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Chapter 34

The lifting theorem

Whenever we have a surjective homomorphism $\phi : P \to Q$, where P and Q are mathematical structures, we can ask whether there is a right inverse of ϕ, a homomorphism $\psi : Q \to P$ such that $\phi\psi$ is the identity on Q. As a general rule, we expect a negative answer; those categories in which epimorphisms always have right inverses (e.g., the category of linear spaces) are rather special, and elsewhere the phenomenon is relatively rare and almost always important. So it is notable that we have a case of this at the very heart of the theory of measure algebras: for any complete probability space (X, Σ, μ) (in fact, for any complete strictly localizable space of non-zero measure) the canonical homomorphism from Σ to the measure algebra of μ has a right inverse (341K). This is the von Neumann-Maharam lifting theorem. Its proof, together with some essentially elementary remarks, takes up the whole of of §341.

As a first application of the theorem (there will be others in Volume 4) I apply it to one of the central problems of measure theory: under what circumstances will a homomorphism between measure algebras be representable by a function between measure spaces? Variations on this question are addressed in §343. For a reasonably large proportion of the measure spaces arising naturally in analysis, homomorphisms are representable (343B). New difficulties arise if we ask for isomorphisms of measure algebras to be representable by isomorphisms of measure spaces, and here we have to work rather hard for rather narrowly applicable results; but in the case of Lebesgue measure and its closest relatives, a good deal can be done, as in 344I-344K.

Returning to liftings, there are many difficult questions concerning the extent to which liftings can be required to have special properties, reflecting the natural symmetries of the standard measure spaces. For instance, Lebesgue measure is translation-invariant; if liftings were in any sense canonical, they could be expected to be automatically translation-invariant in some sense. It seems sure that there is no canonical lifting for Lebesgue measure – all constructions of liftings involve radical use of the axiom of choice – but even so we do have many translation-invariant liftings (§345). We have less luck with product spaces; here the construction of liftings which respect the product structure is fraught with difficulties. I give the currently known results in §346.

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341 The lifting theorem

I embark directly on the principal theorem of this chapter (341K, 'every non-trivial complete strictly localizable measure space has a lifting'), using the minimum of advance preparation. 341A-341B give the definition of 'lifting'; the main argument is in 341F-341K, using the concept of 'lower density' (341C-341E) and a theorem on martingales from §275. In 341P I describe an alternative way of thinking about liftings in terms of the Stone space of the measure algebra.

341A Definition Let (X, Σ, μ) be a measure space, and $\mathfrak{A}$ its measure algebra. By a **lifting** for $\mathfrak{A}$ (or for (X, Σ, μ), or for μ) I shall mean

either a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$

or a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that (i) $\phi E = \emptyset$ whenever $\mu E = 0$ (ii) $\mu(E \triangle \phi E) = 0$ for every $E \in \Sigma$.

341C Definition Let (X, Σ, μ) be a measure space, and $\mathfrak{A}$ its measure algebra. By a **lower density** for $\mathfrak{A}$ (or for (X, Σ, μ), or for μ) I shall mean

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either a function $\underline{\theta} : \mathfrak{A} \to \Sigma$ such that (i) $(\underline{\theta}a)^\bullet = a$ for every $a \in \mathfrak{A}$ (ii) $\underline{\theta}0 = \emptyset$ (iii) $\underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b$ for all a, $b \in \mathfrak{A}$

or a function $\underline{\phi} : \Sigma \to \Sigma$ such that (i) $\underline{\phi}E = \underline{\phi}F$ whenever E, $F \in \Sigma$ and $\mu(E \triangle F) = 0$ (ii) $\mu(E \triangle \underline{\phi}E) = 0$ for every $E \in \Sigma$ (iii) $\underline{\phi}\emptyset = \emptyset$ (iv) $\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all E, $F \in \Sigma$.

341D Remarks (a) As in 341B, there is a natural one-to-one correspondence between lower densities $\underline{\theta} : \mathfrak{A} \to \Sigma$ and lower densities $\underline{\phi} : \Sigma \to \Sigma$ given by the formula

$$\underline{\theta}E^\bullet = \underline{\phi}E \text{ for every } E \in \Sigma.$$

(c) If (X, Σ, μ) is a measure space with measure algebra $\mathfrak{A}$, a **partial lower density** of $\mathfrak{A}$ is a function $\underline{\theta} : \mathfrak{B} \to \Sigma$ such that (i) the domain $\mathfrak{B}$ of $\underline{\theta}$ is a subalgebra of $\mathfrak{A}$ (ii) $(\underline{\theta}b)^\bullet = b$ for every $b \in \mathfrak{B}$ (iii) $\underline{\theta}0 = \emptyset$ (iv) $\underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b$ for all a, $b \in \mathfrak{B}$.

Similarly, if T is a subalgebra of Σ, a function $\underline{\phi} : \mathrm{T} \to \Sigma$ is a **partial lower density** if (i) $\underline{\phi}E = \underline{\phi}F$ whenever E, $F \in \mathrm{T}$ and $\mu(E \triangle F) = 0$ (ii) $\mu(E \triangle \underline{\phi}E) = 0$ for every $E \in \mathrm{T}$ (iii) $\underline{\phi}\emptyset = \emptyset$ (iv) $\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all E, $F \in \mathrm{T}$.

341E Example Let μ be Lebesgue measure on $\mathbb{R}^r$, where $r \ge 1$, and Σ its domain. For $E \in \Sigma$ set

$$\mathrm{int}^*E = \{x : x \in \mathbb{R}^r, \lim_{\delta \downarrow 0} \frac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)} = 1\}.$$

(Here $B(x, \delta)$ is the closed ball with centre x and radius δ.) Then int^* is a lower density for μ; we may call it **lower Lebesgue density**.

341F Lemma Let (X, Σ, μ) be a probability space and $\mathfrak{A}$ its measure algebra. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ and $\underline{\theta} : \mathfrak{B} \to \Sigma$ a partial lower density. Then for any $e \in \mathfrak{A}$ there is a partial lower density $\underline{\theta}_1$, extending $\underline{\theta}$, defined on the subalgebra $\mathfrak{B}_1$ of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{e\}$.

341G Lemma Let (X, Σ, μ) be a probability space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Suppose we have a sequence $\langle \underline{\theta}_n \rangle_{n \in \mathbb{N}}$ of partial lower densities such that, for each n, (i) the domain $\mathfrak{B}_n$ of $\underline{\theta}_n$ is a closed subalgebra of $\mathfrak{A}$ (ii) $\mathfrak{B}_n \subseteq \mathfrak{B}_{n+1}$ and $\underline{\theta}_{n+1}$ extends $\underline{\theta}_n$. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n \in \mathbb{N}} \mathfrak{B}_n$. Then there is a partial lower density $\underline{\theta}$, with domain $\mathfrak{B}$, extending every $\underline{\theta}_n$.

341H Theorem Let (X, Σ, μ) be any strictly localizable measure space. Then it has a lower density $\underline{\phi} : \Sigma \to \Sigma$. If $\mu X > 0$ we can take $\underline{\phi}X = X$.

341I Lemma Let (X, Σ, μ) be a complete measure space with measure algebra $\mathfrak{A}$.

(a) Suppose that $\underline{\theta} : \mathfrak{A} \to \Sigma$ is a lower density and $\underline{\theta}_1 : \mathfrak{A} \to \mathcal{P}X$ is a function such that $\underline{\theta}_1 0 = \emptyset$, $\underline{\theta}_1(a \cap b) = \underline{\theta}_1 a \cap \underline{\theta}_1 b$ for all a, $b \in \mathfrak{A}$ and $\underline{\theta}_1 a \supseteq \underline{\theta}a$ for all $a \in \mathfrak{A}$. Then $\underline{\theta}_1$ is a lower density. If $\underline{\theta}_1$ is a Boolean homomorphism, it is a lifting.

(b) Suppose that $\underline{\phi} : \Sigma \to \Sigma$ is a lower density and $\underline{\phi}_1 : \Sigma \to \mathcal{P}X$ is a function such that $\underline{\phi}_1 E = \underline{\phi}_1 F$ whenever $E \triangle F$ is negligible, $\underline{\phi}_1 \emptyset = \emptyset$, $\underline{\phi}_1(E \cap F) = \underline{\phi}_1 E \cap \underline{\phi}_1 F$ for all E, $F \in \Sigma$ and $\underline{\phi}_1 E \supseteq \underline{\phi}E$ for all $E \in \Sigma$. Then $\underline{\phi}_1$ is a lower density. If $\underline{\phi}_1$ is a Boolean homomorphism, it is a lifting.

341J Proposition Let (X, Σ, μ) be a complete measure space such that $\mu X > 0$, and $\mathfrak{A}$ its measure algebra.

(a) If $\underline{\theta} : \mathfrak{A} \to \Sigma$ is any lower density, there is a lifting $\theta : \mathfrak{A} \to \Sigma$ such that $\theta a \supseteq \underline{\theta}a$ for every $a \in \mathfrak{A}$.

(b) If $\underline{\phi} : \Sigma \to \Sigma$ is any lower density, there is a lifting $\phi : \Sigma \to \Sigma$ such that $\phi E \supseteq \underline{\phi}E$ for every $E \in \Sigma$.

341K The Lifting Theorem Every complete strictly localizable measure space of non-zero measure has a lifting.

341M Proposition Let (X, Σ, μ) be a complete locally determined space with $\mu X > 0$. Then it has a lifting iff it has a lower density iff it is strictly localizable.

341N Extension of partial liftings: Proposition Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ.

(a) Any partial lower density $\underline{\phi}_0 : \mathrm{T} \to \Sigma$ has an extension to a lower density $\underline{\phi} : \Sigma \to \Sigma$.

(b) Suppose now that μ is complete. If $\underline{\phi}_0$ is a Boolean homomorphism, it has an extension to a lifting ϕ for μ.

341O Liftings and Stone spaces Suppose that we have the Stone space (Z,T,ν) of a measure algebra $(\mathfrak{A},\bar{\mu})$; I think of Z as being the set of surjective Boolean homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, so that each $a \in \mathfrak{A}$ corresponds to the open-and-closed set $\widehat{a} = \{z : z(a) = 1\}$. Then we have a lifting $\theta : \mathfrak{A} \to \mathrm{T}$ defined by setting $\theta a = \widehat{a}$ for each $a \in \mathfrak{A}$. The corresponding lifting $\phi : \mathrm{T} \to \mathrm{T}$ is defined by taking ϕE to be that unique open-and-closed set such that $E \triangle \phi E$ is negligible.

341P Proposition Let (X,Σ,μ) be a measure space, $(\mathfrak{A},\bar{\mu})$ its measure algebra, and (Z,T,ν) the Stone space of $(\mathfrak{A},\bar{\mu})$ with its canonical measure.

(a) There is a one-to-one correspondence between liftings $\theta : \mathfrak{A} \to \Sigma$ and functions $f : X \to Z$ such that $f^{-1}[\widehat{a}] \in \Sigma$ and $(f^{-1}[\widehat{a}])^\bullet = a$ for every $a \in \mathfrak{A}$, defined by the formula

$$\theta a = f^{-1}[\widehat{a}] \text{ for every } a \in \mathfrak{A}.$$

(b) If (X,Σ,μ) is complete and locally determined, then a function $f : X \to Z$ satisfies the conditions of (a) iff (α) it is inverse-measure-preserving (β) the homomorphism it induces between the measure algebras of μ and ν is the canonical isomorphism defined by the construction of Z.

341Q Corollary Let (X,Σ,μ) be a strictly localizable measure space, $(\mathfrak{A},\bar{\mu})$ its measure algebra, and Z the Stone space of $\mathfrak{A}$; suppose that $\mu X > 0$. For $E \in \Sigma$ write E^* for the open-and-closed subset of Z corresponding to $E^\bullet \in \mathfrak{A}$. Then there is a function $f : X \to Z$ such that $E \triangle f^{-1}[E^*]$ is negligible for every $E \in \Sigma$. If μ is complete, then f is inverse-measure-preserving.

341Z Problems (a) Can we construct, using the ordinary axioms of mathematics, a probability space (X,Σ,μ) with no lifting?

(b) Set $\kappa = \omega_3$. Let $\mathcal{B}\mathfrak{a}_\kappa$ be the Baire σ-algebra of $\{0,1\}^\kappa$, and μ the restriction to $\mathcal{B}\mathfrak{a}_\kappa$ of the usual measure on $\{0,1\}^\kappa$. Can we show that μ has no lifting?

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342 Compact measure spaces

The next three sections amount to an extended parenthesis, showing how the Lifting Theorem can be used to attack one of the fundamental problems of measure theory: the representation of Boolean homomorphisms between measure algebras by functions between appropriate measure spaces. This section prepares for the main idea by introducing the class of 'locally compact' measures (342Ad), with the associated concepts of 'compact' and 'perfect' measures (342Ac, 342K). These depend on the notions of 'inner regularity' (342Aa, 342B) and 'compact class' (342Ab, 342D). I list the basic permanence properties for compact and locally compact measures (342G-342I) and mention some of the compact measures which we have already seen (342J). Concerning perfect measures, I content myself with the proof that a locally compact measure is perfect (342L). I end the section with two examples (342M, 342N).

342A Definitions (a) Let (X,Σ,μ) be a measure space. If $\mathcal{K} \subseteq \mathcal{P}X$, I will say that μ is **inner regular** with respect to $\mathcal{K}$ if

$$\mu E = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$$

for every $E \in \Sigma$.

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(b) A family $\mathcal{K}$ of sets is a **compact class** if $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property.

Note that any subset of a compact class is again a compact class.

(c) A measure space (X, Σ, μ), or a measure μ, is **compact** if μ is inner regular with respect to some compact class of subsets of X.

μ is a compact measure whenever $\mu X = 0$.

(d) A measure space (X, Σ, μ), or a measure μ, is **locally compact** if the subspace measure μ_E is compact whenever $E \in \Sigma$ and $\mu E < \infty$.

342B Lemma Let (X, Σ, μ) be a measure space, and $\mathcal{K} \subseteq \Sigma$ a set such that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. Let $E \in \Sigma$.

(a) There is a countable disjoint set $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $K \subseteq E$ for every $K \in \mathcal{K}_1$ and $\mu(\bigcup \mathcal{K}_1) = \mu E$.

(b) If $\mu E < \infty$ then $\mu(E \setminus \bigcup \mathcal{K}_1) = 0$.

(c) In any case, there is for any $\gamma < \mu E$ a finite disjoint $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $K \subseteq E$ for every $K \in \mathcal{K}_0$ and $\mu(\bigcup \mathcal{K}_0) \geq \gamma$.

342C Corollary Let (X, Σ, μ) be a measure space and $\mathcal{K} \subseteq \mathcal{P}X$ a family of sets such that (α) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ and $K \cap K' = \emptyset$ (β) whenever $E \in \Sigma$ and $\mu E > 0$, there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$. Then μ is inner regular with respect to $\mathcal{K}$.

342D Lemma Let X be a set and $\mathcal{K}$ a family of subsets of X.

(a) The following are equiveridical:

(i) $\mathcal{K}$ is a compact class;

(ii) there is a topology $\mathfrak{T}$ on X such that X is compact and every member of $\mathcal{K}$ is a closed set for $\mathfrak{T}$.

(b) If $\mathcal{K}$ is a compact class, so are the families $\mathcal{K}_1 = \{K_0 \cup \ldots \cup K_n : K_0, \ldots, K_n \in \mathcal{K}\}$ and $\mathcal{K}_2 = \{\bigcap \mathcal{K}' : \emptyset \neq \mathcal{K}' \subseteq \mathcal{K}\}$.

342E Corollary Suppose that (X, Σ, μ) is a measure space and that $\mathcal{K}$ is a compact class such that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$. Then μ is compact.

342F Corollary A measure space (X, Σ, μ) is compact iff there is a topology on X such that X is compact and μ is inner regular with respect to the closed sets.

342G Proposition (a) Any measurable subspace of a compact measure space is compact.

(b) The completion and c.l.d. version of a compact measure space are compact.

(c) A semi-finite measure space is compact iff its completion is compact iff its c.l.d. version is compact.

(d) The direct sum of a family of compact measure spaces is compact.

(e) The c.l.d. product of two compact measure spaces is compact.

(f) The product of any family of compact probability spaces is compact.

342H Proposition (a) A compact measure space is locally compact.

(b) A strictly localizable locally compact measure space is compact.

(c) Let (X, Σ, μ) be a measure space. Suppose that whenever $E \in \Sigma$ and $\mu E > 0$ there is an $F \in \Sigma$ such that $F \subseteq E$, $\mu F > 0$ and the subspace measure on F is compact. Then μ is locally compact.

342I Proposition (a) Any measurable subspace of a locally compact measure space is locally compact.

(b) A measure space is locally compact iff its completion is locally compact iff its c.l.d. version is locally compact.

(c) The direct sum of a family of locally compact measure spaces is locally compact.

(d) The c.l.d. product of two locally compact measure spaces is locally compact.

342J Examples (a) Lebesgue measure on $\mathbb{R}^r$ is compact.

(b) Similarly, any Radon measure on $\mathbb{R}^r$ is compact.

(c) If $(\mathfrak{A},\bar{\mu})$ is any semi-finite measure algebra, the standard measure on its Stone space Z is compact.

(d) The usual measure on $\{0,1\}^I$ is compact, for any set I.

342K Definition Let (X,Σ,μ) be a measure space. Then (X,Σ,μ), or μ, is **perfect** if whenever $f:X\to\mathbb{R}$ is measurable, $E\in\Sigma$ and $\mu E>0$, then there is a compact set $K\subseteq f[E]$ such that $\mu f^{-1}[K]>0$.

342L Theorem A semi-finite locally compact measure space is perfect.

342M Example Let X be an uncountable set and μ the countable-cocountable measure on X. Then μ is perfect but not compact or locally compact.

***342N Example** There is a complete locally determined localizable locally compact measure space which is not compact.

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343 Realization of homomorphisms

We are now in a position to make progress in one of the basic questions of abstract measure theory. In §324 I have already described the way in which a function between two measure spaces can give rise to a homomorphism between their measure algebras. In this section I discuss some conditions under which we can be sure that a homomorphism can be represented by a function.

The principal theorem of the section is 343B. If a measure space (X,Σ,μ) is locally compact, then many homomorphisms from the measure algebra of μ to other measure algebras will be representable by functions into X; moreover, this characterizes locally compact spaces. In general, a homomorphism between measure algebras can be represented by widely different functions (343I, 343J). But in some of the most important cases (e.g., Lebesgue measure) representing functions are 'almost' uniquely defined; I introduce the concept of 'countably separated' measure space to describe these (343D-343H).

343A Preliminary remarks (a) If (X,Σ,μ) and (Y,T,ν) are measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, I will say that a function $f:X\to Y$ **represents** a homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ if $f^{-1}[F]\in\Sigma$ and $(f^{-1}[F])^\bullet=\pi(F^\bullet)$ for every $F\in\mathrm{T}$.

(b) If (X,Σ,μ) and (Y,T,ν) are measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, $f:X\to Y$ is a function, and $\pi:\mathfrak{B}\to\mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then

$$\{F:F\in\mathrm{T},\ f^{-1}[F]\in\Sigma\text{ and }f^{-1}[F]^\bullet=\pi F^\bullet\}$$

is a σ-subalgebra of T.

(c) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, and $\pi:\mathfrak{B}\to\mathfrak{A}$ a Boolean homomorphism which is represented by a function $f:X\to Y$. Let $(X,\hat{\Sigma},\hat{\mu})$, $(Y,\hat{\mathrm{T}},\hat{\nu})$ be the completions of (X,Σ,μ), (Y,T,ν); then $\mathfrak{A}$ and $\mathfrak{B}$ can be identified with the measure algebras of $\hat{\mu}$ and $\hat{\nu}$. Now f still represents π when regarded as a function from $(X,\hat{\Sigma},\hat{\mu})$ to $(Y,\hat{\mathrm{T}},\hat{\nu})$.

343B Theorem Let (X,Σ,μ) be a non-empty semi-finite measure space, and $(\mathfrak{A},\bar{\mu})$ its measure algebra. Let (Z,Λ,λ) be the Stone space of $(\mathfrak{A},\bar{\mu})$; for $E\in\Sigma$ write E^* for the open-and-closed subset of Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Then the following are equiveridical.

(i) (X,Σ,μ) is locally compact.

(ii) There is a family $\mathcal{K}\subseteq\Sigma$ such that (α) whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu K>0$ (β) whenever $\mathcal{K}'\subseteq\mathcal{K}$ is such that $\mu(\bigcap\mathcal{K}_0)>0$ for every non-empty finite set $\mathcal{K}_0\subseteq\mathcal{K}'$, then $\bigcap\mathcal{K}'\neq\emptyset$.

(iii) There is a family $\mathcal{K} \subseteq \Sigma$ such that $(\alpha)'$ μ is inner regular with respect to $\mathcal{K}$ (β) whenever $\mathcal{K}' \subseteq \mathcal{K}$ is such that $\mu(\bigcap \mathcal{K}_0) > 0$ for every non-empty finite set $\mathcal{K}_0 \subseteq \mathcal{K}'$, then $\bigcap \mathcal{K}' \neq \emptyset$.

(iv) There is a function $f : Z \to X$ such that $f^{-1}[E] \triangle E^*$ is negligible for every $E \in \Sigma$.

(v) Whenever (Y, T, ν) is a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, then there is a $g : Y \to X$ representing π.

(vi) Whenever (Y, T, ν) is a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous measure-preserving Boolean homomorphism, then there is a $g : Y \to X$ representing π.

343C Examples (a) Let I be any set. The usual measure ν_I on $\{0,1\}^I$ is compact. If (X, Σ, μ) is any complete probability space such that the measure algebra $\mathfrak{B}_I$ of ν_I can be embedded as a subalgebra of the measure algebra $\mathfrak{A}$ of μ, there is an inverse-measure-preserving function from X to $\{0,1\}^I$.

(b) In particular, if μ is atomless, there is an inverse-measure-preserving function from X to $\{0,1\}^{\mathbb{N}}$; since this is isomorphic to $[0,1]$ with Lebesgue measure, there is an inverse-measure-preserving function from X to $[0,1]$.

(c) More generally, if (X, Σ, μ) is any complete atomless totally finite measure space, there is an inverse-measure-preserving function from X to the interval $[0, \mu X]$ endowed with Lebesgue measure.

(d) In the other direction, if (X, Σ, μ) is a compact probability space with Maharam type at most $\kappa \geq \omega$, then there is an inverse-measure-preserving function from $\{0,1\}^\kappa$ to X.

343D Uniqueness of realizations: Definition A measure space (X, Σ, μ) is **countably separated** if there is a countable set $\mathcal{A} \subseteq \Sigma$ separating the points of X in the sense that for any distinct x, $y \in X$ there is an $E \in \mathcal{A}$ containing one but not the other.

343E Lemma A measure space (X, Σ, μ) is countably separated iff there is an injective measurable function from X to $\mathbb{R}$.

343F Proposition Let (X, Σ, μ) be a countably separated measure space and (Y, T, ν) any measure space. Let f, $g : Y \to X$ be two functions such that $f^{-1}[E]$ and $g^{-1}[E]$ both belong to T, and $f^{-1}[E] \triangle g^{-1}[E]$ is ν-negligible, for every $E \in \Sigma$. Then $f = g$ ν-almost everywhere, and $\{y : y \in Y,\ f(y) \neq g(y)\}$ is measurable as well as negligible.

343G Corollary If, in 343B, (X, Σ, μ) is countably separated, then the functions $g : Y \to X$ of 343B(v)-(vi) are almost uniquely defined in the sense that if f, g both represent the same homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then $f =_{\text{a.e.}} g$.

343H Examples Leading examples of countably separated measure spaces are

(i) $\mathbb{R}$;

(ii) $\{0,1\}^{\mathbb{N}}$;

(iii) subspaces (measurable or not) of countably separated spaces;

(iv) finite products of countably separated spaces;

(v) countable products of countably separated probability spaces;

(vi) completions and c.l.d. versions of countably separated spaces.

343I Example Let $\nu_{\mathfrak{c}}$ be the usual measure on $X = \{0,1\}^{\mathfrak{c}}$, and $\mathrm{T}_{\mathfrak{c}}$ its domain. Then there is a function $f : X \to X$ such that $f(x) \neq x$ for every $x \in X$, but $E \triangle f^{-1}[E]$ is negligible for every $E \in \mathrm{T}_{\mathfrak{c}}$.

343J The split interval (a) Take $I^{\|}$ to consist of two copies of each point of the unit interval, so that $I^{\|} = \{t^+ : t \in [0,1]\} \cup \{t^- : t \in [0,1]\}$. For $A \subseteq I^{\|}$ write $A_l = \{t : t^- \in A\}$, $A_r = \{t : t^+ \in A\}$. Let Σ be the set

$$\{E : E \subseteq I^{\|},\ E_l \text{ and } E_r \text{ are Lebesgue measurable and } E_l \triangle E_r \text{ is Lebesgue negligible}\}.$$

For $E \in \Sigma$, set

$$\mu E = \mu_L E_l = \mu_L E_r$$

where μ_L is Lebesgue measure on $[0,1]$. $(I^{\|}, \Sigma, \mu)$ is a complete probability space. Also it is compact. The sets $\{t^- : t \in [0,1]\}$ and $\{t^+ : t \in [0,1]\}$ are non-measurable subsets of $I^{\|}$; on both of them the subspace measures correspond exactly to μ_L. We have a canonical inverse-measure-preserving function $h : I^{\|} \to [0,1]$ given by setting $h(t^+) = h(t^-) = t$ for every $t \in [0,1]$; h induces an isomorphism between the measure algebras of μ and μ_L.

$I^{\|}$ is called the **split interval**.

Now we have a map $f : I^{\|} \to I^{\|}$ given by setting

$$f(t^+) = t^-,\ f(t^-) = t^+ \text{ for every } t \in [0,1]$$

such that $f(x) \neq x$ for every x, but $E \triangle f^{-1}[E]$ is negligible for every $E \in \Sigma$, so that f represents the identity homomorphism on the measure algebra of μ. The canonical map from the measure algebra of μ to the measure algebra of μ_L is represented equally by the functions $t \mapsto t^-$ and $t \mapsto t^+$, which are nowhere equal.

(b) Consider the direct sum (Y, ν) of $(I^{\|}, \mu)$ and $([0,1], \mu_L)$; take Y to be $(I^{\|} \times \{0\}) \cup ([0,1] \times \{1\})$. Setting

$$h_1(t^+, 0) = h_1(t^-, 0) = (t, 1), \quad h_1(t, 1) = (t^+, 0),$$

$h_1 : Y \to Y$ induces a measure-preserving involution of the measure algebra $\mathfrak{B}$ of ν. But there is no invertible function from Y to itself which induces this involution of $\mathfrak{B}$.

(c) Thus even with a compact probability space, and an automorphism ϕ of its measure algebra, we cannot be sure of representing ϕ and ϕ^{-1} by functions which will be inverses of each other.

343K Proposition If (X, Σ, μ) is a semi-finite countably separated measure space, it is compact iff it is locally compact iff it is perfect.

343L Proposition Let (X, Σ, μ) be a complete locally determined countably separated measure space, and $A \subseteq X$ a set such that the subspace measure μ_A is perfect. Then A is measurable.

343M Example 343L tells us that any non-measurable set X of $\mathbb{R}^r$, or of $\{0,1\}^{\mathbb{N}}$, with their usual measures, is not perfect, therefore not (locally) compact, when given its subspace measure.

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344 Realization of automorphisms

In 343Jb, I gave an example of a 'good' (compact, complete) probability space X with an automorphism ϕ of its measure algebra such that both ϕ and ϕ^{-1} are representable by functions from X to itself, but there is no such representation in which the two functions are inverses of each other. The present section is an attempt to describe the further refinements necessary to ensure that automorphisms of measure algebras can be represented by automorphisms of the measure spaces. It turns out that in the most important contexts in which this can be done, a little extra work yields a significant generalization: the simultaneous realization of countably many homomorphisms by a consistent family of functions.

I will describe three cases in which such simultaneous realizations can be achieved: Stone spaces (344A), perfect complete countably separated spaces (344C) and suitable measures on $\{0,1\}^I$ (344E-344G). The arguments for 344C, suitably refined, give a complete description of perfect complete countably separated strictly localizable spaces which are not purely atomic (344I). At the same time we find that Lebesgue measure, and the usual measure on $\{0,1\}^I$, are 'homogeneous' in the strong sense that two measurable subspaces (of non-zero measure) are isomorphic iff they have the same measure (344J, 344L).

344A Stone spaces If (Z,Σ,μ) is the Stone space of a measure algebra $(\mathfrak{A},\bar{\mu})$, then every order-continuous Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{A}$ corresponds to a unique continuous function $f_\phi:Z\to Z$ which represents ϕ. $f_{\phi\psi}=f_\psi f_\phi$ for all order-continuous homomorphisms ϕ and ψ; and f_ι is the identity map on Z, so that $f_{\phi^{-1}}$ will have to be f_ϕ^{-1} whenever ϕ is invertible. Thus in this case we can consistently, and canonically, represent all order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself.

344B Theorem Let (X,Σ,μ) be a countably separated measure space with measure algebra $\mathfrak{A}$, and G a countable semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that every member of G can be represented by some function from X to itself. Then a family $\langle f_\phi\rangle_{\phi\in G}$ of such representatives can be chosen in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X.

344C Corollary Let (X,Σ,μ) be a countably separated perfect complete strictly localizable measure space with measure algebra $\mathfrak{A}$, and G a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi\in G$, a function $f_\phi:X\to X$ representing ϕ, in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi\in G$ is invertible, and $\phi^{-1}\in G$, we shall have $f_{\phi^{-1}}=f_\phi^{-1}$; so that if moreover ϕ and ϕ^{-1} are measure-preserving, f_ϕ will be an automorphism of the measure space (X,Σ,μ).

344D Lemma Let X and Y be sets, and $\Sigma\subseteq\mathcal{P}X$, $\mathrm{T}\subseteq\mathcal{P}Y$ σ-algebras. Suppose that there are $f:X\to Y$, $g:Y\to X$ such that $F=f[X]\in\mathrm{T}$, $E=g[Y]\in\Sigma$, f is an isomorphism between (X,Σ) and (F,T_F) and g is an isomorphism between (Y,T) and (E,Σ_E), writing Σ_E, T_F for the subspace σ-algebras. Then (X,Σ) and (Y,T) are isomorphic, and there is an isomorphism $h:X\to Y$ which is covered by f and g in the sense that

$$\{(x,h(x)):x\in X\}\subseteq\{(x,f(x)):x\in X\}\cup\{(g(y),y):y\in Y\}.$$

344E Theorem Let I be any set, and let μ be a σ-finite measure on $X=\{0,1\}^I$ with domain the σ-algebra $\mathcal{B}\mathfrak{a}_I$ generated by the sets $\{x:x(i)=1\}$ as i runs over I; write $\mathfrak{A}$ for the measure algebra of μ. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi\in G$, a function $f_\phi:X\to X$ representing ϕ, in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi\in G$ is invertible and $\phi^{-1}\in G$, we shall have $f_{\phi^{-1}}=f_\phi^{-1}$; so that if moreover ϕ is measure-preserving, f_ϕ will be an automorphism of the measure space $(X,\mathcal{B}\mathfrak{a}_I,\mu)$.

344F Corollary Let I be any set, and let μ be a σ-finite measure on $X=\{0,1\}^I$. Suppose that μ is the completion of its restriction to the σ-algebra $\mathcal{B}\mathfrak{a}_I$ generated by the sets $\{x:x(i)=1\}$ as i runs over I. Write $\mathfrak{A}$ for the measure algebra of μ. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi\in G$, a function $f_\phi:X\to X$ representing ϕ, in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi\in G$ is invertible and $\phi^{-1}\in G$, we shall have $f_{\phi^{-1}}=f_\phi^{-1}$; so that if moreover ϕ is measure-preserving, f_ϕ will be an automorphism of the measure space (X,Σ,μ).

344G Corollary Let I be any set, ν_I the usual measure on $\{0,1\}^I$, and $\mathfrak{B}_I$ its measure algebra. Then any measure-preserving automorphism of $\mathfrak{B}_I$ is representable by a measure space automorphism of $(\{0,1\}^I,\nu_I)$.

344H Lemma Let (X,Σ,μ) be a perfect semi-finite measure space. If $H\in\Sigma$ is a non-negligible set which includes no atom, there is a negligible subset of H with cardinal $\mathfrak{c}$.

344I Theorem Let (X,Σ,μ) and (Y,T,ν) be atomless, perfect, complete, strictly localizable, countably separated measure spaces of the same non-zero magnitude. Then they are isomorphic.

344J Corollary Suppose that E, F are two Lebesgue measurable subsets of $\mathbb{R}^r$ of the same non-zero measure. Then the subspace measures on E and F are isomorphic.

344K Corollary (a) A measure space is isomorphic to Lebesgue measure on $[0,1]$ iff it is an atomless countably separated compact (or perfect) complete probability space; in this case it is also isomorphic to the usual measure on $\{0,1\}^{\mathbb{N}}$.

(b) A measure space is isomorphic to Lebesgue measure on $\mathbb{R}$ iff it is an atomless countably separated compact (or perfect) σ-finite measure space which is not totally finite; in this case it is also isomorphic to Lebesgue measure on any Euclidean space $\mathbb{R}^r$.

(c) Let μ be Lebesgue measure on $\mathbb{R}$. If $0<\mu E<\infty$ and we set $\nu F=\frac{1}{\mu E}\mu F$ for every measurable $F\subseteq E$, then (E,ν) is isomorphic to Lebesgue measure on $[0,1]$.

344L Theorem Let I be an infinite set, and ν_I the usual measure on $\{0,1\}^I$. If $E\subseteq\{0,1\}^I$ is a measurable set of non-zero measure, the subspace measure on E is isomorphic to $(\nu_I E)\nu_I$.

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345 Translation-invariant liftings

In this section and the next I complement the work of §341 by describing some important special properties which can, in appropriate circumstances, be engineered into our liftings. I begin with some remarks on translation-invariance. I restrict my attention to measure spaces which we have already seen, delaying a general discussion of translation-invariant measures on groups until Volume 4.

345A Translation-invariant liftings I shall consider two forms of translation-invariance, as follows.

(a) Let μ be Lebesgue measure on $\mathbb{R}^r$, and Σ its domain. A lifting $\phi:\Sigma\to\Sigma$ is **translation-invariant** if $\phi(E+x)=\phi E+x$ for every $E\in\Sigma$, $x\in\mathbb{R}^r$.

Similarly, writing $\mathfrak{A}$ for the measure algebra of μ, a lifting $\theta:\mathfrak{A}\to\Sigma$ is **translation-invariant** if $\theta(E+x)^\bullet=\theta E^\bullet+x$ for every $E\in\Sigma$, $x\in\mathbb{R}^r$.

(b) Now let I be any set, and let ν_I be the usual measure on $X=\{0,1\}^I$, with T_I its domain and $\mathfrak{B}_I$ its measure algebra. For x, $y\in X$, define $x+y\in X$ by setting $(x+y)(i)=x(i)+_2 y(i)$ for every $i\in I$.

We say that a lifting $\theta:\mathfrak{B}_I\to\mathrm{T}_I$, or $\phi:\mathrm{T}_I\to\mathrm{T}_I$, is **translation-invariant** if

$$\theta(E+x)^\bullet=\theta E^\bullet+x,\quad \phi(E+x)=\phi E+x$$

whenever $E\in\Sigma$ and $x\in X$.

345B Theorem For any $r\ge 1$, there is a translation-invariant lifting for Lebesgue measure on $\mathbb{R}^r$.

345C Theorem For any set I, there is a translation-invariant lifting for the usual measure on $\{0,1\}^I$.

345D Proposition Let (X,Σ,μ) be *either* Lebesgue measure on $\mathbb{R}^r$ *or* the usual measure on $\{0,1\}^I$ for some set I, and let $\phi:\Sigma\to\Sigma$ be a translation-invariant lifting. Then for any open set $G\subseteq X$ we must have $G\subseteq\phi G\subseteq\overline{G}$, and for any closed set F we must have $\operatorname{int}F\subseteq\phi F\subseteq F$.

345E Lemma Give $X=\{0,1\}^{\mathbb{N}}$ its usual measure $\nu_{\mathbb{N}}$, and let $E\subseteq X$ be any non-negligible measurable set. Then there is an $n\in\mathbb{N}$ such that for every $k\ge n$ there are x, $x'\in E$ which differ at k and nowhere else.

345F Proposition Let μ be the restriction of Lebesgue measure to the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$. Then μ is translation-invariant, but has no translation-invariant lifting.

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346 Consistent liftings

I turn now to a different type of condition which we should naturally prefer our liftings to satisfy. If we have a product measure μ on a product $X = \prod_{i \in I} X_i$ of probability spaces, then we can look for liftings ϕ which 'respect coordinates', that is, are compatible with the product structure in the sense that they factor through subproducts (346A). There seem to be obstacles in the way of the natural conjecture (346Za), and I give the partial results which are known. For Maharam-type-homogeneous spaces X_i, there is always a lifting which respects coordinates (346E), and indeed the translation-invariant liftings of §345 on $\{0,1\}^I$ already have this property (346C). There is always a lower density for the product measure which respects coordinates, and we can ask for a little more (346G); using the full strength of 346G, we can enlarge this lower density to a lifting which respects single coordinates and initial segments of a well-ordered product (346H). In the case in which all the factors are copies of each other, we can arrange for the induced liftings on the factors to be copies also (346I, 346J). I end the section with an important fact about Stone spaces which is relevant here (346K-346L).

346A Definition Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, with product (X, Σ, μ). I will say that a lifting $\phi : \Sigma \to \Sigma$ **respects coordinates** if ϕE is determined by coordinates in J whenever $E \in \Sigma$ is determined by coordinates in $J \subseteq I$.

346B Lemma (a) Let (X, Σ, μ) be a measure space with a lifting $\phi : \Sigma \to \Sigma$. Suppose that Y is a set and $f : X \to Y$ a surjective function such that whenever $E \in \Sigma$ is such that $f^{-1}[f[E]] = E$, then $f^{-1}[f[\phi E]] = \phi E$. Then we have a lifting ψ for the image measure μf^{-1} defined by the formula

$$f^{-1}[\psi F] = \phi(f^{-1}[F]) \text{ whenever } F \subseteq Y \text{ and } f^{-1}[F] \in \Sigma.$$

(b) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, with product (Z, Λ, λ). For $J \subseteq I$ let $(Z_J, \Lambda_J, \lambda_J)$ be the product of $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in J}$, and $\pi_J : Z \to Z_J$ the canonical map. Let $\phi : \Lambda \to \Lambda$ be a lifting. If $J \subseteq I$ is such that ϕW is determined by coordinates in J whenever $W \in \Lambda$ is determined by coordinates in J, then ϕ induces a lifting $\phi_J : \Lambda_J \to \Lambda_J$ defined by the formula

$$\pi_J^{-1}[\phi_J E] = \phi(\pi_J^{-1}[E]) \text{ for every } E \in \Lambda_J.$$

346C Theorem Let I be any set, and ν_I the usual measure on $\{0,1\}^I$. Then any translation-invariant lifting for ν_I respects coordinates.

346D Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $\mathfrak{A}$, $\mathfrak{B}$; suppose that $f : X \to Y$ represents an isomorphism $F^\bullet \mapsto f^{-1}[F]^\bullet : \mathfrak{B} \to \mathfrak{A}$. Then if $\phi : \mathrm{T} \to \mathrm{T}$ is a lifting for ν, there is a corresponding lifting $\phi' : \Sigma \to \Sigma$ given by the formula

$$\phi' E = f^{-1}[\phi F] \text{ whenever } \mu(E \triangle f^{-1}[F]) = 0.$$

346E Theorem Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of Maharam-type-homogeneous probability spaces, with product (X, Σ, μ). Then there is a lifting for μ which respects coordinates.

346F Lemma Let (X, Σ, μ) and (Y, T, ν) be complete probability spaces, with product $(X \times Y, \Lambda, \lambda)$. If $\underline{\phi} : \Lambda \to \Lambda$ is a lower density, then we have a lower density $\underline{\phi}_1 : \Sigma \to \Sigma$ defined by saying that

$$\underline{\phi}_1 E = \{x : x \in X,\ \{y : (x, y) \in \underline{\phi}(E \times Y)\} \text{ is conegligible in } Y\}$$

for every $E \in \Sigma$.

346G Theorem Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces with product (X, Σ, μ). For $J \subseteq I$ let Σ_J be the set of members of Σ which are determined by coordinates in J. Then there is a lower density $\underline{\phi} : \Sigma \to \Sigma$ such that

(i) whenever $J \subseteq I$ and $E \in \Sigma_J$ then $\underline{\phi} E \in \Sigma_J$,

(ii) whenever J, $K \subseteq I$ are disjoint, $E \in \Sigma_J$ and $F \in \Sigma_K$ then $\underline{\phi}(E \cup F) = \underline{\phi} E \cup \underline{\phi} F$.

346H Theorem Let ζ be an ordinal, and $\langle(X_\xi,\Sigma_\xi,\mu_\xi)\rangle_{\xi<\zeta}$ a family of probability spaces, with product (Z,Λ,λ). For $J\subseteq\zeta$ let Λ_J be the set of those $W\in\Lambda$ which are determined by coordinates in J. Then there is a lifting $\phi:\Lambda\to\Lambda$ such that $\phi W\in\Lambda_J$ whenever $W\in\Lambda_J$ and J is *either* a singleton subset of ζ *or* an initial segment of ζ.

346I Theorem Let (X,Σ,μ) be a complete probability space. For any set I, write λ_I for the product measure on X^I, Λ_I for its domain and $\pi_{Ii}(x)=x(i)$ for $x\in X^I$, $i\in I$. Then there is a lifting $\psi:\Sigma\to\Sigma$ such that for every set I there is a lifting $\phi:\Lambda_I\to\Lambda_I$ such that $\phi(\pi_{Ii}^{-1}[E])=\pi_{Ii}^{-1}[\psi E]$ whenever $E\in\Sigma$ and $i\in I$.

346J Consistent liftings Let (X,Σ,μ) be a measure space. A lifting $\psi:\Sigma\to\Sigma$ is **consistent** if for every $n\ge 1$ there is a lifting ϕ_n of the product measure on X^n such that $\phi_n(E_1\times\ldots\times E_n)=\psi E_1\times\ldots\times\psi E_n$ for all $E_1,\ldots,E_n\in\Sigma$. Thus every non-trivial complete totally finite measure space has a consistent lifting.

346K Lemma Let (Z,T,ν) be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, and let λ be the product measure on $Z\times Z$, with Λ its domain. Then there is a set $W\in\Lambda$, with $\lambda W<1$, such that $\lambda^*\tilde W=1$, where

$$\tilde W=\bigcup\{G\times H:G,\,H\subseteq Z\text{ are open-and-closed, }(G\times H)\setminus W\text{ is negligible}\}.$$

346L Proposition Let (Z,T,ν) be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$. Let $\psi:\mathrm{T}\to\mathrm{T}$ be the canonical lifting, defined by setting $\psi E=G$ whenever $E\in\mathrm{T}$, G is open-and-closed and $E\triangle G$ is negligible. Then ψ is not consistent.

346Z Problems (a) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X,Σ,μ). Is there always a lifting for μ which respects coordinates in the sense of 346A?

(b) Is there a lower density $\underline{\phi}$ for the usual measure on $\{0,1\}^{\mathbb{N}}$ which is invariant under all permutations of coordinates?

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Concordance for Volume 3

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

341X Exercises 341Xd and 341Xf, referred to in the 2003 and 2006 editions of Volume 4, are now 341Xc and 341Xe.

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