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Chapter 35

Riesz spaces

The next three chapters are devoted to an abstract description of the 'function spaces' described in Chapter 24, this time concentrating on their internal structure and relationships with their associated measure algebras. I find that any convincing account of these must involve a substantial amount of general theory concerning partially ordered linear spaces, and in particular various types of Riesz space or vector lattice. I therefore provide an introduction to this theory, a kind of appendix built into the middle of the volume. The relation of this chapter to the next two is very like the relation of Chapter 31 to Chapter 32. As with Chapter 31, it is not really meant to be read for its own sake; those with a particular interest in Riesz spaces might be better served by LUXEMBURG & ZAANEN 71, SCHAEFER 74, ZAANEN 83 or my own book FREMLIN 74A.

I begin with three sections in an easy gradation towards the particular class of spaces which we need to understand: partially ordered linear spaces (§351), general Riesz spaces (§352) and Archimedean Riesz spaces (§353); the last includes notes on Dedekind (σ-)complete spaces. These sections cover the fragments of the algebraic theory of Riesz spaces which I will use. In the second half of the chapter, I deal with normed Riesz spaces (in particular, L- and M-spaces)(§354), spaces of linear operators (§355) and dual Riesz spaces (§356).

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351 Partially ordered linear spaces

I begin with an account of the most basic structures which involve an order relation on a linear space, partially ordered linear spaces. As often in this volume, I find myself impelled to do some of the work in very much greater generality than is strictly required, in order to show more clearly the nature of the arguments being used. I give the definition (351A) and most elementary properties (351B-351L) of partially ordered linear spaces; then I describe a general representation theorem for arbitrary partially ordered linear spaces as subspaces of reduced powers of $\mathbb{R}$ (351M-351Q). I end with a brief note on Archimedean partially ordered linear spaces (351R).

351A Definition I repeat a definition mentioned in 241E. A **partially ordered linear space** is a linear space $(U,+,\cdot)$ over $\mathbb{R}$ together with a partial order $\le$ on U such that

$$u \le v \Longrightarrow u+w \le v+w,$$

$$u \ge 0,\ \alpha \ge 0 \Longrightarrow \alpha u \ge 0$$

for u, v, $w \in U$ and $\alpha \in \mathbb{R}$.

351B Elementary facts Let U be a partially ordered linear space. We have the following elementary consequences of the definition above, corresponding to the familiar rules for manipulating inequalities among real numbers.

(a) For u, $v \in U$,

$$u \le v \Longrightarrow 0 = u + (-u) \le v + (-u) = v - u \Longrightarrow u = 0 + u \le v - u + u = v,$$

$$u \le v \Longrightarrow -v = u + (-v-u) \le v + (-v-u) = -u.$$

(b) Suppose that u, $v \in U$ and $u \le v$. Then $\alpha u \le \alpha v$ for every $\alpha \ge 0$ and $\alpha v \le \alpha u$ for every $\alpha \le 0$. **P** (i) If $\alpha \ge 0$, then $\alpha(v-u) \ge 0$ so $\alpha v \ge \alpha u$. (ii) If $\alpha \le 0$ then $(-\alpha)u \le (-\alpha)v$ so

$$\alpha v = -(-\alpha)v \le -(-\alpha u) = u. \text{ } \mathbf{Q}$$

(c) If $u \ge 0$ and $\alpha \le \beta$ in $\mathbb{R}$, then $(\beta-\alpha)u \ge 0$, so $\alpha u \le \beta u$. If $0 \le u \le v$ in U and $0 \le \alpha \le \beta$ in $\mathbb{R}$, then $\alpha u \le \beta u \le \beta v$.

351C Positive cones Let U be a partially ordered linear space.

(a) I will write U^+ for the **positive cone** of U, the set $\{u : u \in U,\, u \ge 0\}$.

(b) By 351Ba, the ordering is determined by the positive cone U^+, in the sense that $u \le v \iff v - u \in U^+$.

(c) It is easy to characterize positive cones. If U is a real linear space, a set $C \subseteq U$ is the positive cone for some ordering rendering U a partially ordered linear space iff

$$u+v \in C, \quad \alpha u \in C \text{ whenever } u,\, v \in C \text{ and } \alpha \ge 0,$$

$$0 \in C, \quad u \in C \;\&\; -u \in C \Longrightarrow u = 0.$$

P (i) If $C = U^+$ for some partially ordered linear space ordering $\le$ of U, then

$$u,\, v \in C \Longrightarrow 0 \le u \le u+v \Longrightarrow u+v \in C,$$

$$u \in C,\, \alpha \ge 0 \Longrightarrow \alpha u \ge 0, \text{ i.e., } \alpha u \in C,$$

$$0 \le 0 \text{ so } 0 \in C,$$

$$u,\, -u \in C \Longrightarrow u = 0 + u \le (-u) + u = 0 \le u \Longrightarrow u = 0.$$

(ii) On the other hand, if C satisfies the conditions, define the relation $\le$ by writing $u \le v \iff v - u \in C$; then

$$u - u = 0 \in C \text{ so } u \le u \text{ for every } u \in U,$$

$$\text{if } u \le v \text{ and } v \le w \text{ then } w - u = (w-v) + (v-u) \in C \text{ so } u \le w,$$

$$\text{if } u \le v \text{ and } v \le u \text{ then } u - v,\, v - u \in C \text{ so } u - v = 0 \text{ and } u = v$$

and $\le$ is a partial order; moreover,

$$\text{if } u \le v \text{ and } w \in U \text{ then } (v+w) - (u+w) = v - u \in C \text{ and } u + w \le v + w,$$

$$\text{if } u,\, \alpha \ge 0 \text{ then } \alpha u \in C \text{ and } \alpha u \ge 0,$$

$$u \ge 0 \iff u \in C.$$

So $\le$ makes U a partially ordered linear space in which C is the positive cone. **Q**

(d) An incidental useful fact. Let U be a partially ordered linear space, and $u \in U$. Then $u \ge 0$ iff $u \ge -u$. **P** If $u \ge 0$ then $0 \ge -u$ so $u \ge -u$. If $u \ge -u$ then $2u \ge 0$ so $u = \frac{1}{2} \cdot 2u \ge 0$. **Q**

(e) I have called U^+ a 'positive cone' without defining the term 'cone'. I think this is something we can pass by for the moment; but it will be useful to recognise that U^+ is always convex, for if u, $v \in U^+$ and $\alpha \in [0,1]$ then αu, $(1-\alpha)v \ge 0$ and $\alpha u + (1-\alpha)v \in U^+$, so is a 'convex cone' as defined in 3A5Ba.

351D Suprema and infima Let U be a partially ordered linear space.

(a) The definition of 'partially ordered linear space' implies that $u \mapsto u + w$ is always an order-isomorphism; on the other hand, $u \mapsto -u$ is order-reversing, by 351Ba.

(b) It follows that if $A \subseteq U$ and $v \in U$ then

$$\sup_{u\in A}(v+u) = v + \sup A \text{ if either side is defined,}$$

$$\inf_{u\in A}(v+u) = v + \inf A \text{ if either side is defined,}$$

$$\sup_{u\in A}(v-u) = v - \inf A \text{ if either side is defined,}$$

$$\inf_{u\in A}(v-u) = v - \sup A \text{ if either side is defined.}$$

(c) Moreover, we find that if A, $B \subseteq U$ and $\sup A$ and $\sup B$ are defined, then $\sup(A+B)$ is defined and equal to $\sup A + \sup B$, writing $A + B = \{u + v : u \in A,\ v \in B\}$ as usual. **P** Set $u_0 = \sup A$, $v_0 = \sup B$. Using (b), we have

$$\begin{aligned} u_0 + v_0 &= \sup_{u\in A}(u + v_0) \\ &= \sup_{u\in A}(\sup_{v\in B}(u+v)) = \sup(A+B). \text{ **Q**} \end{aligned}$$

Similarly, if A, $B \subseteq U$ and $\inf A$, $\inf B$ are defined then $\inf(A+B) = \inf A + \inf B$.

(d) If $\alpha > 0$ then $u \mapsto \alpha u$ is an order-isomorphism, so we have $\sup(\alpha A) = \alpha \sup A$ if either side is defined; similarly, $\inf(\alpha A) = \alpha \inf A$.

351E Linear subspaces If U is a partially ordered linear space, and V is any linear subspace of U, then V, with the induced linear and order structures, is a partially ordered linear space; this is obvious from the definition.

351F Positive linear operators Let U and V be partially ordered linear spaces, and write $\mathsf{L}(U;V)$ for the linear space of all linear operators from U to V. For S, $T \in \mathsf{L}(U;V)$ say that $S \le T$ iff $Su \le Tu$ for every $u \in U^+$. Under this ordering, $\mathsf{L}(U;V)$ is a partially ordered linear space; its positive cone is $\{T : Tu \ge 0$ for every $u \in U^+\}$. **P** This is an elementary verification. **Q** Note that, for $T \in \mathsf{L}(U;V)$,

$$\begin{aligned} T \ge 0 &\implies Tu \le Tu + T(v-u) = Tv \text{ whenever } u \le v \text{ in } U \\ &\implies 0 = T0 \le Tu \text{ for every } u \in U^+ \\ &\implies T \ge 0, \end{aligned}$$

so that $T \ge 0$ iff T is order-preserving. In this case we say that T is a **positive** linear operator.

Clearly ST is a positive linear operator whenever U, V and W are partially ordered linear spaces and $T : U \to V$, $S : V \to W$ are positive linear operators (cf. 313Ia).

351G Order-continuous positive linear operators: Proposition Let U and V be partially ordered linear spaces and $T : U \to V$ a positive linear operator.

(a) The following are equiveridical:

(i) T is order-continuous;

(ii) $\inf T[A] = 0$ in V whenever $A \subseteq U$ is a non-empty downwards-directed set with infimum 0 in U;

(iii) $\sup T[A] = Tw$ in V whenever $A \subseteq U^+$ is a non-empty upwards-directed set with supremum w in U.

(b) The following are equiveridical:

(i) T is sequentially order-continuous;

(ii) $\inf_{n\in\mathbb{N}} Tu_n = 0$ in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum 0 in U;

(iii) $\sup_{n\in\mathbb{N}} Tu_n = Tw$ in V whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U^+ with supremum w in U.

proof (a)(i)$\Rightarrow$(iii) is trivial.

(iii)⇒(ii) Assuming (iii), and given that A is non-empty, downwards-directed and has infimum 0, take any $u_0\in A$ and consider $A'=\{u:u\in A,\,u\le u_0\}$, $B=u_0-A'$. Then A' is non-empty, downwards-directed and has infimum 0, so B is non-empty, upwards-directed and has supremum u_0 (using 351Db); by (iii), $\sup T[B]=Tu_0$ and (inverting again)

$$\inf T[A']=\inf T[u_0-B]=\inf(Tu_0-T[B])=Tu_0-\sup T[B]=0.$$

But (because T is positive) 0 is surely a lower bound for $T[A]$, so it is also the infimum of $T[A]$. As A is arbitrary, (ii) is true.

(ii)⇒(i) Suppose now that (ii) is true. **($\boldsymbol{\alpha}$)** If $A\subseteq U$ is non-empty, downwards-directed and has infimum w, then $A-w$ is non-empty, downwards-directed and has infimum 0, so

$$\inf T[A-w]=0,\quad \inf T[A]=\inf(T[A-w]+Tw)=Tw+\inf T[A-w]=Tw.$$

($\boldsymbol{\beta}$) If $A\subseteq U$ is non-empty, upwards-directed and has supremum w, then $-A$ is non-empty, downwards-directed and has infimum $-w$, so

$$\sup T[A]=-\inf(-T[A])=-\inf T[-A]=-T(-w)=Tw.$$

Putting these together, T is order-continuous.

(b) The arguments are identical, replacing each directed set by an appropriate sequence.

351H Riesz homomorphisms (a) For the sake of a representation theorem below (351Q), I introduce the following definition. Let U, V be partially ordered linear spaces. A **Riesz homomorphism** from U to V is a linear operator $T:U\to V$ such that whenever $A\subseteq U$ is a finite non-empty set and $\inf A=0$ in U, then $\inf T[A]=0$ in V. The following facts are now nearly obvious.

(b) Any Riesz homomorphism is a positive linear operator. (For if T is a Riesz homomorphism and $u\ge 0$, then $\inf\{0,u\}=0$ so $\inf\{0,Tu\}=0$ and $Tu\ge 0$.)

(c) Let U and V be partially ordered linear spaces and $T:U\to V$ a Riesz homomorphism. Then

$$\inf T[A]\text{ exists}=T(\inf A),\qquad \sup T[A]\text{ exists}=T(\sup A)$$

whenever $A\subseteq U$ is a finite non-empty set and $\inf A$, $\sup A$ exist. (Apply the definition in (a) to

$$A'=\{u-\inf A:u\in A\},\quad A''=\{\sup A-u:u\in A\}.)$$

(d) If U, V and W are partially ordered linear spaces and $T:U\to V$, $S:V\to W$ are Riesz homomorphisms then $ST:U\to W$ is a Riesz homomorphism.

351I Solid sets Let U be a partially ordered linear space. I will say that a subset A of U is **solid** if

$$A=\{v:v\in U,\,-u\le v\le u\text{ for some }u\in A\}=\bigcup_{u\in A}[-u,u]$$

in the notation of 2A1Ab. (I should perhaps remark that while this definition is well established in the case of Riesz spaces (352J), the extension to general partially ordered linear spaces is not standard. See 351Yb for a warning.)

351J Proposition Let U be a partially ordered linear space and V a solid linear subspace of U. Then the quotient linear space U/V has a partially ordered linear space structure defined by either of the rules

$u^\bullet\le w^\bullet$ iff there is some $v\in V$ such that $u\le v+w$,

$(U/V)^+=\{u^\bullet:u\in U^+\}$,

and for this partial order on U/V the map $u\mapsto u^\bullet:U\to U/V$ is a Riesz homomorphism.

proof (a) I had better start by giving priority to one of the descriptions of the relation $\le$ on U/V; I choose the first. To see that this makes U/V a partially ordered linear space, we have to check the following.

(i) $0\in V$ and $u\le u+0$, so $u^\bullet\le u^\bullet$ for every $u\in U$.

(ii) If u_1, u_2, $u_3\in U$ and $u_1^\bullet\le u_2^\bullet$, $u_2^\bullet\le u_3^\bullet$ then there are v_1, $v_2\in V$ such that $u_1\le u_2+v_1$, $u_2\le u_3+v_2$; in which case $v_1+v_2\in V$ and $u_1\le u_3+v_1+v_2$, so $u_1^\bullet\le u_3^\bullet$.

(iii) If u, $w \in U$ and $u^\bullet \le w^\bullet$, $w^\bullet \le u^\bullet$ then there are v, $v' \in V$ such that $u \le w + v$, $w \le u + v'$. Now there are v_0, $v_0' \in V$ such that $-v_0 \le v \le v_0$, $-v_0' \le v' \le v_0'$, and in this case v_0, $v_0' \ge 0$ (351Cd), so

$$-v_0 - v_0' \le -v' \le u - w \le v \le v_0 + v_0' \in V,$$

.

Accordingly $u - w \in V$ and $u^\bullet = w^\bullet$. Thus U/V is a partially ordered set.

(iv) If u_1, u_2, $w \in U$ and $u_1^\bullet \le u_2^\bullet$, then there is a $v \in V$ such that $u_1 \le u_2 + v$, in which case $u_1 + w \le u_2 + w + v$ and $u_1^\bullet + w^\bullet \le u_2^\bullet + w^\bullet$.

(v) If $u \in U$, $\alpha \in \mathbb{R}$, $u^\bullet \ge 0$ and $\alpha \ge 0$ then there is a $v \in V$ such that $u + v \ge 0$; now $\alpha v \in V$ and $\alpha u + \alpha v \ge 0$, so $\alpha u^\bullet = (\alpha u)^\bullet \ge 0$.

Thus U/V is a partially ordered linear space.

(b) Now $(U/V)^+ = \{u^\bullet : u \ge 0\}$. **P** If $u \ge 0$ then of course $u^\bullet \ge 0$ because $0 \in V$ and $u + 0 \ge 0$. On the other hand, if we have any element p of $(U/V)^+$, there are $u \in U$, $v \in V$ such that $u^\bullet = p$ and $u + v \ge 0$; but now $p = (u + v)^\bullet$ is of the required form. **Q**

(c) Finally, $u \mapsto u^\bullet$ is a Riesz homomorphism. **P** Suppose that $A \subseteq U$ is a non-empty finite set and that $\inf A = 0$ in U. Then $u^\bullet \ge 0$ for every $u \in A$, that is, 0 is a lower bound for $\{u^\bullet : u \in A\}$. Let $p \in U/V$ be any other lower bound for $\{u^\bullet : u \in A\}$. Express p as $w^\bullet$ where $w \in U$. For each $u \in A$, $w^\bullet \le u^\bullet$ so there is a $v_u \in V$ such that $w \le u + v_u$. Next, there is a $v_u' \in V$ such that $-v_u' \le v_u \le v_u'$. Set $v^* = \sum_{u \in A} v_u' \in V$. Then $v_u \le v_u' \le v^*$ so $w \le u + v^*$ for every $u \in A$, and $w - v^*$ is a lower bound for A in U. Accordingly $w - v^* \le 0$, $w \le 0 + v^*$ and $p = w^\bullet \le 0$. As p is arbitrary, $\inf\{u^\bullet : u \in A\} = 0$; as A is arbitrary, $u \mapsto u^\bullet$ is a Riesz homomorphism. **Q**

351K Lemma Suppose that U is a partially ordered linear space, and that W, V are solid linear subspaces of U such that $W \subseteq V$. Then $V_1 = \{v^\bullet : v \in V\}$ is a solid linear subspace of U/W.

proof (i) Because the map $u \mapsto u^\bullet$ is linear, V_1 is a linear subspace of U/W. (ii) If $p \in V_1$, there is a $v \in V$ such that $p = v^\bullet$; because V is solid in U, there is a $v_0 \in V$ such that $-v_0 \le v \le v_0$; now $v_0^\bullet \in V_1$ and $-v_0^\bullet \le p \le v_0^\bullet$. (iii) If $p \in V_1$, $q \in U/W$ and $-p \le q \le p$, take $v_0 \in V$, $u \in U$ such that $v_0^\bullet = p$ and $u^\bullet = q$. Because $-v_0^\bullet \le u^\bullet \le v_0^\bullet$, there are w, $w' \in W$ such that $-v_0 - w \le u \le v_0 + w'$. Now $-v_0 - w$, $v_0 + w'$ both belong to V, which is solid, so $u \in V$ and $q = u^\bullet \in V_1$. (iv) Putting (ii) and (iii) together, V_1 is solid.

351L Products If $\langle U_i \rangle_{i \in I}$ is any family of partially ordered linear spaces, we have a product linear space $U = \prod_{i \in I} U_i$; if we set $u \le v$ in U iff $u(i) \le v(i)$ for every $i \in I$, U becomes a partially ordered linear space, with positive cone $\{u : u(i) \ge 0$ for every $i \in I\}$. For each $i \in I$ the map $u \mapsto u(i) : U \to U_i$ is an order-continuous Riesz homomorphism (in fact, it preserves arbitrary suprema and infima).

351M Reduced powers of $\mathbb{R}$ (a) Let X be any set. Then $\mathbb{R}^X$ is a partially ordered linear space if we say that $f \le g$ means that $f(x) \le g(x)$ for every $x \in X$, as in 351L. If now $\mathcal{F}$ is a filter on X, we have a corresponding set

$$V = \{f : f \in \mathbb{R}^X,\, \{x : f(x) = 0\} \in \mathcal{F}\};$$

it is easy to see that V is a linear subspace of $\mathbb{R}^X$, and is solid because $f \in V$ iff $|f| \in V$. By the **reduced power** $\mathbb{R}^X|\mathcal{F}$ I shall mean the quotient partially ordered linear space $\mathbb{R}^X/V$.

(b) Note that for $f \in \mathbb{R}^X$,

$$f^\bullet \ge 0 \text{ in } \mathbb{R}^X|\mathcal{F} \iff \{x : f(x) \ge 0\} \in \mathcal{F}.$$

P (i) If $f^\bullet \ge 0$, there is a $g \in V$ such that $f + g \ge 0$; now

$$\{x : f(x) \ge 0\} \supseteq \{x : g(x) = 0\} \in \mathcal{F}.$$

(ii) If $\{x : f(x) \ge 0\} \in \mathcal{F}$, then $\{x : (|f| - f)(x) = 0\} \in \mathcal{F}$, so $f^\bullet = |f|^\bullet \ge 0$. **Q**

351N On the way to the next theorem, the main result (in terms of mathematical content) of this section, we need a string of lemmas.

Lemma Let U be a partially ordered linear space. If $u, v_0, \ldots, v_n \in U$ are such that $u \neq 0$ and $v_0, \ldots, v_n \geq 0$ then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) \neq 0$ and $f(v_i) \geq 0$ for every i.

proof The point is that at most one of u, $-u$ can belong to the convex cone C generated by $\{v_0, \ldots, v_n\}$, because this is included in the convex cone set U^+, and since $u \neq 0$ at most one of u, $-u$ can belong to U^+.

Now however the Hahn-Banach theorem, in the form 3A5D, tells us that if $u \notin C$ there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) < 0$ and $f(v_i) \geq 0$ for every i; while if $-u \notin C$ we can get $f(-u) < 0$ and $f(v_i) \geq 0$ for every i. Thus in either case we have a functional of the required type.

351O Lemma Let U be a partially ordered linear space, and u_0 a non-zero member of U. Then there is a solid linear subspace V of U such that $u_0 \notin V$ and whenever $A \subseteq U$ is finite, non-empty and has infimum 0 then $A \cap V \neq \emptyset$.

proof (a) Let $\mathcal{W}$ be the family of all solid linear subspaces of U not containing u_0. Then any non-empty totally ordered $\mathcal{V} \subseteq \mathcal{W}$ has an upper bound $\bigcup \mathcal{V}$ in $\mathcal{W}$. By Zorn's Lemma, $\mathcal{W}$ has a maximal element V say. This is surely a solid linear subspace of U not containing u_0.

(b) Now for any $w \in U^+ \setminus V$ there are $\alpha \geq 0$, $v \in V^+$ such that $-\alpha w - v \leq u_0 \leq \alpha w + v$. **P** Let V_1 be

$$\{u : u \in U, \text{ there are } \alpha \geq 0,\ v \in V^+ \text{ such that } -\alpha w - v \leq u \leq \alpha w + v\}.$$

Then it is easy to check that V_1 is a solid linear subspace of U, including V, and containing w; because $w \notin V$, $V_1 \neq V$, so $V_1 \notin \mathcal{W}$ and $u \in V_1$, as claimed. **Q**

(c) It follows that if $A \subseteq U$ is finite and non-empty and $\inf A = 0$ in U then $A \cap V \neq \emptyset$. **P?** Otherwise, for every $w \in A$ there must be $\alpha_w \geq 0$, $v_w \in V^+$ such that $-\alpha_w w - v_w \leq u_0 \leq \alpha_w w + v_w$. Set $\alpha = 1 + \sum_{w \in A} \alpha_w$, $v = \sum_{w \in A} v_w \in V$; then $-\alpha w - v \leq u_0 \leq \alpha w + v$ for every $w \in A$. Accordingly $\frac{1}{\alpha}(u_0 - v) \leq w$ for every $w \in A$ and $\frac{1}{\alpha}(u_0 - v) \leq 0$, so $u_0 \leq v$. Similarly, $-\frac{1}{\alpha}(v + u_0) \leq w$ for every $w \in A$ and $-v \leq u_0$. But (because V is solid) this means that $u_0 \in V$, which is not so. **XQ**

Accordingly V has the required properties.

351P Lemma Let U be a partially ordered linear space and u a non-zero element of U, and suppose that $A_0, \ldots, A_n$ are finite non-empty subsets of U such that $\inf A_j = 0$ for every $j \leq n$. Then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u) \neq 0$ and $\min f[A_j] = 0$ for every $j \leq n$.

proof By 351O, there is a solid linear subspace V of U such that $u \notin V$ and $A_j \cap V \neq 0$ for every $j \leq n$. Give the quotient space U/V its standard partial ordering (351J), and in U/V set $C = \{v^\bullet : v \in \bigcup_{j \leq n} A_j\}$. Then C is a finite subset of $(U/V)^+$, while $u^\bullet \neq 0$, so by 351N there is a linear functional $g : U/V \to \mathbb{R}$ such that $g(u^\bullet) \neq 0$ but $g(p) \geq 0$ for every $p \in C$. Set $f(v) = g(v^\bullet)$ for $v \in U$; then $f : U \to \mathbb{R}$ is linear, $f(u) \neq 0$ and $f(v) \geq 0$ for every $v \in \bigcup_{j \leq n} A_j$. But also, for each $j \leq n$, there is a $v_j \in A_j \cap V$, so that $f(v_j) = 0$; and this means that $\min f[A_j]$ must be 0, as required.

351Q Now we are ready for the theorem.

Theorem Let U be any partially ordered linear space. Then we can find a set X, a filter $\mathcal{F}$ on X and an injective Riesz homomorphism from U to the reduced power $\mathbb{R}^X|\mathcal{F}$ described in 351M.

proof Let $X = U'$ be the set of all linear functionals $f : U \to \mathbb{R}$; for $u \in U$ define $\hat{u} \in \mathbb{R}^X$ by setting $\hat{u}(f) = f(u)$ whenever $f \in X$ and $u \in U$. Let $\mathcal{A}$ be the family of non-empty finite sets $A \subseteq U$ such that $\inf A = 0$. For $A \in \mathcal{A}$ let F_A be the set of those $f \in X$ such that $\min f[A] = 0$. Since $0 \in F_A$ for every $A \in \mathcal{A}$, the set

$$\mathcal{F} = \{F : F \subseteq X, \text{ there are } A_0, \ldots, A_n \in \mathcal{A} \text{ such that } F \supseteq \textstyle\bigcap_{j \leq n} F_{A_j}\}$$

is a filter on X. Set $\psi(u) = \hat{u}^\bullet \in \mathbb{R}^X|\mathcal{F}$ for $u \in U$. Then $\psi : U \to \mathbb{R}^X|\mathcal{F}$ is an injective Riesz homomorphism.

P (i) ψ is linear because $u \mapsto \hat{u} : U \to \mathbb{R}^X$ and $h \mapsto h^\bullet : \mathbb{R}^X \to \mathbb{R}^X|\mathcal{F}$ are linear. (ii) If $A \in \mathcal{A}$, then $F_A \in \mathcal{F}$. So, first, if $v \in A$, then $\{f : \hat{v}(f) \geq 0\} \in \mathcal{F}$, so that $\psi(v) = \hat{v}^\bullet \geq 0$ in $\mathbb{R}^X|\mathcal{F}$ (351Mb). Next, if $w \in \mathbb{R}^X|\mathcal{F}$ and $w \leq \psi(v)$ for every $v \in A$, we can express w as $h^\bullet$ where $h^\bullet \leq \hat{v}^\bullet$ for every $v \in A$, that is, $H_v = \{f : h(f) \leq \hat{v}(f)\} \in \mathcal{F}$ for every $v \in A$. But now $H = F_A \cap \bigcap_{v \in A} H_v \in \mathcal{F}$, and for $f \in H$ we have

$h(f) \le \min_{v \in A} f(v) = 0$. This means that $w = h^\bullet \le 0$. As w is arbitrary, $\inf \psi[A] = 0$. As A is arbitrary, ψ is a Riesz homomorphism. (iii) Finally, **?** suppose, if possible, that there is a non-zero $u \in U$ such that $\psi(u) = 0$. Then $F = \{f : f(u) = 0\} \in \mathcal{F}$, and there are $A_0, \dots , A_n \in \mathcal{A}$ such that $F \supseteq \bigcap_{j \le n} F_{A_j}$. By 351P, there is an $f \in \bigcap_{j \le n} F_{A_j}$ such that $f(u) \ne 0$; which is impossible. **X** Accordingly ψ is injective, as claimed. **Q**

351R Archimedean spaces (a) For a partially ordered linear space U, the following are equiveridical: (i) if u, $v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$ then $u \le 0$ (ii) if $u \ge 0$ in U then $\inf_{\delta > 0} \delta u = 0$. **P(i)⇒(ii)** If (i) is true and $u \ge 0$, then of course $\delta u \ge 0$ for every $\delta > 0$; on the other hand, if $v \le \delta u$ for every $\delta > 0$, then $nv \le n \cdot \frac{1}{n} u = u$ for every $n \ge 1$, while of course $0v = 0 \le u$, so $v \le 0$. Thus 0 is the greatest lower bound of $\{\delta u : \delta > 0\}$. **(ii)⇒(i)** If (ii) is true and $nu \le v$ for every $n \in \mathbb{N}$, then $0 \le v$ and $u \le \frac{1}{n} v$ for every $n \ge 1$. If now $\delta > 0$, then there is an $n \ge 1$ such that $\frac{1}{n} \le \delta$, so that $u \le \frac{1}{n} v \le \delta v$ (351Bc). Accordingly u is a lower bound for $\{\delta v : \delta > 0\}$ and $u \le 0$. **Q**

(b) I will say that partially ordered linear spaces satisfying the equiveridical conditions of (a) above are **Archimedean**.

(c) Any linear subspace of an Archimedean partially ordered linear space, with the induced partially ordered linear space structure, is Archimedean.

(d) Any product of Archimedean partially ordered linear spaces is Archimedean. **P** If $U = \prod_{i \in I} U_i$ is a product of Archimedean spaces, and $nu \le v$ in U for every $n \in \mathbb{N}$, then for each $i \in I$ we must have $nu(i) \le v(i)$ for every n, so that $u(i) \le 0$; accordingly $u \le 0$. **Q** In particular, $\mathbb{R}^X$ is Archimedean for any set X.

351X Basic exercises >(a) Let ζ be any ordinal. The **lexicographic ordering** of $\mathbb{R}^\zeta$ is defined by saying that $f \le g$ iff either $f = g$ or there is a $\xi < \zeta$ such that $f(\eta) = g(\eta)$ for $\eta < \xi$ and $f(\xi) < g(\xi)$. Show that this is a total order on $\mathbb{R}^\zeta$ which renders $\mathbb{R}^\zeta$ a partially ordered linear space.

(b) Let U be a partially ordered linear space and V a linear subspace of U. Show that the formulae of 351J define a partially ordered linear space structure on the quotient U/V iff V is **order-convex**, that is, $u \in V$ whenever v_1, $v_2 \in V$ and $v_1 \le u \le v_2$.

(c) Let $\langle U_i \rangle_{i \in I}$ be a family of partially ordered linear spaces with product U. For $i \in I$, define $T_i : U_i \to U$ by setting $T_i x = u$ where $u(i) = x$, $u(j) = 0$ for $j \ne i$. Show that T_i is an injective order-continuous Riesz homomorphism.

>(d) Let U be a partially ordered linear space and $\langle V_i \rangle_{i \in I}$ a family of partially ordered linear spaces with product V. Show that $\mathsf{L}(U; V)$ can be identified, as partially ordered linear space, with $\prod_{i \in I} \mathsf{L}(U; V_i)$.

>(e) Show that if U, V are partially ordered linear spaces and V is Archimedean, then $\mathsf{L}(U; V)$ is Archimedean.

351Y Further exercises (a) Give an example of two partially ordered linear spaces U and V and a bijective Riesz homomorphism $T : U \to V$ such that $T^{-1} : V \to U$ is not a Riesz homomorphism.

(b)(i) Let U be a partially ordered linear space. Show that U is a solid subset of itself (on the definition 351I) iff $U = U^+ - U^+$. (ii) Give an example of a partially ordered linear space U satisfying this condition with an element $u \in U$ such that the intersection of the solid sets containing u is not solid.

(c) Show that a reduced power $\mathbb{R}^X | \mathcal{F}$, as described in 351M, is totally ordered iff $\mathcal{F}$ is an ultrafilter, and in this case has a natural structure as a totally ordered field.

(d) Let U be a partially ordered linear space, and suppose that A, $B \subseteq U$ are two non-empty finite sets such that (α) $u \vee v = \sup\{u, v\}$ is defined for every $u \in A$, $v \in B$ (β) $\inf A$ and $\inf B$ and $(\inf A) \vee (\inf B)$ are defined. Show that $\inf\{u \vee v : u \in A, v \in B\} = (\inf A) \vee (\inf B)$. (*Hint*: show that this is true if $U = \mathbb{R}$, if $U = \mathbb{R}^X$ and if $U = \mathbb{R}^X | \mathcal{F}$, and use 351Q.)

(e) Show that a reduced power $\mathbb{R}^X|\mathcal{F}$, as described in 351M, is Archimedean iff $\bigcap_{n\in\mathbb{N}} F_n \in \mathcal{F}$ whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$.

351 Notes and comments The idea of 'partially ordered linear space' is a very natural abstraction from the elementary examples of $\mathbb{R}^X$ and its subspaces, and the only possible difficulty lies in guessing the exact boundary at which one's standard manipulations with such familiar spaces cease to be valid in the general case. (For instance, most people's favourite examples are Archimedean, in the sense of 351R, so it is prudent to check your intuitions against a non-Archimedean space like that of 351Xa.) There is really no room for any deep idea to appear in 351B-351F. When I come to what I call 'Riesz homomorphisms', however (351H), there are some more interesting possibilities in the background.

I shall not discuss the applications of Theorem 351Q to general partially ordered linear spaces; it is here for the sake of its application to Riesz spaces in the next section. But I think it is a very striking fact that not only does any partially ordered linear space U appear as a linear subspace of some reduced power of $\mathbb{R}$, but the embedding can be taken to preserve any suprema and infima of finite sets which exist in U. This is in a sense a result of the same kind as the Stone representation theorem for Boolean algebras; it gives us a chance to confirm that an intuition valid for $\mathbb{R}$ or $\mathbb{R}^X$ may in fact apply to arbitrary partially ordered linear spaces. If you like, this provides a metamathematical foundation for such results as those in 351B. I have to say that for partially ordered linear spaces it is generally quicker to find a proof directly from the definition than to trace through an argument relying on 351Q; but this is not always the case for Riesz spaces. I offer 351Yd as an example of a result where a direct proof does at least call for a moment's thought, while the argument through 351Q is straightforward.

'Reduced powers' are of course of great importance for other reasons; I mention 351Yc as a hint of what can be done.

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352 Riesz spaces

In this section I sketch those fragments of the theory we need which can be expressed as theorems about general Riesz spaces or vector lattices. I begin with the definition (352A) and most elementary properties (352C-352F). In 352G-352J I discuss Riesz homomorphisms and the associated subspaces (Riesz subspaces, solid linear subspaces); I mention product spaces (352K, 352T) and quotient spaces (352Jb, 352U) and the form the representation theorem 351Q takes in the present context (352L-352M). Most of the second half of the section concerns the theory of 'bands' in Riesz spaces, with the algebras of complemented bands (352Q) and projection bands (352S) and a description of bands generated by upwards-directed sets (352V). I conclude with a description of 'f-algebras' (352W).

352A I repeat a definition from 241E.

Definition A **Riesz space** or **vector lattice** is a partially ordered linear space which is a lattice.

352B Lemma If U is a partially ordered linear space, then it is a Riesz space iff $\sup\{0,u\}$ is defined for every $u\in U$.

proof If U is a lattice, then of course $\sup\{u,0\}$ is defined for every u. If $\sup\{u,0\}$ is defined for every u, and v_1, v_2 are any two members of U, consider $w=v_1+\sup\{0,v_2-v_1\}$; by 351Db, $w=\sup\{v_1,v_2\}$. Next,

$$\inf\{v_1,v_2\}=-\sup\{-v_1,-v_2\}$$

must also be defined in U, because $v\mapsto -v$ is order-reversing; as v_1 and v_2 are arbitrary, U is a lattice.

352C Notation In any Riesz space U I will write

$$u^+=u\vee 0,\quad u^-=(-u)\vee 0=(-u)^+,\quad |u|=u\vee(-u)$$

where (as in any lattice) $u\vee v=\sup\{u,v\}$ (and $u\wedge v=\inf\{u,v\}$).

I mention immediately a term which will be useful: a family $\langle u_i\rangle_{i\in I}$ in U is **disjoint** if $|u_i|\wedge|u_j|=0$ for all distinct i, $j\in I$. Similarly, a set $C\subseteq U$ is **disjoint** if $|u|\wedge|v|=0$ for all distinct u, $v\in C$.

352D Elementary identities Let U be a Riesz space. The translation-invariance of the order, and its invariance under positive scalar multiplication, reversal under negative multiplication, lead directly to the following, which are in effect special cases of 351D:

$$u+(v\vee w)=(u+v)\vee(u+w),\quad u+(v\wedge w)=(u+v)\wedge(u+w),$$

$$\alpha(u\vee v)=\alpha u\vee\alpha v \text{ and } \alpha(u\wedge v)=\alpha u\wedge\alpha v \text{ if } \alpha\ge 0,$$

$$-(u\vee v)=(-u)\wedge(-v).$$

Combining and elaborating on these facts, we get

$$u^+-u^-=(u\vee 0)-((-u)\vee 0)=u+(0\vee(-u))-((-u)\vee 0)=u,$$

$$u^++u^-=2u^+-u=(2u\vee 0)-u=u\vee(-u)=|u|,$$

$$u\ge 0\iff -u\le 0\iff u^-=0\iff u=u^+\iff u=|u|,$$

$$|-u|=|u|,\quad |\,|u|\,|=|u|,\quad |\alpha u|=|\alpha||u|$$

(looking at the cases $\alpha\ge 0$, $\alpha\le 0$ separately),

$$\begin{aligned}u\vee v+u\wedge v&=u+(0\vee(v-u))+v+((u-v)\wedge 0)\\&=u+(0\vee(v-u))+v-((v-u)\vee 0)=u+v,\end{aligned}$$

$$u\vee v=u+(0\vee(v-u))=u+(v-u)^+,$$

$$u\wedge v=u+(0\wedge(v-u))=u-(-0\vee(u-v))=u-(u-v)^+,$$

$$u\vee v=\frac{1}{2}(2u\vee 2v)=\frac{1}{2}(u+v+(u-v)\vee(v-u))=\frac{1}{2}(u+v+|u-v|),$$

$$u\wedge v=u+v-u\vee v=\frac{1}{2}(u+v-|u-v|),$$

$$u^+\vee u^-=u\vee(-u)\vee 0=|u|,\quad u^+\wedge u^-=u^++u^--(u^+\vee u^-)=0,$$

$$|u+v|=(u+v)\vee((-u)+(-v))\le(|u|+|v|)\vee(|u|+|v|)=|u|+|v|,$$

$$\begin{aligned}||u|-|v||=(|u|-|v|)\vee(|v|-|u|)&\le(|u-v|+|v|-|v|)\vee(|v-u|+|u|-|u|)\\&=|u-v|\vee|v-u|=|u-v|,\end{aligned}$$

$$|u\vee v|\le|u|+|v|$$

(because $-|u|\le u\vee v\le|u|\vee|v|\le|u|+|v|$)

for u, $v\in U$ and $\alpha\in\mathbb{R}$.

352E Distributive laws Let U be a Riesz space.

(a) If A, $B\subseteq U$ have suprema a, b in U, then $C=\{u\wedge v:u\in A,\,v\in B\}$ has supremum $a\wedge b$. **P** Of course $u\wedge v\le a\wedge b$ for all $u\in A$, $v\in B$, so $a\wedge b$ is an upper bound for C. Now suppose that c is any upper bound for C. If $u\in A$ and $v\in B$ then

$$u-(u-v)^+=u\wedge v\le c,\quad u\le c+(u-v)^+\le c+(a-v)^+$$

(because $(a-v)^+=\sup\{a-v,0\}\ge\sup\{u-v,0\}=(u-v)^+$). As u is arbitrary, $a\le c+(a-v)^+$ and $a\wedge v\le c$. Now turn the argument round:

$$v = (a \wedge v) + (v - a)^+ \le c + (v - a)^+ \le c + (b - a)^+,$$

and this is true for every $v \in B$, so $b \le c + (b - a)^+$ and $a \wedge b \le c$. As c is arbitrary, $a \wedge b = \sup C$, as claimed. **Q**

(b) Similarly, or applying (a) to $-A$ and $-B$, $\inf\{u \vee v : u \in A, v \in B\} = \inf A \vee \inf B$ whenever A, $B \subseteq U$ and the right-hand-side is defined.

(c) In particular, U is a distributive lattice (definition: 3A1Ic).

352F Further identities and inequalities At a slightly deeper level we have the following facts.

Proposition Let U be a Riesz space.

(a)(i) If u, v, $w \ge 0$ in U then $u \wedge (v + w) \le (u \wedge v) + (u \wedge w)$.

(ii) If $v_0, \dots , v_m, w_0, \dots , w_n \in U^+$ then

$$\sum_{i=0}^m v_i \wedge \sum_{j=0}^n w_j \le \sum_{i=0}^m \sum_{j=0}^n v_i \wedge w_j.$$

(b) If $u_0, \dots , u_n \in U$ are disjoint, then $|\sum_{i=0}^n \alpha_i u_i| = \sum_{i=0}^n |\alpha_i||u_i|$ for any $\alpha_0, \dots , \alpha_n \in \mathbb{R}$.

(c) If u, $v \in U$ then

$$u^+ \wedge v^+ \le (u + v)^+ \le u^+ + v^+.$$

(d) If $u_0, \dots , u_m, v_0, \dots , v_n \in U^+$ and $\sum_{i=0}^m u_i = \sum_{j=0}^n v_j$, then there is a family $\langle w_{ij}\rangle_{i\le m,j\le n}$ in U^+ such that $\sum_{i=0}^m w_{ij} = v_j$ for every $j \le n$ and $\sum_{j=0}^n w_{ij} = u_i$ for every $i \le m$.

proof (a)(i)

$$\begin{aligned} u \wedge (v + w) &= \big((u + w) \wedge (v + w)\big) \wedge u \\ &\le \big((u \wedge v) + w\big) \wedge \big((u \wedge v) + u\big) = (u \wedge v) + (u \wedge w). \end{aligned}$$

(ii) Inducing on n, we see that

$$u \wedge \sum_{j=0}^n w_i \le \sum_{j=0}^n u \wedge w_i$$

for every $u \ge 0$; so that

$$\sum_{i=0}^m v_i \wedge \sum_{j=0}^n w_j \le \sum_{j=0}^n (\sum_{i=0}^m v_i) \wedge w_j \le \sum_{j=0}^n \sum_{i=0}^m v_i \wedge w_j.$$

(b)(i)($\boldsymbol{\alpha}$) If $u \wedge v = 0$ then

$$(u - v)^+ = u - (u \wedge v) = u, \quad (u - v)^- = (v - u)^+ = v - (v \wedge u) = v,$$

$$|u - v| = (u - v)^+ + (u - v)^- = u + v = |u + v|,$$

so if $|u| \wedge |v| = 0$ then

$$\begin{aligned} (u^+ + v^+) \wedge (u^- + v^-) &\le (u^+ \wedge u^-) + (u^+ \wedge v^-) + (v^+ \wedge u^-) + (v^+ \wedge v^-) \\ &\le 0 + (|u| \wedge |v|) + (|v| \wedge |u|) + 0 = 0 \end{aligned}$$

and

$$|u + v| = |(u^+ + v^+) - (u^- + v^-)| = u^+ + v^+ + u^- + v^- = |u| + |v|.$$

($\boldsymbol{\beta}$) Now if $|u| \wedge |v| = 0$ and α, $\beta \in \mathbb{R}$,

$$|\alpha u| \wedge |\beta v| = |\alpha||u| \wedge |\beta||v| \le (|\alpha| + |\beta|)|u| \wedge (|\alpha| + |\beta|)|v| = (|\alpha| + |\beta|)(|u| \wedge |v|) = 0.$$

(ii) We may therefore proceed by induction. The case $n = 0$ is trivial. For the inductive step to $n+1$, setting $u_i' = \alpha_i u_i$ we have $|u_i'| \wedge |u_j'| = 0$ for all $i \ne j$, by (i-γ). By (i-α),

$$|u_{n+1}'| \wedge |\sum_{i=0}^n u_i'| \le \sum_{i=0}^n |u_{n+1}'| \wedge |u_i'| = 0,$$

so by (i-β) and the inductive hypothesis

$$|\sum_{i=0}^{n+1} u'_i| = |u'_{n+1}| + |\sum_{i=0}^{n} u'_i| = \sum_{i=0}^{n+1} |u'_i|$$

as required.

(c) By 352E,

$$u^+ \wedge v^+ = (u \vee 0) \wedge (v \vee 0) = (u \wedge v) \vee 0.$$

Now

$$u \wedge v = \frac{1}{2}(u + v - |u - v|) \le \frac{1}{2}(u + v + |u + v|) = (u + v)^+,$$

and of course $0 \le (u + v)^+$, so $u^+ \wedge v^+ \le (u + v)^+$.

For the other inequality we need only note that $u + v \le u^+ + v^+$ (because $u \le u^+$, $v \le v^+$) and $0 \le u^+ + v^+$.

(d) Write w for the common value of $\sum_{i=0}^{m} u_i$ and $\sum_{j=0}^{n} v_j$.

Induce on $k = \#(\{(i,j) : i \le m,\ j \le n,\ u_i \wedge v_j > 0\})$. If $k = 0$, that is, $u_i \wedge v_j = 0$ for all i, j, then, by (a-ii), we must have $w \wedge w = 0$, that is, $w = 0$, and we can take $w_{ij} = 0$ for all i, j. For the inductive step to $k \ge 1$, take i^*, j^* such that $w^* = u_{i^*} \wedge v_{j^*} > 0$. Set

$$\tilde{u}_{i^*} = u_{i^*} - w^*, \quad \tilde{u}_i = u_i \text{ for } i \ne i^*,$$

$$\tilde{v}_{j^*} = v_{j^*} - w^*, \quad \tilde{v}_j = v_j \text{ for } j \ne j^*.$$

Then $\sum_{i=0}^{m} \tilde{u}_i = \sum_{j=0}^{n} \tilde{v}_j = w - w^*$ and $\tilde{u}_i \wedge \tilde{v}_j \le u_i \wedge v_j$ for all i, j, while $\tilde{u}_{i^*} \wedge \tilde{v}_{j^*} = 0$; so that

$$\#(\{(i,j) : \tilde{u}_i \wedge \tilde{v}_j > 0\}) < k.$$

By the inductive hypothesis, there are $\tilde{w}_{ij} \ge 0$, for $i \le m$ and $j \le n$, such that $\tilde{u}_i = \sum_{j=0}^{n} \tilde{w}_{ij}$ for each i and $\tilde{v}_j = \sum_{i=0}^{m} \tilde{w}_{ij}$ for each j. Set $w_{i^*j^*} = \tilde{w}_{i^*j^*} + w^*$, $w_{ij} = \tilde{w}_{ij}$ for $(i,j) \ne (i^*, j^*)$; then $u_i = \sum_{j=0}^{n} w_{ij}$ and $v_j = \sum_{i=0}^{m} w_{ij}$, so the induction proceeds.

352G Riesz homomorphisms: Proposition Let U be a Riesz space, V a partially ordered linear space and $T : U \to V$ a linear operator. Then the following are equiveridical:

(i) T is a Riesz homomorphism in the sense of 351H;

(ii) $(Tu)^+ = \sup\{Tu, 0\}$ is defined and equal to $T(u^+)$ for every $u \in U$;

(iii) $\sup\{Tu, -Tu\}$ is defined and equal to $T|u|$ for every $u \in U$;

(iv) $\inf\{Tu, Tv\} = 0$ in V whenever $u \wedge v = 0$ in U.

proof (i)$\Rightarrow$(iii) and (i)$\Rightarrow$(iv) are special cases of 351Hc. For (iii)$\Rightarrow$(ii) we have

$$\sup\{Tu, 0\} = \frac{1}{2}Tu + \sup\{\frac{1}{2}Tu, -\frac{1}{2}Tu\} = \frac{1}{2}Tu + \frac{1}{2}T|u| = T(u^+).$$

For (ii)$\Rightarrow$(i), argue as follows. If (ii) is true and u, $v \in U$, then

$$Tu \wedge Tv = \inf\{Tu, Tv\} = Tu + \inf\{0, Tv - Tu\} = Tu - \sup\{0, T(u - v)\}$$

is defined and equal to

$$Tu - T((u - v)^+) = T(u - (u - v)^+) = T(u \wedge v).$$

Inducing on n,

$$\inf_{i \le n} Tu_i = T(\inf_{i \le n} u_i)$$

for all $u_0, \dots, u_n \in U$; in particular, if $\inf_{i \le n} u_i = 0$ then $\inf_{i \le n} Tu_i = 0$; which is the definition I gave of Riesz homomorphism.

Finally, for (iv)$\Rightarrow$(ii), we know from (iv) that $0 = \inf\{T(u^+), T(u^-)\}$, so $-T(u^+) = \inf\{0, -Tu\}$ and $T(u^+) = \sup\{0, Tu\}$.

352H Proposition If U and V are Riesz spaces and $T : U \to V$ is a bijective Riesz homomorphism, then T is a partially-ordered-linear-space isomorphism, and $T^{-1} : V \to U$ is a Riesz homomorphism.

proof Use 352G(ii). If $v \in V$, set $u = T^{-1}v$; then $T(u^+) = v^+$ so $T^{-1}(v^+) = u^+ = (T^{-1}v)^+$. Thus T^{-1} is a Riesz homomorphism; in particular, it is order-preserving, so T is an isomorphism for the order structures as well as for the linear structures.

352I Riesz subspaces (a) If U is a partially ordered linear space, a **Riesz subspace** of U is a linear subspace V such that $\sup\{u,v\}$ and $\inf\{u,v\}$ are defined in U and belong to V for every u, $v \in V$. In this case they are the supremum and infimum of $\{u,v\}$ in V, so V, with the induced order and linear structure, is a Riesz space in its own right, and the embedding map $u \mapsto u : V \to U$ is a Riesz homomorphism.

(b) Generally, if U is a Riesz space, V is a partially ordered linear space and $T : U \to V$ is a Riesz homomorphism, then $T[U]$ is a Riesz subspace of V (because, by 351Hc, $Tu \vee Tu' = T(u \vee u')$, $Tu \wedge Tu' = T(u \wedge u')$ are defined in V and belong to $T[U]$ for all u, $u' \in U$).

(c) If U is a Riesz space and V is a linear subspace of U, then V is a Riesz subspace of U iff $|u| \in V$ for every $u \in V$. **P** In this case,

$$u \vee v = \frac{1}{2}(u + v + |u - v|), \quad u \wedge v = \frac{1}{2}(u + v - |u - v|)$$

belong to V for all u, $v \in V$. **Q**

352J Solid subsets (a) If U is a Riesz space, a subset A of U is solid (in the sense of 351I) iff $v \in A$ whenever $u \in A$ and $|v| \le |u|$. **P** (α) If A is solid, $u \in A$ and $|v| \le |u|$, then there is some $w \in A$ such that $-w \le u \le w$; in this case $|v| \le |u| \le w$ and $-w \le v \le w$ and $v \in A$. (β) Suppose that A satisfies the condition. If $u \in A$, then $|u| \in A$ and $-|u| \le u \le |u|$. If $w \in A$ and $-w \le u \le w$ then $-u \le w$, $|u| \le w = |w|$ and $u \in A$. Thus A is solid. **Q** In particular, if A is solid, then $v \in A$ iff $|v| \in A$.

For any set $A \subseteq U$, the set

$$\{u : \text{there is some } v \in A \text{ such that } |u| \le |v|\}$$

is a solid subset of U, the smallest solid set including A; we call it the **solid hull** of A in U.

Any solid linear subspace of U is a Riesz subspace (use 352Ic). If $V \subseteq U$ is a Riesz subspace, then the solid hull of V in U is

$$\{u : \text{there is some } v \in V \text{ such that } |u| \le v\}$$

and is a solid linear subspace of U.

(b) If T is a Riesz homomorphism from a Riesz space U to a partially ordered linear space V, then its kernel W is a solid linear subspace of U. **P** If $u \in W$ and $|v| \le |u|$, then $T|u| = \sup\{Tu, T(-u)\} = 0$, while $-|u| \le v \le |u|$, so that $-0 \le Tv \le 0$ and $v \in W$. **Q**

Now the quotient space U/W, as defined in 351J, isomorphic, as partially ordered linear space, to $T[U]$, and in particular is a Riesz space. **P** Because U/W is the linear space quotient of V by the kernel of the linear operator T, we have an induced linear space isomorphism $S : U/W \to T[U]$ given by setting $Su^\bullet = Tu$ for every $u \in U$. If $p \ge 0$ in U/W there is a $u \in U^+$ such that $u^\bullet = p$ (351J), so that $Sp = Tu \ge 0$. On the other hand, if $p \in U/W$ and $Sp \ge 0$, take $u \in U$ such that $u^\bullet = p$. By 352G, we have

$$T(u^+) = (Tu)^+ = \sup\{Sp, 0\} = Sp = Tu,$$

so that $T(u^-) = Tu^+ - Tu = 0$, $u^- \in W$ and $p = (u^+)^\bullet \ge 0$. Thus $Sp \ge 0$ iff $p \ge 0$, and S is a partially-ordered-linear-space isomorphism. We know from 352Ib that $T[U]$ is a Riesz space, so U/W also is. **Q**

(c) Because a subset of a Riesz space is a solid linear subspace iff it is the kernel of a Riesz homomorphism (see 352U below), such subspaces are sometimes called **ideals**.

352K Products If $\langle U_i \rangle_{i \in I}$ is any family of Riesz spaces, then the product partially ordered linear space $U = \prod_{i \in I} U_i$ (351L) is a Riesz space, with

$$u \vee v = \langle u(i) \vee v(i) \rangle_{i \in I}, \quad u \wedge v = \langle u(i) \wedge v(i) \rangle_{i \in I}, \quad |u| = \langle |u(i)| \rangle_{i \in I}$$

for all u, $v \in U$.

352L Theorem Let U be any Riesz space. Then there are a set X, a filter $\mathcal{F}$ on X and a Riesz subspace of the Riesz space $\mathbb{R}^X|\mathcal{F}$ (definition: 351M) which is isomorphic, as Riesz space, to U.

proof By 351Q, we can find such X and $\mathcal{F}$ and an injective Riesz homomorphism $T : U \to \mathbb{R}^X|\mathcal{F}$. By 352K, or otherwise, $\mathbb{R}^X$ is a Riesz space; by 352Jb, $\mathbb{R}^X|\mathcal{F}$ is a Riesz space (recall that it is a quotient of $\mathbb{R}^X$ by a solid linear subspace, as explained in 351M); by 352Ib, $T[U]$ is a Riesz subspace of $\mathbb{R}^X|\mathcal{F}$; and by 352H it is isomorphic to U.

352M Corollary Any identity involving the operations $+$, $-$, $\vee$, $\wedge$, $^+$, $^-$, $|\ |$ and scalar multiplication, and the relation $\le$, which is valid in $\mathbb{R}$, is valid in all Riesz spaces.

Remark I suppose some would say that a strict proof of this must begin with a formal description of what the phrase 'any identity involving the operations...' means. However I think it is clear in practice what is involved. Given a proposed identity like

$$0 \le \sum_{i=0}^n |\alpha_i||u_i| - |\sum_{i=0}^n \alpha_i u_i| \le \sum_{i\ne j}(|\alpha_i| + |\alpha_j|)(|u_i| \wedge |u_j|),$$

(compare 352Fb), then to check that it is valid in all Riesz spaces you need only check (i) that it is true in $\mathbb{R}$ (ii) that it is true in $\mathbb{R}^X$ (iii) that it is true in any $\mathbb{R}^X|\mathcal{F}$ (iv) that it is true in any Riesz subspace of $\mathbb{R}^X|\mathcal{F}$; and you can hope that the arguments for (ii)-(iv) will be nearly trivial, since (ii) is generally nothing but a coordinate-by-coordinate repetition of (i), and (iii) and (iv) involve only transformations of the formula by Riesz homomorphisms which preserve its structure.

352N Order-density and order-continuity Let U be a Riesz space.

(a) Definition A Riesz subspace V of U is **quasi-order-dense** if for every $u > 0$ in U there is a $v \in V$ such that $0 < v \le u$; it is **order-dense** if $u = \sup\{v : v \in V,\ 0 \le v \le u\}$ for every $u \in U^+$.

(b) If U is a Riesz space and V is a quasi-order-dense Riesz subspace of U, then the embedding $V \hookrightarrow U$ is order-continuous. **P** Let $A \subseteq V$ be a non-empty set such that $\inf A = 0$ in V. **?** If 0 is not the infimum of A in U, then there is a $u > 0$ such that u is a lower bound for A in U; now there is a $v \in V$ such that $0 < v \le u$, and v is a lower bound for A in V which is strictly greater than 0. **X** Thus $0 = \inf A$ in U. As A is arbitrary, the embedding is order-continuous, by 351Ga. **Q**

(c)(i) If $V \subseteq U$ is an order-dense Riesz subspace, it is quasi-order-dense. (ii) If V is a quasi-order-dense Riesz subspace of U and W is a quasi-order-dense Riesz subspace of V, then W is a quasi-order-dense Riesz subspace of U. (iii) If V is an order-dense Riesz subspace of U and W is an order-dense Riesz subspace of V, then W is an order-dense Riesz subspace of U. (Use (b).) (iv) If V is a quasi-order-dense solid linear subspace of U and W is a quasi-order-dense Riesz subspace of U then $V \cap W$ is quasi-order-dense in V, therefore in U.

(d) I ought somewhere to remark that a Riesz homomorphism, being a lattice homomorphism, is order-continuous iff it preserves arbitrary suprema and infima; compare 313L(b-iv) and (b-v).

(e) If V is a Riesz subspace of U, we say that it is **regularly embedded** in U if the identity map from V to U is order-continuous, that is, whenever $A \subseteq V$ is non-empty and has infimum 0 in V, then 0 is still its greatest lower bound in U. Thus quasi-order-dense Riesz subspaces and solid linear subspaces are regularly embedded.

352O Bands Let U be a Riesz space.

(a) Definition A **band** or **normal subspace** of U is an order-closed solid linear subspace.

(b) If $V \subseteq U$ is a solid linear subspace then it is a band iff $\sup A \in V$ whenever $A \subseteq V^+$ is a non-empty, upwards-directed subset of V with a supremum in U. **P** Of course the condition is necessary; I have to show that it is sufficient. (i) Let $A \subseteq V$ be any non-empty upwards-directed set with a supremum in V. Take any $u_0 \in A$ and set $A_1 = \{u - u_0 : u \in A,\ u \ge u_0\}$. Then A_1 is a non-empty upwards-directed subset of V^+, and $u_0 + A_1 = \{u : u \in A,\ u \ge u_0\}$ has the same upper bounds as A, so $\sup A_1 = \sup A - u_0$ is defined in U and belongs to V. Now $\sup A = u_0 + \sup A_1$ also belongs to V. (ii) If $A \subseteq V$ is non-empty, downwards-directed and has an infimum in U, then $-A \subseteq V$ is upwards-directed, so $\inf A = \sup(-A)$ belongs to V. Thus V is order-closed. **Q**

(c) For any set $A \subseteq U$ set $A^\perp = \{v : v \in U, |u| \wedge |v| = 0$ for every $u \in A\}$. Then $A^\perp$ is a band. **P** (i) Of course $0 \in A^\perp$. (ii) If v, $w \in A^\perp$ and $u \in A$, then

$$0 \le |u| \wedge |v + w| \le (|u| \wedge |v|) + (|u| \wedge |w|) = 0,$$

so $v + w \in A^\perp$. (iii) If $v \in A^\perp$ and $|w| \le |v|$ then

$$0 \le |u| \wedge |w| \le |u| \wedge |v| = 0$$

for every $u \in A$, so $w \in A^\perp$. (iv) If $v \in A^\perp$ then $nv \in A^\perp$ for every n, by (ii). So if $\alpha \in \mathbb{R}$, take $n \in \mathbb{N}$ such that $|\alpha| \le n$; then

$$|\alpha v| = |\alpha||v| \le n|v| \in A^\perp$$

and $\alpha v \in A^\perp$. Thus $A^\perp$ is a solid linear subspace of U. (v) If $B \subseteq (A^\perp)^+$ is non-empty and upwards-directed and has a supremum w in U, then

$$|u| \wedge |w| = |u| \wedge w = \sup_{v \in B} |u| \wedge v = 0$$

by 352Ea, so $w \in A^\perp$. Thus $A^\perp$ is a band. **Q**

(d) For any $A \subseteq U$, $A \subseteq (A^\perp)^\perp$. Also $B^\perp \subseteq A^\perp$ whenever $A \subseteq B$. So

$$A^{\perp\perp\perp} \subseteq A^\perp \subseteq A^{\perp\perp\perp}$$

and $A^\perp = A^{\perp\perp\perp}$.

(e) If W is another Riesz space and $T : U \to W$ is an order-continuous Riesz homomorphism then its kernel is a band. (For $\{0\}$ is order-closed in W and the inverse image of an order-closed set under an order-continuous order-preserving function is order-closed (313Id).)

352P Complemented bands Let U be a Riesz space. A band $V \subseteq U$ is **complemented** if $V^{\perp\perp} = V$, that is, if V is of the form $A^\perp$ for some $A \subseteq U$ (352Od). In this case its **complement** is the complemented band $V^\perp$.

352Q Theorem In any Riesz space U, the set $\mathfrak{C}$ of complemented bands forms a Dedekind complete Boolean algebra, with

$$V \cap_{\mathfrak{C}} W = V \cap W, \quad V \cup_{\mathfrak{C}} W = (V + W)^{\perp\perp},$$

$$1_{\mathfrak{C}} = U, \quad 0_{\mathfrak{C}} = \{0\}, \quad 1_{\mathfrak{C}} \setminus_{\mathfrak{C}} V = V^\perp,$$

$$V \subseteq_{\mathfrak{C}} W \iff V \subseteq W$$

for V, $W \in \mathfrak{C}$.

proof To show that $\mathfrak{C}$ is a Boolean algebra, I use the identification of Boolean algebras with complemented distributive lattices (311L).

(a) Of course $\mathfrak{C}$ is partially ordered by $\subseteq$. If V, $W \in \mathfrak{C}$ then

$$V \cap W = V^{\perp\perp} \cap W^{\perp\perp} = (V^\perp \cup W^\perp)^\perp \in \mathfrak{C},$$

and $V \cap W$ must be $\inf\{V, W\}$ in $\mathfrak{C}$. The map $V \mapsto V^\perp : \mathfrak{C} \to \mathfrak{C}$ is an order-reversing permutation, so that $V \subseteq W$ iff $W^\perp \subseteq V^\perp$ and $V \vee W = \sup\{V, W\}$ will be $(V^\perp \cap W^\perp)^\perp$; thus $\mathfrak{C}$ is a lattice. Note also that $V \vee W$ must be the smallest complemented band including $V + W$, that is, it is $(V + W)^{\perp\perp}$.

(b) If V_1, V_2, $W \in \mathfrak{C}$ then $(V_1 \vee V_2) \wedge W = (V_1 \wedge W) \vee (V_2 \wedge W)$. **P** Of course $(V_1 \vee V_2) \wedge W \supseteq (V_1 \wedge W) \vee (V_2 \wedge W)$. **?** Suppose, if possible, that there is a $u \in (V_1 \vee V_2) \cap W \setminus ((V_1 \cap W) \vee (V_2 \cap W))$. Then $u \notin ((V_1 \cap W)^\perp \cap (V_2 \cap W)^\perp)^\perp$, so there is a $v \in (V_1 \cap W)^\perp \cap (V_2 \cap W)^\perp$ such that $u_1 = |u| \wedge |v| > 0$. Now $u_1 \in V_1 \vee V_2 = (V_1^\perp \cap V_2^\perp)^\perp$ so $u_1 \notin V_1^\perp \cap V_2^\perp$; say $u_1 \notin V_j^\perp$, and there is a $v_j \in V_j$ such that $u_2 = u_1 \wedge |v_j| > 0$. In this case we still have $u_2 \in (V_j \cap W)^\perp$, because $u_2 \le |v|$, but also $u_2 \in V_j$ and $u_2 \in W$ because $u_2 \le |u|$; but this means that $u_2 = u_2 \wedge u_2 = 0$, which is absurd. **X** Thus $(V_1 \vee V_2) \wedge W \subseteq (V_1 \wedge W) \vee (V_2 \wedge W)$ and the two are equal. **Q**

(c) Now if $V \in \mathfrak{C}$,

$$V \wedge V^\perp = \{0\}$$

is the least member of $\mathfrak{C}$, because if $v \in V \cap V^\perp$ then $|v| = |v| \wedge |v| = 0$. By 311L, $\mathfrak{C}$ has a Boolean algebra structure, with the Boolean relations described; by 312M, this structure is uniquely defined.

(d) Finally, if $\mathcal{V} \subseteq \mathfrak{C}$ is non-empty, then

$$\bigcap \mathcal{V} = (\bigcup_{V \in \mathcal{V}} V^\perp)^\perp \in \mathfrak{C}$$

and is $\inf \mathcal{V}$ in $\mathfrak{C}$. So $\mathfrak{C}$ is Dedekind complete.

352R Projection bands Let U be a Riesz space.

(a) A **projection band** in U is a set $V \subseteq U$ such that $V + V^\perp = U$. In this case V is a complemented band. **P** If $v \in V^{\perp\perp}$ then v is expressible as $v_1 + v_2$ where $v_1 \in V$ and $v_2 \in V^\perp$. Now $|v| = |v_1| + |v_2| \ge |v_2|$ (352Fb), so

$$|v_2| = |v_2| \wedge |v_2| \le |v_2| \wedge |v| = 0$$

and $v = v_1 \in V$. Thus $V = V^{\perp\perp}$ is a complemented band. **Q** Observe that $U = V^\perp + V^{\perp\perp}$ so $V^\perp$ also is a projection band.

(b) Because $V \cap V^\perp$ is always $\{0\}$, we must have $U = V \oplus V^\perp$ for any projection band $V \subseteq U$; accordingly there is a corresponding **band projection** $P_V : U \to U$ defined by setting $P(v + w) = v$ whenever $v \in V$, $w \in V^\perp$. In this context I will say that v is the **component** of $v + w$ in V. The kernel of P is $V^\perp$, the set of values is V, and $P^2 = P$. Moreover, P is an order-continuous Riesz homomorphism. **P** (i) P is a linear operator because V and $V^\perp$ are linear subspaces. (ii) If $v \in V$ and $w \in V^\perp$ then $|v + w| = |v| + |w|$, by 352Fb, so $P|v + w| = |v| = |P(v + w)|$; consequently P is a Riesz homomorphism (352G). (iii) If $A \subseteq U$ is downwards-directed and has infimum 0, then $Pu \le u$ for every $u \in A$, so $\inf P[A] = 0$; thus P is order-continuous. **Q**

(c) Note that for any band projection P, and any $u \in U$, we have $|Pu| \wedge |u - Pu| = 0$, so that $|u| = |Pu| + |u - Pu|$ and (in particular) $|Pu| \le |u|$; consequently $P[W] \subseteq W$ for any solid linear subspace W of U.

(d) A linear operator $P : U \to U$ is a band projection iff $Pu \wedge (u - Pu) = 0$ for every $u \in U^+$. **P** I remarked in (c) that the condition is satisfied for any band projection. Now suppose that P has the property. (i) For any $u \in U^+$, $Pu \ge 0$ and $u - Pu \ge 0$; in particular, P is a positive linear operator. (ii) If $u, v \in U^+$ then $u - Pu \le (u + v) - P(u + v)$, so

$$Pv \wedge (u - Pu) \le P(u + v) \wedge ((u + v) - P(u + v)) = 0$$

and $Pv \wedge (u - Pu) = 0$. (iii) If $u, v \in U$ then $|Pv| \le P|v|$, $|u - Pu| \le |u| - P|u|$ (because $w \mapsto w - Pw$ is a positive linear operator), so

$$|Pv| \wedge |u - Pu| \le P|v| \wedge (|u| - P|u|) = 0.$$

(iv) Setting $V = P[U]$, we see that $u - Pu \in V^\perp$ for every $u \in U$, so that

$$u = u + (u - Pu) \in V + V^\perp$$

for every u, and $U = V + V^\perp$; thus V is a projection band. (v) Since $Pu \in V$ and $u - Pu \in V^\perp$ for every $u \in U$, P is the band projection onto V. **Q**

352S Proposition Let U be any Riesz space.

(a) The family $\mathfrak{B}$ of projection bands in U is a subalgebra of the Boolean algebra $\mathfrak{C}$ of complemented bands in U.

(b) For $V \in \mathfrak{B}$ let $P_V : U \to V$ be the corresponding projection. Then for any $e \in U^+$,

$$P_{V \cap W} e = P_V e \wedge P_W e = P_V P_W e, \quad P_{V \vee W} e = P_V e \vee P_W e$$

for all V, $W \in \mathfrak{B}$. In particular, band projections commute.

(c) If $V \in \mathfrak{B}$ then the algebra of projection bands in V is just the principal ideal of $\mathfrak{B}$ generated by V.

proof (a) Of course $0_{\mathfrak{C}} = \{0\} \in \mathfrak{B}$. If $V \in \mathfrak{B}$ then $V^\perp = 1_{\mathfrak{C}} \setminus V$ belongs to $\mathfrak{B}$. If now W is another member of $\mathfrak{B}$, then

$$(V \cap W) + (V \cap W)^\perp \supseteq (V \cap W) + V^\perp + W^\perp.$$

But if $u \in U$ then we can express u as $v + v'$, where $v \in V$ and $v' \in V^\perp$, and v as $w + w'$, where $w \in W$ and $w' \in W^\perp$; and as $|w| \le |v|$, we also have $w \in V$, so that

$$u = w + v' + w' \in (V \cap W) + V^\perp + W^\perp.$$

This shows that $V \cap W \in \mathfrak{B}$. Thus $\mathfrak{B}$ is closed under intersection and complements and is a subalgebra of $\mathfrak{C}$.

(b) If V, $W \in \mathfrak{B}$ and $e \in U^+$, we have $e = e_1 + e_2 + e_3 + e_4$ where

$$e_1 = P_W P_V e \in V \cap W, \quad e_2 = P_{W^\perp} P_V e \in V \cap W^\perp,$$

$$e_3 = P_W P_{V^\perp} e \in V^\perp \cap W, \quad e_4 = P_{W^\perp} P_{V^\perp} e \in V^\perp \cap W^\perp,$$

$$e_1 + e_2 = P_V e, \quad e_1 + e_3 = P_W e.$$

Now $e_2 + e_3 + e_4$ belongs to $(V \cap W)^\perp$, so e_1 must be the component of e in $V \cap W$; similarly e_4 is the component of e in $V^\perp \cap W^\perp$, and $e_1 + e_2 + e_3$ is the component of e in $V \vee W$. But as $e_2 \wedge e_3 = 0$, we have

$$P_{V \cap W} e = e_1 = (e_1 + e_2) \wedge (e_1 + e_3) = P_V e \wedge P_W e,$$

$$P_{V \vee W} e = e_1 + e_2 + e_3 = (e_1 + e_2) \vee (e_1 + e_3) = P_V e \vee P_W e,$$

as required.

It follows that

$$P_V P_W = P_{V \cap W} = P_{W \cap V} = P_W P_V.$$

(c) If V, $W \in \mathfrak{B}$ and $W \subseteq V$, then of course W is a band in the Riesz space V (because V is order-closed in U, so that for any set $A \subseteq W$ its supremum in U will be its supremum in V if either is defined). For any $v \in V$, we have an expression of it as $w + w'$, where $w \in W$ and $w' \in W^\perp$, taken in U; but as $|w| + |w'| = |w + w'| = |v| \in V$, w' belongs to V, and is in $W_V^\perp$, the band in V orthogonal to W. Thus $W + W_V^\perp = V$ and W is a projection band in V. Conversely, if W is a projection band in V, then $W^\perp$ (taken in U) includes $W_V^\perp + V^\perp$, so that

$$W + W^\perp \supseteq W + W_V^\perp + V^\perp = V + V^\perp = U$$

and $W \in \mathfrak{B}$.

Thus the algebra of projection bands in V is, as a set, equal to the principal ideal $\mathfrak{B}_V$; because their orderings agree, or otherwise, their Boolean algebra structures coincide.

352T Products again (a) If $U = \prod_{i \in I} U_i$ is a product of Riesz spaces, then for any $J \subseteq I$ we have a subspace

$$V_J = \{u : u \in U,\ u(i) = 0 \text{ for all } i \in I \setminus J\}$$

of U, canonically isomorphic to $\prod_{i \in J} U_i$. Each V_J is a projection band, its complement being $V_{I \setminus J}$; the map $J \mapsto V_J$ is a Boolean homomorphism from $\mathcal{P}I$ to the algebra $\mathfrak{B}$ of projection bands in U, and $\langle V_{\{i\}} \rangle_{i \in I}$ is a partition of unity in $\mathfrak{B}$.

(b) Conversely, if U is a Riesz space and $(V_0, \dots, V_n)$ is a *finite* partition of unity in the algebra $\mathfrak{B}$ of projection bands in U, then every element of U is uniquely expressible as $\sum_{i=0}^n u_i$ where $u_i \in V_i$ for each i. (Induce on n, using 352Rb for the case $n = 2$, and 352Sc in the inductive step.) This decomposition corresponds to a Riesz space isomorphism between U and $\prod_{i \le n} V_i$.

352U Quotient spaces (a) If U is a Riesz space and V is a solid linear subspace, then the canonical map from U onto the quotient partially ordered linear space U/V is a Riesz homomorphism (351J), so U/V is a Riesz space (352Ib). I have already noted that if U and W are Riesz spaces and $T : U \to W$ a Riesz homomorphism, then the kernel V of T is a solid linear subspace of U and the Riesz subspace $T[U]$ of W is isomorphic to U/V (352Jb).

(b) Suppose that U is a Riesz space and V a solid linear subspace. Then the canonical map from U to U/V is order-continuous iff V is a band. **P** (i) If $u \mapsto u^\bullet$ is order-continuous, its kernel V is a band, by 352Oe. (ii) If V is a band, and $A \subseteq U$ is non-empty and downwards-directed and has infimum 0, let $p \in U/V$ be any lower bound for $\{u^\bullet : u \in A\}$. Express p as $w^\bullet$. Then $((w-u)^+)^\bullet = (w^\bullet - u^\bullet)^+ = 0$, that is, $(w-u)^+ \in V$ for every $u \in A$. But this means that

$$w^+ = \sup_{u\in A}(w-u)^+ \in V, \quad p^+ = (w^+)^\bullet = 0,$$

that is, $p \le 0$. As p is arbitrary, $\inf_{u\in A} u^\bullet = 0$; as A is arbitrary, $u \mapsto u^\bullet$ is order-continuous. **Q**

352V Principal bands Let U be a Riesz space. Evidently the intersection of any family of Riesz subspaces of U is a Riesz subspace, the intersection of any family of solid linear subspaces is a solid linear subspace and the intersection of any family of bands is a band; we may therefore speak of the band generated by a subset A of U, the intersection of all the bands including A. Now we have the following description of the band generated by a single element.

Lemma Let U be a Riesz space.

(a) If $A \subseteq U^+$ is upwards-directed and $2w \in A$ for every $w \in A$, then an element u of U belongs to the band generated by A iff $|u| = \sup_{w\in A} |u| \wedge w$.

(b) If $u \in U$ and $w \in U^+$, then u belongs to the band in U generated by w iff $|u| = \sup_{n\in\mathbb{N}} |u| \wedge nw$.

proof (a) Let W be the band generated by A and W' the set of elements of U satisfying the condition.

(i) If $u \in W'$ then $|u| \wedge w \in W$ for every $w \in A$, because W is a solid linear subspace; because W is also order-closed, $|u|$ and u belong to W. Thus $W' \subseteq W$.

(ii) Now W' is a band.

P$(\boldsymbol{\alpha})$ If $u \in W'$ and $|v| \le |u|$ then

$$\sup_{w\in A} |v| \wedge w = \sup_{w\in A} |v| \wedge |u| \wedge w = |v| \wedge \sup_{w\in A} |u| \wedge w = |v| \wedge |u| = |v|$$

by 352Ea, so $v \in W'$.

$(\boldsymbol{\beta})$ If u, $v \in W'$ then, for any w_1, $w_2 \in A$ there is a $w \in A$ such that $w \ge w_1 \vee w_2$. Now $w_1 + w_2 \le 2w \in A$, and

$$(|u| + |v|) \wedge 2w \ge (|u| \wedge w_1) + (|v| \wedge w_2).$$

So any upper bound for $\{(|u|+|v|)\wedge w : w \in A\}$ must also be an upper bound for $\{|u|\wedge w : w \in A\} + \{|v|\wedge w : w \in A\}$ and therefore greater than or equal to

$$\begin{aligned}\sup(\{|u| \wedge w : w \in A\} + \{|v| \wedge w : w \in A\}) &= \sup_{w\in A} |u| \wedge w + \sup_{w\in A} |v| \wedge w \\ &= |u| + |v|\end{aligned}$$

(351Dc). But this means that $\sup_{w\in A}(|u| + |v|) \wedge w$ must be $|u| + |v|$, and $|u| + |v|$ belongs to W'; it follows from (α) that $u + v$ belongs to W'.

$(\boldsymbol{\gamma})$ Just as in 352Oc, we now have

$$nu \in W' \text{ for every } n \in \mathbb{N},\ u \in W',$$

and therefore $\alpha u \in W'$ for every $\alpha \in \mathbb{R}$, $u \in W'$, since $|\alpha u| \le |nu|$ if $|\alpha| \le n$. Thus W' is a solid linear subspace of U.

$(\boldsymbol{\delta})$ Now suppose that $C \subseteq (W')^+$ has a supremum v in U. Then any upper bound of $\{v \wedge w : w \in A\}$ must also be an upper bound of $\{u \wedge w : u \in C,\ w \in A\}$ and greater than or equal to $u = \sup_{w\in A} u \wedge w$ for every $u \in C$, therefore greater than or equal to $v = \sup C$. Thus $v = \sup_{w \in A} v \wedge w$ and $v \in W'$. As C is arbitrary, W' is a band (352Ob). **Q**

(iii) Since A is obviously included in W', W' must include W; putting this together with (i), $W = W'$, as claimed.

(b) Apply (a) with $A = \{nw : n \in \mathbb{N}\}$.

352W f-algebras Some of the most important Riesz spaces have multiplicative structures as well as their order and linear structures. A particular class of these structures appears sufficiently often for it to be useful to develop a little of its theory. The following definition is a common approach.

(a) Definition An **f-algebra** is a Riesz space U with a multiplication $\times : U \times U \to U$ such that

$$u \times (v \times w) = (u \times v) \times w,$$

$$(u + v) \times w = (u \times w) + (v \times w), \quad u \times (v + w) = (u \times v) + (u \times w),$$

$$\alpha(u \times v) = (\alpha u) \times v = u \times (\alpha v)$$

for all u, v, $w \in U$ and $\alpha \in \mathbb{R}$, and

$$u \times v \geq 0 \text{ whenever } u,\ v \geq 0,$$

$$\text{if } u \wedge v = 0 \text{ then } (u \times w) \wedge v = (w \times u) \wedge v = 0 \text{ for every } w \geq 0.$$

An f-algebra is **commutative** if $u \times v = v \times u$ for all u, v.

(b) Let U be an f-algebra.

(i) If $u \wedge v = 0$ in U, then $u \times v = 0$. **P** $v \geq 0$ so $v \wedge (u \times v) = (u \times v) \wedge v = 0$ and $u \times v = (u \times v) \wedge (u \times v) = 0$. **Q**

(ii) $u \times u \geq 0$ for every $u \in U$. **P**

$$\begin{aligned}(u^+ - u^-) \times (u^+ - u^-) &= u^+ \times u^+ - u^+ \times u^- - u^- \times u^+ + u^- \times u^- \\ &= u^+ \times u^+ + u^- \times u^- \geq 0. \ \mathbf{Q}\end{aligned}$$

(iii) If u, $v \in U$ then $|u \times v| = |u| \times |v|$. **P** $u^+ \times v^+$, $u^+ \times v^-$, $u^- \times v^+$ and $u^+ \times v^-$ are disjoint, so

$$\begin{aligned}|u \times v| &= |u^+ \times v^+ - u^+ \times v^- - u^- \times v^+ + u^- \times v^-| \\ &= u^+ \times v^+ + u^+ \times v^- + u^- \times v^+ + u^- \times v^- \\ &= |u| \times |v|\end{aligned}$$

by 352Fb. **Q**

(iv) If $v \in U^+$ the maps $u \mapsto u \times v$, $u \mapsto v \times u : U \to U$ are Riesz homomorphisms. **P** The first four clauses of the definition in (a) ensure that they are linear operators. If $u \in U$, then

$$|u| \times v = |u \times v|, \quad v \times |u| = |v \times u|$$

by (iii), so we have Riesz homomorphisms, by 352G(iii). **Q**

(c) Let $\langle U_i \rangle_{i \in I}$ be a family of f-algebras, with Riesz space product U (352K). If we set $u \times v = \langle u(i) \times v(i) \rangle_{i \in I}$ for all u, $v \in U$, then U becomes an f-algebra.

352X Basic exercises >(a) Let U be any Riesz space. Show that $|u^+ - v^+| \leq |u - v|$ for all u, $v \in U$.

>(b) Let U, V be Riesz spaces and $T : U \to V$ a linear operator. Show that the following are equiveridical: (i) T is a Riesz homomorphism; (ii) $T(u \vee v) = Tu \vee Tv$ for all u, $v \in U$; (iii) $T(u \wedge v) = Tu \wedge Tv$ for all u, $v \in U$; (iv) $|Tu| = T|u|$ for every $u \in U$.

(c) Let U be a Riesz space and V a solid linear subspace; for $u \in U$ write $u^\bullet$ for the corresponding element of U/V. Show that if $A \subseteq U$ is solid then $\{u^\bullet : u \in A\}$ is solid in U/W.

(d) Let U be a Riesz space. Show that $\operatorname{med}(\alpha u, \alpha v, \alpha w) = \alpha \operatorname{med}(u, v, w)$ for all u, v, $w \in U$ and all $\alpha \in \mathbb{R}$. (*Hint*: 3A1Ic, 352M.)

(e) Let U and V be Riesz spaces and $T : U \to V$ a Riesz homomorphism with kernel W. Show that if W is a band in U and $T[U]$ is regularly embedded in V then T is order-continuous.

(f) Give $U = \mathbb{R}^2$ its lexicographic ordering (351Xa). Show that it has a band V which is not complemented.

(g) Let U be a Riesz space and $\mathfrak{C}$ the algebra of complemented bands in U. Show that for any $V \in \mathfrak{C}$ the algebra of complemented bands in V is just the principal ideal of $\mathfrak{C}$ generated by V.

>(h) Let $U = C([0, 1])$ be the space of continuous functions from $[0, 1]$ to $\mathbb{R}$, with its usual linear and order structures, so that it is a Riesz subspace of $\mathbb{R}^{[0,1]}$. Set $V = \{u : u \in U,\ u(t) = 0 \text{ if } t \le \frac{1}{2}\}$. Show that V is a band in U and that $V^\perp = \{u : u(t) = 0 \text{ if } t \ge \frac{1}{2}\}$, so that V is complemented but is not a projection band.

(i) Show that the Boolean homomorphism $J \mapsto V_J : \mathcal{P}I \to \mathfrak{B}$ of 352Ta is order-continuous.

(j) Let U be a Riesz space and $A \subseteq U^+$ an upwards-directed set. Show that the band generated by A is $\{u : |u| = \sup_{n\in\mathbb{N},w\in A} |u| \wedge nw\}$.

>(k)(i) Let X be any set. Setting $(u \times v)(x) = u(x)v(x)$ for u, $v \in \mathbb{R}^X$, $x \in X$, show that $\mathbb{R}^X$ is a commutative f-algebra. (ii) With the same definition of $\times$, show that $\ell^\infty(X)$ is an f-algebra. (iii) If X is a topological space, show that $C(X)$, $C_b(X)$ (definition: 281A, 354Hb) are f-algebras. (iv) If (X, Σ, μ) is a measure space, show that $L^0(\mu)$ and $L^\infty(\mu)$ (§241, §243) are f-algebras.

(l) Let $U \subseteq \mathbb{R}^\mathbb{Z}$ be the set of sequences u such that $\{n : u(n) \ne 0\}$ is bounded above in $\mathbb{Z}$. For u, $v \in U$ (i) say that $u \le v$ if either $u = v$ or there is an $n \in \mathbb{Z}$ such that $u(n) < v(n)$, $u(i) = v(i)$ for every $i > n$ (ii) say that $(u * v)(n) = \sum_{i=-\infty}^{\infty} u(i)v(n-i)$ for every $n \in \mathbb{Z}$. Show that U is a commutative f-algebra under this ordering and multiplication, and that $(U, +, *)$ is a field.

(m) Let U be an f-algebra. (i) Show that any complemented band in U is an ideal in the ring $(U, +, \times)$. (ii) Show that if $P : U \to U$ is a band projection, then $P(u \times v) = Pu \times Pv$ for every u, $v \in U$.

(n) Let U be an f-algebra with multiplicative identity e. Show that $u - \gamma e \le \dfrac{1}{2\gamma}u^2$ for every $u \in U$, $\gamma > 0$. (*Hint*: $(u^+ - \gamma e)^2 \ge 0$.)

(o) Let U be a Riesz space. (i) Show that if u, v, u', $v' \in U$ then $|u \vee v - u' \vee v'| \le |u - u'| \vee |v - v'|$. (ii) Show that if u, v, w, u', v', $w' \in U$ then $|\operatorname{med}(u, v, w) - \operatorname{med}(u', v', w')| \le |u - u'| \vee |v - v'| \vee |w - w'|$.

352Y Further exercises (a) Find an f-algebra with a non-commutative multiplication.

(b) Let U be an f-algebra. Show that the multiplication of U is commutative iff $u \times v = (u \wedge v) \times (u \vee v)$ for all u, $v \in U$.

(c) Let U be an f-algebra. Show that $u \times \operatorname{med}(v_1, v_2, v_3) = \operatorname{med}(u \times v_1, u \times v_2, u \times v_3)$ whenever u, v_1, v_2, $v_3 \in U$.

352 Notes and comments In this section we begin to see a striking characteristic of the theory of Riesz spaces: repeated reflections of results in Boolean algebra. Without spelling out a complete list, I mention the distributive laws (313Bc, 352Ea) and the behaviour of order-continuous homomorphisms (313Pa, 313Qa, 352N, 352Oe, 352Ub, 352Xe). Riesz subspaces correspond to subalgebras, solid linear subspaces to ideals and Riesz homomorphisms to Boolean homomorphisms. We even have a correspondence, though a weaker one, between the representation theorems available; every Boolean algebra is isomorphic to a subalgebra of a power of $\mathbb{Z}_2$ (311D-311E), while every Riesz space is isomorphic to a Riesz subspace of a quotient of a power of $\mathbb{R}$ (352L). It would be a closer parallel if every Riesz space were embeddable in some $\mathbb{R}^X$; I must emphasize that the differences are as important as the agreements. Subspaces of $\mathbb{R}^X$ are of great importance, but are by no means adequate for our needs. And of course the details – for instance, the identities in 352D-352F, or 352V – frequently involve new techniques in the case of Riesz spaces. Elsewhere, as in 352G, I find myself arguing rather from the opposite side, when applying results from the theory of general partially ordered linear spaces, which has little to do with Boolean algebra.

In the theory of bands in Riesz spaces – corresponding to order-closed ideals in Boolean algebras – we have a new complication in the form of bands which are not complemented, which does not arise in the Boolean algebra context; but it will disappear again when we come to specialize to Archimedean Riesz spaces (353B). (Similarly, order-density and quasi-order-density coincide in both Boolean algebras (313K) and Archimedean Riesz spaces (353A).) Otherwise the algebra of complemented bands in a Riesz space looks very like the algebra of order-closed ideals in a Boolean algebra (314Yh, 352Q). The algebra of projection bands in a Riesz space (352S) would correspond, in a Boolean algebra, to the algebra itself.

I draw your attention to 352H. The result is nearly trivial, but it amounts to saying that the theory of Riesz spaces will be 'algebraic', like the theories of groups or linear spaces, rather than 'analytic', like the theories of partially ordered linear spaces or topological spaces, in which we can have bijective morphisms which are not isomorphisms.

Version of 16.2.17

353 Archimedean and Dedekind complete Riesz spaces

I take a few pages over elementary properties of Archimedean and Dedekind (σ)-complete Riesz spaces.

353A Proposition Let U be an Archimedean Riesz space. Then every quasi-order-dense Riesz subspace of U is order-dense.

proof Let $V \subseteq U$ be a quasi-order-dense Riesz subspace, and $u \geq 0$ in U. Set $A = \{v : v \in V, v \leq u\}$. **?** Suppose, if possible, that u is not the least upper bound of A. Then there is a $u_1 < u$ such that $v \leq u_1$ for every $v \in A$. Because $0 \in A$, $u_1 \geq 0$. Because V is quasi-order-dense, there is a $v \in V$ such that $0 < v \leq u - u_1$. Now $nv \leq u_1$ for every $n \in \mathbb{N}$. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step, given $nv \leq u_1$, then $(n+1)v \leq u_1 + v \leq u$, so $(n+1)v \in A$ and $(n+1)v \leq u_1$. Thus the induction proceeds. **Q** But this is impossible, because $v > 0$ and U is supposed to be Archimedean. **X**

So $u = \sup A$. As u is arbitrary, V is order-dense.

353B Proposition Let U be an Archimedean Riesz space. Then

(a) for every $A \subseteq U$, the band generated by A is $A^{\perp\perp}$,

(b) every band in U is complemented.

proof (a) Let V be the band generated by A. Then V is surely included in $A^{\perp\perp}$, because this is a band including A (352O). **?** Suppose, if possible, that $V \neq A^{\perp\perp}$. Then there is a $w \in A^{\perp\perp} \setminus V$, so that $|w| \notin V$. Set $B = \{v : v \in V, v \leq |w|\}$; then B is upwards-directed and non-empty. Because V is order-closed, $|w|$ cannot be the supremum of A, and there is a $u_0 > 0$ such that $|w| - u_0 \geq v$ for every $v \in B$. Now $u_0 \wedge |w| \neq 0$, so $u_0 \notin A^{\perp}$, and there is a $u_1 \in A$ such that $v = u_0 \wedge |u_1| > 0$. In this case $nv \in B$ for every $n \in \mathbb{N}$. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step, given that $nv \in B$, then $nv \leq |w| - u_0$ so $(n+1)v \leq nv + u_0 \leq |w|$; but also $(n+1)v \leq nv + |u_1| \in V$, so $(n+1)v \in B$. **Q** But this means that $nv \leq |w|$ for every n, which is impossible, because U is Archimedean. **X**

(b) Now if $V \subseteq U$ is any band, it is surely the band generated by itself, so is equal to $V^{\perp\perp}$, and is complemented (352P).

Remark We may therefore speak of the **band algebra** of an Archimedean Riesz space, rather than the 'complemented band algebra' (352Q).

353C Corollary Let U be an Archimedean Riesz space and $v \in U$. Let V be the band in U generated by v. If $u \in U$, then $u \in V$ iff there is no w such that $0 < w \le |u|$ and $w \wedge |v| = 0$.

proof By 353B, $V = \{v\}^{\perp\perp}$. Now, for $u \in U$,

$$u \notin V \iff \exists\, w \in \{v\}^{\perp},\, |u| \wedge |w| > 0 \iff \exists\, w \in \{v\}^{\perp},\, 0 < w \le |u|.$$

Turning this round, we have the condition announced.

353D Proposition Let U be an Archimedean Riesz space and V an order-dense Riesz subspace of U. Then the map $W \mapsto W \cap V$ is an isomorphism between the band algebras of U and V.

proof If $W \subseteq U$ is a band, then $W \cap V$ is surely a band in V (it is order-closed in V because it is the inverse image of the order-closed set W under the embedding $V \subsetneqq U$, which is order-continuous by 352Nc and 352Nb). If W, W' are distinct bands in U, say $W' \not\subseteq W$, then $W' \not\subseteq W^{\perp\perp}$, by 353B, so $W' \cap W^{\perp} \ne \{0\}$; because V is order-dense, $V \cap W' \cap W^{\perp} \ne \{0\}$, and $V \cap W' \ne V \cap W$. Thus $W \mapsto W \cap V$ is injective.

If $Q \subseteq V$ is a band in V, then its complementary band in V is just $Q^{\perp} \cap V$, where $Q^{\perp}$ is taken in U. So (because V, like U, is Archimedean, by 351Rc) $Q = (Q^{\perp} \cap V)^{\perp} \cap V = W \cap V$, where $W = (Q^{\perp} \cap V)^{\perp}$ is a band in U. Thus the map $W \mapsto W \cap V$ is an order-preserving bijection between the two band algebras. By 312M, it is a Boolean isomorphism, as claimed.

353E Lemma Let U be an Archimedean Riesz space and $V \subseteq U$ a band such that $\sup\{v : v \in V,\, 0 \le v \le u\}$ is defined for every $u \in U^+$. Then V is a projection band.

proof Take any $u \in U^+$ and set $v = \sup\{v' : v' \in V^+,\, v' \le u\}$, $w = u - v$. $v \in V$ because V is a band. Also $w \in V^{\perp}$. **P?** If not, there is some $v_0 \in V^+$ such that $w \wedge v_0 > 0$. Now for any $n \in \mathbb{N}$ we see that

$$n(w \wedge v_0) \le u \Longrightarrow n(w \wedge v_0) \le v \Longrightarrow (n+1)(w \wedge v_0) \le v + w = u,$$

so an induction on n shows that $n(w \wedge v_0) \le u$ for every n; which is impossible, because U is supposed to be Archimedean. **XQ** Accordingly $u = v + w \in V + V^{\perp}$. As u is arbitrary, $U^+ \subseteq V + V^{\perp}$, and V is a projection band (352R).

353F Lemma Let U be an Archimedean Riesz space. If $A \subseteq U$ is non-empty and bounded above and B is the set of its upper bounds, then $\inf(B - A) = 0$.

proof ? If not, let $w > 0$ be a lower bound for $B - A$. If $u \in A$ and $v \in B$, then $v - u \ge w$, that is, $u \le v - w$; as u is arbitrary, $v - w \in B$. Take any $u_0 \in A$ and $v_0 \in B$. Inducing on n, we see that $v_0 - nw \in B$ for every $n \in \mathbb{N}$, so that $v_0 - nw \ge u_0$, $nw \le v_0 - u_0$ for every n; but this is impossible, because U is supposed to be Archimedean. **X**

353G Proposition Let U be a Riesz space and V an order-dense Riesz subspace of U. If V is Archimedean, so is U.

proof ? Otherwise, let u', $u \in U$ be such that $u' > 0$ and $nu' \le u$ for every $n \in \mathbb{N}$. Let $v' \in V$ be such that $0 < v' \le u'$; set $\tilde{u} = u - v'$; let $v \in V$ be such that $v \le u$ but $v \not\le \tilde{u}$. (This is where we need V to be order-dense rather than just quasi-order-dense.) Let $w \in V$ be such that $w > 0$ and $w \le (v - \tilde{u})^+$; note that $w \le u - \tilde{u} = v'$.

Because V is Archimedean, there is an $n \ge 1$ such that $nw \not\le v$. In this case,

$$0 < (nw - v)^+ \le ((n+1)v' - (v + v'))^+ \le (u - (v + v'))^+ = (\tilde{u} - v)^+$$

but

$$(nw - v)^+ \wedge (\tilde{u} - v)^+ \le nw \wedge n(\tilde{u} - v)^+ \le n((v - \tilde{u})^+ \wedge (\tilde{u} - v)^+) = 0,$$

which is impossible. **X**

353H Dedekind completeness Recall that a partially ordered set P is Dedekind (σ)-complete if (countable) non-empty sets with upper and lower bounds have suprema and infima in P (314A). For a Riesz space U, U is Dedekind complete iff every non-empty upwards-directed subset of U^+ with an upper bound has a least upper bound, and is Dedekind σ-complete iff every non-decreasing sequence in U^+ with an upper bound has a least upper bound. **P** (Compare 314Bc.) (i) Suppose that any non-empty upwards-directed order-bounded subset of U^+ has an upper bound, and that $A \subseteq U$ is any non-empty set with an upper bound. Take $u_0 \in A$ and set

$$B = \{u_0 \vee u_1 \vee \ldots \vee u_n - u_0 : u_1, \ldots , u_n \in A\}.$$

Then B is an upwards-directed subset of U^+, and if w is an upper bound of A then $w - u_0$ is an upper bound of B. So $\sup B$ is defined in U, and in this case $u_0 + \sup B = \sup A$. As A is arbitrary, U is Dedekind complete. (ii) Suppose that order-bounded non-decreasing sequences in U^+ have suprema, and that $A \subseteq U$ is any countable non-empty set with an upper bound. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A, and set $v_n = \sup_{i\le n} u_i - u_0$ for each n. Then $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in U^+, and $u_0 + \sup_{n\in\mathbb{N}} v_n = \sup A$. (iii) Finally, still supposing that order-bounded non-decreasing sequences in U^+ have suprema, if $A \subseteq U$ is non-empty, countable and bounded below, $\inf A$ will be defined and equal to $-\sup(-A)$. **Q**

353I Proposition Let U be a Dedekind σ-complete Riesz space.

(a) U is Archimedean.

(b) For any $v \in U$ the band generated by v is a projection band.

(c) If u, $v \in U$, then u is uniquely expressible as $u_1 + u_2$, where u_1 belongs to the band generated by v and $|u_2| \wedge |v| = 0$.

proof (a) Suppose that u, $v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$. Then $nu^+ \le v^+$ for every n, and $A = \{nu^+ : n \in \mathbb{N}\}$ is a countable non-empty upwards-directed set with an upper bound; say $w = \sup A$. Since $A + u^+ \subseteq A$, $w + u^+ = \sup(A + u^+) \le w$, and $u \le u^+ \le 0$. As u, v are arbitrary, U is Archimedean.

(b) Let V be the band generated by v. Take any $u \in U^+$ and set $A = \{v' : v' \in V, 0 \le v' \le u\}$. Then $\{u \wedge n|v| : n \in \mathbb{N}\}$ is a countable set with an upper bound, so has a supremum u_1 say in U. Now u_1 is an upper bound for A. **P** If $v' \in A$, then

$$v' = \sup_{n\in\mathbb{N}} v' \wedge n|v| \le u_1$$

by 352Vb. **Q** Since $u \wedge n|v| \in A$ for every n, $u_1 = \sup A$.

As u is arbitrary, 353E tells us that V is a projection band.

(c) Again let V be the band generated by v. Then $\{v\}^{\perp\perp}$ is a band containing v, so

$$\{v\} \subseteq V \subseteq \{v\}^{\perp\perp}, \quad \{v\}^{\perp} \supseteq V^{\perp} \supseteq \{v\}^{\perp\perp\perp} = \{v\}^{\perp}$$

(352Od), and $V^{\perp} = \{v\}^{\perp}$.

Now, if $u \in U$, u is uniquely expressible in the form $u_1 + u_2$ where $u_1 \in V$ and $u_2 \in V^{\perp}$, by (b). But

$$u_2 \in V^{\perp} \iff u_2 \in \{v\}^{\perp} \iff |u_2| \wedge |v| = 0.$$

So we have the result.

353J Proposition In a Dedekind complete Riesz space, all bands are projection bands.

proof Use 353E, noting that the sets $\{v : v \in V, 0 \le v \le u\}$ there are always non-empty, upwards-directed and bounded above, so always have suprema.

353K Proposition (a) Let U be a Dedekind σ-complete Riesz space.

(i) If V is a solid linear subspace of U, then V is (in itself) Dedekind σ-complete.

(ii) If V is a sequentially order-closed Riesz subspace of U then V is Dedekind σ-complete.

(iii) If V is a sequentially order-closed solid linear subspace of U, the canonical map from U to the quotient space U/V is sequentially order-continuous, and U/V also is Dedekind σ-complete.

(b) Let U be a Dedekind complete Riesz space.

(i) If V is a solid linear subspace of U, then V is Dedekind complete.

(ii) If $V \subseteq U$ is an order-closed Riesz subspace then V is Dedekind complete.

proof (a)(i) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in V^+ with an upper bound $v \in V$, then $w = \sup_{n\in\mathbb{N}} u_n$ is defined in U; but as $0 \le w \le v$, $w \in V$ and $w = \sup_{n\in\mathbb{N}} u_n$ in V. Thus V is Dedekind σ-complete.

(ii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in W, then $u = \sup_{n\in\mathbb{N}} u_n$ is defined in U; but because V is sequentially order-closed, $u \in V$ and $u = \sup_{n\in\mathbb{N}} u_n$ in V.

(iii) Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in U with supremum u. Then of course $u^\bullet$ is an upper bound for $A = \{u_n^\bullet : n \in \mathbb{N}\}$ in U/V. Now let p be any other upper bound for A. Express p as $v^\bullet$. Then for each $n \in \mathbb{N}$ we have $u_n^\bullet \le p$, so that $(u_n - v)^+ \in V$. Because V is sequentially order-closed, $(u - v)^+ = \sup_{n\in\mathbb{N}}(u_n - v)^+ \in V$ and $u^\bullet \le p$. Thus $u^\bullet$ is the least upper bound of A. By 351Gb, $u \mapsto u^\bullet$ is sequentially order-continuous.

Now suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $(U/V)^+$ with an upper bound $p \in (U/V)^+$. Let $u \in U^+$ be such that $u^\bullet = p$, and for each $n \in \mathbb{N}$ let $u_n \in U^+$ be such that $u_n^\bullet = p_n$. Set $v_n = u \wedge \sup_{i\le n} u_i$ for each n; then $v_n^\bullet = p_n$ for each n, and $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing order-bounded sequence in U. Set $v = \sup_{n\in\mathbb{N}} v_n$; by the last paragraph, $v^\bullet = \sup_{n\in\mathbb{N}} p_n$ in U/V. As $\langle p_n\rangle_{n\in\mathbb{N}}$ is arbitrary, U/V is Dedekind σ-complete, as claimed.

(b) The argument is the same as parts (i) and (ii) of the proof of (a).

353L Proposition Let U be a Riesz space and V a quasi-order-dense Riesz subspace of U which is (in itself) Dedekind complete. Then V is a solid linear subspace of U.

proof Suppose that $v \in V$, $u \in U$ and $|u| \le |v|$. Consider $A = \{w : w \in V, 0 \le w \le u^+\}$. Then A is a non-empty subset of V with an upper bound in V (viz., $|v|$). So A has a supremum v_0 in V. Because the embedding $V \subseteq U$ is order-continuous (352Nb), v_0 is the supremum of A in U. But as V is order-dense (353A), $v_0 = u^+$ and $u^+ \in V$. Similarly, $u^- \in V$ and $u \in V$. As u and v are arbitrary, V is solid.

353M Order units Let U be a Riesz space.

(a) An element e of U^+ is an **order unit** in U if U is the solid linear subspace of itself generated by e; that is, if for every $u \in U$ there is an $n \in \mathbb{N}$ such that $|u| \le ne$. (For the solid linear subspace generated by $v \in U^+$ is $\bigcup_{n\in\mathbb{N}}[-nv, nv]$.)

(b) An element e of U^+ is a **weak order unit** in U if U is the principal band generated by e; that is, if $u = \sup_{n\in\mathbb{N}} u \wedge ne$ for every $u \in U^+$ (352Vb).

Of course an order unit is a weak order unit.

(c) If U is Archimedean, then an element e of U^+ is a weak order unit iff $\{e\}^{\perp\perp} = U$ (353B), that is, iff $\{e\}^\perp = \{0\}$ (because

$$\{e\}^\perp = \{0\} \implies \{e\}^{\perp\perp} = \{0\}^\perp = U \implies \{e\}^\perp = \{e\}^{\perp\perp\perp} = U^\perp = \{0\},)$$

that is, iff $u \wedge e > 0$ whenever $u > 0$.

353N Theorem Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that e corresponds to χX; moreover, this embedding is essentially unique.

Remark Here $C(X)$ is the space of all continuous functions from X to $\mathbb{R}$; because X is compact, they are all bounded, so that χX is an order unit in $C(X)$.

proof (a) Let X be the set of Riesz homomorphisms x from U to $\mathbb{R}$ such that $x(e) = 1$. Define $T : U \to \mathbb{R}^X$ by setting $(Tu)(x) = x(u)$ for $x \in X$, $u \in U$; then it is easy to check that T is a Riesz homomorphism, just because every member of X is a Riesz homomorphism, and of course $Te = \chi X$.

(b) The key to the proof is the fact that X separates the points of U, that is, that T is injective. I choose the following method to show this. Suppose that $w \in U$ and $w > 0$. Because U is Archimedean, there is a $\delta > 0$ such that $(w - \delta e)^+ \ne 0$. Now there is an $x \in X$ such that $x(w) \ge \delta$. **P** (i) By 351O, there is a solid linear subspace V of U such that $(w - \delta e)^+ \notin V$ and whenever $u \wedge v = 0$ in U then one of u, v belongs to V.

(ii) Because $V \neq U$, $e \notin V$, so no non-zero multiple of e can belong to V. Also observe that if u, $v \in U \setminus V$, then one of $(u-v)^+$, $(v-u)^+$ must belong to V, while neither $u = u \wedge v + (u-v)^+$ nor $v = u \wedge v + (v-u)^+$ does; so $u \wedge v \notin V$. (iii) For each $u \in U$ set $A_u = \{\alpha : \alpha \in \mathbb{R}, (u - \alpha e)^+ \in V\}$. Then

$$\alpha \geq \beta \in A_u \Longrightarrow 0 \leq (u - \alpha e)^+ \leq (u - \beta e)^+ \in V \Longrightarrow \alpha \in A_u.$$

Also A_u is non-empty and bounded below, because if $\alpha \geq 0$ is such that $-\alpha e \leq u \leq \alpha e$ then $\alpha \in A_u$ and $-\alpha - 1 \notin A_u$ (since $(u - (-\alpha - 1)e)^+ \geq e \notin V$). (iv) Set $x(u) = \inf A_u$ for every $u \in U$; then $\alpha \in A_u$ for every $\alpha > x(u)$, $\alpha \notin A_u$ for every $\alpha < x(u)$. (v) If u, $v \in U$, $\alpha > x(u)$ and $\beta > x(v)$ then

$$((u+v) - (\alpha + \beta)e)^+ \leq (u - \alpha e)^+ + (v - \beta e)^+ \in V$$

(352Fc), so $\alpha + \beta \in A_{u+v}$; as α and β are arbitrary, $x(u+v) \leq x(u) + x(v)$. (vi) If u, $v \in U$, $\alpha < x(u)$ and $\beta < x(v)$ then

$$((u+v) - (\alpha + \beta)e)^+ \geq (u - \alpha e)^+ \wedge (v - \beta e)^+ \notin V,$$

using (ii) of this argument and the other part of 352Fc, so $\alpha + \beta \notin A_{u+v}$. As α and β are arbitrary, $x(u+v) \geq x(u) + x(v)$. (vii) Thus $x : U \to \mathbb{R}$ is additive. (viii) If $u \in U$ and $\gamma > 0$ then

$$\alpha \in A_u \Longrightarrow (\gamma u - \alpha\gamma e)^+ = \gamma(u - \alpha e)^+ \in V \Longrightarrow \gamma\alpha \in A_{\gamma u};$$

thus $A_{\gamma u} \supseteq \gamma A_u$; similarly, $A_u \supseteq \gamma^{-1} A_{\gamma u}$ so $A_{\gamma u} = \gamma A_u$ and $x(\gamma u) = \gamma x(u)$. (ix) Consequently x is linear, since we know already from (vii) that $x(0u) = 0.x(u)$, $x(-u) = -x(u)$. (x) If $u \geq 0$ then $u + \alpha e \geq \alpha e \notin V$ for every $\alpha > 0$, that is, $-\alpha \notin A_u$ for every $\alpha > 0$, and $x(u) \geq 0$; thus x is a positive linear functional. (xi) If $u \wedge v = 0$, then one of u, v belongs to V, so $\min(x(u), x(v)) \leq 0$ and (using (x)) $\min(x(u), x(v)) = 0$; thus x is a Riesz homomorphism (352G(iv)). (xii) $A_e = [1, \infty[$ so $x(e) = 1$. Thus $x \in X$. (xiii) $\delta \notin A_w$ so $x(w) \geq \delta$. **Q**

(c) Thus $Tw \neq 0$ whenever $w > 0$; consequently $|Tw| = T|w| \neq 0$ whenever $w \neq 0$, and T is injective. I now have to define the topology of X. This is just the subspace topology on X if we regard X as a subset of $\mathbb{R}^U$ with its product topology. To see that X is compact, observe that if for each $u \in U$ we choose an α_u such that $|u| \leq \alpha_u e$, then X is a subspace of $Q = \prod_{u \in U} [-\alpha_u, \alpha_u]$. Because Q is a product of compact spaces, it is compact, by Tychonoff's theorem (3A3J). Now X is a closed subset of Q. **P** X is just the intersection of the sets

$$\{x : x(u+v) = x(u) + x(v)\}, \quad \{x : x(\alpha u) = \alpha x(u)\},$$

$$\{x : x(u^+) = \max(x(u), 0)\}, \quad \{x : x(e) = 1\}$$

as u, v run over U and α over $\mathbb{R}$; and each of these is closed, so X is an intersection of closed sets and therefore itself closed. **Q** Consequently X also is compact. Moreover, the coordinate functionals $x \mapsto x(u)$ are continuous on Q, therefore on X also, that is, $Tu : X \to \mathbb{R}$ is a continuous function for every $u \in U$.

Note also that because Q is a product of Hausdorff spaces, Q and X are Hausdorff (3A3Id, 3A3Bh).

(d) So T is a Riesz homomorphism from U to $C(X)$. Now $T[U]$ is a Riesz subspace of $C(X)$, containing χX, and such that if x, $y \in X$ are distinct there is an $f \in T[U]$ such that $f(x) \neq f(y)$ (because there is surely a $u \in U$ such that $x(u) \neq y(u)$). By the Stone-Weierstrass theorem (281A), $T[U]$ is $\| \|_\infty$-dense in $C(X)$.

Consequently it is also order-dense. **P** If $f > 0$ in $C(X)$, set $\epsilon = \frac{1}{3}\|f\|_\infty$, and let $u \in U$ be such that $\|f - Tu\|_\infty \leq \epsilon$; set $v = (u - \epsilon e)^+$. Since

$$0 < (f - 2\epsilon\chi X)^+ \leq (Tu - \epsilon\chi X)^+ \leq f^+ = f,$$

$0 < Tv \leq f$. As f is arbitrary, $T[U]$ is quasi-order-dense, therefore order-dense (353A). **Q**

(e) I have still to show that the representation is (essentially) unique. Suppose, then, that we have another representation of U as a norm-dense Riesz subspace of $C(Z)$, with e this time corresponding to χZ; to simplify the notation, let us suppose that U is actually a subspace of $C(Z)$. Then for each $z \in Z$, we have a functional $\hat{z} : U \to \mathbb{R}$ defined by setting $\hat{z}(u) = u(z)$ for every $u \in U$; of course $\hat{z}$ is a Riesz homomorphism such that $\hat{z}(e) = 1$, that is, $\hat{z} \in X$. Thus we have a function $z \mapsto \hat{z} : Z \to X$. For any $u \in U$, the function $z \mapsto \hat{z}(u) = u(z)$ is continuous, so the function $z \mapsto \hat{z}$ is continuous (3A3Ib). If z_1, z_2 are distinct members of Z, there is an $f \in C(Z)$ such that $f(z_1) \neq f(z_2)$ (3A3Bf); now there is a $u \in U$

such that $\|f-u\|_\infty \le \frac{1}{3}|f(z_1)-f(z_2)|$, so that $u(z_1)\ne u(z_2)$ and $\hat{z}_1\ne\hat{z}_2$. Thus $z\mapsto\hat{z}$ is injective. Finally, it is also surjective. **P** Suppose that $x\in X$. Set $V=\{u:u\in U, x(u)=0\}$; then V is a solid linear subspace of U (352Jb), not containing e. For $z\in V^+$ set $G_v=\{z:v(z)>1\}$. Because $e\notin V$, $G_v\ne Z$. $\mathcal{G}=\{G_v:v\in V^+\}$ is an upwards-directed family of open sets in Z, not containing Z; consequently, because Z is compact, $\mathcal{G}$ cannot be an open cover of Z. Take $z\in Z\setminus\bigcup\mathcal{G}$. Then $v(z)\le 1$ for every $v\in V^+$; because $\alpha|v|\in V^+$ whenever $v\in V$ and $\alpha\ge 0$, we must have $v(z)=0$ for every $v\in V$. Now, given any $u\in U$, consider $v=u-x(u)e$. Then $x(v)=0$ so $v\in V$ and $v(z)=0$, that is,

$$u(z)=(v+x(u)e)(z)=v(z)+x(u)e(z)=x(u).$$

As u is arbitrary, $\hat{z}=x$; as x is arbitrary, we have the result. **Q**

Thus $z\mapsto\hat{z}$ is a continuous bijection from the compact Hausdorff space Z to the compact Hausdorff space X; it must therefore be a homeomorphism (3A3Dd).

This argument shows that if U is embedded as a norm-dense Riesz subspace of $C(Z)$, where Z is compact and Hausdorff, then Z must be homeomorphic to X. But it shows also that a homeomorphism is canonically defined by the embedding; $z\in Z$ corresponds to the Riesz homomorphism $u\mapsto u(z)$ in X.

353O Lemma Let U be a Riesz space, V an Archimedean Riesz space and S, $T:U\to V$ Riesz homomorphisms such that $Su\wedge Tu'=0$ in V whenever $u\wedge u'=0$ in U. Set $W=\{u:Su=Tu\}$. Then W is a solid linear subspace of U; if S and T are order-continuous, W is a band.

proof (a) It is easy to check that, because S and T are Riesz homomorphisms, W is a Riesz subspace of U.

(b) If $w\in W$ and $0\le u\le w$ in U, then $Su\le Tu$. **P?** Otherwise, set $e=Sw=Tw$, and let V_e be the solid linear subspace of V generated by e, so that V_e is an Archimedean Riesz space with order unit, containing both Su and Tu. By 353N (or its proof), there is a Riesz homomorphism $x:V_e\to\mathbb{R}$ such that $x(e)=1$ and $x(Su)>x(Tu)$. Take α such that $x(Su)>\alpha>x(Tu)$, and consider $u'=(u-\alpha w)^+$, $u''=(\alpha w-u)^+$. Then

$$x(Su')=\max(0,x(Su)-\alpha x(Sw))=\max(0,x(Su)-\alpha)>0,$$

$$x(Tu'')=\max(0,\alpha x(Tw)-x(Tu))=\max(0,\alpha-x(Tu))>0,$$

so

$$x(Su'\wedge Tu'')=\min(x(Su'),x(Tu''))>0$$

and $Su'\wedge Tu''>0$, while $u'\wedge u''=0$. **XQ**

Similarly, $Tu\le Su$ and $u\in W$. As u and w are arbitrary, W is a solid linear subspace.

(c) Finally, suppose that S and T are order-continuous, and that $A\subseteq W$ is a non-empty upwards-directed set with supremum u in U. Then

$$Su=\sup S[A]=\sup T[A]=Tu$$

and $u\in W$. As u and A are arbitrary, W is a band (352Ob).

353P *f*-algebras I give two results on f-algebras, intended to clarify the connexions between the multiplicative and lattice structures of the Riesz spaces in Chapter 36.

Proposition Let U be an Archimedean f-algebra (352W). Then

(a) the multiplication is separately order-continuous in the sense that the maps $u\mapsto u\times w$, $u\mapsto w\times u$ are order-continuous for every $w\in U^+$;

(b) the multiplication is commutative.

proof (a) Let $A\subseteq U$ be a non-empty set with infimum 0, and $v_0\in U^+$ a lower bound for $\{u\times w:u\in A\}$. Fix $u_0\in A$. If $u\in A$ and $\delta>0$, then $v_0\wedge(u_0-\frac{1}{\delta}u)^+\le\delta u_0\times w$. **P** Set $v=v_0\wedge(u_0-\frac{1}{\delta}u)^+$. Then

$$\delta v\wedge(u-\delta u_0)^+\le(\delta u_0-u)^+\wedge(u-\delta u_0)^+=0,$$

so $v\wedge(u-\delta u_0)^+=0$ and $v\wedge((u-\delta u_0)^+\times w)=0$. But

$$v\le v_0\le u\times w\le(u-\delta u_0)^+\times w+\delta u_0\times w,$$

so

$$v \le ((u - \delta u_0)^+ \times w) \wedge v + (\delta u_0 \times w) \wedge v \le \delta u_0 \times w,$$

by 352Fa. **Q**

Taking the infimum over u, and using the distributive laws (352E), we get

$$v_0 \wedge u_0 \le \delta u_0 \times w.$$

Taking the infimum over δ, and using the hypothesis that U is Archimedean,

$$v_0 \wedge u_0 = 0.$$

But this means that $v_0 \wedge (u_0 \times w) = 0$, while $v_0 \le u_0 \times w$, so $v_0 = 0$. As v_0 is arbitrary, $\inf_{u \in A} u \times w = 0$; as A is arbitrary, $u \mapsto u \times w$ is order-continuous. Similarly, $u \mapsto w \times u$ is order-continuous.

(b)(i) Fix $v \in U^+$, and for $u \in U$ set

$$Su = u \times v, \quad Tu = v \times u.$$

Then S and T are both order-continuous Riesz homomorphisms from U to itself (352W(b-iv) and (a) above). Also, $Su \wedge Tu' = 0$ whenever $u \wedge u' = 0$. **P**

$$0 = (u \times v) \wedge u' = (u \times v) \wedge (v \times u'). \text{ } \mathbf{Q}$$

So $W = \{u : u \times v = v \times u\}$ is a band in U (353O). Of course $v \in W$ (because $Sv = Tv = v^2$). If $u \in W^\perp$, then $v \wedge |u| = 0$ so $Su = Tu = 0$ (352W(b-i)), and $u \in W$; but this means that $W^\perp = \{0\}$ and $W = W^{\perp\perp} = U$ (353Bb). Thus $v \times u = u \times v$ for every $u \in U$.

(ii) This is true for every $v \in U^+$. Of course it follows that $v \times u = u \times v$ for every u, $v \in U$, so that multiplication is commutative.

353Q Proposition Let U be an Archimedean f-algebra with multiplicative identity e.

(a) e is a weak order unit in U.

(b) If u, $v \in U$ then $u \times v = 0$ iff $|u| \wedge |v| = 0$.

(c) If $u \in U$ has a multiplicative inverse u^{-1} then $|u|$ also has a multiplicative inverse; if $u \ge 0$ then $u^{-1} \ge 0$ and u is a weak order unit.

(d) If V is another Archimedean f-algebra with multiplicative identity e', and $T : U \to V$ is a positive linear operator such that $Te = e'$, then T is a Riesz homomorphism iff $T(u \times v) = Tu \times Tv$ for all u, $v \in U$.

proof (a) $e = e^2 \ge 0$ by 352W(b-ii). If $u \in U$ and $e \wedge |u| = 0$ then $|u| = (e \times |u|) \wedge |u| = 0$; by 353Mc, e is a weak order unit.

(b) If $|u| \wedge |v| = 0$ then $u \times v = 0$, by 352Wb. If $w = |u| \wedge |v| > 0$, then $w^2 \le |u| \times |v|$. Let $n \in \mathbb{N}$ be such that $nw \not\le e$, and set $w_1 = (nw - e)^+$, $w_2 = (e - nw)^+$. Then

$$\begin{aligned} 0 \ne w_1 = w_1 \times e &= w_1 \times w_2 + w_1 \times (e \wedge nw) \\ &= w_1 \times (e \wedge nw) \le (nw)^2 \le n^2 |u| \times |v| = n^2 |u \times v|, \end{aligned}$$

so $u \times v \ne 0$.

(c) $u \times u^{-1} = e$ so $|u| \times |u^{-1}| = |e| = e$ (352W(b-iii)), and $|u^{-1}| = |u|^{-1}$. (Recall that inverses in any semigroup with identity are unique, so that we need have no inhibitions in using the formulae u^{-1}, $|u|^{-1}$.)

Now suppose that $u \ge 0$. Then $u^{-1} = |u^{-1}| \ge 0$. If $u \wedge |v| = 0$ then

$$e \wedge |v| = (u \times u^{-1}) \wedge |v| = 0,$$

so $v = 0$; accordingly u is a weak order unit.

(d)(i) If T is multiplicative, and $u \wedge v = 0$ in U, then $Tu \times Tv = T(u \times v) = 0$ and $Tu \wedge Tv = 0$, by (b). So T is a Riesz homomorphism, by 352G.

(ii) Accordingly I shall henceforth assume that T is a Riesz homomorphism and seek to show that it is multiplicative.

If u, $v \in U^+$, then $T(u \times v)$ and $Tu \times Tv$ both belong to the band generated by Tu. **P** Write W for this band. (α) For any $n \ge 1$ we have $(v - ne)^2 \ge 0$, that is, $2nv \le v^2 + n^2 e$, so

$$n(v - ne) \le 2nv - n^2 e \le v^2.$$

Consequently

$$T(u \times v) - nTu = T(u \times v) - nT(u \times e) = T(u \times (v - ne)) \le \frac{1}{n} T(u \times v^2)$$

because $v' \mapsto T(u \times v')$ is a positive linear operator; as V is Archimedean, $\inf_{n \in \mathbb{N}} (T(u \times v) - nTu)^+ = 0$ and $T(u \times v) = \sup_{n \in \mathbb{N}} T(u \times v) \wedge nTu$ belongs to W. (β) If $w \wedge |Tu| = 0$ then

$$w \wedge |Tu \times Tv| = w \wedge (|Tu| \times |Tv|) = 0;$$

so $Tu \times Tv \in W^{\perp\perp} = W$. **Q**

(iii) Fix $v \in U^+$. For $u \in U$, set $S_1 u = Tu \times Tv$ and $S_2 u = T(u \times v)$. (Cf. (b-i) of the proof of 353P.) Then S_1 and S_2 are both Riesz homomorphisms from U to V. If $u \wedge u' = 0$ in U, then $S_1 u \wedge S_2 u' = 0$ in V, because (by (ii) just above) $S_1 u$ belongs to the band generated by Tu, while $S_2 u'$ belongs to the band generated by Tu', and $Tu \wedge Tu' = T(u \wedge u') = 0$. By 353O, $W = \{u : S_1 u = S_2 u\}$ is a solid linear subspace of U. Of course it contains e, since

$$S_1 e = Te \times Tv = e' \times Tv = Tv = T(e \times v) = S_2 e.$$

In fact $u \in W$ for every $u \in U^+$. **P** As noted in (ii) just above, $u - ne \le \frac{1}{n} u^2$ for every $n \ge 1$. So

$$\begin{aligned} |S_1 u - S_2 u| &= |S_1(u - ne)^+ + S_1(u \wedge ne) - S_2(u - ne)^+ - S_2(u \wedge ne)| \\ &\le S_1(u - ne)^+ + S_2(u - ne)^+ \le \frac{1}{n}(S_1 u^2 + S_2 u^2) \end{aligned}$$

for every $n \ge 1$, and $|S_1 u - S_2 u| = 0$, that is, $S_1 u = S_2 u$. **Q**

So $W = U$, that is, $Tu \times Tv = T(u \times v)$ for every $u \in U$. And this is true for every $v \in U^+$. It follows at once that it is true for every $v \in U$, so that T is multiplicative, as claimed.

353X Basic exercises >(a) Let U be a Riesz space in which every band is complemented. Show that U is Archimedean.

(b) A Riesz space U has the **principal projection property** iff the band generated by any single member of U is a projection band. Show that any Dedekind σ-complete Riesz space has the principal projection property, and that any Riesz space with the principal projection property is Archimedean.

>(c) Fill in the missing part (b-iii) of 353K.

(d) Let U be an Archimedean f-algebra with an order-unit which is a multiplicative identity. Show that U can be identified, as f-algebra, with a subspace of $C(X)$ for some compact Hausdorff space X.

353Y Further exercises (a) Let U be a Riesz space in which every quasi-order-dense solid linear subspace is order-dense. Show that U is Archimedean.

(b) Let X be a completely regular Hausdorff space. Show that $C(X)$ is Dedekind complete iff $C_b(X)$ is Dedekind complete iff X is extremally disconnected.

(c) Let X be a compact Hausdorff space. Show that $C(X)$ is Dedekind σ-complete iff $\overline{G}$ is open for every cozero set $G \subseteq X$. (Cf. 314Yf.) Show that in this case X is zero-dimensional.

(d) Let U be an Archimedean Riesz space. Show that the following are equiveridical: (i) U has the countable sup property (241Ye) (ii) for every $A \subseteq U$ there is a countable $B \subseteq A$ such that A and B have the same upper bounds (iii) every order-bounded disjoint subset of U^+ is countable.

(e) Let U be an Archimedean Riesz space such that every order-bounded disjoint sequence in U^+ has a supremum in U. Show that U has the principal projection property, but need not be Dedekind σ-complete.

(f) Let U be an Archimedean f-algebra. Show that an element e of U is a multiplicative identity iff $e^2 = e$ and e is a weak order unit.

(g) Let U be an Archimedean f-algebra with a multiplicative identity. Show that if $u \in U$ then u is invertible iff $|u|$ is invertible.

353 Notes and comments As in the last section, many of the results above have parallels in the theory of Boolean algebras; thus 353A corresponds to 313K, 353H corresponds in part to remarks in 314Bc and 314Xa, and 353K corresponds to 314C-314E. Riesz spaces are more complicated; for instance, principal ideals in Boolean algebras are straightforward, while in Riesz spaces we have to distinguish between the solid linear subspace generated by an element and the band generated by the same element. Thus an 'order unit' in a Boolean ring would just be an identity, while in a Riesz space we must distinguish between 'order unit' and 'weak order unit'. As this remark may suggest to you, (Archimedean) Riesz spaces are actually closer in spirit to arbitrary Boolean rings than to the Boolean algebras we have been concentrating on so far; to the point that in §361 below I will return briefly to general Boolean rings.

Note that the standard definition of 'order-dense' in Boolean algebras, as given in 313J, corresponds to the definition of 'quasi-order-dense' in Riesz spaces (352Na); the point here being that Boolean algebras behave like Archimedean Riesz spaces, in which there is no need to make a distinction.

I give the representation theorem 353N more for completeness than because we need it in any formal sense. In 351Q and 352L I have given representation theorems for general partially ordered linear spaces, and general Riesz spaces, as quotients of spaces of functions; in 368F below I give a theorem for Archimedean Riesz spaces corresponding rather more closely to the expressions of the L^p spaces as quotients of spaces of measurable functions. In 353N, by contrast, we have a theorem expressing Archimedean Riesz spaces with order units as true spaces of functions, rather than as spaces of equivalence classes of functions. All these theorems are important in forming an appropriate mental picture of ordered linear spaces, as in 352M.

I give a bare-handed proof of 353N, using only the Riesz space structure of $C(X)$; if you know a little about extreme points of dual unit balls you can approach from that direction instead, using 354Yj. The point is that (as part (d) of the proof of 353N makes clear) the space X can be regarded as a subset of the normed space dual U^* of U with its weak* topology. In this treatise generally, and in the present chapter in particular, I allow myself to be slightly prejudiced against normed-space methods; you can find them in any book on functional analysis, and I prefer here to develop techniques like those in part (b) of the proof of 353N, which will be a useful preparation for such theorems as 368E.

There is a very close analogy between 353N and the Stone representation of Boolean algebras (311E, 311I-311K). Just as the proof of 311E looked at the set of ring homomorphisms from $\mathfrak{A}$ to the elementary Boolean algebra $\mathbb{Z}_2$, so the proof of 353N looks at Riesz homomorphisms from U to the elementary M-space $\mathbb{R}$. Later on, the most important M-spaces, from the point of view of this treatise, will be the L^∞ spaces of §363, explicitly defined in terms of Stone representations (363A).

Of the two parts of 353P, it is (a) which is most important for the purposes of this book. The f-algebras we shall encounter in Chapter 36 can be seen to be commutative for different, and more elementary, reasons. The (separate) order-continuity of multiplication, however, is not always immediately obvious. Similarly, the uniferent Riesz homomorphisms we shall encounter can generally be seen to be multiplicative without relying on the arguments of 353Qd.

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354 Banach lattices

The next step is a brief discussion of norms on Riesz spaces. I start with the essential definitions (354A, 354D) with the principal properties of general Riesz norms (354B-354C) and order-continuous norms (354E). I then describe two of the most important classes of Banach lattice: M-spaces (354F-354L) and L-spaces

(354M-354R), with their elementary properties. For M-spaces I give the basic representation theorem (354K-354L), and for L-spaces I give a note on uniform integrability (354P-354R).

354A Definitions (a) If U is a Riesz space, a **Riesz norm** or **lattice norm** on U is a norm $\| \;\|$ such that $\|u\| \le \|v\|$ whenever $|u| \le |v|$; that is, a norm such that $\||u|\| = \|u\|$ for every u and $\|u\| \le \|v\|$ whenever $0 \le u \le v$.

(b) A **Banach lattice** is a Riesz space with a Riesz norm under which it is complete.

Remark We have already seen many examples of Banach lattices; I list some in 354Xa below.

354B Lemma Let U be a Riesz space with a Riesz norm $\| \;\|$.

(a) U is Archimedean.

(b) The maps $u \mapsto |u|$ and $u \mapsto u^+$ are uniformly continuous.

(c) For any $u \in U$, the sets $\{v : v \le u\}$ and $\{v : v \ge u\}$ are closed; in particular, the positive cone of U is closed.

(d) Any band in U is closed.

(e) If V is a norm-dense Riesz subspace of U, then $V^+ = \{v : v \in V,\, v \ge 0\}$ is norm-dense in the positive cone U^+ of U.

proof (a) If u, $v \in U$ are such that $nu \le v$ for every $n \in \mathbb{N}$, then $nu^+ \le v^+$ so $n\|u^+\| \le \|v^+\|$ for every n, and $\|u^+\| = 0$, that is, $u^+ = 0$ and $u \le 0$. As u, v are arbitrary, U is Archimedean.

(b) For any u, $v \in U$, $||u| - |v|| \le |u - v|$ (352D), so $\||u| - |v|\| \le \|u - v\|$; thus $u \mapsto |u|$ is uniformly continuous. Consequently $u \mapsto \frac{1}{2}(u + |u|) = u^+$ is uniformly continuous.

(c) Now $\{v : v \le u\} = \{v : (v - u)^+ = 0\}$ is closed because the function $v \mapsto (v - u)^+$ is continuous and $\{0\}$ is closed. Similarly $\{v : v \ge u\} = \{v : (u - v)^+ = 0\}$ is closed.

(d) If $V \subseteq U$ is a band, then $V = V^{\perp\perp}$ (353B), that is, $V = \{v : |v| \wedge |w| = 0$ for every $w \in V^\perp\}$. Because the function $v \mapsto |v| \wedge |w| = \frac{1}{2}(|v| + |w| - ||v| - |w||)$ is continuous, all the sets $\{v : |v| \wedge |w| = 0\}$ are closed, and so is their intersection V.

(e) Observe that $V^+ = \{v^+ : v \in V\}$ and $U^+ = \{u^+ : u \in U\}$; recall that $u \mapsto u^+$ is continuous, and apply 3A3Eb.

354C Lemma If U is a Banach lattice and $\langle u_n \rangle_{n\in\mathbb{N}}$ is a sequence in U such that $\sum_{n=0}^\infty \|u_n\| < \infty$, then $\sup_{n\in\mathbb{N}} u_n$ is defined in U, with $\|\sup_{n\in\mathbb{N}} u_n\| \le \sum_{n=0}^\infty \|u_n\|$.

proof Set $v_n = \sup_{i\le n} u_i$ for each n. Then

$$0 \le v_{n+1} - v_n \le (u_{n+1} - u_n)^+ \le |u_{n+1} - u_n|$$

for each $n \in \mathbb{N}$, so

$$\sum_{n=0}^\infty \|v_{n+1} - v_n\| \le \sum_{n=0}^\infty \|u_{n+1} - u_n\| \le \sum_{n=0}^\infty \|u_{n+1}\| + \|u_n\|$$

is finite, and $\langle v_n \rangle_{n\in\mathbb{N}}$ is Cauchy. Let u be its limit; because $\langle v_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, and the sets $\{v : v \ge v_n\}$ are all closed, $u \ge v_n$ for each $n \in \mathbb{N}$. On the other hand, if $v \ge v_n$ for every n, then

$$(u - v)^+ = \lim_{n\to\infty} (v_n - v)^+ = 0,$$

and $u \le v$. So

$$u = \sup_{n\in\mathbb{N}} v_n = \sup_{n\in\mathbb{N}} u_n$$

is the required supremum.

To estimate its norm, observe that $|v_n| \le \sum_{i=0}^n |u_i|$ for each n (induce on n, using the last item in 352D for the inductive step), so that

$$\|u\| = \lim_{n\to\infty} \|v_n\| \le \sum_{i=0}^\infty \||u_i|\| = \sum_{i=0}^\infty \|u_i\|.$$

354D I come now to the basic properties according to which we classify Riesz norms.

Definitions (a) A **Fatou norm** on a Riesz space U is a Riesz norm on U such that whenever $A \subseteq U^+$ is non-empty and upwards-directed and has a least upper bound in U, then $\|\sup A\| = \sup_{u \in A} \|u\|$. (Observe that, once we know that $\| \|$ is a Riesz norm, we can be sure that $\|u\| \le \|\sup A\|$ for every $u \in A$, so that all we shall need to check is that $\|\sup A\| \le \sup_{u \in A} \|u\|$.)

(b) A Riesz norm on a Riesz space U has the **Levi property** if every upwards-directed norm-bounded set is bounded above.

(c) A Riesz norm on a Riesz space U is **order-continuous** if $\inf_{u \in A} \|u\| = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set with infimum 0.

354E Proposition Let U be a Riesz space with an order-continuous Riesz norm $\| \|$.

(a) If $A \subseteq U$ is non-empty and upwards-directed and has a supremum, then $\sup A \in \overline{A}$.

(b) $\| \|$ is Fatou.

(c) If $A \subseteq U$ is non-empty and upwards-directed and bounded above, then for every $\epsilon > 0$ there is a $u \in A$ such that $\|(v-u)^+\| \le \epsilon$ for every $v \in A$; that is, the filter $\mathcal{F}(A\uparrow)$ on U generated by $\{\{v : v \in A, u \le v\} : u \in A\}$ is a Cauchy filter.

(d) Any non-decreasing order-bounded sequence in U is Cauchy.

(e) If U is a Banach lattice it is Dedekind complete.

(f) Every order-dense Riesz subspace of U is norm-dense.

(g) Every norm-closed solid linear subspace of U is a band.

proof (a) Suppose that $A \subseteq U$ is non-empty and upwards-directed and has a least upper bound u_0. Then $B = \{u_0 - u : u \in A\}$ is downwards-directed and has infimum 0. So $\inf_{u \in A} \|u_0 - u\| = 0$, and $u_0 \in \overline{A}$.

(b) If, in (a), $A \subseteq U^+$, then we must have

$$\|u_0\| \le \inf_{u \in A} \|u\| + \|u - u_0\| \le \sup_{u \in A} \|u\|.$$

As A is arbitary, $\| \|$ is a Fatou norm.

(c) Let B be the set of upper bounds for A. Then B is downwards-directed; because A is upwards-directed, $B - A = \{v - u : v \in B, u \in A\}$ is downwards-directed. By 353F, $\inf(B - A) = 0$. So there are $w \in B$, $u \in A$ such that $\|w - u\| \le \epsilon$. Now if $v \in A$,

$$(v-u)^+ = (v \vee u) - u \le w - u,$$

so $\|(v-u)^+\| \le \epsilon$.

In terms of the filter $\mathcal{F}(A\uparrow)$, this tells us that if v_0, v_1 belong to $F_u = \{v : v \in A, v \ge u\}$ then $|v_0 - v_1| \le w - u$ so $\|v_0 - v_1\| \le \epsilon$ and the diameter of F_u is at most ϵ. As ϵ is arbitrary, $\mathcal{F}(A\uparrow)$ is a Cauchy filter.

(d) If $\langle u_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing order-bounded sequence, and $\epsilon > 0$, then, applying (c) to $\{u_n : n \in \mathbb{N}\}$, we find that there is an $m \in \mathbb{N}$ such that $\|u_m - u_n\| \le \epsilon$ whenever $m \ge n$.

(e) Now suppose that U is a Banach lattice. Let $A \subseteq U$ be any non-empty set with an upper bound. Set $B = \{u_0 \vee \ldots \vee u_n : u_0, \ldots, u_n \in A\}$, so that B is upwards-directed and has the same upper bounds as A. Let $\mathcal{F}(B\uparrow)$ be the filter on U generated by $\{B \cap [v, \infty[\, : v \in B\}$. By (c), this is a Cauchy filter with a limit u^* say. For every $u \in A$, $[u, \infty[$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains u^*; thus u^* is an upper bound for A. If w is any upper bound for A, $]-\infty, w]$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains u^*; thus $u^* = \sup A$ and A has a supremum.

(f) If V is an order-dense Riesz subspace of U and $u \in U^+$, set $A = \{v : v \in V, v \le u\}$. Then A is upwards-directed and has supremum u, so $u \in \overline{A} \subseteq \overline{V}$, by (a). Thus $U^+ \subseteq \overline{V}$; it follows at once that $U = U^+ - U^+ \subseteq \overline{V}$.

(g) If V is a norm-closed solid linear subspace of U, and $A \subseteq V^+$ is a non-empty, upwards-directed subset of V with a supremum in U, then $\sup A \in \overline{A} \subseteq V$, by (a); by 352Ob, V is a band.

354F Lemma If U is an Archimedean Riesz space with an order unit e (definition: 353M), there is a Riesz norm $\| \|_e$ defined on U by the formula

$$\|u\|_e = \min\{\alpha : \alpha \geq 0,\, |u| \leq \alpha e\}$$

for every $u \in U$.

proof This is a routine verification. Because e is an order-unit, $\{\alpha : \alpha \geq 0,\, |u| \leq \alpha e\}$ is always non-empty, so always has an infimum α_0 say; now $|u| - \alpha_0 e \leq \delta e$ for every $\delta > 0$, so (because U is Archimedean) $|u| - \alpha_0 e \leq 0$ and $|u| \leq \alpha_0 e$, so that the minimum is attained. In particular, $\|u\|_e = 0$ iff $u = 0$. The subadditivity and homogeneity of $\| \|_e$ are immediate from the facts that $|u + v| \leq |u| + |v|$, $|\alpha u| = |\alpha||u|$.

354G Definitions (a) If U is an Archimedean Riesz space and e an order unit in U, the norm $\| \|_e$ as defined in 354F is the **order-unit norm** on U associated with e.

(b) An M**-space** is a Banach lattice in which the norm is an order-unit norm.

(c) If U is an M-space, its **standard order unit** is the order unit e such that $\| \|_e$ is the norm of U. (To see that e is uniquely defined, observe that it is $\sup\{u : u \in U,\, \|u\| \leq 1\}$.)

354H Examples (a) For any set X, $\ell^\infty(X)$ is an M-space with standard order unit χX. (As remarked in 243Xl, the completeness of $\ell^\infty(X)$ can be regarded as the special case of 243E in which X is given counting measure.)

(b) For any topological space X, the space $C_b(X)$ of bounded continuous real-valued functions on X is an M-space with standard order unit χX. (It is a Riesz subspace of $\ell^\infty(X)$ containing the order unit of $\ell^\infty(X)$, therefore in its own right an Archimedean Riesz space with order unit. To see that it is complete, it is enough to observe that it is closed in $\ell^\infty(X)$ because a uniform limit of continuous functions is continuous (3A3Nb).)

(c) For any measure space (X, Σ, μ), the space $L^\infty(\mu)$ is an M-space with standard order unit $\chi X^\bullet$.

354I Lemma Let U be an Archimedean Riesz space with order unit e, and V a subset of U which is dense for the order-unit norm $\| \|_e$. Then for any $u \in U$ there are sequences $\langle v_n \rangle_{n\in\mathbb{N}}$, $\langle w_n \rangle_{n\in\mathbb{N}}$ in V such that $v_n \leq v_{n+1} \leq u \leq w_{n+1} \leq w_n$ and $\|w_n - v_n\|_e \leq 2^{-n}$ for every n; so that $u = \sup_{n\in\mathbb{N}} v_n = \inf_{n\in\mathbb{N}} w_n$ in U.

If V is a Riesz subspace of U, and $u \geq 0$, we may suppose that $v_n \geq 0$ for every n. Consequently V is order-dense in U.

proof For each $n \in \mathbb{N}$, take v_n, $w_n \in V$ such that

$$\|u - \frac{3}{2^{n+3}}e - v_n\|_e \leq \frac{1}{2^{n+3}}, \quad \|u + \frac{3}{2^{n+3}}e - w_n\|_e \leq \frac{1}{2^{n+3}}.$$

Then

$$u - \frac{1}{2^{n+1}}e \leq v_n \leq u - \frac{1}{2^{n+2}}e \leq u \leq u + \frac{1}{2^{n+2}}e \leq w_n \leq u + \frac{1}{2^{n+1}}e.$$

Accordingly $\langle v_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, $\langle w_n \rangle_{n\in\mathbb{N}}$ is non-increasing and $\|w_n - v_n\|_e \leq 2^{-n}$ for every n. Because U is Archimedean, $\sup_{n\in\mathbb{N}} v_n = \inf_{n\in\mathbb{N}} w_n = u$.

If V is a Riesz subspace of U, then replacing v_n by v_n^+ if necessary we may suppose that every v_n is non-negative; and V is order-dense by the definition in 352Na.

354J Proposition Let U be an Archimedean Riesz space with an order unit e. Then $\| \|_e$ is Fatou and has the Levi property.

proof This is elementary. If $A \subseteq U^+$ is non-empty, upwards-directed and norm-bounded, then it is bounded above by αe, where $\alpha = \sup_{u\in A} \|u\|_e$. This is all that is called for in the Levi property. If moreover $\sup A$ is defined, then $\sup A \leq \alpha e$ so $\|\sup A\| \leq \alpha$, as required in the Fatou property.

354K Theorem Let U be an Archimedean Riesz space with order unit e. Then it can be embedded as an order-dense and norm-dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space, in such a way that e corresponds to χX and $\| \|_e$ corresponds to $\| \|_\infty$; moreover, this embedding is essentially unique.

proof This is nearly word-for-word a repetition of 353N. The only addition is the mention of the norms. Let X and $T : U \to C(X)$ be as in 353N. Then, for any $u \in U$, $|u| \le \|u\|_e e$, so that

$$|Tu| = T|u| \le \|u\|_e Te = \|u\|_e \chi X,$$

and $\|Tu\|_\infty \le \|u\|_e$. On the other hand, if $0 < \delta < \|u\|_e$ then $u_1 = (|u| - \delta e)^+ > 0$, so that $Tu_1 = (|Tu| - \delta\chi X)^+ > 0$ and $\|Tu\|_\infty \ge \delta$; as δ is arbitrary, $\|Tu\|_\infty \ge \|u\|_e$.

354L Corollary Any M-space U is isomorphic, as Banach lattice, to $C(X)$ for some compact Hausdorff X, and the isomorphism is essentially unique. X can be identified with the set of Riesz homomorphisms $x : U \to \mathbb{R}$ such that $x(e) = 1$, where e is the standard order unit of U, with the topology induced by the product topology on $\mathbb{R}^U$.

proof By 354K, there are a compact Hausdorff space X and an embedding of U as a norm-dense Riesz subspace of $C(X)$ matching $\| \|_e$ to $\| \|_\infty$. Since U is complete under $\| \|_e$, its image is closed in $C(X)$ (3A4Ff), and must be the whole of $C(X)$. The expression is unique just in so far as the expression of 353N/354K is unique. In particular, we may, if we wish, take X to be the set of normalized Riesz homomorphisms from U to $\mathbb{R}$, as in the proof of 353N.

Remark The set of uniferent Riesz homomorphisms from U to $\mathbb{R}$ is sometimes called the **spectrum** of U.

354M I come now to a second fundamental class of Banach lattices, in a strong sense 'dual' to the class of M-spaces, as will appear in §356.

Definition An **L-space** is a Banach lattice U such that $\|u + v\| = \|u\| + \|v\|$ whenever $u, v \in U^+$.

Example If (X, Σ, μ) is any measure space, then $L^1(\mu)$, with its norm $\| \|_1$, is an L-space (242D, 242F). In particular, taking μ to be counting measure on $\mathbb{N}$, ℓ^1 is an L-space (242Xa).

354N Theorem If U is an L-space, then its norm is order-continuous and has the Levi property.

proof (a) Both of these are consequences of the following fact: if $A \subseteq U$ is norm-bounded and non-empty and upwards-directed, then $\sup A$ is defined in U and belongs to the norm-closure of A in U. **P** Fix $u_0 \in A$; set $B = \{u - u_0 : u \in A, u \ge u_0\}$. Then $B \subseteq U^+$ is norm-bounded, non-empty and upwards-directed. Set $\gamma = \sup_{u \in B} \|u\|$. Consider the filter $\mathcal{F}(B\uparrow)$ on U generated by sets of the form $\{v : v \in B, v \ge u\}$ for $u \in B$. If $\epsilon > 0$ there is a $u \in B$ such that $\|u\| \ge \gamma - \epsilon$; now if $v, v' \in B \cap [u, \infty[$, there is a $w \in B$ such that $v \vee v' \le w$, so that

$$\|v - v'\| \le \|w - u\| = \|w\| - \|u\| \le \epsilon.$$

As ϵ is arbitrary, $\mathcal{F}(B\uparrow)$ is Cauchy and has a limit u^* say. If $u \in B$, $[u, \infty[$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains u^*; thus u^* is an upper bound for B. If w is an upper bound for B, then $]-\infty, w]$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains u^*; thus u^* is the least upper bound of B. And $B \in \mathcal{F}(B\uparrow)$, so $u^* \in \overline{B}$.

Because $u \mapsto u_0$ is an order-preserving homeomorphism,

$$u^* + u_0 = \sup\{u : u_0 \le u \in A\} = \sup A$$

and $u^* + u_0 \in \overline{A}$, as required. **Q**

(b) This shows immediately that the norm has the Levi property. But also it must be order-continuous. **P** If $A \subseteq U$ is non-empty and downwards-directed and has infimum 0, take any $u_0 \in A$ and consider $B = \{u_0 - u : u \in A, u \le u_0\}$. Then B is upwards-directed and has supremum u_0, so $u_0 \in \overline{B}$ and

$$\inf_{u \in A} \|u\| \le \inf_{v \in B} \|u_0 - v\| = 0. \text{ } \mathbf{Q}$$

354O Proposition If U is an L-space and V is a norm-closed Riesz subspace of U, then V is an L-space in its own right. In particular, any band in U is an L-space.

proof For any Riesz subspace V of U, we surely have $\|u+v\| = \|u\| + \|v\|$ whenever u, $v \in V^+$; so if V is norm-closed, therefore a Banach lattice, it must be an L-space. But in any Banach lattice, a band is norm-closed (354Bd), so a band in an L-space is again an L-space.

354P Uniform integrability in L-spaces Some of the ideas of §246 can be readily expressed in this abstract context.

Definition Let U be an L-space. A set $A \subseteq U$ is **uniformly integrable** if for every $\epsilon > 0$ there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$.

354Q Since I have already used the phrase 'uniformly integrable' based on a different formula, I had better check instantly that the two definitions are consistent.

Proposition If (X, Σ, μ) is any measure space, then a subset of $L^1 = L^1(\mu)$ is uniformly integrable in the sense of 354P iff it is uniformly integrable in the sense of 246A.

proof (a) If $A \subseteq L^1$ is uniformly integrable in the sense of 246A, then for any $\epsilon > 0$ there are $M \ge 0$, $E \in \Sigma$ such that $\mu E < \infty$ and $\int (|u| - M\chi E^\bullet)^+ \le \epsilon$ for every $u \in A$; now $w = M\chi E^\bullet$ belongs to $(L^1)^+$ and $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$. As ϵ is arbitrary, A is uniformly integrable in the sense of 354P.

(b) Now suppose that A is uniformly integrable in the sense of 354P. Let $\epsilon > 0$. Then there is a $w \in (L^1)^+$ such that $\|(|u| - w)^+\| \le \frac{1}{2}\epsilon$ for every $u \in A$. There is a simple function $h : X \to \mathbb{R}$ such that $\|w - h^\bullet\| \le \frac{1}{2}\epsilon$ (242Mb); now take $E = \{x : h(x) \ne 0\}$, $M = \sup_{x \in X} |h(x)|$ (I pass over the trivial case $X = \emptyset$), so that $h \le M\chi E$ and

$$(|u| - M\chi E^\bullet)^+ \le (|u| - w)^+ + (w - M\chi E^\bullet)^+ \le (|u| - w)^+ + (w - h^\bullet)^+,$$

$$\int (|u| - M\chi E^\bullet)^+ \le \|(|u| - w)^+\| + \|w - h^\bullet\| \le \epsilon$$

for every $u \in A$. As ϵ is arbitrary, A is uniformly integrable in the sense of 354P.

354R I give abstract versions of the easiest results from §246.

Theorem Let U be an L-space.

(a) If $A \subseteq U$ is uniformly integrable, then

(i) A is norm-bounded;

(ii) every subset of A is uniformly integrable;

(iii) for any $\alpha \in \mathbb{R}$, αA is uniformly integrable;

(iv) there is a uniformly integrable, solid, convex, norm-closed set $C \supseteq A$;

(v) for any other uniformly integrable set $B \subseteq U$, $A \cup B$ and $A + B$ are uniformly integrable.

(b) For any set $A \subseteq U$, the following are equiveridical:

(i) A is uniformly integrable;

(ii) $\lim_{n\to\infty} (|u_n| - \sup_{i<n} |u_i|)^+ = 0$ for every sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in A;

(iii) *either* A is empty *or* for every $\epsilon > 0$ there are $u_0, \dots, u_n \in A$ such that $\|(|u| - \sup_{i\le n} |u_i|)^+\| \le \epsilon$ for every $u \in A$;

(iv) A is norm-bounded and any disjoint sequence in the solid hull of A is norm-convergent to 0.

(c) If $V \subseteq U$ is a closed Riesz subspace then a subset of V is uniformly integrable when regarded as a subset of V iff it is uniformly integrable when regarded as a subset of U.

proof (a)(i) There must be a $w \in U^+$ such that $\int (|u| - w)^+ \le 1$ for every $u \in A$; now

$$|u| \le |u| - w + |w| \le (|u| - w)^+ + |w|, \quad \|u\| \le \|(|u| - w)^+\| + \|w\| \le 1 + \|w\|$$

for every $u \in A$, so A is norm-bounded.

(ii) This is immediate from the definition.

(iii) Given $\epsilon > 0$, we can find $w \in U^+$ such that $|\alpha|\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$; now $\|(|v| - |\alpha|w)^+\| \le \epsilon$ for every $v \in \alpha A$.

(iv) If A is empty, take $C = A$. Otherwise, try

$$C = \{v : v \in U,\ \|(|v| - w)^+\| \le \sup_{u\in A} \|(|u| - w)^+\| \text{ for every } w \in U^+\}.$$

Evidently $A \subseteq C$, and C satisfies the definition 354M because A does. The functionals

$$v \mapsto \|(|v| - w)^+\| : U \to \mathbb{R}$$

are all continuous for $\| \|$ (because the operators $v \mapsto |v|$, $v \mapsto v - w$, $v \mapsto v^+$, $v \mapsto \|v\|$ are continuous), so C is closed. If $|v'| \le |v|$ and $v \in C$, then

$$\|(|v'| - w)^+\| \le \|(|v| - w)^+\| \le \sup_{u\in A} \|(|u| - w)^+\|$$

for every w, and $v' \in C$. If $v = \alpha v_1 + \beta v_2$ where v_1, $v_2 \in C$, $\alpha \in [0, 1]$ and $\beta = 1-\alpha$, then $|v| \le \alpha|v_1| + \beta|v_2|$, so

$$|v| - w \le (\alpha|v_1| - \alpha w) + (\beta|v_2| - \beta w) \le (\alpha|v_1| - \alpha w)^+ + (\beta|v_2| - \beta w)^+$$

and

$$(|v| - w)^+ \le \alpha(|v_1| - w)^+ + \beta(|v_2| - w)^+$$

for every w; accordingly

$$\begin{aligned}\|(|v| - w)^+\| &\le \alpha\|(|v_1| - w)^+\| + \beta\|(|v_2| - w)^+\| \\ &\le (\alpha + \beta) \sup_{u\in A} \|(|u| - w)^+\| = \sup_{u\in A} \|(|u| - w)^+\|\end{aligned}$$

for every w, and $v \in C$.

Thus C has all the required properties.

(v) I show first that $A \cup B$ is uniformly integrable. **P** Given $\epsilon > 0$, let w_1, $w_2 \in U^+$ be such that

$$\|(|u| - w_1)^+\| \le \epsilon \text{ for every } u \in A, \quad \|(|u| - w_2)^+\| \le \epsilon \text{ for every } u \in B.$$

Set $w = w_1 \vee w_2$; then $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A \cup B$. As ϵ is arbitrary, $A \cup B$ is uniformly integrable. **Q**

Now (iv) tells us that there is a convex uniformly integrable set C including $A \cup B$, and in this case $A + B \subseteq 2C$, so $A + B$ is also uniformly integrable, using (ii) and (iii).

(b)(i)⇒(ii)&(iv) Suppose that A is uniformly integrable and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is any sequence in the solid hull of A. Set $v_n = \sup_{i\le n} |u_i|$ for $n \in \mathbb{N}$ and

$$v'_0 = v_0 = |u_0|, \quad v'_n = v_n - v_{n-1} = (|u_n| - \sup_{i<n} |u_i|)^+$$

for $n \ge 1$. Given $\epsilon > 0$, there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$, and therefore for every u in the solid hull of A. Of course $\sup_{n\in\mathbb{N}} \|v_n \wedge w\| \le \|w\|$ is finite, so there is an $n \in \mathbb{N}$ such that $\|v_i \wedge w\| \le \epsilon + \|v_n \wedge w\|$ for every $i \in \mathbb{N}$. But now, for $m > n$,

$$\begin{aligned}v'_m \le (|u_m| - v_n)^+ &\le (|u_m| - |u_m| \wedge w)^+ + ((|u_m| \wedge w) - v_n)^+ \\ &\le (|u_m| - w)^+ + (v_m \wedge w) - (v_n \wedge w),\end{aligned}$$

so that

$$\begin{aligned}\|v'_m\| &\le \|(|u_m| - w)^+\| + \|(v_m \wedge w) - (v_n \wedge w)\| \\ &= \|(|u_m| - w)^+\| + \|v_m \wedge w\| - \|v_n \wedge w\| \le 2\epsilon,\end{aligned}$$

using the L-space property of the norm for the equality in the middle. As ϵ is arbitrary, $\lim_{n\to\infty} v'_n = 0$. As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, condition (ii) is satisfied; but so is condition (iv), because we know from (a-i) that A is norm-bounded, and if $\langle u_n\rangle_{n\in\mathbb{N}}$ is disjoint then $v'_n = |u_n|$ for every n, so that in this case $\lim_{n\to\infty} u_n = 0$.

(ii)⇒(iii)⇒(i) are elementary.

not-(i)⇒not-(iv) Now suppose that A is not uniformly integrable. If it is not norm-bounded, we can stop. Otherwise, there is some $\epsilon > 0$ such that $\sup_{u\in A}\|(|u|-w)^+\| > \epsilon$ for every $w \in U^+$. Consequently we shall be able to choose inductively a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A such that $\|(|u_n| - 2^n\sup_{i<n}|u_i|)^+\| > \epsilon$ for every $n \ge 1$. Because A is norm-bounded, $\sum_{i=0}^\infty 2^{-i}\|u_i\|$ is finite, and we can set

$$v_n = (|u_n| - 2^n\sup_{i<n}|u_i| - \sum_{i=n+1}^\infty 2^{-i}|u_i|)^+$$

for each n. (The sum $\sum_{i=n+1}^\infty 2^{-i}|u_i|$ is defined because $\langle\sum_{i=n+1}^m 2^{-i}|u_i|\rangle_{m\ge n+1}$ is a Cauchy sequence.) We have $v_m \le |u_m|$,

$$\begin{aligned} v_m \wedge v_n &\le (|u_m| - 2^{-n}|u_n|)^+ \wedge (|u_n| - 2^n|u_m|)^+ \\ &\le (2^n|u_m| - |u_n|)^+ \wedge (|u_n| - 2^n|u_m|)^+ = 0 \end{aligned}$$

whenever $m < n$, so $\langle v_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in the solid hull of A; while

$$\|v_n\| \ge \|(|u_n| - 2^n\sup_{i<n}|u_i|)^+\| - \sum_{i=n+1}^\infty 2^{-i}\|u_i\| \ge \epsilon - 2^{-n}\sup_{u\in A}\|u\| \to \epsilon$$

as $n\to\infty$, so condition (iv) is not satisfied.

(c) Now this follows at once, because conditions (b-ii) and (b-iv) are satisfied in V iff they are satsified in U.

354X Basic exercises >(a) Work through the proofs that the following are all Banach lattices. (i) $\mathbb{R}^r$ with (α) $\|x\|_1 = \sum_{i=1}^r|\xi_i|$ (β) $\|x\|_2 = \sqrt{\sum_{i=1}^r|\xi_i|^2}$ (γ) $\|x\|_\infty = \max_{i\le r}|\xi_i|$, where $x = (\xi_1,\dots,\xi_r)$. (ii) $\ell^p(X)$, for any set X and any $p\in[1,\infty]$ (242Xa, 243Xl, 244Xn). (iii) $L^p(\mu)$, for any measure space (X,Σ,μ) and any $p\in[1,\infty]$ (242F, 243E, 244G). (iv) $\boldsymbol{c}_0$, the space of sequences convergent to 0, with the norm $\|\;\|_\infty$ inherited from ℓ^∞.

(b) Let $\langle U_i\rangle_{i\in I}$ be any family of Banach lattices. Write U for their Riesz space product (352K), and in U set

$$\|u\|_1 = \sum_{i\in I}\|u(i)\|, \quad V_1 = \{u : \|u\|_1 < \infty\},$$

$$\|u\|_\infty = \sup_{i\in I}\|u(i)\| \text{ (counting } \sup\emptyset \text{ as 0)}, \quad V_\infty = \{u : \|u\|_\infty < \infty\}.$$

Show that V_1, V_∞ are solid linear subspaces of U and are Banach lattices under their norms $\|\;\|_1$, $\|\;\|_\infty$.

>(c) Let U be a Riesz space with a Riesz norm. Show that the maps $\wedge : U^2 \to U$, $\vee : U^2 \to U$ and $\mathrm{med} : U^3 \to U$ are all uniformly continuous.

>(d) Let U be a Riesz space with a Riesz norm. (i) Show that any order-bounded set in U is norm-bounded. (ii) Show that in $\mathbb{R}^r$, with any of the standard Riesz norms (354Xa(i)), norm-bounded sets are order-bounded. (iii) Show that in $\ell^1(\mathbb{N})$ there is a sequence converging to 0 (for the norm) which is not order-bounded. (iv) Show that in $\boldsymbol{c}_0$ any sequence converging to 0 is order-bounded, but there is a norm-bounded set which is not order-bounded.

(e) Let U be a Riesz space with a Riesz norm. Show that it is a Banach lattice iff non-decreasing Cauchy sequences are convergent. (*Hint*: if $\|u_{n+1} - u_n\| \le 2^{-n}$ for every n, show that $\langle\sup_{i\le n}u_i\rangle_{n\in\mathbb{N}}$ is Cauchy, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ converges to $\inf_{n\in\mathbb{N}}\sup_{m\ge n}u_m$.)

(f) Let U be a Riesz space with a Riesz norm. Show that U is a Banach lattice iff every non-decreasing Cauchy sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U^+ has a least upper bound u with $\|u\| = \lim_{n\to\infty}\|u_n\|$.

(g) Let U be a Banach lattice. Suppose that $B \subseteq U$ is solid and $\sup_{n\in\mathbb{N}}u_n \in B$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in B with a supremum in U. Show that B is closed. (*Hint*: show first that $u\in B$ whenever there is a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $B\cap U^+$ such that $\|u-u_n\| \le 2^{-n}$ for every n; do this by considering $v_m = \inf_{n\ge m}u_n$.)

(h) Let U be any Riesz space with a Riesz norm. Show that the Banach space completion of U (3A5Jb) has a unique partial ordering under which it is a Banach lattice.

>(i) Show that $\boldsymbol{c}_0$ is a Banach lattice with an order-continuous norm which does not have the Levi property.

>(j) Show that ℓ^∞, with $\| \|_\infty$, is a Banach lattice with a Fatou norm which has the Levi property but is not order-continuous.

(k) Let U be a Riesz space with a Fatou norm. Show that if $V \subseteq U$ is a regularly embedded Riesz subspace then the induced norm on V is a Fatou norm.

(l) Let U be a Riesz space and $\| \|$ a Riesz norm on U which is order-continuous in the sense of 354Dc. Show that its restriction to U^+ is order-continuous in the sense of 313H.

(m) Let U be a Riesz space with an order-continuous norm. Show that if $V \subseteq U$ is a regularly embedded Riesz subspace then the induced norm on V is order-continuous.

(n) Let U be a Dedekind σ-complete Riesz space with a Fatou norm which has the Levi property. Show that it is a Banach lattice. (*Hint*: 354Xf.)

(o) Let $\langle U_i\rangle_{i\in I}$ be any family of Banach lattices and let V_1, V_∞ be the subspaces of $U = \prod_{i\in I} U_i$ as described in 354Xb. (i) Show that V_1, V_∞ have norms which are Fatou, or have the Levi property, iff every U_i has. (ii) Show that the norm of V_1 is order-continuous iff the norm of every U_i is. (iii) Show that V_∞ is an M-space iff every U_i is. (iv) Show that V_1 is an L-space iff every U_i is.

(p) Let U be a Banach lattice with an order-continuous norm. (i) Show that a sublattice of U is norm-closed iff it is order-closed in the sense of 313Da. (ii) Show that a norm-closed Riesz subspace of U is itself a Banach lattice with an order-continuous norm.

>(q) Let U be an M-space and V a norm-closed Riesz subspace of U containing the standard order unit of U. (i) Show that V, with the induced norm, is an M-space. (ii) Deduce that the space $\boldsymbol{c}$ of convergent sequences is an M-space if given the norm $\| \|_\infty$ inherited from ℓ^∞.

(r) Show that a Banach lattice U is an M-space iff (α) its norm is a Fatou norm with the Levi property (β) $\|u \vee v\| = \max(\|u\|, \|v\|)$ for all u, $v \in U^+$.

>(s) Describe a topological space X such that the space $\boldsymbol{c}$ of convergent sequences (354Xq) can be identified with $C(X)$.

(t) Let $D \subseteq \mathbb{R}$ be any non-empty set, and V the space of functions $f : D \to \mathbb{R}$ of bounded variation (§224). For $f \in V$ set $\|f\| = \sup\{|f(t_0)| + \sum_{i=1}^n |f(t_i) - f(t_{i-1})| : t_0 \le t_1 \le \ldots \le t_n \text{ in } D\}$ (224Yb). Let C be the set of bounded non-decreasing functions from D to $[0, \infty[$. Show that C is the positive cone of V for a Riesz space ordering under which V is an L-space.

354Y Further exercises (a) Let U be a Riesz space with a Riesz norm, and V a norm-dense Riesz subspace of U. Suppose that the induced norm on V is Fatou, when regarded as a norm on the Riesz space V. Show (i) that V is order-dense in U (ii) that the norm of U is Fatou. (*Hint*: for (i), show that if $u \in U^+$, $v_n \in V^+$ and $\|u - v_n\| \le 2^{-n-2}\|u\|$ for every n, then $\|v_0 - \inf_{i\le n} v_i\| \le \frac{1}{2}\|u\|$ for every n, so that 0 cannot be $\inf_{n\in\mathbb{N}} v_n$ in V.)

(b) Let U be a Riesz space with a Riesz norm. Show that the following are equiveridical: (i) $\lim_{n\to\infty} u_n = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in U^+ (ii) $\lim_{n\to\infty} u_{n+1} - u_n = 0$ for every order-bounded non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U (iii) whenever $A \subseteq U^+$ is a non-empty downwards-directed set in U^+ with infimum 0, $\inf_{u\in A} \sup_{v\in A, v\le u} \|u - v\| = 0$. (*Hint*: for (i)⇒(ii), show by induction that $\lim_{n\to\infty} u_n = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is an order-bounded sequence such that, for some fixed $k \ge 1$, $\inf_{i\in K} u_i = 0$ for every $K \subseteq \mathbb{N}$ with k members; now show that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and $0 \le u_n \le u$ for every n, then $\inf_{i\in K}(u_{i+1} - u_i - \frac{1}{k}u)^+ = 0$ whenever $K \subseteq \mathbb{N}$ and $\#(K) = k \ge 1$. For (iii)⇒(i), set $A = \{u : \exists\, n,\ u \ge u_i\ \forall\ i \ge n\}$. See Fremlin 74a, 24H.)

(c) Show that any Riesz space with an order-continuous norm has the countable sup property (definition: 241Ye).

(d) Let U be a Banach lattice. Show that the following are equiveridical: (i) the norm on U is order-continuous; (ii) U satisfies the conditions of 354Yb; (iii) every order-bounded monotonic sequence in U is Cauchy.

(e) Let U be a Riesz space with a Fatou norm. Show that the norm on U is order-continuous iff it satisfies the conditions of 354Yb.

(f) For $f \in C([0,1])$, set $\|f\|_1 = \int |f(x)|dx$. Show that $\| \|_1$ is a Riesz norm on $C([0,1])$ satisfying the conditions of 354Yb, but is not order-continuous.

(g) Let U be a Riesz space with a Riesz norm $\| \|$. Show that $(U, \| \|)$ satisfies the conditions of 354Yb iff the norm of its completion is order-continuous.

(h) Let U be a Riesz space with a Riesz norm, and $V \subseteq U$ a norm-dense Riesz subspace such that the induced norm on V is order-continuous. Show that the norm of U is order-continuous. (*Hint*: use 354Ya.)

(i) Let U be an Archimedean Riesz space. For any $e \in U^+$, let U_e be the solid linear subspace of U generated by e, so that e is an order unit in U_e, and let $\| \|_e$ be the corresponding order-unit norm on U_e. We say that U is **uniformly complete** if U_e is complete under $\| \|_e$ for every $e \in U^+$. (i) Show that any Banach lattice is uniformly complete. (ii) Show that any Dedekind σ-complete Riesz space is uniformly complete (cf. 354Xn). (iii) Show that if U is a uniformly complete Riesz space with a Riesz norm which has the Levi property, then U is a Banach lattice. (iv) Show that if U is a Banach lattice then a set $A \subseteq U$ is closed, for the norm topology, iff $A \cap U_e$ is $\| \|_e$-closed for every $e \in U^+$. (v) Let V be a solid linear subspace of U. Show that the quotient Riesz space U/V is Archimedean iff $V \cap U_e$ is $\| \|_e$-closed for every $e \in U^+$. (vi) Show that if U is uniformly complete and $V \subseteq U$ is a solid linear subspace such that U/V is Archimedean, then U/V is uniformly complete. (vii) Show that U is Dedekind σ-complete iff it is uniformly complete and has the principal projection property (353Xb).

(j) Let U be an Archimedean Riesz space with an order unit, endowed with its order-unit norm. Let Z be the unit ball of U^*. Show that for a linear functional $f : U \to \mathbb{R}$ the following are equiveridical: (i) f is an **extreme point** of Z, that is, $f \in Z$ and $Z \setminus \{f\}$ is convex (ii) $|f(e)| = 1$ and one of f, $-f$ is a Riesz homomorphism.

(k) Let U be a Banach lattice such that $\|u + v\| = \|u\| + \|v\|$ whenever $u \wedge v = 0$. Show that U is an L-space. (*Hint*: by 354Yd, the norm is order-continuous, so U is Dedekind complete. If u, $v \geq 0$, set $e = u + v$, and represent U_e as $C(X)$ where X is extremally disconnected (353Yb); now approximate u and v by functions taking only finitely many values to show that $\|u + v\| = \|u\| + \|v\|$.)

(l) Let U be a uniformly complete Archimedean Riesz space (354Yi). Set $U_{\mathbb{C}} = U \times U$ with the complex linear structure defined by identifying $(u, v) \in U \times U$ as $u + iv \in U_{\mathbb{C}}$, so that $u = \mathcal{R}\mathrm{e}(u + iv)$, $v = \mathcal{I}\mathrm{m}(u + iv)$ and $(\alpha + i\beta)(u + iv) = (\alpha u - \beta v) + i(\alpha v + \beta u)$. (i) Show that for $w \in U_{\mathbb{C}}$ we can define $|w| \in U$ by setting $|w| = \sup_{|\zeta|=1} \mathcal{R}\mathrm{e}(\zeta w)$. (ii) Show that if U is a uniformly complete Riesz subspace of $\mathbb{R}^X$ for some set X, then we can identify $U_{\mathbb{C}}$ with the linear subspace of $\mathbb{C}^X$ generated by U. (iii) Show that $|w + w'| \leq |w| + |w'|$, $|\gamma w| = |\gamma||w|$ for all $w \in U_{\mathbb{C}}$, $\gamma \in \mathbb{C}$. (iv) Show that if $w \in U_{\mathbb{C}}$ and $|w| \leq u_1 + u_2$, where u_1, $u_2 \in U^+$, then w is expressible as $w_1 + w_2$ where $|w_j| \leq u_j$ for both j. (*Hint*: set $e = u_1 + u_2$ and represent U_e as $C(X)$.) (v) Show that if U_0 is a solid linear subspace of U, then, for $w \in U_{\mathbb{C}}$, $|w| \in U_0$ iff $\mathcal{R}\mathrm{e}\, w$, $\mathcal{I}\mathrm{m}\, w$ both belong to U_0. (vi) Show that if U has a Riesz norm then we have a norm on $U_{\mathbb{C}}$ defined by setting $\|w\| = \||w|\|$, and that if U is a Banach lattice then $U_{\mathbb{C}}$ is a (complex) Banach space. (vii) Show that if $U = L^p(\mu)$, where (X, Σ, μ) is a measure space and $p \in [1, \infty]$, then $U_{\mathbb{C}}$ can be identified with $L^p_{\mathbb{C}}(\mu)$ as defined in 242P, 243K and 244P. (We may call $U_{\mathbb{C}}$ the **complexification** of the Riesz space U.)

(m) Let (X, Σ, μ) be a measure space and V a Banach lattice. Write $\mathcal{L}^1_V$ for the space of Bochner integrable functions from conegligible subsets of X to V, and L^1_V for the corresponding set of equivalence classes (253Yf). (i) Show that L^1_V is a Banach lattice under the ordering defined by saying that $f^\bullet \le g^\bullet$ iff $f(x) \le g(x)$ in V for μ-almost every $x \in X$. (ii) Show that when $V = L^1(\nu)$, for some other measure space (Y, T, ν), then this ordering of L^1_V agrees with the ordering of $L^1(\lambda)$ where λ is the (c.l.d.) product measure on $X \times Y$ and we identify L^1_V with $L^1(\lambda)$, as in 253Yi. (iii) Show that if V has an order-continuous norm, so has L^1_V. (*Hint*: 354Yd.) (iv) Show that if μ is Lebesgue measure on $[0,1]$ and $V = \ell^\infty$, then L^1_V is not Dedekind σ-complete.

354 Notes and comments Apart from some of the exercises, the material of this section is pretty strictly confined to ideas which will be useful later in this volume. The basic Banach lattices of measure theory are the L^p spaces of Chapter 24; these all have Fatou norms with the Levi property (244Yf-244Yg), and for $p < \infty$ their norms are order-continuous (244Ye). In Chapter 36 I will return to these spaces in a more abstract context. Here I am mostly concerned to establish a vocabulary in which their various properties, and the relationships between these properties, can be expressed.

In normed Riesz spaces we have a very rich mixture of structures, and must take particular care over the concepts of 'boundedness', 'convergence' and 'density', which have more than one possible interpretation. In particular, we must scrupulously distinguish between 'order-bounded' and 'norm-bounded' sets. I have not yet formally introduced any of the various concepts of order-convergence (see §367), but I think that even so it is best to get into the habit of reminding oneself, when a convergent sequence appears, that it is convergent for the norm topology, rather than in any sense related directly to the order structure.

I should perhaps warn you that for the study of M-spaces 354L is not as helpful as it may look. The trouble is that apart from a few special cases (as in 354Xs) the topological space used in the representation is actually more complicated and mysterious than the M-space it is representing.

After the introduction of M-spaces, this section becomes a natural place for 'uniformly complete' spaces (354Yi). For the moment I leave these in the exercises. But I mention them now because they offer a straightforward route towards a theory of 'complex Riesz spaces' (354Yl). In large parts of functional analysis it is natural, and in some parts it is necessary, to work with normed spaces over $\mathbb{C}$ rather than over $\mathbb{R}$, and for L^2 spaces in particular it is useful to have a proper grasp of the complex case. And while the insights offered by the theory of Riesz spaces are not especially important in such areas, I think we should always seek connexions between different topics. So it is worth remembering that uniformly complete Riesz spaces have complexifications.

I shall have a great deal more to say about L-spaces when I come to spaces of additive functionals (§362) and to L^1 spaces again (§365) and to linear operators on them (§371); and before that, there will be something in the next section on their duals, and on L-spaces which are themselves dual spaces. For the moment I just give some easy results, direct translations of the corresponding facts in §246, which have natural expressions in the language of this section, holding deeper ideas over. In particular, the characterization of uniformly integrable sets as relatively weakly compact sets (247C) is valid in general L-spaces (356Q).

For an extensive treatment of Banach lattices, going very much deeper than I have space for in this volume, see Lindenstrauss & Tzafriri 79. For a careful exposition of a great deal of useful information, see Schaefer 74.

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355 Spaces of linear operators

We come now to a discussion of linear operators between Riesz spaces. Linear operators are central to any kind of functional analysis, and a feature of the theory of Riesz spaces is the way the order structure picks out certain classes of operators for special consideration. Here I concentrate on positive and order-continuous operators, with a brief mention of sequential order-continuity. It turns out, in fact, that we need to work with operators which are differences of positive operators or of order-continuous positive operators. I define the basic spaces $\mathrm{L}^\sim$, $\mathrm{L}^\times$ and $\mathrm{L}^\sim_c$ (355A, 355G), with their most important properties (355B, 355E, 355H-355I) and some remarks on the special case of Banach lattices (355C, 355K). At the same time I give an important theorem on extension of operators (355F) and a corollary (355J).

The most important case is of course that in which the codomain is $\mathbb{R}$, so that our operators become real-valued functionals; I shall come to these in the next section.

355A Definition Let U and V be Riesz spaces. A linear operator $T : U \to V$ is **order-bounded** if $T[A]$ is order-bounded in V for every order-bounded $A \subseteq U$.

I will write $\mathsf{L}^{\sim}(U;V)$ for the set of order-bounded linear operators from U to V.

355B Lemma If U and V are Riesz spaces,

(a) a linear operator $T : U \to V$ is order-bounded iff $\{Tu : 0 \le u \le w\}$ is bounded above in V for every $w \in U^+$;

(b) in particular, any positive linear operator from U to V belongs to $\mathsf{L}^{\sim} = \mathsf{L}^{\sim}(U;V)$;

(c) $\mathsf{L}^{\sim}$ is a linear space;

(d) if W is another Riesz space and $T : U \to V$ and $S : V \to W$ are order-bounded linear operators, then $ST : U \to W$ is order-bounded.

proof (a) This is elementary. If $T \in \mathsf{L}^{\sim}$ and $w \in U^+$, $[0,w]$ is order-bounded, so its image must be order-bounded in V, and in particular bounded above. On the other hand, if T satisfies the condition, and A is order-bounded, then $A \subseteq [u_1, u_2]$ for some $u_1 \le u_2$, and

$$T[A] \subseteq T[u_1 + [0, u_2 - u_1]] = Tu_1 + T[[0, u_2 - u_1]]$$

is bounded above; similarly, $T[-A]$ is bounded above, so $T[A]$ is bounded below; as A is arbitrary, T is order-bounded.

(b) If T is positive then $\{Tu : 0 \le u \le w\}$ is bounded above by Tw for every $w \ge 0$, so $T \in \mathsf{L}^{\sim}$.

(c) If $T_1, T_2 \in \mathsf{L}^{\sim}$, $\alpha \in \mathbb{R}$ and $A \subseteq U$ is order-bounded, then there are v_1, $v_2 \in V$ such that $T_i[A] \subseteq [-v_i, v_i]$ for both i. Setting $v = (1 + |\alpha|)v_1 + v_2$, $(\alpha T_1 + T_2)[A] \subseteq [-v, v]$; as A is arbitrary, $\alpha T_1 + T_2$ belongs to $\mathsf{L}^{\sim}$; as α, T_1, T_2 are arbitrary, and since the zero operator surely belongs to $\mathsf{L}^{\sim}$, $\mathsf{L}^{\sim}$ is a linear subspace of the space of all linear operators from U to V.

(d) This is immediate from the definition; if $A \subseteq U$ is order-bounded, then $T[A] \subseteq V$ and $(ST)[A] = S[T[A]] \subseteq W$ are order-bounded.

355C Theorem If U and V are Banach lattices then every order-bounded linear operator (in particular, every positive linear operator) from U to V is continuous.

proof ? Suppose, if possible, that $T : U \to V$ is an order-bounded linear operator which is not continuous. Then for each $n \in \mathbb{N}$ we can find a $u_n \in U$ such that $\|u_n\| \le 2^{-n}$ but $\|Tu_n\| \ge n$. Now $u = \sup_{n \in \mathbb{N}} |u_n|$ is defined in U (354C), and there is a $v \in V$ such that $-v \le Tw \le v$ whenever $-u \le w \le u$; but this means that $\|v\| \ge \|Tu_n\| \ge n$ for every n, which is impossible. **X**

355D Lemma Let U be a Riesz space and V any linear space over $\mathbb{R}$. Then a function $T : U^+ \to V$ extends to a linear operator from U to V iff

$$T(u + u') = Tu + Tu', \quad T(\alpha u) = \alpha Tu$$

for all u, $u' \in U^+$ and every $\alpha > 0$, and in this case the extension is unique.

proof For in this case we can, and must, set

$$T_1 u = Tu_1 - Tu_2 \text{ whenever } u_1,\, u_2 \in U^+ \text{ and } u = u_1 - u_2;$$

it is elementary to check that this defines $T_1 u$ uniquely for every $u \in U$, and that T_1 is a linear operator extending T.

355E Theorem Let U be a Riesz space and V a Dedekind complete Riesz space.

(a) The space $\mathsf{L}^{\sim}$ of order-bounded linear operators from U to V is a Dedekind complete Riesz space; its positive cone is the set of positive linear operators from U to V. In particular, every order-bounded linear operator from U to V is expressible as the difference of positive linear operators.

(b) For $T \in \mathsf{L}^\sim$, T^+ and $|T|$ are defined in the Riesz space $\mathsf{L}^\sim$ by the formulae

$$T^+(w) = \sup\{Tu : 0 \le u \le w\},$$

$$|T|(w) = \sup\{Tu : |u| \le w\} = \sup\{\sum_{i=0}^n |Tu_i| : \sum_{i=0}^n |u_i| = w\}$$

for every $w \in U^+$.

(c) If S, $T \in \mathsf{L}^\sim$ then

$$(S \vee T)(w) = \sup_{0 \le u \le w} Su + T(w-u), \quad (S \wedge T)(w) = \inf_{0 \le u \le w} Su + T(w-u)$$

for every $w \in U^+$.

(d) Suppose that $A \subseteq \mathsf{L}^\sim$ is non-empty and upwards-directed. Then A is bounded above in $\mathsf{L}^\sim$ iff $\{Tu : T \in A\}$ is bounded above in V for every $u \in U^+$, and in this case $(\sup A)(u) = \sup_{T \in A} Tu$ for every $u \ge 0$.

(e) Suppose that $A \subseteq (\mathsf{L}^\sim)^+$ is non-empty and downwards-directed. Then $\inf A = 0$ in $\mathsf{L}^\sim$ iff $\inf_{T \in A} Tu = 0$ in V for every $u \in U^+$.

proof (a)(i) Suppose that $T \in \mathsf{L}^\sim$. For $w \in U^+$ set $R_T(w) = \sup\{Tu : 0 \le u \le w\}$; this is defined because V is Dedekind complete and $\{Tu : 0 \le u \le w\}$ is bounded above in V. Then $R_T(w_1 + w_2) = R_T w_1 + R_T w_2$ for all w_1, $w_2 \in U^+$. **P** Setting $A_i = [0, w_i]$ for each i, $w = w_1 + w_2$ and $A = [0, w]$, then of course $A_1 + A_2 \subseteq A$; but also $A \subseteq A_1 + A_2$, because if $u \in A$ then $u = (u \wedge w_1) + (u - w_1)^+$, and $0 \le (u - w_1)^+ \le (w - w_1)^+ = w_2$, so $u \in A_1 + A_2$. Consequently

$$\begin{aligned} R_T w &= \sup T[A] = \sup T[A_1 + A_2] = \sup(T[A_1] + T[A_2]) \\ &= \sup T[A_1] + \sup T[A_2] = R_T w_1 + R_T w_2 \end{aligned}$$

by 351Dc. **Q** Next, it is easy to see that $R_T(\alpha w) = \alpha R_T w$ for $w \in U^+$ and $\alpha > 0$, just because $u \mapsto \alpha u$, $v \mapsto \alpha v$ are isomorphisms of the partially ordered linear spaces U and V. It follows from 355D that we can extend R_T to a linear operator from U to V.

Because $R_T u \ge T0 = 0$ for every $u \in U^+$, R_T is a positive linear operator. But also $R_T u \ge Tu$ for every $u \in U^+$, so $R_T - T$ is also positive, and $T = R_T - (R_T - T)$ is the difference of two positive linear operators.

(ii) This shows that every order-bounded operator is a difference of positive operators. But of course if T_1 and T_2 are positive, then $(T_1 - T_2)u \le T_1 w$ whenever $0 \le u \le w$ in U, so that $T_1 - T_2$ is order-bounded, by the criterion in 355Ba. Thus $\mathsf{L}^\sim$ is precisely the set of differences of positive operators.

(iii) Just as in 351F, $\mathsf{L}^\sim$ is a partially ordered linear space if we say that $S \le T$ iff $Su \le Tu$ for every $u \in U^+$. Now it is a Riesz space. **P** Take any $T \in \mathsf{L}^\sim$. Then R_T, as defined in (i), is an upper bound for $\{0, T\}$ in $\mathsf{L}^\sim$. If $S \in \mathsf{L}^\sim$ is any other upper bound for $\{0, T\}$, then for any $w \in U^+$ we must have $Sw \ge Su \ge Tu$ whenever $u \in [0, w]$, so that $Sw \ge R_T w$; as w is arbitrary, $S \ge R_T$; as S is arbitrary, $R_T = \sup\{0, T\}$ in $\mathsf{L}^\sim$. Thus $\sup\{0, T\}$ is defined in $\mathsf{L}^\sim$ for every $T \in \mathsf{L}^\sim$; by 352B, $\mathsf{L}^\sim$ is a Riesz space. **Q**

(I defer the proof that it is Dedekind complete to (d-ii) below.)

(b) As remarked in (a-iii), $R_T = T^+$ for each $T \in \mathsf{L}^\sim$; but this is just the formula given for T^+. Now, if $T \in \mathsf{L}^\sim$ and $w \in U^+$,

$$\begin{aligned} |T|(w) &= 2T^+ w - Tw = 2 \sup_{u \in [0,w]} Tu - Tw \\ &= \sup_{u \in [0,w]} T(2u - w) = \sup_{u \in [-w,w]} Tu, \end{aligned}$$

which is the first formula offered for $|T|$. In particular, if $|u| \le w$ then Tu, $-Tu = T(-u)$ are both less than or equal to $|T|(w)$, so that $|Tu| \le |T|(w)$. So if $u_0, \dots, u_n$ are such that $\sum_{i=0}^n |u_i| = w$, then

$$\sum_{i=0}^n |Tu_i| \le \sum_{i=0}^n |T|(|u_i|) = |T|(w).$$

Thus $B = \{\sum_{i=0}^n |Tu_i| : \sum_{i=0}^n |u_i| = w\}$ is bounded above by $|T|(w)$. On the other hand, if v is an upper bound for B and $|u| \le w$, then

$$Tu \le |Tu| + |T(w - |u|)| \le v;$$

as u is arbitrary, $|T|(w) \le v$; thus $|T|(w)$ is the least upper bound for B. This completes the proof of part (b) of the theorem.

(c) We know that $S \vee T = T + (S-T)^+$ (352D), so that

$$(S \vee T)(w) = Tw + (S-T)^+(w) = Tw + \sup_{0 \le u \le w} (S-T)(u)$$
$$= \sup_{0 \le u \le w} Tw + (S-T)(u) = \sup_{0 \le u \le w} Su + T(w-u)$$

for every $w \in U^+$, by the formula in (b). Also from 352D we have $S \wedge T = S + T - T \vee S$, so that

$$(S \wedge T)(w) = Sw + Tw - \sup_{0 \le u \le w} Tu + S(w-u)$$
$$= \inf_{0 \le u \le w} Sw + Tw - Tu - S(w-u)$$

(351Db)

$$= \inf_{0 \le u \le w} Su + T(w-u)$$

for $w \in U^+$.

(d)(i) Now suppose that $A \subseteq \mathsf{L}^\sim$ is non-empty and upwards-directed and that $\{Tu : T \in A\}$ is bounded above in V for every $u \in U^+$. In this case, because V is Dedekind complete, we may set $Ru = \sup_{T \in A} Tu$ for every $u \in U^+$. Now $R(u_1 + u_2) = Ru_1 + Ru_2$ for all u_1, $u_2 \in U^+$. **P** Set $B_i = \{Tu_i : T \in A\}$ for each i, $B = \{T(u_1 + u_2) : T \in A\}$. Then $B \subseteq B_1 + B_2$, so

$$R(u_1 + u_2) = \sup B \le \sup(B_1 + B_2) = \sup B_1 + \sup B_2 = Ru_1 + Ru_2.$$

On the other hand, if $v_i \in B_i$ for both i, there are $T_i \in A$ such that $v_i = T_i u_i$ for each i; because A is upwards-directed, there is a $T \in A$ such that $T \ge T_i$ for both i, and now

$$R(u_1 + u_2) \ge T(u_1 + u_2) = Tu_1 + Tu_2 \ge T_1 u_1 + T_2 u_2 = v_1 + v_2.$$

As v_1, v_2 are arbitrary,

$$R(u_1 + u_2) \ge \sup(B_1 + B_2) = \sup B_1 + \sup B_2 = Ru_1 + Ru_2. \text{ **Q**}$$

It is also easy to see that $R(\alpha u) = \alpha Ru$ for every $u \in U^+$ and $\alpha > 0$. So, using 355D again, R has an extension to a linear operator from U to V.

If we fix any $T_0 \in A$, we have $T_0 u \le Ru$ for every $u \in U^+$, so $R - T_0$ is a positive linear operator, and $R = (R - T_0) + T_0$ belongs to $\mathsf{L}^\sim$. Again, $Tu \le Ru$ for every $T \in A$ and $u \in U^+$, so R is an upper bound for A in $\mathsf{L}^\sim$; and, finally, if S is any upper bound for A in $\mathsf{L}^\sim$, then Su is an upper bound for $\{Tu : T \in A\}$, and must be greater than or equal to Ru, for every $u \in U^+$; so that $R \le S$ and $R = \sup A$ in $\mathsf{L}^\sim$.

(ii) Consequently $\mathsf{L}^\sim$ is Dedekind complete. **P** If $A \subseteq \mathsf{L}^\sim$ is non-empty and bounded above by S say, then $A' = \{T_0 \vee T_1 \vee \ldots \vee T_n : T_0, \ldots, T_n \in A\}$ is upwards-directed and bounded above by S, so $\{Tu : T \in A'\}$ is bounded above by Su for every $u \in U^+$; by (i) just above, A' has a supremum in $\mathsf{L}^\sim$, which will also be the supremum of A. **Q**

(e) Suppose that $A \subseteq (\mathsf{L}^\sim)^+$ is non-empty and downwards-directed. Then $-A = \{-T : T \in A\}$ is non-empty and upwards-directed, so

$$\inf A = 0 \iff \sup(-A) = 0$$
$$\iff \sup_{T \in A}(-Tu) = 0 \text{ for every } u \in U^+$$
$$\iff \inf_{T \in A} Tu = 0 \text{ for every } u \in U^+.$$

355F Theorem Let U and V be Riesz spaces, $U_0 \subseteq U$ a Riesz subspace and $T_0 : U_0 \to V$ a positive linear operator such that $Su = \sup\{T_0 w : w \in U_0,\ 0 \le w \le u\}$ is defined in V for every $u \in U^+$. Suppose *either* that U_0 is order-dense and that T_0 is order-continuous *or* that U_0 is solid.

(a) There is a unique positive linear operator $T : U \to V$, extending T_0, which agrees with S on U^+.
(b) If T_0 is a Riesz homomorphism so is T.
(c) If T_0 is order-continuous so is T.
(d) If U_0 is order-dense and T_0 is an injective Riesz homomorphism, then T is injective.
(e) If U_0 is order-dense and T_0 is order-continuous then T is the only order-continuous positive linear operator from U to V extending T_0.

proof (a)(i) (The key.) If u, $u' \in U^+$ then $S(u+u') = Su + Su'$. **P** If w, $w' \in U_0^+$, $w \le u$ and $w' \le u'$, then $w + w' \le u + u'$, so

$$T_0 w + T_0 w' = T_0(w + w') \le S(u + u');$$

as w and w' are arbitrary, $Su + Su' \le S(u + u')$ (351Dc). In the other direction, suppose that $w \in U_0^+$ and $w \le u + u'$.

case 1 Suppose that U_0 is solid. Then $w \wedge u$ and $(w - u)^+$ belong to U_0, while $w \wedge u \le u$ and $(w - u)^+ \le (u + u' - u)^+ = u'$; so

$$T_0 w = T_0(w \wedge u + (w - u)^+) = T_0(w \wedge u) + T_0(w - u)^+ \le Su + Su';$$

as w is arbitrary, $S(u + u') \le Su + Su'$ and we must have equality.

case 2 Suppose that U_0 is order-dense and T_0 is order-continuous. Set $A = \{v : v \in U_0^+,\, v \le w \wedge u\}$ and $B = \{v : v \in U_0^+,\, v \le (w-u)^+\}$. Then (taking the suprema in U) $w \wedge u = \sup A$ and $(w-u)^+ = \sup B$, because U_0 is order-dense; by 351Dc again, $w = \sup(A+B)$ in U and therefore $w = \sup(A+B)$ in U_0. Also both A and B are upwards-directed, so $A + B$ also is. Because T_0 is order-continuous,

$$T_0 w = \sup T_0[A + B] = \sup(T_0[A] + T_0[B]) \le Su + Su'.$$

So once again we must have $S(u + u') \le Su + Su'$ and therefore $S(u + u') = Su + Su'$. **Q**

(ii) Of course $S(\alpha u) = \alpha Su$ whenever $u \in U^+$ and $\alpha \ge 0$. By 355D, S has a unique extension to a linear operator $T : U \to V$. As $Tu = Su \ge 0$ whenever $u \ge 0$, T is positive. If $u \in U_0^+$ then $Su = T_0 u$, so T extends T_0.

(b) Suppose that T_0 is a Riesz homomorphism. If $u \wedge u' = 0$ in U, then $w \wedge w' = 0$ and $T_0 w \wedge T_0 w' = 0$ whenever $w \in U_0 \cap [0, u]$ and $w' \in U_0 \cap [0, u']$. By 352Ea, $Tu \wedge Tu' = Su \wedge Su' = 0$ in V. By 352G(iv), T is a Riesz homomorphism.

(c) Now suppose that T_0 is order-continuous. Suppose that $B \subseteq U^+$ is non-empty and upwards-directed and has a supremum $u_0 \in U$. Of course $Tu \le Tu_0$ for every $u \in B$, so Tu_0 is an upper bound for $T[B]$. On the other hand, suppose that v is an upper bound for $T[B]$. If $w \in U_0^+$ and $u \in U^+$, $w \wedge u = \sup\{w' : w' \in U_0,\ 0 \le w' \le w \wedge u\}$. **P** If U_0 is solid, $w \wedge u \in U_0$; and otherwise U_0 is order-dense. **Q** So if $w \in U_0$ and $0 \le w \le u_0$,

$$w = w \wedge \sup B = \sup_{u \in B} w \wedge u = \sup_{u \in B} \sup(U_0 \cap [0, w \wedge u]) = \sup C,$$

where

$$C = \{w' : w' \in U_0^+,\ w' \le w \wedge u \text{ for some } u \in B\}.$$

Since C is upwards-directed,

$$T_0 w = \sup T_0[C] \le v.$$

As w is arbitrary, $Tu_0 \le v$; as v is arbitrary, $Tu_0 = \sup T[B]$; as B is arbitrary, T is order-continuous (351Ga).

(d) If U_0 is order-dense and T_0 is an injective Riesz homomorphism, then for any non-zero $u \in U$ there is a non-zero $w \in U_0$ such that $|w| \le |u|$; so that

$$|Tu| = T|u| \ge T_0|w| > 0$$

because T is a Riesz homomorphism, by (b). As u is arbitrary, T is injective.

(e) Finally, if U_0 is order-dense then any order-continuous positive linear operator extending T_0 must agree with S on U^+ and is therefore equal to T.

355G Definition Let U be a Riesz space and V a Dedekind complete Riesz space. Then $\mathsf{L}^{\times}(U;V)$ will be the set of those $T \in \mathsf{L}^{\sim}(U;V)$ expressible as the difference of order-continuous positive linear operators, and $\mathsf{L}^{\sim}_c(U;V)$ will be the set of those $T \in \mathsf{L}^{\sim}(U;V)$ expressible as the difference of sequentially order-continuous positive linear operators.

Because a composition of (sequentially) order-continuous functions is (sequentially) order-continuous, we shall have

$$ST \in \mathsf{L}^{\times}(U;W) \text{ whenever } S \in \mathsf{L}^{\times}(V;W),\, T \in \mathsf{L}^{\times}(U;V),$$

$$ST \in \mathsf{L}^{\sim}_c(U;W) \text{ whenever } S \in \mathsf{L}^{\sim}_c(V;W),\, T \in \mathsf{L}^{\sim}_c(U;V),$$

for all Riesz spaces U and all Dedekind complete Riesz spaces V, W.

355H Theorem Let U be a Riesz space and V a Dedekind complete Riesz space. Then

(i) $\mathsf{L}^{\times} = \mathsf{L}^{\times}(U;V)$ is a projection band in $\mathsf{L}^{\sim} = \mathsf{L}^{\sim}(U;V)$, therefore a Dedekind complete Riesz space in its own right;

(ii) a member T of $\mathsf{L}^{\sim}$ belongs to $\mathsf{L}^{\times}$ iff $|T|$ is order-continuous.

proof There is a fair bit to check, but each individual step is easy enough.

(a) Suppose that S, T are order-continuous positive linear operators from U to V. Then $S+T$ is order-continuous. **P** If $A \subseteq U$ is non-empty, downwards-directed and has infimum 0, then for any u_1, $u_2 \in A$ there is a $u \in A$ such that $u \le u_1$, $u \le u_2$, and now $(S+T)(u) \le Su_1 + Tu_2$. Consequently any lower bound for $(S+T)[A]$ must also be a lower bound for $S[A]+T[A]$. But since

$$\inf(S[A]+T[A]) = \inf S[A] + \inf T[A] = 0$$

(351Dc), $\inf(S+T)[A]$ must also be 0; as A is arbitrary, $S+T$ is order-continuous, by 351Ga. **Q**

(b) Consequently $S+T \in \mathsf{L}^{\times}$ for all S, $T \in \mathsf{L}^{\times}$. Since $-T$ and αT belong to $\mathsf{L}^{\times}$ for every $T \in \mathsf{L}^{\times}$ and $\alpha \ge 0$, we see that $\mathsf{L}^{\times}$ is a linear subspace of $\mathsf{L}^{\sim}$.

(c) If $T : U \to V$ is an order-continuous linear operator, $S : U \to V$ is linear and $0 \le S \le T$, then S is order-continuous. **P** If $A \subseteq U$ is non-empty, downwards-directed and has infimum 0, then any lower bound of $S[A]$ must also be a lower bound of $T[A]$, so $\inf S[A] = 0$; as A is arbitrary, S is order-continuous. **Q**

It follows that $\mathsf{L}^{\times}$ is a solid linear subspace of $\mathsf{L}^{\sim}$. **P** If $T \in \mathsf{L}^{\times}$ and $|S| \le |T|$ in $\mathsf{L}^{\sim}$, express T as $T_1 - T_2$ where T_1, T_2 are order-continuous positive linear operators. Then

$$S^+,\, S^- \le |S| \le |T| \le T_1 + T_2,$$

so S^+ and S^- are order-continuous and $S = S^+ - S^- \in \mathsf{L}^{\times}$. **Q**

Accordingly $\mathsf{L}^{\times}$ is a Dedekind complete Riesz space in its own right (353K(b-i)).

(d) The argument of (c) also shows that if $T \in \mathsf{L}^{\times}$ then $|T|$ is order-continuous; so that for $T \in \mathsf{L}^{\sim}$,

$$T \in \mathsf{L}^{\times} \iff |T| \in \mathsf{L}^{\times} \iff |T| \text{ is order-continuous.}$$

(e) If $C \subseteq (\mathsf{L}^{\times})^+$ is non-empty, upwards-directed and has a supremum $T \in L^{\sim}$, then T is order-continuous, so belongs to $\mathsf{L}^{\times}$. **P** Suppose that $A \subseteq U^+$ is non-empty, upwards-directed and has supremum w. Then

$$Tw = \sup_{S \in C} Sw = \sup_{S \in C} \sup_{u \in A} Su = \sup_{u \in A} Tu,$$

putting 355Ed and 351G(a-iii) together. So (using 351Ga again) T is order-continuous. **Q** Consequently $\mathsf{L}^{\times}$ is a band in $\mathsf{L}^{\sim}$ (352Ob), and it is a projection band because $\mathsf{L}^{\sim}$ is Dedekind complete (353J).

This completes the proof.

355I Theorem Let U be a Riesz space and V a Dedekind complete Riesz space. Then $\mathsf{L}^{\sim}_c(U;V)$ is a projection band in $\mathsf{L}^{\sim}(U;V)$, and a member T of $\mathsf{L}^{\sim}(U;V)$ belongs to $\mathsf{L}^{\sim}_c(U;V)$ iff $|T|$ is sequentially order-continuous.

proof Copy the arguments of 355H.

355J Proposition Let U be a Riesz space and V a Dedekind complete Riesz space. Let $U_0 \subseteq U$ be an order-dense Riesz subspace; then $T \mapsto T\restriction U_0$ is an embedding of $\mathsf{L}^\times(U;V)$ as a solid linear subspace of $\mathsf{L}^\times(U_0;V)$. In particular, any operator in $\mathsf{L}^\times(U_0;V)$ has at most one extension in $\mathsf{L}^\times(U;V)$.

proof (a) Because the embedding $U_0 \subsetneqq U$ is positive and order-continuous (352Nb), $T\restriction U_0$ is positive and order-continuous whenever T is; so $T\restriction U_0 \in \mathsf{L}^\times(U_0;V)$ whenever $T \in \mathsf{L}^\times(U;V)$. Because the map $T \mapsto T\restriction U_0$ is linear, the image W of $\mathsf{L}^\times(U;V)$ is a linear subspace of $\mathsf{L}^\times(U_0;V)$.

(b) If $T \in \mathsf{L}^\times(U;V)$ and $T\restriction U_0 \ge 0$, then $T \ge 0$. **P?** Suppose, if possible, that there is a $u \in U^+$ such that $Tu \not\ge 0$. Because $|T| \in \mathsf{L}^\times(U;V)$ is order-continuous and $A = \{v : v \in U_0,\, v \le u\}$ is an upwards-directed set with supremum u, $\inf\{|T|(u-v) : v \in A\} = 0$ and there is a $v \in A$ such that $Tu+|T|(u-v) \not\ge 0$. But $Tv = Tu + T(v-u) \le Tu + |T|(u-v)$ so $Tv \not\ge 0$ and $T\restriction U_0 \not\ge 0$. **XQ**

This shows that the map $T \mapsto T\restriction U_0$ is an order-isomorphism between $\mathsf{L}^\times(U;V)$ and W, and in particular is injective.

(c) Now suppose that $S_0 \in W$ and that $|S| \le |S_0|$ in $\mathsf{L}^\times(U_0;V)$. Then $S \in W$. **P** Take $T_0 \in \mathsf{L}^\times(U;V)$ such that $T_0\restriction U_0 = S_0$. Then $S_1 = |T_0|\restriction U_0$ is a positive member of W such that $S_0 \le S_1$ and $-S_0 \le S_1$, so $S^+ \le S_1$. Consequently, for any $u \in U^+$,

$$\sup\{S^+v : v \in U_0,\, 0 \le v \le u\} \le \sup\{S_1 v : v \in U_0,\, 0 \le v \le u\} \le |T_0|(u)$$

is defined in V (recall that we are assuming that V is Dedekind complete). But this means that S^+ has an extension to an order-continuous positive linear operator from U to V (355F), and belongs to W. Similarly, $S^- \in W$, so $S \in W$. **Q**

This shows that W is a solid linear subspace of $\mathsf{L}^\times(U_0;V)$, as claimed.

355K Proposition Let U be a Banach lattice with an order-continuous norm.

(a) If V is any Archimedean Riesz space and $T : U \to V$ is a positive linear operator, then T is order-continuous.

(b) If V is a Dedekind complete Riesz space then $\mathsf{L}^\times(U;V) = \mathsf{L}^\sim(U;V)$.

proof (a) Suppose that $A \subseteq U^+$ is non-empty and downwards-directed and has infimum 0. Then for each $n \in \mathbb{N}$ there is a $u_n \in A$ such that $\|u_n\| \le 4^{-n}$. By 354C, $u = \sup_{n\in\mathbb{N}} 2^n u_n$ is defined in U. Now $Tu_n \le 2^{-n}Tu$ for every n, so any lower bound for $T[A]$ must also be a lower bound for $\{2^{-n}Tu : n \in \mathbb{N}\}$ and therefore (because V is Archimedean) less than or equal to 0. Thus $\inf T[A] = 0$; as A is arbitrary, T is order-continuous.

(b) This is now immediate from 355Ea and the definition of $\mathsf{L}^\times$.

355X Basic exercises >(a) Let U and V be arbitrary Riesz spaces. (i) Show that the set $\mathsf{L}(U;V)$ of all linear operators from U to V is a partially ordered linear space if we say that $S \le T$ whenever $Su \le Tu$ for every $u \in U^+$. (ii) Show that if U and V are Banach lattices then the set of positive operators is closed in the normed space $\mathsf{B}(U;V)$ of bounded linear operators from U to V.

>(b) If U is a Riesz space and $\|\,\|$, $\|\,\|'$ are two norms on U both rendering it a Banach lattice, show that they are equivalent, that is, give rise to the same topology.

(c) Let U be a Riesz space with a Riesz norm, V an Archimedean Riesz space with an order unit, and $T : U \to V$ a linear operator which is continuous for the given norm on U and the order-unit norm on V. Show that T is order-bounded.

(d) Let U be a Riesz space, V an Archimedean Riesz space, and $T : U^+ \to V^+$ a map such that $T(u_1 + u_2) = Tu_1 + Tu_2$ for all u_1, $u_2 \in U^+$. Show that T has an extension to a linear operator from U to V.

>(e) Show that if r, $s \ge 1$ are integers then the Riesz space $\mathsf{L}^\sim(\mathbb{R}^r;\mathbb{R}^s)$ can be identified with the space of real $s \times r$ matrices, saying that a matrix is positive iff every coefficient is positive, so that if $T = \langle \tau_{ij}\rangle_{1\le i\le s, 1\le j\le r}$ then $|T|$, taken in $\mathsf{L}^\sim(\mathbb{R}^r;\mathbb{R}^s)$, is $\langle|\tau_{ij}|\rangle_{1\le i\le s,1\le j\le r}$. Show that a positive matrix represents a Riesz homomorphism iff each row has at most one non-zero coefficient.

>(f) Let U be a Riesz space and V a Dedekind complete Riesz space. Show that if $T_0,\dots,T_n\in\mathsf{L}^\sim(U;V)$ then

$$(T_0\vee\ldots\vee T_n)(w)=\sup\{\sum_{i=0}^n T_iu_i : u_i\ge 0\ \forall\, i\le n,\ \sum_{i=0}^n u_i=w\}$$

for every $w\in U^+$.

>(g) Let U be a Riesz space, V a Dedekind complete Riesz space, and $A\subseteq\mathsf{L}^\sim(U;V)$ a non-empty set. Show that A is bounded above in $\mathsf{L}^\sim(U;V)$ iff $C_w=\{\sum_{i=0}^n T_iu_i : T_0,\dots,T_n\in A,\ u_0,\dots,u_n\in U^+,\ \sum_{i=0}^n u_i=w\}$ is bounded above in V for every $w\in U^+$, and in this case $(\sup A)(w)=\sup C_w$ for every $w\in U^+$.

355Y Further exercises (a) Let U and V be Banach lattices. For $T\in\mathsf{L}^\sim=\mathsf{L}^\sim(U;V)$, set

$$\|T\|_\sim=\sup_{w\in U^+,\|w\|\le 1}\inf\{\|v\| : |Tu|\le v \text{ whenever } |u|\le w\}.$$

Show that $\|\ \|_\sim$ is a norm on $\mathsf{L}^\sim$ under which $\mathsf{L}^\sim$ is a Banach space, and that the set of positive linear operators is closed in $\mathsf{L}^\sim$.

(b) Give an example of a continuous linear operator from ℓ^2 to itself which is not order-bounded.

(c) Let U and V be Riesz spaces and $T:U\to V$ a linear operator. (i) Show that for any $w\in U^+$, $C_w=\{\sum_{i=0}^n|Tu_i| : u_0,\dots,u_n\in U^+,\ \sum_{i=0}^n u_i=w\}$ is upwards-directed, and has the same upper bounds as $\{Tu : |u|\le w\}$. (*Hint*: 352Fd.) (ii) Show that if $\sup C_w$ is defined for every $w\in U^+$, then $S=T\vee(-T)$ is defined in the partially ordered linear space $\mathsf{L}^\sim(U;V)$ and $Sw=\sup C_w$ for every $w\in U^+$.

(d) Let U, V and W be Riesz spaces, of which V and W are Dedekind complete. (i) Show that for any $S\in\mathsf{L}^\times(V;W)$, the map $T\mapsto ST : L^\sim(U;V)\to\mathsf{L}^\sim(U;W)$ belongs to $\mathsf{L}^\times(\mathsf{L}^\sim(U;V);\mathsf{L}^\sim(U;W))$, and is a Riesz homomorphism if S is. (*Hint*: 355Yc.) (ii) Show that for any $T\in\mathsf{L}^\sim(U;V)$, the map $S\mapsto ST : L^\sim(V;W)\to\mathsf{L}^\sim(U;W)$ belongs to $\mathsf{L}^\times(L^\sim(V;W);\mathsf{L}^\sim(U;W))$.

(e) Let $\nu_{\mathbb N}$ be the usual measure on $\{0,1\}^{\mathbb N}$ and $\boldsymbol{c}$ the Banach lattice of convergent sequences. Find a linear operator $T:L^2(\nu_{\mathbb N})\to\boldsymbol{c}$ which is norm-continuous, therefore order-bounded, such that 0 and T have no common upper bound in the partially ordered linear space of all linear operators from $L^2(\nu_{\mathbb N})$ to $\boldsymbol{c}$.

(f) Let U and V be Banach lattices. Let $\mathsf{L}^{\mathrm{reg}}$ be the linear space of operators from U to V expressible as the difference of positive operators. For $T\in\mathsf{L}^{\mathrm{reg}}$ let $\|T\|_{\mathrm{reg}}$ be

$$\inf\{\|T_1+T_2\| : T_1,\,T_2:U\to V \text{ are positive, } T=T_1-T_2\}.$$

Show that $\|\ \|_{\mathrm{reg}}$ is a norm under which $\mathsf{L}^{\mathrm{reg}}$ is complete.

(g) Let U and V be Riesz spaces. For this exercise only, say that $\mathsf{L}^\times(U;V)$ is to be the set of linear operators $T:U\to V$ such that whenever $A\subseteq U$ is non-empty, downwards-directed and has infimum 0 then $\{v : v\in V^+,\ \exists\, w\in A,\ |Tu|\le v \text{ whenever } |u|\le w\}$ has infimum 0 in V. (i) Show that $\mathsf{L}^\times(U;V)$ is a linear space. (ii) Show that if U is Archimedean then $\mathsf{L}^\times(U;V)\subseteq\mathsf{L}^\sim(U;V)$. (iii) Show that if U is Archimedean and V is Dedekind complete then this definition agrees with that of 355G. (iv) Show that for any Riesz spaces U, V and W, $ST\in\mathsf{L}^\times(U;W)$ for every $S\in\mathsf{L}^\times(V;W)$ and $T\in\mathsf{L}^\times(U;V)$. (v) Show that if U and V are Banach lattices, then $\mathsf{L}^\times(U;V)$ is closed in $\mathsf{L}^\sim(U;V)$ for the norm $\|\ \|_\sim$ of 355Ya. (vi) Show that if V is Archimedean and U is a Banach lattice with an order-continuous norm, then $\mathsf{L}^\times(U;V)=\mathsf{L}^\sim(U;V)$.

(h) Let U be a Riesz space and V a Dedekind complete Riesz space. Show that the band projection $P:\mathsf{L}^\sim(U;V)\to\mathsf{L}^\times(U;V)$ is given by the formula

$$(PT)(w)=\inf\{\sup_{u\in A}Tu : A\subseteq U^+ \text{ is non-empty, upwards-directed and has supremum } w\}$$

for every $w\in U^+$, $T\in(\mathsf{L}^\sim(U;V))^+$. (Cf. 362Bd.)

(i) Show that if U is a Riesz space with the countable sup property (241Ye), then $\mathsf{L}^{\sim}_c(U;V) = \mathsf{L}^{\times}(U;V)$ for every Dedekind complete Riesz space V.

(j) Let U and V be Riesz spaces, of which V is Dedekind complete, and U_0 a solid linear subspace of U. Show that the map $T \mapsto T\restriction U_0$ is an order-continuous Riesz homomorphism from $\mathsf{L}^{\times}(U;V)$ onto a solid linear subspace of $\mathsf{L}^{\times}(U_0;V)$.

(k) Let U be a uniformly complete Riesz space (354Yi) and V a Dedekind complete Riesz space. Let $U_{\mathbb{C}}$, $V_{\mathbb{C}}$ be their complexifications (354Yl). Show that the complexification of $\mathsf{L}^{\sim}(U;V)$ can be identified with the complex linear space of linear operators $T : U_{\mathbb{C}} \to V_{\mathbb{C}}$ such that $B_T(w) = \{|Tu| : |u| \le w\}$ is bounded above in V for every $w \in U^+$, and that now $|T|(w) = \sup B_T(w)$ for every $T \in \mathsf{L}^{\sim}(U;V)_{\mathbb{C}}$ and $w \in U^+$. (*Hint*: if u, $v \in U$ and $|u + iv| = w$, then u and v can be simultaneously approximated for the order-unit norm $\| \|_w$ on the solid linear subspace generated by w by finite sums $\sum_{j=0}^n (\cos\theta_j)w_j$, $\sum_{j=0}^n (\sin\theta_j)w_j$ where $w_j \in U^+$, $\sum_{j=0}^n w_j = w$. Consequently $|T(u + iv)| \le |T|(w)$ for every $T \in \mathsf{L}^{\sim}_{\mathbb{C}}$.)

355 Notes and comments I have had to make some choices in the basic definitions of this chapter (355A, 355G). For Dedekind complete codomains V, there is no doubt what $\mathsf{L}^{\sim}(U;V)$ should be, since the order-bounded operators (in the sense of 355A) are just the differences of positive operators (355Ea). (These are sometimes called '**regular**' operators.) When V is not Dedekind complete, we have to choose between the two notions, as not every order-bounded operator need be regular (355Ye). In my previous book (Fremlin 74a) I chose the regular operators; I have still not encountered any really persuasive reason to settle definitively on either class. In 355G the technical complications in dealing with any natural equivalent of the larger space (see 355Yg) are such that I have settled for the narrower class, but explicitly restricting the definition to the case in which V is Dedekind complete. In the applications in this book, the codomains are nearly always Dedekind complete, so we can pass these questions by.

The elementary extension technique in 355D may recall the definition of the Lebesgue integral (122L-122M). In the same way, 351G may remind you of the theorem that a linear operator between normed spaces is continuous everywhere if it is continuous anywhere, or of the corresponding results about Boolean homomorphisms and additive functionals on Boolean algebras (313L, 326Ka, 326R).

Of course 355Ea is the central fact about the space $\mathsf{L}^{\sim}(U;V)$ for Dedekind complete V; because we get a new Riesz space from old ones, the prospect of indefinite recursion immediately presents itself. For Banach lattices, $\mathsf{L}^{\sim}(U;V)$ is a linear subspace of the space $\mathsf{B}(U;V)$ of bounded linear operators (355C); the question of when the two are equal will be of great importance to us. I give only the vaguest hints on how to show that they can be different (355Yb, 355Ye), but these should be enough to make it plain that equality is the exception rather than the rule. It is also very useful that we have effective formulae to describe the Riesz space operations on $\mathsf{L}^{\sim}(U;V)$ (355E, 355Xf-355Xg, 355Yc). You may wish to compare these with the corresponding formulae for additive functionals on Boolean algebras in 326Yd and 362B.

If we think of $\mathsf{L}^{\sim}$ as somehow corresponding to the space of bounded additive functionals on a Boolean algebra, the bands $\mathsf{L}^{\sim}_c$ and $\mathsf{L}^{\times}$ correspond to the spaces of countably additive and completely additive functionals. In fact (as will appear in §362) this correspondence is very close indeed. For the moment, all I have sought to establish is that $\mathsf{L}^{\sim}_c$ and $\mathsf{L}^{\times}$ are indeed bands. Of course any case in which $\mathsf{L}^{\sim}(U;V) = \mathsf{L}^{\sim}_c(U;V)$ or $\mathsf{L}^{\sim}_c(U;V) = \mathsf{L}^{\times}(U;V)$ is of interest (355Kb, 355Yi).

Between Banach lattices, positive linear operators are continuous (355C); it follows at once that the Riesz space structure determines the topology (355Xb), so that it is not to be wondered at that there are further connexions between the norm and the spaces $\mathsf{L}^{\sim}$ and $\mathsf{L}^{\times}$, as in 355K.

355F will be a basic tool in the theory of representations of Riesz spaces; if we can represent an order-dense Riesz subspace of U as a subspace of a Dedekind complete space V, we have at least some chance of expressing U also as a subspace of V. Of course it has other applications, starting with analysis of the dual spaces.

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356 Dual spaces

As always in functional analysis, large parts of the theory of Riesz spaces are based on the study of linear functionals. Following the scheme of the last section, I define spaces $U^\sim$, $U^\sim_c$ and $U^\times$, the 'order-bounded', 'sequentially order-continuous' and 'order-continuous' duals of a Riesz space U (356A). These are Dedekind complete Riesz spaces (356B). If U carries a Riesz norm they are closely connected with the normed space dual U^*, which is itself a Banach lattice (356D). For each of them, we have a canonical Riesz homomorphism from U to the corresponding bidual. The map from U to $U^{\times\times}$ is particularly important (356I); when this map is an isomorphism we call U 'perfect' (356J). The last third of the section deals with L- and M-spaces and the duality between them (356N, 356P), with two important theorems on uniform integrability (356O, 356Q).

356A Definition Let U be a Riesz space.

(a) I write $U^\sim$ for the space $\mathcal{L}^\sim(U;\mathbb{R})$ of order-bounded real-valued linear functionals on U, the **order-bounded dual** of U.

(b) $U^\sim_c$ will be the space $\mathcal{L}^\sim_c(U;\mathbb{R})$ of differences of sequentially order-continuous positive real-valued linear functionals on U, the **sequentially order-continuous dual** of U.

(c) $U^\times$ will be the space $\mathcal{L}^\times(U;\mathbb{R})$ of differences of order-continuous positive real-valued linear functionals on U, the **order-continuous dual** of U.

Remark It is easy to check that the three spaces $U^\sim$, $U^\sim_c$ and $U^\times$ are in general different (356Xa-356Xc). But the examples there leave open the question: can we find a Riesz space U, for which $U^\sim_c \ne U^\times$, and which is actually Dedekind complete, rather than just Dedekind σ-complete, as in 356Xc? This leads to unexpectedly deep water; it is yet another form of the Banach-Ulam problem. Really this is a question for Volume 5, but in 363S below I collect the relevant ideas which are within the scope of the present volume.

356B Theorem For any Riesz space U, its order-bounded dual $U^\sim$ is a Dedekind complete Riesz space in which $U^\sim_c$ and $U^\times$ are projection bands, therefore Dedekind complete Riesz spaces in their own right. For $f \in U^\sim$, f^+ and $|f| \in U^\sim$ are defined by the formulae

$$f^+(w) = \sup\{f(u) : 0 \le u \le w\}, \quad |f|(w) = \sup\{f(u) : |u| \le w\}$$

for every $w \in U^+$. A non-empty upwards-directed set $A \subseteq U^\sim$ is bounded above iff $\sup_{f\in A} f(u)$ is finite for every $u \in U$, and in this case $(\sup A)(u) = \sup_{f\in A} f(u)$ for every $u \in U^+$.

proof 355E, 355H, 355I.

356C Proposition Let U be any Riesz space and P a band projection on U. Then its adjoint $P' : U^\sim \to U^\sim$, defined by setting $P'(f) = fP$ for every $f \in U^\sim$, is a band projection on $U^\sim$.

proof Because $P : U \to U$ is a positive linear operator, $P'f \in U^\sim$ for every $f \in U^\sim$ (355Bd), and P' is a positive linear operator from $U^\sim$ to itself. Set $Q = I - P$, the complementary band projection on U; then Q' is another positive linear operator on $U^\sim$, and $P'f + Q'f = f$ for every f. Now $P'f \wedge Q'f = 0$ for every $f \ge 0$. **P** For any $w \in U^+$,

$$\begin{aligned}(P'f - Q'f)^+(w) &= \sup_{0\le u\le w} (P'f - Q'f)(u)\\ &= \sup_{0\le u\le w} f(Pu - Qu) = f(Pw)\end{aligned}$$

(because $Pu - Qu \le Pu \le Pw = P(Pw) - Q(Pw)$ whenever $0 \le u \le w$)

$$= (P'f)(w),$$

so $(P'f - Q'f)^+ = P'f$, that is, $P'f \wedge Q'f = 0$. **Q** By 352Rd, P' is a band projection.

356D Proposition Let U be a Riesz space with a Riesz norm.

(a) The normed space dual U^* of U is a solid linear subspace of $U^\sim$, and in itself is a Banach lattice with a Fatou norm and has the Levi property.

(b) The norm of U is order-continuous iff $U^* \subseteq U^\times$.

(c) If U is a Banach lattice, then $U^* = U^\sim$, so that $U^\sim$, $U^\times$ and $U_c^\sim$ are all Banach lattices.

(d) If U is a Banach lattice with order-continuous norm then $U^* = U^\times = U^\sim$.

proof (a)(i) If $f \in U^*$ then

$$\sup_{|u| \le w} f(u) \le \sup_{|u| \le w} \|f\|\|u\| = \|f\|\|w\| < \infty$$

for every $w \in U^+$, so $f \in U^\sim$ (355Ba). Thus $U^* \subseteq U^\sim$.

(ii) If $f \in U^\sim$, $g \in U^*$ and $|f| \le |g|$, then for any $w \in U$

$$|f(w)| \le |f|(|w|) \le |g|(|w|) = \sup_{|u| \le |w|} g(u) \le \sup_{|u| \le |w|} \|g\|\|u\| \le \|g\|\|w\|.$$

As w is arbitrary, $f \in U^*$ and $\|f\| \le \|g\|$; as f and g are arbitrary, U^* is a solid linear subspace of $U^\sim$ and the norm of U^* is a Riesz norm. Because U^* is a Banach space it is also a Banach lattice.

(iii) If $A \subseteq (U^*)^+$ is non-empty and upwards-directed and $M = \sup_{f \in A} \|f\|$ is finite, then $\sup_{f \in A} f(u) \le M\|u\|$ is finite for every $u \in U^+$, so $g = \sup A$ is defined in $U^\sim$ (355Ed). Now $g(u) = \sup_{f \in A} f(u)$ for every $u \in U^+$, as also noted in 355Ed, so

$$|g(u)| \le g(|u|) \le M\||u|\| = M\|u\|$$

for every $u \in U$, and $\|g\| \le M$. But as A is arbitrary, this proves simultaneously that the norm of $U^\sim$ is Fatou and has the Levi property.

(b)(i) Suppose that the norm of U is order-continuous. If $f \in U^*$ and $A \subseteq U$ is a non-empty downwards-directed set with infimum 0, then

$$\inf_{u \in A} |f|(u) \le \inf_{u \in A} \|f\|\|u\| = 0,$$

so $|f| \in U^\times$ and $f \in U^\times$. Thus $U^* \subseteq U^\times$.

(ii) Now suppose that the norm is not order-continuous. Then there is a non-empty downwards-directed set $A \subseteq U$, with infimum 0, such that $\inf_{u \in A} \|u\| = \delta > 0$. Set

$$B = \{v : v \ge u \text{ for some } u \in A\}.$$

Then B is convex. **P** If v_1, $v_2 \in B$ and $\alpha \in [0,1]$, there are u_1, $u_2 \in A$ such that $v_i \ge u_i$ for both i; now there is a $u \in A$ such that $u \le u_1 \wedge u_2$, so that

$$u = \alpha u + (1-\alpha)u \le \alpha v_1 + (1-\alpha)v_2,$$

and $\alpha v_1 + (1-\alpha)v_2 \in B$. **Q** Also $\inf_{v \in B} \|v\| = \delta > 0$. By the Hahn-Banach theorem (3A5Cb), there is an $f \in U^*$ such that $\inf_{v \in B} f(v) > 0$. But now

$$\inf_{u \in A} |f|(u) \ge \inf_{u \in A} f(u) > 0$$

and $|f|$ is not order-continuous; so $U^* \not\subseteq U^\times$.

(c) By 355C, $U^\sim \subseteq U^*$, so $U^\sim = U^*$. Now $U^\times$ and $U_c^\sim$, being bands, are closed linear subspaces (354Bd), so are Banach lattices in their own right.

(d) Put (b) and (c) together.

356E Biduals If you have studied any functional analysis at all, it will come as no surprise that duals-of-duals are important in the theory of Riesz spaces. I start with a simple lemma.

Lemma Let U be a Riesz space and $f : U \to \mathbb{R}$ a positive linear functional. Then for any $u \in U^+$ there is a positive linear functional $g : U \to \mathbb{R}$ such that $0 \le g \le f$, $g(u) = f(u)$ and $g(v) = 0$ whenever $u \wedge v = 0$.

proof Set $g(v) = \sup_{\alpha \ge 0} f(v \wedge \alpha u)$ for every $v \in U^+$. Then it is easy to see that $g(\beta v) = \beta g(v)$ for every $v \in U^+$, $\beta \in [0, \infty[$. If v, $w \in U^+$ then

$$(v \wedge \alpha u) + (w \wedge \alpha u) \leq (v + w) \wedge 2\alpha u \leq (v \wedge 2\alpha u) + (w \wedge 2\alpha u)$$

for every $\alpha \geq 0$ (352Fa), so $g(v + w) = g(v) + g(w)$. Accordingly g has an extension to a linear functional from U to $\mathbb{R}$ (355D). Of course $0 \leq g(v) \leq f(v)$ for $v \geq 0$, so $0 \leq g \leq f$ in $U^\sim$. We have $g(u) = f(u)$, while if $u \wedge v = 0$ then $\alpha u \wedge v = 0$ for every $\alpha \geq 0$, so $g(v) = 0$.

356F Theorem Let U be a Riesz space and V a solid linear subspace of $U^\sim$. For $u \in U$ define $\hat{u} : V \to \mathbb{R}$ by setting $\hat{u}(f) = f(u)$ for every $f \in V$. Then $u \mapsto \hat{u}$ is a Riesz homomorphism from U to $V^\times$.

proof (a) By the definition of addition and scalar multiplication in V, $\hat{u}$ is linear for every u; also $\widehat{\alpha u} = \alpha\hat{u}$ and $(u_1 + u_2)\hat{} = \hat{u}_1 + \hat{u}_2$ for all u, u_1, $u_2 \in U$ and $\alpha \in \mathbb{R}$. If $u \geq 0$ then $\hat{u}(f) = f(u) \geq 0$ for every $f \in V^+$, so $\hat{u} \geq 0$; accordingly every $\hat{u}$ is the difference of two positive functionals, and $u \mapsto \hat{u}$ is a linear operator from U to $V^\sim$.

(b) If $B \subseteq V$ is a non-empty downwards-directed set with infimum 0, then $\inf_{f \in B} f(u) = 0$ for every $u \in U^+$, by 355Ee. But this means that $\hat{u}$ is order-continuous for every $u \in U^+$, so that $\hat{u} \in V^\times$ for every $u \in U$.

(c) If $u \wedge v = 0$ in U, then for any $f \in V^+$ there is a $g \in [0, f]$ such that $g(u) = f(u)$ and $g(v) = 0$ (356E). So

$$(\hat{u} \wedge \hat{v})(f) \leq \hat{u}(f - g) + \hat{v}(g) = f(u) - g(u) + g(v) = 0.$$

As f is arbitrary, $\hat{u} \wedge \hat{v} = 0$. As u and v are arbitrary, $u \mapsto \hat{u}$ is a Riesz homomorphism (352G).

356G Lemma Suppose that U is a Riesz space such that $U^\sim$ separates the points of U. Then U is Archimedean.

proof ? Otherwise, there are u, $v \in U$ such that $v > 0$ and $nv \leq u$ for every $n \in \mathbb{N}$. Now there is an $f \in U^\sim$ such that $f(v) \neq 0$; but $|f(v)| \leq |f|(v) \leq \frac{1}{n}|f|(u)$ for every n, so this is impossible. **X**

356H Lemma Let U be an Archimedean Riesz space and $f > 0$ in $U^\times$. Then there is a $u \in U$ such that (i) $u > 0$ (ii) $f(v) > 0$ whenever $0 < v \leq u$ (iii) $g(u) = 0$ whenever $g \wedge f = 0$ in $U^\times$. Moreover, if $u_0 \in U^+$ is such that $f(u_0) > 0$, we can arrange that $u \leq u_0$.

proof (a) Because $f > 0$ there is certainly some $u_0 \in U$ such that $f(u_0) > 0$. Set $A = \{v : 0 \leq v \leq u_0, f(v) = 0\}$. Then $(v_1 + v_2) \wedge u_0 \in A$ for all v_1, $v_2 \in A$, so A is upwards-directed. Because $f(u_0) > 0 = \sup f[A]$ and f is order-continuous, u_0 cannot be the least upper bound of A, and there is another upper bound u_1 of A strictly less than u_0.

Set $u = u_0 - u_1 > 0$. If $0 \leq v \leq u$ and $f(v) = 0$, then

$$w \in A \Longrightarrow w \leq u_1 \Longrightarrow w + v \leq u_0 \Longrightarrow w + v \in A;$$

consequently $nv \in A$ and $nv \leq u_0$ for every $n \in \mathbb{N}$, so $v = 0$. Thus u has properties (i) and (ii).

(b) Now suppose that $g \wedge f = 0$ in $U^\times$. Let $\epsilon > 0$. Then for each $n \in \mathbb{N}$ there is a $v_n \in [0, u]$ such that $f(v_n) + g(u - v_n) \leq 2^{-n}\epsilon$ (355Ec). If $v \leq v_n$ for every $n \in \mathbb{N}$ then $f(v) = 0$ so $v = 0$; thus $\inf_{n \in \mathbb{N}} v_n = 0$. Set $w_n = \inf_{i \leq n} v_i$ for each $n \in \mathbb{N}$; then $\langle w_n \rangle_{n \in \mathbb{N}}$ is non-increasing and has infimum 0 so (because g is order-continuous) $\inf_{n \in \mathbb{N}} g(w_n) = 0$. But

$$u - w_n = \sup_{i \leq n} u - v_i \leq \sum_{i=0}^{n} u - v_i,$$

so

$$g(u - w_n) \leq \sum_{i=0}^{n} g(u - v_i) \leq 2\epsilon$$

for every n, and

$$g(u) \leq 2\epsilon + \inf_{n \in \mathbb{N}} g(w_n) = 2\epsilon.$$

As ϵ is arbitrary, $g(u) = 0$; as g is arbitrary, u has the third required property.

356I Theorem Let U be any Archimedean Riesz space. Then the canonical map from U to $U^{\times\times}$ (356F) is an order-continuous Riesz homomorphism from U onto an order-dense Riesz subspace of $U^{\times\times}$. If U is Dedekind complete, its image in $U^{\times\times}$ is solid.

proof (a) By 356F, $u \mapsto \hat{u} : U \to U^{\times\times}$ is a Riesz homomorphism.

To see that it is order-continuous, take any non-empty downwards-directed set $A \subseteq U$ with infimum 0. Then $C = \{\hat{u} : u \in A\}$ is downwards-directed, and for any $f \in (U^{\times})^+$

$$\inf_{\phi\in C} \phi(f) = \inf_{u\in A} f(u) = 0$$

because f is order-continuous. As f is arbitrary, $\inf C = 0$ (355Ee); as A is arbitrary, $u \mapsto \hat{u}$ is order-continuous (351Ga).

(b) Now suppose that $\phi > 0$ in $U^{\times\times}$. By 356H, there is an $f > 0$ in $U^{\times}$ such that $\phi(f) > 0$ and $\phi(g) = 0$ whenever $g \wedge f = 0$. Next, there is a $u > 0$ in U such that $f(u) > 0$. Since $u \geq 0$, $\hat{u} \geq 0$; since $\hat{u}(f) > 0$, $\hat{u} \wedge \phi > 0$.

Because $U^{\times\times}$ (being Dedekind complete) is Archimedean, $\inf_{\alpha>0} \alpha\hat{u} = 0$, and there is an $\alpha > 0$ such that

$$\psi = (\hat{u} \wedge \phi - \alpha\hat{u})^+ > 0.$$

Let $g \in (U^{\times})^+$ be such that $\psi(g) > 0$ and $\theta(g) = 0$ whenever $\theta \wedge \psi = 0$ in $U^{\times\times}$. Let $v \in U^+$ be such that $g(v) > 0$ and $h(v) = 0$ whenever $h \wedge g = 0$ in $U^{\times}$.

Because $\hat{v}(g) = g(v) > 0$, $\hat{v} \wedge \psi > 0$. As $\psi \leq \hat{u}$, $\hat{v} \wedge \hat{u} > 0$ and $\hat{v} \wedge \alpha\hat{u} > 0$. Set $w = v \wedge \alpha u$; then $\hat{w} = \hat{v} \wedge \alpha\hat{u}$, by 356F, so $\hat{w} > 0$.

? Suppose, if possible, that $\hat{w} \not\leq \phi$. Then $\theta = (\hat{w} - \phi)^+ > 0$, so there is an $h \in (U^{\times})^+$ such that $\theta(h) > 0$ and $\theta(h') > 0$ whenever $0 < h' \leq h$ (356H, for the fourth and last time). Now examine

$$\theta(h \wedge g) \leq (\alpha\hat{u} - \phi \wedge \hat{u})^+(g)$$

(because $\hat{w} \leq \alpha\hat{u}$, $\phi \wedge \hat{u} \leq \phi$, $h \wedge g \leq g$)

$$= 0$$

because $(\alpha\hat{u} - \phi \wedge \hat{u})^+ \wedge \psi = 0$. So $h \wedge g = 0$ and $h(v) = 0$. But this means that

$$\theta(h) \leq \hat{w}(h) \leq \hat{v}(h) = 0,$$

which is impossible. **X**

Thus $0 < \hat{w} \leq \phi$. As ϕ is arbitrary, the image $\hat{U}$ of U is quasi-order-dense in $U^{\times\times}$, therefore order-dense (353A).

(c) Now suppose that U is Dedekind complete and that $0 \leq \phi \leq \psi \in \hat{U}$. Express ψ as $\hat{u}$ where $u \in U$, and set $A = \{v : v \in U, v \leq u^+, \hat{v} \leq \phi\}$. If $v \in U$ and $0 \leq \hat{v} \leq \phi$, then $w = v^+ \wedge u^+ \in A$ and $\hat{w} = \hat{v}$; thus $\phi = \sup\{\hat{v} : v \in A\} = \hat{v}_0$, where $v_0 = \sup A$. So $\phi \in \hat{U}$. As ϕ and ψ are arbitrary, $\hat{U}$ is solid in $U^{\times\times}$.

356J Definition A Riesz space U is **perfect** if the canonical map from U to $U^{\times\times}$ is an isomorphism.

356K Proposition A Riesz space U is perfect iff (i) it is Dedekind complete (ii) $U^{\times}$ separates the points of U (iii) whenever $A \subseteq U$ is non-empty and upwards-directed and $\{f(u) : u \in A\}$ is bounded for every $f \in U^{\times}$, then A is bounded above in U.

proof (a) Suppose that U is perfect. Because it is isomorphic to $U^{\times\times}$, which is surely Dedekind complete, U also is Dedekind complete. Because the map $u \mapsto \hat{u} : U \to U^{\times\times}$ is injective, $U^{\times}$ separates the points of U. If $A \subseteq U$ is non-empty and upwards-directed ad $\{f(u) : u \in A\}$ is bounded above for every $f \in U^{\times}$, then $B = \{\hat{u} : u \in A\}$ is non-empty and upwards-directed and $\sup_{\phi\in B} \phi(f) < \infty$ for every $f \in U^{\times}$, so $\sup B$ is defined in $U^{\times\sim}$ (355Ed); but $U^{\times\times}$ is a band in $U^{\times\sim}$, so $\sup B$ belongs to $U^{\times\times}$ and is of the form $\hat{w}$ for some $w \in U$. Because $u \mapsto \hat{u}$ is a Riesz space isomorphism, $w = \sup A$ in U. Thus U satisfies the three conditions.

(b) Suppose that U satisfies the three conditions. We know that $u \mapsto \hat{u}$ is an order-continuous Riesz homomorphism onto an order-dense Riesz subspace of $U^{\times\times}$ (356I). It is injective because $U^{\times}$ separates the points of U. If $\phi \geq 0$ in $U^{\times\times}$, set $A = \{u : u \in U^+,\ \hat{u} \leq \phi\}$. Then A is non-empty and upwards-directed and for any $f \in U^{\times}$

$$\sup_{u\in A} f(u) \leq \sup_{u\in A} |f|(u) \leq \sup_{u\in A} \hat{u}(|f|) \leq \phi(|f|) < \infty,$$

so by condition (iii) A has an upper bound in U. Since U is Dedekind complete, $w = \sup A$ is defined in U. Now

$$\hat{w} = \sup_{u\in A} \hat{u} = \phi.$$

As ϕ is arbitrary, the image of U includes $(U^{\times\times})^+$, therefore is the whole of $U^{\times\times}$, and $u \mapsto \hat{u}$ is a bijective Riesz homomorphism, that is, a Riesz space isomorphism.

356L Proposition (a) Any band in a perfect Riesz space is a perfect Riesz space in its own right.
(b) For any Riesz space U, $U^{\sim}$ is perfect; consequently $U^{\sim}_c$ and $U^{\times}$ are perfect.

proof (a) I use the criterion of 356K. Let U be a perfect Riesz space and V a band in U. Then V is Dedekind complete because U is (353Kb). If $v \in V \setminus \{0\}$ there is an $f \in U^{\times}$ such that $f(v) \neq 0$; but the embedding $V \subsetneqq U$ is order-continuous (352N), so $g = f{\restriction}V$ belongs to $V^{\times}$, and $g(v) \neq 0$. Thus $V^{\times}$ separates the points of V. If $A \subseteq V$ is non-empty and upwards-directed and $\sup_{v\in A} g(v)$ is finite for every $g \in V^{\times}$, then $\sup_{v\in A} f(v) < \infty$ for every $f \in U^{\times}$ (again because $f{\restriction}V \in V^{\times}$), so A has an upper bound in U; because U is Dedekind complete, $\sup A$ is defined in U; because V is a band, $\sup A \in V$ and is an upper bound for A in V. Thus V satisfies the conditions of 356K and is perfect.

(b) $U^{\sim}$ is Dedekind complete, by 355Ea. If $f \in U^{\sim} \setminus \{0\}$, there is a $u \in U$ such that $f(u) \neq 0$; now $\hat{u}(f) \neq 0$, where $\hat{u} \in U^{\sim\times}$ (356F). Thus $U^{\sim\times}$ separates the points of $U^{\sim}$. If $A \subseteq U^{\sim}$ is non-empty and upwards-directed and $\sup_{f\in A} \phi(f)$ is finite for every $\phi \in U^{\sim\times}$, then, in particular,

$$\sup_{f\in A} f(u) = \sup_{f\in A} \hat{u}(f) < \infty$$

for every $u \in U$, so A is bounded above in $U^{\sim}$, by 355Ed. Thus $U^{\sim}$ satisfies the conditions of 356K and is perfect.

By (a), it follows at once that $U^{\times}$ and $U^{\sim}_c$ are perfect.

356M Proposition If U is a Banach lattice in which the norm is order-continuous and has the Levi property, then U is perfect.

proof By 356Db, $U^* = U^{\times}$; since U^* surely separates the points of U, so does $U^{\times}$. By 354Ee, U is Dedekind complete. If $A \subseteq U$ is non-empty and upwards-directed and $f[A]$ is bounded for every $f \in U^{\times}$, then A is norm-bounded, by the Uniform Boundedness Theorem (3A5Hb). Because the norm is supposed to have the Levi property, A is bounded above in U. Thus U satisfies all the conditions of 356K and is perfect.

356N *L*- and *M*-spaces I come now to the duality between L-spaces and M-spaces which I hinted at in §354.

Proposition Let U be an Archimedean Riesz space with an order-unit norm.
(a) $U^* = U^{\sim}$ is an L-space.
(b) If e is the standard order unit of U, then $\|f\| = |f|(e)$ for every $f \in U^*$.
(c) A linear functional $f : U \to \mathbb{R}$ is positive iff it belongs to U^* and $\|f\| = f(e)$.
(d) If $e \neq 0$ there is a positive linear functional f on U such that $f(e) = 1$.

proof (a)-(b) We know already that $U^* \subseteq U^{\sim}$ is a Banach lattice (356Da). If $f \in U^{\sim}$ then

$$\sup\{|f(u)| : \|u\| \leq 1\} = \sup\{|f(u)| : |u| \leq e\} = |f|(e),$$

so $f \in U^*$ and $\|f\| = |f|(e)$; thus $U^{\sim} = U^*$. If $f,\ g \geq 0$ in U^*, then

$$\|f + g\| = (f + g)(e) = f(e) + g(e) = \|f\| + \|g\|;$$

thus U^* is an L-space.

(c) As already remarked, if f is positive then $f \in U^*$ and $\|f\| = f(e)$. On the other hand, if $f \in U^*$ and $\|f\| = f(e)$, take any $u \geq 0$. Set $v = (1 + \|u\|)^{-1}u$. Then $0 \leq v \leq e$ and $\|e - v\| \leq 1$ and

$$f(e-v) \le |f(e-v)| \le \|f\| = f(e).$$

But this means that $f(v) \ge 0$ so $f(u) \ge 0$. As u is arbitrary, $f \ge 0$.

(d) By the Hahn-Banach theorem (3A5Ac), there is an $f \in U^*$ such that $f(e) = \|f\| = 1$; by (c), f is positive.

356O Theorem Let U be an Archimedean Riesz space with order-unit norm. Then a set $A \subseteq U^* = U^\sim$ is uniformly integrable iff it is norm-bounded and $\lim_{n\to\infty} \sup_{f\in A} |f(u_n)| = 0$ for every order-bounded disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U^+.

proof (a) Suppose that A is uniformly integrable. Then it is surely norm-bounded (354Ra). If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in U^+ bounded above by w, then for any $\epsilon > 0$ we can find an $h \ge 0$ in U^* such that $\|(|f|-h)^+\| \le \epsilon$ for every $f \in A$. Now $\sum_{i=0}^n h(u_i) \le h(w)$ for every n, and $\lim_{n\to 0} h(u_n) = 0$; since at the same time

$$|f(u_n)| \le |f|(u_n) \le h(u_n) + (|f|-h)^+(u_n) \le h(u_n) + \epsilon\|u_n\| \le h(u_n) + \epsilon\|w\|$$

for every $f \in A$ and $n \in \mathbb{N}$, $\limsup_{n\to\infty} \sup_{f\in A} |f|(u_n) \le \epsilon\|w\|$. As ϵ is arbitrary,

$$\lim_{n\to\infty} \sup_{f\in A} |f|(u_n) = 0,$$

and the conditions are satisfied.

(b)(i) Now suppose that A is norm-bounded but not uniformly integrable. Write B for the solid hull of A, M for $\sup_{f\in A} \|f\| = \sup_{f\in B} \|f\|$; then there is a disjoint sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in $B \cap (U^*)^+$ which is not norm-convergent to 0 (354R(b-iv)), that is,

$$\delta = \tfrac{1}{2} \limsup_{n\to\infty} g_n(e) = \tfrac{1}{2} \limsup_{n\to\infty} \|g_n\| > 0,$$

where e is the standard order unit of U.

(ii) Set

$$C = \{v : 0 \le v \le e,\ \sup_{g\in B} g(v) \ge \delta\},$$

$$D = \{w : 0 \le w \le e,\ \limsup_{n\to\infty} g_n(w) > \delta\}.$$

Then for any $u \in D$ we can find $v \in C$ and $w \in D$ such that $v \wedge w = 0$. **P** Set $\delta' = \limsup_{n\to\infty} g_n(u)$, $\eta = (\delta' - \delta)/(3 + M) > 0$; take $k \in \mathbb{N}$ so large that $k\eta \ge M$.

Because $g_n(u) \ge \delta' - \eta$ for infinitely many n, we can find a set $K \subseteq \mathbb{N}$, with k members, such that $g_i(u) \ge \delta' - \eta$ for every $i \in K$. Now we know that, for each $i \in K$, $g_i \wedge k\sum_{j\in K, j\ne i} g_j = 0$, so there is a $v_i \le u$ such that $g_i(u - v_i) + k\sum_{j\in K, j\ne i} g_j(v_i) \le \eta$ (355Ec). Now

$$g_i(v_i) \ge g_i(u) - \eta \ge \delta' - 2\eta, \quad g_i(v_j) \le \frac{\eta}{k} \text{ for } i,\, j \in K,\ i \ne j.$$

Set $v'_i = (v_i - \sum_{j\in K, j\ne i} v_j)^+$ for each $i \in K$; then

$$g_i(v'_i) \ge g_i(v_i) - \textstyle\sum_{j\in K, j\ne i} g_i(v_j) \ge \delta' - 3\eta$$

for every $i \in K$, while $v'_j \wedge v'_i = 0$ for distinct i, $j \in K$.

For each $n \in \mathbb{N}$,

$$\textstyle\sum_{i\in K} g_n(u \wedge \frac{1}{\eta} v'_i) \le g_n(u) \le \|g_n\| \le \eta k,$$

so there is some $i(n) \in K$ such that

$$g_n(u \wedge \frac{1}{\eta} v'_{i(n)}) \le \eta, \quad g_n(u - \frac{1}{\eta} v'_{i(n)})^+ \ge g_n(u) - \eta.$$

Since $\{n : g_n(u) \ge \delta + 2\eta\}$ is infinite, there is some $m \in K$ such that $J = \{n : g_n(u) \ge \delta + 2\eta,\ i(n) = m\}$ is infinite. Try

$$v = (v'_m - \eta u)^+, \quad w = (u - \frac{1}{\eta} v'_m)^+.$$

Then v, $w \in [0,u]$ and $v \wedge w = 0$. Next,

$$g_m(v) \geq g_m(v'_m) - \eta M \geq \delta' - 3\eta - \eta M = \delta,$$

so $v \in C$, while for any $n \in J$

$$g_n(w) = g_n(u - \frac{1}{\eta} v'_{i(n)})^+ \geq g_n(u) - \eta \geq \delta + \eta;$$

since J is infinite,

$$\limsup_{n\to\infty} g_n(w) \geq \delta + \eta > \delta$$

and $w \in D$. **Q**

(iii) Since $e \in D$, we can choose inductively sequences $\langle w_n\rangle_{n\in\mathbb{N}}$ in D, $\langle v_n\rangle_{n\in\mathbb{N}}$ in C such that $w_0 = e$, $v_n \wedge w_{n+1} = 0$, $v_n \vee w_{n+1} \leq w_n$ for every $n \in \mathbb{N}$. But in this case $\langle v_n\rangle_{n\in\mathbb{N}}$ is a disjoint order-bounded sequence in $[0,u]$, while for each $n \in \mathbb{N}$, we can find $f_n \in A$ such that $|f_n|(v_n) > \frac{2}{3}\delta$. Now there is a $u_n \in [0,v_n]$ such that $|f_n(u_n)| \geq \frac{1}{3}\delta$. **P** Set $\gamma = \sup_{0\leq v\leq v_n} |f_n(v)|$. Then $f_n^+(v_n)$, $f_n^-(v_n)$ are both less than or equal to γ, so $|f_n|(v_n) \leq 2\gamma$ and $\gamma > \frac{1}{3}\delta$; so there is a $u_n \in [0,v_n]$ such that $|f_n(u_n)| \geq \frac{1}{3}\delta$. **Q**

Accordingly we have a disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $[0,e]$ such that $\sup_{f\in A} |f(u_n)| \geq \frac{1}{3}\delta$ for every $n \in \mathbb{N}$.

(iv) All this is on the assumption that A is norm-bounded and not uniformly integrable. So, turning it round, we see that if A is norm-bounded and $\lim_{n\to\infty} \sup_{f\in A} |f(u_n)| = 0$ for every order-bounded disjoint sequence $\langle u_n\rangle_{n\in\mathbb{N}}$, A must be uniformly integrable.

This completes the proof.

356P Proposition Let U be an L-space.

(a) U is perfect.

(b) $U^* = U^\sim = U^\times$ is an M-space; its standard order unit is the functional $\int$ defined by setting $\int u = \|u^+\| - \|u^-\|$ for every $u \in U$.

(c) If $A \subseteq U$ is non-empty and upwards-directed and $\sup_{u\in A} \int u$ is finite, then $\sup A$ is defined in U and $\int \sup A = \sup_{u\in A} \int u$.

proof (a) By 354N we know that the norm on U is order-continuous and has the Levi property, so 356M tells us that U is perfect.

(b) 356Dd tells us that $U^* = U^\sim = U^\times$.

The L-space property tells us that the functional $u \mapsto \|u\| : U^+ \to \mathbb{R}$ is additive; of course it is also homogeneous, so by 355D it has an extension to a linear functional $\int : U \to \mathbb{R}$ satisfying the given formula. Because $\int u = \|u\| \geq 0$ for $u \geq 0$, $\int \in (U^\sim)^+$. For $f \in U^\sim$,

$$\begin{aligned} |f| \leq \int &\iff |f|(u) \leq \int u \text{ for every } u \in U^+ \\ &\iff |f(v)| \leq \|u\| \text{ whenever } |v| \leq u \in U \\ &\iff |f(v)| \leq \|v\| \text{ for every } v \in U \\ &\iff \|f\| \leq 1, \end{aligned}$$

so the norm on $U^* = U^\sim$ is the order-unit norm defined from $\int$, and $U^\sim$ is an M-space, as claimed.

(c) Fix $u_0 \in A$, and set $B = \{u^+ : u \in A,\, u \geq u_0\}$. Then $B \subseteq U^+$ is upwards-directed, and

$$\begin{aligned} \sup_{v\in B} \|v\| = \sup_{u\in A, u\geq u_0} \int u^+ &= \sup_{u\in A, u\geq u_0} \int u + \int u^- \\ &\leq \sup_{u\in A, u\geq u_0} \int u + \int u_0^- < \infty. \end{aligned}$$

Because $\| \|$ has the Levi property, B is bounded above. But (because A is upwards-directed) every member of A is dominated by some member of B, so A also is bounded above. Because U is Dedekind complete, $\sup A$ is defined in U. Finally, $\int \sup A = \sup_{u\in A} \int u$ because $\int$, being a positive member of $U^\times$, is order-continuous.

356Q Theorem Let U be any L-space. Then a subset of U is uniformly integrable iff it is relatively weakly compact.

proof (a) Let $A \subseteq U$ be a uniformly integrable set.

(i) Suppose that $\mathcal{F}$ is an ultrafilter on X containing A. Then $A \neq \emptyset$. Because A is norm-bounded, $\sup_{u\in A}|f(u)| < \infty$ and $\phi(f) = \lim_{u\to\mathcal{F}} f(u)$ is defined in $\mathbb{R}$ for every $f \in U^*$ (2A3Se).

If f, $g \in U^*$ then

$$\phi(f+g) = \lim_{u\to\mathcal{F}} f(u) + g(u) = \lim_{u\to\mathcal{F}} f(u) + \lim_{u\to\mathcal{F}} g(u) = \phi(f) + \phi(g)$$

(2A3Sf). Similarly,

$$\phi(\alpha f) = \lim_{u\to\mathcal{F}} \alpha f(u) = \alpha\phi(f)$$

whenever $f \in U^*$ and $\alpha \in \mathbb{R}$. Thus $\phi : U^* \to \mathbb{R}$ is linear. Also

$$|\phi(f)| \le \sup_{u\in A}|f(u)| \le \|f\| \sup_{u\in A}\|u\|,$$

so $\phi \in U^{**} = U^{*\sim}$.

(ii) Now the point of this argument is that $\phi \in U^{*\times}$. **P** Suppose that $B \subseteq U^*$ is non-empty and downwards-directed and has infimum 0. Fix $f_0 \in B$. Let $\epsilon > 0$. Then there is a $w \in U^+$ such that $\|(|u| - w)^+\| \le \epsilon$ for every $u \in A$, which means that

$$|f(u)| \le |f|(|u|) \le |f|(w) + |f|(|u| - w)^+ \le |f|(w) + \epsilon\|f\|$$

for every $f \in U^*$ and every $u \in A$. Accordingly $|\phi(f)| \le |f|(w)+\epsilon\|f\|$ for every $f \in U^*$. Now $\inf_{f\in B} f(w) = 0$ (using 355Ee, as usual), so there is an $f_1 \in B$ such that $f_1 \le f_0$ and $f_1(w) \le \epsilon$. In this case

$$|\phi|(f_1) = \sup_{|f|\le f_1}|\phi(f)| \le \sup_{|f|\le f_1}|f|(w) + \epsilon\|f\| \le f_1(w) + \epsilon\|f_1\| \le \epsilon(1 + \|f_0\|).$$

As ϵ is arbitrary, $\inf_{f\in B}|\phi|(f) = 0$; as B is arbitrary, $|\phi|$ is order-continuous and $\phi \in U^{*\times}$. **Q**

(iii) At this point, we recall that $U^* = U^\times$ and that the canonical map from U to $U^{\times\times}$ is surjective (356P). So there is a $u_0 \in U$ such that $\hat{u}_0 = \phi$. But now we see that

$$f(u_0) = \phi(f) = \lim_{u\to\mathcal{F}} f(u)$$

for every $f \in U^*$; which is just what is meant by saying that $\mathcal{F} \to u_0$ for the weak topology on U (2A3Sd).

Accordingly every ultrafilter on U containing A has a limit in U. But because the weak topology on U is regular (3A3Be), it follows that the closure of A for the weak topology is compact (3A3De), so that A is relatively weakly compact.

(b) For the converse I use the criterion of 354R(b-iv). Suppose that $A \subseteq U$ is relatively weakly compact. Then A is norm-bounded, by the Uniform Boundedness Theorem. Now let $\langle u_n\rangle_{n\in\mathbb{N}}$ be any disjoint sequence in the solid hull of A. For each n, let U_n be the band in U generated by u_n. Let P_n be the band projection from U onto U_n (353Ib). Let $v_n \in A$ be such that $|u_n| \le |v_n|$; then

$$|u_n| = P_n|u_n| \le P_n|v_n| = |P_n v_n|,$$

so $\|u_n\| \le \|P_n v_n\|$ for each n. Let $g_n \in U^*$ be such that $\|g_n\| = 1$ and $g_n(P_n v_n) = \|P_n v_n\|$.

Define $T : U \to \mathbb{R}^{\mathbb{N}}$ by setting $Tu = \langle g_n(P_n u)\rangle_{n\in\mathbb{N}}$ for each $u \in U$. Then T is a continuous linear operator from U to ℓ^1. **P** For $m \neq n$, $U_m \cap U_n = \{0\}$, because $|u_m| \wedge |u_n| = 0$. So, for any $u \in U$, $\langle P_n u\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in U, and

$$\sum_{i=0}^n \|P_i u\| = \|\sum_{i=0}^n |P_i u|\| = \|\sup_{i\le n}|P_i u|\| \le \|u\|$$

for every n; accordingly

$$\|Tu\|_1 = \sum_{i=0}^\infty |g_i P_i u| \le \sum_{i=0}^\infty \|P_i u\| \le \|u\|.$$

Since T is certainly a linear operator (because every coordinate functional $g_i P_i$ is linear), we have the result. **Q**

Consequently $T[A]$ is relatively weakly compact in ℓ^1, because T is continuous for the weak topologies (2A5If). But ℓ^1 can be identified with $L^1(\mu)$, where μ is counting measure on $\mathbb{N}$. So $T[A]$ is uniformly integrable in ℓ^1, by 247C, and in particular $\lim_{n\to\infty}\sup_{w\in T[A]}|w(n)| = 0$. But this means that

$$\lim_{n\to\infty}\|u_n\|\le\lim_{n\to\infty}|g_n(P_nv_n)|=\lim_{n\to\infty}|(Tv_n)(n)|=0.$$

As $\langle u_n\rangle_{n\in\mathbb{N}}$ is arbitrary, A satisfies the conditions of 354R(b-iv) and is uniformly integrable.

356X Basic exercises (a) Show that if $U=\ell^\infty$ then $U^\times=U_c^\sim$ can be identified with ℓ^1, and is properly included in $U^\sim$. (*Hint*: show that if $f\in U_c^\sim$ then $f(u)=\sum_{n=0}^\infty u(n)f(e_n)$, where $e_n(n)=1$, $e_n(i)=0$ for $i\ne n$.)

(b) Show that if $U=C([0,1])$ then $U^\times=U_c^\sim=\{0\}$. (*Hint*: show that if $f\in(U_c^\sim)^+$ and $\langle q_n\rangle_{n\in\mathbb{N}}$ enumerates $\mathbb{Q}\cap[0,1]$, then for each $n\in\mathbb{N}$ there is a $u_n\in U^+$ such that $u_n(q_n)=1$ and $f(u_n)\le2^{-n}$.)

(c) Let X be an uncountable set, μ the countable-cocountable measure on X and Σ its domain (211R). Let U be the space of bounded Σ-measurable real-valued functions on X. Show that U is a Dedekind σ-complete Banach lattice if given the supremum norm $\|\,\|_\infty$. Show that $U^\times$ can be identified with $\ell^1(X)$ (cf. 356Xa), and that $u\mapsto\int u\,d\mu$ belongs to $U_c^\sim\setminus U^\times$.

(d) Let U be a Dedekind σ-complete Riesz space and $f\in U_c^\sim$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be an order-bounded sequence in U which is order-convergent to $u\in U$ in the sense that $u=\inf_{n\in\mathbb{N}}\sup_{m\ge n}u_m=\sup_{n\in\mathbb{N}}\inf_{m\ge n}u_m$. Show that $\lim_{n\to\infty}f(u_n)$ exists and is equal to $f(u)$.

(e) Let U be any Riesz space. Show that the band projection $P:U^\sim\to U^\times$ is defined by the formula

$$(Pf)(u)=\inf\{\sup_{v\in A}f(v):A\subseteq U\text{ is non-empty, upwards-directed and has supremum }u\}$$

for every $f\in(U^\sim)^+$, $u\in U^+$. (*Hint*: show that the formula for Pf always defines an order-continuous linear functional. Compare 355Yh, 356Yb and 362Bd.)

(f) Let U be any Riesz space. Show that the band projection $P:U^\sim\to U_c^\sim$ is defined by the formula

$$(Pf)(u)=\inf\{\sup_{n\in\mathbb{N}}f(v_n):\langle v_n\rangle_{n\in\mathbb{N}}\text{ is a non-decreasing sequence with supremum }u\}$$

for every $f\in(U^\sim)^+$, $u\in U^+$.

(g) Let U be a Riesz space with a Riesz norm. Show that U^* is perfect.

(h) Let U be a Riesz space with a Riesz norm. Show that the canonical map from U to U^{**} is a Riesz homomorphism.

(i) Let V be a perfect Riesz space and U any Riesz space. Show that $\mathcal{L}^\sim(U;V)$ is perfect. (*Hint*: show that if $u\in U$ and $g\in V^\times$ then $T\mapsto g(Tu)$ belongs to $\mathcal{L}^\sim(U;V)^\times$.)

(j) Let U be an M-space. Show that it is perfect iff it is Dedekind complete and $U^\times$ separates the points of U.

(k) Let U be a Banach lattice which, as a Riesz space, is perfect. Show that its norm has the Levi property.

(l) Write out a proof from first principles that if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in ℓ^1 such that $|u_n(n)|\ge\delta>0$ for every $n\in\mathbb{N}$, then $\{u_n:n\in\mathbb{N}\}$ is not relatively weakly compact.

(m) Let U be an L-space and $A\subseteq U$ a non-empty set. Show that the following are equiveridical: (i) A is uniformly integrable (ii) $\inf_{f\in B}\sup_{u\in A}|f(u)|$ for every non-empty downwards-directed set $B\subseteq U^\times$ with infimum 0 (iii) $\inf_{n\in\mathbb{N}}\sup_{u\in A}|f_n(u)|=0$ for every non-increasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $U^\times$ with infimum 0 (iv) A is norm-bounded and $\lim_{n\to\infty}\sup_{u\in A}|f_n(u)|=0$ for every disjoint order-bounded sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $U^\times$.

356Y Further exercises (a) Let U be a Riesz space with the countable sup property. Show that $U^\times = U^\sim_c$.

(b) Let U be a Riesz space, and $\mathcal{A}$ a family of non-empty downwards-directed subsets of U^+ all with infimum 0. (i) Show that $U^\sim_{\mathcal{A}} = \{f : f \in U^\sim, \inf_{u \in A} |f|(u) = 0$ for every $A \in \mathcal{A}\}$ is a band in $U^\sim$. (ii) Set $\mathcal{A}^* = \{A_0 + \ldots + A_n : A_0, \ldots , A_n \in \mathcal{A}\}$. Show that $U^\sim_{\mathcal{A}} = U^\sim_{\mathcal{A}^*}$. (iii) Take any $f \in (U^\sim)^+$, and let g, h be the components of f in $U^\sim_{\mathcal{A}}$, $(U^\sim_{\mathcal{A}})^\perp$ respectively. Show that

$$g(u) = \inf_{A \in \mathcal{A}^*} \sup_{v \in A} f(u - v)^+, \quad h(u) = \sup_{A \in \mathcal{A}^*} \inf_{v \in A} f(u \wedge v)$$

for every $u \in U^+$. (Cf. 362Xi.)

(c) Let U be a Riesz space. For any band $V \subseteq U$ write V° for $\{f : f \in U^\times, f(v) \le 1$ for every $v \in V\} = \{f : f \in U^\times, f(v) = 0$ for every $v \in V\}$. Show that $V \mapsto (V^\perp)^\circ$ is a surjective order-continuous Boolean homomorphism from the algebra of complemented bands of U onto the band algebra of $U^\times$, and that it is injective iff $U^\times$ separates the points of U.

(d) Let U be a Dedekind complete Riesz space such that $U^\times$ separates the points of U and U is the solid linear subspace of itself generated by a countable set. Show that U is perfect.

(e) Let U be an L-space and $\langle u_n \rangle_{n \in \mathbb{N}}$ a sequence in U such that $\langle f(u_n) \rangle_{n \in \mathbb{N}}$ is Cauchy for every $f \in U^*$. Show that $\langle u_n \rangle_{n \in \mathbb{N}}$ is convergent for the weak topology of U. (*Hint*: use 356Xm(iv) to show that $\{u_n : n \in \mathbb{N}\}$ is relatively weakly compact.)

(f) Let U be a perfect Banach lattice with order-continuous norm and $\langle u_n \rangle_{n \in \mathbb{N}}$ a sequence in U such that $\langle f(u_n) \rangle_{n \in \mathbb{N}}$ is Cauchy for every $f \in U^*$. Show that $\langle u_n \rangle_{n \in \mathbb{N}}$ is convergent for the weak topology of U. (*Hint*: set $\phi(f) = \lim_{n \to \infty} f_n(u)$. For any $g \in (U^*)^+$ let V_g be the solid linear subspace of U^* generated by g, $W_g = \{u : g(|u|) = 0\}^\perp$, $\|u\|_g = g(|u|)$ for $u \in W_g$. Show that the completion of W_g under $\| \; \|_g$ is an L-space with dual isomorphic to V_g, and hence (using 356Ye) that $\phi \restriction V_g$ belongs to $V_g^\times$; as g is arbitrary, $\phi \in V^\times$ and may be identified with an element of U.)

(g) Let U be a uniformly complete Archimedean Riesz space with complexification V (354Yl). (i) Show that the complexification of $U^\sim$ can be identified with the space of linear functionals $f : V \to \mathbb{C}$ such that $\sup_{|v| \le u} |f(v)|$ is finite for every $u \in U^+$. (ii) Show that if U is a Banach lattice, then the complexification of $U^\sim = U^*$ can be identified (as normed space) with V^*. (See 355Yk.)

(h) Let U be a perfect Banach lattice. Show that the family of closed balls in U is a compact class. (*Hint*: 342Ya.)

356 Notes and comments The section starts easily enough, with special cases of results in §355 (356B). When U has a Riesz norm, the identification of U^* as a subspace of $U^\sim$, and the characterization of order-continuous norms (356D) are pleasingly comprehensive and straightforward. Coming to biduals, we need to think a little (356F), but there is still no real difficulty at first. In 356H-356I, however, something more substantial is happening. I have written these arguments out in what seems to be the shortest route to the main theorem, at the cost perhaps of neglecting any intuitive foundation. What I think we are really doing is matching bands in U, $U^\times$ and $U^{\times\times}$, as in 356Yc.

From now on, almost the first thing we shall ask of any new Riesz space will be whether it is perfect, and if not, which of the three conditions of 356K it fails to satisfy. For reasons which will I hope appear in the next chapter, perfect Riesz spaces are especially important in measure theory; in particular, all L^p spaces for $p \in [1, \infty[$ are perfect (366Dd), as are the L^∞ spaces of localizable measure spaces (365M). Further examples will be discussed in §369 and §374. Of course we have to remember that there are also important Riesz spaces which are not perfect, of which $C([0, 1])$ and $\boldsymbol{c}_0$ are two of the simplest examples.

The duality between L- and M-spaces (356N, 356P) is natural and satisfying. We are now in a position to make a determined attempt to tidy up the notion of 'uniform integrability'. I give two major theorems. The first is yet another 'disjoint-sequence' characterization of uniformly integrable sets, to go with 246G and 354R. The essential difference here is that we are looking at disjoint sequences in a predual; in a sense, this means that the result is a sharper one, because the M-space U need not be Dedekind complete (for instance, it could be $C([0,1])$ – this indeed is the archetype for applications of the theorem) and therefore need not have as many disjoint sequences as its dual. (For instance, in the dual of $C([0,1])$ we have all the point masses δ_t, where $\delta_t(u) = u(t)$; these form a disjoint family in $C([0,1])^\sim$ not corresponding to any disjoint family in $C([0,1])$.) The essence of the proof is a device to extract a disjoint sequence in U to match approximately a subsequence of a given disjoint sequence in $U^\sim$. In the example just suggested, this would correspond, given a sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ of distinct points in $[0,1]$, to finding a subsequence $\langle t_{n(i)}\rangle_{i\in\mathbb{N}}$ which is discrete, so that we can find disjoint $u_i \in C([0,1])$ with $u_i(t_{n(i)}) = 1$ for each i.

The second theorem, 356Q, is a new version of a result already given in §247: in any L-space, uniform integrability is the same as relative weak compactness. I hope you are not exasperated by having been asked, in Volume 2, to master a complex argument (one of the more difficult sections of that volume) which was going to be superseded. Actually it is worse than that. A theorem of Kakutani (369E) tells us that every L-space is isomorphic to an L^1 space. So 356Q is itself a consequence of 247C. I do at least owe you an explanation for writing out two proofs. The first point is that the result is sufficiently important for it to be well worth while spending time in its neighbourhood, and the contrasts and similarities between the two arguments are instructive. The second is that the proof I have just given was not really accessible at the level of Volume 2. It does not rely on every single page of this chapter, but the key idea (that U is isomorphic to $U^{\times\times}$, so it will be enough if we can show that A is relatively compact in $U^{\times\times}$) depends essentially on 356I, which lies pretty deep in the abstract theory of Riesz spaces. The third is an aesthetic one: a theorem about L-spaces ought to be proved in the category of normed Riesz spaces, without calling on a large body of theory outside. Of course this is a book on measure theory, so I did the measure theory first, but if you look at everything that went into it, the proof in §247 is I believe longer, in the formal sense, than the one here, even setting aside the labour of proving Kakutani's theorem.

Let us examine the ideas in the two proofs. First, concerning the proof that uniformly integrable sets are relatively compact, the method here is very smooth and natural; the definition I chose of 'uniform integrability' is exactly adapted to showing that uniformly integrable sets are relatively compact in the order-continuous bidual; all the effort goes into the proof that L-spaces are perfect. The previous argument depended on identifying the dual of L^1 as L^∞ – and was disagreeably complicated by the fact that the identification is not always valid, so that I needed to reduce the problem to the σ-finite case (part (b-ii) of the proof of 247C). After that, the Radon-Nikodým theorem did the trick. Actually Kakutani's theorem shows that the side-step to σ-finite spaces is irrelevant. It directly represents an abstract L-space as $L^1(\mu)$ for a localizable measure μ, in which case $(L^1)^* \cong L^\infty$ exactly.

In the other direction, both arguments depend on a disjoint-sequence criterion for uniform integrability (246G(iii) or 354R(b-iv)). These criteria belong to the 'easy' side of the topic; straightforward Riesz space arguments do the job, whether written out in that language or not. (Of course the new one in this section, 356O, lies a little deeper.) I go a bit faster this time because I feel that you ought by now to be happy with the Hahn-Banach theorem and the Uniform Boundedness Theorem, which I was avoiding in Volume 2. And then of course I quote the result for ℓ^1. This looks like cheating. But ℓ^1 really is easier, as you will find if you just write out part (a) of the proof of 247C for this case. It is not exactly that you can dispense with any particular element of the argument; rather it is that the formulae become much more direct when you can write $u(i)$ in place of $\int_{F_i} u$, and 'cluster points for the weak topology' become pointwise limits of subsequences, so that the key step (the 'sliding hump', in which $u_{k(j)}(n(k(j)))$ is the only significant coordinate of $u_{k(j)}$), is easier to find.

We now have a wide enough variety of conditions equivalent to uniform integrability for it to be easy to find others; I give a couple in 356Xm, corresponding in a way to those in 246G. You may have noticed, in the proof of 247C, that in fact the full strength of the hypothesis 'relatively weakly compact' is never used; all that is demanded is that a couple of sequences should have cluster points for the weak topology. So we see that a set A is uniformly integrable iff every sequence in A has a weak cluster point. But this extra refinement is nothing to do with L-spaces; it is generally true, in any normed space U, that a set $A \subseteq U$ is relatively weakly compact iff every sequence in A has a cluster point in U for the weak topology ('Eberlein's theorem'; see 462D in Volume 4, Köthe 69, 24.2.1, or Dunford & Schwartz 57, V.6.1).

There is a very rich theory concerning weak compactness in perfect Riesz spaces, based on the ideas here; some of it is explored in Fremlin 74a. As a sample, I give one of the basic properties of perfect Banach lattices with order-continuous norms: they are 'weakly sequentially complete' (356Yf).

Version of 16.2.17

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

353H Principal bands This proposition, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 353I.

353I Projection bands This proposition, referred to in the 2003, 2006 and 2013 printings of Volume 4 and the 2008 and 2015 printings of Volume 5, is now 353J.

353K Solid linear subspaces This proposition, referred to in the 2008 and 2015 printings of Volume 5, is now 353L.

353M Riesz spaces with order units This theorem, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 353N.

353P f-algebras with identity This proposition, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 353Q.

354Yk Complexifications of normed Riesz spaces This exercise, referred to in the 2003 edition of Volume 4, is now 354Yl.

Version of 25.8.17

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