$, $v \in U^\perp$; thinking of L^∞ as the space of continuous real-valued functions on the Stone space Z of $\mathfrak{A}$, u and v must be the indicator functions of complementary subsets E, F of Z, which must be open-and-closed, so that $E = \widehat{a}$, $F = \widehat{1 \setminus a}$. In this case $V_a \subseteq U$ and $V_{1\setminus a} \subseteq U^\perp$, so U must be V_a precisely. Thus $a \mapsto V_a$ is surjective. Finally, just as in 361K, $a \subseteq b \iff V_a \subseteq V_b$, so we have a Boolean isomorphism.

363K Dual spaces of L^∞ The questions treated in §362 yield nothing new in the present context. I spell out the details.

Proposition Let $\mathfrak{A}$ be a Boolean algebra. Let M, M_σ and M_τ be the L-spaces of bounded finitely additive functionals, bounded countably additive functionals and completely additive functionals on $\mathfrak{A}$. Then the embedding $S(\mathfrak{A}) \subsetneqq L^\infty(\mathfrak{A})$ induces Riesz space isomorphisms between $S(\mathfrak{A})^\sim \cong M$ and $L^\infty(\mathfrak{A})^\sim = L^\infty(\mathfrak{A})^*$, $S(\mathfrak{A})^\sim_c \cong M_\sigma$ and $L^\infty(\mathfrak{A})^\sim_c$, and $S(\mathfrak{A})^\times \cong M_\tau$ and $L^\infty(\mathfrak{A})^\times$.

proof Write $S = S(\mathfrak{A})$, $L^\infty = L^\infty(\mathfrak{A})$.

(a) For the identifications $S^\sim \cong M$, $S^\sim_c \cong M_\sigma$ and $S^\times \cong M_\tau$ see 362A.

(b) $L^{\infty*} = L^{\infty\sim}$ either because L^∞ is a Banach lattice (356Dc) or because L^∞ has an order-unit norm, so that a linear functional on L^∞ is order-bounded iff it is bounded on the unit ball.

(c) If f is a positive linear functional on L^∞, then $f\restriction S$ is a positive linear functional. Because S is order-dense in L^∞ (363C), the embedding is order-continuous (352Nb); so if f is (sequentially) order-continuous, so is $f\restriction S$. Accordingly the restriction operator $f \mapsto f\restriction S$ gives maps from $L^{\infty\sim}$ to $S^\sim$, $(L^\infty)^\sim_c$ to $S^\sim_c$ and $L^{\infty\times}$ to $S^\times$. If $f \in L^{\infty\sim}$ and $f\restriction S \ge 0$, then $f(u^+) \ge 0$ for every $u \in S$ and therefore for every $u \in L^\infty$, and $f \ge 0$; so all these restriction maps are injective positive linear operators.

(d) I need to show that they are surjective.

(i) If $g \in S^\sim$, then g is bounded on the unit ball $\{u : u \in S, \|u\|_\infty \le 1\}$, so has an extension to a continuous linear $f : L^\infty \to \mathbb{R}$ (2A4I); thus $S^\sim = \{f\restriction S : f \in L^{\infty\sim}\}$. This means that $f \mapsto f\restriction S$ is actually a Riesz space isomorphism between $L^{\infty\sim}$ and $S^\sim$. In particular, $|f|\restriction S = |f\restriction S|$ for any $f \in L^{\infty\sim}$.

(ii) If $f : L^\infty \to \mathbb{R}$ is a positive linear operator and $f\restriction S \in S^\sim_c$, let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence in L^∞ with infimum 0. For each n, $k \in \mathbb{N}$ there is a $v_{nk} \in S$ such that $u_n \le v_{nk} \le u_n + 2^{-k}e$, where e is the standard order unit of L^∞ (354I, as usual); set $w_n = \inf_{i,k\le n} v_{ik}$; then $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in S with infimum 0, so

$$0 \le \inf_{n\in\mathbb{N}} f(u_n) \le \inf_{n\in\mathbb{N}} f(w_n) = 0.$$

As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $f \in (L^\infty)^\sim_c$. Consequently, for general $f \in L^{\infty\sim}$,

$$f \in (L^\infty)^\sim_c \iff |f| \in (L^\infty)^\sim_c \iff |f\restriction S| \in S^\sim_c \iff f\restriction S \in S^\sim_c,$$

and the map $f \mapsto f\restriction S : (L^\infty)^\sim_c \to S^\sim_c$ is a Riesz space isomorphism.

(iii) Similarly, if $f \in L^{\infty\sim}$ is non-negative and $f\restriction S \in S^\times$, then whenever $A \subseteq L^\infty$ is non-empty, downwards-directed and has infimum 0, $B = \{w : w \in S, \exists\, u \in A, w \ge u\}$ has infimum 0, so $\inf_{u\in A} f(u) \le \inf_{w\in B} f(w) \le 0$ and $f \in L^{\infty\times}$. As in (ii), it follows that $f \mapsto f\restriction S$ is a surjection from $L^{\infty\times}$ onto $S^\times$.

***363L Integration with respect to a finitely additive functional (a)** If $\mathfrak{A}$ is a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ is a bounded additive functional, then by 363K we have a corresponding functional $f_\nu \in L^\infty(\mathfrak{A})^*$ defined by saying that $f_\nu(\chi a) = \nu a$ for every $a \in \mathfrak{A}$. There are contexts in which it is convenient, and even helpful, to use the formula $⨍ u\,d\nu$ in place of $f_\nu(u)$ for $u \in L^\infty = L^\infty(\mathfrak{A})$. When doing so, we must of course remember that we may have lost some of the standard properties of 'integration'. But enough of our intuitions (including, for instance, the idea of stochastic independence) remain valid to make the formula a guide to interesting ideas.

(b) Let M be the L-space of bounded finitely additive functionals on $\mathfrak{A}$ (362B). Then we have a function $(u,\nu) \mapsto ⨍ u\,d\nu : L^\infty \times M \to \mathbb{R}$. Now this map is bilinear. **P** For μ, $\nu \in M$, u, $v \in L^\infty$ and $\alpha \in \mathbb{R}$,

$$⨍ u + v\,d\nu = ⨍ u\,d\nu + ⨍ v\,d\nu, \quad ⨍ \alpha u\,d\nu = \alpha ⨍ u\,d\nu$$

just because f_ν is linear. On the other side, we have

$$(f_\mu + f_\nu)(\chi a) = f_\mu(\chi a) + f_\nu(\chi a) = \mu a + \nu a = (\mu+\nu)(a) = f_{\mu+\nu}(\chi a)$$

for every $a \in \mathfrak{A}$, so that $f_\mu + f_\nu$ and $f_{\mu+\nu}$ must agree on $S(\mathfrak{A})$ and therefore on L^∞. But this means that $⨍ u\,d(\mu+\nu) = ⨍ u\,d\mu + ⨍ u\,d\nu$. Similarly, $⨍ u\,d(\alpha\mu) = \alpha ⨍ u\,d\mu$. **Q**

(c) If ν is non-negative, we have $⨍ u\,d\nu \ge 0$ whenever $u \ge 0$, as in part (c) of the proof of 363K. Consequently, for any $\nu \in M$ and $u \in L^\infty$,

$$\begin{aligned}|\int u\,d\nu| &= |\int u^+\,d\nu^+ - \int u^-d\nu^+ - \int u^+d\nu^- + \int u^-d\nu^-|\\ &\le \int u^+\,d\nu^+ + \int u^-d\nu^+ + \int u^+d\nu^- + \int u^-d\nu^-\\ &= \int |u|d|\nu| \le \int \|u\|_\infty\chi 1\,d|\nu| = \|u\|_\infty|\nu|(1) = \|u\|_\infty\|\nu\|.\end{aligned}$$

So $(u,\nu)\mapsto\int u\,d\nu$ has norm (as defined in 253Ab) at most 1. If $\mathfrak{A}\ne 0$, the norm is exactly 1. (For this we need to know that there is a $\nu\in M^+$ such that $\nu 1=1$. Take any z in the Stone space of $\mathfrak{A}$ and set $\nu a=1$ if $z\in\widehat{a}$, 0 otherwise.)

(d) We do not have any result corresponding to B.Levi's theorem in this language, because (even if ν is non-negative and countably additive) there is no reason to suppose that $\sup_{n\in\mathbb{N}}u_n$ is defined in L^∞ just because $\sup_{n\in\mathbb{N}}\int u_n d\nu$ is finite. But if ν is countably additive and $\mathfrak{A}$ is Dedekind σ-complete, we have something corresponding to Lebesgue's Dominated Convergence Theorem (363Yg).

(e) One formula which we can imitate in the present context is that of 252O, where the ordinary integral is represented in the form

$$\int f d\mu = \int_0^\infty \mu\{x: f(x)\ge t\}dt$$

for non-negative f. In the context of general Boolean algebras, we cannot directly represent the set $[\![f\ge t]\!]=\{x: f(x)\ge t\}$ (though in the next section I will show that in Dedekind σ-complete Boolean algebras there is an effective expression of this idea). But what we can say is the following. If $\mathfrak{A}$ is any Boolean algebra, and $\nu:\mathfrak{A}\to[0,\infty[$ is a non-negative additive functional, and $u\in L^\infty(\mathfrak{A})^+$, then

$$\int u\,d\nu = \int_0^\infty \sup\{\nu a: t\chi a\le u\}dt,$$

where the right-hand integral is taken with respect to Lebesgue measure. **P** (i) For $t\ge 0$ set $h(t)=\sup\{\nu a: t\chi a\le u\}$. Then h is non-increasing and zero for $t>\|u\|_\infty$, so $\int_0^\infty h(t)dt$ is defined in $\mathbb{R}$. If we set $h_n(t)=h(2^{-n}(k+1))$ whenever k, $n\in\mathbb{N}$ and $2^{-n}k\le t<2^{-n}(k+1)$, then $\langle h_n(t)\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence which converges to $h(t)$ whenever h is continuous at t, which is almost everywhere (222A, or otherwise); so $\int_0^\infty h(t)dt=\lim_{n\to\infty}\int_0^\infty h_n(t)dt$. Next, given $n\in\mathbb{N}$ and $\epsilon>0$, we can choose for each $k\le k^*=\lfloor 2^n\|u\|_\infty\rfloor$ an a_k such that $2^{-n}(k+1)\chi a_k\le u$ and $\nu a_k\ge h(2^{-n}(k+1))-\epsilon$. In this case $\sum_{k=0}^{k^*}2^{-n}\chi a_k\le u$, so

$$\begin{aligned}\int_0^\infty h_n(t)dt &= 2^{-n}\sum_{k=0}^{k^*}h(2^{-n}(k+1)) \le \|u\|_\infty\epsilon + 2^{-n}\sum_{k=0}^{k^*}\nu a_k\\ &= \|u\|_\infty\epsilon + \int\sum_{k=0}^{k^*}2^{-n}\chi a_k d\nu \le \|u\|_\infty\epsilon + \int u\,d\nu.\end{aligned}$$

As n and ϵ are arbitrary, $\int_0^\infty h(t)dt\le\int u\,d\nu$. (ii) In the other direction, there is for any $\epsilon>0$ a $v\in S(\mathfrak{A})$ such that $v\le u\le v+\epsilon\chi 1$. If we express v as $\sum_{j=0}^m\gamma_j\chi c_j$ where $c_0\supseteq\ldots\supseteq c_m$ and $\gamma_j\ge 0$ for every j (361Ec), then we shall have $h(t)\ge\nu c_k$ whenever $t\le\sum_{j=0}^k\gamma_j$, so

$$\int_0^\infty h(t)dt \ge \sum_{k=0}^m\gamma_k\nu c_k = \int v\,d\nu \ge \int u\,d\nu - \epsilon\nu 1.$$

As ϵ is arbitrary, $\int_0^\infty h(t)dt\ge\int u\,d\nu$ and the two 'integrals' are equal. **Q**

(f) The formula $\int d\nu$ is especially natural when $\mathfrak{A}$ is an algebra of sets, so that L^∞ can be directly interpreted as a space of functions (363Ha); better still, when $\mathfrak{A}$ is actually a σ-algebra of subsets of a set X, L^∞ can be identified with the space of bounded $\mathfrak{A}$-measurable functions on X, as in 363Hb. So in such contexts I may write $\int g\,d\nu$ or even $\int g(x)\nu(dx)$ when $g:X\to\mathbb{R}$ is bounded and $\mathfrak{A}$-measurable, and $\nu:\mathfrak{A}\to\mathbb{R}$ is a bounded additive functional. But I will try to take care to signal any such deviation from the normal principle that the symbol $\int$ refers to the sequentially order-continuous integral defined in §122 with the minor modifications introduced in §§133 and 135.

363M Now I come to a fundamental fact underlying a number of theorems in both this volume and the last.

Theorem Let $\mathfrak{A}$ be a Boolean algebra.

(a) $\mathfrak{A}$ is Dedekind σ-complete iff $L^\infty(\mathfrak{A})$ is Dedekind σ-complete.

(b) $\mathfrak{A}$ is Dedekind complete iff $L^\infty(\mathfrak{A})$ is Dedekind complete.

proof (a)(i) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. By 314M, we may identify $\mathfrak{A}$ with a quotient $\Sigma/\mathcal{M}$, where $\mathcal{M}$ is the ideal of meager subsets of the Stone space Z of $\mathfrak{A}$, and $\Sigma = \{E\triangle A : E \in \mathcal{E},\, A \in \mathcal{M}\}$, writing $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$ for the algebra of open-and-closed subsets of Z. By 363Hb, $L^\infty = L^\infty(\mathfrak{A})$ can be identified with $\mathcal{L}^\infty/\mathcal{V}$, where $\mathcal{L}^\infty$ is the space of bounded Σ-measurable functions from Z to $\mathbb{R}$, and $\mathcal{V}$ is the space of functions zero except on a member of $\mathcal{M}$.

Now suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^∞ with an upper bound $u \in L^\infty$. Express u_n, u as $f_n^\bullet$, $f^\bullet$ where f_n, $f \in \mathcal{L}^\infty$. Set $g(z) = \sup_{n\in\mathbb{N}} \min(f_n(z), f(z))$ for every $z \in Z$; then $g \in \mathcal{L}^\infty$ (121F), so we have a corresponding member $v = g^\bullet$ of L^∞. For each $n \in \mathbb{N}$, $u \ge u_n$ so $(f_n - f)^+ \in \mathcal{V}$,

$$\{z : f_n(z) > g(z)\} \subseteq \{z : f_n(z) > f(z)\} \in \mathcal{M}$$

and $v \ge u_n$. If $w \in L^\infty$ and $w \ge u_n$ for every n, then express w as $h^\bullet$ where $h \in \mathcal{L}^\infty$; we have $(f_n - h)^+ \in \mathcal{V}$ for every n, so

$$\{z : g(z) > h(z)\} \subseteq \bigcup_{n\in\mathbb{N}}\{z : f_n(z) > h(z)\} \in \mathcal{M}$$

because $\mathcal{M}$ is a σ-ideal, and $(g - h)^+ \in \mathcal{V}$, so $w \ge v$. Thus $v = \sup_{n\in\mathbb{N}} u_n$ in L^∞. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, L^∞ is Dedekind σ-complete (using 353H).

(ii) Now suppose that L^∞ is Dedekind σ-complete, and that A is a countable non-empty set in $\mathfrak{A}$. In this case $\{\chi a : a \in A\}$ has a least upper bound u in L^∞. Take $v \in S(\mathfrak{A})$ such that $0 \le v \le u$ and $\|u - v\|_\infty \le \frac{1}{3}$; set $b = [\![v > \frac{1}{3}]\!]$, as defined in 361Eg. If $a \in A$, then $\|(\chi a - v)^+\|_\infty \le \|u - v\|_\infty \le \frac{1}{3}$, so $\frac{2}{3}\chi a \le v$ and $a \subseteq b$. If $c \in \mathfrak{A}$ is any upper bound for A, then $v \le u \le \chi c$ so $b \subseteq c$. Thus $b = \sup A$ in $\mathfrak{A}$. As A is arbitrary, $\mathfrak{A}$ is Dedekind σ-complete.

(b)(i) For the second half of this theorem I use an argument which depends on joining the representation described in (a-i) above with the original definition of L^∞ in 363A. The point is that $C(Z) \subseteq \mathcal{L}^\infty$, and for any $f \in C(Z) = L^\infty(\mathfrak{A})$ its equivalence class $f^\bullet$ in $\mathcal{L}^\infty/\mathcal{V}$ corresponds to f itself. **P** Perhaps it will help to give a name T to the canonical isomorphism from $\mathcal{L}^\infty/\mathcal{V}$ to L^∞. Then $V = \{f : Tf^\bullet = f\}$ is a closed linear subspace of $C(Z)$, because $f \mapsto f^\bullet$ and T are continuous linear operators. But if $a \in \mathfrak{A}$, then $(\widehat{a})^\bullet$, the equivalence class of $\widehat{a} \in \Sigma$ in $\Sigma/\mathcal{M}$, corresponds to a (see the proof of 314M), so $(\chi\widehat{a})^\bullet \in \mathcal{L}^\infty/\mathcal{V}$ corresponds to χa; that is, $T(\chi\widehat{a})^\bullet = \chi\widehat{a}$, if we identify $\chi a \in L^\infty$ with $\chi\widehat{a} : Z \to \{0, 1\}$. So V contains $\chi\widehat{a}$ for every $a \in \mathfrak{A}$; because V is a linear subspace, $S(\mathfrak{A}) \subseteq V$; because V is closed, $L^\infty \subseteq V$. **Q**

For a general $f \in \mathcal{L}^\infty$, $g = Tf^\bullet$ must be the unique member of $C(Z)$ such that $g^\bullet = f^\bullet$, that is, such that $\{z : g(z) \ne f(z)\}$ is meager.

(ii) Suppose now that $\mathfrak{A}$ is actually Dedekind complete. In this case Z is extremally disconnected (314S). Consequently every open set belongs to Σ. **P** If G is open, then $\overline{G}$ is open-and-closed; but $A = \overline{G}\setminus G$ is a closed set with empty interior, so is meager, and $G = \overline{G}\triangle A \in \Sigma$. **Q**

Let $A \subseteq L^\infty = C(Z)$ be any non-empty set with an upper bound in $C(Z)$. For each $z \in Z$ set $g(z) = \sup_{u\in A} u(z)$. Then

$$G_\alpha = \{z : g(z) > \alpha\} = \bigcup_{u\in A}\{z : u(z) > \alpha\}$$

is open for every $\alpha \in \mathbb{R}$ (that is, g is lower semi-continuous). Thus $G_\alpha \in \Sigma$ for every α, so $g \in \mathcal{L}^\infty$, and $v = Tg^\bullet$ is defined in $C(Z)$. For any $u \in A$, $g \ge u$ in $\mathcal{L}^\infty$, so

$$v = Tg^\bullet \ge Tu^\bullet = u$$

in L^∞; thus v is an upper bound for A in L^∞. On the other hand, if w is any upper bound for A in $L^\infty = C(Z)$, then surely $w(z) \ge u(z)$ for every $z \in Z$ and $u \in A$, so $w \ge g$ and

$$w = Tw^\bullet \ge Tg^\bullet = v.$$

This means that v is the least upper bound of A. As A is arbitrary, L^∞ is Dedekind complete.

(iii) Finally, if L^∞ is Dedekind complete, then the argument of (a-ii), applied to arbitrary non-empty subsets A of $\mathfrak{A}$, shows that $\mathfrak{A}$ also is Dedekind complete.

363N Much of the importance of L^∞ spaces in the theory of Riesz spaces arises from the next result.

Proposition Let U be a Dedekind σ-complete Riesz space with an order unit. Then U is isomorphic, as Riesz space, to $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of projection bands in U.

proof (a) By 353N, U is isomorphic to a norm-dense Riesz subspace of $C(X)$ for some compact Hausdorff space X; for the rest of this argument, therefore, we may suppose that U actually is such a subspace.

(b) $U = C(X)$. **P** If $g \in C(X)$ then by 354I there are sequences $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle f'_n\rangle_{n\in\mathbb{N}}$ in U such that $f_n \le g \le g_n$ and $\|g_n - f_n\|_\infty \le 2^{-n}$ for every n. Now $\{f_n : n \in \mathbb{N}\}$ has a least upper bound f in U; since we must have $f_n \le f \le g_n$ for every n, $f = g$ and $g \in U$. **Q**

(c) Next, X is zero-dimensional. **P** Suppose that $G \subseteq X$ is open and $x \in G$. Then there is an open set G_1 such that $x \in G_1 \subseteq \overline{G}_1 \subseteq G$ (3A3Bb). There is an $f \in C(X)$ such that $0 \le f \le \chi G_1$ and $f(x) > 0$ (also by 3A3Bb); write H for $\{y : f(y) > 0\}$. Set $g = \sup_{n\in\mathbb{N}}(nf \wedge \chi X)$, the supremum being taken in $U = C(X)$. For each $y \in H$, we must have $g(y) \ge \min(1, nf(y))$ for every n, so that $g(y) = 1$. On the other hand, if $y \in X \setminus \overline{H}$, there is an $h \in C(X)$ such that $h(y) > 0$ and $0 \le h \le \chi(X \setminus \overline{H})$; now $h \wedge f = 0$ so $h \wedge g = 0$ and $g(y) = 0$. Thus $\chi H \le g \le \chi\overline{H}$. The set $\{y : g(y) \in \{0,1\}\}$ is closed and includes $H \cup (X \setminus \overline{H})$ so must be the whole of X; thus $G_2 = \{y : g(y) > \frac{1}{2}\} = \{y : g(y) \ge \frac{1}{2}\}$ is open-and-closed, and we have

$$x \in H \subseteq G_2 \subseteq \overline{H} \subseteq \overline{G}_1 \subseteq G.$$

As x, G are arbitrary, the set of open-and-closed subsets of X is a base for the topology of X, and X is zero-dimensional. **Q**

(d) We can therefore identify X with the Stone space of its algebra $\mathcal{E}$ of open-and-closed sets (311J). But in this case 363A immediately identifies $U = C(X)$ with $L^\infty(\mathcal{E})$. By 363J, $\mathcal{E}$ is isomorphic to $\mathfrak{A}$, so $U \cong L^\infty(\mathfrak{A})$.

Remark Note that in part (c) of the argument above, we have to take care over the interpretation of 'sup'. In the space of all real-valued functions on X, the supremum of $\{nf \wedge \chi X : n \in \mathbb{N}\}$ is just χH. But g is supposed to be the least *continuous* function greater than or equal to $nf \wedge \chi X$ for every n, and is therefore likely to be strictly greater than χH, even though sandwiched between χH and $\chi\overline{H}$.

363O Corollary Let U be a Dedekind σ-complete M-space. Then U is isomorphic, as Banach lattice, to $L^\infty(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of projection bands of U.

proof This is merely the special case of 363N in which U is known from the start to be complete under an order-unit norm.

363P Corollary Let U be any Dedekind σ-complete Riesz space and $e \in U^+$. Then the solid linear subspace U_e of U generated by e is isomorphic, as Riesz space, to $L^\infty(\mathfrak{A})$ for some Dedekind σ-complete Boolean algebra $\mathfrak{A}$; and if U is Dedekind complete, so is $\mathfrak{A}$.

proof Because U is Dedekind σ-complete, so is U_e (353K(a-i)). Apply 363N to U_e to see that $U_e \cong L^\infty(\mathfrak{A})$ for some $\mathfrak{A}$. Because U_e is Dedekind σ-complete, so is $\mathfrak{A}$, by 363Ma; while if U is Dedekind complete, so are U_e and $\mathfrak{A}$, by 353K(b-i) and 363Mb.

363Q The next theorem will be a striking characterization of the Dedekind complete L^∞ spaces as normed spaces. As a warming-up exercise I give a much simpler result concerning their nature as Banach lattices.

Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Then for any Banach lattice U, a linear operator $T : U \to L^\infty = L^\infty(\mathfrak{A})$ is continuous iff it is order-bounded, and in this case $\|T\| = \||T|\|$, where the modulus $|T|$ is taken in $\mathsf{L}^\sim(U; L^\infty)$.

proof It is generally true that order-bounded operators between Banach lattices are continuous (355C). If $T : U \to L^\infty$ is continuous, then for any $w \in U^+$

$$|u| \le w \Longrightarrow \|u\| \le \|w\| \Longrightarrow \|Tu\|_\infty \le \|T\|\|w\| \Longrightarrow |Tu| \le \|T\|\|w\|ee,$$

where e is the standard order unit of L^∞. So T is order-bounded. As L^∞ is Dedekind complete (363Mb), $|T|$ is defined in $\mathrm{L}^\sim(U; L^\infty)$ (355Ea). For any $w \in U$,

$$|T||w| = \sup\{|Tu| : |u| \le |w|\} \le \|T\|\|w\|e,$$

so $\||T|(w)\| \le \|T\|\|w\|$; accordingly $\||T|\| \le \|T\|$. On the other hand, of course,

$$|Tw| \le |T||w| \le \||T|\|\|w\|e$$

for every $w \in U$, so $\|T\| \le \||T|\|$ and the two norms are equal.

Remark Of course what is happening here is that the spaces $L^\infty(\mathfrak{A})$, for Dedekind complete $\mathfrak{A}$, are just the Dedekind complete M-spaces; this is an elementary consequence of 363N and 363M.

363R Now for something much deeper.

Theorem Let U be a normed space over $\mathbb{R}$. Then the following are equiveridical:

(i) there is a Dedekind complete Boolean algebra $\mathfrak{A}$ such that U is isomorphic, as normed space, to $L^\infty(\mathfrak{A})$;

(ii) whenever V is a normed space, V_0 a linear subspace of V, and $T_0 : V_0 \to U$ is a bounded linear operator, there is an extension of T_0 to a bounded linear operator $T : V \to U$ with $\|T\| = \|T_0\|$.

proof For the purposes of the argument below, let us say that a normed space U satisfying the condition (ii) has the 'Hahn-Banach property'.

Part A: (i)⇒(ii) I have to show that $L^\infty(\mathfrak{A})$ has the Hahn-Banach property for every Dedekind complete Boolean algebra $\mathfrak{A}$. Let V be a normed space, V_0 a linear subspace of V, and $T_0 : V_0 \to L^\infty = L^\infty(\mathfrak{A})$ a bounded linear operator. Set $\gamma = \|T_0\|$.

Let $\mathfrak{P}$ be the set of all functions T such that $\operatorname{dom} T$ is a linear subspace of V including V_0 and $T : \operatorname{dom} T \to U$ is a bounded linear operator extending T_0 and with norm at most γ. Order $\mathfrak{P}$ by saying that $T_1 \le T_2$ if T_2 extends T_1. Then any non-empty totally ordered subset $\mathfrak{Q}$ of $\mathfrak{P}$ has an upper bound in $\mathfrak{P}$. **P** Set $\operatorname{dom} T = \bigcup\{\operatorname{dom} T_1 : T_1 \in \mathfrak{Q}\}$, $Tv = T_1 v$ whenever $T_1 \in \mathfrak{Q}$ and $v \in \operatorname{dom} T_1$; it is elementary to check that $T \in \mathfrak{P}$, so that T is an upper bound for $\mathfrak{Q}$ in $\mathfrak{P}$. **Q**

By Zorn's Lemma, $\mathfrak{P}$ has a maximal element $\tilde{T}$. Now $\operatorname{dom} \tilde{T} = V$. **P?** Suppose, if possible, otherwise. Write $\tilde{V} = \operatorname{dom} \tilde{T}$ and take any $\tilde{v} \in V \setminus \tilde{V}$; let V_1 be the linear span of $\tilde{V} \cup \{\tilde{v}\}$, that is, $\{v + \alpha\tilde{v} : v \in \tilde{V}, \alpha \in \mathbb{R}\}$.

If $v_1, v_2 \in \tilde{V}$ then, writing e for the standard order unit of L^∞,

$$\begin{aligned}\tilde{T}v_1 + \tilde{T}v_2 = \tilde{T}(v_1 + v_2) &\le \|\tilde{T}(v_1 + v_2)\|_\infty e \\ &\le \gamma\|v_1 + v_2\|e \le \gamma\|v_1 - \tilde{v}\|e + \gamma\|v_2 + \tilde{v}\|e,\end{aligned}$$

so

$$\tilde{T}v_1 - \gamma\|v_1 - \tilde{v}\|e \le \gamma\|v_2 + \tilde{v}\|e - \tilde{T}v_2.$$

Because L^∞ is Dedekind complete (363Mb),

$$\tilde{u} = \sup_{v_1 \in \tilde{V}} \tilde{T}v_1 - \gamma\|v_1 - \tilde{v}\|e$$

is defined in L^∞ and $\tilde{u} \le \gamma\|v_2 + \tilde{v}\|e - Tv_2$ for every $v_2 \in \tilde{V}$. Putting these together, we have

$$\tilde{T}v + \tilde{u} \le \gamma\|v + \tilde{v}\|e, \quad \tilde{T}v - \tilde{u} \le \gamma\|v - \tilde{v}\|e$$

for all $v \in \tilde{V}$. Consequently, if $v \in \tilde{V}$, then for $\alpha > 0$

$$\tilde{T}v + \alpha\tilde{u} = \alpha(\tilde{T}(\tfrac{1}{\alpha}v) + \tilde{u}) \le \alpha\gamma\|\tfrac{1}{\alpha}v + \tilde{v}\|e = \gamma\|v + \alpha\tilde{v}\|e,$$

while for $\alpha < 0$

$$\tilde{T}v + \alpha\tilde{u} = |\alpha|(\tilde{T}(-\tfrac{1}{\alpha}v) - \tilde{u}) \le |\alpha|\gamma\| - \tfrac{1}{\alpha}v - \tilde{v}\|e = \gamma\|v + \alpha\tilde{v}\|e,$$

and of course

$$\tilde{T}v \le \|\tilde{T}v\|_\infty e \le \gamma\|v\|e.$$

So we have

$$\tilde{T}v + \alpha\tilde{u} \le \gamma\|v + \alpha\tilde{v}\|e$$

for every $v \in \tilde{V}$, $\alpha \in \mathbb{R}$.

Define $T_1 : V_1 \to L^\infty$ by setting $T_1(v + \alpha\tilde{v}) = \tilde{T}v + \alpha\tilde{u}$ for every $v \in \tilde{V}$, $\alpha \in \mathbb{R}$. (This is well-defined because $\tilde{v} \notin \tilde{V}$, so any member of V_1 is uniquely expressible as $v + \alpha\tilde{v}$ where $v \in \tilde{V}$ and $\alpha \in \mathbb{R}$.) Then T_1 is a linear operator, extending T_0, from a linear subspace of V to L^∞. But from the calculations above we know that $T_1 v \le \gamma\|v\|e$ for every $v \in V_1$; since we also have

$$T_1 v = -T_1(-v) \ge -\gamma\|-v\|e = -\gamma\|v\|e,$$

$\|T_1 v\|_\infty \le \gamma\|v\|$ for every $v \in V_1$, and $T_1 \in \mathfrak{P}$. But now T_1 is a member of $\mathfrak{P}$ properly extending $\tilde{T}$, which is supposed to be impossible. **XQ**

Accordingly $\tilde{T} : V \to L^\infty$ is an extension of T_0 to the whole of V, with the same norm as T_0. As V and T_0 are arbitrary, L^∞ has the Hahn-Banach property.

Part B: (ii)⇒(i) Now let U be a normed space with the Hahn-Banach property. If $U = \{0\}$ then of course it is isomorphic to $L^\infty(\mathfrak{A})$, where $\mathfrak{A} = \{0\}$, so henceforth I will take it for granted that $U \ne \{0\}$.

(a) Let Z be the unit ball of the dual U^* of U, with the weak* topology. Then Z is a compact Hausdorff space (3A5F). For $u \in U$ set $Z_u = \{z : z \in Z, |z(u)| = \|u\|\}$; then Z_u is a closed subset of Z (because $f \mapsto f(u)$ is continuous), and is non-empty, by the Hahn-Banach theorem (3A5Ab, or Part A above!) Now let $\mathfrak{P}$ be the set of those closed sets $X \subseteq Z$ such that $X \cap Z_u \ne \emptyset$ for every $u \in U$. If $\mathfrak{Q} \subseteq \mathfrak{P}$ is non-empty and totally ordered, then $\bigcap\mathfrak{Q} \in \mathfrak{P}$, because for any $u \in U$

$$\{X \cap Z_u : X \in \mathfrak{Q}\}$$

is a downwards-directed family of non-empty compact sets, so must have non-empty intersection. By Zorn's Lemma, upside down, $\mathfrak{P}$ has a minimal element X; with its relative topology, X is a compact Hausdorff space.

(b) We have a linear operator $R : U \to C(X)$ given by setting $(Ru)(x) = x(u)$ for every $u \in U$, $x \in X$; because $X \subseteq Z$, $\|Ru\|_\infty \le \|u\|$, and because $X \in \mathfrak{P}$, $\|Ru\|_\infty = \|u\|$, for every $u \in U$. Moreover, if $G \subseteq X$ is a non-empty open set (in the relative topology of X) then $X \setminus G$ cannot belong to $\mathfrak{P}$, because X is minimal, so there is a (non-zero) $u \in U$ such that $|x(u)| < \|u\|$ for every $x \in X \setminus G$. Replacing u by $\|u\|^{-1}u$ if need be, we may suppose that $\|u\| = 1$.

What this means is that $W = R[U]$ is a linear subspace of $C(X)$ which is isomorphic, as normed space, to U, and has the property that whenever $G \subseteq X$ is a non-empty relatively open set there is an $f \in W$ such that $\|f\|_\infty = 1$ and $|f(x)| < 1$ for every $x \in X \setminus G$. Observe that, because $X \setminus G$ is compact, there is now some $\alpha < 1$ such that $|f(u)| \le \alpha$ for every $f \in X \setminus G$.

Because W is isomorphic to U, it has the Hahn-Banach property.

(c) Now consider $V = \ell^\infty(X)$, $V_0 = W$, $T_0 : V_0 \to W$ the identity map. Because W has the Hahn-Banach property, there is a linear operator $T : \ell^\infty(X) \to W$, extending T_0, and of norm $\|T_0\| = 1$.

(d) If $h \in \ell^\infty(X)$ and $x_0 \in X \setminus \overline{\{x : h(x) \ne 0\}}$, then $(Th)(x_0) = 0$. **P?** Otherwise, set $G = \{y : y \in X \setminus \overline{\{x : h(x) \ne 0\}}, (Th)(y) \ne 0\}$. This is a non-empty open set in X, so there are $f \in W$, $\alpha < 1$ such that $\|f\|_\infty = 1$ and $|f(x)| \le \alpha$ for every $x \in X \setminus G$.

Because $\|f\|_\infty = 1$, there must be some $x_1 \in X$ such that $|f(x_1)| = 1$, and of course $x_1 \in G$, so that $(Th)(x_1) \ne 0$. But let $\delta > 0$ be such that $\delta\|h\|_\infty \le 1 - \alpha$. Then, because $h(x) = 0$ for $x \in G$, $|f(x)| + |\delta h(x)| \le 1$ for every $x \in X$, and $\|f + \delta h\|_\infty$, $\|f - \delta h\|_\infty$ are both less than or equal to 1. As $Tf = f$ and $\|T\| = 1$, this means that

$$\|f + \delta Th\|_\infty \le 1, \quad \|f - \delta Th\|_\infty \le 1;$$

consequently

$$|f(x_1)| + \delta|(Th)(x_1)| = \max(|(f + \delta Th)(x_1)|, |(f - \delta Th)(x_1)|) \le 1.$$

But $|f(x_1)| = 1$ and $\delta(Th)(x_1) \ne 0$, so this is impossible. **XQ**

(e) It follows that $Th = h$ for every $h \in C(X)$. **P?** Suppose, if possible, otherwise. Then there is a $\delta > 0$ such that $G = \{x : |(Th)(x) - h(x)| > \delta\}$ is not empty. Let $f \in W$ be such that $\|f\| = 1$ but $|f(x)| < 1$ for every $x \in X \setminus G$. Then there is an $x_0 \in X$ such that $|f(x_0)| = 1$; of course x_0 must belong to G. Set $f_1 = \frac{h(x_0)}{f(x_0)} f$, so that $f_1 \in W$ and $f_1(x_0) = h(x_0)$. Set

$$h_1(x) = \text{med}(h(x) - \delta, f_1(x), h(x) + \delta)$$

for $x \in X$. Then $h_1 \in C(X)$. Setting

$$H = \{x : |h(x) - h(x_0)| + |f_1(x) - f_1(x_0)| < \delta\},$$

H is an open set containing x_0 and

$$|f_1(x) - h(x)| \le |f_1(x_0) - h(x_0)| + \delta = \delta, \quad h_1(x) = f_1(x)$$

for every $x \in H$. Consequently $x_0 \notin \overline{\{x : (f_1 - h_1)(x) \ne 0\}}$, and $T(f_1 - h_1)(x_0) = 0$, by (d). But this means that

$$(Th_1)(x_0) = (Tf_1)(x_0) = f_1(x_0) = h(x_0),$$

so that

$$|h(x_0) - (Th)(x_0)| = |T(h_1 - h)(x_0)| \le \|T(h_1 - h)\|_\infty \le \|h_1 - h\|_\infty \le \delta,$$

which is impossible, because $x_0 \in G$. **XQ**

(f) This tells us at once that $W = C(X)$. But (d) also tells us that X is extremally disconnected. **P** Let $G \subseteq X$ be any open set. Then $\chi X = \chi G + \chi(X \setminus G)$, so

$$\chi X = T(\chi X) = h_1 + h_2,$$

where $h_1 = T(\chi G)$, $h_2 = T(\chi(X \setminus G))$. Now from (d) we see that h_1 must be zero on $X \setminus \overline{G}$ while h_2 must be zero on G. Thus we have $h_1(x) = 1$ for $x \in G$; as h_1 is continuous, $h_1(x) = 1$ for $x \in \overline{G}$, and $h_1 = \chi\overline{G}$. Of course it follows that $\overline{G}$ is open. As G is arbitrary, X is extremally disconnected. **Q**

(g) Being also compact and Hausdorff, therefore regular (3A3Bb), X is zero-dimensional (3A3Bd). We may therefore identify X with the Stone space of its regular open algebra $\text{RO}(X)$ (314S), and $W = C(X)$ with $L^\infty(\text{RO}(X))$. Thus $R : U \to C(X)$ is a Banach space isomorphism between U and $C(X) \cong L^\infty(\text{RO}(X))$; so U is of the type declared.

363S The Banach-Ulam problem At a couple of points already (232Hc, the notes to §326) I have remarked on a problem which was early recognised as a fundamental question in abstract measure theory. I now set out some formulations of the problem which arise naturally from the work done so far. I will do this by writing down a list of equiveridical statements; the 'Banach-Ulam problem' asks whether they are true.

I should remark that this is not generally counted as an 'open' problem. It is in fact believed by most of us that these statements are independent of the usual axioms of Zermelo-Fraenkel set theory, including the axiom of choice and even the continuum hypothesis. As such, this problem belongs to Volume 5 rather than anywhere earlier, but its manifestations will become steadily more obtrusive as we continue through this volume and the next, and I think it will be helpful to begin collecting them now. The ideas needed to show that the statements here imply each other are already accessible; in particular, they involve no set theory beyond Zorn's Lemma. These implications constitute the following theorem, derived from LUXEMBURG 67A.

Theorem The following statements are equiveridical.

(i) There are a set X and a probability measure ν, with domain $\mathcal{P}X$, such that $\nu\{x\} = 0$ for every $x \in X$.

(ii) There are a localizable measure space (X, Σ, μ) and an absolutely continuous countably additive functional $\nu : \Sigma \to \mathbb{R}$ which is not truly continuous, so has no Radon-Nikodým derivative (definitions: 232Ab, 232Hf).

(iii) There are a Dedekind complete Boolean algebra $\mathfrak{A}$ and a countably additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ which is not completely additive.

(iv) There is a Dedekind complete Riesz space U such that $U_c^\sim \ne U^\times$.

proof (a)(i)⇒(ii) Let X be a set with a probability measure ν, defined on $\mathcal{P}X$, such that $\nu\{x\}=0$ for every $x \in X$. Let μ be counting measure on X. Then $(X,\mathcal{P}X,\mu)$ is strictly localizable, and $\nu:\mathcal{P}X\to\mathbb{R}$ is countably additive; also $\nu E=0$ whenever μE is finite, so ν is absolutely continuous with respect to μ. But if $\mu E<\infty$ then E is finite and $\nu(X\setminus E)=1$, so ν is not truly continuous, and has no Radon-Nikodým derivative (232D).

(b)(ii)⇒(iii) Let (X,Σ,μ) be a localizable measure space and $\nu:\Sigma\to\mathbb{R}$ an absolutely continuous countably additive functional which is not truly continuous. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ; then we have an absolutely continuous countably additive functional $\bar\nu:\mathfrak{A}\to\mathbb{R}$ defined by setting $\bar\nu E^\bullet=\nu E$ for every $E\in\Sigma$ (327C). Since ν is not truly continuous, $\bar\nu$ is not completely additive (327Ce). Also $\mathfrak{A}$ is Dedekind complete, because μ is localizable, so $\mathfrak{A}$ and $\bar\nu$ witness (iii).

(c)(iii)⇒(i) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ a countably additive functional which is not completely additive. Because ν is bounded (326M), therefore expressible as the difference of non-negative countably additive functionals (326L), there must be a non-negative countably additive functional ν' on $\mathfrak{A}$ which is not completely additive.

By 326R, there is a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that $\sum_{i\in I}\nu' a_i<\nu' 1$. Set $K=\{i:i\in I,\,\nu' a_i>0\}$; then K must be countable, so

$$\nu'(\sup_{i\in I\setminus K}a_i)=\nu'1-\nu'(\sup_{i\in K}a_i)=\nu'1-\textstyle\sum_{i\in K}\nu' a_i>0.$$

For $J\subseteq I$ set $\mu J=\nu'(\sup_{i\in J\setminus K}a_i)$; the supremum is always defined because $\mathfrak{A}$ is Dedekind complete. Because ν' is countably additive and non-negative, so is μ; because $\nu' a_i=0$ for $i\in J\setminus K$, $\mu\{i\}=0$ for every $i\in I$. Multiplying μ by a suitable scalar, if need be, $(I,\mathcal{P}I,\mu)$ witnesses that (i) is true.

(d)(iii)⇒(iv) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra with a countably additive functional which is not completely additive, then $U=L^\infty(\mathfrak{A})$ is a Dedekind complete Riesz space (363Mb) and $U_c^\sim\ne U^\times$, by 363K (recalling, as in (c) above, that the functional must be bounded).

(e)(iv)⇒(iii) Let U be a Dedekind complete Riesz space such that $U^\times\ne U_c^\sim$. Take $f\in U_c^\sim\setminus U^\times$; replacing f by $|f|$ if need be, we may suppose that $f\ge 0$ is sequentially order-continuous but not order-continuous (355H, 355I). Let A be a non-empty downwards-directed set in U, with infimum 0, such that $\inf_{u\in A}f(u)>0$ (351Ga). Take $e\in A$, and consider the solid linear subspace U_e of U generated by e; write g for the restriction of f to U_e. Because the embedding of U_e in U is order-continuous, $g\in(U_e)_c^\sim$; because $A\cap U_e$ is downwards-directed and has infimum 0, and

$$\inf_{u\in A\cap U_e}g(u)=\inf_{u\in A}f(u)>0,$$

$g\notin U_e^\times$. But U_e is a Riesz space with order unit e, and is Dedekind complete because U is; so it can be identified with $L^\infty(\mathfrak{A})$ for some Boolean algebra $\mathfrak{A}$ (363N), and $\mathfrak{A}$ is Dedekind complete, by 363M.

Accordingly we have a Dedekind complete Boolean algebra $\mathfrak{A}$ such that $L^\infty(\mathfrak{A})_c^\sim\ne L^\infty(\mathfrak{A})^\times$. By 363K, there is a (bounded) countably additive functional on $\mathfrak{A}$ which is not completely additive, and (iii) is true.

363X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and U a Banach algebra. Let $\nu:\mathfrak{A}\to U$ be a bounded additive function and $T:L^\infty(\mathfrak{A})\to U$ the corresponding bounded linear operator. Show that T is multiplicative iff $\nu(a\cap b)=\nu a\times\nu b$ for all a, $b\in\mathfrak{A}$.

>(b) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T:L^\infty(\mathfrak{A})\to L^\infty(\mathfrak{A})$ a linear operator. Show that the following are equiveridical: (i) there is a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that $T=T_\pi$ (ii) $T(u\times v)=Tu\times Tv$ for all u, $v\in L^\infty(\mathfrak{A})$ (iii) T is a Riesz homomorphism and $Te_{\mathfrak{A}}=e_{\mathfrak{B}}$, where $e_{\mathfrak{A}}$ is the standard order unit of $L^\infty(\mathfrak{A})$.

(c) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and $T:L^\infty(\mathfrak{A})\to L^\infty(\mathfrak{B})$ a Riesz homomorphism. Show that there are a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ and a $v\ge 0$ in $L^\infty(\mathfrak{B})$ such that $Tu=v\times T_\pi u$ for every $u\in L^\infty(\mathfrak{A})$, where T_π is the operator associated with π (363F).

(d) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. Show that $L^\infty(\mathfrak{C})$, regarded as a subspace of $L^\infty(\mathfrak{A})$ (363Ga), is order-dense in $L^\infty(\mathfrak{A})$ iff $\mathfrak{C}$ is order-dense in $\mathfrak{A}$.

>(e) Let (X,Σ,μ) be a measure space with measure algebra $\mathfrak{A}$, and $\mathcal{L}^\infty$ the space of bounded Σ-measurable real-valued functions on X. A **linear lifting** of μ is a positive linear operator $T:L^\infty(\mathfrak{A})\to\mathcal{L}^\infty$ such that $T(\chi 1_{\mathfrak{A}})=\chi X$ and $(Tu)^\bullet=u$ for every $u\in L^\infty(\mathfrak{A})$, writing $f\mapsto f^\bullet$ for the canonical map from $\mathcal{L}^\infty$ to $L^\infty(\mathfrak{A})$ (363H-363I). (i) Show that if $\theta:\mathfrak{A}\to\Sigma$ is a lifting in the sense of 341A then T_θ, as defined in 363F, is a linear lifting. (ii) Show that if $T:L^\infty(\mathfrak{A})\to\mathcal{L}^\infty$ is a linear lifting, then there is a corresponding lower density $\underline{\theta}:\mathfrak{A}\to\Sigma$ defined by setting $\underline{\theta}a=\{x:T(\chi a)(x)=1\}$ for each $a\in\mathfrak{A}$. (iii) Show that $\underline{\theta}$, as defined in (ii), is a lifting iff T is a Riesz homomorphism iff T is multiplicative.

(f) Let U be any commutative ring with multiplicative identity 1. Show that the set A of **idempotents** in U (that is, elements $a\in U$ such that $a^2=a$) is a Boolean algebra with identity 1, writing $a\cap b=ab$, $1\setminus a=1-a$ for a, $b\in A$.

(g) Let $\mathfrak{A}$ be a Boolean algebra. Show that $\mathfrak{A}$ is isomorphic to the Boolean algebras of multiplicative idempotents of $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that for any $u\in L^\infty(\mathfrak{A})$, $\alpha\in\mathbb{R}$ there are elements $[\![u\ge\alpha]\!]$, $[\![u>\alpha]\!]\in\mathfrak{A}$, where $[\![u\ge\alpha]\!]$ is the largest $a\in\mathfrak{A}$ such that $u\times\chi a\ge\alpha\chi a$, and $[\![u>\alpha]\!]=\sup_{\beta>\alpha}[\![u\ge\beta]\!]$. (ii) Show that in the context of 363Hb, if u corresponds to $f^\bullet$ for $f\in\mathcal{L}^\infty$, then $[\![u\ge\alpha]\!]=\{x:f(x)\ge\alpha\}^\bullet$, $[\![u>\alpha]\!]=\{x:f(x)>\alpha\}^\bullet$. (iii) Show that if $A\subseteq L^\infty$ is non-empty and $v\in L^\infty$, then $v=\sup A$ iff $[\![v>\alpha]\!]=\sup_{u\in A}[\![u>\alpha]\!]$ for every $\alpha\in\mathbb{R}$; in particular, $v=u$ iff $[\![v>\alpha]\!]=[\![u>\alpha]\!]$ for every α. (iv) Show that a function $\phi:\mathbb{R}\to\mathfrak{A}$ is of the form $\phi(\alpha)=[\![u>\alpha]\!]$ iff (α) $\phi(\alpha)=\sup_{\beta>\alpha}\phi(\beta)$ for every $\alpha\in\mathbb{R}$ (β) there is an M such that $\phi(M)=0$, $\phi(-M)=1$. (v) Put (iii) and (iv) together to give a proof that L^∞ is Dedekind σ-complete if $\mathfrak{A}$ is.

(i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $U\subseteq L^\infty(\mathfrak{A})$ a (sequentially) order-closed Riesz subspace containing $\chi 1$. Show that U can be identified with $L^\infty(\mathfrak{B})$ for some (sequentially) order-closed subalgebra $\mathfrak{B}\subseteq\mathfrak{A}$. (*Hint*: set $\mathfrak{B}=\{b:\chi b\in U\}$ and use 363N.)

363Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean algebra. Given the linear structure, ordering, multiplication and norm of $S(\mathfrak{A})$ as described in §361, show that a norm completion of $S(\mathfrak{A})$ will serve for $L^\infty(\mathfrak{A})$ in the sense that all the results of 363B-363Q can be proved with no use of the axiom of choice except an occasional appeal to countably many choices in sequential forms of the theorems.

(b) Let $\mathfrak{A}$ be a Boolean algebra. Show that $\mathfrak{A}$ is ccc iff $L^\infty(\mathfrak{A})$ has the countable sup property (241Ye, 353Yd).

(c) Let X be an extremally disconnected topological space, and $\mathrm{RO}(X)$ its regular open algebra. Show that there is a natural isomorphism between $L^\infty(\mathrm{RO}(X))$ and $C_b(X)$.

(d) Let $\mathfrak{A}$ be a Boolean algebra. (i) If $u\in L^\infty=L^\infty(\mathfrak{A})$, show that $|u|=e$, the standard order unit of L^∞, iff $\max(\|u+v\|_\infty,\|u-v\|_\infty)>1$ whenever $v\in L^\infty\setminus\{0\}$. (ii) Show that if u, $v\in L^\infty$ then $|u|\wedge|v|=0$ iff $\|\alpha u+v+w\|_\infty\le\max(\|\alpha u+w\|_\infty,\|v+w\|_\infty)$ whenever $\alpha=\pm1$ and $w\in L^\infty$. (iii) Show that if $T:L^\infty\to L^\infty$ is a normed space automorphism then there are a Boolean automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ and a $w\in L^\infty$ such that $|w|=e$ and $Tu=w\times T_\pi u$ for every $u\in L^\infty$.

(e) Let X be a set, Σ an algebra of subsets of X, and $\mathcal{I}$ an ideal in Σ, and $\mathcal{L}^\infty$ the set of bounded functions $f:X\to\mathbb{R}$ such that whenever $\alpha<\beta$ in $\mathbb{R}$ there is an $E\in\Sigma$ such that $\{x:f(x)\le\alpha\}\subseteq E\subseteq\{x:f(x)\le\beta\}$, as in 363H. (i) Show that $\mathcal{L}^\infty=\{g\phi:g\in C(Z)\}$, where Z is the Stone space of Σ and $\phi:X\to Z$ is a function (to be described). (ii) Show that $L^\infty(\Sigma/\mathcal{I})$ can be identified, as Banach lattice and Banach algebra, with $\mathcal{L}^\infty/\mathcal{V}$, where $\mathcal{V}$ is the set of those functions $f\in\mathcal{L}^\infty$ such that for every $\epsilon>0$ there is a member of $\mathcal{I}$ including $\{x:|f(x)|\ge\epsilon\}$.

(f) Let (X,Σ,μ) be a complete probability space with measure algebra $\mathfrak{A}$. Let $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$, and set $\Sigma_n=\{F:F^\bullet\in\mathfrak{B}_n\}$ for each n. Let $P_n:L^1(\mu)\to L^1(\mu\restriction\Sigma_n)$ be the conditional expectation operator for each n, so that $P_n\restriction L^\infty(\mu)$ is a positive linear operator from $L^\infty(\mu)\cong L^\infty(\mathfrak{A})$ to

$L^\infty(\mu\restriction\Sigma_n) \cong L^\infty(\mathfrak{B}_n)$. Suppose that we are given for each n a lifting $\theta_n : \mathfrak{B}_n \to \Sigma_n$ and that $\theta_{n+1}b = \theta_n b$ whenever $n \in \mathbb{N}$ and $b \in \mathfrak{B}_n$. Let $T_n : L^\infty(\mathfrak{B}_n) \to \mathcal{L}^\infty$ be the corresponding linear liftings (363Xe), and $\mathcal{F}$ any non-principal ultrafilter on $\mathbb{N}$. (i) Show that for any $u \in L^\infty(\mathfrak{A})$, $\langle T_n P_n u\rangle_{n\in\mathbb{N}}$ converges almost everywhere. (ii) For $u \in L^\infty(\mathfrak{A})$ set $(Tu)(x) = \lim_{n\to\mathcal{F}}(T_n P_n u)(x)$ for $x \in X$, $u \in L^\infty(\mathfrak{A})$. Show that T is a linear lifting for μ. (iii) Use 363Xe(ii) and 341J to show that there is a lifting θ of μ extending every θ_n. (iv) Use this as the countable-cofinality inductive step in a proof of the Lifting Theorem (using partial liftings rather than partial lower densities, as suggested in 341Li).

(g) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a bounded countably additive functional. Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in $L^\infty(\mathfrak{A})$ such that $\inf_{n\in\mathbb{N}}\sup_{m\ge n} u_m$ and $\sup_{n\in\mathbb{N}}\inf_{m\ge n} u_m$ are defined in $L^\infty(\mathfrak{A})$ and equal to u say. Show that $\int u\,d\nu = \lim_{n\to\infty}\int u_n d\nu$.

(h) Let Σ be the family of those sets $E \subseteq [0,1]$ such that $\mu(\operatorname{int} E) = \mu\overline{E}$, where μ is Lebesgue measure. (i) Show that Σ is an algebra of subsets of $[0,1]$ and that every member of Σ is Lebesgue measurable. (ii) Show that if we identify $L^\infty(\Sigma)$ with a set of real-valued functions on $[0,1]$, as in 363H, then we get just the space of Riemann integrable functions. (iii) Show that if we write ν for $\mu\restriction\Sigma$, then $⨍ d\nu$, as defined in 363L, is just the Riemann integral.

(i) Let X be a compact Hausdorff space. Let us say that a linear subspace U of $C(X)$ is **ℓ^∞-complemented** in $C(X)$ if there is a linear subspace V such that $C(X) = U \oplus V$ and $\|u+v\|_\infty = \max(\|u\|_\infty, \|v\|_\infty)$ for all $u \in U$, $v \in V$. Show that there is a one-to-one correspondence between such subspaces U and open-and-closed subsets E of X, given by setting $U = \{u : u \in C(X),\ u(x) = 0\ \forall\ x \in X \setminus E\}$. Hence show that if $\mathfrak{A}$ is any Boolean algebra, there is a canonical isomorphism between $\mathfrak{A}$ and the partially ordered set of ℓ^∞-complemented subspaces of $L^\infty(\mathfrak{A})$.

363 Notes and comments As with $S(\mathfrak{A})$, I have chosen a definition of $L^\infty(\mathfrak{A})$ in terms of the Stone space of $\mathfrak{A}$; but as with $S(\mathfrak{A})$, this is optional (363Ya). By and large the basic properties of L^∞ are derived very naturally from those of S. The spaces $L^\infty(\mathfrak{A})$, for general Boolean algebras $\mathfrak{A}$, are not in fact particularly important; they have too few properties not shared by all the spaces $C(X)$ for compact Hausdorff X. The point at which it becomes helpful to interpret $C(X)$ as $L^\infty(\mathfrak{A})$ is when $C(X)$ is Dedekind σ-complete. The spaces X for which this is true are difficult to picture, and alternative representations of L^∞ along the lines of 363H-363I can be easier on the imagination.

For Dedekind σ-complete $\mathfrak{A}$, there is an alternative description of members of $L^\infty(\mathfrak{A})$ in terms of objects '$[\![u > \alpha]\!]$' (363Xh); I will return to this idea in the next section. For the moment I remark only that it gives an alternative approach to 363M not necessarily depending on the representation of L^∞ as a quotient $\mathcal{L}^\infty/\mathcal{V}$ nor on an analysis of a Stone space. I used a version of such an argument in the proof of 363M which I gave in Fremlin 74a, 43D.

I spend so much time on 363M not only because Dedekind completeness is one of the basic properties of any lattice, but because it offers an abstract expression of one of the central results of Chapter 24. In 243H I showed that $L^\infty(\mu)$ is always Dedekind σ-complete, and that it is Dedekind complete if μ is localizable. We can now relate this to the results of 321H and 322Be: the measure algebra of any measure is Dedekind σ-complete, and the measure algebra of a localizable measure is Dedekind complete.

The ideas of the proof of 363M can of course be rearranged in various ways. One uses 353Yb: for completely regular spaces X, $C(X)$ is Dedekind complete iff X is extremally disconnected; while for compact Hausdorff spaces, X is extremally disconnected iff it is the Stone space of a Dedekind complete algebra. With the right modification of the concept 'extremally disconnected' (314Yf), the same approach works for Dedekind σ-completeness.

363R is the 'Nachbin-Kelley theorem'; it is commonly phrased 'a normed space U has the Hahn-Banach extension property iff it is isomorphic, as normed space, to $C(X)$ for some compact extremally disconnected Hausdorff space X', but the expression in terms of L^∞ spaces seems natural in the present context. The implication in one direction (Part A of the proof) calls for nothing but a check through one of the standard proofs of the Hahn-Banach theorem to make sure that the argument applies in the generalized form. Part B of the proof has ideas in it; I have tried to set it out in a way suggesting that if you can remember the construction of the set X the rest is just a matter of a little ingenuity.

One way of trying to understand the multiple structures of L^∞ spaces is by looking at the corresponding automorphisms. We observe, for instance, that an operator T from $L^\infty(\mathfrak{A})$ to itself is a Banach algebra automorphism iff it is a Banach lattice automorphism preserving the standard order unit iff it corresponds to an automorphism of the algebra $\mathfrak{A}$ (363Xb). Of course there are Banach space automorphisms of L^∞ which do not respect the order or multiplicative structure; but they have to be closely related to algebra isomorphisms (363Yd).

I devote a couple of exercises (363Xe, 363Yf) to indications of how the ideas here are relevant to the Lifting Theorem. If you found the formulae of the proof of 341G obscure it may help to work through the parallel argument.

A lecture by W.A.J.Luxemburg on the equivalence between (i) and (iv) in 363S was one of the turning points in my mathematical apprenticeship. I introduce it here, even though the real importance of the Banach-Ulam problem lies in the metamathematical ideas it has nourished, because these formulations provide a focus for questions which arise naturally in this volume and which otherwise might prove distracting. The next group of significant ideas in this context will appear in §438.

Version of 16.7.11

364 L^0

My next objective is to develop an abstract construction corresponding to the $L^0(\mu)$ spaces of §241. These generalized L^0 spaces will form the basis of the work of the rest of this chapter and also the next; partly because their own properties are remarkable, but even more because they form a framework for the study of Archimedean Riesz spaces in general (see §368). There seem to be significant new difficulties, and I take the space to describe an approach which can be made essentially independent of the route through Stone spaces used in the last three sections (364Ya). I embark directly on a definition in the new language (364A), and relate it to the constructions of §241 (364B-364D, 364I) and §§361-363 (364J). The ideas of Chapter 27 can also be expressed in this language; I make a start on developing the machinery for this in 364F-364G, with the formula '$[\![u \in E]\!]$', 'the region in which u belongs to E', and some exercises (364Xe-364Xf). Following through the questions addressed in §363, I discuss Dedekind completeness in L^0 (364L-364M), properties of its multiplication (364N), the expression of the original algebra in terms of L^0 (364O), the action of Boolean homomorphisms on L^0 (364P) and product spaces (364R). In 364S-364V I describe representations of the L^0 space of a regular open algebra.

364A The set $L^0(\mathfrak{A})$ (a) Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. I will write $L^0(\mathfrak{A})$ for the set of all functions $\alpha \mapsto [\![u > \alpha]\!] : \mathbb{R} \to \mathfrak{A}$ such that

(α) $[\![u > \alpha]\!] = \sup_{\beta > \alpha} [\![u > \beta]\!]$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$,

(β) $\inf_{\alpha \in \mathbb{R}} [\![u > \alpha]\!] = 0$,

(γ) $\sup_{\alpha \in \mathbb{R}} [\![u > \alpha]\!] = 1$.

(b) My reasons for using the notation '$[\![u > \alpha]\!]$' rather than '$u(\alpha)$' will I hope become clear in the next few paragraphs. For the moment, if you think of $\mathfrak{A}$ as a σ-algebra of sets and of $L^0(\mathfrak{A})$ as the family of $\mathfrak{A}$-measurable real-valued functions, then $[\![u > \alpha]\!]$ corresponds to the set $\{x : u(x) > \alpha\}$ (364Ia).

(c) Some readers will recognise the formula '$[\![\ldots]\!]$' as belonging to the language of forcing, so that $[\![u > \alpha]\!]$ could be read as 'the Boolean value of the proposition "$u > \alpha$"'. But a vocalisation closer to my intention might be 'the region where $u > \alpha$'.

(d) Note that condition (α) of (a) automatically ensures that $[\![u > \alpha]\!] \subseteq [\![u > \alpha']\!]$ whenever $\alpha' \le \alpha$ in $\mathbb{R}$.

(e) In fact it will sometimes be convenient to note that the conditions of (a) can be replaced by

(α') $[\![u > \alpha]\!] = \sup_{q \in \mathbb{Q}, q > \alpha} [\![u > q]\!]$ for every $\alpha \in \mathbb{R}$,

(β') $\inf_{n \in \mathbb{N}} [\![u > n]\!] = 0$,

(γ') $\sup_{n \in \mathbb{N}} [\![u > -n]\!] = 1$;

the point being that we need look only at suprema and infima of countable subsets of $\mathfrak{A}$.

***(f)** Indeed, because the function $\alpha \mapsto [\![u > \alpha]\!]$ is determined by its values on $\mathbb{Q}$, we have the option of declaring $L^0(\mathfrak{A})$ to be the set of functions $\alpha \mapsto [\![u > \alpha]\!] : \mathbb{Q} \to \mathfrak{A}$ such that

(α'') $[\![u > q]\!] = \sup_{q' \in \mathbb{Q}, q' > q} [\![u > q']\!]$ for every $q \in \mathbb{Q}$,

(β') $\inf_{n \in \mathbb{N}} [\![u > n]\!] = 0$,

(γ') $\sup_{n \in \mathbb{N}} [\![u > -n]\!] = 1$.

However I shall hold this in reserve until I come to forcing constructions in Chapter 55 of Volume 5.

(g) In order to integrate this construction into the framework of the rest of this book, I match it with an alternative route to the same object, based on σ-algebras and σ-ideals of sets, as follows.

364B Proposition Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ.

(a) Write $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ for the space of all Σ-measurable functions from X to $\mathbb{R}$. Then $\mathcal{L}^0$, with its linear structure, ordering and multiplication inherited from $\mathbb{R}^X$, is a Dedekind σ-complete f-algebra with multiplicative identity.

(b) Set

$$\mathcal{W} = \mathcal{W}_\mathcal{I} = \{f : f \in \mathcal{L}^0, \{x : f(x) \neq 0\} \in \mathcal{I}\}.$$

Then

(i) $\mathcal{W}$ is a sequentially order-closed solid linear subspace and ideal of $\mathcal{L}^0$;

(ii) the quotient space $\mathcal{L}^0/\mathcal{W}$, with its inherited linear, order and multiplicative structures, is a Dedekind σ-complete Riesz space and an f-algebra with a multiplicative identity;

(iii) for $f, g \in \mathcal{L}^0$, $f^\bullet \le g^\bullet$ in $\mathcal{L}^0/\mathcal{W}$ iff $\{x : f(x) > g(x)\} \in \mathcal{I}$, and $f^\bullet = g^\bullet$ in $\mathcal{L}^0/\mathcal{W}$ iff $\{x : f(x) \neq g(x)\} \in \mathcal{I}$.

proof (Compare 241A-241H.)

(a) The point is just that $\mathcal{L}^0$ is a sequentially order-closed Riesz subspace and subalgebra of $\mathbb{R}^X$. The facts we need to know – that constant functions belong to $\mathcal{L}^0$, that $f+g$, αf, $f \times g$, $\sup_{n\in\mathbb{N}} f_n$ belong to $\mathcal{L}^0$ whenever f, g, f_n do and $\{f_n : n \in \mathbb{N}\}$ is bounded above – are all covered by 121E-121F. Its multiplicative identity is of course the constant function χX.

(b)(i) The necessary verifications are all elementary.

(ii) Because $\mathcal{W}$ is a solid linear subspace of the Riesz space $\mathcal{L}^0$, the quotient inherits a Riesz space structure (351J, 352Jb); because $\mathcal{W}$ is an ideal of the ring $(\mathcal{L}^0, +, \times)$, $\mathcal{L}^0/\mathcal{W}$ inherits a multiplication; it is a commutative algebra because $\mathcal{L}^0$ is; and has a multiplicative identity $e = \chi X^\bullet$ because χX is the identity of $\mathcal{L}^0$.

To check that $\mathcal{L}^0/\mathcal{W}$ is an f-algebra it is enough to observe that, for any non-negative f, g , $h \in \mathcal{L}^0$,

$$f^\bullet \times g^\bullet = (f \times g)^\bullet \ge 0,$$

and if $f^\bullet \wedge g^\bullet = 0$ then $\{x : f(x) > 0\} \cap \{x : g(x) > 0\} \in \mathcal{I}$, so that $\{x : f(x)h(x) > 0\} \cap \{x : g(x) > 0\} \in \mathcal{I}$ and

$$(f^\bullet \times h^\bullet) \wedge g^\bullet = (h^\bullet \times f^\bullet) \wedge g^\bullet = 0.$$

Finally, $\mathcal{L}^0/\mathcal{W}$ is Dedekind σ-complete, by 353K(a-iii).

(iii) For $f, g \in \mathcal{L}^0$,

$$f^\bullet \le g^\bullet \iff (f-g)^+ \in \mathcal{W} \iff \{x : f(x) > g(x)\} = \{x : (f-g)^+(x) \neq 0\} \in \mathcal{I}$$

(using the fact that the canonical map from $\mathcal{L}^0$ to $\mathcal{L}^0/\mathcal{W}$ is a Riesz homomorphism, so that $((f-g)^+)^\bullet = (f^\bullet - g^\bullet)^+$). Similarly

$$f^\bullet = g^\bullet \iff f - g \in \mathcal{W} \iff \{x : f(x) \neq g(x)\} = \{x : (f-g)(x) \neq 0\} \in \mathcal{I}.$$

364C Theorem Let X be a set and Σ a σ-algebra of subsets of X. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \Sigma \to \mathfrak{A}$ a surjective Boolean homomorphism, with kernel a σ-ideal $\mathcal{I}$; define $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ and $\mathcal{W} = \mathcal{W}_\mathcal{I}$ as in 364B, so that $U = \mathcal{L}^0/\mathcal{W}$ is a Dedekind σ-complete f-algebra with multiplicative identity.

(a) We have a canonical bijection $T : U \to L^0 = L^0(\mathfrak{A})$ defined by the formula

$$[\![Tf^\bullet > \alpha]\!] = \pi\{x : f(x) > \alpha\}$$

for every $f \in \mathcal{L}^0$ and $\alpha \in \mathbb{R}$.

(b)(i) For any u, $v \in U$,

$$[\![T(u+v) > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!]$$

for every $\alpha \in \mathbb{R}$.

(ii) For any $u \in U$ and $\gamma > 0$,

$$[\![T(\gamma u) > \alpha]\!] = [\![Tu > \tfrac{\alpha}{\gamma}]\!]$$

for every $\alpha \in \mathbb{R}$.

(iii) For any u, $v \in U$,

$$u \le v \iff [\![Tu > \alpha]\!] \subseteq [\![Tv > \alpha]\!] \text{ for every } \alpha \in \mathbb{R}.$$

(iv) For any u, $v \in U^+$,

$$[\![T(u \times v) > \alpha]\!] = \sup_{q\in\mathbb{Q},q>0} [\![Tu > q]\!] \cap [\![Tv > \tfrac{\alpha}{q}]\!]$$

for every $\alpha \ge 0$.

(v) Writing $e = (\chi X)^\bullet$ for the multiplicative identity of U, we have

$$[\![Te > \alpha]\!] = 1 \text{ if } \alpha < 1,\ 0 \text{ if } \alpha \ge 1.$$

proof (a)(i) Given $f \in \mathcal{L}^0$, set $\zeta_f(\alpha) = \pi\{x : f(x) > \alpha\}$ for $\alpha \in \mathbb{R}$. Then it is easy to see that ζ_f satisfies the conditions $(\alpha)'$-$(\gamma)'$ of 364Ae, because π is sequentially order-continuous (313Qb). Moreover, if $f^\bullet = g^\bullet$ in U, then

$$\zeta_f(\alpha) \triangle \zeta_g(\alpha) = \pi(\{x : f(x) > \alpha\} \triangle \{x : g(x) > \alpha\}) = 0$$

for every $\alpha \in \mathbb{R}$, because

$$\{x : f(x) > \alpha\} \triangle \{x : g(x) > \alpha\} \subseteq \{x : f(x) \ne g(x)\} \in \mathcal{I},$$

and $\zeta_f = \zeta_g$. So we have a well-defined member Tu of L^0 defined by the given formula, for any $u \in U$.

(ii) Next, given $w \in L^0$, there is a $u \in \mathcal{L}^0/\mathcal{W}$ such that $Tu = w$. **P** For each $q \in \mathbb{Q}$, choose $F_q \in \Sigma$ such that $\pi F_q = [\![w > q]\!]$ in $\mathfrak{A}$. Note that if $q' \ge q$ then

$$\pi(F_{q'} \setminus F_q) = [\![u > q']\!] \setminus [\![u > q]\!] = 0,$$

so $F_{q'} \setminus F_q \in \mathcal{I}$. Set

$$H = \bigcup_{q\in\mathbb{Q}} F_q \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{q\in\mathbb{Q},q\ge n} F_q \in \Sigma,$$

and for $x \in X$ set

$$\begin{aligned} f(x) &= \sup\{q : q \in \mathbb{Q},\ x \in F_q\} \text{ if } x \in H,\\ &= 0 \text{ otherwise.} \end{aligned}$$

(H is chosen just to make the formula here give a finite value for every x.) We have

$$\begin{aligned} \pi H &= \sup_{q\in\mathbb{Q}} [\![w > q]\!] \setminus \inf_{n\in\mathbb{N}} \sup_{q\in\mathbb{Q},q\ge n} [\![w > q]\!]\\ &= 1_\mathfrak{A} \setminus \inf_{n\in\mathbb{N}} [\![w > n]\!] = 1_\mathfrak{A} \setminus 0_\mathfrak{A} = 1_\mathfrak{A}, \end{aligned}$$

so $X \setminus H \in \mathcal{I}$. Now, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}\{x : f(x) > \alpha\} &= \bigcup_{q\in\mathbb{Q},q>\alpha} F_q \cup (X \setminus H) \text{ if } \alpha < 0,\\ &= \bigcup_{q\in\mathbb{Q},q>\alpha} F_q \setminus (X \setminus H) \text{ if } \alpha \geq 0,\end{aligned}$$

and in either case belongs to Σ; so that $f \in \mathcal{L}^0$ and $f^\bullet$ is defined in L^0. Next, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}[\![Tf^\bullet > \alpha]\!] &= \pi\{x : f(x) > \alpha\} = \pi(\bigcup_{q\in\mathbb{Q},q>\alpha} F_q)\\ &= \sup_{q\in\mathbb{Q},q>\alpha} [\![w > q]\!] = [\![w > \alpha]\!],\end{aligned}$$

and $Tf^\bullet = w$. **Q**

(iii) Thus T is surjective. To see that it is injective, observe that if f, $g \in \mathcal{L}^0$, then

$$\begin{aligned}Tf^\bullet = Tg^\bullet &\implies [\![Tf^\bullet > \alpha]\!] = [\![Tg^\bullet > \alpha]\!] \text{ for every } \alpha \in \mathbb{R}\\ &\implies \pi\{x : f(x) > \alpha\} = \pi\{x : g(x) > \alpha\} \text{ for every } \alpha \in \mathbb{R}\\ &\implies \{x : f(x) > \alpha\}\triangle\{x : g(x) > \alpha\} \in \mathcal{I} \text{ for every } \alpha \in \mathbb{R}\\ &\implies \{x : f(x) \neq g(x)\} = \bigcup_{q\in\mathbb{Q}}(\{x : f(x) > q\}\triangle\{x : g(x) > q\}) \in \mathcal{I}\\ &\implies f^\bullet = g^\bullet.\end{aligned}$$

So we have the claimed bijection.

(b)(i) Let f, $g \in \mathcal{L}^0$ be such that $u = f^\bullet$ and $v = g^\bullet$, so that $u + v = (f + g)^\bullet$. For any $\alpha \in \mathbb{R}$,

$$\begin{aligned}[\![T(u + v) > \alpha]\!] &= \pi\{x : f(x) + g(x) > \alpha\}\\ &= \pi(\bigcup_{q\in\mathbb{Q}}\{x : f(x) > q\} \cap \{x : g(x) > \alpha - q\})\\ &= \sup_{q\in\mathbb{Q}} \pi\{x : f(x) > q\} \cap \pi\{x : g(x) > \alpha - q\}\end{aligned}$$

(because π is a sequentially order-continuous Boolean homomorphism)

$$= \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!].$$

(ii) Let $f \in \mathcal{L}^0$ be such that $f^\bullet = u$, so that $(\gamma f)^\bullet = \gamma u$. For any $\alpha \in \mathbb{R}$,

$$[\![T(\gamma u) > \alpha]\!] = \pi\{x : \gamma f(x) > \alpha\} = \pi\{x : f(x) > \tfrac{\alpha}{\gamma}\} = [\![Tu > \tfrac{\alpha}{\gamma}]\!].$$

(iii) Let f, $g \in \mathcal{L}^0$ be such that $f^\bullet = u$ and $g^\bullet = v$. Then

$$u \leq v \iff \{x : f(x) > g(x)\} \in \mathcal{I}$$

(see 364B(b-iii))

$$\begin{aligned}&\iff \bigcup_{q\in\mathbb{Q}}\{x : f(x) > q \geq g(x)\} \in \mathcal{I}\\ &\iff \{x : f(x) > \alpha\} \setminus \{x : g(x) > \alpha\} \in \mathcal{I} \text{ for every } \alpha \in \mathbb{R}\\ &\iff \pi\{x : f(x) > \alpha\} \setminus \pi\{x : g(x) > \alpha\} = 0 \text{ for every } \alpha\\ &\iff [\![Tu > \alpha]\!] \subseteq [\![Tv > \alpha]\!] \text{ for every } \alpha.\end{aligned}$$

(iv) Now suppose that u, $v \geq 0$, so that they can be expressed as $f^\bullet$, $g^\bullet$ where f, $g \geq 0$ in $\mathcal{L}^0$ (351J), and $u \times v = (f \times g)^\bullet$. If $\alpha \geq 0$, then

$$\begin{aligned}[\![T(u \times v) > \alpha]\!] &= \pi(\bigcup_{q\in\mathbb{Q},q>0} \{x : f(x) > q\} \cap \{x : g(x) > \frac{\alpha}{q}\}) \\ &= \sup_{q\in\mathbb{Q},q>0} \pi\{x : f(x) > q\} \cap \pi\{x : g(x) > \frac{\alpha}{q}\} \\ &= \sup_{q\in\mathbb{Q},q>0} [\![Tu > q]\!] \cap [\![Tv > \tfrac{\alpha}{q}]\!].\end{aligned}$$

(v) This is trivial, because

$$\begin{aligned}[\![T(\chi X)^\bullet > \alpha]\!] &= \pi\{x : (\chi X)(x) > \alpha\} \\ &= \pi X = 1 \text{ if } \alpha < 1, \\ &= \pi\emptyset = 0 \text{ if } \alpha \geq 1.\end{aligned}$$

364D Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then $L^0 = L^0(\mathfrak{A})$ has the structure of a Dedekind σ-complete f-algebra with multiplicative identity e, defined by saying

$$[\![u + v > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![u > q]\!] \cap [\![v > \alpha - q]\!],$$

whenever u, $v \in L^0$ and $\alpha \in \mathbb{R}$,

$$[\![\gamma u > \alpha]\!] = [\![u > \tfrac{\alpha}{\gamma}]\!]$$

whenever $u \in L^0$, $\gamma \in \,]0, \infty[$ and $\alpha \in \mathbb{R}$,

$$u \leq v \iff [\![u > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } \alpha \in \mathbb{R},$$

$$[\![u \times v > \alpha]\!] = \sup_{q\in\mathbb{Q},q>0} [\![u > q]\!] \cap [\![v > \tfrac{\alpha}{q}]\!]$$

whenever u, $v \geq 0$ in L^0 and $\alpha \geq 0$,

$$[\![e > \alpha]\!] = 1 \text{ if } \alpha < 1, \ 0 \text{ if } \alpha \geq 1.$$

proof (a) By the Loomis-Sikorski theorem (314M), we can find a set Z (the Stone space of $\mathfrak{A}$), a σ-algebra Σ of subsets of Z (the algebra generated by the open-and-closed sets and the ideal $\mathcal{M}$ of meager sets) and a surjective sequentially order-continuous Boolean homomorphism $\pi : \Sigma \to \mathfrak{A}$ (corresponding to the identification between $\mathfrak{A}$ and the quotient $\Sigma/\mathcal{M}$). Consequently, defining $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ and $\mathcal{W} = \mathcal{W}_\mathcal{M}$ as in 364B, we have a bijection between the Dedekind σ-complete f-algebra $\mathcal{L}^0/\mathcal{W}$ and L^0 (364Ca). Of course this endows L^0 itself with the structure of a Dedekind σ-complete f-algebra; and 364Cb tells us that the description of the algebraic operations above is consistent with this structure.

(b) In fact the f-algebra structure is completely defined by the description offered. For while scalar multiplication is not described for $\gamma \leq 0$, the assertion that L^0 is a Riesz space implies that $0u = 0$ and that $\gamma u = (-\gamma)(-u)$ for $\gamma < 0$; so if we have formulae to describe $u + v$ and γu for $\gamma > 0$, this suffices to define the linear structure of L^0. Note that we have an element $\underline{0}$ in L^0 defined by setting

$$[\![\underline{0} > \alpha]\!] = 0 \text{ if } \alpha \geq 0, \ 1 \text{ if } \alpha < 0,$$

and the formula for $u + v$ shows us that

$$[\![\underline{0} + u > \alpha]\!] = \sup_{q\in\mathbb{Q}} [\![\underline{0} > q]\!] \cap [\![u > \alpha - q]\!] = \sup_{q\in\mathbb{Q},q<0} [\![u > \alpha - q]\!] = [\![u > \alpha]\!]$$

for every α, so that $\underline{0}$ is the zero of L^0. As for multiplication, if L^0 is to be an f-algebra we must have

$$[\![u \times v > \alpha]\!] \supseteq [\![\underline{0} > \alpha]\!] = 1$$

whenever u, $v \in (L^0)^+$ and $\alpha < 0$, because $u \times v \geq \underline{0}$. So the formula offered is sufficient to determine $u \times v$ for non-negative u and v; and for others we know that

$$u \times v = (u^+ \times v^+) - (u^+ \times v^-) - (u^- \times v^+) + (u^- \times v^-),$$

so the whole of the multiplication of L^0 is defined.

364E The rest of this section will be devoted to understanding the structure just established. I start with a pair of elementary facts.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) If u, $v \in L^0 = L^0(\mathfrak{A})$ and α, $\beta \in \mathbb{R}$,

$$[\![u + v > \alpha + \beta]\!] \subseteq [\![u > \alpha]\!] \cup [\![v > \beta]\!].$$

(b) If u, $v \geq 0$ in L^0 and α, $\beta \geq 0$ in $\mathbb{R}$,

$$[\![u \times v > \alpha\beta]\!] \subseteq [\![u > \alpha]\!] \cup [\![v > \beta]\!].$$

proof (a) For any $q \in \mathbb{Q}$, either $q \geq \alpha$ and $[\![u > q]\!] \subseteq [\![u > \alpha]\!]$, or $q \leq \alpha$ and $[\![v > \alpha + \beta - q]\!] \subseteq [\![v > \beta]\!]$; thus in all cases

$$[\![u > q]\!] \cap [\![v > \alpha + \beta - q]\!] \subseteq [\![u > \alpha]\!] \cup [\![v > \beta]\!];$$

taking the supremum over q, we have the result.

(b) The same idea works, replacing $\alpha + \beta - q$ by $\alpha\beta/q$ for $q > 0$.

364F Yet another description of L^0 is sometimes appropriate, and leads naturally to an important construction (364H).

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then there is a bijection between $L^0 = L^0(\mathfrak{A})$ and the set Φ of sequentially order-continuous Boolean homomorphisms from the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ to $\mathfrak{A}$, defined by saying that $u \in L^0$ corresponds to $\phi \in \Phi$ iff $[\![u > \alpha]\!] = \phi(]\alpha, \infty[)$ for every $\alpha \in \mathbb{R}$.

proof (a) If $\phi \in \Phi$, then the map $\alpha \mapsto \phi(]\alpha, \infty[)$ satisfies the conditions of 364Ae, so corresponds to an element u_ϕ of L^0.

(b) If ϕ, $\psi \in \Phi$ and $u_\phi = u_\psi$, then $\phi = \psi$. **P** Set $\mathcal{A} = \{E : E \in \mathcal{B}, \phi(E) = \psi(E)\}$. Then $\mathcal{A}$ is a σ-subalgebra of $\mathcal{B}$, because ϕ and ψ are both sequentially order-continuous Boolean homomorphisms, and contains $]\alpha, \infty[$ for every $\alpha \in \mathbb{R}$. Now $\mathcal{A}$ contains $]-\infty, \alpha]$ for every α, and therefore includes $\mathcal{B}$ (121J). But this means that $\phi = \psi$. **Q**

(c) Thus $\phi \mapsto u_\phi$ is injective. But it is also surjective. **P** As in 364D, take a set Z, a σ-algebra Σ of subsets of Z and a surjective sequentially order-continuous Boolean homomorphism $\pi : \Sigma \to \mathfrak{A}$; let $T : \mathcal{L}^0_\Sigma/\mathcal{W}_{\pi^{-1}[\{0\}]} \to L^0$ be the bijection described in 364C. If $u \in L^0$, there is an $f \in \mathcal{L}^0_\Sigma$ such that $Tf^\bullet = u$. Now consider $\phi E = \pi f^{-1}[E]$ for $E \in \mathcal{B}$. $f^{-1}[E]$ always belongs to Σ (121Ef), so ϕE is always well-defined; $E \mapsto f^{-1}[E]$ and π are sequentially order-continuous, so ϕ also is; and

$$\phi(]\alpha, \infty[) = \pi\{z : f(z) > \alpha\} = [\![u > \alpha]\!]$$

for every α, so $u = u_\phi$. **Q**

Thus we have the declared bijection.

364G Definitions (a) In the context of 364F, I will write $[\![u \in E]\!]$, 'the region where u takes values in E', for $\phi(E)$, where $\phi : \mathcal{B} \to \mathfrak{A}$ is the homomorphism corresponding to $u \in L^0$. Thus $[\![u > \alpha]\!] = [\![u \in]\alpha, \infty[\,]\!]$. In the same spirit I write $[\![u \geq \alpha]\!]$ for $[\![u \in [\alpha, \infty[\,]\!] = \inf_{\beta<\alpha} [\![u > \beta]\!]$, $[\![u \neq 0]\!] = [\![|u| > 0]\!] = [\![u > 0]\!] \cup [\![u < 0]\!]$ and so on, so that (for instance) $[\![u = \alpha]\!] = [\![u \in \{\alpha\}]\!] = [\![u \geq \alpha]\!] \setminus [\![u > \alpha]\!]$ for $u \in L^0$ and $\alpha \in \mathbb{R}$.

(b) If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, $\bar{\mu}\phi : \mathcal{B} \to [0, 1]$ is a probability measure, so that its completion ν is a Radon probability measure on $\mathbb{R}$ (256C); I will call ν the **distribution** of u (cf. 271C).

364H Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $E \subseteq \mathbb{R}$ a Borel set, and $h : E \to \mathbb{R}$ a Borel measurable function. Then whenever $u \in L^0 = L^0(\mathfrak{A})$ is such that $[\![u \in E]\!] = 1$, there is an element $\bar{h}(u)$ of L^0 defined by saying that $[\![\bar{h}(u) \in F]\!] = [\![u \in h^{-1}[F]]\!]$ for every Borel set $F \subseteq \mathbb{R}$.

proof All we have to observe is that $F \mapsto [\![u \in h^{-1}[F]]\!]$ is a sequentially order-continuous Boolean homomorphism. (The condition '$[\![u \in E]\!] = 1$' ensures that $[\![u \in h^{-1}[\mathbb{R}]]\!] = 1$.)

364I Examples Perhaps I should spell out the most important contexts in which we apply these ideas, even though they have in effect already been mentioned.

(a) Let X be a set and Σ a σ-algebra of subsets of X. Then we may identify $L^0(\Sigma)$ with the space $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ of Σ-measurable real-valued functions on X. (This is the case $\mathfrak{A} = \Sigma$ of 364C.) For $f \in \mathcal{L}^0$, $[\![f \in E]\!]$ (364G) is just $f^{-1}[E]$, for any Borel set $E \subseteq \mathbb{R}$; and if h is a Borel measurable function, $\bar{h}(f)$ (364H) is just the composition hf, for any $f \in \mathcal{L}^0$.

(b) Now suppose that $\mathcal{I}$ is a σ-ideal of Σ and that $\mathfrak{A} = \Sigma/\mathcal{I}$. Then, as in 364C, we identify $L^0(\mathfrak{A})$ with a quotient $\mathcal{L}^0/\mathcal{W}_\mathcal{I}$. For $f \in \mathcal{L}^0$, $[\![f^\bullet \in E]\!] = f^{-1}[E]^\bullet$, and $\bar{h}(f^\bullet) = (hf)^\bullet$, for any Borel set E and any Borel measurable function $h : \mathbb{R} \to \mathbb{R}$.

(c) In particular, if (X, Σ, μ) is a measure space with measure algebra $\mathfrak{A}$, then $L^0(\mathfrak{A})$ becomes identified with $L^0(\mu)$ as defined in §241, and the distribution of $f \in \mathcal{L}^0(\mu)$, as defined in 271C, is the same as the distribution of $f^\bullet \in L^0(\mu) \cong L^0(\mathfrak{A})$, as defined in 364Gb.

The same remarks as in 363I apply here; the space $\mathcal{L}^0(\mu)$ of 241A is larger than the space $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$ considered here. But for every $f \in \mathcal{L}^0(\mu)$ there is a $g \in \mathcal{L}^0_\Sigma$ such that $g =_{\text{a.e.}} f$ (241Bk), so that $L^0(\mu) = \mathcal{L}^0(\mu)/=_{\text{a.e.}}$ can be identified with $\mathcal{L}^0_\Sigma/\mathcal{N}$, where $\mathcal{N}$ is the set of functions in $\mathcal{L}^0$ which are zero almost everywhere (241Yc).

364J Embedding S and L^∞ in L^0: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.

(a) We have a canonical embedding of $L^\infty = L^\infty(\mathfrak{A})$ as an order-dense solid linear subspace of $L^0 = L^0(\mathfrak{A})$; it is the solid linear subspace generated by the multiplicative identity e of L^0. Consequently $S = S(\mathfrak{A})$ also is embedded as an order-dense Riesz subspace and subalgebra of L^0.

(b) This embedding respects the linear, lattice and multiplicative structures of L^∞ and S, and the definition of $[\![u > \delta]\!]$, for $u \in S^+$ and $\delta \geq 0$, given in 361Eg.

(c) For $a \in \mathfrak{A}$, χa, when regarded as a member of L^0, can be described by the formula

$$\begin{aligned}[\![\chi a > \alpha]\!] &= 1 \text{ if } \alpha < 0,\\ &= a \text{ if } 0 \leq \alpha < 1,\\ &= 0 \text{ if } 1 \leq \alpha.\end{aligned}$$

The function $\chi : \mathfrak{A} \to L^0$ is additive, injective, order-continuous and a lattice homomorphism.

(d) For every $u \in (L^0)^+$ there is a non-decreasing sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in S such that $u_0 \geq 0$ and $\sup_{n\in\mathbb{N}} u_n = u$.

proof Let Z, Σ, $\mathcal{M}$, $\mathcal{L}^0 = \mathcal{L}^0_\Sigma$, $\mathcal{W} = \mathcal{W}_\mathcal{M}$ and π be as in the proof of 364D. I defined L^∞ to be the space $C(Z)$ of continuous real-valued functions on Z (363A); but because $\mathfrak{A}$ is Dedekind σ-complete, there is an alternative representation as $\mathcal{L}^\infty/\mathcal{W} \cap \mathcal{L}^\infty$, where $\mathcal{L}^\infty$ is the space of bounded Σ-measurable functions from Z to $\mathbb{R}$ (363Hb). Put like this, we clearly have an embedding of $L^\infty \cong \mathcal{L}^\infty/\mathcal{W} \cap \mathcal{L}^\infty$ in $L^0 \cong \mathcal{L}^0/\mathcal{W}$; and this embedding represents L^∞ as a Riesz subspace and subalgebra of L^0, because $\mathcal{L}^\infty$ is a Riesz subspace and subalgebra of $\mathcal{L}^0$. L^∞ becomes the solid linear subspace of L^0 generated by $(\chi Z)^\bullet = e$, because $\mathcal{L}^\infty$ is the solid linear subspace of $\mathcal{L}^0$ generated by χZ. To see that L^∞ is order-dense in L^0, we have only to note that $f = \sup_{n\in\mathbb{N}} f \wedge n\chi Z$ in $\mathcal{L}^0$ for every $f \in \mathcal{L}^0$, and therefore (because the map $f \mapsto f^\bullet$ is sequentially order-continuous) $u = \sup_{n\in\mathbb{N}} u \wedge ne$ in L^0 for every $u \in L^0$.

To identify χa, we have the formula $\chi(\pi E) = (\chi E)^\bullet$, as in 363H(b-iii); but this means that, if $a = \pi E$,

$$\begin{aligned}[\![\chi a > \alpha]\!] = \pi\{z : \chi E(z) > \alpha\} &= \pi Z = 1 \text{ if } \alpha < 0,\\ &= \pi E = a \text{ if } 0 \leq \alpha < 1,\\ &= \pi\emptyset = 0 \text{ if } \alpha \geq 1,\end{aligned}$$

using the formula in 364Ca. Evidently χ is injective.

Because S is an order-dense Riesz subspace and subalgebra of L^∞ (363C), the same embedding represents it as an order-dense Riesz subspace and subalgebra of L^0. (For 'order-dense', use 352N(c-iii).) Concerning the formula $[\![u>\delta]\!]$, suppose that $u\in S^+$ and $\delta\ge 0$; express u as $\sum_{j=0}^m\beta_j\chi b_j$, where $b_0,\dots,b_m\in\mathfrak{A}$ are disjoint and $\beta_j\ge 0$ for every j. Then we have disjoint sets $F_0,\dots,F_m\in\Sigma$ such that $\pi F_j=b_j$ for every j, and u is identified with $(\sum_{j=0}^m\beta_j\chi F_j)^\bullet$. Using 364Ca, we have

$$[\![u>\delta]\!]=\pi\{z:\sum_{j=0}^m\beta_j\chi F_j(z)>\delta\}=\pi(\bigcup\{F_j:\beta_j>\delta\})=\sup\{b_j:\beta_j>\delta\},$$

matching the expression in the proof of 361Eg. So the new interpretation of $[\![\dots]\!]$ matches the former definition in the special case envisaged in 361E.

Because $\chi:\mathfrak{A}\to L^\infty$ is additive, order-continuous and a lattice homomorphism (363D), and the embedding map $L^\infty\subsetneqq L^0$ also is, $\chi:\mathfrak{A}\to L^0$ has the same properties.

Finally, if $u\ge 0$ in L^0, we can represent it as $f^\bullet$ where $f\ge 0$ in $\mathcal{L}^0$. For $n\in\mathbb{N}$ set

$$\begin{aligned}f_n(z)&=2^{-n}k \text{ if } 2^{-n}k\le f(z)<2^{-n}(k+1) \text{ where } 0\le k<4^n,\\ &=0 \text{ if } f(z)\ge 2^n;\end{aligned}$$

then $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in S^+ with supremum u.

364K Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Then $S(\mathfrak{A}^f)$ can be embedded as a Riesz subspace of $L^0(\mathfrak{A})$, which is order-dense iff $(\mathfrak{A},\bar\mu)$ is semi-finite.

proof (Recall that $\mathfrak{A}^f$ is the ring $\{a:\bar\mu a<\infty\}$.) The embedding $\mathfrak{A}^f\subsetneqq\mathfrak{A}$ is an injective ring homomorphism, so induces an embedding of $S(\mathfrak{A}^f)$ as a Riesz subspace of $S(\mathfrak{A})$, by 361J. Now $S(\mathfrak{A}^f)$ is order-dense in $S(\mathfrak{A})$ iff $(\mathfrak{A},\bar\mu)$ is semi-finite. **P** (i) If $(\mathfrak{A},\bar\mu)$ is semi-finite and $v>0$ in $S(\mathfrak{A})$, then v is expressible as $\sum_{j=0}^n\beta_j\chi b_j$ where $\beta_j\ge 0$ for each j and some $\beta_j\chi b_j$ is non-zero; now there is a non-zero $a\in\mathfrak{A}^f$ such that $a\subseteq b_j$, so that $0<\beta_j\chi a\in S(\mathfrak{A}^f)$ and $\beta_j\chi a\le v$. As v is arbitrary, $S(\mathfrak{A}^f)$ is quasi-order-dense, therefor order-dense (353A). (ii) If $S(\mathfrak{A}^f)$ is order-dense in $S(\mathfrak{A})$ and $b\in\mathfrak{A}\setminus\{0\}$, there is a $u>0$ in $S(\mathfrak{A}^f)$ such that $u\le\chi b$; now there are $\alpha>0$, $a\in\mathfrak{A}^f\setminus\{0\}$ such that $\alpha\chi a\le u$, in which case $a\subseteq b$. **Q**

Now because $S(\mathfrak{A}^f)\subseteq S(\mathfrak{A})$ and $S(\mathfrak{A})$ is order-dense in $L^0(\mathfrak{A})$, we must have

$$\begin{aligned}S(\mathfrak{A}^f)\text{ is order-dense in }L^0(\mathfrak{A})&\iff S(\mathfrak{A}^f)\text{ is order-dense in }S(\mathfrak{A})\\ &\iff(\mathfrak{A},\bar\mu)\text{ is semi-finite.}\end{aligned}$$

364L Suprema and infima in L^0 We know that any $L^0(\mathfrak{A})$ is a Dedekind σ-complete partially ordered set. There is a useful description of suprema for this ordering in (a) of the next result. We do not have such a simple formula for general infima (though see 364Xm), but facts in (b) are useful.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $L^0=L^0(\mathfrak{A})$.

(a) Let A be a subset of L^0.

(i) A is bounded above in L^0 iff there is a sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$, with infimum 0, such that $[\![u>n]\!]\subseteq c_n$ for every $u\in A$.

(ii) If A is non-empty, then A has a supremum in L^0 iff $c_\alpha=\sup_{u\in A}[\![u>\alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha\in\mathbb{R}$ and $\inf_{n\in\mathbb{N}}c_n=0$; and in this case $c_\alpha=[\![\sup A>\alpha]\!]$ for every α.

(iii) If A is non-empty and bounded above, then A has a supremum in L^0 iff $\sup_{u\in A}[\![u>\alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha\in\mathbb{R}$.

(b)(i) If u, $v\in L^0$, then $[\![u\wedge v>\alpha]\!]=[\![u>\alpha]\!]\cap[\![v>\alpha]\!]$ for every $\alpha\in\mathbb{R}$.

(ii) If A is a non-empty subset of $(L^0)^+$, then $\inf A=0$ in L^0 iff $\inf_{u\in A}[\![u>\alpha]\!]=0$ in $\mathfrak{A}$ for every $\alpha>0$.

proof (a)(i)($\boldsymbol{\alpha}$) If A has an upper bound u_0, set $c_n=[\![u_0>n]\!]$ for each n; then $\langle c_n\rangle_{n\in\mathbb{N}}$ satisfies the conditions.

($\boldsymbol{\beta}$) If $\langle c_n\rangle_{n\in\mathbb{N}}$ satisfies the conditions, set

$$\begin{aligned}\phi(\alpha) &= 1 \text{ if } \alpha < 0,\\ &= \inf_{i\le n} c_i \text{ if } n\in\mathbb{N},\ \alpha\in [n,n+1[\,.\end{aligned}$$

Then it is easy to check that ϕ satisfies the conditions of 364Aa, since $\inf_{n\in\mathbb{N}} c_n = 0$. So there is a $u_0 \in L^0$ such that $\phi(\alpha) = [\![u_0 > \alpha]\!]$ for each α. Now, given $u \in A$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}[\![u > \alpha]\!] &\subseteq 1 = [\![u_0 > \alpha]\!] \text{ if } \alpha < 0,\\ &\subseteq \inf_{i\le n} [\![u > i]\!] \subseteq \inf_{i\le n} c_i = [\![u_0 > \alpha]\!] \text{ if } n\in\mathbb{N},\ \alpha\in [n,n+1[\,.\end{aligned}$$

Thus u_0 is an upper bound for A in L^0.

(ii)($\boldsymbol{\alpha}$) Suppose that $c_\alpha = \sup_{u\in A} [\![u > \alpha]\!]$ is defined in $\mathfrak{A}$ for every α, and that $\inf_{n\in\mathbb{N}} c_n = 0$. Then, for any α,

$$\sup_{q\in\mathbb{Q},q>\alpha} c_q = \sup_{u\in A,q\in\mathbb{Q},q>\alpha} [\![u > q]\!] = \sup_{u\in A} [\![u > \alpha]\!] = c_\alpha.$$

Also, we are supposing that A contains some u_0, so that

$$\sup_{n\in\mathbb{N}} c_{-n} \supseteq \sup_{n\in\mathbb{N}} [\![u_0 > -n]\!] = 1.$$

Accordingly there is a $u^* \in L^0$ such that $[\![u^* > \alpha]\!] = c_\alpha$ for every $\alpha \in \mathbb{R}$. But now, for $v \in L^0$,

$$\begin{aligned}v \text{ is an upper bound for } A &\iff [\![u > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } u\in A,\ \alpha\in\mathbb{R}\\ &\iff [\![u^* > \alpha]\!] \subseteq [\![v > \alpha]\!] \text{ for every } \alpha\\ &\iff u^* \le v,\end{aligned}$$

so that $u^* = \sup A$ in L^0.

($\boldsymbol{\beta}$) Now suppose that $u^* = \sup A$ is defined in L^0. Of course $[\![u^* > \alpha]\!]$ must be an upper bound for $\{[\![u > \alpha]\!] : u \in A\}$ for every α. **?** Suppose we have an α for which it is not the least upper bound, that is, there is a $c \subset [\![u^* > \alpha]\!]$ which is an upper bound for $\{[\![u > \alpha]\!] : u \in A\}$. Define $\phi : \mathbb{R} \to \mathfrak{A}$ by setting

$$\begin{aligned}\phi(\beta) &= c \cap [\![u^* > \beta]\!] \text{ if } \beta \ge \alpha,\\ &= [\![u^* > \beta]\!] \text{ if } \beta < \alpha.\end{aligned}$$

It is easy to see that ϕ satisfies the conditions of 364Aa (we need the distributive law 313Ba to check that $\phi(\beta) = \sup_{\gamma>\beta} \phi(\gamma)$ if $\beta \ge \alpha$), so corresponds to a member v of L^0. But we now find that v is an upper bound for A (because if $u \in A$ and $\beta \ge \alpha$ then

$$[\![u > \beta]\!] \subseteq [\![u > \alpha]\!] \cap [\![u^* > \beta]\!] \subseteq c \cap [\![u^* > \beta]\!] = [\![v > \beta]\!],)$$

that $v \le u^*$ and that $v \ne u^*$ (because $[\![v > \alpha]\!] = c \ne [\![u^* > \alpha]\!]$); but this is impossible, because u^* is supposed to be the supremum of A. **X** Thus if $u^* = \sup A$ is defined in L^0, then $\sup_{u\in A} [\![u > \alpha]\!] = [\![u^* > \alpha]\!]$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$. Also, of course,

$$\inf_{n\in\mathbb{N}} \sup_{u\in A} [\![u > n]\!] = \inf_{n\in\mathbb{N}} [\![u^* > n]\!] = 0.$$

(iii) This is now easy. If A has a supremum, then surely it satisfies the condition, by (b). If A satisfies the condition, then we have a family $\langle c_\alpha\rangle_{\alpha\in\mathbb{R}}$ as required in (b); but also, by (a) or otherwise, there is a sequence $\langle c'_n\rangle_{n\in\mathbb{N}}$ such that $c_n \subseteq c'_n$ for every n and $\inf_{n\in\mathbb{N}} c'_n = 0$, so $\inf_{n\in\mathbb{N}} c_n$ also is 0, and both conditions in (b) are satisfied, so A has a supremum.

(b)(i) Take Z, $\mathcal{L}^0$ and π as in the proof of 364D. Express u as $f^\bullet$, v as $g^\bullet$ where f, $g \in \mathcal{L}^0$, so that $u \wedge v = (f \wedge g)^\bullet$, because the canonical map from $\mathcal{L}^0$ to L^0 is a Riesz homomorphism (351J). Then

$$\begin{aligned}[\![u\wedge v > \alpha]\!] &= \pi\{z : \min(f(z), g(z)) > \alpha\} = \pi(\{z : f(z) > \alpha\} \cap \{z : g(z) > \alpha\})\\ &= \pi\{z : f(z) > \alpha\} \cap \pi\{z : g(z) > \alpha\} = [\![u > \alpha]\!] \cap [\![v > \alpha]\!]\end{aligned}$$

for every α.

(ii)(α) If $\inf_{u\in A}[\![u>\alpha]\!]=0$ for every $\alpha>0$, and v is any lower bound for A, then $[\![v>\alpha]\!]$ must be 0 for every $\alpha>0$, so that $[\![v>0]\!]=0$; now since $[\![0>\alpha]\!]=0$ for $\alpha\ge 0$, 1 for $\alpha<0$, $v\le 0$. As v is arbitrary, $\inf A=0$.

(β) If $\alpha>0$ is such that $\inf_{u\in A}[\![u>\alpha]\!]$ is undefined, or not equal to 0, let $c\in\mathfrak{A}$ be such that $0\ne c\subseteq[\![u>\alpha]\!]$ for every $u\in A$, and consider $v=\alpha\chi c$. Then $[\![v>\beta]\!]=[\![\chi c>\frac{\beta}{\alpha}]\!]$ is 1 if $\beta<0$, c if $0\le\beta<\alpha$ and 0 if $\beta\ge\alpha$. If $u\in A$ then $[\![u>\beta]\!]$ is 1 if $\beta<0$ (since $u\ge 0$), at least $[\![u>\alpha]\!]\supseteq c$ if $0\le\beta<\alpha$, and always includes 0; so that $v\le u$. As u is arbitrary, $\inf A$ is either undefined in L^0 or not 0.

364M Now we have a reward for our labour, in that the following basic theorem is easy.

Theorem For a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, $L^0=L^0(\mathfrak{A})$ is Dedekind complete iff $\mathfrak{A}$ is.

proof The description of suprema in 364L(a-iii) makes it obvious that if $\mathfrak{A}$ is Dedekind complete, so that $\sup_{u\in A}[\![u>\alpha]\!]$ is always defined, then L^0 must be Dedekind complete. On the other hand, if L^0 is Dedekind complete, then so is $L^\infty(\mathfrak{A})$ (by 364J and 353K(b-i)), so that $\mathfrak{A}$ also is Dedekind complete, by 363Mb.

364N The multiplication of L^0 I have already observed that L^0 is always an f-algebra with identity; in particular (because L^0 is surely Archimedean) the map $u\mapsto u\times v$ is order-continuous for every $v\ge 0$ (353Pa), and multiplication is commutative (353Pb, or otherwise). The multiplicative identity is $\chi 1$ (364D, 364Jc). By 353Qb, or otherwise, $u\times v=0$ iff $|u|\wedge|v|=0$. There is one special feature of multiplication in L^0 which I can mention here.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then an element u of $L^0=L^0(\mathfrak{A})$ has a multiplicative inverse in L^0 iff $|u|$ is a weak order unit in L^0 iff $[\![|u|>0]\!]=1$.

proof If u is invertible, then $|u|$ is a weak order unit, by 353Qc or otherwise. In this case, setting $c=1\setminus[\![|u|>0]\!]$, we see that

$$[\![|u|\wedge\chi c>0]\!]=[\![|u|>0]\!]\cap c=0$$

(364L(b-i)), so that $|u|\wedge\chi c\le 0$ and $\chi c=0$, that is, $c=0$; so $[\![|u|>0]\!]$ must be 1. To complete the circuit, suppose that $[\![|u|>0]\!]=1$. Let Z, Σ, $\mathcal{L}^0=\mathcal{L}^0_\Sigma$, π, $\mathcal{M}$ be as in the proof of 364D, and $S:\mathcal{L}^0\to L^0$ the canonical map, so that $[\![Sh>\alpha]\!]=\pi\{z:h(z)>\alpha\}$ for every $h\in\mathcal{L}^0$, $\alpha\in\mathbb{R}$. Express u as Sf where $f\in\mathcal{L}^0$. Then $\pi\{z:|f(z)|>0\}=[\![S|f|>0]\!]=1$, so $\{z:f(z)=0\}\in\mathcal{M}$. Set

$$g(z)=\frac{1}{f(z)}\text{ if }f(z)\ne 0,\quad g(z)=0\text{ if }f(z)=0.$$

Then $\{z:f(z)g(z)\ne 1\}\in\mathcal{M}$ so

$$u\times Sg=S(f\times g)=S(\chi Z)=\chi 1$$

and u is invertible.

Remark The repeated phrase 'by 353x or otherwise' reflects the fact that the abstract methods there can all be replaced in this case by simple direct arguments based on the construction in 364B-364D.

364O Recovering the algebra: Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. For $a\in\mathfrak{A}$ write V_a for the band in $L^0=L^0(\mathfrak{A})$ generated by χa. Then $a\mapsto V_a$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra of projection bands in L^0.

proof I copy from the argument for 363J, itself based on 361K. If $a\in\mathfrak{A}$ and $w\in L^0$ then $w\times\chi a\in V_a$. **P** If $v\in V_a^\perp$ then $|\chi a|\wedge|v|=0$, so $\chi a\times v=0$, so $(w\times\chi a)\times v=0$, so $|w\times\chi a|\wedge|v|=0$; thus $w\times\chi a\in V_a^{\perp\perp}$, which is equal to V_a because L^0 is Archimedean (353Ba). **Q** Now, if $a\in\mathfrak{A}$, $u\in V_a$ and $v\in V_{1\setminus a}$, then $|u|\wedge|v|=0$ because $\chi a\wedge\chi(1\setminus a)=0$; and if $w\in L^0(\mathfrak{A})$ then

$$w=(w\times\chi a)+(w\times\chi(1\setminus a))\in V_a+V_{1\setminus a}.$$

So V_a and $V_{1\setminus a}$ are complementary projection bands in L^0. Next, if $U\subseteq L^0$ is a projection band, then $\chi 1$ is expressible as $u+v=u\vee v$ where $u\in U$, $v\in U^\perp$. Setting $a=[\![u>\frac{1}{2}]\!]$, $a'=[\![v>\frac{1}{2}]\!]$ we must have $a\cup a'=1$ and $a\cap a'=0$ (using 364L), so that $a'=1\setminus a$; also $\frac{1}{2}\chi a\le u$, so that $\chi a\in U$, and similarly $\chi(1\setminus a)\in U^\perp$. In this case $V_a\subseteq U$ and $V_{1\setminus a}\subseteq U^\perp$, so U must be V_a precisely. Thus $a\mapsto V_a$ is surjective. Finally, just as in 361K, $a\subseteq b\iff V_a\subseteq V_b$, so we have a Boolean isomorphism.

364P I come at last to the result corresponding to 361J and 363F.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism.

(a) We have a multiplicative sequentially order-continuous Riesz homomorphism $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ defined by the formula

$$[\![T_\pi u > \alpha]\!] = \pi[\![u > \alpha]\!]$$

whenever $\alpha \in \mathbb{R}$ and $u \in L^0(\mathfrak{A})$.

(b) Defining $\chi a \in L^0(\mathfrak{A})$ as in 364J, $T_\pi(\chi a) = \chi(\pi a)$ in $L^0(\mathfrak{B})$ for every $a \in \mathfrak{A}$. If we regard $L^\infty(\mathfrak{A})$ and $L^\infty(\mathfrak{B})$ as embedded in $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$ respectively, then T_π, as defined here, agrees on $L^\infty(\mathfrak{A})$ with T_π as defined in 363F.

(c) T_π is order-continuous iff π is order-continuous, injective iff π is injective, surjective iff π is surjective.

(d) $[\![T_\pi u \in E]\!] = \pi[\![u \in E]\!]$ for every $u \in L^0(\mathfrak{A})$ and every Borel set $E \subseteq \mathbb{R}$; consequently $\bar{h}T_\pi = T_\pi\bar{h}$ for every Borel measurable $h : \mathbb{R} \to \mathbb{R}$, writing $\bar{h}$ indifferently for the associated maps from $L^0(\mathfrak{A})$ to itself and from $L^0(\mathfrak{B})$ to itself (364H).

(e) If $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ another sequentially order-continuous Boolean homomorphism then $T_{\theta\pi} = T_\theta T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$.

proof I write T for T_π.

(a)(i) To see that Tu is well-defined in $L^0(\mathfrak{B})$ for every $u \in L^0(\mathfrak{A})$, all we need to do is to check that the map $\alpha \mapsto \pi[\![u > \alpha]\!] : \mathbb{R} \to \mathfrak{B}$ satisfies the conditions of 364Ae, and this is easy, because π preserves all countable suprema and infima.

(ii) To see that T is linear and order-preserving and multiplicative, we can use the formulae of 364D. For instance, if u, $v \in L^0(\mathfrak{A})$, then

$$\begin{aligned}[\![Tu + Tv > \alpha]\!] &= \sup_{q\in\mathbb{Q}} [\![Tu > q]\!] \cap [\![Tv > \alpha - q]\!] = \sup_{q\in\mathbb{Q}} \pi[\![u > q]\!] \cap \pi[\![v > \alpha - q]\!] \\ &= \pi(\sup_{q\in\mathbb{Q}} [\![u > q]\!] \cap [\![v > \alpha - q]\!]) = \pi[\![u + v > \alpha]\!] = [\![T(u + v) > \alpha]\!]\end{aligned}$$

for every $\alpha \in \mathbb{R}$, so that $Tu + Tv = T(u + v)$. In the same way,

$$T(\gamma u) = \gamma Tu \text{ whenever } \gamma > 0,$$

$$Tu \le Tv \text{ whenever } u \le v,$$

$$Tu \times Tv = T(u \times v) \text{ whenever } u,\ v \ge 0,$$

so that, using the distributive laws, T is linear and multiplicative.

To see that T is a sequentially order-continuous Riesz homomorphism, suppose that $A \subseteq L^0(\mathfrak{A})$ is a countable non-empty set with a supremum $u^* \in L^0(\mathfrak{A})$; then $T[A]$ is a non-empty subset of $L^0(\mathfrak{B})$ with an upper bound Tu^*, and

$$\sup_{u\in A} [\![Tu > \alpha]\!] = \sup_{u\in A} \pi[\![u > \alpha]\!] = \pi(\sup_{u\in A} [\![u > \alpha]\!]) = \pi[\![u^* > \alpha]\!]$$

(using 364La)

$$= [\![Tu^* > \alpha]\!]$$

for every $\alpha \in \mathbb{R}$. So using 364La again, $Tu^* = \sup_{u\in A} Tu$. Now this is true, in particular, for doubleton sets A, so that T is a Riesz homomorphism; and also for non-decreasing sequences, so that T is sequentially order-continuous.

(b) The identification of $T(\chi a)$ with $\chi(\pi a)$ is another almost trivial verification. It follows that T agrees with the map of 363F on $S(\mathfrak{A})$, if we think of $S(\mathfrak{A})$ as a subspace of $L^0(\mathfrak{A})$. Next, if $u \in L^\infty(\mathfrak{A}) \subseteq L^0(\mathfrak{A})$, and $\gamma = \|u\|_\infty$, then $|u| \le \gamma\chi 1_{\mathfrak{A}}$, so that $|Tu| \le \gamma\chi 1_{\mathfrak{B}}$, and $Tu \in L^\infty(\mathfrak{B})$, with $\|Tu\|_\infty \le \gamma$. Thus $T{\restriction}L^\infty(\mathfrak{A})$

has norm at most 1. As it agrees with the map of 363F on $S(\mathfrak{A})$, which is $\| \|_\infty$-dense in $L^\infty(\mathfrak{A})$ (363C), and both are continuous, they must agree on the whole of $L^\infty(\mathfrak{A})$.

(c)(i)(*α*) Suppose that π is order-continuous, and that $A \subseteq L^0(\mathfrak{A})$ is a non-empty set with a supremum $u^* \in L^0(\mathfrak{A})$. Then for any $\alpha \in \mathbb{R}$,

$$[\![Tu^* > \alpha]\!] = \pi[\![u^* > \alpha]\!] = \pi(\sup_{u \in A} [\![u > \alpha]\!])$$

(by 364La)

$$= \sup_{u \in A} \pi[\![u > \alpha]\!]$$

(because π is order-continuous)

$$= \sup_{u \in A} [\![Tu > \alpha]\!].$$

As α is arbitrary, $Tu^* = \sup T[A]$, by 364La again. As A is arbitrary, T is order-continuous (351Ga).

(*β*) Now suppose that T is order-continuous and that $A \subseteq \mathfrak{A}$ is a non-empty set with supremum c in $\mathfrak{A}$. Then $\chi c = \sup_{a \in A} \chi a$ (364Jc) so

$$\chi(\pi c) = T(\chi c) = \sup_{a \in A} T(\chi a) = \sup_{a \in A} \chi(\pi a).$$

But now

$$\pi c = [\![\chi(\pi c) > 0]\!] = \sup_{a \in A} [\![\chi(\pi a) > 0]\!] = \sup_{a \in A} \pi a.$$

As A is arbitrary, π is order-continuous.

(ii)(*α*) If π is injective and u, v are distinct elements of $L^0(\mathfrak{A})$, then there must be some α such that $[\![u > \alpha]\!] \ne [\![v > \alpha]\!]$, in which case $[\![Tu > \alpha]\!] \ne [\![Tv > \alpha]\!]$ and $Tu \ne Tv$.

(*β*) Now suppose that T is injective. It is easy to see that $\chi : \mathfrak{A} \to L^0(\mathfrak{A})$ is injective, so that $T\chi : \mathfrak{A} \to L^0(\mathfrak{B})$ is injective; but this is the same as $\chi\pi$ (by (b)), so π must also be injective.

(iii)(*α*) Suppose that π is surjective. Let Σ be a σ-algebra of sets such that there is a sequentially order-continuous Boolean surjection $\phi : \Sigma \to \mathfrak{A}$. Then $\pi\phi : \Sigma \to \mathfrak{B}$ is surjective. So given $w \in L^0(\mathfrak{B})$, there is an $f \in \mathcal{L}^0_\Sigma$ such that $[\![w > \alpha]\!] = \pi\phi\{x : f(x) > \alpha\}$ for every $\alpha \in \mathbb{R}$ (364C). But, also by 364C, there is a $u \in L^0(\mathfrak{A})$ such that $[\![u > \alpha]\!] = \phi\{x : f(x) > \alpha\}$ for every α. And now of course $Tu = w$. As w is arbitrary, T is surjective.

(*β*) If T is surjective, and $b \in \mathfrak{B}$, there must be some $u \in L^0(\mathfrak{A})$ such that $Tu = \chi b$. Now set $a = [\![u > 0]\!]$ and see that $\pi a = [\![\chi b > 0]\!] = b$. As b is arbitrary, π is surjective.

(d) The map $E \mapsto \pi[\![u \in E]\!]$ is a sequentially order-continuous Boolean homomorphism, equal to $[\![Tu \in E]\!]$ when E is of the form $]\alpha, \infty[$; so by 364F the two are equal for all Borel sets E.

If $h : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, $u \in L^0(\mathfrak{A})$ and $E \subseteq \mathbb{R}$ is a Borel set, then

$$\begin{aligned}[\![\bar{h}(Tu) \in E]\!] &= [\![Tu \in h^{-1}[E]]\!] = \pi[\![u \in h^{-1}[E]]\!] \\ &= \pi[\![\bar{h}(u) \in E]\!] = [\![T(\bar{h}(u)) \in E]\!].\end{aligned}$$

As E and u are arbitrary, $T\bar{h} = \bar{h}T$.

(e) This is immediate from (a).

364Q Proposition Let X and Y be sets, Σ, T σ-algebras of subsets of X, Y respectively, and $\mathcal{I}$, $\mathcal{J}$ σ-ideals of Σ, T. Set $\mathfrak{A} = \Sigma/\mathcal{I}$ and $\mathfrak{B} = \mathrm{T}/\mathcal{J}$. Suppose that $\phi : X \to Y$ is a function such that $\phi^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$ and $\phi^{-1}[F] \in \mathcal{I}$ for every $F \in \mathcal{J}$.

(a) There is a sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by saying that $\pi F^\bullet = \phi^{-1}[F]^\bullet$ for every $F \in \mathrm{T}$.

(b) Let $T_\pi : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ be the Riesz homomorphism corresponding to π, as defined in 364P. If we identify $L^0(\mathfrak{B})$ with $\mathcal{L}^0_{\mathrm{T}}/\mathcal{W}_{\mathcal{J}}$ and $L^0(\mathfrak{A})$ with $\mathcal{L}^0_{\Sigma}/\mathcal{W}_{\mathcal{I}}$ in the manner of 364B-364C, then $T_\pi(g^\bullet) = (g\phi)^\bullet$ for every $g \in \mathcal{L}^0_{\mathrm{T}}$.

(c) Let Z be a third set, Υ a σ-algebra of subsets of Z, $\mathcal{K}$ a σ-ideal of Υ, and $\psi : Y \to Z$ a function such that $\psi^{-1}[G] \in \mathrm{T}$ for every $G \in \Upsilon$ and $\psi^{-1}[G] \in \mathcal{J}$ for every $G \in \mathcal{K}$. Let $\theta : \mathfrak{C} \to \mathfrak{B}$ and $T_\theta : L^0(\mathfrak{C}) \to L^0(\mathfrak{B})$ be the homomorphisms corresponding to ψ as in (a)-(b). Then $\pi\theta : \mathfrak{C} \to \mathfrak{A}$ and $T_\pi T_\theta : L^0(\mathfrak{C}) \to L^0(\mathfrak{A})$ correspond to $\psi\phi : X \to Y$ in the same way.

(d) Now suppose that μ and ν are measures with domains Σ, T and null ideals $\mathcal{N}(\mu)$, $\mathcal{N}(\nu)$ respectively, and that $\mathcal{I} = \Sigma \cap \mathcal{N}(\mu)$ and $\mathcal{J} = \mathrm{T} \cap \mathcal{N}(\nu)$. In this case, identifying $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ with $L^0(\mu)$ and $L^0(\nu)$ as in 364Ic, we have $g\phi \in \mathcal{L}^0(\mu)$ and $T_\pi(g^\bullet) = (g\phi)^\bullet$ for every $g \in \mathcal{L}^0(\nu)$.

proof (a) The argument is essentially that of 324A-324B, somewhat simplified. Explicitly: if F_1, $F_2 \in \mathrm{T}$ and $F_1^\bullet = F_2^\bullet$, then $F_1 \triangle F_2 \in \mathcal{J}$ so $\phi^{-1}[F_1] \triangle \phi^{-1}[F_2] = \phi^{-1}[F_1 \triangle F_2]$ belongs to $\mathcal{I}$ and $\phi^{-1}[F_1]^\bullet = \phi^{-1}[F_2]^\bullet$. So the formula offered defines a map $\pi : \mathfrak{B} \to \mathfrak{A}$. It is a Boolean homomorphism, because if F_1, $F_2 \in \mathrm{T}$ then

$$\begin{aligned}\pi F_1^\bullet \mathbin{\triangle} \pi F_2^\bullet &= \phi^{-1}[F_1]^\bullet \mathbin{\triangle} \phi^{-1}[F_2]^\bullet = (\phi^{-1}[F_1] \triangle \phi^{-1}[F_2])^\bullet \\ &= \phi^{-1}[F_1 \triangle F_2]^\bullet = \pi(F_1 \triangle F_2)^\bullet = \pi(F_1^\bullet \mathbin{\triangle} F_2^\bullet),\end{aligned}$$

so $\pi(b_1 \mathbin{\triangle} b_2) = \pi b_1 \mathbin{\triangle} b_2$ for all b_1, $b_2 \in \mathfrak{B}$. Similarly $\pi(b_1 \cap b_2) = \pi b_1 \cap b_2$ for all b_1, $b_2 \in \mathfrak{B}$, and of course

$$\pi 1_{\mathfrak{B}} = \pi Y^\bullet = \phi^{-1}[Y]^\bullet = X^\bullet = 1_{\mathfrak{A}}.$$

To see that π is sequentially order-continuous, let $\langle b_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$. For each n we may choose an $F_n \in \mathrm{T}$ such that $F_n^\bullet = b_n$, and set $F = \bigcup_{n\in\mathbb{N}} F_n$. As the map $H \mapsto H^\bullet : \mathrm{T} \to \mathfrak{B}$ is sequentially order-continuous (313Qb), $F^\bullet = \sup_{n\in\mathbb{N}} b_n$ in $\mathfrak{B}$. Now

$$\begin{aligned}\pi(\sup_{n\in\mathbb{N}} b_n) &= \pi F^\bullet = \phi^{-1}[F]^\bullet = (\bigcup_{n\in\mathbb{N}} \phi^{-1}[F_n])^\bullet \\ &= \sup_{n\in\mathbb{N}} \phi^{-1}[F_n]^\bullet = \sup_{n\in\mathbb{N}} \pi F_n^\bullet = \sup_{n\in\mathbb{N}} \pi b_n.\end{aligned}$$

So π is sequentially order-continuous, by 313Lc.

(b) Now suppose that $g : Y \to \mathbb{R}$ is T-measurable; write v for $g^\bullet$ in $\mathcal{L}^0_{\mathrm{T}}/\mathcal{W}_{\mathcal{J}} \cong L^0(\mathfrak{B})$. Set $f = g\phi$; then

$$\{x : f(x) > \alpha\} = \phi^{-1}[\{y : g(y) > \alpha\}]$$

belongs to Σ for every $\alpha \in \mathbb{R}$, so f is Σ-measurable and we can speak of $u = f^\bullet$ in $\mathcal{L}^0_{\Sigma}/\mathcal{W}_{\mathcal{I}} \cong L^0(\mathfrak{A})$. Now, by 364Ca,

$$\begin{aligned}[\![u > \alpha]\!] &= \{x : f(x) > \alpha\}^\bullet = \phi^{-1}[\{y : g(y) > \alpha\}]^\bullet \\ &= \pi\{y : g(y) > \alpha\}^\bullet = \pi[\![v > \alpha]\!] = [\![T_\pi v > \alpha]\!]\end{aligned}$$

for every $\alpha \in \mathbb{R}$, and

$$(g\phi)^\bullet = f^\bullet = u = T_\pi v = T_\pi g^\bullet,$$

as claimed.

(c) Starting from the facts that $(\psi\phi)^{-1}[G] = \phi^{-1}[\psi^{-1}[G]]$ for every $G \in \Upsilon$ and $h(\psi\phi) = (h\psi)\phi$ for every $h \in \mathcal{L}^0_{\Upsilon}$, we just have to run through the formulae.

(d) If $g \in \mathcal{L}^0(\nu)$, there are a $g_0 \in \mathcal{L}^0_{\mathrm{T}}$ and an $F \in \mathcal{J}$ such that $g(y)$ is defined and equal to $g_0(y)$ for every $y \in Y \setminus F$. In this case, $\phi^{-1}[F] \in \mathcal{I}$ and $g\phi(x)$ is defined and equal to $g_0\phi(x)$ for every $x \in X \setminus \phi^{-1}[F]$, so $g\phi \in \mathcal{L}^0(\mu)$ and

$$(g\phi)^\bullet = (g_0\phi)^\bullet = T_\pi(g_0^\bullet) = T_\pi(g^\bullet)$$

by (b).

364R Products: Proposition Let $\langle \mathfrak{A}_i \rangle_{i\in I}$ be a family of Dedekind σ-complete Boolean algebras, with simple product $\mathfrak{A}$. If $\pi_i : \mathfrak{A} \to \mathfrak{A}_i$ is the coordinate map for each i, and $T_i : L^0(\mathfrak{A}) \to L^0(\mathfrak{A}_i)$ the corresponding homomorphism, then $u \mapsto Tu = \langle T_i u\rangle_{i\in I} : L^0(\mathfrak{A}) \to \prod_{i\in I} L^0(\mathfrak{A}_i)$ is a multiplicative Riesz space isomorphism, so $L^0(\mathfrak{A})$ may be identified with the f-algebra product $\prod_{i\in I} L^0(\mathfrak{A}_i)$ (352Wc).

proof Because each π_i is a surjective order-continuous Boolean homomorphism, 364P assures us that there are corresponding surjective multiplicative Riesz homomorphisms T_i. So all we need to check is that the multiplicative Riesz homomorphism $T : L^0(\mathfrak{A}) \to \prod_{i\in I} L^0(\mathfrak{A}_i)$ is a bijection.

If u, $v \in L^0(\mathfrak{A})$ are distinct, there must be some $\alpha \in \mathbb{R}$ such that $[\![u > \alpha]\!] \ne [\![v > \alpha]\!]$. In this case there must be an $i \in I$ such that $\pi_i[\![u > \alpha]\!] \ne \pi_i[\![v > \alpha]\!]$, that is, $[\![T_i u > \alpha]\!] \ne [\![T_i v > \alpha]\!]$. So $T_i u \ne T_i v$ and $Tu \ne Tv$. As u, v are arbitrary, T is injective.

If $w = \langle w_i\rangle_{i\in I}$ is any member of $\prod_{i\in I} L^0(\mathfrak{A}_i)$, then for $\alpha \in \mathbb{R}$ set

$$\phi(\alpha) = \langle [\![w_i > \alpha]\!]\rangle_{i\in I} \in \mathfrak{A}.$$

It is easy to check that ϕ satisfies the conditions of 364Aa, because, for instance,

$$\sup_{\beta>\alpha} \pi_i\phi(\beta) = \sup_{\beta>\alpha} [\![w_i > \beta]\!] = [\![w_i > \alpha]\!] = \pi_i\phi(\alpha)$$

for every i, so that $\sup_{\beta>\alpha} \phi(\beta) = \phi(\alpha)$, for every $\alpha \in \mathbb{R}$; and the other two conditions are also satisfied because they are satisfied coordinate-by-coordinate. So there is a $u \in L^0(\mathfrak{A})$ such that $\phi(\alpha) = [\![u > \alpha]\!]$ for every α, that is, $\pi_i[\![u > \alpha]\!] = [\![w_i > \alpha]\!]$ for all α, i, that is, $T_i u = w_i$ for every i, that is, $Tu = w$. As w is arbitrary, T is surjective and we are done.

***364S Regular open algebras** I noted in 314P that for every topological space X there is a corresponding Dedekind complete Boolean algebra $\mathrm{RO}(X)$ of regular open sets. We have an identification of $L^0(\mathrm{RO}(X))$ as a space of equivalence classes of functions, different in kind from the representations above, as follows. This is hard work (especially if we do it in full generality), but instructive. I start with a temporary definition.

Definition Let $(X, \mathfrak{T})$ be a topological space and $f : X \to \mathbb{R}$ a function. For $x \in X$ write

$$\omega(f, x) = \inf_{G\in\mathfrak{T}, x\in G} \sup_{y,z\in G} |f(y) - f(z)|$$

(allowing ∞).

***364T Theorem** Let X be any topological space, and $\mathrm{RO}(X)$ its regular open algebra. Let U be the set of functions $f : X \to \mathbb{R}$ such that $\{x : \omega(f, x) < \epsilon\}$ is dense in X for every $\epsilon > 0$. Then U is a Riesz subspace of $\mathbb{R}^X$, closed under multiplication, and we have a surjective multiplicative Riesz homomorphism $T : U \to L^0(\mathrm{RO}(X))$ defined by writing

$$[\![Tf > \alpha]\!] = \sup_{\beta>\alpha} \operatorname{int} \overline{\{x : f(x) > \beta\}},$$

the supremum being taken in $\mathrm{RO}(X)$, for every $\alpha \in \mathbb{R}$ and $f \in U$. The kernel of T is the set W of functions $f : X \to \mathbb{R}$ such that $\operatorname{int}\{x : |f(x)| \le \epsilon\}$ is dense for every $\epsilon > 0$, so $L^0(\mathrm{RO}(X))$ can be identified, as f-algebra, with the quotient space U/W.

proof (a)(i)(α) The first thing to observe is that for any $f \in \mathbb{R}^X$ and $\epsilon > 0$ the set

$$\{x : \omega(f, x) < \epsilon\} = \bigcup\{G : G \subseteq X \text{ is open and non-empty and } \sup_{y,z\in G} |f(y) - f(z)| < \epsilon\}$$

is open.

(β) Next, it is easy to see that

$$\omega(f + g, x) \le \omega(f, x) + \omega(g, x),$$

$$\omega(\gamma f, x) = |\gamma|\omega(f, x),$$

$$\omega(|f|, x) \le \omega(f, x),$$

for all f, $g \in \mathbb{R}^X$ and $\gamma \in \mathbb{R}$.

(γ) Thirdly, it is useful to know that if $f \in U$ and $G \subseteq X$ is a non-empty open set, then there is a non-empty open set $G' \subseteq G$ on which f is bounded. **P** Take any $x_0 \in G$ such that $\omega(f, x_0) < 1$; then there is a non-empty open set G' containing x_0 such that $|f(y) - f(z)| < 1$ for all y, $z \in G'$, and we may suppose that $G' \subseteq G$. But now $|f(x)| \leq 1 + |f(x_0)|$ for every $x \in G'$. **Q**

(ii) So if f, $g \in U$ and $\gamma \in \mathbb{R}$ then

$$\{x : \omega(f + g, x) < \epsilon\} \supseteq \{x : \omega(f, x) < \frac{1}{2}\epsilon\} \cap \{x : \omega(g, x) < \frac{1}{2}\epsilon\}$$

is the intersection of two dense open sets and is therefore dense, while

$$\{x : \omega(\gamma f, x) < \epsilon\} \supseteq \{x : \omega(f, x) < \tfrac{\epsilon}{1+|\gamma|}\},$$

$$\{x : \omega(|f|, x) < \epsilon\} \supseteq \{x : \omega(f, x) < \epsilon\}$$

are also dense. As ϵ is arbitrary, $f + g$, γf and $|f|$ all belong to U; as f, g and γ are arbitrary, U is a Riesz subspace of $\mathbb{R}^X$.

(iii) If f, $g \in U$ then $f \times g \in U$. **P** Take $\epsilon > 0$ and let G_0 be a non-empty open subset of X. By the last remark in (i) above, there is a non-empty open set $G_1 \subseteq G_0$ such that $|f| \vee |g|$ is bounded on G_1; say $\max(|f(x)|, |g(x)|) \leq \gamma$ for every $x \in G_1$.

Set $\delta = \frac{\epsilon}{2\gamma+1} > 0$. Then there is an $x \in G_1$ such that $\omega(f, x) < \delta$ and $\omega(g, x) < \delta$. Let H, H' be open sets containing x such that $|f(y) - f(z)| \leq \delta$ for all y, $z \in H$ and $|g(y) - g(z)| \leq \delta$ for all y, $z \in H'$. Consider $G = G_1 \cap H \cap H'$. This is an open set containing x, and if y, $z \in G$ then

$$\begin{aligned} |f(y)g(y) - f(z)g(z)| &\leq |f(y) - f(z)||g(z)| + |f(z)||g(y) - g(z)| \\ &\leq \delta\gamma + \gamma\delta. \end{aligned}$$

Accordingly

$$\omega(f \times g, x) \leq 2\delta\gamma < \epsilon,$$

while $x \in G_0$. As G_0 is arbitrary, $\{x : \omega(f \times g, x) < \epsilon\}$ is dense; as ϵ is arbitrary, $f \times g \in U$. **Q**

Thus U is a subalgebra of $\mathbb{R}^X$.

(b) Now, for $f \in U$, consider the map $\phi_f : \mathbb{R} \to \mathrm{RO}(X)$ defined by setting

$$\phi_f(\alpha) = \sup_{\beta > \alpha} \operatorname{int} \overline{\{x : f(x) > \beta\}}$$

for every $\alpha \in \mathbb{R}$. Then ϕ_f satisfies the conditions of 364Aa. **P** (See 314P for the calculation of suprema and infima in $\mathrm{RO}(X)$.) (i) If $\alpha \in \mathbb{R}$ then

$$\begin{aligned} \phi_f(\alpha) &= \sup_{\beta > \alpha} \operatorname{int} \overline{\{x : f(x) > \beta\}} = \sup_{\gamma > \beta > \alpha} \operatorname{int} \overline{\{x : f(x) > \gamma\}} \\ &= \sup_{\beta > \alpha} \sup_{\gamma > \beta} \operatorname{int} \overline{\{x : f(x) > \gamma\}} = \sup_{\beta > \alpha} \phi_f(\beta). \end{aligned}$$

(ii) If $G_0 \subseteq X$ is a non-empty open set, then there is a non-empty open set $G_1 \subseteq G_0$ such that f is bounded on G_1; say $|f(x)| < \gamma$ for every $x \in G_1$. If $\beta > \gamma$ then G_1 does not meet $\{x : f(x) > \beta\}$, so $G_1 \cap \operatorname{int} \overline{\{x : f(x) > \gamma\}} = \emptyset$; as β is arbitrary, $G_1 \cap \phi_f(\gamma) = \emptyset$ and $G_0 \not\subseteq \inf_{\alpha \in \mathbb{R}} \phi_f(\alpha)$. On the other hand, $G_1 \subseteq \{x : f(x) > -\gamma\}$, so

$$G_1 \subseteq \operatorname{int} \overline{\{x : f(x) > -\gamma\}} \subseteq \phi_f(-\gamma)$$

and $G_0 \cap \sup_{\alpha \in \mathbb{R}} \phi_f(\alpha) \neq \emptyset$. As G_0 is arbitrary, $\inf_{\alpha \in \mathbb{R}} \phi_f(\alpha) = \emptyset$ and $\sup_{\alpha \in \mathbb{R}} \phi_f(\alpha) = X$. **Q**

(c) Thus we have a map $T : U \to L^0 = L^0(\mathrm{RO}(X))$ defined by setting $[\![Tf > \alpha]\!] = \phi_f(\alpha)$ whenever $\alpha \in \mathbb{R}$ and $f \in U$.

It is worth noting that

$$\{x : f(x) > \alpha + \omega(f, x)\} \subseteq [\![Tf > \alpha]\!] \subseteq \{x : f(x) + \omega(f, x) \geq \alpha\}$$

for every $f \in U$ and $\alpha \in \mathbb{R}$. **P** (i) If $f(x) > \alpha + \omega(f, x)$, set $\delta = \frac{1}{2}(f(x) - \alpha - \omega(f, x)) > 0$. Then there is an open set G containing x such that $|f(y) - f(z)| < \omega(f, x) + \delta$ for every y, $z \in G$, so that $f(y) > \alpha + \delta$ for every $y \in G$, and

$$x \in \operatorname{int}\{y : f(y) > \alpha + \delta\} \subseteq [\![Tf > \alpha]\!].$$

(ii) If $f(x) + \omega(f, x) < \alpha$, set $\delta = \frac{1}{2}(\alpha - f(x) - \omega(f, x)) > 0$; then there is an open neighbourhood G of x such that $|f(y) - f(z)| < \omega(f, x) + \delta$ for every y, $z \in G$, so that $f(y) < \alpha$ for every $y \in G$. Accordingly G does not meet $\{y : f(y) > \beta\}$ nor $\overline{\{y : f(y) > \beta\}}$ for any $\beta > \alpha$, $G \cap [\![Tf > \alpha]\!] = \emptyset$ and $x \notin [\![Tf > \alpha]\!]$. **Q**

(d) T is additive. **P** Let f, $g \in U$ and $\alpha < \beta \in \mathbb{R}$. Set $\delta = \frac{1}{5}(\beta - \alpha) > 0$, $H = \{x : \omega(f, x) < \delta, \omega(g, x) < \delta\}$; then H is the intersection of two dense open sets, so is itself dense and open.

(i) If $x \in H \cap [\![T(f + g) > \beta]\!]$, then $(f + g)(x) + \omega(f + g, x) \ge \beta$; but $\omega(f + g, x) \le 2\delta$ (see (a-i-β) above), so $f(x) + g(x) \ge \beta - 2\delta > \alpha + 2\delta$ and there is a $q \in \mathbb{Q}$ such that

$$f(x) > q + \delta \ge q + \omega(f, x), \quad g(x) > \alpha - q + \delta \ge \alpha - q + \omega(g, x).$$

Accordingly

$$x \in [\![Tf > q]\!] \cap [\![Tg > \alpha - q]\!] \subseteq [\![Tf + Tg > \alpha]\!].$$

Thus $H \cap [\![T(f + g) > \beta]\!] \subseteq [\![Tf + Tg > \alpha]\!]$. Because H is dense, $[\![T(f + g) > \beta]\!] \subseteq [\![Tf + Tg > \alpha]\!]$.

(ii) If $x \in H$, then

$$\begin{aligned} x \in \bigcup_{q \in \mathbb{Q}} &([\![Tf > q]\!] \cap [\![Tg > \beta - q]\!]) \\ &\Longrightarrow \exists\, q \in \mathbb{Q},\ f(x) + \omega(f, x) \ge q,\ g(x) + \omega(g, x) \ge \beta - q \\ &\Longrightarrow f(x) + g(x) + 2\delta \ge \beta \\ &\Longrightarrow (f + g)(x) \ge \alpha + 3\delta > \alpha + \omega(f + g, x) \\ &\Longrightarrow x \in [\![T(f + g) > \alpha]\!]. \end{aligned}$$

Thus

$$H \cap \bigcup\nolimits_{q \in \mathbb{Q}}([\![Tf > q]\!] \cap [\![Tg > \beta - q]\!]) \subseteq [\![T(f + g) > \alpha]\!].$$

Because H is dense and $\bigcup_{q \in \mathbb{Q}}([\![Tf > q]\!] \cap [\![Tg > \beta - q]\!])$ is open,

$$\begin{aligned} [\![Tf + Tg > \beta]\!] &= \operatorname{int} \overline{\bigcup_{q \in \mathbb{Q}} [\![Tf > q]\!] \cap [\![Tg > \beta - q]\!]} \\ &\subseteq \operatorname{int} \overline{[\![T(f + g) > \alpha]\!]} = [\![T(f + g) > \alpha]\!]. \end{aligned}$$

(iii) Now let $\beta \downarrow \alpha$; we have

$$\begin{aligned} [\![T(f + g) > \alpha]\!] &= \sup_{\beta > \alpha} [\![T(f + g) > \beta]\!] \subseteq [\![Tf + Tg > \alpha]\!] \\ &= \sup_{\beta > \alpha} [\![Tf + Tg > \beta]\!] \subseteq [\![T(f + g) > \alpha]\!], \end{aligned}$$

so $[\![T(f + g) > \alpha]\!] = [\![Tf + Tg > \alpha]\!]$. As α is arbitrary, $T(f + g) = Tf + Tg$; as f and g are arbitrary, T is additive. **Q**

(e) It is now easy to see that T is linear. **P** If $\gamma > 0$, $f \in U$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned} [\![T(\gamma f) > \alpha]\!] &= \sup_{\beta > \alpha} \operatorname{int} \overline{\{x : \gamma f(x) > \beta\}} = \sup_{\beta > \alpha} \operatorname{int} \overline{\{x : f(x) > \frac{\beta}{\gamma}\}} \\ &= \sup_{\beta > \alpha/\gamma} \operatorname{int} \overline{\{x : f(x) > \beta\}} = [\![Tf > \frac{\alpha}{\gamma}]\!] = [\![\gamma Tf > \alpha]\!]. \end{aligned}$$

As α is arbitrary, $T(\gamma f)=\gamma Tf$; because we already know that T is additive, this is enough to show that T is linear. **Q**

(f) In fact T is a Riesz homomorphism. **P** If $f\in U$ and $\alpha\ge 0$ then

$$\begin{aligned}[\![T(f^+)>\alpha]\!]&=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f^+(x)>\beta\}}=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f(x)>\beta\}}\\&=[\![Tf>\alpha]\!]=[\![(Tf)^+>\alpha]\!].\end{aligned}$$

If $\alpha<0$ then

$$[\![T(f^+)>\alpha]\!]=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:f^+(x)>\beta\}}=X=[\![(Tf)^+>\alpha]\!].\ \mathbf{Q}$$

(g) Of course the constant function χX belongs to U, and is its multiplicative identity; and $T(\chi X)$ is the multiplicative identity of L^0, because

$$\begin{aligned}[\![T(\chi X)>\alpha]\!]&=\sup_{\beta>\alpha}\operatorname{int}\overline{\{x:(\chi X)(x)>\beta\}}\\&=X\text{ if }\alpha<1,\ \emptyset\text{ if }\alpha\ge 1.\end{aligned}$$

By 353Qd, or otherwise, T is multiplicative.

(h) The kernel of T is W. **P** (i) For $f\in U$,

$$\begin{aligned}Tf=0&\implies[\![T|f|>0]\!]=[\![|Tf|>0]\!]=\emptyset\\&\implies\{x:|f(x)|>\omega(|f|,x)\}=\emptyset\\&\implies\operatorname{int}\{x:|f(x)|\le\epsilon\}\supseteq\{x:\omega(|f|,x)<\epsilon\}\text{ is dense for every }\epsilon>0\\&\implies f\in W.\end{aligned}$$

(ii) If $f\in W$, then, first,

$$\{x:\omega(f,x)<\epsilon\}\supseteq\operatorname{int}\{x:|f(x)|\le\tfrac{1}{3}\epsilon\}$$

is dense for every $\epsilon>0$, so $f\in U$; and next, for any $\beta>0$, $\overline{\{x:|f(x)|>\beta\}}$ does not meet the dense open set $\operatorname{int}\{x:|f(x)|\le\beta\}$, so

$$[\![|Tf|>0]\!]=[\![T|f|>0]\!]=\sup_{\beta>0}\operatorname{int}\overline{\{x:|f(x)|>\beta\}}=\emptyset$$

and $Tf=0$. **Q**

(i) Finally, T is surjective. **P** Take any $u\in L^0$. Define $\tilde f:X\to[-\infty,\infty]$ by setting $\tilde f(x)=\sup\{\alpha:x\in[\![u>\alpha]\!]\}$ for each x, counting $\inf\emptyset$ as $-\infty$. Then

$$\{x:\tilde f(x)>\alpha\}=\bigcup_{\beta>\alpha}[\![u>\beta]\!]$$

is open, for every $\alpha\in\mathbb{R}$. The set

$$\{x:\tilde f(x)=\infty\}=\bigcap_{\alpha\in\mathbb{R}}[\![u>\alpha]\!]$$

is nowhere dense, because $\inf_{\alpha\in\mathbb{R}}[\![u>\alpha]\!]=\emptyset$ in $\mathrm{RO}(X)$; while

$$\{x:\tilde f(x)=-\infty\}=X\setminus\bigcup_{\alpha\in\mathbb{R}}[\![u>\alpha]\!]$$

also is nowhere dense, because $\sup_{\alpha\in\mathbb{R}}[\![u>\alpha]\!]=X$ in $\mathrm{RO}(X)$. Accordingly $E=\operatorname{int}\{x:\tilde f(x)\in\mathbb{R}\}$ is dense. Set $f(x)=\tilde f(x)$ for $x\in E$, 0 for $x\in X\setminus E$.

Let $\epsilon>0$. If $G\subseteq X$ is a non-empty open set, there is an $\alpha\in\mathbb{R}$ such that $G\not\subseteq[\![u>\alpha]\!]$, so $G_1=G\setminus\overline{[\![u>\alpha]\!]}\ne\emptyset$, and $\tilde f(x)\le\alpha$ for every $x\in G_1$. Set

$$\alpha'=\sup_{x\in G_1}\tilde f(x)\le\alpha<\infty.$$

Because E meets G_1, $\alpha' > -\infty$. Then $G_2 = G_1 \cap [\![u > \alpha' - \frac{1}{2}\epsilon]\!]$ is a non-empty open subset of G and $\alpha' - \frac{1}{2}\epsilon \le \tilde{f}(x) \le \alpha'$ for every $x \in G_2$. Accordingly $|f(y) - f(z)| \le \frac{1}{2}\epsilon$ for all y, $z \in G_2$, and $\omega(f, x) < \epsilon$ for all $x \in G_2$. As G is arbitrary, $\{x : \omega(f, x) < \epsilon\}$ is dense; as ϵ is arbitrary, $f \in U$.

Take $\alpha < \beta$ in $\mathbb{R}$, and set $\delta = \frac{1}{2}(\beta - \alpha)$. Then $H = E \cap \{x : \omega(f, x) < \delta\}$ is a dense open set, and

$$H \cap [\![Tf > \beta]\!] \subseteq H \cap \{x : f(x) + \omega(f, x) \ge \beta\} \subseteq E \cap \{x : f(x) > \alpha\}$$
$$\subseteq \{x : \tilde{f}(x) > \alpha\} \subseteq [\![u > \alpha]\!].$$

As H is dense, $[\![Tf > \beta]\!] \subseteq [\![u > \alpha]\!]$. In the other direction

$$H \cap [\![u > \beta]\!] \subseteq H \cap \{x : \tilde{f}(x) \ge \beta\} = H \cap \{x : f(x) \ge \beta\}$$
$$\subseteq \{x : f(x) > \alpha + \omega(f, x)\} \subseteq [\![Tf > \alpha]\!],$$

so $[\![u > \beta]\!] \subseteq [\![Tf > \alpha]\!]$. Just as in (d) above, this is enough to show that $Tf = u$. As u is arbitrary, T is surjective. **Q**

This completes the proof.

***364U Compact spaces** Suppose now that X is a compact Hausdorff topological space. In this case the space U of 364T is just the space of functions $f : X \to \mathbb{R}$ such that $\{x : f$ is continuous at $x\}$ is dense in X. **P** It is easy to see that

$$\{x : f \text{ is continuous at } x\} = \{x : \omega(f, x) = 0\} = \bigcap_{n\in\mathbb{N}} H_n$$

where $H_n = \{x : \omega(f, x) < 2^{-n}\}$ for each n. Each H_n is an open set (see part (a-i-α) of the proof of 364T), so by Baire's theorem (3A3G) $\bigcap_{n\in\mathbb{N}} H_n$ is dense iff every H_n is dense, that is, iff $f \in U$. **Q**

Now W, as defined in 364T, becomes $\{f : f \in U, \{x : f(x) = 0\}$ is dense$\}$. **P** (i) If $f \in W$, then $T|f| = 0$, so (by the formula in (c) of the proof of 364T) $|f(x)| \le \omega(|f|, x)$ for every x. But $\{x : \omega(f, x) = 0\}$ is dense, because $f \in U$, so $\{x : f(x) = 0\}$ also is dense. (ii) If $f \in U$ and $\{x : f(x) = 0\}$ is dense, then

$$\omega(f, x) \ge \inf_{x\in G \text{ is open}} \sup_{y\in G} |f(y) - f(x)| \ge |f(x)|$$

for every $x \in X$. So for any $\epsilon > 0$, $\mathrm{int}\{x : |f(x)| \le \epsilon\} \supseteq \{x : \omega(f, x) < \epsilon\}$ is dense, and $f \in W$. **Q**

In the case of extremally disconnected spaces, we can go farther.

***364V Theorem** Let X be a compact Hausdorff extremally disconnected space, and $\mathrm{RO}(X)$ its regular open algebra. Write $C^\infty = C^\infty(X)$ for the space of continuous functions $g : X \to [-\infty, \infty]$ such that $\{x : g(x) = \pm\infty\}$ is nowhere dense. Then we have a bijection $S : C^\infty \to L^0 = L^0(\mathrm{RO}(X))$ defined by saying that

$$[\![Sg > \alpha]\!] = \overline{\{x : g(x) > \alpha\}}$$

for every $\alpha \in \mathbb{R}$. Addition and multiplication in L^0 correspond to the operations $\dot{+}$, $\dot{\times}$ on C^∞ defined by saying that $g \dot{+} h$, $g \dot{\times} h$ are the unique elements of C^∞ agreeing with $g + h$, $g \times h$ on $\{x : g(x), h(x)$ are both finite$\}$. Scalar multiplication in L^0 corresponds to the operation

$$(\gamma g)(x) = \gamma g(x) \text{ for } x \in X,\ g \in C^\infty,\ \gamma \in \mathbb{R}$$

on C^∞ (counting $0 \cdot \infty$ as 0), while the ordering of L^0 corresponds to the relation

$$g \le h \iff g(x) \le h(x) \text{ for every } x \in X.$$

proof (a) For $g \in C^\infty$, set $H_g = \{x : g(x) \in \mathbb{R}\}$, so that H_g is a dense open set, and define $Rg : X \to \mathbb{R}$ by setting $(Rg)(x) = g(x)$ if $x \in H_g$, 0 if $x \in X \setminus H_g$. Then Rg is continuous at every point of H_g, so belongs to the space U of 364T-364U. Set $Sg = T(Rg)$, where $T : U \to L^0$ is the map of 364T. Then

$$[\![Sg > \alpha]\!] = \overline{\{x : g(x) > \alpha\}}$$

for every $\alpha \in \mathbb{R}$. **P** (i) $\omega(g, x) = 0$ for every $x \in H_g$, so, if $\beta > \alpha$,

$$H_g \cap [\![Sg > \beta]\!] \subseteq \{x : x \in H_g,\ (Rg)(x) \ge \beta\} \subseteq \{x : g(x) \ge \beta\}$$

by the formula in part (c) of the proof of 364T. As $[\![Sg > \beta]\!]$ is open and H_g is dense,

$$[\![Sg>\beta]\!] \subseteq \overline{H_g \cap [\![Sg>\beta]\!]} \subseteq \{x : g(x) \ge \beta\} \subseteq \{x : g(x) > \alpha\}.$$

Now

$$[\![Sg>\alpha]\!] = \sup_{\beta>\alpha} [\![Sg>\beta]\!] = \operatorname{int}\overline{\bigcup_{\beta>\alpha} [\![Sg>\beta]\!]} \subseteq \overline{\{x : g(x) > \alpha\}}.$$

(ii) In the other direction, $H_g \cap \{x : g(x) > \alpha\} \subseteq [\![Sg>\alpha]\!]$, by the other half of the formula in the proof of 364T. Again because $\{x : g(x) > \alpha\}$ is open and H_g is dense,

$$\overline{\{x : g(x) > \alpha\}} \subseteq \overline{[\![Sg>\alpha]\!]} = [\![Sg>\alpha]\!]$$

because X is extremally disconnected (see 314S). **Q**

(b) Thus $S = TR$ defined by the formula offered. Now if g, $h \in C^\infty$ and $g \le h$, we surely have $\{x : g(x) > \alpha\} \subseteq \{x : h(x) > \alpha\}$ for every α, so $[\![Sg>\alpha]\!] \subseteq [\![Sh>\alpha]\!]$ for every α and $Sg \le Sh$. On the other hand, if $g \not\le h$ then $Sg \not\le Sh$. **P** Take x_0 such that $g(x_0) > h(x_0)$, and $\alpha \in \mathbb{R}$ such that $g(x_0) > \alpha > h(x_0)$; set $H = \{x : g(x) > \alpha > h(x)\}$; this is a non-empty open set and $H \subseteq [\![Sg>\alpha]\!]$. On the other hand, $H \cap \{x : h(x) > \alpha\} = \emptyset$ so $H \cap [\![Sh>\alpha]\!] = \emptyset$. Thus $[\![Sg>\alpha]\!] \not\subseteq [\![Sh>\alpha]\!]$ and $Sg \not\le Sh$. **Q** In particular, S is injective.

(c) S is surjective. **P** If $u \in L^0$, set

$$g(x) = \sup\{\alpha : x \in [\![u>\alpha]\!]\} \in [-\infty, \infty]$$

for every $x \in X$, taking $\sup\emptyset = -\infty$. Then, for any $\alpha \in \mathbb{R}$, $\{x : g(x) > \alpha\} = \bigcup_{\beta>\alpha} [\![u>\alpha]\!]$ is open. On the other hand,

$$\{x : g(x) < \alpha\} = \bigcup_{\beta<\alpha} \{x : x \notin [\![u>\beta]\!]\}$$

also is open, because all the sets $[\![u>\beta]\!]$ are open-and-closed. So $g : X \to [-\infty, \infty]$ is continuous. Also

$$\{x : g(x) > -\infty\} = \bigcup_{\alpha\in\mathbb{R}} [\![u>\alpha]\!],$$

$$\{x : g(x) < \infty\} = \bigcup_{\alpha\in\mathbb{R}} X \setminus [\![u>\alpha]\!]$$

are dense, so $g \in C^\infty$. Now, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned}[\![Sg>\alpha]\!] &= \overline{\{x : g(x) > \alpha\}} = \overline{\bigcup_{\beta>\alpha} [\![u>\beta]\!]} \\ &= \operatorname{int}\overline{\bigcup_{\beta>\alpha} [\![u>\beta]\!]} = \sup_{\beta>\alpha} [\![u>\beta]\!] = [\![u>\alpha]\!].\end{aligned}$$

So $Sg = u$. As u is arbitrary, S is surjective. **Q**

(d) Accordingly S is a bijection. I have already checked (in part (b)) that it is an isomorphism of the order structures. For the algebraic operations, observe that if g, $h \in C^\infty$ then there are f_1, $f_2 \in C^\infty$ such that $Sg + Sh = Sf_1$ and $Sg \times Sh = Sf_2$, that is,

$$T(Rg + Rh) = TRg + TRh = TRf_1, \quad T(Rg \times Rh) = TRg \times TRh = TRf_2.$$

But this means that

$$T(Rg + Rh - Rf_1) = T((Rg \times Rh) - Rf_2) = 0,$$

so that $Rg + Rh - Rf_1$, $(Rg \times Rh) - Rf_2$ belong to W, as defined in 364T-364U, and are zero on dense sets (364U). Since we know also that the set $G = \{x : g(x), h(x) \text{ are both finite}\}$ is a dense open set, while g, h, f_1 and f_2 are all continuous, we must have $f_1(x) = g(x) + h(x)$, $f_2(x) = g(x)h(x)$ for every $x \in G$. And of course this uniquely specifies f_1 and f_2 as members of C^∞.

Thus we do have operations $\dot{+}$, $\dot{\times}$ as described, rendering S additive and multiplicative. As for scalar multiplication, it is easy to check that $R(\gamma g) = \gamma Rg$ (at least, unless $\gamma = 0$, which is trivial), so that $S(\gamma g) = \gamma Sg$ for every $g \in C^\infty$.

364X Basic exercises >(a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. For u, $v \in L^0 = L^0(\mathfrak{A})$ set $[\![u<v]\!] = [\![v>u]\!] = [\![v-u>0]\!]$, $[\![u\le v]\!] = [\![v\ge u]\!] = 1 \setminus [\![v<u]\!]$, $[\![u=v]\!] = [\![u\le v]\!] \cap [\![v\le u]\!]$. (i) Show

that $(\llbracket u<v\rrbracket, \llbracket u=v\rrbracket, \llbracket u>v\rrbracket)$ is always a partition of unity in $\mathfrak{A}$. (ii) Show that for any u, u', v, $v' \in L^0$, $\llbracket u\le u'\rrbracket \cap \llbracket v\le v'\rrbracket \subseteq \llbracket u+v\le u'+v'\rrbracket$ and $\llbracket u=u'\rrbracket \cap \llbracket v=v'\rrbracket \subseteq \llbracket u\times v=u'\times v'\rrbracket$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that if u, $v \in L^0 = L^0(\mathfrak{A})$ and α, $\beta \in \mathbb{R}$ then $\llbracket u+v\ge\alpha+\beta\rrbracket \subseteq \llbracket u\ge\alpha\rrbracket \cup \llbracket v\ge\beta\rrbracket$. (ii) Show that if u, $v \in (L^0)^+$ and α, $\beta \ge 0$ then $\llbracket u\times v\ge\alpha\beta\rrbracket \subseteq \llbracket u\ge\alpha\rrbracket \cup \llbracket v\ge\beta\rrbracket$.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $u \in L^0(\mathfrak{A})$. Show that $\{\llbracket u\in E\rrbracket : E \subseteq \mathbb{R}$ is Borel$\}$ is the σ-subalgebra of $\mathfrak{A}$ generated by $\{\llbracket u>\alpha\rrbracket : \alpha \in \mathbb{R}\}$.

>(e) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\langle u_i\rangle_{i\in I}$ any family in $L^0(\mathfrak{A})$; for each $i \in I$ let $\mathfrak{B}_i$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{\llbracket u_i>\alpha\rrbracket : \alpha \in \mathbb{R}\}$. Show that the following are equiveridical: (i) $\bar\mu(\inf_{i\in J}\llbracket u_i>\alpha_i\rrbracket) = \prod_{i\in J}\bar\mu\llbracket u_i>\alpha_i\rrbracket$ whenever $J \subseteq I$ is finite and $\alpha_i \in \mathbb{R}$ for each $i \in J$ (ii) $\langle\mathfrak{B}_i\rangle_{i\in I}$ is stochastically independent in the sense of 325L. (In this case we may call $\langle u_i\rangle_{i\in I}$ **independent**.)

>(f) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and u, v two $\bar\mu$-independent members of $L^0(\mathfrak{A})$. Show that the distribution of their sum is the convolution of their distributions. (*Hint*: 272T).

>(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and g, $h : \mathbb{R} \to \mathbb{R}$ Borel measurable functions. (i) Show that $\bar g\bar h = \overline{gh}$, where $\bar g$, $\bar h : L^0 \to L^0$ are defined as in 364H. (ii) Show that $\overline{g+h}(u) = \bar g(u) + \bar h(u)$, $\overline{g\times h}(u) = \bar g(u)\times\bar h(u)$ for every $u \in L^0 = L^0(\mathfrak{A})$. (iii) Show that if $\langle h_n\rangle_{n\in\mathbb{N}}$ is a sequence of Borel measurable functions on $\mathbb{R}$ and $\sup_{n\in\mathbb{N}} h_n = h$, then $\sup_{n\in\mathbb{N}}\bar h_n(u) = \bar h(u)$ for every $u \in L^0$. (iv) Show that if h is non-decreasing and continuous on the left, then $\bar h(\sup A) = \sup \bar h[A]$ whenever $A \subseteq L^0$ is a non-empty set with a supremum in L^0.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that $S(\mathfrak{A})$ can be identified (α) with the set of those $u \in L^0 = L^0(\mathfrak{A})$ such that $\{\llbracket u>\alpha\rrbracket : \alpha \in \mathbb{R}\}$ is finite (β) with the set of those $u \in L^0$ such that $\llbracket u\in I\rrbracket = 1$ for some finite $I \subseteq \mathbb{R}$. (ii) Show that $L^\infty(\mathfrak{A})$ can be identified with the set of those $u \in L^0$ such that $\llbracket u\in[-\alpha,\alpha]\rrbracket = 1$ for some $\alpha \ge 0$, and that $\|u\|_\infty$ is the smallest such α.

(i) Show that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, and $u \in L^0(\mathfrak{A})$, then for any $\alpha \in \mathbb{R}$

$$\llbracket u>\alpha\rrbracket = \inf_{\beta>\alpha}\sup\{a : a \in \mathfrak{A},\ u\times\chi a \ge \beta\chi a\}$$

(compare 363Xh).

>(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a non-negative finitely additive functional. Let $\int : L^\infty(\mathfrak{A}) \to \mathbb{R}$ be the corresponding linear functional, as in 363L. Write U for the set of those $u \in L^0(\mathfrak{A})$ such that $\sup\{\int v : v \in L^\infty(\mathfrak{A}),\ v \le |u|\}$ is finite. Show that $\int$ has an extension to a non-negative linear functional on U.

(k) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $u \ge 0$ in $L^0 = L^0(\mathfrak{A})$. Show that $u = \sup_{q\in\mathbb{Q}} q\chi\llbracket u>q\rrbracket$ in L^0.

(l)(i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $A \subseteq L^0(\mathfrak{A})$ a non-empty countable set with supremum w. Show that $\llbracket w\in G\rrbracket \subseteq \sup_{u\in A}\llbracket u\in G\rrbracket$ for every open set $G \subseteq \mathbb{R}$. (ii) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $A \subseteq L^0(\mathfrak{A})$ a non-empty set with supremum w. Show that $\llbracket w\in G\rrbracket \subseteq \sup_{u\in A}\llbracket u\in G\rrbracket$ for every open set $G \subseteq \mathbb{R}$.

(m) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $A \subseteq L^0 = L^0(\mathfrak{A})$ a non-empty set which is bounded below in L^0. Suppose that $\phi_0(\alpha) = \inf_{u\in A}\llbracket u>\alpha\rrbracket$ is defined in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$. Show that $v = \inf A$ is defined in L^0, and that $\llbracket v>\alpha\rrbracket = \sup_{\beta>\alpha}\phi_0(\beta)$ for every $\alpha \in \mathbb{R}$.

>(n) Let (X,Σ,μ) be a measure space and $f : X \to [0,\infty[$ a function; set $A = \{g^\bullet : g \in \mathcal{L}^0(\mu),\ g \le_{\text{a.e.}} f\}$. (i) Show that if (X,Σ,μ) either is localizable or has the measurable envelope property (213Xl), then $\sup A$ is defined in $L^0(\mu)$. (ii) Show that if (X,Σ,μ) is complete and locally determined and $w = \sup A$ is defined in $L^0(\mu)$, then $w \in A$.

(o) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that if u, $v \in L^0 = L^0(\mathfrak{A})$ then the following are equiveridical: (α) $[\![|v|>0]\!] \subseteq [\![|u|>0]\!]$ (β) v belongs to the band in L^0 generated by u (γ) there is a $w \in L^0$ such that $u \times w = v$.

>(p) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $a \in \mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Show that $L^0(\mathfrak{A}_a)$ can be identified, as f-algebra, with the band in $L^0(\mathfrak{A})$ generated by χa.

(q) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be the corresponding Riesz homomorphism (364P). Show that (i) the kernel of T is the sequentially order-closed solid linear subspace of $L^0(\mathfrak{A})$ generated by $\{\chi a : a \in \mathfrak{A},\, \pi a = 0\}$ (ii) the set of values of T is the sequentially order-closed linear subspace of $L^0(\mathfrak{B})$ generated by $\{\chi(\pi a) : a \in \mathfrak{A}\}$.

(r) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism, with $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ the associated operator. Suppose that h is a Borel measurable real-valued function defined on a Borel subset of $\mathbb{R}$. Show that $\bar{h}(Tu) = T\bar{h}(u)$ whenever $u \in L^0(\mathfrak{A})$ and $\bar{h}(u)$ is defined in the sense of 364H.

(s) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be probability algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving Boolean homomorphism; let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be the corresponding Riesz homomorphism. Show that if $\langle u_i \rangle_{i \in I}$ is a family in $L^0(\mathfrak{A})$, it is $\bar{\mu}$-independent iff $\langle Tu_i \rangle_{i \in I}$ is $\bar{\nu}$-independent.

>(t) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Show that $L^0(\mathfrak{B})$ can be identified with the sequentially order-closed Riesz subspace of $L^0(\mathfrak{A})$ generated by $\{\chi b : b \in \mathfrak{B}\}$.

(u) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism; let $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ be the corresponding Riesz homomorphism. Let $\mathfrak{C}$ be the fixed-point subalgebra of π. Show that $\{u : u \in L^0(\mathfrak{A}),\, T_\pi u = u\}$ can be identified with $L^0(\mathfrak{C})$.

(v) Use the ideas of part (d) of the proof of 364T to show that the operator T there is multiplicative, without appealing to 353Q.

(w) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$. Show that $L^0(\mathfrak{B})$, regarded as a subset of $L^0(\mathfrak{A})$, is order-closed in $L^0(\mathfrak{A})$.

(x) (W.Ricker) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ such that $\Sigma/\mathcal{I}$ is Dedekind complete. Suppose that Φ is a family of Σ-measurable real-valued functions, all with domains belonging to Σ, such that $\{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\, f(x) \neq g(x)\} \in \mathcal{I}$ whenever f, $g \in \Phi$. Show that there is a Σ-measurable function $h : X \to \mathbb{R}$ such that $\{x : x \in \operatorname{dom} f,\ f(x) \neq h(x)\} \in \mathcal{I}$ for every $f \in \Phi$. (*Hint*: 213N.)

364Y Further exercises >(a)(i) Show directly, without using the Loomis-Sikorski theorem or the Stone representation, that if $\mathfrak{A}$ is any Dedekind σ-complete Boolean algebra then the formulae of 364D define a group operation $+$ on $L^0(\mathfrak{A})$, and generally an f-algebra structure. (ii) Defining $\chi : \mathfrak{A} \to L^0(\mathfrak{A})$ by the formula in 364Jc, show that $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ can be identified with the linear span of $\{\chi a : a \in \mathfrak{A}\}$ and the solid linear subspace of $L^0(\mathfrak{A})$ generated by $e = \chi 1$. (iii) Still without using the Loomis-Sikorski theorem, explain how to define $\bar{h} : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ for continuous functions $h : \mathbb{R} \to \mathbb{R}$. (iv) Check that these ideas are sufficient to yield 364L-364R, except that in 364Pd we may have difficulty with arbitrary Borel functions h.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\boldsymbol{u} = (u_1, \dots, u_n)$ a member of $L^0(\mathfrak{A})^n$. Write $\mathcal{B}_n$ for the algebra of Borel sets in $\mathbb{R}^n$. (i) Show that there is a unique sequentially order-continuous Boolean homomorphism $E \mapsto [\![\boldsymbol{u} \in E]\!] : \mathcal{B}_n \to \mathfrak{A}$ such that $[\![\boldsymbol{u} \in E]\!] = \inf_{i \le n} [\![u_i > \alpha_i]\!]$ when $E = \prod_{i \le n}]\alpha_i, \infty[$. (ii) Show that for every sequentially order-continuous Boolean homomorphism $\phi : \mathcal{B}_n \to \mathfrak{A}$ there is a unique $\boldsymbol{u} \in L^0(\mathfrak{A})^n$ such that $\phi E = [\![\boldsymbol{u} \in E]\!]$ for every $E \in \mathcal{B}_n$.

(c) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $n \ge 1$ and $h : \mathbb{R}^n \to \mathbb{R}$ a Borel measurable function. Show that we have a corresponding function $\bar{h} : L^0(\mathfrak{A})^n \to L^0(\mathfrak{A})$ defined by saying that $[\![\bar{h}(\boldsymbol{u}) \in E]\!] = [\![\boldsymbol{u} \in h^{-1}[E]]\!]$ for every Borel set $E \subseteq \mathbb{R}$ and $\boldsymbol{u} \in L^0(\mathfrak{A})^n$.

(d) Suppose that $h_1(x, y) = x + y$, $h_2(x, y) = xy$, $h_3(x, y) = \max(x, y)$ for all x, $y \in \mathbb{R}$. Show that, in the language of 364Yc, $\bar{h}_1(u, v) = u + v$, $\bar{h}_2(u, v) = u \times v$, $\bar{h}_3(u, v) = u \vee v$ for all u, $v \in L^0$.

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\mathfrak{A}$ is ccc iff $L^0(\mathfrak{A})$ has the countable sup property.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, and $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ a Riesz homomorphism such that $Te = e'$, where e, e' are the multiplicative identities of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively. Show that there is a unique sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $T = T_\pi$ in the sense of 364P. (Compare 375A below.)

(g) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous ring homomorphism. (i) Show that we have a multiplicative sequentially order-continuous Riesz homomorphism $T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ defined by the formula

$$[\![T_\pi u > \alpha]\!] = \pi[\![u > \alpha]\!]$$

whenever $u \in L^0(\mathfrak{A})$ and $\alpha > 0$. (ii) Show that T_π is order-continuous iff π is order-continuous, injective iff π is injective, and surjective iff π is surjective. (iii) Show that if $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ another sequentially order-continuous ring homomorphism then $T_{\theta\pi} = T_\theta T_\pi : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$.

(h) Suppose, in 364T, that $X = \mathbb{Q}$. (i) Show that there is an $f \in W$ such that $f(q) > 0$ for every $q \in \mathbb{Q}$. (ii) Show that there is a $u \in L^0$ such that no $f \in U$ representing u can be continuous at any point of $\mathbb{Q}$.

(i) Let X and Y be topological spaces and $\phi : X \to Y$ a continuous function such that $\phi^{-1}[M]$ is nowhere dense in X for every nowhere dense subset M of Y. (Cf. 313R.) (i) Show that we have an order-continuous Boolean homomorphism π from the regular open algebra $\mathrm{RO}(Y)$ of Y to the regular open algebra $\mathrm{RO}(X)$ of X defined by the formula $\pi G = \operatorname{int} \overline{\phi^{-1}[G]}$ for every $G \in \mathrm{RO}(Y)$. (ii) Show that if U_X, U_Y are the function spaces of 364T then $g\phi \in U_X$ for every $g \in U_Y$. (iii) Show that if $T_X : U_X \to L^0(\mathrm{RO}(X))$, $T_Y : U_Y \to L^0(\mathrm{RO}(Y))$ are the canonical surjections, and $T : L^0(\mathrm{RO}(Y)) \to L^0(\mathrm{RO}(X))$ is the homomorphism corresponding to π, then $T(T_Y g) = T_X(g\phi)$ for every $g \in U_Y$. (iv) Rewrite these ideas for the special case in which X is a dense subset of Y and ϕ is the identity map, showing that in this case π and T are isomorphisms.

(j) Let X be a Baire space, $\mathrm{RO}(X)$ its algebra of regular open sets, $\mathcal{M}$ its ideal of meager sets, and $\widehat{\mathcal{B}}$ the Baire-property σ-algebra $\{G \triangle A : G \subseteq X \text{ is open}, A \in \mathcal{M}\}$, so that $\mathrm{RO}(X)$ can be identified with $\widehat{\mathcal{B}}/\mathcal{M}$ (314Yd). (i) Repeat the arguments of 364U in this context. (ii) Show that the space U of 364T-364U is a subspace of $\mathcal{L}^0 = \mathcal{L}^0_{\widehat{\mathcal{B}}}$, and that $W = U \cap \mathcal{W}$ where $\mathcal{W} = \{f : f \in \mathbb{R}^X, \{x : f(x) \ne 0\} \in \mathcal{M}\}$, so that the representations of $L^0(\mathrm{RO}(X))$ as U/W, $\mathcal{L}^0/\mathcal{W}$ are consistent.

(k) Work through the arguments of 364T and 364Yj for the case of compact Hausdorff X, seeking simplifications based on 364U.

(l) Let X be an extremally disconnected compact Hausdorff space with regular open algebra $\mathrm{RO}(X)$. Let U_0 be the space of real-valued functions $f : X \to \mathbb{R}$ such that $\operatorname{int}\{x : f \text{ is continuous at } x\}$ is dense. Show that U_0 is a Riesz subspace of the space U of 364T, and that every member of $L^0(\mathrm{RO}(X))$ is represented by a member of U_0.

(m) Let X be a Baire space. Let Q be the set of all continuous real-valued functions defined on subsets of X, and Q^* the set of all members of Q which are maximal in the sense that there is no member of Q properly extending them. (i) Show that the domain of any member of Q^* is a dense G_δ set. (ii) Show that we can define addition and multiplication and scalar multiplication on Q^* by saying that $f \dot{+} g$, $f \dot{\times} g$, $\gamma.f$

are to be the unique members of Q^* extending the partially-defined functions $f+g$, $f\times g$, γf, and that these definitions render Q^* an f-algebra if we say that $f\le g$ iff $f(x)\le g(x)$ for every $x\in\operatorname{dom}f\cap\operatorname{dom}g$. (iii) Show that every member of Q^* has an extension to a member of U, as defined in 364T, and that these extensions define an isomorphism between Q^* and $L^0(\mathrm{RO}(X))$, where $\mathrm{RO}(X)$ is the regular open algebra of X. (iv) Show that if X is compact, Hausdorff and extremally disconnected, then every member of Q^* has a unique extension to a member of $C^\infty(X)$, as defined in 364V.

(n) Let X be an extremally disconnected Hausdorff space, and Z any compact Hausdorff space. Show that if $D\subseteq X$ is dense and $f:D\to Z$ is continuous, there is a continuous $g:X\to Z$ extending f.

(o) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. (i) Show that for any $\boldsymbol{u}=(u_1,\dots,u_n)\in L^0(\mathfrak{A})^n$ there is a unique Radon probability measure ν on $\mathbb{R}^n$ such that $\nu(\prod_{1\le i\le n}]\alpha,\infty[)=\bar\mu(\inf_{1\le i\le n}[\![u_i>\alpha_i]\!])$ for all $\alpha_1,\dots,\alpha_n\in\mathbb{R}$, and that now $\nu E=\bar\mu[\![\boldsymbol{u}\in E]\!]$ for every Borel set $E\subseteq\mathbb{R}^n$. I will call ν the **distribution** of $\boldsymbol{u}$. (ii) Show that $(u_1,\dots,u_n)$ is independent iff ν is expressible as $\prod_{1\le i\le n}\nu_i$ where ν_i is a Radon probability measure on $\mathbb{R}$ for each i. (iii) Write $\mathfrak{A}_{\boldsymbol{u}}$ for the closed subalgebra $\{[\![\boldsymbol{u}\in E]\!]:E\subseteq\mathbb{R}^n$ is a Borel set$\}$; check that $u_i\in L^0(\mathfrak{A}_{\boldsymbol{u}})$ for every i. Suppose that $(\mathfrak{B},\bar\nu)$ is another probability algebra and that $\boldsymbol{v}=(v_1,\dots,v_n)\in(L^0(\mathfrak{B}))^n$. Show that the following are equiveridical: (α) there is a measure-preserving isomorphism $\pi:\mathfrak{A}_{\boldsymbol{u}}\to\mathfrak{B}_{\boldsymbol{v}}$ such that $T_\pi u_i=v_i$ for every i (β) $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same distribution.

(p) X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I}$ a σ-ideal of Σ; let $\mathfrak{A}$ be the quotient algebra $\Sigma/\mathcal{I}$. Set $\mathcal{L}^0=\mathcal{L}^0_\Sigma$ as as in 364B-364C; for $f\in\mathcal{L}^0$ write $Tf^\bullet$ for the corresponding member of $L^0=L^0(\mathfrak{A})$ (364C). Suppose that $\mathfrak{A}$ is ccc. Let $g:X\to[0,\infty[$ be any function. Show that $\{Tf^\bullet:f\in\mathcal{L}^0,\ f\le g\}$ is bounded above in L^0.

364 Notes and comments This has been a long section, and so far all we have is a supposedly thorough grasp of the construction of L^0 spaces; discussion of their properties still lies ahead. The difficulties seem to stem from a variety of causes. First, L^0 spaces have a rich structure, being linear ordered spaces with multiplications; consequently all the main theorems have to check rather a lot of different aspects. Second, unlike L^∞ spaces, they are not accessible by means of the theory of normed spaces, so I must expect to do more of the work here rather than in an appendix. But this is in fact a crucial difference, because it affects the proof of the central theorem 364D. The point is that a given algebra $\mathfrak{A}$ will be expressible in the form $\Sigma/\mathcal{I}$ for a variety of algebras Σ of sets. Consequently any definition of $L^0(\mathfrak{A})$ as a quotient $\mathcal{L}^0_\Sigma/\mathcal{W}_\mathcal{I}$ must include a check that the structure produced is independent of the particular pair Σ, $\mathcal{I}$ chosen.

The same question arises with $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$. But in the case of S, I was able to use a general theory of additive functions on $\mathfrak{A}$ (see the proof of 361L), while in the case of L^∞ I could quote the result for S and a little theory of normed spaces (see the proof of 363H). The theorems of §368 will show, among other things, that a similar approach (describing L^0 as a special kind of extension of S or L^∞) can be made to work in the present situation. I have chosen, however, an alternative route using a novel technique. The price is the time required to develop skill in the technique, and to relate it to the earlier approach (364C, 364D, 364J). The reward is a construction which is based directly on the algebra $\mathfrak{A}$, independent of any representation (364A), and methods of dealing with it which are complementary to those of the previous three sections. In particular, they can be used in the absence of the full axiom of choice (364Ya).

I have deliberately chosen the notation $[\![u>\alpha]\!]$ from the theory of forcing. I do not propose to try to explain myself here, but I remark that much of the labour of this section is a necessary basis for understanding real analysis in Boolean-valued models of set theory. The idea is that just as a function $f:X\to\mathbb{R}$ can be described in terms of the sets $\{x:f(x)>\alpha\}$, so can an element u of $L^0(\mathfrak{A})$ be described in terms of the regions $[\![u>\alpha]\!]$ of $\mathfrak{A}$ where in some sense u is greater than α. This description is well adapted to discussion of the order struction of $L^0(\mathfrak{A})$ (see 364L-364M), but rather ill-adapted to discussion of its linear and multiplicative structures, which leads to a large part of the length of the work above. Once we have succeeded in describing the algebraic operations on L^0 in terms of the values of $[\![u>\alpha]\!]$, however, as in 364D, the fundamental result on the action of Boolean homomorphisms (364P) is elegant and reasonably straightforward.

The concept '$[\![u>\alpha]\!]$' can be dramatically generalized to the concept '$[\![(u_1,\dots,u_n)\in E]\!]$', where E is a Borel subset of $\mathbb{R}^n$ and $u_1,\dots,u_n\in L^0(\mathfrak{A})$ (364G, 364Yb). This is supposed to recall the notation

$\Pr(X \in E)$, already used in Chapter 27. If, as sometimes seems reasonable, we wish to regard a random variable as a member of $L^0(\mu)$ rather than of $\mathcal{L}^0(\mu)$, then '$[\![u \in E]\!]$' is the appropriate translation of '$X^{-1}[E]$'. The reasons why we can reach all Borel sets E here, but then have to stop, seem to lie fairly deep; I will return to this question in 566O in Volume 5. We see that we have here another potential definition of $L^0(\mathfrak{A})$, as the set of sequentially order-continuous Boolean homomorphisms from the Borel σ-algebra of $\mathbb{R}$ to $\mathfrak{A}$. This is suitably independent of realizations of $\mathfrak{A}$, but makes the f-algebra structure of L^0 difficult to elucidate, unless we move to a further level of abstraction in the definitions, as in 364Yd.

I take the space to describe the L^0 spaces of general regular open algebras in detail (364T) partly to offer a demonstration of an appropriate technique, and partly to show that we are not limited to σ-algebras of sets and their quotients. This really is a new representation; for instance, it does not meld in any straightforward way with the constructions of 364F-364H. Of course the most important examples are compact Hausdorff spaces, for which alternative methods are available (364U-364V, 364Yj, 364Yl, 364Ym); from the point of view of applications, indeed, it is worth working through the details of the theory for compact Hausdorff spaces (364Yk). The version in 364V is derived from Vulikh 67. But I have starred everything from 364S on, because I shall not rely on this work later for anything essential.

Version of 20.1.15

365 L^1

Continuing my programme of developing the ideas of Chapter 24 at a deeper level of abstraction, I arrive at last at L^1. As usual, the first step is to establish a definition which can be matched both with the constructions of the previous sections and with the definition of $L^1(\mu)$ in §242 (365A-365C, 365F). Next, I give what I regard as the most characteristic internal properties of L^1 spaces, including versions of the Radon-Nikodým theorem (365E), before turning to abstract versions of theorems in §235 (365H, 365S) and the duality between L^1 and L^∞ (365K-365M). As in §§361 and 363, the construction is associated with universal mapping theorems (365I-365J) which define the Banach lattice structure of L^1. As in §§361, 363 and 364, homomorphisms between measure algebras correspond to operators between their L^1 spaces; but now the duality theory gives us two types of operators (365N-365P), of which one class can be thought of as abstract conditional expectations (365Q). For localizable measure algebras, the underlying algebra can be recovered from its L^1 space (365R), but the measure cannot.

365A Definition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. For $u \in L^0(\mathfrak{A})$, write

$$\|u\|_1 = \int_0^\infty \bar{\mu}[\![|u| > \alpha]\!]\,d\alpha,$$

the integral being with respect to Lebesgue measure on $\mathbb{R}$, and allowing ∞ as a value of the integral. (Because the integrand is monotonic, it is certainly measurable.) Set $L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu}) = \{u : u \in L^0(\mathfrak{A}),\ \|u\|_1 < \infty\}$.

It is convenient to note at once that if $u \in L^1_{\bar{\mu}}$, then $\mu[\![|u| > \alpha]\!]$ must be finite for almost every $\alpha > 0$, and therefore for every $\alpha > 0$, since it is a non-increasing function of α; so that $[\![u > \alpha]\!]$ also belongs to the Boolean ring $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$ for every $\alpha > 0$.

365B Theorem Let (X, Σ, μ) be a measure space with measure algebra $(\mathfrak{A}, \bar{\mu})$. Then the canonical isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ (364Ic) matches $L^1(\mu) \subseteq L^0(\mu)$, defined in §242, with $L^1(\mathfrak{A}, \bar{\mu}) \subseteq L^0(\mathfrak{A})$, and the standard norm of $L^1(\mu)$ with $\|\;\|_1 : L^1(\mathfrak{A}, \bar{\mu}) \to [0, \infty[$, as defined in 365A.

proof Take any Σ-measurable function $f : X \to \mathbb{R}$ (364B); write $f^\bullet$ for its equivalence class in $L^0(\mu)$, and u for the corresponding element of $L^0(\mathfrak{A})$. Then $[\![|u| > \alpha]\!] = \{x : |f(x)| > \alpha\}^\bullet$ in $\mathfrak{A}$ for every $\alpha \in \mathbb{R}$, and

$$\|u\|_1 = \int_0^\infty \mu\{x : |f(x)| > \alpha\}\,d\alpha = \int |f| d\mu$$

by 252O. In particular, $u \in L^1(\mathfrak{A}, \bar{\mu})$ iff $f \in \mathcal{L}^1(\mu)$ iff $f^\bullet \in L^1(\mu)$, and in this case $\|u\|_1 = \|f^\bullet\|_1$.

365C Accordingly we can apply everything we know about $L^1(\mu)$ spaces to $L^1_{\bar{\mu}}$ spaces. For instance:

Theorem For any measure algebra $(\mathfrak{A},\bar\mu)$, $L^1(\mathfrak{A},\bar\mu)$ is a solid linear subspace of $L^0(\mathfrak{A})$, and $\| \|_1$ is a norm on $L^1(\mathfrak{A},\bar\mu)$ under which $L^1(\mathfrak{A},\bar\mu)$ is an L-space. Consequently $L^1(\mathfrak{A},\bar\mu)$ is a perfect Riesz space with an order-continuous norm which has the Levi property, and if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing norm-bounded sequence in $L^1(\mathfrak{A},\bar\mu)$ then it converges for $\| \|_1$ to $\sup_{n\in\mathbb{N}} u_n$.

proof $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra of some measure space (X,Σ,μ) (321J). $L^1(\mu)$ is a solid linear subspace of $L^0(\mu)$ (242Cb), so $L^1_{\bar\mu}$ is a solid linear subspace of $L^0(\mathfrak{A})$. $L^1(\mu)$ is an L-space (354M), so $L^1_{\bar\mu}$ also is. The rest of the properties claimed are general features of L-spaces (354N, 354E, 356P).

365D Integration Let $(\mathfrak{A},\bar\mu)$ be any measure algebra.

(a) If $u\in L^1=L^1(\mathfrak{A},\bar\mu)$, then u^+ and u^-, calculated in $L^0=L^0(\mathfrak{A})$, belong to L^1, and we may set

$$\int u=\|u^+\|_1-\|u^-\|_1=\int_0^\infty\bar\mu[\![u>\alpha]\!]\,d\alpha-\int_0^\infty\bar\mu[\![-u>\alpha]\!]\,d\alpha.$$

Now $\int:L^1\to\mathbb{R}$ is an order-continuous positive linear functional (356Pc), and under the translation of 365B matches the integral on $L^1(\mu)$ as defined in 242Ab. Note that if $a\in\mathfrak{A}$ then

$$\int\chi a=\int_0^\infty\bar\mu[\![\chi a>\alpha]\!]d\alpha=\int_0^1\bar\mu a\,d\alpha=\bar\mu a,$$

so that if $\bar\mu$ is totally finite then the integral here agrees with that of 363L on $L^\infty(\mathfrak{A})$. I will sometimes write $\int u\,d\bar\mu$ if it seems helpful to indicate the measure.

(b) Of course $\|u\|_1=\int|u|\ge|\int u|$ for every $u\in L^1$.

(c) If $u\in L^1$ and $a\in\mathfrak{A}$ we may set $\int_a u=\int u\times\chi a$. (Compare 242Ac.) If $\gamma>0$ and $0\ne a\subseteq[\![u>\gamma]\!]$ then there is a $\delta>\gamma$ such that $a'=a\cap[\![u>\delta]\!]\ne0$, so that

$$\int_a u=\int_0^\infty\bar\mu(a\cap[\![u>\alpha]\!])d\alpha\ge\int_0^\gamma\bar\mu a\,d\alpha+\int_\gamma^\delta\bar\mu a'>\gamma\bar\mu a.$$

In particular, setting $a=[\![u>\gamma]\!]$, $\bar\mu[\![u>\gamma]\!]$ must be finite.

(d)(i) If $u\in L^1$ then $u\ge0$ iff $\int_a u\ge0$ for every $a\in\mathfrak{A}^f$, writing $\mathfrak{A}^f=\{a:\bar\mu a<\infty\}$, as usual. **P** If $u\ge0$ then $u\times\chi a\ge0$ and $\int_a u\ge0$ for every $a\in\mathfrak{A}$. If $u\not\ge0$, then $[\![u^->0]\!]\ne0$ and there is an $\alpha>0$ such that $a=[\![u^->\alpha]\!]\ne0$. But now $\bar\mu a$ is finite ((c) above) and

$$\int u\times\chi a=-\int u^-\times\chi a=-\int\bar\mu(a\cap[\![u^-\ge\beta]\!])d\beta\le-\alpha\bar\mu a<0,$$

so $\int_a u<0$. **Q**

(ii) If u, $v\in L^1$ and $\int_a u=\int_a v$ for every $a\in\mathfrak{A}^f$ then $u=v$ (cf. 242Ce).

(iii) If $u\ge0$ in L^1 then $\int u=\sup\{\int_a u:a\in\mathfrak{A}^f\}$. **P** Of course $u\times\chi a\le u$ so $\int_a u\le u$ for every $a\in\mathfrak{A}$. On the other hand, setting $a_n=[\![u>2^{-n}]\!]$, $\langle u\times\chi a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u, so $\int u=\lim_{n\to\infty}\int_{a_n}u$, while $\bar\mu a_n$ is finite for every n. **Q**

(e) If $u\in L^1$, $u\ge0$ and $\int u=0$ then $u=0$ (put 365B and 122Rc together). If $u\in L^1$, $u\ge0$ and $\int_a u=0$ then $u\times\chi a=0$, that is, $a\cap[\![u>0]\!]=0$.

(f) If $C\subseteq L^1$ is non-empty and upwards-directed and $\sup_{v\in C}\int v$ is finite, then $\sup C$ is defined in L^1 and $\int\sup C=\sup_{v\in C}\int v$ (356Pc).

(g) It will occasionally be convenient to adapt the conventions of §133 to the new context; so that I may write $\int u=\infty$ if $u\in L^0$, $u^-\in L^1$ and $u^+\notin L^1$, while $\int u=-\infty$ if $u^+\in L^1$ and $u^-\notin L^1$.

(h) On this convention, we can restate (f) as follows: if $C\subseteq(L^0)^+$ is non-empty and upwards-directed and has a supremum u in L^0, then $\int u=\sup_{v\in C}\int v$ in $[0,\infty]$. **P** For if $\sup_{v\in C}\int v$ is infinite, then surely $\int u=\infty$; while otherwise we can apply (f). **Q**

365E The Radon-Nikodým theorem again (a) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra and $\nu:\mathfrak{A}\to\mathbb{R}$ an additive functional. Then the following are equiveridical:

(i) there is a $u\in L^1=L^1(\mathfrak{A},\bar\mu)$ such that $\nu a=\int_a u$ for every $a\in\mathfrak{A}$;

(ii) ν is additive and continuous for the measure-algebra topology on $\mathfrak{A}$;

(iii) ν is completely additive.

(b) Let $(\mathfrak{A},\bar\mu)$ be any measure algebra, and $\nu:\mathfrak{A}^f\to\mathbb{R}$ a function. Then the following are equiveridical:

(i) ν is additive and bounded and $\inf_{a\in A}|\nu a|=0$ whenever $A\subseteq\mathfrak{A}^f$ is downwards-directed and has infimum 0;

(ii) there is a $u\in L^1$ such that $\nu a=\int_a u$ for every $a\in\mathfrak{A}^f$.

proof (a) The equivalence of (ii) and (iii) is 327Bd. The equivalence of (i) and (iii) is just a translation of 327D into the new context, since $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra of a measure space which by 322Bd will be semi-finite.

(b)(i)⇒(ii)(α) Set $M=\sup_{a\in\mathfrak{A}^f}|\nu a|<\infty$.

Let $D\subseteq\mathfrak{A}^f$ be a maximal disjoint set. For each $d\in D$, write $\mathfrak{A}_d$ for the principal ideal of $\mathfrak{A}$ generated by d, and $\bar\mu_d$ for the restriction of $\bar\mu$ to $\mathfrak{A}_d$, so that $(\mathfrak{A}_d,\bar\mu_d)$ is a totally finite measure algebra. Set $\nu_d=\nu\restriction\mathfrak{A}_d$; then $\nu_d:\mathfrak{A}_d\to\mathbb{R}$ is completely additive. By (a), there is a $u_d\in L^1(\mathfrak{A}_d,\bar\mu_d)$ such that $\int_a u_d=\nu_d a=\nu a$ for every $a\subseteq d$.

Now $u_d^+\in L^0(\mathfrak{A}_d)$ corresponds to a member $\tilde u_d^+$ of $L^0(\mathfrak{A})^+$ defined by saying

$$\begin{aligned}[\![\tilde u_d^+>\alpha]\!]&=[\![u_d^+>\alpha]\!]=[\![u_d>\alpha]\!] \text{ if } \alpha\ge 0,\\ &=1 \text{ if } \alpha<0.\end{aligned}$$

If $a\in\mathfrak{A}$, then

$$\int_a\tilde u_d^+d\bar\mu=\int_0^\infty\bar\mu(a\cap[\![\tilde u_d^+>\alpha]\!])d\alpha=\int_0^\infty\bar\mu_d(a\cap[\![u_d^+>\alpha]\!])d\alpha=\int_{a\cap d}u_d^+d\bar\mu_d;$$

taking $a=1$, we see that $\|\tilde u_d^+\|_1=\|u_d^+\|_1=\nu[\![u_d>0]\!]$ is finite, so that $\tilde u_d^+\in L^1$.

(β) For any finite $I\subseteq D$, set $v_I=\sum_{d\in I}\tilde u_d^+$. Then

$$\int v_I=\nu(\sup_{d\in I}[\![u_d>0]\!])\le M;$$

consequently the upwards-directed set $A=\{v_I:I\subseteq D \text{ is finite}\}$ is bounded above in L^1, and we can set $v=\sup A$ in L^1. If $a\in\mathfrak{A}$, then $\int_a v_I=\sum_{d\in I}\int_{a\cap d}u_d^+$ for each finite $I\subseteq D$, so $\int_a v=\sum_{d\in D}\int_{a\cap d}u_d^+$.

Applying the same arguments to $-\nu$, there is a $w\in L^1$ such that

$$\int_a w=\sum_{d\in D}\int_{a\cap d}u_d^-$$

for every $a\in\mathfrak{A}$. Try $u=v-w$; then

$$\int_a u=\sum_{d\in D}\int_{a\cap d}u_d^+-\int_{a\cap d}u_d^-=\sum_{d\in D}\int_{a\cap d}u_d=\sum_{d\in D}\nu(a\cap d)$$

for every $a\in\mathfrak{A}$.

(γ) Now take any $a\in\mathfrak{A}^f$. For $J\subseteq D$ set $a_J=\sup_{d\in J}a\cap d$. Let $\epsilon>0$. Then there is a finite $I\subseteq D$ such that

$$|\int_a u-\nu a_J|=|\sum_{d\in D}\nu(a\cap d)-\sum_{d\in J}\nu(a\cap d)|\le\epsilon$$

whenever $I\subseteq J\subseteq D$ and J is finite. But now consider

$$A=\{a\setminus a_J:I\subseteq J\subseteq D,\ J \text{ is finite}\}.$$

Then $\inf A=0$, so there is a finite J such that $I\subseteq J\subseteq D$ and

$$|\nu a-\nu a_J|=|\nu(a\setminus a_J)|\le\epsilon.$$

Consequently

$$|\nu a-\int_a u|\le|\nu a-\nu a_J|+|\int_a u-\nu a_J|\le 2\epsilon.$$

As ϵ is arbitrary, $\nu a=\int_a u$. As a is arbitrary, (ii) is proved.

(ii)⇒(i) From where we now are, this is nearly trivial. Thinking of νa as $\int u \times \chi a$, ν is surely additive and bounded. Also $|\nu a| \le \int |u| \times \chi a$. If $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, the same is true of $\{|u| \times \chi a : a \in A\}$, because $a \mapsto |u| \times \chi a$ is order-continuous, so

$$\inf_{a\in A} |\nu a| \le \inf_{a\in A} \int |u| \times \chi a = \inf_{a\in A} \||u| \times \chi a\|_1 = 0.$$

365F It will be useful later to have spelt out the following elementary facts.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Write S^f for the intersection $S(\mathfrak{A}) \cap L^1(\mathfrak{A}, \bar\mu)$. Then S^f is a norm-dense and order-dense Riesz subspace of $L^1(\mathfrak{A}, \bar\mu)$, and can be identified with $S(\mathfrak{A}^f)$. The function $\chi : \mathfrak{A}^f \to S^f \subseteq L^1(\mathfrak{A}, \bar\mu)$ is an injective order-continuous additive lattice homomorphism. If $u \ge 0$ in $L^1(\mathfrak{A}, \bar\mu)$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $(S^f)^+$ such that $u = \sup_{n\in\mathbb{N}} u_n = \lim_{n\to\infty} u_n$.

proof As in 364K, we can think of $S(\mathfrak{A}^f)$ as a Riesz subspace of $S = S(\mathfrak{A})$, embedded in $L^0(\mathfrak{A})$. If $u \in S$, it is expressible as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n \in \mathfrak{A}$ are disjoint and no α_i is zero. Now $|u| = \sum_{i=0}^n |\alpha_i| \chi a_i$, so $u \in L^1$ iff $\bar\mu a_i < \infty$ for every i, that is, iff $u \in S(\mathfrak{A}^f)$; thus $S^f \cong S(\mathfrak{A}^f)$.

Now suppose that $u \ge 0$ in L^1. By 364Jd, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A})^+$ such that $u_0 \ge 0$ and $u = \sup_{n\in\mathbb{N}} u_n$ in L^0. Because L^1 is a solid linear subspace of L^0, every u_n belongs to L^1 and therefore to S^f. By 365C, $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u. This shows also that S^f is order-dense in L^1.

The map $\chi : \mathfrak{A}^f \to S^f$ is an injective order-continuous additive lattice homomorphism; because S^f is regularly embedded in L^1 (352Ne), χ has the same properties when regarded as a map into L^1.

For general $u \in L^1$, there are sequences in S^f converging to u^+ and to u^-, so that their difference is a sequence in S^f converging to u, and u belongs to the closure of S^f; thus S^f is norm-dense in L^1.

Remark Of course S^f here corresponds to the space of (equivalence classes of) simple functions, as in 242Mb.

365G Semi-finite algebras: Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) $(\mathfrak{A}, \bar\mu)$ is semi-finite iff $L^1 = L^1(\mathfrak{A}, \bar\mu)$ is order-dense in $L^0 = L^0(\mathfrak{A})$.

(b) In this case, writing $S^f = S(\mathfrak{A}) \cap L^1$ (as in 365F), $\int u = \sup\{\int v : v \in S^f,\ 0 \le v \le u\}$ in $[0, \infty]$ for every $u \in (L^0)^+$.

proof (a) If $(\mathfrak{A}, \bar\mu)$ is semi-finite then S^f is order-dense in L^0 (364K), so L^1 also must be. If L^1 is order-dense in L^0, then so is S^f, by 365F and 352Nc, so $(\mathfrak{A}, \bar\mu)$ is semi-finite, by 364K in the other direction.

(b) Set $C = \{v : v \in S^f,\ 0 \le v \le u\}$. Then C is an upwards-directed set with supremum u, because S^f is order-dense in L^0. So $\int u = \sup_{v\in C} \int v$ by 365Dh.

365H Measurable transformations We have a generalization of the ideas of §235 in this abstract context.

Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be the sequentially order-continuous Riesz homomorphism associated with π (364P).

(a) Suppose that $w \ge 0$ in $L^0(\mathfrak{B})$ is such that $\int_{\pi a} w\, d\bar\nu = \bar\mu a$ whenever $a \in \mathfrak{A}$ and $\bar\mu a < \infty$. Then for any $u \in L^1(\mathfrak{A}, \bar\mu)$ and $a \in \mathfrak{A}$, $\int_{\pi a} Tu \times w\, d\bar\nu$ is defined and equal to $\int_a u\, d\bar\mu$.

(b) Suppose that $w' \ge 0$ in $L^0(\mathfrak{A})$ is such that $\int_a w' d\bar\mu = \bar\nu(\pi a)$ for every $a \in \mathfrak{A}$. Then $\int Tu\, d\bar\nu = \int u \times w'\, d\bar\mu$ whenever $u \in L^0(\mathfrak{A})$ and either integral is defined in $[-\infty, \infty]$.

Remark I am using the convention of 365Dg concerning '∞' as the value of an integral.

proof (a) If $u \in S^f = L^1_{\bar\mu} \cap S(\mathfrak{A})$ then u is expressible as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \dots, a_n$ have finite measure, so that $Tu = \sum_{i=0}^n \alpha_i \chi(\pi a_i)$ and

$$\int Tu \times w\, d\bar\nu = \sum_{i=0}^n \alpha_i \int w \times \chi\pi a_i\, d\bar\nu = \sum_{i=0}^n \alpha_i \bar\mu a_i = \int u\, d\bar\mu.$$

If $u \ge 0$ in $L^1_{\bar\mu}$ there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S^f with supremum u, so that $Tu = \sup_{n\in\mathbb{N}} Tu_n$ and $w \times Tu = \sup_{n\in\mathbb{N}} w \times Tu_n$ in $L^0(\mathfrak{B})$, and

$$\int Tu \times w = \sup_{n\in\mathbb{N}} \int Tu_n \times w = \sup_{n\in\mathbb{N}} \int u_n = \int u.$$

(365Df tells us that in this context $Tu \times w \in L^1_{\bar{\nu}}$.) Finally, for general $u \in L^1_{\bar{\mu}}$,

$$\int Tu \times w = \int Tu^+ \times w - \int Tu^- \times w = \int u^+ - \int u^- = \int u.$$

(b) The argument follows the same lines: start with $u = \chi a$ for $a \in \mathfrak{A}$, then with $u \in S(\mathfrak{A})$, then with $u \in L^0(\mathfrak{A})^+$ and conclude with general $u \in L^0(\mathfrak{A})$. The point is that T is a Riesz homomorphism, so that at the last step

$$\begin{aligned}\int Tu &= \int (Tu)^+ - \int (Tu)^- = \int T(u^+) - \int T(u^-)\\ &= \int u^+ \times w' - \int u^- \times w' = \int (u \times w')^+ - \int (u \times w')^- = \int u \times w'\end{aligned}$$

whenever either side is defined in $[-\infty, \infty]$.

365I Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and U a Banach space. Let $\nu : \mathfrak{A}^f \to U$ be a function. Then the following are equiveridical:

(i) there is a continuous linear operator T from $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ to U such that $\nu a = T(\chi a)$ for every $a \in \mathfrak{A}^f$;

(ii)(α) ν is additive

(β) there is an $M \ge 0$ such that $\|\nu a\| \le M\bar{\mu}a$ for every $a \in \mathfrak{A}^f$.

Moreover, in this case, T is unique and $\|T\|$ is the smallest number M satisfying the condition in (ii-β).

proof (a)(i)$\Rightarrow$(ii) If $T : L^1 \to U$ is a continuous linear operator, then $\chi a \in L^1$ for every $a \in \mathfrak{A}^f$, so $\nu = T\chi$ is a function from $\mathfrak{A}^f$ to U. If $a, b \in \mathfrak{A}^f$ and $a \cap b = 0$, then $\chi(a \cup b) = \chi a + \chi b$ in $L^0 = L^0(\mathfrak{A})$ and therefore in L^1, so

$$\nu(a \cup b) = T\chi(a \cup b) = T(\chi a + \chi b) = T(\chi a) + T(\chi b) = \nu a + \nu b.$$

If $a \in \mathfrak{A}^f$ then $\|\chi a\|_1 = \bar{\mu}a$ (using the formula in 365A, or otherwise), so

$$\|\nu a\| = \|T(\chi a)\| \le \|T\|\|\chi a\|_1 = \|T\|\bar{\mu}a.$$

(b)(ii)$\Rightarrow$(i) Now suppose that $\nu : \mathfrak{A}^f \to U$ is additive and that $\|\nu a\| \le M\bar{\mu}a$ for every $a \in \mathfrak{A}^f$. Write S^f for $L^1 \cap S(\mathfrak{A})$, as in 365F. Then there is a linear operator $T_0 : S^f \to U$ such that $T_0(\chi a) = \nu a$ for every $a \in \mathfrak{A}^f$ (361F). Next, $\|T_0 u\| \le M\|u\|_1$ for every $u \in S^f$. **P** If $u \in S^f \cong S(\mathfrak{A}^f)$, then u is expressible as $\sum_{j=0}^m \beta_j \chi b_j$ where $b_0, \dots, b_m \in \mathfrak{A}^f$ are disjoint (361Eb). So

$$\|T_0 u\| = \|\sum_{j=0}^m \beta_j \nu b_j\| \le M\sum_{j=0}^m |\beta_j|\bar{\mu}b_j = M\|u\|_1. \text{ **Q**}$$

There is therefore a continuous linear operator $T : L^1 \to U$, extending T_0, and with $\|T\| \le \|T_0\| \le M$ (2A4I). Of course we still have $\nu = T\chi$.

(c) The argument in (b) shows that $T_0 = T{\restriction}S^f$ and T are uniquely defined from ν. We have also seen that if T, ν correspond to each other then

$$\|\nu a\| \le \|T\|\bar{\mu}a \text{ for every } a \in \mathfrak{A}^f,$$

$$\|T\| \le M \text{ whenever } \|\nu a\| \le M\bar{\mu}a \text{ for every } a \in \mathfrak{A}^f,$$

so that

$$\|T\| = \min\{M : M \ge 0,\ \|\nu a\| \le M\bar{\mu}a \text{ for every } a \in \mathfrak{A}^f\}.$$

365J Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, U a Banach lattice, and T a bounded linear operator from $L^1 = L^1(\mathfrak{A}, \bar{\mu})$ to U. Let $\nu : \mathfrak{A}^f \to U$ be the corresponding additive function, as in 365I.

(a) T is a positive linear operator iff $\nu a \ge 0$ in U for every $a \in \mathfrak{A}^f$; in this case, T is order-continuous.

(b) If U is Dedekind complete and $T \in \mathrm{L}^{\sim}(L^1; U)$, then $|T| : L^1 \to U$ corresponds to $|\nu| : \mathfrak{A}^f \to U$, where

$$|\nu|(a) = \sup\{\sum_{i=0}^n |\nu a_i| : a_0, \dots , a_n \subseteq a \text{ are disjoint}\}$$

for every $a \in \mathfrak{A}^f$.

(c) T is a Riesz homomorphism iff ν is a lattice homomorphism.

proof As in 365F, let S^f be $L^1 \cap S(\mathfrak{A})$, identified with $S(\mathfrak{A}^f)$.

(a)(i) If T is a positive linear operator and $a \in \mathfrak{A}^f$, then $\chi a \ge 0$ in L^1, so $\nu a = T(\chi a) \ge 0$ in U.

(ii) Now suppose that $\nu a \ge 0$ in U for every $a \in \mathfrak{A}^f$, and take $u \ge 0$ in L^1, $\epsilon > 0$ in $\mathbb{R}$. Then there is a $v \in S^f$ such that $0 \le v \le u$ and $\|u - v\|_1 \le \epsilon$ (365F). Express v as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_i \in \mathfrak{A}^f$, $\alpha_i \ge 0$ for each i. Now

$$\|Tu - Tv\| \le \|T\|\|u - v\|_1 \le \epsilon\|T\|.$$

On the other hand,

$$Tv = \sum_{i=0}^n \alpha_i \nu a_i \in U^+.$$

As U^+ is norm-closed in U (354Bc), and ϵ is arbitrary, $Tu \in U^+$. As u is arbitrary, T is a positive linear operator.

(iii) By 355Ka, T is order-continuous.

(b) If $a \in \mathfrak{A}^f$, then

$$|\nu b| = |T(\chi b)| \le |T|(\chi b) \le |T|(\chi a)$$

for every $b \subseteq a$, so $\{\nu b : b \subseteq a\}$ is order-bounded in U. As a is arbitrary, we have an additive function $|\nu| : \mathfrak{A}^f \to U$ given by the proposed formula, by 361H. Next, $|T| : L^1 \to U$ is a bounded linear operator (355C), so we can speak of $\||T|\|$; and we also have an additive function $\nu_1 : \mathfrak{A}^f \to U$ corresponding to $|T|$.

If $b \subseteq a \in \mathfrak{A}^f$, then

$$\nu b - \nu(a \setminus b) = T(\chi b) - T(\chi(a \setminus b)) \le |T|(\chi b) + |T|(\chi(a \setminus b)) = |T|(\chi a) = \nu_1 a;$$

taking the supremum over b, the other formula in 361H tells us that $|\nu|a \le \nu_1 a$, so

$$\||\nu|a\| \le \|\nu_1 a\| = \||T|(\chi a)\| \le \||T|\|\|\chi a\|_1 = \||T|\|\bar\mu a.$$

By 365I, there is a bounded linear operator $S : L^1 \to U$ such that $S(\chi a) = |\nu|a$ for every $a \in \mathfrak{A}^f$.

We now have $(S - T)(\chi a) = |\nu|a - \nu a \ge 0$ for every $a \in \mathfrak{A}^f$, so $S - T \ge 0$ in $\mathsf{L}^\sim(L^1; U)$, by (a) above, and $T \le S$; similarly, $-T \le S$ and $|T| \le S$. On the other hand, $|\nu|a \le \nu_1 a$ for every a, so the same argument shows that $S \le |T|$. Thus $S = |T|$ and $|\nu|$ corresponds to $|T|$, as claimed.

(c)(i) If T is a lattice homomorphism, then so is $\nu = T\chi$, because $\chi : \mathfrak{A}^f \to S^f$ is a lattice homomorphism.

(ii) Now suppose that χ is a lattice homomorphism. In this case $T\restriction S^f$ is a Riesz homomorphism (361Gc), that is, $|Tv| = T|v|$ for every $v \in S^f$. Because S^f is norm-dense in L^1 and the map $u \mapsto |u|$ is continuous both in L^1 and in U (354Bb), $|Tu| = T|u|$ for every $u \in L^1$, and T is a Riesz homomorphism.

365K The duality between L^1 and L^∞ Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and set $L^1 = L^1(\mathfrak{A}, \bar\mu)$, $L^\infty = L^\infty(\mathfrak{A})$. If we identify L^∞ with the solid linear subspace of $L^0 = L^0(\mathfrak{A})$ generated by $e = \chi 1_{\mathfrak{A}}$ (364J), then we have a bilinear operator $(u, v) \mapsto u \times v : L^1 \times L^\infty \to L^1$, because $|u \times v| \le \|v\|_\infty |u|$ and L^1 is a solid linear subspace of L^0. Note that $\|u \times v\|_1 \le \|u\|_1\|v\|_\infty$, so that the bilinear operator $(u, v) \mapsto u \times v$ has norm at most 1 (253Ab, 253E). Consequently we have a bilinear functional $(u, v) \mapsto \int u \times v : L^1 \times L^\infty \to \mathbb{R}$, which also has norm at most 1, corresponding to linear operators $S : L^1 \to (L^\infty)^*$ and $T : L^\infty \to (L^1)^*$, both of norm at most 1, defined by the formula

$$(Su)(v) = (Tv)(u) = \int u \times v \text{ for } u \in L^1,\ v \in L^\infty.$$

Because L^1 and L^∞ are both Banach lattices, we have $(L^1)^* = (L^1)^\sim$ and $(L^\infty)^* = (L^\infty)^\sim$ (356Dc). Because the norm of L^1 is order-continuous, $(L^1)^* = (L^1)^\times$ (356Dd).

365L Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and set $L^1 = L^1(\mathfrak{A}, \bar\mu)$, $L^\infty = L^\infty(\mathfrak{A})$. Let $S : L^1 \to (L^\infty)^* = (L^\infty)^\sim$, $T : L^\infty \to (L^1)^* = (L^1)^\sim = (L^1)^\times$ be the canonical maps defined by the duality between L^1 and L^∞, as in 365K. Then

(a) S and T are order-continuous Riesz homomorphisms, $S[L^1] \subseteq (L^\infty)^\times$, S is norm-preserving and $T[L^\infty]$ is order-dense in $(L^1)^\sim$.

(b) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff T is injective, and in this case T is norm-preserving, while S is a normed Riesz space isomorphism between L^1 and $(L^\infty)^\times$.

(c) $(\mathfrak{A}, \bar{\mu})$ is localizable iff T is bijective, and in this case T is a normed Riesz space isomorphism between L^∞ and $(L^1)^* = (L^1)^\sim = (L^1)^\times$.

proof (a)(i) If $u \geq 0$ in L^1 and $v \geq 0$ in L^∞ then $u \times v \geq 0$ and

$$(Tv)(u) = \int u \times v \geq 0.$$

As u is arbitrary, $Tv \geq 0$ in $(L^1)^\times$; as v is arbitrary, T is a positive linear operator.

If $v \in L^\infty$, set $a = [\![v > 0]\!] \in \mathfrak{A}$. (Remember that we are identifying $L^0(\mu)$, as defined in §241, with $L^0(\mathfrak{A})$, as defined in §364.) Then $v^+ = v \times \chi a$, so for any $u \geq 0$ in L^1

$$(Tv^+)(u) = \int u \times v \times \chi a = (Tv)(u \times \chi a) \leq (Tv)^+(u).$$

As u is arbitrary, $Tv^+ \leq (Tv)^+$. On the other hand, because T is a positive linear operator, $Tv^+ \geq Tv$ and $Tv^+ \geq 0$, so $Tv^+ \geq (Tv)^+$. Thus $Tv^+ = (Tv)^+$. As v is arbitrary, T is a Riesz homomorphism (352G).

(ii) Exactly the same arguments show that S is a Riesz homomorphism.

(iii) Given $u \in L^1$, set $a = [\![u > 0]\!]$; then

$$\|Su\| \geq (Su)(\chi a - \chi(1 \setminus a)) = \int_a u - \int_{1\setminus a} u = \int |u| = \|u\|_1 \geq \|Su\|.$$

So S is norm-preserving.

(iv) By 355Ka, S is order-continuous.

(v) If $A \subseteq L^\infty$ is a non-empty downwards-directed set with infimum 0, and $u \in (L^1)^+$, then $\inf_{v\in A} u \times v = 0$ for every $u \in (L^1)^+$, because $v \mapsto u \times v : L^0 \to L^0$ is order-continuous. So

$$\inf_{v\in A}(Tv)(u) = \inf_{v\in A} \int u \times v = \inf_{v\in A} \|u \times v\|_1 = 0.$$

As a is arbitrary, the only possible non-negative lower bound for $T[A]$ in $(L^1)^\times$ is 0. As A is arbitrary, T is order-continuous.

(vi) The ideas of (v) show also that $S[L^1] \subseteq (L^\infty)^\times$. **P** If $u \in (L^1)^+$ and $A \subseteq L^\infty$ is non-empty, downwards-directed and has infimum 0, then

$$\inf_{v\in A}(Su)(v) = \inf_{v\in A} \int u \times v = 0.$$

As A is arbitrary, Su is order-continuous. For general $u \in L^1$, $Su = Su^+ - Su^-$ belongs to $(L^\infty)^\times$. **Q**

(vii) Now suppose that $h > 0$ in $(L^1)^\sim = (L^1)^* = (L^1)^\times$. By 365Ja, applied to $-h$, there must be an $a \in \mathfrak{A}^f$ such that $h(\chi a) > 0$. Set $\nu b = h(\chi(a \cap b))$ for $b \in \mathfrak{A}^f$. Then ν is additive and non-negative and bounded by $\|h\|\bar{\mu}a$. If $A \subseteq \mathfrak{A}^f$ is a non-empty downwards-directed set with infimum 0, then $C = \{\chi b : b \in A\}$ is downwards-directed and has infimum 0 in $L^0(\mathfrak{A})$ (364Jc), so $\inf_{b\in A} \nu b = \inf_{u\in C} h(u) = 0$. By 365Eb, there is a $v \in L^1$ such that $\nu b = \int_b v$ for every $b \in \mathfrak{A}^f$. As $\int_b v \geq 0$ for every $b \in \mathfrak{A}^f$, $v \geq 0$ (365C(d-i)). Setting $b = [\![v > \|h\|]\!]$, we have

$$\int_b v \leq h(\chi b) \leq \|h\|\|\chi b\|_1 = \|h\|\bar{\mu}b;$$

so $b = 0$ (365Cc). Accordingly $0 \leq v \leq \|h\|\chi 1$ and $v \in L^\infty$. Consider $Tv \in (L^1)^\times$. We have $Tv \geq 0$ because T is positive; also

$$(Tv)(\chi a) = \int_a v = \nu a = h(\chi a) > 0,$$

so $Tv > 0$. Next, for every $b \in \mathfrak{A}^f$,

$$(Tv)(\chi b) = \int_b v = h(\chi(a \cap b)) \leq h(\chi b).$$

By 365Ja again, $h - Tv \geq 0$, that is, $Tv \leq h$. As h is arbitrary, $T[L^\infty]$ is quasi-order-dense in $(L^1)^*$, therefore order-dense (353A).

(b)(i) If $(\mathfrak{A},\bar\mu)$ is not semi-finite, let $a\in\mathfrak{A}$ be such that $\bar\mu a=\infty$ and $\bar\mu b=\infty$ whenever $0\ne b\subseteq a$. If $u\in L^1$, then $[\![|u|>\frac{1}{n}]\!]$ has finite measure for every $n\ge 1$, so must be disjoint from a; accordingly

$$a\cap[\![|u|>0]\!]=\sup_{n\ge 1}a\cap[\![|u|>\tfrac{1}{n}]\!]=0.$$

This means that $\int u\times\chi a=0$ for every $u\in L^1$. Accordingly $T(\chi a)=0$ and T is not injective.

(ii) If $(\mathfrak{A},\bar\mu)$ is semi-finite, take any $v\in L^\infty$. Then if $0\le\delta<\|v\|_\infty$, $a=[\![|v|>\delta]\!]\ne 0$. Let $b\subseteq a$ be such that $0<\bar\mu b<\infty$. Then $\chi b\in L^1$, and

$$\|Tv\|=\||Tv|\|=\|T|v|\|\ge\frac{(T|v|)(\chi b)}{\|\chi b\|_1}\ge\delta$$

because $|v|\times\chi b\ge\delta\chi b$, so

$$(T|v|)(\chi b)\ge\delta\bar\mu b=\delta\|\chi b\|_1.$$

As δ is arbitrary, $\|Tv\|\ge\|v\|_\infty$. But we already know that $\|Tv\|\le\|v\|_\infty$, so the two are equal. As v is arbitrary, T is norm-preserving (and, in particular, is injective).

(iii) Still supposing that $(\mathfrak{A},\bar\mu)$ is semi-finite, $S[L^1]=(L^\infty)^\times$. **P** Take any $h\in(L^\infty)^\times$. For $a\in\mathfrak{A}$, set $\nu a=h(\chi a)$. By 363K, $\nu:\mathfrak{A}\to\mathbb{R}$ is completely additive. By 365Ea, there is a $u\in L^1$ such that

$$(Su)(\chi a)=\int u\times\chi a=\int_a u=\nu a=h(\chi a)$$

for every $a\in\mathfrak{A}$. Because Su and h are both linear functionals on L^∞, they must agree on $S(\mathfrak{A})$; because they are continuous and $S(\mathfrak{A})$ is dense in L^∞ (363C), $Su=h$. As h is arbitrary, S is surjective. **Q**

(c) Using (b), we know that if either T is bijective or $(\mathfrak{A},\bar\mu)$ is localizable, then $(\mathfrak{A},\bar\mu)$ is semi-finite. Given this, if T is bijective, then it is a Riesz space isomorphism between L^∞ and $(L^1)^\sim$, which is Dedekind complete (356B); so 363Mb tells us that $\mathfrak{A}$ is Dedekind complete and $(\mathfrak{A},\bar\mu)$ is localizable. In the other direction, if $(\mathfrak{A},\bar\mu)$ is localizable, then L^∞ is Dedekind complete. As T is injective, $T[L^\infty]$ is, in itself, Dedekind complete; being an order-dense Riesz subspace of $(L^1)^\sim$ (by (a) here) it must be solid (353L); as it contains $T(\chi 1)$, which is the standard order unit of the M-space $(L^1)^\sim$, it is the whole of $(L^1)^\sim$, and T is bijective.

365M Corollary If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra, $L^\infty(\mathfrak{A})$ is a perfect Riesz space.

proof By 365L(b)-(c), we can identify L^∞ with $(L^1_{\bar\mu})^\times\cong(L^\infty)^{\times\times}$.

365N Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Let $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ be a measure-preserving ring homomorphism.

(a) There is a unique order-continuous norm-preserving Riesz homomorphism $T_\pi:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{B},\bar\nu)$ such that $T_\pi(\chi a)=\chi(\pi a)$ whenever $a\in\mathfrak{A}^f$. We have $T_\pi(u\times\chi a)=T_\pi u\times\chi(\pi a)$ whenever $a\in\mathfrak{A}^f$ and $u\in L^1(\mathfrak{A},\bar\mu)$.

(b) $\int T_\pi u=\int u$ and $\int_{\pi a}T_\pi u=\int_a u$ for every $u\in L^1(\mathfrak{A},\bar\mu)$ and $a\in\mathfrak{A}^f$.

(c) $[\![T_\pi u>\alpha]\!]=\pi[\![u>\alpha]\!]$ for every $u\in L^1(\mathfrak{A},\bar\mu)$ and $\alpha>0$.

(d) T_π is surjective iff π is.

(e) If $(\mathfrak{C},\bar\lambda)$ is another measure algebra and $\theta:\mathfrak{B}^f\to\mathfrak{C}^f$ another measure-preserving ring homomorphism, then $T_{\theta\pi}=T_\theta T_\pi:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{C},\bar\lambda)$.

proof Throughout the proof I will write T for T_π and S^f for $S(\mathfrak{A})\cap L^1_{\bar\mu}\cong S(\mathfrak{A}^f)$ (see 365F).

(a)(i) We have a map $\psi:\mathfrak{A}^f\to L^1_{\bar\nu}$ defined by writing $\psi a=\chi(\pi a)$ for $a\in\mathfrak{A}^f$. Because

$$\chi\pi(a\cup b)=\chi(\pi a\cup\pi b)=\chi\pi a+\chi\pi b,\quad\|\chi(\pi a)\|_1=\bar\nu(\pi a)=\bar\mu a$$

whenever a, $b\in\mathfrak{A}^f$ and $a\cap b=0$, we get a (unique) corresponding bounded linear operator $T:L^1_{\bar\mu}\to L^1_{\bar\nu}$ such that $T\chi=\chi\pi$ on $\mathfrak{A}^f$ (365I). Because $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ and $\chi:\mathfrak{B}^f\to L^1_{\bar\nu}$ are lattice homomorphisms, so is ψ, and T is a Riesz homomorphism (365Jc).

(ii) If $u\in S^f$, express it as $\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\dots,a_n$ are disjoint in $\mathfrak{A}^f$. Then $Tu=\sum_{i=0}^n\alpha_i\chi(\pi a_i)$ and $\pi a_0,\dots,\pi a_n$ are disjoint in $\mathfrak{B}^f$, so

$$\|Tu\|_1 = \sum_{i=0}^n |\alpha_i|\bar\nu(\pi a_i) = \sum_{i=0}^n |\alpha_i|\bar\mu a_i = \|u\|_1.$$

Because S^f is norm-dense in $L^1_{\bar\mu}$ and $u \mapsto \|u\|_1$ is continuous (in both $L^1_{\bar\mu}$ and $L^1_{\bar\nu}$), $\|Tu\|_1 = \|u\|_1$ for every $u \in L^1_{\bar\mu}$, that is, T is norm-preserving. As noted in 365Ja, T is order-continuous.

(iii) If a, $b \in \mathfrak{A}^f$ then

$$T(\chi a \times \chi b) = T(\chi(a \cap b)) = \chi\pi(a \cap b) = \chi(\pi a \cap \pi b) = \chi\pi a \times \chi\pi b = \chi\pi a \times T(\chi b).$$

Because T is linear and $\times$ is bilinear, $T(\chi a \times u) = \chi\pi a \times Tu$ for every $u \in S^f$. Because the maps $u \mapsto u \times \chi a : L^1_{\bar\mu} \to L^1_{\bar\mu}$, $T : L^1_{\bar\mu} \to L^1_{\bar\nu}$ and $v \mapsto v \times \chi\pi a : L^1_{\bar\nu} \to L^1_{\bar\nu}$ are all continuous, $Tu \times \chi\pi a = T(u \times \chi a)$ for every $u \in L^1_{\bar\mu}$.

(iv) T is unique because the formula $T(\chi a) = \chi\pi a$ defines T on the norm-dense and order-dense subspace S^f.

(b) Because T is positive,

$$\int Tu = \|Tu^+\|_1 - \|Tu^-\|_1 = \|u^+\|_1 - \|u^-\|_1 = \int u.$$

For $a \in \mathfrak{A}^f$,

$$\int_{\pi a} Tu = \int Tu \times \chi\pi a = \int T(u \times \chi a) = \int u \times \chi a = \int_a u.$$

(c) If $u \in S^f$, express it as $\sum_{i=0}^n \alpha_i\chi a_i$ where $a_0, \dots , a_n$ are disjoint; then

$$\pi[\![u > \alpha]\!] = \pi(\sup_{i\in I} a_i) = \sup_{i\in I} \pi a_i = [\![Tu > \alpha]\!]$$

where $I = \{i : i \le n,\, \alpha_i > \alpha\}$. For $u \in (L^1_{\bar\mu})^+$, take a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in S^f with supremum u; then $\sup_{n\in\mathbb{N}} Tu_n = Tu$, so

$$\pi[\![u > \alpha]\!] = \pi(\sup_{n\in\mathbb{N}} [\![u_n > \alpha]\!])$$

(364L(a-ii); $[\![u > \alpha]\!] \in \mathfrak{A}^f$ by 365A)

$$= \sup_{n\in\mathbb{N}} \pi[\![u_n > \alpha]\!]$$

(because π is order-continuous, see 361Ad)

$$= \sup_{n\in\mathbb{N}} [\![Tu_n > \alpha]\!] = [\![Tu > \alpha]\!]$$

because T is order-continuous. For general $u \in L^1_{\bar\mu}$,

$$\pi[\![u > \alpha]\!] = \pi[\![u^+ > \alpha]\!] = [\![T(u^+) > \alpha]\!] = [\![(Tu)^+ > \alpha]\!] = [\![Tu > \alpha]\!]$$

because T is a Riesz homomorphism.

(d)(i) Suppose that T is surjective and that $b \in \mathfrak{B}^f$. Then there is a $u \in L^1_{\bar\mu}$ such that $Tu = \chi b$. Now

$$b = [\![Tu > \tfrac{1}{2}]\!] = \pi[\![u > \tfrac{1}{2}]\!] \in \pi[\mathfrak{A}^f];$$

as b is arbitrary, π is surjective.

(ii) Suppose now that π is surjective. Then $T[L^1_{\bar\mu}]$ is a linear subspace of $L^1_{\bar\nu}$ containing χb for every $b \in \mathfrak{B}^f$, so includes $S(\mathfrak{B}^f)$. If $v \in (L^1_{\bar\nu})^+$ there is a sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{B}^f)^+$ with supremum v. For each n, choose u_n such that $Tu_n = v_n$. Setting $u'_n = \sup_{i\le n} u_i$, we get a non-decreasing sequence $\langle u'_n\rangle_{n\in\mathbb{N}}$ such that $v_n \le Tu'_n \le v$ for every $n \in \mathbb{N}$. So

$$\sup_{n\in\mathbb{N}} \|u'_n\|_1 = \sup_{n\in\mathbb{N}} \|Tu'_n\|_1 \le \|v\|_1 < \infty$$

and $u = \sup_{n\in\mathbb{N}} u'_n$ is defined in $L^1_{\bar\mu}$, with

$$Tu = \sup_{n\in\mathbb{N}} Tu'_n = v.$$

Thus $(L^1_{\bar\nu})^+ \subseteq T[L^1_{\bar\mu}]$; consequently $L^1_{\bar\nu} \subseteq T[L^1_{\bar\mu}]$ and T is surjective.

(e) This is an immediate consequence of the 'uniqueness' assertion in (i), because for any $a \in \mathfrak{A}^f$

$$T_\theta T_\pi(\chi a) = T_\theta \chi(\pi a) = \chi(\theta\pi a),$$

so that $T_\theta T_\pi : L^1_{\bar\mu} \to L^1_{\bar\lambda}$ is a bounded linear operator taking the right values at elements χa, and must therefore be equal to $T_{\theta\pi}$.

365O Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be measure algebras, and $\pi : \mathfrak{A}^f \to \mathfrak{B}$ an order-continuous ring homomorphism.

(a) There is a unique positive linear operator $P_\pi : L^1(\mathfrak{B}, \bar\nu) \to L^1(\mathfrak{A}, \bar\mu)$ such that $\int_a P_\pi v = \int_{\pi a} v$ for every $v \in L^1(\mathfrak{B}, \bar\nu)$ and $a \in \mathfrak{A}^f$.

(b) P_π is order-continuous and norm-continuous, and $\|P_\pi\| \le 1$.

(c) If $a \in \mathfrak{A}^f$ and $v \in L^1(\mathfrak{B}, \bar\nu)$ then $P_\pi(v \times \chi\pi a) = P_\pi v \times \chi a$.

(d) If $\pi[\mathfrak{A}^f]$ is order-dense in $\mathfrak{B}$ then P_π is a norm-preserving Riesz homomorphism; in particular, P_π is injective.

(e) If $(\mathfrak{B}, \bar\nu)$ is semi-finite and π is injective, then P_π is surjective, and there is for every $u \in L^1(\mathfrak{A}, \bar\mu)$ a $v \in L^1(\mathfrak{B}, \bar\nu)$ such that $P_\pi v = u$ and $\|v\|_1 = \|u\|_1$.

(f) Suppose again that $(\mathfrak{B}, \bar\nu)$ is semi-finite. If $(\mathfrak{C}, \bar\lambda)$ is another measure algebra and $\theta : \mathfrak{B} \to \mathfrak{C}$ an order-continuous Boolean homomorphism, then $P_{\theta\pi} = P_\pi P_{\theta'} : L^1(\mathfrak{C}, \bar\lambda) \to L^1(\mathfrak{A}, \bar\mu)$, where I write θ' for the restriction of θ to $\mathfrak{B}^f$.

proof I write P for P_π.

(a)-(b) For $v \in L^1_{\bar\nu}$ and $a \in \mathfrak{A}^f$ set $\nu_v(a) = \int_{\pi a} v$. Then $\nu_v : \mathfrak{A}^f \to \mathbb{R}$ is additive, bounded (by $\|v\|_1$) and if $A \subseteq \mathfrak{A}^f$ is non-empty, downwards-directed and has infimum 0, then

$$\inf_{a\in A} |\nu_v(a)| \le \inf_{a\in A} \int |v| \times \chi\pi a = 0$$

because $a \mapsto \int |v| \times \chi\pi a$ is a composition of order-continuous functions, therefore order-continuous. So 365Eb tells us that there is a $Pv \in L^1_{\bar\mu}$ such that $\int_a Pv = \nu_v(a) = \int_{\pi a} v$ for every $a \in \mathfrak{A}^f$. By 365D(d-ii), this formula defines Pv uniquely. Consequently P must be linear (since $Pv_1 + Pv_2$, αPv will always have the properties defining $P(v_1 + v_2)$, $P(\alpha v)$).

If $v \ge 0$ in $L^1_{\bar\nu}$, then $\int_a Pv = \int_{\pi a} v \ge 0$ for every $a \in \mathfrak{A}^f$, so $Pv \ge 0$ (365D(d-i)); thus P is positive. It must therefore be norm-continuous and order-continuous (355C, 355Ka).

Again supposing that $v \ge 0$, we have

$$\|Pv\|_1 = \int Pv = \sup_{a\in\mathfrak{A}^f} \int_a Pv = \sup_{a\in\mathfrak{A}^f} \int_{\pi a} v \le \|v\|_1$$

(using 365D(d-iii)). For general $v \in L^1_{\bar\nu}$,

$$\|Pv\|_1 = \||Pv|\|_1 \le \|P|v|\|_1 \le \|v\|_1.$$

(c) For any $c \in \mathfrak{A}^f$,

$$\int_c Pv \times \chi a = \int_{c\cap a} Pv = \int_{\pi(c\cap a)} v = \int_{\pi c} v \times \chi\pi a = \int_c P(v \times \chi\pi a).$$

(d) Now suppose that $\pi[\mathfrak{A}^f]$ is order-dense. Take any v, $v' \in L^1_{\bar\nu}$ such that $v \wedge v' = 0$. **?** Suppose, if possible, that $u = Pv \wedge Pv' > 0$. Take $\alpha > 0$ such that $a = [\![u > \alpha]\!]$ is non-zero. Since

$$\int_{\pi a} v = \int_a Pv \ge \int_a u > 0,$$

$b = \pi a \cap [\![v > 0]\!]$ is non-zero. Let $c \in \mathfrak{A}^f$ be such that $0 \ne \pi c \subseteq b$; then $\pi(a \cap c) = \pi c \ne 0$, so $a \cap c \ne 0$, and

$$0 < \int_{a\cap c} u \le \int_{a\cap c} Pv' \le \int_{\pi c} v'.$$

But $\pi c \subseteq [\![v > 0]\!]$ and $v \wedge v' = 0$ so $\int_{\pi c} v' = 0$. **X**

So $Pv \wedge Pv' = 0$. As v, v' are arbitrary, P is a Riesz homomorphism (352G).

Next, if $v \ge 0$ in $L^1_{\bar\nu}$,

$$\int Pv = \sup_{a\in\mathfrak{A}^f} \int_a Pv = \sup_{a\in\mathfrak{A}^f} \int_{\pi a} v = \int v$$

because $\pi[\mathfrak{A}^f]$ is upwards-directed and has supremum 1 in $\mathfrak{B}$. So, for general $v \in L^1_{\bar\nu}$,

$$\|Pv\|_1 = \int |Pv| = \int P|v| = \int |v| = \|v\|_1,$$

and P is norm-preserving.

(e) Next suppose that $(\mathfrak{B}, \bar{\nu})$ is semi-finite and that π is injective.

(i) If $u > 0$ in $L^1_{\bar{\mu}}$, there is a $v > 0$ in $L^1_{\bar{\nu}}$ such that $Pv \leq u$ and $\int Pv \geq \int v$. **P** Let $\delta > 0$ be such that $a = [\![u > \delta]\!] \neq 0$. Then $\pi a \neq 0$. Because $(\mathfrak{B}, \bar{\nu})$ is semi-finite, there is a non-zero $b \in \mathfrak{B}^f$ such that $b \subseteq \pi a$. Set $u_1 = P(\chi b)$. Then $u_1 \geq 0$, $\int_a u_1 = \bar{\nu} b > 0$ and

$$\int_{1 \setminus a} u_1 = \sup_{c \in \mathfrak{A}^f} \int_{c \setminus a} u_1 = \sup_{c \in \mathfrak{A}^f} \int_{\pi c \setminus \pi a} \chi b = 0.$$

So $u_1 \times \chi(1 \setminus a) = 0$ and $0 \neq [\![u_1 > 0]\!] \subseteq a$. Let $\gamma > 0$ be such that $[\![u_1 > \gamma]\!] \neq [\![u_1 > 0]\!]$, and set $a_1 = a \setminus [\![u_1 > \gamma]\!]$, $v = \frac{\delta}{\gamma}\chi(b \cap \pi a_1)$. Then

$$Pv = \frac{\delta}{\gamma}P(\chi b \times \chi(\pi a_1)) = \frac{\delta}{\gamma}P(\chi b) \times \chi a_1 = \frac{\delta}{\gamma}u_1 \times \chi a_1 \leq \delta \chi a \leq u,$$

because

$$[\![u_1 \times \chi a_1 > \gamma]\!] \subseteq [\![u_1 > \gamma]\!] \cap a_1 = 0$$

so

$$u_1 \times \chi a_1 \leq \gamma \chi [\![u_1 > 0]\!] \leq \gamma \chi a.$$

Also $a_1 \cap [\![u_1 > 0]\!] \neq 0$, so Pv and v are non-zero; and

$$\int Pv \geq \int_{a_1} Pv = \int_{\pi a_1} v = \int v. \ \mathbf{Q}$$

(ii) Now take any $u \geq 0$ in $L^1_{\bar{\mu}}$, and set $B = \{v : v \in L^1_{\bar{\nu}},\ v \geq 0,\ Pv \leq u,\ \int v \leq \int Pv\}$. B is not empty because it contains 0. If $C \subseteq B$ is non-empty and upwards-directed, then $\sup_{v \in C} \int v \leq \int u$ is finite, so C has a supremum in $L^1_{\bar{\nu}}$ (365Df). Because P is order-continuous, $P(\sup C) = \sup P[C] \leq u$; also

$$\int \sup C = \sup_{v \in C} \int v \leq \sup_{v \in C} \int Pv \leq \int P(\sup C).$$

Thus $\sup C \in B$. As C is arbitrary, B satisfies the conditions of Zorn's Lemma, and has a maximal element v_0 say.

? Suppose, if possible, that $Pv_0 \neq u$. By (α), there is a $v_1 > 0$ such that $Pv_1 \leq u - Pv_0$ and $\int v_1 \leq \int Pv_1$. In this case, $v_0 < v_0 + v_1 \in B$, which is impossible. **X** Thus $Pv_0 = u$; also

$$\|v_0\|_1 = \int v_0 \leq \int Pv_0 = \|Pv_0\|_1.$$

(iii) Finally, take any $u \in L^1_{\bar{\mu}}$. By (ii), there are non-negative v_1, $v_2 \in L^1_{\bar{\nu}}$ such that $Pv_1 = u^+$, $Pv_2 = u^-$, $\|v_1\|_1 \leq \|u^+\|_1$ and $\|v_2\|_1 \leq \|u^-\|_1$. Setting $v = v_1 - v_2$, we have $Pv = u$. Also we must have

$$\|v\|_1 \leq \|v_1\|_1 + \|v_2\|_1 \leq \|u^+\|_1 + \|u^-\|_1 = \|u\|_1 \leq \|P\|\|v\|_1 = \|v\|_1,$$

so $\|v\|_1 = \|u\|_1$, as required.

(f) As usual, this is a consequence of the uniqueness of P. However (because I do not assume that $\pi[\mathfrak{A}^f] \subseteq \mathfrak{B}^f$) there is an extra refinement: we need to know that $\int_b P_{\theta'} w = \int_{\theta b} w$ for every $b \in \mathfrak{B}$ and $w \in L^1_{\bar{\lambda}}$. **P** Because θ is order-continuous and $(\mathfrak{B}, \bar{\nu})$ is semi-finite, $\theta b = \sup\{\theta b' : b' \in \mathfrak{B}^f,\ b' \subseteq b\}$, so if $w \geq 0$ then

$$\int_{\theta b} w = \sup_{b' \in \mathfrak{B}^f, b' \subseteq b} \int_{\theta b'} w = \sup_{b' \in \mathfrak{B}^f, b' \subseteq b} \int_{b'} P_{\theta'} w = \int_b P_{\theta'} w.$$

Expressing w as $w^+ - w^-$, we see that the same is true for every $w \in L^1_{\bar{\nu}}$. **Q**

Now we can say that $PP_{\theta'}$ is a positive linear operator from $L^1_{\bar{\lambda}}$ to $L^1_{\bar{\mu}}$ such that

$$\int_a PP_{\theta'} w = \int_{\pi a} P_{\theta'} w = \int_{\theta \pi a} w = \int_a P_{\theta \pi} w$$

whenever $a \in \mathfrak{A}^f$ and $w \in L^1_{\bar{\lambda}}$, and must be equal to $P_{\theta\pi}$.

365P Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\mu)$ be measure algebras and $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ a measure-preserving ring homomorphism.

(a) In the language of 365N-365O above, $P_\pi T_\pi$ is the identity operator on $L^1(\mathfrak{A},\bar\mu)$.

(b) If π is surjective (so that it is an isomorphism between $\mathfrak{A}^f$ and $\mathfrak{B}^f$) then $P_\pi=T_\pi^{-1}=T_{\pi^{-1}}$ and $T_\pi=P_\pi^{-1}=P_{\pi^{-1}}$.

proof (a) If $u\in L^1_{\bar\mu}$ and $a\in\mathfrak{A}^f$ then

$$\int_a P_\pi T_\pi u=\int_{\pi a}T_\pi u=\int_a u.$$

So $u=P_\pi T_\pi u$, by 365D(d-ii).

(b) From 365Nd, we know that T_π is surjective, while $P_\pi T_\pi$ is the identity, so that $P_\pi=T_\pi^{-1}$ and $T_\pi=P_\pi^{-1}$. As for $T_{\pi^{-1}}$, 365Ne tells us that $T_{\pi^{-1}}=T_\pi^{-1}$; so

$$P_{\pi^{-1}}=T_{\pi^{-1}}^{-1}=T_\pi.$$

365Q Conditional expectations It is a nearly universal rule that any investigation of L^1 spaces must include a look at conditional expectations. In the present context, they take the following form.

(a) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\mathfrak{B}$ a closed subalgebra; write $\bar\nu$ for the restriction $\bar\mu\restriction\mathfrak{B}$. The identity map from $\mathfrak{B}$ to $\mathfrak{A}$ induces operators $T:L^1(\mathfrak{B},\bar\nu)\to L^1(\mathfrak{A},\bar\mu)$ and $P:L^1(\mathfrak{A},\bar\mu)\to L^1(\mathfrak{B},\bar\nu)$. If we take $L^0(\mathfrak{A})$ to be defined as the set of functions from $\mathbb{R}$ to $\mathfrak{A}$ described in 364Aa, then $L^0(\mathfrak{B})$ becomes a subset of $L^0(\mathfrak{A})$ in the literal sense, and T is actually the identity operator associated with the subset $L^1(\mathfrak{B},\bar\nu)\subseteq L^1(\mathfrak{A},\bar\mu)$; $L^1(\mathfrak{B},\bar\nu)$ is a norm-closed and order-closed Riesz subspace of $L^1(\mathfrak{A},\bar\mu)$. P is a positive linear operator, while PT is the identity, so P is a projection from $L^1(\mathfrak{A},\bar\mu)$ onto $L^1(\mathfrak{B},\bar\nu)$. P is defined by the familiar formula

$$\int_b Pu=\int_b u\text{ for every }u\in L^1(\mathfrak{A},\bar\mu),\ b\in\mathfrak{B},$$

so is the conditional expectation operator in the sense of 242J. Observe that the formula in 365A tells us that $L^1(\mathfrak{B},\bar\nu)$ is just $L^1(\mathfrak{A},\bar\mu)\cap L^0(\mathfrak{B})$. Translating 233K into this language, we see that $P(u\times v)=Pu\times v$ whenever $u\in L^1(\mathfrak{A},\bar\mu)$, $v\in L^0(\mathfrak{B})$ and $u\times v\in L^1(\mathfrak{A},\bar\mu)$.

(b) Just as in 233I-233J and 242K, we have a version of Jensen's inequality. Let $h:\mathbb{R}\to\mathbb{R}$ be a convex function and $\bar h:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ the corresponding map (364H). If $u\in L^1(\mathfrak{A},\bar\mu)$, then $h(\int u)\le\int\bar h(u)$; and if $\bar h(u)\in L^1(\mathfrak{A},\bar\mu)$, then $\bar h(Pu)\le P(\bar h(u))$. **P** I repeat the proof of 233I-233J. For each $q\in\mathbb{Q}$, take $\beta_q\in\mathbb{R}$ such that $h(t)\ge h_q(t)=h(q)+\beta_q(t-q)$ for every $t\in\mathbb{R}$, so that $h(t)=\sup_{q\in\mathbb{Q}}h_q(t)$ for every $t\in\mathbb{R}$, and $\bar h(u)=\sup_{q\in\mathbb{Q}}\bar h_q(u)$ for every $u\in L^0(\mathfrak{A})$. (This is because

$$\begin{aligned}[\![\bar h(u)>\alpha]\!]&=[\![u\in h^{-1}[\,]\alpha,\infty[\,]]\!]=[\![u\in\textstyle\bigcup_{q\in\mathbb{Q}}h_q^{-1}[\,]\alpha,\infty[\,]]\!]\\&=\sup_{q\in\mathbb{Q}}[\![u\in h_q^{-1}[\,]\alpha,\infty[\,]]\!]=\sup_{q\in\mathbb{Q}}[\![\bar h_q(u)>\alpha]\!]\end{aligned}$$

for every $\alpha\in\mathbb{R}$.) But setting $e=\chi 1$, we see that $\bar h_q(u)=h(q)e+\beta_q(u-qe)$ for every $u\in L^0(\mathfrak{A})$, so that

$$\int\bar h_q(u)=h(q)+\beta_q(\int u-q)=h_q(\int u),$$

$$P(\bar h_q(u))=h(q)e+\beta_q(Pu-qe)=\bar h_q(Pu)$$

because $\int e=1$ and $Pe=e$. Taking the supremum over q, we get

$$h(\int u)=\sup_{q\in\mathbb{Q}}h_q(\int u)=\sup_{q\in\mathbb{Q}}\int\bar h_q(u)\le\int\bar h(u),$$

and if $\bar h(u)\in L^1_{\bar\mu}$ then

$$\bar h(Pu)=\sup_{q\in\mathbb{Q}}\bar h_q(Pu)=\sup_{q\in\mathbb{Q}}P(\bar h_q(u))\le P(\bar h(u)).\ \mathbf{Q}$$

Of course the result in this form can also be deduced from 233I-233J if we represent $(\mathfrak{A},\bar\mu)$ as the measure algebra of a probability space (X,Σ,μ) and set $\mathrm{T}=\{E:E\in\Sigma,\ E^\bullet\in\mathfrak{B}\}$.

(c) I note here a fact which is occasionally useful. If $u \in L^1(\mathfrak{A},\bar{\mu})$ is non-negative, then $[\![Pu > 0]\!] = \mathrm{upr}([\![u > 0]\!], \mathfrak{B})$, the upper envelope of $[\![u > 0]\!]$ in $\mathfrak{B}$ as defined in 313S. **P** We have only to observe that, for $b \in \mathfrak{B}$,

$$b \cap [\![Pu > 0]\!] = 0 \iff \chi b \times Pu = 0 \iff \int_b Pu = 0$$

$$\iff \int_b u = 0 \iff b \cap [\![u > 0]\!] = 0.$$

Taking complements, $b \supseteq [\![Pu > 0]\!]$ iff $b \supseteq [\![u > 0]\!]$. **Q**

(d) Suppose now that $(\mathfrak{C},\bar{\lambda})$ is another probability algebra and $\pi : \mathfrak{A} \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism. Then $\mathfrak{D} = \pi[\mathfrak{B}]$ is a closed subalgebra of $\mathfrak{C}$ (314F(a-i)). Let $Q : L^1(\mathfrak{C},\bar{\lambda}) \to L^1(\mathfrak{D},\bar{\lambda}\restriction\mathfrak{D}) \subseteq L^1(\mathfrak{C},\bar{\lambda})$ be the conditional expectation associated with $\mathfrak{D}$, and $T_\pi : L^1(\mathfrak{A},\bar{\mu}) \to L^1(\mathfrak{C},\bar{\lambda})$ the norm-preserving Riesz homomorphism defined by π. Then $T_\pi P = QT_\pi$. **P** Take $u \in L^1(\mathfrak{A},\bar{\mu})$. Then

$$[\![T_\pi Pu > \alpha]\!] = \pi[\![Pu > \alpha]\!] \in \pi[\mathfrak{B}] = \mathfrak{D}$$

for every $\alpha \in \mathbb{R}$, so $T_\pi Pu \in L^0(\mathfrak{D})$. If $d \in \mathfrak{D}$, set $b = \pi^{-1}d \in \mathfrak{B}$; then

$$\begin{aligned}\int_d T_\pi Pu &= \int T_\pi Pu \times \chi d = \int T_\pi Pu \times T_\pi \chi b = \int T_\pi(Pu \times \chi b)\\ &= \int Pu \times \chi b = \int_b Pu = \int_b u = \int u \times \chi b\\ &= \int T_\pi(u \times \chi b) = \int T_\pi u \times T_\pi \chi b = \int T_\pi u \times \chi d = \int_d T_\pi u.\end{aligned}$$

As d is arbitrary, $T_\pi Pu$ satisfies the defining formula for $QT_\pi u$ and $T_\pi Pu = QT_\pi u$; as u is arbitrary, $T_\pi P = QT_\pi$. **Q**

365R Recovering the algebra: Proposition (a) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. Then $\mathfrak{A}$ is isomorphic to the band algebra of $L^1(\mathfrak{A},\bar{\mu})$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\bar{\mu}$, $\bar{\nu}$ two measures on $\mathfrak{A}$ such that $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{A},\bar{\nu})$ are both semi-finite measure algebras. Then $L^1(\mathfrak{A},\bar{\mu})$ is isomorphic, as Banach lattice, to $L^1(\mathfrak{A},\bar{\nu})$.

proof (a) Because $(\mathfrak{A},\bar{\mu})$ is semi-finite, $L^1_{\bar{\mu}}$ is order-dense in $L^0 = L^0(\mathfrak{A})$ (365G). Consequently, $L^1_{\bar{\mu}}$ and L^0 have isomorphic band algebras (353D). But the band algebra of L^0 is just its algebra of projection bands (because $\mathfrak{A}$ and therefore L^0 are Dedekind complete, see 364M and 353J), which is isomorphic to $\mathfrak{A}$ (364O).

(b) Let $\pi : \mathfrak{A} \to \mathfrak{A}$ be the identity map. Regarding π as an order-continuous Boolean homomorphism from $\mathfrak{A}^f_{\bar{\mu}} = \{a : \bar{\mu}a < \infty\}$ to $(\mathfrak{A},\bar{\nu})$, we have an associated positive linear operator $P = P_\pi : L^1_{\bar{\nu}} \to L^1_{\bar{\mu}}$; similarly, we have $Q = P_{\pi^{-1}} : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$, and both P and Q have norm at most 1 (365Ob). Now 365Of assures us that QP is the identity operator on $L^1_{\bar{\nu}}$ and PQ is the identity operator on $L^1_{\bar{\mu}}$. So P and Q are the two halves of a Banach lattice isomorphism between $L^1_{\bar{\mu}}$ and $L^1_{\bar{\nu}}$.

365S Having opened the question of varying measures on a single Boolean algebra, this seems an appropriate moment for a general description of how they interact.

Proposition Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, and $\bar{\mu} : \mathfrak{A} \to [0,\infty]$, $\bar{\nu} : \mathfrak{A} \to [0,\infty]$ two functions such that $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{A},\bar{\nu})$ are both semi-finite (therefore localizable) measure algebras.

(a) There is a unique $u \in L^0 = L^0(\mathfrak{A})$ such that (if we allow ∞ as a value of the integral) $\int_a u\,d\bar{\mu} = \bar{\nu}a$ for every $a \in \mathfrak{A}$.

(b) For $v \in L^0(\mathfrak{A})$, $\int v\,d\bar{\nu} = \int u \times v\,d\bar{\mu}$ if either is defined in $[-\infty,\infty]$.

(c) u is strictly positive (i.e., $[\![u > 0]\!] = 1$) and, writing $\frac{1}{u}$ for the multiplicative inverse of u, $\int_a \frac{1}{u} d\bar{\nu} = \bar{\mu}a$ for every $a \in \mathfrak{A}$.

proof (a) Because $(\mathfrak{A},\bar{\nu})$ is semi-finite, there is a partition of unity $D \subseteq \mathfrak{A}$ such that $\bar{\nu}d < \infty$ for every $d \in D$. For each $d \in D$, the functional $a \mapsto \bar{\nu}(a \cap d) : \mathfrak{A} \to \mathbb{R}$ is completely additive, so there is a $u_d \in L^1_{\bar{\mu}}$

such that $\int_a u_d d\bar\mu = \bar\nu(a \cap d)$ for every $a \in \mathfrak{A}$. Because $\int_a u_d d\bar\mu \ge 0$ for every a, $u_d \ge 0$. Because $\int_{1\setminus d} u_d = 0$, $[\![u_d > 0]\!] \subseteq d$. Now $u = \sup_{d\in D} u_d$ is defined in L^0. **P** (This is a special case of 368K below.) For $n \in \mathbb{N}$, set $c_n = \sup_{d\in D}[\![u_d > n]\!]$. If d, $d' \in D$ are distinct, then $d \cap [\![u_{d'} > n]\!] = 0$, so $d \cap c_n = [\![u_d > n]\!]$. Set $c = \inf_{n\in\mathbb{N}} c_n$. If $d \in D$, then

$$d \cap c = \inf_{n\in\mathbb{N}} d \cap c_n = \inf_{n\in\mathbb{N}}[\![u_d > n]\!] = 0.$$

But $c \subseteq c_0 \subseteq \sup D$, so $c = 0$. By 364L(a-i), $\{u_d : d \in D\}$ is bounded above in L^0, so has a supremum, because L^0 is Dedekind complete, by 364M. **Q**

For finite $I \subseteq D$ set $\tilde u_I = \sum_{d\in I} u_d = \sup_{d\in I} u_d$ (because $u_d \wedge u_c = 0$ for distinct c, $d \in D$). Then $u = \sup\{\tilde u_I : I \subseteq D, I \text{ is finite}\}$. So, for any $a \in \mathfrak{A}$,

$$\int_a u\,d\bar\mu = \sup_{I\subseteq D \text{ is finite}} \int_a \tilde u_I d\bar\mu$$

(365Dh)

$$= \sup_{I\subseteq D \text{ is finite}} \sum_{d\in I}\int_a u_d d\bar\mu = \sup_{I\subseteq D \text{ is finite}} \sum_{d\in I} \bar\nu(a\cap d) = \bar\nu a.$$

Note that if $a \in \mathfrak{A}$ is non-zero, then $\bar\nu a > 0$, so $a \cap [\![u > 0]\!] \ne 0$; consequently $[\![u > 0]\!] = 1$.

To see that u is unique, observe that if u' has the same property then for any $d \in D$

$$\int_a u \times \chi d\,d\bar\mu = \bar\nu(a \cap d) = \int_a u' \times \chi d\,d\bar\mu$$

for every $a \in \mathfrak{A}$, so that $u \times \chi d = u' \times \chi d$; because $\sup D = 1$ in $\mathfrak{A}$, u must be equal to u'.

(b) Use 365Hb, with π and T the identity maps.

(c) In the same way, there is a $w \in L^0$ such that $\int_a w\,d\bar\nu = \bar\mu a$ for every $a \in \mathfrak{A}$. To relate u and w, observe that applying (b) above we get

$$\int w \times \chi a \times u\,d\bar\mu = \int w \times \chi a\,d\bar\nu$$

for every $a \in \mathfrak{A}$, that is, $\int_a w \times u\,d\bar\mu = \bar\mu a$ for every a. But from this we see that $w \times u \times \chi b = \chi b$ at least when $\bar\mu b < \infty$, so that $w \times u = \chi 1$ is the multiplicative identity of L^0, and $w = \frac{1}{u}$.

365T Uniform integrability Continuing the programme in 365C, we can transcribe the ideas of §§246, 247, 354 and 356 into the new context.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Set $L^1 = L^1(\mathfrak{A}, \bar\mu)$.

(a) For a non-empty subset A of L^1, the following are equiveridical:

(i) A is uniformly integrable in the sense of 354P;

(ii) for every $\epsilon > 0$ there are an $a \in \mathfrak{A}^f$ and an $M \ge 0$ such that $\int(|u| - M\chi a)^+ \le \epsilon$ for every $u \in \mathfrak{A}$;

(iii)(α) $\sup_{u\in A}|\int_a u|$ is finite for every atom $a \in \mathfrak{A}$,

(β) for every $\epsilon > 0$ there are $c \in \mathfrak{A}^f$ and $\delta > 0$ such that $|\int_a u| \le \epsilon$ whenever $u \in A$, $a \in \mathfrak{A}$ and $\bar\mu(a \cap c) \le \delta$;

(iv)(α) $\sup_{u\in A}|\int_a u|$ is finite for every atom $a \in \mathfrak{A}$,

(β) $\lim_{n\to\infty}\sup_{u\in A}|\int_{a_n} u| = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$;

(v) A is relatively weakly compact in L^1.

(b) If $(\mathfrak{A}, \bar\mu)$ is a probability algebra and $A \subseteq L^1$ is uniformly integrable, then there is a solid convex norm-closed uniformly integrable set $C \supseteq A$ such that $P[C] \subseteq C$ whenever $P : L^1 \to L^1$ is the conditional expectation operator associated with a closed subalgebra of $\mathfrak{A}$.

proof 354Q, 354R, 356Q and 246D, with a little help from 246C and 246G.

365X Basic exercises (a) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $u \in L^1_{\bar\mu}$. Show that

$$\int u = \int_0^\infty \bar\mu[\![u > \alpha]\!]\,d\alpha - \int_{-\infty}^0 \bar\mu(1 \setminus [\![u > \alpha]\!])\,d\alpha.$$

>(b) Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra, and $u \in L^1_{\bar{\mu}}$. (i) Show that $\|u\|_1 \le 2\sup_{a\in\mathfrak{A}^f}|\int_a u|$. (*Hint*: 246F.) (ii) Show that for any $\epsilon > 0$ there is an $a \in \mathfrak{A}^f$ such that $|\int u - \int_b u| \le \epsilon$ whenever $a \subseteq b \in \mathfrak{A}$.

>(c) Let U be an L-space. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is any norm-bounded sequence in U^+, show that

$$\liminf_{n\to\infty} u_n = \sup_{n\in\mathbb{N}} \inf_{m\ge n} u_m$$

is defined in U, and that $\int \liminf_{n\to\infty} u_n \le \liminf_{n\to\infty} \int u_n$.

(d) Let U be an L-space. Let $\mathcal{F}$ be a filter on U such that $\{u : u \ge 0, \|u\| \le k\}$ belongs to $\mathcal{F}$ for some $k \in \mathbb{N}$. Show that $u_0 = \sup_{F\in\mathcal{F},F\subseteq U^+} \inf F$ is defined in U, and that $\int u_0 \le \sup_{F\in\mathcal{F}} \inf_{u\in F} \int u$.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $A \subseteq L^1_{\bar{\mu}}$ a non-empty set. Show that A is bounded above in $L^1_{\bar{\mu}}$ iff

$$\sup\{\textstyle\sum_{i=0}^n \int_{a_i} u_i : a_0,\dots,a_n \text{ is a partition of unity in } \mathfrak{A},\ u_0,\dots,u_n \in A\}$$

is finite, and that in this case the supremum is $\int \sup A$. (*Hint*: given $u_0,\dots,u_n \in A$, set $b_i = \inf_j [\![u_i \ge u_j]\!]$, $a_i = b_i \setminus \sup_{j<i} b_j$, and show that $\int \sup_{i\le n} u_i = \sum_{i=0}^n \int_{a_i} u_i$.)

(f) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and u, $v \in L^0(\mathfrak{A})^+$. Show that $\int u \times v\,d\bar{\mu} = \int_0^\infty (\int_{[\![u>\alpha]\!]} v\,d\bar{\mu})d\alpha$. (*Hint*: start with $u \in S(\mathfrak{A})^+$.)

(g) Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra and $\nu : \mathfrak{A}^f \to \mathbb{R}$ a bounded additive functional. Show that the following are equiveridical: (i) there is a $u \in L^1_{\bar{\mu}}$ such that $\nu a = \int_a u$ for every $a \in \mathfrak{A}^f$; (ii) for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \le \epsilon$ whenever $\bar{\mu} a \le \delta$; (iii) for every $\epsilon > 0$, $c \in \mathfrak{A}^f$ there is a $\delta > 0$ such that $\nu a \le \epsilon$ whenever $a \subseteq c$ and $\bar{\mu} a \le \delta$; (iv) for every $\epsilon > 0$ there are $c \in \mathfrak{A}^f$, $\delta > 0$ such that $|\nu a| \le \epsilon$ whenever $a \in \mathfrak{A}^f$ and $\bar{\mu}(a \cap c) \le \delta$; (v) $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}^f$ with infimum 0.

(h) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let $T : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ be the Riesz homomorphism associated with π (364P). Suppose that $w \ge 0$ in $L^0(\mathfrak{B})$ is such that $\int_{\pi a} w\,d\bar{\nu} = \bar{\mu} a$ whenever $a \in \mathfrak{A}$. Show that for any $u \in L^0(\mathfrak{A},\bar{\mu})$, $\int Tu \times w\,d\bar{\nu} = \int u\,d\bar{\mu}$ whenever either is defined in $[-\infty,\infty]$.

>(i) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and $a \in \mathfrak{A}$; write $\mathfrak{A}_a$ for the principal ideal it generates. Show that if π is the identity embedding of $\mathfrak{A}^f \cap \mathfrak{A}_a$ into $\mathfrak{A}^f$, then T_π, as defined in 365N, identifies $L^1(\mathfrak{A}_a,\bar{\mu}\restriction\mathfrak{A}_a)$ with a band in $L^1_{\bar{\mu}}$.

>(j) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$. Let $\phi : X \to Y$ be an inverse-measure-preserving function and $\pi : \mathfrak{B} \to \mathfrak{A}$ the corresponding measure-preserving homomorphism (324M). Show that $T_\pi : L^1_{\bar{\nu}} \to L^1_{\bar{\mu}}$ (365N) corresponds to the map $g^\bullet \mapsto (g\phi)^\bullet : L^1(\nu) \to L^1(\mu)$ of 242Xd.

(k) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras. Let $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ be a ring homomorphism such that, for some $\gamma > 0$, $\bar{\nu}(\pi a) \le \gamma\bar{\mu} a$ for every $a \in \mathfrak{A}^f$. (i) Show that there is a unique order-continuous Riesz homomorphism $T : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$ such that $T(\chi a) = \chi(\pi a)$ whenever $a \in \mathfrak{A}^f$, and that $\|T\| \le \gamma$. (ii) Show that $[\![Tu > \alpha]\!] = \pi[\![u > \alpha]\!]$ whenever $u \in L^1_{\bar{\mu}}$ and $\alpha > 0$. (iii) Show that T is surjective iff π is, injective iff π is. (iv) Show that T is norm-preserving iff $\bar{\nu}(\pi a) = \bar{\mu} a$ for every $a \in \mathfrak{A}^f$.

(l) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving Boolean homomorphism. Let $T : L^1_{\bar{\mu}} \to L^1_{\bar{\nu}}$ and $P : L^1_{\bar{\nu}} \to L^1_{\bar{\mu}}$ be the operators corresponding to $\pi\restriction\mathfrak{A}^f$, as described in 365N-365O, and $\tilde{T} : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{B})$ the operator corresponding to π, as described in 363F. (i) Show that $T(u \times v) = Tu \times \tilde{T}v$ for every $u \in L^1_{\bar{\mu}}$, $v \in L^\infty(\mathfrak{A})$. (ii) Show that if π is order-continuous, then $\int Pv \times u = \int v \times \tilde{T}u$ for every $u \in L^\infty(\mathfrak{A})$, $v \in L^1_{\bar{\nu}}$.

>(m) Let (X,Σ,μ) be a probability space, with measure algebra $(\mathfrak{A},\bar{\mu})$, and let T be a σ-subalgebra of Σ. Set $\nu=\mu\restriction\mathrm{T}$, $\mathfrak{B}=\{F^\bullet:F\in\mathrm{T}\}\subseteq\mathfrak{A}$, $\bar{\nu}=\bar{\mu}\restriction\mathfrak{B}$, so that $(\mathfrak{B},\bar{\nu})$ is a measure algebra. Let $\pi:\mathfrak{B}\to\mathfrak{A}$ be the identity homomorphism. Show that $T_\pi:L^1_{\bar{\nu}}\to L^1_{\bar{\mu}}$ (365N) corresponds to the canonical embedding of $L^1(\nu)$ in $L^1(\mu)$ described in 242Jb, while $P_\pi:L^1_{\bar{\mu}}\to L^1_{\bar{\nu}}$ (365O) corresponds to the conditional expectation operator described in 242Jd.

(n) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be probability algebras, $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism, and $T:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ the corresponding Riesz homomorphism. Let $\mathfrak{C}$ be a closed subalgebra of $\mathfrak{A}$ and $P:L^1(\mathfrak{A},\bar{\mu})\to L^1(\mathfrak{C},\bar{\mu}\restriction\mathfrak{C})\subseteq L^1(\mathfrak{A},\bar{\mu})$, $Q:L^1(\mathfrak{B},\bar{\nu})\to L^1(\mathfrak{B},\bar{\nu})$ the conditional expectation operators defined from $\mathfrak{C}\subsetneqq\mathfrak{A}$ and $\pi[\mathfrak{C}]\subsetneqq\mathfrak{B}$. Show that $TP=QT$.

(o) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and $(\widehat{\mathfrak{A}},\hat{\mu})$ its localization (322Q). Show that the natural embedding of $\mathfrak{A}$ in $\widehat{\mathfrak{A}}$ induces a Banach lattice isomorphism between $L^1_{\bar{\mu}}$ and $L^1_{\hat{\mu}}$, so that the band algebra of $L^1_{\bar{\mu}}$ can be identified with the Dedekind completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$.

(p) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\bar{\mu}$, $\bar{\nu}$ two functions such that $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{A},\bar{\nu})$ are measure algebras. Show that $L^1_{\bar{\mu}}\subseteq L^1_{\bar{\nu}}$ (as subsets of $L^0(\mathfrak{A})$) iff there is a $\gamma>0$ such that $\bar{\nu}a\le\gamma\bar{\mu}a$ for every $a\in\mathfrak{A}$. (*Hint*: show that the identity operator from $L^1_{\bar{\mu}}$ to $L^1_{\bar{\nu}}$ is bounded.)

(q) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, I_∞ the ideal of 'purely infinite' elements of $\mathfrak{A}$ together with 0 and $\bar{\mu}_{\mathrm{sf}}$ the measure on $\mathfrak{B}=\mathfrak{A}/I_\infty$ as defined in 322Xa. Let $\pi:\mathfrak{A}\to\mathfrak{B}$ be the canonical map. Show that T_π, as defined in 365N, is a Banach lattice isomorphism between $L^1_{\bar{\mu}}$ and $L^1(\mathfrak{B},\bar{\mu}_{\mathrm{sf}})$.

(r) Let (X,Σ,μ) be a a semi-finite measure space. Show that $L^1(\mu)$ is separable iff μ is σ-finite and has countable Maharam type.

365Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, not $\{0\}$. Show that the topological density of $L^1_{\bar{\mu}}$ is $\max(\omega,\tau(\mathfrak{A}),c(\mathfrak{A}))$, where $\tau(\mathfrak{A})$, $c(\mathfrak{A})$ are the Maharam type and cellularity of $\mathfrak{A}$.

365 Notes and comments You should not suppose that L^1 spaces appear in the second half of this chapter because they are of secondary importance. Indeed I regard them as the most important of all function spaces. I have delayed the discussion of them for so long because it is here that for the first time we need measure algebras in an essential way.

The actual definition of $L^1_{\bar{\mu}}$ which I give is designed for speed rather than illumination; I seek only a formula, visibly independent of any particular representation of $(\mathfrak{A},\bar{\mu})$ as the measure algebra of a measure space, from which I can prove 365B. 365C-365D and 365Ea are now elementary. In 365Eb I take a page to describe a form of the Radon-Nikodým theorem which is applicable to arbitrary measure algebras, at the cost of dealing with functionals on the ring $\mathfrak{A}^f$ rather than on the whole algebra $\mathfrak{A}$. This is less for the sake of applications than to emphasize one of the central properties of L^1: it depends only on $\mathfrak{A}^f$ and $\bar{\mu}\restriction\mathfrak{A}^f$. For alternative versions of the condition 365Eb(i) see 365Xg.

The convergence theorems (B.Levi's theorem, Fatou's lemma and Lebesgue's dominated convergence theorem) are so central to the theory of integrable functions that it is natural to look for versions in the language here. Corresponding to B.Levi's theorem is the Levi property of a norm in an L-space; note how the abstract formulation makes it natural to speak of general upwards-directed families rather than of non-decreasing sequences, though the sequential form is so often used that I have spelt it out (365C). In the same way, the integral becomes order-continuous rather than just sequentially order-continuous (365Da). Corresponding to Fatou's lemma we have 365Xc-365Xd. For abstract versions of Lebesgue's theorem I will wait until §367.

In 365H I have deliberately followed the hypotheses of 235A and 235R. Of course 365H can be deduced from these if we use the Stone representations of $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$, so that π can be represented by a function between the Stone spaces (312Q). But 365H is essentially simpler, because the technical problems concerning measurability which took up so much of §235 have been swept under the carpet. In the same way, 365Xh corresponds to 235E. Here we have a fair example of the way in which the abstract expression in terms of measure algebras can be tidier than the expression in terms of measure spaces. But in my view this is because here, at least, some of the mathematics has been left out.

365I-365J correspond closely to 361F-361H and 363E. 365L is a re-run of 243G, but with the additional refinement that I examine the action of L^1 on L^∞ (the operator S) as well as the action of L^∞ on L^1 (the operator T). Of course 365Lc is just the abstract version of 243Hb, and can easily be proved from it. Note that while the proof of 365L does not itself involve any representation of $(\mathfrak{A},\bar{\mu})$ as the measure algebra of a measure space, (a-vii) and (b-iii) of the proof of 365L depend on the Radon-Nikodým theorem through 327D and 365E. For a development of the theory of $L^1(\mathfrak{A},\bar{\mu})$ which does not (in a formal sense) depend on measure spaces, see Fremlin 74a, 63J.

Theorems 365N-365P lie at the centre of my picture of L^1 spaces, and are supposed to show their dual nature. Starting from a semi-finite measure algebra $(\mathfrak{A},\bar{\mu})$ we have two essentially different routes to the L^1-space: we can either build it up from indicator functions of elements of finite measure, so that it is naturally embedded in $L^0(\mathfrak{A})$, or we can think of it as the order-continuous dual of $L^\infty(\mathfrak{A})$. The first is a 'covariant' construction (signalled by the formula $T_{\theta\pi}=T_\theta T_\pi$ in 365Ne) and the second is 'contravariant' (so that $P_{\theta\pi}=P_\pi P_{\theta'}$ in 365Of). The first construction is the natural one if we are seeking to copy the ideas of §242, but the second arises inevitably if we follow the ordinary paths of functional analysis and study dual spaces whenever they appear. The link between them is the Radon-Nikodým theorem.

I have deliberately written out 365N and 365O with different hypotheses on the homomorphism π in the hope of showing that the two routes to L^1 really are different, and can be expected to tell us different things about it. I use the letter P in 365O in order to echo the language of 242J; in the most important context, in which $\mathfrak{A}$ is actually a subalgebra of $\mathfrak{B}$ and π is the identity map, P is a kind of conditional expectation operator (365Q). I note that in the proof of 365Oe I have returned to first principles, using some of the ideas of the Radon-Nikodým theorem (232E), but a different approach to the exhaustion step (converting 'for every $u>0$ there is a $v>0$ such that $Pv\le u$' into 'P is surjective'). I chose the somewhat cruder method in 232E (part (c) of the proof) in order to use the weakest possible form of the axiom of choice. In the present context such scruples seem absurd.

I used the words 'covariant' and 'contravariant' above; of course this distinction depends on the side of the mirror on which we are standing; if our measure-preserving homomorphism is derived (contravariantly) from an inverse-measure-preserving transformation, then the T's become contravariant (365Xj). An important component of this work, for me, is the fact that not all measure-preserving homomorphisms between measure algebras can be represented by inverse-measure-preserving functions (343Jb, 343M).

I have noted at various points (e.g., 242Yd) that the properties of $L^1(\mu)$ are not much affected by peculiarities in a measure space (X,Σ,μ). In this section I offer an explanation: unlike L^0 or L^∞, L^1 really depends only on $\mathfrak{A}^f$, the ring of elements of finite measure in the measure algebra. (See 365N-365P, 365Xo and 365Xq.) Note that while the algebra $\mathfrak{A}$ is uniquely determined (given that $(\mathfrak{A},\bar{\mu})$ is localizable, 365Ra), the measure $\bar{\mu}$ is not; if $\mathfrak{A}$ is any algebra carrying two non-isomorphic semi-finite measures, the corresponding L^1 spaces are still isomorphic (365Rb). For instance, the L^1-spaces of Lebesgue measure μ on $\mathbb{R}$, and the subspace measure $\mu_{[0,1]}$ on $[0,1]$, are isomorphic, though their measure algebras are not.

In 365T I have recapitulated the results in §§246, 247, 354 and 356 concerning uniform integrability and weak compactness, but I make no attempt to add to them. Once we have left measure spaces behind, these ideas belong to the theory of Banach lattices, and there is little to relate them to the questions dealt with in this section. But see 373Xj and 373Xn below.

Version of 10.11.08

366 L^p

In this section I apply the methods of this chapter to L^p spaces, where $1<p<\infty$. The constructions proceed without surprises up to 366E, translating the ideas of §244 by the methods used in §365. Turning to the action of Boolean homomorphisms on L^p spaces, I introduce a space M^0, which can be regarded as the part of L^0 that can be determined from the ring $\mathfrak{A}^f$ of elements of $\mathfrak{A}$ of finite measure (366F), and which includes L^p whenever $1\le p<\infty$. Now a measure-preserving ring homomorphism from $\mathfrak{A}^f$ to $\mathfrak{B}^f$ acts on the M^0 spaces in a way which includes injective Riesz homomorphisms from $L^p(\mathfrak{A},\bar{\mu})$ to $L^p(\mathfrak{B},\bar{\nu})$ and surjective positive linear operators from $L^p(\mathfrak{B},\bar\nu)$ to $L^p(\mathfrak{A},\bar\mu)$ (366H). The latter may be regarded as conditional expectation operators (366J). The case $p=2$ (366K-366L) is of course by far the most important. As with the familiar spaces $L^p(\mu)$ of Chapter 24, we have complex versions $L^p_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ with the expected properties (366M).

366A Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and suppose that $1<p<\infty$. For $u\in L^0(\mathfrak{A})$, define $|u|^p\in L^0(\mathfrak{A})$ by setting

$$\begin{aligned}[\![|u|^p>\alpha]\!]&=[\![|u|>\alpha^{1/p}]\!] \text{ if } \alpha\ge 0,\\ &=1 \text{ if } \alpha<0.\end{aligned}$$

(In the language of 364H, $|u|^p=\bar h(u)$, where $h(t)=|t|^p$ for $t\in\mathbb{R}$.) Set

$$L^p_{\bar\mu}=L^p(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\ |u|^p\in L^1(\mathfrak{A},\bar\mu)\},$$

and for $u\in L^0(\mathfrak{A})$ set

$$\|u\|_p=(\int|u|^p)^{1/p}=\||u|^p\|_1^{1/p},$$

counting $\infty^{1/p}$ as ∞, so that $L^p_{\bar\mu}=\{u:u\in L^0(\mathfrak{A}),\ \|u\|_p<\infty\}$.

366B Theorem Let (X,Σ,μ) be a measure space, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Then the canonical isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ (364Ic) makes $L^p(\mu)$, as defined in §244, correspond to $L^p(\mathfrak{A},\bar\mu)$.

proof What we really have to check is that if $w\in L^0(\mu)$ corresponds to $u\in L^0(\mathfrak{A})$, then $|w|^p$, as defined in 244A, corresponds to $|u|^p$ as defined in 366A. But this was noted in 364Ib.

Now, because the isomorphism between $L^0(\mu)$ and $L^0(\mathfrak{A})$ matches $L^1(\mu)$ with $L^1_{\bar\mu}$ (365B), we can be sure that $|w|^p\in L^1(\mu)$ iff $|u|^p\in L^1_{\bar\mu}$, and that in this case

$$\|w\|_p=\Bigl(\int|w|^p\Bigr)^{1/p}=\Bigl(\int|u|^p\Bigr)^{1/p}=\|u\|_p,$$

as required.

366C Corollary For any measure algebra $(\mathfrak{A},\bar\mu)$ and $p\in\,]1,\infty[$, $L^p=L^p(\mathfrak{A},\bar\mu)$ is a solid linear subspace of $L^0(\mathfrak{A})$. It is a Dedekind complete Banach lattice under its uniformly convex norm $\|\;\|_p$. Setting $q=p/(p-1)$, $(L^p)^*$ is identified with $L^q(\mathfrak{A},\bar\mu)$ by the duality $(u,v)\mapsto\int u\times v$. Writing $\mathfrak{A}^f$ for the ring $\{a:a\in\mathfrak{A},\ \bar\mu a<\infty\}$, $S(\mathfrak{A}^f)$ is norm-dense in L^p.

proof Because we can find a measure space (X,Σ,μ) such that $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra of μ (321J), this is just a digest of the results in 244B, 244E-244H, 244K, 244L and 244O[1]. (Of course $S(\mathfrak{A}^f)$ corresponds to the space S of equivalence classes of simple functions in 244Ha, just as in 365F.)

366D I can add a little more, corresponding to 365C and 365L.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $p\in\,]1,\infty[$.

(a) The norm $\|\;\|_p$ on $L^p=L^p(\mathfrak{A},\bar\mu)$ is order-continuous.

(b) L^p has the Levi property.

(c) Setting $q=p/(p-1)$, the canonical identification of $L^q=L^q(\mathfrak{A},\bar\mu)$ with $(L^p)^*$ is a Riesz space isomorphism between L^q and $(L^p)^\sim=(L^p)^\times$.

(d) L^p is a perfect Riesz space.

proof (a) Suppose that $A\subseteq L^p$ is non-empty, downwards-directed and has infimum 0. For u, $v\ge 0$ in L^p, $u\le v\Rightarrow u^p\le v^p$ (by the definition in 366A, or otherwise), so $B=\{u^p:u\in A\}$ is downwards-directed. If $v_0=\inf B$ in $L^1=L^1(\mathfrak{A},\bar\mu)$, then $v_0^{1/p}$ (defined by the formula in 366A, or otherwise) is less than or equal to every member of A, so must be 0, and $v_0=0$. Accordingly $\inf B=0$ in L^1. Because $\|\;\|_1$ is order-continuous (365C),

[1] Later editions only.

$$\inf_{u\in A}\|u\|_p=\inf_{u\in A}\|u^p\|_1^{1/p}=(\inf_{v\in B}\|v\|_1)^{1/p}=0.$$

As A is arbitrary, $\| \|_p$ is order-continuous.

(b) Now suppose that $A\subseteq(L^p)^+$ is non-empty, upwards-directed and norm-bounded. Then $B=\{u^p:u\in A\}$ is non-empty, upwards-directed and norm-bounded in L^1. So $v_0=\sup B$ is defined in L^1, and $v_0^{1/p}$ is an upper bound for A in L^p.

(c) By 356Dd, $(L^p)^*=(L^p)^\sim=(L^p)^\times$. The extra information we need is that the identification of L^q with $(L^p)^*$ is an order-isomorphism. **P** (α) If $w\in(L^q)^+$ and $u\in(L^p)^+$ then $u\times w\ge 0$ in L^1, so $(Tw)(u)=\int u\times w\ge 0$, writing $T:L^q\to(L^p)^*$ for the canonical bijection. As u is arbitrary, $Tw\ge 0$. As w is arbitrary, T is a positive linear operator. (β) If $w\in L^q$ and $Tw\ge 0$, consider $u=(w^-)^{q/p}$. Then $u\ge 0$ in L^p and $w^+\times u=0$ (because $[\![w^+>0]\!]\cap[\![u>0]\!]=[\![w^+>0]\!]\cap[\![w^->0]\!]=0$), so

$$0\le(Tw)(u)=\int w\times u=-\int w^-\times u=-\int(w^-)^q\le 0,$$

and $\int(w^-)^q=0$. But as $(w^-)^q\ge 0$ in L^1, this means that $(w^-)^q$ and w^- must be 0, that is, $w\ge 0$. As w is arbitrary, T^{-1} is positive and T is an order-isomorphism. **Q**

(d) This is an immediate consequence of (c), since $p=q/(q-1)$, so that L^p can be identified with $(L^q)^*=(L^q)^\times$. From 356M we see that it is also a consequence of (a) and (b).

366E Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $p\in[1,\infty]$. Set $q=p/(p-1)$ if $1<p<\infty$, $q=\infty$ if $p=1$ and $q=1$ if $p=\infty$. Then

$$L^q(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\ u\times v\in L^1(\mathfrak{A},\bar\mu)\text{ for every }v\in L^p(\mathfrak{A},\bar\mu)\}.$$

proof (a) We already know that if $u\in L^p=L^p(\mathfrak{A},\bar\mu)$ and $v\in L^q=L^q(\mathfrak{A},\bar\mu)$ then $u\times v\in L^1=L^1(\mathfrak{A},\bar\mu)$; this is elementary if $p\in\{1,\infty\}$ and otherwise is covered by 366C.

(b) So suppose that $u\in L^0\setminus L^p$. If $p=1$ then of course $\chi 1\in L^\infty=L^q$ and $u\times\chi 1\notin L^1$. If $p>1$ set

$$A=\{w:w\in S(\mathfrak{A}^f),\ 0\le w\le|u|\}.$$

Because $\bar\mu$ is semi-finite, $S(\mathfrak{A}^f)$ is order-dense in L^0 (364K), and $|u|=\sup A$. Because the norm on L^p has the Levi property (365C, 366Db, 363Ba) and A is not bounded above in L^p, $\sup_{w\in A}\|w\|_p=\infty$.

For each $n\in\mathbb{N}$ choose $w_n\in A$ with $\|w_n\|_p>4^n$. Then there is a $v_n\in L^q$ such that $\|v_n\|_q=1$ and $\int w_n\times v_n\ge 4^n$. **P** (α) If $p<\infty$ this is covered by 366C, since $\|w_n\|_p=\sup\{\int w_n\times v:\|v\|_q\le 1\}$. (β) If $p=\infty$ then $[\![w_n>4^n]\!]\ne 0$; because $\bar\mu$ is semi-finite, there is a $b\subseteq[\![w_n>4^n]\!]$ such that $0<\bar\mu b<\infty$, and $\|\frac{1}{\bar\mu b}\chi b\|_1=1$, while $\int w_n\times\frac{1}{\bar\mu b}\chi b\ge 4^n$. **Q**

Because L^q is complete (363Ba, 366C), $v=\sum_{n=0}^\infty 2^{-n}|v_n|$ is defined in L^q. But now

$$\int|u|\times v\ge 2^{-n}\int w_n\times v_n\ge 2^n$$

for every n, so $u\times v\notin L^1$.

Remark This result is characteristic of perfect subspaces of L^0; see 369C and 369J.

366F The next step is to look at the action of Boolean homomorphisms, as in 365N. It will be convenient to be able to deal with all L^p spaces at once by introducing names for a pair of spaces which include all of them.

Definition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Write

$$M^0_{\bar\mu}=M^0(\mathfrak{A},\bar\mu)=\{u:u\in L^0(\mathfrak{A}),\ \bar\mu[\![|u|>\alpha]\!]<\infty\text{ for every }\alpha>0\},$$

$$M^{1,0}_{\bar\mu}=M^{1,0}(\mathfrak{A},\bar\mu)=\{u:u\in M^0_{\bar\mu},\ u\times\chi a\in L^1(\mathfrak{A},\bar\mu)\text{ whenever }\bar\mu a<\infty\}.$$

366G Lemma Let $(\mathfrak{A},\bar\mu)$ be any measure algebra. Write $M^0=M^0(\mathfrak{A},\bar\mu)$, etc.

(a) M^0 and $M^{1,0}$ are Dedekind complete solid linear subspaces of L^0 which include L^p for every $p\in[1,\infty[$; moreover, M^0 is closed under multiplication.

(b) If $u \in M^0$ and $u \ge 0$, there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)$ such that $u = \sup_{n\in\mathbb{N}} u_n$.

(c) $M^{1,0} = \{u : u \in L^0,\ (|u| - \epsilon\chi 1)^+ \in L^1 \text{ for every } \epsilon > 0\} = L^1 + (L^\infty \cap M^0)$.

(d) If $u,\ v \in M^{1,0}$ and $\int_a u \le \int_a v$ whenever $\bar\mu a < \infty$, then $u \le v$; so if $\int_a u = \int_a v$ whenever $\bar\mu a < \infty$, $u = v$.

proof (a) If $u,\ v \in M^0$ and $\gamma \in \mathbb{R}$, then for any $\alpha > 0$

$$[\![|u+v| > \alpha]\!] \subseteq [\![|u| > \tfrac12\alpha]\!] \cup [\![|v| > \tfrac12\alpha]\!],$$

$$[\![|\gamma u| > \alpha]\!] \subseteq [\![|u| > \tfrac{\alpha}{1+|\gamma|}]\!],$$

$$[\![|u \times v| > \alpha]\!] \subseteq [\![|u| > \sqrt{\alpha}\,]\!] \cup [\![|v| > \sqrt{\alpha}\,]\!]$$

(364E) are of finite measure. So $u+v$, γu and $u \times v$ belong to M^0. Thus M^0 is a linear subspace of L^0 closed under multiplication. If $u \in M^0$, $|v| \le |u|$ and $\alpha > 0$, then $[\![|v| > \alpha]\!] \subseteq [\![|u| > \alpha]\!]$ has finite measure; thus $v \in M^0$ and M^0 is a solid linear subspace of L^0. It follows that $M^{1,0}$ also is. If $u \in L^p = L^p(\mathfrak{A},\bar\mu)$, where $p < \infty$, and $\alpha > 0$, then $[\![|u| > \alpha]\!] = [\![|u|^p > \alpha^p]\!]$ has finite measure, so $u \in M^0$; moreover, if $\bar\mu a < \infty$, then $\chi a \in L^q$, where $q = p/(p-1)$, so $u \times \chi a \in L^1$; thus $u \in M^{1,0}$.

To see that M^0 is Dedekind complete, observe that if $A \subseteq (M^0)^+$ is non-empty and bounded above by $u_0 \in M^0$, and $\alpha > 0$, then $\{[\![u > \alpha]\!] : u \in A\}$ is bounded above by $[\![u_0 > \alpha]\!] \in \mathfrak{A}^f$, so has a supremum in $\mathfrak{A}$ (321C). Accordingly $\sup A$ is defined in L^0 (364L(a-iii)) and belongs to M^0. Finally, $M^{1,0}$, being a solid linear subspace of M^0, must also be Dedekind complete.

(b) If $u \ge 0$ in M^0, then there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $S = S(\mathfrak{A})$ such that $u = \sup_{n\in\mathbb{N}} u_n$ and $u_0 \ge 0$ (364Jd). But now every u_n belongs to $S \cap M^0 = S(\mathfrak{A}^f)$, just as in 365F.

(c)(i) If $u \in M^{1,0}$ and $\epsilon > 0$, then $a = [\![|u| > \epsilon]\!] \in \mathfrak{A}^f$, so $u \times \chi a \in L^1 = L^1(\mathfrak{A},\bar\mu)$; but $(|u| - \epsilon\chi 1)^+ \le |u| \times \chi a$, so $(|u| - \epsilon\chi 1)^+ \in L^1$.

(ii) Suppose that $u \in L^0$ and $(|u| - \epsilon\chi 1)^+ \in L^1$ for every $\epsilon > 0$. Then, given $\epsilon > 0$, $v = (|u| - \frac12\epsilon\chi 1)^+ \in L^1$, and $\bar\mu[\![v > \frac12\epsilon]\!] < \infty$; but $[\![|u| > \epsilon]\!] \subseteq [\![v > \frac12\epsilon]\!]$, so also has finite measure. Thus $u \in M^0$. Next, if $a \in \mathfrak{A}^f$, then $|u \times \chi a| \le \chi a + (|u| - \chi 1)^+ \in L^1$, so $u \in M^{1,0}$.

(iii) Of course L^1 and $L^\infty \cap M^0$ are included in $M^{1,0}$, so their linear sum also is. On the other hand, if $u \in M^{1,0}$, then

$$u = (u^+ - \chi 1)^+ - (u^- - \chi 1)^+ + (u^+ \wedge \chi 1) - (u^- \wedge \chi 1) \in L^1 + (L^\infty \cap M^0).$$

(d) Take $\alpha > 0$ and set $a = [\![u - v > \alpha]\!]$. Because both u and v belong to $M^{1,0}_{\bar\mu}$, $\bar\mu a < \infty$ and $\int_a u \le \int_a v$, that is, $\int_a u - v \le 0$; so a must be 0 (365Dc). As α is arbitrary, $u - v \le 0$ and $u \le v$. If $\int_a u = \int_a v$ for every $a \in \mathfrak{A}^f$, then $v \le u$ so $u = v$.

366H Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. Let $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ be a measure-preserving ring homomorphism.

(a)(i) We have a unique order-continuous Riesz homomorphism $T = T_\pi : M^0(\mathfrak{A},\bar\mu) \to M^0(\mathfrak{B},\bar\nu)$ such that $T(\chi a) = \chi(\pi a)$ for every $a \in \mathfrak{A}^f$.

(ii) $[\![Tu > \alpha]\!] = \pi[\![u > \alpha]\!]$ for every $u \in M^0(\mathfrak{A},\bar\mu)$ and $\alpha > 0$.

(iii) T is injective and multiplicative.

(iv) For $p \in [1,\infty]$ and $u \in M^0(\mathfrak{A},\bar\mu)$, $\|Tu\|_p = \|u\|_p$; in particular, $Tu \in L^p(\mathfrak{B},\bar\nu)$ iff $u \in L^p(\mathfrak{A},\bar\mu)$. Consequently $\int Tu = \int u$ whenever $u \in L^1(\mathfrak{A},\bar\mu)$.

(v) For $u \in M^0(\mathfrak{A},\bar\mu)$, $Tu \in M^{1,0}(\mathfrak{B},\bar\nu)$ iff $u \in M^{1,0}(\mathfrak{A},\bar\mu)$.

(b)(i) We have a unique order-continuous positive linear operator $P = P_\pi : M^{1,0}(\mathfrak{B},\bar\nu) \to M^{1,0}(\mathfrak{A},\bar\mu)$ such that $\int_a Pv = \int_{\pi a} v$ whenever $v \in M^{1,0}(\mathfrak{B},\bar\nu)$ and $a \in \mathfrak{A}^f$.

(ii) If $u \in M^0(\mathfrak{A},\bar\mu)$, $v \in M^{1,0}(\mathfrak{B},\bar\nu)$ and $v \times Tu \in M^{1,0}(\mathfrak{B},\bar\nu)$, then $P(v \times Tu) = u \times Pv$.

(iii) If $q \in [1,\infty[$ and $v \in L^q(\mathfrak{B},\bar\nu)$, then $Pv \in L^q(\mathfrak{A},\bar\mu)$ and $\|Pv\|_q \le \|v\|_q$; if $v \in L^\infty(\mathfrak{B}) \cap M^0(\mathfrak{B},\bar\nu)$, then $Pv \in L^\infty(\mathfrak{A})$ and $\|Pv\|_\infty \le \|v\|_\infty$.

(iv) $PTu = u$ for every $u \in M^{1,0}(\mathfrak{A},\bar\mu)$; in particular, $P[L^p(\mathfrak{B},\bar\nu)] = L^p(\mathfrak{A},\bar\mu)$ for every $p \in [1,\infty[$.

(c) If $(\mathfrak{C},\lambda)$ is another measure algebra and $\theta:\mathfrak{B}^f\to\mathfrak{C}^f$ another measure-preserving ring homomorphism, then $T_{\theta\pi}=T_\theta T_\pi:M^0(\mathfrak{A},\bar\mu)\to M^0(\mathfrak{C},\bar\lambda)$ and $P_{\theta\pi}=P_\pi P_\theta:M^{1,0}(\mathfrak{C},\bar\lambda)\to M^{1,0}(\mathfrak{A},\bar\mu)$.

(d) Now suppose that $\pi[\mathfrak{A}^f]=\mathfrak{B}^f$, so that π is a measure-preserving isomorphism between the rings $\mathfrak{A}^f$ and $\mathfrak{B}^f$.

(i) T is a Riesz space isomorphism between $M^0(\mathfrak{A},\bar\mu)$ and $M^0(\mathfrak{B},\bar\nu)$, and its inverse is $T_{\pi^{-1}}$.

(ii) P is a Riesz space isomorphism between $M^{1,0}(\mathfrak{B},\bar\nu)$ and $M^{1,0}(\mathfrak{A},\bar\mu)$, and its inverse is $P_{\pi^{-1}}$.

(iii) The restriction of T to $M^{1,0}(\mathfrak{A},\bar\mu)$ is $P^{-1}=P_{\pi^{-1}}$; the restriction of $T^{-1}=T_{\pi^{-1}}$ to $M^{1,0}(\mathfrak{B},\bar\nu)$ is P.

(iv) For any $p\in[1,\infty[$, $T\restriction L^p(\mathfrak{A},\bar\mu)=P_{\pi^{-1}}\restriction L^p(\mathfrak{A},\bar\mu)$ and $P\restriction L^p(\mathfrak{B},\bar\nu)=T_{\pi^{-1}}\restriction L^p(\mathfrak{B},\bar\nu)$ are the two halves of a Banach lattice isomorphism between $L^p(\mathfrak{A},\bar\mu)$ and $L^p(\mathfrak{B},\bar\nu)$.

proof (a)(i) By 361J, π induces a multiplicative Riesz homomorphism $T_0:S(\mathfrak{A}^f)\to S(\mathfrak{B}^f)$ which is order-continuous because π is (361Ad, 361Je). If $u\in S(\mathfrak{A}^f)$ and $\alpha>0$, then $[\![T_0u>\alpha]\!]=\pi[\![u>\alpha]\!]$. **P** Express u as $\sum_{i=0}^n\alpha_i\chi a_i$ where $a_0,\dots,a_n$ are disjoint in $\mathfrak{A}^f$; then $T_0u=\sum_{i=0}^n\alpha_i\chi(\pi a_i)$, so

$$[\![T_0u>\alpha]\!]=\sup\{\pi a_i:i\le n,\,\alpha_i>\alpha\}=\pi(\sup\{a_i:i\le n,\,\alpha_i>\alpha\})=\pi[\![u>\alpha]\!].\ \mathbf{Q}$$

Now if $u_0\ge0$ in $M^0_{\bar\mu}$, $\sup\{T_0u:u\in S(\mathfrak{A}^f),\,0\le u\le u_0\}$ is defined in $M^0_{\bar\nu}$. **P** Set $A=\{u:u\in S(\mathfrak{A}^f),\,0\le u\le u_0\}$. Because $u_0=\sup A$ (366Gb),

$$\sup_{u\in A}[\![Tu>\alpha]\!]=\sup_{u\in A}\pi[\![u>\alpha]\!]=\pi(\sup_{u\in A}[\![u>\alpha]\!])=\pi[\![u_0>\alpha]\!]$$

is defined and belongs to $\mathfrak{B}^f$ for any $\alpha>0$. Also

$$\inf_{n\ge1}\sup_{u\in A}[\![Tu>n]\!]=\pi(\inf_{n\ge1}[\![u_0>n]\!])=0.$$

By 364L(a-ii), $v_0=\sup T_0[A]$ is defined in $L^0(\mathfrak{B})$, and $[\![v_0>\alpha]\!]=\pi[\![u_0>\alpha]\!]\in\mathfrak{B}^f$ for every $\alpha>0$, so $v_0\in M^0_{\bar\nu}$, as required. **Q**

Consequently T_0 has a unique extension to an order-continuous Riesz homomorphism $T:M^0_{\bar\mu}\to M^0_{\bar\nu}$ (355F).

(ii) If $u_0\in M^0_{\bar\mu}$ and $\alpha>0$, then

$$[\![Tu_0>\alpha]\!]=[\![Tu_0^+>\alpha]\!]$$

(because T is a Riesz homomorphism)

$$=\sup_{u\in S(\mathfrak{A}^f),0\le u\le u_0^+}[\![Tu>\alpha]\!]$$

(because T is order-continuous and $S(\mathfrak{A}^f)$ is order-dense in $M^0_{\bar\mu}$)

$$=\pi[\![u_0>\alpha]\!]$$

by the argument used in (i).

(iii) I have already remarked, at the beginning of the proof of (i), that $T(u\times u')=Tu\times Tu'$ for u, $u'\in S(\mathfrak{A}^f)$. Because both T and $\times$ are order-continuous and $S(\mathfrak{A}^f)$ is order-dense in $M^0_{\bar\mu}$,

$$\begin{aligned}T(u_0\times u_1)&=\sup\{T(u\times u'):u,\,u'\in S(\mathfrak{A}^f),\,0\le u\le u_0,\,0\le u'\le u_1\}\\&=\sup_{u,u'}Tu\times Tu'=Tu_0\times Tu_1\end{aligned}$$

whenever u_0, $u_1\ge0$ in $M^0_{\bar\mu}$. Because T is linear and $\times$ is bilinear, it follows that T is multiplicative on $M^0_{\bar\mu}$.

To see that it is injective, observe that if $u\ne0$ in $M^0_{\bar\mu}$ then there is some $\alpha>0$ such that $a=[\![|u|>\alpha]\!]\ne0$, so that $0<\alpha\chi\pi a\le T|u|=|Tu|$ and $Tu\ne0$.

(iv)(α) For any $\alpha>0$,

$$[\![|Tu|^p>\alpha]\!]=[\![T|u|>\alpha^{1/p}]\!]=\pi[\![|u|>\alpha^{1/p}]\!]=\pi[\![|u|^p>\alpha]\!].$$

So

$$\||Tu|^p\|_1=\int_0^\infty\bar\nu[\![|Tu|^p>\alpha]\!]\,d\alpha=\int_0^\infty\bar\mu[\![|u|^p>\alpha]\!]\,d\alpha=\||u|^p\|_1.$$

If $p<\infty$ then, taking pth roots, $\|Tu\|_p=\|u\|_p$.

(β) As for the case $p=\infty$, if $u\in L^\infty(\mathfrak{A})$ and $\gamma=\|u\|_\infty>0$ then $[\![|u|>\gamma]\!]=0$, so $[\![|Tu|>\gamma]\!]=\pi[\![|u|>\gamma]\!]=0$. This shows that $\|Tu\|_\infty\le\gamma$. On the other hand, if $0<\alpha<\gamma$ then $a=[\![|u|>\alpha]\!]\ne0$, and $\alpha\chi a\le|u|$ so $\alpha\chi(\pi a)\le|Tu|$; as $\pi a\ne0$ (because $\bar\nu(\pi a)=\bar\mu a>0$), $\|Tu\|_\infty>\alpha$. This shows that $\|Tu\|_\infty=\|u\|_\infty$, at least when $u\ne0$; but the case $u=0$ is trivial.

(γ) If $u\in L^1_{\bar\mu}$, then

$$\int Tu=\|(Tu)^+\|_1-\|(Tu)^-\|_1=\|Tu^+\|_1-\|Tu^-\|_1=\|u^+\|_1-\|u^-\|_1=\int u.$$

(v) If $u\in M^{1,0}_{\bar\mu}$ and $\epsilon>0$, then $T(|u|\wedge\epsilon\chi1_{\mathfrak{A}})=|Tu|\wedge\epsilon\chi1_{\mathfrak{B}}$. **P** Set $a=[\![|u|>\epsilon]\!]\in\mathfrak{A}^f$. Then $|u|\wedge\epsilon\chi1_{\mathfrak{A}}=\epsilon\chi a+|u|-|u|\times\chi a$ and $[\![|Tu|>\epsilon]\!]=\pi a$. So

$$\begin{aligned}T(|u|\wedge\epsilon\chi1_{\mathfrak{A}})&=T(\epsilon\chi a)+T|u|-T(|u|\times\chi a)\\&=\epsilon\chi(\pi a)+|Tu|-|Tu|\times\chi(\pi a)=|Tu|\wedge\epsilon\chi1_{\mathfrak{B}}.\ \mathbf{Q}\end{aligned}$$

Consequently

$$T(|u|-\epsilon\chi1_{\mathfrak{A}})^+=T(|u|-|u|\wedge\epsilon\chi1_{\mathfrak{A}})=(|Tu|-\epsilon\chi1_{\mathfrak{B}})^+.$$

But this means that $(|u|-\epsilon\chi1_{\mathfrak{A}})^+\in L^1_{\bar\mu}$ iff $(|Tu|-\epsilon\chi1_{\mathfrak{B}})^+\in L^1_{\bar\nu}$. Since this is true for every $\epsilon>0$, 366Gc tells us that $u\in M^{1,0}_{\bar\mu}$ iff $Tu\in M^{1,0}_{\bar\nu}$.

(b)(i)(α) By 365Oa, we have an order-continuous positive linear operator $P_0:L^1_{\bar\nu}\to L^1_{\bar\mu}$ such that $\int_a P_0v=\int_{\pi a}v$ for every $v\in L^1_{\bar\nu}$ and $a\in\mathfrak{A}^f$.

(β) We now find that if $v_0\ge0$ in $M^{1,0}_{\bar\nu}$ and $B=\{v:v\in L^1_{\bar\nu},\ 0\le v\le v_0\}$, then $P_0[B]$ has a supremum in $L^0(\mathfrak{A})$ which belongs to $M^{1,0}_{\bar\mu}$. **P** Because B is upwards-directed and P_0 is order-preserving, $P_0[B]$ is upwards-directed. If $\alpha>0$ and $v\in B$ and $a=[\![P_0v>\alpha]\!]$, then

$$v\le(v_0-\tfrac{\alpha}{2}\chi1_{\mathfrak{B}})^++\frac{\alpha}{2}\chi1_{\mathfrak{B}},$$

so

$$\begin{aligned}\alpha\bar\mu a\le\int_a P_0v=\int_{\pi a}v&\le\int(v_0-\frac{\alpha}{2}\chi1_{\mathfrak{B}})^++\frac{\alpha}{2}\bar\nu(\pi a)\\&=\int(v_0-\frac{\alpha}{2}\chi1_{\mathfrak{B}})^++\frac{\alpha}{2}\bar\mu a\end{aligned}$$

and

$$\bar\mu[\![P_0v>\alpha]\!]\le\frac{2}{\alpha}\int(v_0-\tfrac{\alpha}{2}\chi1_{\mathfrak{B}})^+.$$

Thus $\{[\![P_0v>\alpha]\!]:v\in B\}$ is an upwards-directed set in $\mathfrak{A}^f$ with measures bounded above in $\mathbb{R}$, and

$$c_\alpha=\sup_{v\in B}[\![P_0v>\alpha]\!]$$

is defined in $\mathfrak{A}^f$. Also

$$\inf_{n\ge1}\bar\mu c_n\le\inf_{n\ge1}\frac{2}{n}\int(v_0-\tfrac{n}{2}\chi1_{\mathfrak{B}})^+=0.$$

So $\inf_{n\in\mathbb{N}}c_n=0$ and $P_0[B]$ has a supremum $u_0\in L^0(\mathfrak{A})$ (364L(a-ii)). As $[\![u_0>\alpha]\!]=c_\alpha\in\mathfrak{A}^f$ for every $\alpha>0$, $u_0\in M^0_{\bar\mu}$. If $c\in\mathfrak{A}^f$, then

$$\int_c u_0=\sup_{v\in B}\int_c P_0v=\sup_{v\in B}\int_{\pi c}v\le\int_{\pi c}v_0<\infty,$$

so $u_0\in M^{1,0}_{\bar\mu}$. **Q**

(γ) Now 355F tells us that P_0 has a unique extension to an order-continuous positive linear operator $P:M^{1,0}_{\bar\nu}\to M^{1,0}_{\bar\mu}$. If $v_0\ge0$ in $M^{1,0}_{\bar\nu}$ and $a\in\mathfrak{A}^f$, then, as remarked above,

$$\int_a Pv_0 = \sup\{\int_a P_0 v : v \in L^1_{\bar\nu},\, 0 \le v \le v_0\}$$
$$= \sup\{\int_{\pi a} v : v \in L^1_{\bar\nu},\, 0 \le v \le v_0\} = \int_{\pi a} v_0;$$

because P is linear, $\int_a Pv = \int_{\pi a} v$ for every $v \in M^{1,0}_{\bar\nu}$, $a \in \mathfrak{A}^f$.

(δ) By 366Gd, P is uniquely defined by the formula

$$\int_a Pv = \int_{\pi a} v \text{ whenever } v \in M^{1,0}_{\bar\nu} \text{ and } a \in \mathfrak{A}^f.$$

(ii) Because $M^0_{\bar\mu}$ is closed under multiplication, $u \times Pv$ certainly belongs to $M^0_{\bar\mu}$.

(α) Suppose that u, $v \ge 0$. Fix $c \in \mathfrak{A}^f$ for the moment. Suppose that $u' \in S(\mathfrak{A}^f)$. Then we can express u' as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_i \in \mathfrak{A}^f$ for every $i \le n$. Accordingly

$$\int_c u' \times Pv = \sum_{i=0}^n \alpha_i \int_{c\cap a_i} Pv = \sum_{i=0}^n \alpha_i \int v \times \chi(\pi a_i) \times \chi(\pi c) = \int_{\pi c} v \times Tu'.$$

Next, we can find a non-decreasing sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $S(\mathfrak{A}^f)^+$ with supremum u, and

$$\sup_{n\in\mathbb{N}} \int_c u_n \times Pv = \sup_{n\in\mathbb{N}} \int_{\pi c} v \times Tu_n = \int_{\pi c} \sup_{n\in\mathbb{N}} v \times Tu_n$$
$$= \int_{\pi c} v \times \sup_{n\in\mathbb{N}} Tu_n = \int_{\pi c} v \times Tu,$$

using the order-continuity of T, $\int$ and $\times$. But this means that $u \times Pv = \sup_{n\in\mathbb{N}} u_n \times Pv$ is integrable over c and that $\int_c u \times Pv = \int_{\pi c} v \times Tu$. As c is arbitrary, $u \times Pv = P(v \times Tu) \in M^{1,0}_{\bar\mu}$.

(β) For general u, v,

$$v^+ \times Tu^+ + v^+ \times Tu^- + v^- \times Tu^+ + v^- \times Tu^- = |v| \times T|u| = |v \times Tu| \in M^{1,0}_{\bar\nu}$$

(because T is a Riesz homomorphism), so we may apply (α) to each of the four products; combining them, we get $P(v \times Tu) = u \times Pv$, as required.

(iii) Because P is a positive operator, we surely have $|Pv| \le P|v|$, so it will be enough to show that $\|Pv\|_q \le \|v\|_q$ for $v \ge 0$ in $L^q_{\bar\nu}$.

(α) I take the case $q = 1$ first. In this case, for any $a \in \mathfrak{A}^f$, we have $\int_a Pv = \int_{\pi a} v \le \|v\|_1$. In particular, setting $a_n = [\![Pv > 2^{-n}]\!]$, $\int_{a_n} Pv \le \|v\|_1$. But $Pv = \sup_{n\in\mathbb{N}} Pv \times \chi a_n$, so

$$\|Pv\|_1 = \sup_{n\in\mathbb{N}} \int_{a_n} Pv \le \|v\|_1.$$

(β) Next, suppose that $q = \infty$, so that $v \in L^\infty(\mathfrak{B})^+$; say $\|v\|_\infty = \gamma$. **?** If $\gamma > 0$ and $a = [\![Pv > \gamma]\!] \ne 0$, then

$$\gamma\bar\mu a < \int_a Pv = \int_{\pi a} v \le \gamma\bar\nu(\pi a) = \gamma\bar\mu a. \text{ **X**}$$

So $[\![Pv > \gamma]\!] = 0$ and $Pv \in L^\infty(\mathfrak{A})$, with $\|Pv\|_\infty \le \|v\|_\infty$, at least when $\|v\|_\infty > 0$; but the case $\|v\|_\infty = 0$ is trivial.

(γ) I come at last to the 'general' case $q \in]1,\infty[$, $v \in L^q_{\bar\nu}$. In this case set $p = q/(q-1)$. If $u \in L^p_{\bar\mu}$ then $Tu \in L^p_{\bar\nu}$ so $Tu \times v \in L^1_{\bar\nu}$ and

$$|\int u \times Pv| \le \|u \times Pv\|_1 = \|P(Tu \times v)\|_1$$

(by (ii))

$$\le \|Tu \times v\|_1$$

(by (α) just above)

$$= \int |Tu| \times |v| \le \|Tu\|_p \|v\|_q = \|u\|_p \|v\|_q$$

by (a-iii) of this theorem. But this means that $u \mapsto \int u \times Pv$ is a bounded linear functional on $L^p_{\bar\mu}$, and is therefore represented by some $w \in L^q_{\bar\mu}$ with $\|w\|_q \le \|v\|_q$. If $a \in \mathfrak{A}^f$ then $\chi a \in L^p_{\bar\mu}$, so $\int_a w = \int_a Pv$; accordingly Pv is actually equal to w (by 366Gd) and $\|Pv\|_q = \|w\|_q \le \|v\|_q$, as claimed.

(iv) If $u \in M^{1,0}_{\bar\mu}$ and $a \in \mathfrak{A}^f$, we must have

$$\int_a PTu = \int_{\pi a} Tu = \int T(\chi a) \times Tu = \int T(\chi a \times u) = \int \chi a \times u = \int_a u,$$

using (a-iv) to see that $\int \chi a \times u$ is defined and equal to $\int T(\chi a \times u)$. As a is arbitrary, $u \in M^{1,0}_{\bar\mu}$ and $PTu = u$.

(c) As usual, in view of the uniqueness of $T_{\theta\pi}$ and $P_{\theta\pi}$, all we have to check is that

$$T_\theta T(\chi a) = T_\theta \chi(\pi a) = \chi(\theta\pi a) = T_{\theta\pi}(\chi a),$$

$$\int_a PP_\theta w = \int_{\pi a} P_\theta w = \int_{\theta\pi a} w = \int_a P_{\theta\pi} w$$

whenever $a \in \mathfrak{A}^f$ and $w \in M^{1,0}_{\bar\lambda}$.

(d)(i) By (c), $T_{\pi^{-1}}T = T_{\pi^{-1}\pi}$ must be the identity operator on $M^0_{\bar\mu}$; similarly, $TT_{\pi^{-1}}$ is the identity operator on $M^0_{\bar\nu}$. Because T and $T_{\pi^{-1}}$ are Riesz homomorphisms, they must be the two halves of a Riesz space isomorphism.

(ii) In the same way, P and $P_{\pi^{-1}}$ must be the two halves of an ordered linear space isomorphism between $M^{1,0}_{\bar\mu}$ and $M^{1,0}_{\bar\nu}$, and are therefore both Riesz homomorphisms.

(iii) By (b-iv), $PTu = u$ for every $u \in M^{1,0}_{\bar\mu}$, so $T{\restriction}M^{1,0}_{\bar\mu}$ must be P^{-1}. Similarly $P = P^{-1}_{\pi^{-1}}$ is the restriction of $T^{-1} = T_{\pi^{-1}}$ to $M^{1,0}_{\bar\nu}$.

(iv) Because $T^{-1}[L^p_{\bar\nu}] = L^p_{\bar\mu}$ (by (a-iv)), and T is a bijection between $M^0_{\bar\mu}$ and $M^0_{\bar\nu}$, $T{\restriction}L^p_{\bar\mu}$ must be a Riesz space isomorphism between $L^p_{\bar\mu}$ and $L^p_{\bar\nu}$; (a-iv) also tells us that it is norm-preserving. Now its inverse is $P{\restriction}L^p_{\bar\nu}$, by (iii) here.

366I Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Then, for any $p \in [1, \infty[$, $L^p(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$ can be identified, as Banach lattice, with the closed linear subspace of $L^p(\mathfrak{A}, \bar\mu)$ generated by $\{\chi b : b \in \mathfrak{B},\ \bar\mu b < \infty\}$.

proof The identity map $b \mapsto b : \mathfrak{B} \to \mathfrak{A}$ induces an injective Riesz homomorphism $T : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ (364P) such that $Tu \in L^p_{\mathfrak{A}} = L^p(\mathfrak{A}, \bar\mu)$ and $\|Tu\|_p = \|u\|_p$ whenever $p \in [1, \infty[$ and $u \in L^p_{\mathfrak{B}} = L^p(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$ (366H(a-iv)). Because $S(\mathfrak{B}^f)$, the linear span of $\{\chi b : b \in \mathfrak{B},\ \bar\mu b < \infty\}$, is dense in $L^p_{\mathfrak{B}}$ (366C), the image of $L^p_{\mathfrak{B}}$ in $L^p_{\mathfrak{A}}$ must be the closure of the image of $S(\mathfrak{B}^f)$ in $L^p_{\mathfrak{A}}$, that is, the closed linear span of $\{\chi b : b \in \mathfrak{B}^f\}$ interpreted as a subset of $L^p_{\mathfrak{A}}$.

366J Corollary If $(\mathfrak{A}, \bar\mu)$ is a probability algebra, $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, and $P : L^1(\mathfrak{A}, \bar\mu) \to L^1(\mathfrak{B}, \bar\mu{\restriction}\mathfrak{B})$ is the conditional expectation operator (365Q), then $\|Pu\|_p \le \|u\|_p$ whenever $p \in [1, \infty]$ and $u \in L^p(\mathfrak{A}, \bar\mu)$.

proof Because $(\mathfrak{A}, \bar\mu)$ is totally finite, $M^{1,0}(\mathfrak{A}, \bar\mu) = L^1_{\bar\mu}$, so that the operator P of 366Hb can be identified with the conditional expectation operator of 365Q. Now 366H(b-iii) gives the result.

Remark Of course this is also covered by 244M.

366K Corollary Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ a measure-preserving ring homomorphism. Let $T:L^2(\mathfrak{A},\bar{\mu})\to L^2(\mathfrak{B},\bar{\nu})$ and $P:L^2(\mathfrak{B},\bar{\nu})\to L^2(\mathfrak{A},\bar{\mu})$ be the corresponding operators, as in 366H. Then $TP:L^2(\mathfrak{B},\bar{\nu})\to L^2(\mathfrak{B},\bar{\nu})$ is an orthogonal projection, its range $TP[L^2(\mathfrak{B},\bar{\nu})]$ being isomorphic, as Banach lattice, to $L^2(\mathfrak{A},\bar{\mu})$. The kernel of TP is just

$$\{v:v\in L^2(\mathfrak{B},\bar{\nu}),\ \int_{\pi a}v=0\text{ for every }a\in\mathfrak{A}^f\}.$$

proof Most of this is simply because T is a norm-preserving Riesz homomorphism (so that $T[L^2_{\bar{\mu}}]$ is isomorphic to $L^2_{\bar{\mu}}$), PT is the identity on $L^2_{\bar{\mu}}$ (so that $(TP)^2=TP$) and $\|P\|\le 1$ (so that $\|TP\|\le 1$). These are enough to ensure that TP is a projection of norm at most 1, that is, an orthogonal projection. Also

$$\begin{aligned}TPv=0\iff Pv=0\iff \int_a Pv=0\text{ for every }a\in\mathfrak{A}^f\\ \iff\int_{\pi a}v=0\text{ for every }a\in\mathfrak{A}^f.\end{aligned}$$

366L Corollary Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\pi:\mathfrak{A}^f\to\mathfrak{A}^f$ a measure-preserving ring automorphism. Then there is a corresponding Banach lattice isomorphism T of $L^2=L^2(\mathfrak{A},\bar{\mu})$ defined by writing $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}^f$. Its inverse is defined by the formula

$$\int_a T^{-1}u=\int_{\pi a}u\text{ for every }u\in L^2,\ a\in\mathfrak{A}^f.$$

proof In the language of 366H, $T=T_\pi$ and $T^{-1}=P_\pi$.

***366M Complex L^p spaces (a)** Just as in §§241-244, we have 'complex' versions of all the spaces considered in this chapter. Using the representation theorems for Boolean algebras, we can get effective descriptions of these matching the ones in Chapter 24. Thus for any Boolean algebra $\mathfrak{A}$ with Stone space Z, we can identify $L^\infty_{\mathbb{C}}(\mathfrak{A})$ with the space $C(Z;\mathbb{C})$ of continuous functions from Z to $\mathbb{C}$; inside this, we have a $\|\,\|_\infty$-dense subspace $S_{\mathbb{C}}(\mathfrak{A})$ consisting of complex linear combinations of indicator functions of open-and-closed sets. If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, identified with a quotient $\Sigma/\mathcal{M}$ where Σ is a σ-algebra of subsets of a set Z and $\mathcal{M}$ is a σ-ideal of Σ, then we can write $\mathcal{L}^0_{\mathbb{C}}$ for the set of functions from Z to $\mathbb{C}$ such that their real and imaginary parts are both Σ-measurable, $\mathcal{W}_{\mathbb{C}}$ for the set of those $f\in\mathcal{L}^0_{\mathbb{C}}$ such that $\{z:f(z)\ne 0\}$ belongs to $\mathcal{M}$, and $L^0_{\mathbb{C}}=L^0_{\mathbb{C}}(\mathfrak{A})$ for the linear space quotient $\mathcal{L}^0_{\mathbb{C}}/\mathcal{W}_{\mathbb{C}}$. As in 241J, we find that we have a natural embedding of $L^0=L^0(\mathfrak{A})$ in $L^0_{\mathbb{C}}$ and functions

$$\mathcal{R}\mathrm{e}:L^0_{\mathbb{C}}\to L^0,\quad \mathcal{I}\mathrm{m}:L^0_{\mathbb{C}}\to L^0,\quad |\ |:L^0_{\mathbb{C}}\to L^0,\quad {}^-:L^0_{\mathbb{C}}\to L^0_{\mathbb{C}}$$

such that

$$u=\mathcal{R}\mathrm{e}(u)+i\,\mathcal{I}\mathrm{m}(u),\quad \mathcal{R}\mathrm{e}(u+v)=\mathcal{R}\mathrm{e}(u)+\mathcal{R}\mathrm{e}(v),\quad \mathcal{I}\mathrm{m}(u+v)=\mathcal{I}\mathrm{m}(u)+\mathcal{I}\mathrm{m}(v),$$

$$\mathcal{R}\mathrm{e}(\alpha u)=\mathcal{R}\mathrm{e}(\alpha)\,\mathcal{R}\mathrm{e}(u)-\mathcal{I}\mathrm{m}(\alpha)\,\mathcal{I}\mathrm{m}(u),\quad \mathcal{I}\mathrm{m}(\alpha u)=\mathcal{R}\mathrm{e}(\alpha)\,\mathcal{I}\mathrm{m}(u)+\mathcal{I}\mathrm{m}(\alpha)\,\mathcal{R}\mathrm{e}(u),$$

$$|\alpha u|=|\alpha||u|,\quad |u+v|\le|u|+|v|,\quad |u|=\sup_{|\gamma|=1}\mathcal{R}\mathrm{e}(\gamma u),$$

$$\bar{u}=\mathcal{R}\mathrm{e}(u)-i\,\mathcal{I}\mathrm{m}(u),\quad \overline{u+v}=\bar{u}+\bar{v},\quad \overline{\alpha u}=\bar{\alpha}\bar{u}$$

for all u, $v\in L^0_{\mathbb{C}}$ and $\alpha\in\mathbb{C}$.

I seem to have omitted to mention it in 241J, but of course we also have a multiplication

$$u\times v=(\mathcal{R}\mathrm{e}(u)\times\mathcal{R}\mathrm{e}(v)-\mathcal{I}\mathrm{m}(u)\times\mathcal{I}\mathrm{m}(v))+i(\mathcal{R}\mathrm{e}(u)\times\mathcal{I}\mathrm{m}(v)+\mathcal{I}\mathrm{m}(u)\times\mathcal{R}\mathrm{e}(v)),$$

for which we have the expected formulae

$$u\times v=v\times u,\quad u\times(v\times w)=(u\times v)\times w,\quad u\times(v+w)=(u\times v)+(u\times w),$$

$$(\alpha u)\times v=u\times(\alpha v)=\alpha(u\times v),$$

$$\overline{u\times v}=\bar{u}\times\bar{v},\quad |u\times v|=|u|\times|v|,\quad u\times\bar{u}=|u|^2=(\mathcal{R}\mathrm{e}(u))^2+(\mathcal{I}\mathrm{m}(u))^2$$

for u, $v\in L^0_{\mathbb{C}}$ and $\alpha\in\mathbb{C}$.

(b) If $(\mathfrak{A},\bar\mu)$ is a measure algebra and $1\le p<\infty$, we can think of $L^p_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ as the set of those $u\in L^0_{\mathbb{C}}$ such that $|u|\in L^p(\mathfrak{A},\bar\mu)$, with its norm defined by the formula $\|u\|_p=\||u|\|_p$; this will make $L^p_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ a Banach space (cf. 242Pb, 244Pb[2]), with dual $L^q(\mathfrak{A},\bar\mu)$ where $\frac{1}{p}+\frac{1}{q}=1$ if $p>1$ (244Pb again). (Similarly, if $(\mathfrak{A},\bar\mu)$ is localizable, the dual of $L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ can be identified with $L^\infty_{\mathbb{C}}$, as in 365Lc.)

Writing $S_{\mathbb{C}}(\mathfrak{A}^f)$ for the space of linear combinations of indicator functions of elements of $\mathfrak{A}$ of finite measure, $S_{\mathbb{C}}(\mathfrak{A}^f)$ is dense in $L^p_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ whenever $1\le p<\infty$, as in 366C.

(c) Of course L^1- and L^2-spaces have special additional features, their integrals and inner products. Here we can set

$$\int u=\int\mathcal{R}\mathrm{e}(u)+i\int\mathcal{I}\mathrm{m}(u)$$

for $u\in L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu)$, and $\int:L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu)\to\mathbb{C}$ becomes a $\mathbb{C}$-linear functional. As for L^2, we see at once from the formulae above that

$$|u\times v|=|u|\times|v|\in L^1(\mathfrak{A},\bar\mu),\quad u\times v\in L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu),\quad \int u\times\bar u=\|u\|_2^2$$

for u, $v\in L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$. So if we set

$$(u|v)=\int u\times\bar v$$

for u, $v\in L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$, $L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$ becomes a complex Hilbert space.

(d) In the language of the present chapter we have something else to look at. If $\mathfrak{A}$, $\mathfrak{B}$ are Dedekind σ-complete Boolean algebras and $\pi:\mathfrak{A}\to\mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, then we have a linear operator $T_\pi:L^0_{\mathbb{C}}(\mathfrak{A})\to L^0_{\mathbb{C}}(\mathfrak{B})$ defined by setting $T_\pi u=T^{\mathrm{real}}_\pi(\mathcal{R}\mathrm{e}(u))+iT^{\mathrm{real}}_\pi(\mathcal{I}\mathrm{m}(u))$, where $T^{\mathrm{real}}_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{B})$ is the Riesz homomorphism described in 364P. Of course T_π, like T^{real}_π, will be multiplicative; hence, or otherwise, $T_\pi|u|=|T_\pi u|$ for every $u\in L^0_{\mathbb{C}}(\mathfrak{A})$. Observe that $T_\pi\bar u=\overline{T_\pi u}$ for every $u\in L^0_{\mathbb{C}}(\mathfrak{A})$. Also, as in 364Pe, if $\mathfrak{C}$ is another Dedekind σ-complete Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ and $\phi:\mathfrak{B}\to\mathfrak{C}$ are sequentially order-continuous Boolean homomorphisms, $T_{\phi\pi}=T_\phi T_\pi$. So if $\pi:\mathfrak{A}\to\mathfrak{A}$ is a Boolean automorphism, T_π will be a bijection with inverse $T_{\pi^{-1}}$.

(e) Similarly, if $(\mathfrak{A},\bar\mu)$ is a measure algebra and $\pi:\mathfrak{A}\to\mathfrak{A}$ is a measure-preserving Boolean homomorphism, $\int T_\pi u=\int u$ for every $u\in L^1_{\mathbb{C}}(\mathfrak{A},\bar\mu)$. If u, $v\in L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$, then

$$(T_\pi u|T_\pi v)=\int T_\pi u\times\overline{T_\pi v}=\int T_\pi u\times T_\pi\bar v=\int T_\pi(u\times\bar v)=\int u\times\bar v=(u|v).$$

If π is actually a measure-preserving Boolean automorphism, we shall have

$$(T_\pi u|v)=(T_{\pi^{-1}}T_\pi u|T_{\pi^{-1}}v)=(u|T_\pi^{-1}v)$$

for all u, $v\in L^2_{\mathbb{C}}(\mathfrak{A},\bar\mu)$.

366X Basic exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $p\in\,]1,\infty[$. Show that $\|u\|_p^p=p\int_0^\infty\alpha^{p-1}\bar\mu[\![|u|>\alpha]\!]\,d\alpha$ for every $u\in L^0(\mathfrak{A})$. (Cf. 263Xa.)

>(b) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $p\in[1,\infty]$. Show that the band algebra of $L^p_{\bar\mu}$ is isomorphic to $\mathfrak{A}$. (Cf. 365R.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $p\in\,]1,\infty[$. Show that $L^p_{\bar\mu}$ is separable iff $L^1_{\bar\mu}$ is.

(d) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. (i) Show that $L^\infty(\mathfrak{A})\cap M^0_{\bar\mu}$ and $L^\infty(\mathfrak{A})\cap M^{1,0}_{\bar\mu}$, as defined in 366F, are equal. (ii) Call this intersection $M^{\infty,0}_{\bar\mu}$. Show that it is a norm-closed solid linear subspace of $L^\infty(\mathfrak{A})$, therefore a Banach lattice in its own right.

(e) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra and $(\widehat{\mathfrak{A}},\hat\mu)$ its localization (322Q). Show that the natural embedding of $\mathfrak{A}$ in $\widehat{\mathfrak{A}}$ induces a Banach lattice isomorphism between $L^p_{\bar\mu}$ and $L^p_{\hat\mu}$ for every $p\in[1,\infty[$, so that the band algebra of $L^p_{\bar\mu}$ can be identified with $\widehat{\mathfrak{A}}$.

[2]Formerly 244O.

(f) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra which is not localizable (cf. 211Ye, 216D), and $(\widehat{\mathfrak{A}},\hat{\mu})$ its localization. Let $\pi:\mathfrak{A}\to\widehat{\mathfrak{A}}$ be the identity embedding, so that π is an order-continuous measure-preserving Boolean homomorphism. Show that if we set $v=\chi b$ where $b\in\widehat{\mathfrak{A}}\setminus\mathfrak{A}$, then there is no $u\in L^\infty(\mathfrak{A})$ such that $\int_a u=\int_{\pi a}v$ whenever $\bar{\mu}a<\infty$.

(g) In 366H, show that $[\![Tu\in E]\!]=\pi[\![u\in E]\!]$ (notation: 364G) whenever $u\in M^0_{\bar{\mu}}$ and $E\subseteq\mathbb{R}$ is a Borel set such that $0\notin\overline{E}$.

>(h) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and let G be the group of all measure-preserving ring automorphisms of $\mathfrak{A}^f$. Let H be the group of all Banach lattice automorphisms of $L^2_{\bar{\mu}}$. Show that the map $\pi\mapsto T$ of 366L is an injective group homomorphism from G to H, so that G is represented as a subgroup of H.

(i) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be any family of measure algebras, with simple product $(\mathfrak{A},\bar{\mu})$ (322L). Show that for any $p\in[1,\infty[$, $L^p_{\bar{\mu}}$ can be identified, as normed Riesz space, with the solid linear subspace

$$\{u:\|u\|=\bigl(\sum_{i\in I}\|u(i)\|_p^p\bigr)^{1/p}<\infty\}$$

of $\prod_{i\in I}L^p_{\bar{\mu}_i}$.

(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\bar{\mu}$, $\bar{\nu}$ two functionals rendering $\mathfrak{A}$ a semi-finite measure algebra. Show that for any $p\in[1,\infty[$, $L^p_{\bar{\mu}}$ and $L^p_{\bar{\nu}}$ are isomorphic as normed Riesz spaces. (*Hint*: use 366Xe to reduce to the case in which $\mathfrak{A}$ is Dedekind complete. Take $w\in L^0(\mathfrak{A})$ such that $\int_a w\,d\bar{\mu}=\bar{\nu}a$ for every $a\in\mathfrak{A}$ (365S). Set $Tu=w^{1/p}\times u$ for $u\in L^p_{\bar{\mu}}$.)

(k) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras, and $p\in[1,\infty[$. Show that the following are equiveridical: (i) $L^p_{\bar{\mu}}$ and $L^p_{\bar{\nu}}$ are isomorphic as Banach lattices; (ii) $L^p_{\bar{\mu}}$ and $L^p_{\bar{\nu}}$ are isomorphic as Riesz spaces; (iii) $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic Dedekind completions.

(l) For a Boolean algebra $\mathfrak{A}$, state and prove results corresponding to 363C, 363Ea and 363F-363I for $L^\infty_{\mathbb{C}}(\mathfrak{A})$ as defined in 366Ma.

366Y Further exercises (a) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and suppose that $0<p<1$. Write $L^p=L^p_{\bar{\mu}}=L^p(\mathfrak{A},\bar{\mu})$ for $\{u:u\in L^0(\mathfrak{A}),\,|u|^p\in L^1_{\bar{\mu}}\}$, and for $u\in L^p$ set $\tau(u)=\int|u|^p$. (i) Show that τ is an F-seminorm (2A5B[3]) and defines a Hausdorff linear space topology on L^p. (ii) Show that if $A\subseteq L^p$ is non-empty, downwards-directed and has infimum 0 then $\inf_{u\in A}\tau(u)=0$. (iii) Show that if $A\subseteq L^p$ is non-empty, upwards-directed and bounded in the linear topological space sense then A is bounded above. (iv) Show that $(L^p)^\sim=(L^p)^\times$ is just the set of continuous linear functionals from L^p to $\mathbb{R}$, and is $\{0\}$ iff $\mathfrak{A}$ has no atom of finite measure.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Show that $M^0(\mathfrak{A},\bar{\mu})$ has the countable sup property.

(c) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra and define $M^{\infty,0}_{\bar{\mu}}$ as in 366Xd. Show that $(M^{\infty,0}_{\bar{\mu}})^\times$ can be identified with $L^1_{\bar{\mu}}$.

(d) In 366H, show that if $\tilde{T}:M^0(\mathfrak{A},\bar{\mu})\to M^0(\mathfrak{B},\bar{\nu})$ is any positive linear operator such that $\tilde{T}(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}^f$, then $\tilde{T}$ is order-continuous, so is equal to T_π.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. (i) Show that there is a natural one-to-one correspondence between $M^{1,0}(\mathfrak{A},\bar{\mu})$ and the set of additive functionals $\nu:\mathfrak{A}^f\to\mathbb{R}$ such that $\nu\ll\mu$ in the double sense that for every $\epsilon>0$ there are δ, $M>0$ such that $|\nu a|\le\epsilon$ whenever $\mu a\le\delta$ and $|\nu a|\le\epsilon\mu a$ whenever $\mu a\ge M$. (ii) Use this description of $M^{1,0}$ to prove 366H(b-i).

(f) In 366H, show that the following are equiveridical: (α) $\pi[\mathfrak{A}^f]=\mathfrak{B}^f$; (β) $T=T_\pi$ is surjective; (γ) $P=P_\pi$ is injective; (δ) P is a Riesz homomorphism; (ϵ) there is some $q\in[1,\infty]$ such that $\|Pv\|_q=\|v\|_q$ for every $v\in L^q_{\bar{\nu}}$; (ζ) $TPv=v$ for every $v\in M^{1,0}_{\bar{\nu}}$.

[3]Later editions only.

(g) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and suppose that $\pi:\mathfrak{A}^f\to\mathfrak{B}^f$ is a measure-preserving ring homomorphism, as in 366H; let $T:M^0_{\bar{\mu}}\to M^0_{\bar{\nu}}$ be the associated linear operator. Show that if $0<p<1$ (as in 366Ya) then $L^p_{\bar{\mu}}\subseteq M^0_{\bar{\mu}}$ and $T^{-1}[L^p_{\bar{\nu}}]=L^p_{\bar{\mu}}$.

(h) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra. (i) For each Boolean automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$, let $T_\pi:L^0(\mathfrak{A})\to L^0(\mathfrak{A})$ be the associated Riesz space isomorphism, and let $w_\pi\in(L^1_{\bar{\mu}})^+$ be such that $\int_a w_\pi=\mu(\pi^{-1}a)$ for every $a\in\mathfrak{A}$ (365Ea). Set $Q_\pi u=T_\pi u\times\sqrt{w_\pi}$ for $u\in L^0(\mathfrak{A})$. Show that $\|Q_\pi u\|_2=\|u\|_2$ for every $u\in L^2_{\bar{\mu}}$. (ii) Show that if π, $\phi:\mathfrak{A}\to\mathfrak{A}$ are Boolean automorphisms then $Q_{\pi\phi}=Q_\pi Q_\phi$.

(i) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\pi:\mathfrak{A}^f\to\mathfrak{A}^f$ a measure-preserving Boolean homomorphism, with associated linear operator $T_\pi:M^0_{\bar{\mu}}\to M^0_{\bar{\mu}}$. Show that the following are equiveridical: (i) there is some $p\in[1,\infty[$ such that $\{T^n_\pi\restriction L^p_{\bar{\mu}}:n\in\mathbb{N}\}$ is relatively compact in $\mathsf{B}(L^p_{\bar{\mu}};L^p_{\bar{\mu}})$ for the strong operator topology; (ii) for every $p\in[1,\infty[$, $\{T^n_\pi\restriction L^p_{\bar{\mu}}:n\in\mathbb{N}\}$ is relatively compact in $\mathsf{B}(L^p_{\bar{\mu}};L^p_{\bar{\mu}})$ for the strong operator topology; (iii) $\{\pi^n a:n\in\mathbb{N}\}$ is relatively compact in $\mathfrak{A}^f$, for the strong measure-algebra topology, for every $a\in\mathfrak{A}^f$.

(j) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $L^0_{\mathbb{C}}(\mathfrak{A})$ can be identified with the complexification of $L^0(\mathfrak{A})$ as defined in 354Yl.

(k) Write $\mathcal{B}(\mathbb{C})$ for the Borel σ-algebra of $\mathbb{C}\cong\mathbb{R}^2$ as defined in 111Gd. Show that if $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, we have a unique function $(u,E)\mapsto[\![u\in E]\!]:L^0_{\mathbb{C}}(\mathfrak{A})\times\mathcal{B}(\mathbb{C})\to\mathfrak{A}$ such that (i) for any $u\in L^0_{\mathbb{C}}(\mathfrak{A})$, the function $E\mapsto[\![u\in E]\!]$ is a sequentially order-continuous Boolean homomorphism from $\mathcal{B}(\mathbb{C})$ to $\mathfrak{A}$ (ii) if E_0, $E_1\subseteq\mathbb{R}$ are Borel sets, then $[\![u\in E_0\times E_1]\!]=[\![\mathcal{R}\mathrm{e}(u)\in E_0]\!]\cap[\![\mathcal{I}\mathrm{m}(u)\in E_1]\!]$ for every $u\in L^0_{\mathbb{C}}(\mathfrak{A})$ (iii) if $\phi:\mathcal{B}(\mathbb{C})\to\mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then there is a unique $u\in L^0_{\mathbb{C}}(\mathfrak{A})$ such that $\phi(E)=[\![u\in E]\!]$ for every $E\in\mathcal{B}(\mathbb{C})$.

(l) A function $h:\mathbb{C}\to\mathbb{C}$ is called **Borel measurable** if its real and imaginary parts are $\mathcal{B}(\mathbb{C})$-measurable, where $\mathcal{B}(\mathbb{C})$ is the Borel σ-algebra of $\mathbb{C}$. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. (i) Show that for every Borel measurable $h:\mathbb{C}\to\mathbb{C}$ and $u\in L^0_{\mathbb{C}}(\mathfrak{A})$ we have an element $\bar{h}(u)\in L^0_{\mathbb{C}}(\mathfrak{A})$ such that $[\![\bar{h}(u)\in E]\!]=[\![u\in h^{-1}[E]]\!]$ for every $E\in\mathcal{B}(\mathbb{C})$. (ii) Show that if $\pi:\mathfrak{A}\to\mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism and $T:L^0_{\mathbb{C}}(\mathfrak{A})\to L^0_{\mathbb{C}}(\mathfrak{A})$ the corresponding linear operator (366Mc), then $T\bar{h}=\bar{h}T$ for every Borel measurable $h:\mathbb{C}\to\mathbb{C}$.

(m) Show that a normed space over $\mathbb{C}$ has the Hahn-Banach property of 363R for complex spaces iff it is isomorphic to $L^\infty_{\mathbb{C}}(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$.

366 Notes and comments The L^p spaces, for $1\le p\le\infty$, constitute the most important family of leading examples for the theory of Banach lattices, and it is not to be wondered at that their properties reflect a wide variety of general results. Thus 366Dd and 366E can both be regarded as special cases of theorems about perfect Riesz spaces (356M and 369D). In a different direction, the concept of 'Orlicz space' (369Xd below) generalizes the L^p spaces if they are regarded as normed subspaces of L^0 invariant under measure-preserving automorphisms of the underlying algebra. Yet another generalization looks at the (non-locally-convex) spaces L^p for $0<p<1$ (366Ya).

In 366H and its associated results I try to emphasize the way in which measure-preserving homomorphisms of the underlying algebras induce both 'direct' and 'dual' operators on L^p spaces. We have already seen the phenomenon in 365N-365O. I express this in a slightly different form in 366H, noting that we really do need the homomorphisms to be measure-preserving, for the dual operators as well as the direct operators, so we no longer have the shift in the hypotheses which appears between 365N and 365O. Of course all these refinements in the hypotheses are irrelevant to the principal applications of the results, and they make substantial demands on the reader; but I believe that the demands are actually demands to expand one's imagination, to encompass the different ways in which the spaces depend on the underlying measure algebras.

In the context of 366H, L^∞ is set apart from the other L^p spaces, because $L^\infty(\mathfrak{A})$ is not in general determined by the ideal $\mathfrak{A}^f$, and the hypotheses of 366H do not look outside $\mathfrak{A}^f$. 366H(a-iv) and 366H(b-iii) reach only the space $M^{\infty,0}$ as defined in 366Xd. To deal with L^∞ we need slightly stronger hypotheses. If we are given a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, rather than from $\mathfrak{A}^f$ to $\mathfrak{B}^f$, then of course the direct operator T has a natural version acting on $L^\infty(\mathfrak{A})$ and indeed on $M^{1,\infty}_{\bar\mu}$, as in 363F and 369Xi. If we know that $(\mathfrak{A},\bar\mu)$ is localizable, then $\mathfrak{A}$ can be recovered from $\mathfrak{A}^f$, and the dual operator P acts on $L^\infty(\mathfrak{B})$, as in 369Xi. But in general we can't expect this to work (366Xf).

Of course 366H can be applied to many other spaces; for reasons which will appear in §§371 and 374, the archetypes are not really L^p spaces at all, but the spaces $M^{1,0}$ (366F) and $M^{1,\infty}$.

I include 366L and 366Yh as pointers to one of the important applications of these ideas: the investigation of properties of a measure-preserving homomorphism in terms of its action on L^p spaces. The case $p=2$ is the most useful because the group of unitary operators (that is, the normed space automorphisms) of L^2 has been studied intensively.

Version of 2.5.16/7.9.18

367 Convergence in measure

Continuing through the ideas of Chapter 24, I come to 'convergence in measure'. The basic results of §245 all translate easily into the new language (367L-367M, 367P). The associated concept of (sequential) order-convergence can also be expressed in abstract terms (367A), and I take the trouble to do this in the context of general lattices (367A-367B), since the concept can be applied in many ways (367C-367E, 367K, 367Xa-367Xp). In the particular case of L^0 spaces, which are the first aim of this section, the idea is most naturally expressed by 367F. It enables us to express some of the basic theorems in Volumes 1 and 2 in the language of this chapter (367I-367J).

In 367N and 367O I give two of the most characteristic properties of the topology of convergence in measure on L^0; it is one of the fundamental types of topological Riesz space. Another striking fact is the way it is determined by the Riesz space structure (367T). In 367U I set out a theorem which is the basis of many remarkable applications of the concept; for the sake of a result in §369 I give one such application (367V).

367A Order*-convergence As I have remarked before, the function spaces of measure theory have three interdependent structures: they are linear spaces, they have a variety of interesting topologies, and they are ordered spaces. Ordinary elementary functional analysis studies interactions between topologies and linear structures, in the theory of normed spaces and, more generally, of linear topological spaces. Chapter 35 in this volume looked at interactions between linear and order structures. It is natural to seek to complete the triangle with a theory of topological ordered spaces. The relative obscurity of any such theory is in part due to the difficulty of finding convincing definitions; that is, isolating concepts which lead to elegant and useful general theorems. Among the many rival ideas, however, I believe it is possible to identify one which is particularly important in the context of measure theory.

In its natural home in the theory of L^0 spaces, this notion of 'order*-convergence' has a particularly straightforward expression (367F). But, suitably interpreted, the same idea can be applied in other contexts, some of which will be very useful to us, and I therefore begin with a definition which is applicable in any lattice.

Definition Let P be a lattice, p an element of P and $\langle p_n\rangle_{n\in\mathbb{N}}$ a sequence in P. I will say that $\langle p_n\rangle_{n\in\mathbb{N}}$ **order*-converges** to p if

$$\begin{aligned} p &= \inf\{q : \exists\, n\in\mathbb{N},\, q\ge (p'\vee p_i)\wedge p''\ \forall\, i\ge n\}\\ &= \sup\{q : \exists\, n\in\mathbb{N},\, q\le p'\vee(p_i\wedge p'')\ \forall\, i\ge n\}\end{aligned}$$

whenever $p'\le p\le p''$ in P.

Remark In the formulae above, we always have $p'\vee(p_i\wedge p'')\le(p'\vee p_i)\wedge p''$, because $p'\le p''$. If P is a distributive lattice, both are equal to $\mathrm{med}(p',p_i,p'')$.

367B Lemma Let P be a lattice.

(a) A sequence in P can order*-converge to at most one point.

(b) A constant sequence order*-converges to its constant value.

(c) Any subsequence of an order*-convergent sequence is order*-convergent, with the same limit.

(d) If $\langle p_n\rangle_{n\in\mathbb{N}}$ and $\langle p'_n\rangle_{n\in\mathbb{N}}$ both order*-converge to p, and $p_n \le q_n \le p'_n$ for every n, then $\langle q_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(e) If $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in P, then it order*-converges to $p \in P$ iff

$$\begin{aligned} p &= \inf\{q : \exists\, n \in \mathbb{N},\ q \ge p_i\ \forall\ i \ge n\} \\ &= \sup\{q : \exists\, n \in \mathbb{N},\ q \le p_i\ \forall\ i \ge n\}. \end{aligned}$$

(f) If P is a Dedekind σ-complete lattice (314Ab) and $\langle p_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence in P, then it order*-converges to $p \in P$ iff

$$p = \sup_{n\in\mathbb{N}} \inf_{i\ge n} p_i = \inf_{n\in\mathbb{N}} \sup_{i\ge n} p_i.$$

proof (a) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to both p and $\tilde{p}$. Set $p' = p \wedge \tilde{p}$, $p'' = p \vee \tilde{p}$; then

$$p = \inf\{q : \exists\, n \in \mathbb{N},\ q \ge (p' \vee p_i) \wedge p''\ \forall\ i \ge n\} = \tilde{p}.$$

(b) is trivial.

(c) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to p, and that $\langle p'_n\rangle_{n\in\mathbb{N}}$ is a subsequence of $\langle p_n\rangle_{n\in\mathbb{N}}$. Take p', p'' such that $p' \le p \le p''$, and set

$$B = \{q : \exists\, n \in \mathbb{N},\ q \le p' \vee (p_i \wedge p'')\ \forall\ i \ge n\},$$

$$B' = \{q : \exists\, n \in \mathbb{N},\ q \le p' \vee (p'_i \wedge p'')\ \forall\ i \ge n\},$$

$$C = \{q : \exists\, n \in \mathbb{N},\ q \ge (p' \vee p_i) \wedge p''\ \forall\ i \ge n\},$$

$$C' = \{q : \exists\, n \in \mathbb{N},\ q \ge (p' \vee p'_i) \wedge p''\ \forall\ i \ge n\}.$$

If $q \in B'$ and $q' \in C$, then for all sufficiently large i

$$q \le p' \vee (p'_i \wedge p'') \le (p' \vee p'_i) \wedge p'' \le q'.$$

As $p = \inf C$, we must have $q \le p$; thus p is an upper bound for B'. On the other hand, $\{p'_i : i \ge n\} \subseteq \{p_i : i \ge n\}$ for every n, so $B \subseteq B'$ and p must be the least upper bound of B', since $p = \sup B$.

Similarly, $p = \inf C'$. As p' and p'' are arbitrary, $\langle p'_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(d) Take p', p'' such that $p' \le p \le p''$, and set

$$B = \{q : \exists\, n \in \mathbb{N},\ q \le p' \vee (p_i \wedge p'')\ \forall\ i \ge n\},$$

$$B' = \{q : \exists\, n \in \mathbb{N},\ q \le p' \vee (q_i \wedge p'')\ \forall\ i \ge n\},$$

$$C = \{q : \exists\, n \in \mathbb{N},\ q \ge (p' \vee p'_i) \wedge p''\ \forall\ i \ge n\},$$

$$C' = \{q : \exists\, n \in \mathbb{N},\ q \ge (p' \vee q_i) \wedge p''\ \forall\ i \ge n\}.$$

If $q \in B'$ and $q' \in C$, then for all sufficiently large i

$$q \le p' \vee (q_i \wedge p'') \le (p' \vee p'_i) \wedge p'' \le q'.$$

As $p = \inf C$, we must have $q \le p$; thus p is an upper bound for B'. On the other hand, $p' \vee (p_i \wedge p'') \le p' \vee (q_i \wedge p'')$ for every i, so $B \subseteq B'$ and $p = \sup B'$. Similarly, $p = \inf C'$. As p' and p'' are arbitrary, $\langle q_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(e) Set

$$B = \{q : \exists\, n \in \mathbb{N},\ q \le p_i\ \forall\ i \ge n\},$$

$$C = \{q : \exists\, n \in \mathbb{N},\ q \ge p_i\ \forall\ i \ge n\}.$$

(i) Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ order*-converges to p. Let p', p'' be such that $p'\le p_n\le p''$ for every $n\in\mathbb{N}$ and $p'\le p\le p''$. Then

$$B=\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(p_i\wedge p'')\;\forall\,i\ge n\},$$

so $\sup B=p$. Similarly, $\inf C=p$, so the condition is satisfied.

(ii) Suppose that $\sup B=\inf C=p$. Take any p', p'' such that $p'\le p\le p''$ and set

$$B'=\{q:\exists\,n\in\mathbb{N},\,q\le p'\vee(p_i\wedge p'')\;\forall\,i\ge n\},$$

$$C'=\{q:\exists\,n\in\mathbb{N},\,q\ge(p'\vee p_i)\wedge p''\;\forall\,i\ge n\}.$$

If $q\in B'$ and $q'\in C$, then for all large enough i

$$q\le p'\vee(p_i\wedge p'')\le p'\vee q'=q'$$

because $p\le q'$. As $\inf C=p$, p is an upper bound for B'. On the other hand, if $q\in B$, then $q\le p$, so $q\le p'\vee(p_i\wedge p'')$ whenever $q\le p_i$, which is so for all sufficiently large i, and $q\in B'$. Thus $B'\supseteq B$ and p must be the supremum of B'. Similarly, $p=\inf C'$; as p' and p'' are arbitrary, $\langle p_n\rangle_{n\in\mathbb{N}}$ order*-converges to p.

(f) This follows at once from (e). Setting

$$B=\{q:\exists\,n\in\mathbb{N},\,q\le p_i\;\forall\,i\ge n\},\quad B'=\{\inf_{i\ge n}p_i:i\in\mathbb{N}\},$$

then $B'\subseteq B$ and for every $q\in B$ there is a $q'\in B'$ such that $q\le q'$; so $\sup B=\sup B'$ if either is defined. Similarly,

$$\inf\{q:\exists\,n\in\mathbb{N},\,q\ge p_i\;\forall\,i\ge n\}=\inf_{n\in\mathbb{N}}\sup_{i\ge n}p_i$$

if either is defined.

Remark Part (b) above tells us that we may speak of 'the' order*-limit of an order*-convergent sequence.

367C Proposition Let U be a Riesz space.

(a) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ are two sequences in U order*-converging to u, v respectively.

(i) $\langle u_n+w\rangle_{n\in\mathbb{N}}$ order*-converges to $u+w$ for every $w\in U$, and αu_n order*-converges to αu for every $\alpha\in\mathbb{R}$.

(ii) $\langle u_n\vee v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u\vee v$ and $\langle u_n\wedge v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u\wedge v$.

(iii) If $\langle w_n\rangle_{n\in\mathbb{N}}$ is any sequence in U, then it order*-converges to $w\in U$ iff $\langle|w_n-w|\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

(iv) $\langle u_n+v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u+v$.

(v) If $\langle w_n\rangle_{n\in\mathbb{N}}$ and $\langle z_n\rangle_{n\in\mathbb{N}}$ are sequences in U, $\langle w_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 and $|z_n|\le|w_n|$ for every n, then $\langle z_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

(b) Now suppose that U is Archimedean.

(i) If $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ converging to $\alpha\in\mathbb{R}$, and $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U order*-converging to $u\in U$, then $\langle\alpha_n u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu.

(ii) A sequence $\langle w_n\rangle_{n\in\mathbb{N}}$ in U^+ is *not* order*-convergent to 0 iff there is a $\tilde w>0$ such that $\tilde w=\sup_{i\ge n}\tilde w\wedge w_i$ for every $n\in\mathbb{N}$.

(iii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U^+ such that $\{\sum_{i=0}^n u_i:n\in\mathbb{N}\}$ is bounded above, then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

proof (a)(i)($\boldsymbol{\alpha}$) $\langle u_n+w\rangle_{n\in\mathbb{N}}$ order*-converges to $u+w$ because the ordering of U is translation-invariant; the map $w'\mapsto w'+w$ is an order-isomorphism.

($\boldsymbol{\beta}$) If $\alpha>0$, then the map $w'\mapsto\alpha w'$ is an order-isomorphism, so $\langle\alpha u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu.

($\boldsymbol{\gamma}$) If $\alpha=0$ then $\langle\alpha u_n\rangle_{n\in\mathbb{N}}$ order*-converges to $\alpha u=0$ by 367Bb.

($\boldsymbol{\delta}$) If $w'\le -u\le w''$ then $-w''\le u\le -w'$ so

$$\begin{aligned}u&=\inf\{w:\exists\,n\in\mathbb{N},\,w\ge((-w'')\vee u_i)\wedge(-w')\;\forall\,i\ge n\}\\&=\sup\{w:\exists\,n\in\mathbb{N},\,w\le(-w'')\vee(u_i\wedge(-w'))\;\forall\,i\ge n\}.\end{aligned}$$

Turning these formulae upside down,

$$\begin{aligned}-u &= \sup\{w : \exists\, n \in \mathbb{N},\ w \le (w'' \wedge (-u_i)) \vee w' \ \forall\, i \ge n\}\\ &= \inf\{w : \exists\, n \in \mathbb{N},\ w \ge w'' \wedge ((-u_i) \vee w') \ \forall\, i \ge n\}.\end{aligned}$$

As w' and w'' are arbitrary, $\langle -u_n\rangle_{n\in\mathbb{N}}$ order*-converges to $-u$.

(ϵ) Putting (β) and (δ) together, $\langle \alpha u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu for every $\alpha < 0$.

(ii) Suppose that $w' \le u \vee v \le w''$. Set

$$B = \{w : \exists\, n \in \mathbb{N},\ w \le w' \vee ((u_i \vee v_i) \wedge w'') \ \forall\, i \ge n\},$$

$$C = \{w : \exists\, n \in \mathbb{N},\ w \ge (w' \vee (u_i \vee v_i)) \wedge w'' \ \forall\, i \ge n\},$$

$$B_1 = \{w : \exists\, n \in \mathbb{N},\ w \le (w' \wedge u) \vee (u_i \wedge w'') \ \forall\, i \ge n\},$$

$$B_2 = \{w : \exists\, n \in \mathbb{N},\ w \le (w' \wedge v) \vee (v_i \wedge w'') \ \forall\, i \ge n\},$$

$$C_1 = \{w : \exists\, n \in \mathbb{N},\ w \ge ((w' \wedge u) \vee u_i) \wedge w'' \ \forall\, i \ge n\},$$

$$C_2 = \{w : \exists\, n \in \mathbb{N},\ w \ge ((w' \wedge v) \vee v_i) \wedge w'' \ \forall\, i \ge n\},$$

If $w_1 \in B_1$ and $w_2 \in B_2$ then $w_1 \vee w_2 \in B$. **P** There is an $n \in \mathbb{N}$ such that $w_1 \le (w' \wedge u) \vee (u_i \wedge w'')$ for every $i \ge n$, while $w_2 \le (w' \wedge v) \vee (v_i \wedge w'')$ for every $i \ge n$. So

$$\begin{aligned}w_1 \vee w_2 &\le (w' \wedge u) \vee (w' \wedge v) \vee (u_i \wedge w'') \vee (v_i \wedge w'')\\ &= (w' \wedge (u \vee v)) \vee ((u_i \vee v_i) \wedge w'')\end{aligned}$$

(352Ec)

$$= w' \vee ((u_i \vee v_i) \wedge w'')$$

for every $i \ge n$, and $w_1 \vee w_2 \in B$. **Q**

Similarly, if $w_1 \in C_1$ and $w_2 \in C_2$ then $w_1 \vee w_2 \in C$. **P** There is an $n \in \mathbb{N}$ such that $w_1 \ge ((w' \wedge u) \vee u_i) \wedge w''$ and $w_2 \ge ((w' \wedge v) \vee v_i) \wedge w''$ for every $i \ge n$. So

$$\begin{aligned}w_1 \vee w_2 &\ge (((w' \wedge u) \vee u_i) \wedge w'') \vee (((w' \wedge v) \vee v_i) \wedge w'')\\ &= ((w' \wedge u) \vee u_i \vee (w' \wedge v) \vee v_i) \wedge w''\\ &= ((w' \wedge (u \vee v)) \vee (u_i \vee v_i)) \wedge w''\\ &= (w' \vee (u_i \vee v_i)) \wedge w''\end{aligned}$$

for every $i \ge n$, so $w_1 \vee w_2 \in C$. **Q**

At the same time, of course, $w \le \tilde{w}$ whenever $w \in B$ and $\tilde{w} \in C$, since there is some $i \in \mathbb{N}$ such that

$$w \le w' \vee ((u_i \vee v_i) \wedge w'') \le (w' \vee (u_i \vee v_i)) \wedge w'' \le \tilde{w}.$$

Since

$$\sup\{w_1 \vee w_2 : w_1 \in B_1,\ w_2 \in B_2\} = (\sup B_1) \vee (\sup B_2) = u \vee v,$$

$$\inf\{w_1 \vee w_2 : w_1 \in C_1,\ w_2 \in C_2\} = (\inf C_1) \vee (\inf C_2) = u \vee v$$

(using the generalized distributive laws in 352E), we must have $\sup B = \inf C = u \vee v$. As w' and w'' are arbitrary, $\langle u_n \vee v_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to $u \vee v$.

Putting this together with (i), we see that $\langle u_n \wedge v_n\rangle_{n\in\mathbb{N}} = \langle -((-u_n) \vee (-v_n))\rangle_{n\in\mathbb{N}}$ order*-converges to $-((-u) \vee (-v)) = u \wedge v$.

(iii) The hard parts are over. (α) If $\langle w_n\rangle_{n\in\mathbb{N}}$ order*-converges to w, then $\langle w_n - w\rangle_{n\in\mathbb{N}}$, $\langle w - w_n\rangle_{n\in\mathbb{N}}$ and $\langle |w_n - w|\rangle_{n\in\mathbb{N}} = \langle (w_n - w) \vee (w - w_n)\rangle_{n\in\mathbb{N}}$ all order*-converge to 0, putting (i) and (ii) together. (β)

If $\langle|w_n - w|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, then so do $\langle -|w_n - w|\rangle_{n\in\mathbb{N}}$ and $\langle w_n - w\rangle_{n\in\mathbb{N}}$, by (i) and 367Bd; so $\langle w_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by (i) again.

(iv) $\langle|u_n - u|\rangle_{n\in\mathbb{N}}$ and $\langle|v_n - v|\rangle_{n\in\mathbb{N}}$ order*-converge to 0, by (iii), so $\langle 2(|u_n - u| \vee |v_n - v|)\rangle_{n\in\mathbb{N}}$ also order*-converges to 0, by (ii) and (i). But

$$0 \le |(u_n + v_n) - (u + v)| \le |u_n - u| + |v_n - v| \le 2(|u_n - u| \vee |v_n - v|)$$

for every n, so $\langle|(u_n + v_n) - (u + v)|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Bb and 367Bd, and $\langle u_n + v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u + v$.

(v) By (iii), $\langle|w_n|\rangle_{n\in\mathbb{N}}$ order*-converges to 0. So $\inf_{n\in\mathbb{N}}\sup_{m\ge n}|w_m|$ is defined and equal to 0; consequently $\inf_{n\in\mathbb{N}}\sup_{m\ge n}|z_m| = 0$, $\langle|z_n|\rangle_{n\in\mathbb{N}}$ order*-converges to 0 and $\langle z_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by (iii) again.

(b)(i) Set $\beta_n = \sup_{i\ge n}|\alpha_i - \alpha|$ for each n. Then $\langle\beta_n\rangle_{n\in\mathbb{N}} \to 0$, so $\inf_{n\in\mathbb{N}}\beta_n|u| = 0$, because U is Archimedean. Consequently $\langle\beta_n|u|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Be. But we also have $\beta_0|u_n - u|$ order*-converging to 0, by (a-iii) and (a-i), so $\langle\beta_0|u_n - u| + \beta_n|u|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by (a-iv). As $|\alpha_n u_n - \alpha u| \le \beta_0|u_n - u| + \beta_n|u|$ for every n, $\langle\alpha_n u_n\rangle_{n\in\mathbb{N}}$ order*-converges to αu, as required.

(ii)($\boldsymbol{\alpha}$) Suppose that $\langle w_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0. Then there are w', w'' such that $w' \le 0 \le w''$ and either

$$B = \{w : \exists\, n \in \mathbb{N},\, w \le w' \vee (w_i \wedge w'')\ \forall\, i \ge n\}$$

does not have supremum 0, or

$$C = \{w : \exists\, n \in \mathbb{N},\, w \ge (w' \vee w_i) \wedge w''\ \forall\, i \ge n\}$$

does not have infimum 0. Now $0 \in B$, because every $w_i \ge 0$, and every member of B is a lower bound for C; so 0 cannot be the greatest lower bound of C. Let $\tilde{w} > 0$ be a lower bound for C.

Let $n \in \mathbb{N}$, and set

$$C_n = \{w : w \ge (w' \vee w_i) \wedge w''\ \forall\, i \ge n\} = \{w : w \ge w_i \wedge w''\ \forall\, i \ge n\}.$$

(Recall that U is a distributive lattice.) Because U is Archimedean, we know that $\inf(C_n - A_n) = 0$, where $A_n = \{w_i \wedge w'' : i \ge n\}$ (353F). Now $\tilde{w}$ is a lower bound for C_n, so

$$\begin{aligned}\inf_{i\ge n}(\tilde{w} - w_i)^+ &\le \inf\{(w - w_i)^+ : w \in C,\, i \ge n\}\\ &\le \inf\{(w - (w_i \wedge w''))^+ : w \in C,\, i \ge n\}\\ &= \inf\{w - (w_i \wedge w'') : w \in C,\, i \ge n\} = \inf(C_n - A_n) = 0.\end{aligned}$$

As this is true for every $n \in \mathbb{N}$, $\tilde{w}$ has the property declared.

($\boldsymbol{\beta}$) If $\tilde{w} > 0$ is such that $\tilde{w} = \sup_{i\ge n}\tilde{w} \wedge w_i$ for every $n \in \mathbb{N}$, then

$$\{w : \exists\, n \in \mathbb{N},\, w \ge (0 \vee w_i) \wedge \tilde{w}\ \forall\, i \ge n\}$$

cannot have infimum 0, and $\langle w_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0.

(iii) Set $v_n = \sum_{i=0}^n u_i$ for $n \in \mathbb{N}$. Let C be the set of upper bounds of $\{v_n : n \in \mathbb{N}\}$, and write B for $\{w - v_n : w \in C,\, n \in \mathbb{N}\}$. Then $\inf B = 0$ (353F). But if $n \in \mathbb{N}$ and $w \in C$ then $u_i = v_{i+1} - v_i \le w - v_n$ for every $i \ge n$. So

$$\{u : \exists\, n \in \mathbb{N},\, u \ge u_i\ \forall\, i \ge n\}$$

includes B and must have infimum 0. On the other side,

$$\{u : \exists\, n \in \mathbb{N},\, u \le u_i\ \forall\, i \ge n\}$$

contains 0 and must have supremum 0. By 367Be, $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.

367D As an example of the use of this concept in a moderately general setting, I offer the following.

Proposition Let U be a Riesz space with a Riesz norm $\|\ \|$.

(a) If a sequence in U is both order*-convergent and norm-convergent, the two limits are the same.

(b) $\| \|$ is order-continuous iff every order-bounded order*-convergent sequence in U is norm-convergent.

proof (a) Let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a sequence in U which is order*-convergent to v and norm-convergent to w. Then $\langle \mathrm{med}(u_n, v, w)\rangle_{n\in\mathbb{N}} = \langle ((v \wedge w) \vee u_n) \wedge (v \vee w)\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\mathrm{med}(v, v, w) = v$ (367C(a-ii)) and norm-convergent to $\mathrm{med}(w, v, w) = w$ (354Xc, or 354B with 352D). So

$$v = \inf\{u : \exists\, n \in \mathbb{N},\ u \ge \mathrm{med}(u_i, v, w)\ \forall\ i \ge n\}$$
$$= \sup\{u : \exists\, n \in \mathbb{N},\ u \le \mathrm{med}(u_i, v, w)\ \forall\ i \ge n\}.$$

But if $n \in \mathbb{N}$ and $u \ge \mathrm{med}(u_i, v, w)$ for every $i \ge n$, then $u \ge \lim_{i\to\infty} \mathrm{med}(u_i, v, w) = w$, because $\{u' : u' \le u\}$ is norm-closed (354Bc). As u is arbitrary, $w \le v$. Similarly, because $\{u' : u' \ge u\}$ is norm-closed for every u, $w \ge v$. So $w = v$, as claimed.

(b)(i) Suppose that $\| \|$ is order-continuous, and that an order-bounded sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to u. Then $\langle |u_n - u| \rangle_{n\in\mathbb{N}}$ is order-bounded and order*-convergent to 0 (367C(a-iii)), so

$$C = \{v : \exists\, n \in \mathbb{N},\ v \ge |u_i - u|\ \forall\ i \ge n\}$$

has infimum 0 (367Be). Because U is a lattice, C is downwards-directed, so $\inf_{v\in C} \|v\| = 0$. But

$$\inf_{v\in C} \|v\| \ge \inf_{n\in\mathbb{N}} \sup_{i\ge n} \|u_i - u\|,$$

so $\lim_{n\to\infty} \|u_n - u\| = 0$, that is, $\langle u_n \rangle_{n\in\mathbb{N}}$ is norm-convergent to u.

(ii) Suppose that all order-bounded order*-convergent sequences in U are norm-convergent,

(α) Let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a non-increasing sequence in U^+, and set $v_n = u_n - u_{n+1}$ for each n. Then $\langle v_n \rangle_{n\in\mathbb{N}}$ is norm-convergent to 0. **P** By 367C(b-iii), applied to $\langle -v_n \rangle_{n\in\mathbb{N}}$, $\langle v_n \rangle_{n\in\mathbb{N}}$ is order*-convergent to 0, so must be norm-convergent, and (a) here tells us that the norm limit is 0. **Q**

(β) Now suppose that $A \subseteq U^+$ is a non-empty downwards-directed set with infimum 0. Choose $\langle u_n \rangle_{n\in\mathbb{N}}$, $\langle \gamma_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with any $u_0 \in A$. Given $u_n \in A$, set $\gamma_n = \sup_{u\in A\cap[0,u_n]} \|u_n - u\|$ and choose $u_{n+1} \in A \cap [0, u_n]$ such that $\|u_n - u_{n+1}\| \ge \frac{1}{2}\gamma_n$; continue.

By (α), $\lim_{n\to\infty} \|u_n - u_{n+1}\| = 0$, so $\lim_{n\to\infty} \gamma_n = 0$. Now suppose that $v \in U$ is any lower bound of $\{u_n : n \in \mathbb{N}\}$. Then $v \le 0$. **P** Take $u \in A$ and $n \in \mathbb{N}$. Then there is a $u' \in A$ such that $u' \le u \wedge u_n$, because A is downwards-directed. So

$$\|v - u \wedge v\| \le \|v - u' \wedge v\| = \|u' \vee v - u'\|$$

(because $u' \vee v + u' \wedge v = u' + v$, as noted in 352D)

$$\le \|u_n - u'\| \le \gamma_n.$$

As n is arbitrary, $v - u \wedge v = 0$ and $v \le u$. As u is arbitrary, v is a lower bound of A and must be less than or equal to $\inf A = 0$. **Q**

Thus $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, and order*-converges to 0, by 367Be. Accordingly it norm-converges to 0, and $\inf_{u\in A} \|u\| = \inf_{n\in\mathbb{N}} \|u_n\| = 0$. As A is arbitrary, $\| \|$ is order-continuous,

As A is arbitrary, the norm of U is order-continuous.

367E One of the fundamental obstacles to the development of any satisfying general theory of ordered topological spaces is the erratic nature of the relations between subspace topologies of order topologies and order topologies on subspaces. The particular virtue of order*-convergence in the context of function spaces is that it is relatively robust when transferred to the subspaces we are interested in.

Proposition Let U be an Archimedean Riesz space and V a regularly embedded Riesz subspace. (For instance, V might be either solid or order-dense.) If $\langle v_n \rangle_{n\in\mathbb{N}}$ is a sequence in V and $v \in V$, then $\langle v_n \rangle_{n\in\mathbb{N}}$ order*-converges to v when regarded as a sequence in V, iff it order*-converges to v when regarded as a sequence in U.

proof (a) Since, in either V or U, $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to v iff $\langle|v_n-v|\rangle_{n\in\mathbb{N}}$ order*-converges to 0 (367C(a-iii)), it is enough to consider the case $v_n\ge 0$, $v=0$.

(b) If $\langle v_n\rangle_{n\in\mathbb{N}}$ is *not* order*-convergent to 0 in U, then, by 367C(b-ii), there is a $u>0$ in U such that $u=\sup_{i\ge n}u\wedge v_i$ for every $n\in\mathbb{N}$ (the supremum being taken in U, of course). In particular, there is a $k\in\mathbb{N}$ such that $u\wedge v_k>0$. Now consider the set

$$C=\{w:w\in V,\,\exists\,n\in\mathbb{N},\,w\ge v_i\wedge v_k\;\forall\,i\ge n\}.$$

Then for any $w\in C$,

$$u\wedge v_k=\sup_{i\ge n}u\wedge v_i\wedge v_k\le w,$$

using the generalized distributive law in U, so 0 is not the greatest lower bound of C in U. But as the embedding of V in U is order-continuous, 0 is not the greatest lower bound of C in V, and $\langle v_n\rangle_{n\in\mathbb{N}}$ cannot be order*-convergent to 0 in V.

(c) Now suppose that $\langle v_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0 in V. Because V, like U, is Archimedean (351Rc), there is a $w>0$ in V such that $w=\sup_{i\ge n}w\wedge v_i$ for every $n\in\mathbb{N}$, the suprema being taken in V. Again because V is regularly embedded in U, we have the same suprema in U, so, by 367C(b-ii) in the other direction, $\langle v_n\rangle_{n\in\mathbb{N}}$ is not order*-convergent to 0 in U.

367F I now spell out the connexion between the definition above and the concepts introduced in 245C.

Proposition Let X be a set, Σ a σ-algebra of subsets of X, $\mathfrak{A}$ a Boolean algebra and $\pi:\Sigma\to\mathfrak{A}$ a sequentially order-continuous surjective Boolean homomorphism; let $\mathcal{I}$ be its kernel. Write $\mathcal{L}^0$ for the space of Σ-measurable functions from X to $\mathbb{R}$, and let $T=T_\pi:\mathcal{L}^0\to L^0=L^0(\mathfrak{A})$ be the canonical Riesz homomorphism (364C, 364P). Then for any $\langle f_n\rangle_{n\in\mathbb{N}}$ and f in $\mathcal{L}^0$, $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf in L^0 iff $X\setminus\{x:f(x)=\lim_{n\to\infty}f_n(x)\}\in\mathcal{I}$.

proof Set $H=\{x:\lim_{n\to\infty}f_n(x)\text{ exists}=f(x)\}$; of course $H\in\Sigma$. Set $g_n(x)=|f_n(x)-f(x)|$ for $n\in\mathbb{N}$ and $x\in X$.

(a) If $X\setminus H\in\mathcal{I}$, set $h_n(x)=\sup_{i\ge n}g_i(x)$ for $x\in H$ and $h_n(x)=0$ for $x\in X\setminus H$. Then $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0 in $\mathcal{L}^0$, so $\inf_{n\in\mathbb{N}}Th_n=0$ in L^0, because T is sequentially order-continuous (364Pa). But as $X\setminus H\in\mathcal{I}$, $Th_n\ge Tg_i=|Tf_i-Tf|$ whenever $i\ge n$, so $\langle|Tf_n-Tf|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, by 367Be or 367Bf, and $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf, by 367C(a-iii).

(b) Now suppose that $\langle Tf_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tf. Set $g_n'(x)=\min(1,g_n(x))$ for $n\in\mathbb{N}$, $x\in X$; then $\langle Tg_n'\rangle_{n\in\mathbb{N}}=\langle e\wedge|Tf_n-Tf|\rangle_{n\in\mathbb{N}}$ order*-converges to 0, where $e=T(\chi X)$. By 367Bf, $\inf_{n\in\mathbb{N}}\sup_{i\ge n}Tg_i'=0$ in L^0. But T is a sequentially order-continuous Riesz homomorphism, so $T(\inf_{n\in\mathbb{N}}\sup_{i\ge n}g_i')=0$, that is,

$$X\setminus H=\{x:\inf_{n\in\mathbb{N}}\sup_{i\ge n}g_i'>0\}$$

belongs to $\mathcal{I}$.

367G Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra.
(a) Any order*-convergent sequence in $L^0=L^0(\mathfrak{A})$ is order-bounded.
(b) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0, then it is order*-convergent to $u\in L^0$ iff

$$u=\inf_{n\in\mathbb{N}}\sup_{i\ge n}u_i=\sup_{n\in\mathbb{N}}\inf_{i\ge n}u_i.$$

proof (a) We can express $\mathfrak{A}$ as a quotient $\Sigma/\mathcal{I}$ of a σ-algebra of sets, in which case L^0 can be identified with the canonical image of $\mathcal{L}^0=\mathcal{L}^0(\Sigma)$ (364C). If $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order*-convergent sequence in L^0, then it is expressible as $\langle Tf_n\rangle_{n\in\mathbb{N}}$, where $T:\mathcal{L}^0\to L^0$ is the canonical map, and 367F tells us that $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ converges for every $x\in H$, where $X\setminus H\in\mathcal{I}$. If we set $h(x)=\sup_{n\in\mathbb{N}}|f_n(x)|$ for $x\in H$, 0 for $x\in X\setminus H$, then we see that $|u_n|\le Th$ for every $n\in\mathbb{N}$, so that $\langle u_n\rangle_{n\in\mathbb{N}}$ is order-bounded in L^0.

(b) This now follows from 367Bf, because L^0 is Dedekind σ-complete.

367H Proposition Suppose that $E\subseteq\mathbb{R}$ is a Borel set and $h:E\to\mathbb{R}$ is a continuous function. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and set $Q_E=\{u:u\in L^0,\,[\![u\in E]\!]=1\}$, where $L^0=L^0(\mathfrak{A})$.

Let $\bar{h} : Q_E \to L^0$ be the function defined by h (364H). Then $\langle \bar{h}(u_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{h}(u)$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in Q_E order*-converging to $u \in Q_E$.

proof This is an easy consequence of 367F. We can represent $\mathfrak{A}$ as $\Sigma/\mathcal{I}$ where Σ is a σ-algebra of subsets of some set X and $\mathcal{I}$ is a σ-ideal of Σ (314M); let $T : \mathcal{L}^0 \to L^0(\mathfrak{A})$ be the corresponding homomorphism (364C, 367F). Now we can find Σ-measurable functions $\langle f_n\rangle_{n\in\mathbb{N}}$, f such that $Tf_n = u_n$, $Tf = u$, as in 367F; and the hypothesis $[\![u_n \in E]\!] = 1$, $[\![u \in E]\!] = 1$ means just that, adjusting f_n and f on a member of $\mathcal{I}$ if necessary, we can suppose that $f_n(x)$, $f(x) \in E$ for every $x \in X$. (I am passing over the trivial case $E = \emptyset$, $X \in \mathcal{I}$, $\mathfrak{A} = \{0\}$.) Accordingly $\bar{h}(u_n) = T(hf_n)$ and $\bar{h}(u) = T(hf)$, and (because h is continuous)

$$\{x : h(f(x)) \neq \lim_{n\to\infty} h(f_n(x))\} \subseteq \{x : f(x) \neq \lim_{n\to\infty} f_n(x)\} \in \mathcal{I},$$

so $\langle \bar{h}(u_n)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar{h}(u)$.

367I Dominated convergence We now have a suitable language in which to express an abstract version of Lebesgue's Dominated Convergence Theorem.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^1 = L^1_{\bar{\mu}} = L^1(\mathfrak{A}, \bar{\mu})$ which is order-bounded and order*-convergent in L^1, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to some $u \in L^1$; in particular, $\int u = \lim_{n\to\infty} \int u_n$.

proof The norm of L^1 is order-continuous (365C), so $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u, by 367Da. As $\int$ is norm-continuous, $\int u = \lim_{n\to\infty} \int u_n$.

367J The Martingale Theorem In the same way, we can re-write theorems from §275 in this language.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and $\langle \mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$. For each $n \in \mathbb{N}$ let $P_n : L^1 = L^1_{\bar{\mu}} \to L^1 \cap L^0(\mathfrak{B}_n)$ be the conditional expectation operator (365Q); let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$, and P the conditional expectation operator onto $L^1 \cap L^0(\mathfrak{B})$.

(a) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a norm-bounded sequence in L^1 such that $P_n(u_{n+1}) = u_n$ for every $n \in \mathbb{N}$, then $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent in L^1.

(b) If $u \in L^1$ then $\langle P_n u\rangle_{n\in\mathbb{N}}$ is order*-convergent and $\| \|_1$-convergent to Pu.

proof If we represent $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a probability space, these become mere translations of 275G and 275I. (Note that this argument relies on the description of order*-convergence in L^0 in terms of a.e. convergence of functions, as in 367F; so that we need to know that order*-convergence in L^1 matches order*-convergence in L^0, which is what 367E is for.)

Remark See also 367Q below.

367K Some of the most important applications of these ideas concern spaces of continuous functions. I do not think that this is the time to go very far along this road, but one particular fact will be useful in §376.

Proposition Let X be a locally compact Hausdorff space, and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in $C(X)$, the space of continuous real-valued functions on X. Then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$ iff $\{x : x \in X, \limsup_{n\to\infty} |u_n(x)| > 0\}$ is meager. In particular, $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 if $\lim_{n\to\infty} u_n(x) = 0$ for every x.

proof (a) The following elementary fact is worth noting: if $A \subseteq C(X)^+$ is non-empty and $\inf A = 0$ in $C(X)$, then $G = \bigcup_{u\in A}\{x : u(x) < \epsilon\}$ is dense for every $\epsilon > 0$. **P?** If not, take $x_0 \in X \setminus \overline{G}$. Because X is completely regular (3A3Bb), there is a continuous function $w : X \to [0, 1]$ such that $w(x_0) = 1$ and $w(x) = 0$ for every $x \in \overline{G}$. But in this case $0 < \epsilon w \leq u$ for every $u \in A$, which is impossible. **XQ**

(b) Suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0. Set $v_n = |u_n| \wedge \chi X$, so that $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 (using 367Ca, as usual). Set

$$B = \{v : v \in C(X), \exists\, n \in \mathbb{N}, v_i \leq v \;\forall\, i \geq n\},$$

so that $\inf B = 0$ in $C(X)$ (367Be). For each $k \in \mathbb{N}$, set $G_k = \bigcup_{v\in B}\{x : v(x) < 2^{-k}\}$; then G_k is dense, by (a), and of course is open. So $H = \bigcup_{k\in\mathbb{N}} X \setminus G_k$ is a countable union of nowhere dense sets and is meager. But this means that

$$\begin{aligned}\{x : \limsup_{n\to\infty} |u_n(x)| > 0\} &= \{x : \limsup_{n\to\infty} v_n(x) > 0\}\\ &\subseteq \{x : \inf_{v\in B} v(x) > 0\} \subseteq H\end{aligned}$$

is meager.

(c) Now suppose that $\langle u_n\rangle_{n\in\mathbb{N}}$ does not order*-converge to 0. By 367C(b-ii), there is a $w > 0$ in $C(X)$ such that $w = \sup_{i\ge n} w \wedge |u_i|$ for every $n \in \mathbb{N}$; that is, $\inf_{i\ge n}(w - |u_i|)^+ = 0$ for every n. Set

$$G_n = \{x : \inf_{i\ge n}(w - |u_i|)^+(x) < 2^{-n}\} = \{x : \sup_{i\ge n} |u_i(x)| > w(x) - 2^{-n}\}$$

for each n. Then

$$H = \bigcap_{n\in\mathbb{N}} G_n = \{x : \limsup_{n\to\infty} u_n(x) \ge w(x)\}$$

is the intersection of a sequence of dense open sets, and its complement is meager.

Let G be the non-empty open set $\{x : w(x) > 0\}$. Then G is not meager, by Baire's theorem (3A3Ha); so $G \cap H$ cannot be meager. But $\{x : \limsup_{n\to\infty} |u_n(x)| > 0\}$ includes $G \cap H$, so is also not meager.

Remark Unless the topology of X is discrete, $C(X)$ is not regularly embedded in $\mathbb{R}^X$, and we expect to find sequences in $C(X)$ which order*-converge to 0 in $C(X)$ but not in $\mathbb{R}^X$. But the proposition tells us that if we have a sequence in $C(X)$ which order*-converges in $\mathbb{R}^X$ to a member of $C(X)$, then it order*-converges in $C(X)$.

367L Everything above concerns a particular notion of sequential convergence. There is inevitably a suggestion that there ought to be a topological interpretation of this convergence (see 367Yb, 367Yk, 3A3P), but I have taken care to avoid spelling one out at this stage; I will return to the point in §393. (For a general discussion in the context of Boolean algebras, see Vladimirov 02, chap. 4.) I come now to something which really is a topology, and is as closely involved with order-convergence as any.

Convergence in measure Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. For $a \in \mathfrak{A}^f = \{a : \bar\mu a < \infty\}$, $u \in L^0 = L^0(\mathfrak{A})$ and $\epsilon > 0$ set $\tau_a(u) = \int |u| \wedge \chi a$ and $\tau_{a\epsilon}(u) = \bar\mu(a \cap [\![|u| > \epsilon]\!])$. Then the **topology of convergence in measure** on L^0 is defined *either* as the topology generated by the F-seminorms τ_a *or* by saying that $G \subseteq L^0$ is open iff for every $u \in G$ there are $a \in \mathfrak{A}^f$ and $\epsilon > 0$ such that $v \in G$ whenever $\tau_{a\epsilon}(u - v) \le \epsilon$.

Remark The sentences above include a number of assertions which need proving. But at this point, rather than write out any of the relevant arguments, I refer you to §245. Since we know that $L^0(\mathfrak{A})$ can be identified with $L^0(\mu)$ for a suitable measure space (X, Σ, μ) (321J, 364Ic), everything we know about general spaces $L^0(\mu)$ can be applied directly to $L^0(\mathfrak{A})$ for measure algebras $(\mathfrak{A}, \bar\mu)$; and that is what I will do for the next few paragraphs. So far, all I have done is to write τ_a in place of the $\bar\tau_F$ of 245Ac, and call on the remarks in 245Bb and 245F.

367M Theorem (a) For any measure algebra $(\mathfrak{A}, \bar\mu)$, the topology $\mathfrak{T}$ of convergence in measure on $L^0 = L^0(\mathfrak{A})$ is a linear space topology, and any order*-convergent sequence in L^0 is $\mathfrak{T}$-convergent to the same limit.

(b) $u \mapsto |u| : L^0 \to L^0$ and $(u, v) \mapsto u \vee v$, $(u, v) \mapsto u \times v : L^0 \times L^0 \to L^0$ are continuous.

(c) $(\mathfrak{A}, \bar\mu)$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(d) $(\mathfrak{A}, \bar\mu)$ is localizable iff $\mathfrak{T}$ is Hausdorff and L^0 is complete under the uniformity corresponding to $\mathfrak{T}$.

(e) $(\mathfrak{A}, \bar\mu)$ is σ-finite iff $\mathfrak{T}$ is metrizable.

proof 245D, 245Cb, 245E. Of course we need 322B to assure us that the phrases 'semi-finite', 'localizable', 'σ-finite' here correspond to the same phrases used in §245, and 367F to identify order*-convergence in L^0 with the order-convergence studied in §245.

367N Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure.

(a) If $A \subseteq L^0$ is a non-empty, downwards-directed set with infimum 0, then for every neighbourhood G of 0 in L^0 there is a $u \in A$ such that $v \in G$ whenever $|v| \le u$.

(b) If $U \subseteq L^0$ is an order-dense Riesz subspace, it is topologically dense.

(c) In particular, $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ are topologically dense.

proof (a) Let $a \in \mathfrak{A}^f$, $\epsilon > 0$ be such that $u \in G$ whenever $\int |u| \wedge \chi a \le \epsilon$ (see 245Bb). Since $\{u \wedge \chi a : u \in A\}$ is a downwards-directed set in $L^1 = L^1_{\bar\mu}$ with infimum 0 in L^1, there must be a $u \in A$ such that $\int u \wedge \chi a \le \epsilon$ (365Da). But now $[-u, u] \subseteq G$, as required.

(b) Write $\overline{U}$ for the closure of U. Then $(L^0)^+ \subseteq \overline{U}$. **P** If $v \in (L^0)^+$, then $\{u : u \in U,\ u \le v\}$ is an upwards-directed set with supremum u, so $A = \{v - u : u \in U,\ u \le v\}$ is a downwards-directed set with infimum 0 (351Db). By (a), every neighbourhood of 0 meets A, and (because subtraction is continuous) every neighbourhood of v meets U, that is, $v \in \overline{U}$. **Q**

Since $\overline{U}$ is a linear subspace of L^0 (2A5Ec), it includes $(L^0)^+ - (L^0)^+ = L^0$ (352D).

(c) By 364Ja, $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ are order-dense Riesz subspaces of L^0.

367O Theorem Let U be a Banach lattice and $(\mathfrak{A}, \bar\mu)$ a measure algebra. Give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. If $T : U \to L^0$ is a positive linear operator, then it is continuous.

proof Take any open set $G \subseteq L^0$. **?** Suppose, if possible, that $T^{-1}[G]$ is not open. Then we can find u, $\langle u_n \rangle_{n\in\mathbb{N}} \in U$ such that $Tu \in G$ and $\|u_n - u\| \le 2^{-n}$, $Tu_n \notin G$ for every n. Set $H = G - Tu$; then H is an open set containing 0 but not $T(u_n - u)$, for any $n \in \mathbb{N}$. Since $\sum_{n=0}^\infty n\|u_n - u\| < \infty$, $v = \sum_{n=0}^\infty n|u_n - u|$ is defined in U, and $|T(u_n - u)| \le \frac{1}{n} Tv$ for every $n \ge 1$. But by 367Na (or otherwise) we know that there is some n such that $w \in H$ whenever $|w| \le \frac{1}{n} Tv$, so that $T(u_n - u) \in H$ for some n, which is impossible. **X**

367P Proposition Let $(\mathfrak{A}, \bar\mu)$ be a σ-finite measure algebra.

(a) A sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $L^0 = L^0(\mathfrak{A})$ converges in measure to $u \in L^0$ iff every subsequence of $\langle u_n \rangle_{n\in\mathbb{N}}$ has a sub-subsequence which order*-converges to u.

(b) A set $F \subseteq L^0$ is closed for the topology of convergence in measure iff $u \in F$ whenever there is a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in F order*-converging to $u \in L^0$.

proof 245K, 245L.

367Q As an example of the power of the language we now have available, I give abstract versions of some martingale convergence theorems.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra; for each closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, let $P_{\mathfrak{B}}$ be the corresponding conditional expectation operator from $L^1 = L^1_{\bar\mu}$ to $L^1 \cap L^0(\mathfrak{B}) = L^1_{\bar\mu\restriction\mathfrak{B}}$.

(a) If $\mathbb{B}$ is a non-empty downwards-directed family of closed subalgebras of $\mathfrak{A}$ with intersection $\mathfrak{C}$, and $u \in L^1 = L^1_{\bar\mu}$, then $P_{\mathfrak{C}} u$ is the $\| \|_1$-limit $\lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\downarrow)} P_{\mathfrak{B}} u$, where $\mathcal{F}(\mathbb{B}\downarrow)$ is the filter on $\mathbb{B}$ generated by $\{\{\mathfrak{B} : \mathfrak{B}_0 \supseteq \mathfrak{B} \in \mathbb{B}\} : \mathfrak{B}_0 \in \mathbb{B}\}$.

(b) If $\mathbb{B}$ is a non-empty upwards-directed family of closed subalgebras of $\mathfrak{A}$ and $\mathfrak{C}$ is the closed subalgebra generated by $\bigcup \mathbb{B}$, then for every $u \in L^1$, $P_{\mathfrak{C}} u$ is the $\| \|_1$-limit $\lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)} P_{\mathfrak{B}} u$, where $\mathcal{F}(\mathbb{B}\uparrow)$ is the filter on $\mathbb{B}$ generated by $\{\{\mathfrak{B} : \mathfrak{B}_0 \subseteq \mathfrak{B} \in \mathbb{B}\} : \mathfrak{B}_0 \in \mathbb{B}\}$. as $\mathfrak{B}$ decreases through $\mathbb{B}$.

(c) Suppose that $\mathbb{B}$ is a non-empty upwards-directed family of closed subalgebras of $\mathfrak{A}$, and $\langle u_{\mathfrak{B}} \rangle_{\mathfrak{B}\in\mathbb{B}}$ is a $\| \|_1$-bounded family in L^1 such that $u_{\mathfrak{B}} = P_{\mathfrak{B}} u_{\mathfrak{C}}$ whenever $\mathfrak{B}$, $\mathfrak{C} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{C}$. Then $\lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)} u_{\mathfrak{B}}$ is defined for the topology of convergence in measure and belongs to L^1.

proof (a)(i) Note first that $\{P_{\mathfrak{B}} u : \mathfrak{B} \in \mathbb{B}\}$ is uniformly integrable (246D, or directly), therefore relatively weakly compact in L^1 (247C/356Q). Consequently there must be a $v \in L^1$ which is a weak cluster point of $P_{\mathfrak{B}} u$ as $\mathfrak{B}$ decreases through $\mathbb{B}$, in the sense that v belongs to the weak closure $\overline{\{P_{\mathfrak{B}} u : \mathfrak{B} \in \mathbb{B},\ \mathfrak{B} \subseteq \mathfrak{B}_0\}}$ for every $\mathfrak{B}_0 \in \mathbb{B}$.

It follows that $v = P_{\mathfrak{C}} u$. **P** For every $\mathfrak{B}_0 \in \mathbb{B}$, $L^1 \cap L^0(\mathfrak{B}_0) = L^1_{\bar\mu\restriction\mathfrak{B}_0}$ is a norm-closed linear subspace of L^1 containing $P_{\mathfrak{B}} u$ whenever $\mathfrak{B} \subseteq \mathfrak{B}_0$. It is therefore weakly closed (3A5Ee) and contains v. Consequently $[\![v > \alpha]\!] \in \mathfrak{B}_0$ for every $\alpha \in \mathbb{R}$. As $\mathfrak{B}_0$ is arbitrary, $[\![v > \alpha]\!] \in \mathfrak{C}$ for every $\alpha \in \mathbb{R}$, and $v \in L^1_{\bar\mu\restriction\mathfrak{C}}$. Next, if $c \in \mathfrak{C}$, then

$$\int_c v \in \overline{\{\int_c P_{\mathfrak{B}}u : \mathfrak{B} \in \mathbb{B}\}} = \{\int_c u\};$$

so $v = P_{\mathfrak{C}}u$. **Q**

(ii) Now take $\epsilon > 0$. Then there is a $\mathfrak{B}_0 \in \mathbb{B}$ such that $\|P_{\mathfrak{B}}u - P_{\mathfrak{B}_0}u\|_1 \le \frac{1}{2}\epsilon$ whenever $\mathfrak{B} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$. **P?** Otherwise, we can find a non-increasing sequence $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{B}$ such that $\|P_{\mathfrak{B}_{n+1}}u - P_{\mathfrak{B}_n}u\|_1 > \frac{1}{2}\epsilon$ for every $n \in \mathbb{N}$. By the reverse martingale theorem (275K), $\langle P_{\mathfrak{B}_n}u \rangle_{n\in\mathbb{N}}$ is order*-convergent to w say. But as $\{P_{\mathfrak{B}_n}u : n \in \mathbb{N}\}$ is uniformly integrable, $\langle P_{\mathfrak{B}_n}u \rangle_{n\in\mathbb{N}}$ is $\| \|_1$-convergent to w (246Ja), and $\lim_{n\to\infty} \|P_{\mathfrak{B}_{n+1}}u - P_{\mathfrak{B}_n}u\|_1 = 0$. **XQ**

At this point, however, observe that $C = \{w : \|w - P_{\mathfrak{B}_0}u\|_1 \le \frac{1}{2}\epsilon\}$ is convex and $\| \|_1$-closed, therefore weakly closed, in L^1. Since it contains $P_{\mathfrak{B}}u$ whenever $\mathfrak{B} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$, it contains $v = P_{\mathfrak{C}}u$. Consequently

$$\|P_{\mathfrak{B}}u - P_{\mathfrak{C}}u\|_1 \le \|P_{\mathfrak{B}}u - P_{\mathfrak{B}_0}u\|_1 + \|P_{\mathfrak{B}_0}u - v\|_1 \le \epsilon$$

whenever $\mathfrak{B} \in \mathbb{B}$ and $\mathfrak{B} \subseteq \mathfrak{B}_0$. As ϵ and u are arbitrary, (a) is true.

(b) We can use the same method. Again take any $u \in L^1$.

(i) This time, observe that $P_{\mathfrak{B}}u$ must have a weak cluster point v as $\mathfrak{B}$ increases through $\mathbb{B}$. Since $P_{\mathfrak{B}}u$ belongs to $L^1 \cap L^0(\mathfrak{C})$ for every $\mathfrak{B} \in \mathbb{B}$, so does v. Next, if $b \in \mathfrak{B}_0 \in \mathbb{B}$, then $\int_b P_{\mathfrak{B}}u = \int_b u$ whenever $\mathfrak{B} \supseteq \mathfrak{B}_0$, so $\int_b v = \int_b u$. Thus $\mathfrak{D} = \{b : b \in \mathfrak{A}, \int_b v = \int_b u\}$ includes $\bigcup \mathbb{B}$. But $\mathfrak{D}$ is closed for the measure algebra topology of $\mathfrak{A}$, so $\mathfrak{D} \supseteq \mathfrak{C}$ and $\int_c v = \int_c u$ for every $c \in \mathfrak{C}$. Thus once again we have $v = P_{\mathfrak{C}}u$.

(ii) Now repeat the argument of (a-ii) almost word for word, but taking '$\mathfrak{B} \supseteq \mathfrak{B}'$' in place of every '$\mathfrak{B} \subseteq \mathfrak{B}'$', and quoting the ordinary martingale theorem instead of the reverse martingale theorem.

(c)(i) If $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathbb{B}$, then $\langle u_{\mathfrak{B}_n} \rangle_{n\in\mathbb{N}}$ is order*-convergent, by Doob's martingale theorem (367Ja).

(ii) It follows that the image $\mathcal{G}$ of $\mathcal{F}(\mathbb{B}\uparrow)$ under the map $\mathfrak{B} \mapsto u_{\mathfrak{B}} : \mathbb{B} \to L^0$ is Cauchy for the linear space topology $\mathfrak{T}$ of convergence in measure. **P?** Otherwise, set $\tau(v) = \int |v| \wedge \chi 1$ for $v \in L^0$. There is an $\epsilon > 0$ such that $\sup_{v,v'\in C} \tau(v - v') > 2\epsilon$ for every $C \in \mathcal{G}$; in which case, for any $\mathfrak{B} \in \mathbb{B}$, there must be a $\mathfrak{C} \in \mathbb{B}$ such that $\mathfrak{C} \supseteq \mathfrak{B}$ $\tau(u_{\mathfrak{C}} - u_{\mathfrak{B}}) \ge \epsilon$. But now there will be a non-decreasing sequence $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{B}$ such that $\tau(u_{\mathfrak{B}_{n+1}} - u_{\mathfrak{B}_n}) \ge \epsilon$ for every $n \in \mathbb{N}$ and $\langle u_{\mathfrak{B}_n} \rangle_{n\in\mathbb{N}}$ cannot be order*-convergent. **XQ**

(iii) By 367Mc, $u = \lim \mathcal{G} = \lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)} u_{\mathfrak{B}}$ is defined in L^0 for $\mathfrak{T}$. But as u belongs to the $\mathfrak{T}$-closure of the $\| \|_1$-bounded set $\{u_{\mathfrak{B}} : \mathfrak{B} \in \mathbb{B}\}$, $u \in L^1$, by 245J(b-i).

367R It will be useful later to be able to quote the following straightforward facts.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Give $\mathfrak{A}$ its measure-algebra topology (323A) and $L^0 = L^0(\mathfrak{A})$ the topology of convergence in measure.

(a) The map $\chi : \mathfrak{A} \to L^0$ is a homeomorphism between $\mathfrak{A}$ and its image in L^0.

(b) If $\mathfrak{A}$ has countable Maharam type, then L^0 is separable.

(c) Suppose that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ which is closed for the measure-algebra topology. Then $L^0(\mathfrak{B})$ is closed in $L^0(\mathfrak{A})$.

(d) A non-empty set $A \subseteq L^0$ is bounded in the linear topological space sense (3A5N) iff $\inf_{k\in\mathbb{N}} \sup_{u\in A} \bar{\mu}(a \cap [\![|u| > k]\!]) = 0$ for every $a \in \mathfrak{A}^f$.

proof (a) Of course χ is injective (364Jc). The measure-algebra topology of $\mathfrak{A}$ is defined by the pseudometrics $\rho_a(b, c) = \bar{\mu}(a \cap (b \triangle c))$, while the topology of L^0 is defined by the pseudometrics $\sigma_a(u, v) = \int |u-v| \wedge \chi a$, in both cases taking a to run over elements of $\mathfrak{A}$ of finite measure; as $\sigma_a(\chi b, \chi c)$ is always equal to $\rho_a(b, c)$, we have the result.

(b) By 331O, $\mathfrak{A}$ is separable in its measure-algebra topology; let $B \subseteq \mathfrak{A}$ be a countable dense set. Set

$$B^* = \{\sum_{i=0}^n \alpha_i \chi b_i : n \in \mathbb{N},\, \alpha_0, \dots, \alpha_n \in \mathbb{Q},\, b_0, \dots, b_n \in B\}.$$

B^* is a countable subset of L^0; let V be its closure. Then V includes $S(\mathfrak{A})$. **P** For any $n \in \mathbb{N}$, the function $(\alpha_0, \dots, \alpha_n, a_0, \dots, a_n) \mapsto \sum_{i=0}^n \alpha_i \chi a_i : \mathbb{R}^{n+1} \times \mathfrak{A}^{n+1} \to L^0$ is continuous, just because $\chi : \mathfrak{A} \to L^0$ and addition and scalar multiplication in L^0 are continuous ((a) above, 367M). So

$$D_n = \{(\alpha_0, \dots, \alpha_n, a_0, \dots, a_n) : \sum_{i=0}^{n} \alpha_i \chi a_i \in V\}$$

is a closed subset of $\mathbb{R}^{n+1} \times \mathfrak{A}^{n+1}$ including $\mathbb{Q}^{n+1} \times B^{n+1}$. But $\mathbb{Q}^{n+1} \times B^{n+1}$ is dense in $\mathbb{R}^{n+1} \times \mathfrak{A}^{n+1}$ (3A3Ie), so $D_n = \mathbb{R}^{n+1} \times \mathfrak{A}^{n+1}$, that is, $\sum_{i=0}^{n} \alpha_i \chi a_i \in V$ whenever $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ and $a_0, \dots, a_n \in V$. As n is arbitrary, $S(\mathfrak{A}) \subseteq V$. **Q**

Since $S(\mathfrak{A})$ is dense in L^0 (367Nc), $V = L^0$, B^* is dense in L^0 and L^0 is separable.

(c) Note first that $\mathfrak{B}$ is order-closed in $\mathfrak{A}$ (323D(c-i)), so that $L^0(\mathfrak{B})$, defined as in 364A, is a subset of $L^0(\mathfrak{A})$ (cf. 364Xt). Applying 364P to the identity map $\mathfrak{B} \subsetneqq \mathfrak{A}$, we see that the map $L^0(\mathfrak{B}) \subsetneqq L^0(\mathfrak{A})$ identifies the operations of addition, scalar multiplication and supremum in $L^0(\mathfrak{B})$ with the restrictions of the corresponding operations on $L^0(\mathfrak{A})$.

Suppose that $u \in L^0(\mathfrak{A})$ is in the closure of $L^0(\mathfrak{B})$, and $\alpha \in \mathbb{R}$; let $n \in \mathbb{N}$ be such that $|\alpha| < n$, and fix $a \in \mathfrak{A}^f$ for the moment. For each $k \in \mathbb{N}$, choose $v_k \in L^0(\mathfrak{B})$ such that $\int |u - v_k| \wedge \chi a \le 2^{-k}$ (367L). Consider $v'_k = \operatorname{med}(-n\chi 1, v_k, n\chi 1)$ for $k \in \mathbb{N}$, and $v = \inf_{k\in\mathbb{N}} \sup_{j\ge k} v'_k$. We do not need to ask whether the operations here are being performed in $L^0(\mathfrak{A})$ or in $L^0(\mathfrak{B})$, and v will belong to $L^0(\mathfrak{B})$. Accordingly, now necessarily working in $L^0(\mathfrak{A})$, we shall have

$$v \times \chi a = \inf_{k\in\mathbb{N}} \sup_{j\ge k} v'_k \times \chi a.$$

Now observe that, for each k, $w_k = 2n \sup_{j\ge k} |u - v_j| \wedge \chi a$ is defined in $L^1_{\bar\mu}$ and $\int w_k \le 2^{-k+2} n$. Set $u' = \operatorname{med}(-n\chi 1, u, n\chi 1)$. For $j \ge k$,

$$\begin{aligned} |u' \times \chi a - v'_j \times \chi a| &= |\operatorname{med}(-n\chi 1, u \times \chi a, n\chi 1) - \operatorname{med}(-n\chi 1, v_j \times \chi a, n\chi 1)| \\ &\le |u - v_j| \wedge 2n\chi a \le w_k. \end{aligned}$$

So, for any $m \in \mathbb{N}$,

$$\begin{aligned} u' \times \chi a - v \times \chi a &= \sup_{k\in\mathbb{N}} \inf_{j\ge k} u' \times \chi a - v'_k \times \chi a \\ &= \sup_{k\ge m} \inf_{j\ge k} u' \times \chi a - v'_k \times \chi a \le \sup_{k\ge m} w_k = w_m, \\ v \times \chi a - u' \times \chi a &= \inf_{k\in\mathbb{N}} \sup_{j\ge k} v'_k \times \chi a - u' \times \chi a \\ &\le \sup_{j\ge m} v'_k \times \chi a - u' \times \chi a \le w_m. \end{aligned}$$

Putting these together,

$$|u' \times \chi a - v \times \chi a| \le w_m$$

for every $m \in \mathbb{N}$, and $u' \times \chi a = v \times \chi a$. But this means that $a \cap [\![u' > \alpha]\!] = a \cap [\![v > \alpha]\!]$; at the same time, because $-n < \alpha < n$, $[\![u' > \alpha]\!] = [\![u > \alpha]\!]$.

Thus we see that for every $a \in \mathfrak{A}^f$ there is a $b \in \mathfrak{B}$ such that $a \cap (b \triangle [\![u > \alpha]\!]) = 0$. It follows at once that $[\![u > \alpha]\!]$ belongs to the closure of $\mathfrak{B}$, which is $\mathfrak{B}$ itself. As α is arbitrary, $u \in L^0(\mathfrak{B})$; as u is arbitrary, $L^0(\mathfrak{B})$ is closed.

(d)(i) Suppose that A is topologically bounded, $a \in \mathfrak{A}^f$ and $\epsilon > 0$. Then $G = \{v : v \in L^0, \bar\mu(a \cap [\![|v| > 1]\!]) \le \epsilon$ is a neighbourhood of 0, so there is a $k \in \mathbb{N}$ such that $A \subseteq kG$. If $u \in A$, there is a $v \in U$ such that $u = kv$, so that

$$\bar\mu(a \cap [\![|u| > k]\!]) = \bar\mu(a \cap [\![|v| > 1]\!]) \le \epsilon.$$

Thus $\inf_{k\in\mathbb{N}} \sup_{u\in A} \bar\mu(a \cap [\![|u| > k]\!]) \le \epsilon$; as a and ϵ are arbitrary, the condition is satisfied.

(ii) Suppose that the condition is satisfied. Let G be a neighbourhood of of 0 in L^0. Then there are an $a \in \mathfrak{A}^f$ and an $\epsilon > 0$ such that $v \in G$ whenever $\bar\mu(a \cap [\![|v| > \epsilon]\!]) \le \epsilon$. Now there is a $k \in \mathbb{N}$ such that $\bar\mu(a \cap [\![|u| > k]\!]) \le \epsilon$ for every $u \in A$. Let $n \ge 1$ be such that $n\epsilon \ge k$; then

$$\bar\mu(a \cap [\![\tfrac{1}{n}|u| > \epsilon]\!]) = \bar\mu(a \cap [\![|u| > n\epsilon]\!]) \le \bar\mu(a \cap [\![|u| > k]\!]) \le \epsilon$$

for every $u \in A$, so $\dfrac{1}{n} A \subseteq G$ and $A \subseteq nG$. As G is arbitrary, A is topologically bounded.

367S Proposition Let $E \subseteq \mathbb{R}$ be a Borel set, and $h : E \to \mathbb{R}$ a continuous function. Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\bar{h} : Q_E \to L^0 = L^0(\mathfrak{A})$ the associated function, where $Q_E = \{u : u \in L^0, [\![u \in E]\!] = 1\}$ (364H). Then $\bar{h}$ is continuous for the topology of convergence in measure.

proof (Compare 245Dd.) Express $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a measure space (X, Σ, μ). Take any $u \in Q_E$, any $a \in \mathfrak{A}$ such that $\bar{\mu}a < \infty$, and any $\epsilon > 0$. Express u as $f^\bullet$ where $f : X \to E$ is a measurable function, and a as $F^\bullet$ where $F \in \Sigma$. Set $\eta = \epsilon/(2 + \mu F)$. For each $n \in \mathbb{N}$, write E_n for

$$\{t : t \in E, |h(s) - h(t)| \le \eta \text{ whenever } s \in E \text{ and } |s - t| \le 2^{-n}\}.$$

Then $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of Borel sets with union E, so there is an n such that $\mu\{x : x \in F, f(x) \notin E_n\} \le \eta$.

Now suppose that $v \in Q_E$ is such that $\int |v - u| \wedge \chi a \le 2^{-n}\eta$. Express v as $g^\bullet$ where $g : X \to E$ is a measurable function. Then

$$\int_F \min(1, |g(x) - f(x)|)\mu(dx) \le 2^{-n}\eta,$$

so $\mu\{x : x \in F, |f(x) - g(x)| > 2^{-n}\} \le \eta$, and

$$\begin{aligned}\{x : x \in F, |h(g(x)) - h(f(x))| > \eta\}& \\ \subseteq \{x : x \in F, f(x) \notin E_n\} \cup \{x &: x \in F, |f(x) - g(x)| > 2^{-n}\}\end{aligned}$$

has measure at most 2η. But this means that

$$\int |\bar{h}(v) - \bar{h}(u)| \wedge \chi a = \int_F \min(1, |hg(x) - hf(x)|)\mu(dx) \le 2\eta + \eta\mu F = \epsilon.$$

As u, a and ϵ are arbitrary, $\bar{h}$ is continuous.

367T Intrinsic description of convergence in measure It is a remarkable fact that the topology of convergence in measure, not only on L^0 but on its order-dense Riesz subspaces, can be described in terms of the Riesz space structure alone, without referring at all to the underlying measure algebra or to integration. (Compare 324H.) There is more than one way of doing this. As far as I know, none is outstandingly convincing; I present a formulation which seems to me to exhibit some, at least, of the essence of the phenomenon.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and U an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$. Suppose that $A \subseteq U$ and $u^* \in U$. Then u^* belongs to the closure of A for the topology of convergence in measure iff

there is an order-dense Riesz subspace V of U such that
for every $v \in V^+$ there is a non-empty downwards-directed $B \subseteq U$, with infimum 0, such that
for every $w \in B$ there is a $u \in A$ such that

$$|u - u^*| \wedge v \le w.$$

proof (a) Suppose first that $u^* \in \overline{A}$. Take V to be $U \cap L^1_{\bar{\mu}}$; then V is an order-dense Riesz subspace of L^0, by 352Nc and 353A, and is therefore order-dense in U. (This is where I use the hypothesis that $(\mathfrak{A}, \bar{\mu})$ is semi-finite, so that $L^1_{\bar{\mu}}$ is order-dense in L^0, by 365Ga.)

Take any $v \in V^+$. For each $n \in \mathbb{N}$, set $a_n = [\![v > 2^{-n}]\!] \in \mathfrak{A}^f$. Because $u^* \in \overline{A}$, there is a $u_n \in A$ such that $\bar{\mu}b_n \le 2^{-n}$, where

$$b_n = a_n \cap [\![|u_n - u^*| > 2^{-n}]\!] = [\![|u_n - u^*| \wedge v > 2^{-n}]\!].$$

Set $c_n = \sup_{i\ge n} b_i$; then $\bar{\mu}c_n \le 2^{-n+1}$ for each n, so $\inf_{n\in\mathbb{N}} c_n = 0$ and $\inf_{n\in\mathbb{N}} w_n = 0$ in L^0, where $w_n = v \times \chi c_n + 2^{-n}\chi 1$. Also $|u_n - u^*| \wedge v \le w_n$ for each n.

The w_n need not belong to U, so we cannot set $B = \{w_n : n \in \mathbb{N}\}$. But if instead we write

$$B = \{w : w \in U, w \ge v \wedge w_n \text{ for some } n \in \mathbb{N}\},$$

then B is non-empty and downwards-directed (because $\langle w_n\rangle_{n\in\mathbb{N}}$ is non-increasing); and

$$\begin{aligned}\inf B &= v - \sup\{v - w : w \in B\}\\ &= v - \sup\{w : w \in U,\, w \le (v - w_n)^+ \text{ for some } n \in \mathbb{N}\}\\ &= v - \sup_{n\in\mathbb{N}}(v - w_n)^+\end{aligned}$$

(because U is order-dense in L^0)

$$= 0.$$

Since for every $w \in B$ there is an n such that $w \ge v \wedge w_n \ge v \wedge |u_n - u^*|$, B witnesses that the condition is satisfied.

(b) Now suppose that the condition is satisfied. Fix $a \in \mathfrak{A}^f$ and $\epsilon > 0$. Because V is order-dense in U and therefore in L^0, there is a $v \in V$ such that $0 \le v \le \chi a$ and $\int v \ge \bar\mu a - \epsilon$. Let B be a downwards-directed set, with infimum 0, such that for every $w \in B$ there is a $u \in A$ with $v \wedge |u - u^*| \le w$. Then there is a $w \in B$ such that $\int w \wedge v \le \epsilon$. Now there is a $u \in A$ such that $|u - u^*| \wedge v \le w$, so that

$$\int |u - u^*| \wedge \chi a \le \epsilon + \int |u - u^*| \wedge v \le \epsilon + \int w \wedge v \le 2\epsilon.$$

As a and ϵ are arbitrary, $u^* \in \overline{A}$.

***367U Theorem** Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra; write L^1 for $L^1_{\bar\mu}$. Let $P : (L^1)^{**} \to L^1$ be the linear operator corresponding to the band projection from $(L^1)^{**} = (L^1)^{\times\sim}$ onto $(L^1)^{\times\times}$ and the canonical isomorphism between L^1 and $(L^1)^{\times\times}$. For $A \subseteq L^1$ write A^* for the weak* closure of the image of A in $(L^1)^{**}$. Then for every $A \subseteq L^1$

$$P[A^*] \subseteq \overline{\Gamma(A)},$$

where $\Gamma(A)$ is the convex hull of A and $\overline{\Gamma(A)}$ is the closure of $\Gamma(A)$ in $L^0 = L^0(\mathfrak{A})$ for the topology of convergence in measure.

proof (a) The statement of the theorem includes a number of assertions: that $(L^1)^* = (L^1)^\times$; that $(L^1)^{**} = ((L^1)^*)^\sim$; that the natural embedding of L^1 into $(L^1)^{**} = (L^1)^{\times\sim}$ identifies L^1 with $(L^1)^{\times\times}$; and that $(L^1)^{\times\times}$ is a projection band in $(L^1)^{\times\sim}$. For proofs of these see 365C, 356P, 356B and 356D.

Now for the new argument. First, observe that the statement of the theorem involves the measure algebra $(\mathfrak{A}, \bar\mu)$ and the space L^0 only in the definition of 'convergence in measure'; everything else depends only on the Banach lattice structure of L^1. And since we are concerned only with the question of whether members of $P[A^*]$, which is surely a subset of L^1, belong to $\overline{\Gamma(A)}$, 367T shows that this also can be answered in terms of the Riesz space structure of L^1. What this means is that we can suppose that $(\mathfrak{A}, \bar\mu)$ is localizable. **P** Let $(\widehat{\mathfrak{A}}, \tilde\mu)$ be the localization of $(\mathfrak{A}, \bar\mu)$ (322Q). The natural expression of $\mathfrak{A}$ as an order-dense subalgebra of $\widehat{\mathfrak{A}}$ identifies $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\, \bar\mu a < \infty\}$ with $\widehat{\mathfrak{A}}^f$ (322P), so that $L^1_{\bar\mu}$ becomes identified with $L^1_{\tilde\mu}$, by 365Nd. Thus we can think of L^1 as $L^1_{\tilde\mu}$, and $(\widehat{\mathfrak{A}}, \tilde\mu)$ is localizable. **Q**

(b) We need a version of a result in §362. As we are supposing that $(\mathfrak{A}, \bar\mu)$ is localizable, we can identify $(L^1)^*$, as Banach lattice, with $L^\infty = L^\infty(\mathfrak{A})$ (365Lc). Take any $\phi \in (L^1)^{**} \cong (L^\infty)^*$ such that $\phi \ge 0$ and $P\phi = 0$. Then $C = \{c : c \in \mathfrak{A},\, \phi(\chi c) = 0\}$ is an order-dense ideal of $\mathfrak{A}$. **P** Just because ϕ is a positive linear operator, C is an ideal of $\mathfrak{A}$. We have an L-space isomorphism between $(L^\infty)^* = (L^\infty)^\sim$ and the space M of bounded additive functionals on $\mathfrak{A}$, and this isomorphism matches $(L^\infty)^\times$ with the projection band M_τ of completely additive functionals (363K, 362B). So if we write $P_\tau : M \to M_\tau$ for the band projection onto M_τ, P_τ must correspond to the band projection $P : (L^\infty)^\sim \to (L^\infty)^\times$. Let ν be the member of M corresponding to ϕ, so that $\nu a = \phi(\chi a)$ for every $a \in \mathfrak{A}$. Then $(P_\tau\nu)(1) = (P\phi)(\chi 1) = 0$, that is, $P_\tau\nu = 0$. Now $\mathfrak{A}$ is weakly (σ, ∞)-distributive (322F), so there is an upwards-directed set $D \subseteq \mathfrak{A}$, with supremum 1, such that $\sup_{d\in D} \nu d = (P_\tau\nu)(1) = 0$, that is, $0 = \nu d = \phi(\chi d)$ for every $d \in D$, and $D \subseteq C$. So $\sup C = 1$, that is, C is order-dense, as claimed. **Q**

(c) Now take $\phi \in A^*$ and set $u_0 = P\phi$; I have to show that $u_0 \in \overline{\Gamma(A)}$. Write R for the canonical map from L^1 to $(L^1)^{**}$, so that ϕ belongs to the weak* closure of $R[A]$.

(i) Consider first the case $u_0 = 0$, that is, $P\phi = 0$. Since P is a band projection, $P|\phi| = 0$. By (b), $C = \{c : (P|\phi|)(\chi c) = 0\}$ is an order-dense ideal in $\mathfrak{A}$. Take any $a \in \mathfrak{A}^f$ and $\epsilon > 0$. Then $\{c : c \in C,\ c \subseteq a\}$ is upwards-directed and has supremum a, so there is a $c \in C$ such that $\bar\mu(a \setminus c) \le \epsilon$.

Consider the map $Q : L^1 \to L^1$ defined by setting $Qw = w \times \chi c$ for every $w \in L^1$. Then its adjoint $Q' : L^\infty \to L^\infty$ (3A5Ed) can be defined by the same formula: $Q'v = v \times \chi c$ for every $v \in L^\infty$. Now

$$|\phi(Q'v)| \le \|v\|_\infty |\phi|(\chi c) = 0$$

for every $v \in L^\infty$, and $Q''\phi = 0$, where $Q'' : (L^\infty)^* \to (L^\infty)^*$ is the adjoint of Q'. Since Q'' is continuous for the weak* topology on $(L^\infty)^*$, $0 \in \overline{Q''[R[A]]}$, where $\overline{Q''R[A]}$ is the closure for the weak* topology of $(L^\infty)^*$. But of course $Q''R = RQ$, while the weak* topology of $(L^\infty)^*$ corresponds, on the image $R[L^1]$ of L^1, to the weak topology of L^1; so that 0 belongs to the closure of $Q[A]$ for the weak topology of L^1.

Because Q is linear, $Q[\Gamma(A)]$ is convex. Since 0 belongs to the closure of $Q[\Gamma(A)]$ for the weak topology of L^1, it belongs to the closure of $Q[\Gamma(A)]$ for the norm topology (3A5Ee). So there is a $w \in \Gamma(A)$ such that $\|w \times \chi c\|_1 \le \epsilon^2$. But this means that $\bar\mu(c \cap [\![|w| \ge \epsilon]\!]) \le \epsilon$ and $\bar\mu(a \cap [\![|w| \ge \epsilon]\!]) \le 2\epsilon$. Since a and ϵ are arbitrary, $0 \in \overline{\Gamma(A)}$.

(ii) This deals with the case $u_0 = 0$. Now the general case follows at once if we set $B = A - u_0$ and observe that $\phi - Ru_0 \in B^*$, so

$$0 = P(\phi - Ru_0) \in \overline{\Gamma(B)} = \overline{\Gamma(A) - u_0} = \overline{\Gamma(A)} - u_0$$

because the topology of convergence in measure is a linear space topology.

Remark This is a version of a theorem from Bukhvalov 95.

***367V Corollary** Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. Let $\mathcal{C}$ be a family of convex subsets of $L^0 = L^0(\mathfrak{A})$, all closed for the topology of convergence in measure, with the finite intersection property, and suppose that for every non-zero $a \in \mathfrak{A}$ there are a non-zero $b \subseteq a$ and a $C \in \mathcal{C}$ such that $\sup_{u \in C} \int_b |u| < \infty$. Then $\bigcap \mathcal{C} \ne \emptyset$.

proof Because $\mathcal{C}$ has the finite intersection property, there is an ultrafilter $\mathcal{F}$ on L^0 including $\mathcal{C}$. Set

$$I = \{a : a \in \mathfrak{A},\ \inf_{F \in \mathcal{F}} \sup_{u \in F} \int_a |u| < \infty\};$$

because $\mathcal{F}$ is a filter, I is an ideal in $\mathfrak{A}$, and the condition on $\mathcal{C}$ tells us that I is order-dense. For each $a \in I$, define $Q_a : L^0 \to L^0$ by setting $Q_a u = u \times \chi a$. Then there is an $F \in \mathcal{F}$ such that $Q_a[F]$ is a norm-bounded set in $L^1 = L^1_{\bar\mu}$, so $\phi_a = \lim_{u \to \mathcal{F}} RQ_a u$ is defined in $(L^\infty)^* = L^\infty(\mathfrak{A})^*$ for the weak* topology on $(L^\infty)^*$, writing R for the canonical map from L^1 to $(L^\infty)^* \cong (L^1)^{**}$. (Once again, we can identify $(L^1)^*$ with L^∞ because $(\mathfrak{A}, \bar\mu)$ is localizable.) If $P : (L^\infty)^* \to L^1$ is the map corresponding to the band projection $\tilde P$ from $(L^\infty)^\sim$ onto $(L^\infty)^\times$, as in 367U, and $C \in \mathcal{C}$, then 367U tells us that $P\phi_a$ must belong to the closure of the convex set $Q_a[C]$ for the topology of convergence in measure. Moreover, if $a \subseteq b \in I$, so that $Q_a = Q_a Q_b$, then $P\phi_a = Q_a P\phi_b$. **P** Observe that

$$\tilde P = RP : (L^\infty)^* \to (L^\infty)^\times, \quad Q''_a R = RQ_a{\restriction}L^1, \quad Q''_a{\restriction}(L^\infty)^\times = RQ_a R^{-1}.$$

$Q_a{\restriction}L^1$ is a band projection on L^1, so its adjoint Q'_a is a band projection on $L^\infty \cong (L^1)^\sim$ (356C) and Q''_a is a band projection on $(L^\infty)^* \cong (L^\infty)^\sim$. This means that Q''_a will commute with $\tilde P$ (352Sb). But also Q''_a is continuous for the weak* topology of $(L^\infty)^*$, so

$$Q''_a \phi_b = \lim_{u \to \mathcal{F}} Q''_a RQ_b u = \lim_{u \to \mathcal{F}} RQ_a Q_b u = \phi_a$$

and

$$P\phi_a = R^{-1}\tilde P\phi_a = R^{-1}\tilde P Q''_a(\phi_b) = R^{-1}Q''_a \tilde P\phi_b = Q_a R^{-1}\tilde P\phi_b = Q_a P\phi_b. \ \mathbf{Q}$$

Generally, if a, $b \in I$, then

$$Q_a P\phi_b = Q_a Q_b P\phi_b = Q_{a \cap b} P\phi_b = P\phi_{a \cap b} = Q_b P\phi_a.$$

What this means is that if we take a partition D of unity included in I (313K), so that $L^0 \cong \prod_{d \in D} L^0(\mathfrak{A}_d)$ (315F(iii), 364R), and define $w \in L^0$ by saying that $w \times \chi d = P\phi_d$ for every $d \in D$, then we shall have

$$w \times \chi a \times \chi d = Q_a P\phi_d = Q_d P\phi_a = P\phi_a \times \chi d$$

whenever $a \in I$ and $d \in D$. Consequently

$$w \times \chi a = P\phi_a \in \overline{Q_a[C]}$$

for every $a \in I$ and $C \in \mathcal{C}$. But now, given $a \in \mathfrak{A}^f$ and $\epsilon > 0$ and $C \in \mathcal{C}$, there is a $b \in I$ such that $\bar{\mu}(a \setminus b) \le \epsilon$; $w \times \chi b \in \overline{Q_b[C]}$, so there is a $u \in C$ such that $\bar{\mu}(b \cap [\![|w-u| \ge \epsilon]\!]) \le \epsilon$; and $\bar{\mu}(a \cap [\![|w-u| \ge \epsilon]\!]) \le 2\epsilon$. As a and ϵ are arbitrary and C is closed, $w \in C$; as C is arbitrary, $w \in \bigcap \mathcal{C}$ and $\bigcap \mathcal{C} \ne \emptyset$.

***367W Independence** I have given myself very little room in this chapter to discuss stochastic independence. There are direct translations of results from §272 in 364Xe-364Xf. However the language here is adapted to a significant result not presented in §272. I had better begin by repeating a definition from 364Xe. Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. Then a family $\langle u_i \rangle_{i \in I}$ in $L^0(\mathfrak{A})$ is **stochastically independent** if $\bar{\mu}(\inf_{i \in J} [\![u_i > \alpha_i]\!]) = \prod_{i \in J} \bar{\mu}[\![u_i > \alpha_i]\!]$ whenever $J \subseteq I$ is a non-empty finite set and $\alpha_i \in \mathbb{R}$ for every $i \in I$. (The direct translation of the definition in 272Ac would rather be '$\bar{\mu}(\inf_{i \in J} [\![u_i \le \alpha_i]\!]) = \prod_{i \in J} \bar{\mu}[\![u_i \le \alpha_i]\!]$ whenever $J \subseteq I$ is a non-empty finite set and $\alpha_i \in \mathbb{R}$ for every $i \in I$', intepreting $[\![u_i \le \alpha_i]\!]$ as in 364Xa. Of course 272F tells us that this comes to the same thing.) Now the new fact is the following.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and I any set. Give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. Then the collection of independent families $\langle u_i \rangle_{i \in I}$ is a closed set in $(L^0)^I$.

proof Suppose that $\langle u_i \rangle_{i \in I} \in (L^0)^I$ is not independent. Then there are a finite set $J \subseteq I$ and a family $\langle \alpha_i \rangle_{i \in J}$ of real numbers such that $\bar{\mu}(\inf_{i \in J} [\![u_i > \alpha_i]\!]) \ne \prod_{i \in J} \bar{\mu}[\![u_i > \alpha_i]\!]$. Set $a_i = [\![u_i > \alpha_i]\!]$ for each i. Let $\delta > 0$ be such that $\gamma \ne \prod_{i \in J} \gamma_i$ whenever $|\gamma - \bar{\mu}(\inf_{i \in J} a_i)| \le 2\delta\#(J)$ and $|\gamma_i - \bar{\mu}a_i| \le 2\delta$ for every $i \in J$. Let $\eta \in \,]0,1]$ be such that $\bar{\mu}[\![u_i > \alpha_i + 2\eta]\!] \ge \bar{\mu}a_i - \delta$ for every $i \in J$.

Now if $\langle v_i \rangle_{i \in I} \in (L^0)^I$ and $\bar{\mu}[\![|v_i - u_i| > \eta]\!] \le \delta$ for each $i \in J$, $\langle v_i \rangle_{i \in I}$ is not independent. **P** For each $i \in J$, consider $b_i = [\![v_i > \alpha_i + \eta]\!]$, $a'_i = [\![u_i > \alpha_i + 2\eta]\!]$. We have

$$a'_i = [\![u_i > \alpha_i + 2\eta]\!] \subseteq [\![v_i > \alpha_i + \eta]\!] \cup [\![u_i - v_i > \eta]\!] \subseteq b_i \cup [\![|u_i - v_i| > \eta]\!]$$

(364Ea), and

$$b_i = [\![v_i > \alpha_i + \eta]\!] \subseteq [\![u_i > \alpha_i]\!] \cup [\![v_i - u_i > \eta]\!] \subseteq a_i \cup [\![|v_i - u_i| > \eta]\!],$$

so

$$b_i \triangle a_i = (b_i \setminus a_i) \cup (a_i \setminus b_i) \subseteq [\![|v_i - u_i| > \eta]\!] \cup (a_i \setminus a'_i)$$

has measure at most 2δ. It follows that $(\inf_{i \in J} b_i) \triangle (\inf_{i \in J} a_i)$ has measure at most $2\delta\#(J)$, and $|\bar{\mu}(\inf_{i \in J} b_i) - \bar{\mu}(\inf_{i \in J} a_i)| \le 2\delta\#(J)$. At the same time, for each $i \in J$, $|\bar{\mu}b_i - \bar{\mu}a_i| \le 2\delta$. By the choice of δ, $\bar{\mu}(\inf_{i \in J} b_i) \ne \prod_{i \in J} \bar{\mu}b_i$, and $\langle v_i \rangle_{i \in I}$ is not independent. **Q**

This shows that the set of non-independent families is open in $(L^0)^I$, so that the set of independent families is closed, as claimed.

367X Basic exercises >(a) Let P be a lattice. (i) Show that if $p \in P$ and $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in P, then $\langle p_n \rangle_{n \in \mathbb{N}}$ is order*-convergent to p iff $p = \sup_{n \in \mathbb{N}} p_n$. (ii) Suppose that $\langle p_n \rangle_{n \in \mathbb{N}}$ is a sequence in P order*-converging to $p \in P$. Show that $p = \sup_{n \in \mathbb{N}} p \wedge p_n = \inf_{n \in \mathbb{N}} p \vee p_n$. (iii) Let $\langle p_n \rangle_{n \in \mathbb{N}}$, $\langle q_n \rangle_{n \in \mathbb{N}}$ be two sequences in P which are order*-convergent to p, q respectively. Show that if $p_n \le q_n$ for every n then $p \le q$. (iv) Let $\langle p_n \rangle_{n \in \mathbb{N}}$ be a sequence in P. Show that $\langle p_n \rangle_{n \in \mathbb{N}}$ order*-converges to $p \in P$ iff $\langle p_n \vee p \rangle_{n \in \mathbb{N}}$ and $\langle p_n \wedge p \rangle_{n \in \mathbb{N}}$ both order*-converge to p.

(b) Let P and Q be lattices, and $f : P \to Q$ an order-preserving function. Suppose that $\langle p_n \rangle_{n \in \mathbb{N}}$ is an order-bounded sequence which order*-converges to p in P. Show that $\langle f(p_n) \rangle_{n \in \mathbb{N}}$ order*-converges to $f(p)$ in Q if *either* f is order-continuous *or* P is Dedekind σ-complete and f is sequentially order-continuous.

(c) Let P be *either* a Boolean algebra *or* a Riesz space. Suppose that $\langle p_n \rangle_{n \in \mathbb{N}}$ is a sequence in P such that $\langle p_{2n} \rangle_{n \in \mathbb{N}}$ and $\langle p_{2n+1} \rangle_{n \in \mathbb{N}}$ are both order*-convergent to $p \in P$. Show that $\langle p_n \rangle_{n \in \mathbb{N}}$ is order*-convergent to p. (*Hint*: 313B, 352E.)

(d) Let $\langle P_i \rangle_{i \in I}$ be a family of lattices with product P (315Xd). Show that a sequence $\langle p_n \rangle_{n \in \mathbb{N}}$ in P order*-converges to $p \in P$ iff $\langle p_n(i) \rangle_{n \in \mathbb{N}}$ order*-converges to $p(i)$ in P_i for every $i \in I$.

>(e) Let $\mathfrak{A}$ be a Boolean algebra and $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ two sequences in $\mathfrak{A}$ order*-converging to a, b respectively. Show that $\langle a_n\cup b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\cap b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\setminus b_n\rangle_{n\in\mathbb{N}}$, $\langle a_n\triangle b_n\rangle_{n\in\mathbb{N}}$ order*-converge to $a\cup b$, $a\cap b$, $a\setminus b$ and $a\triangle b$ respectively.

(f) Let $\mathfrak{A}$ be a Boolean algebra and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Show that $\langle a_n\rangle_{n\in\mathbb{N}}$ does not order*-converge to 0 iff there is a non-zero $a\in\mathfrak{A}$ such that $a=\sup_{i\ge n}a\wedge a_i$ for every $n\in\mathbb{N}$.

>(g)(i) Let U be a Riesz space and $\langle u_n\rangle_{n\in\mathbb{N}}$ an order*-convergent sequence in U^+ with limit u. Show that $h(u)\le\liminf_{n\to\infty}h(u_n)$ for every $h\in(U^\times)^+$. (ii) Let U be a Riesz space and $\langle u_n\rangle_{n\in\mathbb{N}}$ an order-bounded order*-convergent sequence in U with limit u. Show that $h(u)=\lim_{n\to\infty}h(u_n)$ for every $h\in U^\times$. (Compare 356Xd.)

(h)(i) Show that if U is a Banach lattice, every norm-convergent sequence has a subsequence which is order-bounded and order*-convergent. (*Hint*: consider the case in which $\sum_{n=0}^\infty\|u_n-u\|$ is finite.) (ii) Find a Riesz norm on C([0,1]) for which there is an order-bounded norm-convergent sequence which has no order*-convergent subsequence.

>(i) Let U be a Riesz space with a Fatou norm $\|\ \|$. (i) Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order*-convergent sequence in U with limit u, then $\|u\|\le\liminf_{n\to\infty}\|u_n\|$. (*Hint*: $\langle|u_n|\wedge|u|\rangle_{n\in\mathbb{N}}$ is order*-convergent to $|u|$.) (ii) Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a norm-convergent sequence in U it has an order*-convergent subsequence. (*Hint*: if $\sum_{n=0}^\infty\|u_n\|<\infty$ then $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0.)

(j) Let U and V be Archimedean Riesz spaces and $T:U\to V$ an order-continuous Riesz homomorphism. Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U which order*-converges to $u\in U$, then $\langle Tu_n\rangle_{n\in\mathbb{N}}$ order*-converges to Tu in V.

(k) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a regularly embedded subalgebra. Show that if $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $b\in\mathfrak{B}$, then $\langle b_n\rangle_{n\in\mathbb{N}}$ order*-converges to b in $\mathfrak{B}$ iff it order*-converges to b in $\mathfrak{A}$.

(l) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ two sequences in $L^0(\mathfrak{A})$ which are order*-convergent to u, v respectively. Show that $\langle u_n\times v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u\times v$. Show that if u, u_n all have multiplicative inverses u^{-1}, u_n^{-1} then $\langle u_n^{-1}\rangle_{n\in\mathbb{N}}$ order*-converges to u^{-1}.

(m) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathcal{I}$ a σ-ideal of $\mathfrak{A}$. Show that for any $\langle a_n\rangle_{n\in\mathbb{N}}\in\mathfrak{A}^{\mathbb{N}}$ and $a\in\mathfrak{A}$, $\langle a_n^\bullet\rangle_{n\in\mathbb{N}}$ order*-converges to $a^\bullet$ in $\mathfrak{A}/\mathcal{I}$ iff $\inf_{n\in\mathbb{N}}\sup_{m\ge n}a_m\triangle a\in\mathcal{I}$.

>(n) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle h_n\rangle_{n\in\mathbb{N}}$ a sequence of Borel measurable functions from $\mathbb{R}$ to itself such that $h(t)=\lim_{n\to\infty}h_n(t)$ is defined for every $t\in\mathbb{R}$. Show that $\langle\bar h_n(u)\rangle_{n\in\mathbb{N}}$ order*-converges to $\bar h(u)$ for every $u\in L^0=L^0(\mathfrak{A})$, where $\bar h_n$, $\bar h:L^0\to L^0$ are defined as in 364H.

(o) Let U be an L-space and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in U which is order*-convergent to $u\in U$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ is norm-convergent to u iff $\{u_n:n\in\mathbb{N}\}$ is uniformly integrable iff $\|u\|_1=\lim_{n\to\infty}\|u_n\|_1$. (*Hint*: 245H, 246J.)

(p) Let U be an L-space and $\langle u_n\rangle_{n\in\mathbb{N}}$ a norm-bounded sequence in U. Show that there are a $v\in U$ and a subsequence $\langle v_n\rangle_{n\in\mathbb{N}}$ of $\langle u_n\rangle_{n\in\mathbb{N}}$ such that $\langle\frac{1}{n+1}\sum_{i=0}^n w_i\rangle_{n\in\mathbb{N}}$ order*-converges to v for every subsequence $\langle w_n\rangle_{n\in\mathbb{N}}$ of $\langle v_n\rangle_{n\in\mathbb{N}}$. (*Hint*: 276H.)

(q) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $p\in[1,\infty[$. For $v\in(L^p)^+=(L^p_{\bar\mu})^+$ define $\tau_v:L^0\to[0,\infty[$ by setting $\tau_v(u)=\||u|\wedge v\|_p$ for $u\in U$. Show that each τ_v is an F-seminorm and that the topology on $L^0(\mathfrak{A})$ defined by $\{\tau_v:v\in(L^p)^+\}$ is the topology of convergence in measure.

(r) Let $(\mathfrak{A},\bar\mu)$ be a σ-finite measure algebra. Suppose we have a double sequence $\langle u_{ij}\rangle_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in $L^0=L^0(\mathfrak{A})$ such that $\langle u_{ij}\rangle_{j\in\mathbb{N}}$ order*-converges to u_i in L^0 for each i, while $\langle u_i\rangle_{i\in\mathbb{N}}$ order*-converges to u. Show that there is a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle u_{i,n(i)}\rangle_{i\in\mathbb{N}}$ order*-converges to u.

(s) Let (X, Σ, μ) be a semi-finite measure space. Show that $L^0(\mu)$ is separable for the topology of convergence in measure iff μ is σ-finite and has countable Maharam type. (Cf. 365Xr.)

(t) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and give $L^0 = L^0(\mathfrak{A})$ its topology of convergence in measure. Show that if $(\mathfrak{A}, \bar{\mu})$ is semi-finite, then a set $A \subseteq L^0$ is bounded in the linear-topological-space sense iff $\{\alpha_n x_n : n \in \mathbb{N}\}$ is order-bounded for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in A and every sequence $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ in $\mathbb{R}$ converging to 0.

(u) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and write $\mathfrak{T}$ for its measure-algebra topology. (i) Show that if $\langle a_n \rangle_{n \in \mathbb{N}}$ is order*-convergent to $a \in \mathfrak{A}$, then $\langle a_n \rangle_{n \in \mathbb{N}} \to a$ for $\mathfrak{T}$. (ii) Show that if $(\mathfrak{A}, \bar{\mu})$ is σ-finite, then (α) a sequence converges to a for $\mathfrak{T}$ iff every subsequence has a sub-subsequence which is order*-convergent to to a (β) a set $F \subseteq \mathfrak{A}$ is $\mathfrak{T}$-closed iff $a \in F$ whenever there is a sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in F which is order*-convergent to $a \in \mathfrak{A}$. (iii) Show that if $(\mathfrak{A}, \bar{\mu})$ is semi-finite but not σ-finite, there is an $A \subseteq L^0$ such that the limit of any order*-convergent sequence in A belongs to A, but A is not $\mathfrak{T}$-closed.

(v) Let U be a Banach lattice with an order-continuous norm. (i) Show that a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ is norm-convergent to $u \in U$ iff every subsequence has a sub-subsequence which is order-bounded and order*-convergent to u. (ii) Show that a set $F \subseteq U$ is closed for the norm topology iff $u \in F$ whenever there is an order-bounded sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in F order*-converging to $u \in U$.

>(w) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. For $u \in L^0 = L^0(\mathfrak{A})$ let ν_u be the distribution of u (364Gb). Show that $u \mapsto \nu_u$ is continuous when L^0 is given the topology of convergence in measure and the space of probability distributions on $\mathbb{R}$ is given the vague topology (274Ld).

(x) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\langle u_n \rangle_{n \in \mathbb{N}}$ a stochastically independent sequence in $L^0(\mathfrak{A})$, all with the Cauchy distribution $\nu_{C,1}$ with centre 0 and scale parameter 1 (285Xp). For each n let C_n be the convex hull of $\{u_i : i \geq n\}$, and $\overline{C_n}$ its closure for the topology of convergence in measure. Show that every $u \in \overline{C_0}$ has distribution $\nu_{C,1}$. (*Hint*: consider first $u \in C_0$.) Show that $\overline{C_0}$ is bounded for the topology of convergence in measure. Show that $\bigcap_{n \in \mathbb{N}} \overline{C_n} = \emptyset$.

(y) If U is a linear space and $C \subseteq U$ is a convex set, a function $f : C \to \mathbb{R}$ is **convex** if $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$ whenever x, $y \in C$ and $\alpha \in [0, 1]$. Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra and $C \subseteq L^1_{\bar{\mu}}$ a non-empty convex norm-bounded set which is closed in $L^0(\mathfrak{A})$ for the topology of convergence in measure. Show that any convex function $f : C \to \mathbb{R}$ which is lower semi-continuous for the topology of convergence in measure is bounded below and attains its infimum.

(z) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0, 1]$. Show that there are a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $L^1 = L^1_{\bar{\mu}}$ and u, $v \in L^1$ such that u_n and v are independent for every n, $\langle u_n \rangle_{n \in \mathbb{N}}$ converges weakly to u, but u and v are not independent.

367Y Further exercises (a) Give an example of an Archimedean Riesz space U and an order-bounded sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in U which is order*-convergent to 0, but such that there is no non-increasing sequence $\langle v_n \rangle_{n \in \mathbb{N}}$, with infimum 0, such that $u_n \leq v_n$ for every $n \in \mathbb{N}$.

(b) Let P be any lattice. (i) Show that there is a topology on P for which a set $A \subseteq P$ is closed iff $p \in A$ whenever there is a sequence in A which is order*-convergent to p. Show that any closed set for this topology is sequentially order-closed. (ii) Now let Q be another lattice, with the topology defined in the same way, and $f : P \to Q$ an order-preserving function. Show that if f is topologically continuous it is sequentially order-continuous.

(c) Give an example of a distributive lattice P with p, $q \in P$ and a sequence $\langle p_n \rangle_{n \in \mathbb{N}}$, order*-convergent to p, such that $\langle p_n \wedge q \rangle_{n \in \mathbb{N}}$ is not order*-convergent to $p \wedge q$.

(d) Let us say that a lattice P is $(2, \infty)$**-distributive** if (α) whenever A, $B \subseteq P$ are non-empty sets with infima p, q respectively, then $\inf\{a \vee b : a \in A, b \in B\} = p \vee q$ (β) whenever A, $B \subseteq P$ are non-empty sets with suprema p, q respectively, then $\sup\{a \wedge b : a \in A, b \in B\} = p \wedge q$. Show that, in this case, if $\langle p_n \rangle_{n \in \mathbb{N}}$ order*-converges to p and $\langle q_n \rangle_{n \in \mathbb{N}}$ order*-converges to q, $\langle p_n \vee q_n \rangle_{n \in \mathbb{N}}$ order*-converges to $p \vee q$.

(e)(i) Give an example of a Riesz space U with an order-dense Riesz subspace V of U and a sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in V such that $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in V but does not order*-converge in U. (ii) Give an example of a Riesz space U with an order-dense Riesz subspace V of U and a sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in V, order-bounded in V, such that $\langle v_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in U but does not order*-converge in V.

(f) Let U be an Archimedean f-algebra. Show that if $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ are sequences in U order*-converging to u, v respectively, then $\langle u_n\times v_n\rangle_{n\in\mathbb{N}}$ order*-converges to $u\times v$.

(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $r\geq 1$. Let $E\subseteq\mathbb{R}^r$ be a Borel set and write $Q_E=\{(u_1,\ldots,u_r):[\![(u_1,\ldots,u_r)\in E]\!]=1\}\subseteq L^0(\mathfrak{A})^r$ (364Yb). Let $h:E\to\mathbb{R}$ be a continuous function and $\bar h:Q_E\to L^0=L^0(\mathfrak{A})$ the corresponding map (364Yc). Show that if $\langle\mathbf{u}_n\rangle_{n\in\mathbb{N}}$ is a sequence in Q_E which is order*-convergent to $\mathbf{u}\in Q_E$ (in the lattice $(L^0)^r$), then $\langle\bar h(\mathbf{u}_n)\rangle_{n\in\mathbb{N}}$ is order*-convergent to $\bar h(\mathbf{u})$.

(h) Let X be a completely regular Baire space (definition: 314Yd), and $\langle u_n\rangle_{n\in\mathbb{N}}$ a sequence in $C(X)$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C(X)$ iff $\{x:\limsup_{n\to\infty}|u_n(x)|>0\}$ is meager in X.

(i)(i) Give an example of a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $C([0,1])$ such that $\lim_{n\to\infty}u_n(x)=0$ for every $x\in[0,1]$, but $\{u_n:n\in\mathbb{N}\}$ is not order-bounded in $C([0,1])$. (ii) Give an example of an order-bounded sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $C(\mathbb{Q})$ such that $\lim_{n\to\infty}u_n(q)=0$ for every $q\in\mathbb{Q}$, but $\sup_{i\geq n}u_i=\chi\mathbb{Q}$ in $C(\mathbb{Q})$ for every $n\in\mathbb{N}$. (iii) Give an example of a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $C([0,1])$ such that $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in $C([0,1])$, but $\lim_{n\to\infty}u_n(q)>0$ for every $q\in\mathbb{Q}\cap[0,1]$.

(j) Write out an alternative proof of 367J/367Yh based on the fact that, for a Baire space X, $C(X)$ can be identified with an order-dense Riesz subspace of a quotient of the space of $\widehat{\mathcal{B}}$-measurable functions, where $\widehat{\mathcal{B}}$ is the Baire-property algebra of X.

(k) Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Boolean algebra. Show that there is a topology on $\mathfrak{A}$ such that the closure of any $A\subseteq\mathfrak{A}$ is precisely the set of limits of order*-convergent sequences in A.

(l) Give an example of a set X and a double sequence $\langle u_{mn}\rangle_{m,n\in\mathbb{N}}$ in $\mathbb{R}^X$ such that $\lim_{n\to\infty}u_{mn}(x)=u_m(x)$ exists for every $m\in\mathbb{N}$ and $x\in X$, $\lim_{m\to\infty}u_m(x)=0$ for every $x\in X$, but there is no sequence $\langle v_k\rangle_{k\in\mathbb{N}}$ in $\{u_{mn}:m,n\in\mathbb{N}\}$ such that $\lim_{k\to\infty}v_k(x)=0$ for every x.

(m) Let U be a Riesz space with a Riesz norm $\|\,\|$. For $v\in U^+$ define $\tau_v:U\to[0,\infty[$ by setting $\tau_v(u)=\||u|\wedge v\|$ for every $u\in U$. Show that every τ_v is an F-seminorm on U, and that $\{\tau_v:v\in U^+\}$ defines a Hausdorff linear space topology on U.

(n) Let U be any Riesz space. For $h\in U^{\sim+}$ (356Ab) and $v\in U^+$ define $\tau_{vh}:U\to[0,\infty[$ by setting $\tau_{vh}(u)=h(|u|\wedge v)$ for every $u\in U$. Show that τ_{vh} is an F-seminorm on U.

(o) Let $(\mathfrak{A},\bar\mu)$ be a σ-finite measure algebra. Show that the function $(\alpha,u)\mapsto[\![u>\alpha]\!]:\mathbb{R}\times L^0\to\mathfrak{A}$ is Borel measurable when $L^0=L^0(\mathfrak{A})$ is given the topology of convergence in measure and $\mathfrak{A}$ is given its measure-algebra topology.

(p) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$. Show that there is no Hausdorff topology $\mathfrak{T}$ on $L^0(\mathfrak{G})$ such that $\langle u_n\rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}$-convergent to u whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to u.

(q) In 367Qc, show that $u=\lim_{\mathfrak{B}\to\mathcal{F}(\mathbb{B}\uparrow)}u_{\mathfrak{B}}$ for the norm topology of L^1 iff $\{u_{\mathfrak{B}}:\mathfrak{B}\in\mathbb{B}\}$ is uniformly integrable, and that in this case $u_{\mathfrak{B}}=P_{\mathfrak{B}}u$ for every $\mathfrak{B}\in\mathbb{B}$.

(r) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $r\geq 1$. Let $E\subseteq\mathbb{R}^r$ be a Borel set and write $Q_E=\{(u_1,\ldots,u_r):[\![(u_1,\ldots,u_r)\in E]\!]=1\}\subseteq L^0(\mathfrak{A})^r$, as in 367Yg. Let $h:E\to\mathbb{R}$ be a continuous function and $\bar h:Q_E\to L^0=L^0(\mathfrak{A})$ the corresponding map. Show that if $\bar h$ is continuous if L^0 is given its topology of convergence in measure and $(L^0)^r$ the product topology.

(s) Show that 367U is true for all measure algebras, whether semi-finite or not.

(t) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. For $a\in\mathfrak{A}^f$, $n\in\mathbb{N}$ and u, $v\in L^0(\mathfrak{A})$, set $\rho_{an}(u,v)=\int_{-n}^n\bar\mu(a\cap([\![u>\alpha]\!]\triangle[\![v>\alpha]\!]))$, the integral being with respect to Lebesgue measure. (i) Show that the integral is always defined. (ii) Show that each $\rho_{an}:L^0(\mathfrak{A})\times L^0(\mathfrak{A})\to[0,\infty[$ is a pseudometric. (iii) Show that $\{\rho_{an}:a\in\mathfrak{A}^f,\,n\in\mathbb{N}\}$ defines the topology of convergence in measure on $L^0(\mathfrak{A})$.

367 Notes and comments I have given a very general definition of 'order*-convergence'. The general theory of convergence structures on ordered spaces is complex and has many traps for the unwary. I have tried to lay out a safe path to the results which are important in the context of this book. But the propositions here are necessarily full of little conditions (e.g., the requirement that U should be Archimedean in 367E) whose significance may not be immediately obvious. In particular, the definition is very much better adapted to distributive lattices than to others (367Yc, 367Yd). It is useful in the study of Riesz spaces and Boolean algebras largely because these satisfy strong distributive laws (313B, 352E). The special feature which distinguishes the definition here from other definitions of order-convergence is the fact that it can be applied to sequences which are not order-bounded. For order-bounded sequences there are useful simplifications (367Be-f), but the Martingale Theorem (367J), for instance, if we want to express it in terms of its natural home in the Riesz space L^1, refers to sequences which are hardly ever order-bounded.

The * in the phrase 'order*-convergent' is supposed to be a warning that it may not represent exactly the concept you expect. I think nearly any author using the phrase 'order-convergent' would accept sequences fulfilling the conditions of 367Bf; but beyond this no standard definitions have taken root.

The fact that order*-convergent sequences in an L^0 space are order-bounded (367G) is actually one of the characteristic properties of L^0. Related ideas will be important in the next section (368A, 368M).

It is one of the outstanding characteristics of measure algebras in this context that they provide non-trivial linear space topologies on their L^0 spaces, related in striking ways to the order structure. Not all L^0 spaces have such topologies (367Yp). A topology corresponding to 'convergence in measure' can be defined on $L^0(\mathfrak{A})$ for any Maharam algebra $\mathfrak{A}$; see 393K below.

367T shows that the topology of convergence in measure on $L^0(\mathfrak{A})$ is (at least for semi-finite measure algebras) determined by the Riesz space structure of L^0; and that indeed the same is true of its order-dense Riesz subspaces. This fact is important for a full understanding of the representation theorems in §369 below. If a Riesz space U can be embedded as an order-dense subspace of any such L^0, then there is already a 'topology of convergence in measure' on U, independent of the embedding. It is therefore not surprising that there should be alternative descriptions of the topology of convergence in measure on the important subspaces of L^0 (367Xq, 367Ym).

For σ-finite measure algebras, the topology of convergence in measure is easily described in terms of order-convergence (367P). For other measure algebras, the formula fails (367Xu(iii)). 367Yp shows that trying to apply the same ideas to Riesz spaces in general gives rise to some very curious phenomena.

367V enables us to prove results which would ordinarily be associated with some form of compactness. Of course compactness is indeed involved, as the proof through 367U makes clear; but it is weak* compactness in $(L^1)^{**}$, rather than in the space immediately to hand.

I hardly mention 'uniform integrability' in this section, not because it is uninteresting, but because I have nothing to add at this point to 246J and the exercises in §246. But I do include translations of Lebesgue's Dominated Convergence Theorem (367I) and the Martingale Theorem (367J) to show how these can be expressed in the language of this chapter.

Version of 16.9.09

368 Embedding Riesz spaces in L^0

In this section I turn to the representation of Archimedean Riesz spaces as function spaces. Any Archimedean Riesz space U can be represented as an order-dense subspace of $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is its band algebra (368E). Consequently we get representations of Archimedean Riesz spaces as quotients of subspaces of $\mathbb{R}^X$ (368F) and as subspaces of $C^\infty(X)$ (368G), and a notion of 'Dedekind completion' (368I-368J). Closely associated with these is the fact that we have a very general extension theorem for order-continuous Riesz homomorphisms into L^0 spaces (368B). I give a characterization of L^0 spaces in terms of lateral completeness (368M, 368Yd), and I discuss weakly (σ,∞)-distributive Riesz spaces (368N-368S).

368A Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $A\subseteq(L^0)^+$ a set with no upper bound in L^0, where $L^0=L^0(\mathfrak{A})$. If *either* A is countable *or* $\mathfrak{A}$ is Dedekind complete, there is a $v>0$ in L^0 such that $nv=\sup_{u\in A}u\wedge nv$ for every $n\in\mathbb{N}$.

proof The hypothesis 'A is countable or $\mathfrak{A}$ is Dedekind complete' ensures that $c_\alpha=\sup_{u\in A}[\![u>\alpha]\!]$ is defined for each α. By 364L(a-i), $c=\inf_{n\in\mathbb{N}}c_n=\inf_{\alpha\in\mathbb{R}}c_\alpha$ is non-zero. Now for any $n\ge 1$, $\alpha\in\mathbb{R}$

$$[\![\sup_{u\in A}(u\wedge n\chi c)>\alpha]\!]=\sup_{u\in A}[\![u>\alpha]\!]\cap[\![\chi c>\tfrac{\alpha}{n}]\!]=[\![\chi c>\tfrac{\alpha}{n}]\!],$$

because if $\alpha\ge 0$ then

$$\sup_{u\in A}[\![u>\alpha]\!]=c_\alpha\supseteq c\supseteq[\![\chi c>\tfrac{\alpha}{n}]\!],$$

while if $\alpha<0$ then (because A is a non-empty subset of $(L^0)^+$)

$$\sup_{u\in A}[\![u>\alpha]\!]=1=[\![\chi c>\tfrac{\alpha}{n}]\!].$$

So $\sup_{u\in A}u\wedge n\chi c=n\chi c$ for every $n\ge 1$, and we can take $v=\chi c$. (The case $n=0$ is of course trivial.)

368B Theorem Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, U an Archimedean Riesz space, V an order-dense Riesz subspace of U and $T:V\to L^0=L^0(\mathfrak{A})$ an order-continuous Riesz homomorphism. Then T has a unique extension to an order-continuous Riesz homomorphism $\tilde{T}:U\to L^0$.

proof (a) The key to the proof is the following: if $u\ge 0$ in U, then $\{Tv:v\in V,0\le v\le u\}$ is bounded above in L^0. **P?** Suppose, if possible, otherwise. Then by 368A there is a $w>0$ in L^0 such that $nw=\sup_{v\in A}nw\wedge Tv$ for every $n\in\mathbb{N}$, where $A=\{v:v\in V,\,0\le v\le u\}$. In particular, there is a $v_0\in A$ such that $w_0=w\wedge Tv_0>0$. Because U is Archimedean, $\inf_{k\ge 1}\frac{1}{k}u=0$, so $v_0=\sup_{k\ge 1}(v_0-\frac{1}{k}u)^+$. Because V is order-dense in U, $v_0=\sup B$ where

$$B=\{v:v\in V,\,0\le v\le(v_0-\frac{1}{k}u)^+\text{ for some }k\ge 1\}.$$

Because T is order-continuous, $Tv_0=\sup T[B]$ in L^0, and there is a $v_1\in B$ such that $w_1=w_0\wedge Tv_1>0$. Let $k\ge 1$ be such that $v_1\le(v_0-\frac{1}{k}u)^+$. Then for any $m\in\mathbb{N}$,

$$mv_1\wedge u\le(mv_1\wedge kv_0)+(mv_1\wedge(u-kv_0)^+)$$

(352F(a-i))

$$\le kv_0+(m+k)(v_1\wedge(\frac{1}{k}u-v_0)^+)=kv_0.$$

So for any $v\in A$, $m\in\mathbb{N}$,

$$mw_1\wedge Tv=mw_1\wedge mTv_1\wedge Tv\le T(mv_1\wedge v)\le T(mv_1\wedge u)\le T(kv_0)=kTv_0.$$

But this means that, for $m\in\mathbb{N}$,

$$mw_1=mw_1\wedge mw=\sup_{v\in A}mw_1\wedge(mw\wedge Tv)=\sup_{v\in A}mw_1\wedge Tv\le kTv_0,$$

which is impossible because L^0 is Archimedean and $w_1>0$. **XQ**

(b) Because L^0 is Dedekind complete, $\sup\{Tv:v\in V,\,0\le v\le u\}$ is defined in L^0 for every $u\in U$. By 355F, T has a unique extension to an order-continuous Riesz homomorphism from U to L^0.

368C Corollary Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind complete Boolean algebras and U, V order-dense Riesz subspaces of $L^0(\mathfrak{A})$, $L^0(\mathfrak{B})$ respectively. Then any Riesz space isomorphism between U and V extends uniquely to a Riesz space isomorphism between $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$; and in this case $\mathfrak{A}$ and $\mathfrak{B}$ must be isomorphic as Boolean algebras.

proof If $T : U \to V$ is a Riesz space isomorphism, then 368B tells us that we have (unique) order-continuous Riesz homomorphisms $\tilde{T} : L^0(\mathfrak{A}) \to L^0(\mathfrak{B})$ and $\tilde{T}' : L^0(\mathfrak{B}) \to L^0(\mathfrak{A})$ extending T, T^{-1} respectively. Now $\tilde{T}'\tilde{T} : L^0(\mathfrak{A}) \to L^0(\mathfrak{A})$ is an order-continuous Riesz homomorphism agreeing with the identity on U, so must be the identity on $L^0(\mathfrak{A})$; similarly $\tilde{T}\tilde{T}'$ is the identity on $L^0(\mathfrak{B})$, and $\tilde{T}$ is a Riesz space isomorphism. To see that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, recall that by 364O they can be identified with the algebras of projection bands of $L^0(\mathfrak{A})$ and $L^0(\mathfrak{B})$, which must be isomorphic.

368D Corollary Suppose that $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, and that U is an order-dense Riesz subspace of $L^0(\mathfrak{A})$ which is isomorphic, as Riesz space, to $L^0(\mathfrak{B})$ for some Dedekind complete Boolean algebra $\mathfrak{B}$. Then $U = L^0(\mathfrak{A})$ and $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ (so, in particular, is Dedekind complete).

proof The identity mapping $U \to U$ is surely an order-continuous Riesz homomorphism, so by 368B extends to an order-continuous Riesz homomorphism $\tilde{T} : L^0(\mathfrak{A}) \to U$. Now $\tilde{T}$ must be injective, because if $u \neq 0$ in $L^0(\mathfrak{A})$ there is a $u' \in U$ such that $0 < u' \le |u|$, so that $0 < u' \le |\tilde{T}u|$. So we must have $U = L^0(\mathfrak{A})$ and $\tilde{T}$ the identity map. By 364O again, $\mathfrak{A} \cong \mathfrak{B}$.

368E Theorem Let U be any Archimedean Riesz space, and $\mathfrak{A}$ its band algebra (353B). Then U can be embedded as an order-dense Riesz subspace of $L^0(\mathfrak{A})$.

proof (a) If $U = \{0\}$ then $\mathfrak{A} = \{0\}$, $L^0 = L^0(\mathfrak{A}) = \{0\}$ and the result is trivial; I shall therefore suppose henceforth that U is non-trivial. Note that by 352Q $\mathfrak{A}$ is Dedekind complete.

Let $C \subseteq U^+ \setminus \{0\}$ be a maximal disjoint set (in the sense of 352C); to obtain such a set apply Zorn's lemma to the family of all disjoint subsets of $U^+ \setminus \{0\}$. Now I can write down the formula for the embedding $T : U \to L^0$ immediately, though there will be a good deal of work to do in justification: for $u \in U$ and $\alpha \in \mathbb{R}$, $[\![Tu > \alpha]\!]$ will be the band in U generated by

$$\{e \wedge (u - \alpha e)^+ : e \in C\}.$$

(For once, I allow myself to use the formula $[\![\ldots]\!]$ without checking immediately that it represents a member of L^0; all I claim for the moment is that $[\![Tu > \alpha]\!]$ is a member of $\mathfrak{A}$ determined by u and α.)

(b) Before getting down to the main argument, I make some remarks which will be useful later.

(i) If $u > 0$ in U, then there is some $e \in C$ such that $u \wedge e > 0$, since otherwise we ought to have added u to C. Thus $C^\perp = \{0\}$.

(ii) If $u \in U$ and $e \in C$ and $\alpha \in \mathbb{R}$, then $v = e \wedge (\alpha e - u)^+$ belongs to $[\![Tu > \alpha]\!]^\perp$. **P** If $e' \in C$, then either $e' \neq e$ so

$$v \wedge e' \wedge (u - \alpha e')^+ \le e \wedge e' = 0,$$

or $e' = e$ and

$$v \wedge e' \wedge (u - \alpha e')^+ \le (\alpha e - u)^+ \wedge (u - \alpha e)^+ = 0.$$

Accordingly $[\![Tu > \alpha]\!]$ is included in the band $\{v\}^\perp$ and $v \in [\![Tu > \alpha]\!]^\perp$. **Q**

(c) Now I must confirm that the formula given for $[\![Tu > \alpha]\!]$ is consistent with the conditions laid down in 364Aa. **P** Take $u \in U$.

(i) If $\alpha \le \beta$ then

$$0 \le e \wedge (u - \beta e)^+ \le e \wedge (u - \alpha e)^+ \in [\![Tu > \alpha]\!]$$

so $e \wedge (u - \beta e)^+ \in [\![Tu > \alpha]\!]$, for every $e \in C$, and $[\![Tu > \beta]\!] \subseteq [\![Tu > \alpha]\!]$.

(ii) Given $\alpha \in \mathbb{R}$, set $W = \sup_{\beta > \alpha} [\![Tu > \beta]\!]$ in $\mathfrak{A}$, that is, the band in U generated by $\{e \wedge (u - \beta e)^+ : e \in C, \beta > \alpha\}$. Then for each $e \in C$,

$$\sup_{\beta > \alpha} e \wedge (u - \beta e)^+ = e \wedge (u - \inf_{\beta > \alpha} \beta e)^+ = e \wedge (u - \alpha e)^+$$

using the general distributive laws in U (352E), the translation-invariance of the order (351D) and the fact that U is Archimedean (to see that $\alpha e = \inf_{\beta > \alpha} \beta e$). So $e \wedge (u - \alpha e)^+ \in W$; as e is arbitrary, $[\![Tu > \alpha]\!] \subseteq W$ and $[\![Tu > \alpha]\!] = W$.

(iii) Now set $W = \inf_{n\in\mathbb{N}} [\![Tu > n]\!]$. For any $e \in C$, $n \in \mathbb{N}$ we have

$$e \wedge (ne - u)^+ \in [\![Tu > n]\!]^\perp \subseteq W^\perp,$$

so that

$$e \wedge (e - \tfrac{1}{n}u^+)^+ \le e \wedge (e - \tfrac{1}{n}u)^+ \in W^\perp$$

for every $n \ge 1$ and

$$e = \sup_{n\ge 1} e \wedge (e - \tfrac{1}{n}u^+)^+ \in W^\perp.$$

Thus $C \subseteq W^\perp$ and $W \subseteq C^\perp = \{0\}$. So we have $\inf_{n\in\mathbb{N}} [\![Tu > n]\!] = 0$.

(iv) Finally, set $W = \sup_{n\in\mathbb{N}} [\![Tu > -n]\!]$. Then

$$e \wedge (e - \tfrac{1}{n}u^-)^+ \le e \wedge (e + \tfrac{1}{n}u)^+ \le e \wedge (u + ne)^+ \in W$$

for every $n \ge 1$ and $e \in C$, so

$$e = \sup_{n\ge 1} e \wedge (e - \tfrac{1}{n}u^-)^+ \in W$$

for every $e \in C$ and $W^\perp = \{0\}$, $W = U$. Thus all three conditions of 364Aa are satisfied. **Q**

(d) Thus we have a well-defined map $T : U \to L^0$. I show next that $T(u+v) = Tu + Tv$ for all $u, v \in U$. **P** I rely on the formulae in 364D and 364Ea, and on partitions of unity in $\mathfrak{A}$, constructed as follows. Fix $n \ge 1$ for the moment. Then we know that

$$\sup_{i\in\mathbb{Z}} [\![Tu > \tfrac{i}{n}]\!] = 1, \quad \inf_{i\in\mathbb{Z}} [\![Tu > \tfrac{i}{n}]\!] = 0.$$

So setting

$$V_i = [\![Tu > \tfrac{i}{n}]\!] \setminus [\![Tu > \tfrac{i+1}{n}]\!] = [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tu > \tfrac{i+1}{n}]\!]^\perp,$$

$\langle V_i \rangle_{i\in\mathbb{Z}}$ is a partition of unity in $\mathfrak{A}$. Similarly, $\langle W_i \rangle_{i\in\mathbb{Z}}$ is a partition of unity, where

$$W_i = [\![Tv > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{i+1}{n}]\!]^\perp.$$

Now, for any i, j, $k \in \mathbb{Z}$ such that $i + j \ge k$,

$$V_i \cap W_j \subseteq [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{j}{n}]\!] \subseteq [\![Tu + Tv > \tfrac{i+j}{n}]\!] \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!];$$

thus

$$[\![Tu + Tv > \tfrac{k}{n}]\!] \supseteq \sup_{i+j\ge k} V_i \cap W_j.$$

On the other hand, if $q \in \mathbb{Q}$ and $k \in \mathbb{Z}$, there is an $i \in \mathbb{Z}$ such that $\frac{i}{n} \le q < \frac{i+1}{n}$, so that

$$[\![Tu > q]\!] \cap [\![Tv > \tfrac{k+1}{n} - q]\!] \subseteq [\![Tu > \tfrac{i}{n}]\!] \cap [\![Tv > \tfrac{k-i}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j;$$

thus for any $k \in \mathbb{Z}$

$$[\![Tu + Tv > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!].$$

Also, if $0 < w \in V_i \cap W_j$ and $e \in C$ then

$$w \wedge e \wedge (u - \frac{i+1}{n}e)^+ = w \wedge e \wedge (v - \frac{j+1}{n}e)^+ = 0,$$

so that

$$w \wedge e \wedge (u + v - \frac{i+j+2}{n}e)^+ = 0$$

because

$$(u + v - \frac{i+j+2}{n}e)^+ \le (u - \frac{i+1}{n}e)^+ + (v - \frac{j+1}{n}e)^+$$

by 352Fc. But this means that $V_i \cap W_j \cap [\![T(u+v) > \frac{i+j+2}{n}]\!] = \{0\}$. Turning this round,

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \cap \sup_{i+j\le k-1} V_i \cap W_j = 0,$$

and because $\sup_{i,j\in\mathbb{Z}} V_i \cap W_j = U$ in $\mathfrak{A}$,

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j.$$

Finally, if $i+j \ge k$ and $0 < w \in V_i \cap V_j$, then there is an $e \in C$ such that $w_1 = w \wedge e \wedge (u - \frac{i}{n}e)^+ > 0$; there is an $e' \in C$ such that $w_2 = w_1 \wedge e' \wedge (v - \frac{j}{n}e')^+ > 0$; of course $e = e'$, and

$$0 < w_2 \le e \wedge (u - \frac{i}{n}e)^+ \wedge (v - \frac{j}{n})^+ \le e \wedge (u + v - \frac{i+j}{n}e)^+$$

$$\in [\![T(u+v) > \frac{i+j}{n}]\!] \subseteq [\![T(u+v) > \frac{k}{n}]\!]$$

using 352Fc again. This shows that $w \notin [\![T(u+v) > \frac{k}{n}]\!]^\perp$; as w is arbitrary, $V_i \cap W_j \subseteq [\![T(u+v) > \frac{k}{n}]\!]$; so we get

$$\sup_{i+j\ge k} V_i \cap W_j \subseteq [\![T(u+v) > \tfrac{k}{n}]\!].$$

Putting these four facts together, we see that

$$[\![T(u+v) > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![Tu + Tv > \tfrac{k}{n}]\!],$$

$$[\![Tu + Tv > \tfrac{k+1}{n}]\!] \subseteq \sup_{i+j\ge k} V_i \cap W_j \subseteq [\![T(u+v) > \tfrac{k}{n}]\!]$$

for all $n \ge 1$ and $k \in \mathbb{Z}$. But this means that we must have

$$[\![T(u+v) > \beta]\!] \subseteq [\![Tu + Tv > \alpha]\!], \quad [\![Tu + Tv > \beta]\!] \subseteq [\![T(u+v) > \alpha]\!]$$

whenever $\alpha < \beta$. Consequently

$$\begin{aligned}[\![Tu + Tv > \alpha]\!] &= \sup_{\beta>\alpha} [\![Tu + Tv > \beta]\!] \subseteq [\![T(u+v) > \alpha]\!] \\ &= \sup_{\beta>\alpha} [\![T(u+v) > \beta]\!] \subseteq [\![Tu + Tv > \alpha]\!]\end{aligned}$$

and $[\![Tu + Tv > \alpha]\!] = [\![T(u+v) > \alpha]\!]$ for every α, that is, $T(u+v) = Tu + Tv$. **Q**

(e) The hardest part is over. If $u \in U$, $\gamma > 0$ and $\alpha \in \mathbb{R}$, then for any $e \in C$

$$\min(1, \frac{1}{\gamma})(e \wedge (\gamma u - \alpha e)^+) \le e \wedge (u - \frac{\alpha}{\gamma}e)^+ \le \max(1, \frac{1}{\gamma})(e \wedge (\gamma u - \alpha e)^+),$$

so

$$[\![T(\gamma u) > \alpha]\!] = [\![Tu > \tfrac{\alpha}{\gamma}]\!] = [\![\gamma Tu > \alpha]\!];$$

as α is arbitrary, $\gamma Tu = T(\gamma u)$; as γ and u are arbitrary, T is linear. (We need only check linearity for $\gamma > 0$ because we know from the additivity of T that $T(-u) = -Tu$ for every u.)

(f) To see that T is a Riesz homomorphism, take any $u \in U$ and $\alpha \in \mathbb{R}$ and consider the band $[\![Tu > \alpha]\!] \cup [\![-Tu > \alpha]\!] = [\![|Tu| > \alpha]\!]$ (by 364L(a-ii)). This is the band generated by $\{e \wedge (u - \alpha e)^+ : e \in C\} \cup \{e \wedge (-u - \alpha e)^+ : e \in C\}$. But this must also be the band generated by

$$\{(e \wedge (u - \alpha e)^+) \vee (e \wedge (-u - \alpha e)^+) : e \in C\} = \{e \wedge (|u| - \alpha e)^+ : e \in C\},$$

which is $[\![T|u| > \alpha]\!]$. Thus $[\![|Tu| > \alpha]\!] = [\![T|u| > \alpha]\!]$ for every α and $|Tu| = T|u|$. As u is arbitrary, T is a Riesz homomorphism.

(g) To see that T is injective, take any non-zero $u \in U$. Then there must be some $e \in C$ such that $|u| \wedge e \ne 0$, and some $\alpha > 0$ such that $|u| \wedge e \not\le \alpha e$, so that $e \wedge (|u| - \alpha e)^+ \ne 0$ and $[\![T|u| > \alpha]\!] \ne \{0\}$ and $T|u| \ne 0$ and $Tu \ne 0$.

Thus T embeds U as a Riesz subspace of L^0.

(h) Finally, I must check that $T[U]$ is order-dense in L^0. **P** Let $p > 0$ in L^0. Then there is some $\alpha > 0$ such that $V = [\![p > \alpha]\!] \ne 0$. Take $u > 0$ in V. Let $e \in C$ be such that $u \wedge e > 0$. Then $v = u \wedge \alpha e > 0$. Now $e \wedge (v - \alpha e)^+ = 0$; but also $e' \wedge v = 0$ for every $e' \in C$ distinct from e, so that $[\![Tv > \alpha]\!] = \{0\}$. Next, $v \in V$, so $e' \wedge (v - \beta e')^+ \in V$ whenever $e' \in C$ and $\beta \ge 0$, and $[\![Tv > \beta]\!] \subseteq V$ for every $\beta \ge 0$. Accordingly we have

$$\begin{aligned}[\![Tv > \beta]\!] &= \{0\} \subseteq [\![p > \beta]\!] \text{ if } \beta \ge \alpha,\\ &\subseteq V \subseteq [\![p > \beta]\!] \text{ if } 0 \le \beta < \alpha,\\ &= U = [\![p > \beta]\!] \text{ if } \beta < 0,\end{aligned}$$

and $Tv \le p$. Also $Tv > 0$, by (g). As p is arbitrary, $T[U]$ is order-dense in L^0. **Q**

368F Corollary A Riesz space U is Archimedean iff it is isomorphic to a Riesz subspace of some reduced power $\mathbb{R}^X|\mathcal{F}$, where X is a set and $\mathcal{F}$ is a filter on X such that $\bigcap_{n\in\mathbb{N}} F_n \in \mathcal{F}$ whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$.

proof (a) If U is an Archimedean Riesz space, then by 368E there is a space of the form $L^0 = L^0(\mathfrak{A})$ such that U can be embedded into L^0. As in the proof of 364D, L^0 is isomorphic to some space of the form $\mathcal{L}^0(\Sigma)/\mathcal{W}$, where Σ is a σ-algebra of subsets of some set X and $\mathcal{W} = \{f : f \in \mathcal{L}^0, \{x : f(x) \ne 0\} \in \mathcal{I}\}$, $\mathcal{I}$ being a σ-ideal of Σ. But now $\mathcal{F} = \{A : A \cup E = X \text{ for some } E \in \mathcal{I}\}$ is a filter on X such that $\bigcap_{n\in\mathbb{N}} F_n \in \mathcal{F}$ for every sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$. (I am passing over the trivial case $X \in \mathcal{I}$, since then U must be $\{0\}$.) And $\mathcal{L}^0/\mathcal{W}$ is (isomorphic to) the image of $\mathcal{L}^0$ in $\mathbb{R}^X|\mathcal{F}$, since $\mathcal{W} = \{f : f \in \mathcal{L}^0, \{x : f(x) = 0\} \in \mathcal{F}\}$. Thus U is isomorphic to a Riesz subspace of $\mathbb{R}^X|\mathcal{F}$.

(b) On the other hand, if $\mathcal{F}$ is a filter on X closed under countable intersections, then $\mathcal{W} = \{f : f \in \mathbb{R}^X, \{x : f(x) = 0\} \in \mathcal{F}\}$ is a sequentially order-closed solid linear subspace of the Dedekind σ-complete Riesz space $\mathbb{R}^X$, so that $\mathbb{R}^X|\mathcal{F} = \mathbb{R}^X/\mathcal{W}$ is Dedekind σ-complete (353K(a-iii)) and all its Riesz subspaces must be Archimedean (353Ia, 351Rc).

368G Corollary Every Archimedean Riesz space U is isomorphic to an order-dense Riesz subspace of some space $C^\infty(X)$ (definition: 364V), where X is an extremally disconnected compact Hausdorff space.

proof Let Z be the Stone space of the band algebra $\mathfrak{A}$ of U. Because $\mathfrak{A}$ is Dedekind complete (352Q again), Z is extremally disconnected and $\mathfrak{A}$ can be identified with the regular open algebra $\mathrm{RO}(Z)$ of Z (314S). By 364V, $L^0(\mathrm{RO}(Z))$ can be identified with $C^\infty(Z)$. So an embedding of U as an order-dense Riesz subspace of $L^0(\mathfrak{A})$ (368E) can be regarded as an embedding of U as an order-dense Riesz subspace of $C^\infty(Z)$.

368H Corollary Any Dedekind complete Riesz space U is isomorphic to an order-dense solid linear subspace of $L^0(\mathfrak{A})$ for some Dedekind complete Boolean algebra $\mathfrak{A}$.

proof Embed U in $L^0 = L^0(\mathfrak{A})$ as in 368E; because U is order-dense in L^0 and (in itself) Dedekind complete, it is solid (353L).

368I Corollary Let U be an Archimedean Riesz space. Then U can be embedded as an order-dense Riesz subspace of a Dedekind complete Riesz space V in such a way that the solid linear subspace of V generated by U is V itself, and this can be done in essentially only one way. If W is any other Dedekind complete Riesz space and $T : U \to W$ is an order-continuous positive linear operator, there is a unique positive linear operator $\tilde{T} : V \to W$ extending T.

proof By 368E, we may suppose that U is actually an order-dense Riesz subspace of $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is a Dedekind complete Boolean algebra. In this case, we can take V to be the solid linear subspace generated by U, that is, $\{v : |v| \le u \text{ for some } u \in U\}$; being a solid linear subspace of the Dedekind complete Riesz space $L^0(\mathfrak{A})$, V is Dedekind complete, and of course U is order-dense in V.

If W is any other Dedekind complete Riesz space and $T : U \to W$ is an order-continuous positive linear operator, then for any $v \in V^+$ there is a $u_0 \in U$ such that $v \le u_0$, so that Tu_0 is an upper bound for $\{Tu : u \in U, 0 \le u \le v\}$; as W is Dedekind complete, $\sup_{u\in U, 0\le u\le v} Tu$ is defined in W. By 355Fa, T has a unique extension to an order-continuous positive linear operator from V to W.

In particular, if V_1 is another Dedekind complete Riesz space in which U can be embedded as an order-dense Riesz subspace, this embedding of U extends to an embedding of V; since V is Dedekind complete, its copy in V_1 must be a solid linear subspace, so if V_1 is the solid linear subspace of itself generated by U, we get an identification between V and V_1, uniquely determined by the embeddings of U in V and V_1.

368J Definition If U is an Archimedean Riesz space, a **Dedekind completion** of U is a Dedekind complete Riesz space V together with an embedding of U in V as an order-dense Riesz subspace of V such that the solid linear subspace of V generated by U is V itself. 368I tells us that every Archimedean Riesz space U has an essentially unique Dedekind completion, so that we may speak of 'the' Dedekind completion of U.

368K This is a convenient point at which to give a characterization of the Riesz spaces $L^0(\mathfrak{A})$.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Suppose that $A \subseteq L^0(\mathfrak{A})^+$ is disjoint. If *either* A is countable *or* $\mathfrak{A}$ is Dedekind complete, A is bounded above in $L^0(\mathfrak{A})$.

proof If $A = \emptyset$, this is trivial; suppose that A is not empty. For $n \in \mathbb{N}$, set $a_n = \sup_{u \in A} [\![u > n]\!]$; this is always defined; set $a = \inf_{n\in\mathbb{N}} a_n$. Now $a = 0$. **P?** Otherwise, there must be a $u \in A$ such that $a' = a \cap [\![u > 0]\!] \neq 0$, since $a \subseteq a_0$. But now, for any n, and any $v \in A \setminus \{u\}$,

$$a' \cap [\![v > n]\!] \subseteq [\![u > 0]\!] \cap [\![v > 0]\!] = 0,$$

so that $a' \subseteq [\![u > n]\!]$. As n is arbitrary, $\inf_{n\in\mathbb{N}} [\![u > n]\!] \neq 0$, which is impossible. **XQ**

By 364L(a-i), A is bounded above.

368L Definition A Riesz space U is called **laterally complete** or **universally complete** if A is bounded above whenever $A \subseteq U^+$ is disjoint.

368M Theorem Let U be an Archimedean Riesz space. Then the following are equiveridical:

(i) there is a Dedekind complete Boolean algebra $\mathfrak{A}$ such that U is isomorphic to $L^0(\mathfrak{A})$;

(ii) U is Dedekind σ-complete and laterally complete;

(iii) whenever V is an Archimedean Riesz space, V_0 is an order-dense Riesz subspace of V and $T : V_0 \to U$ is an order-continuous Riesz homomorphism, there is a positive linear operator $\tilde{T} : V \to U$ extending T.

proof (a)(i)⇒(ii) and **(i)⇒(iii)** are covered by 368K and 368B.

(b)(ii)⇒(i) Assume (ii). By 368E, we may suppose that U is actually an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$ for a Dedekind complete Boolean algebra $\mathfrak{A}$.

(α) If $u \in U^+$ and $a \in \mathfrak{A}$ then $u \times \chi a \in U$. **P** Set $A = \{v : v \in U,\ 0 \leq v \leq \chi a\}$, and let $C \subseteq A$ be a maximal disjoint set; then $w = \sup C$ is defined in U, and is also the supremum in L^0. Set $b = [\![w > 0]\!]$. As $w \leq \chi a$, $b \subseteq a$. **?** If $b \neq a$, then $\chi(a \setminus b) > 0$, and there is a $v' \in U$ such that $0 < v' \leq \chi(a \setminus b)$; but now $v' \in A$ and $v' \wedge w = 0$, so $v' \wedge v = 0$ for every $v \in C$, and we ought to have added v' to C. **X** Thus $[\![w > 0]\!] = a$.

Now consider $u' = \sup_{n\in\mathbb{N}} u \wedge nw$; as U is Dedekind σ-complete, $u' \in U$. Since $[\![u' > 0]\!] \subseteq a$, $u' \leq u \times \chi a$. On the other hand,

$$u \times \chi[\![w > \tfrac{1}{n}]\!] \times \chi[\![u \leq n]\!] \leq u \wedge n^2 w \leq u'$$

for every $n \geq 1$, so, taking the supremum over n, $u \times \chi a \leq u'$. Accordingly

$$u \times \chi a = u' \in U,$$

as required. **Q**

(β) If $w \geq 0$ in L^0, there is a $u \in U$ such that $\frac{1}{2}w \leq u \leq w$. **P** Set

$$A = \{u : u \in U,\ 0 \leq u \leq w\},$$

$$C = \{a : a \in \mathfrak{A},\ a \subseteq [\![u - \tfrac{1}{2}w \geq 0]\!] \text{ for some } u \in A\}.$$

Then $\sup A = w$, so C is order-dense in $\mathfrak{A}$. (If $a \in \mathfrak{A} \setminus \{0\}$, either $a \cap [\![w > 0]\!] = 0$ and $a \subseteq [\![0 - \frac{1}{2}w \geq 0]\!]$, so $a \in C$, or there is a $u \in U$ such that $0 < u \leq w \times \chi a$. In the latter case there is some n such that $2^n u \leq w$ and $2^{n+1}u \not\leq w$, and now $c = a \cap [\![2^n u - \frac{1}{2}w \geq 0]\!]$ is a non-zero member of C included in a.) Let $D \subseteq C$ be a partition of unity and for each $d \in D$ choose $u_d \in A$ such that $d \subseteq [\![u_d - \frac{1}{2}w \geq 0]\!]$. By ($\alpha$), $u_d \times \chi d \in U$ for every $d \in D$, so $u = \sup_{d\in D} u_d \times \chi d \in U$. Now $u \leq w$, but also $[\![u - \frac{1}{2}w \geq 0]\!] \supseteq d$ for every $d \in D$, so is equal to 1, and $u \geq \frac{1}{2}w$, as required. **Q**

(γ) Given $w \geq 0$ in L^0, we can therefore choose $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ inductively such that $v_0 = 0$ and

$$u_n \in U, \quad \frac{1}{2}(w - v_n) \leq u_n \leq w - v_n, \quad v_{n+1} = v_n + u_n$$

for every $n \in \mathbb{N}$. Now $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U and $w - v_n \leq 2^{-n}w$ for every n, so $w = \sup_{n\in\mathbb{N}} v_n \in U$.

As w is arbitrary, $(L^0)^+ \subseteq U$ and $U = L^0$ is of the right form.

(c)(iii)⇒(i) As in (b), we may suppose that U is an order-dense Riesz subspace of L^0. But now apply condition (iii) with $V = L^0$, $V_0 = U$ and T the identity operator. There is an extension $\tilde{T} : L^0 \to U$. If $v \geq 0$ in L^0, $\tilde{T}v \geq Tu = u$ whenever $u \in U$ and $u \leq v$, so $\tilde{T}v \geq v$, since $v = \sup\{u : u \in U, 0 \leq u \leq v\}$ in L^0. Similarly, $\tilde{T}(\tilde{T}v - v) \geq \tilde{T}v - v$. But as $\tilde{T}v \in U$, $\tilde{T}(\tilde{T}v) = T(\tilde{T}v) = \tilde{T}v$ and $\tilde{T}(\tilde{T}v - v) = 0$, so $v = \tilde{T}v \in U$. As v is arbitrary, $U = L^0$.

368N Weakly (σ,∞)-distributive Riesz spaces We are now ready to look at the class of Riesz spaces corresponding to the weakly (σ,∞)-distributive Boolean algebras of §316.

Definition Let U be a Riesz space. Then U is **weakly (σ,∞)-distributive** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty downwards-directed subsets of U^+, each with infimum 0, and $\bigcup_{n\in\mathbb{N}} A_n$ has an upper bound in U, then

$$\{u : u \in U, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } v \leq u\}$$

has infimum 0 in U.

Remark Because the definition looks only at sequences $\langle A_n\rangle_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} A_n$ is order-bounded, we can invert it, as follows: a Riesz space U is weakly (σ,∞)-distributive iff whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty upwards-directed subsets of U^+, all with supremum u_0, then

$$\{u : u \in U^+, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } u \leq v\}$$

also has supremum u_0.

368O Lemma Let U be an Archimedean Riesz space. Then the following are equiveridical:

(i) U is not weakly (σ,∞)-distributive;

(ii) there are a $u > 0$ in U and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\sup_{n\in\mathbb{N}} u_n = u$ whenever $u_n \in A_n$ for every $n \in \mathbb{N}$.

proof (ii)⇒(i) is immediate from the definition of 'weakly (σ,∞)-distributive'. For (i)⇒(ii), suppose that U is not weakly (σ,∞)-distributive. Then there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\bigcup_{n\in\mathbb{N}} A_n$ is bounded above, but

$$A = \{w : w \in U, \text{ for every } n \in \mathbb{N} \text{ there is a } v \in A_n \text{ such that } v \leq w\}$$

does not have infimum 0. Let $u > 0$ be a lower bound for A, and set $A'_n = \{u \wedge v : v \in A_n\}$ for each $n \in \mathbb{N}$. Then each A'_n is a non-empty downwards-directed set with infimum 0. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a sequence such that $u_n \in A'_n$ for every n. Express each u_n as $u \wedge v_n$ where $v_n \in A_n$. Let B be the set of upper bounds of $\{v_n : n \in \mathbb{N}\}$. Then $\inf_{w\in B,n\in\mathbb{N}} w - v_n = 0$, because U is Archimedean (353F), while $B \subseteq A$, so $u \leq w$ for every $w \in B$. If u' is any upper bound for $\{u_n : n \in \mathbb{N}\}$, then

$$u - u' \leq u - u \wedge v_n = (u - v_n)^+ \leq (w - v_n)^+ = w - v_n$$

whenever $n \in \mathbb{N}$ and $w \in B$. So $u' \geq u$. Thus $u = \sup_{n\in\mathbb{N}} u_n$. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, u and $\langle A'_n\rangle_{n\in\mathbb{N}}$ witness that (ii) is true.

368P Proposition (a) A regularly embedded Riesz subspace of an Archimedean weakly (σ,∞)-distributive Riesz space is weakly (σ,∞)-distributive.

(b) An Archimedean Riesz space with a weakly (σ,∞)-distributive order-dense Riesz subspace is weakly (σ,∞)-distributive.

(c) If U is a Riesz space such that $U^\times$ separates the points of U, then U is weakly (σ,∞)-distributive; in particular, $U^\sim$ and $U^\times$ are weakly (σ,∞)-distributive for every Riesz space U.

proof (a) Suppose that U is an Archimedean Riesz space and that $V \subseteq U$ is a regularly embedded Riesz subspace which is not weakly (σ,∞)-distributive. Then 368O tells us that there are a $v > 0$ in V and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed subsets of V, all with infimum 0 in V, such that $\sup_{n\in\mathbb{N}} v_n = v$ in V whenever $v_n \in A_n$ for every $n \in \mathbb{N}$. Because V is regularly embedded in U, $\inf A_n = 0$ in U for every n and $\sup_{n\in\mathbb{N}} v_n = v$ in U for every sequence $\langle v_n\rangle_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$, so U is not weakly (σ,∞)-distributive. Turning this round, we have (a).

(b) Let U be an Archimedean Riesz space which is not weakly (σ,∞)-distributive, and V an order-dense Riesz subspace of U. By 368O again, there are a $u^* > 0$ in U and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets in U, all with infimum 0, such that $\sup_{n\in\mathbb{N}} u_n = u^*$ whenever $u_n \in A_n$ for every n. Let $v \in V$ be such that $0 < v \le u^*$. Set

$$B_n = \{w : w \in V, \text{ there is some } u \in A_n \text{ such that } v \wedge u \le w \le v\}$$

for each $n \in \mathbb{N}$. Because A_n is downwards-directed, $w \wedge w' \in B_n$ for all w, $w' \in B_n$; $v \in B_n$, so $B_n \ne \emptyset$; and $\inf B_n = 0$ in V. **P** Setting

$$C = \{w : w \in V^+, \text{ there is some } u \in A_n \text{ such that } w \le (v-u)^+\},$$

then (because V is order-dense) any upper bound for C in U is also an upper bound of $\{(v-u)^+ : u \in A_n\}$. But

$$\sup_{u\in A_n}(v-u)^+ = (v - \inf A_n)^+ = v,$$

so $v = \sup C$ in U and $\inf B_n = \inf\{v - w : w \in C\} = 0$ in U and in V. **Q**

Now if $v_n \in B_n$ for every $n \in \mathbb{N}$, we can choose $u_n \in A_n$ such that $v \wedge u_n \le v_n \le v$ for every n, so that

$$v = v \wedge u^* = v \wedge \sup_{n\in\mathbb{N}} u_n = \sup_{n\in\mathbb{N}} v \wedge u_n \le \sup_{n\in\mathbb{N}} v_n \le v,$$

and $v = \sup_{n\in\mathbb{N}} v_n$. Thus $\langle B_n\rangle_{n\in\mathbb{N}}$ witnesses that V is not weakly (σ,∞)-distributive.

(c) Now suppose that $U^\times$ separates the points of U. In this case U is surely Archimedean (356G). **?** If U is not weakly (σ,∞)-distributive, there are a $u > 0$ in U and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets, all with infimum 0, such that $\sup_{n\in\mathbb{N}} u_n = u$ whenever $u_n \in A_n$ for each n. Take $f \in U^\times$ such that $f(u) \ne 0$; replacing f by $|f|$ if necessary, we may suppose that $f > 0$. Set $\delta = f(u) > 0$. For each $n \in \mathbb{N}$, there is a $u_n \in A_n$ such that $f(u_n) \le 2^{-n-2}\delta$. But in this case $\langle \sup_{i<n} u_i\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum u, so

$$f(u) = \lim_{n\to\infty} f(\sup_{i\le n} u_i) \le \sum_{i=0}^{\infty} f(u_i) \le \tfrac{1}{2}\delta < f(u),$$

which is absurd. **X** Thus U is weakly (σ,∞)-distributive.

For any Riesz space U, U acts on $U^\sim$ as a subspace of $U^{\sim\times}$ (356F); as U surely separates the points of $U^\sim$, so does $U^{\sim\times}$. So $U^\sim$ is weakly (σ,∞)-distributive. Now $U^\times$ is a band in $U^\sim$ (356B), so is regularly embedded, and must also be weakly (σ,∞)-distributive, by (a) above.

368Q Theorem (a) For any Boolean algebra $\mathfrak{A}$, $\mathfrak{A}$ is weakly (σ,∞)-distributive iff $S(\mathfrak{A})$ is weakly (σ,∞)-distributive iff $L^\infty(\mathfrak{A})$ is weakly (σ,∞)-distributive.

(b) For a Dedekind σ-complete Boolean algebra $\mathfrak{A}$, $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is weakly (σ,∞)-distributive.

proof (a)(i) ? Suppose, if possible, that $\mathfrak{A}$ is weakly (σ,∞)-distributive but $S = S(\mathfrak{A})$ is not. By 368O, as usual, we have a $u > 0$ in S and a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of non-empty downwards-directed sets in S, all with infimum 0, such that $u = \sup_{n\in\mathbb{N}} u_n$ whenever $u_n \in A_n$ for every n. Let $\alpha > 0$ be such that $c = [\![u > \alpha]\!] \ne 0$ (361Eg), and consider

$$B_n = \{[\![v > \alpha]\!] : v \in A_n\} \subseteq \mathfrak{A}$$

for each $n \in \mathbb{N}$. Then each B_n is downwards-directed (because A_n is), and $\inf B_n = 0$ in $\mathfrak{A}$ (because if b is a lower bound of B_n, $\alpha\chi b \le v$ for every $v \in A_n$). Because $\mathfrak{A}$ is weakly (σ,∞)-distributive, there must be some $a \in \mathfrak{A}$ such that $a \not\supseteq c$ but there is, for every $n \in \mathbb{N}$, a $b_n \in B_n$ such that $a \supseteq b_n$. Take $v_n \in A_n$ such that $b_n = [\![v_n > \alpha]\!]$, so that

$$v_n \le \alpha\chi 1 \vee \|v_n\|_\infty \chi b_n \le \alpha\chi 1 \vee \|u\|_\infty \chi a.$$

Since $u = \sup_{n\in\mathbb{N}} v_n$, $u \le \alpha\chi 1 \vee \|u\|_\infty \chi a$. But in this case

$$c = [\![u > \alpha]\!] \subseteq a,$$

contradicting the choice of a. **X**

Thus S must be weakly (σ,∞)-distributive if $\mathfrak{A}$ is.

(ii) Now suppose that S is weakly (σ,∞)-distributive, and let $\langle B_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Set $A_n = \{\chi b : b \in B_n\}$ for each n; then $A_n \subseteq S$ is non-empty, downwards-directed and has infimum 0 in S, because $\chi : \mathfrak{A} \to S$ is order-continuous (361Ef). Set

$$A = \{v : v \in S, \text{ for every } n \in \mathbb{N} \text{ there is a } u \in A_n \text{ such that } u \le v\},$$

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in B_n \text{ such that } a \subseteq b\}.$$

? If 0 is not the greatest lower bound of B, take a non-zero lower bound c. Because S is weakly (σ,∞)-distributive, $\inf A = 0$, and there is a $v \in A$ such that $\chi c \not\le v$. Express v as $\sum_{i=0}^n \alpha_i \chi a_i$, where $\langle a_i\rangle_{i\le n}$ is disjoint, and set $a = \sup\{a_i : i \le n,\ \alpha_i \ge 1\}$; then $\chi a \le v$, so $c \not\subseteq a$. For each n there is a $b_n \in B_n$ such that $\chi b_n \le v$. But in this case $b_n \subseteq a$ for each $n \in \mathbb{N}$, so that $a \in B$; which means that c is not a lower bound for B. **X**

Thus $\inf B = 0$ in $\mathfrak{A}$. As $\langle B_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive.

(iii) Thus S is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is. But S is order-dense in $L^\infty = L^\infty(\mathfrak{A})$ (363C), therefore regularly embedded (352Ne), so 368Pa-b tell us that S is weakly (σ,∞)-distributive iff L^∞ is.

(b) In the same way, because S can be regarded as an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$ (364Ja), L^0 is weakly (σ,∞)-distributive iff S is, that is, iff $\mathfrak{A}$ is.

368R Corollary An Archimedean Riesz space is weakly (σ,∞)-distributive iff its band algebra is weakly (σ,∞)-distributive.

proof Let U be an Archimedean Riesz space and $\mathfrak{A}$ its band algebra. By 368E, U is isomorphic to an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$. By 368P, U is weakly (σ,∞)-distributive iff L^0 is; and by 368Qb L^0 is weakly (σ,∞)-distributive iff $\mathfrak{A}$ is.

368S Corollary If $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra, any regularly embedded Riesz subspace (in particular, any solid linear subspace and any order-dense Riesz subspace) of $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive.

proof By 322F, $\mathfrak{A}$ is weakly (σ,∞)-distributive; by 368Qb, $L^0(\mathfrak{A})$ is weakly (σ,∞)-distributive; by 368Pa, any regularly embedded Riesz subspace is weakly (σ,∞)-distributive.

368X Basic exercises (a) Let X be an uncountable set and Σ the countable-cocountable σ-algebra of subsets of X. Show that there is a family $A \subseteq L^0 = L^0(\Sigma)$ such that $u \wedge v = 0$ for all distinct u, $v \in A$ but A has no upper bound in L^0. Show moreover that if $w > 0$ in L^0 then there is an $n \in \mathbb{N}$ such that $nw \ne \sup_{u\in A} u \wedge nw$.

(b) Let U be a linear space, $\mathfrak{A}$ a Dedekind complete Boolean algebra, and $p : U \to L^0 = L^0(\mathfrak{A})$ a function such that $p(u+v) \le p(u) + p(v)$ and $p(\alpha u) = \alpha p(u)$ whenever u, $v \in U$ and $\alpha \ge 0$. Suppose that $V \subseteq U$ is a linear subspace and $T : V \to L^0$ is a linear operator such that $Tv \le p(v)$ for every $v \in V$. Show that there is a linear operator $\tilde T : U \to L^0$, extending T, such that $\tilde T u \le p(u)$ for every $u \in U$. (*Hint*: part A of the proof of 363R.)

(c) Let $\mathfrak{A}$ be any Boolean algebra, and $\widehat{\mathfrak{A}}$ its Dedekind completion (314U). Show that $L^\infty(\widehat{\mathfrak{A}})$ can be identified with the Dedekind completions of $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$.

(d) Explain how to prove 368K from 368A.

(e) Show that any product of weakly (σ,∞)-distributive Riesz spaces is weakly (σ,∞)-distributive.

(f) Let $\mathfrak{A}$ be a Dedekind complete weakly (σ,∞)-distributive Boolean algebra. Show that a set $A \subseteq L^0 = L^0(\mathfrak{A})$ is order-bounded iff $\langle 2^{-n}u_n\rangle_{n\in\mathbb{N}}$ order*-converges to 0 in L^0 whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in A. (*Hint*: use 368A. If $v > 0$ and $v = \sup_{u\in A} v \wedge 2^{-n}u$ for every n, we can find a $w > 0$ and a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A such that $w \le 2^{-n}u_n$ for every n.)

(g) Give a direct proof of 368S, using the ideas of 322F, but not relying on it or on 368Q.

368Y Further exercises (a) (i) Use 364T-364U to show that if X is any compact Hausdorff space then $C(X)$ can be regarded as an order-dense Riesz subspace of $L^0(\mathrm{RO}(X))$, where $\mathrm{RO}(X)$ is the regular open algebra of X. (ii) Use 353N to show that any Archimedean Riesz space with order unit can be embedded as an order-dense Riesz subspace of some $L^0(\mathrm{RO}(X))$. (iii) Let U be an Archimedean Riesz space and $C \subseteq U^+$ a maximal disjoint set, as in part (a) of the proof of 368E. For $e \in C$ let U_e be the solid linear subspace of U generated by e, and let V be the solid linear subspace of U generated by C. Show that V can be embedded as an order-dense Riesz subspace of $\prod_{e\in C} U_e$ and therefore in $\prod_{e\in C} L^0(\mathrm{RO}(X_e)) \cong L^0(\prod_{e\in C}\mathrm{RO}(X_e))$ for a suitable family of regular open algebras $\mathrm{RO}(X_e)$. (iv) Now use 368B to complete a proof of 368E.

(b) Let U be any Archimedean Riesz space. Let $\mathcal{V}$ be the family of pairs (A,B) of non-empty subsets of U such that B is the set of upper bounds of A and A is the set of lower bounds of B. Show that $\mathcal{V}$ can be given the structure of a Dedekind complete Riesz space defined by the formulae

$$(A_1,B_1)+(A_2,B_2)=(A,B) \text{ iff } A_1+A_2\subseteq A,\ B_1+B_2\subseteq B,$$

$$\alpha(A,B)=(\alpha A,\alpha B) \text{ if } \alpha>0,$$

$$(A_1,B_1)\le(A_2,B_2) \text{ iff } A_1\subseteq A_2.$$

Show that $u \mapsto (\left]-\infty,u\right],\left[u,\infty\right[)$ defines an embedding of U as an order-dense Riesz subspace of $\mathcal{V}$, so that $\mathcal{V}$ may be identified with the Dedekind completion of U.

(c) Work through the proof of 364T when X is compact, Hausdorff and extremally disconnected, and show that it is easier than the general case. Hence show that 368Yb can be used to shorten the proof of 368E sketched in 368Ya.

(d) Let U be a Riesz space. Show that the following are equiveridical: (i) U is isomorphic, as Riesz space, to $L^0(\mathfrak{A})$ for some Dedekind σ-complete Boolean algebra $\mathfrak{A}$ (ii) U is Dedekind σ-complete and has a weak order unit and whenever $A \subseteq U^+$ is countable and disjoint then A is bounded above in U.

(e) Let U be a weakly (σ,∞)-distributive Riesz space and V a Riesz subspace of U which is *either* solid *or* order-dense. Show that V is weakly (σ,∞)-distributive.

(f) Show that $C([0,1])$ is not weakly (σ,∞)-distributive. (Compare 316J.)

(g) Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Boolean algebra. Suppose we have a double sequence $\langle a_{ij}\rangle_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in $\mathfrak{A}$ such that $\langle a_{ij}\rangle_{j\in\mathbb{N}}$ order*-converges to a_i in $\mathfrak{A}$ for each i, while $\langle a_i\rangle_{i\in\mathbb{N}}$ order*-converges to a. Show that there is a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle a_{i,n(i)}\rangle_{i\in\mathbb{N}}$ order*-converges to a.

(h) Let U be a weakly (σ,∞)-distributive Riesz space with the countable sup property. Suppose we have an order-bounded double sequence $\langle u_{ij}\rangle_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ in U such that $\langle u_{ij}\rangle_{j\in\mathbb{N}}$ order*-converges to u_i in U for each i, while $\langle u_i\rangle_{i\in\mathbb{N}}$ order*-converges to u. Show that there is a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that $\langle u_{i,n(i)}\rangle_{i\in\mathbb{N}}$ order*-converges to u.

(i) Let $\mathfrak{A}$ be a ccc weakly (σ,∞)-distributive Dedekind complete Boolean algebra. Show that there is a topology on $L^0 = L^0(\mathfrak{A})$ such that the closure of any $A \subseteq L^0$ is precisely the set of order*-limits of sequences in A. (Cf. 367Yk.)

(j) Let U be a weakly (σ,∞)-distributive Riesz space and $f : U \to \mathbb{R}$ a positive linear functional; write f_τ for the component of f in $U^\times$. (i) Show that for any $u \in U^+$ there is an upwards-directed $A \subseteq [0,u]$, with supremum u, such that $f_\tau(u) = \sup_{v\in A} f(v)$. (See 356Xe, 362D.) (ii) Show that if f is strictly positive, so is f_τ. (Compare 391D.)

368 Notes and comments 368A-368B are manifestations of a principle which will reappear in §375: Dedekind complete L^0 spaces are in some sense 'maximal'. If we have an order-dense subspace U of such an L^0, then any Archimedean Riesz space including U as an order-dense subspace can itself be embedded in L^0 (368B). In fact this property characterizes Dedekind complete L^0 spaces (368M). Moreover, any Archimedean Riesz space U can be embedded in this way (368E); by 368C, the L^0 space (though not the embedding) is unique up to isomorphism. If U and V are Archimedean Riesz spaces, each embedded as an order-dense Riesz subspace of a Dedekind complete L^0 space, then any order-continuous Riesz homomorphism from U to V extends uniquely to the L^0 spaces (368B). If one Dedekind complete L^0 space is embedded as an order-dense Riesz subspace of another, they must in fact be the same (368D). Thus we can say that every Archimedean Riesz space U can be extended to a Dedekind complete L^0 space, in a way which respects order-continuous Riesz homomorphisms, and that this extension is maximal, in that U cannot be order-dense in any larger space.

The proof of 368E which I give is long because I am using a bare-hands approach. Alternative methods shift the burdens. For instance, if we take the trouble to develop a direct construction of the 'Dedekind completion' of a Riesz space (368Yb), then we need prove the theorem only for Dedekind complete Riesz spaces. A more substantial aid is the representation theorem for Archimedean Riesz spaces with order unit (353N); I sketch an argument in 368Ya. The drawback to this approach is the proof of Theorem 364T, which seems to be quite as long as the direct proof of 368E which I give here. Of course we need 364T only for compact Hausdorff spaces, which are usefully easier than the general case (364U, 368Yc).

368G is a version of Ogasawara's representation theorem for Archimedean Riesz spaces. Both this and 368F can be regarded as expressions of the principle that an Archimedean Riesz space is 'nearly' a space of functions.

I have remarked before on the parallels between the theories of Boolean algebras and Archimedean Riesz spaces. The notion of 'weak (σ,∞)-distributivity' is one of the more striking correspondences. (Compare, for instance, 316Xi(i) with 368Pa.) What is really important to us, of course, is the fact that the function spaces of measure theory are mostly weakly (σ,∞)-distributive, by 368S. Of course this is easy to prove directly (368Xg), but I think that the argument through 368Q gives a better idea of what is really happening here. Some of the features of 'order*-convergence', as defined in §367, are related to weak (σ,∞)-distributivity (compare 367Yi, 367Yp); in 368Yi I describe a topology which can be thought of as an abstract version of the topology of convergence in measure on the L^0 space of a σ-finite measure algebra (367M).

Version of 23.11.16

369 Banach function spaces

In this section I continue the work of §368 with results which involve measure algebras. The first step is a modification of the basic representation theorem for Archimedean Riesz spaces. If U is any Archimedean Riesz space, it can be represented as a subspace of $L^0 = L^0(\mathfrak{A})$, where $\mathfrak{A}$ is its band algebra (368E); now if $U^\times$ separates the points of U, there is a measure rendering $\mathfrak{A}$ a localizable measure algebra (369A, 369Xa). Moreover, we get a simultaneous representation of $U^\times$ as a subspace of L^0 (369C-369D), the duality between U and $U^\times$ corresponding exactly to the familiar duality between L^p and L^q. In particular, every L-space can be represented as an L^1-space (369E).

Still drawing inspiration from the classical L^p spaces, we have a general theory of 'associated Fatou norms' (369F-369M, 369R). I include notes on the spaces $M^{1,\infty}$, $M^{\infty,1}$ and $M^{1,0}$ (369N-369Q), which will be particularly useful in the next chapter.

369A Theorem Let U be a Riesz space such that $U^\times$ separates the points of U. Then U can be embedded as an order-dense Riesz subspace of $L^0 = L^0(\mathfrak{A})$ for some localizable measure algebra $(\mathfrak{A},\bar{\mu})$.

proof (a) Consider the canonical map $S : U \to U^{\times\times}$. We know that this is a Riesz homomorphism onto an order-dense Riesz subspace of $U^{\times\times}$ (356I). Because $U^\times$ separates the points of U, S is injective. Let $\mathfrak{A}$ be the band algebra of $U^{\times\times}$ and $T : U^{\times\times} \to L^0$ an injective Riesz homomorphism onto an order-dense Riesz subspace V of L^0, as in 368E. The composition $TS : U \to L^0$ is now an injective Riesz homomorphism, so embeds U as a Riesz subspace of L^0, which is order-dense because V is order-dense in L^0 and $TS[U]$ is order-dense in V (352N(c-iii)). Thus all that we need to find is a measure $\bar\mu$ on $\mathfrak{A}$ rendering it a localizable measure algebra.

(b) Note that V is isomorphic, as Riesz space, to $U^{\times\times}$, which is Dedekind complete (356B), so V must be solid in L^0 (353L). Also $V^\times$ must separate the points of V (356Lb).

Let D be the set of those $d \in \mathfrak{A}$ such that the principal ideal $\mathfrak{A}_d$ is measurable in the sense that there is some $\bar\nu$ for which $(\mathfrak{A}_d, \bar\nu)$ is a totally finite measure algebra. Then D is order-dense in $\mathfrak{A}$. **P** Take any non-zero $a \in \mathfrak{A}$. Because V is order-dense, there is a non-zero $v \in V$ such that $v \le \chi a$. Take $h \ge 0$ in $V^\times$ such that $h(v) > 0$. Then there is a v' such that $0 < v' \le v$ and $h(w) > 0$ whenever $0 < w \le v'$ in V (356H). Let $\alpha > 0$ be such that $d = [\![v' > \alpha]\!] \ne 0$. Then $\chi b \le \frac{1}{\alpha} v' \in V$ whenever $b \in \mathfrak{A}_d$. Set $\bar\nu b = h(\chi b) \in [0, \infty[$ for $b \in \mathfrak{A}_d$. Because the map $b \mapsto \chi b : \mathfrak{A} \to L^0$ is additive and order-continuous, the map $b \mapsto \chi b : \mathfrak{A}_d \to V$ also is, and $\bar\nu = h\chi$ must be additive and order-continuous; in particular, $\bar\nu(\sup_{n\in\mathbb{N}} b_n) = \sum_{n=0}^\infty \bar\nu b_n$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_d$. Moreover, if $b \in \mathfrak{A}_d$ is non-zero, then $0 < \alpha\chi b \le v'$, so $\bar\nu b = h(\chi b) > 0$. Thus $(\mathfrak{A}_d, \bar\nu)$ is a totally finite measure algebra, and $d \in D$, while $0 \ne d \subseteq a$. As a is arbitrary, D is order-dense. **Q**

(c) By 313K, there is a partition of unity $C \subseteq D$. For each $c \in C$, let $\bar\nu_c : \mathfrak{A}_c \to [0, \infty[$ be a functional such that $(\mathfrak{A}_c, \bar\nu_c)$ is a totally finite measure algebra. Define $\bar\mu : \mathfrak{A} \to [0, \infty]$ by setting $\bar\mu a = \sum_{c\in C} \bar\nu_c(a \cap c)$ for every $a \in \mathfrak{A}$. Then $(\mathfrak{A}, \bar\mu)$ is a localizable measure algebra. **P** (i) $\bar\mu 0 = \sum_{c\in C} \bar\nu 0 = 0$. (ii) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, then

$$\bar\mu a = \sum_{c\in C} \bar\nu_c(a \cap c) = \sum_{c\in C, n\in\mathbb{N}} \bar\nu_c(a_n \cap c) = \sum_{n=0}^\infty \bar\mu a_n.$$

(iii) If $a \in \mathfrak{A} \setminus \{0\}$, then there is a $c \in C$ such that $a \cap c \ne 0$, so that $\bar\mu a \ge \bar\nu_c(a \cap c) > 0$. Thus $(\mathfrak{A}, \bar\mu)$ is a measure algebra. (iv) Moreover, in (iii), $\bar\mu(a \cap c) = \bar\nu_c(a \cap c)$ is finite. So $(\mathfrak{A}, \bar\mu)$ is semi-finite. (v) $\mathfrak{A}$ is Dedekind complete, being a band algebra (352Q), so $(\mathfrak{A}, \bar\mu)$ is localizable. **Q**

369B Corollary Let U be a Banach lattice with order-continuous norm. Then U can be embedded as an order-dense solid linear subspace of $L^0(\mathfrak{A})$ for some localizable measure algebra $(\mathfrak{A}, \bar\mu)$.

proof By 356Dd, $U^\times = U^*$, which separates the points of U, by the Hahn-Banach theorem (3A5Ae). So 369A tells us that U can be embedded as an order-dense Riesz subspace of an appropriate $L^0(\mathfrak{A})$. But also U is Dedekind complete (354Ee), so its copy in $L^0(\mathfrak{A})$ must be solid, as in 368H.

369C The representation in 369A is complemented by the following result, which is a kind of generalization of 365L and 366Dc.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, and $U \subseteq L^0 = L^0(\mathfrak{A})$ an order-dense Riesz subspace. Set

$$V = \{v : v \in L^0,\ v \times u \in L^1 \text{ for every } u \in U\},$$

writing L^1 for $L^1(\mathfrak{A}, \bar\mu) \subseteq L^0$. Then V is a solid linear subspace of L^0, and we have an order-continuous injective Riesz homomorphism $T : V \to U^\times$ defined by setting

$$(Tv)(u) = \int u \times v \text{ for all } u \in U,\ v \in V.$$

The image of V is order-dense in $U^\times$. If $(\mathfrak{A}, \bar\mu)$ is localizable, then T is surjective, so is a Riesz space isomorphism between V and $U^\times$.

proof (a)(i) Because $\times : L^0 \times L^0 \to L^0$ is bilinear and L^1 is a linear subspace of L^0, V is a linear subspace of L^0. If $u \in U$, $v \in V$, $w \in L^0$ and $|w| \le |v|$, then

$$|w \times u| = |w| \times |u| \le |v| \times |u| = |v \times u| \in L^1;$$

as L^1 is solid, $w \times u \in L^1$; as u is arbitrary, $w \in V$; this shows that V is solid.

(ii) By the definition of V, $(Tv)(u)$ is defined in $\mathbb{R}$ whenever $u \in U$ and $v \in V$. Because $\times$ is bilinear and $\int$ is linear, $Tv : U \to \mathbb{R}$ is linear for every $v \in V$, and T is a linear functional from V to the space of linear operators from U to $\mathbb{R}$.

(iii) If $u \geq 0$ in U and $v \geq 0$ in V, then $u \times v \geq 0$ in L^1 and $(Tv)(u) = \int u \times v \geq 0$. This shows that T is a positive linear operator from V to $U^{\sim}$.

(iv) If $v \geq 0$ in V and $A \subseteq U$ is a non-empty downwards-directed set with infimum 0 in U, then $\inf A = 0$ in L^0, because U is order-dense (352Nb). Consequently $\inf_{u \in A} u \times v = 0$ in L^0 and in L^1 (364B(b-ii), 353Pa), and

$$\inf_{u \in A}(Tv)(u) = \inf_{u \in A} \int u \times v = 0$$

(because $\int$ is order-continuous). As A is arbitrary, Tv is order-continuous. As v is arbitrary, $T[V] \subseteq U^{\times}$.

(v) If $v \in V$ and $u_0 \geq 0$ in U, set $a = [\![v > 0]\!]$. Then $v^+ = v \times \chi a$. Set $A = \{u : u \in U,\, 0 \leq u \leq u_0 \times \chi a\}$. Because U is order-dense in L^0, $u_0 \times \chi a = \sup A$ in L^0. Because $\times$ and $\int$ are order-continuous,

$$\begin{aligned}(Tv)^+(u_0) &\geq \sup_{u \in A}(Tv)(u) = \sup_{u \in A} \int v \times u \\ &= \int v \times u_0 \times \chi a = \int v^+ \times u_0 = T(v^+)(u_0).\end{aligned}$$

As u_0 is arbitrary, $(Tv)^+ \geq Tv^+$. But because T is a positive linear operator, we must have $Tv^+ \geq (Tv)^+$, so that $Tv^+ = (Tv)^+$. As v is arbitrary, T is a Riesz homomorphism.

(vi) Now T is injective. **P** If $v \neq 0$ in V, there is a $u > 0$ in U such that $u \leq |v|$, because U is order-dense. In this case $u \times |v| > 0$ so $\int u \times |v| > 0$. Accordingly $|Tv| = T|v| \neq 0$ and $Tv \neq 0$. **Q**

(b) Putting (a-i) to (a-vi) together, we see that T is an injective Riesz homomorphism from V to $U^{\times}$. All this is easy. The point of the theorem is the fact that $T[V]$ is order-dense in $U^{\times}$.

P Take $h > 0$ in $U^{\times}$. Let U_1 be the solid linear subspace of L^0 generated by U. Then U is an order-dense Riesz subspace of U_1, $h : U \to \mathbb{R}$ is an order-continuous positive linear functional, and $\sup\{h(u) : u \in U,\, 0 \leq u \leq v\}$ is defined in $\mathbb{R}$ for every $v \geq 0$ in U_1; so we have an extension $\tilde{h}$ of h to U_1 such that $\tilde{h} \in U_1^{\times}$ (355F).

Set $S_1 = S(\mathfrak{A}) \cap U_1$; then S_1 is an order-dense Riesz subspace of U_1, because $S(\mathfrak{A})$ is order-dense in L^0 and U_1 is solid in L^0. Note that S_1 is the linear span of $\{\chi c : c \in I\}$, where $I = \{c : c \in \mathfrak{A},\, \chi c \in U_1\}$, and that I is an ideal in $\mathfrak{A}$.

Because $h \neq 0$, $\tilde{h} \neq 0$; there must therefore be a $u_0 \in S_1$ such that $\tilde{h}(u_0) > 0$, and a $d \in I$ such that $\tilde{h}(\chi d) > 0$. For $a \in \mathfrak{A}$, set $\nu a = \tilde{h}\chi(d \cap a)$. Because $\cap$, χ and $\tilde{h}$ are all order-continuous, so is ν, and $\nu : \mathfrak{A} \to \mathbb{R}$ is a non-negative completely additive functional.

By 365Ea, there is a $v \in L^1$ such that

$$\int_a v = \nu a = \tilde{h}\chi(d \cap a)$$

for every $a \in \mathfrak{A}$; of course $v \geq 0$. We have $\int u \times v \leq \tilde{h}(u)$ whenever $u = \chi a$ for $a \in I$, and therefore for every $u \in S_1^+$. If $u \in U^+$, then $A = \{u' : u' \in S_1,\, 0 \leq u' \leq u\}$ is upwards-directed, $\sup A = u$ and

$$\sup_{u' \in A} \int v \times u' \leq \sup_{u' \in A} \tilde{h}(u') = \tilde{h}(u) = h(u)$$

is finite, so $v \times u = \sup_{u \in A'} v \times u'$ belongs to L^1 (365Df) and $\int v \times u \leq h(u)$. As u is arbitrary, $v \in V$ and $Tv \leq h$. At the same time, because $\chi d \in U_1$, there is a $w \in U$ such that $\chi d \leq w$ and

$$(Tv)(w) = \int v \times w \geq \int_d v = \tilde{h}(\chi d) > 0$$

and $Tv > 0$. As h is arbitrary, $T[V]$ is order-dense. **Q**

It follows that T is order-continuous (352Nb again), as can also be easily proved by the argument of (a-iv) above.

(c) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable, that is, that $\mathfrak{A}$ is Dedekind complete. $T^{-1} : T[V] \to V$ is a Riesz space isomorphism, so certainly an order-continuous Riesz homomorphism; because V is a solid linear subspace of L^0, T^{-1} is still an injective order-continuous Riesz homomorphism when regarded as a map from $T[V]$ to L^0. Since $T[V]$ is order-dense in $U^\times$, T^{-1} has an extension to an order-continuous Riesz homomorphism $Q : U^\times \to L^0$ (368B). But $Q[U^\times] \subseteq V$. **P** Take $h \geq 0$ in $U^\times$ and $u \geq 0$ in U. Then $B = \{g : g \in T[V], 0 \leq g \leq h\}$ is upwards-directed and has supremum h. For $g \in B$, we know that $u \times T^{-1}g \in L^1$ and $\int u \times T^{-1}g = g(u)$, by the definition of T. But this means that

$$\sup_{g\in B} \int u \times T^{-1}g = \sup_{g\in B} g(u) = h(u) < \infty.$$

Since $\{u \times T^{-1}g : g \in B\}$ is upwards-directed, it follows that

$$u \times Qh = \sup_{g\in B} u \times Qg = \sup_{g\in B} u \times T^{-1}g \in L^1$$

by 365Df again. As u is arbitrary, $Qh \in V$. As h is arbitrary (and Q is linear), $Q[U^\times] \subseteq V$. **Q**

Also Q is injective. **P** If $h \in U^\times$ is non-zero, there is a $v \in V$ such that $0 < Tv \leq |h|$, so that

$$|Qh| = Q|h| \geq QTv = v > 0$$

and $Qh \neq 0$. **Q** Since QT is the identity on V, Q and T must be the two halves of a Riesz space isomorphism between V and $U^\times$.

369D Corollary Let U be any Riesz space such that $U^\times$ separates the points of U. Then there is a localizable measure algebra $(\mathfrak{A}, \bar\mu)$ such that the pair $(U, U^\times)$ can be represented by a pair (V, W) of order-dense Riesz subspaces of $L^0 = L^0(\mathfrak{A})$ such that $W = \{w : w \in L^0,\ v \times w \in L^1 \text{ for every } v \in V\}$, writing L^1 for $L^1(\mathfrak{A}, \bar\mu)$. In this case, $U^{\times\times}$ becomes represented by $\tilde V = \{v : v \in L^0,\ v \times w \in L^1 \text{ for every } w \in W\} \supseteq V$.

proof Put 369A and 369C together. The construction of 369A finds $(\mathfrak{A}, \bar\mu)$ and an order-dense V which is isomorphic to U, and 369C identifies W with $V^\times$ and $W^\times$ with $\tilde V$. To check that W is order-dense, take any $u > 0$ in L^0. There is a $v \in V$ such that $0 < v \leq u$. There is an $h \in (V^\times)^+$ such that $h(v) > 0$, so there is a $w \in W^+$ such that $w \times v \neq 0$, that is, $w \wedge v \neq 0$. But now $w \wedge v \in W$, because W is solid, and $0 < w \wedge v \leq u$.

Remark Thus the canonical embedding of U in $U^{\times\times}$ (356I) is represented by the embedding $V \subsetneqq \tilde V$; U, or V, is 'perfect' iff $V = \tilde V$.

369E Kakutani's theorem (Kakutani 1941) If U is any L-space, there is a localizable measure algebra $(\mathfrak{A}, \bar\mu)$ such that U is isomorphic, as Banach lattice, to $L^1 = L^1(\mathfrak{A}, \bar\mu)$.

proof U is a perfect Riesz space, and $U^\times = U^*$ has an order unit $\int$ defined by saying that $\int u = \|u\|$ for $u \geq 0$ (356P). By 369D, we can find a localizable measure algebra $(\mathfrak{A}, \bar\mu)$ and an identification of the pair $(U, U^\times)$, as dual Riesz spaces, with a pair (V, W) of subspaces of $L^0 = L^0(\mathfrak{A})$; and V will be $\{v : v \times w \in L^1$ for every $w \in W\}$. But W, like $U^\times$, must have an order unit; call it e. Because W is order-dense, $[\![e > 0]\!]$ must be 1 and e must have a multiplicative inverse $\frac{1}{e}$ in L^0 (364N). This means that V must be $\{v : v \times e \in L^1\}$, so that $v \mapsto v \times e$ is a Riesz space isomorphism between V and L^1, which gives a Riesz space isomorphism between U and L^1. Moreover, if we write $\| \|'$ for the norm on V corresponding to the norm of U, we have

$$\|v\|' = \int |v| \times e = \int |v \times e| = \|v \times e\|_1 \text{ for } v \in V.$$

Thus the Riesz space isomorphism between U and L^1 is norm-preserving, and U and L^1 are isomorphic as Banach lattices.

369F The L^p spaces are leading examples for a general theory of normed subspaces of L^0, which I proceed to sketch in the rest of the section.

Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. An **extended Fatou norm** on $L^0 = L^0(\mathfrak{A})$ is a function $\tau : L^0 \to [0, \infty]$ such that

(i) $\tau(u + v) \leq \tau(u) + \tau(v)$ for all $u, v \in L^0$;

(ii) $\tau(\alpha u) = |\alpha|\tau(u)$ whenever $u \in L^0$ and $\alpha \in \mathbb{R}$ (counting $0 \cdot \infty$ as 0, as usual);

(iii) $\tau(u) \leq \tau(v)$ whenever $|u| \leq |v|$ in L^0;

(iv) $\sup_{u\in A} \tau(u) = \tau(v)$ whenever $A \subseteq (L^0)^+$ is a non-empty upwards-directed set with supremum v in L^0;

(v) $\tau(u) > 0$ for every non-zero $u \in L^0$;

(vi) whenever $u > 0$ in L^0 there is a $v \in L^0$ such that $0 < v \leq u$ and $\tau(v) < \infty$.

369G Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then $L^\tau = \{u : u \in L^0, \tau(u) < \infty\}$ is an order-dense solid linear subspace of L^0, and τ, restricted to L^τ, is a Fatou norm under which L^τ is a Banach lattice. If $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing norm-bounded sequence in $(L^\tau)^+$, then it has a supremum in L^τ; if $\mathfrak{A}$ is Dedekind complete, then L^τ has the Levi property.

proof (a) By (i), (ii) and (iii) of 369F, L^τ is a solid linear subspace of L^0; by (vi), it is order-dense. Hypotheses (i), (ii), (iii) and (v) show that τ is a Riesz norm on L^τ, while (iv) shows that it is a Fatou norm.

(b)(i) Suppose that $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing norm-bounded sequence in $(L^\tau)^+$. Then $u = \sup_{n\in\mathbb{N}} u_n$ is defined in L^0. **P?** Otherwise, there is a $v > 0$ in L^0 such that $kv = \sup_{n\in\mathbb{N}} kv \wedge u_n$ for every $k \in \mathbb{N}$ (368A). By (v)-(vi) of 369F, there is a v' such that $0 < v' \le v$ and $0 < \tau(v') < \infty$. Now $kv' = \sup_{n\in\mathbb{N}} kv' \wedge u_n$ for every k, so

$$k\tau(v') = \tau(kv') = \sup_{n\in\mathbb{N}} \tau(kv' \wedge u_n) \le \sup_{n\in\mathbb{N}} \tau(u_n)$$

for every k, using 369F(iv), and $\sup_{n\in\mathbb{N}} \tau(u_n) = \infty$, contrary to hypothesis. **XQ** By 369F(iv) again, $\tau(u) = \sup_{n\in\mathbb{N}} \tau(u_n) < \infty$, so that $u \in L^\tau$ and $u = \sup_{n\in\mathbb{N}} u_n$ in L^τ.

(ii) It follows that L^τ is complete under τ. **P** Let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a sequence in L^τ such that $\tau(u_{n+1} - u_n) \le 2^{-n}$ for every $n \in \mathbb{N}$. Set $v_{mn} = \sum_{i=m}^{n} |u_{i+1} - u_i|$ for $m \le n$; then $\tau(v_{mn}) \le 2^{-m+1}$ for every n, so by (i) just above $v_m = \sup_{n\in\mathbb{N}} v_{mn}$ is defined in L^τ, and $\tau(v_m) \le 2^{-m+1}$. Now $v_m = |u_{m+1} - u_m| + v_{m+1}$ for each m, so $\langle u_m - v_m \rangle_{m\in\mathbb{N}}$ is non-decreasing and $\langle u_m + v_m \rangle_{m\in\mathbb{N}}$ is non-increasing, while $u_m - v_m \le u_m \le u_m + v_m$ for every m. Accordingly $u = \sup_{m\in\mathbb{N}} u_m - v_m$ is defined in L^τ and $|u - u_m| \le v_m$ for every m. But this means that $\lim_{m\to\infty} \tau(u - u_m) \le \lim_{m\to\infty} \tau(v_m) = 0$ and $u = \lim_{m\to\infty} u_m$ in L^τ. As $\langle u_n \rangle_{n\in\mathbb{N}}$ is arbitrary, L^τ is complete. **Q**

(c) Now suppose that $\mathfrak{A}$ is Dedekind complete and $A \subseteq (L^\tau)^+$ is a non-empty upwards-directed norm-bounded set in L^τ. By the argument of (b-i) above, using the other half of 368A, $\sup A$ is defined in L^0 and belongs to L^τ. As A is arbitrary, L^τ has the Levi property.

369H Associate norms: Definition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Define $\tau' : L^0 \to [0, \infty]$ by setting

$$\tau'(u) = \sup\{\|u \times v\|_1 : v \in L^0, \tau(v) \le 1\}$$

for every $u \in L^0$; then τ' is the **associate** of τ. (The word suggests a symmetric relationship; it is justified by the next theorem.)

369I Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$. Then

(i) its associate τ' is also an extended Fatou norm on L^0;

(ii) τ is the associate of τ';

(iii) $\|u \times v\|_1 \le \tau(u)\tau'(v)$ for all u, $v \in L^0$.

proof (a) Before embarking on the proof that τ' is an extended Fatou seminorm on L^0, I give the greater part of the argument needed to show that $\tau = \tau''$, where

$$\tau''(u) = \sup\{\|u \times w\|_1 : w \in L^0, \tau'(w) \le 1\}$$

for every $u \in L^0$.

(α) Set

$$B = \{u : u \in L^1, \tau(u) \le 1\},$$

writing L^1 for $L^1(\mathfrak{A}, \bar{\mu})$. Then B is a convex set in L^1 and is closed for the norm topology of L^1. **P** Suppose that u belongs to the closure of B in L^1. Then for each $n \in \mathbb{N}$ we can choose $u_n \in B$ such that $\|u - u_n\|_1 \le 2^{-n}$. Set $v_{mn} = \inf_{m\le i\le n} |u_i|$ for $m \le n$, and

$$v_m = \inf_{n\ge m} v_{mn} = \inf_{n\ge m} |u_n| \le |u|$$

for $m \in \mathbb{N}$. The sequence $\langle v_m \rangle_{m\in\mathbb{N}}$ is non-decreasing, $\tau(v_m) \le \tau(u_m) \le 1$ for every m, and

$$\||u| - v_m\|_1 \le \sup_{n\ge m} \||u| - v_{mn}\|_1 \le \sum_{i=m}^{\infty} \||u| - |u_i|\|_1 \le \sum_{i=m}^{\infty} \|u - u_i\|_1 \to 0$$

as $m \to \infty$. So $|u| = \sup_{m\in\mathbb{N}} v_m$ in L^0,

$$\tau(u) = \tau(|u|) = \sup_{m\in\mathbb{N}} \tau(v_m) \le 1$$

and $u \in B$. **Q**

(β) Now take any $u_0 \in L^0$ such that $\tau(u_0) > 1$. Then, writing $\mathfrak{A}^f$ for $\{a : \bar\mu a < \infty\}$,

$$A = \{u : u \in S(\mathfrak{A}^f),\, 0 \le u \le u_0\}$$

is an upwards-directed set with supremum u_0 (this is where I use the hypothesis that $(\mathfrak{A}, \bar\mu)$ is semi-finite, so that $S(\mathfrak{A}^f)$ is order-dense in L^0), and $\sup_{u\in A} \tau(u) = \tau(u_0) > 1$. Take $u_1 \in A$ such that $\tau(u_1) > 1$, that is, $u_1 \notin B$. By the Hahn-Banach theorem (3A5Cc), there is a continuous linear functional $f : L^1 \to \mathbb{R}$ such that $f(u_1) > 1$ but $f(u) \le 1$ for every $u \in B$. Because $(L^1)^* = (L^1)^\sim$ (356Dc), $|f|$ is defined in $(L^1)^*$, and of course

$$|f|(u_1) \ge f(u_1) > 1, \quad |f|(u) = \sup\{f(v) : |v| \le u\} \le 1$$

whenever $u \in B$ and $u \ge 0$. Set $c = [\![u_1 > 0]\!]$, so that $\bar\mu c < \infty$, and define

$$\nu a = |f|(\chi(a \cap c))$$

for every $a \in \mathfrak{A}$. Then ν is a completely additive real-valued functional on $\mathfrak{A}$, so there is a $w \in L^1$ such that $\nu a = \int_a w$ for every $a \in \mathfrak{A}$ (365Ea). Because $\nu a \ge 0$ for every a, $w \ge 0$. Now

$$\int_a w = |f|(\chi a \times \chi c)$$

for every $a \in \mathfrak{A}$, so

$$\int w \times u = |f|(u \times \chi c) \le |f|(u) \le 1$$

for every $u \in S(\mathfrak{A})^+ \cap B$. But if $\tau(v) \le 1$, then

$$A_v = \{u : u \in S(\mathfrak{A})^+ \cap B,\, u \le |v|\}$$

is an upwards-directed set with supremum $|v|$, so that

$$\|w \times v\|_1 = \sup_{u\in A_v} \int w \times u \le 1.$$

Thus $\tau'(w) \le 1$. On the other hand,

$$\|w \times u_0\|_1 \ge \int w \times u_0 \ge \int w \times u_1 = |f|(u_1) > 1,$$

so $\tau''(u_0) > 1$.

(γ) This shows that, for $u \in L^0$,

$$\tau''(u) \le 1 \Longrightarrow \tau(u) \le 1.$$

(c) Now I return to the proof that τ' is an extended Fatou norm. It is easy to check that it satisfies conditions (i)-(iv) of 369F; in effect, these depend only on the fact that $\|\ \|_1$ is an extended Fatou norm. For (v)-(vi), take $v > 0$ in L^0. Then there is a u such that $0 \le u \le v$ and $0 < \tau(u) < \infty$; set $\alpha = 1/\tau(u)$. Then $\tau(2\alpha u) > 1$, so that $\tau''(2\alpha u) > 1$ and there is a $w \in L^0$ such that $\tau'(w) \le 1$ and $\|2\alpha u \times w\|_1 > 1$. But now set $v_1 = v \wedge |w|$; then

$$v \ge v_1 \ge u \wedge |w| > 0,$$

while $\tau'(v_1) < \infty$. Also $v \wedge \alpha u \ne 0$ so

$$\tau'(v) \ge \|v \times \alpha u\|_1 > 0.$$

As v is arbitrary, τ' satisfies 369F(v)-(vi).

(d) Accordingly τ'' also is an extended Fatou norm. Now in (a) I showed that

$$\tau''(u) \le 1 \Longrightarrow \tau(u) \le 1.$$

It follows easily that $\tau(u) \le \tau''(u)$ for every u (since otherwise there would be some α such that

$$\tau''(\alpha u) = \alpha\tau''(u) < 1 < \alpha\tau(u) = \tau(\alpha u)).$$

On the other hand, we surely have

$$\tau(u) \le 1 \Longrightarrow \|u \times v\|_1 \le 1 \text{ whenever } \tau'(v) \le 1 \Longrightarrow \tau''(u) \le 1,$$

so we must also have $\tau''(u) \le \tau(u)$ for every u. Thus $\tau'' = \tau$, as claimed.

(e) Of course we have $\|u \times v\|_1 \le 1$ whenever $\tau(u) \le 1$ and $\tau'(v) \le 1$. It follows easily that $\|u \times v\|_1 \le \tau(u)\tau'(v)$ whenever u, $v \in L^0$ and both $\tau(u)$, $\tau'(v)$ are non-zero. But if one of them is zero, then $u \times v = 0$, because both τ and τ' satisfy (v) of 369F, so the result is trivial.

369J Theorem Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$, with associate θ. Then

$$L^\theta = \{v : v \in L^0,\ u \times v \in L^1(\mathfrak{A}, \bar\mu) \text{ for every } u \in L^\tau\}.$$

proof (a) If $v \in L^\theta$ and $u \in L^\tau$, then $\|u \times v\|_1$ is finite, by 369I(iii), so $u \times v \in L^1 = L^1(\mathfrak{A}, \bar\mu)$.

(b) If $v \notin L^\theta$ then for every $n \in \mathbb{N}$ there is a u_n such that $\tau(u_n) \le 1$ and $\|u_n \times v\|_1 \ge 2^n$. Set $w_n = \sum_{i=0}^n 2^{-i}|u_i|$ for each n. Then $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence and $\tau(w_n) \le 2$ for each n, so $w = \sup_{n\in\mathbb{N}} w_n$ is defined in L^τ, by 369G; now $\int w \times |v| \ge n+1$ for every n, so $w \times v \notin L^1$.

369K Corollary Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra, and τ an extended Fatou norm on $L^0 = L^0(\mathfrak{A})$, with associate θ. Then L^θ may be identified, as normed Riesz space, with $(L^\tau)^\times \subseteq (L^\tau)^*$, and L^τ is a perfect Riesz space.

proof Putting 369J and 369C together, we have an identification between L^θ and $(L^\tau)^\times$. Now 369I tells us that τ is the associate of θ, so that we can identify L^τ with $(L^\theta)^\times$, and L^τ is perfect, as in 369D.

By the definition of θ, we have, for any $v \in L^\theta$,

$$\begin{aligned}\theta(v) &= \sup_{\tau(u)\le 1} \|u \times v\|_1 \\ &= \sup_{\tau(u)\le 1, \|w\|_\infty \le 1} \int u \times v \times w = \sup_{\tau(u)\le 1} \int u \times v,\end{aligned}$$

which is the norm of the linear functional on L^τ corresponding to v.

369L *L^p* I remarked above that the L^p spaces are leading examples for this theory; perhaps I should spell out the details. Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra and $p \in [1, \infty]$. Then $\| \|_p$ is an extended Fatou norm. **P** Conditions (i)-(iii) and (v) of 369F are satisfied just because $L^p = L^p_{\bar\mu}$ is a solid linear subspace of $L^0(\mathfrak{A})$ on which $\| \|_p$ is a Riesz norm, (iv) because $\| \|_p$ is a Fatou norm with the Levi property (363Ba, 365C, 366D), and (vi) because $S(\mathfrak{A}^f)$ is included in L^p and order-dense in $L^0 = L^0(\mathfrak{A})$ (364K). **Q**

As usual, set $q = p/(p-1)$ if $1 < p < \infty$, ∞ if $p = 1$, and 1 if $p = \infty$. Then $\| \|_q$ is the associate extended Fatou norm of $\| \|_p$. **P** By 365Lb and 366C, $\|v\|_q = \sup\{\|u \times v\|_1 : \|u\|_p \le 1\}$ for every $v \in L^q = L^q_{\bar\mu}$. But as L^q is order-dense in L^0,

$$\begin{aligned}\|v\|_q = \sup_{w\in L^q, |w|\le v} \|w\|_q &= \sup\{\int |u| \times |w| : w \in L^q,\ w \le |v|,\ \|u\|_p \le 1\} \\ &= \sup\{\int |u| \times |v| : \|u\|_p \le 1\}\end{aligned}$$

for every $v \in L^0$. **Q**

369M Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and τ an extended Fatou norm on $L^0=L^0(\mathfrak{A})$. Then

(a) the embedding $L^\tau \subsetneqq L^0$ is continuous for the norm topology of L^τ and the topology of convergence in measure on L^0;

(b) $\tau: L^0 \to [0,\infty]$ is lower semi-continuous, that is, all the balls $\{u:\tau(u)\le\gamma\}$ are closed for the topology of convergence in measure;

(c) if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^0 which is order*-convergent to $u\in L^0$ (definition: 367A), then $\tau(u)$ is at most $\liminf_{n\to\infty}\tau(u_n)$.

proof (a) This is a special case of 367O.

(b) Set $B_\gamma=\{u:\tau(u)\le\gamma\}$. If $u\in L^0\setminus B_\gamma$, then

$$A=\{|u|\times\chi a : a\in\mathfrak{A}^f\}$$

is an upwards-directed set with supremum $|u|$, so there is an $a\in\mathfrak{A}^f$ such that $\tau(u\times\chi a)>\gamma$. **?** If u is in the closure of B_γ for the topology of convergence in measure, then for every $k\in\mathbb{N}$ there is a $v_k\in B_\gamma$ such that $\bar{\mu}(a\cap[\![|u-v_k|>2^{-k}]\!])\le 2^{-k}$ (see the formulae in 367L). Set

$$v'_k=|u|\wedge\inf_{i\ge k}|v_i|$$

for each k, and $v^*=\sup_{k\in\mathbb{N}}v'_k$. Then $\tau(v'_k)\le\tau(v_k)\le\gamma$ for each k, and $\langle v_k\rangle_{k\in\mathbb{N}}$ is non-decreasing, so $\tau(v^*)\le\gamma$. But

$$a\cap[\![|u|-v^*>2^{-k}]\!]\subseteq a\cap\sup_{i\ge k}[\![|u-v_i|>2^{-k}]\!]$$

has measure at most $\sum_{i=k}^\infty 2^{-i}$ for each k, so $a\cap[\![|u|-v^*>0]\!]$ must be 0, that is, $|u|\times\chi a\le v^*$ and $\tau(|u|\times\chi a)\le\gamma$; contrary to the choice of a. **X** Thus u cannot belong to the closure of B_γ. As u is arbitrary, B_γ is closed.

(c) If $\langle u_n\rangle_{n\in\mathbb{N}}$ order*-converges to u, it converges in measure (367Ma). If $\gamma>\liminf_{n\to\infty}\tau(u_n)$, there is a subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ in B_γ, and $\tau(u)\le\gamma$, by (b). As γ is arbitary, $\tau(u)\le\liminf_{n\to\infty}\tau(u_n)$.

369N I now turn to another special case which we have already had occasion to consider in other contexts.

Definition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Set

$$M^{\infty,1}_{\bar{\mu}}=M^{\infty,1}(\mathfrak{A},\bar{\mu})=L^1(\mathfrak{A},\bar{\mu})\cap L^\infty(\mathfrak{A}),$$

$$M^{1,\infty}_{\bar{\mu}}=M^{1,\infty}(\mathfrak{A},\bar{\mu})=L^1(\mathfrak{A},\bar{\mu})+L^\infty(\mathfrak{A}),$$

and

$$\|u\|_{\infty,1}=\max(\|u\|_1,\|u\|_\infty)$$

for $u\in L^0(\mathfrak{A})$.

Remark I hope that the notation I have chosen here will not completely overload your short-term memory. The idea is that in $M^{p,q}$ the symbol p is supposed to indicate the 'local' nature of the space, that is, the nature of $u\times\chi a$ where $u\in M^{p,q}$ and $\bar{\mu}a<\infty$, while q indicates the nature of $|u|\wedge\chi 1$ for $u\in M^{p,q}$. Thus $M^{1,\infty}$ is the space of u such that $u\times\chi a\in L^1$ for every $a\in\mathfrak{A}^f$ and $|u|\wedge\chi 1\in L^\infty$; in $M^{1,0}$ we demand further that $|u|\wedge\chi 1\in M^0$ (366F); while in $M^{\infty,1}$ we ask that $|u|\wedge\chi 1\in L^1$ and that $u\times\chi a\in L^\infty$ for every $a\in\mathfrak{A}^f$.

369O Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra.

(a) $\|\ \|_{\infty,1}$ is an extended Fatou norm on $L^0=L^0(\mathfrak{A})$, and the corresponding Banach lattice is $M^{\infty,1}(\mathfrak{A},\bar{\mu})$.

(b) The associate of $\|\ \|_{\infty,1}$ is $\|\ \|_{1,\infty}$, which may be defined by any of the formulae

$$\begin{aligned}\|u\|_{1,\infty} &= \sup\{\|u \times v\|_1 : v \in L^0,\ \|v\|_{\infty,1} \le 1\}\\ &= \min\{\|v\|_1 + \|w\|_\infty : v \in L^1,\ w \in L^\infty,\ v + w = u\}\\ &= \min\{\alpha + \int (|u| - \alpha\chi 1)^+ : \alpha \ge 0\}\\ &= \int_0^\infty \min(1, \bar{\mu}[\![|u| > \alpha]\!])d\alpha\end{aligned}$$

for every $u \in L^0$, writing $L^1 = L^1(\mathfrak{A}, \bar{\mu})$, $L^\infty = L^\infty(\mathfrak{A})$.

(c)

$$\{u : u \in L^0,\ \|u\|_{1,\infty} < \infty\} = M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar{\mu}),$$

$$\{u : u \in L^0,\ \|u\|_{\infty,1} < \infty\} = M^{\infty,1} = M^{\infty,1}(\mathfrak{A}, \bar{\mu}).$$

(d) Writing $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$, $S(\mathfrak{A}^f)$ is norm-dense in $M^{\infty,1}$ and $S(\mathfrak{A})$ is norm-dense in $M^{1,\infty}$.

(e) For any $p \in [1, \infty]$,

$$\|u\|_{1,\infty} \le \|u\|_p \le \|u\|_{\infty,1}$$

for every $u \in L^0$.

Remark By writing 'min' rather than 'inf' in the formulae of part (b) I mean to assert that the infima are attained.

proof (a) This is easy; to see that $\|\ \|_{\infty,1}$ is an extended Fatou norm all we need to know is that $\|\ \|_1$ and $\|\ \|_\infty$ are extended Fatou norms, and work through the clauses of 369F. And obviously

$$M^{\infty,1} = \{u : \|u\|_1 < \infty,\ \|u\|_\infty < \infty\} = \{u : \|u\|_{\infty,1} < \infty\}.$$

(b) We have four functionals on L^0 to look at; let me give them names:

$$\tau_1(u) = \sup\{\|u \times v\|_1 : \|v\|_{\infty,1} \le 1\},$$

$$\tau_2(u) = \inf\{\|u'\|_1 + \|u''\|_\infty : u = u' + u''\},$$

$$\tau_3(u) = \inf_{\alpha \ge 0}(\alpha + \textstyle\int (|u| - \alpha\chi 1)^+),$$

$$\tau_4(u) = \textstyle\int_0^\infty \min(1, \bar{\mu}[\![|u| > \alpha]\!])d\alpha.$$

(I write 'inf' here to avoid the question of attainment for the moment.) Now we have the following.

(i) $\tau_1(u) \le \tau_2(u)$. **P** If $\|v\|_{\infty,1} \le 1$ and $u = u' + u''$, then

$$\|u \times v\|_1 \le \|u' \times v\|_1 + \|u'' \times v\|_1 \le \|u'\|_1\|v\|_\infty + \|u''\|_\infty\|v\|_1 \le \|u'\|_1 + \|u''\|_\infty.$$

Taking the supremum over v and the infinum over u' and u'', $\tau_1(u) \le \tau_2(u)$. **Q**

(ii) $\tau_2(u) \le \tau_4(u)$. **P** If $\tau_4(u) = \infty$ this is trivial. Otherwise, take w such that $\|w\|_\infty \le 1$ and $u = |u| \times w$. Set $\alpha_0 = \inf\{\alpha : \bar{\mu}[\![|u| > \alpha]\!] \le 1\}$, and try

$$u' = w \times (|u| - \alpha_0\chi 1)^+, \quad u'' = w \times (|u| \wedge \alpha_0\chi 1).$$

Then $u = u' + u''$, $|u'| \le (|u| - \alpha_0\chi 1)^+$,

$$\begin{aligned}\|u'\|_1 = \int_0^\infty \bar{\mu}[\![|u'| > \alpha]\!]d\alpha\ &= \int_0^\infty \bar{\mu}[\![|u| > \alpha + \alpha_0]\!]d\alpha\\ = \int_{\alpha_0}^\infty \bar{\mu}[\![|u| > \alpha]\!]d\alpha &= \int_{\alpha_0}^\infty \min(1, \bar{\mu}[\![|u| > \alpha]\!])d\alpha\end{aligned}$$

and

$$\|u''\|_\infty \le \alpha_0 = \textstyle\int_0^{\alpha_0} \min(1, [\![|u| > \alpha]\!])d\alpha,$$

so

$$\tau_2(u) \le \|u'\|_1 + \|u''\|_\infty \le \tau_4(u). \text{ } \mathbf{Q}$$

(iii) $\tau_4(u) \le \tau_3(u)$. **P** For any $\alpha \ge 0$,

$$\begin{aligned}\tau_4(u) &= \int_0^\alpha \min(1, \bar\mu[\![|u| > \beta]\!])d\beta + \int_\alpha^\infty \min(1, \bar\mu[\![|u| > \beta]\!])d\beta \\ &\le \alpha + \int_0^\infty \bar\mu[\![|u| > \alpha + \beta]\!]d\beta \\ &= \alpha + \int_0^\infty \bar\mu[\![(|u| - \alpha\chi 1)^+ > \beta]\!]d\beta = \alpha + \int (|u| - \alpha\chi 1)^+.\end{aligned}$$

Taking the infimum over α, $\tau_4(u) \le \tau_3(u)$. **Q**

(iv) $\tau_3(u) \le \tau_1(u)$.

P$(\boldsymbol{\alpha})$ It is enough to consider the case $0 < \tau_1(u) < \infty$, because if $\tau_1(u) = 0$ then $u = 0$ and evidently $\tau_3(0) = 0$, while if $\tau_1(u) = \infty$ the required inequality is trivial. Furthermore, since $\tau_3(u) = \tau_3(|u|)$ and $\tau_1(u) = \tau_1(|u|)$, it is enough to consider the case $u \ge 0$.

$(\boldsymbol{\beta})$ Note next that if $\bar\mu a < \infty$, then $\|\frac{1}{\max(1,\bar\mu a)}\chi a\|_{\infty,1} \le 1$, so that $\int_a u \le \max(1, \bar\mu a)\tau_1(u)$.

$(\boldsymbol{\gamma})$ Set $c = [\![u > 2\tau_1(u)]\!]$. If $a \subseteq c$ and $\bar\mu a < \infty$, then

$$2\tau_1(u)\bar\mu a \le \int_a u \le \max(1, \bar\mu a)\tau_1(u),$$

so $\bar\mu a \le \frac{1}{2}$. As $(\mathfrak{A}, \bar\mu)$ is semi-finite, it follows that $\bar\mu c \le \frac{1}{2}$ (322Eb).

$(\boldsymbol{\delta})$ I may therefore write

$$\alpha_0 = \inf\{\alpha : \alpha \ge 0, \ \bar\mu[\![u > \alpha]\!] \le 1\}.$$

Now $[\![u > \alpha_0]\!] = \sup_{\alpha > \alpha_0} [\![u > \alpha]\!]$, so

$$\bar\mu[\![u > \alpha_0]\!] = \sup_{\alpha > \alpha_0} \bar\mu[\![u > \alpha]\!] \le 1.$$

$(\boldsymbol{\epsilon})$ If $\alpha \ge \alpha_0$ then

$$(u - \alpha_0\chi 1)^+ \le (\alpha - \alpha_0)\chi[\![u > \alpha_0]\!] + (u - \alpha\chi 1)^+,$$

so

$$\begin{aligned}\alpha_0 + \int (u - \alpha_0\chi 1)^+ &\le \alpha_0 + (\alpha - \alpha_0)\bar\mu[\![u > \alpha_0]\!] + \int (u - \alpha\chi 1)^+ \\ &\le \alpha + \int (u - \alpha\chi 1)^+.\end{aligned}$$

If $0 \le \alpha < \alpha_0$ then, for every $\beta \in [0, \alpha_0 - \alpha[$,

$$(u - \alpha_0\chi 1)^+ + \beta[\![u > \alpha + \beta]\!] \le (u - \alpha\chi 1)^+,$$

while $\bar\mu[\![u > \alpha + \beta]\!] > 1$, so

$$\int (u - \alpha_0\chi 1)^+ + \beta + \alpha \le \alpha + \int (u - \alpha\chi 1)^+;$$

taking the supremum over β,

$$\alpha_0 + \int (u - \alpha_0\chi 1)^+ \le \alpha + \int (u - \alpha\chi 1)^+.$$

Thus $\alpha_0 + \int (u - \alpha_0\chi 1)^+ = \tau_3(u)$.

$(\boldsymbol{\zeta})$ If $\alpha_0 = 0$, take $v = \chi[\![u > 0]\!]$; then $\|v\|_{\infty,1} = \bar\mu[\![u > 0]\!] \le 1$ and

$$\tau_3(u) = \int u = \|u \times v\|_1 \le \tau_1(u).$$

(η) If $\alpha_0 > 0$, set $\gamma = \bar{\mu}[\![u > \alpha_0]\!]$. Take any $\beta \in [0, \alpha_0[$. Then $\bar{\mu}([\![u > \beta]\!] \setminus [\![u > \alpha_0]\!]) > 1 - \gamma$, so there is a $b \subseteq [\![u > \beta]\!] \setminus [\![u > \alpha_0]\!]$ such that $1 - \gamma < \bar{\mu}b < \infty$. Set $v = \chi[\![u > \alpha_0]\!] + \frac{1-\gamma}{\bar{\mu}b}\chi b$. Then $\|v\|_{\infty,1} = 1$ so

$$\tau_1(u) \ge \int u \times v \ge \int (u - \alpha_0\chi 1)^+ + \alpha_0\gamma + \beta\frac{1-\gamma}{\bar{\mu}b}\bar{\mu}b = \tau_3(u) - (1-\gamma)(\alpha_0 - \beta).$$

As β is arbitrary, $\tau_1(u) \ge \tau_3(u)$ in this case also. **Q**

(v) Thus $\tau_1(u) = \tau_2(u) = \tau_3(u) = \tau_4(u)$ for every $u \in L^0$, and I may write $\|u\|_{1,\infty}$ for their common value; being the associate of $\| \|_{\infty,1}$, $\| \|_{1,\infty}$ is an extended Fatou norm. As for the attainment of the infima, the argument of (iv-ϵ) above shows that, at least when $0 < \|u\|_{1,\infty} < \infty$, there is an α_0 such that $\alpha_0 + \int (|u| - \alpha_0)^+ = \|u\|_{1,\infty}$. This omits the cases $\|u\|_{1,\infty} \in \{0, \infty\}$; but in either of these cases we can set $\alpha_0 = 0$ to see that the infimum is attained for trivial reasons. For the other infimum, observe that the argument of (ii) produces u', u'' such that $u = u' + u''$ and $\|u'\|_1 + \|u''\|_\infty \le \tau_4(u)$.

(c) This is now obvious from the definition of $\| \|_{\infty,1}$ and the characterization of $\| \|_{1,\infty}$ in terms of $\| \|_1$ and $\| \|_\infty$.

(d) To see that $S = S(\mathfrak{A})$ is norm-dense in $M^{1,\infty}$, we need only note that S is dense in L^∞ and $S \cap L^1$ is dense in L^1; so that given $v \in L^1$, $w \in L^\infty$ and $\epsilon > 0$ there are v', $w' \in S$ such that

$$\|(v + w) - (v' + w')\|_{1,\infty} \le \|v - v'\|_1 + \|w - w'\|_\infty \le \epsilon.$$

As for $M^{\infty,1}$, if $u \ge 0$ in $M^{\infty,1}$ and $r \in \mathbb{N}$, set $v_r = \sup_{k\in\mathbb{N}} 2^{-r}k\chi[\![u > 2^{-r}k]\!]$; then each v_r belongs to $S^f = S(\mathfrak{A}^f)$ and $\|u - v_r\|_\infty \le 2^{-r}$, while $\langle v_r\rangle_{r\in\mathbb{N}}$ is a non-decreasing sequence with supremum u, so that $\lim_{r\to\infty} \int v_r = \int u$ and $\lim_{r\to\infty} \|u - v_r\|_{\infty,1} = 0$. Thus $(S^f)^+$ is dense in $(M^{\infty,1})^+$. As usual, it follows that $S^f = (S^f)^+ - (S^f)^+$ is dense in $M^{\infty,1} = (M^{\infty,1})^+ - (M^{\infty,1})^+$.

(e)(i) If $p = 1$ or $p = \infty$ this is immediate from the definition of $\| \|_{\infty,1}$ and the characterization of $\| \|_{1,\infty}$ in (b). So suppose henceforth that $1 < p < \infty$.

(ii) If $\|u\|_{\infty,1} \le 1$ then $\|u\|_p \le 1$. **P** Because $\|u\|_\infty \le 1$, $|u|^p \le |u|$, so that $\int |u|^p \le \|u\|_1 \le 1$ and $\|u\|_p \le 1$. **Q**

On considering scalar multiples of u, we see at once that $\|u\|_p \le \|u\|_{\infty,1}$ for every $u \in L^0$.

(ii) Now set $q = p/(p-1)$. Then

$$\|u\|_p = \sup\{\|u \times v\|_1 : \|v\|_q \le 1\}$$

(369L)

$$\ge \sup\{\|u \times v\|_1 : \|v\|_{\infty,1} \le 1\} = \|u\|_{1,\infty}$$

because $\| \|_{1,\infty}$ is the associate of $\| \|_{\infty,1}$. This completes the proof.

369P In preparation for some ideas in §372, I go a little farther with $M^{1,0}$, as defined in 366F.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) $M^{1,0} = M^{1,0}(\mathfrak{A}, \bar{\mu})$ is a norm-closed solid linear subspace of $M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar{\mu})$.

(b) The norm $\| \|_{1,\infty}$ is order-continuous on $M^{1,0}$.

(c) $S(\mathfrak{A}^f)$ and $L^1(\mathfrak{A}, \bar{\mu})$ are norm-dense and order-dense in $M^{1,0}$.

proof (a) Of course $M^{1,0}$, being a solid linear subspace of $L^0 = L^0(\mathfrak{A})$ included in $M^{1,\infty}$, is a solid linear subspace of $M^{1,\infty}$. To see that it is norm-closed, take any point u of its closure. Then for any $\epsilon > 0$ there is a $v \in M^{1,0}$ such that $\|u - v\|_{1,\infty} \le \epsilon$; now $(|u - v| - \epsilon\chi 1)^+ \in L^1 = L^1_{\bar{\mu}}$, so $[\![|u - v| > 2\epsilon]\!]$ has finite measure; also $[\![|v| > \epsilon]\!]$ has finite measure, so

$$[\![|u| > 3\epsilon]\!] \subseteq [\![|u - v| > 2\epsilon]\!] \cup [\![|v| > \epsilon]\!]$$

(364Ea) has finite measure. As ϵ is arbitrary, $u \in M^{1,0}$; as u is arbitrary, $M^{1,0}$ is closed.

(b) Suppose that $A \subseteq M^{1,0}$ is non-empty and downwards-directed and has infimum 0. Let $\epsilon > 0$. Set $B = \{(u - \epsilon\chi 1)^+ : u \in A\}$. Then $B \subseteq L^1$ (by 366Gc); B is non-empty and downwards-directed and has infimum 0. Because $\| \|_1$ is order-continuous (365C), $\inf_{v\in B} \|v\|_1 = 0$ and there is a $u \in A$ such that $\|(u - \epsilon\chi 1)^+\|_1 \le \epsilon$, so that $\|u\|_{1,\infty} \le 2\epsilon$. As ϵ is arbitrary, $\inf_{u\in A} \|u\|_{1,\infty} = 0$; as A is arbitrary, $\| \|_{1,\infty}$ is order-continuous on $M^{1,0}$.

(c) By 366Gb, $S(\mathfrak{A}^f)$ is order-dense in $M^{1,0}$. Because the norm of $M^{1,0}$ is order-continuous, $S(\mathfrak{A}^f)$ is also norm-dense (354Ef). Now $S(\mathfrak{A}^f) \subseteq L^1 \subseteq M^{1,0}$, so L^1 must also be norm-dense and order-dense.

369Q Corollary Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. Set $M^{1,\infty} = M^{1,\infty}(\mathfrak{A}, \bar\mu)$, etc.
(a) $(M^{1,\infty})^\times$ and $(M^{1,0})^\times$ can both be identified with $M^{\infty,1}$.
(b) $(M^{\infty,1})^\times$ can be identified with $M^{1,\infty}$; $M^{1,\infty}$ and $M^{\infty,1}$ are perfect Riesz spaces.

proof Everything is covered by 369O and 369K except the identification of $(M^{1,0})^\times$ with $M^{\infty,1}$. For this I return to 369C. Of course $M^{1,0}$ is order-dense in L^0, because it includes L^1, or otherwise. Setting

$$V = \{v : v \in L^0,\ u \times v \in L^1 \text{ for every } u \in M^{1,0}\},$$

369C identifies V with $(M^{1,0})^\times$. Of course $M^{\infty,1} \subseteq V$ just because $M^{1,0} \subseteq M^{1,\infty}$.

Also $V \subseteq M^{\infty,1}$. **P** Because $L^1 \subseteq M^{1,0}$ and $\| \|_\infty$ is the associate of $\| \|_1$, $V \subseteq L^\infty$. **?** If there is a $v \in V \setminus L^1$, then (because $(\mathfrak{A}, \bar\mu)$ is semi-finite, so that $|v| = \sup_{a\in\mathfrak{A}^f} |v| \times \chi a$) $\sup_{a\in\mathfrak{A}^f} \int_a |v| = \infty$. For each $n \in \mathbb{N}$ choose $a_n \in \mathfrak{A}^f$ such that $\int_{a_n} |v| \ge 4^n$, and set $u = \sup_{n\in\mathbb{N}} 2^{-n}\chi a_n \in M^{1,0}$; then $\int u \times |v| \ge 2^n$ for each n, so $v \notin V$. **X** Thus $V \subseteq L^1$ and $V \subseteq M^{\infty,1}$. **Q**

So $M^{\infty,1} = V$ can be identified with $(M^{1,0})^\times$.

369R The detailed formulae of 369O are of course special to the norms $\| \|_1$, $\| \|_\infty$, but the general phenomenon is not.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra, and τ_1, τ_2 two extended Fatou norms on $L^0 = L^0(\mathfrak{A})$ with associates τ_1', τ_2'. Then we have an extended Fatou norm τ defined by the formula

$$\tau(u) = \min\{\tau_1(v) + \tau_2(w) : v,\ w \in L^0,\ v + w = u\}$$

for every $u \in L^0$, and its associate τ' is given by the formula

$$\tau'(u) = \max(\tau_1'(u), \tau_2'(u))$$

for every $u \in L^0$. Moreover, the corresponding function spaces are

$$L^\tau = L^{\tau_1} + L^{\tau_2}, \quad L^{\tau'} = L^{\tau_1'} \cap L^{\tau_2'}.$$

proof (a) For the moment, define τ by setting

$$\tau(u) = \inf\{\tau_1(v) + \tau_2(w) : v + w = u\}$$

for $u \in L^0$. It is easy to check that, for u, $u' \in L^0$ and $\alpha \in \mathbb{R}$,

$$\tau(u + u') \le \tau(u) + \tau(u'), \quad \tau(\alpha u) = |\alpha|\tau(u), \quad \tau(u) \le \tau(u') \text{ if } |u| \le |u'|.$$

(For the last, remember that in this case $u = u' \times z$ where $\|z\|_\infty \le 1$.) Note also that if $u \ge 0$ then $\tau(u) = \inf\{\tau_1(v) + \tau_2(u - v) : 0 \le v \le u\}$.

(b) Take any non-empty, upwards-directed set $A \subseteq (L^0)^+$, with supremum u_0. Suppose that $\gamma = \sup_{u\in A} \tau(u) < \infty$. For $u \in A$ and $n \in \mathbb{N}$ set

$$C_{un} = \{v : v \in L^0,\ 0 \le v \le u_0,\ \tau_1(v) + \tau_2(u - v)^+ \le \gamma + 2^{-n}\}.$$

(i) Every C_{un} is non-empty (because $\tau(u) \le \gamma$).

(ii) Every C_{un} is convex (because if v_1, $v_2 \in C_{un}$ and $\alpha \in [0, 1]$ and $v = \alpha v_1 + (1 - \alpha)v_2$, then

$$(u - v)^+ = (\alpha(u - v_1) + (1 - \alpha)(u - v_2))^+ \le \alpha(u - v_1)^+ + (1 - \alpha)(u - v_2)^+,$$

so

$$\begin{aligned}\tau_1(v) + \tau_2(u - v)^+ &\le \alpha\tau_1(v_1) + (1 - \alpha)\tau_1(v_2) + \alpha\tau_2(u - v_1)^+ + (1 - \alpha)\tau_2(u - v_2)^+ \\ &\le \gamma + 2^{-n}).\end{aligned}$$

(iii) if u, $u' \in A$, m, $n \in \mathbb{N}$, $u \le u'$ and $m \le n$ then $C_{u'n} \subseteq C_{um}$.

(iv) Every C_{un} is closed for the topology of convergence in measure. **P?** Suppose otherwise. Then we can find a v in the closure of C_{un} for the topology of convergence in measure, but such that $\tau_1(v)+\tau_2(u-v)^+ > \gamma + 2^{-n}$. In this case

$$\tau_1(v) = \sup\{\tau_1(v \times \chi a) : a \in \mathfrak{A}^f\}, \quad \tau_2(u-v)^+ = \sup\{\tau_2((u-v)^+ \times \chi a) : a \in \mathfrak{A}^f\},$$

so there is an $a \in \mathfrak{A}^f$ such that

$$\tau_1(v \times \chi a) + \tau_2((u-v)^+ \times \chi a) > \gamma + 2^{-n}.$$

Now there is a sequence $\langle v_k \rangle_{k\in\mathbb{N}}$ in C_{un} such that $\bar\mu(a \cap [\![|v - v_k| \ge 2^{-k}]\!]) \le 2^{-k}$ for every k. Setting

$$v'_k = \inf_{i\ge k} v_i, \quad w_k = \inf_{i\ge k}(u - v_i)^+$$

we have

$$\tau_1(v'_k) + \tau_2(w_k) \le \tau_1(v_k) + \tau_2(u - v_k)^+ \le \gamma + 2^{-n}$$

for each k, and $\langle v'_k \rangle_{k\in\mathbb{N}}$, $\langle w_k \rangle_{k\in\mathbb{N}}$ are non-decreasing. So setting $v^* = \sup_{k\in\mathbb{N}} v \wedge v'_k$, $w^* = \sup_{k\in\mathbb{N}}(u-v)^+ \wedge w_k$, we get

$$\tau_1(v^*) + \tau_2(w^*) \le \gamma + 2^{-n}.$$

But $v^* \ge v \times \chi a$ and $w^* \ge (u - v)^+ \times \chi a$, so

$$\tau_1(v \times \chi a) + \tau_2((u - v)^+ \times \chi a) \le \gamma + 2^{-n},$$

contrary to the choice of a. **XQ**

(v) If $a \in \mathfrak{A} \setminus \{0\}$, there is a non-zero $b \subseteq a$ such that $\bar\mu b < \infty$ and $b \subseteq [\![u_0 \le \alpha]\!]$ for some $\alpha > 0$. Take any $u \in A$; then $\sup_{v\in C_{u0}} \int_b |v|$ is finite.

(c) Thus $\{C_{un} : u \in A,\ n \in \mathbb{N}\}$ satisfies all the conditions of 367V, and $\bigcap_{u\in A,n\in\mathbb{N}} C_{un}$ is non-empty. If v belongs to the intersection, then

$$\tau_1(v) + \tau_2(u - v)^+ \le \gamma$$

for every $u \in A$; since $\{(u - v)^+ : u \in A\}$ is an upwards-directed set with supremum $(u_0 - v)^+$, and τ_2 is an extended Fatou norm,

$$\tau_1(v) + \tau_2(u_0 - v)^+ \le \gamma.$$

(d) This shows both that the infimum in the definition of $\tau(u)$ is always attained (since this is trivial if $\tau(u) = \infty$, and otherwise we consider $A = \{|u|\}$), and also that $\tau(\sup A) = \sup_{u\in A} \tau(u)$ whenever $A \subseteq (L^0)^+$ is a non-empty upwards-directed set with a supremum. Thus τ satisfies conditions (i)-(iv) of 369F. Condition (vi) there is trivial, since (for instance) $\tau(v) \le \tau_1(v)$ for every v. As for 369F(v), suppose that $u > 0$ in L^0. Take u_1 such that $0 < u_1 \le u$ and $\tau'_1(u_1) \le 1$, and u_2 such that $0 < u_2 \le u_1$ and $\tau'_2(u_2) \le 1$. In this case, if $u_2 = v + w$, we must have

$$\tau_1(v) + \tau_2(w) \ge \|v \times u_1\|_1 + \|w \times u_2\|_1 \ge \|u_2 \times u_2\|_1;$$

so that

$$\tau(u) \ge \|u_2 \times u_2\|_1 > 0.$$

Thus all the conditions of 369F are satisfied, and τ is an extended Fatou norm on L^0.

(e) The calculation of τ' is now very easy. Since surely we have $\tau \le \tau_i$ for both i, we must have $\tau' \ge \tau'_i$ for both i. On the other hand, if u, $z \in L^0$, then there are v, w such that $u = v+w$ and $\tau(u) = \tau_1(v)+\tau_2(w)$, so that

$$\|u \times z\|_1 \le \|v \times z\|_1 + \|w \times z\|_1 \le \tau_1(v)\tau'_1(z) + \tau_2(w)\tau'_2(z) \le \tau(u)\max(\tau'_1(z), \tau'_2(z));$$

as u is arbitrary, $\tau'(z) \le \max(\tau'_1(z), \tau'_2(z))$. So $\tau' = \max(\tau'_1, \tau'_2)$, as claimed.

(f) Finally, it is obvious that

$$L^{\tau'} = \{z : \tau'(z) < \infty\} = \{z : \tau'_1(z) < \infty,\ \tau'_2(z) < \infty\} = L^{\tau'_1} \cap L^{\tau'_2},$$

while the fact that the infimum in the definition of τ is always attained means that $L^\tau \subseteq L^{\tau_1} + L^{\tau_2}$, so that we have equality here also.

369X Basic exercises >(a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that the following are equiveridical: (i) there is a function $\bar\mu$ such that $(\mathfrak{A}, \bar\mu)$ is a semi-finite measure algebra; (ii) $(L^\infty)^\times$ separates the points of $L^\infty = L^\infty(\mathfrak{A})$; (iii) for every non-zero $a \in \mathfrak{A}$ there is a completely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu a \neq 0$; (iv) there is some order-dense Riesz subspace U of $L^0 = L^0(\mathfrak{A})$ such that $U^\times$ separates the points of U; (v) for every order-dense Riesz subspace U of L^0 there is an order-dense Riesz subspace V of U such that $V^\times$ separates the points of V.

(b) Let us say that a function $\phi : \mathbb{R} \to \left]-\infty, \infty\right]$ is **convex** if $\phi(\alpha s + (1-\alpha)t) \le \alpha\phi(s) + (1-\alpha)\phi(t)$ for all s, $t \in \mathbb{R}$ and $\alpha \in [0,1]$, interpreting $0 \cdot \infty$ as 0, as usual. For any convex function $\phi : \mathbb{R} \to \left]-\infty, \infty\right]$ which is not always infinite, set $\phi^*(t) = \sup_{s\in\mathbb{R}} st - \phi(s)$ for every $t \in \mathbb{R}$. (i) Show that $\phi^* : \mathbb{R} \to \left]-\infty, \infty\right]$ is convex and lower semi-continuous and not always infinite. (*Hint*: 233Xh.) (ii) Show that if ϕ is lower semi-continuous then $\phi = \phi^{**}$. (*Hint*: It is easy to check that $\phi^{**} \le \phi$. For the reverse inequality, consider first the case $\phi(t) = \alpha t + \beta$, and use 233Ha).)

>(c) For the purposes of this exercise and the next, say that a **Young's function** is a non-negative non-constant lower semi-continuous convex function $\phi : [0, \infty[\to [0, \infty]$ such that $\phi(0) = 0$ and $\phi(t)$ is finite for some $t > 0$. (**Warning!** the phrase 'Young's function' has other meanings.) (i) Show that in this case ϕ is non-decreasing and continuous on the left and ϕ^*, defined by saying that $\phi^*(t) = \sup_{s\ge0} st - \phi(s)$ for every $t \ge 0$, is again a Young's function. (ii) Show that $\phi^{**} = \phi$. Say that ϕ and ϕ^* are **complementary**. (iii) Compute ϕ^* in the cases (α) $\phi(t) = t$ (β) $\phi(t) = \max(0, t-1)$ (γ) $\phi(t) = t^2$ (δ) $\phi(t) = t^p$ where $1 < p < \infty$.

>(d) Let ϕ, $\psi = \phi^*$ be complementary Young's functions in the sense of 369Xc, and $(\mathfrak{A}, \bar\mu)$ a semi-finite measure algebra. Set

$$B = \{u : u \in L^0, \int \bar\phi(|u|) \le 1\}, \quad C = \{v : v \in L^0, \int \bar\psi(|v|) \le 1\}.$$

(For finite-valued ϕ, $\bar\phi : (L^0)^+ \to L^0$ is given by 364H. Devise an appropriate convention for the case in which ϕ takes the value ∞.) (i) Show that B and C are order-closed solid convex sets, and that $\int |u \times v| \le 2$ for all $u \in B$, $v \in C$. (*Hint*: for 'order-closed', use 364Xg(iv).) (ii) Show that there is a unique extended Fatou norm τ_ϕ on L^0 for which B is the unit ball. (iii) Show that if $u \in L^0 \setminus B$ there is a $v \in C$ such that $\int |u \times v| > 1$. (*Hint*: start with the case in which $u \in S(\mathfrak{A})^+$.) (iv) Show that $\tau_\psi \le \tau'_\phi \le 2\tau_\psi$, where τ_ψ is the extended Fatou norm corresponding to ψ and τ'_ϕ is the associate of τ_ϕ, so that τ_ψ and τ'_ϕ can be interpreted as equivalent norms on the same Banach space.

(U and V are complementary **Orlicz spaces**; I will call τ_ϕ, τ_ψ **Orlicz norms**.)

(e) Let U be a Riesz space such that $U^\times$ separates the points of U, and suppose that $\| \|$ is a Fatou norm on U. (i) Show that there is a localizable measure algebra $(\mathfrak{A}, \bar\mu)$ with an extended Fatou norm τ on $L^0(\mathfrak{A})$ such that U can be identified, as normed Riesz space, with an order-dense Riesz subspace of L^τ. (ii) Hence, or otherwise, show that $\|u\| = \sup_{f\in U^\times, \|f\|\le1} |f(u)|$ for every $u \in U$. (iii) Show that if U is Dedekind complete and has the Levi property, then U becomes identified with L^τ itself, and in particular is a Banach lattice (cf. 354Xn).

(f) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, and τ an extended Fatou norm on $L^0(\mathfrak{A})$. Show that the norm of L^τ is order-continuous iff the norm topology of L^τ agrees with the topology of convergence in measure on any order-bounded subset of L^τ.

(g) Let $(\mathfrak{A}, \bar\mu)$ be a σ-finite measure algebra of countable Maharam type, and τ an extended Fatou norm on $L^0(\mathfrak{A})$ such that the norm of L^τ is order-continuous. Show that L^τ is separable in its norm topology.

(h) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be semi-finite measure algebras, and $\pi : \mathfrak{A}^f \to \mathfrak{B}^f$ a measure-preserving ring homomorphism, as in 366H, with associated maps $T : M^0_{\bar\mu} \to M^0_{\bar\nu}$ and $P : M^{1,0}_{\bar\nu} \to M^{1,0}_{\bar\mu}$. Show that $\|Tu\|_{\infty,1} = \|u\|_{\infty,1}$ for every $u \in M^{\infty,1}_{\bar\mu}$ and $\|Pv\|_{\infty,1} \le \|v\|_{\infty,1}$ for every $v \in M^{\infty,1}_{\bar\nu}$.

(i) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism. (i) Show that there is a unique Riesz homomorphism $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$ such that $T(\chi a)=\chi(\pi a)$ for every $a\in\mathfrak{A}$ and $\|Tu\|_{1,\infty}=\|u\|_{1,\infty}$ for every $u\in M^{1,\infty}_{\bar\mu}$. (ii) Now suppose that $(\mathfrak{A},\bar\mu)$ is localizable and π is order-continuous. Show that there is a unique positive linear operator $P:M^{1,\infty}_{\bar\nu}\to M^{1,\infty}_{\bar\mu}$ such that $\int_a Pv=\int_{\pi a}v$ for every $a\in\mathfrak{A}^f$ and $v\in M^{1,\infty}_{\bar\nu}$, and that $\|Pv\|_\infty\le\|v\|_\infty$ for every $v\in L^\infty(\mathfrak{B})$, $\|Pv\|_{\infty,1}\le\|v\|_{\infty,1}$ for every $v\in M^{\infty,1}_{\bar\nu}$, $\|Pv\|_{1,\infty}\le\|v\|_{1,\infty}$ for every $v\in M^{1,\infty}_{\bar\nu}$. (Compare 365O.)

(j) Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra. Show that $\|u\|_{1,\infty}=\max\{\int_a|u|:a\in\mathfrak{A},\,\bar\mu a\le 1\}$ for every $u\in L^0(\mathfrak{A})$. (*Hint*: take $a\supseteq[\![|u|>\alpha_0]\!]$ in part (b-iv) of the proof of 369O.)

(k) Let $(\mathfrak{A},\bar\mu)$ be any semi-finite measure algebra. Show that if τ_ϕ is any Orlicz norm on $L^0=L^0(\mathfrak{A})$, then there is a $\gamma>0$ such that $\|u\|_{1,\infty}\le\gamma\tau_\phi(u)\le\gamma^2\|u\|_{\infty,1}$ for every $u\in L^0$, so that $M^{\infty,1}_{\bar\mu}\subseteq L^{\tau_\phi}\subseteq M^{1,\infty}_{\bar\mu}$.

(l) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra. Show that the subspaces $M^{1,\infty}_{\bar\mu}$, $M^{\infty,1}_{\bar\mu}$ of $L^0(\mathfrak{A})$ can be expressed as a complementary pair of Orlicz spaces, and that the norm $\|\;\|_{\infty,1}$ can be represented as an Orlicz norm, but that if $\mathfrak{A}$ is atomless and $\bar\mu$ is not totally finite. $\|\;\|_{1,\infty}$ cannot be represented as an Orlicz norm.

>(m) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and U a Banach space. (i) Suppose that $\nu:\mathfrak{A}\to U$ is an additive function such that $\|\nu a\|\le\min(1,\bar\mu a)$ for every $a\in\mathfrak{A}$. Show that there is a unique bounded linear operator $T:M^{1,\infty}_{\bar\mu}\to U$ such that $T(\chi a)=\nu a$ for every $a\in\mathfrak{A}$. (ii) Suppose that $\nu:\mathfrak{A}^f\to U$ is an additive function such that $\|\nu a\|\le\max(1,\bar\mu a)$ for every $a\in\mathfrak{A}^f$. Show that there is a unique bounded linear operator $T:M^{\infty,1}_{\bar\mu}\to U$ such that $T(\chi a)=\nu a$ for every $a\in\mathfrak{A}^f$.

(n) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, and $\phi:[0,\infty[\,\to[0,\infty]$ a Young's function; write τ_ϕ for the corresponding Orlicz norm on either $L^0(\mathfrak{A})$ or $L^0(\mathfrak{B})$. Let $\pi:\mathfrak{A}\to\mathfrak{B}$ be a measure-preserving Boolean homomorphism, with associated map $T:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\nu}$, as in 369Xi. (i) Show that $\tau_\phi(Tu)=\tau_\phi(u)$ for every $u\in M^{1,\infty}_{\bar\mu}$. (ii) Show that if $(\mathfrak{A},\bar\mu)$ is localizable, π is order-continuous and $P:M^{1,\infty}_{\bar\nu}\to M^{1,\infty}_{\bar\mu}$ is the map of 369Xi(ii), then $\tau_\phi(Pv)\le\tau_\phi(v)$ for every $v\in M^{1,\infty}_{\bar\nu}$. (*Hint*: 365Q.)

>(o) Let $(\mathfrak{A},\bar\mu)$ be any semi-finite measure algebra and τ_1, τ_2 two extended Fatou norms on $L^0(\mathfrak{A})$. Show that $u\mapsto\max(\tau_1(u),\tau_2(u))$ is an extended Fatou norm.

(p) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and $(\widehat{\mathfrak{A}},\tilde\mu)$ its localization (322Q). Show that the Dedekind completion of $M^{1,\infty}(\mathfrak{A},\bar\mu)$ can be identified with $M^{1,\infty}(\widehat{\mathfrak{A}},\tilde\mu)$.

(q) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra. (i) Show that if $\mathfrak{B}$ is any closed subalgebra of $\mathfrak{A}$ such that $\sup\{b:b\in\mathfrak{B},\,\bar\mu b<\infty\}=1$ in $\mathfrak{A}$, we have an order-continuous positive linear operator $P_{\mathfrak{B}}:M^{1,\infty}_{\bar\mu}\to M^{1,\infty}_{\bar\mu\restriction\mathfrak{B}}$ such that $\int_b P_{\mathfrak{B}}u=\int_b u$ whenever $u\in M^{1,\infty}_{\bar\mu}$, $b\in\mathfrak{B}$ and $\bar\mu b<\infty$. (ii) Show that if $\langle\mathfrak{B}_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of closed subalgebras of $\mathfrak{A}$ such that $\sup\{b:b\in\mathfrak{B}_0,\,\bar\mu b<\infty\}=1$ in $\mathfrak{A}$, and $\mathfrak{B}$ is the closure of $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$, then $\langle P_{\mathfrak{B}_n}u\rangle_{n\in\mathbb{N}}$ is order*-convergent to $P_{\mathfrak{B}}u$ for every $u\in M^{1,\infty}_{\bar\mu}$. (Cf. 367J.)

(r) Let ϕ_1 and ϕ_2 be Young's functions and $(\mathfrak{A},\bar\mu)$ a semi-finite measure algebra. Set $\phi(t)=\max(\phi_1(t),\phi_2(t))$ for $t\in[0,\infty[$. (i) Show that ϕ is a Young's function. (ii) Writing τ_{ϕ_1} τ_{ϕ_2}, τ_ϕ for the corresponding extended Fatou norms on $L^0(\mathfrak{A})$ (369Xd), show that $\tau_\phi\ge\max(\tau_{\phi_1},\tau_{\phi_2})\ge\frac{1}{2}\tau_\phi$, so that $L^{\tau_\phi}=L^{\tau_{\phi_1}}\cap L^{\tau_{\phi_2}}$ and $L^{\tau_{\phi^*}}=L^{\tau_{\phi_1^*}}+L^{\tau_{\phi_2^*}}$, writing ϕ^* for the Young's function complementary to ϕ. (iii) Repeat with $\psi=\phi_1+\phi_2$ in place of ϕ.

369Y Further exercises (a) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $A\subseteq L^0=L^0(\mathfrak{A})$ a countable set. Show that the solid linear subspace U of L^0 generated by A is a perfect Riesz space. (*Hint*: reduce to the case in which U is order-dense. If $A=\{u_n:n\in\mathbb{N}\}$, $w\in(L^0)^+\setminus U$ find $v_n\in(L^0)^+$ such that $\int v_n\times w\ge 2^n\ge 4^n\int v_n\times|u_i|$ for every $i\le n$. Show that $v=\sup_{n\in\mathbb{N}}v_n$ is defined in L^0 and corresponds to a member of $U^\times$.)

(b) Let U be a Banach lattice and suppose that $p \in [1,\infty[$ is such that $\|u+v\|^p = \|u\|^p + \|v\|^p$ whenever $u, v \in U$ and $|u| \wedge |v| = 0$. Show that U is isomorphic, as Banach lattice, to $L^p_{\bar\mu}$ for some localizable measure algebra $(\mathfrak{A},\bar\mu)$. (*Hint*: start by using 354Yb to show that the norm of U is order-continuous, as in 354Yk.)

(c) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra, and τ an Orlicz norm on $L^0(\mathfrak{A})$. Show that L^τ has the Levi property, whether or not $\mathfrak{A}$ is Dedekind complete.

(d) Let $\phi : [0,\infty[\to [0,\infty[$ be a strictly increasing Young's function such that $\sup_{t>0} \phi(2t)/\phi(t)$ is finite. Show that the associated Orlicz norms τ_ϕ are always order-continuous on their function spaces.

(e) Let $\phi : [0,\infty[\to [0,\infty]$ be a Young's function, and suppose that the corresponding Orlicz norm on $L^0(\mathfrak{A})$, where $(\mathfrak{A},\bar\mu)$ is an atomless measure algebra which is not totally finite, is order-continuous on its function space L^{τ_ϕ}. Show that there is an $M \ge 0$ such that $\phi(2t) \le M\phi(t)$ for every $t \ge 0$.

(f) Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra and $\phi : [0,\infty[\to [0,\infty[$ be a strictly increasing Young's function such that $\sup_{t>0} \phi(2t)/\phi(t)$ is finite. Show that if $\mathcal{F}$ is a filter on L^{τ_ϕ}, then $\mathcal{F} \to u \in L^{\tau_\phi}$ for the norm τ_ϕ iff (i) $\mathcal{F} \to u$ for the topology of convergence in measure (ii) $\limsup_{v\to\mathcal{F}} \tau_\phi(v) \le \tau_\phi(u)$. (Compare 245Xl.)

(g) Give examples of extended Fatou norms τ on measure spaces $L^0(\mathfrak{A})$, where $(\mathfrak{A},\bar\mu)$ is a semi-finite measure algebra, such that (α) $\tau{\restriction}L^\tau$ is order-continuous (β) there is a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in L^τ, converging in measure to $u \in L^\tau$, such that $\lim_{n\to\infty} \tau(u_n) = \tau(u)$ but $\langle u_n\rangle_{n\in\mathbb{N}}$ does not converge to u for the norm on L^τ. Do this (i) with τ an Orlicz norm (ii) with $(\mathfrak{A},\bar\mu)$ the measure algebra of Lebesgue measure on $[0,1]$.

(h) Let $(\mathfrak{A},\bar\mu)$ be any measure algebra. Show that $(M^{1,0}_{\bar\mu})^\times$ can be identified with $M^{\infty,1}_{\bar\mu}$. (*Hint*: show that neither $M^{1,0}$ nor $M^{\infty,1}$ is changed by moving first to the semi-finite version of $(\mathfrak{A},\bar\mu)$, as described in 322Xa, and then to its localization.)

(i) Give an example to show that the result of 369R may fail if $(\mathfrak{A},\bar\mu)$ is only semi-finite, not localizable.

369 Notes and comments The representation theorems 369A-369D give a concrete form to the notion of 'perfect' Riesz space: it is just one which can be expressed as a subspace of $L^0(\mathfrak{A})$, for some localizable measure algebra $(\mathfrak{A},\bar\mu)$, in such a way that it is its own second dual, where the duality here is between subspaces of L^0, taking $V = \{v : u\times v \in L^1 \text{ for every } u \in U\}$. (I see that in this expression I ought somewhere to mention that both U and V are assumed to be order-dense in L^0.) Indeed I believe that the original perfect spaces were the 'vollkommene Räume' of G.Köthe, which were subspaces of $\mathbb{R}^{\mathbb{N}}$, corresponding to the measure algebra $\mathcal{P}\mathbb{N}$ with counting measure, so that V or $U^\times$ was $\{v : u \times v \in \ell^1 \text{ for every } u \in U\}$.

I have presented Kakutani's theorem on the representation of L-spaces as a corollary of 369A and 369C. As usual in such things, this is a reversal of the historical relationship; Kakutani's theorem was one of the results which led to the general theory. The complete list of localizable measure algebras provided by Maharam's theorem (332B, 332J) now gives us a complete list of L-spaces.

Just as perfect Riesz spaces come in dual pairs, so do some of the most important Banach lattices: those with Fatou norms and the Levi property for which the order-continuous dual separates the points. (Note that the dual of any space with a Riesz norm has these properties; see 356Da.) I leave the details of representing such spaces to you (369Xe). The machinery of 369F-369K gives a solid basis for studying such pairs.

Among the extended Fatou norms of 369F the Orlicz norms (369Xd, 369Yd-369Yf) form a significant subfamily. Because they are defined in a way which is to some extent independent of the measure algebra involved, these spaces have some of the same properties as L^p spaces in relation to measure-preserving homomorphisms (369Xi-369Xn). In §§373-374 I will elaborate on these ideas. Among the Orlicz spaces, we have a largest and a smallest; these are just $M^{1,\infty} = L^1 + L^\infty$ and $M^{\infty,1} = L^1 \cap L^\infty$ (369N-369O, 369Xk, 369Xl). Of course these two are particularly important.

There is an interesting phenomenon here. It is easy to see that $\| \|_{\infty,1} = \max(\| \|_1, \| \|_\infty)$ is an extended Fatou norm and that the corresponding Banach lattice is $L^1 \cap L^\infty$; and that the same ideas work for any pair of extended Fatou norms (369Xo). To check that the dual of $L^1 \cap L^\infty$ is precisely the linear sum $L^\infty + L^1$ a little more is needed, and the generalization of this fact to other extended Fatou norms (369R) seems to go quite deep. In view of our ordinary expectation that properties of these normed function spaces should be reflected in perfect Riesz spaces in general, I mention that I believe I have found an example, dependent on the continuum hypothesis, of two perfect Riesz subspaces U, V of $\mathbb{R}^{\mathbb{N}}$ such that their linear sum $U+V$ is not perfect.

Version of 20.7.11

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

364Be $L^0(\mathfrak{A})$ This re-phrasing of the definition of $L^0(\mathfrak{A})$, referred to in the 2008 edition of Volume 5, is now 364Af.

364D L^0 as f-algebra This paragraph, referred to in the 2008 edition of Volume 5, is now 364C.

364E Algebraic operations on L^0 This paragraph, referred to in the 2008 edition of Volume 5, is now 364D.

364G The identification of $L^0(\mathfrak{A})$ with the set of sequentially order-continuous Boolean homomorphisms from $\mathcal{B}(\mathbb{R})$ to $\mathfrak{A}$, referred to in the 2008 edition of Volume 5, is now 364F.

364I Action of Borel functions on L^0 This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 364H.

364J $L^0(\Sigma/\mathcal{I})$ The identification of $L^0(\Sigma/\mathcal{I})$ as a space of equivalence classes of functions, referred to in the 2003 and 2206 editions of Volume 4 and the 2008 edition of Volume 5, is now 364I.

364K Embedding S and L^∞ in L^0 This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 364J.

364M-364N Suprema and infima in $L^0(\mathfrak{A})$ These paragraphs, referred to in the 2003 and 2006 editions of Volume 4 and the 2008 edition of Volume 5, have now been amalgamated as 364L.

364O Dedekind completeness of L^0 This paragraph, referred to in the 2008 edition of Volume 5, is now 364M.

364P Multiplicative inverses in L^0 This paragraph, referred to in the 2003 and 2006 editions of Volume 4, is now 364J.

364R Action of Boolean homomorphisms on L^0 This paragraph, referred to the 2003 and 2006 editions of Volume 4 and in the 2008 edition of Volume 5, is now 364P.

364Xw Extension of $⨍$ This exercise, referred to in the 2008 edition of Volume 5, is now 364Xj.

364Yn $L^0_{\mathbb{C}}(\mathfrak{A})$ This exercise on complex L^0 spaces, referred to in the 2003 and 2006 editions of Volume 4, has been moved to 366M.

365K Additive functions on $\mathfrak{A}^f$ and linear operators on L^1 This theorem, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 365J.

365M L^1 and L^∞ This theorem, referred to in the 2008 printing of Volume 5, is now 365L.

365O Ring homomorphisms on $\mathfrak{A}^f$ and Riesz homomorphisms on L^1 This theorem, referred to in the 2013 printing of Volume 4 and the 2008 printing of Volume 5, is now 365N.

365P Order-continuous ring homomorphisms on $\mathfrak{A}^f$ and conditional expectations This theorem, referred to in the 2008 printing of Volume 5, is now 365O.

365R Conditional expectations These notes, referred to in the 2006 and 2013 printings of Volume 4 and the 2008 printing of Volume 5, is now 365Q.

365T Change of measure This proposition, referred to in the 2008 printing of Volume 5, is now 365S.

Version of 25.8.17

References for Volume 3

Anderson I. [87] *Combinatorics of Finite Sets.* Oxford U.P., 1987. [332Xk, 3A1K.]

Anzai H. [51] 'On an example of a measure-preserving transformation which is not conjugate to its inverse', Proc. Japanese Acad. Sci. 27 (1951) 517-522. [§382 *notes*.]

Balcar B., Główczyński W. & Jech T. [98] 'The sequential topology on complete Boolean algebras', Fundamenta Math. 155 (1998) 59-78. [393L, 393Q.]

Balcar B., Jech T. & Pazák T. [05] 'Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann', Bull. London Math. Soc. 37 (2005) 885-898. [393L, 393Q.]

Becker H. & Kechris A.S. [96] *The descriptive set theory of Polish group actions.* Cambridge U.P., 1996 (London Math. Soc. Lecture Note Series 232). [§395 *notes*.]

Bekkali M. & Bonnet R. [89] 'Rigid Boolean algebras', pp. 637-678 in Monk 89. [384L.]

Bellow A. & Kölzow D. [76] *Measure Theory, Oberwolfach 1975.* Springer, 1976 (Lecture Notes in Mathematics 541).

Bhaskara Rao, K.P.S. & Bhaskara Rao, M. *Theory of Charges.* Academic, 1983.

Billingsley P. [65] *Ergodic Theory and Information.* Wiley, 1965. [§372 *notes*.]

Bollobás B. [79] *Graph Theory.* Springer, 1979. [332Xk, 3A1K.]

Bose R.C. & Manvel B. [84] *Introduction to Combinatorial Theory.* Wiley, 1984. [3A1K.]

Bourbaki N. [66] *General Topology.* Hermann/Addison-Wesley, 1968. [§3A3, §3A4.]

Bourbaki N. [68] *Theory of Sets.* Hermann/Addison-Wesley, 1968. [§315 *notes*.]

Bourbaki N. [87] *Topological Vector Spaces.* Springer, 1987. [§3A5.]

Bukhvalov A.V. [95] 'Optimization without compactness, and its applications', pp. 95-112 in Huijsmans Kaashoek Luxemburg & Pagter 95. [367U.]

Burke M.R. [93] 'Liftings for Lebesgue measure', pp. 119-150 in Judah 93. [341L, 345F.]

Burke M.R. [n95] 'Consistent liftings', privately circulated, 1995. [346Ya.]

Burnside W. [1911] *Theory of Groups of Finite Order.* Cambridge U.P., 1911 (reprinted by Dover, 1955). [§384 *notes*.]

Chacon R.V. [69] 'Weakly mixing transformations which are not strongly mixing', Proc. Amer. Math. Soc. 22 (1969) 559-562. [372R.]

Chacon R.V. & Krengel U. [64] 'Linear modulus of a linear operator', Proc. Amer. Math. Soc. 15 (1964) 553-559. [§371 *notes*.]

Choksi J.R. & Prasad V.S. [82] 'Ergodic theory of homogeneous measure algebras', pp. 367-408 of Kölzow & Maharam-Stone 82. [383I.]

Cohn H. [06] 'A short proof of the simple continued fraction expansion of e', Amer. Math. Monthly 113 (2006) 57-62; `arXiv:math.NT/0601660`. [372L.]

Coleman A.J. & Ribenboim P. [67] (eds.) *Proceedings of the Symposium in Analysis, Queen's University, June 1967.* Queen's University, Kingston, Ontario, 1967.

Comfort W.W. & Negrepontis S. [82] *Chain Conditions in Topology.* Cambridge U.P., 1982. [§391 *notes*.]

Cziszár I. [67] 'Information-type measures of difference of probability distributions and indirect observations', Studia Scientiarum Math. Hungarica 2 (1967) 299-318. [386G.]

Davey B.A. & Priestley H.A. [90] 'Introduction to Lattices and Order', Cambridge U.P., 1990. [3A6C.]

Dugundji J. [66] *Topology.* Allyn & Bacon, 1966. [§3A3, 3A4Bb.]

Dunford N. [1936] 'Integration and linear operators', Trans. Amer. Math. Soc. 40 (1936) 474-494. [§376 *notes*.]

Dunford N. & Schwartz J.T. [57] *Linear Operators I.* Wiley, 1957 (reprinted 1988). [§356 *notes*, §371 *notes*, §376 *notes*, §3A5.]

Dye H.A. [59] 'On groups of measure preserving transformations I', Amer. J. Math. 81 (1959) 119-159. [§388 *notes*.]

Eigen S.J. [82] 'The group of measure-preserving transformations of $[0,1]$ has no outer automorphisms', Math. Ann. 259 (1982) 259-270. [384D.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [§3A3, §3A4.]

Enderton H.B. [77] *Elements of Set Theory.* Academic, 1977. [§3A1.]

Erdős P. & Oxtoby J.C. [55] 'Partitions of the plane into sets having positive measure in every non-null product set', Trans. Amer. Math. Soc. 79 (1955) 91-102. [§325 *notes*.]

Fathi A. [78] 'Le groupe des transformations de $[0,1]$ qui préservent la mesure de Lebesgue est un groupe simple', Israel J. Math. 29 (1978) 302-308. [§382 *notes*, 383I.]

Fremlin D.H. [74a] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [Chap. 35 *intro.*, 354Yb, §355 *notes*, §356 *notes*, §363 *notes*, §365 *notes*, §371 *notes*, §376 *notes*.]

Fremlin D.H. [74b] 'A characterization of L-spaces', Indag. Math. 36 (1974) 270-275. [§371 *notes*.]

Fremlin D.H., de Pagter B. & Ricker W.J. [05] 'Sequential closedness of Boolean algebras of projections in Banach spaces', Studia Math. 167 (2005) 45-62. [§323 *notes*.]

Frolík Z. [68] 'Fixed points of maps of extremally disconnected spaces and complete Boolean algebras', Bull. Acad. Polon. Sci. 16 (1968) 269-275. [382E.]

Gaal S.A. [64] *Point Set Topology.* Academic, 1964. [§3A3, §3A4.]

Gaifman H. [64] 'Concerning measures on Boolean algebras', Pacific J. Math. 14 (1964) 61-73. [§391 *notes*.]

Gale D. [60] *The theory of linear economic models.* McGraw-Hill, 1960. [3A5D.]

Gnedenko B.V. & Kolmogorov A.N. [54] *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, 1954. [§342 *notes*.]

Graf S. & Weizsäcker H.von [76] 'On the existence of lower densities in non-complete measure spaces', pp. 155-158 in Bellow & Kölzow 76. [341L.]

Hajian A. & Ito Y. [69] 'Weakly wandering sets and invariant measures for a group of transformations', J. of Math. and Mech. 18 (1969) 1203-1216. [396B.]

Hajian A., Ito Y. & Kakutani S. [75] 'Full groups and a theorem of Dye', Advances in Math. 17 (1975) 48-59. [§388 *notes*.]

Halmos P.R. [1948] 'The range of a vector measure', Bull. Amer. Math. Soc. 54 (1948) 416-421. [326Yk.]

Halmos P.R. [60] *Naive Set Theory.* Van Nostrand, 1960. [3A1D.]

Huijsmans C.B., Kaashoek M.A., Luxemburg W.A.J. & de Pagter B. [95] (eds.) *Operator Theory in Function Spaces and Banach Lattices.* Birkhäuser, 1995.

Ionescu Tulcea C. & Ionescu Tulcea A. [69] *Topics in the Theory of Lifting.* Springer, 1969. [§341 *notes*.]

James I.M. [87] *Topological and Uniform Spaces.* Springer, 1987. [§3A3, §3A4.]

Jech T. [03] *Set Theory, Millennium Edition.* Springer, 2002. [§3A1.]

Jech T. [08] 'Algebraic characterizations of measure algebras', Proc. Amer. Math. Soc. 136 (2008) 1285-1294. [393Xj.]

Johnson R.A. [80] 'Strong liftings which are not Borel liftings', Proc. Amer. Math. Soc. 80 (1980) 234-236. [345F.]

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Kakutani S. [1941] 'Concrete representation of abstract L-spaces and the mean ergodic theorem', Annals of Math. 42 (1941) 523-537. [369E.]

Kalton N.J., Peck N.T. & Roberts J.W. [84] 'An F-space sampler', Cambridge U.P., 1984. [§375 *notes*.]

Kalton N.J. & Roberts J.W. [83] 'Uniformly exhaustive submeasures and nearly additive set functions', Trans. Amer. Math. Soc. 278 (1983) 803-816. [392D, §392 *notes*.]

Kantorovich L.V., Vulikh B.Z. & Pinsker A.G. [50] *Functional Analysis in Partially Ordered Spaces*, Gostekhizdat, 1950. [391D.]

Kawada Y. [1944] 'Über die Existenz der invarianten Integrale', Jap. J. Math. 19 (1944) 81-95. [§395 *notes*.]

Kelley J.L. [59] 'Measures on Boolean algebras', Pacific J. Math. 9 (1959) 1165-1177. [§391 *notes*.]

Kolmogorov A.N. [58] 'New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces', Dokl. Akad. Nauk SSSR 119 (1958) 861-864. [385P.]

Kölzow D. & Maharam-Stone D. [82] (eds.) *Measure Theory Oberwolfach 1981.* Springer, 1982 (Lecture Notes in Math. 945).

Koppelberg S. [89] *General Theory of Boolean Algebras*, vol. 1 of Monk 89. [Chap. 31 *intro.*, §332 *notes.*]

Köthe G. [69] *Topological Vector Spaces I.* Springer, 1969. [§356 *notes*, 3A5N.]

Kranz P. & Labuda I. [93] (eds.) *Proceedings of the Orlicz Memorial Conference, 1991*, unpublished manuscript, available from first editor (`mmkranz@olemiss.edu`).

Krengel, U. [63] '"Uber den Absolutbetrag stetiger linearer Operatoren und seine Anwendung auf ergodische Zerlegungen', Math. Scand. 13 (1963) 151-187. [371Xb.]

Krieger W. [76] 'On ergodic flows and the isomorphism of factors', Math. Ann. 223 (1976) 19-70. [§388 *notes.*]

Krivine J.-L. [71] *Introduction to Axiomatic Set Theory.* D.Reidel, 1971. [§3A1.]

Kullback S. [67] 'A lower bound for discrimination information in terms of variation', IEEE Trans. on Information Theory 13 (1967) 126-127. [386G.]

Kunen K. [80] *Set Theory.* North-Holland, 1980. [§3A1.]

Kwapien S. [73] 'On the form of a linear operator on the space of all measurable functions', Bull. Acad. Polon. Sci. 21 (1973) 951-954. [§375 *notes.*]

Lang S. [93] *Real and Functional Analysis.* Springer, 1993. [§3A5.]

Liapounoff A.A. [1940] 'Sur les fonctions-vecteurs complètement additives', Bull. Acad. Sci. URSS (Izvestia Akad. Nauk SSSR) 4 (1940) 465-478. [326H.]

Lindenstrauss J. & Tzafriri L. [79] *Classical Banach Spaces II.* Springer, 1979, reprinted in Lindenstrauss & Tzafriri 96. [§354 *notes*, 374Xj.]

Lindenstrauss J. & Tzafriri L. [96] *Classical Banach Spaces I & II.* Springer, 1996.

Lipschutz S. [64] *Set Theory and Related Topics.* McGraw-Hill, 1964 (Schaum's Outline Series). [3A1D.]

Luxemburg W.A.J. [67a] 'Is every integral normal?', Bull. Amer. Math. Soc. 73 (1967) 685-688. [363S.]

Luxemburg W.A.J. [67b] 'Rearrangement-invariant Banach function spaces', pp. 83-144 in Coleman & Ribenboim 67. [§374 *notes.*]

Luxemburg W.A.J. & Zaanen A.C. [71] *Riesz Spaces I.* North-Holland, 1971. [Chap. 35 *intro.*]

Macheras N.D., Musiał K. & Strauss W. [99] 'On products of admissible liftings and densities', J. for Analysis and its Applications 18 (1999) 651-668. [346G.]

Macheras N.D. & Strauss W. [95] 'Products of lower densities', J. for Analysis and its Applications 14 (1995) 25-32. [346Xf.]

Macheras N.D. & Strauss W. [96a] 'On products of almost strong liftings', J. Australian Math. Soc. (A) 60 (1996) 1-23. [346Yc.]

Macheras N.D. & Strauss W. [96b] 'The product lifting for arbitrary products of complete probability spaces', Atti Sem. Math. Fis. Univ. Modena 44 (1996) 485-496. [346H, 346Yd.]

Maharam D. [1942] 'On homogeneous measure algebras', Proc. Nat. Acad. Sci. U.S.A. 28 (1942) 108-111. [331F, 332B.]

Maharam D. [1947] 'An algebraic characterization of measure algebras', Ann. Math. 48 (1947) 154-167. [393J.]

Maharam D. [58] 'On a theorem of von Neumann', Proc. Amer. Math. Soc. 9 (1958) 987-994. [§341 *notes*, §346 *notes.*]

Marczewski E. [53] 'On compact measures', Fund. Math. 40 (1953) 113-124. [342A.]

McCune W. [97] 'Solution of the Robbins problem', J. Automated Reasoning 19 (1997) 263-276. [311Yc.]

Miller B.D. [04] PhD Thesis, University of California, Berkeley, 2004. [382Xc, §382 *notes.*]

Monk J.D. [89] (ed.) *Handbook of Boolean Algebra.* North-Holland, 1989.

Nadkarni M.G. [90] 'On the existence of a finite invariant measure', Proc. Indian Acad. Sci., Math. Sci. 100 (1990) 203-220. [§395 *notes.*]

Ornstein D.S. [74] *Ergodic Theory, Randomness and Dynamical Systems.* Yale U.P., 1974. [§387 *notes.*]

Ornstein D.S. & Shields P.C. [73] 'An uncountable family of K-automorphisms', Advances in Math. 10 (1973) 63-88. [§382 *notes.*]

Perović Ž. & Veličković B. [18] 'Ranks of Maharam algebras', Advances in Math. 330 (2018) 253-279. [394A.]

Petersen K. [83] *Ergodic Theory.* Cambridge U.P., 1983. [328Xa, 385C, §385 *notes*, 386E.]

Roberts J.W. [93] 'Maharam's problem', in Kranz & Labuda 93. [§394 *notes*.]

Rotman J.J. [84] 'An Introduction to the Theory of Groups', Allyn & Bacon, 1984. [§384 *notes*, 3A6B.]

Rudin W. [91] *Functional Analysis.* McGraw-Hill, 1991. [§3A5.]

Ryzhikov V.V. [93] 'Factorization of an automorphism of a complete Boolean algebra into a product of three involutions', Mat. Zametki (=Math. Notes of Russian Acad. Sci.) 54 (1993) 79-84. [§382 *notes*.]

Sazonov V.V. [66] 'On perfect measures', A.M.S. Translations (2) 48 (1966) 229-254. [§342 *notes*.]

Schaefer H.H. [66] *Topological Vector Spaces.* MacMillan, 1966; reprinted with corrections Springer, 1971. [3A4A, 3A5J, 3A5N.]

Schaefer H.H. [74] *Banach Lattices and Positive Operators.* Springer, 1974. [Chap. 35 *intro*., §354 *notes*.]

Schubert H. [68] *Topology.* Allyn & Bacon, 1968. [§3A3, §3A4.]

Shelah S. [98] 'The lifting problem with the full ideal', J. Applied Analysis 4 (1998) 1-17. [341L.]

Sikorski R. [64] *Boolean Algebras.* Springer, 1964. [Chap. 31 *intro*.]

Sinaĭ Ya.G. [59] 'The notion of entropy of a dynamical system', Dokl. Akad. Nauk SSSR 125 (1959) 768-771. [385P.]

Sinaĭ Ya.G. [62] 'Weak isomorphism of transformations with an invariant measure', Soviet Math. 3 (1962) 1725-1729. [387E.]

Smorodinsky M. [71] *Ergodic Theory, Entropy.* Springer, 1971 (Lecture Notes in Math., 214). [§387 *notes*.]

Štěpánek P. & Rubin M. [89] 'Homogeneous Boolean algebras', pp. 679-715 in Monk 89. [382S, §382 *notes*.]

Talagrand M. [82a] 'Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations', Ann. Institut Fourier (Grenoble) 32 (1982) 39-69. [§346 *notes*.]

Talagrand M. [82b] 'La pathologie des relèvements invariants', Proc. Amer. Math. Soc. 84 (1982) 379-382. [345F.]

Talagrand M. [84] *Pettis integral and measure theory.* Mem. Amer. Math. Soc. 307 (1984). [§346 *notes*.]

Talagrand M. [08] 'Maharam's problem', Annals of Math. 168 (2008) 981-1009. [§394.]

Taylor A.E. [64] *Introduction to Functional Analysis.* Wiley, 1964. [§3A5.]

Todorčević S. [04] 'A problem of von Neumann and Maharam about algebras supporting continuous submeasures', Fund. Math. 183 (2004) 169-183. [393S.]

Truss J.K. [89] 'Infinite permutation groups I: products of conjugacy classes', J. Algebra 120 (1989) 454-493. [§382 *notes*.]

Vladimirov D.A. [02] *Boolean algebras in analysis.* Kluwer, 2002 (Math. and its Appl. 540). [367L.]

Vulikh B.C. [67] *Introduction to the Theory of Partially Ordered Vector Spaces.* Wolters-Noordhoff, 1967. [§364 *notes*.]

Wagon S. [85] *The Banach-Tarski Paradox.* Cambridge U.P., 1985. [§395 *notes*.]

Wilansky A. [64] *Functional Analysis.* Blaisdell, 1964. [§3A5.]

Zaanen A.C. [83] *Riesz Spaces II.* North-Holland, 1983. [Chap. 35 *intro*., 376K, §376 *notes*.]