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Chapter 35

Riesz spaces

The next three chapters are devoted to an abstract description of the 'function spaces' described in Chapter 24, this time concentrating on their internal structure and relationships with their associated measure algebras. I find that any convincing account of these must involve a substantial amount of general theory concerning partially ordered linear spaces, and in particular various types of Riesz space or vector lattice. I therefore provide an introduction to this theory, a kind of appendix built into the middle of the volume. The relation of this chapter to the next two is very like the relation of Chapter 31 to Chapter 32. As with Chapter 31, it is not really meant to be read for its own sake; those with a particular interest in Riesz spaces might be better served by LUXEMBURG & ZAANEN 71, SCHAEFER 74, ZAANEN 83 or my own book FREMLIN 74A.

I begin with three sections in an easy gradation towards the particular class of spaces which we need to understand: partially ordered linear spaces (§351), general Riesz spaces (§352) and Archimedean Riesz spaces (§353); the last includes notes on Dedekind (σ-)complete spaces. These sections cover the fragments of the algebraic theory of Riesz spaces which I will use. In the second half of the chapter, I deal with normed Riesz spaces (in particular, L- and M-spaces)(§354), spaces of linear operators (§355) and dual Riesz spaces (§356).

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351 Partially ordered linear spaces

I begin with an account of the most basic structures which involve an order relation on a linear space, partially ordered linear spaces. As often in this volume, I find myself impelled to do some of the work in very much greater generality than is strictly required, in order to show more clearly the nature of the arguments being used. I give the definition (351A) and most elementary properties (351B-351L) of partially ordered linear spaces; then I describe a general representation theorem for arbitrary partially ordered linear spaces as subspaces of reduced powers of $\mathbb{R}$ (351M-351Q). I end with a brief note on Archimedean partially ordered linear spaces (351R).

351A Definition A **partially ordered linear space** is a linear space $(U, +, \cdot)$ over $\mathbb{R}$ together with a partial order $\le$ on U such that

$$u \le v \Longrightarrow u + w \le v + w,$$

$$u \ge 0,\ \alpha \ge 0 \Longrightarrow \alpha u \ge 0$$

for u, v, $w \in U$ and $\alpha \in \mathbb{R}$.

351B Elementary facts Let U be a partially ordered linear space.

(a) For u, $v \in U$,

$$u \le v \iff 0 \le v - u \iff -v \le -u.$$

(b) Suppose that u, $v \in U$ and $u \le v$. Then $\alpha u \le \alpha v$ for every $\alpha \ge 0$ and $\alpha v \le \alpha u$ for every $\alpha \le 0$.

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(c) If $u \geq 0$ and $\alpha \leq \beta$ in $\mathbb{R}$, then $\alpha u \leq \beta u$. If $0 \leq u \leq v$ in U and $0 \leq \alpha \leq \beta$ in $\mathbb{R}$, then $\alpha u \leq \beta v$.

351C Positive cones Let U be a partially ordered linear space.

(a) I will write U^+ for the **positive cone** of U, the set $\{u : u \in U,\, u \geq 0\}$.

(b) $u \leq v \iff v - u \in U^+$.

(c) If U is a real linear space, a set $C \subseteq U$ is the positive cone for some ordering rendering U a partially ordered linear space iff

$$u + v \in C, \quad \alpha u \in C \text{ whenever } u,\, v \in C \text{ and } \alpha \geq 0,$$

$$0 \in C, \quad u \in C \,\&\, -u \in C \Longrightarrow u = 0.$$

(d) Let U be a partially ordered linear space, and $u \in U$. Then $u \geq 0$ iff $u \geq -u$.

(e) U^+ is always convex.

351D Suprema and infima Let U be a partially ordered linear space.

(a) $u \mapsto u + w$ is always an order-isomorphism; $u \mapsto -u$ is order-reversing.

(b) If $A \subseteq U$ and $v \in U$ then

$$\sup_{u \in A}(v + u) = v + \sup A \text{ if either side is defined,}$$

$$\inf_{u \in A}(v + u) = v + \inf A \text{ if either side is defined,}$$

$$\sup_{u \in A}(v - u) = v - \inf A \text{ if either side is defined,}$$

$$\inf_{u \in A}(v - u) = v - \sup A \text{ if either side is defined.}$$

(c) If A, $B \subseteq U$ and $\sup A$ and $\sup B$ are defined, then $\sup(A+B)$ is defined and equal to $\sup A + \sup B$. Similarly, if A, $B \subseteq U$ and $\inf A$, $\inf B$ are defined then $\inf(A + B) = \inf A + \inf B$.

(d) If $\alpha > 0$ then $\sup(\alpha A) = \alpha \sup A$ if either side is defined; similarly, $\inf(\alpha A) = \alpha \inf A$.

351E Linear subspaces If U is a partially ordered linear space, and V is any linear subspace of U, then V, with the induced linear and order structures, is a partially ordered linear space.

351F Positive linear operators Let U and V be partially ordered linear spaces, and write $\mathsf{L}(U; V)$ for the linear space of all linear operators from U to V. For S, $T \in \mathsf{L}(U; V)$ say that $S \leq T$ iff $Su \leq Tu$ for every $u \in U^+$. Under this ordering, $\mathsf{L}(U; V)$ is a partially ordered linear space; its positive cone is $\{T : Tu \geq 0$ for every $u \in U^+\}$. For $T \in \mathsf{L}(U; V)$, $T \geq 0$ iff T is order-preserving. In this case we say that T is a **positive** linear operator.

ST is a positive linear operator whenever U, V and W are partially ordered linear spaces and $T : U \to V$, $S : V \to W$ are positive linear operators.

351G Order-continuous positive linear operators: Proposition Let U and V be partially ordered linear spaces and $T : U \to V$ a positive linear operator.

(a) The following are equiveridical:

(i) T is order-continuous;

(ii) $\inf T[A] = 0$ in V whenever $A \subseteq U$ is a non-empty downwards-directed set with infimum 0 in U;

(iii) $\sup T[A] = Tw$ in V whenever $A \subseteq U^+$ is a non-empty upwards-directed set with supremum w in U.

(b) The following are equiveridical:

(i) T is sequentially order-continuous;

(ii) $\inf_{n\in\mathbb{N}} Tu_n = 0$ in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum 0 in U;

(iii) $\sup_{n\in\mathbb{N}} Tu_n = Tw$ in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U^+ with supremum w in U.

351H Riesz homomorphisms (a) Let U, V be partially ordered linear spaces. A **Riesz homomorphism** from U to V is a linear operator $T : U \to V$ such that whenever $A \subseteq U$ is a finite non-empty set and $\inf A = 0$ in U, then $\inf T[A] = 0$ in V.

(b) Any Riesz homomorphism is a positive linear operator.

(c) Let U and V be partially ordered linear spaces and $T : U \to V$ a Riesz homomorphism. Then

$$\inf T[A] \text{ exists} = T(\inf A), \quad \sup T[A] \text{ exists} = T(\sup A)$$

whenever $A \subseteq U$ is a finite non-empty set and $\inf A$, $\sup A$ exist.

(d) If U, V and W are partially ordered linear spaces and $T : U \to V$, $S : V \to W$ are Riesz homomorphisms then $ST : U \to W$ is a Riesz homomorphism.

351I Solid sets Let U be a partially ordered linear space. I will say that a subset A of U is **solid** if

$$A = \bigcup_{u \in A} [-u, u].$$

351J Proposition Let U be a partially ordered linear space and V a solid linear subspace of U. Then the quotient linear space U/V has a partially ordered linear space structure defined by either of the rules

$u^\bullet \le w^\bullet$ iff there is some $v \in V$ such that $u \le v + w$,

$(U/V)^+ = \{u^\bullet : u \in U^+\}$,

and for this partial order on U/V the map $u \mapsto u^\bullet : U \to U/V$ is a Riesz homomorphism.

351K Lemma Suppose that U is a partially ordered linear space, and that W, V are solid linear subspaces of U such that $W \subseteq V$. Then $V_1 = \{v^\bullet : v \in V\}$ is a solid linear subspace of U/W.

351L Products If $\langle U_i \rangle_{i \in I}$ is any family of partially ordered linear spaces, we have a product linear space $U = \prod_{i \in I} U_i$; if we set $u \le v$ in U iff $u(i) \le v(i)$ for every $i \in I$, U becomes a partially ordered linear space, with positive cone $\{u : u(i) \ge 0 \text{ for every } i \in I\}$. For each $i \in I$ the map $u \mapsto u(i) : U \to U_i$ is an order-continuous Riesz homomorphism.

351M Reduced powers of $\mathbb{R}$ (a) Let X be any set. Then $\mathbb{R}^X$ is a partially ordered linear space if we say that $f \le g$ means that $f(x) \le g(x)$ for every $x \in X$. If now $\mathcal{F}$ is a filter on X, we have a corresponding set

$$V = \{f : f \in \mathbb{R}^X, \{x : f(x) = 0\} \in \mathcal{F}\};$$

V is a linear subspace of $\mathbb{R}^X$, and is solid. By the **reduced power** $\mathbb{R}^X|\mathcal{F}$ I shall mean the quotient partially ordered linear space $\mathbb{R}^X/V$.

(b) For $f \in \mathbb{R}^X$,

$$f^\bullet \ge 0 \text{ in } \mathbb{R}^X|\mathcal{F} \iff \{x : f(x) \ge 0\} \in \mathcal{F}.$$

351N Lemma Let U be a partially ordered linear space. If $u, v_0, \dots, v_n \in U$ are such that $u \ne 0$ and $v_0, \dots, v_n \ge 0$ then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) \ne 0$ and $f(v_i) \ge 0$ for every i.

351O Lemma Let U be a partially ordered linear space, and u_0 a non-zero member of U. Then there is a solid linear subspace V of U such that $u_0 \notin V$ and whenever $A \subseteq U$ is finite, non-empty and has infimum 0 then $A \cap V \ne \emptyset$.

351P Lemma Let U be a partially ordered linear space and u a non-zero element of U, and suppose that $A_0, \dots, A_n$ are finite non-empty subsets of U such that $\inf A_j = 0$ for every $j \le n$. Then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) \ne 0$ and $\min f[A_j] = 0$ for every $j \le n$.

351Q Theorem Let U be any partially ordered linear space. Then we can find a set X, a filter $\mathcal{F}$ on X and an injective Riesz homomorphism from U to the reduced power $\mathbb{R}^X|\mathcal{F}$.

351R Archimedean spaces (a) For a partially ordered linear space U, the following are equiveridical: (i) if u, $v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$ then $u \le 0$ (ii) if $u \ge 0$ in U then $\inf_{\delta>0} \delta u = 0$.

(b) I will say that partially ordered linear spaces satisfying the equiveridical conditions of (a) above are **Archimedean**.

(c) Any linear subspace of an Archimedean partially ordered linear space, with the induced partially ordered linear space structure, is Archimedean.

(d) Any product of Archimedean partially ordered linear spaces is Archimedean. In particular, $\mathbb{R}^X$ is Archimedean for any set X.

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352 Riesz spaces

In this section I sketch those fragments of the theory we need which can be expressed as theorems about general Riesz spaces or vector lattices. I begin with the definition (352A) and most elementary properties (352C-352F). In 352G-352J I discuss Riesz homomorphisms and the associated subspaces (Riesz subspaces, solid linear subspaces); I mention product spaces (352K, 352T) and quotient spaces (352Jb, 352U) and the form the representation theorem 351Q takes in the present context (352L-352M). Most of the second half of the section concerns the theory of 'bands' in Riesz spaces, with the algebras of complemented bands (352Q) and projection bands (352S) and a description of bands generated by upwards-directed sets (352V). I conclude with a description of 'f-algebras' (352W).

352A Definition A **Riesz space** or **vector lattice** is a partially ordered linear space which is a lattice.

352B Lemma If U is a partially ordered linear space, then it is a Riesz space iff $\sup\{0, u\}$ is defined for every $u \in U$.

352C Notation In any Riesz space U I will write

$$u^+ = u \vee 0, \quad u^- = (-u) \vee 0 = (-u)^+, \quad |u| = u \vee (-u)$$

where $u \vee v = \sup\{u, v\}$ (and $u \wedge v = \inf\{u, v\}$).

A family $\langle u_i \rangle_{i \in I}$ in U is **disjoint** if $|u_i| \wedge |u_j| = 0$ for all distinct i, $j \in I$. Similarly, a set $C \subseteq U$ is **disjoint** if $|u| \wedge |v| = 0$ for all distinct u, $v \in C$.

352D Elementary identities Let U be a Riesz space.

$$u + (v \vee w) = (u + v) \vee (u + w), \qquad u + (v \wedge w) = (u + v) \wedge (u + w),$$

$$\alpha(u \vee v) = \alpha u \vee \alpha v \text{ and } \alpha(u \wedge v) = \alpha u \wedge \alpha v \text{ if } \alpha \ge 0,$$

$$-(u \vee v) = (-u) \wedge (-v).$$

Combining and elaborating on these facts, we get

$$u^+ - u^- = u,$$

$$u^+ + u^- = |u|,$$

$$u \ge 0 \iff -u \le 0 \iff u^- = 0 \iff u = u^+ \iff u = |u|,$$

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$$|-u| = |u|, \quad |\,|u|\,| = |u|, \quad |\alpha u| = |\alpha||u|,$$

$$u \vee v + u \wedge v = u + v,$$

$$u \vee v = u + (v-u)^+,$$

$$u \wedge v = u - (u-v)^+,$$

$$u \vee v = \frac{1}{2}(u + v + |u-v|),$$

$$u \wedge v = \frac{1}{2}(u + v - |u-v|),$$

$$u^+ \vee u^- = |u|, \quad u^+ \wedge u^- = 0,$$

$$|u+v| \le |u| + |v|, \quad ||u| - |v|| \le |u-v|, \quad |u \vee v| \le |u| + |v|$$

for u, $v \in U$ and $\alpha \in \mathbb{R}$.

352E Distributive laws Let U be a Riesz space.

(a) If A, $B \subseteq U$ have suprema a, b in U, then $C = \{u \wedge v : u \in A, v \in B\}$ has supremum $a \wedge b$.

(b) $\inf\{u \vee v : u \in A, v \in B\} = \inf A \vee \inf B$ whenever A, $B \subseteq U$ and the right-hand-side is defined.

(c) U is a distributive lattice.

352F Further identities and inequalities: Proposition Let U be a Riesz space.
(a)(i) If u, v, $w \ge 0$ in U then $u \wedge (v + w) \le (u \wedge v) + (u \wedge w)$.
(ii) If $v_0, \dots, v_m, w_0, \dots, w_n \in U^+$ then

$$\sum_{i=0}^m v_i \wedge \sum_{j=0}^n w_j \le \sum_{i=0}^m \sum_{j=0}^n v_i \wedge w_j.$$

(b) If $u_0, \dots, u_n \in U$ are disjoint, then $|\sum_{i=0}^n \alpha_i u_i| = \sum_{i=0}^n |\alpha_i||u_i|$ for any $\alpha_0, \dots, \alpha_n \in \mathbb{R}$.
(c) If u, $v \in U$ then

$$u^+ \wedge v^+ \le (u+v)^+ \le u^+ + v^+.$$

(d) If $u_0, \dots, u_m, v_0, \dots, v_n \in U^+$ and $\sum_{i=0}^m u_i = \sum_{j=0}^n v_j$, then there is a family $\langle w_{ij} \rangle_{i \le m, j \le n}$ in U^+ such that $\sum_{i=0}^m w_{ij} = v_j$ for every $j \le n$ and $\sum_{j=0}^n w_{ij} = u_i$ for every $i \le m$.

352G Riesz homomorphisms: Proposition Let U be a Riesz space, V a partially ordered linear space and $T : U \to V$ a linear operator. Then the following are equiveridical:
(i) T is a Riesz homomorphism in the sense of 351H;
(ii) $(Tu)^+ = \sup\{Tu, 0\}$ is defined and equal to $T(u^+)$ for every $u \in U$;
(iii) $\sup\{Tu, -Tu\}$ is defined and equal to $T|u|$ for every $u \in U$;
(iv) $\inf\{Tu, Tv\} = 0$ in V whenever $u \wedge v = 0$ in U.

352H Proposition If U and V are Riesz spaces and $T : U \to V$ is a bijective Riesz homomorphism, then T is a partially-ordered-linear-space isomorphism, and $T^{-1} : V \to U$ is a Riesz homomorphism.

352I Riesz subspaces (a) If U is a partially ordered linear space, a **Riesz subspace** of U is a linear subspace V such that $\sup\{u, v\}$ and $\inf\{u, v\}$ are defined in U and belong to V for every u, $v \in V$. In this case V, with the induced order and linear structure, is a Riesz space in its own right, and the embedding map $u \mapsto u : V \to U$ is a Riesz homomorphism.

(b) Generally, if U is a Riesz space, V is a partially ordered linear space and $T : U \to V$ is a Riesz homomorphism, then $T[U]$ is a Riesz subspace of V.

(c) If U is a Riesz space and V is a linear subspace of U, then V is a Riesz subspace of U iff $|u| \in V$ for every $u \in V$.

352J Solid subsets (a) If U is a Riesz space, a subset A of U is solid iff $v \in A$ whenever $u \in A$ and $|v| \le |u|$. In particular, if A is solid, then $v \in A$ iff $|v| \in A$.

For any set $A \subseteq U$, the set

$$\{u : \text{there is some } v \in A \text{ such that } |u| \le |v|\}$$

is a solid subset of U, the smallest solid set including A; we call it the **solid hull** of A in U.

Any solid linear subspace of U is a Riesz subspace. If $V \subseteq U$ is a Riesz subspace, then the solid hull of V in U is

$$\{u : \text{there is some } v \in V \text{ such that } |u| \le v\}$$

and is a solid linear subspace of U.

(b) If T is a Riesz homomorphism from a Riesz space U to a partially ordered linear space V, then its kernel W is a solid linear subspace of U.

Now the quotient space U/W isomorphic, as partially ordered linear space, to $T[U]$, and is a Riesz space.

352K Products If $\langle U_i\rangle_{i\in I}$ is any family of Riesz spaces, then the product partially ordered linear space $U = \prod_{i\in I} U_i$ is a Riesz space, with

$$u \vee v = \langle u(i) \vee v(i)\rangle_{i\in I}, \quad u \wedge v = \langle u(i) \wedge v(i)\rangle_{i\in I}, \quad |u| = \langle |u(i)|\rangle_{i\in I}$$

for all u, $v \in U$.

352L Theorem Let U be any Riesz space. Then there are a set X, a filter $\mathcal{F}$ on X and a Riesz subspace of the Riesz space $\mathbb{R}^X|\mathcal{F}$ which is isomorphic, as Riesz space, to U.

352M Corollary Any identity involving the operations $+$, $-$, $\vee$, $\wedge$, $^+$, $^-$, $|\;|$ and scalar multiplication, and the relation $\le$, which is valid in $\mathbb{R}$, is valid in all Riesz spaces.

352N Order-density and order-continuity Let U be a Riesz space.

(a) Definition A Riesz subspace V of U is **quasi-order-dense** if for every $u > 0$ in U there is a $v \in V$ such that $0 < v \le u$; it is **order-dense** if $u = \sup\{v : v \in V,\, 0 \le v \le u\}$ for every $u \in U^+$.

(b) If U is a Riesz space and V is a quasi-order-dense Riesz subspace of U, then the embedding $V \hookrightarrow U$ is order-continuous.

(c)(i) If $V \subseteq U$ is an order-dense Riesz subspace, it is quasi-order-dense. (ii) If V is a quasi-order-dense Riesz subspace of U and W is a quasi-order-dense Riesz subspace of V, then W is a quasi-order-dense Riesz subspace of U. (iii) If V is an order-dense Riesz subspace of U and W is an order-dense Riesz subspace of V, then W is an order-dense Riesz subspace of U. (iv) If V is a quasi-order-dense solid linear subspace of U and W is a quasi-order-dense Riesz subspace of U then $V \cap W$ is quasi-order-dense in U.

(d) A Riesz homomorphism is order-continuous iff it preserves arbitrary suprema and infima.

(e) If V is a Riesz subspace of U, we say that it is **regularly embedded** in U if the identity map from V to U is order-continuous.

352O Bands Let U be a Riesz space.

(a) Definition A **band** or **normal subspace** of U is an order-closed solid linear subspace.

(b) If $V \subseteq U$ is a solid linear subspace then it is a band iff $\sup A \in V$ whenever $A \subseteq V^+$ is a non-empty, upwards-directed subset of V with a supremum in U.

(c) For any set $A \subseteq U$ set $A^\perp = \{v : v \in U, |u| \wedge |v| = 0 \text{ for every } u \in A\}$. Then $A^\perp$ is a band.

(d) For any $A \subseteq U$, $A \subseteq (A^\perp)^\perp$. Also $B^\perp \subseteq A^\perp$ whenever $A \subseteq B$. So $A^\perp = A^{\perp\perp\perp}$.

(e) If W is another Riesz space and $T : U \to W$ is an order-continuous Riesz homomorphism then its kernel is a band.

352P Complemented bands Let U be a Riesz space. A band $V \subseteq U$ is **complemented** if $V^{\perp\perp} = V$, that is, if V is of the form $A^\perp$ for some $A \subseteq U$. In this case its **complement** is the complemented band $V^\perp$.

352Q Theorem In any Riesz space U, the set $\mathfrak{C}$ of complemented bands forms a Dedekind complete Boolean algebra, with

$$V \cap_{\mathfrak{C}} W = V \cap W, \quad V \cup_{\mathfrak{C}} W = (V + W)^{\perp\perp},$$

$$1_{\mathfrak{C}} = U, \quad 0_{\mathfrak{C}} = \{0\}, \quad 1_{\mathfrak{C}} \setminus_{\mathfrak{C}} V = V^\perp,$$

$$V \subseteq_{\mathfrak{C}} W \iff V \subseteq W$$

for V, $W \in \mathfrak{C}$.

352R Projection bands Let U be a Riesz space.

(a) A **projection band** in U is a set $V \subseteq U$ such that $V + V^\perp = U$. In this case V is a complemented band. Observe that $U = V^\perp + V^{\perp\perp}$ so $V^\perp$ also is a projection band.

(b) $U = V \oplus V^\perp$ for any projection band $V \subseteq U$; there is a corresponding **band projection** $P_V : U \to U$ defined by setting $P(v+w) = v$ whenever $v \in V$, $w \in V^\perp$. In this context I will say that v is the **component** of $v + w$ in V. The kernel of P is $V^\perp$, the set of values is V, and $P^2 = P$. P is an order-continuous Riesz homomorphism.

(c) Note that for any band projection P, and any $u \in U$, we have $|Pu| \le |u|$; $P[W] \subseteq W$ for any solid linear subspace W of U.

(d) A linear operator $P : U \to U$ is a band projection iff $Pu \wedge (u - Pu) = 0$ for every $u \in U^+$.

352S Proposition Let U be any Riesz space.

(a) The family $\mathfrak{B}$ of projection bands in U is a subalgebra of the Boolean algebra $\mathfrak{C}$ of complemented bands in U.

(b) For $V \in \mathfrak{B}$ let $P_V : U \to V$ be the corresponding projection. Then for any $e \in U^+$,

$$P_{V\cap W}e = P_V e \wedge P_W e = P_V P_W e, \quad P_{V\vee W}e = P_V e \vee P_W e$$

for all V, $W \in \mathfrak{B}$. In particular, band projections commute.

(c) If $V \in \mathfrak{B}$ then the algebra of projection bands in V is just the principal ideal of $\mathfrak{B}$ generated by V.

352T Products again (a) If $U = \prod_{i\in I} U_i$ is a product of Riesz spaces, then for any $J \subseteq I$ we have a subspace

$$V_J = \{u : u \in U, u(i) = 0 \text{ for all } i \in I \setminus J\}$$

of U, canonically isomorphic to $\prod_{i\in J} U_i$. Each V_J is a projection band, its complement being $V_{I\setminus J}$; the map $J \mapsto V_J$ is a Boolean homomorphism from $\mathcal{P}I$ to the algebra $\mathfrak{B}$ of projection bands in U, and $\langle V_{\{i\}}\rangle_{i\in I}$ is a partition of unity in $\mathfrak{B}$.

(b) Conversely, if U is a Riesz space and $(V_0,\dots,V_n)$ is a *finite* partition of unity in the algebra $\mathfrak{B}$ of projection bands in U, then every element of U is uniquely expressible as $\sum_{i=0}^n u_i$ where $u_i\in V_i$ for each i. This decomposition corresponds to a Riesz space isomorphism between U and $\prod_{i\le n}V_i$.

352U Quotient spaces (a) If U is a Riesz space and V is a solid linear subspace, then the canonical map from U onto the quotient partially ordered linear space U/V is a Riesz homomorphism, so U/V is a Riesz space. I have already noted that if U and W are Riesz spaces and $T:U\to W$ a Riesz homomorphism, then the kernel V of T is a solid linear subspace of U and the Riesz subspace $T[U]$ of W is isomorphic to U/V.

(b) Suppose that U is a Riesz space and V a solid linear subspace. Then the canonical map from U to U/V is order-continuous iff V is a band.

352V Principal bands Let U be a Riesz space. Evidently the intersection of any family of Riesz subspaces of U is a Riesz subspace, the intersection of any family of solid linear subspaces is a solid linear subspace and the intersection of any family of bands is a band; we may speak of the band generated by a subset A of U, the intersection of all the bands including A.

Lemma Let U be a Riesz space.

(a) If $A\subseteq U^+$ is upwards-directed and $2w\in A$ for every $w\in A$, then an element u of U belongs to the band generated by A iff $|u|=\sup_{w\in A}|u|\wedge w$.

(b) If $u\in U$ and $w\in U^+$, then u belongs to the band in U generated by w iff $|u|=\sup_{n\in\mathbb{N}}|u|\wedge nw$.

352W f-algebras (a) Definition An **f-algebra** is a Riesz space U with a multiplication $\times:U\times U\to U$ such that

$$u\times(v\times w)=(u\times v)\times w,$$

$$(u+v)\times w=(u\times w)+(v\times w),\quad u\times(v+w)=(u\times v)+(u\times w),$$

$$\alpha(u\times v)=(\alpha u)\times v=u\times(\alpha v)$$

for all u, v, $w\in U$ and $\alpha\in\mathbb{R}$, and

$$u\times v\ge 0 \text{ whenever } u,\,v\ge 0,$$

$$\text{if } u\wedge v=0 \text{ then } (u\times w)\wedge v=(w\times u)\wedge v=0 \text{ for every } w\ge 0.$$

An f-algebra is **commutative** if $u\times v=v\times u$ for all u, v.

(b) Let U be an f-algebra.

(i) If $u\wedge v=0$ in U, then $u\times v=0$.

(ii) $u\times u\ge 0$ for every $u\in U$.

(iii) If u, $v\in U$ then $|u\times v|=|u|\times|v|$.

(iv) If $v\in U^+$ the maps $u\mapsto u\times v$, $u\mapsto v\times u:U\to U$ are Riesz homomorphisms.

(c) Let $\langle U_i\rangle_{i\in I}$ be a family of f-algebras, with Riesz space product U (352K). If we set $u\times v=\langle u(i)\times v(i)\rangle_{i\in I}$ for all u, $v\in U$, then U becomes an f-algebra.

Version of 16.2.17

353 Archimedean and Dedekind complete Riesz spaces

I take a few pages over elementary properties of Archimedean and Dedekind (σ)-complete Riesz spaces.

353A Proposition Let U be an Archimedean Riesz space. Then every quasi-order-dense Riesz subspace of U is order-dense.

353B Proposition Let U be an Archimedean Riesz space. Then
(a) for every $A \subseteq U$, the band generated by A is $A^{\perp\perp}$,
(b) every band in U is complemented.

353C Corollary Let U be an Archimedean Riesz space and $v \in U$. Let V be the band in U generated by v. If $u \in U$, then $u \in V$ iff there is no w such that $0 < w \leq |u|$ and $w \wedge |v| = 0$.

353D Proposition Let U be an Archimedean Riesz space and V an order-dense Riesz subspace of U. Then the map $W \mapsto W \cap V$ is an isomorphism between the band algebras of U and V.

353E Lemma Let U be an Archimedean Riesz space and $V \subseteq U$ a band such that $\sup\{v : v \in V, 0 \leq v \leq u\}$ is defined for every $u \in U^+$. Then V is a projection band.

353F Lemma Let U be an Archimedean Riesz space. If $A \subseteq U$ is non-empty and bounded above and B is the set of its upper bounds, then $\inf(B - A) = 0$.

353G Proposition Let U be a Riesz space and V an order-dense Riesz subspace of U. If V is Archimedean, so is U.

353H Dedekind completeness For a Riesz space U, U is Dedekind complete iff every non-empty upwards-directed subset of U^+ with an upper bound has a least upper bound, and is Dedekind σ-complete iff every non-decreasing sequence in U^+ with an upper bound has a least upper bound.

353I Proposition Let U be a Dedekind σ-complete Riesz space.
(a) U is Archimedean.
(b) For any $v \in U$ the band generated by v is a projection band.
(c) If u, $v \in U$, then u is uniquely expressible as $u_1 + u_2$, where u_1 belongs to the band generated by v and $|u_2| \wedge |v| = 0$.

353J Proposition In a Dedekind complete Riesz space, all bands are projection bands.

353K Proposition (a) Let U be a Dedekind σ-complete Riesz space.
(i) If V is a solid linear subspace of U, then V is (in itself) Dedekind σ-complete.
(ii) If V is a sequentially order-closed Riesz subspace of U then V is Dedekind σ-complete.
(iii) If V is a sequentially order-closed solid linear subspace of U, the canonical map from U to the quotient space U/V is sequentially order-continuous, and U/V also is Dedekind σ-complete.
(b) Let U be a Dedekind complete Riesz space.
(i) If V is a solid linear subspace of U, then V is Dedekind complete.
(ii) If $V \subseteq U$ is an order-closed Riesz subspace then V is Dedekind complete.

353L Proposition Let U be a Riesz space and V a quasi-order-dense Riesz subspace of U which is (in itself) Dedekind complete. Then V is a solid linear subspace of U.

353M Order units Let U be a Riesz space.

(a) An element e of U^+ is an **order unit** in U if U is the solid linear subspace of itself generated by e.

(b) An element e of U^+ is a **weak order unit** in U if U is the principal band generated by e.
Of course an order unit is a weak order unit.

(c) If U is Archimedean, then an element e of U^+ is a weak order unit iff $u \wedge e > 0$ whenever $u > 0$.

353N Theorem Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that e corresponds to χX; moreover, this embedding is essentially unique.

353O Lemma Let U be a Riesz space, V an Archimedean Riesz space and S, $T : U \to V$ Riesz homomorphisms such that $Su \wedge Tu' = 0$ in V whenever $u \wedge u' = 0$ in U. Set $W = \{u : Su = Tu\}$. Then W is a solid linear subspace of U; if S and T are order-continuous, W is a band.

353P f-algebras: Proposition Let U be an Archimedean f-algebra. Then

(a) the multiplication is separately order-continuous in the sense that the maps $u \mapsto u \times w$, $u \mapsto w \times u$ are order-continuous for every $w \in U^+$;

(b) the multiplication is commutative.

353Q Proposition Let U be an Archimedean f-algebra with multiplicative identity e.

(a) e is a weak order unit in U.

(b) If u, $v \in U$ then $u \times v = 0$ iff $|u| \wedge |v| = 0$.

(c) If $u \in U$ has a multiplicative inverse u^{-1} then $|u|$ also has a multiplicative inverse; if $u \geq 0$ then $u^{-1} \geq 0$ and u is a weak order unit.

(d) If V is another Archimedean f-algebra with multiplicative identity e', and $T : U \to V$ is a positive linear operator such that $Te = e'$, then T is a Riesz homomorphism iff $T(u \times v) = Tu \times Tv$ for all u, $v \in U$.

Version of 18.8.08

354 Banach lattices

The next step is a brief discussion of norms on Riesz spaces. I start with the essential definitions (354A, 354D) with the principal properties of general Riesz norms (354B-354C) and order-continuous norms (354E). I then describe two of the most important classes of Banach lattice: M-spaces (354F-354L) and L-spaces (354M-354R), with their elementary properties. For M-spaces I give the basic representation theorem (354K-354L), and for L-spaces I give a note on uniform integrability (354P-354R).

354A Definitions (a) If U is a Riesz space, a **Riesz norm** or **lattice norm** on U is a norm $\| \,\|$ such that $\|u\| \leq \|v\|$ whenever $|u| \leq |v|$.

(b) A **Banach lattice** is a Riesz space with a Riesz norm under which it is complete.

354B Lemma Let U be a Riesz space with a Riesz norm $\| \,\|$.

(a) U is Archimedean.

(b) The maps $u \mapsto |u|$ and $u \mapsto u^+$ are uniformly continuous.

(c) For any $u \in U$, the sets $\{v : v \leq u\}$ and $\{v : v \geq u\}$ are closed; in particular, the positive cone of U is closed.

(d) Any band in U is closed.

(e) If V is a norm-dense Riesz subspace of U, then $V^+ = \{v : v \in V, v \geq 0\}$ is norm-dense in the positive cone U^+ of U.

354C Lemma If U is a Banach lattice and $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\sum_{n=0}^{\infty} \|u_n\| < \infty$, then $\sup_{n\in\mathbb{N}} u_n$ is defined in U, with $\|\sup_{n\in\mathbb{N}} u_n\| \leq \sum_{n=0}^{\infty} \|u_n\|$.

354D Definitions (a) A **Fatou norm** on a Riesz space U is a Riesz norm on U such that whenever $A \subseteq U^+$ is non-empty and upwards-directed and has a least upper bound in U, then $\|\sup A\| = \sup_{u\in A} \|u\|$.

(b) A Riesz norm on a Riesz space U has the **Levi property** if every upwards-directed norm-bounded set is bounded above.

(c) A Riesz norm on a Riesz space U is **order-continuous** if $\inf_{u\in A} \|u\| = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set with infimum 0.

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354E Proposition Let U be a Riesz space with an order-continuous Riesz norm $\| \|$.

(a) If $A \subseteq U$ is non-empty and upwards-directed and has a supremum, then $\sup A \in \overline{A}$.

(b) $\| \|$ is Fatou.

(c) If $A \subseteq U$ is non-empty and upwards-directed and bounded above, then for every $\epsilon > 0$ there is a $u \in A$ such that $\|(v-u)^+\| \le \epsilon$ for every $v \in A$; that is, the filter $\mathcal{F}(A\uparrow)$ on U generated by $\{\{v : v \in A, u \le v\} : u \in A\}$ is a Cauchy filter.

(d) Any non-decreasing order-bounded sequence in U is Cauchy.

(e) If U is a Banach lattice it is Dedekind complete.

(f) Every order-dense Riesz subspace of U is norm-dense.

(g) Every norm-closed solid linear subspace of U is a band.

354F Lemma If U is an Archimedean Riesz space with an order unit e, there is a Riesz norm $\| \|_e$ defined on U by the formula

$$\|u\|_e = \min\{\alpha : \alpha \ge 0,\ |u| \le \alpha e\}$$

for every $u \in U$.

354G Definitions (a) If U is an Archimedean Riesz space and e an order unit in U, the norm $\| \|_e$ as defined in 354F is the **order-unit norm** on U associated with e.

(b) An **M-space** is a Banach lattice in which the norm is an order-unit norm.

(c) If U is an M-space, its **standard order unit** is the order unit e such that $\| \|_e$ is the norm of U.

354H Examples (a) For any set X, $\ell^\infty(X)$ is an M-space with standard order unit χX.

(b) For any topological space X, the space $C_b(X)$ of bounded continuous real-valued functions on X is an M-space with standard order unit χX.

(c) For any measure space (X, Σ, μ), the space $L^\infty(\mu)$ is an M-space with standard order unit $\chi X^\bullet$.

354I Lemma Let U be an Archimedean Riesz space with order unit e, and V a subset of U which is dense for the order-unit norm $\| \|_e$. Then for any $u \in U$ there are sequences $\langle v_n\rangle_{n\in\mathbb{N}}$, $\langle w_n\rangle_{n\in\mathbb{N}}$ in V such that $v_n \le v_{n+1} \le u \le w_{n+1} \le w_n$ and $\|w_n - v_n\|_e \le 2^{-n}$ for every n; so that $u = \sup_{n\in\mathbb{N}} v_n = \inf_{n\in\mathbb{N}} w_n$ in U.

If V is a Riesz subspace of U, and $u \ge 0$, we may suppose that $v_n \ge 0$ for every n. Consequently V is order-dense in U.

354J Proposition Let U be an Archimedean Riesz space with an order unit e. Then $\| \|_e$ is Fatou and has the Levi property.

354K Theorem Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that e corresponds to χX and $\| \|_e$ corresponds to $\| \|_\infty$; moreover, this embedding is essentially unique.

354L Corollary Any M-space U is isomorphic, as Banach lattice, to $C(X)$ for some compact Hausdorff X, and the isomorphism is essentially unique. X can be identified with the set of Riesz homomorphisms $x : U \to \mathbb{R}$ such that $x(e) = 1$, where e is the standard order unit of U, with the topology induced by the product topology on $\mathbb{R}^U$.

354M Definition An **L-space** is a Banach lattice U such that $\|u+v\| = \|u\| + \|v\|$ whenever $u, v \in U^+$.

Example If (X, Σ, μ) is any measure space, then $L^1(\mu)$, with its norm $\| \|_1$, is an L-space. ℓ^1 is an L-space.

354N Theorem If U is an L-space, then its norm is order-continuous and has the Levi property.

354O Proposition If U is an L-space and V is a norm-closed Riesz subspace of U, then V is an L-space in its own right. In particular, any band in U is an L-space.

354P Uniform integrability in L-spaces: Definition Let U be an L-space. A set $A \subseteq U$ is **uniformly integrable** if for every $\epsilon > 0$ there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$.

354Q Proposition If (X, Σ, μ) is any measure space, then a subset of $L^1 = L^1(\mu)$ is uniformly integrable in the sense of 354P iff it is uniformly integrable in the sense of 246A.

354R Theorem Let U be an L-space.

(a) If $A \subseteq U$ is uniformly integrable, then

(i) A is norm-bounded;

(ii) every subset of A is uniformly integrable;

(iii) for any $\alpha \in \mathbb{R}$, αA is uniformly integrable;

(iv) there is a uniformly integrable, solid, convex, norm-closed set $C \supseteq A$;

(v) for any other uniformly integrable set $B \subseteq U$, $A \cup B$ and $A + B$ are uniformly integrable.

(b) For any set $A \subseteq U$, the following are equiveridical:

(i) A is uniformly integrable;

(ii) $\lim_{n\to\infty}(|u_n| - \sup_{i<n} |u_i|)^+ = 0$ for every sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A;

(iii) *either* A is empty *or* for every $\epsilon > 0$ there are $u_0, \dots , u_n \in A$ such that $\|(|u| - \sup_{i\le n} |u_i|)^+\| \le \epsilon$ for every $u \in A$;

(iv) A is norm-bounded and any disjoint sequence in the solid hull of A is norm-convergent to 0.

(c) If $V \subseteq U$ is a closed Riesz subspace then a subset of V is uniformly integrable when regarded as a subset of V iff it is uniformly integrable when regarded as a subset of U.

Version of 1.12.07

355 Spaces of linear operators

We come now to a discussion of linear operators between Riesz spaces. Linear operators are central to any kind of functional analysis, and a feature of the theory of Riesz spaces is the way the order structure picks out certain classes of operators for special consideration. Here I concentrate on positive and order-continuous operators, with a brief mention of sequential order-continuity. It turns out, in fact, that we need to work with operators which are differences of positive operators or of order-continuous positive operators. I define the basic spaces $\mathsf{L}^\sim$, $\mathsf{L}^\times$ and $\mathsf{L}^\sim_c$ (355A, 355G), with their most important properties (355B, 355E, 355H-355I) and some remarks on the special case of Banach lattices (355C, 355K). At the same time I give an important theorem on extension of operators (355F) and a corollary (355J).

The most important case is of course that in which the codomain is $\mathbb{R}$, so that our operators become real-valued functionals; I shall come to these in the next section.

355A Definition Let U and V be Riesz spaces. A linear operator $T : U \to V$ is **order-bounded** if $T[A]$ is order-bounded in V for every order-bounded $A \subseteq U$.

I will write $\mathsf{L}^\sim(U; V)$ for the set of order-bounded linear operators from U to V.

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355B Lemma If U and V are Riesz spaces,

(a) a linear operator $T : U \to V$ is order-bounded iff $\{Tu : 0 \le u \le w\}$ is bounded above in V for every $w \in U^+$;

(b) in particular, any positive linear operator from U to V belongs to $\mathsf{L}^\sim = \mathsf{L}^\sim(U;V)$;

(c) $\mathsf{L}^\sim$ is a linear space;

(d) if W is another Riesz space and $T : U \to V$ and $S : V \to W$ are order-bounded linear operators, then $ST : U \to W$ is order-bounded.

355C Theorem If U and V are Banach lattices then every order-bounded linear operator from U to V is continuous.

355D Lemma Let U be a Riesz space and V any linear space over $\mathbb{R}$. Then a function $T : U^+ \to V$ extends to a linear operator from U to V iff

$$T(u + u') = Tu + Tu', \quad T(\alpha u) = \alpha Tu$$

for all u, $u' \in U^+$ and every $\alpha > 0$, and in this case the extension is unique.

355E Theorem Let U be a Riesz space and V a Dedekind complete Riesz space.

(a) The space $\mathsf{L}^\sim$ of order-bounded linear operators from U to V is a Dedekind complete Riesz space; its positive cone is the set of positive linear operators from U to V. In particular, every order-bounded linear operator from U to V is expressible as the difference of positive linear operators.

(b) For $T \in \mathsf{L}^\sim$, T^+ and $|T|$ are defined in the Riesz space $\mathsf{L}^\sim$ by the formulae

$$T^+(w) = \sup\{Tu : 0 \le u \le w\},$$

$$|T|(w) = \sup\{Tu : |u| \le w\} = \sup\{\sum_{i=0}^n |Tu_i| : \sum_{i=0}^n |u_i| = w\}$$

for every $w \in U^+$.

(c) If S, $T \in \mathsf{L}^\sim$ then

$$(S \vee T)(w) = \sup_{0 \le u \le w} Su + T(w - u), \quad (S \wedge T)(w) = \inf_{0 \le u \le w} Su + T(w - u)$$

for every $w \in U^+$.

(d) Suppose that $A \subseteq \mathsf{L}^\sim$ is non-empty and upwards-directed. Then A is bounded above in $\mathsf{L}^\sim$ iff $\{Tu : T \in A\}$ is bounded above in V for every $u \in U^+$, and in this case $(\sup A)(u) = \sup_{T\in A} Tu$ for every $u \ge 0$.

(e) Suppose that $A \subseteq (\mathsf{L}^\sim)^+$ is non-empty and downwards-directed. Then $\inf A = 0$ in $\mathsf{L}^\sim$ iff $\inf_{T\in A} Tu = 0$ in V for every $u \in U^+$.

355F Theorem Let U and V be Riesz spaces, $U_0 \subseteq U$ a Riesz subspace and $T_0 : U_0 \to V$ a positive linear operator such that $Su = \sup\{T_0 w : w \in U_0,\ 0 \le w \le u\}$ is defined in V for every $u \in U^+$. Suppose *either* that U_0 is order-dense and that T_0 is order-continuous *or* that U_0 is solid.

(a) There is a unique positive linear operator $T : U \to V$, extending T_0, which agrees with S on U^+.

(b) If T_0 is a Riesz homomorphism so is T.

(c) If T_0 is order-continuous so is T.

(d) If U_0 is order-dense and T_0 is an injective Riesz homomorphism, then T is injective.

(e) If U_0 is order-dense and T_0 is order-continuous then T is the only order-continuous positive linear operator from U to V extending T_0.

355G Definition Let U be a Riesz space and V a Dedekind complete Riesz space. Then $\mathsf{L}^\times(U;V)$ will be the set of those $T \in \mathsf{L}^\sim(U;V)$ expressible as the difference of order-continuous positive linear operators, and $\mathsf{L}^\sim_c(U;V)$ will be the set of those $T \in \mathsf{L}^\sim(U;V)$ expressible as the difference of sequentially order-continuous positive linear operators.

$$ST \in \mathsf{L}^\times(U;W) \text{ whenever } S \in \mathsf{L}^\times(V;W),\ T \in \mathsf{L}^\times(U;V),$$

$$ST \in \mathsf{L}^\sim_c(U;W) \text{ whenever } S \in \mathsf{L}^\sim_c(V;W),\ T \in \mathsf{L}^\sim_c(U;V),$$

for all Riesz spaces U and all Dedekind complete Riesz spaces V, W.

355H Theorem Let U be a Riesz space and V a Dedekind complete Riesz space. Then

(i) $\mathsf{L}^\times = \mathsf{L}^\times(U;V)$ is a projection band in $\mathsf{L}^\sim = \mathsf{L}^\sim(U;V)$, therefore a Dedekind complete Riesz space in its own right;

(ii) a member T of $\mathsf{L}^\sim$ belongs to $\mathsf{L}^\times$ iff $|T|$ is order-continuous.

355I Theorem Let U be a Riesz space and V a Dedekind complete Riesz space. Then $\mathsf{L}^\sim_c(U;V)$ is a projection band in $\mathsf{L}^\sim(U;V)$, and a member T of $\mathsf{L}^\sim(U;V)$ belongs to $\mathsf{L}^\sim_c(U;V)$ iff $|T|$ is sequentially order-continuous.

355J Proposition Let U be a Riesz space and V a Dedekind complete Riesz space. Let $U_0 \subseteq U$ be an order-dense Riesz subspace; then $T \mapsto T\restriction U_0$ is an embedding of $\mathsf{L}^\times(U;V)$ as a solid linear subspace of $\mathsf{L}^\times(U_0;V)$. In particular, any operator in $\mathsf{L}^\times(U_0;V)$ has at most one extension in $\mathsf{L}^\times(U;V)$.

355K Proposition Let U be a Banach lattice with an order-continuous norm.

(a) If V is any Archimedean Riesz space and $T : U \to V$ is a positive linear operator, then T is order-continuous.

(b) If V is a Dedekind complete Riesz space then $\mathsf{L}^\times(U;V) = \mathsf{L}^\sim(U;V)$.

Version of 5.10.04

356 Dual spaces

As always in functional analysis, large parts of the theory of Riesz spaces are based on the study of linear functionals. Following the scheme of the last section, I define spaces $U^\sim$, $U^\sim_c$ and $U^\times$, the 'order-bounded', 'sequentially order-continuous' and 'order-continuous' duals of a Riesz space U (356A). These are Dedekind complete Riesz spaces (356B). If U carries a Riesz norm they are closely connected with the normed space dual U^*, which is itself a Banach lattice (356D). For each of them, we have a canonical Riesz homomorphism from U to the corresponding bidual. The map from U to $U^{\times\times}$ is particularly important (356I); when this map is an isomorphism we call U 'perfect' (356J). The last third of the section deals with L- and M-spaces and the duality between them (356N, 356P), with two important theorems on uniform integrability (356O, 356Q).

356A Definition Let U be a Riesz space.

(a) I write $U^\sim$ for the space $\mathcal{L}^\sim(U;\mathbb{R})$ of order-bounded real-valued linear functionals on U, the **order-bounded dual** of U.

(b) $U^\sim_c$ will be the space $\mathcal{L}^\sim_c(U;\mathbb{R})$ of differences of sequentially order-continuous positive real-valued linear functionals on U, the **sequentially order-continuous dual** of U.

(c) $U^\times$ will be the space $\mathcal{L}^\times(U;\mathbb{R})$ of differences of order-continuous positive real-valued linear functionals on U, the **order-continuous dual** of U.

356B Theorem For any Riesz space U, $U^\sim$ is a Dedekind complete Riesz space in which $U^\sim_c$ and $U^\times$ are projection bands, therefore Dedekind complete Riesz spaces in their own right. For $f \in U^\sim$, f^+ and $|f| \in U^\sim$ are defined by the formulae

$$f^+(w) = \sup\{f(u) : 0 \le u \le w\}, \quad |f|(w) = \sup\{f(u) : |u| \le w\}$$

for every $w \in U^+$. A non-empty upwards-directed set $A \subseteq U^\sim$ is bounded above iff $\sup_{f\in A} f(u)$ is finite for every $u \in U$, and in this case $(\sup A)(u) = \sup_{f\in A} f(u)$ for every $u \in U^+$.

356C Proposition Let U be any Riesz space and P a band projection on U. Then its adjoint $P' : U^\sim \to U^\sim$, defined by setting $P'(f) = fP$ for every $f \in U^\sim$, is a band projection on $U^\sim$.

356D Proposition Let U be a Riesz space with a Riesz norm.

(a) The normed space dual U^* of U is a solid linear subspace of $U^\sim$, and in itself is a Banach lattice with a Fatou norm and has the Levi property.

(b) The norm of U is order-continuous iff $U^* \subseteq U^\times$.

(c) If U is a Banach lattice, then $U^* = U^\sim$, so that $U^\sim$, $U^\times$ and $U_c^\sim$ are all Banach lattices.

(d) If U is a Banach lattice with order-continuous norm then $U^* = U^\times = U^\sim$.

356E Biduals: Lemma Let U be a Riesz space and $f : U \to \mathbb{R}$ a positive linear functional. Then for any $u \in U^+$ there is a positive linear functional $g : U \to \mathbb{R}$ such that $0 \le g \le f$, $g(u) = f(u)$ and $g(v) = 0$ whenever $u \wedge v = 0$.

356F Theorem Let U be a Riesz space and V a solid linear subspace of $U^\sim$. For $u \in U$ define $\hat{u} : V \to \mathbb{R}$ by setting $\hat{u}(f) = f(u)$ for every $f \in V$. Then $u \mapsto \hat{u}$ is a Riesz homomorphism from U to $V^\times$.

356G Lemma Suppose that U is a Riesz space such that $U^\sim$ separates the points of U. Then U is Archimedean.

356H Lemma Let U be an Archimedean Riesz space and $f > 0$ in $U^\times$. Then there is a $u \in U$ such that (i) $u > 0$ (ii) $f(v) > 0$ whenever $0 < v \le u$ (iii) $g(u) = 0$ whenever $g \wedge f = 0$ in $U^\times$. Moreover, if $u_0 \in U^+$ is such that $f(u_0) > 0$, we can arrange that $u \le u_0$.

356I Theorem Let U be any Archimedean Riesz space. Then the canonical map from U to $U^{\times\times}$ is an order-continuous Riesz homomorphism from U onto an order-dense Riesz subspace of $U^{\times\times}$. If U is Dedekind complete, its image in $U^{\times\times}$ is solid.

356J Definition A Riesz space U is **perfect** if the canonical map from U to $U^{\times\times}$ is an isomorphism.

356K Proposition A Riesz space U is perfect iff (i) it is Dedekind complete (ii) $U^\times$ separates the points of U (iii) whenever $A \subseteq U$ is non-empty and upwards-directed and $\{f(u) : u \in A\}$ is bounded for every $f \in U^\times$, then A is bounded above in U.

356L Proposition (a) Any band in a perfect Riesz space is a perfect Riesz space in its own right.

(b) For any Riesz space U, $U^\sim$ is perfect; consequently $U_c^\sim$ and $U^\times$ are perfect.

356M Proposition If U is a Banach lattice in which the norm is order-continuous and has the Levi property, then U is perfect.

356N L- and M-spaces: Proposition Let U be an Archimedean Riesz space with an order-unit norm.

(a) $U^* = U^\sim$ is an L-space.

(b) If e is the standard order unit of U, then $\|f\| = |f|(e)$ for every $f \in U^*$.

(c) A linear functional $f : U \to \mathbb{R}$ is positive iff it belongs to U^* and $\|f\| = f(e)$.

(d) If $e \ne 0$ there is a positive linear functional f on U such that $f(e) = 1$.

356O Theorem Let U be an Archimedean Riesz space with order-unit norm. Then a set $A \subseteq U^* = U^\sim$ is uniformly integrable iff it is norm-bounded and $\lim_{n\to\infty} \sup_{f\in A} |f(u_n)| = 0$ for every order-bounded disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U^+.

356P Proposition Let U be an L-space.

(a) U is perfect.

(b) $U^* = U^\sim = U^\times$ is an M-space; its standard order unit is the functional $\int$ defined by setting $\int u = \|u^+\| - \|u^-\|$ for every $u \in U$.

(c) If $A \subseteq U$ is non-empty and upwards-directed and $\sup_{u\in A} \int u$ is finite, then $\sup A$ is defined in U and $\int \sup A = \sup_{u\in A} \int u$.

356Q Theorem Let U be any L-space. Then a subset of U is uniformly integrable iff it is relatively weakly compact.

Version of 16.2.17

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

353H Principal bands This proposition, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 353I.

353I Projection bands This proposition, referred to in the 2003, 2006 and 2013 printings of Volume 4 and the 2008 and 2015 printings of Volume 5, is now 353J.

353K Solid linear subspaces This proposition, referred to in the 2008 and 2015 printings of Volume 5, is now 353L.

353M Riesz spaces with order units This theorem, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 353N.

353P f-algebras with identity This proposition, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 353Q.

354Yk Complexifications of normed Riesz spaces This exercise, referred to in the 2003 edition of Volume 4, is now 354Yl.

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