Version of 17.11.10

Chapter 33

Maharam's theorem

We are now ready for the astonishing central fact about measure algebras: there are very few of them. Any localizable measure algebra has a canonical expression as a simple product of measure algebras of easily described types. This complete classification necessarily dominates all further discussion of measure algebras; to the point that all the results of Chapter 32 have to be regarded as 'elementary', since however complex their formulation they have been proved by techniques not involving, nor providing, any particular insight into the special nature of measure algebras. The proof depends, of course, on developing methods of defining measure-preserving homomorphisms and isomorphisms; I give a number of results, progressively more elaborate, but all based on the same idea. These techniques are of great power, leading, for instance, to an effective classification of closed subalgebras and their embeddings.

'Maharam's theorem' itself, the classification of localizable measure algebras, is in §332. I devote §331 to the definition and description of 'homogeneous' probability algebras. In §333 I turn to the problem of describing pairs $(\mathfrak{A},\mathfrak{C})$ where $\mathfrak{A}$ is a probability algebra and $\mathfrak{C}$ is a closed subalgebra. Finally, in §334, I give some straightforward results on the classification of free products of probability algebras.

Version of 1.2.05

331 Maharam types and homogeneous measure algebras

I embark directly on the principal theorem of this chapter (331I), split between 331B, 331D and 331I; 331B and 331D will be the basis of many of the results in later sections of this chapter. In 331E-331H I introduce the concepts of 'Maharam type' and 'Maharam-type-homogeneity'. I discuss the measure algebras of products $\{0,1\}^\kappa$, showing that these provide a complete set of examples of Maharam-type-homogeneous probability algebras (331J-331L).

331A Definition Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$. A non-zero element a of $\mathfrak{A}$ is a **relative atom** over $\mathfrak{B}$ if every $c\subseteq a$ is of the form $a\cap b$ for some $b\in\mathfrak{B}$. $\mathfrak{A}$ is **relatively atomless** over $\mathfrak{B}$ if there are no relative atoms in $\mathfrak{A}$ over $\mathfrak{B}$.

At some point I ought to remark that if a is an atom of $\mathfrak{A}$ then it is a relative atom over $\mathfrak{B}$; if $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$, then it is atomless.

331B Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$, and $a_0\in\mathfrak{A}$. Let $\nu:\mathfrak{B}\to\mathbb{R}$ be an additive functional such that $0\le\nu b\le\bar\mu(b\cap a_0)$ for every $b\in\mathfrak{B}$. Then there is a $c\in\mathfrak{A}$ such that $c\subseteq a_0$ and $\nu b=\bar\mu(b\cap c)$ for every $b\in\mathfrak{B}$.

331C Corollary Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra, and $a\in\mathfrak{A}$. Suppose that $0\le\gamma\le\bar\mu a$. Then there is a $c\in\mathfrak{A}$ such that $c\subseteq a$ and $\bar\mu c=\gamma$.

331D Lemma Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be totally finite measure algebras and $\mathfrak{C}\subseteq\mathfrak{A}$ a closed subalgebra. Suppose that $\pi:\mathfrak{C}\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism such that $\mathfrak{B}$ is relatively atomless over $\pi[\mathfrak{C}]$. Take any $a\in\mathfrak{A}$, and let $\mathfrak{C}_1$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\{a\}$. Then there is a measure-preserving homomorphism from $\mathfrak{C}_1$ to $\mathfrak{B}$ extending π.

331E Generating sets If $\mathfrak{A}$ is a Boolean algebra and B is a subset of $\mathfrak{A}$ we can look at any of the three algebras

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– $\mathfrak{B}$, the smallest subalgebra of $\mathfrak{A}$ including B;
– $\mathfrak{B}_\sigma$, the smallest σ-subalgebra of $\mathfrak{A}$ including B;
– $\mathfrak{B}_\tau$, the smallest order-closed subalgebra of $\mathfrak{A}$ including B.

I will say henceforth that
– $\mathfrak{B}$ is the subalgebra of $\mathfrak{A}$ generated by B, and B **generates** $\mathfrak{A}$ if $\mathfrak{A}=\mathfrak{B}$;
– $\mathfrak{B}_\sigma$ is the σ-subalgebra of $\mathfrak{A}$ generated by B, and B **σ-generates** $\mathfrak{A}$ if $\mathfrak{A}=\mathfrak{B}_\sigma$;
– $\mathfrak{B}_\tau$ is the order-closed subalgebra of $\mathfrak{A}$ generated by B, and B **τ-generates** or **completely generates** $\mathfrak{A}$ if $\mathfrak{A}=\mathfrak{B}_\tau$.

331F Maharam types (a) If $\mathfrak{A}$ is a Boolean algebra, its **Maharam type** $\tau(\mathfrak{A})$ is the smallest cardinal of any subset of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$.

(b) A Boolean algebra $\mathfrak{A}$ is **Maharam-type-homogeneous** if $\tau(\mathfrak{A}_a)=\tau(\mathfrak{A})$ for every non-zero $a\in\mathfrak{A}$, writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a.

(c) Let (X,Σ,μ) be a measure space, with measure algebra $(\mathfrak{A},\bar\mu)$. Then the **Maharam type** of (X,Σ,μ), or of μ, is the Maharam type of $\mathfrak{A}$; and (X,Σ,μ), or μ, is **Maharam-type-homogeneous** if $\mathfrak{A}$ is.

331G Proposition Let $\mathfrak{A}$ be a Boolean algebra, B a subset of $\mathfrak{A}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by B, $\mathfrak{B}_\sigma$ the σ-subalgebra of $\mathfrak{A}$ generated by B, and $\mathfrak{B}_\tau$ the order-closed subalgebra of $\mathfrak{A}$ generated by B.

(a) $\mathfrak{B}\subseteq\mathfrak{B}_\sigma\subseteq\mathfrak{B}_\tau$.

(b) If B is finite, so is $\mathfrak{B}$, and in this case $\mathfrak{B}=\mathfrak{B}_\sigma=\mathfrak{B}_\tau$.

(c) For every $a\in\mathfrak{B}$, there is a finite $B'\subseteq B$ such that a belongs to the subalgebra of $\mathfrak{A}$ generated by B'. Consequently $\#(\mathfrak{B})\le\max(\omega,\#(B))$.

(d) For every $a\in\mathfrak{B}_\sigma$, there is a countable $B'\subseteq B$ such that a belongs to the σ-subalgebra of $\mathfrak{A}$ generated by B'.

(e) If $\mathfrak{A}$ is ccc, then $\mathfrak{B}_\sigma=\mathfrak{B}_\tau$.

331H Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a)(i) $\tau(\mathfrak{A})=0$ iff $\mathfrak{A}$ is either $\{0\}$ or $\{0,1\}$.

(ii) $\tau(\mathfrak{A})$ is finite iff $\mathfrak{A}$ is finite.

(b) If $\mathfrak{B}$ is another Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\tau(\mathfrak{B})\le\tau(\mathfrak{A})$.

(c) If $a\in\mathfrak{A}$ then $\tau(\mathfrak{A}_a)\le\tau(\mathfrak{A})$, where $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a.

(d) If $\mathfrak{A}$ has an atom and is Maharam-type-homogeneous, then $\mathfrak{A}=\{0,1\}$.

331I Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be Maharam-type-homogeneous measure algebras of the same Maharam type, with $\bar\mu 1=\bar\nu 1<\infty$. Then they are isomorphic as measure algebras.

331J Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and κ an infinite cardinal.

(a) If there is a family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}$ such that $\inf_{\xi\in I}a_\xi=0$ and $\sup_{\xi\in I}a_\xi=1$ for every infinite $I\subseteq\kappa$, then $\tau(\mathfrak{A}_d)\ge\kappa$ for every non-zero $d\in\mathfrak{A}$.

(b) Let ν_κ be the usual measure on $\{0,1\}^\kappa$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. If there is an order-continuous Boolean homomorphism from $\mathfrak{B}_\kappa$ to $\mathfrak{A}$, $\tau(\mathfrak{A}_d)\ge\kappa$ for every non-zero $d\in\mathfrak{A}$.

331K Theorem Let κ be any infinite cardinal. Let ν_κ be the usual measure on $\{0,1\}^\kappa$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. Then $\mathfrak{B}_\kappa$ is Maharam-type-homogeneous, with Maharam type κ.

331L Theorem Let $(\mathfrak{A},\bar\mu)$ be a Maharam-type-homogeneous probability algebra. Then there is exactly one κ, either 0 or an infinite cardinal, such that $(\mathfrak{A},\bar\mu)$ is isomorphic, as measure algebra, to the measure algebra $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ of the usual measure on $\{0,1\}^\kappa$.

331M Homogeneous Boolean algebras Of course a homogeneous Boolean algebra must be Maharam-type-homogeneous, since $\tau(\mathfrak{A})=\tau(\mathfrak{A}_c)$ whenever $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_c$. In general, a Boolean algebra can be Maharam-type-homogeneous without being homogeneous (331Xj, 331Yg). But for σ-finite measure algebras this doesn't happen.

331N Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a Maharam-type-homogeneous σ-finite measure algebra. Then it is homogeneous as a Boolean algebra.

331O I will wait until Chapter 52 of Volume 5 for a systematic discussion of properties of measure algebras which depend on their Maharam types. There are however a couple of results which are easy, useful and expressible in terms already introduced.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra with countable Maharam type. Then $\mathfrak{A}$ is separable in its measure-algebra topology.

et $B \subseteq \mathfrak{A}$ be a countable set which τ-generates $\mathfrak{A}$. Then the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by B is countable (331Gc). Now $\mathfrak{B}$ is dense for the measure-algebra topology. **P** Let G be a non-empty open subset of $\mathfrak{A}$, and c any element of G. Let $\mathrm{P} = \{\rho_a : a \in \mathfrak{A}^f\}$ be the upwards-directed family of pseudometrics defining the topology of $\mathfrak{A}$, as described in 323A. Then there must be an $a \in \mathfrak{A}^f$ and an $\epsilon > 0$ such that $\{b : \rho_a(b, c) \leq \epsilon\} \subseteq G$. Let $\mathfrak{C}$ be the order-closed subalgebra of the principal ideal $\mathfrak{A}_a$ generated by $\mathfrak{B}_a = \{b \cap a : b \in \mathfrak{B}\}$. Because $b \mapsto b \cap a : \mathfrak{A} \to \mathfrak{A}_a$ is an order-continuous Boolean homomorphism, $\{b : b \in \mathfrak{A},\, b \cap a \in \mathfrak{C}\}$ is an order-closed subalgebra of $\mathfrak{A}$, and must be the whole of $\mathfrak{A}$, because it includes B. So $\mathfrak{C} = \mathfrak{A}_a$. By 323J, $\mathfrak{C}$ is the topological closure of $\mathfrak{B}_a$ in $\mathfrak{A}_a$, and there must be a $b \in \mathfrak{B}_a$ such that $\bar{\mu}(b \,\triangle\, (c \cap a)) \leq \epsilon$; that is, there is a $b \in \mathfrak{B}$ such that $\bar{\mu}(a \cap (b \,\triangle\, c)) \leq \epsilon$ and $b \in G$. Thus $\mathfrak{B}$ meets G; as G is arbitrary, $\mathfrak{B}$ is dense. **Q**

So $\mathfrak{B}$ is a countable dense subset of $\mathfrak{A}$ and $\mathfrak{A}$ is separable.

331P Proposition Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra of countable Maharam type. Then it is isomorphic to the measure algebras of the usual measure on $\{0,1\}^{\mathbb{N}}$ and of Lebesgue measure on $[0,1]$.

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332 Classification of localizable measure algebras

In this section I present what I call 'Maharam's theorem', that every localizable measure algebra is expressible as a weighted simple product of measure algebras of spaces of the form $\{0,1\}^\kappa$ (332B). Among its many consequences is a complete description of the isomorphism classes of localizable measure algebras (332J). This description needs the concepts of 'cellularity' of a Boolean algebra (332D) and its refinement, the 'magnitude' of a measure algebra (332G). I end this section with a discussion of those pairs of measure algebras for which there is a measure-preserving homomorphism from one to the other (332P-332Q), and a general formula for the Maharam type of a localizable measure algebra (332S).

332A Lemma Let $\mathfrak{A}$ be any Boolean algebra. Writing $\mathfrak{A}_a$ for the principal ideal generated by $a \in \mathfrak{A}$, $\{a : a \in \mathfrak{A},\, \mathfrak{A}_a \text{ is Maharam-type-homogeneous}\}$ is order-dense in $\mathfrak{A}$.

332B Maharam's theorem Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Then it is isomorphic to the simple product of a family $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i \in I}$ of measure algebras, where for each $i \in I$ $(\mathfrak{A}_i, \bar{\mu}_i)$ is isomorphic, up to a re-normalization of the measure, to the measure algebra of the usual measure on $\{0,1\}^{\kappa_i}$, where κ_i is either 0 or an infinite cardinal.

332C Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. For any cardinal κ, write ν_κ for the usual measure on $\{0,1\}^\kappa$, and T_κ for its domain. Then we can find families $\langle\kappa_i\rangle_{i \in I}$, $\langle\gamma_i\rangle_{i \in I}$ such that every κ_i is either 0 or an infinite cardinal, every γ_i is a strictly positive real number, and $(\mathfrak{A}, \bar{\mu})$ is isomorphic to the measure algebra of (X, Σ, ν), where

$$X = \{(x, i) : i \in I,\, x \in \{0,1\}^{\kappa_i}\},$$

$$\Sigma = \{E : E \subseteq X,\, \{x : (x, i) \in E\} \in \mathrm{T}_{\kappa_i} \text{ for every } i \in I\},$$

$$\nu E = \sum_{i \in I} \gamma_i \nu_{\kappa_i} \{x : (x, i) \in E\}$$

for every $E \in \Sigma$.

332D The cellularity of a Boolean algebra If $\mathfrak{A}$ is any Boolean algebra, write

$$c(\mathfrak{A}) = \sup\{\#(C) : C \subseteq \mathfrak{A} \setminus \{0\} \text{ is disjoint}\},$$

the **cellularity** of $\mathfrak{A}$.

332E Proposition Let $(\mathfrak{A}, \bar{\mu})$ be any semi-finite measure algebra, and C any partition of unity in $\mathfrak{A}$ consisting of elements of finite measure. Then $\max(\omega, \#(C)) = \max(\omega, c(\mathfrak{A}))$.

332F Corollary Let $(\mathfrak{A}, \bar{\mu})$ be any semi-finite measure algebra. Then there is a disjoint set in $\mathfrak{A} \setminus \{0\}$ of cardinal $c(\mathfrak{A})$.

332G Definitions(a) If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, let us say that the **magnitude** of an $a \in \mathfrak{A}$ is $\bar{\mu}a$ if $\bar{\mu}a$ is finite, and otherwise is the cellularity of the principal ideal $\mathfrak{A}_a$ generated by a. If we take it that any real number is less than any infinite cardinal, then the class of possible magnitudes is totally ordered.

I shall sometimes speak of the **magnitude** of the measure algebra $(\mathfrak{A}, \bar{\mu})$ itself, meaning the magnitude of $1_{\mathfrak{A}}$. Similarly, if (X, Σ, μ) is a semi-finite measure space, the **magnitude** of (X, Σ, μ), or of μ, is the magnitude of its measure algebra.

(b) Next, for any Dedekind complete Boolean algebra $\mathfrak{A}$, and any cardinal κ, we can look at the element

$$e_\kappa = \sup\{a : a \in \mathfrak{A} \setminus \{0\}, \mathfrak{A}_a \text{ is Maharam-type-homogeneous with Maharam type } \kappa\},$$

writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a, as usual. I will call this the **Maharam-type-κ component** of $\mathfrak{A}$.

$\{e_\kappa : \kappa \text{ is a cardinal}\}$ is a partition of unity in $\mathfrak{A}$.

332H Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and κ an infinite cardinal. Let e be the Maharam-type-κ component of $\mathfrak{A}$. If $0 \neq d \subseteq e$ and the principal ideal $\mathfrak{A}_d$ generated by d is ccc, then it is Maharam-type-homogeneous with Maharam type κ.

332I Lemma Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra which is not totally finite. Then it has a partition of unity consisting of elements of measure 1.

332J Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras. For each cardinal κ, let e_κ, f_κ be the Maharam-type-κ components of $\mathfrak{A}$, $\mathfrak{B}$ respectively. Then $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are isomorphic, as measure algebras, iff (i) e_κ and f_κ have the same magnitude for every infinite cardinal κ (ii) for every $\gamma \in]0, \infty[$, $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ have the same number of atoms of measure γ.

332L Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $a, b \in \mathfrak{A}$ two elements of finite measure. Suppose that $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$ is a measure-preserving isomorphism, where $\mathfrak{A}_a$, $\mathfrak{A}_b$ are the principal ideals generated by a and b. Then there is a measure-preserving automorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ which extends π.

332M Lemma Suppose that $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are homogeneous measure algebras, with $\tau(\mathfrak{A}) \leq \tau(\mathfrak{B})$ and $\bar{\mu}1 = \bar{\nu}1 < \infty$. Then there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

332N Lemma If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra and $\kappa \geq \max(\omega, \tau(\mathfrak{A}))$, then there is a measure-preserving Boolean homomorphism from $(\mathfrak{A}, \bar{\mu})$ to the measure algebra $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ of the usual measure ν on $\{0,1\}^\kappa$; that is, $(\mathfrak{A}, \bar{\mu})$ is isomorphic to a closed subalgebra of $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$.

332O Lemma Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be localizable measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components, and for $\gamma \in]0, \infty[$ let e_γ, f_γ be the suprema of the atoms of measure γ in $\mathfrak{A}$, $\mathfrak{B}$ respectively. If there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then the magnitude of $\sup_{\kappa \geq \lambda} e_\kappa$ is not greater than the magnitude of $\sup_{\kappa \geq \lambda} f_\kappa$ whenever λ is an infinite cardinal, while the magnitude of $\sup_{\kappa \geq \omega} e_\kappa \cup \sup_{\gamma \leq \delta} e_\gamma$ is not greater than the magnitude of $\sup_{\kappa \geq \omega} f_\kappa \cup \sup_{\gamma \leq \delta} f_\gamma$ for any $\delta \in]0, \infty[$.

332P Proposition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be atomless totally finite measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar{\mu})$ is isomorphic to a closed subalgebra of a principal ideal of $(\mathfrak{B},\bar{\nu})$;

(ii) for every cardinal λ,

$$\bar{\mu}(\sup_{\kappa\ge\lambda} e_\kappa) \le \bar{\nu}(\sup_{\kappa\ge\lambda} f_\kappa).$$

332Q Proposition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be totally finite measure algebras, and suppose that there are measure-preserving Boolean homomorphisms $\pi_1 : \mathfrak{A} \to \mathfrak{B}$ and $\pi_2 : \mathfrak{B} \to \mathfrak{A}$. Then $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ are isomorphic.

332R Lemma Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra. Then $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$.

332S Theorem Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra. Then $\tau(\mathfrak{A})$ is the least cardinal λ such that (α) $c(\mathfrak{A}) \le 2^\lambda$ (β) $\tau(\mathfrak{A}_a) \le \lambda$ for every Maharam-type-homogeneous principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$.

332T Proposition Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra and $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$. Then

(a) there is a function $\bar{\nu} : \mathfrak{B} \to [0,\infty]$ such that $(\mathfrak{B},\bar{\nu})$ is a localizable measure algebra;

(b) $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

Version of 27.6.08

333 Closed subalgebras

Proposition 332P tells us, in effect, which totally finite measure algebras can be embedded as closed subalgebras of each other. Similar techniques make it possible to describe the possible forms of such embeddings. In this section I give the fundamental theorems on extension of measure-preserving homomorphisms from closed subalgebras (333C, 333D); these rely on the concept of 'relative Maharam type' (333A). I go on to describe possible canonical forms for structures $(\mathfrak{A},\bar{\mu},\mathfrak{C})$, where $(\mathfrak{A},\bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (333K, 333N). I end the section with a description of fixed-point subalgebras (333R).

333A Definitions (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. The **relative Maharam type of $\mathfrak{A}$ over $\mathfrak{C}$**, $\tau_{\mathfrak{C}}(\mathfrak{A})$, is the smallest cardinal of any set $A \subseteq \mathfrak{A}$ such that $A \cup \mathfrak{C}$ τ-generates $\mathfrak{A}$.

(b) In this section, I will regularly use the following notation: if $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, and $a \in \mathfrak{A}$, then I will write $\mathfrak{C}_a$ for $\{c \cap a : c \in \mathfrak{C}\}$. Observe that $\mathfrak{C}_a$ is a subalgebra of the principal ideal $\mathfrak{A}_a$.

(c) I will say that an element a of $\mathfrak{A}$ is **relatively Maharam-type-homogeneous over $\mathfrak{C}$** if $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b) = \tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ for every non-zero $b \subseteq a$.

(d) If κ is a cardinal, I will write $(\mathfrak{B}_\kappa,\bar{\nu}_\kappa)$ for the measure algebra of the usual measure ν_κ on $\{0,1\}^\kappa$.

333B Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$.

(a) If $a \subseteq b$ in $\mathfrak{A}$, then $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \tau_{\mathfrak{C}_b}(\mathfrak{A}_b)$. In particular, $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \tau_{\mathfrak{C}}(\mathfrak{A})$ for every $a \in \mathfrak{A}$.

(b) The set $\{a : a \in \mathfrak{A}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}\}$ is order-dense in $\mathfrak{A}$.

(c) If $\mathfrak{A}$ is Dedekind complete and $\mathfrak{C}$ is order-closed in $\mathfrak{A}$, then $\mathfrak{C}_a$ is order-closed in $\mathfrak{A}_a$.

(d) If $a \in \mathfrak{A}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}$ then either $\mathfrak{A}_a = \mathfrak{C}_a$, so that $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = 0$ and a is a relative atom of $\mathfrak{A}$ over $\mathfrak{C}$, or $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \ge \omega$.

(e) If $\mathfrak{D}$ is another subalgebra of $\mathfrak{A}$ and $\mathfrak{D} \subseteq \mathfrak{C}$, then

$$\tau(\mathfrak{A}_a) = \tau_{\{0,a\}}(\mathfrak{A}_a) \ge \tau_{\mathfrak{D}_a}(\mathfrak{A}_a) \ge \tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \ge \tau_{\mathfrak{A}_a}(\mathfrak{A}_a) = 0$$

for every $a \in \mathfrak{A}$.

333C Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be totally finite measure algebras, and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Let $\phi:\mathfrak{C}\to\mathfrak{B}$ be a measure-preserving Boolean homomorphism.

(a) If $\tau_{\mathfrak{C}}(\mathfrak{A})\le\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$ for every non-zero $b\in\mathfrak{B}$, there is a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ extending ϕ.

(b) If $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)=\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$ for every non-zero $a\in\mathfrak{A}$, $b\in\mathfrak{B}$, then there is a measure algebra isomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ extending ϕ.

333D Corollary Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be totally finite measure algebras and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Suppose that

$$\tau(\mathfrak{C})<\max(\omega,\tau(\mathfrak{A}))\le\min\{\tau(\mathfrak{B}_b):b\in\mathfrak{B}\setminus\{0\}\}.$$

Then any measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ can be extended to a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$.

333E Theorem Let $(\mathfrak{C},\bar\mu)$ be a totally finite measure algebra and κ an infinite cardinal. Let $(\mathfrak{A},\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu)$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$, and $\varepsilon:\mathfrak{C}\to\mathfrak{A}$ the corresponding homomorphism. Then for any non-zero $a\in\mathfrak{A}$,

$$\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)=\kappa.$$

333F Corollary Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ and κ an infinite cardinal.

(a) Suppose that $\kappa\ge\tau_{\mathfrak{C}}(\mathfrak{A})$. Let $(\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa,\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$, and $\varepsilon:\mathfrak{C}\to\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa$ the corresponding homomorphism. Then there is a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa$ extending ε.

(b) Suppose further that $\kappa=\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ for every non-zero $a\in\mathfrak{A}$. Then π can be taken to be an isomorphism.

333G Corollary Let $(\mathfrak{C},\bar\mu)$ be a totally finite measure algebra. Suppose that $\kappa\ge\max(\omega,\tau(\mathfrak{C}))$ is a cardinal. Let $(\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa,\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu)$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$. Then

(a) $\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa$ is Maharam-type-homogeneous, with Maharam type κ if $\mathfrak{C}\ne\{0\}$;

(b) for every measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{C}$ there is a measure-preserving automorphism $\pi:\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa\to\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa$ such that $\pi(c\otimes 1)=\phi c\otimes 1$ for every $c\in\mathfrak{C}$, writing $c\otimes 1$ for the canonical image in $\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa$ of any $c\in\mathfrak{C}$.

333H Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Then there are $\langle\mu_i\rangle_{i\in I}$, $\langle c_i\rangle_{i\in I}$, $\langle\kappa_i\rangle_{i\in I}$ such that

for each $i\in I$, μ_i is a non-negative completely additive functional on $\mathfrak{C}$,

$c_i=[\![\mu_i>0]\!]\in\mathfrak{C}$,

κ_i is 0 or an infinite cardinal,

$(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ is a totally finite measure algebra, writing $\mathfrak{C}_{c_i}$ for the principal ideal of $\mathfrak{C}$ generated by c_i,

$\sum_{i\in I}\mu_i c=\bar\mu c$ for every $c\in\mathfrak{C}$,

there is a measure-preserving isomorphism π from $\mathfrak{A}$ to the simple product $\prod_{i\in I}\mathfrak{C}_{c_i}\widehat\otimes\mathfrak{B}_{\kappa_i}$ of the localizable measure algebra free products $\mathfrak{C}_{c_i}\widehat\otimes\mathfrak{B}_{\kappa_i}$ of $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ and $(\mathfrak{B}_{\kappa_i},\bar\nu_{\kappa_i})$.

Moreover, π may be taken such that

for every $c\in\mathfrak{C}$, $\pi c=\langle(c\cap c_i)\otimes 1\rangle_{i\in I}$, writing $c\otimes 1$ for the image in $\mathfrak{C}_{c_i}\widehat\otimes\mathfrak{B}_{\kappa_i}$ of $c\in\mathfrak{C}_{c_i}$.

333I Remarks Whenever $(\mathfrak{C},\bar\mu)$ is a Dedekind complete measure algebra, $\langle\mu_i\rangle_{i\in I}$ is a family of non-negative completely additive functionals on $\mathfrak{C}$ such that $\sum_{i\in I}\mu_i=\bar\mu$, and $\langle\kappa_i\rangle_{i\in I}$ is a family of cardinals all infinite or zero, then the construction above can be applied to give a measure algebra $(\mathfrak{A},\bar\lambda)$, the product of the family $\langle\mathfrak{C}_{c_i}\widehat\otimes\mathfrak{B}_{\kappa_i}\rangle_{i\in I}$, together with an order-continuous measure-preserving homomorphism $\pi:\mathfrak{C}\to\mathfrak{A}$; and the partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ corresponding to this product has $\mu_i c=\bar\lambda(a_i\cap\pi c)$ for every $c\in\mathfrak{C}$ and

$i \in I$, while each principal ideal $\mathfrak{A}_{a_i}$ can be identified with $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, so that a_i is relatively Maharam-type-homogeneous over $\pi[\mathfrak{C}]$. Thus any structure $(\mathfrak{C}, \bar{\mu}, \langle\mu_i\rangle_{i\in I}, \langle\kappa_i\rangle_{i\in I})$ of the type described here corresponds to an embedding of $\mathfrak{C}$ as a closed subalgebra of a localizable measure algebra.

333J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\mathfrak{C}$ a closed subalgebra. Let A be the set of relative atoms of $\mathfrak{A}$ over $\mathfrak{C}$. Then there is a unique sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ of additive functionals on $\mathfrak{C}$ such that (i) $\mu_{n+1} \le \mu_n$ for every n (ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $\sup_{n\in\mathbb{N}} a_n = \sup A$ and $\mu_n c = \bar{\mu}(a_n \cap c)$ for every $n \in \mathbb{N}$ and $c \in \mathfrak{C}$.

333K Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Then there are unique families $\langle\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\mu_\kappa\rangle_{\kappa\in K}$ such that

K is a countable set of infinite cardinals,

for $i \in \mathbb{N} \cup K$, μ_i is a non-negative countably additive functional on $\mathfrak{C}$, and $\sum_{i\in\mathbb{N}\cup K} \mu_i c = \bar{\mu}c$ for every $c \in \mathfrak{C}$,

$\mu_{n+1} \le \mu_n$ for every $n \in \mathbb{N}$, and $\mu_\kappa \ne 0$ for $\kappa \in K$,

setting $e_i = [\![\mu_i > 0]\!] \in \mathfrak{C}$, and giving the principal ideal $\mathfrak{C}_{e_i}$ generated by e_i the measure $\mu_i\restriction\mathfrak{C}_{e_i}$ for each $i \in \mathbb{N} \cup K$, we have a measure algebra isomorphism

$$\pi : \mathfrak{A} \to \prod_{n\in\mathbb{N}} \mathfrak{C}_{e_n} \times \prod_{\kappa\in K} \mathfrak{C}_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$$

such that

$$\pi c = (\langle c \cap e_n\rangle_{n\in\mathbb{N}}, \langle(c \cap e_\kappa) \otimes 1\rangle_{\kappa\in K})$$

for each $c \in C$, writing $c \otimes 1$ for the canonical image in $C_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$ of $c \in \mathfrak{C}_{e_\kappa}$.

333L Remark Thus for the classification of structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$, where $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra, it will be enough to classify objects $(\mathfrak{C}, \bar{\mu}, \langle\mu_n\rangle_{n\in\mathbb{N}}, \langle\mu_\kappa\rangle_{\kappa\in K})$, where

$(\mathfrak{C}, \bar{\mu})$ is a totally finite measure algebra,

$\langle\mu_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-negative countably additive functionals on $\mathfrak{C}$,

K is a countable set of infinite cardinals,

$\langle\mu_\kappa\rangle_{\kappa\in K}$ is a family of non-zero non-negative countably additive functionals on $\mathfrak{C}$,

$\sum_{n=0}^{\infty} \mu_n + \sum_{\kappa\in K} \mu_\kappa = \bar{\mu}$.

333M Lemma Let $(\mathfrak{C}, \bar{\mu})$ be a totally finite measure algebra and $\langle\mu_i\rangle_{i\in I}$ a family of countably additive functionals on $\mathfrak{C}$. For $i \in I$, $\alpha \in \mathbb{R}$ set $e_{i\alpha} = [\![\mu_i > \alpha\bar{\mu}]\!]$, and let $\mathfrak{C}_0$ be the closed subalgebra of $\mathfrak{C}$ generated by $\{e_{i\alpha} : i \in I, \alpha \in \mathbb{R}\}$. Write Σ for the σ-algebra of subsets of $\mathbb{R}^I$ generated by sets of the form $E_{i\alpha} = \{x : x(i) > \alpha\}$ as i runs over I and α runs over $\mathbb{R}$. Then

(a) there is a measure μ, with domain Σ, such that there is a measure-preserving isomorphism $\pi : \Sigma/\mathcal{N}_\mu \to \mathfrak{C}_0$ for which $\pi E^\bullet_{i\alpha} = e_{i\alpha}$ for every $i \in I$ and $\alpha \in \mathbb{R}$, writing $\mathcal{N}_\mu$ for $\mu^{-1}[\{0\}]$;

(b) this formula determines both μ and π;

(c) for every $E \in \Sigma$ and $i \in I$, we have

$$\mu_i\pi E^\bullet = \int_E x(i)\mu(dx);$$

(d) for every $i \in I$, μ_i is the standard extension of $\mu_i\restriction\mathfrak{C}_0$ to $\mathfrak{C}$;

(e) for every $i \in I$, $\mu_i \ge 0$ iff $x(i) \ge 0$ for μ-almost every x;

(f) for every i, $j \in I$, $\mu_i \ge \mu_j$ iff $x(i) \ge x(j)$ for μ-almost every x;

(g) for every $i \in I$, $\mu_i = 0$ iff $x(i) = 0$ for μ-almost every x.

333N A canonical form for closed subalgebras We now have all the elements required to describe a canonical form for structures

$$(\mathfrak{A}, \bar{\mu}, \mathfrak{C}),$$

where $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$. The first step is the matching of such structures with structures

$$(\mathfrak{C}, \bar{\mu}, \langle\mu_n\rangle_{n\in\mathbb{N}}, \langle\mu_\kappa\rangle_{\kappa\in K}),$$

where $(\mathfrak{C}, \bar{\mu})$ is a totally finite measure algebra, $\langle\mu_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-negative countably additive functionals on $\mathfrak{C}$, K is a countable set of infinite cardinals, $\langle\mu_\kappa\rangle_{\kappa\in K}$ is a family of non-zero non-negative countably additive functionals on $\mathfrak{C}$, and $\sum_{n=0}^{\infty}\mu_n + \sum_{\kappa\in K}\mu_\kappa = \bar{\mu}$.

Next, given any structure of this second kind, we have a corresponding closed subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$, a measure μ on $\mathbb{R}^I$, where $I = \mathbb{N}\cup K$, and an isomorphism π from the measure algebra $\mathfrak{C}_0^*$ of μ to $\mathfrak{C}_0$, all uniquely defined from the family $\langle\mu_i\rangle_{i\in I}$. $\mu_i\restriction\mathfrak{C}_0$ is fixed by π and μ. Moreover, the functionals μ_i can be recovered from their restrictions to $\mathfrak{C}_0$. Thus from $(\mathfrak{C}, \bar{\mu}, \langle\mu_i\rangle_{i\in I})$ we are led, by a canonical and reversible process, to the structure

$$(\mathfrak{C}, \bar{\mu}, \mathfrak{C}_0, I, \mu, \pi).$$

But the extension $\mathfrak{C}$ of $\mathfrak{C}_0 = \pi[\mathfrak{C}_0^*]$ can be described, up to isomorphism, by the same process as before; that is, it corresponds to a sequence $\langle\theta'_n\rangle_{n\in\mathbb{N}}$ and a family $\langle\theta'_\kappa\rangle_{\kappa\in L}$ of countably additive functionals on $\mathfrak{C}_0$ satisfying the conditions of 333K. We can transfer these to $\mathfrak{C}_0^*$, where they correspond to families $\langle\theta_n\rangle_{n\in\mathbb{N}}$, $\langle\theta_\kappa\rangle_{\kappa\in L}$ of absolutely continuous countably additive functionals defined on Σ, setting

$$\theta_j E = \theta'_j \pi E^\bullet$$

for $E\in\Sigma$, $j\in\mathbb{N}\cup L$. This process too is reversible; every absolutely continuous countably additive functional ν on Σ corresponds to countably additive functionals on $\mathfrak{C}_0^*$ and $\mathfrak{C}_0$.

Putting all this together, a structure $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ leads, in a canonical and (up to isomorphism) reversible way, to a structure

$$(K, \mu, L, \langle\theta_j\rangle_{j\in\mathbb{N}\cup L})$$

such that

K and L are countable sets of infinite cardinals,

μ is a totally finite measure on $\mathbb{R}^I$, where $I = \mathbb{N}\cup K$, and its domain Σ is precisely the σ-algebra of subsets of $\mathbb{R}^I$ defined by the coordinate functionals,

for μ-almost every $x\in\mathbb{R}^I$ we have $x(i)\ge 0$ for every $i\in I$, $x(n)\ge x(n+1)$ for every $n\in\mathbb{N}$ and $\sum_{i\in I}x(i) = 1$,

for $\kappa\in K$, $\mu\{x : x(\kappa) > 0\} > 0$,

for $j\in J = \mathbb{N}\cup L$, θ_j is a non-negative countably additive functional on Σ,

$\theta_n\ge\theta_{n+1}$ for every $n\in\mathbb{N}$, $\theta_\kappa\ne 0$ for every $\kappa\in L$, $\sum_{j\in J}\theta_j = \mu$.

333P Proposition Let $(\mathfrak{B}, \bar{\nu})$ be a homogeneous probability algebra. Then there is a measure-preserving automorphism $\phi : \mathfrak{B}\to\mathfrak{B}$ such that

$$\lim_{n\to\infty}\bar{\nu}(c\cap\phi^n(b)) = \bar{\nu}c\cdot\bar{\nu}b$$

for all b, $c\in\mathfrak{B}$.

333Q Corollary Let $(\mathfrak{C}, \bar{\mu}_0)$ be a totally finite measure algebra and $(\mathfrak{B}, \bar{\nu})$ a probability algebra which is *either* homogeneous *or* purely atomic with finitely many atoms all of the same measure. Let $(\mathfrak{A}, \bar{\mu})$ be the localizable measure algebra free product of $(\mathfrak{C}, \bar{\mu}_0)$ and $(\mathfrak{B}, \bar{\nu})$. Then there is a measure-preserving automorphism $\pi : \mathfrak{A}\to\mathfrak{A}$ such that

$$\{a : a\in\mathfrak{A},\ \pi a = a\} = \{c\otimes 1 : c\in\mathfrak{C}\}.$$

333R For an integer $n\ge 1$, I will write $\mathfrak{B}_n$ for the power set of $\{0,\dots,n\}$ and set $\bar{\nu}_n b = \frac{1}{n+1}\#(b)$ for $b\in\mathfrak{B}_n$.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C}$ a subset of $\mathfrak{A}$. Then the following are equiveridical:

(i) there is some set G of measure-preserving automorphisms of $\mathfrak{A}$ such that

$$\mathfrak{C} = \{c : c\in\mathfrak{A},\ \pi c = c \text{ for every } \pi\in G\};$$

(ii) $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ and there is a partition of unity $\langle e_i\rangle_{i\in I}$ in $\mathfrak{C}$, where I is a countable set of cardinals, such that $\mathfrak{A}$ is isomorphic to $\prod_{i\in I}\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$, writing $\mathfrak{C}_{e_i}$ for the principal ideal of $\mathfrak{C}$ generated by e_i and endowed with $\bar{\mu}\restriction\mathfrak{C}_{e_i}$, and $\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$ for the localizable measure algebra free product of $\mathfrak{C}_{e_i}$ and $\mathfrak{B}_i$ – the isomorphism being one which takes any $c\in\mathfrak{C}$ to $\langle(c\cap e_i)\otimes 1\rangle_{i\in I}$;

(iii) there is a single measure-preserving automorphism π of $\mathfrak{A}$ such that

$$\mathfrak{C}=\{c:c\in\mathfrak{A},\,\pi c=c\}.$$

Version of 26.9.08

334 Products

I devote a short section to results on the Maharam classification of the measure algebras of product measures, or, if you prefer, of the free products of measure algebras. The complete classification, even for probability algebras, is complex, so I content myself with a handful of the most useful results. I start with upper bounds for the Maharam type of the c.l.d. product of two measure spaces (334A) and the localizable measure algebra free product of two semi-finite measure algebras (334B), and go on to the corresponding results for general products of probability spaces and algebras (334C-334D). Finally, I show that any infinite power of a probability space is Maharam-type-homogeneous (334E).

In this section I will write $\tau(\mu)$ for the Maharam type of a measure μ, defined as the Maharam type of its measure algebra (331Fc).

334A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and λ the c.l.d. product measure on $X\times Y$. Then $\tau(\lambda)\le\max(\omega,\tau(\mu),\tau(\nu))$.

334B Corollary Let $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{B},\bar{\nu})$ be semi-finite measure algebras, with localizable measure algebra free product $(\mathfrak{C},\bar{\lambda})$. Then $\tau(\mathfrak{C})\le\max(\omega,\tau(\mathfrak{A}),\tau(\mathfrak{B}))$.

334C Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X,Λ,λ). Then

$$\tau(\lambda)\le\max(\omega,\#(I),\sup_{i\in I}\tau(\mu_i)).$$

334D Corollary Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C},\bar{\lambda})$. Then

$$\tau(\mathfrak{C})\le\max(\omega,\#(I),\sup_{i\in I}\tau(\mathfrak{A}_i)).$$

334E Theorem Let (X,Σ,μ) be a probability space and I an infinite set; let λ be the product measure on X^I. Then λ is Maharam-type-homogeneous. If $\tau(\mu)=0$ then $\tau(\lambda)=0$; otherwise $\tau(\lambda)=\max(\tau(\mu),\#(I))$.