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Chapter 33

Maharam's theorem

We are now ready for the astonishing central fact about measure algebras: there are very few of them. Any localizable measure algebra has a canonical expression as a simple product of measure algebras of easily described types. This complete classification necessarily dominates all further discussion of measure algebras; to the point that all the results of Chapter 32 have to be regarded as 'elementary', since however complex their formulation they have been proved by techniques not involving, nor providing, any particular insight into the special nature of measure algebras. The proof depends, of course, on developing methods of defining measure-preserving homomorphisms and isomorphisms; I give a number of results, progressively more elaborate, but all based on the same idea. These techniques are of great power, leading, for instance, to an effective classification of closed subalgebras and their embeddings.

'Maharam's theorem' itself, the classification of localizable measure algebras, is in §332. I devote §331 to the definition and description of 'homogeneous' probability algebras. In §333 I turn to the problem of describing pairs $(\mathfrak{A},\mathfrak{C})$ where $\mathfrak{A}$ is a probability algebra and $\mathfrak{C}$ is a closed subalgebra. Finally, in §334, I give some straightforward results on the classification of free products of probability algebras.

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331 Maharam types and homogeneous measure algebras

I embark directly on the principal theorem of this chapter (331I), split between 331B, 331D and 331I; 331B and 331D will be the basis of many of the results in later sections of this chapter. In 331E-331H I introduce the concepts of 'Maharam type' and 'Maharam-type-homogeneity'. I discuss the measure algebras of products $\{0,1\}^\kappa$, showing that these provide a complete set of examples of Maharam-type-homogeneous probability algebras (331J-331L).

331A Definition The following idea is almost the key to the whole chapter. Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$. A non-zero element a of $\mathfrak{A}$ is a **relative atom** over $\mathfrak{B}$ if every $c \subseteq a$ is of the form $a \cap b$ for some $b \in \mathfrak{B}$; that is, $\{a \cap b : b \in \mathfrak{B}\}$ is the principal ideal generated by a. We say that $\mathfrak{A}$ is **relatively atomless** over $\mathfrak{B}$ if there are no relative atoms in $\mathfrak{A}$ over $\mathfrak{B}$.

(I'm afraid the phrases 'relative atom', 'relatively atomless' are bound to seem opaque at this stage. I hope that after the structure theory of §333 they will seem more natural. For the moment, note only that a is an atom in $\mathfrak{A}$ iff it is a relative atom over the smallest subalgebra $\{0,1\}$, and every element of $\mathfrak{A}$ is a relative atom over the largest subalgebra $\mathfrak{A}$. In a way, a is a relative atom over $\mathfrak{B}$ if its image is an atom in a kind of quotient $\mathfrak{A}/\mathfrak{B}$. But we are two volumes away from any prospect of making sense of this kind of quotient.)

At some point I ought to remark that if a is an atom of $\mathfrak{A}$ then it is surely a relative atom over $\mathfrak{B}$; so if $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$, then it is atomless in the sense of 316Kb.

331B The first lemma is the heart of Maharam's theorem.

Lemma Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$, and $a_0 \in \mathfrak{A}$. Let $\nu : \mathfrak{B} \to \mathbb{R}$ be an additive functional such that $0 \le \nu b \le \bar\mu(b \cap a_0)$ for every $b \in \mathfrak{B}$. Then there is a $c \in \mathfrak{A}$ such that $c \subseteq a_0$ and $\nu b = \bar\mu(b \cap c)$ for every $b \in \mathfrak{B}$.

Remark Recall that by 323H we need not distinguish between 'order-closed' and 'topologically closed' subalgebras.

proof (a) It is worth noting straight away that ν is necessarily countably additive. This is easy to check from first principles, but if you want to trace the underlying ideas they are in 313O (the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is order-continuous), 326Jf (so $\mu\restriction\mathfrak{B}:\mathfrak{B}\to\mathbb{R}$ is countably additive) and 326Kb (therefore ν is countably additive).

(b) For each $a\in\mathfrak{A}$ set $\nu_a b=\bar\mu(b\cap a)$ for every $b\in\mathfrak{B}$; then $\nu_a:\mathfrak{B}\to\mathbb{R}$ is countably additive (326Jd). Note that $\nu_{c\cup d}=\nu_c+\nu_d$ whenever $c.\ d\in\mathfrak{A}$ are disjoint. The key idea is the following fact: for every non-zero $a\in\mathfrak{A}$ there is a non-zero $d\subseteq a$ such that $\nu_d\le\frac{1}{2}\nu_a$. **P** Because $\mathfrak{A}$ is relatively atomless over $\mathfrak{B}$, there is an $e\subseteq a$ such that $e\ne a\cap b$ for any $b\in\mathfrak{B}$. Consider the countably additive functional $\lambda=\nu_a-2\nu_e:\mathfrak{B}\to\mathbb{R}$. By 326M, there is a $b_0\in\mathfrak{B}$ such that $\lambda b\ge 0$ whenever $b\in\mathfrak{B}$ and $b\subseteq b_0$, while $\lambda b\le 0$ whenever $b\in\mathfrak{B}$ and $b\cap b_0=0$.

If $e\cap b_0\ne 0$, try $d=e\cap b_0$. Then $0\ne d\subseteq a$, and for every $b\in\mathfrak{B}$

$$\nu_d b=\nu_e(b\cap b_0)=\frac{1}{2}(\nu_a(b\cap b_0)-\lambda(b\cap b_0))\le\frac{1}{2}\nu_a b$$

(because $\lambda(b\cap b_0)\ge 0$) so $\nu_d\le\frac{1}{2}\nu_a$.

If $e\cap b_0=0$, then (by the choice of e) $e\ne a\cap(1\setminus b_0)$, so $d=a\setminus(e\cup b_0)\ne 0$, and of course $d\subseteq a$. In this case, for every $b\in\mathfrak{B}$,

$$\nu_d b=\nu_a(b\setminus b_0)-\nu_e(b\setminus b_0)=\frac{1}{2}(\lambda(b\setminus b_0)+\nu_a(b\setminus b_0))\le\frac{1}{2}\nu_a b$$

(because $\lambda(b\setminus b_0)\le 0$), so once again $\nu_d\le\frac{1}{2}\nu_a$.

Thus in either case we have a suitable d. **Q**

(c) It follows at once, by induction on n, that if a is any non-zero element of $\mathfrak{A}$ and $n\in\mathbb{N}$ then there is a non-zero $d\subseteq a$ such that $\nu_d\le 2^{-n}\nu_a$.

(d) Now suppose that $a\in\mathfrak{A}$ and that $\lambda:\mathfrak{B}\to[0,\infty[$ is a non-zero countably additive functional such that $\lambda\le\nu_a$. Then there is a non-zero $d\subseteq a$ such that $\nu_d\le\lambda$. **P** Let $b^*\in\mathfrak{B}$ be such that $\lambda b^*>0$; then

$$\lambda(b^*\setminus a)\le\nu_a(b^*\setminus a)=0,$$

so $\lambda b^*=\lambda(b^*\cap a)$. Take $n\in\mathbb{N}$ such that $2^{-n}\nu_a b^*<\lambda b^*$; set $\lambda_1=\lambda-2^{-n}\nu_a$. By 326M (for the second time), there is a $b_1\in\mathfrak{B}$ such that $\lambda_1 b\ge 0$ if $b\subseteq b_1$ and $\lambda_1 b\le 0$ if $b\cap b_1=0$. Set $c=a\cap b_1$. Now

$$2^{-n}\nu_a(a\cap b^*)=2^{-n}\nu_a b^*<\lambda b^*=\lambda(a\cap b^*),$$

so $\lambda_1(a\cap b^*)>0$ and $a\cap b^*\cap b_1\ne 0$ and $c\ne 0$. By (c), we have a non-zero $d\subseteq c$ such that $\nu_d\le 2^{-n}\nu_c$. If $b\in\mathfrak{B}$ then

$$\nu_d b\le 2^{-n}\nu_c b=2^{-n}\nu_a(b\cap b_1)=\lambda(b\cap b_1)-\lambda_1(b\cap b_1)\le\lambda(b\cap b_1)\le\lambda b,$$

so $\nu_d\le\lambda$, as required. **Q**

(e) Let C be the set

$$\{a:a\in\mathfrak{A},\ a\subseteq a_0,\ \nu_a\le\nu\}.$$

Then $0\in C$, so $C\ne\emptyset$. If $D\subseteq C$ is upwards-directed and not empty, then $a=\sup D$ is defined in $\mathfrak{A}$ and included in a_0, and

$$\nu_{\sup D}b=\bar\mu(b\cap\sup D)=\bar\mu(\sup_{d\in D}b\cap d)=\sup_{d\in D}\bar\mu(b\cap d)=\sup_{d\in D}\nu_d b\le\nu b$$

using 313Ba and 321C. So $a\in C$ and is an upper bound for D in C. In particular, any non-empty totally ordered subset of C has an upper bound in C. By Zorn's Lemma, C has a maximal element c say. Of course $c\subseteq a_0$.

(e) ? Suppose, if possible, that $\nu_c\ne\nu$. Set $\lambda=\nu-\nu_c$; note that

$$\lambda b=\nu b-\nu_c b\le\bar\mu(b\cap a_0)-\bar\mu(b\cap c)=\nu_{a_0\setminus c}b$$

for every $b\in\mathfrak{B}$, so $\lambda\le\nu_{a_0\setminus c}$. Because $c\in C$, $\lambda\ge 0$, and we are supposing that $\lambda\ne 0$. By (d), there is a non-zero $d\subseteq a_0\setminus c$ such that $\nu_d\le\lambda$. But now $c\cup d\subseteq a_0$,

$$\nu_{c\cup d}=\nu_c+\nu_d\le\nu_c+\lambda=\nu$$

and $c \cup d \in C$, so c is not maximal in C. **X**

Thus c is an element of $\mathfrak{A}$, included in a_0, giving a representation of ν.

331C Corollary Let $(\mathfrak{A}, \bar{\mu})$ be an atomless semi-finite measure algebra, and $a \in \mathfrak{A}$. Suppose that $0 \le \gamma \le \bar{\mu}a$. Then there is a $c \in \mathfrak{A}$ such that $c \subseteq a$ and $\bar{\mu}c = \gamma$.

proof If $\gamma = \bar{\mu}a$, take $c = a$. If $\gamma < \bar{\mu}a$, there is a $d \in \mathfrak{A}$ such that $d \subseteq a$ and $\gamma \le \bar{\mu}d < \infty$ (322Eb). Apply 331B to the principal ideal $\mathfrak{A}_d$ generated by d, with $a_0 = d$, $\mathfrak{B} = \{0, d\}$ and $\nu d = \gamma$. (The point is that because $\mathfrak{A}$ is atomless, no non-trivial principal ideal of $\mathfrak{A}_d$ can be of the form $\{c \cap b : b \in \mathfrak{B}\} = \{0, c\}$, so $\mathfrak{A}_d$ is relatively atomless over $\{0, d\}$.)

Remark Of course this is also an easy consequence of either 215D or the one-dimensional case of 326H.

331D Lemma Let $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras and $\mathfrak{C} \subseteq \mathfrak{A}$ a closed subalgebra. Suppose that $\pi : \mathfrak{C} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism such that $\mathfrak{B}$ is relatively atomless over $\pi[\mathfrak{C}]$. Take any $a \in \mathfrak{A}$, and let $\mathfrak{C}_1$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a\}$. Then there is a measure-preserving homomorphism from $\mathfrak{C}_1$ to $\mathfrak{B}$ extending π.

proof We know that $\pi[\mathfrak{C}]$ is a closed subalgebra of $\mathfrak{B}$ (324Kb), and that π is a Boolean isomorphism between $\mathfrak{C}$ and $\pi[\mathfrak{C}]$. Consequently the countably additive functional $c \mapsto \bar{\mu}(c \cap a) : \mathfrak{C} \to \mathbb{R}$ is transferred to a countably additive functional $\lambda : \pi[\mathfrak{C}] \to \mathbb{R}$, writing $\lambda(\pi c) = \bar{\mu}(c \cap a)$ for every $c \in \mathfrak{C}$. Of course $\lambda(\pi c) \le \bar{\mu}c = \bar{\nu}(\pi c)$ for every $c \in \mathfrak{C}$. So by 331B there is a $b \in \mathfrak{B}$ such that $\lambda(\pi c) = \bar{\nu}(b \cap \pi c)$ for every $c \in \mathfrak{C}$.

If $c \in \mathfrak{C}$ and $c \subseteq a$ then

$$\bar{\nu}(b \cap \pi c) = \lambda(\pi c) = \bar{\mu}(a \cap c) = \bar{\mu}c = \bar{\nu}(\pi c),$$

so $\pi c \subseteq b$. Similarly, if $a \subseteq c \in \mathfrak{C}$, then

$$\bar{\nu}(b \cap \pi c) = \bar{\mu}(a \cap c) = \bar{\mu}(a \cap 1) = \bar{\nu}(b \cap \pi 1) = \bar{\nu}b,$$

so $b \subseteq \pi c$. It follows from 312O that there is a Boolean homomorphism $\pi_1 : \mathfrak{C}_1 \to \mathfrak{B}$, extending π, such that $\pi_1 a = b$.

To see that π_1 is measure-preserving, take any member of $\mathfrak{C}_1$. By 312N, this is expressible as $e = (c_1 \cap a) \cup (c_2 \setminus a)$, where c_1, $c_2 \in \mathfrak{C}$. Now

$$\begin{aligned}\bar{\nu}(\pi_1 e) &= \bar{\nu}((\pi c_1 \cap b) \cup (\pi c_2 \setminus b)) = \bar{\nu}(\pi c_1 \cap b) + \bar{\nu}(\pi c_2) - \bar{\nu}(\pi c_2 \cap b) \\ &= \bar{\mu}(c_1 \cap a) + \bar{\mu}c_2 - \bar{\mu}(c_2 \cap a) = \bar{\mu}e.\end{aligned}$$

As e is arbitrary, π_1 is measure-preserving.

331E Generating sets For the sake of the next definition, we need a language a little more precise than I have felt the need to use so far. The point is that if $\mathfrak{A}$ is a Boolean algebra and B is a subset of $\mathfrak{A}$, there is more than one subalgebra of $\mathfrak{A}$ which can be said to be 'generated' by B, because we can look at any of the three algebras

– $\mathfrak{B}$, the smallest subalgebra of $\mathfrak{A}$ including B;

– $\mathfrak{B}_\sigma$, the smallest σ-subalgebra of $\mathfrak{A}$ including B;

– $\mathfrak{B}_\tau$, the smallest order-closed subalgebra of $\mathfrak{A}$ including B.

(See 313Fb.) Now I will say henceforth, in this context, that

– $\mathfrak{B}$ is the subalgebra of $\mathfrak{A}$ generated by B, and B **generates** $\mathfrak{A}$ if $\mathfrak{A} = \mathfrak{B}$;

– $\mathfrak{B}_\sigma$ is the σ-subalgebra of $\mathfrak{A}$ generated by B, and B **σ-generates** $\mathfrak{A}$ if $\mathfrak{A} = \mathfrak{B}_\sigma$;

– $\mathfrak{B}_\tau$ is the order-closed subalgebra of $\mathfrak{A}$ generated by B, and B **τ-generates** or **completely generates** $\mathfrak{A}$ if $\mathfrak{A} = \mathfrak{B}_\tau$.

There is a danger inherent in these phrases, because if we have $B \subseteq \mathfrak{A}'$, where $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$, it is possible that the smallest order-closed subalgebra of $\mathfrak{A}'$ including B might not be recoverable from the smallest order-closed subalgebra of $\mathfrak{A}$ including B. (See 331Yb-331Yc.) This problem will not seriously interfere with the ideas below; but for definiteness let me say that the phrases 'B σ-generates $\mathfrak{A}$', 'B τ-generates $\mathfrak{A}$' will always refer to suprema and infima taken in $\mathfrak{A}$ itself, not in any larger algebra in which it may be embedded.

331F Maharam types (a) With the language of 331E established, I can now define the **Maharam type** or **complete generation number** $\tau(\mathfrak{A})$ of any Boolean algebra $\mathfrak{A}$; it is the smallest cardinal of any subset of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$.

(I think that this is the first 'cardinal function' which I have mentioned in this treatise. All you need to know, to confirm that the definition is well-conceived, is that there is *some* set which τ-generates $\mathfrak{A}$; and obviously $\mathfrak{A}$ τ-generates itself. For this means that (assuming the axiom of choice) the set $A = \{\#(B) : B \subseteq \mathfrak{A}\ \tau\text{-generates } \mathfrak{A}\}$ is a non-empty class of cardinals, and therefore has a least member (2A1Lf). In 331Ye-331Yf I mention a further function, the 'density' of a topological space, which is closely related to Maharam type.)

(b) A Boolean algebra $\mathfrak{A}$ is **Maharam-type-homogeneous** if $\tau(\mathfrak{A}_a) = \tau(\mathfrak{A})$ for every non-zero $a \in \mathfrak{A}$, writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a.

(c) Let (X, Σ, μ) be a measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Then the **Maharam type** of (X, Σ, μ), or of μ, is the Maharam type of $\mathfrak{A}$; and (X, Σ, μ), or μ, is **Maharam-type-homogeneous** if $\mathfrak{A}$ is.

Remark I should perhaps remark that the phrases 'Maharam type' and 'Maharam-type-homogeneous', while well established in the context of probability algebras, are not in common use for general Boolean algebras. But the cardinal $\tau(\mathfrak{A})$ is important in the general context, and is such an obvious extension of Maharam's idea (Maharam 1942) that I am happy to propose this extension of terminology.

331G For the sake of those who have not mixed set theory and algebra before, I had better spell out some basic facts.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, B a subset of $\mathfrak{A}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by B, $\mathfrak{B}_\sigma$ the σ-subalgebra of $\mathfrak{A}$ generated by B, and $\mathfrak{B}_\tau$ the order-closed subalgebra of $\mathfrak{A}$ generated by B.

(a) $\mathfrak{B} \subseteq \mathfrak{B}_\sigma \subseteq \mathfrak{B}_\tau$.

(b) If B is finite, so is $\mathfrak{B}$, and in this case $\mathfrak{B} = \mathfrak{B}_\sigma = \mathfrak{B}_\tau$.

(c) For every $a \in \mathfrak{B}$, there is a finite $B' \subseteq B$ such that a belongs to the subalgebra of $\mathfrak{A}$ generated by B'. Consequently $\#(\mathfrak{B}) \le \max(\omega, \#(B))$.

(d) For every $a \in \mathfrak{B}_\sigma$, there is a countable $B' \subseteq B$ such that a belongs to the σ-subalgebra of $\mathfrak{A}$ generated by B'.

(e) If $\mathfrak{A}$ is ccc, then $\mathfrak{B}_\sigma = \mathfrak{B}_\tau$.

proof (a) All we need to know is that $\mathfrak{B}_\sigma$ is a subalgebra of $\mathfrak{A}$ including B, and that $\mathfrak{B}_\tau$ is a σ-subalgebra of $\mathfrak{A}$ including B.

(b) Induce on $\#(B)$, using 312N for the inductive step, to see that $\mathfrak{B}$ is finite. In this case it must be order-closed, so is equal to $\mathfrak{B}_\tau$.

(c)(i) For $I \subseteq B$, let $\mathfrak{C}_I$ be the subalgebra of $\mathfrak{A}$ generated by I. If I, $J \subseteq B$ then $\mathfrak{C}_I \cup \mathfrak{C}_J \subseteq \mathfrak{C}_{I\cup J}$. So $\bigcup\{\mathfrak{C}_I : I \subseteq B \text{ is finite}\}$ is a subalgebra of $\mathfrak{A}$, and must be equal to $\mathfrak{B}$, as claimed.

(ii) To estimate the size of $\mathfrak{B}$, recall that the set $[B]^{<\omega}$ of all finite subsets of B has cardinal at most $\max(\omega, \#(B))$ (3A1Cd). For each $I \in [B]^{<\omega}$, $\mathfrak{C}_I$ is finite, so

$$\#(\mathfrak{B}) = \#(\bigcup_{I\in[B]^{<\omega}} \mathfrak{C}_I) \le \max(\omega, \#(I), \sup_{I\in[B]^{<\omega}} \#(\mathfrak{C}_I)) \le \max(\omega, \#(B))$$

by 3A1Cc.

(d) For $I \subseteq B$, let $\mathfrak{D}_I \subseteq \mathfrak{B}_\sigma$ be the σ-subalgebra of $\mathfrak{A}$ generated by I. If I, $J \subseteq B$ then $\mathfrak{D}_I \cup \mathfrak{D}_J \subseteq \mathfrak{D}_{I\cup J}$, so $\mathfrak{B}'_\sigma = \bigcup\{\mathfrak{D}_I : I \subseteq B \text{ is countable}\}$ is a subalgebra of $\mathfrak{A}$. But also it is sequentially order-closed in $\mathfrak{A}$. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{B}'_\sigma$ with supremum a in $\mathfrak{A}$. For each $n \in \mathbb{N}$ there is a countable $I(n) \subseteq B$ such that $a_n \in \mathfrak{C}_{I(n)}$. Set $K = \bigcup_{n\in\mathbb{N}} I(n)$; then K is a countable subset of B and every a_n belongs to $\mathfrak{D}_K$, so $a \in \mathfrak{D}_K \subseteq \mathfrak{B}'_\sigma$. **Q** So $\mathfrak{B}'_\sigma$ is a σ-subalgebra of $\mathfrak{A}$ including B and must be the whole of $\mathfrak{B}_\sigma$.

(e) By 316Fb, $\mathfrak{B}_\sigma$ is order-closed in $\mathfrak{A}$, so must be equal to $\mathfrak{B}_\tau$.

331H Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a)(i) $\tau(\mathfrak{A}) = 0$ iff $\mathfrak{A}$ is either $\{0\}$ or $\{0,1\}$.

(ii) $\tau(\mathfrak{A})$ is finite iff $\mathfrak{A}$ is finite.

(b) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

(c) If $a \in \mathfrak{A}$ then $\tau(\mathfrak{A}_a) \le \tau(\mathfrak{A})$, where $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a.

(d) If $\mathfrak{A}$ has an atom and is Maharam-type-homogeneous, then $\mathfrak{A} = \{0,1\}$.

proof (a)(i) $\tau(\mathfrak{A}) = 0$ iff $\mathfrak{A}$ has no proper subalgebras.

(ii) If $\mathfrak{A}$ is finite, then $\tau(\mathfrak{A}) \le \#(\mathfrak{A})$ is finite. If $\tau(\mathfrak{A})$ is finite, then there is a finite set $B \subseteq \mathfrak{A}$ which τ-generates $\mathfrak{A}$; by 331Gb, $\mathfrak{A}$ is finite.

(b) We know that there is a set $A \subseteq \mathfrak{A}$, τ-generating $\mathfrak{A}$, with $\#(A) = \tau(\mathfrak{A})$. Now $\pi[A]$ τ-generates $\pi[\mathfrak{A}] = \mathfrak{B}$ (313Mb), so

$$\tau(\mathfrak{B}) \le \#(\pi[A]) \le \#(A) = \tau(\mathfrak{A}).$$

(c) Apply (b) to the map $b \mapsto a \cap b : \mathfrak{A} \to \mathfrak{A}_a$.

(d) If $a \in \mathfrak{A}$ is an atom, then $\tau(\mathfrak{A}_a) = 0$, so if $\mathfrak{A}$ is Maharam-type-homogeneous then $\tau(\mathfrak{A}) = 0$ and $\mathfrak{A} = \{0, a\} = \{0,1\}$.

331I We are now ready for the theorem.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be Maharam-type-homogeneous measure algebras of the same Maharam type, with $\bar{\mu}1 = \bar{\nu}1 < \infty$. Then they are isomorphic as measure algebras.

proof (a) If $\tau(\mathfrak{A}) = \tau(\mathfrak{B}) = 0$, this is trivial. So let us take $\kappa = \tau(\mathfrak{A}) = \tau(\mathfrak{B}) > 0$. In this case, because $\mathfrak{A}$ and $\mathfrak{B}$ are Maharam-type-homogeneous, they can have no atoms and must be infinite, so κ is infinite (331H). Let $\langle a_\xi \rangle_{\xi<\kappa}$ and $\langle b_\xi \rangle_{\xi<\kappa}$ enumerate τ-generating subsets of $\mathfrak{A}$, $\mathfrak{B}$ respectively.

The strategy of the proof is to define a measure-preserving isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ as the last of an increasing family $\langle \pi_\xi \rangle_{\xi \le \kappa}$ of isomorphisms between closed subalgebras $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ of $\mathfrak{A}$ and $\mathfrak{B}$. The inductive hypothesis will be that, for some families $\langle a'_\xi \rangle_{\xi<\kappa}$, $\langle b'_\xi \rangle_{\xi<\kappa}$ to be determined,

$\mathfrak{C}_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta : \eta < \xi\} \cup \{a'_\eta : \eta < \xi\}$,

$\mathfrak{D}_\xi$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta < \xi\}$,

$\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{D}_\xi$ is a measure-preserving isomorphism,

π_ξ extends π_η whenever $\eta < \xi$.

(Formally speaking, this will be a transfinite recursion, defining a function $\xi \mapsto f(\xi) = (\mathfrak{C}_\xi, \mathfrak{D}_\xi, \pi_\xi, a'_\xi, b'_\xi)$ on the ordinal $\kappa + 1$ by a rule which chooses $f(\xi)$ in terms of $f \restriction \xi$, as described in 2A1B. The construction of an actual function F for which $f(\xi) = F(f \restriction \xi)$ will necessitate the axiom of choice.)

(b) The induction starts with $\mathfrak{C}_0 = \{0,1\}$, $\mathfrak{D}_0 = \{0,1\}$, $\pi_0(0) = 0$, $\pi_0(1) = 1$. (The hypothesis $\bar{\mu}1 = \bar{\nu}1$ is what we need to ensure that π_0 is measure-preserving.)

(c) For the inductive step to a successor ordinal $\xi + 1$, where $\xi < \kappa$, suppose that $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ and π_ξ have been defined.

(i) For any non-zero $b \in \mathfrak{B}$, the principal ideal $\mathfrak{B}_b$ of $\mathfrak{B}$ generated by b has Maharam type κ, because $\mathfrak{B}$ is Maharam-type-homogeneous. On the other hand, the Maharam type of $\mathfrak{D}_\xi$ is at most

$$\#(\{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta < \xi\}) \le \#(\xi \times \{0,1\}) < \kappa,$$

because if ξ is finite so is $\xi \times \{0,1\}$, while if ξ is infinite then $\#(\xi \times \{0,1\}) = \#(\xi) \le \xi < \kappa$. Consequently $\mathfrak{B}_b$ cannot be an order-continuous image of $\mathfrak{D}_\xi$ (331Hb). Now the map $c \mapsto c \cap b : \mathfrak{D}_\xi \to \mathfrak{B}_b$ is order-continuous, because $\mathfrak{D}_\xi$ is closed, so that the embedding $\mathfrak{D}_\xi \subsetneqq \mathfrak{B}$ is order-continuous. It therefore cannot be surjective, and

$$\{b \cap \pi_\xi a : a \in \mathfrak{C}_\xi\} = \{b \cap d : d \in \mathfrak{D}_\xi\} \ne \mathfrak{B}_b.$$

This means that $\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{D}_\xi$ satisfies the conditions of 331D, and must have an extension ϕ_ξ to a measure-preserving homomorphism from the subalgebra $\mathfrak{C}'_\xi$ of $\mathfrak{A}$ generated by $\mathfrak{C}_\xi \cup \{a_\xi\}$ to $\mathfrak{B}$. We know that $\mathfrak{C}'_\xi$ is a closed subalgebra of $\mathfrak{A}$ (314Ja), so it must be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta : \eta \le \xi\} \cup \{a'_\eta : \eta < \xi\}$. Also $\mathfrak{D}'_\xi = \phi_\xi[\mathfrak{C}'_\xi]$ will be the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}_\xi \cup \{b'_\xi\}$, where $b'_\xi = \phi_\xi(a_\xi)$, so is closed in $\mathfrak{B}$, and is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta \le \xi\}$.

(ii) The next step is to repeat the whole of the argument above, but applying it to $\phi_\xi^{-1} : \mathfrak{D}'_\xi \to \mathfrak{C}_\xi$, b_ξ in place of $\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{D}_\xi$ and a_ξ. Once again, we have $\tau(\mathfrak{D}'_\xi) < \kappa = \tau(\mathfrak{A}_a)$ for every $a \in \mathfrak{A}$, so we can use Lemma 331D to find a measure-preserving isomorphism $\psi_\xi : \mathfrak{D}_{\xi+1} \to \mathfrak{C}_{\xi+1}$ extending ϕ_ξ^{-1}, where $\mathfrak{D}_{\xi+1}$ is the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}'_\xi \cup \{b_\xi\}$, and $\mathfrak{C}_{\xi+1}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}'_\xi \cup \{a'_\xi\}$, setting $a'_\xi = \psi_\xi(b_\xi)$. As in (i), we find that $\mathfrak{C}_{\xi+1}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta : \eta \le \xi\} \cup \{a'_\eta : \eta \le \xi\}$, while $\mathfrak{D}_{\xi+1}$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{b_\eta : \eta \le \xi\} \cup \{b'_\eta : \eta \le \xi\}$.

(iii) We can therefore take $\pi_{\xi+1} = \psi_\xi^{-1} : \mathfrak{C}_{\xi+1} \to \mathfrak{D}_{\xi+1}$, and see that $\pi_{\xi+1}$ is a measure-preserving isomorphism, extending π_ξ, such that $\pi_{\xi+1}(a_\xi) = b'_\xi$ and $\pi_{\xi+1}(a'_\xi) = b_\xi$. Evidently $\pi_{\xi+1}$ extends π_η for every $\eta \le \xi$ because it extends π_ξ and (by the inductive hypothesis) π_ξ extends π_η for every $\eta < \xi$.

(d) For the inductive step to a limit ordinal ξ, where $0 < \xi \le \kappa$, suppose that $\mathfrak{C}_\eta$, $\mathfrak{D}_\eta$, a'_η, b'_η, π_η have been defined for $\eta < \xi$. Set $\mathfrak{C}^*_\xi = \bigcup_{\eta<\xi} \mathfrak{C}_\xi$. Then $\mathfrak{C}^*_\xi$ is a subalgebra of $\mathfrak{A}$, because it is the union of an upwards-directed family of subalgebras; similarly, $\mathfrak{D}^*_\xi = \bigcup_{\eta<\xi} \mathfrak{D}_\xi$ is a subalgebra of $\mathfrak{B}$. Next, we have a function $\pi^*_\xi : \mathfrak{C}^*_\xi \to \mathfrak{D}^*_\xi$ defined by setting $\pi^*_\xi a = \pi_\eta a$ whenever $\eta < \xi$ and $a \in \mathfrak{C}_\eta$; for if η, $\zeta < \xi$ and $a \in \mathfrak{C}_\eta \cap \mathfrak{C}_\zeta$, then $\pi_\eta a = \pi_{\max(\eta,\zeta)} a = \pi_\zeta a$. Clearly

$$\pi^*_\xi[\mathfrak{C}^*_\xi] = \bigcup_{\eta<\xi} \pi_\eta[\mathfrak{C}_\eta] = \mathfrak{D}^*_\xi.$$

Moreover, $\bar\nu(\pi^*_\xi a) = \bar\mu a$ for every $a \in \mathfrak{C}^*_\xi$, since $\bar\nu(\pi_\eta a) = \bar\mu a$ whenever $\eta < \xi$ and $a \in \mathfrak{C}_\eta$.

Now let $\mathfrak{C}_\xi$ be the smallest closed subalgebra of $\mathfrak{A}$ including $\mathfrak{C}^*_\xi$, that is, the topological closure of $\mathfrak{C}^*_\xi$ in $\mathfrak{A}$ (323J). Since $\mathfrak{C}_\xi$ is the smallest closed subalgebra of $\mathfrak{A}$ including $\mathfrak{C}_\eta$ for every $\eta < \xi$, it must be the closed subalgebra of $\mathfrak{A}$ generated by $\{a_\eta : \eta < \xi\} \cup \{a'_\eta : \eta < \xi\}$. By 324O, π^*_ξ has an extension to a measure-preserving homomorphism $\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{B}$. Set $\mathfrak{D}_\xi = \pi_\xi[\mathfrak{C}_\xi]$; by 324Kb again, $\mathfrak{D}_\xi$ is a closed subalgebra of $\mathfrak{B}$. Because $\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{B}$ is continuous (also noted in 324Kb),

$$\mathfrak{D}^*_\xi = \pi^*_\xi[\mathfrak{C}^*_\xi] = \pi_\xi[\mathfrak{C}^*_\xi]$$

is topologically dense in $\mathfrak{D}_\xi$ (3A3Eb), and $\mathfrak{D}_\xi = \overline{\mathfrak{D}^*_\xi}$ is the closed subalgebra of $\mathfrak{B}$-generated by $\{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta < \xi\}$. Finally, if $\eta < \xi$, π_ξ extends π_η because π^*_ξ extends π_η. Thus the induction continues.

(e) The induction ends with $\xi = \kappa$, $\mathfrak{C}_\kappa = \mathfrak{A}$, $\mathfrak{D}_\kappa = \mathfrak{B}$ and $\pi = \pi_\kappa : \mathfrak{A} \to \mathfrak{B}$ the required measure algebra isomorphism.

331J Lemma Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, and κ an infinite cardinal.

(a) If there is a family $\langle a_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{A}$ such that $\inf_{\xi\in I} a_\xi = 0$ and $\sup_{\xi\in I} a_\xi = 1$ for every infinite $I \subseteq \kappa$, then $\tau(\mathfrak{A}_d) \ge \kappa$ for every non-zero $d \in \mathfrak{A}$.

(b) Let ν_κ be the usual measure on $\{0,1\}^\kappa$ (254J) and $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ its measure algebra. If there is an order-continuous Boolean homomorphism from $\mathfrak{B}_\kappa$ to $\mathfrak{A}$, $\tau(\mathfrak{A}_d) \ge \kappa$ for every non-zero $d \in \mathfrak{A}$.

proof (a)(i) To begin with (down to the end of (iii)), let us take it that $d = 1$. For $a \in \mathfrak{A}$, $\delta > 0$ set $U(a, \delta) = \{a' : \bar\mu(a' \triangle a) < \delta\}$, the ordinary open δ-neighbourhood of a. If $a \in \mathfrak{A}$, then there is a $\delta > 0$ such that $\{\xi : \xi < \kappa, a_\xi \in U(a, \delta)\}$ is finite. **P?** Suppose, if possible, otherwise. Then there is a sequence $\langle \xi_n \rangle_{n\in\mathbb{N}}$ of distinct elements of κ such that $\bar\mu(a \triangle a_{\xi_n}) \le 2^{-n-2}\bar\mu 1$ for every $n \in \mathbb{N}$. Now $\inf_{n\in\mathbb{N}} a_{\xi_n} = 0$, so

$$\bar\mu a = \bar\mu(a \setminus \inf_{n\in\mathbb{N}} a_{\xi_n}) \le \sum_{n=0}^\infty \bar\mu(a \setminus a_{\xi_n}).$$

Similarly

$$\bar\mu(1 \setminus a) = \bar\mu(\sup_{n\in\mathbb{N}} a_{\xi_n} \setminus a) \le \sum_{n=0}^\infty \bar\mu(a_{\xi_n} \setminus a).$$

Putting these together,

$$\bar\mu 1 = \bar\mu a + \bar\mu(1\setminus a) \le \sum_{n=0}^{\infty}\bar\mu(a\setminus a_{\xi_n}) + \sum_{n=0}^{\infty}\bar\mu(a_{\xi_n}\setminus a)$$
$$= \sum_{n=0}^{\infty}\bar\mu(a \triangle a_{\xi_n}) \le \sum_{n=0}^{\infty} 2^{-n-2}\bar\mu 1 < \bar\mu 1,$$

which is impossible. **XQ**

(ii) Note that $\mathfrak{A}$ is infinite; for if $a\in\mathfrak{A}$ the set $\{\xi : a_\xi = a\}$ must be finite, and κ is supposed to be infinite. So $\tau(\mathfrak{A})$ must be infinite.

(iii) Now take a set $C\subseteq\mathfrak{A}$, with cardinal $\tau(\mathfrak{A})$, which τ-generates $\mathfrak{A}$. By (ii), C is infinite. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by C; then $\#(\mathfrak{C}) = \#(C) = \tau(\mathfrak{A})$, by 331Gc, and $\mathfrak{C}$ is topologically dense in $\mathfrak{A}$ (323J again). If $a\in\mathfrak{A}$, there are $c\in\mathfrak{C}$ and $k\in\mathbb{N}$ such that $a\in U(c,2^{-k})$ and $\{\xi : a_\xi \in U(c,2^{-k})\}$ is finite. **P** By (b), there is a $\delta>0$ such that $\{\xi : a_\xi\in U(a,\delta)\}$ is finite. Take $k\in\mathbb{N}$ such that $2\cdot 2^{-k}\le\delta$, and $c\in\mathfrak{C}\cap U(a,2^{-k})$; then $U(c,2^{-k})\subseteq U(a,\delta)$ can contain only finitely many a_ξ, so these c, k serve. **Q**

Consider

$$\mathcal{U} = \{U(c,2^{-k}) : c\in\mathfrak{C},\, k\in\mathbb{N},\, \{\xi : a_\xi\in U(c,2^{-k})\}\text{ is finite}\}.$$

Then $\#(\mathcal{U}) \le \max(\#(\mathfrak{C}),\omega) = \tau(\mathfrak{A})$. Also $\mathcal{U}$ is a cover of $\mathfrak{A}$. In particular, $\kappa = \bigcup_{U\in\mathcal{U}} J_U$, where $J_U = \{\xi : a_\xi\in U\}$. But this means that

$$\kappa = \#(\kappa) \le \max(\omega, \#(\mathcal{U}), \sup_{U\in\mathcal{U}}\#(J_U)) = \tau(\mathfrak{A}).$$

This proves the result when $d=1$.

(iv) For the general case, given $d\in\mathfrak{A}\setminus\{0\}$, set $a'_\xi = a_\xi\cap d$ for each ξ. Since $\inf_{\xi\in I} a_\xi\cap d = 0$ and $\sup_{\xi\in I} a_\xi\cap d = d$ for every infinite $I\subseteq\kappa$, we can apply (i)-(iii) to $(\mathfrak{A}_d, \bar\mu\restriction\mathfrak{A}_d, \langle a'_\xi\rangle_{\xi<\kappa})$ to see that $\tau(\mathfrak{A}_d)\ge\kappa$, as required.

(b) Let $\pi:\mathfrak{B}_\kappa\to\mathfrak{A}$ be an order-continuous Boolean homomorphism. Set $E_\xi = \{x : x\in\{0,1\}^\kappa,\, x(\xi)=1\}$, $e_\xi = E_\xi^\bullet\in\mathfrak{B}_\kappa$ and $a_\xi = \pi e_\xi\in\mathfrak{A}$ for each $\xi<\kappa$. If $\langle\xi_n\rangle_{n\in\mathbb{N}}$ is any sequence of distinct elements of κ,

$$\nu_\kappa(\bigcap_{n\in\mathbb{N}} E_{\xi_n}) = \lim_{n\to\infty}\nu_\kappa(\bigcap_{i\le n} E_{\xi_n}) = \lim_{n\to\infty} 2^{-n-1} = 0,$$

so that $\bar\nu_\kappa(\inf_{n\in\mathbb{N}} e_{\xi_n}) = 0$ and $\inf_{n\in\mathbb{N}} e_{\xi_n} = 0$. Because π is order-continuous, $\inf_{n\in\mathbb{N}} a_{\xi_n} = 0$ in $\mathfrak{A}$. Similarly, $\nu_\kappa(\bigcup_{n\in\mathbb{N}} E_{\xi_n}) = 1$ and $\sup_{n\in\mathbb{N}} a_{\xi_n} = 1$. As $\langle\xi_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\inf_{\xi\in I} a_\xi = 0$ and $\sup_{\xi\in I} a_\xi = 1$ for every infinite $I\subseteq\kappa$. So we can apply (a) to get the result.

331K Theorem Let κ be any infinite cardinal. Let ν_κ be the usual measure on $\{0,1\}^\kappa$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ its measure algebra. Then $\mathfrak{B}_\kappa$ is Maharam-type-homogeneous, with Maharam type κ.

proof Set $X=\{0,1\}^\kappa$ and write Σ for the domain of ν_κ.

(a) To see that $\tau(\mathfrak{B}_\kappa)\le\kappa$, set $E_\xi = \{x : x\in X,\, x(\xi)=1\}$ and $e_\xi = E_\xi^\bullet$ for each $\xi<\kappa$. Writing $\mathcal{E}$ for the algebra of subsets of X generated by $\{E_\xi : \xi<\kappa\}$, we see that every measurable cylinder in X, as defined in 254Aa, belongs to $\mathcal{E}$, so that every member of Σ is approximated, in measure, by members of $\mathcal{E}$ (254Fe), that is, $\{E^\bullet : E\in\mathcal{E}\}$ is topologically dense in $\mathfrak{A}$. But this means just that the subalgebra $\mathfrak{C}$ of $\mathfrak{B}_\kappa$ generated by $\{e_\xi : \xi<\kappa\}$ is topologically dense in $\mathfrak{B}_\kappa$, so that $\{e_\xi : \xi<\kappa\}$ τ-generates $\mathfrak{B}_\kappa$, and $\tau(\mathfrak{B}_\kappa)\le\kappa$.

(b) Next, if $c\in\mathfrak{B}_\kappa\setminus\{0\}$ and $(\mathfrak{B}_\kappa)_c$ is the principal ideal of $\mathfrak{B}_\kappa$ generated by c, the map $b\mapsto b\cap c$ is an order-continuous Boolean homomorphism from $\mathfrak{B}_\kappa$ to $(\mathfrak{B}_\kappa)_c$, so by 331Jb we must have $\tau((\mathfrak{B}_\kappa)_c)\ge\kappa$. Thus

$$\kappa \le \tau((\mathfrak{B}_\kappa)_c) \le \tau(\mathfrak{B}_\kappa) \le \kappa.$$

As c is arbitrary, $\mathfrak{B}_\kappa$ is Maharam-type-homogeneous of Maharam type κ.

331L Theorem Let $(\mathfrak{A},\bar\mu)$ be a Maharam-type-homogeneous probability algebra. Then there is exactly one κ, either 0 or an infinite cardinal, such that $(\mathfrak{A},\bar\mu)$ is isomorphic, as measure algebra, to the measure algebra $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ of the usual measure on $\{0,1\}^\kappa$.

proof If $\tau(\mathfrak{A})$ is finite, it is zero, and $\mathfrak{A} = \{0,1\}$ (331Ha, 331Hd) so that (interpreting $\{0,1\}^0$ as $\{\emptyset\}$) we have the case $\kappa = 0$. If $\kappa = \tau(\mathfrak{A})$ is infinite, then by 331K we know that $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ also is Maharam-type-homogeneous with Maharam type κ, so 331I gives the required isomorphism. Of course κ is uniquely defined by $\mathfrak{A}$.

331M Homogeneous Boolean algebras Of course a homogeneous Boolean algebra (definition: 316N) must be Maharam-type-homogeneous, since $\tau(\mathfrak{A}) = \tau(\mathfrak{A}_c)$ whenever $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_c$. In general, a Boolean algebra can be Maharam-type-homogeneous without being homogeneous (331Xj, 331Yg). But for σ-finite measure algebras this doesn't happen.

331N Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a Maharam-type-homogeneous σ-finite measure algebra. Then it is homogeneous as a Boolean algebra.

proof If $\mathfrak{A} = \{0\}$ this is trivial; so suppose that $\mathfrak{A} \neq \{0\}$. By 322G, there is a measure $\bar{\nu}$ on $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{\nu})$ is a probability algebra. Now let c be any non-zero member of $\mathfrak{A}$, and set $\gamma = \bar{\nu}c$, $\bar{\nu}_c' = \gamma^{-1}\bar{\nu}_c$, where $\bar{\nu}_c$ is the restriction of $\bar{\nu}$ to the principal ideal $\mathfrak{A}_c$ of $\mathfrak{A}$ generated by c. Then $(\mathfrak{A}, \bar{\nu})$ and $(\mathfrak{A}_c, \bar{\nu}_c')$ are Maharam-type-homogeneous probability algebras of the same Maharam type, so are isomorphic as measure algebras, and *a fortiori* as Boolean algebras.

331O I will wait until Chapter 52 of Volume 5 for a systematic discussion of properties of measure algebras which depend on their Maharam types. There are however a couple of results which are easy, useful and expressible in terms already introduced.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra with countable Maharam type. Then $\mathfrak{A}$ is separable in its measure-algebra topology.

proof Let $B \subseteq \mathfrak{A}$ be a countable set which τ-generates $\mathfrak{A}$. Then the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by B is countable (331Gc). Now $\mathfrak{B}$ is dense for the measure-algebra topology. **P** Let G be a non-empty open subset of $\mathfrak{A}$, and c any element of G. Let $\mathrm{P} = \{\rho_a : a \in \mathfrak{A}^f\}$ be the upwards-directed family of pseudometrics defining the topology of $\mathfrak{A}$, as described in 323A. Then there must be an $a \in \mathfrak{A}^f$ and an $\epsilon > 0$ such that $\{b : \rho_a(b,c) \le \epsilon\} \subseteq G$. Let $\mathfrak{C}$ be the order-closed subalgebra of the principal ideal $\mathfrak{A}_a$ generated by $\mathfrak{B}_a = \{b \cap a : b \in \mathfrak{B}\}$. Because $b \mapsto b \cap a : \mathfrak{A} \to \mathfrak{A}_a$ is an order-continuous Boolean homomorphism, $\{b : b \in \mathfrak{A}, \, b \cap a \in \mathfrak{C}\}$ is an order-closed subalgebra of $\mathfrak{A}$, and must be the whole of $\mathfrak{A}$, because it includes B. So $\mathfrak{C} = \mathfrak{A}_a$. By 323J, $\mathfrak{C}$ is the topological closure of $\mathfrak{B}_a$ in $\mathfrak{A}_a$, and there must be a $b \in \mathfrak{B}_a$ such that $\bar{\mu}(b \triangle (c \cap a)) \le \epsilon$; that is, there is a $b \in \mathfrak{B}$ such that $\bar{\mu}(a \cap (b \triangle c)) \le \epsilon$ and $b \in G$. Thus $\mathfrak{B}$ meets G; as G is arbitrary, $\mathfrak{B}$ is dense. **Q**

So $\mathfrak{B}$ is a countable dense subset of $\mathfrak{A}$ and $\mathfrak{A}$ is separable.

331P Proposition Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra of countable Maharam type. Then it is isomorphic to the measure algebras of the usual measure on $\{0,1\}^{\mathbb{N}}$ and of Lebesgue measure on $[0,1]$.

proof The point is that $\mathfrak{A}$ is Maharam-type-homogeneous. **P** For any non-zero $a \in \mathfrak{A}$, $\mathfrak{A}_a$ is atomless, so must be infinite, and $\tau(\mathfrak{A}_a) \ge \omega$ (331H(a-ii)); as also $\tau(\mathfrak{A}_a) \le \tau(\mathfrak{A})$ (331Hc), and we are supposing that $\tau(\mathfrak{A}) \le \omega$, we have $\tau(\mathfrak{A}_a) = \tau(\mathfrak{A})$. **Q** So $(\mathfrak{A}, \bar{\mu}) \cong (\mathfrak{B}_\kappa, \nu_\kappa)$ for some κ (331L); but as $\tau(\mathfrak{B}_\kappa) = \kappa$ (331K), $\kappa = \omega$ and $(\mathfrak{A}, \bar{\mu})$ is isomorphic to the measure algebra of the usual measure on $\{0,1\}^{\mathbb{N}}$.

Since Lebesgue measure on $[0,1]$ is isomorphic to the usual measure on $\{0,1\}^{\mathbb{N}}$ (254K), they surely have isomorphic measure algebras.

331X Basic exercises (a) Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ such that for any non-negligible $E \in \Sigma$ there is an $F \in \Sigma$ such that $F \subseteq E$ and $\mu(F \triangle (E \cap H)) > 0$ for every $H \in \mathrm{T}$. Suppose that $f : X \to [0,1]$ is a measurable function. Show that there is an $F \in \Sigma$ such that $\int_H f = \mu(H \cap F)$ for every $H \in \mathrm{T}$.

>(b) Write out a direct proof of 331C not relying on 331B or 321J.

(c) Let $\mathfrak{A}$ be a finite Boolean algebra with n atoms. Show that $\tau(\mathfrak{A})$ is the least k such that $n \le 2^k$.

>(d) Show that the measure algebra of Lebesgue measure on $\mathbb{R}$ is Maharam-type-homogeneous with Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty,q]^\bullet : q \in \mathbb{Q}\}$.)

(e) Show that the measure algebra of Lebesgue measure on $\mathbb{R}^r$ is Maharam-type-homogeneous with Maharam type ω, for any $r \ge 1$. (*Hint*: show that it is τ-generated by $\{]-\infty,q]^\bullet : q \in \mathbb{Q}^r\}$.)

(f) Show that the measure algebra of any Radon measure on $\mathbb{R}^r$ (256Ad) has countable Maharam type. (*Hint*: show that it is τ-generated by $\{]-\infty,q]^\bullet : q \in \mathbb{Q}^r\}$.)

>(g) Show that $\mathcal{P}\mathbb{R}$ has Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty,q] : q \in \mathbb{Q}\}$.)

>(h) Show that the regular open algebra of $\mathbb{R}$ is Maharam-type-homogeneous with Maharam type ω. (*Hint*: show that it is τ-generated by $\{]-\infty,q]^\bullet : q \in \mathbb{Q}\}$.)

(i) Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, and κ an infinite cardinal. Suppose that there is a family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}$ such that $\inf_{\xi\in I} a_\xi = 0$, $\sup_{\xi\in I} a_\xi = 1$ for every infinite $I \subseteq \kappa$. Show that $\tau(\mathfrak{A}_a) \ge \kappa$ for every non-zero principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$.

(j) Let $\mathfrak{A}$ be the measure algebra of Lebesgue measure on $\mathbb{R}$, and $\mathfrak{G}$ the regular open algebra of $\mathbb{R}$. Show that the simple product $\mathfrak{A}\times\mathfrak{G}$ is Maharam-type-homogeneous of Maharam type ω, but is not homogeneous. (*Hint*: $\mathfrak{A}$ is weakly (σ,∞)-distributive, but $\mathfrak{G}$ is not, so they are not isomorphic.)

(k) Show that a homogeneous semi-finite measure algebra is σ-finite.

(l) Let (X,Σ,μ) be a measure space, and A a subset of X which has a measurable envelope. Show that the Maharam type of the subspace measure on A is less than or equal to the Maharam type of μ.

(m) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Show that $\tau(\mathfrak{A}) \le \tau(\mathfrak{B})$.

(n) Let μ be a semi-finite measure, and $\tilde\mu$ the c.l.d. version of μ. Show that the Maharam type of $\tilde\mu$ is at most the Maharam type of μ. (*Hint*: 322Db.)

(o) Let $\langle\mu_i\rangle_{i\in I}$ be a non-empty countable family of σ-finite measures all with the same domain; let μ be the sum measure $\sum_{i\in I}\mu_i$. Writing $\tau(\mu)$, $\tau(\mu_i)$ for the Maharam types of the measures, show that $\sup_{i\in I}\tau(\mu_i) \le \tau(\mu) \le \max(\omega, \sup_{i\in I}\tau(\mu_i))$.

331Y Further exercises (a) Suppose that $\mathfrak{A}$ is a Dedekind complete Boolean algebra, $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$ and $\mathfrak{C}$ is an order-closed subalgebra of $\mathfrak{B}$. Show that if $a \in \mathfrak{A}$ is a relative atom in $\mathfrak{A}$ over $\mathfrak{C}$, then $\mathrm{upr}(a,\mathfrak{B})$ is a relative atom in $\mathfrak{B}$ over $\mathfrak{C}$. So if $\mathfrak{B}$ is relatively atomless over $\mathfrak{C}$, then $\mathfrak{A}$ is relatively atomless over $\mathfrak{C}$.

(b) Give an example of a Boolean algebra $\mathfrak{A}$ with a subalgebra $\mathfrak{A}'$ and a proper subalgebra $\mathfrak{B}$ of $\mathfrak{A}'$ which is order-closed in $\mathfrak{A}'$, but τ-generates $\mathfrak{A}$. (*Hint*: take $\mathfrak{A}$ to be the measure algebra $\mathfrak{A}_L$ of Lebesgue measure on $\mathbb{R}$ and $\mathfrak{B}$ the subalgebra $\mathfrak{B}_{\mathbb{Q}}$ of $\mathfrak{A}$ generated by $\{[a,b]^\bullet : a,\, b \in \mathbb{Q}\}$. Take $E \subseteq \mathbb{R}$ such that $I\cap E$, $I\setminus E$ have non-zero measure for every non-trivial interval $I \subseteq \mathbb{R}$ (134Jb), and let $\mathfrak{A}'$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}\cup\{E^\bullet\}$.)

(c) Give an example of a Boolean algebra $\mathfrak{A}$ with a subalgebra $\mathfrak{A}'$ and a proper subalgebra $\mathfrak{B}$ of $\mathfrak{A}'$ which is order-closed in $\mathfrak{A}$, but τ-generates $\mathfrak{A}'$. (*Hint*: in the notation of 331Yb, take Z to be the Stone space of $\mathfrak{A}_L$, and set $\mathfrak{A}' = \{\widehat{a} : a \in \mathfrak{A}_L\}$, $\mathfrak{B} = \{\widehat{a} : a \in \mathfrak{B}_{\mathbb{Q}}\}$; let $\mathfrak{A}$ be the subalgebra of $\mathcal{P}Z$ generated by $\mathfrak{A}'\cup\{\{z\} : z \in Z\}$.)

(d) Let $\mathfrak{A}$ be a Dedekind complete purely atomic Boolean algebra, and A the set of its atoms. Show that $\tau(\mathfrak{A})$ is the least cardinal κ such that $\#(A) \le 2^\kappa$.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Write $d(\mathfrak{A})$ for the smallest cardinal of any subset of $\mathfrak{A}$ which is dense for the measure-algebra topology. Show that $d(\mathfrak{A}) \le \max(\omega,\tau(\mathfrak{A}))$. Show that if $(\mathfrak{A},\bar{\mu})$ is semi-finite, then $\tau(\mathfrak{A}) \le d(\mathfrak{A})$.

(f) Let (X,ρ) be a metric space. Write $d(X)$ for the **density** of X, the smallest cardinal of any dense subset of X. (i) Show that if $\mathcal{G}$ is any family of open subsets of X, there is a family $\mathcal{H} \subseteq \mathcal{G}$ such that $\bigcup\mathcal{H} = \bigcup\mathcal{G}$ and $\#(\mathcal{H}) \le \max(\omega, d(X))$. (ii) Show that if $\kappa > \max(\omega, d(X))$ and $\langle x_\xi\rangle_{\xi<\kappa}$ is any family in X, then there is an $x \in X$ such that $\#(\{\xi : x_\xi \in G\}) > \max(\omega, d(X))$ for every open set G containing x, and that there is a strictly increasing sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ in κ such that $x = \lim_{n\to\infty} x_{\xi_n}$.

(g) Let $(\mathfrak{A},\bar{\mu})$ be the simple product (322L) of ω_1 copies of the measure algebra of the usual measure on $\{0,1\}^{\omega_1}$. Show that $\mathfrak{A}$ is Maharam-type-homogeneous but not homogeneous.

(h) Let κ be an infinite cardinal, ν_κ the usual measure on $\{0,1\}^\kappa$ and $(\mathfrak{B}_\kappa,\bar{\nu}_\kappa)$ its measure algebra. Suppose that $(\mathfrak{A},\bar{\mu})$ is a totally finite measure algebra and such that $\tau(\mathfrak{A}) < \kappa$, and $\pi : \mathfrak{B}_\kappa \to \mathfrak{A}$ a Boolean homomorphism. Show that (i) for every $\epsilon > 0$ there is a $b \in \mathfrak{B}_\kappa$ such that $\bar{\nu}_\kappa b \ge 1-\epsilon$ and $\bar{\mu}(\pi b) \le \epsilon$ (ii) π is not injective.

(i) Give an example of a semi-finite measure space (X,Σ,μ) such that the Maharam type of μ is greater than the Maharam type of its c.l.d. version.

(j) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra which is separable when given its measure-algebra topology. Show that it has countable Maharam type.

(k) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be a non-empty family of homogeneous probability algebras, and $\mathcal{F}$ an ultrafilter on I. Show that the probability algebra reduced product $\prod_{i\in I}(\mathfrak{A}_i,\bar{\mu}_i)|\mathcal{F}$ (328C) is homogeneous.

331 Notes and comments Maharam's theorem belongs with the Radon-Nikodým theorem, Fubini's theorem and the strong law of large numbers as one of the theorems which make measure theory what it is. Once you have this theorem and its consequences in the next section properly absorbed, you will never again look at a measure space without trying to classify its measure algebra in terms of the Maharam types of its homogeneous principal ideals. As one might expect, a very large proportion of the important measure algebras of analysis are homogeneous, and indeed a great many are homogeneous with Maharam type ω.

In this section I have contented myself with the basic statement of Theorem 331I on the isomorphism of Maharam-type-homogeneous measure algebras and the identification of representative homogeneous probability algebras (331K). The same techniques lead to an enormous number of further facts, some of which I will describe in the rest of the chapter. For the moment, it gives us a complete description of Maharam-type-homogeneous probability algebras (331L). There is the atomic algebra $\{0,1\}$, with Maharam type 0, and for each infinite cardinal κ there is the measure algebra of $\{0,1\}^\kappa$, with Maharam type κ; these are all non-isomorphic, and every Maharam-type-homogeneous probability algebra is isomorphic to exactly one of them. The isomorphisms here are not unique; indeed, it is characteristic of measure algebras that they have very large automorphism groups (see Chapter 38 below), and there are correspondingly large numbers of isomorphisms between any isomorphic pair. The proof of 331I already suggests this, since we have such a vast amount of choice concerning the lists $\langle a_\xi\rangle_{\xi<\kappa}$ and $\langle b_\xi\rangle_{\xi<\kappa}$, and even with these fixed there remains a good deal of scope in the choice of $\langle a'_\xi\rangle_{\xi<\kappa}$ and $\langle b'_\xi\rangle_{\xi<\kappa}$.

The isomorphisms described in Theorem 331I are measure algebra isomorphisms, that is, measure-preserving Boolean isomorphisms. Obvious questions arise concerning Boolean isomorphisms which are not necessarily measure-preserving; the theorem also helps us to settle many of these (see 331N). But we can observe straight away the remarkable fact that two homogeneous probability algebras which are isomorphic as Boolean algebras are also isomorphic as probability algebras, since they must have the same Maharam type.

I have already mentioned certain measure space isomorphisms (254K, 255A). Of course any isomorphism between measure spaces must induce an isomorphism between their measure algebras (see 324M), and any isomorphism between measure algebras corresponds to an isomorphism between their Stone spaces (see 324N). But there are many important examples of isomorphisms between measure algebras which do not correspond to isomorphisms between the measure spaces most naturally involved. (I describe one in 343J.) Maharam's theorem really is a theorem about measure *algebras* rather than measure *spaces*.

The particular method I use to show that the measure algebra of the usual measure on $\{0,1\}^\kappa$ is homogeneous for infinite κ (331J-331K) is chosen with a view to a question in the next section (332O). There are other ways of doing it. But I recommend study of this particular one because of the way in which it involves the topological, algebraic and order properties of the algebra $\mathfrak{B}$. I have extracted some of the elements of the argument in 331Xi and 331Ye-331Yf. These use the concept of 'density' of a topological space. This does not seem the moment to go farther along this road, but I hope you can see that there are likely to be many further 'cardinal functions' to provide useful measures of complexity in both algebraic and topological structures.

Version of 19.3.05

332 Classification of localizable measure algebras

In this section I present what I call 'Maharam's theorem', that every localizable measure algebra is expressible as a weighted simple product of measure algebras of spaces of the form $\{0,1\}^\kappa$ (332B). Among its many consequences is a complete description of the isomorphism classes of localizable measure algebras (332J). This description needs the concepts of 'cellularity' of a Boolean algebra (332D) and its refinement, the 'magnitude' of a measure algebra (332G). I end this section with a discussion of those pairs of measure algebras for which there is a measure-preserving homomorphism from one to the other (332P-332Q), and a general formula for the Maharam type of a localizable measure algebra (332S).

332A Lemma Let $\mathfrak{A}$ be any Boolean algebra. Writing $\mathfrak{A}_a$ for the principal ideal generated by $a \in \mathfrak{A}$, the set $\{a : a \in \mathfrak{A},\ \mathfrak{A}_a$ is Maharam-type-homogeneous$\}$ is order-dense in $\mathfrak{A}$.

proof Take any $a \in \mathfrak{A} \setminus \{0\}$. Then $A = \{\tau(\mathfrak{A}_b) : 0 \ne b \subseteq a\}$ has a least member; take $c \subseteq a$ such that $c \ne 0$ and $\tau(\mathfrak{A}_c) = \min A$. If $0 \ne b \subseteq c$, then $\tau(\mathfrak{A}_b) \le \tau(\mathfrak{A}_c)$, by 331Hc, while $\tau(\mathfrak{A}_b) \in A$, so $\tau(\mathfrak{A}_c) \le \tau(\mathfrak{A}_b)$. Thus $\tau(\mathfrak{A}_b) = \tau(\mathfrak{A}_c)$ for every non-zero $b \subseteq c$, and $\mathfrak{A}_c$ is Maharam-type-homogeneous.

332B Maharam's theorem Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. Then it is isomorphic to the simple product of a family $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ of measure algebras, where for each $i \in I$ $(\mathfrak{A}_i, \bar\mu_i)$ is isomorphic, up to a re-normalization of the measure, to the measure algebra of the usual measure on $\{0,1\}^{\kappa_i}$, where κ_i is either 0 or an infinite cardinal.

proof (a) For $a \in \mathfrak{A}$, let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Then

$$D = \{a : a \in \mathfrak{A},\ 0 < \bar\mu a < \infty,\ \mathfrak{A}_a \text{ is Maharam-type-homogeneous}\}$$

is order-dense in $\mathfrak{A}$. **P** If $a \in \mathfrak{A} \setminus \{0\}$, then (because $(\mathfrak{A}, \bar\mu)$ is semi-finite) there is a $b \subseteq a$ such that $0 < \bar\mu b < \infty$; now by 332A there is a non-zero $d \subseteq b$ such that $\mathfrak{A}_d$ is Maharam-type-homogeneous. **Q**

(b) By 313K, there is a partition of unity $\langle e_i\rangle_{i\in I}$ consisting of members of D; by 322L(d-i), $(\mathfrak{A}, \bar\mu)$ is isomorphic, as measure algebra, to the simple product of the principal ideals $\mathfrak{A}_i = \mathfrak{A}_{e_i}$.

(c) For each $i \in I$, $(\mathfrak{A}_i, \bar\mu_i)$ is a non-trivial totally finite Maharam-type-homogeneous measure algebra, writing $\bar\mu_i = \bar\mu\restriction\mathfrak{A}_i$. Take $\gamma_i = \bar\mu_i(1_{\mathfrak{A}_i}) = \bar\mu e_i$, and set $\bar\mu_i' = \gamma_i^{-1}\bar\mu_i$. Then $(\mathfrak{A}_i, \bar\mu_i')$ is a Maharam-type-homogeneous probability algebra, so by 331L is isomorphic to the measure algebra $(\mathfrak{B}_{\kappa_i}, \bar\nu_{\kappa_i})$ of the usual measure on $\{0,1\}^{\kappa_i}$, where κ_i is either 0 or an infinite cardinal. Thus $(\mathfrak{A}_i, \bar\mu_i)$ is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_{\kappa_i}, \bar\nu_{\kappa_i})$.

Remark For the case of totally finite measure algebras, this is Theorem 2 of Maharam 1942.

332C Corollary Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra. For any cardinal κ, write ν_κ for the usual measure on $\{0,1\}^\kappa$, and T_κ for its domain. Then we can find families $\langle\kappa_i\rangle_{i\in I}$, $\langle\gamma_i\rangle_{i\in I}$ such that every κ_i is either 0 or an infinite cardinal, every γ_i is a strictly positive real number, and $(\mathfrak{A}, \bar{\mu})$ is isomorphic to the measure algebra of (X, Σ, ν), where

$$X = \{(x, i) : i \in I,\, x \in \{0,1\}^{\kappa_i}\},$$

$$\Sigma = \{E : E \subseteq X,\, \{x : (x, i) \in E\} \in \mathrm{T}_{\kappa_i} \text{ for every } i \in I\},$$

$$\nu E = \sum_{i \in I} \gamma_i \nu_{\kappa_i} \{x : (x, i) \in E\}$$

for every $E \in \Sigma$.

proof Take the family $\langle \kappa_i \rangle_{i \in I}$ from the last theorem, take the $\gamma_i = \bar{\mu} e_i$ to be the normalizing factors of the proof there, and apply 322Lb to identify the simple product of the measure algebras of $(\{0,1\}^{\kappa_i}, \mathrm{T}_{\kappa_i}, \gamma_i \nu_{\kappa_i})$ with the measure algebra of their direct sum (X, Σ, ν).

332D The cellularity of a Boolean algebra In order to properly describe non-sigma-finite measure algebras, we need the following concept. If $\mathfrak{A}$ is any Boolean algebra, write

$$c(\mathfrak{A}) = \sup\{\#(C) : C \subseteq \mathfrak{A} \setminus \{0\} \text{ is disjoint}\},$$

the **cellularity** of $\mathfrak{A}$. (If $\mathfrak{A} = \{0\}$, take $c(\mathfrak{A}) = 0$.) Thus $\mathfrak{A}$ is ccc (316A) iff $c(\mathfrak{A}) \le \omega$.

332E Proposition Let $(\mathfrak{A}, \bar{\mu})$ be any semi-finite measure algebra, and C any partition of unity in $\mathfrak{A}$ consisting of elements of finite measure. Then $\max(\omega, \#(C)) = \max(\omega, c(\mathfrak{A}))$.

proof Of course $\#(C \setminus \{0\}) \le c(\mathfrak{A})$, because $C \setminus \{0\}$ is disjoint, so

$$\max(\omega, \#(C)) = \max(\omega, \#(C \setminus \{0\}) \le \max(\omega, c(\mathfrak{A})).$$

Now suppose that D is any disjoint set in $\mathfrak{A} \setminus \{0\}$. For $c \in C$, $\{d \cap c : d \in D\}$ is a disjoint set in the principal ideal $\mathfrak{A}_c$ generated by c. But $\mathfrak{A}_c$ is ccc (322G), so $\{d \cap c : d \in D\}$ must be countable, and $D_c = \{d : d \in D,\, d \cap c \ne 0\}$ is countable. Because $\sup C = 1$, $D = \bigcup_{c \in C} D_c$, so

$$\#(D) \le \max(\omega, \#(C), \sup_{c \in C} \#(D_c)) = \max(\omega, \#(C)).$$

As D is arbitrary, $c(\mathfrak{A}) \le \max(\omega, \#(C))$ and $\max(\omega, c(\mathfrak{A})) = \max(\omega, \#(C))$.

332F Corollary Let $(\mathfrak{A}, \bar{\mu})$ be any semi-finite measure algebra. Then there is a disjoint set in $\mathfrak{A} \setminus \{0\}$ of cardinal $c(\mathfrak{A})$.

proof Start by taking any partition of unity C consisting of non-zero elements of finite measure. If $\#(C) = c(\mathfrak{A})$ we can stop, because C is a disjoint set in $\mathfrak{A} \setminus \{0\}$. Otherwise, by 332E, we must have C finite and $c(\mathfrak{A}) = \omega$. Let A be the set of atoms in $\mathfrak{A}$. If A is infinite, it is a disjoint set with cardinal ω, so we can stop. Otherwise, since there is certainly a disjoint set $D \subseteq \mathfrak{A} \setminus \{0\}$ with cardinal greater than $\#(A)$, and since each member of A can meet at most one member of D, there must be a member d of D which does not include any atom. Accordingly we can choose inductively a sequence $\langle d_n \rangle_{n \in \mathbb{N}}$ such that $d_0 = d$, $0 \ne d_{n+1} \subset d_n$ for every n. Now $\{d_n \setminus d_{n+1} : n \in \mathbb{N}\}$ is a disjoint set in $\mathfrak{A} \setminus \{0\}$ with cardinal $\omega = c(\mathfrak{A})$.

332G Definitions For the next theorem, it will be convenient to have some special terminology.

(a) The first word I wish to introduce is a variant of the idea of 'cellularity', adapted to measure algebras. If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, let us say that the **magnitude** of an $a \in \mathfrak{A}$ is $\bar{\mu} a$ if $\bar{\mu} a$ is finite, and otherwise is the cellularity of the principal ideal $\mathfrak{A}_a$ generated by a. (This is necessarily infinite, since any partition of a into sets of finite measure must be infinite.) If we take it that any real number is less than any infinite cardinal, then the class of possible magnitudes is totally ordered.

I shall sometimes speak of the **magnitude** of the measure algebra $(\mathfrak{A}, \bar{\mu})$ itself, meaning the magnitude of $1_{\mathfrak{A}}$. Similarly, if (X, Σ, μ) is a semi-finite measure space, the **magnitude** of (X, Σ, μ), or of μ, is the magnitude of its measure algebra.

(b) Next, for any Dedekind complete Boolean algebra $\mathfrak{A}$, and any cardinal κ, we can look at the element

$$e_\kappa = \sup\{a : a \in \mathfrak{A} \setminus \{0\},\, \mathfrak{A}_a \text{ is Maharam-type-homogeneous with Maharam type } \kappa\},$$

writing $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a, as usual. I will call this the **Maharam-type-κ component** of $\mathfrak{A}$. Of course $e_\kappa \cap e_\lambda = 0$ whenever λ, κ are distinct cardinals. **P** $a \cap b = 0$ whenever $\mathfrak{A}_a$, $\mathfrak{A}_b$ are Maharam-type-homogeneous of different Maharam types, since $\tau(\mathfrak{A}_{a\cap b})$ cannot be equal simultaneously to $\tau(\mathfrak{A}_a)$ and $\tau(\mathfrak{A}_b)$. **Q**

Also $\{e_\kappa : \kappa \text{ is a cardinal}\}$ is a partition of unity in $\mathfrak{A}$, because

$$\sup\{e_\kappa : \kappa \text{ is a cardinal}\} = \sup\{a : \mathfrak{A}_a \text{ is Maharam-type-homogeneous}\} = 1$$

by 332A. Note that there is no claim that $\mathfrak{A}_{e_\kappa}$ itself is homogeneous; but we do have a useful result in this direction.

332H Lemma Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and κ an infinite cardinal. Let e be the Maharam-type-κ component of $\mathfrak{A}$. If $0 \ne d \subseteq e$ and the principal ideal $\mathfrak{A}_d$ generated by d is ccc, then it is Maharam-type-homogeneous with Maharam type κ.

proof (a) The point is that $\tau(\mathfrak{A}_d) \le \kappa$. **P** Set

$$A = \{a : a \in \mathfrak{A} \setminus \{0\}, \mathfrak{A}_a \text{ is Maharam-type-homogeneous of Maharam type } \kappa\}.$$

Then $d = \sup\{a \cap d : a \in A\}$. Because $\mathfrak{A}_d$ is ccc, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $d = \sup_{n\in\mathbb{N}} d \cap a_n$ (316E); set $b_n = d \cap a_n$. We have $\tau(\mathfrak{A}_{b_n}) \le \tau(\mathfrak{A}_{a_n}) = \kappa$ for each n; let D_n be a subset of $\mathfrak{A}_{b_n}$, with cardinal at most κ, which τ-generates $\mathfrak{A}_{b_n}$. Set

$$D = \bigcup_{n\in\mathbb{N}} D_n \cup \{b_n : n \in \mathbb{N}\} \subseteq \mathfrak{A}_d.$$

If $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}_d$ generated by D, then $\mathfrak{C} \cap \mathfrak{A}_{b_n}$ is an order-closed subalgebra of $\mathfrak{A}_{b_n}$ including D_n, so is equal to $\mathfrak{A}_{b_n}$, for every n. But $a = \sup_{n\in\mathbb{N}} a \cap b_n$ for every $a \in \mathfrak{A}_d$, so $\mathfrak{C} = \mathfrak{A}_d$. Thus D τ-generates $\mathfrak{A}_d$, and

$$\tau(\mathfrak{A}_d) \le \#(D) \le \max(\omega, \sup_{n\in\mathbb{N}} \#(D_n)) = \kappa. \textbf{ Q}$$

(b) If now b is any non-zero member of $\mathfrak{A}_d$, there is some $a \in A$ such that $b \cap a \ne 0$, so that

$$\kappa = \tau(\mathfrak{A}_{b\cap a}) \le \tau(\mathfrak{A}_b) \le \tau(\mathfrak{A}_d) \le \kappa.$$

Thus we must have $\tau(\mathfrak{A}_b) = \kappa$ for every non-zero $b \in \mathfrak{A}_d$, and $\mathfrak{A}_d$ is Maharam-type-homogeneous with type κ, as claimed.

332I Lemma Let $(\mathfrak{A}, \bar\mu)$ be an atomless semi-finite measure algebra which is not totally finite. Then it has a partition of unity consisting of elements of measure 1.

proof Let A be the set $\{a : \bar\mu a = 1\}$, and $\mathcal{C}$ the family of disjoint subsets of A. By Zorn's lemma, $\mathcal{C}$ has a maximal member C_0 (compare the proof of 313K). Set $D = \{d : d \in \mathfrak{A},\ d \cap c = 0 \text{ for every } c \in C_0\}$. Then D is upwards-directed. If $d \in D$, then $\bar\mu a \ne 1$ for every $a \subseteq d$, so $\bar\mu d < 1$, by 331C. So $d_0 = \sup D$ is defined in $\mathfrak{A}$ (321C); of course $d_0 \in D$, so $\bar\mu d_0 < 1$. Observe that $\sup C_0 = 1 \setminus d_0$.

Because $\bar\mu 1 = \infty$, C_0 must be infinite; let $\langle a_n\rangle_{n\in\mathbb{N}}$ be any sequence of distinct elements of C_0. For each $n \in \mathbb{N}$, use 331C again to choose an $a'_n \subseteq a_n$ such that $\bar\mu a'_n = \bar\mu d_0$. Set

$$b_0 = d_0 \cup (a_0 \setminus a'_0), \quad b_n = a'_{n-1} \cup (a_n \setminus a'_n)$$

for every $n \ge 1$. Then $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of elements of measure 1 and $\sup_{n\in\mathbb{N}} b_n = \sup_{n\in\mathbb{N}} a_n \cup d_0$. Now

$$(C_0 \setminus \{a_n : n \in \mathbb{N}\}) \cup \{b_n : n \in \mathbb{N}\}$$

is a partition of unity consisting of elements of measure 1.

332J Now I can formulate a complete classification theorem for localizable measure algebras, refining the expression in 332B.

Theorem Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be localizable measure algebras. For each cardinal κ, let e_κ, f_κ be the Maharam-type-κ components of $\mathfrak{A}$, $\mathfrak{B}$ respectively. Then $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are isomorphic, as measure algebras, iff (i) e_κ and f_κ have the same magnitude for every infinite cardinal κ (ii) for every $\gamma \in]0,\infty[$, $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ have the same number of atoms of measure γ.

proof Throughout the proof, write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a, and $\bar\mu_a$ for the restriction of $\bar\mu$ to $\mathfrak{A}_a$; and define $\mathfrak{B}_b$, $\bar\nu_b$ similarly for $b \in \mathfrak{B}$.

(a) If $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are isomorphic, then of course the isomorphism matches their Maharam-type components together and retains their magnitudes, and matches atoms of the same measure together; so the conditions are surely satisfied.

(b) Now suppose that the conditions are satisfied. Set

$$K = \{\kappa : \kappa \text{ is an infinite cardinal}, e_\kappa \neq 0\} = \{\kappa : \kappa \text{ is an infinite cardinal}, f_\kappa \neq 0\}.$$

For $\gamma \in]0,\infty[$, let A_γ be the set of atoms of measure γ in $\mathfrak{A}$, and set $e_\gamma = \sup A_\gamma$. Write $I = K \cup]0,\infty[$. Then $\langle e_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, so $(\mathfrak{A},\bar\mu)$ is isomorphic to the simple product of $\langle(\mathfrak{A}_{e_i},\bar\mu_{e_i})\rangle_{i\in I}$, writing $\mathfrak{A}_{e_i}$ for the principal ideal generated by e_i and $\bar\mu_{e_i}$ for the restriction $\bar\mu\restriction\mathfrak{A}_{e_i}$.

In the same way, writing B_γ for the set of atoms of measure γ in $\mathfrak{B}$, f_γ for $\sup B_\gamma$, $\mathfrak{B}_{f_i}$ for the principal ideal generated by f_i and $\bar\nu_{f_i}$ for the restriction of $\bar\nu$ fo $\mathfrak{B}_{f_i}$, we have $(\mathfrak{B},\bar\nu)$ isomorphic to the simple product of $\langle(\mathfrak{B}_{f_i},\bar\nu_{f_i})\rangle_{i\in I}$.

(c) It will therefore be enough if I can show that $(\mathfrak{A}_{e_i},\bar\mu_{e_i}) \cong (\mathfrak{B}_{f_i},\bar\nu_{f_i})$ for every $i \in I$.

(i) For $\kappa \in K$, the hypothesis is that e_κ and f_κ have the same magnitude. If they are both of finite magnitude, that is, $\bar\mu e_\kappa = \bar\nu f_\kappa < \infty$, then both $(\mathfrak{A}_{e_\kappa},\bar\mu_{e_\kappa})$ and $(\mathfrak{B}_{f_\kappa},\bar\nu_{f_\kappa})$ are homogeneous and of Maharam type κ, by 332H. So 331I tells us that they are isomorphic. If they are both of infinite magnitude λ, then 332I tells us that both $\mathfrak{A}_{e_\kappa}$, $\mathfrak{B}_{f_\kappa}$ have partitions of unity C, D consisting of sets of measure 1. So $(\mathfrak{A}_{e_\kappa},\bar\mu_{e_\kappa})$ is isomorphic to the simple product of $\langle(\mathfrak{A}_c,\bar\mu_c)\rangle_{c\in C}$, while $(\mathfrak{B}_{f_\kappa},\bar\nu_{f_\kappa})$ is isomorphic to the simple product of $\langle(\mathfrak{B}_d,\bar\nu_d)\rangle_{d\in D}$. But we know also that every $(\mathfrak{A}_c,\bar\mu_c)$, $(\mathfrak{B}_d,\bar\nu_d)$ is a homogeneous probability algebra with Maharam type κ, by 332H again, so by Maharam's theorem again they are all isomorphic. Since C, D and λ are all infinite,

$$\#(C) = c(\mathfrak{A}_{e_\kappa}) = \lambda = c(\mathfrak{B}_{f_\kappa}) = \#(D)$$

by 332E. So we are taking the same number of factors in each product and $(\mathfrak{A}_{e_\kappa},\bar\mu_{e_\kappa})$ must be isomorphic to $(\mathfrak{B}_{f_\kappa},\bar\nu_{f_\kappa})$.

(ii) For $\gamma \in]0,\infty[$, our hypothesis is that $\#(A_\gamma) = \#(B_\gamma)$. Now A_γ is a partition of unity in $\mathfrak{A}_{e_\gamma}$, so $(\mathfrak{A}_{e_\gamma},\bar\mu_{e_\gamma})$ is isomorphic to the simple product of $\langle(\mathfrak{A}_a,\bar\mu_a)\rangle_{a\in A_\gamma}$. Similarly, $(\mathfrak{B}_{f_\gamma},\bar\nu_{f_\gamma})$ is isomorphic to the simple product of $\langle(\mathfrak{B}_b,\bar\nu_b)\rangle_{b\in B_\gamma}$. Since every $(\mathfrak{A}_a,\bar\mu_a)$, $(\mathfrak{B}_b,\bar\nu_b)$ is just a simple atom of measure γ, these are all isomorphic; since we are taking the same number of factors in each product, $(\mathfrak{A}_{e_\gamma},\bar\mu_{e_\gamma})$ must be isomorphic to $(\mathfrak{B}_{f_\gamma},\bar\nu_{f_\gamma})$.

(iii) Thus we have the full set of required isomorphisms, and $(\mathfrak{A},\bar\mu)$ is isomorphic to $(\mathfrak{B},\bar\nu)$.

332K Remarks (a) The partition of unity $\{e_i : i \in I\}$ of $\mathfrak{A}$ used in the above theorem is in some sense canonical. (You might feel it more economical to replace I by $K \cup \{\gamma : A_\gamma \neq \emptyset\}$.) The further partition of the atomic part into individual atoms (part (c-ii) of the proof) is also canonical. But of course the partition of the e_κ of infinite magnitude into elements of measure 1 requires a degree of arbitrary choice.

The value of the expressions in 332C is that the parameters κ_i, γ_i there are sufficient to identify the measure algebra up to isomorphism. For, amalgamating the language of 332C and 332J, we see that the magnitude of e_κ in 332J is just $\sum_{\kappa_i=\kappa}\gamma_i$ if this is finite, $\#(\{i : \kappa_i = \kappa\})$ otherwise (using 332E, as usual); while the number of atoms of measure γ is $\#(\{i : \kappa_i = 0, \gamma_i = \gamma\})$.

(b) The classification which Maharam's theorem gives us is not merely a listing. It involves a real insight into the nature of the algebras, enabling us to answer a very wide variety of natural questions. I give the next couple of results as a sample of what we can expect these methods to do for us.

332L Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and a, $b \in \mathfrak{A}$ two elements of finite measure. Suppose that $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$ is a measure-preserving isomorphism, where $\mathfrak{A}_a$, $\mathfrak{A}_b$ are the principal ideals generated by a and b. Then there is a measure-preserving automorphism $\phi : \mathfrak{A} \to \mathfrak{A}$ which extends π.

proof The point is that $\mathfrak{A}_{b\setminus a}$ is isomorphic, as measure algebra, to $\mathfrak{A}_{a\setminus b}$. **P** Set $c = a \cup b$. For each infinite cardinal κ, let e_κ be the Maharam-type-κ component of $\mathfrak{A}_c$. Then $e_\kappa \cap a$ is the Maharam-type-κ component of $\mathfrak{A}_a$, because if $d \subseteq c$ and $\mathfrak{A}_d$ is Maharam homogeneous with Maharam type κ, then $\mathfrak{A}_{d\cap a}$ is either $\{0\}$ or again Maharam-type-homogeneous with Maharam type κ. Similarly, $e_\kappa \setminus a$ is the Maharam-type-κ component of $\mathfrak{A}_{c\setminus a} = \mathfrak{A}_{b\setminus a}$, $e_\kappa \cap b$ is the Maharam-type-κ component of $\mathfrak{A}_b$ and $e_\kappa \setminus b$ is the Maharam-type-κ component of $\mathfrak{A}_{a\setminus b}$. Now $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism, so $\pi(e_\kappa \cap a)$ must be $e_\kappa \cap b$, and

$$\begin{aligned}\bar\mu(e_\kappa \setminus a) &= \bar\mu e_\kappa - \bar\mu(e_\kappa \cap a) = \bar\mu e_\kappa - \bar\mu\pi(e_\kappa \cap a)\\ &= \bar\mu e_\kappa - \bar\mu(e_\kappa \cap b) = \bar\mu(e_\kappa \setminus b).\end{aligned}$$

In the same way, if we write $n_\gamma(d)$ for the number of atoms of measure γ in $\mathfrak{A}_d$, then

$$n_\gamma(b \setminus a) = n_\gamma(c) - n_\gamma(a) = n_\gamma(c) - n_\gamma(b) = n_\gamma(a \setminus b)$$

for every $\gamma \in]0,\infty[$. By 332J, there is a measure-preserving isomorphism $\pi_1 : \mathfrak{A}_{b\setminus a} \to \mathfrak{A}_{a\setminus b}$. **Q**

If we now set

$$\phi d = \pi(d \cap a) \cup \pi_1(d \cap b \setminus a) \cup (d \setminus c)$$

for every $d \in \mathfrak{A}$, $\phi : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving isomorphism which agrees with π on $\mathfrak{A}_a$.

Remark There is an elementary proof of this result not relying on Maharam's theorem; see 381Ye(ii).

332M Lemma Suppose that $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are homogeneous measure algebras, with $\tau(\mathfrak{A}) \le \tau(\mathfrak{B})$ and $\bar\mu 1 = \bar\nu 1 < \infty$. Then there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

proof The case $\tau(\mathfrak{A}) = 0$ is trivial. Otherwise, considering normalized versions of the measures, we are reduced to the case $\bar\mu 1 = \bar\nu 1 = 1$, $\tau(\mathfrak{A}) = \kappa \ge \omega$, $\tau(\mathfrak{B}) = \lambda \ge \kappa$, so that $(\mathfrak{A}, \bar\mu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ of the usual measure $\bar\nu_\kappa$ on $\{0,1\}^\kappa$; and similarly $(\mathfrak{B}, \bar\nu)$ is isomorphic to the measure algebra $(\mathfrak{B}_\lambda, \bar\nu_\lambda)$ of the usual measure on $\{0,1\}^\lambda$. Now (identifying the cardinals κ, λ with von Neumann ordinals, as usual), $\kappa \subseteq \lambda$, so we have an inverse-measure-preserving map $x \mapsto x\restriction\kappa : \{0,1\}^\lambda \to \{0,1\}^\kappa$ (254Oa), which induces a measure-preserving Boolean homomorphism from $\mathfrak{B}_\kappa$ to $\mathfrak{B}_\lambda$ (324M), and hence a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

332N Lemma If $(\mathfrak{A}, \bar\mu)$ is a probability algebra and $\kappa \ge \max(\omega, \tau(\mathfrak{A}))$, then there is a measure-preserving Boolean homomorphism from $(\mathfrak{A}, \bar\mu)$ to the measure algebra $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ of the usual measure ν on $\{0,1\}^\kappa$; that is, $(\mathfrak{A}, \bar\mu)$ is isomorphic to a closed subalgebra of $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$.

proof Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that every principal ideal $\mathfrak{A}_{c_i}$ is homogeneous and no c_i is zero. Then I is countable and $\sum_{i\in I} \bar\mu c_i = 1$. Accordingly there is a partition of unity $\langle d_i\rangle_{i\in I}$ in $\mathfrak{B}_\kappa$ such that $\bar\nu d_i = \bar\mu c_i$ for every i. **P** Because I is countable, we may suppose that it is either $\mathbb{N}$ or an initial segment of $\mathbb{N}$. In this case, choose $\langle d_i\rangle_{i\in I}$ inductively such that $d_i \subseteq 1 \setminus \sup_{j<i} d_j$ and $\bar\nu d_i = \bar\mu d_i$ for each $i \in I$, using 331C. **Q**

If $i \in I$, then $\tau(\mathfrak{A}_{c_i}) \le \kappa = \tau((\mathfrak{B}_\kappa)_{d_i})$, so there is a measure-preserving Boolean homomorphism $\pi_i : \mathfrak{A}_{c_i} \to (\mathfrak{B}_\kappa)_{d_i}$. Setting $\pi a = \sup_{i\in I} \pi_i(a \cap c_i)$ for $a \in \mathfrak{A}$, we have a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}_\kappa$. By 324Kb, $\pi[\mathfrak{A}]$ is a closed subalgebra of $\mathfrak{B}_\kappa$, and of course $(\pi[\mathfrak{A}], \bar\nu_\kappa\restriction\pi[\mathfrak{A}])$ is isomorphic to $(\mathfrak{A}, \bar\mu)$.

332O Lemma Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be localizable measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components, and for $\gamma \in]0,\infty[$ let e_γ, f_γ be the suprema of the atoms of measure γ in $\mathfrak{A}$, $\mathfrak{B}$ respectively. If there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then the magnitude of $\sup_{\kappa\ge\lambda} e_\kappa$ is not greater than the magnitude of $\sup_{\kappa\ge\lambda} f_\kappa$ whenever λ is an infinite cardinal, while the magnitude of $\sup_{\kappa\ge\omega} e_\kappa \cup \sup_{\gamma\le\delta} e_\gamma$ is not greater than the magnitude of $\sup_{\kappa\ge\omega} f_\kappa \cup \sup_{\gamma\le\delta} f_\gamma$ for any $\delta \in]0,\infty[$.

proof Suppose that $\pi : \mathfrak{A} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism. For infinite cardinals λ, set $e^*_\lambda = \sup_{\kappa\ge\lambda} e_\kappa$, $f^*_\lambda = \sup_{\kappa\ge\lambda} f_\kappa$, while for $\delta \in]0,\infty[$ set $e^*_\delta = \sup_{\kappa\ge\omega} e_\kappa \cup \sup_{\gamma\le\delta} e_\gamma$, $f^*_\delta = \sup_{\kappa\ge\omega} f_\kappa \cup \sup_{\gamma\le\delta} f_\gamma$. Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak{A}$ such that all the principal ideals

$\mathfrak{A}_{c_i}$ are totally finite and homogeneous, as in 332B. Then $c_i \subseteq e_\kappa$ whenever $\kappa = \tau(\mathfrak{A}_{c_i})$ is infinite, and $c_i \subseteq e_\gamma$ if c_i is an atom of measure γ. Take v to be either an infinite cardinal or a strictly positive real number. Set

$$J = \{i : i \in I,\, c_i \subseteq e_v^*\};$$

then $e_v^* = \sup_{i\in J} c_i$.

Now the point is that if $i \in J$ then $\pi c_i \subseteq f_v^*$. **P** We need to consider two cases. (i) If c_i is an atom, then $v \in]0,\infty[$ and $\bar\mu c_i \le v$. So we need only observe that $1 \setminus f_v^*$ is just the supremum in $\mathfrak{B}$ of the atoms of measure greater than v, none of which can meet πc_i, since this has measure at most v. (ii) Now suppose that $\mathfrak{A}_{c_i}$ is atomless, with $\tau(\mathfrak{A}_{c_i}) = \kappa \ge v$. If $0 \ne b \subseteq \pi c_i$, then $a \mapsto b \cap \pi a : \mathfrak{A}_{c_i} \to \mathfrak{B}_b$ is an order-continuous Boolean homomorphism, while $\mathfrak{A}_{c_i}$ is isomorphic (as Boolean algebra) to the measure algebra of $\{0,1\}^\kappa$, so 331J tells us that $\tau(\mathfrak{B}_b) \ge \kappa$. This means, first, that b cannot be an atom, so that πc_i cannot meet $\sup_{\gamma\in]0,\infty[} f_\gamma$; and also that b cannot be included in $f_{\kappa'}$ for any infinite $\kappa' < \kappa$, so that πc_i cannot meet $\sup_{\omega\le\kappa'<\kappa} f_\kappa$. Thus πc_i must be included in $\sup_{\kappa'\ge\kappa} f_\kappa \subseteq f_v^*$. **Q**

Of course $\langle \pi c_i\rangle_{i\in J}$ is disjoint. So if e_v^* has finite magnitude, the magnitude of f_v^* is at least

$$\sum_{i\in J} \bar\nu\pi c_i = \sum_{i\in J} \bar\mu c_i = \bar\mu e_v^*,$$

the magnitude of e_v^*. While if e_v^* has infinite magnitude, this is $\#(J)$, by 332E, which is not greater than the magnitude of f_v^*.

332P Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be atomless totally finite measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar\mu)$ is isomorphic to a closed subalgebra of a principal ideal of $(\mathfrak{B},\bar\nu)$;

(ii) for every cardinal λ,

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa) \le \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa).$$

proof (a)(i)⇒(ii) Suppose that $\pi : \mathfrak{A} \to \mathfrak{B}_d$ is a measure-preserving isomorphism between $\mathfrak{A}$ and a closed subalgebra of a principal ideal $\mathfrak{B}_d$ of $\mathfrak{B}$. The Maharam-type-κ component of $\mathfrak{B}_d$ is just $d \cap f_\kappa$, so 332O tells us that

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa) \le \bar\nu(\sup_{\kappa\ge\lambda} d \cap f_\kappa) \le \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa)$$

for every λ.

(b)(ii)⇒(i) Now suppose that the condition is satisfied.

(α) Let P be the set of all measure-preserving Boolean homomorphisms π from principal ideals $\mathfrak{A}_{c_\pi}$ of $\mathfrak{A}$ to principal ideals $\mathfrak{B}_{d_\pi}$ of $\mathfrak{B}$ such that

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa \setminus c_\pi) \le \bar\nu(\sup_{\kappa\ge\lambda} \bar\nu f_\kappa \setminus d_\pi)$$

for every cardinal $\lambda \ge \omega$. Then the trivial homomorphism from $\mathfrak{A}_0$ to $\mathfrak{B}_0$ belongs to P, so P is not empty. Order P by saying that $\pi \le \pi'$ if π' extends π, that is, if $c_\pi \subseteq c_{\pi'}$ and $\pi' a = \pi a$ for every $a \in \mathfrak{A}_{c_\pi}$. Then P is a partially ordered set.

(β) If $Q \subseteq P$ is non-empty and totally ordered, it is bounded above in P. **P** Set $c^* = \sup_{\pi\in Q} c_\pi$, $d^* = \sup_{\pi\in Q} d_\pi$. For $a \subseteq c^*$ set $\pi^* a = \sup_{\pi\in Q} \pi(a \cap c_\pi)$. Because Q is totally ordered, π^* extends all the functions in Q. It is also easy to check that $\pi^* 0 = 0$, $\pi^*(a \cap a') = \pi^* a \cap \pi^* a'$ and $\pi^*(a \cup a') = \pi^* a \cup \pi^* a'$ for all a, $a' \in \mathfrak{A}_{c^*}$, $\pi^* c^* = d^*$ and that $\bar\nu\pi^* a = \bar\mu a$ for every $a \in \mathfrak{A}_{c^*}$; so that π^* is a measure-preserving Boolean homomorphism from $\mathfrak{A}_{c^*}$ to $\mathfrak{B}_{d^*}$.

Now suppose that λ is any cardinal; then

$$\bar\mu(\sup_{\kappa\ge\lambda} e_\kappa \setminus c^*) = \inf_{\pi\in Q} \bar\mu(\sup_{\kappa\ge\lambda} e_\kappa \setminus c_\pi) \le \inf_{\pi\in Q} \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa \setminus d_\pi) = \bar\nu(\sup_{\kappa\ge\lambda} f_\kappa \setminus d^*).$$

So $\pi^* \in P$ and is the required upper bound of Q. **Q**

(γ) By Zorn's Lemma, P has a maximal element $\tilde\pi$ say. Now $c_{\tilde\pi} = 1$. **P?** If not, then let κ_0 be the least cardinal such that $e_{\kappa_0} \setminus c_{\tilde\pi} \ne 0$. Then

$$0 < \bar{\mu}(\sup_{\kappa \ge \kappa_0} e_\kappa \setminus c_{\tilde{\pi}}) \le \bar{\nu}(\sup_{\kappa \ge \kappa_0} \bar{\nu} f_\kappa \setminus d_{\tilde{\pi}}),$$

so there is a least $\kappa_1 \ge \kappa_0$ such that $f_{\kappa_1} \setminus d_{\tilde{\pi}} \ne 0$. Set $\delta = \min(\bar{\mu}(e_{\kappa_0} \setminus c_{\tilde{\pi}}), \bar{\nu}(f_{\kappa_1} \setminus d_{\tilde{\pi}})) > 0$. Because $\mathfrak{A}$ and $\mathfrak{B}$ are atomless, there are $a \subseteq e_{\kappa_0} \setminus c_{\tilde{\pi}}$ and $b \subseteq f_{\kappa_1} \setminus d_{\tilde{\pi}}$ such that $\bar{\mu}a = \bar{\nu}b = \delta$ (331C). Now $\mathfrak{A}_a$ is homogeneous with Maharam type κ_0, while $\mathfrak{B}_b$ is homogeneous with Maharam type κ_1 (332H), so there is a measure-preserving Boolean homomorphism $\phi : \mathfrak{A}_a \to \mathfrak{B}_b$ (332M). Set

$$c^* = c_{\tilde{\pi}} \cup a, \quad d^* = d_{\tilde{\pi}} \cup b,$$

and define $\pi^* : \mathfrak{A}_{c^*} \to \mathfrak{B}_{d^*}$ by setting $\pi^*(g) = \tilde{\pi}(g \cap c_{\tilde{\pi}}) \cup \phi(g \cap a)$ for every $g \subseteq c^*$. It is easy to check that π^* is a measure-preserving Boolean homomorphism.

If λ is a cardinal and $\lambda \le \kappa_0$,

$$\bar{\mu}(\sup_{\kappa \ge \lambda} e_\kappa \setminus c^*) = \bar{\mu}(\sup_{\kappa \ge \lambda} e_\kappa \setminus c_{\tilde{\pi}}) - \delta \le \bar{\nu}(\sup_{\kappa \ge \lambda} f_\kappa \setminus d_{\tilde{\pi}}) - \delta = \bar{\nu}(\sup_{\kappa \ge \lambda} \bar{\nu} f_\kappa \setminus d^*).$$

If $\kappa_0 < \lambda \le \kappa_1$,

$$\begin{aligned}\bar{\mu}(\sup_{\kappa \ge \lambda} e_\kappa \setminus c^*) &= \bar{\mu}(\sup_{\kappa \ge \lambda} e_\kappa \setminus c_{\tilde{\pi}}) \le \bar{\mu}(\sup_{\kappa \ge \kappa_0} e_\kappa \setminus c_{\tilde{\pi}}) - \bar{\mu}(e_{\kappa_0} \setminus c_{\tilde{\pi}}) \\ &\le \bar{\mu}(\sup_{\kappa \ge \kappa_0} e_\kappa \setminus c_{\tilde{\pi}}) - \delta \le \bar{\nu}(\sup_{\kappa \ge \kappa_0} f_\kappa \setminus d_{\tilde{\pi}}) - \delta \\ &= \bar{\nu}(\sup_{\kappa \ge \kappa_1} f_\kappa \setminus d_{\tilde{\pi}}) - \delta\end{aligned}$$

(by the choice of κ_1)

$$= \bar{\nu}(\sup_{\kappa \ge \kappa_1} f_\kappa \setminus d^*) \le \bar{\nu}(\sup_{\kappa \ge \lambda} f_\kappa \setminus d^*).$$

If $\lambda > \kappa_1$,

$$\bar{\mu}(\sup_{\kappa \ge \lambda} e_\kappa \setminus c^*) = \bar{\mu}(\sup_{\kappa \ge \lambda} e_\kappa \setminus c_{\tilde{\pi}}) \le \bar{\nu}(\sup_{\kappa \ge \lambda} f_\kappa \setminus d_{\tilde{\pi}}) = \bar{\nu}(\sup_{\kappa \ge \lambda} f_\kappa \setminus d^*).$$

But this means that $\pi^* \in P$, and evidently it is a proper extension of $\tilde{\pi}$, which is supposed to be impossible. **XQ**

(δ) Thus $\tilde{\pi}$ has domain $\mathfrak{A}$ and is the required measure-preserving homomorphism from $\mathfrak{A}$ to the principal ideal $\mathfrak{B}_{d_{\tilde{\pi}}}$ of $\mathfrak{B}$.

332Q Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras, and suppose that there are measure-preserving Boolean homomorphisms $\pi_1 : \mathfrak{A} \to \mathfrak{B}$ and $\pi_2 : \mathfrak{B} \to \mathfrak{A}$. Then $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are isomorphic.

proof Writing e_κ, f_κ for their Maharam-type-κ components, 332O (applied to both π_1 and π_2) tells us that

$$\bar{\mu}(\sup_{\kappa \ge \lambda} e_\kappa) = \bar{\nu}(\sup_{\kappa \ge \lambda} f_\kappa)$$

for every λ. Because all these measures are finite,

$$\begin{aligned}\bar{\mu} e_\lambda &= \bar{\mu}(\sup_{\kappa \ge \lambda} e_\kappa) - \bar{\mu}(\sup_{\kappa > \lambda} e_\kappa) \\ &= \bar{\nu}(\sup_{\kappa \ge \lambda} f_\kappa) - \bar{\nu}(\sup_{\kappa > \lambda} f_\kappa) = \bar{\nu} f_\lambda\end{aligned}$$

for every λ.

Similarly, writing e_γ, f_γ for the suprema in $\mathfrak{A}$, $\mathfrak{B}$ of the atoms of measure γ, 332O tells us that

$$\bar{\mu}(\sup_{\gamma \le \delta} e_\gamma) = \bar{\nu}(\sup_{\gamma \le \delta} f_\gamma)$$

for every $\delta \in]0, \infty[$, and hence that $\bar{\mu} e_\gamma = \bar{\nu} f_\gamma$ for every γ, that is, that $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of atoms of measure γ.

So $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are isomorphic, by 332J.

332R 332J tells us that if we know the magnitudes of the Maharam-type-κ components of a localizable measure algebra, we shall have specified the algebra completely, so that all its properties are determined. The calculation of its Maharam type is straightforward and useful, so I give the details.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Then $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$.

proof Let $C \subseteq \mathfrak{A} \setminus \{0\}$ be a disjoint set, and $B \subseteq \mathfrak{A}$ a τ-generating set with cardinal $\tau(\mathfrak{A})$.

(a) If $\mathfrak{A}$ is purely atomic, then for each $c \in C$ choose an atom $c' \subseteq c$, and set $f(c) = \{b : b \in B,\, c' \subseteq b\}$. If c_1, c_2 are distinct members of C, the set

$$\{a : a \in \mathfrak{A},\, c_1' \subseteq a \iff c_2' \subseteq a\}$$

is an order-closed subalgebra of $\mathfrak{A}$ not containing either c_1' or c_2', so cannot include B, and $f(c_1) \ne f(c_2)$. Thus f is injective, and

$$\#(C) \le \#(\mathcal{P}B) = 2^{\tau(\mathfrak{A})}.$$

(b) Now suppose that $\mathfrak{A}$ is not purely atomic; in this case $\tau(\mathfrak{A})$ is infinite. For each $c \in C$ choose an element $c' \subseteq c$ of non-zero finite measure. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by B. Then the topological closure of $\mathfrak{B}$ is $\mathfrak{A}$ itself (323J), and $\#(\mathfrak{B}) = \tau(\mathfrak{A})$ (331Gc). For $c \in C$ set

$$f(c) = \{b : b \in \mathfrak{B},\, \bar{\mu}(b \cap c') \ge \tfrac{1}{2}\bar{\mu}c'\}.$$

Then $f : C \to \mathcal{P}\mathfrak{B}$ is injective. **P** If c_1, c_2 are distinct members of C, then (because $\mathfrak{B}$ is topologically dense in $\mathfrak{A}$) there is a $b \in \mathfrak{B}$ such that

$$\bar{\mu}((c_1' \cup c_2') \cap (c_1' \mathbin{\triangle} b)) \le \frac{1}{3}\min(\bar{\mu}c_1', \bar{\mu}c_2').$$

But in this case

$$\bar{\mu}(c_1' \setminus b) \le \frac{1}{3}\bar{\mu}c_1', \quad \bar{\mu}(c_2' \cap b) \le \frac{1}{3}\bar{\mu}c_2',$$

and $b \in f(c_1) \triangle f(c_2)$, so $f(c_1) \ne f(c_2)$. **Q** Accordingly $\#(C) \le 2^{\#(\mathfrak{B})} = 2^{\tau(\mathfrak{A})}$ in this case also.

As C is arbitrary, $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$.

332S Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Then $\tau(\mathfrak{A})$ is the least cardinal λ such that (α) $c(\mathfrak{A}) \le 2^\lambda$ (β) $\tau(\mathfrak{A}_a) \le \lambda$ for every Maharam-type-homogeneous principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$.

proof Fix λ as the least cardinal satisfying (α) and (β).

(a) By 331Hc, $\tau(\mathfrak{A}_a) \le \tau(\mathfrak{A})$ for every $a \in \mathfrak{A}$, while $c(\mathfrak{A}) \le 2^{\tau(\mathfrak{A})}$ by 332R; so $\lambda \le \tau(\mathfrak{A})$.

(b) Let C be a partition of unity in $\mathfrak{A}$ consisting of elements of non-zero finite measure generating Maharam-type-homogeneous principal ideals (as in the proof of 332B); then $\#(C) \le c(\mathfrak{A}) \le 2^\lambda$, and there is an injective function $f : C \to \mathcal{P}\lambda$. For each $c \in C$, let $B_c \subseteq \mathfrak{A}_c$ be a τ-generating set with cardinal $\tau(\mathfrak{A}_c)$, and $f_c : B_c \to \lambda$ an injection. Set

$$b_\xi = \sup\{c : c \in C,\, \xi \in f(c)\},$$

$$b_\xi' = \sup\{b : \text{there is some } c \in C \text{ such that } b \in B_c \text{ and } f_c(b) = \xi\}$$

for $\xi < \lambda$. Set $B = \{b_\xi : \xi < \lambda\} \cup \{b_\xi' : \xi < \lambda\}$ if λ is infinite, $\{b_\xi : \xi < \lambda\}$ if λ is finite; then $\#(B) \le \lambda$. Note that if $c \in C$ and $b \in B_c$ there is a $b' \in B$ such that $b = b' \cap c$. **P** Since $B_c \ne \emptyset$, $\tau(\mathfrak{A}_c) > 0$; but this means that $\tau(\mathfrak{A}_c)$ is infinite (see 331H) so λ is infinite and $b_\xi' \in B$, where $\xi = f_c(b)$; now $b = b_\xi' \cap c$. **Q**

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by B. Then $C \subseteq \mathfrak{B}$. **P** For $c \in C$, we surely have $c \subseteq b_\xi$ if $\xi \in f(c)$; but also, because C is disjoint, $c \cap b_\xi = 0$ if $\xi \in \lambda \setminus f(c)$. Consequently

$$c^* = \inf_{\xi \in f(c)} b_\xi \cap \inf_{\xi \in \lambda \setminus f(c)} (1 \setminus b_\xi)$$

includes c. On the other hand, if d is any other member of C, there is some $\xi \in f(c) \triangle f(d)$, so that

$$d^* \cap c^* \subseteq b_\xi \cap (1 \setminus b_\xi) = 0.$$

Since $\sup C = 1$, it follows that $c = c^*$; but $c^* \in \mathfrak{B}$, so $c \in \mathfrak{B}$. **Q**

For any $c \in C$, look at $\{b \cap c : b \in \mathfrak{B}\} \subseteq \mathfrak{B}$. This is a closed subalgebra of $\mathfrak{A}_c$ (314F(a-i)) including B_c, so must be the whole of $\mathfrak{A}_c$. Thus $\mathfrak{A}_c \subseteq \mathfrak{B}$ for every $c \in C$. But $\sup C = 1$, so $a = \sup_{c\in C} a \cap c \in \mathfrak{B}$ for every $a \in \mathfrak{A}$, and $\mathfrak{A} = \mathfrak{B}$. Consequently $\tau(\mathfrak{A}) \le \#(B) \le \lambda$, and $\tau(\mathfrak{A}) = \lambda$.

332T Proposition Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra and $\mathfrak{B}$ a closed subalgebra of $\mathfrak{A}$. Then

(a) there is a function $\bar\nu : \mathfrak{B} \to [0,\infty]$ such that $(\mathfrak{B}, \bar\nu)$ is a localizable measure algebra;

(b) $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

proof (a) Let D be the set of those $b \in \mathfrak{B}$ such that the principal ideal $\mathfrak{B}_b$ has Maharam type at most $\tau(\mathfrak{A})$ and is a totally finite measure algebra when endowed with an appropriate measure. Then D is order-dense in $\mathfrak{B}$. **P** Take any non-zero $b_0 \in \mathfrak{B}$. Then there is an $a \in \mathfrak{A}$ such that $a \subseteq b_0$ and $0 < \bar\mu a < \infty$. Set $c = \operatorname{upr}(a, \mathfrak{B}) = \min\{b : b \in \mathfrak{B},\, a \subseteq b\}$; then $c \in \mathfrak{B}$ and $a \subseteq c \subseteq b_0$. If $0 \ne b \in \mathfrak{B}_c$, then $c \setminus b$ belongs to $\mathfrak{B}$ and is properly included in c, so cannot include a; accordingly $a \cap b \ne 0$. For $b \in \mathfrak{B}_c$, set $\bar\nu b = \bar\mu(a \cap b)$. Because the map $b \mapsto a \cap b$ is an injective order-continuous Boolean homomorphism, $\bar\nu$ is countably additive and strictly positive, that is, $(\mathfrak{B}_c, \bar\nu)$ is a measure algebra. It is totally finite because $\bar\nu c = \bar\mu a < \infty$.

Let $d \in \mathfrak{B}_c \setminus \{0\}$ be such that $\mathfrak{B}_d$ is Maharam-type-homogeneous; suppose that its Maharam type is κ. The map $b \mapsto b \cap a$ is a measure-preserving Boolean homomorphism from $\mathfrak{B}_d$ to $\mathfrak{A}_{a\cap d}$, so by 332O $\mathfrak{A}_{a\cap d}$ must have a non-zero Maharam-type-κ' component for some $\kappa' \ge \kappa$; but this means that

$$\tau(\mathfrak{B}_d) \le \kappa \le \kappa' \le \tau(\mathfrak{A}_{a\cap d}) \le \tau(\mathfrak{A}).$$

Thus $d \in D$, while $0 \ne d \subseteq c \subseteq b_0$. As b_0 is arbitrary, D is order-dense. **Q**

Accordingly there is a partition of unity C in $\mathfrak{B}$ such that $C \subseteq D$. For each $c \in C$ we have a functional $\bar\nu_c$ such that $(\mathfrak{B}_c, \bar\nu_c)$ is a totally finite measure algebra with Maharam type at most $\tau(\mathfrak{A})$; define $\bar\nu : \mathfrak{B} \to [0,\infty]$ by setting $\bar\nu b = \sum_{c\in C} \bar\nu_c(b \cap c)$ for every $b \in \mathfrak{B}$. It is easy to check that $(\mathfrak{B}, \bar\nu)$ is a measure algebra (compare 322La); it is localizable because $\mathfrak{B}$ (being order-closed in a Dedekind complete partially ordered set) is Dedekind complete.

(b) The construction above ensures that every homogeneous principal ideal of $\mathfrak{B}$ can have Maharam type at most $\tau(\mathfrak{A})$, since it must share a principal ideal with some $\mathfrak{B}_c$ for $c \in C$. Moreover, any disjoint set in $\mathfrak{B}$ is also a disjoint set in $\mathfrak{A}$, so $c(\mathfrak{B}) \le c(\mathfrak{A})$. So 332S tells us that $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$.

Remark I think the only direct appeal I shall make to this result will be when $(\mathfrak{A}, \bar\mu)$ is a probability algebra, in which case (a) above becomes trivial, and the proof of (b) can be shortened to some extent, though I think we still need some of the ideas of 332S.

332X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra. Show that it is isomorphic to a simple product of Maharam-type-homogeneous Boolean algebras.

(b) Let $\mathfrak{A}$ be a Boolean algebra of finite cellularity. Show that $\mathfrak{A}$ is purely atomic.

(c) Let $\mathfrak{A}$ be a purely atomic Boolean algebra. Show that $c(\mathfrak{A})$ is the number of atoms in $\mathfrak{A}$.

(d) Let $\mathfrak{A}$ be any Boolean algebra, and Z its Stone space. Show that $c(\mathfrak{A})$ is equal to

$$c(Z) = \sup\{\#(\mathcal{G}) : G \text{ is a disjoint family of non-empty open subsets of } Z\},$$

the **cellularity** of the topological space Z.

(e) Let X be a topological space, and $\mathrm{RO}(X)$ its regular open algebra. Show that $c(\mathrm{RO}(X)) = c(X)$ as defined in 332Xd.

(f) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ any subalgebra of $\mathfrak{A}$. Show that $c(\mathfrak{B}) \le c(\mathfrak{A})$.

(g) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with simple product $\mathfrak{A}$. Show that the cellularity of $\mathfrak{A}$ is at most $\max(\omega, \#(I), \sup_{i\in I} c(\mathfrak{A}_i))$. Devise an elegant expression of a necessary and sufficient condition for equality.

(h) Let $\mathfrak{A}$ be any Boolean algebra, and $a \in \mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal generated by a. Show that $c(\mathfrak{A}_a) \le c(\mathfrak{A})$.

(i) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Show that it has a partition of unity with cardinal $c(\mathfrak{A})$.

(j) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be localizable measure algebras. For each cardinal κ let e_κ, f_κ be their Maharam-type-κ components, and $\mathfrak{A}_{e_\kappa}$, $\mathfrak{B}_{f_\kappa}$ the corresponding principal ideals. Show that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, as Boolean algebras, iff $c(\mathfrak{A}_{e_\kappa}) = c(\mathfrak{B}_{f_\kappa})$ for every κ.

(k) Let ζ be an ordinal, and $\langle\alpha_\xi\rangle_{\xi<\zeta}$, $\langle\beta_\xi\rangle_{\xi<\zeta}$ two families of non-negative real numbers such that $\sum_{\theta\le\xi<\zeta}\alpha_\xi \le \sum_{\theta\le\eta<\zeta}\beta_\eta < \infty$ for every $\theta \le \zeta$. Show that there is a family $\langle\gamma_{\xi\eta}\rangle_{\xi\le\eta<\zeta}$ of non-negative real numbers such that $\alpha_\xi = \sum_{\xi\le\eta<\zeta}\gamma_{\xi\eta}$ for every $\xi < \zeta$ and $\beta_\eta \ge \sum_{\xi\le\eta}\gamma_{\xi\eta}$ for every $\eta < \zeta$. (If only finitely many of the α_ξ, β_ξ are non-zero, this is an easy special case of the max-flow min-cut theorem; see BOLLOBÁS 79, §III.1 or ANDERSON 87, 12.3.1; there is a statement of the theorem in 4A4N in the next volume.) Show that the $\gamma_{\xi\eta}$ can be chosen in such a way that if $\xi < \xi' \le \eta' < \eta$ then at least one of $\gamma_{\xi\eta}$, $\gamma_{\xi'\eta'}$ is zero.

(l) Use 332Xk and 332M to give another proof of 332P.

(m) For each cardinal κ, write $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. Let $(\mathfrak{A}, \bar\mu)$ be the simple product of $\langle(\mathfrak{B}_{\omega_n}, \bar\nu_{\omega_n})\rangle_{n\in\mathbb{N}}$ and $(\mathfrak{B}, \bar\nu)$ the simple product of $(\mathfrak{A}, \bar\mu)$ with $(\mathfrak{B}_{\omega_\omega}, \bar\nu_{\omega_\omega})$. (See 3A1E if you are puzzled by the names ω_n, ω_ω.) Show that there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, but that no such homomorphism can be order-continuous.

(n) For each cardinal κ, write $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. Let $(\mathfrak{A}, \bar\mu)$ be the simple product of $\langle(\mathfrak{B}_{\kappa_n}, \bar\nu_{\kappa_n})\rangle_{n\in\mathbb{N}}$ and $(\mathfrak{B}, \bar\nu)$ the simple product of $\langle(\mathfrak{B}_{\lambda_n}, \bar\nu_{\lambda_n})\rangle_{n\in\mathbb{N}}$, where $\kappa_n = \omega$ for even n, ω_n for odd n, while $\lambda_n = \omega$ for odd n, ω_n for even n. Show that there are order-continuous measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ and from $\mathfrak{B}$ to $\mathfrak{A}$, but that these two measure algebras are not isomorphic.

(o) Let $\mathfrak{C}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{C}$ is isomorphic (as Boolean algebra) to a closed subalgebra of a localizable measure algebra; (ii) there is a $\bar\mu$ such that $(\mathfrak{C}, \bar\mu)$ is itself a localizable measure algebra; (iii) $\mathfrak{C}$ is Dedekind complete and for every non-zero $c \in \mathfrak{C}$ there is a completely additive real-valued functional ν on $\mathfrak{C}$ such that $\nu c \ne 0$. (*Hint for (iii)⇒(ii)*: show that the set of supports of non-negative completely additive functionals is order-dense in $\mathfrak{C}$, so includes a partition of unity.)

332Y Further exercises (a) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be atomless localizable measure algebras. For each infinite cardinal κ let e_κ, f_κ be their Maharam-type-κ components. Show that the following are equiveridical: (i) $(\mathfrak{A}, \bar\mu)$ is isomorphic to a closed subalgebra of a principal ideal of $(\mathfrak{B}, \bar\nu)$; (ii) for every cardinal λ, the magnitude of $\sup_{\kappa\ge\lambda} e_\kappa$ is not greater than the magnitude of $\sup_{\kappa\ge\lambda} f_\kappa$.

(b) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be any semi-finite measure algebras, and $(\widehat{\mathfrak{A}}, \hat\mu)$, $(\widehat{\mathfrak{B}}, \hat\nu)$ their localizations (322P-322Q). Let $\langle e_i\rangle_{i\in I}$, $\langle f_j\rangle_{j\in J}$ be partitions of unity in $\mathfrak{A}$, $\mathfrak{B}$ respectively into elements of finite measure generating homogeneous principal ideals $\mathfrak{A}_{e_i}$, $\mathfrak{B}_{f_j}$. For each infinite cardinal κ set $I_\kappa = \{i : \tau(\mathfrak{A}_{e_i}) = \kappa\}$, $J_\kappa = \{j : \tau(\mathfrak{B}_{f_j}) = \kappa\}$; for $\gamma \in \,]0,\infty[$, set $I_\gamma = \{i : e_i$ is an atom, $\bar\mu e_i = \gamma\}$, $J_\gamma = \{j : f_j$ is an atom, $\bar\nu f_j = \gamma\}$. Show that $(\widehat{\mathfrak{A}}, \hat\mu)$ and $(\widehat{\mathfrak{B}}, \hat\nu)$ are isomorphic iff for each u, either $\sum_{i\in I_u}\bar\mu e_i = \sum_{j\in J_u}\bar\nu f_j < \infty$ or $\sum_{i\in I_u}\bar\mu e_i = \sum_{j\in J_u}\bar\nu f_j = \infty$ and $\#(I_u) = \#(J_u)$.

(c) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ be non-zero localizable measure algebras; let e_κ, f_κ be their Maharam-type-κ components. Show that the following are equiveridical: (i) $\mathfrak{A}$ is isomorphic, as Boolean algebra, to an order-closed subalgebra of a principal ideal of $\mathfrak{B}$; (ii) $c(\mathfrak{A}^*_\lambda) \le c(\mathfrak{B}^*_\lambda)$ for every cardinal λ, where $\mathfrak{A}^*_\lambda$, $\mathfrak{B}^*_\lambda$ are the principal ideals generated by $\sup_{\kappa\ge\lambda} e_\kappa$ and $\sup_{\kappa\ge\lambda} f_\kappa$ respectively.

332 Notes and comments Maharam's theorem tells us that all localizable measure algebras – in particular, all σ-finite measure algebras – can be obtained from the basic algebra $\mathfrak{A} = \{0, a, 1 \setminus a, 1\}$, with $\bar{\mu}a = \bar{\mu}(1 \setminus a) = \frac{1}{2}$, by combining the constructions of probability algebra free products, scalar multiples of measures and simple products. But what is much more important is the fact that we get a description of our measure algebras in terms sufficiently explicit to make a very wide variety of questions resolvable. The description I offer in 332J hinges on the complementary concepts of 'Maharam type' and 'magnitude'. If you like, the magnitude of a measure algebra is a measure of its width, while its Maharam type is a measure of its depth. The latter is more important just because, for localizable algebras, we have such a simple decomposition into algebras of finite magnitude. Of course there is a good deal of scope for further complications if we seek to consider non-localizable semi-finite algebras. For these, the natural starting point is a proper description of their localizations, which is not difficult (332Yb).

Observe that 332C gives a representation of a localizable measure algebra as the measure algebra of a measure space which is completely different from the Stone representation in 321K. It is less canonical (since there is a degree of choice about the partition $\langle e_i \rangle_{i \in I}$) but very much more informative, since the κ_i, γ_i carry enough information to identify the measure algebra up to isomorphism (332K).

'Cellularity' is the second cardinal function I have introduced in this chapter. It refers as much to topological spaces as to Boolean algebras (see 332Xd-332Xe). There is an interesting question in this context. If $\mathfrak{A}$ is an arbitrary Boolean algebra, is there necessarily a disjoint set in $\mathfrak{A}$ with cardinal $c(\mathfrak{A})$? This is believed to be undecidable from the ordinary axioms of set theory (including the axiom of choice); see the 'Erdős-Tarski theorem' in Volume 5. But for semi-finite measure algebras we have a definite answer (332F).

Maharam's classification not only describes the isomorphism classes of localizable measure algebras, but also tells us when to expect Boolean homomorphisms between them (332P, 332Yc). I have given 332P only for atomless totally finite measure algebras because the non-totally-finite case (332Ya, 332Yc) seems to require a new idea, while atoms introduce combinatorial complications.

I offer 332T as an example of the kind of result which these methods make very simple. It fails for general Boolean algebras; in fact, there is for any κ a countably τ-generated Dedekind complete Boolean algebra $\mathfrak{A}$ with cellularity κ (514Xi in Volume 5, or Koppelberg 89, 13.1), so that $\mathcal{P}\kappa$ is isomorphic to an order-closed subalgebra of $\mathfrak{A}$, and if $\kappa > \mathfrak{c}$ then $\tau(\mathcal{P}\kappa) > \omega$ (332R).

For totally finite measure algebras we have a kind of weak Schröder-Bernstein theorem: if we have two of them, each isomorphic to a closed subalgebra of the other, they are isomorphic (332Q). This fails for σ-finite algebras (332Xn). I call it a 'weak' Schröder-Bernstein theorem because it is not clear how to build the isomorphism from the two injections; 'strong' Schröder-Bernstein theorems include definite recipes for constructing the isomorphisms declared to exist (see, for instance, 344D below).

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333 Closed subalgebras

Proposition 332P tells us, in effect, which totally finite measure algebras can be embedded as closed subalgebras of each other. Similar techniques make it possible to describe the possible forms of such embeddings. In this section I give the fundamental theorems on extension of measure-preserving homomorphisms from closed subalgebras (333C, 333D); these rely on the concept of 'relative Maharam type' (333A). I go on to describe possible canonical forms for structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$, where $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ (333K, 333N). I end the section with a description of fixed-point subalgebras (333R).

333A Definitions (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. The **relative Maharam type of $\mathfrak{A}$ over $\mathfrak{C}$**, $\tau_{\mathfrak{C}}(\mathfrak{A})$, is the smallest cardinal of any set $A \subseteq \mathfrak{A}$ such that $A \cup \mathfrak{C}$ τ-generates $\mathfrak{A}$.

(b) In this section, I will regularly use the following notation: if $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, and $a \in \mathfrak{A}$, then I will write $\mathfrak{C}_a$ for $\{c \cap a : c \in \mathfrak{C}\}$. Observe that $\mathfrak{C}_a$ is a subalgebra of the principal ideal $\mathfrak{A}_a$ (because $c \mapsto c \cap a : \mathfrak{C} \to \mathfrak{A}_a$ is a Boolean homomorphism); it is included in $\mathfrak{C}$ iff $a \in \mathfrak{C}$.

(c) Still taking $\mathfrak{A}$ to be a Boolean algebra and $\mathfrak{C}$ to be a subalgebra of $\mathfrak{A}$, I will say that an element a of $\mathfrak{A}$ is **relatively Maharam-type-homogeneous over $\mathfrak{C}$** if $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b) = \tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ for every non-zero $b \subseteq a$.

(d) If κ is a cardinal which is either infinite or zero, I will write $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ for the measure algebra of the usual measure ν_κ on $\{0,1\}^\kappa$. I hope that there will be no confusion between this notation and the use, in 333C-333F, of the formula $\mathfrak{B}_b$ for the principal ideal generated by b in an arbitrary Boolean algebra $\mathfrak{B}$.

333B Evidently this is a generalization of the ordinary concept of Maharam type as used in §§331-332; if $\mathfrak{C} = \{0,1\}$ then $\tau_{\mathfrak{C}}(\mathfrak{A}) = \tau(\mathfrak{A})$. The first step is naturally to check the results corresponding to 331H.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$.

(a) If $a \subseteq b$ in $\mathfrak{A}$, then $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \tau_{\mathfrak{C}_b}(\mathfrak{A}_b)$. In particular, $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \tau_{\mathfrak{C}}(\mathfrak{A})$ for every $a \in \mathfrak{A}$.

(b) The set $\{a : a \in \mathfrak{A}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}\}$ is order-dense in $\mathfrak{A}$.

(c) If $\mathfrak{A}$ is Dedekind complete and $\mathfrak{C}$ is order-closed in $\mathfrak{A}$, then $\mathfrak{C}_a$ is order-closed in $\mathfrak{A}_a$.

(d) If $a \in \mathfrak{A}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}$ then either $\mathfrak{A}_a = \mathfrak{C}_a$, so that $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = 0$ and a is a relative atom of $\mathfrak{A}$ over $\mathfrak{C}$ (definition: 331A), or $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \ge \omega$.

(e) If $\mathfrak{D}$ is another subalgebra of $\mathfrak{A}$ and $\mathfrak{D} \subseteq \mathfrak{C}$, then

$$\tau(\mathfrak{A}_a) = \tau_{\{0,a\}}(\mathfrak{A}_a) \ge \tau_{\mathfrak{D}_a}(\mathfrak{A}_a) \ge \tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \ge \tau_{\mathfrak{A}_a}(\mathfrak{A}_a) = 0$$

for every $a \in \mathfrak{A}$.

proof (a) Let $D \subseteq \mathfrak{A}_b$ be a set with cardinal $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b)$ such that $D \cup \mathfrak{C}_b$ τ-generates $\mathfrak{A}_b$. Set $D' = \{d \cap a : d \in D\}$. Then $D' \cup \mathfrak{C}_a$ τ-generates $\mathfrak{A}_a$. **P** Apply 313Mc to the map $d \mapsto d \cap a : \mathfrak{A}_b \to \mathfrak{A}_a$, as in 331Hc. **Q** Consequently

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \#(D') \le \#(D) = \tau_{\mathfrak{C}_b}(\mathfrak{A}_b),$$

as claimed. Setting $b = 1$ we get $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) \le \tau_{\mathfrak{C}}(\mathfrak{A})$.

(b) Just as in the proof of 332A, given $b \in \mathfrak{A} \setminus \{0\}$, there is an $a \in \mathfrak{A}_b \setminus \{0\}$ minimising $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$, and this a must be relatively Maharam-type-homogeneous over $\mathfrak{C}$.

(c) $\mathfrak{C}_a$ is the image of the Dedekind complete Boolean algebra $\mathfrak{C}$ under the order-continuous Boolean homomorphism $c \mapsto c \cap a$, so must be order-closed (314F(a-i)).

(d) Suppose that $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ is finite. Let $D \subseteq \mathfrak{A}_a$ be a finite set such that $D \cup \mathfrak{C}_a$ τ-generates $\mathfrak{A}_a$. Then there is a non-zero $b \in \mathfrak{A}_a$ such that $b \cap d$ is either 0 or b for every $d \in D$. But this means that $\mathfrak{C}_b = \{d \cap b : d \in D \cup \mathfrak{C}_a\}$, which τ-generates $\mathfrak{A}_b$; so that $\tau_{\mathfrak{C}_b}(\mathfrak{A}_b) = 0$. Since a is relatively Maharam-type-homogeneous over $\mathfrak{C}$, $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ must be zero, that is, $\mathfrak{A}_a = \mathfrak{C}_a$.

(e) The middle inequality is true just because $\mathfrak{A}_a$ will be τ-generated by $D \cup \mathfrak{C}_a$ whenever it is τ-generated by $D \cup \mathfrak{D}_a$. The neighbouring inequalities are special cases of the middle one, and the outer equalities are elementary.

333C Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras, and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Let $\phi : \mathfrak{C} \to \mathfrak{B}$ be a measure-preserving Boolean homomorphism.

(a) If, in the notation of 333A, $\tau_{\mathfrak{C}}(\mathfrak{A}) \le \tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$ for every non-zero $b \in \mathfrak{B}$, there is a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ extending ϕ.

(b) If $\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = \tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$ for every non-zero $a \in \mathfrak{A}$, $b \in \mathfrak{B}$, then there is a measure algebra isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ extending ϕ.

proof In both parts, the idea is to use the technique of the proof of 331I to construct π as the last of an increasing family $\langle \pi_\xi \rangle_{\xi \le \kappa}$ of measure-preserving homomorphisms from closed subalgebras $\mathfrak{C}_\xi$ of $\mathfrak{A}$, where $\kappa = \tau_{\mathfrak{C}}(\mathfrak{A})$. Let $\langle a_\xi \rangle_{\xi < \kappa}$ be a family in $\mathfrak{A}$ such that $\mathfrak{C} \cup \{a_\xi : \xi < \kappa\}$ τ-generates $\mathfrak{A}$. Write $\mathfrak{D}$ for $\phi[\mathfrak{C}]$; remember that $\mathfrak{D}$ is a closed subalgebra of $\mathfrak{B}$ (324L).

(a)(i) In this case, we can describe the $\mathfrak{C}_\xi$ immediately; $\mathfrak{C}_\xi$ will be the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a_\eta : \eta < \xi\}$. The induction starts with $\mathfrak{C}_0 = \mathfrak{C}$, $\pi_0 = \phi$.

(ii) For the inductive step to a successor ordinal $\xi + 1$, where $\xi < \kappa$, suppose that $\mathfrak{C}_\xi$ and π_ξ have been defined. Take any non-zero $b \in \mathfrak{B}$. We are supposing that $\tau_{\mathfrak{D}_b}(\mathfrak{B}_b) \ge \kappa > \#(\xi)$, so $\mathfrak{B}_b$ cannot be τ-generated by

$$D = \mathfrak{D}_b \cup \{b \cap \pi_\xi a_\eta : \eta < \xi\} = \pi_\xi[\mathfrak{C}]_b \cup \{b \cap \pi_\xi a_\eta : \eta < \xi\} = \psi[\mathfrak{C} \cup \{a_\eta : \eta < \xi\}],$$

writing $\psi c = b \cap \pi_\xi c$ for $c \in \mathfrak{C}_\xi$. As ψ is order-continuous, $\psi[\mathfrak{C}_\xi]$ is precisely the closed subalgebra of $\mathfrak{B}_b$ generated by D (314H), and is therefore not the whole of $\mathfrak{B}_b$.

But this means that $\mathfrak{B}_b \neq \{b \cap \pi_\xi c : c \in \mathfrak{C}_\xi\}$. As b is arbitrary, π_ξ satisfies the conditions of 331D, and has an extension to a measure-preserving Boolean homomorphism $\pi_{\xi+1} : \mathfrak{C}_{\xi+1} \to \mathfrak{B}$, since $\mathfrak{C}_{\xi+1}$ is just the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a_\xi\}$.

(iii) For the inductive step to a non-zero limit ordinal $\xi \leq \kappa$, we can argue exactly as in part (d) of the proof of 331I; $\mathfrak{C}_\xi$ will be the metric closure of $\mathfrak{C}^*_\xi = \bigcup_{\eta<\xi} \mathfrak{C}_\eta$, so we can take $\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{B}$ to be the unique measure-preserving homomorphism extending $\pi^*_\xi = \bigcup_{\eta<\xi} \pi_\eta$.

Thus the induction proceeds, and evidently $\pi = \pi_\kappa$ will be a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ extending ϕ.

(b) (This is rather closer to the proof of 331I, being indeed a direct generalization of it.) Observe that the hypothesis (b) implies that $1_{\mathfrak{A}}$ is relatively Maharam-type-homogeneous over $\mathfrak{C}$; so either $\kappa = 0$, in which case $\mathfrak{A} = \mathfrak{C}$, $\mathfrak{B} = \phi[\mathfrak{C}]$ and the result is trivial, or $\kappa \geq \omega$, by 333Bd. Let us therefore take it that κ is infinite.

We are supposing, among other things, that $\tau_{\mathfrak{D}}(\mathfrak{B}) = \kappa$; let $\langle b_\xi \rangle_{\xi<\kappa}$ be a family in $\mathfrak{B}$ such that $\mathfrak{B}$ is τ-generated by $\mathfrak{D} \cup \{b_\xi : \xi < \kappa\}$. This time, as in 331I, we shall have to choose further families $\langle a'_\xi \rangle_{\xi<\kappa}$ and $\langle b'_\xi \rangle_{\xi<\kappa}$, and

$\mathfrak{C}_\xi$ will be the closed subalgebra of $\mathfrak{A}$ generated by

$$\mathfrak{C} \cup \{a_\eta : \eta < \xi\} \cup \{a'_\eta : \eta < \xi\},$$

$\mathfrak{D}_\xi$ will be the closed subalgebra of $\mathfrak{B}$ generated by

$$\mathfrak{D} \cup \{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta < \xi\},$$

$\pi_\xi : \mathfrak{C}_\xi \to \mathfrak{D}_\xi$ will be a measure-preserving homomorphism.

The induction will start with $\mathfrak{C}_0 = \mathfrak{C}$, $\mathfrak{D}_0 = \mathfrak{D}$ and $\pi_0 = \phi$, as in (a).

(i) For the inductive step to a successor ordinal $\xi + 1$, where $\xi < \kappa$, suppose that $\mathfrak{C}_\xi$, $\mathfrak{D}_\xi$ and π_ξ have been defined.

($\boldsymbol{\alpha}$) Let $b \in \mathfrak{B} \setminus \{0\}$. Because

$$\tau_{\mathfrak{D}_b}(\mathfrak{B}_b) = \kappa > \#(\{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta < \xi\}),$$

$\mathfrak{B}_b$ cannot be τ-generated by $\mathfrak{D}_b \cup \{b \cap b_\eta : \eta < \xi\} \cup \{b \cap b'_\eta : \eta < \xi\}$, and cannot be equal to $\{b \cap d : d \in \mathfrak{D}_\xi\}$. As b is arbitrary, there is an extension of π_ξ to a measure-preserving homomorphism ϕ_ξ from $\mathfrak{C}'_\xi$ to $\mathfrak{B}$, where $\mathfrak{C}'_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a_\eta : \eta \leq \xi\} \cup \{a'_\eta : \eta < \xi\}$. Setting $b'_\xi = \phi_\xi(a_\xi)$, the image $\mathfrak{D}'_\xi = \phi_\xi[\mathfrak{C}'_\xi]$ will be the closed subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D} \cup \{b_\eta : \eta < \xi\} \cup \{b'_\eta : \eta \leq \xi\}$.

($\boldsymbol{\beta}$) Next, as in 331I, we must repeat the argument of (α), applying it now to $\phi_\xi^{-1} : \mathfrak{D}_\xi \to \mathfrak{A}$. If $a \in \mathfrak{A} \setminus \{0\}$,

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = \kappa > \#(\{a_\eta : \eta \leq \xi\} \cup \{a'_\eta : \eta < \xi\}),$$

so that $\mathfrak{A}_a$ cannot be $\{a \cap c : c \in \mathfrak{C}'_\xi\}$. As a is arbitrary, ϕ_ξ^{-1} has an extension to a measure-preserving homomorphism $\psi_\xi : \mathfrak{D}_{\xi+1} \to \mathfrak{C}_{\xi+1}$, where $\mathfrak{D}_{\xi+1}$ is the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D}'_\xi \cup \{b_\xi\}$, that is, the closed subalgebra of $\mathfrak{B}$ generated by $\mathfrak{D} \cup \{b_\eta : \eta \leq \xi\} \cup \{b'_\eta : \eta \leq \xi\}$, and $\mathfrak{C}_{\xi+1}$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}'_\xi \cup \{a'_\xi\}$, setting $a'_\xi = \psi_\xi(b_\xi)$.

We can therefore take $\pi_{\xi+1} = \psi_\xi^{-1} : \mathfrak{C}_{\xi+1} \to \mathfrak{D}_{\xi+1}$, as in 331I.

(ii) The inductive step to a non-zero limit ordinal $\xi \leq \kappa$ is exactly the same as in (a) above or in 331I; $\mathfrak{C}_\xi$ is the metric closure of $\mathfrak{C}^*_\xi = \bigcup_{\eta<\xi} \mathfrak{C}_\eta$, $\mathfrak{D}_\xi$ is the metric closure of $\mathfrak{D}^*_\xi = \bigcup_{\eta<\xi} \mathfrak{D}_\eta$, and π_ξ is the unique measure-preserving homomorphism from $\mathfrak{C}_\xi$ to $\mathfrak{D}_\xi$ extending every π_η for $\eta < \xi$.

(iii) The induction stops, as before, with $\pi = \pi_\kappa : \mathfrak{C}_\kappa \to \mathfrak{D}_\kappa$, where $\mathfrak{C}_\kappa = \mathfrak{A}$, $\mathfrak{D}_\kappa = \mathfrak{B}$.

333D Corollary Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Suppose that

$$\tau(\mathfrak{C}) < \max(\omega, \tau(\mathfrak{A})) \le \min\{\tau(\mathfrak{B}_b) : b \in \mathfrak{B} \setminus \{0\}\}.$$

Then any measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ can be extended to a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$.

proof Set $\kappa = \min\{\tau(\mathfrak{B}_b) : b \in \mathfrak{B} \setminus \{0\}\}$. Then for any non-zero $b \in \mathfrak{B}$,

$$\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b) \ge \kappa.$$

P There is a set $C \subseteq \mathfrak{C}$, with cardinal $\tau(\mathfrak{C})$, which τ-generates $\mathfrak{C}$, so that $C' = \{b \cap \phi c : c \in C\}$ τ-generates $\phi[\mathfrak{C}]_b$. Now there is a set $D \subseteq \mathfrak{B}_b$, with cardinal $\tau_{\phi[\mathfrak{C}]_b}(\mathfrak{B}_b)$, such that $\phi[\mathfrak{C}]_b \cup D$ τ-generates $\mathfrak{B}_b$. In this case $C' \cup D$ must τ-generate $\mathfrak{B}_b$, so $\kappa \le \#(C' \cup D)$. But $\#(C') \le \#(C) < \kappa$ and κ is infinite, so we must have $\#(D) \ge \kappa$, as claimed. **Q**

On the other hand, $\tau_{\mathfrak{C}}(\mathfrak{A}) \le \tau(\mathfrak{A}) \le \kappa$. So we can apply 333Ca to give the result.

333E Theorem Let $(\mathfrak{C}, \bar{\mu})$ be a totally finite measure algebra and κ an infinite cardinal. Let $(\mathfrak{A}, \bar{\lambda})$ be the localizable measure algebra free product of $(\mathfrak{C}, \bar{\mu})$ and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ (notation: 333Ad), and $\varepsilon : \mathfrak{C} \to \mathfrak{A}$ the corresponding homomorphism. Then for any non-zero $a \in \mathfrak{A}$,

$$\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a) = \kappa,$$

in the notation of 333A above.

proof Recall from 325Dd that $\varepsilon[\mathfrak{C}]$ is a closed subalgebra of $\mathfrak{A}$.

(a) Let $\langle e_\xi \rangle_{\xi<\kappa}$ be the standard generating family in $\mathfrak{B}_\kappa$, corresponding to the sets $\{x : x \in \{0,1\}^\kappa, x(\xi) = 1\}$. Let $\varepsilon' : \mathfrak{B}_\kappa \to \mathfrak{A}$ be the canonical map, and set $e'_\xi = \varepsilon' e_\xi$ for each ξ.

We know that $\{e_\xi : \xi < \kappa\}$ τ-generates $\mathfrak{B}_\kappa$ (see part (a) of the proof of 331K). Consequently $\varepsilon[\mathfrak{C}] \cup \{e'_\xi : \xi < \kappa\}$ τ-generates $\mathfrak{A}$. **P** Let $\mathfrak{A}_1$ be the closed subalgebra of $\mathfrak{A}$ generated by $\varepsilon[\mathfrak{C}] \cup \{e'_\xi : \xi < \kappa\}$. Because $\varepsilon' : \mathfrak{B}_\kappa \to \mathfrak{A}$ is order-continuous (325Da), $\varepsilon'[\mathfrak{B}_\kappa] \subseteq \mathfrak{A}_1$ (313Mb). But this means that $\mathfrak{A}_1$ includes $\varepsilon[\mathfrak{C}] \cup \varepsilon'[\mathfrak{B}_\kappa]$ and therefore includes the image of $\mathfrak{C} \otimes \mathfrak{B}_\kappa$ in $\mathfrak{A}$; because this is topologically dense in $\mathfrak{A}$ (325Dc), $\mathfrak{A}_1 = \mathfrak{A}$, as claimed. **Q**

(b) It follows that

$$\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a) \le \tau_{\varepsilon[\mathfrak{C}]}(\mathfrak{A}) \le \kappa$$

(333Ba).

(c) We need to know that if $\xi < \kappa$ and a belongs to the closed subalgebra $\mathfrak{C}_\xi$ of $\mathfrak{A}$ generated by $\varepsilon[\mathfrak{C}] \cup \{e'_\eta : \eta \ne \xi\}$, then $\bar{\lambda}(a \cap e'_\xi) = \frac{1}{2}\bar{\lambda}a$. **P** Set

$$E = \varepsilon[\mathfrak{C}] \cup \{e'_\eta : \eta \ne \xi\}, \quad F = \{a_0 \cap \ldots \cap a_n : a_0, \ldots, a_n \in E\}.$$

Then every member of F is expressible in the form

$$a = \varepsilon c \cap \inf_{\eta \in J} e'_\eta,$$

where $c \in \mathfrak{C}$ and $J \subseteq \kappa \setminus \{\xi\}$ is finite. Now

$$\bar{\lambda}a = \bar{\mu}c \cdot \bar{\nu}(\inf_{\eta \in J} e_\eta) = 2^{-\#(J)}\bar{\mu}c,$$

$$\bar{\lambda}(e'_\xi \cap a) = \bar{\mu}c \cdot \bar{\nu}(e_\xi \cap \inf_{\eta \in J} e_\eta) = 2^{-\#(J \cup \{\xi\})}\bar{\mu}c = \frac{1}{2}\bar{\lambda}a.$$

Now consider the set

$$G = \{a : a \in \mathfrak{A},\ \bar{\lambda}(e_\xi \cap a) = \frac{1}{2}\bar{\lambda}a\}.$$

We have $1_{\mathfrak{A}} \in F \subseteq G$, and F is closed under $\cap$. Secondly, if a, $a' \in G$ and $a \subseteq a'$, then

$$\bar{\lambda}(e_\xi \cap (a' \setminus a)) = \bar{\lambda}(e_\xi \cap a') - \bar{\lambda}(e_\xi \cap a) = \frac{1}{2}\bar{\lambda}a' - \frac{1}{2}\bar{\lambda}a = \frac{1}{2}\bar{\lambda}(a' \setminus a),$$

so $a' \setminus a \in G$. Also, if $H \subseteq G$ is non-empty and upwards-directed,

$$\bar\lambda(e_\xi\cap\sup H)=\bar\lambda(\sup_{a\in H}e_\xi\cap a)=\sup_{a\in H}\bar\lambda(e_\xi\cap a)=\sup_{a\in H}\frac12\bar\lambda a=\frac12\bar\lambda(\sup H),$$

so $\sup H\in G$. By the Monotone Class Theorem (313Gc), G includes the order-closed subalgebra of $\mathfrak{D}$ generated by F. But this is just $\mathfrak{E}_\xi$. **Q**

(d) The next step is to see that $\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)>0$. **P** By (a) and 323J, $\mathfrak{A}$ is the metric closure of the subalgebra $\mathfrak{A}_0$ generated by $\varepsilon[\mathfrak{C}]\cup\{e'_\eta:\eta<\kappa\}$, so there must be an $a_0\in\mathfrak{A}_0$ such that $\bar\lambda(a_0\triangle a)\le\frac14\bar\lambda a$. Now there is a finite $J\subseteq\kappa$ such that a_0 belongs to the subalgebra $\mathfrak{A}_1$ generated by $\varepsilon[\mathfrak{C}]\cup\{e'_\eta:\eta\in J\}$. Take any $\xi\in\kappa\setminus J$ (this is where I use the hypothesis that κ is infinite). If $c\in\mathfrak{C}$, then by (c) we have

$$\begin{aligned}\bar\lambda((a\cap\varepsilon c)\triangle(a\cap e'_\xi))&=\bar\lambda(a\cap(\varepsilon c\triangle e'_\xi))\ge\bar\lambda(a_0\cap(\varepsilon c\triangle e'_\xi))-\bar\lambda(a\triangle a_0)\\&=\bar\lambda(a_0\cap e'_\xi)+\bar\lambda(a_0\cap\varepsilon c)-2\bar\lambda(a_0\cap\varepsilon c\cap e'_\xi)-\bar\lambda(a\triangle a_0)\\&=\frac12\bar\lambda a_0-\bar\lambda(a\triangle a_0)\end{aligned}$$

(because both a_0 and $a_0\cap\varepsilon c$ belong to $\mathfrak{E}_\xi$)

$$\ge\frac12\bar\lambda a-\frac32\bar\lambda(a\triangle a_0)>0.$$

Thus $a\cap e'_\xi$ is not of the form $a\cap\varepsilon c$ for any $c\in\mathfrak{C}$, and $\mathfrak{A}_a\ne\varepsilon[\mathfrak{C}]_a$, so that $\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)>0$. **Q**

(e) It follows that $\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)$ is infinite. **P** There is a non-zero $d\subseteq a$ which is relatively Maharam-type-homogeneous over $\varepsilon[\mathfrak{C}]$. By (d), applied to d, $\tau_{\varepsilon[\mathfrak{C}]_d}(\mathfrak{A}_d)>0$; but now 333Bd tells us that $\tau_{\varepsilon[\mathfrak{C}]_d}(\mathfrak{A}_d)$ must be infinite, so $\tau_{\varepsilon[\mathfrak{C}]_a}(\mathfrak{A}_a)$ is infinite. **Q**

(f) If $\kappa=\omega$, we can stop here. If $\kappa>\omega$, we continue, as follows. Let $D\subseteq\mathfrak{A}_a$ be any set with cardinal less than κ. Each $d\in D\cup\{a\}$ belongs to the closed subalgebra of $\mathfrak{A}$ generated by $C=\varepsilon[\mathfrak{C}]\cup\{e'_\xi:\xi<\kappa\}$. But because $\mathfrak{A}$ is ccc, this is just the σ-subalgebra of $\mathfrak{A}$ generated by C (331Ge). So d belongs to the closed subalgebra of $\mathfrak{A}$ generated by some countable subset C_d of C, by 331Gd. Now $J_d=\{\eta:e'_\eta\in C_d\}$ is countable. Set $J=\bigcup_{d\in D\cup\{a\}}J_d$; then

$$\#(J)\le\max(\omega,\#(D\cup\{a\}))=\max(\omega,\#(D))<\kappa,$$

so $J\ne\kappa$, and there is a $\xi\in\kappa\setminus J$. Accordingly $\varepsilon[\mathfrak{C}]\cup D\cup\{a\}$ is included in $\mathfrak{E}_\xi$, as defined in (c) above, and $\varepsilon[\mathfrak{C}]_a\cup D\subseteq\mathfrak{E}_\xi$. As $\mathfrak{A}_a\cap\mathfrak{E}_\xi$ is a closed subalgebra of $\mathfrak{A}_a$, it includes the closed subalgebra generated by $\varepsilon[\mathfrak{C}]_a\cup D$. But $a\cap e'_\xi$ surely does not belong to $\mathfrak{E}_\xi$, since

$$\bar\lambda(a\cap e'_\xi\cap e'_\xi)=\bar\lambda(a\cap e'_\xi)=\frac12\bar\lambda a>0,$$

and $\bar\lambda(a\cap e'_\xi\cap e'_\xi)\ne\frac12\bar\lambda(a\cap e'_\xi)$. Thus $a\cap e'_\xi$ cannot belong to the closed subalgebra of $\mathfrak{A}_a$ generated by $\varepsilon[\mathfrak{C}]_a\cup D$, and $\varepsilon[\mathfrak{C}]_a\cup D$ does not τ-generate $\mathfrak{A}_a$. As D is arbitrary, $\tau_{\phi[\mathfrak{C}]_a}(\mathfrak{A}_a)\ge\kappa$.

This completes the proof.

333F Corollary Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ and κ an infinite cardinal.

(a) Suppose that $\kappa\ge\tau_{\mathfrak{C}}(\mathfrak{A})$. Let $(\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa,\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu\restriction\mathfrak{C})$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$, and $\varepsilon:\mathfrak{C}\to\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa$ the corresponding homomorphism. Then there is a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa$ extending ε.

(b) Suppose further that $\kappa=\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)$ for every non-zero $a\in\mathfrak{A}$. Then π can be taken to be an isomorphism.

proof All we have to do is apply 333C with $\mathfrak{B}=\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa$, using 333E to see that the hypothesis

$$\tau_{\varepsilon[\mathfrak{C}]_b}(\mathfrak{B}_b)=\kappa\text{ for every non-zero }b\in\mathfrak{B}$$

is satisfied.

333G Corollary Let $(\mathfrak{C},\bar\mu)$ be a totally finite measure algebra. Suppose that $\kappa\ge\max(\omega,\tau(\mathfrak{C}))$ is a cardinal. Let $(\mathfrak{C}\widehat\otimes\mathfrak{B}_\kappa,\bar\lambda)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu)$ and $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$. Then

(a) $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ is Maharam-type-homogeneous, with Maharam type κ if $\mathfrak{C} \neq \{0\}$;

(b) for every measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{C}$ there is a measure-preserving automorphism $\pi : \mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa \to \mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ such that $\pi(c \otimes 1) = \phi c \otimes 1$ for every $c \in \mathfrak{C}$, writing $c \otimes 1$ for the canonical image in $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$ of any $c \in \mathfrak{C}$.

proof Write $\mathfrak{A}$ for $\mathfrak{C}\widehat{\otimes}\mathfrak{B}_\kappa$, as in 333E, and $\mathfrak{D}$ for $\{c \otimes 1 : c \in \mathfrak{C}\} \subseteq \mathfrak{A}$.

(a) If $C \subseteq \mathfrak{C}$ is a set with cardinal $\tau(\mathfrak{C})$ which τ-generates $\mathfrak{C}$, and $B \subseteq \mathfrak{B}_\kappa$ a set with cardinal κ which τ-generates $\mathfrak{B}_\kappa$ (331K), then $\{c \otimes b : c \in C,\, b \in B\}$ is a set with cardinal at most $\max(\omega, \tau(\mathfrak{C}), \kappa) = \kappa$ which τ-generates $\mathfrak{A}$ (because the subalgebra it generates is topologically dense in $\mathfrak{A}$, by 325Dc). So $\tau(\mathfrak{A}) \leq \kappa$. On the other hand, if $a \in \mathfrak{A}$ is non-zero, then $\tau(\mathfrak{A}_a) \geq \tau_{\mathfrak{D}_a}(\mathfrak{A}_a) \geq \kappa$, by 333E; so $\mathfrak{A}$ is Maharam-type-homogeneous, with Maharam type κ unless $\mathfrak{C} = \{0\}$.

(b) We have a measure-preserving automorphism $\phi_1 : \mathfrak{D} \to \mathfrak{D}$ defined by setting $\phi_1(c \otimes 1) = \phi c \otimes 1$ for every $c \in \mathfrak{C}$. Because $\phi_1[\mathfrak{D}] \subseteq \mathfrak{D}$, 333Be and 333E tell us that

$$\kappa = \tau(\mathfrak{A}_a) \geq \tau_{\phi_1[\mathfrak{D}]_a}(\mathfrak{A}_a) \geq \tau_{\mathfrak{D}_a}(\mathfrak{A}_a) = \kappa$$

for every non-zero $a \in \mathfrak{A}$, so we can use 333Cb, with $\mathfrak{B} = \mathfrak{A}$, to see that ϕ_1 can be extended to a measure-preserving automorphism on $\mathfrak{A}$.

333H I turn now to the classification of closed subalgebras.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Then there are $\langle \mu_i \rangle_{i \in I}$, $\langle c_i \rangle_{i \in I}$, $\langle \kappa_i \rangle_{i \in I}$ such that

for each $i \in I$, μ_i is a non-negative completely additive functional on $\mathfrak{C}$,

$c_i = [\![\mu_i > 0]\!] \in \mathfrak{C}$,

κ_i is 0 or an infinite cardinal,

$(\mathfrak{C}_{c_i}, \mu_i \restriction \mathfrak{C}_{c_i})$ is a totally finite measure algebra, writing $\mathfrak{C}_{c_i}$ for the principal ideal of $\mathfrak{C}$ generated by c_i,

$\sum_{i \in I} \mu_i c = \bar\mu c$ for every $c \in \mathfrak{C}$,

there is a measure-preserving isomorphism π from $\mathfrak{A}$ to the simple product $\prod_{i \in I} \mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of the localizable measure algebra free products $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of $(\mathfrak{C}_{c_i}, \mu_i \restriction \mathfrak{C}_{c_i})$ and $(\mathfrak{B}_{\kappa_i}, \bar\nu_{\kappa_i})$.

Moreover, π may be taken such that

for every $c \in \mathfrak{C}$, $\pi c = \langle (c \cap c_i) \otimes 1 \rangle_{i \in I}$, writing $c \otimes 1$ for the image in $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ of $c \in \mathfrak{C}_{c_i}$.

Remark Recall that $[\![\mu_i > 0]\!]$ is that element of $\mathfrak{C}$ such that $\mu_i c > 0$ whenever $c \in \mathfrak{C}$ and $0 \neq c \subseteq [\![\mu_i > 0]\!]$, $\mu_i c \leq 0$ whenever $c \in \mathfrak{C}$ and $c \cap [\![\mu_i > 0]\!] = 0$ (326S).

proof (a) Let A be the set of those elements of $\mathfrak{A}$ which are relatively Maharam-type-homogeneous over $\mathfrak{C}$ (see 333Ac). By 333Bb, A is order-dense in $\mathfrak{A}$ (compare part (a) of the proof of 332B), and consequently $A' = \{a : a \in A,\, \bar\mu a < \infty\}$ is order-dense in $\mathfrak{A}$. So there is a partition of unity $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ consisting of members of A' (313K). For each $i \in I$, set $\mu_i c = \bar\mu(a_i \cap c)$ for every $c \in \mathfrak{C}$; then μ_i is non-negative, and it is completely additive by 327E. Because $\langle a_i \rangle_{i \in I}$ is a partition of unity in $\mathfrak{A}$,

$$\bar\mu c = \sum_{i \in I} \bar\mu(c \cap a_i) = \sum_{i \in I} \mu_i c$$

for every $c \in \mathfrak{C}$. Next, $(\mathfrak{C}_{c_i}, \mu_i \restriction \mathfrak{C}_{c_i})$ is a totally finite measure algebra. **P** $\mathfrak{C}_{c_i}$ is a Dedekind σ-complete Boolean algebra because $\mathfrak{C}$ is. $\mu_i \restriction \mathfrak{C}_{c_i}$ is a non-negative countably additive functional because μ_i is. If $c \in \mathfrak{C}_{c_i}$ and $\mu_i c = 0$, then $c = 0$ by the choice of c_i. **Q** Note also that

$$\bar\mu(a_i \setminus c_i) = \mu_i(1 \setminus c_i) = 0,$$

so that $a_i \subseteq c_i$.

(b) By 333Bd, any finite κ_i must actually be zero. The next element we need is the fact that, for each $i \in I$, we have a measure-preserving isomorphism $c \mapsto c \cap a_i$ from $(\mathfrak{C}_{c_i}, \mu_i \restriction \mathfrak{C}_{c_i})$ to $(\mathfrak{C}_{a_i}, \bar\mu \restriction \mathfrak{C}_{a_i})$. **P** Of course this is a ring homomorphism. Because $a_i \subseteq c_i$, it is a surjective Boolean homomorphism. It is measure-preserving by the definition of μ_i, and therefore injective. **Q**

(c) Still focusing on a particular $i \in I$, let $\mathfrak{A}_{a_i}$ be the principal ideal of $\mathfrak{A}$ generated by a_i. Then we have a measure-preserving isomorphism $\tilde{\pi}_i : \mathfrak{A}_{a_i} \to \mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, extending the canonical homomorphism $c \mapsto c \otimes 1 : \mathfrak{C}_{a_i} \to \mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$. **P** When κ_i is infinite, this is just 333Fb. But the only other case is when $\kappa_i = 0$, that is, $\mathfrak{C}_{a_i} = \mathfrak{A}_{a_i}$, while $\mathfrak{B}_{\kappa_i} = \{0, 1\}$ and $\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i} \cong \mathfrak{C}_{c_i}$. **Q**

The isomorphism between $(\mathfrak{C}_{c_i}, \mu_i{\restriction}\mathfrak{C}_{c_i})$ and $(\mathfrak{C}_{a_i}, \bar{\mu}{\restriction}\mathfrak{C}_{a_i})$ induces an isomorphism between $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ and $\mathfrak{C}_{a_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$. So we have a measure-preserving isomorphism $\pi_i : \mathfrak{A}_{a_i} \to \mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$ such that $\pi_i(c \cap a_i) = c \otimes 1$ for every $c \in \mathfrak{C}_{c_i}$.

(d) By 322Le, we have a measure-preserving isomorphism $a \mapsto \langle a \cap a_i\rangle_{i\in I} : \mathfrak{A} \to \prod_{i\in I} \mathfrak{A}_{a_i}$. Putting this together with the isomorphisms of (c), we have a measure-preserving isomorphism π from $\mathfrak{A}$ to $\prod_{i\in I} \mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, setting $\pi a = \langle \pi_i(a \cap a_i)\rangle_{i\in I}$ for $a \in \mathfrak{A}$. Observe that, for $c \in \mathfrak{C}$,

$$\pi c = \langle \pi_i(c \cap a_i)\rangle_{i\in I} = \langle (c \cap c_i) \otimes 1\rangle_{i\in I},$$

as required.

333I Remarks (a) I hope it is clear that whenever $(\mathfrak{C}, \bar{\mu})$ is a Dedekind complete measure algebra, $\langle \mu_i\rangle_{i\in I}$ is a family of non-negative completely additive functionals on $\mathfrak{C}$ such that $\sum_{i\in I} \mu_i = \bar{\mu}$, and $\langle \kappa_i\rangle_{i\in I}$ is a family of cardinals all infinite or zero, then the construction above can be applied to give a measure algebra $(\mathfrak{A}, \bar{\lambda})$, the product of the family $\langle \mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}\rangle_{i\in I}$, together with an order-continuous measure-preserving homomorphism $\pi : \mathfrak{C} \to \mathfrak{A}$; and that the partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ corresponding to this product (315E) has $\mu_i c = \bar{\lambda}(a_i \cap \pi c)$ for every $c \in \mathfrak{C}$ and $i \in I$, while each principal ideal $\mathfrak{A}_{a_i}$ can be identified with $\mathfrak{C}_{c_i}\widehat{\otimes}\mathfrak{B}_{\kappa_i}$, so that a_i is relatively Maharam-type-homogeneous over $\pi[\mathfrak{C}]$. Thus any structure $(\mathfrak{C}, \bar{\mu}, \langle \mu_i\rangle_{i\in I}, \langle \kappa_i\rangle_{i\in I})$ of the type described here corresponds to an embedding of $\mathfrak{C}$ as a closed subalgebra of a localizable measure algebra.

(b) The obvious next step is to seek a complete classification of objects $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$, where $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra and $\mathfrak{C}$ is a closed subalgebra, corresponding to the classification of localizable measure algebras in terms of the magnitudes of their Maharam-type-κ components in 332J. The general case seems to be complex. But I can deal with the special case in which $(\mathfrak{A}, \bar{\mu})$ is totally finite. In this case, we have the following facts.

333J Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\mathfrak{C}$ a closed subalgebra. Let A be the set of relative atoms of $\mathfrak{A}$ over $\mathfrak{C}$. Then there is a unique sequence $\langle \mu_n\rangle_{n\in\mathbb{N}}$ of additive functionals on $\mathfrak{C}$ such that (i) $\mu_{n+1} \le \mu_n$ for every n (ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $\sup_{n\in\mathbb{N}} a_n = \sup A$ and $\mu_n c = \bar{\mu}(a_n \cap c)$ for every $n \in \mathbb{N}$ and $c \in \mathfrak{C}$.

Remark I hope it is plain from my wording that it is the μ_n which are unique, not the a_n.

proof (a) For each $a \in \mathfrak{A}$ set $\theta_a(c) = \bar{\mu}(c \cap a)$ for $c \in \mathfrak{C}$. Then θ_a is a non-negative completely additive real-valued functional on $\mathfrak{C}$ (see 326Od).

The key step is I suppose in (c) below; I approach by a two-stage argument. For each $b \in \mathfrak{A}$ write $A_b^\perp$ for $\{a : a \in A, a \cap b = 0\}$.

(b) For every $b \in \mathfrak{A}$ and non-zero $c \in \mathfrak{C}$ there are $a \in A_b^\perp$, $c' \in \mathfrak{C}$ such that $0 \ne c' \subseteq c$ and $\theta_a(d) \ge \theta_e(d)$ whenever $d \in \mathfrak{C}$, $e \in A_b^\perp$ and $d \subseteq c'$. **P?** Otherwise, choose $\langle a_n\rangle_{n\in\mathbb{N}}$ and $\langle c_n\rangle_{n\in\mathbb{N}}$ as follows. Since 0, c won't serve for a, c', there must be an $a_0 \in A_b^\perp$ such that $\theta_{a_0}(c) > 0$. Let $\delta > 0$ be such that $\theta_{a_0}(c) > \delta\bar{\mu}c$ and set $c_0 = c \cap [\![\theta_{a_0} > \delta\bar{\mu}{\restriction}\mathfrak{C}]\!]$; then $c_0 \in \mathfrak{C}$ and $0 \ne c_0 \subseteq c$. Given that $a_n \in A_b^\perp$, $c_n \in \mathfrak{C}$ and $0 \ne c_n \subseteq c$, then there must be $a_{n+1} \in A_b^\perp$, $d_n \in \mathfrak{C}$ such that $d_n \subseteq c_n$ and $\theta_{a_{n+1}}(d_n) > \theta_{a_n}(d_n)$. Set $c_{n+1} = d_n \cap [\![\theta_{a_{n+1}} > \theta_{a_n}]\!]$, so that $c_{n+1} \in \mathfrak{C}$ and $0 \ne c_{n+1} \subseteq c_n$, and continue.

There is some $n \in \mathbb{N}$ such that $n\delta \ge 1$. For any $i < n$, the construction ensures that

$$0 \ne c_{n+1} \subseteq c_{i+1} \subseteq [\![\theta_{a_{i+1}} > \theta_{a_i}]\!],$$

so $\theta_{a_i}(c_{n+1}) < \theta_{a_{i+1}}(c_{n+1})$; also $c_{n+1} \subseteq c_0$ so

$$\bar{\mu}(a_i \cap c_{n+1}) = \theta_{a_i}(c_{n+1}) \ge \theta_{a_0}(c_{n+1}) > \delta\bar{\mu}c_{n+1}.$$

But this means that $\sum_{i=0}^{n-1}\bar\mu(a_i\cap c_{n+1})>\bar\mu c_{n+1}$ and there must be distinct j, $k<n$ such that $a_j\cap a_k\cap c_{n+1}$ is non-zero. Because a_j, $a_k\in A$ there are d', $d''\in\mathfrak C$ such that $a_j\cap a_k=a_j\cap d'=a_k\cap d''$; set $d=c_{n+1}\cap d'\cap d''$, so that $d\in\mathfrak C$ and

$$a_j\cap d=a_j\cap a_k\cap c_{n+1}=a_k\cap d,\quad \theta_{a_j}(d)=\bar\mu(a_j\cap a_k\cap c_{n+1})=\theta_{a_k}(d).$$

But as $0\ne d\subseteq[\![\theta_{a_{i+1}}>\theta_{a_i}]\!]$ for every $i<n$, $\theta_{a_0}(d)<\theta_{a_1}(d)<\ldots<\theta_{a_n}(d)$, so this is impossible. **XQ**

(c) Now for a global, rather than local, version of the same idea. For every $b\in\mathfrak A$ there is an $a\in A_b^\perp$ such that and $\theta_a\ge\theta_e$ whenever $e\in A_b^\perp$. **P** (i) By (b), the set C of those $c\in\mathfrak C$ such that there is an $a\in A_b^\perp$ such that $\theta_a{\restriction}\mathfrak C_c\ge\theta_e{\restriction}\mathfrak C_c$ for every $e\in A_b^\perp$ is order-dense in $\mathfrak C$. Let $\langle c_i\rangle_{i\in I}$ be a partition of unity in $\mathfrak C$ consisting of members of C, and for each $i\in I$ choose $a_i\in A_b^\perp$ such that $\theta_{a_i}{\restriction}\mathfrak C_{c_i}\ge\theta_e{\restriction}\mathfrak C_{c_i}$ for every $e\in A_b^\perp$. Consider $a=\sup_{i\in I}a_i\cap c_i$. (ii) If $a'\in\mathfrak A$ and $a'\subseteq a$, then for each $i\in I$ there is a $d_i\in\mathfrak C$ such that $a_i\cap a'=a_i\cap d_i$. Set $d'=\sup_{i\in I}c_i\cap d_i$; then (because $\langle c_i\rangle_{i\in I}$ is disjoint)

$$a\cap d'=\sup_{i\in I}a_i\cap c_i\cap d_i=\sup_{i\in I}a_i\cap c_i\cap a'=a\cap a'=a'.$$

As a' is arbitrary, this shows that $a\in A$. (iii) Of course $a\cap b=0$, so $a\in A_b^\perp$. Now take any $e\in A_b^\perp$ and $d\in\mathfrak C$. Then

$$\theta_a(d)=\sum_{i\in I}\theta_{a_i}(c_i\cap d)\ge\sum_{i\in I}\theta_e(c_i\cap d)=\theta_e(d).$$

So this a has the required property. **Q**

(d) Choose $\langle a_n\rangle_{n\in\mathbb N}$ inductively in A so that, for each n, $a_n\cap\sup_{i<n}a_i=0$ and $\theta_{a_n}\ge\theta_e$ whenever $e\in A$ and $e\cap\sup_{i<n}a_i=0$. Set $\mu_n=\theta_{a_n}$. Because $a_{n+1}\cap\sup_{i<n}a_i=0$, $\mu_{n+1}\le\mu_n$ for each n. Also $\sup_{n\in\mathbb N}a_n=\sup A$. **P** Take any $a\in A$ and set $e=a\setminus\sup_{n\in\mathbb N}a_n$. Then $e\in A$ and, for any $n\in\mathbb N$, $e\cap\sup_{i<n}a_i=0$, so $\theta_e\le\theta_{a_n}$ and

$$\bar\mu e=\theta_e(1)\le\theta_{a_n}(1)=\bar\mu a_n.$$

But as $\langle a_n\rangle_{n\in\mathbb N}$ is disjoint, this means that $e=0$, that is, $a\subseteq\sup_{n\in\mathbb N}a_n$. As a is arbitrary, $\sup A\subseteq\sup_{n\in\mathbb N}a_n$. **Q**

(e) Thus we have a sequence $\langle\mu_n\rangle_{n\in\mathbb N}$ of the required type, witnessed by $\langle a_n\rangle_{n\in\mathbb N}$. To see that it is unique, suppose that $\langle\mu'_n\rangle_{n\in\mathbb N}$, $\langle a'_n\rangle_{n\in\mathbb N}$ are another pair of sequences with the same properties. Note first that if $c\in\mathfrak C$ and $0\ne c\subseteq[\![\mu'_i>0]\!]$ there is some $k\in\mathbb N$ such that $c\cap a'_i\cap a_k\ne0$; this is because $\bar\mu(a'_i\cap c)=\mu'_i(c)>0$, so that $a'_i\cap c\ne0$, while $a'_i\subseteq\sup A=\sup_{k\in\mathbb N}a_k$. **?** Suppose, if possible, that there is some n such that $\mu_n\ne\mu'_n$; since the situation is symmetric, there is no loss of generality in supposing that $\mu'_n\not\le\mu_n$, that is, that $c=[\![\mu'_n>\mu_n]\!]\ne0$. For any $i\le n$, $\mu'_i\ge\mu'_n$ so $c\subseteq[\![\mu'_i>0]\!]$. We may therefore choose $c_0,\ldots,c_{n+1}\in\mathfrak C_c\setminus\{0\}$ and $k(0),\ldots,k(n)\in\mathbb N$ such that $c_0=c$ and, for $i\le n$,

$$c_i\cap a'_i\cap a_{k(i)}\ne0$$

(choosing $k(i)$, recalling that $0\ne c_i\subseteq c\subseteq[\![\mu'_i>0]\!]$),

$$c_{i+1}\in\mathfrak C,\quad c_{i+1}\subseteq c_i,\quad c_{i+1}\cap a'_i=c_{i+1}\cap a_{k(i)}=c_i\cap a'_i\cap a_{k(i)}$$

(choosing c_{i+1}, using the fact that a'_i and $a_{k(i)}$ both belong to A – see the penultimate sentence in part (b) of the proof). On reaching c_{n+1}, we have $0\ne c_{n+1}\subseteq c$ so $\mu_n(c_{n+1})<\mu'_n(c_{n+1})$. On the other hand, for each $i\le n$,

$$c_{n+1}\cap a'_i\cap a_{k(i)}=c_{n+1}\cap c_{i+1}\cap a'_i\cap a_{k(i)}=c_{n+1}\cap a'_i=c_{n+1}\cap a_{k(i)},$$

so

$$\mu_n(c_{n+1})<\mu'_n(c_{n+1})\le\mu'_i(c_{n+1})=\bar\mu(c_{n+1}\cap a'_i)=\bar\mu(c_{n+1}\cap a_{k(i)})=\mu_{k(i)}(c_{n+1}),$$

and $k(i)$ must be less than n. There are therefore distinct i, $j\le n$ such that $k(i)=k(j)$. But in this case

$$c_{n+1}\cap a'_i=c_{n+1}\cap a_{k(i)}=c_{n+1}\cap a_{k(j)}=c_{n+1}\cap a'_j\ne0$$

because $0\ne c_{n+1}\subseteq[\![\mu'_j>0]\!]$. So a'_i, a'_j cannot be disjoint, breaking one of the rules of the construction. **X** Thus $\mu_n=\mu'_n$ for every $n\in\mathbb N$.

This completes the proof.

333K Theorem Let $(\mathfrak{A},\bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Then there are unique families $\langle\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\mu_\kappa\rangle_{\kappa\in K}$ such that

K is a countable set of infinite cardinals,

for $i\in\mathbb{N}\cup K$, μ_i is a non-negative countably additive functional on $\mathfrak{C}$, and $\sum_{i\in\mathbb{N}\cup K}\mu_i c=\bar\mu c$ for every $c\in\mathfrak{C}$,

$\mu_{n+1}\le\mu_n$ for every $n\in\mathbb{N}$, and $\mu_\kappa\ne 0$ for $\kappa\in K$,

setting $e_i=[\![\mu_i>0]\!]\in\mathfrak{C}$, and giving the principal ideal $\mathfrak{C}_{e_i}$ generated by e_i the measure $\mu_i\restriction\mathfrak{C}_{e_i}$ for each $i\in\mathbb{N}\cup K$, we have a measure algebra isomorphism

$$\pi:\mathfrak{A}\to\prod_{n\in\mathbb{N}}\mathfrak{C}_{e_n}\times\prod_{\kappa\in K}\mathfrak{C}_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$$

such that

$$\pi c=(\langle c\cap e_n\rangle_{n\in\mathbb{N}},\langle(c\cap e_\kappa)\otimes 1\rangle_{\kappa\in K})$$

for each $c\in C$, writing $c\otimes 1$ for the canonical image in $C_{e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$ of $c\in\mathfrak{C}_{e_\kappa}$.

proof (a) I aim to use the construction of 333H, but taking much more care over the choice of $\langle a_i\rangle_{i\in I}$ in part (a) of the proof there. We start by taking $\langle a_n\rangle_{n\in\mathbb{N}}$ as in 333J, and setting $\mu_n c=\bar\mu(a_n\cap c)$ for every $n\in\mathbb{N}$, $c\in\mathfrak{C}$; then these a_n will deal with the relative atoms over $\mathfrak{C}$.

(b) The further idea required here concerns the treatment of infinite κ. Let $\langle b_i\rangle_{i\in I}$ be any partition of unity in $\mathfrak{A}$ consisting of non-zero members of $\mathfrak{A}$ which are relatively Maharam-type-homogeneous over $\mathfrak{C}$, and $\langle\kappa_i\rangle_{i\in I}$ the corresponding cardinals, so that $\kappa_i=0$ iff b_i is a relative atom. Set $I_1=\{i:i\in I,\,\kappa_i\ge\omega\}$. Set $K=\{\kappa_i:i\in I_1\}$, so that K is a countable set of infinite cardinals, and for $\kappa\in K$ set $J_\kappa=\{i:\kappa_i=\kappa\}$, $a_\kappa=\sup_{i\in J_\kappa}b_i$ for $\kappa\in K$. Now every a_κ is relatively Maharam-type-homogeneous over $\mathfrak{C}$. **P** (Compare 332H.) J_κ must be countable, because $\mathfrak{A}$ is ccc. If $0\ne a\subseteq a_\kappa$, there is some $i\in J_\kappa$ such that $a\cap b_i\ne 0$; now

$$\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)\ge\tau_{\mathfrak{C}_{a\cap b_i}}(\mathfrak{A}_{a\cap b_i})=\kappa_i=\kappa$$

(333Ba). At the same time, for each $i\in J_\kappa$, there is a set $D_i\subseteq\mathfrak{A}_{b_i}$ such that $\#(D_i)=\kappa$ and $\mathfrak{C}_{b_i}\cup D_i$ τ-generates $\mathfrak{A}_{b_i}$. Set $D=\bigcup_{i\in J_\kappa}D_i\cup\{b_i:i\in J_\kappa\}$; then

$$\#(D)\le\max(\omega,\#(J_\kappa),\sup_{i\in K}\#(D_i))=\kappa.$$

Let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}_{a_\kappa}$ generated by $\mathfrak{C}_{a_\kappa}\cup D$. Then

$$\mathfrak{C}_{b_i}\cup D_i\subseteq\{b\cap b_i:b\in\mathfrak{B}\}=\mathfrak{B}\cap\mathfrak{A}_{b_i}$$

so $\mathfrak{B}\supseteq\mathfrak{A}_{b_i}$ for each $i\in J_\kappa$, and $\mathfrak{B}=\mathfrak{A}_{a_\kappa}$. Thus $\mathfrak{C}_{a_\kappa}\cup D$ τ-generates $\mathfrak{A}_{a_\kappa}$, and

$$\tau_{\mathfrak{C}_{a_\kappa}}(\mathfrak{A}_{a_\kappa})\le\kappa\le\min_{0\ne a\subseteq a_\kappa}\tau_{\mathfrak{C}_a}(\mathfrak{A}_a).$$

This shows that a_κ is relatively Maharam-type-homogeneous over $\mathfrak{C}$, with $\tau_{\mathfrak{C}_{a_\kappa}}(\mathfrak{A}_{a_\kappa})=\kappa$. **Q**

Since evidently $\langle J_\kappa\rangle_{\kappa\in K}$ and $\langle a_\kappa\rangle_{\kappa\in K}$ are disjoint, and $\sup_{\kappa\in K}a_\kappa=\sup_{i\in I_1}b_i$, this process yields a partition $\langle a_i\rangle_{i\in\mathbb{N}\cup K}$ of unity in $\mathfrak{A}$. Now the arguments of 333H show that we get an isomorphism π of the kind described.

(c) To see that the families $\langle\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\mu_\kappa\rangle_{\kappa\in K}$ (and therefore the e_i and the $(\mathfrak{C}_{e_i},\mu_i\restriction\mathfrak{C}_{e_i})$, but not π) are uniquely defined, argue as follows. Let A be the set of those $a\in\mathfrak{A}$ which are relatively Maharam-type-homogeneous over $\mathfrak{C}$. Take families $\langle\tilde\mu_n\rangle_{n\in\mathbb{N}}$, $\langle\tilde\mu_\kappa\rangle_{\kappa\in\tilde K}$ which correspond to an isomorphism

$$\tilde\pi:\mathfrak{A}\to\mathfrak{D}=\prod_{n\in\mathbb{N}}\mathfrak{C}_{\tilde e_n}\times\prod_{\kappa\in\tilde K}\mathfrak{C}_{\tilde e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa,$$

writing $\tilde e_i=[\![\tilde\mu_i>0]\!]$ for $i\in\mathbb{N}\cup\tilde K$. In the simple product $\prod_{n\in\mathbb{N}}\mathfrak{C}_{\tilde e_n}\times\prod_{\kappa\in\tilde K}\mathfrak{C}_{\tilde e_\kappa}\widehat{\otimes}\mathfrak{B}_\kappa$, we have a partition of unity $\langle e_i^*\rangle_{i\in\mathbb{N}\cup\tilde K}$ corresponding to the product structure. Now for $d\subseteq e_i^*$, we have

$$\begin{aligned}\tau_{\tilde\pi[\mathfrak{C}]_d}(\mathfrak{D}_d)&=0\text{ if }i\in\mathbb{N},\\&=\kappa\text{ if }i=\kappa\in\tilde K.\end{aligned}$$

So $\tilde K$ must be

$$\{\kappa:\kappa\ge\omega,\,\exists\,a\in A,\,\tau_{\mathfrak{C}_a}(\mathfrak{A}_a)=\kappa\}=K,$$

and for $\kappa \in \tilde{K}$,

$$\tilde{\pi}^{-1}e^*_\kappa = \sup\{a : a \in A,\, \tau_{\mathfrak{C}_a}(\mathfrak{A}_a) = \kappa\} = a_\kappa,$$

so that $\tilde{\mu}_\kappa = \mu_\kappa$. On the other hand, $\langle \tilde{\pi}^{-1}e^*_n\rangle_{n\in\mathbb{N}}$ must be a disjoint sequence with supremum $\sup A$, and the corresponding functionals $\tilde{\mu}_n$ are supposed to form a non-increasing sequence, so must be equal to the μ_n by 333J.

333L Remark Thus for the classification of structures $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$, where $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra, it will be enough to classify objects $(\mathfrak{C}, \bar{\mu}, \langle \mu_n\rangle_{n\in\mathbb{N}}, \langle \mu_\kappa\rangle_{\kappa\in K})$, where

$(\mathfrak{C}, \bar{\mu})$ is a totally finite measure algebra,

$\langle \mu_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-negative countably additive functionals on $\mathfrak{C}$,

K is a countable set of infinite cardinals (possibly empty),

$\langle \mu_\kappa\rangle_{\kappa\in K}$ is a family of non-zero non-negative countably additive functionals on $\mathfrak{C}$,

$\sum_{n=0}^{\infty} \mu_n + \sum_{\kappa\in K} \mu_\kappa = \bar{\mu}$.

To do this we need the concept of 'standard extension' of a countably additive functional on a closed subalgebra of a measure algebra, treated in 327F-327G, together with the following idea.

333M Lemma Let $(\mathfrak{C}, \bar{\mu})$ be a totally finite measure algebra and $\langle \mu_i\rangle_{i\in I}$ a family of countably additive functionals on $\mathfrak{C}$. For $i \in I$, $\alpha \in \mathbb{R}$ set $e_{i\alpha} = [\![\mu_i > \alpha\bar{\mu}]\!]$ (326T), and let $\mathfrak{C}_0$ be the closed subalgebra of $\mathfrak{C}$ generated by $\{e_{i\alpha} : i \in I, \alpha \in \mathbb{R}\}$. Write Σ for the σ-algebra of subsets of $\mathbb{R}^I$ generated by sets of the form $E_{i\alpha} = \{x : x(i) > \alpha\}$ as i runs over I and α runs over $\mathbb{R}$. Then

(a) there is a measure μ, with domain Σ, such that there is a measure-preserving isomorphism $\pi : \Sigma/\mathcal{N}_\mu \to \mathfrak{C}_0$ for which $\pi E^\bullet_{i\alpha} = e_{i\alpha}$ for every $i \in I$ and $\alpha \in \mathbb{R}$, writing $\mathcal{N}_\mu$ for $\mu^{-1}[\{0\}]$;

(b) this formula determines both μ and π;

(c) for every $E \in \Sigma$ and $i \in I$, we have

$$\mu_i \pi E^\bullet = \int_E x(i)\mu(dx);$$

(d) for every $i \in I$, μ_i is the standard extension of $\mu_i{\restriction}\mathfrak{C}_0$ to $\mathfrak{C}$;

(e) for every $i \in I$, $\mu_i \ge 0$ iff $x(i) \ge 0$ for μ-almost every x;

(f) for every i, $j \in I$, $\mu_i \ge \mu_j$ iff $x(i) \ge x(j)$ for μ-almost every x;

(g) for every $i \in I$, $\mu_i = 0$ iff $x(i) = 0$ for μ-almost every x.

proof (a) Express $(\mathfrak{C}, \bar{\mu})$ as the measure algebra of a measure space (Y, T, ν); write $\phi : \mathrm{T} \to \mathfrak{C}$ for the corresponding homomorphism. For each $i \in I$ let $f_i : Y \to \mathbb{R}$ be a T-measurable, ν-integrable function such that $\int_H f_i = \mu_i\phi H$ for every $H \in \mathrm{T}$. Define $\psi : Y \to \mathbb{R}^I$ by setting $\psi(y) = \langle f_i(y)\rangle_{i\in I}$; then $\psi^{-1}[E_{i\alpha}] \in \Sigma$, and $e_{i\alpha} = \phi(\psi^{-1}[E_{i\alpha}])$ for every $i \in I$ and $\alpha \in \mathbb{R}$. (See part (a) of the proof of 327F.) So $\{E : E \subseteq \mathbb{R}^I,\, \psi^{-1}[E] \in \mathrm{T}\}$, which is a σ-algebra of subsets of $\mathbb{R}^I$, contains every $E_{i\alpha}$, and therefore includes Σ; that is, $\psi^{-1}[E] \in \mathrm{T}$ for every $E \in \Sigma$. Accordingly we may define μ by setting $\mu E = \nu\psi^{-1}[E]$ for every $E \in \Sigma$, and μ will be a measure on $\mathbb{R}^I$ with domain Σ. The Boolean homomorphism $E \mapsto \phi\psi^{-1}[E] : \Sigma \to \mathfrak{C}$ has kernel $\mathcal{N}_\mu$, so descends to a homomorphism $\pi : \Sigma/\mathcal{N}_\mu \to \mathfrak{C}$, which is measure-preserving. To see that $\pi[\Sigma/\mathcal{N}_\mu] = \mathfrak{C}_0$, observe that because Σ is the σ-algebra generated by $\{E_{i\alpha} : i \in I, \alpha \in \mathbb{R}\}$, $\pi[\Sigma/\mathcal{N}_\mu]$ must be the closed subalgebra of $\mathfrak{C}$ generated by $\{\pi E^\bullet_{i\alpha} : i \in I, \alpha \in \mathbb{R}\} = \{e_{i\alpha} : i \in I, \alpha \in \mathbb{R}\}$, which is $\mathfrak{C}_0$.

(b) Now suppose that μ', π' have the same properties. Consider

$$\mathcal{A} = \{E : E \in \Sigma,\, \pi E^\bullet = \pi' E^\circ\},$$

where I write $E^\bullet$ for the equivalence class of E in $\Sigma/\mathcal{N}_\mu$, and E° for the equivalence class of E in $\Sigma/\mathcal{N}_{\mu'}$. Then $\mathcal{A}$ is a σ-subalgebra of Σ, because $E \mapsto \pi E^\bullet$, $E \mapsto \pi' E^\circ$ are both sequentially order-continuous Boolean homomorphisms, and contains every $E_{i\alpha}$, so must be the whole of Σ. Consequently

$$\mu E = \bar{\mu}\pi E^\bullet = \bar{\mu}\pi' E^\circ = \mu' E$$

for every $E \in \Sigma$, and $\mu' = \mu$; it follows at once that $\pi' = \pi$. So μ and π are uniquely determined.

(c) If $E \in \Sigma$ and $i \in I$,

$$\int_E x(i)\mu(dx) = \int x(i)\chi E(x)\mu(dx) = \int \psi(y)(i)\chi E(\psi(y))\nu(dy)$$

(applying 235G[1] to the inverse-measure-preserving function $\psi : Y \to \mathbb{R}^I$)

$$= \int_{\psi^{-1}[E]} f_i(y)\nu(dy)$$

(by the definition of ψ)

$$= \mu_i\phi(\psi^{-1}[E])$$

(by the choice of f_i)

$$= \mu_i\pi E^\bullet$$

by the definition of π.

(d) For every $\alpha \in \mathbb{R}$, $[\![\mu_i > \alpha\bar{\mu}]\!]$ belongs to $\mathfrak{C}_0$, so must be equal to $[\![\mu_i\restriction\mathfrak{C}_0 > \alpha\bar{\mu}\restriction\mathfrak{C}_0]\!]$. Thus μ_i is the standard extension of $\mu_i\restriction\mathfrak{C}_0$ (327G).

(e)-(g) The point is that, because the standard-extension operator is order-preserving (327F(b-ii)),

$$\begin{aligned}
\mu_i \ge 0 &\iff \mu_i\restriction\mathfrak{C}_0 \ge 0 \\
&\iff \int_E x(i)\mu(dx) \ge 0 \text{ for every } E \in \Sigma \\
&\iff x(i) \ge 0 \ \mu\text{-a.e.}, \\
\mu_i \ge \mu_j &\iff \mu_i\restriction\mathfrak{C}_0 \ge \mu_j\restriction\mathfrak{C}_0 \\
&\iff \int_E x(i)\mu(dx) \ge \int_E x(j)\mu(dx) \text{ for every } E \in \Sigma \\
&\iff x(i) \ge x(j) \ \mu\text{-a.e.}, \\
\mu_i = 0 &\iff \mu_i\restriction\mathfrak{C}_0 = 0 \\
&\iff \int_E x(i)\mu(dx) = 0 \text{ for every } E \in \Sigma \\
&\iff x(i) = 0 \ \mu\text{-a.e.}.
\end{aligned}$$

333N A canonical form for closed subalgebras We now have all the elements required to describe a canonical form for structures

$$(\mathfrak{A}, \bar{\mu}, \mathfrak{C}),$$

where $(\mathfrak{A}, \bar{\mu})$ is a totally finite measure algebra and $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$. The first step is the matching of such structures with structures

$$(\mathfrak{C}, \bar{\mu}, \langle\mu_n\rangle_{n\in\mathbb{N}}, \langle\mu_\kappa\rangle_{\kappa\in K}),$$

where $(\mathfrak{C}, \bar{\mu})$ is a totally finite measure algebra, $\langle\mu_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-negative countably additive functionals on $\mathfrak{C}$, K is a countable set of infinite cardinals, $\langle\mu_\kappa\rangle_{\kappa\in K}$ is a family of non-zero non-negative countably additive functionals on $\mathfrak{C}$, and $\sum_{n=0}^\infty \mu_n + \sum_{\kappa\in K}\mu_\kappa = \bar{\mu}$; this is the burden of 333K.

Next, given any structure of this second kind, we have a corresponding closed subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$, a measure μ on $\mathbb{R}^I$, where $I = \mathbb{N} \cup K$, and an isomorphism π from the measure algebra $\mathfrak{C}_0^*$ of μ to $\mathfrak{C}_0$, all uniquely defined from the family $\langle\mu_i\rangle_{i\in I}$ by the process of 333M. For any E belonging to the domain Σ of μ, and $i \in I$, we have

$$\mu_i\pi E^\bullet = \int_E x(i)\mu(dx)$$

[1] Formerly 235I.

(333Mc), so that $\mu_i\restriction\mathfrak{C}_0$ is fixed by π and μ. Moreover, the functionals μ_i can be recovered from their restrictions to $\mathfrak{C}_0$ by the formulae of 327F (333Md). Thus from $(\mathfrak{C},\bar\mu,\langle\mu_i\rangle_{i\in I})$ we are led, by a canonical and reversible process, to the structure

$$(\mathfrak{C},\bar\mu,\mathfrak{C}_0,I,\mu,\pi).$$

But the extension $\mathfrak{C}$ of $\mathfrak{C}_0=\pi[\mathfrak{C}_0^*]$ can be described, up to isomorphism, by the same process as before; that is, it corresponds to a sequence $\langle\theta'_n\rangle_{n\in\mathbb{N}}$ and a family $\langle\theta'_\kappa\rangle_{\kappa\in L}$ of countably additive functionals on $\mathfrak{C}_0$ satisfying the conditions of 333K. We can transfer these to $\mathfrak{C}_0^*$, where they correspond to families $\langle\theta_n\rangle_{n\in\mathbb{N}}$, $\langle\theta_\kappa\rangle_{\kappa\in L}$ of absolutely continuous countably additive functionals defined on Σ, setting

$$\theta_jE=\theta'_j\pi E^\bullet$$

for $E\in\Sigma$, $j\in\mathbb{N}\cup L$. This process too is reversible; every absolutely continuous countably additive functional ν on Σ corresponds to countably additive functionals on $\mathfrak{C}_0^*$ and $\mathfrak{C}_0$. Let me repeat that the results of 327F mean that the whole structure $(\mathfrak{C},\bar\mu,\langle\mu_i\rangle_{i\in I})$ can be recovered from $(\mathfrak{C}_0,\bar\mu\restriction\mathfrak{C}_0,\langle\mu_i\restriction\mathfrak{C}_0\rangle_{i\in I})$ if we can get the description of $(\mathfrak{C},\bar\mu)$ right, and that the requirements $\mu_i\ge0$, $\mu_n\ge\mu_{n+1}$, $\mu_\kappa\ne0$, $\sum_{i\in I}\mu_i=\bar\mu$ imposed in 333K will survive the process (327F(b-iv)).

Putting all this together, a structure $(\mathfrak{A},\bar\mu,\mathfrak{C})$ leads, in a canonical and (up to isomorphism) reversible way, to a structure

$$(K,\mu,L,\langle\theta_j\rangle_{j\in\mathbb{N}\cup L})$$

such that

K and L are countable sets of infinite cardinals,

μ is a totally finite measure on $\mathbb{R}^I$, where $I=\mathbb{N}\cup K$, and its domain Σ is precisely the σ-algebra of subsets of $\mathbb{R}^I$ defined by the coordinate functionals,

for μ-almost every $x\in\mathbb{R}^I$ we have $x(i)\ge0$ for every $i\in I$, $x(n)\ge x(n+1)$ for every $n\in\mathbb{N}$ and $\sum_{i\in I}x(i)=1$,

for $\kappa\in K$, $\mu\{x:x(\kappa)>0\}>0$,

(these two clauses corresponding to the requirements $\mu_i\ge0$, $\mu_n\ge\mu_{n+1}$, $\sum_{i\in I}\mu_i=\bar\mu$, $\mu_\kappa\ne0$ – see 333M(e)-(g))

for $j\in J=\mathbb{N}\cup L$, θ_j is a non-negative countably additive functional on Σ,

$\theta_n\ge\theta_{n+1}$ for every $n\in\mathbb{N}$, $\theta_\kappa\ne0$ for every $\kappa\in L$, $\sum_{j\in J}\theta_j=\mu$.

333O Remark I do not envisage quoting the result above very often. Indeed I do not claim that its final form adds anything to the constituent results 333K, 327F and 333M. I have taken the trouble to spell it out, however, because it does not seem to me obvious that the trail is going to end quite as quickly as it does. We need to use 333K twice, but only twice. The most important use of the ideas expressed here, I suppose, is in constructing examples to strengthen our intuition for the structures $(\mathfrak{A},\bar\mu,\mathfrak{C})$ under consideration, and I hope that you will experiment in this direction.

333P At the risk of trespassing on the province of Chapter 38, I turn now to a special type of closed subalgebra, in which there is a particularly elegant alternative form for a canonical description. The first step is an important result concerning automorphisms of homogeneous probability algebras.

Proposition Let $(\mathfrak{B},\bar\nu)$ be a homogeneous probability algebra. Then there is a measure-preserving automorphism $\phi:\mathfrak{B}\to\mathfrak{B}$ such that

$$\lim_{n\to\infty}\bar\nu(c\cap\phi^n(b))=\bar\nu c\cdot\bar\nu b$$

for all b, $c\in\mathfrak{B}$.

proof (a) The case $\mathfrak{B}=\{0,1\}$ is trivial (ϕ is, and must be, the identity map) so we may take it that $(\mathfrak{B},\bar\nu)=(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ for some infinite cardinal κ is an infinite cardinal. Because $\#(\kappa\times\mathbb{Z})=\max(\omega,\kappa)=\kappa$, there must be a permutation $\theta:\kappa\to\kappa$ such that every orbit of θ in κ is infinite (take θ to correspond to the bijection $(\xi,n)\mapsto(\xi,n+1):\kappa\times\mathbb{Z}\to\kappa\times\mathbb{Z}$). This induces a permutation $\hat\theta:\{0,1\}^\kappa\to\{0,1\}^\kappa$ through the formula $\hat\theta(x)=x\theta$ for every $x\in\{0,1\}^\kappa$, and of course $\hat\theta$ is an automorphism of the measure space $(\{0,1\}^\kappa,\nu_\kappa)$. We therefore have a corresponding automorphism ϕ of $\mathfrak{B}$, setting $\phi E^\bullet = (\hat\theta^{-1}[E])^\bullet$ for every E in the domain T_κ of ν_κ.

(b) Let $\mathcal{E}$ be the family of subsets E of $\{0,1\}^\kappa$ which are determined by coordinates in finite sets, that is, are expressible in the form $E=\{x:x\restriction J\in\tilde E\}$ for some finite set $J\subseteq\kappa$ and some $\tilde E\subseteq\{0,1\}^J$; equivalently, expressible as a finite union of basic cylinder sets $\{x:x\restriction J=y\}$. Then $\mathcal{E}$ is a subalgebra of T_κ, so $\mathfrak{C}=\{E^\bullet:E\in\mathcal{E}\}$ is a subalgebra of $\mathfrak{B}$.

(c) Now if b, $c\in\mathfrak{C}$, there is an $n\in\mathbb{N}$ such that $\bar\nu(c\cap\phi^m(b))=\bar\nu c\cdot\bar\nu b$ for every $m\ge n$. **P** Express b, c as $E^\bullet$, $F^\bullet$ where $E=\{x:x\restriction J\in\tilde E\}$, $F=\{x:x\restriction K\in\tilde F\}$ and J, K are finite subsets of κ. For $\xi\in K$, all the $\theta^n(\xi)$ are distinct, so only finitely many of them can belong to J; as K is also finite, there is an n such that $\theta^m[J]\cap K=\emptyset$ for every $m\ge n$. Fix $m\ge n$. Then $\phi^m(b)=H^\bullet$ where

$$H=\{x:x\theta^m\in E\}=\{x:x\theta^m\restriction J\in\tilde E\}=\{x:x\restriction L\in\tilde H\},$$

where $L=\theta^m[J]$ and $\tilde H=\{z\theta^{-m}:z\in\tilde E\}$. So $\bar\nu(c\cap\phi^m(b))=\nu(F\cap H)$. But L and K are disjoint, because $m\ge n$, so F and H must be independent (cf. 272K), and

$$\bar\nu(c\cap\phi^m(b))=\nu F\cdot\nu H=\nu F\cdot\nu E=\bar\nu c\cdot\bar\nu b,$$

as claimed. **Q**

(d) Now recall that for every $E\in\mathrm{T}_\kappa$ and $\epsilon>0$ there is an $E'\in\mathcal{E}$ such that $\nu(E\triangle E')\le\epsilon$ (254Fe). So, given b, $c\in\mathfrak{B}$ and $\epsilon>0$, we can find b', $c'\in\mathfrak{C}$ such that $\bar\nu(b\triangle b')\le\epsilon$ and $\bar\nu(c\triangle c')\le\epsilon$, and in this case

$$\begin{aligned}
&\limsup_{n\to\infty}|\bar\nu(c\cap\phi^n(b))-\bar\nu c\cdot\bar\nu b|\\
&\qquad\le\limsup_{n\to\infty}|\bar\nu(c\cap\phi^n(b))-\bar\nu(c'\cap\phi^n(b'))|\\
&\qquad\qquad\qquad+|\bar\nu(c'\cap\phi^n(b'))-\bar\nu c'\cdot\bar\nu b'|+|\bar\nu c\cdot\bar\nu b-\bar\nu c'\cdot\bar\nu b'|\\
&\qquad=\limsup_{n\to\infty}|\bar\nu(c\cap\phi^n(b))-\bar\nu(c'\cap\phi^n(b'))|+|\bar\nu c\cdot\bar\nu b-\bar\nu c'\cdot\bar\nu b'|\\
&\qquad\le\limsup_{n\to\infty}\bar\nu(c\triangle c')+\bar\nu(\phi^n(b)\triangle\phi^n(b'))\\
&\qquad\qquad\qquad+\bar\nu c|\bar\nu b-\bar\nu b'|+|\bar\nu c-\bar\nu c'|\bar\nu b'\\
&\qquad\le\bar\nu(c\triangle c')+\bar\nu(b\triangle b')+\bar\nu c\cdot\bar\nu(b\triangle b')+\bar\nu(c\triangle c')\cdot\bar\nu b'\le 4\epsilon.
\end{aligned}$$

As ϵ is arbitrary,

$$\lim_{n\to\infty}\bar\nu(c\cap\phi^n(b))=\bar\nu c\cdot\bar\nu b,$$

as required.

Remark Automorphisms of this type are called **mixing** (see 372O below).

333Q Corollary Let $(\mathfrak{C},\bar\mu_0)$ be a totally finite measure algebra and $(\mathfrak{B},\bar\nu)$ a probability algebra which is *either* homogeneous *or* purely atomic with finitely many atoms all of the same measure. Let $(\mathfrak{A},\bar\mu)$ be the localizable measure algebra free product of $(\mathfrak{C},\bar\mu_0)$ and $(\mathfrak{B},\bar\nu)$. Then there is a measure-preserving automorphism $\pi:\mathfrak{A}\to\mathfrak{A}$ such that

$$\{a:a\in\mathfrak{A},\ \pi a=a\}=\{c\otimes 1:c\in\mathfrak{C}\}.$$

Remark I am following 315N in using the notation $c\otimes b$ for the intersection in $\mathfrak{A}$ of the canonical images of $c\in\mathfrak{C}$ and $b\in\mathfrak{B}$. By 325D(c-i) I need not distinguish between the free product $\mathfrak{C}\otimes\mathfrak{B}$ and its image in $\mathfrak{A}$.

proof Set $\gamma=\bar\mu 1=\bar\mu_0 1$.

(a) Let me deal with the case of atomic $\mathfrak{B}$ first. In this case, if $\mathfrak{B}$ has $n+1$ atoms $b_0,\dots,b_n$, let $\phi:\mathfrak{B}\to\mathfrak{B}$ be the measure-preserving homomorphism cyclically permuting these atoms, so that $\phi b_0=b_1,\dots,\phi b_n=b_0$. Because ϕ is an automorphism of $(\mathfrak{B},\bar\nu)$, it induces an automorphism π of $(\mathfrak{A},\bar\mu)$; any member of $\mathfrak{A}$ is uniquely expressible as $a=\sup_{i\le n}c_i\otimes b_i$, and now $\pi a=\sup_{i\le n}c_i\otimes b_{i+1}$, if we set $b_{n+1}=b_0$. So $\pi a=a$ iff

$c_i = c_{i+1}$ for $i < n$ and $c_n = c_0$, that is, iff all the c_i are the same and $a = \sup_{i\le n} c\otimes b_i = c\otimes 1$ for some $c \in \mathfrak{C}$.

(b) If $\mathfrak{B}$ is homogeneous, then take a mixing measure-preserving automorphism $\phi : \mathfrak{B} \to \mathfrak{B}$ as described in 333P. As in (a), this corresponds to an automorphism π of $\mathfrak{A}$, defined by saying that $\pi(c\otimes b) = c\otimes\phi(b)$ for every $c \in \mathfrak{C}$, $b \in \mathfrak{A}$. Of course $\pi(c\otimes 1) = c\otimes 1$ for every $c \in \mathfrak{C}$.

Now suppose that $a \in \mathfrak{A}$ and $\pi a = a$; I need to show that $a \in \mathfrak{C}_1 = \{c\otimes 1 : c \in \mathfrak{C}\}$. Take any $\epsilon \in \left]0, \frac{1}{4}\right]$. We know that $\mathfrak{C}\otimes\mathfrak{B}$ is topologically dense in $\mathfrak{A}$ (325Dc), so there is an $a' \in \mathfrak{C}\otimes\mathfrak{B}$ such that $\bar\mu(a \mathbin{\triangle} a') \le \epsilon^2$. Express a' as $\sup_{i\in I} c_i\otimes b_i$, where $\langle c_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{C}$ (315Oa). Then

$$\pi a' = \sup_{i\in I} c_i\otimes\phi(b_i), \quad \pi^n(a') = \sup_{i\in I} c_i\otimes\phi^n(b_i) \text{ for every } n \in \mathbb{N}.$$

So we can get a formula for

$$\begin{aligned}\lim_{n\to\infty}\bar\mu(a'\cap\pi^n(a')) &= \lim_{n\to\infty}\bar\mu(\sup_{i\in I} c_i\otimes(b_i\cap\phi^n(b_i)))\\ &= \lim_{n\to\infty}\sum_{i\in I}\bar\mu_0 c_i\cdot\bar\nu(b_i\cap\phi^n(b_i)) = \sum_{i\in I}\bar\mu_0 c_i\cdot(\bar\nu b_i)^2.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{i\in I}\bar\mu_0 c_i\cdot(\bar\nu b_i)^2 &= \lim_{n\to\infty}\bar\mu(a'\cap\pi^n(a'))\\ &\ge \limsup_{n\to\infty}\bar\mu(a\cap\pi^n(a)) - \bar\mu(a\mathbin{\triangle} a') - \bar\mu(\pi^n(a)\mathbin{\triangle}\pi^n(a'))\\ &= \bar\mu a - 2\bar\mu(a\mathbin{\triangle} a') \ge \bar\mu a' - 3\bar\mu(a\mathbin{\triangle} a') \ge \sum_{i\in I}\bar\mu_0 c_i\cdot\bar\nu b_i - 3\epsilon^2,\end{aligned}$$

that is,

$$\textstyle\sum_{i\in I}\bar\mu_0 c_i\cdot\bar\nu b_i\cdot(1-\bar\nu b_i) \le 3\epsilon^2.$$

But this means that, setting $J = \{i : i \in I,\ \bar\nu b_i\cdot(1-\bar\nu b_i) \ge \epsilon\}$, we must have $\sum_{i\in J}\bar\mu_0 c_i \le 3\epsilon$. Set

$$K = \{i : i \in I,\ \bar\nu b_i \ge 1-2\epsilon\}, \quad L = \{i : i \in I\setminus K,\ \bar\nu b_i \le 2\epsilon\}, \quad c = \sup_{i\in K} c_i.$$

Then $I\setminus(K\cup L) \subseteq J$, so

$$\begin{aligned}\bar\mu(a'\mathbin{\triangle}(c\otimes 1)) &= \sum_{i\in I\setminus K}\bar\mu_0 c_i\cdot\bar\nu b_i + \sum_{i\in K}\bar\mu_0 c_i\cdot(1-\bar\nu b_i)\\ &\le \sum_{i\in J}\bar\mu_0 c_i\cdot\bar\nu b_i + \sum_{i\in L}\bar\mu_0 c_i\cdot\bar\nu b_i + \sum_{i\in K}\bar\mu_0 c_i\cdot(1-\bar\nu b_i)\\ &\le \sum_{i\in J}\bar\mu_0 c_i + 2\epsilon\sum_{i\in L}\bar\mu_0 c_i + 2\epsilon\sum_{i\in K}\bar\mu_0 c_i \le 3\epsilon + 2\epsilon\gamma,\end{aligned}$$

and

$$\bar\mu(a\mathbin{\triangle}(c\otimes 1)) \le \epsilon^2 + 3\epsilon + 2\epsilon\gamma.$$

As ϵ is arbitrary, a belongs to the topological closure of $\mathfrak{C}_1$. But of course $\mathfrak{C}_1$ is a closed subalgebra of $\mathfrak{A}$ (325Dd), so must actually contain a.

As a is arbitrary, π has the required property.

333R Now for the promised special type of closed subalgebra. It will be convenient to have the following temporary notation. For an integer $n \ge 1$, I will (for this paragraph only) write $\mathfrak{B}_n$ for the power set of $\{0,\dots,n\}$ and set $\bar\nu_n b = \frac{1}{n+1}\#(b)$ for $b \in \mathfrak{B}_n$. (The natural interpretation of $(\mathfrak{B}_n, \bar\nu_n)$ as defined in 333Ad corresponds to $(\mathfrak{B}_{2^{n+1}-1}, \bar\nu_{2^{n+1}-1})$ here, so we have a match if $n = 0$.)

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a subset of $\mathfrak{A}$. Then the following are equiveridical:

(i) there is some set G of measure-preserving automorphisms of $\mathfrak{A}$ such that

$$\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c \text{ for every } \pi \in G\};$$

(ii) $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ and there is a partition of unity $\langle e_i \rangle_{i\in I}$ in $\mathfrak{C}$, where I is a countable set of cardinals, such that $\mathfrak{A}$ is isomorphic to $\prod_{i\in I} \mathfrak{C}_{e_i} \widehat{\otimes} \mathfrak{B}_i$, writing $\mathfrak{C}_{e_i}$ for the principal ideal of $\mathfrak{C}$ generated by e_i and endowed with $\bar{\mu}\restriction\mathfrak{C}_{e_i}$, and $\mathfrak{C}_{e_i}\widehat{\otimes}\mathfrak{B}_i$ for the localizable measure algebra free product of $\mathfrak{C}_{e_i}$ and $\mathfrak{B}_i$ – the isomorphism being one which takes any $c \in \mathfrak{C}$ to $\langle (c \cap e_i) \otimes 1\rangle_{i\in I}$, as in 333H and 333K;

(iii) there is a single measure-preserving automorphism π of $\mathfrak{A}$ such that

$$\mathfrak{C} = \{c : c \in \mathfrak{A},\ \pi c = c\}.$$

proof (a)(i)⇒(ii)($\boldsymbol{\alpha}$) $\mathfrak{C}$ is a subalgebra because every $\pi \in G$ is a Boolean homomorphism, and it is order-closed because every π is order-continuous (324Kb). (Or, if you prefer, $\mathfrak{C}$ is topologically closed because every π is continuous.)

($\boldsymbol{\beta}$) Because $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$, its embedding can be described in terms of families $\langle \mu_n \rangle_{n\in\mathbb{N}}$, $\langle \mu_\kappa \rangle_{\kappa\in K}$ as in Theorem 333K. Set $I = K \cup \mathbb{N}$. Recall that each μ_i is defined by setting $\mu_i c = \bar{\mu}(c \cap a_i)$, where $\langle a_i \rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$ (see the proofs of 333H and 333K). For $\kappa \in K$, a_κ is the maximal element of $\mathfrak{A}$ which is relatively Maharam-type-homogeneous over $\mathfrak{C}$ with relative Maharam type κ (part (b) of the proof of 333K). Consequently we must have $\pi a_\kappa = a_\kappa$ for any measure algebra automorphism of $(\mathfrak{A}, \bar{\mu})$ which leaves $\mathfrak{C}$ invariant; in particular, for every $\pi \in G$. Thus $a_\kappa \in \mathfrak{C}$ for every $\kappa \in K$.

($\boldsymbol{\gamma}$) Now consider the relatively atomic part of $\mathfrak{A}$. The elements a_n, for $n \in \mathbb{N}$, are not uniquely defined. However, the functionals μ_n and their supports $e'_n = [\![\mu_n > 0]\!]$ are uniquely defined from the structure $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ and therefore invariant under G. Since

$$\begin{aligned} e'_n &= 1 \setminus \sup\{c : c \in \mathfrak{C},\ \mu_n c \le 0\} \\ &= 1 \setminus \sup\{c : c \in \mathfrak{C},\ c \cap a_n = 0\} = \inf\{c : c \in \mathfrak{C},\ c \supseteq a_n\}, \end{aligned}$$

and $\sup_{n\in\mathbb{N}} a_n = 1 \setminus \sup_{\kappa\in K} a_\kappa$ belongs to $\mathfrak{C}$, while $e'_n \supseteq e'_{n+1}$ for every n, we must have $e'_0 = \sup_{n\in\mathbb{N}} a_n$.

Let G^* be the set of all those automorphisms π of the measure algebra $(\mathfrak{A}, \bar{\mu})$ such that $\pi c = c$ for every $c \in \mathfrak{C}$. Then of course G^* is a group including G. Now $\sup_{\pi\in G^*} \pi a_n$ must be invariant under every member of G^*, so belongs to $\mathfrak{C}$; it includes a_n and is included in any member of $\mathfrak{C}$ including a_n, so must be e'_n.

($\boldsymbol{\delta}$) I claim now that if $n \in \mathbb{N}$ then $e'_n \cap [\![\mu_0 > \mu_n]\!] = 0$. **P?** Otherwise, set $c = [\![\mu_0 > \mu_n]\!] \cap e'_n$. Then $\mu_0 c > 0$ so $c \cap a_0 \ne 0$. By the last remark in (γ), there is a $\pi \in G^*$ such that $c \cap a_0 \cap \pi a_n \ne 0$. Now there is a $c' \in \mathfrak{C}$ such that $c \cap a_0 \cap \pi a_n = c' \cap a_0$, and of course we may suppose that $c' \subseteq c$. But this means that

$$\pi(c' \cap a_n) = c' \cap \pi a_n \supseteq c' \cap a_0 \cap \pi a_n = c' \cap a_0,$$

so that

$$\mu_n c' = \bar{\mu}(c' \cap a_n) = \bar{\mu}\pi(c' \cap a_n) \ge \bar{\mu}(c' \cap a_0) = \mu_0 c',$$

which is impossible, because $0 \ne c' \subseteq [\![\mu_0 > \mu_n]\!]$. **XQ**

So $\mu_0 c \le \mu_n c$ whenever $c \in \mathfrak{C}$ and $c \subseteq e'_n$. Because the μ_k have been chosen to be a non-increasing sequence, we must have $\mu_0 c = \mu_1 c = \ldots = \mu_n c$ for every $c \subseteq e'_n$.

($\boldsymbol{\epsilon}$) Recalling now that $\sum_{i\in I} \mu_i = \bar{\mu}\restriction\mathfrak{C}$, we see that $\mu_0 c \le \frac{1}{n+1}\bar{\mu} c$ for every $c \subseteq e'_n$. It follows that if $e^* = \inf_{n\in\mathbb{N}} e'_n$, $\mu_0 e^* = 0$; but this must mean that $e^* = 0$. Consequently, setting $e_n = e'_n \setminus e'_{n+1}$ for $n \in \mathbb{N}$, $e_\kappa = a_\kappa$ for $\kappa \in K$, we find that $\langle e_i \rangle_{i\in I}$ is a partition of unity in $\mathfrak{C}$.

Moreover, for $n \in \mathbb{N}$ and $c \subseteq e_n$, we must have $\mu_{n+1} c = 0$,

$$\bar{\mu} c = \sum_{i\in I} \mu_i c = \sum_{k=0}^{n} \mu_k c = (n+1)\mu_0 c,$$

so that $\mu_k c = \frac{1}{n+1}\mu c$ for every $k \le n$. But this means that we have a measure-preserving homomorphism $\psi_n : \mathfrak{A}_{e_n} \to \mathfrak{C}_{e_n}\widehat{\otimes}\mathfrak{B}_n$ given by setting

$$\psi_n(a_k \cap c) = c \otimes \{k\}$$

whenever $c \in \mathfrak{C}_{e_n}$ and $k \le n$; this is well-defined because $e_n \subseteq e'_k$, so that $a_k \cap c \ne a_k \cap c'$ if c, c' are distinct members of $\mathfrak{C}_{e_n}$, and it is measure-preserving because

$$\bar{\mu}(a_k \cap c) = \mu_k c = \frac{1}{n+1}\bar{\mu}c = \bar{\mu}c \cdot \bar{\nu}_n\{k\}$$

for all relevant k and c. Because $\mathfrak{B}_n$ is finite, ψ_n is surjective.

(ζ) Just as in 333H, we now see that because $\langle e_i \rangle_{i \in I}$ is a partition of unity in $\mathfrak{A}$ as well as in $\mathfrak{C}$, we can identify $\mathfrak{A}$ with $\prod_{i \in I} \mathfrak{A}_{e_i}$ and therefore with $\prod_{i \in I} \mathfrak{C}_{e_i} \widehat{\otimes} \mathfrak{B}_i$.

(b)(ii)⇒(iii) Let us work in $\mathfrak{D} = \prod_{i \in I} \mathfrak{C}_{e_i} \widehat{\otimes} \mathfrak{B}_i$, writing $\psi : \mathfrak{A} \to \mathfrak{D}$ for the given isomorphism. For each $i \in I$, we have a measure-preserving automorphism π_i of $\mathfrak{C}_{e_i} \widehat{\otimes} \mathfrak{B}_i$ with fixed-point subalgebra $\{c \otimes 1 : c \in \mathfrak{C}_{e_i}\}$ (333Q). For $d = \langle d_i \rangle_{i \in I} \in \mathfrak{D}$, set

$$\pi d = \langle \pi_i d_i \rangle_{i \in I}.$$

Then π is a measure-preserving automorphism because every π_i is. If $\pi d = d$, then for every $i \in I$ there must be a $c_i \subseteq e_i$ such that $d_i = c_i \otimes 1$. But this means that $d = \psi c$, where $c = \sup_{i \in I} c_i \in \mathfrak{C}$. Thus the fixed-point subalgebra of π is just $\psi[\mathfrak{C}]$. Transferring the structure $(\mathfrak{D}, \psi[\mathfrak{C}], \pi)$ back to $\mathfrak{A}$, we obtain a measure-preserving automorphism $\psi^{-1}\pi\psi$ of $\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$, as required.

(c)(iii)⇒(i) is trivial.

333X Basic exercises >(a) Show that, in the proof of 333H, $c_i = \operatorname{upr}(a_i, \mathfrak{C})$ (definition: 313S) for every $i \in I$.

(b) In Lemma 333J, show that every relative atom in $\mathfrak{A}$ over $\mathfrak{C}$ belongs to the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C} \cup \{a_n : n \in \mathbb{N}\}$.

(c) In the context of Lemma 333M, show that if I is countable we have a one-to-one correspondence between atoms c of $\mathfrak{C}_0$ and points x of non-zero mass in $\mathbb{R}^I$, given by the formula $\pi\{x\}^\bullet = c$.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be totally finite measure algebra and G a set of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself such that $\pi\phi \in G$ for all π, $\phi \in G$. (i) Show that $a \subseteq \sup_{\pi \in G} \pi a$ for every $a \in \mathfrak{A}$. (*Hint*: if $\pi c \subseteq c$, where $\pi \in G$ and $c \in \mathfrak{A}$, then $\pi c = c$; apply this to $c = \sup_{\pi \in G} \pi a$.) (ii) Set $\mathfrak{C} = \{c : c \in \mathfrak{A}, \pi c = c \text{ for every } \pi \in G\}$. Show that $\sup_{\pi \in G} \pi a = \operatorname{upr}(a, \mathfrak{C})$ for every $a \in \mathfrak{A}$.

333Y Further exercises (a) Show that when $I = \mathbb{N}$ the algebra Σ of subsets of $\mathbb{R}^I$, used in 333M, is precisely the Borel σ-algebra as described in 271Ya.

(b) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and $\mathfrak{B}$, $\mathfrak{C}$ two closed subalgebras of $\mathfrak{A}$ with $\mathfrak{C} \subseteq \mathfrak{B}$. Show that $\tau_{\mathfrak{C}}(\mathfrak{B}) \le \tau_{\mathfrak{C}}(\mathfrak{A})$. (*Hint*: use 333K and the ideas of 332T.)

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra. Show that $\mathfrak{A}$ is homogeneous iff there is a measure-preserving automorphism of $\mathfrak{A}$ which is mixing in the sense of 333P.

(d) Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, and G a set of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. Set $\mathfrak{C} = \{c : c \in \mathfrak{A}, \pi c = c \text{ for every } \pi \in G\}$. Show that $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ of the type described in 333R. (*Hint*: in the language of part (a) of the proof of 333R, show that every a_κ still belongs to $\mathfrak{C}$.)

333 Notes and comments I have done my best, in the first part of this section, to follow the lines already laid out in §§331-332, using what should (once you have seen them) be natural generalizations of the former definitions and arguments. Thus the Maharam type $\tau(\mathfrak{A})$ of an algebra is just the relative Maharam type $\tau_{\{0,1\}}(\mathfrak{A})$, and $\mathfrak{A}$ is Maharam-type-homogeneous iff it is relatively Maharam-type-homogeneous over $\{0, 1\}$. To help you trace through the correspondence, I list the code numbers: 331Fa→333Aa, 331Fb→333Ac, 331Hc→333Ba, 331Hd→333Bd, 332A→333Bb, 331I→333Cb, 331K→333E, 331L→333Fb, 332B→333H, 332J→333K. 333D overlaps with 332P. Throughout, the principle is the same: everything can be built up from products and free products.

Theorem 333Ca does not generalize any explicitly stated result, but overlaps with Proposition 332P. In the proof of 333E I have used a new idea; the same method would of course have worked just as well for

331K, but I thought it worth while to give an example of an alternative technique, which displays a different facet of homogeneous algebras, and a different way in which the algebraic, topological and metric properties of homogeneous algebras interact. The argument of 331K-331L relies (without using the term) on the fact that measure algebras of Maharam type κ have topological density at most $\max(\kappa,\omega)$ (see 331Ye), while the the argument of 333E uses the rather more sophisticated concept of stochastic independence.

Corollary 333Fa is cruder than the more complicated results which follow, but I think that it is invaluable as a first step in forming a picture of the possible embeddings of a given (totally finite) measure algebra $\mathfrak{C}$ in a larger algebra $\mathfrak{A}$. If we think of $\mathfrak{C}$ as the measure algebra of a measure space (X,Σ,μ), then we can be sure that $\mathfrak{A}$ is representable as a closed subalgebra of the measure algebra of $X\times\{0,1\}^\kappa$ for some κ, that is, the measure algebra of $\lambda\restriction\mathrm{T}$ where λ is the product measure on $X\times\{0,1\}^\kappa$ and T is some σ-subalgebra of the domain of λ; the embedding of $\mathfrak{C}$ in $\mathfrak{A}$ being defined by the formula $E^\bullet\to(E\times\{0,1\}^\kappa)^\bullet$ for $E\in\Sigma$ (325A, 325D). Identifying, in our imaginations, both X and $\{0,1\}^\kappa$ with the unit interval, we can try to picture everything in the unit square – and these pictures, although necessarily inadequate for algebras of uncountable Maharam type, already give a great deal of scope for invention.

I said above that everything can be constructed from simple products and free products, judiciously combined; of course some further ideas must be mixed with these. The difference between 332B and 333H, for instance, is partly in the need for the functionals μ_i in the latter, whereas in the former the decomposition involves only principal ideals with the induced measures. Because the μ_i are completely additive, they all have supports c_i (326Xl) and we get measure algebras $(\mathfrak{C}_{c_i},\mu_i\restriction\mathfrak{C}_{c_i})$ to use in the products. (I note that the c_i can be obtained directly from the a_i, without mentioning the functionals μ_i, by the process of 333Xa.) The fact that the c_i can overlap means that the 'relatively atomic' part of the larger algebra $\mathfrak{A}$ needs a much more careful description than before; this is the burden of 333J, and also the principal complication in the proof of 333R. The 'relatively atomless' part is (comparatively) straightforward, since we can use the same kind of amalgamation as before (part (c-i) of the proof of 332J, part (b) of the proof of 333K), simplified because I am no longer seeking to deal with algebras of infinite magnitude.

Theorem 333K gives a canonical form for superalgebras of a given totally finite measure algebra $(\mathfrak{C},\bar\mu)$, taking the structure $(\mathfrak{C},\bar\mu)$ itself for granted. I hope it is clear that while the μ_i and e_i and the algebra $\widehat{\mathfrak{A}}=\prod_{n\in\mathbb{N}}\mathfrak{C}_{e_n}\times\prod_{\kappa\in K}\mathfrak{C}_{e_\kappa}\widehat\otimes\mathfrak{B}_\kappa$ and the embedding of $\mathfrak{C}$ in $\widehat{\mathfrak{A}}$ are uniquely defined, the rest of the isomorphism $\pi:\mathfrak{A}\to\widehat{\mathfrak{A}}$ generally is not. Even when the a_κ are uniquely defined the isomorphisms between $\mathfrak{A}_{a_\kappa}$ and $\mathfrak{C}_{e_\kappa}\widehat\otimes\mathfrak{B}_\kappa$ depend on choosing generating families in the $\mathfrak{A}_{a_\kappa}$; see the proof of 333Cb.

To understand the possible structures $(\mathfrak{C},\langle\mu_i\rangle_{i\in I})$ of that theorem, we have to go rather deeper. The route I have chosen is to pick out the subalgebra $\mathfrak{C}_0$ of $\mathfrak{C}$ determined by $\langle\mu_i\rangle_{i\in I}$ and identify it with the measure algebra of a particular measure on $\mathbb{R}^I$. Perhaps I should apologise for not stating explicitly in the course of 333N that the measure μ there is a 'Borel measure' (see 333Ya); but I am afraid of opening a door to an invasion of ideas which belong in Volume 4. Besides, if I were going to do anything more with these measures than observe that they are uniquely defined by the construction proposed, I would complete them and call them Radon measures. In order to validate this approach, I must show that the μ_i can be recovered from their restrictions to $\mathfrak{C}_0$; this is 333Md, and is the motive for the discussion of 'standard extensions' in §327. No doubt there are other ways of doing it. One temptation which I felt it right to resist was the idea of decomposing $\mathfrak{C}$ into its homogeneous principal ideals; this seemed merely an additional complication. Of course the subalgebra $\mathfrak{C}_0$ has countable Maharam type (being τ-generated by the elements e_{iq}, for $i\in I$ and $q\in\mathbb{Q}$, of 333M), so that its decomposition is relatively simple, being just a matter of picking out the atoms (333Xc).

Another way of looking at the expression in 333N is to observe that $\mathfrak{A}$ is obtained by amalgamating two extensions of the core subalgebra $\mathfrak{C}_0$; one defined by $\langle\mu_i\restriction\mathfrak{C}_0\rangle_{i\in\mathbb{N}\cup K}$, and one by $\langle\theta'_j\rangle_{j\in\mathbb{N}\cup L}$. After using the second family to represent $(\mathfrak{C},\bar\mu\restriction\mathfrak{C},\mathfrak{C}_0)$, we obtain standard extensions μ_i from which we can represent $(\mathfrak{A},\bar\mu,\mathfrak{C})$. Put this way, it seems that the process demands that the steps be performed in the given order. In fact it can be made symmetric; but for that I think we need the theory of 'relative free products', which I will come to in §458 of Volume 4.

In 333P I find myself presenting an important fact about homogeneous measure algebras, rather out of context; but I hope that it will help you to believe that I have by no means finished with the insights which Maharam's theorem provides. I give it here for the sake of 333R. For the moment, I invite you to think of 333R as just a demonstration of the power of the techniques I have developed in this chapter, and of the kind of simplification (in the equivalence of conditions (i) and (iii)) which seems to arise repeatedly in the theory of measure algebras. But you will see that the first step to understanding any automorphism will be a description of its fixed-point subalgebra, so 333R will also be basic to the theory of automorphisms of measure algebras. I note that the hypothesis (i) of 333R can in fact be relaxed (333Yd), but this seems to need an extra idea.

Version of 26.9.08

334 Products

I devote a short section to results on the Maharam classification of the measure algebras of product measures, or, if you prefer, of the free products of measure algebras. The complete classification, even for probability algebras, is complex (334Xc, 334Ya), so I content myself with a handful of the most useful results. I start with upper bounds for the Maharam type of the c.l.d. product of two measure spaces (334A) and the localizable measure algebra free product of two semi-finite measure algebras (334B), and go on to the corresponding results for general products of probability spaces and algebras (334C-334D). Finally, I show that any infinite power of a probability space is Maharam-type-homogeneous (334E).

In this section I will write $\tau(\mu)$ for the Maharam type of a measure μ, defined as the Maharam type of its measure algebra (331Fc).

334A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and λ the c.l.d. product measure on $X\times Y$. Then $\tau(\lambda)\le\max(\omega,\tau(\mu),\tau(\nu))$.

proof Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be the measure algebras of μ, ν and λ, respectively. Recall from 325A that we have order-continuous Boolean homomorphisms $\varepsilon_1:\mathfrak{A}\to\mathfrak{C}$ and $\varepsilon_2:\mathfrak{B}\to\mathfrak{C}$ defined by setting $\varepsilon_1(E^\bullet)=(E\times Y)^\bullet$, $\varepsilon_2(F^\bullet)=(X\times F)^\bullet$ for $E\in\Sigma$ and $F\in\mathrm{T}$. Let $A\subseteq\mathfrak{A}$, $B\subseteq\mathfrak{B}$ be τ-generating sets with $\#(A)=\tau(\mathfrak{A})=\tau(\mu)$, $\#(B)=\tau(\nu)$; set $C=\varepsilon_1[A]\cup\varepsilon_2[B]$. Then C τ-generates $\mathfrak{C}$. **P** Let $\mathfrak{C}_1$ be the order-closed subalgebra of $\mathfrak{C}$ generated by C. Because ε_1 is order-continuous, $\varepsilon_1^{-1}[\mathfrak{C}_1]$ is an order-closed subalgebra of $\mathfrak{A}$, and it includes A, so must be the whole of $\mathfrak{A}$; thus $\varepsilon_1 a\in\mathfrak{C}_1$ for every $a\in\mathfrak{A}$. Similarly, $\varepsilon_2 b\in\mathfrak{C}_1$ for every $b\in\mathfrak{B}$.

This means that

$$\Lambda_1=\{W:W\in\Lambda,\,W^\bullet\in\mathfrak{C}_1\}$$

contains $E\times F$ for every $E\in\Sigma$, $F\in\mathrm{T}$. Also Λ_1 is a σ-algebra of subsets of $X\times Y$, because $\mathfrak{C}_1$ is (sequentially) order-closed in $\mathfrak{C}$. So $\Lambda_1\supseteq\Sigma\widehat{\otimes}\mathrm{T}$ (definition: 251D). But this means that if $W\in\Lambda$ there is a $V\in\Lambda_1$ such that $V\subseteq W$ and $\lambda V=\lambda W$ (251Ib); that is, if $c\in\mathfrak{C}$ there is a $d\in\mathfrak{C}_1$ such that $d\subseteq c$ and $\bar\lambda d=\bar\lambda c$. Thus $\mathfrak{C}_1$ is order-dense in $\mathfrak{C}$, and

$$c=\sup\{d:d\in\mathfrak{C}_1,\,d\subseteq c\}\in\mathfrak{C}_1$$

for every $c\in\mathfrak{C}$. So $\mathfrak{C}_1=\mathfrak{C}$ and C τ-generates $\mathfrak{C}$, as claimed. **Q**

Consequently

$$\tau(\lambda)=\tau(\mathfrak{C})\le\#(C)\le\max(\omega,\tau(\mu),\tau(\nu)).$$

334B Corollary Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ be semi-finite measure algebras, with localizable measure algebra free product $(\mathfrak{C},\bar\lambda)$ (325E). Then $\tau(\mathfrak{C})\le\max(\omega,\tau(\mathfrak{A}),\tau(\mathfrak{B}))$.

proof By the construction of part (a) of the proof of 325D, $\mathfrak{C}$ can be regarded as the measure algebra of the c.l.d. product of the Stone representations of $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$; so the result follows at once from 334A.

334C Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X,Λ,λ). Then

$$\tau(\lambda)\le\max(\omega,\#(I),\sup_{i\in I}\tau(\mu_i)).$$

proof For $i\in I$, let $\mathfrak{A}_i$ be the measure algebra of μ_i; let $\mathfrak{C}$ be the measure algebra of λ. Recall from 325I that we have order-continuous Boolean homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ corresponding to the inverse-measure-preserving maps $x\mapsto\pi_i(x)=x(i):X\to X_i$. For each $i\in I$, let $A_i\subseteq\mathfrak{A}_i$ be a set with cardinal $\tau(\mu_i)$ which

τ-generates $\mathfrak{A}_i$. Set $C = \bigcup_{i \in I} \varepsilon_i[A_i]$. Then C τ-generates $\mathfrak{C}$. **P** Let $\mathfrak{C}_1$ be the order-closed subalgebra of $\mathfrak{C}$ generated by C. Because ε_i is order-continuous, $\varepsilon_i^{-1}[\mathfrak{C}_1]$ is an order-closed subalgebra of $\mathfrak{A}_i$, and it includes A_i, so must be the whole of $\mathfrak{A}_i$; thus $\varepsilon_i a \in \mathfrak{C}_1$ for every $a \in \mathfrak{A}_i$, $i \in I$.

This means that

$$\Lambda_1 = \{W : W \in \Lambda,\, W^\bullet \in \mathfrak{C}_1\}$$

contains $\pi_i^{-1}[E]$ for every $E \in \Sigma_i$, $i \in I$. Also Λ_1 is a σ-algebra of subsets of X, because $\mathfrak{C}_1$ is (sequentially) order-closed in $\mathfrak{C}$. So $\Lambda_1 \supseteq \widehat{\bigotimes}_{i \in I} \Sigma_i$. But this means that if $W \in \Lambda$ there is a $V \in \Lambda_1$ such that $V^\bullet = W^\bullet$ (254Ff), that is, that $\mathfrak{C}_1 = \mathfrak{C}$, and C τ-generates $\mathfrak{C}$, as claimed. **Q**

Consequently

$$\tau(\lambda) = \tau(\mathfrak{C}) \le \#(C) \le \max(\omega, \#(I), \sup_{i \in I} \tau(\mu_i)).$$

334D Corollary Let $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i \in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C}, \bar\lambda)$. Then

$$\tau(\mathfrak{C}) \le \max(\omega, \#(I), \sup_{i \in I} \tau(\mathfrak{A}_i)).$$

proof See 325J-325K.

334E I come now to the question of when a product of probability spaces is Maharam-type-homogeneous. I give just one result in detail, leaving others to the exercises.

Theorem Let (X, Σ, μ) be a probability space and I an infinite set; let λ be the product measure on X^I. Then λ is Maharam-type-homogeneous. If $\tau(\mu) = 0$ then $\tau(\lambda) = 0$; otherwise $\tau(\lambda) = \max(\tau(\mu), \#(I))$.

proof (a) Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ and $(\mathfrak{C}, \bar\lambda)$ the measure algebra of λ. If $\tau(\mu) = 0$, that is, $\mathfrak{A} = \{0, 1\}$, then $\mathfrak{C} = \{0, 1\}$ (by 254Fe, or 325Jc, or otherwise), and in this case is surely homogeneous, with $\tau(\mathfrak{C}) = 0$; so that λ is Maharam-type-homogeneous and $\tau(\lambda) = 0$. So let us suppose hencforth that $\tau(\mu) > 0$. We have

$$\tau(\mathfrak{C}) \le \max(\omega, \#(I), \tau(\mathfrak{A})) = \max(\#(I), \tau(\mathfrak{A})$$

by 334C.

(b) Fix on $b \in \mathfrak{A} \setminus \{0, 1\}$. For each $i \in I$, let $\varepsilon_i : \mathfrak{A} \to \mathfrak{C}$ be the canonical measure-preserving homomorphism corresponding to the inverse-measure-preserving function $x \mapsto x(i) : X^I \to X$. For each $n \in \mathbb{N}$, there is a set $J \subseteq I$ with cardinal n, and now the finite subalgebra of $\mathfrak{C}$ generated by $\{\varepsilon_i b : i \in J\}$ has atoms of measure at most δ^n, where $\delta = \max(\bar\mu b, 1 - \bar\mu b) < 1$. Consequently $\mathfrak{C}$ can have no atom of measure greater than δ^n, for any n, and is therefore atomless.

(c) Because I is infinite, there is a bijection between I and $I \times \mathbb{N}$; that is, there is a partition $\langle J_i \rangle_{i \in I}$ of I into countably infinite sets. Now (X^I, λ) can be identified with the product of the family $\langle(X^{J_i}, \lambda_i)\rangle_{i \in I}$, where λ_i is the product measure on X^{J_i} (254N). By (b), every λ_i is atomless, so there are sets $E_i \subseteq X^{J_i}$ of measure $\frac{1}{2}$. The sets $E_i' = \{x : x{\restriction}J_i \in E_i\}$ are now stochastically independent in X. Accordingly we have an inverse-measure-preserving function $f : X \to \{0, 1\}^I$, endowed with its usual measure ν_I, defined by setting $f(x)(i) = 1$ if $x \in E_i'$, 0 otherwise, and therefore a measure-preserving Boolean homomorphism $\pi : \mathfrak{B}_I \to \mathfrak{C}$, writing $\mathfrak{B}_I$ for the measure algebra of ν_I.

Now if $c \in \mathfrak{C} \setminus \{0\}$ and $\mathfrak{C}_c$ is the corresponding ideal, $b \mapsto c \cap \pi b : \mathfrak{B}_I \to \mathfrak{C}_c$ is an order-continuous Boolean homomorphism. It follows that $\tau(\mathfrak{C}_c) \ge \#(I)$ (331Jb).

(d) Again take any non-zero $c \in \mathfrak{C}$. For each $i \in I$, set $a_i = \inf\{a : \varepsilon_i a \supseteq c\}$. Writing $\mathfrak{A}_{a_i}$ for the corresponding principal ideal of $\mathfrak{A}$, we have an order-continuous Boolean homomorphism $\varepsilon_i' : \mathfrak{A}_{a_i} \to \mathfrak{C}_c$, given by the formula

$$\varepsilon_i' a = \varepsilon_i a \cap c \text{ for every } a \in \mathfrak{A}_{a_i}.$$

Now ε_i' is injective, so is a Boolean isomorphism between $\mathfrak{A}_{a_i}$ and its image $\varepsilon_i'[\mathfrak{A}_{a_i}]$, which by 314F(a-i) is a closed subalgebra of $\mathfrak{C}_c$. So

$$\tau(\mathfrak{A}_{a_i}) = \tau(\varepsilon_i'[\mathfrak{A}_{a_i}]) \le \tau(\mathfrak{C}_c)$$

by 332Tb.

For any finite $J \subseteq I$,

$$0 < \bar{\lambda}c \le \bar{\lambda}(\inf_{i\in J} \varepsilon_i a_i) = \prod_{i\in J} \bar{\lambda}(\varepsilon_i a_i) = \prod_{i\in J} \bar{\mu}a_i.$$

So for any $\delta < 1$, $\{i : \bar{\mu}a_i \le \delta\}$ must be finite, and $\sup_{i\in I} \bar{\mu}a_i = 1$. In particular, $\sup_{i\in I} a_i = 1$ in $\mathfrak{A}$. But this means that if ζ is any cardinal such that the Maharam-type-ζ component e_ζ of $\mathfrak{A}$ is non-zero, then $e_\zeta \cap a_i \ne 0$ for some $i \in I$, so that

$$\zeta \le \tau(\mathfrak{A}_{e_\zeta \cap a_i}) \le \tau(\mathfrak{A}_{a_i}) \le \tau(\mathfrak{C}_c).$$

As ζ is arbitrary, $\tau(\mathfrak{A}) \le \max(\omega, \tau(\mathfrak{C}_c))$ (332S).

(e) Putting (a)-(d) together, we have

$$\max(\tau(\mathfrak{A}), \#(I)) \le \max(\omega, \tau(\mathfrak{C}_c)) = \tau(\mathfrak{C}_c) \le \tau(\mathfrak{C}) \le \max(\tau(\mathfrak{A}), \#(I))$$

for every non-zero $c \in \mathfrak{C}$; so $\mathfrak{C}$ is homogeneous, with $\tau(\mathfrak{C}) = \max(\tau(\mathfrak{A}), \#(I))$. Re-stating this in terms of λ and μ, λ is Maharam-type-homogeneous and $\tau(\lambda) = \max(\tau(\mu), \#(I))$.

334X Basic exercises (a) Let (X, Σ, μ) and (Y, T, ν) be complete locally determined measure spaces with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Show that if $\nu Y > 0$ then $\tau(\mu) \le \tau(\lambda)$.

>(b) Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, with probability algebra free product $(\mathfrak{C}, \bar{\lambda})$. Show that $\tau(\mathfrak{A}_i) \le \tau(\mathfrak{C})$ for every i, and that

$$\#(\{i : i \in I,\ \tau(\mathfrak{A}_i) > 0\}) \le \tau(\mathfrak{C}).$$

(c) Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces, and λ the product measure on $X \times Y$. Show that λ is Maharam-type-homogeneous iff one of μ, ν is Maharam-type-homogeneous with Maharam type at least as great as the Maharam type of the other.

(d) Show that the product of any family of Maharam-type-homogeneous probability spaces is again Maharam-type-homogeneous.

>(e) Let (X, Σ, μ) be a probability space of Maharam type κ, and I any set with cardinal at least $\max(\omega, \kappa)$. Show that the product measure on $X \times \{0, 1\}^I$ is Maharam-type-homogeneous, with Maharam type $\#(I)$.

334Y Further exercises (a) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be an infinite family of probability spaces, with product (X, Λ, λ). Let κ_i be the Maharam type of μ_i for each i; set $\kappa = \max(\#(I), \sup_{i\in I} \kappa_i)$. Show that *either* λ is Maharam-type-homogeneous, with Maharam type κ, *or* there are $\kappa' < \kappa$, $X_i' \in \Sigma_i$ such that $\sum_{i\in I} \mu_i(X_i \setminus X_i') < \infty$, the Maharam type of the subspace measure on X_i' is at most κ' for every $i \in I$ and $\#(\{i : \kappa_i \ne 0\}) \le \kappa'$.

334 Notes and comments The results above are all very natural ones; I have spelt them out partly for completeness and partly for the sake of an application in §346 below. But note the second alternative in 334Ya; it is possible, even in an infinite product, for a kernel of relatively small Maharam type to be preserved.

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