Version of 28.5.07

Chapter 34

The lifting theorem

Whenever we have a surjective homomorphism $\phi : P \to Q$, where P and Q are mathematical structures, we can ask whether there is a right inverse of ϕ, a homomorphism $\psi : Q \to P$ such that $\phi\psi$ is the identity on Q. As a general rule, we expect a negative answer; those categories in which epimorphisms always have right inverses (e.g., the category of linear spaces) are rather special, and elsewhere the phenomenon is relatively rare and almost always important. So it is notable that we have a case of this at the very heart of the theory of measure algebras: for any complete probability space (X, Σ, μ) (in fact, for any complete strictly localizable space of non-zero measure) the canonical homomorphism from Σ to the measure algebra of μ has a right inverse (341K). This is the von Neumann-Maharam lifting theorem. Its proof, together with some essentially elementary remarks, takes up the whole of of §341.

As a first application of the theorem (there will be others in Volume 4) I apply it to one of the central problems of measure theory: under what circumstances will a homomorphism between measure algebras be representable by a function between measure spaces? Variations on this question are addressed in §343. For a reasonably large proportion of the measure spaces arising naturally in analysis, homomorphisms are representable (343B). New difficulties arise if we ask for isomorphisms of measure algebras to be representable by isomorphisms of measure spaces, and here we have to work rather hard for rather narrowly applicable results; but in the case of Lebesgue measure and its closest relatives, a good deal can be done, as in 344I-344K.

Returning to liftings, there are many difficult questions concerning the extent to which liftings can be required to have special properties, reflecting the natural symmetries of the standard measure spaces. For instance, Lebesgue measure is translation-invariant; if liftings were in any sense canonical, they could be expected to be automatically translation-invariant in some sense. It seems sure that there is no canonical lifting for Lebesgue measure – all constructions of liftings involve radical use of the axiom of choice – but even so we do have many translation-invariant liftings (§345). We have less luck with product spaces; here the construction of liftings which respect the product structure is fraught with difficulties. I give the currently known results in §346.

Version of 9.4.10

341 The lifting theorem

I embark directly on the principal theorem of this chapter (341K, 'every non-trivial complete strictly localizable measure space has a lifting'), using the minimum of advance preparation. 341A-341B give the definition of 'lifting'; the main argument is in 341F-341K, using the concept of 'lower density' (341C-341E) and a theorem on martingales from §275. In 341P I describe an alternative way of thinking about liftings in terms of the Stone space of the measure algebra.

341A Definition Let (X, Σ, μ) be a measure space, and $\mathfrak{A}$ its measure algebra. By a **lifting** for $\mathfrak{A}$ (or for (X, Σ, μ), or for μ) I shall mean

either a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$

or a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that (i) $\phi E = \emptyset$ whenever $\mu E = 0$ (ii) $\mu(E \triangle \phi E) = 0$ for every $E \in \Sigma$.

341B Remarks (a) I trust that the ambiguities permitted by this terminology will not cause any confusion. The point is that there is a natural one-to-one correspondence between liftings $\theta : \mathfrak{A} \to \Sigma$ and liftings $\phi : \Sigma \to \Sigma$ given by the formula

$$\theta E^\bullet = \phi E \text{ for every } E \in \Sigma.$$

P (i) Given a lifting $\theta : \mathfrak{A} \to \Sigma$, the formula defines a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that

$$\phi\emptyset = \theta 0 = \emptyset, \quad (E \triangle \phi E)^\bullet = E^\bullet \triangle (\theta E^\bullet)^\bullet = 0 \ \forall \ E \in \Sigma,$$

so that ϕ is a lifting. (ii) Given a lifting $\phi : \Sigma \to \Sigma$, the kernel of ϕ includes $\{E : \mu E = 0\}$, so there is a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $\theta E^\bullet = \phi E$ for every E (3A2G), and now

$$(\theta E^\bullet)^\bullet = (\phi E)^\bullet = E^\bullet$$

for every $E \in \Sigma$, so θ is a lifting. **Q**

I suppose that the word 'lifting' applies most naturally to functions from $\mathfrak{A}$ to Σ; but for applications in measure theory the other type of lifting is used at least equally often.

(b) Note that if $\phi : \Sigma \to \Sigma$ is a lifting then $\phi^2 = \phi$. **P** For any $E \in \Sigma$,

$$\phi^2 E \triangle \phi E = \phi(E \triangle \phi E) = \emptyset. \text{ **Q**}$$

If ϕ is associated with $\theta : \mathfrak{A} \to \Sigma$, then $\phi\theta a = \theta a$ for every $a \in \mathfrak{A}$. **P** $\phi\theta a = \theta((\theta a)^\bullet) = \theta a$. **Q**

(c) In the theorems to follow, there will occasionally intrude a hypothesis '$\mu X > 0$'. The point is that if we have a measure space (X, Σ, μ) which is trivial in the sense that $\mu X = 0$, then the only candidate for a 'lifting' $\phi : \Sigma \to \Sigma$ is the constant function with value $\emptyset$; and if $X \neq \emptyset$ this is not a Boolean homomorphism in the sense of this book. The simplest way of dealing with these cases is to rule them out of the discussion.

341C Definition Let (X, Σ, μ) be a measure space, and $\mathfrak{A}$ its measure algebra. By a **lower density** for $\mathfrak{A}$ (or for (X, Σ, μ), or for μ) I shall mean

either a function $\underline{\theta} : \mathfrak{A} \to \Sigma$ such that (i) $(\underline{\theta} a)^\bullet = a$ for every $a \in \mathfrak{A}$ (ii) $\underline{\theta} 0 = \emptyset$ (iii) $\underline{\theta}(a \cap b) = \underline{\theta} a \cap \underline{\theta} b$ for all a, $b \in \mathfrak{A}$

or a function $\underline{\phi} : \Sigma \to \Sigma$ such that (i) $\underline{\phi} E = \underline{\phi} F$ whenever E, $F \in \Sigma$ and $\mu(E \triangle F) = 0$ (ii) $\mu(E \triangle \underline{\phi} E) = 0$ for every $E \in \Sigma$ (iii) $\underline{\phi}\emptyset = \emptyset$ (iv) $\underline{\phi}(E \cap F) = \underline{\phi} E \cap \underline{\phi} F$ for all E, $F \in \Sigma$.

341D Remarks (a) As in 341B, there is a natural one-to-one correspondence between lower densities $\underline{\theta} : \mathfrak{A} \to \Sigma$ and lower densities $\underline{\phi} : \Sigma \to \Sigma$ given by the formula

$$\underline{\theta} E^\bullet = \underline{\phi} E \text{ for every } E \in \Sigma.$$

(For the requirement $\underline{\phi} E = \underline{\phi} F$ whenever $E^\bullet = F^\bullet$ in $\mathfrak{A}$ means that every $\underline{\phi}$ corresponds to a function $\underline{\theta}$, and the other clauses match each other directly.)

(b) As before, if $\underline{\phi} : \Sigma \to \Sigma$ is a lower density then $\underline{\phi}^2 = \underline{\phi}$. If $\underline{\phi}$ is associated with $\underline{\theta} : \mathfrak{A} \to \Sigma$, then $\underline{\phi}\underline{\theta} = \underline{\theta}$.

(c) It will be convenient, in the course of the proofs of 341F-341H below, to have the following concept available. If (X, Σ, μ) is a measure space with measure algebra $\mathfrak{A}$, a **partial lower density** of $\mathfrak{A}$ is a function $\underline{\theta} : \mathfrak{B} \to \Sigma$ such that (i) the domain $\mathfrak{B}$ of $\underline{\theta}$ is a subalgebra of $\mathfrak{A}$ (ii) $(\underline{\theta} b)^\bullet = b$ for every $b \in \mathfrak{B}$ (iii) $\underline{\theta} 0 = \emptyset$ (iv) $\underline{\theta}(a \cap b) = \underline{\theta} a \cap \underline{\theta} b$ for all a, $b \in \mathfrak{B}$.

Similarly, if T is a subalgebra of Σ, a function $\underline{\phi} : \mathrm{T} \to \Sigma$ is a **partial lower density** if (i) $\underline{\phi} E = \underline{\phi} F$ whenever E, $F \in \mathrm{T}$ and $\mu(E \triangle F) = 0$ (ii) $\mu(E \triangle \underline{\phi} E) = 0$ for every $E \in \mathrm{T}$ (iii) $\underline{\phi}\emptyset = \emptyset$ (iv) $\underline{\phi}(E \cap F) = \underline{\phi} E \cap \underline{\phi} F$ for all E, $F \in \mathrm{T}$.

(d) Note that lower densities and partial lower densities are order-preserving; if $a \subseteq b$ in $\mathfrak{A}$, and $\underline{\theta}$ is a lower density for $\mathfrak{A}$, then

$$\underline{\theta} a = \underline{\theta}(a \cap b) = \underline{\theta} a \cap \underline{\theta} b \subseteq \underline{\theta} b.$$

(e) Of course a Boolean homomorphism from $\mathfrak{A}$ to Σ, or from Σ to itself, is a lifting iff it is a lower density.

341E Example Let μ be Lebesgue measure on $\mathbb{R}^r$, where $r \geq 1$, and Σ its domain. For $E \in \Sigma$ set

$$\text{int*}E = \{x : x \in \mathbb{R}^r,\ \lim_{\delta \downarrow 0} \tfrac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)} = 1\}.$$

(Here $B(x,\delta)$ is the closed ball with centre x and radius δ.) Then int* is a lower density for μ; we may call it **lower Lebesgue density**. **P** (You may prefer at first to suppose that $r = 1$, so that $B(x,\delta) = [x-\delta, x+\delta]$ and $\mu B(x,\delta) = 2\delta$.) By 261Db (or 223B, for the one-dimensional case) $E \triangle \text{int*}E$ is negligible for every E; in particular, $\text{int*}E \in \Sigma$ for every $E \in \Sigma$. If $E \triangle F$ is negligible, then $\mu(E \cap B(x,\delta)) = \mu(F \cap B(x,\delta))$ for every x and δ, so $\text{int*}E = \text{int*}F$. If $E \subseteq F$, then $\mu(E \cap B(x,\delta)) \leq \mu(F \cap B(x,\delta))$ for every x, δ, so $\text{int*}E \subseteq \text{int*}F$; consequently $\text{int*}(E \cap F) \subseteq \text{int*}E \cap \text{int*}F$ for all E, $F \in \Sigma$. If E, $F \in \Sigma$ and $x \in \text{int*}E \cap \text{int*}F$, then

$$\begin{aligned}\mu(E \cap F \cap B(x,\delta)) &= \mu(E \cap B(x,\delta)) + \mu(F \cap B(x,\delta)) - \mu((E \cup F) \cap B(x,\delta)) \\ &\geq \mu(E \cap B(x,\delta)) + \mu(F \cap B(x,\delta)) - \mu(B(x,\delta))\end{aligned}$$

for every δ, so

$$\frac{\mu(E \cap F \cap B(x,\delta))}{\mu B(x,\delta)} \geq \frac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)} + \frac{\mu(F \cap B(x,\delta))}{\mu B(x,\delta)} - 1 \to 1$$

as $\delta \downarrow 0$, and $x \in \text{int*}(E \cap F)$. Thus $\text{int*}(E \cap F) = \text{int*}E \cap \text{int*}F$ for all E, $F \in \Sigma$, and int* is a lower density. **Q**

Remark In Chapter 47 of Volume 4 I will return to the operator int* in a context in which an alternative name, 'essential interior', is more natural.

341F The hard work of this section is in the proof of 341H below. To make it a little more digestible, I extract two parts of the proof as separate lemmas.

Lemma Let (X, Σ, μ) be a probability space and $\mathfrak{A}$ its measure algebra. Let $\mathfrak{B}$ be a closed subalgebra of $\mathfrak{A}$ and $\underline{\theta} : \mathfrak{B} \to \Sigma$ a partial lower density. Then for any $e \in \mathfrak{A}$ there is a partial lower density $\underline{\theta}_1$, extending $\underline{\theta}$, defined on the subalgebra $\mathfrak{B}_1$ of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{e\}$.

proof (a) Because $\mathfrak{B}$ is order-closed, therefore Dedekind complete in itself (314Ea),

$$v = \text{upr}(e, \mathfrak{B}) = \inf\{a : a \in \mathfrak{B},\ a \supseteq e\}, \quad w = \text{upr}(1 \setminus e, \mathfrak{B})$$

are defined in $\mathfrak{B}$. Let $E \in \Sigma$ be such that $E^\bullet = e$.

(b) We have a function $\underline{\theta}_1 : \mathfrak{B}_1 \to \Sigma$ defined by writing

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = \bigl(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\bigr)$$

for a, $b \in \mathfrak{B}$. **P** By 312N, every element of $\mathfrak{B}_1$ is expressible as $(a \cap e) \cup (b \setminus e)$ for some a, $b \in \mathfrak{B}$. If a, a', b, $b' \in \mathfrak{B}$ are such that $(a \cap e) \cup (b \setminus e) = (a' \cap e) \cup (b' \setminus e)$, then $a \cap e = a' \cap e$ and $b \setminus e = b' \setminus e$, that is,

$$a \triangle a' \subseteq 1 \setminus e \subseteq w, \quad b \triangle b' \subseteq e \subseteq v.$$

This means that $e \subseteq 1 \setminus (a \triangle a') \in \mathfrak{B}$ and $1 \setminus e \subseteq 1 \setminus (b \triangle b') \in \mathfrak{B}$. So we also have $v \subseteq 1 \setminus (a \triangle a')$ and $w \subseteq 1 \setminus (b \triangle b')$. Accordingly

$$a \cap v = a' \cap v, \quad b \cap w = b' \cap w. \quad a \setminus w = a' \setminus w, \quad b \setminus v = b' \setminus v.$$

But this means that

$$\begin{aligned}&\bigl(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\bigr) \\ &\quad = \bigl(\underline{\theta}((a' \cap v) \cup (b' \setminus v)) \cap E\bigr) \cup \bigl(\underline{\theta}((a' \setminus w) \cup (b' \cap w)) \setminus E\bigr).\end{aligned}$$

Thus the formula given defines $\underline{\theta}_1$ uniquely. **Q**

(c) Now $\underline{\theta}_1$ is a partial lower density.

P(i) If a, $b\in\mathfrak{B}$,

$$\begin{aligned}(\underline{\theta}_1((a\cap e)\cup(b\setminus e)))^\bullet &= \bigl(\bigl(\underline{\theta}((a\cap v)\cup(b\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a\setminus w)\cup(b\cap w))\setminus E\bigr)\bigr)^\bullet\\ &= \bigl(((a\cap v)\cup(b\setminus v))\cap e\bigr)\cup\bigl(((a\setminus w)\cup(b\cap w))\setminus e\bigr)\\ &= (a\cap e)\cup(b\setminus e).\end{aligned}$$

So $(\underline{\theta}_1 c)^\bullet = c$ for every $c\in\mathfrak{B}_1$.

(ii)

$$\underline{\theta}_1(0) = \bigl(\underline{\theta}((0\cap v)\cup(0\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((0\setminus w)\cup(0\cap w))\setminus E\bigr) = \emptyset.$$

(iii) If a, a', b, $b'\in\mathfrak{B}$, then

$$\begin{aligned}&\underline{\theta}_1(((a\cap e)\cup(b\setminus e))\cap((a'\cap e)\cup(b'\setminus e)))\\ &\quad= \underline{\theta}_1((a\cap a'\cap e)\cup(b\cap b'\setminus e))\\ &\quad= \bigl(\underline{\theta}((a\cap a'\cap v)\cup(b\cap b'\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a\cap a'\setminus w)\cup(b\cap b'\cap w))\setminus E\bigr)\\ &\quad= \bigl(\underline{\theta}(((a\cap v)\cup(b\setminus v))\cap((a'\cap v)\cup(b'\setminus v)))\cap E\bigr)\\ &\qquad\qquad\cup\bigl(\underline{\theta}(((a\setminus w)\cup(b\cap w))\cap((a'\setminus w)\cup(b'\cap w)))\setminus E\bigr)\\ &\quad= \bigl(\underline{\theta}((a\cap v)\cup(b\setminus v))\cap\underline{\theta}((a'\cap v)\cup(b'\setminus v))\cap E\bigr)\\ &\qquad\qquad\cup\bigl(\underline{\theta}((a\setminus w)\cup(b\cap w))\cap\underline{\theta}((a'\setminus w)\cup(b'\cap w))\setminus E\bigr)\\ &\quad= \bigl(\bigl(\underline{\theta}((a\cap v)\cup(b\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a\setminus w)\cup(b\cap w))\setminus E\bigr)\bigr)\\ &\qquad\qquad\cap\bigl(\bigl(\underline{\theta}((a'\cap v)\cup(b'\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a'\setminus w)\cup(b'\cap w))\setminus E\bigr)\bigr)\\ &\quad= \underline{\theta}_1((a\cap e)\cup(b\setminus e))\cap\underline{\theta}_1((a'\cap e)\cup(b'\setminus e)).\end{aligned}$$

So $\underline{\theta}_1(c\cap c') = \underline{\theta}_1(c)\cap\underline{\theta}_1(c')$ for all c, $c'\in\mathfrak{B}_1$. **Q**

(d) If $a\in\mathfrak{B}$, then

$$\begin{aligned}\underline{\theta}_1(a) &= \underline{\theta}_1((a\cap e)\cup(a\setminus e))\\ &= \bigl(\underline{\theta}((a\cap v)\cup(a\setminus v))\cap E\bigr)\cup\bigl(\underline{\theta}((a\setminus w)\cup(a\cap w))\setminus E\bigr)\\ &= \bigl(\underline{\theta}(a)\cap E\bigr)\cup\bigl(\underline{\theta}(a)\setminus E\bigr) = \underline{\theta}a.\end{aligned}$$

Thus $\underline{\theta}_1$ extends $\underline{\theta}$, as required.

341G Lemma Let (X,Σ,μ) be a probability space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Suppose we have a sequence $\langle\underline{\theta}_n\rangle_{n\in\mathbb{N}}$ of partial lower densities such that, for each n, (i) the domain $\mathfrak{B}_n$ of $\underline{\theta}_n$ is a closed subalgebra of $\mathfrak{A}$ (ii) $\mathfrak{B}_n\subseteq\mathfrak{B}_{n+1}$ and $\underline{\theta}_{n+1}$ extends $\underline{\theta}_n$. Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{B}_n$. Then there is a partial lower density $\underline{\theta}$, with domain $\mathfrak{B}$, extending every $\underline{\theta}_n$.

proof (a) For each n, set

$$\Sigma_n = \{E : E\in\Sigma,\ E^\bullet\in\mathfrak{B}_n\},$$

and set

$$\Sigma_\infty = \{E : E\in\Sigma,\ E^\bullet\in\mathfrak{B}\}.$$

Then (because all the $\mathfrak{B}_n$, $\mathfrak{B}$ are σ-subalgebras of $\mathfrak{A}$, and $E\mapsto E^\bullet$ is sequentially order-continuous) all the Σ_n, Σ_∞ are σ-subalgebras of Σ. We need to know that Σ_∞ is just the σ-algebra Σ^*_∞ of subsets of X generated by $\bigcup_{n\in\mathbb{N}}\Sigma_n$. **P** Because Σ_∞ is a σ-algebra including $\bigcup_{n\in\mathbb{N}}\Sigma_n$, $\Sigma^*_\infty\subseteq\Sigma_\infty$. On the other hand, $\mathfrak{B}^* = \{E^\bullet : E\in\Sigma^*_\infty\}$ is a σ-subalgebra of $\mathfrak{A}$ including $\mathfrak{B}_n$ for every $n\in\mathbb{N}$. Because $\mathfrak{A}$ is ccc, $\mathfrak{B}^*$ is (order-)closed (316Fb), so includes $\mathfrak{B}$. This means that if $E\in\Sigma_\infty$ there must be an $F\in\Sigma^*_\infty$ such that $E^\bullet = F^\bullet$. But now $(E\triangle F)^\bullet = 0\in\mathfrak{B}_0$, so $E\triangle F\in\Sigma_0\subseteq\Sigma^*_\infty$, and E also belongs to Σ^*_∞. This shows that $\Sigma_\infty\subseteq\Sigma^*_\infty$ and the two algebras are equal. **Q**

(b) For each $n \in \mathbb{N}$, we have the partial lower density $\underline{\theta}_n : \mathfrak{B}_n \to \Sigma$. Since $(\underline{\theta}_n a)^\bullet = a \in \mathfrak{B}_n$ for every $a \in \mathfrak{B}_n$, $\underline{\theta}_n$ takes all its values in Σ_n. For $n \in \mathbb{N}$, let $\underline{\phi}_n : \Sigma_n \to \Sigma_n$ be the lower density corresponding to $\underline{\theta}_n$ (341Ba), that is, $\underline{\phi}_n E = \underline{\theta}_n E^\bullet$ for every $E \in \Sigma_n$.

(c) For $a \in \mathfrak{A}$, $n \in \mathbb{N}$ choose $G_a \in \Sigma$, g_{an} such that $G_a^\bullet = a$ and g_{an} is a conditional expectation of χG_a on Σ_n; that is,

$$\int_E g_{an} = \int_E \chi G_a = \mu(E \cap G_a) = \bar{\mu}(E^\bullet \cap a)$$

for every $E \in \Sigma_n$. As remarked in 233Db, such a function g_{an} can always be found, and moreover we may take it to be Σ_n-measurable and defined everywhere on X. Now if $a \in \mathfrak{B}$, $\lim_{n\to\infty} g_{an}(x)$ exists and is equal to $\chi G_a(x)$ for almost every x. **P** By Lévy's martingale theorem (275I), $\lim_{n\to\infty} g_{an}$ is defined almost everywhere and is a conditional expectation of χG_a on the σ-algebra generated by $\bigcup_{n\in\mathbb{N}} \Sigma_n$. As observed in (a), this is just Σ_∞; and as χG_a is itself Σ_∞-measurable, it is also a conditional expectation of itself on Σ_∞, and must be equal almost everywhere to $\lim_{n\to\infty} g_{an}$. **Q**

(d) For $a \in \mathfrak{B}$, $k \geq 1$, $n \in \mathbb{N}$ set

$$H_{kn}(a) = \{x : x \in X,\ g_{an}(x) \geq 1 - 2^{-k}\} \in \Sigma_n, \quad \tilde{H}_{kn}(a) = \underline{\phi}_n(H_{kn}(a)),$$

$$\underline{\theta} a = \bigcap_{k\geq 1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \tilde{H}_{km}(a).$$

The rest of the proof is devoted to showing that $\underline{\theta} : \mathfrak{B} \to \Sigma$ has the required properties.

(e) G_0 is negligible, so every g_{0n} is zero almost everywhere, every $H_{kn}(0)$ is negligible and every $\tilde{H}_{kn}(0)$ is empty; so $\underline{\theta}0 = \emptyset$.

(f) If $a \subseteq b$ in $\mathfrak{B}$, then $\underline{\theta}a \subseteq \underline{\theta}b$. **P** $G_a \setminus G_b$ is negligible, $g_{an} \leq g_{bn}$ almost everywhere for every n, every $H_{kn}(a) \setminus H_{kn}(b)$ is negligible, $\tilde{H}_{kn}(a) \subseteq \tilde{H}_{kn}(b)$ for every n and k, and $\underline{\theta}a \subseteq \underline{\theta}b$. **Q**

(g) If a, $b \in \mathfrak{B}$ then $\underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b$. **P** $\chi G_{a\cap b} \geq_{\text{a.e.}} \chi G_a + \chi G_b - 1$ so $g_{a\cap b,n} \geq_{\text{a.e.}} g_{an} + g_{bn} - 1$ for every n. Accordingly

$$H_{k+1,n}(a) \cap H_{k+1,n}(b) \setminus H_{kn}(a \cap b)$$

is negligible, and (because $\underline{\phi}_n$ is a lower density)

$$\tilde{H}_{kn}(a \cap b) \supseteq \underline{\phi}_n(H_{k+1,n}(a) \cap H_{k+1,n}(b)) = \tilde{H}_{k+1,n}(a) \cap \tilde{H}_{k+1,n}(b)$$

for all $k \geq 1$, $n \in \mathbb{N}$. Now, if $x \in \underline{\theta}a \cap \underline{\theta}b$, then, for any $k \geq 1$, there are n_1, $n_2 \in \mathbb{N}$ such that

$$x \in \bigcap_{m\geq n_1} \tilde{H}_{k+1,m}(a), \quad x \in \bigcap_{m\geq n_2} \tilde{H}_{k+1,m}(b).$$

But this means that

$$x \in \bigcap_{m\geq \max(n_1,n_2)} \tilde{H}_{km}(a \cap b).$$

As k is arbitrary, $x \in \underline{\theta}(a \cap b)$; as x is arbitrary, $\underline{\theta}a \cap \underline{\theta}b \subseteq \underline{\theta}(a \cap b)$. We know already from (f) that $\underline{\theta}(a \cap b) \subseteq \underline{\theta}a \cap \underline{\theta}b$, so $\underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b$. **Q**

(h) If $a \in \mathfrak{B}$, then $\underline{\theta}a^\bullet = a$. **P** $\langle g_{an}\rangle_{n\in\mathbb{N}} \to \chi G_a$ a.e., so setting

$$V_a = \bigcap_{k\geq 1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} H_{km}(a) = \{x : \liminf_{n\to\infty} g_{an}(x) \geq 1\},$$

$V_a \triangle G_a$ is negligible, and $V_a^\bullet = a$; but

$$\underline{\theta}a \triangle V_a \subseteq \bigcup_{k\geq 1, n\in\mathbb{N}} H_{kn}(a) \triangle \tilde{H}_{kn}(a)$$

is negligible, so $\underline{\theta}a^\bullet$ is also equal to a. **Q** Thus $\underline{\theta}$ is a partial lower density with domain $\mathfrak{B}$.

(i) Finally, $\underline{\theta}$ extends $\underline{\theta}_n$ for every $n \in \mathbb{N}$. **P** If $a \in \mathfrak{B}_n$, then $G_a \in \Sigma_m$ for every $m \geq n$, so $g_{am} =_{\text{a.e.}} \chi G_a$ for every $m \geq n$; $H_{km}(a) \triangle G_a$ is negligible for $k \geq 1$, $m \geq n$;

$$\tilde{H}_{km} = \underline{\phi}_m G_a = \underline{\theta}_m a = \underline{\theta}_n a$$

for $k \geq 1$, $m \geq n$ (this is where I use the hypothesis that $\underline{\theta}_{m+1}$ extends $\underline{\theta}_m$ for every m); and

$$\underline{\theta}a = \bigcap_{k\ge1}\bigcup_{r\in\mathbb{N}}\bigcap_{m\ge r}\tilde{H}_{km}(a)$$
$$= \bigcap_{k\ge1}\bigcup_{r\ge n}\bigcap_{m\ge r}\tilde{H}_{km}(a) = \bigcap_{k\ge1}\bigcup_{r\ge n}\underline{\theta}_n a = \underline{\theta}_n a. \textbf{ Q}$$

The proof is complete.

341H Now for the first main theorem.

Theorem Let (X,Σ,μ) be any strictly localizable measure space. Then it has a lower density $\underline{\phi}:\Sigma\to\Sigma$. If $\mu X>0$ we can take $\underline{\phi}X=X$.

proof : Part A I deal first with the case of probability spaces. Let (X,Σ,μ) be a probability space, and $(\mathfrak{A},\bar{\mu})$ its measure algebra.

(a) Set $\kappa=\#(\mathfrak{A})$ and enumerate $\mathfrak{A}$ as $\langle a_\xi\rangle_{\xi<\kappa}$. For $\xi\le\kappa$ let $\mathfrak{A}_\xi$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta:\eta<\xi\}$. I seek to define a lower density $\underline{\theta}:\mathfrak{A}\to\Sigma$ as the last of a family $\langle\underline{\theta}_\xi\rangle_{\xi\le\kappa}$, where $\underline{\theta}_\xi:\mathfrak{A}_\xi\to\Sigma$ is a partial lower density for each ξ. The inductive hypothesis will be that $\underline{\theta}_\xi$ extends $\underline{\theta}_\eta$ whenever $\eta\le\xi\le\kappa$.

To start the induction, we have $\mathfrak{A}_0=\{0,1\}$, $\underline{\theta}_0 0=\emptyset$, $\underline{\theta}_0 1=X$.

(b) *Inductive step to a successor ordinal* ξ Given a successor ordinal $\xi\le\kappa$, express it as $\zeta+1$; we are supposing that $\underline{\theta}_\zeta:\mathfrak{A}_\zeta\to\Sigma$ has been defined. Now $\mathfrak{A}_\xi$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\zeta\cup\{a_\zeta\}$ (because this is a closed subalgebra, by 323K). So 341F tells us that $\underline{\theta}_\zeta$ can be extended to a partial lower density $\underline{\theta}_\xi$ with domain $\mathfrak{A}_\xi$.

(c) *Inductive step to a non-zero limit ordinal* ξ *of countable cofinality* In this case, there is a strictly increasing sequence $\langle\zeta(n)\rangle_{n\in\mathbb{N}}$ with supremum ξ. Applying 341G with $\mathfrak{B}_n=\mathfrak{A}_{\zeta(n)}$, we see that there is a partial lower density $\underline{\theta}_\xi$, with domain the closed subalgebra $\mathfrak{B}$ generated by $\bigcup_{n\in\mathbb{N}}\mathfrak{A}_{\zeta(n)}$, extending every $\underline{\theta}_{\zeta(n)}$. Now $\mathfrak{A}_{\zeta(n)}\subseteq\mathfrak{A}_\xi$ for every n, so $\mathfrak{B}\subseteq\mathfrak{A}_\xi$; but also, if $\eta<\xi$, there is an $n\in\mathbb{N}$ such that $\eta<\zeta(n)$, so that $a_\eta\in\mathfrak{A}_{\zeta(n)}\subseteq\mathfrak{B}$; as η is arbitrary, $\mathfrak{A}_\xi\subseteq\mathfrak{B}$ and $\mathfrak{A}_\xi=\mathfrak{B}$. Again, if $\eta<\xi$, there is an n such that $\eta\le\zeta(n)$, so that $\underline{\theta}_{\zeta(n)}$ extends $\underline{\theta}_\eta$ and $\underline{\theta}_\xi$ extends $\underline{\theta}_\eta$. Thus the induction continues.

(d) *Inductive step to a limit ordinal* ξ *of uncountable cofinality* In this case, $\mathfrak{A}_\xi=\bigcup_{\eta<\xi}\mathfrak{A}_\eta$. **P** Because $\mathfrak{A}$ is ccc, every member a of $\mathfrak{A}_\xi$ must be in the closed subalgebra of $\mathfrak{A}$ generated by some countable subset A of $\{a_\eta:\eta<\xi\}$ (331Gd-331Ge). Now A can be expressed as $\{a_\eta:\eta\in I\}$ for some countable $I\subseteq\xi$. As I cannot be cofinal with ξ, there is a $\zeta<\xi$ such that $\eta<\zeta$ for every $\eta\in I$, so that $A\subseteq\mathfrak{A}_\zeta$ and $a\in\mathfrak{A}_\zeta$. **Q**

But now, because $\underline{\theta}_\zeta$ extends $\underline{\theta}_\eta$ whenever $\eta\le\zeta<\xi$, we have a function $\underline{\theta}_\xi:\mathfrak{A}_\xi\to\Sigma$ defined by writing $\underline{\theta}_\xi a=\underline{\theta}_\eta a$ whenever $\eta<\xi$ and $a\in\mathfrak{A}_\eta$. Because the family $\{\mathfrak{A}_\eta:\eta<\xi\}$ is totally ordered and every $\underline{\theta}_\eta$ is a partial lower density, $\underline{\theta}_\xi$ is a partial lower density.

Thus the induction proceeds when ξ is a limit ordinal of uncountable cofinality.

(e) The induction stops when we reach $\underline{\theta}_\kappa:\mathfrak{A}\to\Sigma$, which is a lower density such that $\underline{\theta}_\kappa 1=X$. Setting $\underline{\phi}E=\underline{\theta}_\kappa E^\bullet$, $\underline{\phi}$ is a lower density such that $\underline{\phi}X=X$.

Part B The general case of a strictly localizable measure space follows easily. First, if $\mu X=0$, then $\mathfrak{A}=\{0\}$ and we can set $\underline{\phi}0=\emptyset$. Second, if μ is totally finite but not zero, we can replace it by ν, where $\nu E=\mu E/\mu X$ for every $E\in\Sigma$; a lower density for ν is also a lower density for μ. Third, if μ is not totally finite, let $\langle X_i\rangle_{i\in I}$ be a decomposition of X (211E). There is surely some j such that $\mu X_j>0$; replacing X_j by $X_j\cup\bigcup\{X_i:i\in I,\,\mu X_i=0\}$, we may assume that $\mu X_i>0$ for every $i\in I$. For each $i\in I$, let $\underline{\phi}_i:\Sigma_i\to\Sigma_i$ be a lower density for μ_i, where $\Sigma_i=\Sigma\cap\mathcal{P}X_i$ and $\mu_i=\mu\restriction\Sigma_i$, such that $\underline{\phi}_i X_i=X_i$. Then it is easy to check that we have a lower density $\underline{\phi}:\Sigma\to\Sigma$ given by setting

$$\underline{\phi}E=\bigcup_{i\in I}\underline{\phi}_i(E\cap X_i)$$

for every $E\in\Sigma$, and that $\underline{\phi}X=X$.

341I The next step is to give a method of moving from lower densities to liftings. I start with an elementary remark on lower densities on complete measure spaces.

Lemma Let (X,Σ,μ) be a complete measure space with measure algebra $\mathfrak{A}$.

(a) Suppose that $\underline{\theta}:\mathfrak{A}\to\Sigma$ is a lower density and $\underline{\theta}_1:\mathfrak{A}\to\mathcal{P}X$ is a function such that $\underline{\theta}_1 0=\emptyset$, $\underline{\theta}_1(a\cap b)=\underline{\theta}_1 a\cap\underline{\theta}_1 b$ for all a, $b\in\mathfrak{A}$ and $\underline{\theta}_1 a\supseteq\underline{\theta}a$ for all $a\in\mathfrak{A}$. Then $\underline{\theta}_1$ is a lower density. If $\underline{\theta}_1$ is a Boolean homomorphism, it is a lifting.

(b) Suppose that $\underline{\phi}:\Sigma\to\Sigma$ is a lower density and $\underline{\phi}_1:\Sigma\to\mathcal{P}X$ is a function such that $\underline{\phi}_1 E=\underline{\phi}_1 F$ whenever $E\triangle F$ is negligible, $\underline{\phi}_1\emptyset=\emptyset$, $\underline{\phi}_1(E\cap F)=\underline{\phi}_1 E\cap\underline{\phi}_1 F$ for all E, $F\in\Sigma$ and $\underline{\phi}_1 E\supseteq\underline{\phi}E$ for all $E\in\Sigma$. Then $\underline{\phi}_1$ is a lower density. If $\underline{\phi}_1$ is a Boolean homomorphism, it is a lifting.

proof (a) All I have to check is that $\underline{\theta}_1 a\in\Sigma$ and $(\underline{\theta}_1 a)^\bullet=a$ for every $a\in\mathfrak{A}$. But

$$\underline{\theta}a\subseteq\underline{\theta}_1 a,\quad \underline{\theta}(1\setminus a)\subseteq\underline{\theta}_1(1\setminus a),\quad \underline{\theta}_1 a\cap\underline{\theta}_1(1\setminus a)=\underline{\theta}_1 0=\emptyset.$$

So

$$\underline{\theta}a\subseteq\underline{\theta}_1 a\subseteq X\setminus\underline{\theta}(1\setminus a).$$

Since

$$(\underline{\theta}a)^\bullet=a=(X\setminus\underline{\theta}(1\setminus a))^\bullet,$$

and μ is complete, $\underline{\theta}_1$ is a lower density. If it is a Boolean homomorphism, then it is also a lifting (341De).

(b) This follows by the same argument, or by looking at the functions from $\mathfrak{A}$ to Σ defined by $\underline{\phi}$ and $\underline{\phi}_1$ and using (a).

341J Proposition Let (X,Σ,μ) be a complete measure space such that $\mu X>0$, and $\mathfrak{A}$ its measure algebra.

(a) If $\underline{\theta}:\mathfrak{A}\to\Sigma$ is any lower density, there is a lifting $\theta:\mathfrak{A}\to\Sigma$ such that $\theta a\supseteq\underline{\theta}a$ for every $a\in\mathfrak{A}$.

(b) If $\underline{\phi}:\Sigma\to\Sigma$ is any lower density, there is a lifting $\phi:\Sigma\to\Sigma$ such that $\phi E\supseteq\underline{\phi}E$ for every $E\in\Sigma$.

proof (a) For each $x\in\underline{\theta}1$, set

$$I_x=\{a:a\in\mathfrak{A},\,x\in\underline{\theta}(1\setminus a)\}.$$

Then I_x is a proper ideal of $\mathfrak{A}$. **P** We have

$0\in I_x$, because $x\in\underline{\theta}1$,

if $b\subseteq a\in I_x$ then $b\in I_x$, because $x\in\underline{\theta}(1\setminus a)\subseteq\underline{\theta}(1\setminus b)$,

if a, $b\in I_x$ then $a\cup b\in I_x$, because $x\in\underline{\theta}(1\setminus a)\cap\underline{\theta}(1\setminus b)=\underline{\theta}(1\setminus(a\cup b))$,

$1\notin I_x$ because $x\notin\emptyset=\underline{\theta}0$. **Q**

For $x\in X\setminus\underline{\theta}1$, set $I_x=\{0\}$; this is also a proper ideal of $\mathfrak{A}$, because $\mathfrak{A}\neq\{0\}$. By 311D, there is a surjective Boolean homomorphism $\pi_x:\mathfrak{A}\to\{0,1\}$ such that $\pi_x d=0$ for every $d\in I_x$.

Define $\theta:\mathfrak{A}\to\mathcal{P}X$ by setting

$$\theta a=\{x:x\in X,\,\pi_x(a)=1\}$$

for every $a\in\mathfrak{A}$. It is easy to check that, because every π_x is a surjective Boolean homomorphism, θ is a Boolean homomorphism. Now for any $a\in\mathfrak{A}$, $x\in X$,

$$x\in\underline{\theta}a\Longrightarrow 1\setminus a\in I_x\Longrightarrow\pi_x(1\setminus a)=0\Longrightarrow\pi_x a=1\Longrightarrow x\in\theta a.$$

Thus $\theta a\supseteq\underline{\theta}a$ for every $a\in\mathfrak{A}$. By 341I, θ is a lifting, as required.

(b) Repeat the argument above, or apply it, defining $\underline{\theta}$ by setting $\underline{\theta}(E^\bullet)=\underline{\phi}E$ for every $E\in\Sigma$, and ϕ by setting $\phi E=\theta(E^\bullet)$ for every E.

341K The Lifting Theorem Every complete strictly localizable measure space of non-zero measure has a lifting.

proof By 341H, it has a lower density, so by 341J it has a lifting.

341L Remarks If we count 341F-341K as a single argument, it may be the longest proof, after Carleson's theorem (§286), which I have yet presented in this treatise, and perhaps it will be helpful if I suggest ways of looking at its components.

(a) The first point is that the theorem should be thought of as one about probability spaces. The shift to general strictly localizable spaces (Part B of the proof of 341H) is purely a matter of technique. I would not have presented it if I did not think that it's worth doing, for a variety of reasons, but there is no significant idea needed, and if – for instance – the result were valid only for σ-finite spaces, it would still be one of the great theorems of mathematics. So the rest of these remarks will be directed to the ideas needed in probability spaces.

(b) All the proofs I know of the theorem depend in one way or another on an inductive construction. We do not, of course, need a transfinite induction written out in the way I have presented it in 341H above. Essentially the same proof can be presented as an application of Zorn's Lemma; if we take P to be the set of partial lower densities, then the arguments of 341G and part (A-d) of the proof of 341H can be adapted to prove that any totally ordered subset of P has an upper bound in P, while the argument of 341F shows that any maximal element of P must have domain $\mathfrak{A}$. I think it is purely a matter of taste which form one prefers. I suppose I have used the ordinal-indexed form largely because that seemed appropriate for Maharam's theorem in the last chapter.

(c) There are then three types of inductive step to examine, corresponding to 341F, 341G and (A-d) in 341H. The first and last are easier than the second. Seeking the one-step extension of $\underline{\theta} : \mathfrak{B} \to \Sigma$ to $\underline{\theta}_1 : \mathfrak{B}_1 \to \Sigma$, the natural model to use is the one-step extension of a Boolean homomorphism presented in 312O. The situation here is rather more complicated, as $\underline{\theta}_1$ is not fully specified by the value of $\underline{\theta}_1 e$, and we do in fact have more freedom at this point than is entirely welcome. The formula used in the proof of 341F is derived from Graf & Weizsäcker 76.

(d) At this point I must call attention to the way in which the whole proof is dominated by the choice of *closed* subalgebras as the domains of our partial liftings. This is what makes the inductive step to a limit ordinal ξ of countable cofinality difficult, because $\mathfrak{A}_\xi$ will ordinarily be larger than $\bigcup_{\eta<\xi} \mathfrak{A}_\eta$. But it is absolutely essential in the one-step extensions as treated here. (I will return to this point in §535 of Volume 5. See also 341Ye.)

Because we are dealing with a ccc algebra $\mathfrak{A}$, the requirement that the $\mathfrak{A}_\xi$ should be closed is not a problem when $\mathrm{cf}\,\xi$ is uncountable, since in this case $\bigcup_{\eta<\xi} \mathfrak{A}_\eta$ is already a closed subalgebra; this is the only idea needed in (A-d) of 341H.

(e) So we are left with the inductive step to ξ when $\mathrm{cf}\,\xi = \omega$, which is 341G. Here we actually need some measure theory, and a particularly striking bit. (You will see that the *measure* μ, as opposed to the algebras Σ and $\mathfrak{A}$ and the homomorphism $E \mapsto E^\bullet$ and the ideal of negligible sets, is simply not mentioned anywhere else in the whole argument.)

(i) The central idea is to use the fact that bounded martingales converge to define $\underline{\theta} a$ in terms of a sequence of conditional expectations. Because I have chosen a fairly direct assault on the problem, some of the surrounding facts are not perhaps so clearly visible as they might have been if I had used a more leisurely route. For each $a \in \mathfrak{A}$, I start by choosing a representative $G_a \in \Sigma$; let me emphasize that this is a crude application of the axiom of choice, and that the different sets G_a are in no way coordinated. (The theorem we are proving is that they *can* be coordinated, but we have not reached that point yet.) Next, I choose, arbitrarily, a conditional expectation g_{an} of χG_a on each Σ_n. Once again, the choices are not coordinated; but the martingale theorem assures us that $g_a = \lim_{n\to\infty} g_{an}$ is defined almost everywhere, and is equal almost everywhere to χG_a if $a \in \mathfrak{B}$. Of course I could have gone to the g_{an} directly, without mentioning the G_a; g_{an} is a Radon-Nikodým derivative of the countably additive functional $E \mapsto \bar{\mu}(E^\bullet \cap a) : \Sigma_n \to \mathbb{R}$. Now the g_{an}, like the G_a, are not uniquely defined. But they are defined 'up to a negligible set', so that any alternative functions g'_{an} would have $g'_{an} =_{\text{a.e.}} g_{an}$. This means that the sets $H_{kn}(a) = \{x : g_{an}(x) \ge 1-2^{-k}\}$ are also defined 'up to a negligible set', and consequently the sets $\tilde{H}_{kn}(a) = \underline{\phi}_n(H_{kn}(a))$ are uniquely defined. I point this out to show that it is not a complete miracle that we have formulae

$$\tilde{H}_{kn}(a) \subseteq \tilde{H}_{kn}(b) \text{ if } a \subseteq b,$$

$$\tilde{H}_{kn}(a \cap b) \supseteq \tilde{H}_{k+1,n}(a) \cap \tilde{H}_{k+1,n}(b) \text{ for all } a,\, b \in \mathfrak{A}$$

which do not ask us to turn a blind eye to any negligible sets. I note in passing that I could have defined the $\tilde{H}_{kn}(a)$ without mentioning the g_{an}; in fact

$$\tilde{H}_{kn}(a)=\underline{\theta}_n(\sup\{c:c\in\mathfrak{B}_n,\ \bar{\mu}(a\cap d)\ge 1-2^{-k}\bar{\mu}d \text{ whenever } d\in\mathfrak{B}_n \text{ and } d\subseteq c\}).$$

(ii) Now, with the sets $\tilde{H}_{kn}(a)$ in hand, we can look at

$$\tilde{V}_a=\bigcap_{k\ge1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde{H}_{kn}(a);$$

because $g_{an}\to\chi G_a$ a.e., $\tilde{V}_a\triangle G_a$ is negligible and $\tilde{V}_a^\bullet=a$ for every $a\in\mathfrak{A}_\xi$. The rest of the argument amounts to checking that $a\mapsto\tilde{V}_a$ will serve for $\underline{\theta}$.

(f) The arguments above apply to all probability spaces, and show that every probability space has a lower density. The next step is to convert a lower density into a lifting. It is here that we need to assume completeness. The point is that we can find a Boolean homomorphism $\theta:\mathfrak{A}\to\mathcal{P}X$ such that $\underline{\theta}a\subseteq\theta a$ for every a; this corresponds just to extending the ideals $I_x=\{a:x\in\underline{\theta}(1\setminus a)\}$ to maximal ideals (and giving a moment's thought to $x\in X\setminus\underline{\theta}1$). In order to ensure that $\theta a\in\Sigma$ and $(\theta a)^\bullet=a$, we have to observe that θa is sandwiched between $\underline{\theta}a$ and $X\setminus\underline{\theta}(1\setminus a)$, which differ by a negligible set; so that if μ is complete all will be well.

(g) The fact that completeness is needed at only one point in the argument makes it natural to wonder whether the theorem might be true for probability spaces in general. (I will come later, in 341M, to non-strictly-localizable spaces.) There is as yet no satisfactory answer to this. For Borel measure on $\mathbb{R}$, the question is known to be undecidable from the ordinary axioms of set theory (including the axiom of choice, but not the continuum hypothesis, as usual); I will give the easy part of the argument in §535; see Burke 93 for the rest. But I conjecture that there is a counter-example under the ordinary axioms (see 341Z below).

(h) Quite apart from whether completeness is needed in the argument, it is not absolutely clear why measure theory is required. The general question of whether a lifting exists can be formulated for any triple $(X,\Sigma,\mathcal{I})$ where X is a set, Σ is a σ-algebra of subsets of X, and $\mathcal{I}$ is a σ-ideal of Σ. (See 341Ya below.) S.Shelah has given an example of such a triple without a lifting in which two of the basic properties of the measure-theoretic case are satisfied: $(X,\Sigma,\mathcal{I})$ is 'complete' in the sense that every subset of any member of $\mathcal{I}$ belongs to Σ (and therefore to $\mathcal{I}$), and $\mathcal{I}$ is ω_1-saturated in Σ in the sense of 316C (see Shelah 98). But many other cases are known (e.g., 341Yb) in which liftings do exist.

(i) It is of course possible to prove 341K without mentioning 'lower densities', and there are even some advantages in doing so. The idea is to follow the lines of 341H, but with 'liftings' instead of 'lower densities' throughout. The inductive step to a successor ordinal is actually easier, because we have a Boolean homomorphism θ in 341F to extend, and we can use 312O as it stands if we can choose the pair E, $F=X\setminus E$ correctly. The inductive step to an ordinal of uncountable cofinality remains straightforward. But in the inductive step to an ordinal of countable cofinality, we find that in 341G we get no help from assuming that the $\underline{\theta}_n$ are actually liftings; we are still led to a lower density $\underline{\theta}$. So at this point we have to interpolate the argument of 341J to convert this lower density into a lifting.

I have chosen the more leisurely exposition, with the extra concept, partly in order to get as far as possible without assuming completeness of the measure and partly because lower densities are an important tool for further work (see §§345-346).

(j) For more light on the argument of 341G see also 363Xe and 363Yf below.

341M I remarked above that the shift from probability spaces to general strictly localizable spaces was simply a matter of technique. The question of which spaces have liftings is also primarily a matter concerning probability spaces, as the next result shows.

Proposition Let (X,Σ,μ) be a complete locally determined space with $\mu X>0$. Then it has a lifting iff it has a lower density iff it is strictly localizable.

proof If (X,Σ,μ) is strictly localizable then it has a lifting, by 341K. A lifting is already a lower density, and if (X,Σ,μ) has a lower density it has a lifting, by 341J. So we have only to prove that if it has a lifting then it is strictly localizable.

Let $\theta:\mathfrak{A}\to\Sigma$ be a lifting, where $\mathfrak{A}$ is the measure algebra of (X,Σ,μ). Let C be a partition of unity in $\mathfrak{A}$ consisting of elements of finite measure (322Ea). Set $\mathcal{A}=\{\theta c:c\in C\}$. Because C is disjoint, so is $\mathcal{A}$. Because $\sup C=1$ in $\mathfrak{A}$, every set of positive measure meets some member of $\mathcal{A}$ in a set of positive measure. So the conditions of 213Oa are satisfied, and (X,Σ,μ) is strictly localizable.

341N Extension of partial liftings The following facts are obvious from the proof of 341H, but it will be useful to have them out in the open.

Proposition Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ.

(a) Any partial lower density $\underline{\phi}_0:\mathrm{T}\to\Sigma$ has an extension to a lower density $\underline{\phi}:\Sigma\to\Sigma$.

(b) Suppose now that μ is complete. If $\underline{\phi}_0$ is a Boolean homomorphism, it has an extension to a lifting ϕ for μ.

proof (a) In Part A of the proof of 341H, let $\mathfrak{A}_\xi$ be the closed subalgebra of $\mathfrak{A}$ generated by $\{E^\bullet:E\in\mathrm{T}\}\cup\{a_\eta:\eta<\xi\}$, and set $\underline{\theta}_0E^\bullet=\underline{\phi}_0E$ for every $E\in\mathrm{T}$. Proceed with the induction as before. The only difference is that we no longer have a guarantee that $\underline{\phi}X=X$.

(b) Suppose now that $\underline{\phi}_0$ is a Boolean homomorphism and μ is complete. 341J tells us that there is a lifting $\phi:\Sigma\to\Sigma$ such that $\phi E\supseteq\underline{\phi}E$ for every $E\in\Sigma$. But if $E\in\mathrm{T}$ we must have $\phi E\supseteq\underline{\phi}_0E$,

$$\phi E\setminus\underline{\phi}_0E=\phi E\cap\underline{\phi}_0(X\setminus E)\subseteq\phi E\cap\phi(X\setminus E)=\emptyset,$$

so that $\phi E=\underline{\phi}_0E$, and ϕ extends $\underline{\phi}_0$.

341O Liftings and Stone spaces The arguments of this section so far involve repeated use of the axiom of choice, and offer no suggestion that any liftings (or lower densities) are in any sense 'canonical'. There is however one context in which we have a distinguished lifting. Suppose that we have the Stone space (Z,T,ν) of a measure algebra $(\mathfrak{A},\bar\mu)$; as in 311E, I think of Z as being the set of surjective Boolean homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, so that each $a\in\mathfrak{A}$ corresponds to the open-and-closed set $\widehat{a}=\{z:z(a)=1\}$. Then we have a lifting $\theta:\mathfrak{A}\to\mathrm{T}$ defined by setting $\theta a=\widehat{a}$ for each $a\in\mathfrak{A}$. (I am identifying $\mathfrak{A}$ with the measure algebra of ν, as in 321J.) The corresponding lifting $\phi:\mathrm{T}\to\mathrm{T}$ is defined by taking ϕE to be that unique open-and-closed set such that $E\triangle\phi E$ is negligible (or, if you prefer, meager).

Generally, liftings can be described in terms of Stone spaces, as follows.

341P Proposition Let (X,Σ,μ) be a measure space, $(\mathfrak{A},\bar\mu)$ its measure algebra, and (Z,T,ν) the Stone space of $(\mathfrak{A},\bar\mu)$ with its canonical measure.

(a) There is a one-to-one correspondence between liftings $\theta:\mathfrak{A}\to\Sigma$ and functions $f:X\to Z$ such that $f^{-1}[\widehat{a}]\in\Sigma$ and $(f^{-1}[\widehat{a}])^\bullet=a$ for every $a\in\mathfrak{A}$, defined by the formula

$$\theta a=f^{-1}[\widehat{a}]\text{ for every }a\in\mathfrak{A}.$$

(b) If (X,Σ,μ) is complete and locally determined, then a function $f:X\to Z$ satisfies the conditions of (a) iff (α) it is inverse-measure-preserving (β) the homomorphism it induces between the measure algebras of μ and ν is the canonical isomorphism defined by the construction of Z.

proof Recall that T is just the set $\{\widehat{a}\triangle M:a\in\mathfrak{A},\ M\subseteq Z\text{ is meager}\}$, and that $\nu(\widehat{a}\triangle M)=\bar\mu a$ for all such a, M; while the canonical isomorphism π between $\mathfrak{A}$ and the measure algebra of ν is defined by the formula

$$\pi F^\bullet=a\text{ whenever }F\in\mathrm{T},\ a\in\mathfrak{A}\text{ and }F\triangle\widehat{a}\text{ is meager}$$

(341K).

(a) If $\theta:\mathfrak{A}\to\Sigma$ is any Boolean homomorphism, then for every $x\in X$ we have a surjective Boolean homomorphism $f_\theta(x):\mathfrak{A}\to\mathbb{Z}_2$ defined by saying that $f_\theta(x)(a)=1$ if $x\in\theta a$, 0 otherwise. f_θ is a function from X to Z. We can recover θ from f_θ by the formula

$$\theta a=\{x:f_\theta(x)(a)=1\}=\{x:f_\theta(x)\in\widehat{a}\}=f_\theta^{-1}[\widehat{a}].$$

So $f_\theta^{-1}[\widehat{a}] \in \Sigma$ and, if θ is a lifting,

$$(f_\theta^{-1}[\widehat{a}])^\bullet = (\theta a)^\bullet = a.$$

for every $a \in \mathfrak{A}$.

Similarly, given a function $f : X \to Z$ with this property, then we can set $\theta a = f^{-1}[\widehat{a}]$ for every $a \in \mathfrak{A}$ to obtain a lifting $\theta : \mathfrak{A} \to \Sigma$; and of course we now have

$$f(x)(a) = 1 \iff f(x) \in \widehat{a} \iff x \in \theta a,$$

so $f_\theta = f$.

(b) Assume now that (X, Σ, μ) is complete and locally determined.

(i) Let $f : X \to Z$ be the function associated with a lifting θ, as in (a). I show first that f is inverse-measure-preserving. **P** If $F \in \mathrm{T}$, express it as $\widehat{a}\triangle M$, where $a \in \mathfrak{A}$ and $M \subseteq Z$ is meager. By 322F, $\mathfrak{A}$ is weakly (σ,∞)-distributive, so M is nowhere dense (316I). Consider $f^{-1}[M]$. If $E \subseteq X$ is measurable and of finite measure, then $E \cap f^{-1}[M]$ has a measurable envelope H (132Ee). **?** If $\mu H > 0$, then $b = H^\bullet \neq 0$ and $\widehat{b}$ is a non-empty open set in Z. Because M is nowhere dense, there is a non-zero $a \in \mathfrak{A}$ such that $\widehat{a} \subseteq \widehat{b} \setminus M$. Now $\mu(f^{-1}[\widehat{b}]\triangle H) = 0$, so $f^{-1}[\widehat{a}] \setminus H$ is negligible, and $f^{-1}[\widehat{a}] \cap H$ is a non-negligible measurable set disjoint from $E \cap f^{-1}[M]$ and included in H; which is impossible. **X** Thus H and $E \cap f^{-1}[M]$ are negligible. This is true for every measurable set E of finite measure. Because μ is complete and locally determined, $f^{-1}[M] \in \Sigma$ and $\mu f^{-1}[M] = 0$. So $f^{-1}[F] = f^{-1}[\widehat{a}]\triangle f^{-1}[M]$ is measurable, and

$$\mu f^{-1}[F] = \mu f^{-1}[\widehat{a}] = \mu\theta a = \bar{\mu} a = \nu\widehat{a} = \nu F.$$

As F is arbitrary, f is inverse-measure-preserving. **Q**

It follows at once that for any $F \in \mathrm{T}$,

$$f^{-1}[F]^\bullet = a = \pi F^\bullet$$

where a is that element of $\mathfrak{A}$ such that $M = F\triangle a$ is meager, because in this case $f^{-1}[\widehat{a}]^\bullet = a$, by (a), while $f^{-1}[M]$ is negligible. So π is the homomorphism induced by f.

(ii) Now suppose that $f : X \to Z$ is an inverse-measure-preserving function such that $f^{-1}[F]^\bullet = \pi F^\bullet$ for every $F \in \mathrm{T}$. Then, in particular,

$$f^{-1}[\widehat{a}]^\bullet = \pi\widehat{a}^\bullet = a$$

for every $a \in \mathfrak{A}$, so that f satisfies the conditions of (a).

341Q Corollary Let (X, Σ, μ) be a strictly localizable measure space, $(\mathfrak{A}, \bar{\mu})$ its measure algebra, and Z the Stone space of $\mathfrak{A}$; suppose that $\mu X > 0$. For $E \in \Sigma$ write E^* for the open-and-closed subset of Z corresponding to $E^\bullet \in \mathfrak{A}$. Then there is a function $f : X \to Z$ such that $E\triangle f^{-1}[E^*]$ is negligible for every $E \in \Sigma$. If μ is complete, then f is inverse-measure-preserving.

proof Let $\hat{\mu}$ be the completion of μ, and $\hat{\Sigma}$ its domain. Then we can identify $(\mathfrak{A}, \bar{\mu})$ with the measure algebra of $\hat{\mu}$ (322Da). Let $\theta : \mathfrak{A} \to \hat{\Sigma}$ be a lifting, and $f : X \to Z$ the corresponding function. If $E \in \Sigma$ then $E^* = \widehat{a}$ where $a = E^\bullet$, so $E\triangle f^{-1}[E^*] = E\triangle\theta E^\bullet$ is negligible. If μ is itself complete, so that $\hat{\Sigma} = \Sigma$, then f is inverse-measure-preserving, by 341Pb.

341X Basic exercises (a) Let (X, Σ, μ) be a measure space and $\phi : \Sigma \to \Sigma$ a function. Show that ϕ is a lifting iff it is a lower density and $\phi E \cup \phi(X \setminus E) = X$ for every $E \in \Sigma$.

>(b) Let $\nu_{\mathbb{N}}$ be the usual measure on $X = \{0,1\}^{\mathbb{N}}$, and $\mathrm{T}_{\mathbb{N}}$ its domain. For $x \in X$ and $n \in \mathbb{N}$ set $U_n(x) = \{y : y \in X,\ y{\restriction}n = x{\restriction}n\}$. For $E \in \mathrm{T}_{\mathbb{N}}$ set $\underline{\phi}E = \{x : \lim_{n\to\infty} 2^n\mu(E \cap U_n(x)) = 1\}$. Show that $\underline{\phi}$ is a lower density for $\nu_{\mathbb{N}}$.

>(c) Let $\mathfrak{A}$ be a Boolean algebra, I an ideal of $\mathfrak{A}$, and $\mathfrak{B}$ a countable subalgebra of the quotient algebra $\mathfrak{A}/I$. Show that there is a Boolean homomorphism $\theta : \mathfrak{B} \to \mathfrak{A}$ such that $(\theta b)^\bullet = b$ for every $b \in \mathfrak{B}$. (*Hint*: let $\langle b_n\rangle_{n\in\mathbb{N}}$ run over $\mathfrak{B}$; let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{B}$ generated by $\{b_i : i < n\}$; given $\theta{\restriction}\mathfrak{B}_n$, show that there is an $a_n \in \mathfrak{A}$ such that $a_n^\bullet = b_n$ and $\theta b' \subseteq a_n \subseteq \theta b''$ whenever $b', b'' \in \mathfrak{B}_n$ and $b' \subseteq b_n \subseteq b''$.)

>(d) Let P be the set of all lower densities of a complete measure space (X, Σ, μ), with measure algebra $\mathfrak{A}$, ordered by saying that $\underline{\theta} \le \underline{\theta}'$ if $\underline{\theta}a \subseteq \underline{\theta}'a$ for every $a \in \mathfrak{A}$. Show that any non-empty totally ordered subset of P has an upper bound in P. Show that if $\underline{\theta} \in P$, $a \in \mathfrak{A} \setminus \{0\}$ and $x \in X \setminus (\underline{\theta}a \cup \underline{\theta}(1 \setminus a))$, then $\underline{\theta}' : \mathfrak{A} \to \Sigma$ is a lower density, where $\underline{\theta}'b = \underline{\theta}b \cup \{x\}$ if either $a \subseteq b$ or there is a $c \in \mathfrak{A}$ such that $x \in \underline{\theta}c$ and $a \cap c \subseteq b$, and $\underline{\theta}'b = \underline{\theta}b$ otherwise. Hence prove 341J.

(e) Let (X, Σ, μ) and (Y, T, ν) be measure spaces and suppose that there is an inverse-measure-preserving function $f : X \to Y$ such that the associated homomorphism from the measure algebra of ν to that of μ (324M) is an isomorphism. Show that for every lifting ϕ for (Y, T, ν) we have a corresponding lifting ψ of (X, Σ, μ) defined uniquely by the formula

$$\psi(f^{-1}[F]) = f^{-1}[\phi F] \text{ for every } F \in \mathrm{T}.$$

(f) Let (X, Σ, μ) be a measure space, and write $\mathcal{L}^\infty(\Sigma)$ for the linear space of all bounded Σ-measurable functions from X to $\mathbb{R}$. Show that for any lifting $\phi : \Sigma \to \Sigma$ of μ there is a unique linear operator $T : L^\infty(\mu) \to \mathcal{L}^\infty(\Sigma)$ such that $T(\chi E)^\bullet = \chi(\phi E)$ for every $E \in \Sigma$ and $Tu \ge 0$ in $\mathcal{L}^\infty(\Sigma)$ whenever $u \ge 0$ in $L^\infty(\mu)$. Show that (i) $(Tu)^\bullet = u$ and $\sup_{x \in X} |(Tu)(x)| = \|u\|_\infty$ for every $u \in L^\infty(\mu)$ (ii) $T(u \times v) = Tu \times Tv$ for all u, $v \in L^\infty(\mu)$.

(g) Let μ be Lebesgue measure on $[0, 1]$. Write $\mathcal{L}^1_{\Sigma_L}$ for the linear space of integrable functions $f : [0, 1] \to \mathbb{R}$. Show that there is no operator $T : L^1(\mu) \to \mathcal{L}^1_{\Sigma_L}$ such that (i) $(Tu)^\bullet = u$ for every $u \in L^1(\mu)$ (ii) $Tu \ge Tv$ whenever $u \ge v$ in $L^1(\mu)$. (*Hint*: Let $F \subseteq \mathcal{L}^1_{\Sigma_L}$ be the countable set $\{n\chi[2^{-n}k, 2^{-n}(k+1)] : n \in \mathbb{N}, k < 2^n\}$. Show that if T satisfies (i) then there is an $x \in \{0, 1\}^{\mathbb{N}}$ such that $T(f^\bullet)(x) = f(x)$ for every $f \in F$; find a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in F such that $\{f_n^\bullet : n \in \mathbb{N}\}$ is bounded above in $L^1(\mu)$ but $\sup_{n \in \mathbb{N}} f_n(x) = \infty$.)

341Y Further exercises (a) Let X be a set, Σ an algebra of subsets of X and $\mathcal{I}$ an ideal of Σ; let $\mathfrak{A}$ be the quotient Boolean algebra $\Sigma/\mathcal{I}$. We say that a function $\theta : \mathfrak{A} \to \Sigma$ is a **lifting** if it is a Boolean homomorphism and $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$, and that $\underline{\theta} : \mathfrak{A} \to \Sigma$ is a **lower density** if $\underline{\theta}0 = \emptyset$, $\underline{\theta}(a \cap b) = \underline{\theta}a \cap \underline{\theta}b$ for all a, $b \in \mathfrak{A}$, and $(\underline{\theta}a)^\bullet = a$ for every $a \in \mathfrak{A}$.

Show that if $(X, \Sigma, \mathcal{I})$ is 'complete' in the sense that $F \in \Sigma$ whenever $F \subseteq E \in \mathcal{I}$, and if $X \notin \mathcal{I}$, and $\underline{\theta} : \mathfrak{A} \to \Sigma$ is a lower density, then there is a lifting $\theta : \mathfrak{A} \to \Sigma$ such that $\underline{\theta}a \subseteq \theta a$ for every $a \in \mathfrak{A}$.

(b) Let X be a Baire space, $\widehat{\mathcal{B}}$ the Baire-property algebra of X (314Yd) and $\mathcal{M}$ the ideal of meager subsets of X. Show that there is a lifting θ from $\widehat{\mathcal{B}}/\mathcal{M}$ to $\widehat{\mathcal{B}}$ such that $\theta G^\bullet \supseteq G$ for every open $G \subseteq X$. (*Hint*: in 341Ya, set $\underline{\theta}(G^\bullet) = G$ for every regular open set G.)

(c) Let (X, Σ, μ) be a Maharam-type-homogeneous probability space with Maharam type $\kappa \ge \omega$. Let $\mathcal{B}\mathfrak{a}_\kappa$ be the Baire σ-algebra of $Y = \{0, 1\}^\kappa$, that is, the σ-algebra of subsets of Y generated by the family $\{\{x : x(\xi) = 1\} : \xi < \kappa\}$, and let ν be the restriction to $\mathcal{B}\mathfrak{a}_\kappa$ of the usual measure on $\{0, 1\}^\kappa$. Show that there is an inverse-measure-preserving function $f : X \to Y$ which induces an isomorphism between the measure algebras of μ and ν.

(d) Let (X, Σ, μ) be a complete Maharam-type-homogeneous probability space with Maharam type $\kappa \ge \omega$, and give $Y = \{0, 1\}^\kappa$ its usual measure ν_κ. Show that there is an inverse-measure-preserving function $f : X \to Y$ which induces an isomorphism between the measure algebras of μ and ν_κ.

***(e)** Give an example of a complete probability space (X, Σ, μ), a subalgebra T of Σ, and a partial lower density $\underline{\phi} : \mathrm{T} \to \Sigma$ which has no extension to a lower density for μ. (*Hint*: There is a subset of $\{0, 1\}^{\mathfrak{c}}$, with cardinal $\mathfrak{c}$, which is non-negligible for the usual measure on $\{0, 1\}^{\mathfrak{c}}$.)

(f) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle a_i \rangle_{i \in I}$ a family in $\mathfrak{A}$. Let $\mathcal{B}\mathfrak{a}_I$ be the Baire σ-algebra of $Y = \{0, 1\}^I$, that is, the σ-algebra of subsets of Y generated by the family $\{E_i : i \in I\}$ where $E_i = \{y : y \in Y,\ y(i) = 1\}$ for $i \in I$. Show that there is a unique sequentially order-continuous Boolean homomorphism $\phi : \mathcal{B}\mathfrak{a}_I \to \mathfrak{A}$ such that $\phi E_i = a_i$ for every $i \in I$, and that $\phi[\mathcal{B}\mathfrak{a}_I]$ is the σ-subalgebra of $\mathfrak{A}$ generated by $\{a_i : i \in I\}$.

341Z Problems (a) Can we construct, using the ordinary axioms of mathematics (including the axiom of choice, but not the continuum hypothesis), a probability space (X, Σ, μ) with no lifting?

(b) Set $\kappa = \omega_3$. (There is a reason for taking ω_3 here; see 535E in Volume 5.) Let $\mathcal{B}\mathfrak{a}_\kappa$ be the Baire σ-algebra of $\{0,1\}^\kappa$ (as in 341Yc), and μ the restriction to $\mathcal{B}\mathfrak{a}_\kappa$ of the usual measure on $\{0,1\}^\kappa$. Can we show that μ has no lifting?

341 Notes and comments Innumerable variations of the proof of 341K have been devised, as each author has struggled with the technical complications. I have discussed the reasons for my own choices in 341L.

The theorem has a curious history. It was originally announced by von Neumann, but he seems never to have written his proof down, and the first published proof is that of Maharam 58. That argument is based on Maharam's theorem, 341Xe and 341Yd, which show that it is enough to find liftings for every $\{0,1\}^\kappa$; this requires most of the ideas presented above, but feels more concrete, and some of the details are slightly simpler. The argument as I have written it owes a great deal to Ionescu Tulcea & Ionescu Tulcea 69.

The lifting theorem and Maharam's theorem are the twin pillars of modern abstract measure theory. But there remains a degree of mystery about the lifting theorem which is absent from the other. The first point is that there is nothing canonical about the liftings we can construct, except in the quite exceptional case of Stone spaces (341O). Even when there is a more or less canonical lower density present (341E, 341Xb), the conversion of this into a lifting requires arbitrary choices, as in 341J. While we can distinguish some liftings as being somewhat more regular than others, I know of no criterion which marks out any particular lifting for Lebesgue measure, for instance, among the rest. Perhaps associated with this arbitrariness is the extreme difficulty of deciding whether liftings of any given type exist. Neither positive nor negative results are easily come by (I will present a few in the later sections of this chapter), and the nature of the obstacles remains quite unclear.

Version of 9.7.10

342 Compact measure spaces

The next three sections amount to an extended parenthesis, showing how the Lifting Theorem can be used to attack one of the fundamental problems of measure theory: the representation of Boolean homomorphisms between measure algebras by functions between appropriate measure spaces. This section prepares for the main idea by introducing the class of 'locally compact' measures (342Ad), with the associated concepts of 'compact' and 'perfect' measures (342Ac, 342K). These depend on the notions of 'inner regularity' (342Aa, 342B) and 'compact class' (342Ab, 342D). I list the basic permanence properties for compact and locally compact measures (342G-342I) and mention some of the compact measures which we have already seen (342J). Concerning perfect measures, I content myself with the proof that a locally compact measure is perfect (342L). I end the section with two examples (342M, 342N).

342A Definitions (a) Let (X, Σ, μ) be a measure space. If $\mathcal{K} \subseteq \mathcal{P}X$, I will say that μ is **inner regular** with respect to $\mathcal{K}$ if

$$\mu E = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$$

for every $E \in \Sigma$.

Of course μ is inner regular with respect to $\mathcal{K}$ iff it is inner regular with respect to $\mathcal{K} \cap \Sigma$. It is convenient in this context to interpret $\sup \emptyset$ as 0, so that we have to check the definition only when $\mu E > 0$, and need not insist that $\emptyset \in \mathcal{K}$.

(b) A family $\mathcal{K}$ of sets is a **compact class** if $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property.

Note that any subset of a compact class is again a compact class. (In particular, it is convenient to allow the empty set as a compact class.)

(c) A measure space (X,Σ,μ), or a measure μ, is **compact** if μ is inner regular with respect to some compact class of subsets of X.

Allowing $\emptyset$ as a compact class, and interpreting $\sup\emptyset$ as 0 in (a) above, μ is a compact measure whenever $\mu X=0$.

(d) A measure space (X,Σ,μ), or a measure μ, is **locally compact** if the subspace measure μ_E is compact whenever $E\in\Sigma$ and $\mu E<\infty$.

Remark I ought to point out that the original definitions of 'compact class' and 'compact measure' (Marczewski 53) correspond to what I will call 'countably compact class' and 'countably compact measure' in Volume 4. For another variation on the concept of 'compact class' see condition (β) in 343B(ii)-(iii).

For examples of compact measure spaces see 342J and 342Xf.

342B I prepare the ground with some straightforward lemmas.

Lemma Let (X,Σ,μ) be a measure space, and $\mathcal{K}\subseteq\Sigma$ a set such that whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}$ such that $K\subseteq E$ and $\mu K>0$. Let $E\in\Sigma$.

(a) There is a countable disjoint set $\mathcal{K}_1\subseteq\mathcal{K}$ such that $K\subseteq E$ for every $K\in\mathcal{K}_1$ and $\mu(\bigcup\mathcal{K}_1)=\mu E$.

(b) If $\mu E<\infty$ then $\mu(E\setminus\bigcup\mathcal{K}_1)=0$.

(c) In any case, there is for any $\gamma<\mu E$ a finite disjoint $\mathcal{K}_0\subseteq\mathcal{K}$ such that $K\subseteq E$ for every $K\in\mathcal{K}_0$ and $\mu(\bigcup\mathcal{K}_0)\ge\gamma$.

proof Set $\mathcal{K}'=\{K:K\in\mathcal{K},\,K\subseteq E,\,\mu K>0\}$. Let $\mathcal{K}^*$ be a maximal disjoint subfamily of $\mathcal{K}'$. If $\mathcal{K}^*$ is uncountable, then there is some $n\in\mathbb{N}$ such that $\{K:K\in\mathcal{K}^*,\,\mu K\ge 2^{-n}\}$ is infinite, so that there is a countable $\mathcal{K}_1\subseteq\mathcal{K}^*$ such that $\mu(\bigcup\mathcal{K}_1)=\infty=\mu E$.

If $\mathcal{K}^*$ is countable, set $\mathcal{K}_1=\mathcal{K}^*$. Then $F=\bigcup\mathcal{K}_1$ is measurable, and $F\subseteq E$. Moreover, there is no member of $\mathcal{K}'$ disjoint from F; but this means that $E\setminus F$ must be negligible. So $\mu F=\mu E$, and (a) is true. Now (b) and (c) follow at once, because

$$\mu(\bigcup\mathcal{K}_1)=\sup\{\mu(\bigcup\mathcal{K}_0):\mathcal{K}_0\subseteq\mathcal{K}_1\text{ is finite}\}.$$

Remark This lemma can be thought of as more versions of the principle of exhaustion; compare 215A.

342C Corollary Let (X,Σ,μ) be a measure space and $\mathcal{K}\subseteq\mathcal{P}X$ a family of sets such that (α) $K\cup K'\in\mathcal{K}$ whenever $K,\,K'\in\mathcal{K}$ and $K\cap K'=\emptyset$ (β) whenever $E\in\Sigma$ and $\mu E>0$, there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\mu K>0$. Then μ is inner regular with respect to $\mathcal{K}$.

proof Apply 342Bc to $\mathcal{K}\cap\Sigma$.

342D Lemma Let X be a set and $\mathcal{K}$ a family of subsets of X.

(a) The following are equiveridical:

(i) $\mathcal{K}$ is a compact class;

(ii) there is a topology $\mathfrak{T}$ on X such that X is compact and every member of $\mathcal{K}$ is a closed set for $\mathfrak{T}$.

(b) If $\mathcal{K}$ is a compact class, so are the families $\mathcal{K}_1=\{K_0\cup\ldots\cup K_n:K_0,\ldots,K_n\in\mathcal{K}\}$ and $\mathcal{K}_2=\{\bigcap\mathcal{K}':\emptyset\ne\mathcal{K}'\subseteq\mathcal{K}\}$.

proof (a)(i)$\Rightarrow$(ii) Let $\mathfrak{T}$ be the topology generated by $\{X\setminus K:K\in\mathcal{K}\}$. Then of course every member of $\mathcal{K}$ is closed for $\mathfrak{T}$. Let $\mathcal{F}$ be an ultrafilter on X. Then $\mathcal{K}\cap\mathcal{F}$ has the finite intersection property; because $\mathcal{K}$ is a compact class, it has non-empty intersection; take $x\in X\cap\bigcap(\mathcal{K}\cap\mathcal{F})$. The family

$$\{G:G\subseteq X,\text{ either }G\in\mathcal{F}\text{ or }x\notin G\}$$

is easily seen to be a topology on X, and contains $X\setminus K$ for every $K\in\mathcal{K}$ (because if $X\setminus K\notin\mathcal{F}$ then $K\in\mathcal{F}$ and $x\in K$), so includes $\mathfrak{T}$; but this just means that every $\mathfrak{T}$-open set containing x belongs to $\mathcal{F}$, that is, that $\mathcal{F}\to x$. As $\mathcal{F}$ is arbitrary, X is compact for $\mathfrak{T}$ (2A3R).

(ii)$\Rightarrow$(i) Use 3A3Da.

(b) Let $\mathfrak{T}$ be a topology on X such that X is compact and every member of $\mathcal{K}$ is closed for $\mathfrak{T}$; then the same is true of every member of $\mathcal{K}_1$ or $\mathcal{K}_2$.

342E Corollary Suppose that (X,Σ,μ) is a measure space and that $\mathcal{K}$ is a compact class such that whenever $E\in\Sigma$ and $\mu E>0$ there is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E$ and $\mu K>0$. Then μ is compact.

proof Set $\mathcal{K}_1=\{K_0\cup\ldots\cup K_n:K_0,\ldots,K_n\in\mathcal{K}\}$. By 342Db, $\mathcal{K}_1$ is a compact class, and by 342C μ is inner regular with respect to $\mathcal{K}_1$.

342F Corollary A measure space (X,Σ,μ) is compact iff there is a topology on X such that X is compact and μ is inner regular with respect to the closed sets.

proof (a) If μ is inner regular with respect to a compact class $\mathcal{K}$, then there is a compact topology on X such that every member of $\mathcal{K}$ is closed (342Da); now the family $\mathcal{F}$ of closed sets includes $\mathcal{K}$, so μ is also inner regular with respect to $\mathcal{F}$.

(b) If there is a compact topology on X such that μ is inner regular with respect to the family $\mathcal{K}$ of closed sets, then this is a compact class, so μ is a compact measure.

342G Now I look at the standard questions concerning preservation of the properties of 'compactness' or 'local compactness' under the usual manipulations.

Proposition (a) Any measurable subspace of a compact measure space is compact.

(b) The completion and c.l.d. version of a compact measure space are compact.

(c) A semi-finite measure space is compact iff its completion is compact iff its c.l.d. version is compact.

(d) The direct sum of a family of compact measure spaces is compact.

(e) The c.l.d. product of two compact measure spaces is compact.

(f) The product of any family of compact probability spaces is compact.

proof (a) Let (X,Σ,μ) be a compact measure space, and $E\in\Sigma$. If $\mathcal{K}$ is a compact class such that μ is inner regular with respect to $\mathcal{K}$, then $\mathcal{K}_E=\mathcal{K}\cap\mathcal{P}E$ is a compact class (just because it is a subset of $\mathcal{K}$) and the subspace measure μ_E is inner regular with respect to $\mathcal{K}_E$.

(b) Let (X,Σ,μ) be a compact measure space. Write $(X,\check\Sigma,\check\mu)$ for *either* the completion *or* the c.l.d. version of (X,Σ,μ). Let $\mathcal{K}\subseteq\mathcal{P}X$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$. Then $\check\mu$ also is inner regular with respect to $\mathcal{K}$. **P** If $E\in\check\Sigma$ and $\gamma<\check\mu E$ there is an $E'\in\Sigma$ such that $E'\subseteq E$ and $\mu E'>\gamma$; if $\check\mu$ is the c.l.d. version of μ, we may take $\mu E'$ to be finite. There is a $K\in\mathcal{K}\cap\Sigma$ such that $K\subseteq E'$ and $\mu K\ge\gamma$. Now $\check\mu K=\mu K\ge\gamma$ and $K\subseteq E$ and $K\in\mathcal{K}\cap\check\Sigma$. **Q**

(c) Now suppose that (X,Σ,μ) is semi-finite; again write $(X,\check\Sigma,\check\mu)$ for *either* its completion *or* its c.l.d. version. We already know that if μ is compact, so is $\check\mu$. If $\check\mu$ is compact, let $\mathcal{K}\subseteq\mathcal{P}X$ be a compact class such that $\check\mu$ is inner regular with respect to $\mathcal{K}$. Set $\mathcal{K}^*=\{\bigcap\mathcal{K}':\emptyset\ne\mathcal{K}'\subseteq\mathcal{K}\}$; then $\mathcal{K}^*$ is a compact class (342Db). Now μ is inner regular with respect to $\mathcal{K}^*$. **P** Take $E\in\Sigma$ and $\gamma<\mu E$. Choose $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle K_n\rangle_{n\in\mathbb{N}}$ as follows. Because μ is semi-finite, there is an $E_0\subseteq E$ such that $E_0\in\Sigma$ and $\gamma<\mu E_0<\infty$. Given $E_n\in\Sigma$ such that $\mu E_n>\gamma$, there is a $K_n\in\mathcal{K}\cap\check\Sigma$ such that $K_n\subseteq E_n$ and $\check\mu K_n>\gamma$. Now there is an $E_{n+1}\in\Sigma$ such that $E_{n+1}\subseteq K_n$ and $\mu E_{n+1}>\gamma$. Continue. On completing the induction, set $K=\bigcap_{n\in\mathbb{N}}K_n=\bigcap_{n\in\mathbb{N}}E_n$, so that $K\in\mathcal{K}^*\cap\Sigma$ and $K\subseteq E$ and $\mu K=\lim_{n\to\infty}\mu E_n\ge\gamma$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}^*$. **Q** As $\mathcal{K}^*$ is a compact class, μ is a compact measure.

(d) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of compact measure spaces, with direct sum (X,Σ,μ). We may suppose that each X_i is actually a subset of X, with μ_i the subspace measure. For each $i\in I$ let $\mathcal{K}_i\subseteq\mathcal{P}X_i$ be a compact class such that μ_i is inner regular with respect to $\mathcal{K}_i$. Then $\mathcal{K}=\bigcup_{i\in I}\mathcal{K}_i$ is a compact class, for if $\mathcal{K}'\subseteq\mathcal{K}$ has the finite intersection property, then $\mathcal{K}'\subseteq\mathcal{K}_i$ for some i, so has non-empty intersection. Now if $E\in\Sigma$ and $\mu E>0$ there is some $i\in I$ such that $\mu_i(E\cap X_i)>0$, and we can find a $K\in\mathcal{K}_i\cap\Sigma_i\subseteq\mathcal{K}\cap\Sigma$ such that $K\subseteq E\cap X_i$ and $\mu_iK>0$, in which case $\mu K>0$. By 342E, μ is compact.

(e) Let (X,Σ,μ) and (Y,T,ν) be two compact measure spaces, with c.l.d. product measure $(X\times Y,\Lambda,\lambda)$. Let $\mathfrak{T}$, $\mathfrak{S}$ be topologies on X, Y respectively such that X and Y are compact spaces and μ, ν are inner regular with respect to the closed sets. Then the product topology on $X\times Y$ is compact (3A3J).

The point is that λ is inner regular with respect to the family $\mathcal{K}$ of closed subsets of $X\times Y$. **P** Suppose that $W\in\Lambda$ and $\lambda W>\gamma$. Then there are $E\in\Sigma$, $F\in\mathrm{T}$ such that $\mu E<\infty$, $\nu F<\infty$ and $\lambda(W\cap(E\times F))>\gamma$ (251F). Now there are sequences $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ in Σ, T respectively such that

$$(E \times F) \setminus W \subseteq \bigcup_{n\in\mathbb{N}} E_n \times F_n,$$

$$\sum_{n=0}^{\infty} \mu E_n \cdot \nu F_n < \lambda((E \times F) \setminus W) + \lambda((E \times F) \cap W) - \gamma = \lambda(E \times F) - \gamma$$

(251C). Set

$$W' = (E \times F) \setminus \bigcup_{n\in\mathbb{N}} E_n \times F_n = \bigcap_{n\in\mathbb{N}}((E \times (F \setminus F_n)) \cup ((E \setminus E_n) \times F)).$$

Then $W' \subseteq W$, and

$$\lambda((E \times F) \setminus W') \le \lambda(\bigcup_{n\in\mathbb{N}} E_n \times F_n) \le \sum_{n=0}^{\infty} \mu E_n \cdot \nu F_n < \lambda(E \times F) - \gamma,$$

so $\lambda W' > \gamma$.

Set $\epsilon = \frac{1}{4}(\lambda W' - \gamma)/(1 + \mu E + \mu F)$. For each n, we can find closed measurable sets K_n, $K'_n \subseteq X$ and L_n, $L'_n \subseteq Y$ such that

$$K_n \subseteq E, \quad \mu(E \setminus K_n) \le 2^{-n}\epsilon,$$

$$L'_n \subseteq F \setminus F_n, \quad \nu((F \setminus F_n) \setminus L'_n) \le 2^{-n}\epsilon,$$

$$K'_n \subseteq E \setminus E_n, \quad \mu((E \setminus E_n) \setminus K'_n) \le 2^{-n}\epsilon,$$

$$L_n \subseteq F, \quad \nu(F \setminus L_n) \le 2^{-n}\epsilon.$$

Set

$$V = \bigcap_{n\in\mathbb{N}}(K_n \times L'_n) \cup (K'_n \times L_n) \subseteq W' \subseteq W.$$

Now

$$\begin{aligned} W' \setminus V \subseteq \bigcup_{n\in\mathbb{N}} &((E \setminus K_n) \times F) \cup (E \times ((F \setminus F_n) \setminus L'_n)) \\ &\cup (((E \setminus E_n) \setminus K'_n) \times F) \cup (E \times (F \setminus L_n)), \end{aligned}$$

so

$$\begin{aligned} \lambda(W' \setminus V) &\le \sum_{n=0}^{\infty} \mu(E \setminus K_n) \cdot \nu F + \mu E \cdot \nu((F \setminus F_n) \setminus L'_n) \\ &\qquad + \mu((E \setminus E_n) \setminus K'_n) \cdot \nu F + \mu E \cdot \nu(F \setminus L_n) \\ &\le \sum_{n=0}^{\infty} 2^{-n}\epsilon(2\mu E + 2\mu F) \le \lambda W' - \gamma, \end{aligned}$$

and $\lambda V \ge \gamma$. But V is a countable intersection of finite unions of products of closed measurable sets, so is itself a closed measurable set, and belongs to $\mathcal{K} \cap \Lambda$. **Q**

Accordingly the product topology on $X \times Y$ witnesses that λ is a compact measure.

(f) The same method works. In detail: let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i\in I}$ be a family of compact probability spaces, with product (X, Λ, λ). For each i, let $\mathfrak{T}_i$ be a topology on X_i such that X_i is compact and μ_i is inner regular with respect to the closed sets. Give X the product topology; this is compact. If $W \in \Lambda$ and $\epsilon > 0$, let $\langle C_n \rangle_{n\in\mathbb{N}}$ be a sequence of measurable cylinders (in the sense of 254A) such that $X \setminus W \subseteq \bigcup_{n\in\mathbb{N}} C_n$ and $\sum_{n=0}^{\infty} \lambda C_n \le \lambda(X \setminus W) + \epsilon$. Express each C_n as $\prod_{i\in I} E_{ni}$ where $E_{ni} \in \Sigma_i$ for each i and $J_n = \{i : E_{ni} \ne X_i\}$ is finite. For $n \in \mathbb{N}$ set $\epsilon_n = 2^{-n}\epsilon/(1 + \#(J_n))$. Choose closed measurable sets $K_{ni} \subseteq X_i \setminus E_{ni}$ such that $\mu_i((X_i \setminus E_{ni}) \setminus K_{ni}) \le \epsilon_n$ whenever $n \in \mathbb{N}$ and $i \in J_n$. For each $n \in \mathbb{N}$, set

$$V_n = \bigcup_{i\in J_n} \{x : x \in X,\, x(i) \in K_{ni}\},$$

so that V_n is a closed measurable subset of X. Observe that

$$X \setminus V_n = \{x : x(i) \in X \setminus K_{ni} \text{ for } i \in J_n\}$$

includes C_n, and that

$$\lambda(X \setminus (V_n \cup C_n)) \le \sum_{i\in J_n} \lambda\{x : x(i) \in X_i \setminus (K_{ni} \cup E_{ni})\} \le \sum_{i\in J_n} \epsilon_n \le 2^{-n}\epsilon.$$

Now set $V = \bigcap_{n\in\mathbb{N}} V_n$; then V is again a closed measurable set, and

$$X \setminus V \subseteq \bigcup_{n\in\mathbb{N}} C_n \cup (X \setminus (C_n \cup V_n))$$

has measure at most

$$\sum_{n=0}^{\infty} \lambda C_n + 2^{-n}\epsilon \le 1 - \lambda W + \epsilon + 2\epsilon,$$

so $\lambda V \ge \lambda W - 3\epsilon$. As W and ϵ are arbitrary, λ is inner regular with respect to the closed sets, and is a compact measure.

342H Proposition (a) A compact measure space is locally compact.

(b) A strictly localizable locally compact measure space is compact.

(c) Let (X, Σ, μ) be a measure space. Suppose that whenever $E \in \Sigma$ and $\mu E > 0$ there is an $F \in \Sigma$ such that $F \subseteq E$, $\mu F > 0$ and the subspace measure on F is compact. Then μ is locally compact.

proof (a) This is immediate from 342Ga and the definition of 'locally compact' measure space.

(b) Suppose that (X, Σ, μ) is a strictly localizable locally compact measure space. Let $\langle X_i\rangle_{i\in I}$ be a decomposition of X, and for each $i \in I$ let μ_i be the subspace measure on X_i. Then μ_i is compact. Now μ can be identified with the direct sum of the μ_i, so itself is compact, by 342Gd.

(c) Write $\mathcal{F}$ for the set of measurable sets $F \subseteq X$ such that the subspace measures μ_F are compact. Take $E \in \Sigma$ with $\mu E < \infty$. By 342Bb, there is a countable disjoint family $\langle F_i\rangle_{i\in I}$ in $\mathcal{F}$ such that $F_i \subseteq E$ for each i, and $F' = E \setminus \bigcup_{i\in I} F_i$ is negligible; now this means that $F' \in \mathcal{F}$ (342Ac), so we may take it that $E = \bigcup_{i\in I} F_i$. In this case μ_E is isomorphic to the direct sum of the measures μ_{F_i} and is compact. As E is arbitrary, μ is locally compact.

342I Proposition (a) Any measurable subspace of a locally compact measure space is locally compact.

(b) A measure space is locally compact iff its completion is locally compact iff its c.l.d. version is locally compact.

(c) The direct sum of a family of locally compact measure spaces is locally compact.

(d) The c.l.d. product of two locally compact measure spaces is locally compact.

proof (a) Trivial: if (X, Σ, μ) is locally compact, and $E \in \Sigma$, and $F \subseteq E$ is a measurable set of finite measure for the subspace measure on E, then $F \in \Sigma$ and $\mu F < \infty$, so the subspace measure on F is compact.

(b) Let (X, Σ, μ) be a measure space, and write $(X, \check{\Sigma}, \check{\mu})$ for *either* its completion *or* its c.l.d. version.

(i) Suppose that μ is locally compact, and that $\check{\mu}F < \infty$. Then there is an $E \in \Sigma$ such that $E \subseteq F$ and $\mu E = \check{\mu}F$. Let μ_E be the subspace measure on E induced by the measure μ; then we are assuming that μ_E is compact. Let $\mathcal{K} \subseteq \mathcal{P}E$ be a compact class such that μ_E is inner regular with respect to $\mathcal{K}$. Then, as in the proof of 342Gb, the subspace measure $\check{\mu}_F$ on F induced by $\check{\mu}$ is also inner regular with respect to $\mathcal{K}$, so $\check{\mu}_F$ is compact; as F is arbitrary, $\check{\mu}$ is locally compact.

(ii) Now suppose that $\check{\mu}$ is locally compact, and that $\mu E < \infty$. Then the subspace measure $\check{\mu}_E$ is compact. But this is just the completion of the subspace measure μ_E, so μ_E is compact, by 342Gc; as E is arbitrary, μ is locally compact.

(c) Put (a) and 342Hc together.

(d) Let (X, Σ, μ) and (Y, T, ν) be locally compact measure spaces, with product $(X \times Y, \Lambda, \lambda)$. If $W \in \Lambda$ and $\lambda W > 0$, there are $E \in \Sigma$, $F \in \mathrm{T}$ such that $\mu E < \infty$, $\nu F < \infty$ and $\lambda(W \cap (E \times F)) > 0$. Now the subspace measure $\lambda_{E\times F}$ induced by λ on $E \times F$ is just the product of the subspace measures (251Q(ii-α)), so is compact, and the subspace measure $\lambda_{W\cap(E\times F)}$ is therefore again compact, by 342Ga. By 342Hc, this is enough to show that λ is locally compact.

342J Examples It is time I listed some examples of compact measure spaces.

(a) Lebesgue measure on $\mathbb{R}^r$ is compact. (Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}^r$ which are compact for the usual topology. By 134Fb, Lebesgue measure is inner regular with respect to $\mathcal{K}$.)

(b) Similarly, any Radon measure on $\mathbb{R}^r$ (256A) is compact.

(c) If $(\mathfrak{A}, \bar{\mu})$ is any semi-finite measure algebra, the standard measure ν on its Stone space Z is compact. (By 322Ra, ν is inner regular with respect to the family of open-and-closed subsets of Z, which are all compact for the standard topology of Z, so form a compact class.)

(d) The usual measure on $\{0,1\}^I$ is compact, for any set I. (It is obvious that the usual measure on $\{0,1\}$ is compact; now use 342Gf.)

Remark Actually all these measures are 'Radon' in the sense of Volume 4.

342K One of the most important properties of (locally) compact measure spaces has been studied under the following name.

Definition Let (X,Σ,μ) be a measure space. Then (X,Σ,μ), or μ, is **perfect** if whenever $f : X \to \mathbb{R}$ is measurable, $E \in \Sigma$ and $\mu E > 0$, then there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$.

342L Theorem A semi-finite locally compact measure space is perfect.

proof Let (X,Σ,μ) be a semi-finite locally compact measure space, $f : X \to \mathbb{R}$ a measurable function, and $E \in \Sigma$ a set of non-zero measure. Because μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu F < \infty$. Now the subspace measure μ_F is compact; let $\mathfrak{T}$ be a topology on F such that F is compact and μ_F is inner regular with respect to the family $\mathcal{K}$ of closed sets for $\mathfrak{T}$.

Let $\langle\epsilon_q\rangle_{q\in\mathbb{Q}}$ be a family of strictly positive real numbers such that $\sum_{q\in\mathbb{Q}}\epsilon_q < \frac{1}{2}\mu F$. (For instance, you could set $\epsilon_{q(n)} = 2^{-n-3}\mu F$ where $\langle q(n)\rangle_{n\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$.) For each $q \in \mathbb{Q}$, set $E_q = \{x : x \in F,\, f(x) \le q\}$, $E'_q = \{x : x \in F,\, f(x) > q\}$, and choose K_q, $K'_q \in \mathcal{K} \cap \Sigma$ such that $K_q \subseteq E_q$, $K'_q \subseteq E'_q$, $\mu(E_q \setminus K_q) \le \epsilon_q$ and $\mu(E'_q \setminus K'_q) \le \epsilon_q$. Then $K = \bigcap_{q\in\mathbb{Q}}(K_q \cup K'_q) \in \mathcal{K}\cap\Sigma$, $K \subseteq F$ and

$$\mu(F \setminus K) \le \sum_{q\in\mathbb{Q}}\mu(E_q \setminus K_q) + \mu(E'_q \setminus K'_q) < \mu F,$$

so $\mu K > 0$.

The point is that $f{\restriction}K$ is continuous. **P** For any $q \in \mathbb{Q}$, $\{x : x \in K,\, f(x) \le q\} = K \cap K_q$ and $\{x : x \in K,\, f(x) > q\} = K \cap K'_q$. If $H \subseteq \mathbb{R}$ is open and $x \in K \cap f^{-1}[H]$, take q, $q' \in \mathbb{Q}$ such that $f(x) \in \,]q, q'] \subseteq H$; then $G = K \setminus (K_q \cup K'_{q'})$ is a relatively open subset of K containing x and included in $f^{-1}[H]$. Thus $K \cap f^{-1}[H]$ is relatively open in K; as H is arbitrary, $f{\restriction}K$ is continuous. **Q**

Accordingly $f[K]$ is a continuous image of a compact set, therefore compact; it is a subset of $f[E]$, and $\mu f^{-1}[f[K]] \ge \mu K > 0$. As f and E are arbitrary, μ is perfect.

342M I ought to give examples to distinguish between the concepts introduced here, partly on general principles, but also because it is not obvious that the concept of 'locally compact' measure space is worth spending time on at all. It is easy to distinguish between 'perfect' and '(locally) compact'; 'locally compact' and 'compact' are harder to separate.

Example Let X be an uncountable set and μ the countable-cocountable measure on X (211R). Then μ is perfect but not compact or locally compact.

proof (a) If $f : X \to \mathbb{R}$ is measurable and $E \subseteq X$ is measurable, with measure greater than 0, set $A = \{\alpha : \alpha \in \mathbb{R}, \{x : x \in X,\, f(x) \le \alpha\}$ is negligible$\}$. Then $\alpha \in A$ whenever $\alpha \le \beta \in A$. Since $X = \bigcup_{n\in\mathbb{N}}\{x : f(x) \le n\}$, there is some n such that $n \notin A$, in which case A is bounded above by n. Also there is some $m \in \mathbb{N}$ such that $\{x : f(x) > -m\}$ is non-negligible, in which case it must be conegligible, and $-m \in A$, so A is non-empty. Accordingly $\gamma = \sup A$ is defined in $\mathbb{R}$. Now for any $k \in \mathbb{N}$, $\{x : f(x) \le \gamma - 2^{-k}\}$ is negligible, so $\{x : f(x) < \gamma\}$ is negligible. Also, for any k, $\{x : f(x) \le \gamma + 2^{-k}\}$ is non-negligible, so $\{x : f(x) > \gamma + 2^{-k}\}$ must be negligible; accordingly, $\{x : f(x) > \gamma\}$ is negligible. But this means that $\{x : f(x) = \gamma\}$ is conegligible and has measure 1. Thus we have a compact set $K = \{\gamma\}$ such that $\mu f^{-1}[K] = 1$, and γ must belong to $f[E]$. As f and E are arbitrary, μ is perfect.

(b) μ is not compact. **P?** Suppose, if possible, that $\mathcal{K} \subseteq \mathcal{P}X$ is a compact class such that μ is inner regular with respect to $\mathcal{K}$. Then for every $x \in X$ there is a measurable set $K_x \in \mathcal{K}$ such that $K_x \subseteq X \setminus \{x\}$

and $\mu K_x > 0$, that is, K_x is conegligible. But this means that $\{K_x : x \in X\}$ must have the finite intersection property; as it also has empty intersection, $\mathcal{K}$ cannot be a compact class. **XQ**

(c) Because μ is totally finite, it cannot be locally compact.

Remark See also 342X(n-viii).

***342N Example** There is a complete locally determined localizable locally compact measure space which is not compact.

proof (a) I refer to the example of 216E. In that construction, we have a set I and a family $\langle x_\gamma\rangle_{\gamma\in C}$ in $X = \{0,1\}^I$ such that for every $D \subseteq C$ there is an $i \in I$ such that $D = \{\gamma : x_\gamma(i) = 1\}$; moreover, $\#(C) > \mathfrak{c}$. The σ-algebra Σ is the family of sets $E \subseteq X$ such that for every γ there is a countable set $J \subseteq I$ such that $\{x : x{\restriction}J = x_\gamma{\restriction}J\}$ is a subset of either E or $X \setminus E$; and for $E \in \Sigma$, μE is $\#(\{\gamma : x_\gamma \in E\})$ if this is finite, ∞ otherwise. Note that any subset of X determined by coordinates in a countable set belongs to Σ.

For each $\gamma \in C$, let $i_\gamma \in I$ be such that $x_\gamma(i_\gamma) = 1$, $x_\delta(i_\gamma) = 0$ for $\delta \neq \gamma$. (In 216E I took I to be $\mathcal{P}C$, and i_γ would be $\{\gamma\}$.) Set

$$Y = \{x : x \in X,\ \{\gamma : \gamma \in C,\ x(i_\gamma) = 1\} \text{ is finite}\}.$$

Give Y its subspace measure μ_Y with domain Σ_Y. Then μ_Y is complete, locally determined and localizable (214Ie). Note that $x_\gamma \in Y$ for every $\gamma \in C$.

(b) μ_Y is locally compact. **P** Suppose that $F \in \Sigma_Y$ and $\mu_Y F < \infty$. If $\mu_Y F = 0$ then surely the subspace measure μ_F is compact. Otherwise, we can express F as $E \cap Y$ where $E \in \Sigma$ and $\mu E = \mu_Y F$. Then $D = \{\gamma : x_\gamma \in E\} = \{\gamma : x_\gamma \in F\}$ is finite. For $\gamma \in D$ set

$$G'_\gamma = \{x : x \in X,\ x(i_\gamma) = 1,\ x(i_\delta) = 0 \text{ for every } \delta \in D \setminus \{\gamma\}\} \in \Sigma,$$

$$\mathcal{K}_\gamma = \{K : x_\gamma \in K \subseteq F \cap G'_\gamma\}.$$

Then each $\mathcal{K}_\gamma$ is a compact class, and members of different $\mathcal{K}_\gamma$'s are disjoint, so $\mathcal{K} = \bigcup_{\gamma\in D} \mathcal{K}_\gamma$ is a compact class.

Now suppose that H belongs to the subpsace σ-algebra Σ_F and $\mu_F H > 0$. Then there is a $\gamma \in D$ such that $x_\gamma \in H$, so that $H \cap G'_\gamma \in \mathcal{K} \cap \Sigma_F$ and $\mu_F(H \cap G'_\gamma) > 0$. By 342E, this is enough to show that μ_F is compact. As F is arbitrary, μ_Y is locally compact. **Q**

(c) μ_Y is not compact. **P?** Suppose, if possible, that μ_Y is inner regular with respect to a compact class $\mathcal{K} \subseteq \mathcal{P}Y$. For each $\gamma \in C$ set $G_\gamma = \{x : x \in X,\ x(i_\gamma) = 1\}$, so that $x_\gamma \in G_\gamma \in \Sigma$ and $\mu_Y(G_\gamma \cap Y) = 1$. There must therefore be a $K_\gamma \in \mathcal{K}$ such that $K_\gamma \subseteq G_\gamma \cap Y$ and $\mu_Y K_\gamma = 1$ (since μ_Y takes no value in $]0,1[$). Express K_γ as $Y \cap E_\gamma$, where $E_\gamma \in \Sigma$, and let $J_\gamma \subseteq I$ be a countable set such that

$$E_\gamma \supseteq \{x : x \in X,\ x{\restriction}J_\gamma = x_\gamma{\restriction}J_\gamma\}.$$

At this point I call on the full strength of 2A1P. There is a set $B \subseteq C$, with cardinal greater than $\mathfrak{c}$, such that $x_\gamma{\restriction}J_\gamma \cap J_\delta = x_\delta{\restriction}J_\gamma \cap J_\delta$ for all γ, $\delta \in B$. But this means that, for any finite set $D \subseteq B$, we can define $x \in X$ by setting

$$\begin{aligned} x(i) &= x_\alpha(i) \text{ if } \alpha \in D,\ i \in J_\alpha, \\ &= 0 \text{ if } i \in I \setminus \bigcup_{\alpha\in D} J_\alpha. \end{aligned}$$

It is easy to check that $\{\gamma : \gamma \in C,\ x(i_\gamma) = 1\} = D$, so that $x \in Y$; but now

$$x \in Y \cap \textstyle\bigcap_{\alpha\in D} E_\alpha = \bigcap_{\alpha\in D} K_\alpha.$$

What this shows is that $\{K_\alpha : \alpha \in B\}$ has the finite intersection property. It must therefore have non-empty intersection; say

$$y \in \textstyle\bigcap_{\alpha\in B} K_\alpha \subseteq \bigcap_{\alpha\in B} G_\alpha.$$

But now we have a member y of Y such that $\{\gamma : y(i_\gamma) = 1\} \supseteq B$ is infinite, contrary to the definition of Y. **XQ**

342X Basic exercises >(a) Show that a measure space (X,Σ,μ) is semi-finite iff μ is inner regular with respect to $\{E:\mu E<\infty\}$.

(b) Find a proof of 342B based on 215A.

(c) Let (X,Σ,μ) be a locally compact semi-finite measure space in which all singleton sets are negligible. Show that it is atomless.

(d) Let (X,Σ,μ) be a measure space, and ν an indefinite-integral measure over μ (234J[1]). Show that ν is compact, or locally compact, if μ is. (*Hint*: if $\mathcal{K}$ satisfies the conditions of 342E with respect to μ, then it satisfies them for ν.)

(e) Let $f:\mathbb{R}\to\mathbb{R}$ be any non-decreasing function, and ν_f the corresponding Lebesgue-Stieltjes measure. Show that ν_f is compact. (*Hint*: 256Xg.)

(f) Let μ be Lebesgue measure on $[0,1]$, ν the countable-cocountable measure on $[0,1]$, and λ their c.l.d. product. Show that λ is a compact measure. (*Hint*: let $\mathcal{K}$ be the family of sets $K\times A$ where $A\subseteq[0,1]$ is cocountable and $K\subseteq A$ is compact.)

(g)(i) Give an example of a compact probability space (X,Σ,μ), a set Y and a function $f:X\to Y$ such that the image measure μf^{-1} is not compact. (ii) Give an example of a compact probability space (X,Σ,μ) and a σ-subalgebra T of Σ such that $(X,\mathrm{T},\mu\restriction\mathrm{T})$ is not compact. (*Hint*: 342Xf.)

(h) Let (X,Σ,μ) be a perfect measure space, and $f:X\to\mathbb{R}$ a measurable function. Show that the image measure μf^{-1} is inner regular with respect to the compact subsets of $\mathbb{R}$, so is a compact measure.

(i) Let (X,Σ,μ) be a σ-finite measure space. Show that it is perfect iff for every measurable $f:X\to\mathbb{R}$ there is a Borel set $H\subseteq f[X]$ such that $f^{-1}[H]$ is conegligible in X. (*Hint*: 342Xh for 'only if', 256C for 'if'.)

(j) Let (X,Σ,μ) be a complete totally finite perfect measure space and $f:X\to\mathbb{R}$ a measurable function. Show that the image measure μf^{-1} is a Radon measure, and is the only Radon measure on $\mathbb{R}$ for which f is inverse-measure-preserving. (*Hint*: 256G.)

(k) Suppose that (X,Σ,μ) is a perfect measure space. (i) Show that if (Y,T,ν) is a measure space, and $f:X\to Y$ is a function such that $f^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$ and $f^{-1}[F]$ is μ-negligible for every ν-negligible set F, then (Y,T,ν) is perfect. (ii) Show that if T is a σ-subalgebra of Σ then $(X,\mathrm{T},\mu\restriction\mathrm{T})$ is perfect.

(l) Let (X,Σ,μ) be a perfect measure space such that Σ is the σ-algebra generated by a sequence of sets. Show that μ is compact. (*Hint*: if Σ is generated by $\{E_n:n\in\mathbb{N}\}$, set $f=\sum_{n=0}^{\infty}3^{-n}\chi E_n$ and consider $\{f^{-1}[K]:K\subseteq f[X]\text{ is compact}\}$.)

(m) Let (X,Σ,μ) be a semi-finite measure space. Show that μ is perfect iff $\mu\restriction\mathrm{T}$ is compact for every countably generated σ-subalgebra T of Σ.

(n) Show that (i) a measurable subspace of a perfect measure space is perfect (ii) a semi-finite measure space is perfect iff all its totally finite subspaces are perfect (iii) the direct sum of any family of perfect measure spaces is perfect (iv) the c.l.d. product of two perfect measure spaces is perfect (*hint*: put 342Xm and 342Ge together) (v) the product of any family of perfect probability spaces is perfect (vi) a measure space is perfect iff its completion is perfect (vii) the c.l.d. version of a perfect measure space is perfect (viii) any purely atomic measure space is perfect (ix) an indefinite-integral measure over a perfect measure is perfect (x) a sum (234G[2]) of perfect measures is perfect.

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[1]Formerly 234B.

[2]Later editions only.

(o) Let μ be Lebesgue measure on $\mathbb{R}$, A a subset of $\mathbb{R}$, and μ_A the subspace measure on A. Show that μ_A is compact iff it is perfect iff A is Lebesgue measurable. (*Hint*: if μ_A is perfect, consider the image measure $\mu_A h^{-1}$ on $\mathbb{R}$, where $h(x) = x$ for $x \in A$.)

342Y Further exercises (a) Let U be a Banach space such that there is a linear operator $T : U^{**} \to U$, of norm at most 1, such that $T\hat{u} = u$ for every $u \in U$, writing $\hat{u}$ for the member of U^{**} corresponding to u. Show that the family of closed balls in U is a compact class.

(b) Give an example of a compact class $\mathcal{K}$ of subsets of $\mathbb{N}$ such that there is no compact Hausdorff topology on $\mathbb{N}$ for which every member of $\mathcal{K}$ is closed.

(c) Show that the space (X, Σ, μ) of 216E and 342N is a compact measure space. (*Hint*: use the usual topology on $X = \{0, 1\}^I$.)

(d) Give an example of a compact complete locally determined measure space which is not localizable. (*Hint*: in 216D, add a point to each horizontal and vertical section of X, so that all the sections become compact measure spaces.)

342 Notes and comments The terminology I find myself using in this section – 'compact', 'locally compact', 'perfect' – is not entirely satisfactory, in that it risks collision with the same words applied to topological spaces. For the moment, this is not a serious problem; but when in Volume 4 we come to the systematic analysis of spaces which have both topologies and measures present, it will be necessary to watch our language carefully. Of course there are cases in which a 'compact class' of the sort discussed here can be taken to be the family of compact sets for some familiar topology, as in 342Ja-342Jd, but in others this is not so (see 342Xf); and even when we have a familiar compact class, the topology constructed from it by the method of 342Da need not be one we might expect. (Consider, for instance, the topology on $\mathbb{R}$ for which the closed sets are just the sets which are compact for the usual topology, together with the set $\mathbb{R}$ itself.)

I suppose that 'compact' and 'perfect' measure spaces look reasonably natural objects to study; they offer to illuminate one of the basic properties of Radon measures, the fact that (at least for totally finite Radon measures on Euclidean space) the image measure of a Radon measure under a measurable function is again Radon (256G, 342Xj). Indeed this was the original impetus for the study of perfect measures (Gnedenko & Kolmogorov 54, Sazonov 66). It is not obvious that there is any need to examine 'locally compact' measure spaces, but actually they are the chief purpose of this section, since the main theorem of the next section is an alternative characterization of semi-finite locally compact measure spaces (343B). Of course you may feel that the fact that 'locally compact' and 'compact' coincide for strictly localizable spaces (342Hb) excuses you from troubling about the distinction at first reading.

As with any new classification of measure spaces, it is worth finding out how the classes of 'compact' and 'perfect' measure spaces behave with respect to the standard constructions. I run through the basic facts in 342G-342I, 342Xd, 342Xk and 342Xn. We can also look for relationships between the new properties and those already studied. Here, in fact, there is not much to be said; 342N and 342Yd show that 'compactness' is largely independent of the classification in §211. However there are interactions with the concept of 'atom' (342Xc, 342Xn(viii)).

I give examples to show that perfect measure spaces need not be locally compact, and that locally compact measure spaces need not be compact (342M, 342N). The standard examples of measure spaces which are not perfect are non-measurable subspaces (342Xo); I will return to these in the next section (343L-343M).

Something which is not important to us at the moment, but is perhaps worth taking note of, is the following observation. To determine whether a measure space (X, Σ, μ) is compact, we need only the structure $(X, \Sigma, \mathcal{N})$, where $\mathcal{N}$ is the σ-ideal of negligible sets, since that is all that is referred to in the criterion of 342E. The same is true of local compactness, by 342Hc, and of perfectness, by the definition in 342K. Compare 342Xd, 342Xk and 342Xn(ix).

Much of the material of this section will be repeated in Volume 4 as part of a more systematic analysis of inner regularity.

Version of 17.11.10

343 Realization of homomorphisms

We are now in a position to make progress in one of the basic questions of abstract measure theory. In §324 I have already described the way in which a function between two measure spaces can give rise to a homomorphism between their measure algebras. In this section I discuss some conditions under which we can be sure that a homomorphism can be represented by a function.

The principal theorem of the section is 343B. If a measure space (X, Σ, μ) is locally compact, then many homomorphisms from the measure algebra of μ to other measure algebras will be representable by functions into X; moreover, this characterizes locally compact spaces. In general, a homomorphism between measure algebras can be represented by widely different functions (343I, 343J). But in some of the most important cases (e.g., Lebesgue measure) representing functions are 'almost' uniquely defined; I introduce the concept of 'countably separated' measure space to describe these (343D-343H).

343A Preliminary remarks It will be helpful to establish some vocabulary and a couple of elementary facts.

(a) If (X, Σ, μ) and (Y, T, ν) are measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, I will say that a function $f : X \to Y$ **represents** a homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ if $f^{-1}[F] \in \Sigma$ and $(f^{-1}[F])^\bullet = \pi(F^\bullet)$ for every $F \in \mathrm{T}$.

(Perhaps I should emphasize here that some homomorphisms are representable in this sense, and some are not; see 343M below for examples of non-representable homomorphisms.)

(b) If (X, Σ, μ) and (Y, T, ν) are measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, $f : X \to Y$ is a function, and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then

$$\{F : F \in \mathrm{T},\ f^{-1}[F] \in \Sigma \text{ and } f^{-1}[F]^\bullet = \pi F^\bullet\}$$

is a σ-subalgebra of T. (The verification is elementary.)

(c) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $\mathfrak{A}$ and $\mathfrak{B}$, and $\pi : \mathfrak{B} \to \mathfrak{A}$ a Boolean homomorphism which is represented by a function $f : X \to Y$. Let $(X, \hat{\Sigma}, \hat{\mu})$, $(Y, \hat{\mathrm{T}}, \hat{\nu})$ be the completions of (X, Σ, μ), (Y, T, ν); then $\mathfrak{A}$ and $\mathfrak{B}$ can be identified with the measure algebras of $\hat{\mu}$ and $\hat{\nu}$ (322Da). Now f still represents π when regarded as a function from $(X, \hat{\Sigma}, \hat{\mu})$ to $(Y, \hat{\mathrm{T}}, \hat{\nu})$. **P** If G is ν-negligible, there is a negligible $F \in \mathrm{T}$ such that $G \subseteq F$; since

$$f^{-1}[F]^\bullet = \pi F^\bullet = 0,$$

$f^{-1}[F]$ is μ-negligible, so $f^{-1}[G]$ is negligible, therefore belongs to $\hat{\Sigma}$. If G is any element of $\hat{\mathrm{T}}$, there is an $F \in \mathrm{T}$ such that $G \triangle F$ is negligible, so that

$$f^{-1}[G] = f^{-1}[F] \triangle f^{-1}[G \triangle F] \in \hat{\Sigma},$$

and

$$f^{-1}[G]^\bullet = f^{-1}[F]^\bullet = \pi F^\bullet = \pi G^\bullet. \text{ } \mathbf{Q}$$

343B Theorem Let (X, Σ, μ) be a non-empty semi-finite measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let (Z, Λ, λ) be the Stone space of $(\mathfrak{A}, \bar{\mu})$; for $E \in \Sigma$ write E^* for the open-and-closed subset of Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Then the following are equiveridical.

(i) (X, Σ, μ) is locally compact in the sense of 342Ad.

(ii) There is a family $\mathcal{K} \subseteq \Sigma$ such that (α) whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$ (β) whenever $\mathcal{K}' \subseteq \mathcal{K}$ is such that $\mu(\bigcap \mathcal{K}_0) > 0$ for every non-empty finite set $\mathcal{K}_0 \subseteq \mathcal{K}'$, then $\bigcap \mathcal{K}' \neq \emptyset$.

(iii) There is a family $\mathcal{K} \subseteq \Sigma$ such that $(\alpha)'$ μ is inner regular with respect to $\mathcal{K}$ (β) whenever $\mathcal{K}' \subseteq \mathcal{K}$ is such that $\mu(\bigcap \mathcal{K}_0) > 0$ for every non-empty finite set $\mathcal{K}_0 \subseteq \mathcal{K}'$, then $\bigcap \mathcal{K}' \neq \emptyset$.

(iv) There is a function $f : Z \to X$ such that $f^{-1}[E] \triangle E^*$ is negligible for every $E \in \Sigma$.

(v) Whenever (Y, T, ν) is a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, then there is a $g : Y \to X$ representing π.

(vi) Whenever (Y,T,ν) is a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ is an order-continuous measure-preserving Boolean homomorphism, then there is a $g:Y\to X$ representing π.

proof (a)(i)⇒(ii) Because μ is semi-finite, there is a partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that $\bar\mu a_i<\infty$ for each i. For each $i\in I$, let $E_i\in\Sigma$ be such that $E_i^\bullet=a_i$. Then the subspace measure μ_{E_i} on E_i is compact; let $\mathcal{K}_i\subseteq\mathcal{P}E_i$ be a compact class such that μ_{E_i} is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{K}=\bigcup_{i\in I}\mathcal{K}_i$. If $\mathcal{K}'\subseteq\mathcal{K}$ and $\mu(\bigcap\mathcal{K}_0)>0$ for every non-empty finite $\mathcal{K}_0\subseteq\mathcal{K}$, then $\mathcal{K}'\subseteq\mathcal{K}_i$ for some i, and surely has the finite intersection property, so $\bigcap\mathcal{K}'\ne\emptyset$; thus $\mathcal{K}'$ satisfies (β) of condition (ii). And if $E\in\Sigma$, $\mu E>0$ then there must be some $i\in I$ such that $E_i^\bullet\cap a_i\ne 0$, that is, $\mu(E\cap E_i)>0$, in which case there is a $K\in\mathcal{K}_i\subseteq\mathcal{K}$ such that $K\subseteq E\cap E_i$ and $\mu K>0$; so that $\mathcal{K}$ satisfies condition (α).

(b)(ii)⇒(iii) Suppose that $\mathcal{K}\subseteq\Sigma$ witnesses that (ii) is true. If $\mu X=0$ then $\mathcal{K}$ already witnesses that (iii) is true, so we need consider only the case $\mu X>0$. Set $\mathcal{L}=\{K_0\cup\ldots\cup K_n:K_0,\ldots,K_n\in\mathcal{K}\}$. Then $\mathcal{L}$ witnesses that (iii) is true. **P** By 342Ba, μ is inner regular with respect to $\mathcal{L}$. Let $\mathcal{L}'\subseteq\mathcal{L}$ be such that $\mu(\bigcap\mathcal{L}_0)>0$ for every non-empty finite $\mathcal{L}_0\subseteq\mathcal{L}'$. Then

$$\mathcal{F}_0=\{A:A\subseteq X,\text{ there is a finite }\mathcal{L}_0\subseteq\mathcal{L}'\text{ such that }X\cap\bigcap\mathcal{L}_0\setminus A\text{ is negligible}\}$$

is a filter on X, so there is an ultrafilter $\mathcal{F}$ on X including $\mathcal{F}_0$. Note that every conegligible set belongs to $\mathcal{F}_0$, so no negligible set can belong to $\mathcal{F}$. Set $\mathcal{K}'=\mathcal{K}\cap\mathcal{F}$; then $\bigcap\mathcal{K}_0$ belongs to $\mathcal{F}$, so is not negligible, for every non-empty finite $\mathcal{K}_0\subseteq\mathcal{K}'$. Accordingly there is some $x\in\bigcap\mathcal{K}'$. But any member of $\mathcal{L}'$ is of the form $L=K_0\cup\ldots\cup K_n$ where each $K_i\in\mathcal{K}$; because $\mathcal{F}$ is an ultrafilter and $L\in\mathcal{F}$, there must be some $i\le n$ such that $K_i\in\mathcal{F}$, in which case $x\in K_i\subseteq L$. Thus $x\in\bigcap\mathcal{L}'$. As $\mathcal{L}'$ is arbitrary, $\mathcal{L}$ satisfies the condition (β). **Q**

(c)(iii)⇒(iv) Let $\mathcal{K}\subseteq\Sigma$ witness that (iii) is true. For any $z\in Z$, set $\mathcal{K}_z=\{K:K\in\mathcal{K},\,z\in K^*\}$. If $K_0,\ldots,K_n\in\mathcal{K}_z$, then $z\in\bigcap_{i\le n}K_i^*=(\bigcap_{i\le n}K_i)^*$, so $(\bigcap_{i\le n}K_i)^*\ne\emptyset$ and $\mu(\bigcap_{i\le n}K_i)>0$. By (β) of condition (iii), $\bigcap\mathcal{K}_z\ne\emptyset$; and even if $\mathcal{K}_z=\emptyset$, $X\cap\bigcap\mathcal{K}_z\ne\emptyset$ because X is non-empty. So we may choose $f(z)\in X\cap\bigcap\mathcal{K}_z$. This defines a function $f:Z\to X$. Observe that, for $K\in\mathcal{K}$ and $z\in Z$,

$$z\in K^*\implies K\in\mathcal{K}_z\implies f(z)\in K\implies z\in f^{-1}[K],$$

so that $K^*\subseteq f^{-1}[K]$.

Now take any $E\in\Sigma$. Consider

$$U_1=\bigcup\{K^*:K\in\mathcal{K},\,K\subseteq E\}\subseteq\bigcup\{E^*\cap f^{-1}[K]:K\in\mathcal{K},\,K\subseteq E\}\subseteq E^*\cap f^{-1}[E],$$

$$U_2=\bigcup\{K^*:K\in\mathcal{K},\,K\subseteq X\setminus E\}\subseteq(X\setminus E)^*\cap f^{-1}[X\setminus E]=Z\setminus(f^{-1}[E]\cup E^*),$$

so that $f^{-1}[E]\triangle E^*\subseteq Z\setminus(U_1\cup U_2)$. Now U_1 and U_2 are open subsets of Z, so $M=Z\setminus(U_1\cup U_2)$ is closed, and in fact M is nowhere dense. **P?** Otherwise, there is a non-zero $a\in\mathfrak{A}$ such that the corresponding open-and-closed set $\widehat{a}$ is included in M, and an $F\in\Sigma$ of non-zero measure such that $a=F^\bullet$. At least one of $F\cap E$, $F\setminus E$ is non-negligible and therefore includes a non-negligible member K of $\mathcal{K}$. But in this case K^* is a non-empty open subset of M which is included in either U_1 or U_2, which is impossible. **XQ**

By the definition of λ (321J-321K), M is λ-negligible, so $f^{-1}[E]\triangle E^*\subseteq M$ is negligible, as required.

(d)(iv)⇒(v) Now assume that $f:Z\to X$ witnesses (iv), and let (Y,T,ν) be a complete strictly localizable measure space, with measure algebra $\mathfrak{B}$, and $\pi:\mathfrak{A}\to\mathfrak{B}$ an order-continuous Boolean homomorphism. If $\nu Y=0$ then any function from Y to X will represent π, so we may suppose that $\nu Y>0$. Write W for the Stone space of $\mathfrak{B}$. Then we have a continuous function $\phi:W\to Z$ such that $\phi^{-1}[\widehat{a}]=\widehat{\pi a}$ for every $a\in\mathfrak{A}$ (312Q), and $\phi^{-1}[M]$ is nowhere dense in W for every nowhere dense $M\subseteq Z$ (313R). It follows that $\phi^{-1}[M]$ is meager for every meager $M\subseteq Z$, that is, $\phi^{-1}[M]$ is negligible in W for every negligible $M\subseteq Z$. By 341Q, there is an inverse-measure-preserving function $h:Y\to W$ such that $h^{-1}[\widehat{b}]^\bullet=b$ for every $b\in\mathfrak{B}$. Consider $g=f\phi h:Y\to X$.

If $E\in\Sigma$, set $a=E^\bullet\in\mathfrak{A}$, so that $E^*=\widehat{a}\subseteq Z$, and $M=f^{-1}[E]\triangle E^*$ is λ-negligible; consequently $\phi^{-1}[M]$ is negligible in W. Because h is inverse-measure-preserving,

$$g^{-1}[E]\triangle h^{-1}[\phi^{-1}[E^*]]=h^{-1}[\phi^{-1}[f^{-1}[E]]]\triangle h^{-1}[\phi^{-1}[E^*]]=h^{-1}[\phi^{-1}[M]]$$

is negligible. But $\phi^{-1}[E^*]=\widehat{\pi a}$, so

$$g^{-1}[E]^\bullet = h^{-1}[\phi^{-1}[E^*]]^\bullet = \pi a.$$

As E is arbitrary, g induces the homomorphism π.

(e)(v)⇒(vi) is trivial.

(f)(vi)⇒(iv) Assume (vi). Let ν be the c.l.d. version of λ, T its domain, and $\mathfrak{B}$ its measure algebra; then ν is strictly localizable (322Rb). The embedding $\Lambda \subsetneq \mathrm{T}$ corresponds to an order-continuous measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (322Db). By (vi), there is a function $f : Z \to X$ such that $f^{-1}[E] \in \mathrm{T}$ and $f^{-1}[E]^\bullet = (E^*)^\bullet$ in $\mathfrak{B}$ for every $E \in \Sigma$. But as ν and λ have the same negligible sets (322Rb), $f^{-1}[E]\triangle E^*$ is λ-negligible for every $E \in \Sigma$, as required by (iv).

(g)(iv)⇒(i)(*α*) To begin with (down to the end of (γ) below) I suppose that μ is totally finite. In this case we have a function $g : X \to Z$ such that $E\triangle g^{-1}[E^*]$ is negligible for every $E \in \Sigma$ (341Q again). We are supposing also that there is a function $f : Z \to X$ such that $f^{-1}[E]\triangle E^*$ is negligible for every $E \in \Sigma$. Write $\mathcal{K}$ for the family of sets $K \subseteq X$ such that $K \in \Sigma$ and there is a compact set $L \subseteq Z$ such that $f[L] \subseteq K \subseteq g^{-1}[L]$.

(*β*) μ is inner regular with respect to $\mathcal{K}$. **P** Take $F \in \Sigma$ and $\gamma < \mu F$. Choose $\langle V_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ as follows. $F_0 = F$. Given that $\mu F_n > \gamma$, then

$$\lambda(f^{-1}[F_n] \cap F_n^*) = \lambda F_n^* = \mu F_n > \gamma,$$

so there is an open-and-closed set $V_n \subseteq f^{-1}[F_n] \cap F_n^*$ with $\lambda V_n > \gamma$. Express V_n as F_{n+1}^* where $F_{n+1} \in \Sigma$; since $F_n\triangle g^{-1}[F_n^*]$ is negligible, and $V_n \subseteq F_n^*$, we may take it that $F_{n+1} \subseteq g^{-1}[F_n^*]$. Continue.

At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} F_n \in \Sigma$ and $L = \bigcap_{n\in\mathbb{N}} F_n^*$. Because $F_{n+1} \setminus F_n \subseteq g^{-1}[F_n^*] \setminus F_n$ is negligible for each n, $\mu K = \lim_{n\to\infty} \mu F_n \ge \gamma$, while $K \subseteq F$ and L is surely compact. We have

$$L \subseteq \bigcap_{n\in\mathbb{N}} V_n \subseteq \bigcap_{n\in\mathbb{N}} f^{-1}[F_n] = f^{-1}[K],$$

so $f[L] \subseteq K$. Also

$$K \subseteq \bigcap_{n\in\mathbb{N}} F_{n+1} \subseteq \bigcap_{n\in\mathbb{N}} g^{-1}[F_n^*] = g^{-1}[L].$$

So $K \in \mathcal{K}$. As F and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$. **Q**

(*γ*) Next, $\mathcal{K}$ is a compact class. **P** Suppose that $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property. If $\mathcal{K}' = \emptyset$, of course $\bigcap\mathcal{K}' \ne \emptyset$; suppose that $\mathcal{K}'$ is non-empty. Let $\mathcal{L}$ be the family of closed sets $L \subseteq Z$ such that $g^{-1}[L]$ includes some member of $\mathcal{K}'$. Then $\mathcal{L}$ has the finite intersection property, and Z is compact, so there is some $z \in \bigcap\mathcal{L}$; also $Z \in \mathcal{L}$, so $z \in Z$. For any $K \in \mathcal{K}'$, there is some closed set $L \subseteq Z$ such that $f[L] \subseteq K \subseteq g^{-1}[L]$, so that $L \in \mathcal{L}$ and $z \in L$ and $f(z) \in K$. Thus $f(z) \in \bigcap\mathcal{K}'$. As $\mathcal{K}'$ is arbitrary, $\mathcal{K}$ is a compact class. **Q**

So $\mathcal{K}$ witnesses that μ is a compact measure.

(*δ*) Now consider the general case. Take any $E \in \Sigma$ of finite measure. If $E = \emptyset$ then surely the subspace measure μ_E is compact. Otherwise, we can identify the measure algebra of μ_E with the principal ideal $\mathfrak{A}_{E^\bullet}$ of $\mathfrak{A}$ generated by $E^\bullet$ (322Ja), and $E^* \subseteq Z$ with the Stone space of $\mathfrak{A}_{E^\bullet}$ (312T). Take any $x_0 \in E$ and define $\tilde f : E^* \to E$ by setting $\tilde f(z) = f(z)$ if $z \in E^* \cap f^{-1}[E]$, x_0 if $z \in E^* \setminus f^{-1}[E]$. Then f and $\tilde f$ agree almost everywhere in E^*, so $\tilde f^{-1}[F]\triangle F^*$ is negligible for every $F \in \Sigma_E$, that is, $\tilde f$ represents the canonical isomorphism between the measure algebras of μ_E and the subspace measure λ_{E^*} on E^*. But this means that condition (iv) is true of μ_E, so μ_E is compact, by (α)-(γ) above. As E is arbitrary, μ is locally compact.

This completes the proof.

343C Examples (a) Let I be any set. We know that the usual measure ν_I on $\{0,1\}^I$ is compact (342Jd). It follows that if (X,Σ,μ) is any complete probability space such that the measure algebra $\mathfrak{B}_I$ of ν_I can be embedded as a subalgebra of the measure algebra $\mathfrak{A}$ of μ, there is an inverse-measure-preserving function from X to $\{0,1\}^I$. For infinite I, this is so iff every non-zero principal ideal of $\mathfrak{A}$ has Maharam type at least κ, by 332P. Of course this does not depend in any way on the results of the present chapter. If $\mathfrak{B}_\kappa$ can be embedded in $\mathfrak{A}$, there must be a stochastically independent family $\langle E_\xi\rangle_{\xi<\kappa}$ of sets of measure $\frac{1}{2}$; now we get a map $h : X \to \{0,1\}^\kappa$ by saying that $h(x)(\xi) = 1$ iff $x \in E_\xi$, which by 254G is inverse-measure-preserving.

(b) In particular, if μ is atomless, there is an inverse-measure-preserving function from X to $\{0,1\}^{\mathbb{N}}$; since this is isomorphic, as measure space, to $[0,1]$ with Lebesgue measure (254K), there is an inverse-measure-preserving function from X to $[0,1]$.

(c) More generally, if (X,Σ,μ) is any complete atomless totally finite measure space, there is an inverse-measure-preserving function from X to the interval $[0,\mu X]$ endowed with Lebesgue measure. (If $\mu X>0$, apply (b) to the normalized measure $(\mu X)^{-1}\mu$; or argue directly from 343B, using the fact that Lebesgue measure on $[0,\mu X]$ is compact; or use the idea suggested in 343Xd.)

(d) In the other direction, if (X,Σ,μ) is a compact probability space with Maharam type at most $\kappa\geq\omega$, then there is an inverse-measure-preserving function from $\{0,1\}^\kappa$ to X. **P** By 332N, there is a measure-preserving homomorphism from the measure algebra of μ to the measure algebra of ν_κ; by 343B, this is represented by an inverse-measure-preserving function from $\{0,1\}^\kappa$ to X. **Q**

(e) Throughout the work above – in §254 as well as in 343B – I have taken the measures involved to be complete. It does occasionally happen, in this context, that this restriction is inconvenient. Typical results not depending on completeness in the domain space X are in 343Xc-343Xd. Of course these depend not only on the very special nature of the codomain spaces $\{0,1\}^I$ or $[0,1]$, but also on the measures on these spaces being taken to be incomplete.

343D Uniqueness of realizations The results of 342E-342J, together with 343B, give a respectable number of contexts in which homomorphisms between measure algebras can be represented by functions between measure spaces. They say nothing about whether such functions are unique, or whether we can distinguish, among the possible representations of a homomorphism, any canonical one. In fact the proof of 343B, using the Lifting Theorem as it does, strongly suggests that this is like looking for a canonical lifting, and I am sure that (outside a handful of very special cases) any such search is vain. Nevertheless, we do have a weak kind of uniqueness theorem, valid in a useful number of spaces, as follows.

Definition A measure space (X,Σ,μ) is **countably separated** if there is a countable set $\mathcal{A}\subseteq\Sigma$ separating the points of X in the sense that for any distinct x, $y\in X$ there is an $E\in\mathcal{A}$ containing one but not the other. (Of course this is a property of the structure (X,Σ) rather than of (X,Σ,μ).)

343E Lemma A measure space (X,Σ,μ) is countably separated iff there is an injective measurable function from X to $\mathbb{R}$.

proof If (X,Σ,μ) is countably separated, let $\mathcal{A}\subseteq\Sigma$ be a countable set separating the points of X. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{A}\cup\{\emptyset\}$. Set

$$f=\sum_{n=0}^\infty 3^{-n}\chi E_n:X\to\mathbb{R}.$$

Then f is measurable (because every E_n is measurable) and injective (because if $x\neq y$ in X and $n=\min\{i:\#(E_i\cap\{x,y\})=1\}$ and $x\in E_n$, then

$$f(x)\geq 3^{-n}+\sum_{i<n}3^{-i}\chi E_i(x)>\sum_{i>n}3^{-i}+\sum_{i<n}3^{-i}\chi E_i(y)\geq f(y).)$$

On the other hand, if $f:X\to\mathbb{R}$ is measurable and injective, then $\mathcal{A}=\{f^{-1}[\,]-\infty,q]\,]:q\in\mathbb{Q}\}$ is a countable subset of Σ separating the points of X, so (X,Σ,μ) is countably separated.

Remark The construction of the function f from the sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in the proof above is a standard trick; such f are sometimes called **Marczewski functionals**.

343F Proposition Let (X,Σ,μ) be a countably separated measure space and (Y,T,ν) any measure space. Let f, $g:Y\to X$ be two functions such that $f^{-1}[E]$ and $g^{-1}[E]$ both belong to T, and $f^{-1}[E]\triangle g^{-1}[E]$ is ν-negligible, for every $E\in\Sigma$. Then $f=g$ ν-almost everywhere, and $\{y:y\in Y,\ f(y)\neq g(y)\}$ is measurable as well as negligible.

proof Let $\mathcal{A}\subseteq\Sigma$ be a countable set separating the points of X. Then

$$\{y:f(y)\neq g(y)\}=\bigcup_{E\in\mathcal{A}}f^{-1}[E]\triangle g^{-1}[E]$$

is measurable and negligible.

343G Corollary If, in 343B, (X, Σ, μ) is countably separated, then the functions $g : Y \to X$ of 343B(v)-(vi) are almost uniquely defined in the sense that if f, g both represent the same homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then $f =_{\text{a.e.}} g$.

343H Examples Leading examples of countably separated measure spaces are
(i) $\mathbb{R}$ (take $\mathcal{A} = \{]-\infty, q] : q \in \mathbb{Q}\}$);
(ii) $\{0,1\}^{\mathbb{N}}$ (take $\mathcal{A} = \{E_n : n \in \mathbb{N}\}$, where $E_n = \{x : x(n) = 1\}$);
(iii) subspaces (measurable or not) of countably separated spaces;
(iv) finite products of countably separated spaces;
(v) countable products of countably separated probability spaces;
(vi) completions and c.l.d. versions of countably separated spaces.
As soon as we move away from these elementary ideas, however, some interesting difficulties arise.

343I Example Let $\nu_{\mathfrak{c}}$ be the usual measure on $X = \{0,1\}^{\mathfrak{c}}$, where $\mathfrak{c} = \#(\mathbb{R})$, and $\mathrm{T}_{\mathfrak{c}}$ its domain. Then there is a function $f : X \to X$ such that $f(x) \neq x$ for every $x \in X$, but $E \triangle f^{-1}[E]$ is negligible for every $E \in \mathrm{T}_{\mathfrak{c}}$. **P** The set $\mathfrak{c} \setminus \omega$ is still with cardinal $\mathfrak{c}$, so there is an injection $h : \{0,1\}^{\omega} \to \mathfrak{c} \setminus \omega$. (As usual, I am identifying the cardinal number $\mathfrak{c}$ with the corresponding initial ordinal. But if you prefer to argue without the full axiom of choice, you can express all the same ideas with $\mathbb{R}$ in the place of $\mathfrak{c}$ and $\mathbb{N}$ in the place of ω.) For $x \in X$, set

$$\begin{aligned} f(x)(\xi) &= 1 - x(\xi) \text{ if } \xi = h(x{\restriction}\omega), \\ &= x(\xi) \text{ otherwise .} \end{aligned}$$

Evidently $f(x) \neq x$ for every x. If $E \subseteq X$ is measurable, then we can find a countable set $J \subseteq \mathfrak{c}$ and sets E', E'', both determined by coordinates in J, such that $E' \subseteq E \subseteq E''$ and $E'' \setminus E'$ is negligible (254Oc). Now for any particular $\xi \in \mathfrak{c} \setminus \omega$, $\{x : h(x{\restriction}\omega) = \xi\}$ is negligible, being either empty or of the form $\{x : x(n) = z(n)$ for every $n < \omega\}$ for some $z \in \{0,1\}^{\omega}$. So $H = \{x : h(x{\restriction}\omega) \in J\}$ is negligible. Now we see that for $x \in X \setminus H$, $f(x){\restriction}J = x{\restriction}J$, so for $x \in X \setminus (H \cup (E'' \setminus E'))$,

$$x \in E \Longrightarrow x \in E' \Longrightarrow f(x) \in E' \Longrightarrow f(x) \in E,$$

$$x \notin E \Longrightarrow x \notin E'' \Longrightarrow f(x) \notin E'' \Longrightarrow f(x) \notin E.$$

Thus $E \triangle f^{-1}[E] \subseteq H \cup (E'' \setminus E')$ is negligible. **Q**

343J The split interval I introduce a construction which here will seem essentially elementary, but in other contexts is of great interest, as will appear in Volume 4.

(a) Take $I^{\|}$ to consist of two copies of each point of the unit interval, so that $I^{\|} = \{t^+ : t \in [0,1]\} \cup \{t^- : t \in [0,1]\}$. For $A \subseteq I^{\|}$ write $A_l = \{t : t^- \in A\}$, $A_r = \{t : t^+ \in A\}$. Let Σ be the set

$$\{E : E \subseteq I^{\|},\ E_l \text{ and } E_r \text{ are Lebesgue measurable and } E_l \triangle E_r \text{ is Lebesgue negligible}\}.$$

For $E \in \Sigma$, set

$$\mu E = \mu_L E_l = \mu_L E_r$$

where μ_L is Lebesgue measure on $[0,1]$. It is easy to check that $(I^{\|}, \Sigma, \mu)$ is a complete probability space (cf. 234F, 234Ye). Also it is compact. **P** Take $\mathcal{K}$ to be the family of sets $K \subseteq I^{\|}$ such that $K_l = K_r$ is a compact subset of $[0,1]$, and check that $\mathcal{K}$ is a compact class and that μ is inner regular with respect to $\mathcal{K}$; or use 343Xa below. **Q** The sets $\{t^- : t \in [0,1]\}$ and $\{t^+ : t \in [0,1]\}$ are non-measurable subsets of $I^{\|}$; on both of them the subspace measures correspond exactly to μ_L. We have a canonical inverse-measure-preserving function $h : I^{\|} \to [0,1]$ given by setting $h(t^+) = h(t^-) = t$ for every $t \in [0,1]$; h induces an isomorphism between the measure algebras of μ and μ_L.

$I^{\|}$ is called the **split interval** or (especially when given its standard topology, as in 343Yc below) the **double arrow space** or **two arrows space**.

Now the relevance to the present discussion is this: we have a map $f : I^{\|} \to I^{\|}$ given by setting

$$f(t^+) = t^-,\ f(t^-) = t^+ \text{ for every } t \in [0,1]$$

such that $f(x) \neq x$ for every x, but $E\triangle f^{-1}[E]$ is negligible for every $E \in \Sigma$, so that f represents the identity homomorphism on the measure algebra of μ. The function $h : I^{\parallel} \to [0,1]$ is canonical enough, but is two-to-one, and the canonical map from the measure algebra of μ to the measure algebra of μ_L is represented equally by the functions $t \mapsto t^-$ and $t \mapsto t^+$, which are nowhere equal.

(b) Consider the direct sum (Y,ν) of $(I^{\parallel},\mu)$ and $([0,1],\mu_L)$; for definiteness, take Y to be $(I^{\parallel} \times \{0\}) \cup ([0,1] \times \{1\})$. Setting

$$h_1(t^+,0) = h_1(t^-,0) = (t,1), \quad h_1(t,1) = (t^+,0),$$

we see that $h_1 : Y \to Y$ induces a measure-preserving involution of the measure algebra $\mathfrak{B}$ of ν, corresponding to its expression as a simple product of the isomorphic measure algebras of μ and μ_L. But h_1 is not invertible, and indeed there is no invertible function from Y to itself which induces this involution of $\mathfrak{B}$. **P?** Suppose, if possible, that $g : Y \to Y$ were such a function. Looking at the sets

$$E_q = [0,q] \times \{1\}, \quad F_q = \{(t^+,0) : t \in [0,q]\} \cup \{(t^-,0) : t \in [0,q]\}$$

for $q \in \mathbb{Q}$, we must have $g^{-1}[E_q]\triangle F_q$ negligible for every q, so that we must have $g(t^+,0) = g(t^-,0) = (t,1)$ for almost every $t \in [0,1]$, and g cannot be injective. **XQ**

(c) Thus even with a compact probability space, and an automorphism ϕ of its measure algebra, we cannot be sure of representing ϕ and ϕ^{-1} by functions which will be inverses of each other.

343K 342L has a partial converse.

Proposition If (X,Σ,μ) is a semi-finite countably separated measure space, it is compact iff it is locally compact iff it is perfect.

proof We already know that compact measure spaces are locally compact and locally compact semi-finite measure spaces are perfect (342Ha, 342L). So suppose that (X,Σ,μ) is a perfect semi-finite countably separated measure space. Let $f : X \to \mathbb{R}$ be an injective measurable function (343E). Consider

$$\mathcal{K} = \{f^{-1}[L] : L \subseteq f[X],\ L \text{ is compact in } \mathbb{R}\}.$$

The definition of 'perfect' measure space states exactly that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. And $\mathcal{K}$ is a compact class. **P** If $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property, $\mathcal{L} = \{f[K] : K \in \mathcal{K}'\}$ is a family of compact sets in $\mathbb{R}$ with the finite intersection property, and has non-empty intersection; so that $\bigcap \mathcal{K}'$ is also non-empty, because f is injective. **Q** By 342E, (X,Σ,μ) is compact.

343L The time has come to give examples of spaces which are *not* locally compact, so that we can expect to have measure-preserving homomorphisms not representable by inverse-measure-preserving functions. The most commonly arising ones are covered by the following result.

Proposition Let (X,Σ,μ) be a complete locally determined countably separated measure space, and $A \subseteq X$ a set such that the subspace measure μ_A is perfect. Then A is measurable.

proof ? Otherwise, there is a set $E \in \Sigma$ such that $\mu E < \infty$ and $B = A \cap E \notin \Sigma$. Let $f : X \to \mathbb{R}$ be an injective measurable function (343E again). Then $f{\restriction}B$ is Σ_B-measurable, where Σ_B is the domain of the subspace measure μ_B on B. Set

$$\mathcal{K} = \{f^{-1}[L] : L \subseteq f[B],\ L \text{ is compact in } \mathbb{R}\}.$$

Just as in the proof of 343K, $\mathcal{K}$ is a compact class and μ_B is inner regular with respect to $\mathcal{K}$. By 342Bb, there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $\mu_B(B \setminus \bigcup_{n\in\mathbb{N}} K_n) = 0$. But of course $\mathcal{K} \subseteq \Sigma$, because f is Σ-measurable, so $\bigcup_{n\in\mathbb{N}} K_n \in \Sigma$. Because μ is complete, $B \setminus \bigcup_{n\in\mathbb{N}} K_n \in \Sigma$ and $B \in \Sigma$. **X**

343M Example 343L tells us that any non-measurable set X of $\mathbb{R}^r$, or of $\{0,1\}^{\mathbb{N}}$, with their usual measures, is not perfect, therefore not (locally) compact, when given its subspace measure.

To find a non-representable homomorphism, we do not need to go through the whole apparatus of 343B. Take Y to be a measurable envelope of X (132Ee). Then the identity function from X to Y induces an isomorphism of their measure algebras. But there is no function from Y to X inducing the same isomorphism. **P?** Writing Z for $\mathbb{R}^r$ or $\{0,1\}^{\mathbb{N}}$ and μ for its measure, Z is countably separated; suppose $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of measurable sets in Z separating its points. For each n, $(Y\cap E_n)^\bullet$ in the measure algebra of μ_Y corresponds to $(X\cap E_n)^\bullet$ in the measure algebra of μ_X. So if $f:Y\to X$ were a function representing the isomorphism of the measure algebras, $(Y\cap E_n)\triangle f^{-1}[E_n]$ would have to be negligible for each n, and $A=\bigcup_{n\in\mathbb{N}}(Y\cap E_n)\triangle f^{-1}[E_n]$ would be negligible. But for $y\in Y\setminus A$, $f(y)$ belongs to just the same E_n as y does, so must be equal to y. Accordingly $X\supseteq Y\setminus A$ and X is measurable. **XQ**

343X Basic exercises (a) Let (X,Σ,μ) be a semi-finite measure space. (i) Suppose that there is a set $A\subseteq X$, of full outer measure, such that the subspace measure on A is compact. Show that μ is locally compact. (*Hint*: show that μ satisfies (ii) or (v) of 343B.) (ii) Suppose that for every non-negligible $E\in\Sigma$ there is a non-negligible set $A\subseteq E$ such that the subspace measure on A is compact. Show that μ is locally compact.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty sets, with product X; write $\pi_i:X\to X_i$ for the coordinate map. Suppose we are given a σ-algebra Σ_i of subsets of X_i for each i; let $\Sigma=\widehat{\bigotimes}_{i\in I}\Sigma_i$ be the corresponding σ-algebra of subsets of X generated by $\{\pi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$. Let μ be a totally finite measure with domain Σ, and for $i\in I$ let μ_i be the image measure $\mu\pi_i^{-1}$. Check that the domain of μ_i is Σ_i. Show that if every (X_i,Σ_i,μ_i) is compact, then so is (X,Σ,μ). (*Hint*: *either* show that μ satisfies (v) of 343B *or* adapt the method of 342Gf.)

(c) Let I be any set. Let $\mathcal{B}\mathfrak{a}$ be the σ-algebra of subsets of $\{0,1\}^I$ generated by the sets $F_i=\{z:z(i)=1\}$ for $i\in I$, and ν any probability measure with domain $\mathcal{B}\mathfrak{a}$; let $\mathfrak{B}$ be the measure algebra of ν. Let (X,Σ,μ) be a measure space with measure algebra $\mathfrak{A}$, and $\phi:\mathfrak{B}\to\mathfrak{A}$ an order-continuous Boolean homomorphism. Show that there is an inverse-measure-preserving function $f:X\to\{0,1\}^I$ representing ϕ. (*Hint*: for each $i\in I$, take $E_i\in\Sigma$ such that $E_i^\bullet=\phi F_i^\bullet$; set $f(x)(i)=1$ if $x\in E_i$, and use 343Ab.)

(d) Let (X,Σ,μ) be an atomless probability space. Let $\mu_{\mathcal{B}}$ be the restriction of Lebesgue measure to the σ-algebra of Borel subsets of $[0,1]$. Show that there is a function $g:X\to[0,1]$ which is inverse-measure-preserving for μ and $\mu_{\mathcal{B}}$. (*Hint*: find an $f:X\to\{0,1\}^{\mathbb{N}}$ as in 343Xc, and set $g=hf$ where $h(z)=\sum_{n=0}^\infty 2^{-n-1}g(n)$, as in 254K; or choose $E_q\in\Sigma$ such that $\mu E_q=q$, $E_q\subseteq E_{q'}$ whenever $q\le q'$ in $[0,1]\cap\mathbb{Q}$, and set $f(x)=\inf\{q:x\in E_q\}$ for $x\in E_1$.)

(e) Let (X,Σ,μ) be a countably separated measure space, with measure algebra $\mathfrak{A}$. (i) Show that $\{x\}\in\Sigma$ for every $x\in X$. (ii) Show that every atom of $\mathfrak{A}$ is of the form $\{x\}^\bullet$ for some $x\in X$.

(f) Let (X,Σ,μ) be a semi-finite countably separated measure space. (i) Show that μ is point-supported iff it is complete, strictly localizable and purely atomic. (ii) Show that μ is atomless iff $\mu\{x\}=0$ for every $x\in X$.

(g) Let $I^{\parallel}$ be the split interval, with its usual measure μ described in 343J, and $h:I^{\parallel}\to[0,1]$ the canonical surjection. Show that the canonical isomorphism between the measure algebras of μ and Lebesgue measure on $[0,1]$ is given by the formula '$E^\bullet\mapsto h[E]^\bullet$ for every measurable $E\subseteq I^{\parallel}$'.

(h) Let (X,Σ,μ) and (Y,T,ν) be measure spaces with measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$. Suppose that $X\cap Y=\emptyset$ and that we have a measure-preserving isomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$. Set

$$\Lambda=\{W:W\subseteq X\cup Y,\,W\cap X\in\Sigma,\,W\cap Y\in\mathrm{T},\,\pi(W\cap X)^\bullet=(W\cap Y)^\bullet\},$$

and for $W\in\Lambda$ set $\lambda W=\mu(W\cap X)=\nu(W\cap Y)$. Show that $(X\cup Y,\Lambda,\lambda)$ is a measure space which is locally compact, or perfect, if (X,Σ,μ) is.

>(i) Let (X,Σ,μ) be a complete perfect totally finite measure space, (Y,T,ν) a complete countably separated measure space, and $f:X\to Y$ an inverse-measure-preserving function. Show that $\mathrm{T}=\{F:F\subseteq Y,\,f^{-1}[F]\in\Sigma\}$, so that a function $h:Y\to\mathbb{R}$ is ν-integrable iff hf is μ-integrable. (*Hint*: if $A\subseteq Y$ and $E=f^{-1}[A]\in\Sigma$, $f\restriction E$ is inverse-measure-preserving for the subspace measures μ_E, ν_A; by 342Xk, ν_A is perfect, so by 343L $A\in\mathrm{T}$. Now use 235J.)

(j) Let (X, Σ, μ) be a complete compact measure space, Y a set and $f : Y \to X$ a surjection; set

$$\mathrm{T} = \{F : F \subseteq Y,\, f[F] \in \Sigma,\, \mu(f[F] \cap f[Y \setminus F]) = 0\}, \quad \nu F = \mu f[F] \text{ for } F \in \mathrm{T},$$

so that ν is a measure on Y and f is inverse-measure-preserving (234Ye). Show that ν is a compact measure.

343Y Further exercises (a) Let (X, Σ, μ) be a semi-finite measure space, and suppose that there is a compact class $\mathcal{K} \subseteq \mathcal{P}X$ such that (α) whenever $E \in \Sigma$ and $\mu E > 0$ there is a non-negligible $K \in \mathcal{K}$ such that $K \subseteq E$ (β) whenever $K_0, \dots, K_n \in \mathcal{K}$ and $\bigcap_{i \le n} K_i = \emptyset$ then there are measurable sets $E_0, \dots, E_n$ such that $E_i \supseteq K_i$ for every i and $\bigcap_{i \le n} E_i$ is negligible. Show that μ is locally compact.

(b)(i) Show that a countably separated semi-finite measure space has magnitude and Maharam type at most $2^{\mathfrak{c}}$. (ii) Show that the direct sum of $\mathfrak{c}$ or fewer countably separated measure spaces is countably separated. (iii) Show that a countably separated perfect measure space has countable Maharam type.

(c) Let $I^{\parallel} = \{t^+ : t \in [0,1]\} \cup \{t^- : t \in [0,1]\}$ be the split interval (343J). (i) Show that the rules

$$s^- \le t^- \iff s^+ \le t^+ \iff s \le t, \quad s^+ \le t^- \iff s < t,$$

$$t^- \le t^+ \text{ for all } t \in [0,1]$$

define a Dedekind complete total order on $I^{\parallel}$ with greatest and least elements. (ii) Show that the intervals $[0^-, t^-]$, $[t^+, 1^+]$, interpreted for this ordering, generate a compact Hausdorff topology on $I^{\parallel}$ for which the map $h : I^{\parallel} \to [0,1]$ of 343J is continuous. (iii) Show that a subset E of $I^{\parallel}$ is Borel for this topology iff the sets E_r, $E_l \subseteq [0,1]$, as described in 343Ja, are Borel and $E_r \triangle E_l$ is countable. (iv) Show that if $f : [0,1] \to \mathbb{R}$ is of bounded variation then there is a continuous $g : I^{\parallel} \to \mathbb{R}$ such that $g = fh$ except perhaps at countably many points. (v) Show that the measure μ of 343J is inner regular with respect to the compact subsets of $I^{\parallel}$. (vi) Show that we have a lower density ϕ for μ defined by setting

$$\phi E = \{t^- : 0 < t \le 1,\, \lim_{\delta \downarrow 0} \frac{1}{\delta} \mu(E \cap [(t-\delta)^+, t^-]) = 1\}$$

$$\cup \{t^+ : 0 \le t < 1,\, \lim_{\delta \downarrow 0} \frac{1}{\delta} \mu(E \cap [t^+, (t+\delta)^-]) = 1\}$$

for measurable sets $E \subseteq I^{\parallel}$.

(d) Set $X = \{0,1\}^{\mathfrak{c}}$, with its usual measure $\nu_{\mathfrak{c}}$. Show that there is an inverse-measure-preserving function $f : X \to X$ such that $f[X]$ is non-measurable but f induces the identity automorphism of the measure algebra of $\nu_{\mathfrak{c}}$. (*Hint*: use the idea of 343I.) Show that under these conditions $f[X]$, with its subspace measure, must be compact. (*Hint*: use 343B(iv).)

(e) Let μ_{Hr} be r-dimensional Hausdorff measure on $\mathbb{R}^s$, where $s \ge 1$ is an integer and $r \ge 0$ (§264). (i) Show that μ_{Hr} is countably separated. (ii) Show that the c.l.d. version of μ_{Hr} is compact. (*Hint*: 264Yi.)

(f) Give an example of a countably separated probability space (X, Σ, μ) and a function f from X to a set Y such that the image measure μf^{-1} is not countably separated. (*Hint*: use 223B to show that if $E \subseteq \mathbb{R}$ is Lebesgue measurable and not negligible, then $E + \mathbb{Q}$ is conegligible; or use the zero-one law to show that if $E \subseteq \mathcal{P}\mathbb{N}$ is measurable and not negligible for the usual measure on $\mathcal{P}\mathbb{N}$, then $\{a \triangle b : a \in E,\, b \in [\mathbb{N}]^{<\omega}\}$ is conegligible.)

343 Notes and comments The points at which the Lifting Theorem impinges on the work of this section are in the proofs of (iv)⇒(i) and (iv)⇒(v) in Theorem 343B. In fact the ideas can be rearranged to give a proof of 343B which does not rely on the Lifting Theorem; I give a hint in Volume 4 (413Ye).

I suppose the significant new ideas of this section are in 343B and 343K. The rest is mostly a matter of being thorough and careful. But I take this material at a slow pace because there are some potentially confusing features, and the underlying question is of the greatest importance: when, given a Boolean homomorphism from one measure algebra to another, can we be sure of representing it by a measurable function between measure spaces? The concept of 'compact' measure puts the burden firmly on the measure space corresponding to the *domain* of the Boolean homomorphism, which will be the *codomain* of the measurable function. So the first step is to try to understand properly which measures are compact, and what other properties they can be expected to have; which accounts for much of the length of §342. But having understood that many of our favourite measures are compact, we have to come to terms with the fact that we still cannot count on a measure algebra isomorphism corresponding to a measure space isomorphism. I introduce the split interval (343J, 343Xg, 343Yc) as a close approximation to Lebesgue measure on $[0,1]$ which is not isomorphic to it. Of course we have already seen a more dramatic example: the Stone space of the Lebesgue measure algebra also has the same measure algebra as Lebesgue measure, while being in almost every other way very much more complex, as will appear in Volumes 4 and 5.

As 343C suggests, elementary cases in which 343B can be applied are often amenable to more primitive methods, avoiding not only the concept of 'compact' measure, but also Stone spaces and the Lifting Theorem. For substantial examples in which we can prove that a measure space (X,μ) is compact, without simultaneously finding direct constructions for inverse-measure-preserving functions into X (as in 343Xc-343Xd), I think we shall have to wait until Volume 4.

The concept of 'countably separated' measure space does not involve the measure at all, nor even the null ideal; it belongs to the theory of σ-algebras of sets. Some simple permanence properties are in 343H and 343Yb(ii). Let us note in passing that 343Xi describes some more situations in which the 'image measure catastrophe', described in 235H, cannot arise.

I include the variants 343B(ii), 343B(iii) and 343Ya of the notion of 'local compactness' because they are not obvious and may illuminate it.

Version of 22.3.06

344 Realization of automorphisms

In 343Jb, I gave an example of a 'good' (compact, complete) probability space X with an automorphism ϕ of its measure algebra such that both ϕ and ϕ^{-1} are representable by functions from X to itself, but there is no such representation in which the two functions are inverses of each other. The present section is an attempt to describe the further refinements necessary to ensure that automorphisms of measure algebras can be represented by automorphisms of the measure spaces. It turns out that in the most important contexts in which this can be done, a little extra work yields a significant generalization: the simultaneous realization of countably many homomorphisms by a consistent family of functions.

I will describe three cases in which such simultaneous realizations can be achieved: Stone spaces (344A), perfect complete countably separated spaces (344C) and suitable measures on $\{0,1\}^I$ (344E-344G). The arguments for 344C, suitably refined, give a complete description of perfect complete countably separated strictly localizable spaces which are not purely atomic (344I, 344Xc). At the same time we find that Lebesgue measure, and the usual measure on $\{0,1\}^I$, are 'homogeneous' in the strong sense that two measurable subspaces (of non-zero measure) are isomorphic iff they have the same measure (344J, 344L).

344A Stone spaces The first case is immediate from the work of §§312, 313 and 321, as collected in 324E. If (Z,Σ,μ) is actually the Stone space of a measure algebra $(\mathfrak{A},\bar{\mu})$, then every order-continuous Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{A}$ corresponds to a unique continuous function $f_\phi:Z\to Z$ (312Q) which represents ϕ (324E). The uniqueness of f_ϕ means that we can be sure that $f_{\phi\psi}=f_\psi f_\phi$ for all order-continuous homomorphisms ϕ and ψ; and of course f_ι is the identity map on Z, so that $f_{\phi^{-1}}$ will have to be f_ϕ^{-1} whenever ϕ is invertible. Thus in this special case we can consistently, and canonically, represent all order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself.

Now for two cases where we have to work for the results.

344B Theorem Let (X,Σ,μ) be a countably separated measure space with measure algebra $\mathfrak{A}$, and G a countable semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that every member of G can be represented by some function from X to itself. Then a family $\langle f_\phi\rangle_{\phi\in G}$ of such representatives can be chosen in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X.

proof (a) Because $G \cup \{\iota\}$ satisfies the same conditions as G, we may suppose from the beginning that ι belongs to G itself. Let $\mathcal{A} \subseteq \Sigma$ be a countable set separating the points of X. For each $\phi \in G$ take some representing function $g_\phi : X \to X$; take g_ι to be the identity function. If ϕ, $\psi \in G$, then of course

$$((g_\phi g_\psi)^{-1}[E])^\bullet = (g_\psi^{-1}[g_\phi^{-1}[E]])^\bullet = \psi(g_\phi^{-1}[E])^\bullet = \psi\phi E^\bullet = (g_{\psi\phi}^{-1}[E])^\bullet$$

for every $E \in \Sigma$. By 343F, the set

$$H_{\phi\psi} = \{x : g_{\psi\phi}(x) \neq g_\phi g_\psi(x)\}$$

is negligible and belongs to Σ.

(b) Set

$$H = \bigcup_{\phi,\psi \in G} H_{\phi\psi};$$

because G is countable, H also is measurable and negligible. Try defining $f_\phi : X \to X$ by setting $f_\phi(x) = g_\phi(x)$ if $x \in X \setminus H$, $f_\phi(x) = x$ if $x \in H$. Because H is measurable, $f_\phi^{-1}[E] \in \Sigma$ for every $E \in \Sigma$; because H is negligible,

$$(f_\phi^{-1}[E])^\bullet = (g_\phi^{-1}[E])^\bullet = \phi E^\bullet$$

for every $E \in \Sigma$, and f_ϕ represents ϕ, for every $\phi \in G$. Of course $f_\iota = g_\iota$ is the identity function on X.

(c) If $\theta \in G$ then $f_\theta^{-1}[H] = H$. **P** (i) If $x \in H$ then $f_\theta(x) = x \in H$. (ii) If $f_\theta(x) \in H$ and $f_\theta(x) = x$ then of course $x \in H$. (iii) If $f_\theta(x) = g_\theta(x) \in H$ then there are ϕ, $\psi \in G$ such that $g_\phi g_\psi g_\theta(x) \neq g_{\psi\phi} g_\theta(x)$. So either

$$g_\psi g_\theta(x) \neq g_{\theta\psi}(x),$$

or

$$g_\phi g_{\theta\psi}(x) \neq g_{\theta\psi\phi}(x)$$

or

$$g_{\theta\psi\phi}(x) \neq g_{\psi\phi} g_\theta(x),$$

and in any case $x \in H$. **Q**

(d) It follows that $f_\phi f_\psi = f_{\psi\phi}$ for every ϕ, $\psi \in G$. **P** (i) If $x \in H$ then

$$f_\phi f_\psi(x) = x = f_{\psi\phi}(x).$$

(ii) If $x \in X \setminus H$ then $f_\psi(x) \notin H$, by (c), so

$$f_\phi f_\psi(x) = g_\phi g_\psi(x) = g_{\psi\phi}(x) = f_{\psi\phi}(x). \textbf{ Q}$$

344C Corollary Let (X, Σ, μ) be a countably separated perfect complete strictly localizable measure space with measure algebra $\mathfrak{A}$, and G a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible, and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ and ϕ^{-1} are measure-preserving, f_ϕ will be an automorphism of the measure space (X, Σ, μ).

proof By 343K, (X, Σ, μ) is compact. So 343B(v) tells us that every member of G is representable, and we can apply 344B.

Reminder Spaces satisfying the conditions of this corollary include Lebesgue measure on $\mathbb{R}^r$, the usual measure on $\{0,1\}^\mathbb{N}$, and their measurable subspaces; see also 342J, 342Xe, 343H and 343Ye.

344D The third case I wish to present requires a more elaborate argument. I start with a kind of Schröder-Bernstein theorem for measurable spaces.

Lemma Let X and Y be sets, and $\Sigma \subseteq \mathcal{P}X$, $\mathrm{T} \subseteq \mathcal{P}Y$ σ-algebras. Suppose that there are $f : X \to Y$, $g : Y \to X$ such that $F = f[X] \in \mathrm{T}$, $E = g[Y] \in \Sigma$, f is an isomorphism between (X, Σ) and (F, T_F) and

g is an isomorphism between (Y, T) and (E, Σ_E), writing Σ_E, T_F for the subspace σ-algebras (see 121A). Then (X, Σ) and (Y, T) are isomorphic, and there is an isomorphism $h : X \to Y$ which is covered by f and g in the sense that

$$\{(x, h(x)) : x \in X\} \subseteq \{(x, f(x)) : x \in X\} \cup \{(g(y), y) : y \in Y\}.$$

proof Set $X_0 = X$, $Y_0 = Y$, $X_{n+1} = g[Y_n]$ and $Y_{n+1} = f[X_n]$ for each $n \in \mathbb{N}$; then $\langle X_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ and $\langle Y_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T. Set $X_\infty = \bigcap_{n\in\mathbb{N}} X_n$, $Y_\infty = \bigcap_{n\in\mathbb{N}} Y_n$. Then $f{\restriction}X_{2k} \setminus X_{2k+1}$ is an isomorphism between $X_{2k} \setminus X_{2k+1}$ and $Y_{2k+1} \setminus Y_{2k+2}$, while $g{\restriction}Y_{2k} \setminus Y_{2k+1}$ is an isomorphism between $Y_{2k} \setminus Y_{2k+1}$ and $X_{2k+1} \setminus X_{2k+2}$; and $g{\restriction}Y_\infty$ is an isomorphism between Y_∞ and X_∞. So the formula

$$\begin{aligned} h(x) &= f(x) \text{ if } x \in \bigcup_{k\in\mathbb{N}} X_{2k} \setminus X_{2k+1}, \\ &= g^{-1}(x) \text{ for other } x \in X \end{aligned}$$

gives the required isomorphism between X and Y.

Remark You will recognise the ordinary Schröder-Bernstein theorem (2A1G) as the case $\Sigma = \mathcal{P}X$, $\mathrm{T} = \mathcal{P}Y$.

344E Theorem Let I be any set, and let μ be a σ-finite measure on $X = \{0,1\}^I$ with domain the σ-algebra $\mathcal{B}\mathfrak{a}_I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I; write $\mathfrak{A}$ for the measure algebra of μ. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ is measure-preserving, f_ϕ will be an automorphism of the measure space $(X, \mathcal{B}\mathfrak{a}_I, \mu)$.

proof (a) As in 344C, we may as well suppose from the beginning that $\iota \in G$. The case of finite I is trivial, so I will suppose that I is infinite. For $i \in I$, set $E_i = \{x : x(i) = 1\}$; for $J \subseteq I$, let $\mathcal{B}_J$ be the σ-subalgebra of $\mathcal{B}\mathfrak{a}_I$ generated by $\{E_i : I \in J\}$. For $i \in I$, $\phi \in G$ choose $F_{\phi i} \in \mathcal{B}\mathfrak{a}_I$ such that $F_{\phi i}^\bullet = \phi E_i^\bullet$. (Of course we take $F_{\iota i} = E_i$ for every i.) Let $\mathcal{J}$ be the family of those subsets J of I such that $F_{\phi i} \in \mathcal{B}_J$ for every $i \in J$ and $\phi \in G$.

(b) For the purposes of this proof, I will say that a pair $(J, \langle g_\phi \rangle_{\phi\in G})$ is **consistent** if $J \in \mathcal{J}$ and, for each $\phi \in G$, g_ϕ is a function from X to itself such that

$g_\phi^{-1}[E_i] \in \mathcal{B}_J$ and $(g_\phi^{-1}[E_i])^\bullet = \phi E_i^\bullet$ whenever $i \in J$, $\phi \in G$,

$g_\phi^{-1}[E_i] = E_i$ whenever $i \in I \setminus J$, $\phi \in G$,

$g_\phi g_\psi = g_{\psi\phi}$ whenever ϕ, $\psi \in G$,

$g_\iota(x) = x$ for every $x \in X$.

Now the key to the proof is the following fact: if $(J, \langle g_\phi \rangle_{\phi\in G})$ is consistent, and $\tilde{J}$ is a member of $\mathcal{J}$ such that $\tilde{J} \setminus J$ is countably infinite, then there is a family $\langle tildeg_\phi \rangle_{\phi\in G}$ such that $(\tilde{J}, \langle \tilde{g}_\phi \rangle_{\phi\in G})$ is consistent and $\tilde{g}_\phi^{-1}[E_i] = g_\phi^{-1}[E_i]$ whenever $i \in J$ and $\phi \in G$, that is, $\tilde{g}_\phi(x){\restriction}J = g_\phi(x){\restriction}J$ whenever $\phi \in G$ and $x \in X$. The construction is as follows.

(i) Start by fixing on any infinite set $K \subseteq \tilde{J} \setminus J$ such that $(\tilde{J} \setminus J) \setminus K$ also is infinite. For $z \in \{0,1\}^K$, set $V_z = \{x : x \in X, x{\restriction}K = z\}$; then $V_z \in \mathcal{B}_{\tilde{J}}$. All the sets V_z, as z runs over the uncountable set $\{0,1\}^K$, are disjoint, so they cannot all have non-zero measure (because μ is σ-finite), and we can choose z such that V_z is μ-negligible.

(ii) Define $h_\phi : X \to X$, for $\phi \in G$, by setting

$$\begin{aligned} h_\phi(x)(i) &= g_\phi(x)(i) \text{ if } i \in J, \\ &= x(i) \text{ if } i \in I \setminus \tilde{J}, \\ &= x(i) \text{ if } i \in \tilde{J} \setminus J \text{ and } x \in V_z, \\ &= 1 \text{ if } i \in \tilde{J} \setminus J \text{ and } x \in F_{\phi i} \setminus V_z, \\ &= 0 \text{ if } i \in \tilde{J} \setminus J \text{ and } x \notin F_{\phi i} \cup V_z. \end{aligned}$$

Because $V_z \in \mathcal{B}_{\tilde{J}}$ and $\mu V_z = 0$, we see that

(α) $h_\phi^{-1}[E_i] = g_\phi^{-1}[E_i] \in \mathcal{B}_J$ if $i \in J$,

(β) $h_\phi^{-1}[E_i] \in \mathcal{B}_{\tilde{J}}$ and $h_\phi^{-1}[E_i] \triangle F_{\phi i}$ is negligible if $i \in \tilde{J} \setminus J$,

and consequently

(γ) $(h_\phi^{-1}[E_i])^\bullet = \phi E_i^\bullet$ for every $i \in \tilde{J}$,

(δ) $(h_\phi^{-1}[E])^\bullet = \phi E^\bullet$ for every $E \in \mathcal{B}_{\tilde{J}}$

(by 343Ab); moreover,

(ϵ) $h_\phi^{-1}[E] = g_\phi^{-1}[E]$ for every $E \in \mathcal{B}_J$,

(ζ) $h_\phi^{-1}[E] \in \mathcal{B}_{\tilde{J}}$ for every $E \in \mathcal{B}_{\tilde{J}}$,

(η) $h_\phi^{-1}[E_i] = E_i$ if $i \in I \setminus \tilde{J}$,

so that

(θ) $h_\phi^{-1}[E] \in \mathcal{B}\mathfrak{a}_I$ for every $E \in \mathcal{B}\mathfrak{a}_I$;

finally, because $F_{\iota i} = E_i$,

(ι) $h_\iota(x) = x$ for every $x \in X$.

(iii) The next step is to note that if ϕ, $\psi \in G$ then

$$H_{\phi,\psi} = \{x : x \in X,\, h_\phi h_\psi(x) \neq h_{\psi\phi}(x)\}$$

belongs to $\mathcal{B}_{\tilde{J}}$ and is negligible. **P**

$$H_{\phi,\psi} = \bigcup_{i\in I} h_\psi^{-1}[h_\phi^{-1}[E_i]] \triangle h_{\psi\phi}^{-1}[E_i].$$

Now if $i \in J$, then $h_\phi^{-1}[E_i] = g_\phi^{-1}[E_i] \in \mathcal{B}_J$, so

$$h_\psi^{-1}[h_\phi^{-1}[E_i]] = h_\psi^{-1}[g_\phi^{-1}[E_i]] = g_\psi^{-1}[g_\phi^{-1}[E_i]] = g_{\psi\phi}^{-1}[E_i] = h_{\psi\phi}^{-1}[E_i].$$

Next, for $i \in I \setminus \tilde{J}$,

$$h_\psi^{-1}[h_\phi^{-1}[E_i]] = h_\psi^{-1}[E_i] = E_i = h_{\psi\phi}^{-1}[E_i].$$

So

$$H_{\phi,\psi} = \bigcup_{i\in \tilde{J}\setminus J} h_\psi^{-1}[h_\phi^{-1}[E_i]] \triangle h_{\psi\phi}^{-1}[E_i].$$

But for any particular $i \in \tilde{J} \setminus J$, E_i and $h_\phi^{-1}[E_i]$ belong to $\mathcal{B}_{\tilde{J}}$, so

$$(h_\psi^{-1}[h_\phi^{-1}[E_i]])^\bullet = \psi(h_\phi^{-1}[E_i])^\bullet = \psi\phi E_i^\bullet = (h_{\psi\phi}^{-1}[E_i])^\bullet,$$

and $h_\psi^{-1}[h_\phi^{-1}[E_i]] \triangle h_{\psi\phi}^{-1}[E_i]$ is a negligible set, which by (ii-ζ) belongs to $\mathcal{B}_{\tilde{J}}$. So $H_{\phi,\psi}$ is a countable union of sets of measure 0 in $\mathcal{B}_{\tilde{J}}$ and is itself a negligible member of $\mathcal{B}_{\tilde{J}}$, as claimed. **Q**

(iv) Set

$$H = \bigcup_{\phi,\psi\in G} H_{\phi,\psi} \cup \bigcup_{\phi\in G} h_\phi^{-1}[V_z].$$

Then $H \in \mathcal{B}_{\tilde{J}}$ and $\mu H = 0$. **P** We know that every $H_{\phi,\psi}$ is negligible and belongs to $\mathcal{B}_{\tilde{J}}$ ((iii) above), that every $h_\phi^{-1}[V_z]$ belongs to $\mathcal{B}_{\tilde{J}}$ (by (ii-ζ), and that $(h_\phi^{-1}[V_z])^\bullet = \phi V_z^\bullet = 0$, so that $h_\phi^{-1}[V_z]$ is negligible, for every $\phi \in G$ (by (ii-δ)). Consequently H is negligible and belongs to $\mathcal{B}_{\tilde{J}}$. **Q** Also, of course, $V_z = h_\iota^{-1}[V_z] \subseteq H$.

Next, $h_\phi(x) \notin H$ whenever $x \in X \setminus H$ and $\phi \in G$. **P** If ψ, $\theta \in G$ then

$$h_{\theta\psi} h_\phi(x) = h_{\phi\theta\psi}(x) = h_\psi h_{\phi\theta}(x) = h_\psi h_\theta h_\phi(x),$$

$$h_\psi h_\phi(x) = h_{\phi\psi}(x) \notin V_z$$

because

$$x \notin H_{\theta\psi,\phi} \cup H_{\psi,\phi\theta} \cup H_{\theta,\phi} \cup H_{\psi,\phi} \cup h_{\phi\psi}^{-1}[V_z];$$

thus $h_\phi(x) \notin H_{\psi,\theta} \cup h_\psi^{-1}[V_z]$; as ψ and θ are arbitrary, $h_\phi(x) \notin H$. **Q**

(v) The next fact we need is that there is a bijection $q : X \to H$ such that (α) for $E \subseteq H$, $E \in \mathcal{B}_{\tilde{J}}$ iff $q^{-1}[E] \in \mathcal{B}_{\tilde{J}}$ (β) $q(x)(i) = x(i)$ for every $i \in I\setminus(\tilde{J}\setminus J)$ and $x \in X$. **P** Fix any bijection $r : \tilde{J}\setminus J \to \tilde{J}\setminus(J\cup K)$. Consider the maps $p_1 : X \to H$, $p_2 : H \to X$ given by

$$\begin{aligned} p_1(x)(i) &= x(r^{-1}(i)) \text{ if } i \in \tilde{J} \setminus (J \cup K),\\ &= z(i) \text{ if } i \in K,\\ &= x(i) \text{ if } i \in X \setminus (\tilde{J} \setminus J),\\ p_2(y) &= y \end{aligned}$$

for $x \in X$, $y \in H$. Then p_1 is actually an isomorphism between $(X, \mathcal{B}_{\tilde{J}})$ and $(V_z, \mathcal{B}_{\tilde{J}} \cap \mathcal{P}V_z)$. So p_1, p_2 are isomorphisms between $(X, \mathcal{B}_{\tilde{J}})$, $(H, \mathcal{B}_{\tilde{J}} \cap \mathcal{P}H)$ and measurable subspaces of H, X respectively. By 344D, there is an isomorphism q between X and H such that, for every $x \in X$, either $q(x) = p_1(x)$ or $p_2(q(x)) = x$. Since $p_1(x)\restriction I \setminus (\tilde{J} \setminus J) = x\restriction I \setminus (\tilde{J} \setminus J)$ for every $x \in X$, and $p_2(y)\restriction I \setminus (\tilde{J} \setminus J) = y\restriction I \setminus (\tilde{J} \setminus J)$ for every $y \in H$, $q(x)\restriction I \setminus (\tilde{J} \setminus J) = x\restriction I \setminus (\tilde{J} \setminus J)$ for every $x \in X$. **Q**

(vi) An incidental fact which will be used below is the following: if $i \in \tilde{J}$ and $\phi \in G$ then $g_\phi^{-1}[E_i]$ belongs to $\mathcal{B}_{\tilde{J}}$, because it belongs to $\mathcal{B}_J$ if $i \in J$, and otherwise is equal to E_i. Consequently $g_\phi^{-1}[E] \in \mathcal{B}_{\tilde{J}}$ for every $E \in \mathcal{B}_{\tilde{J}}$.

(vii) I am at last ready to give a formula for $\tilde{g}_\phi$. For $\phi \in G$ set

$$\begin{aligned} \tilde{g}_\phi(x) &= h_\phi(x) \text{ if } x \in X \setminus H,\\ &= q g_\phi q^{-1}(x) \text{ if } x \in H. \end{aligned}$$

Now $(\tilde{J}, \langle \tilde{g}_\phi \rangle_{\phi\in G})$ is consistent. **P**

(α) If $i \in \tilde{J}$ and $\phi \in G$,

$$\tilde{g}_\phi^{-1}[E_i] = (h_\phi^{-1}[E_i] \setminus H) \cup q[g_\phi^{-1}[q^{-1}[E_i \cap H]]] \in \tilde{\mathcal{B}}_J$$

because $H \in \mathcal{B}_{\tilde{J}}$ and $h_\phi^{-1}[E]$, $q^{-1}[H \cap E]$, $g_\phi^{-1}[E]$ and $q[E]$ all belong to $\mathcal{B}_{\tilde{J}}$ for every $E \in \mathcal{B}_{\tilde{J}}$. At the same time, because $\tilde{g}_\phi$ agrees with h_ϕ on the conegligible set $X \setminus H$,

$$(\tilde{g}_\phi^{-1}[E_i])^\bullet = (h_\phi^{-1}[E_i])^\bullet = \phi E_i^\bullet.$$

(β) If $i \in I \setminus \tilde{J}$, $\phi \in G$ and $x \in X$ then

$$g_\phi(x)(i) = h_\phi(x)(i) = q(x)(i) = x(i),$$

and if $x \in H$ then $q^{-1}(x)(i)$ also is equal to $x(i)$; so $\tilde{g}_\phi(x)(i) = x(i)$. But this means that $\tilde{g}_\phi^{-1}[E_i] = E_i$.

(γ) If ϕ, $\psi \in G$ and $x \in X \setminus H$, then

$$\tilde{g}_\psi(x) = h_\psi(x) \in X \setminus H$$

by (iv) above. So

$$\tilde{g}_\phi \tilde{g}_\psi(x) = h_\phi h_\psi(x) = h_{\psi\phi}(x) = \tilde{g}_{\psi\phi}(x)$$

because $x \notin H_{\phi,\psi}$. While if $x \in H$, then

$$\tilde{g}_\psi(x) = q g_\psi q^{-1}(x) \in H,$$

so

$$\tilde{g}_\phi \tilde{g}_\psi(x) = q g_\phi q^{-1} q g_\psi q^{-1}(x) = q g_\phi g_\psi q^{-1}(x) = q g_{\psi\phi} q^{-1}(x) = \tilde{g}_{\psi\phi}(x).$$

Thus $\tilde{g}_\phi \tilde{g}_\psi = \tilde{g}_{\psi\phi}$.

(δ) Because $g_\iota(x)=h_\iota(x)=x$ for every x, $\tilde g_\iota(x)=x$ for every x. **Q**

(viii) Finally, if $i\in J$ and $\phi\in G$, $q^{-1}[E_i]=E_i$, so that $q[E_i]=E_i\cap H$. Accordingly $q(x)\restriction J=x\restriction J$ for every $x\in X$, while $q^{-1}(x)\restriction J=x\restriction J$ for $x\in H$. So $g_\phi q^{-1}(x)\restriction J=g_\phi(x)\restriction J$ for $x\in H$, and

$$\begin{aligned}\tilde g_\phi(x)\restriction J&=h_\phi(x)\restriction J=g_\phi(x)\restriction J \text{ if } x\in X\setminus H,\\ &=qg_\phi q^{-1}(x)\restriction J=g_\phi q^{-1}(x)\restriction J=g_\phi(x)\restriction J \text{ if } x\in H.\end{aligned}$$

Thus $(\tilde J,\langle\tilde g_\phi\rangle_{\phi\in G})$ satisfies all the required conditions.

(c) The remaining idea we need is the following: there is a non-decreasing family $\langle J_\xi\rangle_{\xi\le\kappa}$ in $\mathcal{J}$, for some cardinal κ, such that $J_{\xi+1}\setminus J_\xi$ is countably infinite for every $\xi<\kappa$, $J_\xi=\bigcup_{\eta<\xi}J_\eta$ for every limit ordinal $\eta<\kappa$, and $J_\kappa=I$. **P** Recall that I am already supposing that I is infinite. If I is countable, set $\kappa=1$, $J_0=\emptyset$, $J_1=I$. Otherwise, set $\kappa=\#(I)$ and let $\langle i_\xi\rangle_{\xi<\kappa}$ be an enumeration of I. For $i\in I$, $\phi\in G$ let $K_{\phi i}\subseteq I$ be a countable set such that $F_{\phi i}\in\mathcal{B}_{K_{\phi i}}$. Choose the J_ξ inductively, as follows. The inductive hypothesis must include the requirement that $\#(J_\xi)\le\max(\omega,\#(\xi))$ for every ξ. Start by setting $J_0=\emptyset$. Given $\xi<\kappa$ and $J_\xi\in\mathcal{J}$ with $\#(J_\xi)\le\max(\omega,\#(\xi))<\kappa$, take an infinite set $L\subseteq\kappa\setminus J_\xi$ and set $J_{\xi+1}=J_\xi\cup\bigcup_{n\in\mathbb{N}}L_n$, where

$$L_0=L\cup\{i_\xi\},$$

$$L_{n+1}=\bigcup_{i\in L_n,\phi\in G}K_{\phi i}$$

for $n\in\mathbb{N}$, so that every L_n is countable,

$$F_{\phi i}\in\mathcal{B}_{L_{n+1}} \text{ whenever } i\in L_n,\ \phi\in G$$

and $J_{\xi+1}\in\mathcal{J}$; since $L\subseteq J_{\xi+1}\setminus J_\xi\subseteq\bigcup_{n\in\mathbb{N}}L_n$, $J_{\xi+1}\setminus J_\xi$ is countably infinite, and

$$\#(J_{\xi+1})=\max(\omega,\#(J_\xi))\le\max(\omega,\#(\xi))=\max(\omega,\#(\xi+1)).$$

For non-zero limit ordinals $\xi<\kappa$, set $J_\xi=\bigcup_{\eta<\xi}J_\eta$; then

$$\#(J_\xi)\le\max(\omega,\#(\xi),\sup_{\eta<\xi}\#(J_\eta))\le\max(\omega,\#(\xi)).$$

Thus the induction proceeds. Observing that the construction puts i_ξ into $J_{\xi+1}$ for every ξ, we see that J_κ will be the whole of I, as required. **Q**

(d) Now put (b) and (c) together, as follows. Take $\langle J_\xi\rangle_{\xi\le\kappa}$ from (c). Set $f_{\phi0}(x)=x$ for every $\phi\in G$, $x\in X$; then, because $J_0=\emptyset$, $(J_0,\langle f_{\phi0}\rangle_{\phi\in G})$ is consistent in the sense of (b). Given that $(J_\xi,\langle f_{\phi\xi}\rangle_{\phi\in G})$ is consistent, where $\xi<\kappa$, use the construction of (b) to find a family $\langle f_{\phi,\xi+1}\rangle_{\phi\in G}$ such that $(J_{\xi+1},\langle f_{\phi,\xi+1}\rangle_{\phi\in G})$ is consistent and $f_{\phi,\xi+1}(x)(i)=f_{\phi\xi}(x)(i)$ for every $i\in J_\xi$ and $x\in X$. At a non-zero limit ordinal $\xi\le\kappa$, set

$$\begin{aligned}f_{\phi\xi}(x)(i)&=f_{\phi\eta}(x)(i) \text{ if } x\in X,\ \eta<\xi,\ i\in J_\eta,\\ &=x(i) \text{ if } i\in I\setminus J_\xi.\end{aligned}$$

(The inductive hypothesis includes the requirement that $f_{\phi\eta}(x)\restriction J_\zeta=f_{\phi\zeta}(x)\restriction J_\zeta$ whenever $\phi\in G$, $x\in X$ and $\zeta\le\eta<\xi$.) To see that $(J_\xi,\langle f_{\phi\xi}\rangle_{\phi\in G})$ is consistent, the only non-trivial point to check is that

$$f_{\phi,\xi}f_{\psi,\xi}=f_{\psi\phi,\xi}$$

for all ϕ, $\psi\in G$. But if $i\in J_\xi$ there is some $\eta<\xi$ such that $i\in J_\eta$, and in this case

$$f_{\psi,\xi}^{-1}[E_i]=f_{\psi,\eta}^{-1}[E_i]\in\mathcal{B}_{J_\eta}$$

is determined by coordinates in J_η, so that (because $f_{\phi,\xi}(x)\restriction J_\eta=f_{\phi,\eta}(x)\restriction J_\eta$ for every x)

$$f_{\phi,\xi}^{-1}[f_{\psi,\xi}^{-1}[E_i]]=f_{\phi,\eta}^{-1}[f_{\psi,\eta}^{-1}[E_i]]=f_{\psi\phi,\eta}^{-1}[E_i]=f_{\psi\phi,\xi}^{-1}[E_i];$$

while if $i\in I\setminus J_\xi$ then

$$f_{\psi\phi,\xi}^{-1}[E_i]=E_i=f_{\phi,\xi}^{-1}[E_i]=f_{\psi,\xi}^{-1}[E_i]=f_{\psi,\xi}^{-1}[f_{\phi,\xi}^{-1}[E_i]].$$

Thus

$$f_{\psi,\xi}^{-1}[f_{\phi,\xi}^{-1}[E_i]]=f_{\psi\phi,\xi}^{-1}[E_i]$$

for every i, and $f_{\phi,\xi}f_{\psi,\xi} = f_{\psi\phi,\xi}$.

On completing the induction, set $f_\phi = f_{\phi\kappa}$ for every $\phi \in G$; it is easy to see that $\langle f_\phi\rangle_{\phi\in G}$ satisfies the conditions of the theorem.

344F Corollary Let I be any set, and let μ be a σ-finite measure on $X = \{0,1\}^I$. Suppose that μ is the completion of its restriction to the σ-algebra $\mathcal{B}\mathfrak{a}_I$ generated by the sets $\{x : x(i) = 1\}$ as i runs over I. Write $\mathfrak{A}$ for the measure algebra of μ. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Then we can choose simultaneously, for each $\phi \in G$, a function $f_\phi : X \to X$ representing ϕ, in such a way that $f_{\phi\psi} = f_\psi f_\phi$ for all ϕ, $\psi \in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi \in G$ is invertible and $\phi^{-1} \in G$, we shall have $f_{\phi^{-1}} = f_\phi^{-1}$; so that if moreover ϕ is measure-preserving, f_ϕ will be an automorphism of the measure space (X, Σ, μ).

proof Apply 344E to $\mu\restriction\mathcal{B}\mathfrak{a}_I$; of course $\mathfrak{A}$ is canonically isomorphic to the measure algebra of $\mu\restriction\mathcal{B}\mathfrak{a}_I$ (322Da). The functions f_ϕ provided by 344E still represent the homomorphisms ϕ when re-interpreted as functions on the completed measure space $(\{0,1\}^I, \mu)$, by 343Ac.

344G Corollary Let I be any set, ν_I the usual measure on $\{0,1\}^I$, and $\mathfrak{B}_I$ its measure algebra. Then any measure-preserving automorphism of $\mathfrak{B}_I$ is representable by a measure space automorphism of $(\{0,1\}^I, \nu_I)$.

344H Lemma Let (X, Σ, μ) be a perfect semi-finite measure space. If $H \in \Sigma$ is a non-negligible set which includes no atom, there is a negligible subset of H with cardinal $\mathfrak{c}$.

proof (a) Consider first the case in which μ is atomless, compact and totally finite, and $H = X$. Let $\mathcal{K} \subseteq \mathcal{P}X$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$. Set $S = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$, and choose $\langle K_\sigma\rangle_{\sigma\in S}$ inductively, as follows. $K_\emptyset$ is to be any non-negligible member of $\mathcal{K}\cap\Sigma$. Given that $\mu K_\sigma > 0$, where $\sigma \in \{0,1\}^n$, take F_σ, $F'_\sigma \subseteq K_\sigma$ to be disjoint non-negligible measurable sets both of measure at most 3^{-n}; such exist because μ is atomless (215D). Choose $K_{\sigma^\frown<0>} \subseteq F_\sigma$, $K_{\sigma^\frown<1>} \subseteq F'_\sigma$ to be non-negligible members of $\mathcal{K} \cap \Sigma$.

For each $w \in \{0,1\}^{\mathbb{N}}$, $\langle K_{w\restriction n}\rangle_{n\in\mathbb{N}}$ is a decreasing sequence of members of $\mathcal{K}$ all of non-zero measure, so has non-empty intersection; choose a point $x_w \in \bigcap_{n\in\mathbb{N}} K_{w\restriction n}$. Since $K_{\sigma^\frown<0>} \cap K_{\sigma^\frown<1>} = \emptyset$ for every $\sigma \in S$, all the x_w are distinct, and $A = \{x_w : w \in \{0,1\}^{\mathbb{N}}\}$ has cardinal $\mathfrak{c}$. Also

$$A \subseteq \bigcup_{\sigma\in\{0,1\}^n} K_\sigma$$

which has measure at most $2^n 3^{-(n-1)}$ for every $n \geq 1$, so $\mu^*A = 0$ and A is negligible.

(b) Now consider the case in which μ is atomless and totally finite and perfect, but not necessarily compact, while again $H = X$. In this case, by 215D, we can choose $\langle E_n\rangle_{n\in\mathbb{N}}$ inductively so that $\mu(E_n \cap E) = \frac{1}{2}\mu E$ whenever $n \in \mathbb{N}$ and E is an atom of the subalgebra of $\mathcal{P}X$ generated by $\{E_i : i < n\}$. Now define $f : X \to \{0,1\}^{\mathbb{N}}$ by setting $f(x) = \langle\chi E_n(x)\rangle_{n\in\mathbb{N}}$ for $x \in X$. Consider the image measure $\nu = \mu f^{-1}$ on $Y = f[X] \subseteq \{0,1\}^{\mathbb{N}}$. This is perfect. **P** If $g : Y \to \mathbb{R}$ is T-measurable, where $\mathrm{T} = \operatorname{dom}\nu$, and $\nu F > 0$, then $gf : X \to \mathbb{R}$ is Σ-measurable and $\mu f^{-1}[F] > 0$. There is therefore a compact set $K \subseteq gf[f^{-1}[F]]$ such that $\mu(gf)^{-1}[K] > 0$. In this case, $K \subseteq g[F]$ and $\nu g^{-1}[K] > 0$. **Q**

Next, for every $n \in \mathbb{N}$ and $\sigma \in \{0,1\}^n$,

$$\nu\{y : y \in Y,\, y\restriction n = \sigma\} = \mu\{x : \forall\, i < n,\, x \in E_i \iff \sigma(i) = 1\} = 2^{-n}\mu X.$$

So ν can have no atom of measure greater than $2^{-n}\mu X$; as n is arbitrary, ν is atomless. Thirdly, (Y, T, ν) is countably separated, because $\langle\{y : y \in Y,\, y(n) = 1\}\rangle_{n\in\mathbb{N}}$ is a sequence of measurable sets separating the points of Y. By 343K, ν is compact; by (a) here, there is a ν-negligible set $B \subseteq Y$ of cardinal $\mathfrak{c}$. Now $f^{-1}[B]$ is μ-negligible, and because $B \subseteq f[X]$, $\#(f^{-1}[B]) \geq \#(B) = \mathfrak{c}$. We therefore have a set $A \subseteq f^{-1}[B]$ with cardinal $\mathfrak{c}$, and A is μ-negligible.

(c) Finally, for the general case in which μ is just semi-finite and perfect, and H is a non-negligible subset of X not including an atom, let $E \subseteq H$ be a set of non-zero finite measure. Then the subspace measure μ_E is atomless. Also μ_E is perfect. **P** Let $f : E \to \mathbb{R}$ be a measurable function. Define $g : X \to \mathbb{R}$ by setting

$$g(x) = e^{f(x)} \text{ if } x \in E,$$
$$= 0 \text{ if } x \in X \setminus E.$$

Then g is measurable. There is therefore a compact set $K \subseteq g[E]$ such that $\mu g^{-1}[K] > 0$. Now $\ln[K] \subseteq f[E]$ is compact and $\mu_E f^{-1}[\ln[K]] = \mu g^{-1}[K] > 0$. **Q**

By (b), there is a μ_E-negligible set $A \subseteq E$ with cardinal $\mathfrak{c}$, and of course A is also a μ-negligible subset of H.

Remark I see that in this proof I have slipped into a notation which is a touch more sophisticated than what I have used so far. See 3A1H for a note on the interpretations of '$\{0,1\}^n$', '$\{0,1\}^{\mathbb{N}}$' which make sense of the formulae here.

344I Theorem Let (X,Σ,μ) and (Y,T,ν) be atomless, perfect, complete, strictly localizable, countably separated measure spaces of the same non-zero magnitude. Then they are isomorphic.

proof (a) The point is that the measure algebra $(\mathfrak{A},\bar\mu)$ of μ has Maharam type ω. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ separating the points of X. Let Σ_0 be the σ-subalgebra of Σ generated by $\{E_n : n \in \mathbb{N}\}$, and $\mathfrak{A}_0$ the order-closed subalgebra of $\mathfrak{A}$ generated by $\{E_n^\bullet : n \in \mathbb{N}\}$; then $E^\bullet \in \mathfrak{A}_0$ for every $E \in \Sigma_0$, and $(X,\Sigma_0,\mu\restriction\Sigma_0)$ is countably separated. Let $f : X \to \mathbb{R}$ be Σ_0-measurable and injective (343E). Of course f is also Σ-measurable. If $a \in \mathfrak{A}\setminus\{0\}$, express a as $E^\bullet$ where $E \in \Sigma$. Because (X,Σ,μ) is perfect, there is a compact $K \subseteq \mathbb{R}$ such that $K \subseteq f[E]$ and $\mu f^{-1}[K] > 0$. K is surely a Borel set, so $f^{-1}[K] \in \Sigma_0$ and

$$b = f^{-1}[K]^\bullet \in \mathfrak{A}_0 \setminus \{0\}.$$

But because f is injective, we also have $f^{-1}[K] \subseteq E$ and $b \subseteq a$. As a is arbitrary, $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$; but $\mathfrak{A}_0$ is order-closed, so must be the whole of $\mathfrak{A}$. Thus $\mathfrak{A}$ is τ-generated by the countable set $\{E_n^\bullet : n \in \mathbb{N}\}$, and $\tau(\mathfrak{A}) \le \omega$. **Q**

On the other hand, because $\mathfrak{A}$ is atomless, and not $\{0\}$, none of its principal ideals can have finite Maharam type, and it is Maharam-type-homogeneous, with type ω.

(b) Writing $(\mathfrak{B},\bar\nu)$ for the measure algebra of ν, we see that the argument of (a) applies equally to $(\mathfrak{B},\bar\nu)$, so that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are atomless localizable measure algebras, with Maharam type ω and the same magnitude. Consequently they are isomorphic as measure algebras, by 332J. Let $\phi : \mathfrak{A} \to \mathfrak{B}$ be a measure-preserving isomorphism.

By 343K, both μ and ν are (locally) compact. As they are also complete and strictly localizable, 343B tells us that there are functions $g : Y \to X$ and $f : X \to Y$ representing ϕ and ϕ^{-1}. Now $fg : Y \to Y$ and $gf : X \to X$ represent the identity automorphisms on $\mathfrak{B}$, $\mathfrak{A}$, so by 343F are equal almost everywhere to the identity functions on Y, X respectively. Set

$$E = \{x : x \in X,\ gf(x) = x\}, \quad F = \{y : y \in Y,\ fg(y) = y\};$$

then both E and F are conegligible. Of course $f[E] \subseteq F$ (since $fgf(x) = f(x)$ for every $x \in E$), and similarly $g[F] \subseteq E$; consequently $f\restriction E$, $g\restriction F$ are the two halves of a one-to-one correspondence between E and F. Because ϕ is measure-preserving, $\mu f^{-1}[H] = \nu H$ and $\nu g^{-1}[G] = \mu G$ for every $G \in \Sigma$, $H \in \mathrm{T}$; accordingly $f\restriction E$ is an isomorphism between the subspace measures on E and F.

(c) By 344H, there is a negligible set $A \subseteq E$ with cardinal $\mathfrak{c}$. Now X and Y, being countably separated, both have cardinal at most $\mathfrak{c}$. (There are injective functions from X and Y to $\mathbb{R}$.) Set

$$B = A \cup (X \setminus E), \quad C = f[A] \cup (Y \setminus F).$$

Then B and C are negligible subsets of X, Y respectively, and both have cardinal $\mathfrak{c}$ precisely, so there is a bijection $h : B \to C$. Set

$$f_1(x) = f(x) \text{ if } x \in X \setminus B = E \setminus A,$$
$$= h(x) \text{ if } x \in B.$$

Then, because μ and ν are complete, f_1 is an isomorphism between the measure spaces (X,Σ,μ) and (Y,T,ν), as required.

344J Corollary Suppose that E, F are two Lebesgue measurable subsets of $\mathbb{R}^r$ of the same non-zero measure. Then the subspace measures on E and F are isomorphic.

344K Corollary (a) A measure space is isomorphic to Lebesgue measure on $[0,1]$ iff it is an atomless countably separated compact (or perfect) complete probability space; in this case it is also isomorphic to the usual measure on $\{0,1\}^{\mathbb{N}}$.

(b) A measure space is isomorphic to Lebesgue measure on $\mathbb{R}$ iff it is an atomless countably separated compact (or perfect) σ-finite measure space which is not totally finite; in this case it is also isomorphic to Lebesgue measure on any Euclidean space $\mathbb{R}^r$.

(c) Let μ be Lebesgue measure on $\mathbb{R}$. If $0<\mu E<\infty$ and we set $\nu F=\frac{1}{\mu E}\mu F$ for every measurable $F\subseteq E$, then (E,ν) is isomorphic to Lebesgue measure on $[0,1]$.

344L The homogeneity property of Lebesgue measure described in 344J is repeated in $\{0,1\}^I$ for any infinite I.

Theorem Let I be an infinite set, and ν_I the usual measure on $\{0,1\}^I$. If $E\subseteq\{0,1\}^I$ is a measurable set of non-zero measure, the subspace measure on E is isomorphic to $(\nu_I E)\nu_I$.

proof For $J\subseteq I$ let ν_J be the usual measure on $X_J=\{0,1\}^J$.

(a) If I is countably infinite, then the subspace measure on E is perfect and complete and countably separated, so is isomorphic to Lebesgue measure on the interval $[0,\nu_I E]$, by 344I. But by 344Kc, or otherwise, this is isomorphic, up to a scalar multiple of the measure, to Lebesgue measure on $[0,1]$, which is in turn isomorphic to ν_I.

So henceforth we can suppose that I is uncountable.

(b) By 254Oc there are a countable set $J\subseteq I$ and a set $E'\subseteq E$, determined by coordinates in J, such that $E\setminus E'$ is negligible. Identifying X_I with $X_J\times X_{I\setminus J}$ (254N), we can think of E' as $V\times X_{I\setminus J}$ where V is measured by ν_J (see 254O). Take $v_0\in V$ and set

$$V'=V\setminus\{v_0\},\quad W'=X_J\setminus\{v_0\},\quad E''=V'\times X_{I\setminus J},\quad F''=W'\times X_{I\setminus J}.$$

Then by (a), applied to V' and W' in turn, we have a bijection $g:V'\to W'$ which, up to a scalar multiple of the measure, is an isomorphism between the subspace measures. Now the subspace measure on $V'\times X_{I\setminus J}$ is just the product of the subspace measure on V' with $\nu_{I\setminus J}$ (251Q(ii)), so if we set $f_0(x,z)=(g(x),z)$ for $x\in V'$ and $z\in X_{I\setminus J}$, then $f_0:E''\to F''$ is an isomorphism of the subspace measures on E'' and F'', up to a scalar multiple of the measures as always. On the other hand, $E\setminus E''$ and $X_I\setminus F''$ are negligible and both have cardinal $\#(X_{I\setminus J})=\#(X_I)$, so we have a bijection $f_1:E\setminus E''\to X_I\setminus F''$. Putting f_0 and f_1 together, we have a bijection $f:E\to X_I$ which, up to a scalar multiple of the measure, is an isomorphism of the subspace measure on E with ν_I.

344X Basic exercises (a) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and suppose that there are $E\in\Sigma$, $F\in\mathrm{T}$ such that (X,Σ,μ) is isomorphic to the subspace (F,T_F,ν_F), while (Y,T,ν) is isomorphic to (E,Σ_E,μ_E). Show that (X,Σ,μ) and (Y,T,ν) are isomorphic.

(b) Let (X,Σ,μ) and (Y,T,ν) be perfect countably separated complete strictly localizable measure spaces with isomorphic measure algebras. Show that there are conegligible subsets $X'\subseteq X$, $Y'\subseteq Y$ such that X' and Y', with the subspace measures, are isomorphic.

(c) Let (X,Σ,μ) and (Y,T,ν) be perfect countably separated complete strictly localizable measure spaces with isomorphic measure algebras. Suppose that they are not purely atomic. Show that they are isomorphic.

(d) Give an example of two perfect countably separated complete probability spaces, with isomorphic measure algebras, which are not isomorphic.

(e) Let (Z,Σ,μ) be the Stone space of a homogeneous measure algebra. Show that if E, $F\in\Sigma$ have the same non-zero finite measure, then the subspace measures on E and F are isomorphic.

(f) Let $(I^{\|},\Sigma,\mu)$ be the split interval with its usual measure (343J), and $\mathfrak{A}$ its measure algebra. (i) Show that every measure-preserving automorphism of $\mathfrak{A}$ is represented by a measure space automorphism of $I^{\|}$. (ii) Show that if E, $F\in\Sigma$ and $\mu E=\mu F>0$ then the subspace measures on E and F are isomorphic.

344Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, $\mathcal{I}$ a σ-ideal of Σ, and $\mathfrak{A}$ the quotient $\Sigma/\mathcal{I}$. Suppose that there is a countable set $\mathcal{A}\subseteq\Sigma$ separating the points of X. Let G be a countable semigroup of Boolean homomorphisms from $\mathfrak{A}$ to itself such that every member of G can be represented by some function from X to itself. Show that a family $\langle f_\phi\rangle_{\phi\in G}$ of such representatives can be chosen in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X.

(b) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras. Suppose that each is isomorphic to a principal ideal of the other. Show that they are isomorphic.

(c) Let I be an infinite set, and write $\mathcal{B}\mathfrak{a}_I$ for the σ-algebra of subsets of $X=\{0,1\}^I$ generated by the sets $\{x:x(i)=1\}$ as i runs over I. Let μ and ν be σ-finite measures on X, both with domain $\mathcal{B}\mathfrak{a}_I$, and with measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$. Show that any Boolean isomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$ is represented by a permutation $f:X\to X$ such that f^{-1} represents $\phi^{-1}:\mathfrak{B}\to\mathfrak{A}$, and hence that $(\mathfrak{A},\bar\mu)$ is isomorphic to $(\mathfrak{B},\bar\nu)$ iff $(X,\mathcal{B}\mathfrak{a}_I,\mu)$ is isomorphic to $(X,\mathcal{B}\mathfrak{a}_I,\nu)$.

(d) Let I be any set, and write $\mathcal{B}\mathfrak{a}_I$ for the σ-algebra of subsets of $X=\{0,1\}^I$ generated by the sets $\{x:x(i)=1\}$ as i runs over I. Let $\mathcal{I}$ be an ω_1-saturated σ-ideal of $\mathcal{B}\mathfrak{a}_I$, and write $\mathfrak{A}$ for the quotient Boolean algebra $\mathcal{B}/\mathcal{I}$. Let G be a countable semigroup of order-continuous Boolean homomorphisms from $\mathfrak{A}$ to itself. Show that we can choose simultaneously, for each $\phi\in G$, a function $f_\phi:X\to X$ representing ϕ, in such a way that $f_{\phi\psi}=f_\psi f_\phi$ for all ϕ, $\psi\in G$; and if the identity automorphism ι belongs to G, then we may arrange that f_ι is the identity function on X. In particular, if $\phi\in G$ is invertible and $\phi^{-1}\in G$, f_ϕ will be an automorphism of the structure $(X,\mathcal{B}\mathfrak{a}_I,\mathcal{I})$.

(e) Let I be any set, and write $\mathcal{B}\mathfrak{a}_I$ for the σ-algebra of subsets of $X=\{0,1\}^I$ generated by the sets $\{x:x(i)=1\}$ as i runs over I. Let $\mathcal{I}$, $\mathcal{J}$ be ω_1-saturated σ-ideals of $\mathcal{B}\mathfrak{a}_I$. Show that if the Boolean algebras $\mathcal{B}\mathfrak{a}_I/\mathcal{I}$ and $\mathcal{B}\mathfrak{a}_I/\mathcal{J}$ are isomorphic, so are the structures $(X,\mathcal{B}\mathfrak{a}_I,\mathcal{I})$ and $(X,\mathcal{B}\mathfrak{a}_I,\mathcal{J})$.

344 Notes and comments In this section and the last, I have allowed myself to drift some distance from the avowed subject of this chapter; but it seemed a suitable place for this material, which is fundamental to abstract measure theory. We find that the concepts of §§342-343 are just what is needed to characterize Lebesgue measure (344K), and the characterization shows that among non-negligible measurable subspaces of $\mathbb{R}^r$ the isomorphism classes are determined by a single parameter, the measure of the subspace. Of course a very large number of other spaces – indeed, most of those appearing in ordinary applications of measure theory to other topics – are perfect and countably separated (for example, those of 342Xe and 343Ye), and therefore covered by this classification. I note that it includes, as a special case, the isomorphism between Lebesgue measure on $[0,1]$ and the usual measure on $\{0,1\}^{\mathbb{N}}$ already described in 254K.

In 344I, the first part of the proof is devoted to showing that a perfect countably separated measure space has countable Maharam type; I ought perhaps to note here that we must resist the temptation to suppose that all countably separated measure spaces have countable Maharam type. In fact there are countably separated probability spaces with Maharam type as high as $2^{\mathfrak{c}}$. The arguments are elementary but seem to fit better into §521 of Volume 5 than here.

I have offered three contexts in which automorphisms of measure algebras are represented by automorphisms of measure spaces (344A, 344C, 344E). In the first case, every automorphism can be represented simultaneously in a consistent way. In the other two cases, there is, I am sure, no such consistent family of representations which can be constructed within ZFC; but the theorems I give offer consistent simultaneous representations of countably many homomorphisms. The question arises, whether 'countably many' is the true natural limit of the arguments. In fact it is possible to extend both results to families of at most ω_1 automorphisms.

Having successfully characterized Lebesgue measure – or, what is very nearly the same thing, the usual measure on $\{0,1\}^{\mathbb{N}}$ – it is natural to seek similar characterizations of the usual measures on $\{0,1\}^\kappa$ for uncountable cardinals κ. This seems to be hard. A variety of examples (some touched on in the exercises to §521) show that none of the most natural conjectures can be provable in ZFC.

In fact the principal new ideas of this section do not belong specifically to measure theory; rather, they belong to the general theory of σ-algebras and σ-ideals of sets. In the case of the Schröder-Bernstein-type theorem 344D, this is obvious from the formulation I give. (See also 344Yb.) In the case of 344B and 344E, I offer generalizations in 344Ya-344Ye. Of course the applications of 344B here, in 344C and its corollaries, depend on Maharam's theorem and the concept of 'compact' measure space. The former has no generalization to the wider context, and the value of the latter is based on the equivalences in Theorem 343B, which also do not have simple generalizations.

The property described in 344J – a measure space (X, Σ, μ) in which any two measurable subsets of the same non-zero measure are isomorphic – seems to be a natural concept of 'homogeneity' for measure spaces; it seems unreasonable to ask for all sets of zero measure to be isomorphic, since finite sets of different cardinalities can be expected to be of zero measure. An extra property, shared by Lebesgue measure and the usual measure on $\{0,1\}^I$ and by the measure on the split interval (344Kc, 344L, 344Xf) but not by counting measure, would be the requirement that measurable sets of different non-zero finite measures should be isomorphic up to a scalar multiple of the measure. All these examples have the further property, that all automorphisms of their measure algebras correspond to automorphisms of the measure spaces.

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345 Translation-invariant liftings

In this section and the next I complement the work of §341 by describing some important special properties which can, in appropriate circumstances, be engineered into our liftings. I begin with some remarks on translation-invariance. I restrict my attention to measure spaces which we have already seen, delaying a general discussion of translation-invariant measures on groups until Volume 4.

345A Translation-invariant liftings I shall consider two forms of translation-invariance, as follows.

(a) Let μ be Lebesgue measure on $\mathbb{R}^r$, and Σ its domain. A lifting $\phi : \Sigma \to \Sigma$ is **translation-invariant** if $\phi(E + x) = \phi E + x$ for every $E \in \Sigma$, $x \in \mathbb{R}^r$. (Recall from 134A that $E + x = \{y + x : y \in E\}$ belongs to Σ for every $E \in \Sigma$, $x \in \mathbb{R}^r$.)

Similarly, writing $\mathfrak{A}$ for the measure algebra of μ, a lifting $\theta : \mathfrak{A} \to \Sigma$ is **translation-invariant** if $\theta(E + x)^\bullet = \theta E^\bullet + x$ for every $E \in \Sigma$, $x \in \mathbb{R}^r$.

It is easy to see that if θ and ϕ correspond to each other in the manner of 341B, then one is translation-invariant if and only if the other is.

(b) Now let I be any set, and let ν_I be the usual measure on $X = \{0,1\}^I$, with T_I its domain and $\mathfrak{B}_I$ its measure algebra. For x, $y \in X$, define $x + y \in X$ by setting $(x + y)(i) = x(i) +_2 y(i)$ for every $i \in I$; that is, give X the group structure of the product group $\mathbb{Z}_2^I$. This makes X an abelian group (isomorphic to the additive group $(\mathcal{P}I, \triangle)$ of the Boolean algebra $\mathcal{P}I$, if we match $x \in X$ with $\{i : x(i) = 1\} \subseteq I$).

Recall that the measure ν_I is a product measure (254J), being the product of copies of the fair-coin probability measure on the two-element set $\{0,1\}$. If $x \in X$, then for each $i \in I$ the map $\epsilon \mapsto \epsilon +_2 x(i) : \{0,1\} \to \{0,1\}$ is a measure space automorphism of $\{0,1\}$, since the two singleton sets $\{0\}$ and $\{1\}$ have the same measure $\frac{1}{2}$. It follows at once that the map $y \mapsto y + x : X \to X$ is a measure space automorphism.

Accordingly we can again say that a lifting $\theta : \mathfrak{B}_I \to \mathrm{T}_I$, or $\phi : \mathrm{T}_I \to \mathrm{T}_I$, is **translation-invariant** if

$$\theta(E + x)^\bullet = \theta E^\bullet + x, \quad \phi(E + x) = \phi E + x$$

whenever $E \in \Sigma$ and $x \in X$.

345B Theorem For any $r \geq 1$, there is a translation-invariant lifting for Lebesgue measure on $\mathbb{R}^r$.

proof (a) Write μ for Lebesgue measure on $\mathbb{R}^r$, Σ for its domain. Let $\underline{\phi} : \Sigma \to \Sigma$ be lower Lebesgue density (341E). Then $\underline{\phi}$ is translation-invariant in the sense that $\underline{\phi}(E + x) = \underline{\phi}E + x$ for every $E \in \Sigma$, $x \in \mathbb{R}^r$. **P**

$$\begin{aligned}\underline{\phi}(E+x) &= \{y : y \in \mathbb{R}^r, \lim_{\delta\downarrow 0} \tfrac{\mu((E+x)\cap B(y,\delta))}{\mu(B(y,\delta))} = 1\} \\ &= \{y : y \in \mathbb{R}^r, \lim_{\delta\downarrow 0} \tfrac{\mu(E\cap B(y-x,\delta))}{\mu(B(y-x,\delta))} = 1\}\end{aligned}$$

(because μ is translation-invariant)

$$\begin{aligned}&= \{y+x : y \in \mathbb{R}^r, \lim_{\delta\downarrow 0} \tfrac{\mu(E\cap B(y,\delta))}{\mu(B(y,\delta))} = 1\} \\ &= \underline{\phi}E + x. \textbf{ Q}\end{aligned}$$

(b) Let ϕ_0 be any lifting for μ such that $\phi_0 E \supseteq \underline{\phi}E$ for every $E \in \Sigma$ (341Jb). Consider

$$\phi E = \{y : \mathbf{0} \in \phi_0(E-y)\}$$

for $E \in \Sigma$. It is easy to check that $\phi : \Sigma \to \Sigma$ is a Boolean homomorphism because ϕ_0 is, so that, for instance,

$$\begin{aligned} y \in \phi E \triangle \phi F &\iff \mathbf{0} \in \phi_0(E-y)\triangle\phi_0(F-y) \\ &\iff \mathbf{0} \in \phi_0((E-y)\triangle(F-y)) = \phi_0((E\triangle F)-y) \\ &\iff y \in \phi(E\triangle F).\end{aligned}$$

(c) If $\mu E = 0$, then $E-y$ is negligible for every $y \in \mathbb{R}^r$, so $\phi_0(E-y)$ is always empty and $\phi E = \emptyset$.

(d) Next, $\underline{\phi}E \subseteq \phi E$ for every $E \in \Sigma$. **P** If $y \in \underline{\phi}E$, then

$$\mathbf{0} = y - y \in \underline{\phi}E - y = \underline{\phi}(E-y) \subseteq \phi_0(E-y),$$

so $y \in \phi E$. **Q** By 341Ib, ϕ is a lifting for μ.

(e) Finally, ϕ is translation-invariant, because if $E \in \Sigma$ and x, $y \in \mathbb{R}^r$ then

$$\begin{aligned} y \in \phi(E+x) &\iff \mathbf{0} \in \phi_0(E+x-y) = \phi_0(E-(y-x)) \\ &\iff y - x \in \phi E \\ &\iff y \in \phi E + x.\end{aligned}$$

345C Theorem For any set I, there is a translation-invariant lifting for the usual measure on $\{0,1\}^I$.

proof I base the argument on the same programme as in 345B. This time we have to work rather harder, as we have no simple formula for a translation-invariant lower density. However, the ideas already used in 341F-341H are in fact adequate, if we take care, to produce one.

(a) Since there is certainly a bijection between I and its cardinal $\kappa = \#(I)$, it is enough to consider the case $I = \kappa$. Write ν_κ for the usual measure on $X = \{0,1\}^I = \{0,1\}^\kappa$ and T_κ for its domain. For each $\xi < \kappa$ set $E_\xi = \{x : x \in X, x(\xi) = 1\}$, and let Σ_ξ be the σ-algebra generated by $\{E_\eta : \eta < \xi\}$. Because $x + E_\eta$ is either E_η or $X \setminus E_\eta$, and in either case belongs to Σ_ξ, for every $\eta < \xi$ and $x \in X$, Σ_ξ is translation-invariant. (Consider the algebra

$$\Sigma'_\xi = \{E : E + x \in \Sigma_\xi \text{ for every } x \in X\};$$

this must be Σ_ξ.) Let Φ_ξ be the set of partial lower densities $\underline{\phi} : \Sigma_\xi \to \mathrm{T}_\kappa$ which are translation-invariant in the sense that $\underline{\phi}(E+x) = \underline{\phi}E + x$ for any $E \in \Sigma_\xi$, $x \in X$.

(b)(i) For $\xi < \kappa$, $\Sigma_{\xi+1}$ is just the algebra of subsets of X generated by $\Sigma_\xi \cup \{E_\xi\}$, that is, sets of the form $(F \cap E_\xi) \cup (G \setminus E_\xi)$ where F, $G \in \Sigma_\xi$ (312N). Moreover, the expression is unique. **P** Define $x_\xi \in X$ by setting $x_\xi(\xi) = 1$, $x_\xi(\eta) = 0$ if $\eta \neq \xi$. Then $x_\xi + E_\eta = E_\eta$ for every $\eta < \xi$, so $x_\xi + F = F$ for every $F \in \Sigma_\xi$. If $H = (F \cap E_\xi) \cup (G \setminus E_\xi)$ where F, $G \in \Sigma_\xi$, then

$$x_\xi + H = ((x_\xi + F) \cap (x_\xi + E_\xi)) \cup ((x_\xi + G) \setminus (x_\xi + E_\xi)) = (F \setminus E_\xi) \cup (G \cap E_\xi),$$

so

$$F = (H \cap E_\xi) \cup ((x_\xi + H) \setminus E_\xi) = F_H,$$

$$G = (H \setminus E_\xi) \cup ((x_\xi + H) \cap E_\xi) = G_H$$

are determined by H. **Q**

(ii) The functions $H \mapsto F_H$, $H \mapsto G_H : \Sigma_{\xi+1} \to \Sigma_\xi$ defined above are clearly Boolean homomorphisms; moreover, if H, $H' \in \Sigma_{\xi+1}$ and $H \triangle H'$ is negligible, then

$$(F_H \triangle F_{H'}) \cup (G_H \triangle G_{H'}) \subseteq (H \triangle H') \cup (x_\xi + (H \triangle H'))$$

is negligible. It follows at once that if $\xi < \kappa$ and $\underline{\phi} \in \Phi_\xi$, we can define $\underline{\phi}_1 : \Sigma_{\xi+1} \to \mathrm{T}_\kappa$ by setting

$$\underline{\phi}_1 H = (\underline{\phi} F_H \cap E_\xi) \cup (\underline{\phi} G_H \setminus E_\xi),$$

and $\underline{\phi}_1$ will be a lower density. If $H \in \Sigma_\xi$ then $F_H = G_H = H$, so $\underline{\phi}_1 H = \underline{\phi} H$. Generally, if H, $H' \in \Sigma_\xi$ then

$$\underline{\phi}_1((H \cap E_\xi) \cup (H' \setminus E_\xi)) = (\underline{\phi} F_H \cap E_\xi) \cup (\underline{\phi} G_{H'} \setminus E_\xi) = (\underline{\phi} H \cap E_\xi) \cup (\underline{\phi} H' \setminus E_\xi).$$

(iii) To see that $\underline{\phi}_1$ is translation-invariant, observe that if $x \in X$ and $x(\xi) = 0$ then $x + E_\xi = E_\xi$, so, for any F, $G \in \Sigma_\xi$,

$$\begin{aligned}\underline{\phi}_1(x + ((F \cap E_\xi) \cup (G \setminus E_\xi))) &= \underline{\phi}_1(((F + x) \cap E_\xi) \cup ((G + x) \setminus E_\xi))\\ &= (\underline{\phi}(F + x) \cap E_\xi) \cup (\underline{\phi}(G + x) \setminus E_\xi)\\ &= ((\underline{\phi} F + x) \cap E_\xi) \cup ((\underline{\phi} G + x) \setminus E_\xi)\\ &= x + (\underline{\phi} F \cap E_\xi) \cup (\underline{\phi} G \setminus E_\xi)\\ &= x + \underline{\phi}_1((F \cap E_\xi) \cup (G \setminus E_\xi)).\end{aligned}$$

While if $x(\xi) = 1$ then $x + E_\xi = X \setminus E_\xi$, so

$$\begin{aligned}\underline{\phi}_1(x + ((F \cap E_\xi) \cup (G \setminus E_\xi))) &= \underline{\phi}_1(((F + x) \setminus E_\xi) \cup ((G + x) \cap E_\xi))\\ &= (\underline{\phi}(F + x) \setminus E_\xi) \cup (\underline{\phi}(G + x) \cap E_\xi)\\ &= ((\underline{\phi} F + x) \setminus E_\xi) \cup ((\underline{\phi} G + x) \cap E_\xi)\\ &= x + (\underline{\phi} F \cap E_\xi) \cup (\underline{\phi} G \setminus E_\xi)\\ &= x + \underline{\phi}_1((F \cap E_\xi) \cup (G \setminus E_\xi)).\end{aligned}$$

So $\underline{\phi}_1 \in \Phi_{\xi+1}$.

(iv) Thus every member of Φ_ξ has an extension to a member of $\Phi_{\xi+1}$.

(c) Now suppose that $\langle \zeta(n) \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in κ with supremum $\xi < \kappa$. Then Σ_ξ is just the σ-algebra generated by $\bigcup_{n \in \mathbb{N}} \Sigma_{\zeta(n)}$. If we have a sequence $\langle \underline{\phi}_n \rangle_{n \in \mathbb{N}}$ such that $\underline{\phi}_n \in \Phi_{\zeta(n)}$ and $\underline{\phi}_{n+1}$ extends $\underline{\phi}_n$ for every n, then there is a $\underline{\phi} \in \Phi_\xi$ extending every $\underline{\phi}_n$. **P** I repeat the ideas of 341G.

(i) For $E \in \Sigma_\xi$, $n \in \mathbb{N}$ choose g_{En} such that g_{En} is a conditional expectation of χE on $\Sigma_{\zeta(n)}$; that is,

$$\int_F g_{En} = \int_F \chi E = \nu_\kappa(F \cap E)$$

for every $E \in \Sigma_{\zeta(n)}$. Moreover, make these choices in such a way that (α) every g_{En} is $\Sigma_{\zeta(n)}$-measurable and defined everywhere on X (β) $g_{En} = g_{E'n}$ for every n if $E \triangle E'$ is negligible. Now $\lim_{n\to\infty} g_{En}$ exists and is equal to χE almost everywhere, by Lévy's martingale theorem (275I).

(ii) For $E \in \Sigma_\xi$, $k \geq 1$, $n \in \mathbb{N}$ set

$$H_{kn}(E) = \{x : x \in X,\, g_{En}(x) \geq 1 - 2^{-k}\} \in \Sigma_{\zeta(n)}, \quad \tilde{H}_{kn}(E) = \underline{\phi}_n(H_{kn}(E)),$$

$$\underline{\phi}E=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\tilde{H}_{km}(E).$$

(iii) Every $g_{\emptyset n}$ is zero almost everywhere, every $H_{kn}(\emptyset)$ is negligible and every $\tilde{H}_{kn}(\emptyset)$ is empty; so $\underline{\phi}\emptyset=\emptyset$. If E, $E'\in\Sigma_\xi$ and $E\triangle E'$ is negligible, $g_{En}=g_{E'n}$ for every n, $H_{nk}(E)=H_{nk}(E')$ and $\tilde{H}_{nk}(E)=\tilde{H}_{nk}(E')$ for all n, k, and $\underline{\phi}E=\underline{\phi}E'$.

(iv) If $E\subseteq F$ in Σ_ξ, then $g_{En}\le g_{Fn}$ almost everywhere for every n, every $H_{kn}(E)\setminus H_{kn}(F)$ is negligible, $\tilde{H}_{kn}(E)\subseteq\tilde{H}_{kn}(F)$ for every n, k, and $\underline{\phi}E\subseteq\underline{\phi}F$.

(v) If E, $F\in\Sigma_\xi$ then $\chi(E\cap F)\ge_{\text{a.e.}}\chi E+\chi F-1$ so $g_{E\cap F,n}\ge_{\text{a.e.}}g_{En}+g_{Fn}-1$ for every n. Accordingly

$$H_{k+1,n}(E)\cap H_{k+1,n}(F)\setminus H_{kn}(E\cap F)$$

is negligible, and (because $\underline{\phi}_n$ is a lower density)

$$\tilde{H}_{kn}(E\cap F)\supseteq\underline{\phi}_n(H_{k+1,n}(E)\cap H_{k+1,n}(F))=\tilde{H}_{k+1,n}(E)\cap\tilde{H}_{k+1,n}(F)$$

for all $k\ge 1$, $n\in\mathbb{N}$. Now, if $x\in\underline{\phi}E\cap\underline{\phi}F$, then, for any $k\ge 1$, there are n_1, $n_2\in\mathbb{N}$ such that

$$x\in\bigcap_{m\ge n_1}\tilde{H}_{k+1,m}(E),\quad x\in\bigcap_{m\ge n_2}\tilde{H}_{k+1,m}(F).$$

But this means that

$$x\in\bigcap_{m\ge\max(n_1,n_2)}\tilde{H}_{km}(E\cap F).$$

As k is arbitrary, $x\in\underline{\phi}(E\cap F)$; as x is arbitrary, $\underline{\phi}E\cap\underline{\phi}F\subseteq\underline{\phi}(E\cap F)$. We know already from (iv) that $\underline{\phi}(E\cap F)\subseteq\underline{\phi}E\cap\underline{\phi}F$, so $\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F$.

(vi) If $E\in\Sigma_\xi$, then $g_{En}\to\chi E$ a.e., so setting

$$V=\bigcap_{k\ge 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}H_{km}(E)=\{x:\limsup_{n\to\infty}g_{En}(x)\ge 1\},$$

$V\triangle E$ is negligible; but

$$\underline{\phi}E\triangle V\subseteq\bigcup_{k\ge 1,n\in\mathbb{N}}H_{kn}(E)\triangle\tilde{H}_{kn}(E)$$

is also negligible, so $\underline{\phi}E\triangle E$ is negligible. Thus $\underline{\phi}$ is a partial lower density with domain Σ_ξ.

(vii) If $E\in\Sigma_{\zeta(n)}$, then $E\in\Sigma_{\zeta(m)}$ for every $m\ge n$, so $g_{Em}=_{\text{a.e.}}\chi E$ for every $m\ge n$; $H_{km}(E)\triangle E$ is negligible for $k\ge 1$, $m\ge n$;

$$\tilde{H}_{km}(E)=\underline{\phi}_mE=\underline{\phi}_nE$$

for $k\ge 1$, $m\ge n$; and $\underline{\phi}E=\underline{\phi}_nE$. Thus $\underline{\phi}$ extends every $\underline{\phi}_n$.

(viii) I have still to check the translation-invariance of $\underline{\phi}$. If $E\in\Sigma_\xi$ and $x\in X$, consider g'_n, defined by setting

$$g'_n(y)=g_{En}(y-x)$$

for every $y\in X$, $n\in\mathbb{N}$; that is, g'_n is the composition $g_{En}\psi$, where $\psi(y)=y-x$ for $y\in X$. (I am not sure whether it is more, or less, confusing to distinguish between the operations of addition and subtraction in X. Of course $y-x=y+(-x)=y+x$ for every y.) Because ψ is a measure space automorphism, and in particular is inverse-measure-preserving, we have

$$\int_{F+x}g'_n=\int_{\psi^{-1}[F]}g'_n=\int_F g_{En}=\nu_\kappa(E\cap F)$$

whenever $F\in\Sigma_{\zeta(n)}$ (235Gc[3]). But because $\Sigma_{\zeta(n)}$ is itself translation-invariant, we can apply this to $F-x$ to get

$$\int_F g'_n=\nu_\kappa(E\cap(F-x))=\nu_\kappa((E+x)\cap F)$$

for every $F\in\Sigma_{\zeta(n)}$. Moreover, for any $\alpha\in\mathbb{R}$,

$$\{y:g'_n(y)\ge\alpha\}=\{y:g_{En}(y)\ge\alpha\}+x\in\Sigma_{\zeta(n)}$$

[3]Formerly 235I.

for every α, and g'_n is $\Sigma_{\zeta(n)}$-measurable. So g'_n is a conditional expectation of $\chi(E+x)$ on $\Sigma_{\zeta(n)}$, and must be equal almost everywhere to $g_{E+x,n}$.

This means that if we set

$$H'_{kn} = \{y : g'_n(y) \ge 1 - 2^{-k}\} = H_{kn}(E) + x$$

for k, $n \in \mathbb{N}$, we shall have $H'_{kn} \in \Sigma_{\zeta(n)}$ and $H'_{kn} \triangle H_{kn}(E+x)$ will be negligible, so

$$\begin{aligned}\tilde{H}_{kn}(E+x) &= \underline{\phi}_n(H_{kn}(E+x)) = \underline{\phi}_n(H'_{kn}) \\ &= \underline{\phi}_n(H_{kn}(E)+x) = \underline{\phi}_n(H_{kn}(E)) + x = \tilde{H}_{kn}(E) + x.\end{aligned}$$

Consequently

$$\begin{aligned}\underline{\phi}(E+x) &= \bigcap_{k\ge 1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \tilde{H}_{kn}(E+x) \\ &= \bigcap_{k\ge 1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \tilde{H}_{kn}(E) + x = \underline{\phi}E + x.\end{aligned}$$

As E and x are arbitrary, $\underline{\phi}$ is translation-invariant and belongs to Φ_ξ. **Q**

(d) We are now ready for the proof that there is a translation-invariant lower density for ν_κ. **P** Build inductively a family $\langle \underline{\phi}_\xi \rangle_{\xi \le \kappa}$ such that (α) $\underline{\phi}_\xi \in \Phi_\xi$ for each ξ (β) $\underline{\phi}_\xi$ extends $\underline{\phi}_\eta$ whenever $\eta \le \xi \le \kappa$. The induction starts with $\Sigma_0 = \{\emptyset, X\}$, $\underline{\phi}_0\emptyset = \emptyset$, $\underline{\phi}_0 X = X$. The inductive step to a successor ordinal is dealt with in (b), and the inductive step to a non-zero ordinal of countable cofinality is dealt with in (c). If $\xi \le \kappa$ has uncountable cofinality, then $\Sigma_\xi = \bigcup_{\eta<\xi} \Sigma_\eta$, so we can (and must) take $\underline{\phi}_\xi$ to be the unique common extension of all the previous $\underline{\phi}_\eta$.

The induction ends with $\underline{\phi}_\kappa : \Sigma_\kappa \to \mathrm{T}_\kappa$. Note that Σ_κ is not in general the whole of T_κ. But for every $E \in \mathrm{T}_\kappa$ there is an $F \in \Sigma_\kappa$ such that $E \triangle F$ is negligible (254Ff). So we can extend $\underline{\phi}_\kappa$ to a function $\underline{\phi}$ defined on the whole of T_κ by setting

$$\underline{\phi}E = \underline{\phi}_\kappa F \text{ whenever } E \in \mathrm{T}_\kappa,\ F \in \Sigma_\kappa \text{ and } \nu_\kappa(E \triangle F) = 0$$

(the point being that $\underline{\phi}_\kappa F = \underline{\phi}_\kappa F'$ if F, $F' \in \Sigma_\kappa$ and $\nu_\kappa(E\triangle F) = \nu_\kappa(E \triangle F') = 0$). It is easy to check that $\underline{\phi}$ is a lower density, and it is translation-invariant because if $E \in \mathrm{T}_\kappa$, $x \in X$, $F \in \Sigma_\kappa$ and $E \triangle F$ is negligible, then $(E+x)\triangle(F+x) = (E\triangle F) + x$ is negligible, so

$$\underline{\phi}(E+x) = \underline{\phi}_\kappa(F+x) = \underline{\phi}_\kappa F + x = \underline{\phi}E + x. \text{ **Q**}$$

(e) The rest of the argument is exactly that of parts (b)-(e) of the proof of 345B; you have to change $\mathbb{R}^r$ into X wherever it appears, but otherwise you can use it word for word, interpreting '$\mathbf{0}$' as the identity of the group X, that is, the constant function with value 0.

345D Translation-invariant liftings are of great importance, and I will return to them in §447 with a theorem dramatically generalizing the results above. Here I shall content myself with giving one of their basic properties, set out for the two kinds of translation-invariant lifting we have seen.

Proposition Let (X, Σ, μ) be *either* Lebesgue measure on $\mathbb{R}^r$ *or* the usual measure on $\{0,1\}^I$ for some set I, and let $\phi : \Sigma \to \Sigma$ be a translation-invariant lifting. Then for any open set $G \subseteq X$ we must have $G \subseteq \phi G \subseteq \overline{G}$, and for any closed set F we must have $\operatorname{int} F \subseteq \phi F \subseteq F$.

proof (a) Suppose that $G \subseteq X$ is open and that $x \in G$. Then there is an open set U such that $\mathbf{0} \in U$ and $x+U-U = \{x+y-z : y,\, z \in U\} \subseteq G$. **P** ($\alpha$) If $X = \mathbb{R}^r$, take $\delta > 0$ such that $\{y : \|y-x\| \le \delta\} \subseteq G$, and set $U = \{y : \|y-x\| < \frac{1}{2}\delta\}$. ($\beta$) If $X = \{0,1\}^I$, then there is a finite set $K \subseteq I$ such that $\{y : y\restriction K = x\restriction K\} \subseteq G$ (3A3K); set $U = \{y : y(i) = 0 \text{ for every } i \in K\}$. **Q**

It follows that $x \in \phi G$. **P** Consider $H = x + U$. Then $\mu H = \mu U > 0$ so $H \cap \phi H \ne \emptyset$. Let $y \in U$ be such that $x + y \in \phi H$. Then

$$x = (x+y) - y \in \phi(H - y) \subseteq \phi G$$

because

$$H - y \subseteq x + U - U \subseteq G. \quad \mathbf{Q}$$

(b) Thus $G \subseteq \phi G$ for every open set $G \subseteq X$. But it follows at once that if G is open and F is closed,

$$\operatorname{int} F \subseteq \phi(\operatorname{int} F) \subseteq \phi F,$$

$$\overline{G} = X \setminus \operatorname{int}(X \setminus G) \supseteq X \setminus \phi(X \setminus G) = \phi G,$$

$$F = X \setminus (X \setminus F) \supseteq X \setminus \phi(X \setminus F) = \phi F.$$

345E I remarked in 341Lg that it is undecidable in ordinary set theory whether there is a lifting for Borel measure on $\mathbb{R}$. It is however known that there can be no translation-invariant Borel lifting. The argument depends on the following fact about measurable sets in $\{0,1\}^{\mathbb{N}}$.

Lemma Give $X = \{0,1\}^{\mathbb{N}}$ its usual measure $\nu_{\mathbb{N}}$, and let $E \subseteq X$ be any non-negligible measurable set. Then there is an $n \in \mathbb{N}$ such that for every $k \geq n$ there are x, $x' \in E$ which differ at k and nowhere else.

proof By 254Fe, there is a set F, determined by coordinates in a finite set, such that $\nu_{\mathbb{N}}(E \triangle F) \leq \frac{1}{4}\nu_{\mathbb{N}}E$; we have $\nu_{\mathbb{N}}F \geq \frac{3}{4}\nu_{\mathbb{N}}E$, so $\nu_{\mathbb{N}}(E \triangle F) \leq \frac{1}{3}\nu_{\mathbb{N}}F$. Let $n \in \mathbb{N}$ be such that F is determined by coordinates in $\{0,\dots,n-1\}$. Take any $k \geq n$. Then the map $\psi : X \to X$, defined by setting $(\psi x)(k) = 1 - x(k)$, $(\psi x)(i) = x(i)$ for $i \neq k$, is a measure space automorphism, and

$$\nu_{\mathbb{N}}(\psi^{-1}[E \triangle F] \cup (E \triangle F)) \leq 2\nu_{\mathbb{N}}(E \triangle F) < \nu_{\mathbb{N}}F.$$

Take any $x \in F \setminus ((E \triangle F) \cup \psi^{-1}[E \triangle F])$. Then $x' = \psi x$ differs from x at k, and only there; but also $x' \in F$, by the choice of n, so both x and x' belong to E.

345F Proposition Let μ be the restriction of Lebesgue measure to the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$. Then μ is translation-invariant, but has no translation-invariant lifting.

proof (a) To see that μ is translation-invariant all we have to know is that $\mathcal{B}$ is translation-invariant and that Lebesgue measure is translation-invariant. I have already cited 134A for the proof that Lebesgue measure is invariant, and $\mathcal{B}$ is invariant because $G + x$ is open for every open set G and every $x \in \mathbb{R}$.

(b) The argument below is most easily expressed in terms of the geometry of the Cantor set C. Recall that C is defined as the intersection $\bigcap_{n\in\mathbb{N}} C_n$ of a sequence of closed subsets of $[0,1]$; each C_n consists of 2^n closed intervals of length 3^{-n}; C_{n+1} is obtained from C_n by deleting the middle third of each interval of C_n. Any point of C is uniquely expressible as $f(e) = \frac{2}{3}\sum_{n=0}^{\infty} 3^{-n} e(n)$ for some $e \in \{0,1\}^{\mathbb{N}}$. (See 134Gb.) Let $\nu_{\mathbb{N}}$ be the usual measure of $\{0,1\}^{\mathbb{N}}$. Because the map $e \mapsto e(n) : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ is measurable for each n, $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ is measurable.

We can label the closed intervals constituting C_n as $\langle J_z \rangle_{z\in\{0,1\}^n}$, taking $J_\emptyset$ to be the unit interval $[0,1]$ and, for $z \in \{0,1\}^n$, taking $J_{z^\frown<0>}$ to be the left-hand third of J_z and $J_{z^\frown<1>}$ to be the right-hand third of J_z. (If the notation here seems odd to you, there is an explanation in 3A1H.)

For $n \in \mathbb{N}$ and $z \in \{0,1\}^n$, let J'_z be the open interval with the same centre as J_z and twice the length. Then $J'_z \setminus J_z$ consists of two open intervals of length $3^{-n}/2$ on either side of J_z; call the left-hand one V_z and the right-hand one W_z. Thus $V_{z^\frown<1>}$ is the right-hand half of the middle third of J_z, and $W_{z^\frown<0>}$ is the left-hand half of the middle third of J_z.

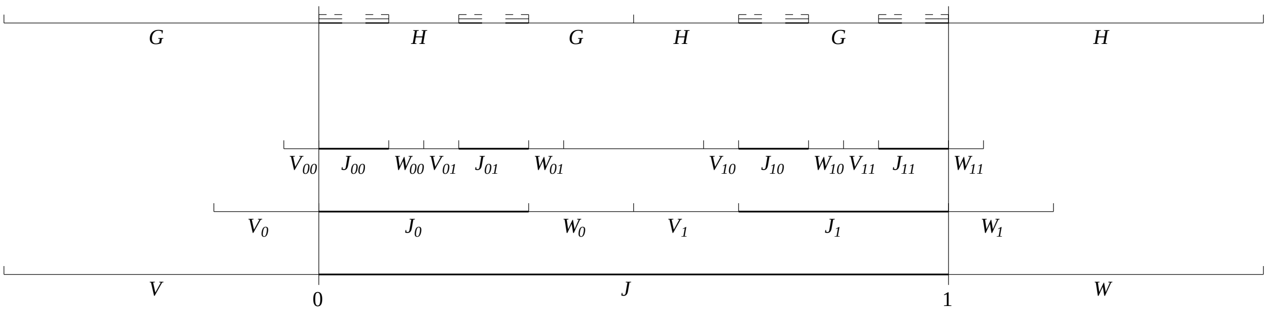
Construct sets G, $H \subseteq \mathbb{R}$ as follows.

> G is to be the union of the intervals V_z where z takes the value 1 an even number of times, together with the intervals W_z where z takes the value 0 an odd number of times;
>
> H is to be the union of the intervals V_z where z takes the value 1 an odd number of times, together with the intervals W_z where z takes the value 0 an even number of times.

G and H are open sets. The intervals V_z, W_z between them cover the whole of the interval $\left]-\frac{1}{2}, \frac{3}{2}\right[$ with the exception of the set C and the countable set of midpoints of the intervals J_z; so that $\left]-\frac{1}{2}, \frac{3}{2}\right[\setminus (G \cup H)$ is negligible. We have to observe that $G \cap H = \emptyset$. $\mathbf{P}$ For each z, $J'_{z^\frown<0>}$ and $J'_{z^\frown<1>}$ are disjoint subsets of J'_z. Consequently $J'_z \cap J'_w$ is non-empty just when one of z, w extends the other, and we need consider only the intersections of the four sets V_z, W_z, V_w, W_w when w is a proper extension of z; say $w \in \{0,1\}^n$

and $z = w\restriction m$, where $m < n$. (α) If in the extension $(w(m),\ldots,w(n-1))$ both values 0 and 1 appear, J'_w will be a subset of J_z, and certainly the four sets will all be disjoint. (β) If $w(i) = 0$ for $m \le i < n$, then $W_w \subseteq J_z$ is disjoint from the rest, while $V_w \subseteq V_z$; but z and w take the value 1 the same number of times, so V_w is assigned to G iff V_z is, and otherwise both are assigned to H. (γ) Similarly, if $w(i) = 1$ for $m \le i < n$, $V_w \subseteq J_z$, $W_w \subseteq W_z$ and z, w take the value 0 the same number of times, so W_z and W_w are assigned to the same set. **Q**

The following diagram may help you to see what is supposed to be happening:



The assignment rule can be restated as follows:

$V = V_\emptyset$ is assigned to G, $W = W_\emptyset$ is assigned to H;

$V_{z^\frown<0>}$ is assigned to the same set as V_z, and $V_{z^\frown<1>}$ to the other;

$W_{z^\frown<1>}$ is assigned to the same set as W_z, and $W_{z^\frown<0>}$ to the other.

(c) Now take any $n \in \mathbb{N}$ and $z \in \{0,1\}^n$. Consider the two open intervals $I_0 = J'_{z^\frown<0>}$, $I_1 = J'_{z^\frown<1>}$. These are both of length $\gamma = 2\cdot 3^{-n-1}$ and abut at the centre of J_z, so I_1 is just the translate $I_0 + \gamma$. I claim that $I_1 \cap H = (I_0 \cap G) + \gamma$. **P** Let A be the set

$$\bigcup_{m>n}\{w : w \in \{0,1\}^m,\ w \text{ extends } z^\frown<0>\},$$

and for $w \in A$ let w' be the finite sequence obtained from w by changing $w(n) = 0$ into $w'(n) = 1$ but leaving the other values of w unaltered. Then $V_{w'} = V_w + \gamma$ and $W_{w'} = W_w + \gamma$ for every $w \in A$. Now

$$\begin{aligned} I_0 \cap G = &\bigcup\{V_w : w \in A,\ w \text{ takes the value 1 an even number of times}\}\\ &\cup \bigcup\{W_w : w \in A,\ w \text{ takes the value 0 an odd number of times}\}, \end{aligned}$$

so

$$\begin{aligned} (I_0 \cap G) + \gamma &= \bigcup\{V_{w'} : w \in A,\ w \text{ takes the value 1 an even number of times}\}\\ &\qquad \cup \bigcup\{W_{w'} : w \in A,\ w \text{ takes the value 0 an odd number of times}\}\\ &= \bigcup\{V_{w'} : w \in A,\ w' \text{ takes the value 1 an odd number of times}\}\\ &\qquad \cup \bigcup\{W_{w'} : w \in A,\ w' \text{ takes the value 0 an even number of times}\}\\ &= I_1 \cap H. \ \mathbf{Q} \end{aligned}$$

(d) ? Now suppose, if possible, that $\phi : \mathcal{B} \to \mathcal{B}$ is a translation-invariant lifting. Note first that $U \subseteq \phi U$ for every open $U \subseteq \mathbb{R}$. **P** The argument is exactly that of 345D as applied to $\mathbb{R} = \mathbb{R}^1$. **Q** Consequently

$$J'_\emptyset = \left]-\tfrac{1}{2}, \tfrac{3}{2}\right[\subseteq \phi J'_\emptyset.$$

But as $J'_\emptyset \setminus (G \cup H)$ is negligible,

$$C \subseteq \left]-\tfrac{1}{2}, \tfrac{3}{2}\right[\subseteq \phi G \cup \phi H.$$

Consider the sets $E = f^{-1}[\phi G]$, $F = \{0,1\}^{\mathbb{N}} \setminus E = f^{-1}[\phi H]$. Because f is measurable and ϕG, ϕH are Borel sets, E and F are measurable subsets of $\{0,1\}^{\mathbb{N}}$, and at least one of them has positive measure for $\nu_{\mathbb{N}}$. There must therefore be e, $e' \in \{0,1\}^{\mathbb{N}}$, differing at exactly one coordinate, such that either both belong to E or both belong to F (345E). Let us suppose that n is such that $e(n) = 0$, $e'(n) = 1$ and $e(i) = e'(i)$ for $i \ne n$. Set $z = e\restriction n = e'\restriction n$. Then $f(e)$ belongs to the open interval $I_0 = J'_{z^\frown<0>}$, so $f(e) \in \phi I_0$ and $f(e) \in \phi G$ iff $f(e) \in \phi(I_0 \cap G)$. But now

$$f(e') = f(e) + 2 \cdot 3^{-n-1} \in I_1 = J'_{z^\frown<1>},$$

so

$$e \in E \iff f(e) \in \phi G \iff f(e) \in \phi(I_0 \cap G)$$
$$\iff f(e') \in \phi((I_0 \cap G) + 2 \cdot 3^{-n-1})$$

(because ϕ is translation-invariant)

$$\iff f(e') \in \phi(I_1 \cap H)$$

(by (c) above)

$$\iff f(e') \in \phi H$$

(because $f(e') \in I_1 \subseteq \phi I_1$)

$$\iff e' \in F.$$

But this contradicts the choice of e. **X**

Thus there is no translation-invariant lifting for μ.

Remark This result is due to JOHNSON 80; the proof here follows TALAGRAND 82B. For references to various generalizations see BURKE 93, §3.

345X Basic exercises (a) In 345Ab I wrote 'It follows at once that the map $y \mapsto y + x : X \to X$ is a measure space automorphism'. Write the details out in full, using 254G or otherwise.

(b) Let S^1 be the unit circle in $\mathbb{R}^2$, and let μ be one-dimensional Hausdorff measure on S^1 (§§264-265). Show that μ is translation-invariant, if S^1 is given its usual group operation corresponding to complex multiplication (255M), and that it has a translation-invariant lifting ϕ. (*Hint*: Identifying S^1 with $]-\pi,\pi]$ with the group operation $+_{2\pi}$, show that we can set $\phi E = \,]-\pi,\pi] \cap \phi'(\bigcup_{n\in\mathbb{Z}} E + 2\pi n)$, where ϕ' is any translation-invariant lifting for Lebesgue measure.)

>(c) Show that there is no lifting ϕ of Lebesgue measure on $\mathbb{R}$ which is 'symmetric' in the sense that $\phi(-E) = -\phi E$ for every measurable set E, writing $-E = \{-x : x \in E\}$. (*Hint*: can 0 belong to $\phi([0,\infty[)$?)

>(d) Let μ be Lebesgue measure on $X = \mathbb{R} \setminus \{0\}$. Show that there is a lifting ϕ of μ such that $\phi(xE) = x\phi E$ for every $x \in X$ and every measurable $E \subseteq X$, writing $xE = \{xy : y \in E\}$.

(e) Let ν_I be the usual measure on $X = \{0,1\}^I$, for some set I, T_I its domain, and $(\mathfrak{B}_I, \bar\nu_I)$ its measure algebra. (i) Show that we can define $\pi_x(a) = a + x$, for $a \in \mathfrak{B}_I$ and $x \in X$, by the formula $E^\bullet + x = (E+x)^\bullet$; and that $x \mapsto \pi_x$ is a group homomorphism from X to the group of measure-preserving automorphisms of $\mathfrak{A}$. (ii) Define Σ_ξ as in the proof of 345C, and set $\mathfrak{A}_\xi = \{E^\bullet : E \in \Sigma_\xi\}$. Say that a partial lifting $\underline{\theta} : \mathfrak{A}_\xi \to \mathrm{T}_I$ is translation-invariant if $\underline{\theta}(a + x) = \underline{\theta}a + x$ for every $a \in \mathfrak{A}_\xi$ and $x \in X$. Show that any such partial lifting can be extended to a translation-invariant partial lifting on $\mathfrak{A}_{\xi+1}$. (iii) Write out a proof of 345C in the language of 341F-341H.

>(f) Let $\underline{\phi}$ be a lower density for Lebesgue measure on $\mathbb{R}^r$ which is translation-invariant in the sense that $\underline{\phi}(E + x) = \underline{\phi}E + x$ for every $x \in \mathbb{R}^r$ and every measurable set E. Show that $\underline{\phi}G \supseteq G$ for every open set $G \subseteq \mathbb{R}^r$.

(g) Let μ be 1-dimensional Hausdorff measure on S^1, as in 345Xb. Show that there is no translation-invariant lifting ϕ of μ such that ϕE is a Borel set for every $E \in \operatorname{dom}\mu$.

345Y Further exercises (a) Let (X, Σ, μ) be a complete measure space, and suppose that X has a group operation $(x, y) \mapsto xy$ (not necessarily abelian!) such that μ is left-translation-invariant, in the sense that $xE = \{xy : y \in E\} \in \Sigma$ and $\mu(xE) = \mu E$ whenever $E \in \Sigma$ and $x \in X$. Suppose that $\underline{\phi} : \Sigma \to \Sigma$ is a lower density which is left-translation-invariant in the sense that $\underline{\phi}(xE) = x(\underline{\phi}E)$ for every $E \in \Sigma$ and $x \in X$. Show that there is a left-translation-invariant lifting $\phi : \Sigma \to \Sigma$ such that $\underline{\phi}E \subseteq \phi E$ for every $E \in \Sigma$.

(b) Write Σ for the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$, and $\mathcal{L}^0(\Sigma)$ for the linear space of Σ-measurable functions from $\mathbb{R}$ to itself. Show that there is a linear operator $T : L^0(\mu) \to \mathcal{L}^0(\Sigma)$ such that (α) $(Tu)^\bullet = u$ for every $u \in L^0(\mu)$ (β) $\sup_{x\in\mathbb{R}} |(Tu)(x)| = \|u\|_\infty$ for every $u \in L^\infty(\mu)$ (γ) $Tu \ge 0$ whenever $u \in L^\infty(\mu)$ and $u \ge 0$ (δ) T is translation-invariant in the sense that $T(S_x f)^\bullet = S_x T f^\bullet$ for every $x \in \mathbb{R}$ and $f \in \mathcal{L}^0(\Sigma)$, where $(S_x f)(y) = f(x + y)$ for $f \in \mathcal{L}^0(\Sigma)$ and x, $y \in \mathbb{R}$ (ϵ) T is reflection-invariant in the sense that $T(Rf)^\bullet = RTf^\bullet$ for every $f \in \mathcal{L}^0(\Sigma)$, where $(Rf)(x) = f(-x)$ for $f \in \mathcal{L}^0(\Sigma)$ and $x \in \mathbb{R}$. (*Hint*: for $f \in \mathcal{L}^0(\Sigma)$, set

$$p(f^\bullet) = \inf\{\alpha : \alpha \in [0, \infty], \lim_{\delta\downarrow 0} \frac{1}{2\delta}\mu\{x : |x| \le \delta, |f(x)| > \alpha\} = 0\}.$$

Set $V = \{u : u \in L^0(\mu), p(u) < \infty\}$ and show that V is a linear subspace of $L^0(\mu)$ and that $p\restriction V$ is a seminorm. Let $h_0 : V \to \mathbb{R}$ be a linear functional such that $h_0(\chi\mathbb{R})^\bullet = 1$ and $h_0(u) \le p(u)$ for every $u \in V$. Extend h_0 arbitrarily to a linear functional $h_1 : L^0(\mu) \to \mathbb{R}$; set $h(f^\bullet) = \frac{1}{2}(h_1(f^\bullet) + h_1(Rf)^\bullet)$. Set $(Tf^\bullet)(x) = h(S_{-x}f)^\bullet$. You will need 223C.) Show that there must be a $u \in L^1(\mu)$ such that $u \ge 0$ but $Tu \not\ge 0$.

(c) Show that there is no translation-invariant lifting ϕ of the usual measure on $\{0, 1\}^{\mathbb{N}}$ such that ϕE is a Borel set for every measurable set E.

345 Notes and comments I have taken a great deal of care over the concept of 'translation-invariance'. I hope that you are already a little impatient with some of the details as I have written them out; but while it is very easy to guess at the structure of such arguments as part (e) of the proof of 345B, or (b-iii) and (c-viii) in the proof of 345C, I am not sure that one can always be certain of guessing correctly. A fair test of your intuition will be how quickly you can generate the formulae appropriate to a non-abelian group operation, as in 345Ya.

Part (b) of the proof of 345C is based on the same idea as the proof of 341F. There is a useful simplification because the set E_ξ in 345C, corresponding to the set E of the proof of 341F, is independent of the algebra Σ_ξ in a very strong sense, so that the expression of an element of $\Sigma_{\xi+1}$ in the form $(F \cap E_\xi) \cup (G \setminus E_\xi)$ is unique. Interpreted in the terms of 341F, we have $w = v = 1$, so that the formula

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = \big(\underline{\theta}((a \cap v) \cup (b \setminus v)) \cap E\big) \cup \big(\underline{\theta}((a \setminus w) \cup (b \cap w)) \setminus E\big)$$

used there becomes

$$\underline{\theta}_1((a \cap e) \cup (b \setminus e)) = (\underline{\theta}a \cap E) \cup (\underline{\theta}b \setminus E),$$

matching the formula for $\underline{\phi}_1$ in the proof of 345C.

The results of this section are satisfying and natural; they have obvious generalizations, many of which are true. The most important measure spaces come equipped with a variety of automorphisms, and we can always ask which of these can be preserved by a lifting. The answers are not always obvious; I offer 345Xc and 346Xc as warnings, and 345Xd as an encouragement. 345Yb is striking (I have made it as striking as I can), but slightly off the most natural target; the sting is in the last sentence (see 341Xg).

Version of 17.12.10

346 Consistent liftings

I turn now to a different type of condition which we should naturally prefer our liftings to satisfy. If we have a product measure μ on a product $X = \prod_{i\in I} X_i$ of probability spaces, then we can look for liftings ϕ which 'respect coordinates', that is, are compatible with the product structure in the sense that they factor through subproducts (346A). There seem to be obstacles in the way of the natural conjecture (346Za), and I give the partial results which are known. For Maharam-type-homogeneous spaces X_i, there is always a lifting which respects coordinates (346E), and indeed the translation-invariant liftings of §345 on $\{0,1\}^I$ already have this property (346C). There is always a lower density for the product measure which respects coordinates, and we can ask for a little more (346G); using the full strength of 346G, we can enlarge this lower density to a lifting which respects single coordinates and initial segments of a well-ordered product (346H). In the case in which all the factors are copies of each other, we can arrange for the induced liftings on the factors to be copies also (346I, 346J, 346Ye). I end the section with an important fact about Stone spaces which is relevant here (346K-346L).

346A Definition Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X, Σ, μ). I will say that a lifting $\phi : \Sigma \to \Sigma$ **respects coordinates** if ϕE is determined by coordinates in J whenever $E \in \Sigma$ is determined by coordinates in $J \subseteq I$.

Remark Recall that a set $E \subseteq X$ is 'determined by coordinates in J' if $x' \in E$ whenever $x \in E$, $x' \in X$ and $x'\restriction J = x\restriction J$; that is, if E is expressible as $\pi_J^{-1}[F]$ for some $F \subseteq \prod_{i\in J} X_i$, where $\pi_J(x) = x\restriction J$ for every $x \in X$; that is, if $E = \pi_J^{-1}[\pi_J[E]]$. See 254M. Recall also that in this case, if E is measured by the product measure on X, then $\pi_J[E]$ is measured by the product measure on $\prod_{i\in J} X_i$ (254Ob).

346B Lemma (a) Let (X, Σ, μ) be a measure space with a lifting $\phi : \Sigma \to \Sigma$. Suppose that Y is a set and $f : X \to Y$ a surjective function such that whenever $E \in \Sigma$ is such that $f^{-1}[f[E]] = E$, then $f^{-1}[f[\phi E]] = \phi E$. Then we have a lifting ψ for the image measure μf^{-1} defined by the formula

$$f^{-1}[\psi F] = \phi(f^{-1}[F]) \text{ whenever } F \subseteq Y \text{ and } f^{-1}[F] \in \Sigma.$$

(b) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (Z, Λ, λ). For $J \subseteq I$ let $(Z_J, \Lambda_J, \lambda_J)$ be the product of $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in J}$, and $\pi_J : Z \to Z_J$ the canonical map. Let $\phi : \Lambda \to \Lambda$ be a lifting. If $J \subseteq I$ is such that ϕW is determined by coordinates in J whenever $W \in \Lambda$ is determined by coordinates in J, then ϕ induces a lifting $\phi_J : \Lambda_J \to \Lambda_J$ defined by the formula

$$\pi_J^{-1}[\phi_J E] = \phi(\pi_J^{-1}[E]) \text{ for every } E \in \Lambda_J.$$

proof (a) Set $\psi F = f[\phi(f^{-1}[F])]$ for $F \in \operatorname{dom}(\mu f^{-1})$. Because f is surjective, $\psi Y = Y$, and it is now elementary to check that ψ is a lifting for μf^{-1}.

(b) By 254Oa, λ_J is the image measure $\lambda \pi_J^{-1}$, so we can use (a).

Remark Of course we frequently wish to use part (b) here with a singleton set $J = \{j\}$. In this case we must remember that $(Z_J, \Sigma_J, \lambda_J)$ corresponds to the *completion* of the probability space (X_j, Σ_j, μ_j).

346C Theorem Let I be any set, and ν_I the usual measure on $X = \{0,1\}^I$. Then any translation-invariant lifting for ν_I respects coordinates.

proof Suppose that $E \subseteq X$ is a measurable set determined by coordinates in $J \subseteq I$; take $x \in \phi E$ and $x' \in X$ such that $x'\restriction J = x\restriction J$. Set $y = x' - x$; then $y(i) = 0$ for $i \in J$, so that $E + y = y$. Now

$$x' = x + y \in \phi E + y = \phi(E + y) = \phi E$$

because ϕ is translation-invariant. As x, x' are arbitrary, ϕE is determined by coordinates in J. As E and J are arbitrary, ϕ respects coordinates.

346D I describe a standard method of constructing liftings from other liftings.

Lemma Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $\mathfrak{A}$, $\mathfrak{B}$; suppose that $f:X\to Y$ represents an isomorphism $F^\bullet\mapsto f^{-1}[F]^\bullet:\mathfrak{B}\to\mathfrak{A}$. Then if $\phi:\mathrm{T}\to\mathrm{T}$ is a lifting for ν, there is a corresponding lifting $\phi':\Sigma\to\Sigma$ given by the formula

$$\phi'E=f^{-1}[\phi F]\text{ whenever }\mu(E\triangle f^{-1}[F])=0.$$

proof If we say that $\pi:\mathfrak{B}\to\mathfrak{A}$ is the isomorphism induced by f, then

$$\phi'E=f^{-1}[\theta(\pi^{-1}E^\bullet)],$$

where $\theta:\mathfrak{B}\to\mathrm{T}$ is the lifting corresponding to $\phi:\mathrm{T}\to\mathrm{T}$. Since θ, π^{-1} and $F\mapsto f^{-1}[F]$ are all Boolean homomorphisms, so is ϕ', and it is easy to check that $(\phi'E)^\bullet=E^\bullet$ for every $E\in\Sigma$ and that $\phi'E=\emptyset$ if $\mu E=0$.

Remark Compare the construction in 341P.

346E Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of Maharam-type-homogeneous probability spaces, with product (X,Σ,μ). Then there is a lifting for μ which respects coordinates.

proof (a) Replacing each μ_i by its completion does not change μ (254I), so we may suppose that all the μ_i are complete. In this case there is for each i an isomorphism between the measure algebra $(\mathfrak{A}_i,\bar\mu_i)$ of μ_i and the measure algebra $(\mathfrak{B}_{J_i},\bar\nu_{J_i})$ of some $\{0,1\}^{J_i}$ with its usual measure ν_{J_i} (331L). We may suppose that the sets J_i are disjoint. Each ν_{J_i} is compact (342Jd), so the isomorphisms are represented by inverse-measure-preserving functions $f_i:X_i\to\{0,1\}^{J_i}$ (343Ca).

Set $K=\bigcup_{i\in I}J_i$, and let ν_K be the usual measure on $Y=\{0,1\}^K$, T_K its domain. We have a natural bijection between $\prod_{i\in I}\{0,1\}^{J_i}$ and Y, so we obtain a function $f:X\to Y$; literally speaking,

$$f(x)(j)=f_i(x(i))(j)$$

for $i\in I$, $j\in J_i$ and $x\in X$.

(b) Now f is inverse-measure-preserving and induces an isomorphism between the measure algebras $\mathfrak{A}$, $\mathfrak{B}_K$ of μ, ν_K.

P(i) If $L\subseteq K$ is finite and $z\in\{0,1\}^L$, then, setting $L_i=L\cap J_i$ for $i\in I$,

$$\begin{aligned}\mu\{x:x\in X,\,f(x)\restriction L=z\}&=\mu(\prod_{i\in I}\{w:w\in X_i,\,f_i(w)\restriction L_i=z\restriction L_i\})\\&=\prod_{i\in I}\mu_i\{w:w\in X_i,\,f_i(w)\restriction L_i=z\restriction L_i\}\\&=\prod_{i\in I}\nu_{J_i}\{v:v\in\{0,1\}^{J_i},\,v\restriction L_i=z\restriction L_i\}\end{aligned}$$

(because every f_i is inverse-measure-preserving)

$$=\prod_{i\in I}2^{-\#(L_i)}=2^{-\#(L)}=\nu_K\{y:y\in Y,\,y\restriction L=z\}.$$

So $\mu f^{-1}[C]=\nu_KC$ for every basic cylinder set $C\subseteq Y$. By 254G, f is inverse-measure-preserving.

(ii) Accordingly f induces a measure-preserving homomorphism $\pi:\mathfrak{B}_K\to\mathfrak{A}$. To see that π is surjective, consider

$$\Lambda'=\{E:E\text{ is }\Sigma\text{-measurable, }E^\bullet\in\pi[\mathfrak{B}_K]\}.$$

Because $\pi[\mathfrak{B}_K]$ is a closed subalgebra of $\mathfrak{A}$ (324Kb), Λ' is a σ-subalgebra of the domain Λ of μ, and of course it contains all μ-negligible sets. If $i\in J$ and $G\in\Sigma_i$, then there is an $H\subseteq\{0,1\}^{J_i}$ such that $G\triangle f_i^{-1}[H]$ is μ_i-negligible. Now if $E=\{x:x\in X,\,x(i)\in G\}$ and $F=\{y:y\in Y,\,y\restriction J_i\in H\}$,

$$E\triangle f^{-1}[F]=\{x:x(i)\in G\triangle f_i^{-1}[H]\}$$

is μ-negligible, and $E \in \Lambda'$. But this means that $\Lambda' \supseteq \widehat{\bigotimes}_{i\in I}\Sigma_i$, and must therefore be the whole of Λ (254Ff). **Q**

(c) By 345C, there is a translation-invariant lifting ϕ for ν_K; by 346C, this respects coordinates. By 346D, we have a corresponding lifting ϕ' for μ such that

$$\phi' f^{-1}[F] = f^{-1}[\phi F]$$

for every $F \in \mathrm{T}_K$. Now suppose that $E \in \Lambda$ is determined by coordinates in $L \subseteq I$. Then there is an E' belonging to the σ-algebra Λ'_L generated by

$$\{\{x : x(i) \in G\} : i \in L,\, G \in \Sigma_i\}$$

such that $\mu(E\triangle E') = 0$ (254Ob). Write T_L for the family of sets in T_K determined by coordinates in $\bigcup_{i\in L} J_i$. Then, just as in (b-ii), every member of Λ'_L differs by a negligible set from some set of the form $f^{-1}[F]$ with $F \in \mathrm{T}_L$. So there is an $F \in \mathrm{T}_L$ such that $E\triangle f^{-1}[F]$ is μ-negligible. Consequently

$$\phi' E = \phi' f^{-1}[F] = f^{-1}[\phi F].$$

But ϕ respects coordinates, so ϕF is determined by coordinates in $\bigcup_{i\in L} J_i$. It follows at once that $f^{-1}[\phi F]$ is determined by coordinates in L; that is, that $\phi' E$ is determined by coordinates in L. As E and L are arbitrary, ϕ' respects coordinates, and witnesses the truth of the theorem.

346F It seems to be unknown whether 346E is true of arbitrary probability spaces (346Za); I give some partial results in this direction. The following general method of constructing lower densities will be useful.

Lemma Let (X, Σ, μ) and (Y, T, ν) be complete probability spaces, with product $(X\times Y, \Lambda, \lambda)$. If $\underline{\phi} : \Lambda \to \Lambda$ is a lower density, then we have a lower density $\underline{\phi}_1 : \Sigma \to \Sigma$ defined by saying that

$$\underline{\phi}_1 E = \{x : x \in X,\, \{y : (x, y) \in \underline{\phi}(E \times Y)\} \text{ is conegligible in } Y\}$$

for every $E \in \Sigma$.

proof For $E \in \Sigma$, $(E \times Y)\triangle\underline{\phi}(E \times Y)$ is negligible, so that

$$H_x = \{y : (x, y) \in (E \times Y)\triangle\underline{\phi}(E \times Y)\}$$

is ν-negligible for almost every $x \in X$ (252D). Now $E\triangle\underline{\phi}_1 E = \{x : H_x \text{ is not negligible}\}$ is negligible, so $\underline{\phi}_1 E \in \Sigma$. If E, $F \in \Sigma$, then

$$\underline{\phi}((E \cap F) \times Y) = \underline{\phi}((E \times Y) \cap (F \times Y)) = \underline{\phi}(E \times Y) \cap \underline{\phi}(F \times Y),$$

so that

$$\{y : (x, y) \in \underline{\phi}((E \cap F) \times Y)\} = \{y : (x, y) \in \underline{\phi}(E \times Y)\} \cap \{y : (x, y) \in \underline{\phi}(F \times Y)\}$$

is conegligible iff both $\{y : (x, y) \in \underline{\phi}(E\times Y)\}$ and $\{y : (x, y) \in \underline{\phi}(F\times Y)\}$ are conegligible, and $\underline{\phi}_1(E\cap F) = \underline{\phi}_1 E \cap \underline{\phi}_1 F$.

The rest is easy. Of course $\underline{\phi}(\emptyset \times Y) = \emptyset$ so $\underline{\phi}_1\emptyset = \emptyset$. If E, $F \in \Sigma$ and $E\triangle F$ is negligible, then $(E\times Y)\triangle(F\times Y)$ is negligible, $\underline{\phi}(E\times Y) = \underline{\phi}(F\times Y)$ and $\underline{\phi}_1 E = \underline{\phi}_1 F$. So $\underline{\phi}_1$ is a lower density, as claimed.

346G Theorem Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces with product (X, Σ, μ). For $J \subseteq I$ let Σ_J be the set of members of Σ which are determined by coordinates in J. Then there is a lower density $\underline{\phi} : \Sigma \to \Sigma$ such that

(i) whenever $J \subseteq I$ and $E \in \Sigma_J$ then $\underline{\phi}E \in \Sigma_J$,

(ii) whenever J, $K \subseteq I$ are disjoint, $E \in \Sigma_J$ and $F \in \Sigma_K$ then $\underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F$.

proof For each $i \in I$, set $Y_i = X_i^{\mathbb{N}}$, with the product measure ν_i; set $Y = \prod_{i\in I} Y_i$, with its product measure ν; set $Z_i = X_i \times Y_i$, with its product measure λ_i, and $Z = \prod_{i\in I} Z_i$, with its product measure λ. Then the natural identification of $Z = \prod_{i\in I} X_i \times Y_i$ with $\prod_{i\in I} X_i \times \prod_{i\in I} Y_i = X \times Y$ makes λ correspond to the product of μ and ν (254N).

Each (Z_i, λ_i) can be identified with an infinite power of (X_i, μ_i), and is therefore Maharam-type-homogeneous (334E). Consequently there is a lifting $\phi : \Lambda \to \Lambda$ which respects coordinates (346E). Regarding

(Z,λ) as the product of (X,μ) and (Y,ν), we see that ϕ induces a lower density $\underline{\phi}:\Sigma\to\Sigma$ by the formula of 346F.

If $J\subseteq I$ and $E\in\Sigma$ is determined by coordinates in J, then $E\times Y$ (regarded as a subset of $\prod_{i\in I}Z_i$) is determined by coordinates in J, so $\phi(E\times Y)$ also is. Now suppose that $x\in\underline{\phi}E$, $x'\in X$ and $x\restriction J=x'\restriction J$. Then for any $y\in Y$, $(x\restriction J,y\restriction J)=(x'\restriction J,y\restriction J)$, so $(x,y)\in\phi(E\times Y)$ iff $(x',y)\in\phi(E\times Y)$. Thus

$$\{y:(x',y)\in\phi(E\times Y)\}=\{y:(x,y)\in\phi(E\times Y)\}$$

is conegligible in Y, and $x'\in\underline{\phi}E$. This shows that $\underline{\phi}E$ is determined by coordinates in J.

Now suppose that J and K are disjoint subsets of I, that E, $F\in\Sigma$ are determined by coordinates in J, K respectively, and that $x\notin\underline{\phi}E\cup\underline{\phi}F$. Then $A=\{y:(x,y)\notin\phi(E\times Y)\}$ and $B=\{y:(x,y)\notin\phi(F\times Y)\}$ are non-negligible. As noted just above, $\phi(E\times Y)$ is determined by coordinates in J, so A is determined by coordinates in J, and can be expressed as $\{y:y\restriction J\in A'\}$, where $A'\subseteq Y_J=\prod_{i\in J}Y_i$. Because $y\mapsto y\restriction J:Y\to Y_J$ is inverse-measure-preserving, A' cannot be negligible in Y_J. Similarly, B can be expressed as $\{y:y\restriction K\in B'\}$ for some non-negligible $B'\subseteq Y_K$.

By 251S/251Wm, $A'\times B'\times Y_{I\setminus(J\cup K)}$, regarded as a subset of Y, is non-negligible, that is,

$$C=\{y:y\in Y,\,y\restriction J\in A',\,y\restriction K\in B'\}$$

is non-negligible. But

$$C=A\cap B=\{y:(x,y)\notin\phi(E\times Y)\cup\phi(F\times Y)\}=\{y:(x,y)\notin\phi((E\cup F)\times Y\}.$$

So $x\notin\underline{\phi}(E\cup F)$. As x is arbitrary, $\underline{\phi}(E\cup F)\subseteq\underline{\phi}E\cup\underline{\phi}F$; but of course $\underline{\phi}E\cup\underline{\phi}F\subseteq\underline{\phi}(E\cup F)$, because $\underline{\phi}$ is a lower density, so that $\underline{\phi}(E\cup F)=\underline{\phi}E\cup\underline{\phi}F$, as required.

Remark See Macheras Musiał & Strauss 99 for an alternative proof.

346H Theorem Let ζ be an ordinal, and $\langle(X_\xi,\Sigma_\xi,\mu_\xi)\rangle_{\xi<\zeta}$ a family of probability spaces, with product (Z,Λ,λ). For $J\subseteq\zeta$ let Λ_J be the set of those $W\in\Lambda$ which are determined by coordinates in J. Then there is a lifting $\phi:\Lambda\to\Lambda$ such that $\phi W\in\Lambda_J$ whenever $W\in\Lambda_J$ and J is *either* a singleton subset of ζ *or* an initial segment of ζ.

proof (a) Let P be the set of all lower densities $\underline{\phi}:\Lambda\to\Lambda$ such that, for every $\xi<\zeta$, (i) whenever $E\in\Lambda_\xi$ then $\underline{\phi}E\in\Lambda_\xi$ (ii) whenever $E\in\Lambda_{\{\xi\}}$ then $\underline{\phi}E\in\Lambda_{\{\xi\}}$ (iii) whenever $E\in\Lambda_\xi$ and $F\in\Lambda_{\zeta\setminus\xi}$ then $\underline{\phi}(E\cup F)=\underline{\phi}E\cup\underline{\phi}F$. By 346G, P is not empty. Order P by saying that $\underline{\phi}\le\underline{\phi}'$ if $\underline{\phi}E\subseteq\underline{\phi}'E$ for every $E\in\Lambda$; then P is a partially ordered set. Note that if $\underline{\phi}\in P$ then $\underline{\phi}Z=Z$ (because $\Lambda_0=\{\emptyset,Z\}$).

(b) Any non-empty totally ordered subset Q of P has an upper bound in P. **P** Define $\underline{\phi}^*:\Lambda\to\mathcal{P}X$ by setting $\underline{\phi}^*E=\bigcup_{\underline{\phi}\in Q}\underline{\phi}E$ for every $E\in\Lambda$. (i)

$$\underline{\phi}^*\emptyset=\bigcup_{\underline{\phi}\in Q}\emptyset=\emptyset.$$

(ii) If E, $F\in\Lambda$ and $\lambda(E\triangle F)=0$ then $\underline{\phi}E=\underline{\phi}F$ for every $\underline{\phi}\in Q$ so $\underline{\phi}^*E=\underline{\phi}^*F$. (iii) If E, $F\in\Lambda$ and $E\subseteq F$ then $\underline{\phi}E\subseteq\underline{\phi}F$ for every $\underline{\phi}\in Q$ so $\underline{\phi}^*E\subseteq\underline{\phi}^*F$. (iv) If E, $F\in\Lambda$ and $x\in\underline{\phi}^*E\cap\underline{\phi}^*F$, then there are $\underline{\phi}_1$, $\underline{\phi}_2\in Q$ such that $x\in\underline{\phi}_1E\cap\underline{\phi}_2F$; now either $\underline{\phi}_1\le\underline{\phi}_2$ or $\underline{\phi}_2\le\underline{\phi}_1$, so that

$$x\in(\underline{\phi}_1E\cap\underline{\phi}_1F)\cup(\underline{\phi}_2E\cap\underline{\phi}_2F)=\underline{\phi}_1(E\cap F)\cup\underline{\phi}_2(E\cap F)\subseteq\underline{\phi}^*(E\cap F).$$

Accordingly $\underline{\phi}^*E\cap\underline{\phi}^*F\subseteq\underline{\phi}^*(E\cap F)$ and $\underline{\phi}^*E\cap\underline{\phi}^*F=\underline{\phi}^*(E\cap F)$. (v) Taking any $\underline{\phi}_0\in Q$, we have $\underline{\phi}_0E\subseteq\underline{\phi}^*E$ for every $E\in\Lambda$, so (because λ is complete) $\underline{\phi}^*$ is a lower density, by 341Ib. (vi) Now suppose that $J\subseteq I$ is either a singleton $\{\xi\}$ or an initial segment ξ, and that $E\in\Lambda_J$. Then $\underline{\phi}E$ is determined by coordinates in J for every $\underline{\phi}\in Q$, so $\underline{\phi}^*E$ is determined by coordinates in J. (vii) Finally, suppose that $\xi<\zeta$ and that $E\in\Lambda_\xi$, $F\in\Lambda_{\zeta\setminus\xi}$. If $x\in\underline{\phi}^*(E\cup F)$ then there is a $\underline{\phi}\in Q$ such that

$$x\in\underline{\phi}(E\cup F)=\underline{\phi}E\cup\underline{\phi}F\subseteq\underline{\phi}^*E\cup\underline{\phi}^*F.$$

So $\underline{\phi}^*(E\cup F)\subseteq\underline{\phi}^*E\cup\underline{\phi}^*F$ and (using (iii) again) $\underline{\phi}^*(E\cup F)=\underline{\phi}^*E\cup\underline{\phi}^*F$. Thus $\underline{\phi}^*$ belongs to P and is an upper bound for Q in P. **Q**

By Zorn's Lemma, P has a maximal element $\tilde{\underline{\phi}}$.

(c) For any $H\in\Lambda$ we may define a function $\underline{\phi}_H$ as follows. Set $A_H=Z\setminus(\tilde{\underline{\phi}}H\cup\tilde{\underline{\phi}}(Z\setminus H))$,

$$\underline{\phi}_H E = \tilde{\underline{\phi}} E \cup (A_H \cap \tilde{\underline{\phi}}(H \cup E))$$

for $E \in \Lambda$. Then $\underline{\phi}_H$ is a lower density. **P** (i) Because $H \triangle \tilde{\underline{\phi}} H$ and $(Z \setminus H) \triangle \tilde{\underline{\phi}}(Z \setminus H)$ are both negligible, A_H is negligible and $\underline{\phi}_H E$ is measurable and $(\underline{\phi}_H E)^\bullet = (\tilde{\underline{\phi}} E)^\bullet = E^\bullet$ for every $E \in \Lambda$. (ii) Because $A_H \cap \tilde{\underline{\phi}} H = \emptyset$, $\underline{\phi}_H \emptyset = \emptyset$. (iii) If E, $F \in \Lambda$ and $\lambda(E \triangle F) = 0$ then $\tilde{\underline{\phi}} E = \tilde{\underline{\phi}} F$ and $\tilde{\underline{\phi}}(E \cup H) = \tilde{\underline{\phi}}(F \cup H)$, so $\underline{\phi}_H E = \underline{\phi}_H F$. (iv) If E, $F \in \Lambda$ and $E \subseteq F$ then $\tilde{\underline{\phi}} E \subseteq \tilde{\underline{\phi}} F$ and $\tilde{\underline{\phi}}(E \cup H) \subseteq \tilde{\underline{\phi}}(F \cup H)$, so $\underline{\phi}_H E \subseteq \underline{\phi}_H F$. (v) If E, $F \in \Lambda$ and $x \in \underline{\phi}_H E \cap \underline{\phi}_H F$, then

(α) if $x \notin A_H$,

$$x \in \tilde{\underline{\phi}} E \cap \tilde{\underline{\phi}} F = \tilde{\underline{\phi}}(E \cap F) \subseteq \underline{\phi}_H (E \cap F),$$

(β) if $x \in A_H$,

$$x \in \tilde{\underline{\phi}}(E \cup H) \cap \tilde{\underline{\phi}}(F \cup H) = \tilde{\underline{\phi}}((E \cap F) \cup H) \subseteq \underline{\phi}_H (E \cap F).$$

Thus $\underline{\phi}_H E \cap \underline{\phi}_H F \subseteq \underline{\phi}_H (E \cap F)$ and $\underline{\phi}_H E \cap \underline{\phi}_H F = \underline{\phi}_H (E \cap F)$. **Q**

(d) It is worth noting the following.

(i) If E, $H \in \Lambda$ and $\tilde{\underline{\phi}}(E \cup H) = \tilde{\underline{\phi}} E \cup \tilde{\underline{\phi}} H$ then $\underline{\phi}_H E = \tilde{\underline{\phi}} E$. **P** We have

$$\underline{\phi}_H E = \tilde{\underline{\phi}} E \cup (A_H \cap \tilde{\underline{\phi}}(E \cup H)) = \tilde{\underline{\phi}} E \cup (A_H \cap \tilde{\underline{\phi}} E) \cup (A_H \cap \tilde{\underline{\phi}} H) = \tilde{\underline{\phi}} E$$

because $A_H \cap \tilde{\underline{\phi}} H = \emptyset$. **Q**

(ii) If $H \in \Lambda$ and $\underline{\phi}_H \in P$ then $\tilde{\underline{\phi}} H \cup \tilde{\underline{\phi}}(Z \setminus H) = Z$. **P** By the maximality of $\tilde{\underline{\phi}}$, we must have $\underline{\phi}_H = \tilde{\underline{\phi}}$. But

$$A_H = \underline{\phi}_H (Z \setminus H) \setminus \tilde{\underline{\phi}}(Z \setminus H),$$

so $A_H = \emptyset$, that is, $\tilde{\underline{\phi}} H \cup \tilde{\underline{\phi}}(Z \setminus H) = Z$. **Q**

(iii) If E, $F \in \Lambda$ and $\tilde{\underline{\phi}} E \cup \tilde{\underline{\phi}}(Z \setminus E) = Z$, then $\tilde{\underline{\phi}}(E \cup F) = \tilde{\underline{\phi}} E \cup \tilde{\underline{\phi}} F$. **P**

$$\tilde{\underline{\phi}}(E \cup F) \setminus \tilde{\underline{\phi}} E = \tilde{\underline{\phi}}(E \cup F) \cap \tilde{\underline{\phi}}(Z \setminus E) = \tilde{\underline{\phi}}((E \cup F) \cap (Z \setminus E)) = \tilde{\underline{\phi}}(F \setminus E) \subseteq \tilde{\underline{\phi}} F,$$

so $\tilde{\underline{\phi}}(E \cup F) \subseteq \tilde{\underline{\phi}} E \cup \tilde{\underline{\phi}} F$; as the reverse inclusion is true for all E and F, we have the result. **Q**

(e) If $\xi < \zeta$ and $H \in \Lambda_{\{\xi\}}$, then $\underline{\phi}_H \in P$.

P (i) If $J \subseteq I$ is either a singleton or an inital segment, and $E \in \Lambda_J$, then

(α) if $\xi \in J$, $E \cup H$ and $\tilde{\underline{\phi}} E$ and $\tilde{\underline{\phi}}(E \cup H)$ and A_H all belong to Λ_J, so $\underline{\phi}_H E \in \Lambda_J$.

(β) If $\xi \notin J$, $\tilde{\underline{\phi}}(E \cup H) = \tilde{\underline{\phi}} E \cup \tilde{\underline{\phi}} H$, because there is some η such that $J \subseteq \eta$ and $\{\xi\} \subseteq \zeta \setminus \eta$; so $\underline{\phi}_H E = \tilde{\underline{\phi}} E \in \Lambda_J$ by (d-i).

(ii) If $\eta < \zeta$, $E \in \Lambda_\eta$ and $F \in \Lambda_{\zeta \setminus \eta}$, then

if $\xi < \eta$, $E \cup H \in \Lambda_\eta$ so $\tilde{\underline{\phi}}(E \cup F \cup H) = \tilde{\underline{\phi}}(E \cup H) \cup \tilde{\underline{\phi}} F$, and

$$\begin{aligned}\underline{\phi}_H (E \cup F) &= \tilde{\underline{\phi}}(E \cup F) \cup (A_H \cap \tilde{\underline{\phi}}(E \cup F \cup H)) \\ &= \tilde{\underline{\phi}} E \cup \tilde{\underline{\phi}} F \cup (A_H \cap \tilde{\underline{\phi}}(E \cup H)) \cup (A_H \cap \tilde{\underline{\phi}} F) \subseteq \underline{\phi}_H E \cup \underline{\phi}_H F;\end{aligned}$$

if $\eta \le \xi$, $F \cup H \in \Lambda_{\zeta \setminus \eta}$ so $\tilde{\underline{\phi}}(E \cup F \cup H) = \tilde{\underline{\phi}}(E) \cup \tilde{\underline{\phi}}(F \cup H)$, and

$$\begin{aligned}\underline{\phi}_H (E \cup F) &= \tilde{\underline{\phi}}(E \cup F) \cup (A_H \cap \tilde{\underline{\phi}}(E \cup F \cup H)) \\ &= \tilde{\underline{\phi}} E \cup \tilde{\underline{\phi}} F \cup (A_H \cap \tilde{\underline{\phi}} E)) \cup (A_H \cap \tilde{\underline{\phi}}(F \cup H)) \subseteq \underline{\phi}_H E \cup \underline{\phi}_H F;\end{aligned}$$

accordingly $\underline{\phi}_H (E \cup F) = \underline{\phi}_H E \cup \underline{\phi}_H F$. **Q**

By (d-ii) we have

$$\tilde{\underline{\phi}} H \cup \tilde{\underline{\phi}}(Z \setminus H) = Z$$

whenever $\xi < \zeta$ and $H \in \Lambda_{\{\xi\}}$.

(f) If $\xi \le \zeta$ and $H \in \Lambda_\xi$, then $\underline{\phi}_H \in P$. **P** Induce on ξ. For $\xi = 0$, $H \in \Lambda_0 = \{\emptyset, Z\}$ so $\underline{\tilde\phi}H$ is either $\emptyset$ or Z, $A_H = \emptyset$ and $\underline{\phi}_H = \underline{\tilde\phi}$ belongs to P. For the inductive step to $\xi \le \zeta$, we have the following.

(i) If $\eta < \zeta$ and $E \in \Lambda_\eta$, then

(α) if $\xi \le \eta$, $E \cup H$ and $\underline{\tilde\phi}E$ and $\underline{\tilde\phi}(E \cup H)$ and A_H all belong to Λ_η, so $\underline{\phi}_H E \in \Lambda_\eta$.

(β) if $\eta < \xi$, then, by the inductive hypothesis, $\underline{\phi}_E \in P$, $\underline{\tilde\phi}E = Z \setminus \underline{\tilde\phi}(Z \setminus E)$ and $\underline{\tilde\phi}(E \cup H) = \underline{\tilde\phi}E \cup \underline{\tilde\phi}H$, by (d-ii) and (d-iii) above; so $\underline{\phi}_H E = \underline{\tilde\phi}E \in \Lambda_\eta$ by (d-i).

(ii) If $\eta < \zeta$ and $E \in \Lambda_{\{\eta\}}$, then, by (e), $\underline{\tilde\phi}E \cup \underline{\tilde\phi}(Z \setminus E) = Z$, so that $\underline{\tilde\phi}(E \cup H) = \underline{\tilde\phi}E \cup \underline{\tilde\phi}H$, by (d-iii), and $\underline{\phi}_H E = \underline{\tilde\phi}E \in \Lambda_{\{\eta\}}$, by (d-i).

(iii) If $\eta < \zeta$, $E \in \Lambda_\eta$ and $F \in \Lambda_{\zeta\setminus\eta}$, then

(α) if $\xi \le \eta$, then $E \cup H \in \Lambda_\eta$ and $F \in \Lambda_{\zeta\setminus\eta}$, so that $\underline{\tilde\phi}(E \cup F \cup H) = \underline{\tilde\phi}(E \cup H) \cup \underline{\tilde\phi}F$, and

$$\begin{aligned}\underline{\phi}_H(E \cup F) &= \underline{\tilde\phi}(E \cup F) \cup (A_H \cap \underline{\tilde\phi}(E \cup F \cup H)) \\ &= \underline{\tilde\phi}E \cup \underline{\tilde\phi}F \cup (A_H \cap \underline{\tilde\phi}(E \cup H)) \cup (A_H \cap \underline{\tilde\phi}F) \subseteq \underline{\phi}_H E \cup \underline{\phi}_H F,\end{aligned}$$

as in (e-ii) above, and accordingly $\underline{\phi}_H(E \cup F) = \underline{\phi}_H E \cup \underline{\phi}_H F$.

(β) If $\eta < \xi$ then, as in (ii), using the inductive hypothesis, we have $\underline{\tilde\phi}(E \cup F \cup H) = \underline{\tilde\phi}E \cup \underline{\tilde\phi}(F \cup H)$, and (just as in ($\alpha$)) we get $\underline{\phi}_H(E \cup F) = \underline{\phi}_H E \cup \underline{\phi}_H F$.

Thus $\underline{\phi}_H \in P$ and the induction continues. **Q**

(g) But the case $\xi = \zeta$ of (f) just tells us that

$$\underline{\tilde\phi}H \cup \underline{\tilde\phi}(Z \setminus H) = Z$$

for every $H \in \Lambda$. This means that $\underline{\tilde\phi}$ is actually a lifting (since it preserves intersections and complements). And the definition of P is just what is needed to ensure that it is a lifting of the right type.

Remark This result is due to Macheras & Strauss 96b.

346I Theorem Let (X, Σ, μ) be a complete probability space. For any set I, write λ_I for the product measure on X^I, Λ_I for its domain and $\pi_{Ii}(x) = x(i)$ for $x \in X^I$, $i \in I$. Then there is a lifting $\psi : \Sigma \to \Sigma$ such that for every set I there is a lifting $\phi : \Lambda_I \to \Lambda_I$ such that $\phi(\pi_{Ii}^{-1}[E]) = \pi_{Ii}^{-1}[\psi E]$ whenever $E \in \Sigma$ and $i \in I$.

proof ? Suppose, if possible, otherwise.

Let Ψ be the set of all liftings for μ. We are supposing that for every $\psi \in \Psi$ there is a set I_ψ for which there is no lifting for λ_{I_ψ} consistent with ψ in the sense above. Let κ be a cardinal greater than $\max(\omega, \#(\Psi), \sup_{\psi\in\Psi} \#(I_\psi))$. Let $\phi_0 : \Lambda_\kappa \to \Lambda_\kappa$ be a lifting satisfying the conditions of 346H. 346Bb tells us that for every $\xi < \kappa$ we have a lifting ψ for μ defined by the formula $\pi_{\kappa\xi}^{-1}[\psi E] = \phi_0(\pi_{\kappa\xi}^{-1}[E])$. For $\psi \in \Psi$ set

$$K_\psi = \{\xi : \xi < \kappa,\ \phi_0(\pi_{\kappa\xi}^{-1}[E]) = \pi_{\kappa\xi}^{-1}[\psi E] \text{ for every } E \in \Sigma\}.$$

Then $\bigcup_{\psi\in\Psi} K_\psi = \kappa$, so $\kappa \le \max(\omega, \#(\Psi), \sup_{\psi\in\Psi} \#(K_\psi))$ and there is some $\psi \in \Psi$ such that $\#(K_\psi) > \#(I_\psi)$. Take $I \subseteq K_\psi$ such that $\#(I) = \#(I_\psi)$.

We may regard X^κ as $X^I \times X^{\kappa\setminus I}$, and in this form we can use the method of 346F to obtain a lower density $\underline{\phi} : \Lambda_I \to \Lambda_I$ from $\phi_0 : \Lambda_\kappa \to \Lambda_\kappa$. Now

$$\underline{\phi}(\pi_{I\xi}^{-1}[E]) = \pi_{I\xi}^{-1}[\psi E] \text{ for every } E \in \Sigma,\ \xi \in I.$$

P The point is that $\pi_{I\xi}^{-1}[E] \times X^{\kappa\setminus I}$ corresponds to $\pi_{\kappa\xi}^{-1}[E] \subseteq X^\kappa$, while $\phi_0(\pi_{\kappa\xi}^{-1}[E]) = \pi_{\kappa\xi}^{-1}[\psi E]$ can be identified with $\pi_{I\xi}^{-1}[\psi E] \times X^{\kappa\setminus I}$. Now the construction of 346F obviously makes $\underline{\phi}(\pi_{I\xi}^{-1}[E])$ equal to $\pi_{I\xi}^{-1}[\psi E]$. **Q**

By 341Jb, there is a lifting $\phi : \Lambda_I \to \Lambda_I$ such that $\phi W \supseteq \underline{\phi}W$ for every $W \in \Lambda_I$. But now we must have

$$\begin{aligned}\pi_{I\xi}^{-1}[\psi E] &= \underline{\phi}(\pi_{I\xi}^{-1}[E]) \subseteq \phi(\pi_{I\xi}^{-1}[E])\\ &= X^I \setminus \phi(\pi_{I\xi}^{-1}[X \setminus E]) \subseteq X^I \setminus \underline{\phi}(\pi_{I\xi}^{-1}[X \setminus E])\\ &= X^I \setminus \pi_{I\xi}^{-1}[\psi(X \setminus E)] = X^I \setminus \pi_{I\xi}^{-1}[X \setminus \psi E] = \pi_{I\xi}^{-1}[\psi E]\end{aligned}$$

and $\phi(\pi_{I\xi}^{-1}[E]) = \pi_{I\xi}^{-1}[\psi E]$ for every $E \in \Sigma$ and $\xi \in I$. But since $\#(I) = \#(I_\psi)$, this must be impossible, by the choice of I_ψ. **X**

This contradiction proves the theorem.

346J Consistent liftings Let (X, Σ, μ) be a measure space. A lifting $\psi : \Sigma \to \Sigma$ is **consistent** if for every $n \geq 1$ there is a lifting ϕ_n of the product measure on X^n such that $\phi_n(E_1 \times \ldots \times E_n) = \psi E_1 \times \ldots \times \psi E_n$ for all $E_1, \ldots , E_n \in \Sigma$. Thus 346I tells us, in part, that every complete probability space has a consistent lifting; it follows that every non-trivial complete totally finite measure space has a consistent lifting.

I do not suppose you will be surprised to be told that not all liftings on probability spaces are consistent. What may be surprising is the fact that one of the standard liftings already introduced is not consistent. This depends on a general fact about Stone spaces of measure algebras which has further important applications, so I present it as a lemma.

346K Lemma Let (Z, T, ν) be the Stone space of the measure algebra of Lebesgue measure on $[0, 1]$, and let λ be the product measure on $Z \times Z$, with Λ its domain. Then there is a set $W \in \Lambda$, with $\lambda W < 1$, such that $\lambda^*\tilde{W} = 1$, where

$$\tilde{W} = \bigcup\{G \times H : G,\, H \subseteq Z \text{ are open-and-closed, } (G \times H) \setminus W \text{ is negligible}\}.$$

Remark For the sake of anybody who has already become acquainted with the alternative measures which can be put on the product of topological measure spaces, I ought to insist that the 'product measure' λ here is, as always in this volume, the ordinary completed product measure as defined in Chapter 25.

proof (a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable subsets of $[0, 1]$, stochastically independent for Lebesgue measure μ on $[0, 1]$, such that $\mu E_n = \frac{1}{n+2}$ for each n. Set $a_n = E_n^\bullet$ in the measure of algebra of μ, and $E_n^* = \widehat{a}_n$ the corresponding compact open subset of Z. Set $W = \bigcup_{n\in\mathbb{N}} E_n^* \times E_n^*$. Then

$$\lambda W \leq \sum_{n=0}^{\infty}(\nu E_n)^2 = \sum_{n=2}^{\infty} \frac{1}{n^2} < 1.$$

? Suppose, if possible, that $\lambda^*\tilde{W} < 1$. Then there are sequences $\langle G_n\rangle_{n\in\mathbb{N}}$, $\langle H_n\rangle_{n\in\mathbb{N}}$ in T such that $\tilde{W} \subseteq \bigcup_{n\in\mathbb{N}} G_n \times H_n$ and $\lambda(\bigcup_{n\in\mathbb{N}} G_n \times H_n) < 1$. Recall from 322Rc that

$$\nu F = \inf\{\nu G : G \text{ is compact and open, } F \subseteq G\}$$

for every $F \in \mathrm{T}$. Accordingly we can find compact open sets $\tilde{G}_n$, $\tilde{H}_n$ such that $G_n \subseteq \tilde{G}_n$, $H_n \subseteq \tilde{H}_n$ for every $n \in \mathbb{N}$ and

$$\sum_{n=0}^{\infty} \nu(\tilde{G}_n \setminus G_n) + \sum_{n=0}^{\infty} \nu(\tilde{H}_n \setminus H_n) < 1 - \lambda(\bigcup_{n\in\mathbb{N}} G_n \times H_n),$$

so that $\lambda(\bigcup_{n\in\mathbb{N}} \tilde{G}_n \times \tilde{H}_n) < 1$.

Let $\mathcal{U}_0$ be the family

$$\{Z\} \cup \{E_n^* : n \in \mathbb{N}\} \cup \{Z \setminus \tilde{G}_n : n \in \mathbb{N}\} \cup \{Z \setminus \tilde{H}_n : n \in \mathbb{N}\},$$

so that $\mathcal{U}_0$ is a countable subset of T. Let $\mathcal{U}$ be the set of finite intersections $U_0 \cap U_1 \cap \ldots \cap U_n$ where $U_0, \ldots , U_n \in \mathcal{U}_0$, so that $\mathcal{U}$ also is a countable subset of T, and $\mathcal{U}$ is closed under $\cap$.

(b) For $U \in \mathcal{U}$, define $Q(U)$ as follows. If $\nu U = 0$, then $Q(U) = U$. Otherwise,

$$Q(U) = Z \setminus \bigcup\{E_n^* : n \in \mathbb{N},\, \nu(E_n^* \cap U) > 0\}.$$

Then $\nu Q(U)$ is always 0. **P** Of course this is true if $\nu U = 0$, so suppose that $\nu U > 0$. Set $I = \{n : \nu(E_n^* \cap U) = 0\}$. Then we have $\nu U' > 0$, where $U' = U \setminus \bigcup_{n\in I} E_n^*$, and $Z \setminus E_n^* \supseteq U'$ for every $n \in I$. Because $\langle E_n\rangle_{n\in\mathbb{N}}$ is stochastically independent for μ, $\langle E_n^*\rangle_{n\in\mathbb{N}}$ is stochastically independent for ν, while

$$\nu(\bigcup_{n\in I} E_n^*) \le 1 - \nu U' < 1.$$

By the Borel-Cantelli lemma (273K), $\sum_{n\in I} \nu E_n^* < \infty$. Consequently $\sum_{n\in\mathbb{N}\setminus I} \nu E_n^* = \infty$, because $\sum_{n=0}^{\infty} \frac{1}{n+2}$ is infinite, so

$$\nu(Z \setminus Q(U)) = \nu(\bigcup_{n\in\mathbb{N}\setminus I} E_n^*) = 1,$$

and $\nu Q(U) = 0$. **Q**

(c) Set $Q_0 = \bigcup_{U\in\mathcal{U}} Q(U)$; because $\mathcal{U}$ is countable, Q_0 is negligible. Accordingly $(Z \setminus Q_0)^2$ has measure 1 and cannot be included in $\bigcup_{n\in\mathbb{N}} \tilde{G}_n \times \tilde{H}_n$; take $(w, z) \in (Z \setminus Q_0)^2 \setminus \bigcup_{n\in\mathbb{N}} \tilde{G}_n \times \tilde{H}_n$.

(d) We can find sequences $\langle C_n\rangle_{n\in\mathbb{N}}$, $\langle D_n\rangle_{n\in\mathbb{N}}$, $\langle U_n\rangle_{n\in\mathbb{N}}$ and $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that

$w \in U_{n+1} \subseteq U_n$, $z \in V_{n+1} \subseteq V_n$, $(U_{n+1} \times V_{n+1}) \cap (\tilde{G}_n \times \tilde{H}_n) = \emptyset$,

$\nu C_n > 0$, $\nu D_n > 0$,

$C_n \subseteq U_n$, $D_n \subseteq V_{n+1}$,

$C_n \times V_{n+1} \subseteq W$, $U_{n+1} \times D_n \subseteq W$

for every $n \in \mathbb{N}$. **P** Build the sequences inductively, as follows. Start with $U_0 = V_0 = Z$. Given that $w \in U_n \in \mathcal{U}$ and $z \in V_n \in \mathcal{U}$, then we know that $(w, z) \notin \tilde{G}_n \times \tilde{H}_n$. If $w \notin \tilde{G}_n$, set $U_n' = U_n \setminus \tilde{G}_n$, $V_n' = V_n$; otherwise set $U_n' = U_n$, $V_n' = V_n \setminus \tilde{H}_n$. In either case, we have $w \in U_n' \in \mathcal{U}$, $z \in V_n' \in \mathcal{U}$ and $(U_n' \times V_n') \cap (\tilde{G}_n \times \tilde{H}_n) = \emptyset$.

Because $U_n' \in \mathcal{U}$, $w \notin Q(U_n')$. But $w \in U_n'$, so this must be because $\nu U_n' > 0$. Now $z \notin Q(U_n')$, so $z \in \bigcup\{E_k^* : k \in \mathbb{N}, \nu(E_k^* \cap U_n') > 0\}$. Take some $k \in \mathbb{N}$ such that $z \in E_k^*$ and $\nu(E_k^* \cap U_n') > 0$, and set

$$V_{n+1} = V_n' \cap E_k^*, \quad C_n = E_k^* \cap U_n',$$

so that

$$z \in V_{n+1} \in \mathcal{U}, \quad C_n \subseteq U_n, \quad C_n \times V_{n+1} \subseteq E_k^* \times E_k^* \subseteq W, \quad \nu C_n > 0.$$

Next, $z \notin Q(V_{n+1})$ and $\nu V_{n+1} > 0$; also $w \notin Q(V_{n+1})$, so there is an l such that $w \in E_l^*$ and $\nu(E_l^* \cap V_{n+1}) > 0$. Set

$$U_{n+1} = U_n' \cap E_l^*, \quad D_n = E_l^* \cap V_{n+1},$$

so that

$$w \in U_{n+1} \in \mathcal{U}, \quad D_n \subseteq V_{n+1}, \quad U_{n+1} \times D_n \subseteq E_l^* \times E_l^* \subseteq W, \quad \nu D_n > 0,$$

$$(U_{n+1} \times V_{n+1}) \cap (\tilde{G}_n \times \tilde{H}_n) \subseteq (U_n' \times V_n') \cap (\tilde{G}_n \times \tilde{H}_n) = \emptyset,$$

and continue the process. **Q**

(e) Setting $C = \bigcup_{n\in\mathbb{N}} C_n$ and $D = \bigcup_{n\in\mathbb{N}} D_n$ we see that $C \times D \subseteq W$. **P** If $m \le n$, $D_n \subseteq V_{n+1} \subseteq V_{m+1}$, so $C_m \times D_n \subseteq W$. If $m > n$, $C_m \subseteq U_m \subseteq U_{n+1}$, so $C_m \times D_n \subseteq W$. **Q**

Recall from 321K that the measurable sets of Z are precisely those of the form $G \triangle M$ where M is nowhere dense and negligible and G is compact and open. There must therefore be compact open sets G, $H \subseteq Z$ such that $G \triangle C$ and $H \triangle D$ are negligible. Consequently

$$(G \times H) \setminus W \subseteq ((G \setminus C) \times Z) \cup (Z \times (H \setminus D))$$

is negligible, and

$$G \times H \subseteq \tilde{W} \subseteq \bigcup_{n\in\mathbb{N}} \tilde{G}_n \times \tilde{H}_n.$$

But because $G \times H$ is compact (3A3J), and all the $\tilde{G}_n \times \tilde{H}_n$ are open, there must be some n such that $G \times H \subseteq \bigcup_{k\le n} \tilde{G}_k \times \tilde{H}_k = S$ say. Now $(U_{k+1} \times V_{k+1}) \cap (\tilde{G}_k \times \tilde{H}_k) = \emptyset$ for every k, so

$$(C_{n+2} \times D_{n+2}) \cap (G \times H) \subseteq (U_{n+1} \times V_{n+1}) \cap S = \emptyset,$$

and either $C_{n+2} \cap G = \emptyset$ or $D_{n+2} \cap H = \emptyset$. Since

$$C_{n+2} \setminus G \subseteq C \setminus G, \quad D_{n+2} \setminus H \subseteq D \setminus H$$

are both negligible, one of C_{n+2}, D_{n+2} is negligible. But the construction took care to ensure that all the C_k, D_k were non-negligible. **X**

(f) Thus $\lambda^*\tilde{W} = 1$, as required.

346L Proposition Let (Z, T, ν) be the Stone space of the measure algebra of Lebesgue measure on $[0, 1]$. Let $\psi : \mathrm{T} \to \mathrm{T}$ be the canonical lifting, defined by setting $\psi E = G$ whenever $E \in \mathrm{T}$, G is open-and-closed and $E \triangle G$ is negligible (341O). Then ψ is not consistent.

proof ? Suppose, if possible, that ϕ is a lifting on $Z \times Z$ such that $\phi(E \times F) = \psi E \times \psi F$ for every E, $F \in \mathrm{T}$. Let $W \subseteq Z \times Z$ be a set as in 346K, and consider ϕW. If G, $H \subseteq Z$ are open-and-closed and $(G \times H) \setminus W$ is negligible, then

$$G \times H = \psi G \times \psi H = \phi(G \times H) \subseteq \phi W;$$

that is, in the language of 346K, we must have $\tilde{W} \subseteq \phi W$. But this means that

$$\lambda(\phi W) \geq \lambda^*\tilde{W} = 1 > \lambda W,$$

which is impossible. **X**

Thus ψ fails the first test and cannot be consistent.

346X Basic exercises (a) Let (X, Σ, μ) be a measure space and $\langle \underline{\phi}_n \rangle_{n\in\mathbb{N}}$ a sequence of lower densities for μ. (i) Show that $E \mapsto \bigcap_{n\in\mathbb{N}} \underline{\phi}_n E$ and $E \mapsto \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \underline{\phi}_m E$ are also lower densities for μ. (ii) Show that if μ is complete and $\mathcal{F}$ is any filter on $\mathbb{N}$, then $E \mapsto \bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} \underline{\phi}_n E$ is a lower density for μ.

(b) Let (X, Σ, μ) be a strictly localizable measure space, and G a countable group of measure space automorphisms from X to itself. Show that there is a lower density $\underline{\phi} : \Sigma \to \Sigma$ which is G-invariant in the sense that $\underline{\phi}(g^{-1}[E]) = g^{-1}[\underline{\phi}E]$ for every $E \in \Sigma$ and $g \in G$. (*Hint*: set $\underline{\phi}E = \bigcap_{g\in G} g[\underline{\phi}_0(g^{-1}[E])]$.)

>(c) Show that there is no lifting ϕ of Lebesgue measure on $[0, 1]^2$ which is 'symmetric' in the sense that $\phi(E^{-1}) = (\phi E)^{-1}$ for every measurable set E, writing $E^{-1} = \{(y, x) : (x, y) \in E\}$. (*Hint*: 345Xc.)

(d) Let (X, Σ, μ) be a measure space and $\underline{\phi}$ a lower density for μ. Take $H \in \Sigma$ and set $A = X \setminus (\underline{\phi}H \cup \underline{\phi}(Z \setminus H))$, $\underline{\phi}'E = \underline{\phi}E \cup (A \cap \underline{\phi}(H \cup E))$ for $E \in \Sigma$. Show that $\underline{\phi}'$ is a lower density.

(e) Describe the connections between 346B, 346D and 346F.

>(f) Suppose, in 341H, that (X, Σ, μ) is a product of probability spaces, and that in the proof, instead of taking $\langle a_\xi \rangle_{\xi<\kappa}$ to run over the whole measure algebra $\mathfrak{A}$, we take it to run over the elements of $\mathfrak{A}$ expressible as $E^\bullet$ where $E \in \Sigma$ is determined by a single coordinate. Show that the resulting lower density $\underline{\theta}$ respects coordinates in the sense that $\underline{\theta}E^\bullet$ is determined by coordinates in J whenever $E \in \Sigma$ is determined by coordinates in J. (Compare Macheras & Strauss 95, Theorem 2.)

>(g) Let $\underline{\phi}$ be lower Lebesgue density on $\mathbb{R}$, and ϕ a translation-invariant lifting for Lebesgue measure on $\mathbb{R}$ such that $\phi E \supseteq \underline{\phi}E$ for every measurable set E. Show that ϕ is consistent. (*Hint*: given $n \geq 1$, let $\underline{\phi}_n$ be lower Lebesgue density on $\mathbb{R}^n$. Let $\mathcal{I}$ be the ideal generated by

$$\{W : \mathbf{0} \in \underline{\phi}_n(\mathbb{R}^n \setminus W)\} \cup \bigcup_{i<n} \{\pi_i^{-1}[E] : 0 \in \phi(\mathbb{R} \setminus E)\};$$

show that $\mathbb{R}^n \notin \mathcal{I}$, so that we can use the method of 345B to construct a lifting for Lebesgue measure on $\mathbb{R}^n$.)

(h) Show that Lemma 346K is valid for any (Z, T, ν) which is the Stone space of an atomless probability space.

346Y Further exercises (a) Let $(X_1, \Sigma_1, \mu_1), \dots, (X_n, \Sigma_n, \mu_n)$ be probability spaces with product (X, Σ, μ). Show that there is a lifting for μ which respects coordinates. (Burke n95.)

(b) Let (X, Σ, μ) be a probability space, I any set, and λ the product measure on X^I. Show that there is a lower density for λ which is invariant under transpositions of pairs of coordinates.

(c) Suppose that (X, Σ, μ) and (Y, T, ν) are complete probability spaces with product $(X \times Y, \Lambda, \lambda)$. Show that for any lifting $\psi_1 : \Sigma \to \Sigma$ there are liftings $\psi_2 : \mathrm{T} \to \mathrm{T}$ and $\phi : \Lambda \to \Lambda$ such that $\phi(E \times F) = \psi_1 E \times \psi_2 F$ for all $E \in \Sigma$, $F \in \mathrm{T}$. (*Hint*: use the methods of §341. In the inductive construction of 341H, start with $\underline{\phi}_0(E \times Y) = (\psi_1 E) \times Y$ for every $E \in \Sigma$. Extend each lower density $\underline{\phi}_\xi$ to the algebra generated by $\mathrm{dom}(\underline{\phi}_\xi) \cup \{X \times F_\xi\}$ for some $F_\xi \in \mathrm{T}$. Make sure that $\underline{\phi}_\xi(X \times F)$ is always of the form $X \times F'$, and that $\underline{\phi}_\xi((E \times Y) \cup (X \times F)) = \underline{\phi}_\xi(E \times Y) \cup \underline{\phi}_\xi(X \times F)$; adapt the construction of 341G to maintain this. Use the method of 346H to generate a lifting from the final lower density $\underline{\phi}$. See Macheras & Strauss 96a, Theorem 4.)

(d) Use 346Yc and induction on ζ to prove 346H. (Macheras & Strauss 96b.)

(e) Let (X, Σ, μ) be a complete probability space. Show that there is a lifting $\psi : \Sigma \to \Sigma$ such that whenever $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ is a family of probability spaces, with product measure λ, there is a lifting ϕ for λ such that $\phi(\pi_i^{-1}[E]) = \pi_i^{-1}[\psi E]$ whenever $E \in \Sigma$ and $i \in I$ is such that $(X_i, \Sigma_i, \mu_i) = (X, \Sigma, \mu)$, writing $\pi_i(x) = x(i)$ for $x \in \prod_{i \in I} X_i$.

346Z Problems (a) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, with product (X, Σ, μ). Is there always a lifting for μ which respects coordinates in the sense of 346A?

(b) Is there a lower density $\underline{\phi}$ for the usual measure on $\{0,1\}^{\mathbb{N}}$ which is invariant under all permutations of coordinates?

346 Notes and comments I ought to say at once that in writing this section I have been greatly assisted by M.R.Burke.

The theorem that every complete probability space has a consistent lifting (346J) is due to Talagrand 82a; it is the inspiration for the whole of the section. 'Consistent' liftings were devised in response to some very interesting questions (see Talagrand 84, §6) which I do not discuss here; one will be mentioned in Theorem 465P in Volume 4. My aim here is rather to suggest further ways in which a lifting on a product space can be consistent with the product structure. The labour is substantial and the results achieved are curiously partial. I offer 346Za as the easiest natural question which does not appear amenable to the methods I describe.

The arguments I use are based on the fact that the translation-invariant measures of 345C already respect coordinates (346C). Maharam's theorem now makes it easy to show that any product of Maharam-type-homogeneous probability spaces has a lifting which respects coordinates (346E). A kind of projection argument (346F) makes it possible to obtain a lower density which respects coordinates on any product of probability spaces (346G). In fact the methods of §341, very slightly refined, automatically produce such lower densities (346Xf). But the extra power of 346G lies in the condition (ii): if E and F are 'fully independent' in the sense of being determined by coordinates in disjoint sets, then $\underline{\phi}(E \cup F) = \underline{\phi}E \cup \underline{\phi}F$, that is, $\underline{\phi}$ is making a tentative step towards being a lifting. (Remember that the difference between a lifting and a lower density is mostly that a lifting preserves finite unions as well as finite intersections; see 341Xa.) This can also be achieved by a modification of the previous method, but we have to work harder at one point in the proof.

The next step is to move to liftings which continue, as far as possible, to respect coordinates. Here there seem to be quite new obstacles, and 346H is the best result I know; the lifting respects *individual* coordinates, and also, for a given well-ordering of the index set, initial segments of the coordinates. The treatment of initial segments makes essential use of the well-ordering, which is what leaves 346Za open.

Finally, if all the factors are identical, we can seek lower densities and liftings which are invariant under permutation of coordinates. I give 345Xc and 346Xc as examples to show that we must not just assume that a symmetry in the underlying measure space can be reflected in a symmetry of a lifting. The problems there concern liftings themselves, not lower densities, since we can frequently find lower densities which share symmetries (346Xb, 346Yb). (Even for lower densities there seem to be difficulties if we are more ambitious (346Zb).) However a very simple argument (346I) shows that at least we can make each individual coordinate look more or less the same, as long as we do not investigate its relations with others.

Still on the question of whether, and when, liftings can be 'good', note 346L/346Xh and 346Xg. The most natural liftings for Lebesgue measure are necessarily consistent; but the only example we have of a truly canonical lifting is not consistent in any non-trivial context.

I have deliberately used a variety of techniques here, even though 346H (for instance) has an alternative proof based on the ideas of §341 (346Yc-346Yd). In particular, I give some of the standard methods of constructing liftings and lower densities (346B, 346D, 346F, 346Xd, 346Xa). In fact 346D was one of the elements of Maharam's original proof of the lifting theorem (Maharam 58).

Version of 10.4.10

Concordance for Volume 3

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

341X Exercises 341Xd and 341Xf, referred to in the 2003 and 2006 editions of Volume 4, are now 341Xc and 341Xe.

Version of 25.8.17

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