Version of 21.7.11

Chapter 32

Measure algebras

I now come to the real work of this volume, the study of the Boolean algebras of equivalence classes of measurable sets. In this chapter I work through the 'elementary' theory, defining this to consist of the parts which do not depend on Maharam's theorem or the lifting theorem or non-trivial set theory.

§321 gives the definition of 'measure algebra', and relates this idea to its origin as the quotient of a σ-algebra of measurable sets by a σ-ideal of negligible sets, both in its elementary properties (following those of measure spaces treated in §112) and in an appropriate version of the Stone representation. §322 deals with the classification of measure algebras according to the scheme already developed in §211 for measure spaces. §323 discusses the standard topology and uniformity of a measure algebra. §324 contains results concerning Boolean homomorphisms between measure algebras, with the relationships between topological continuity, order-continuity and preservation of measure. §325 is devoted to the measure algebras of product measures, and their abstract characterization as completed free products. §§326-327 address the properties of additive functionals on Boolean algebras, generalizing the ideas of Chapter 23. Finally, §328 looks at 'reduced products' of probability algebras and some related constructions, including inductive limits.

Version of 3.1.11

321 Measure algebras

I begin by defining 'measure algebra' and relating this concept to the work of Chapter 31 and to the elementary properties of measure spaces.

321A Definition A **measure algebra** is a pair $(\mathfrak{A}, \bar{\mu})$, where $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ is a function such that

$\bar{\mu}0 = 0$;

whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, $\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) = \sum_{n=0}^{\infty} \bar{\mu}a_n$;

$\bar{\mu}a > 0$ whenever $a \in \mathfrak{A}$ and $a \neq 0$.

321B Elementary properties of measure algebras Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) If a, $b \in \mathfrak{A}$ and $a \cap b = 0$ then $\bar{\mu}(a \cup b) = \bar{\mu}a + \bar{\mu}b$.

(b) If a, $b \in \mathfrak{A}$ and $a \subseteq b$ then $\bar{\mu}a \leq \bar{\mu}b$.

(c) For any a, $b \in \mathfrak{A}$, $\bar{\mu}(a \cup b) \leq \bar{\mu}a + \bar{\mu}b$.

(d) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{A}$, then $\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) \leq \sum_{n=0}^{\infty} \bar{\mu}a_n$.

(e) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$, then $\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) = \lim_{n\to\infty} \bar{\mu}a_n$.

(f) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} \bar{\mu}a_n < \infty$, then $\bar{\mu}(\inf_{n\in\mathbb{N}} a_n) = \lim_{n\to\infty} \bar{\mu}a_n$.

321C Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $A \subseteq \mathfrak{A}$ a non-empty upwards-directed set. If $\sup_{a\in A} \bar{\mu}a < \infty$, then $\sup A$ is defined in $\mathfrak{A}$ and $\bar{\mu}(\sup A) = \sup_{a\in A} \bar{\mu}a$.

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321D Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a non-empty upwards-directed set. If $\sup A$ is defined in $\mathfrak{A}$, then $\bar\mu(\sup A)=\sup_{a\in A}\bar\mu a$.

321E Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a disjoint set. If $\sup A$ is defined in $\mathfrak{A}$, then $\bar\mu(\sup A)=\sum_{a\in A}\bar\mu a$.

321F Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $A\subseteq\mathfrak{A}$ a non-empty downwards-directed set. If $\inf_{a\in A}\bar\mu a<\infty$, then $\inf A$ is defined in $\mathfrak{A}$ and $\bar\mu(\inf A)=\inf_{a\in A}\bar\mu a$.

321G Subalgebras If $(\mathfrak{A},\bar\mu)$ is a measure algebra, and $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, then $(\mathfrak{B},\bar\mu\restriction\mathfrak{B})$ is a measure algebra.

321H The measure algebra of a measure space: Theorem Let (X,Σ,μ) be a measure space, and $\mathcal{N}$ the null ideal of μ. Let $\mathfrak{A}$ be the Boolean algebra quotient $\Sigma/\Sigma\cap\mathcal{N}$. Then we have a functional $\bar\mu:\mathfrak{A}\to[0,\infty]$ defined by setting

$$\bar\mu E^\bullet=\mu E \text{ for every } E\in\Sigma,$$

and $(\mathfrak{A},\bar\mu)$ is a measure algebra. The canonical map $E\mapsto E^\bullet:\Sigma\to\mathfrak{A}$ is sequentially order-continuous.

321I Definition For any measure space (X,Σ,μ) I will call $(\mathfrak{A},\bar\mu)$, as constructed above, the **measure algebra of** (X,Σ,μ).

321J The Stone representation of a measure algebra: Theorem Let $(\mathfrak{A},\bar\mu)$ be any measure algebra. Then it is isomorphic, as measure algebra, to the measure algebra of some measure space.

321K Definition I will call the measure space (Z,Σ,ν) constructed in the proof of 321J the **Stone space** of the measure algebra $(\mathfrak{A},\bar\mu)$.

Z is a compact Hausdorff space, being the Stone space of $\mathfrak{A}$. $\mathfrak{A}$ can be identified with the algebra of open-and-closed sets in Z. The null ideal of ν coincides with the ideal of meager subsets of Z; ν is complete. The measurable sets are precisely those expressible in the form $E=\widehat{a}\triangle M$ where $a\in\mathfrak{A}$, $\widehat{a}\subseteq Z$ is the corresponding open-and-closed set, and M is meager; in this case $\nu E=\bar\mu a$ and a is the member of $\mathfrak{A}$ corresponding to E.

Version of 24.4.06

322 Taxonomy of measure algebras

Before going farther with the general theory of measure algebras, I run through those parts of the classification of measure spaces in §211 which have expressions in terms of measure algebras. The most important concepts at this stage are those of 'semi-finite', 'localizable' and 'σ-finite' measure algebra (322Ac-322Ae); these correspond exactly to the same terms applied to measure spaces (322B). I briefly investigate the Boolean-algebra properties of semi-finite and σ-finite measure algebras (322F, 322G), with mentions of completions and c.l.d. versions (322D), subspace measures (322I-322J), indefinite-integral measures (322K), direct sums of measure spaces (322L, 322M) and subalgebras of measure algebras (322N). It turns out that localizability of a measure algebra is connected in striking ways to the properties of the canonical measure on its Stone space (322O). I end the section with a description of the 'localization' of a semi-finite measure algebra (322P-322Q) and with some further properties of Stone spaces (322R).

322A Definitions Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) $(\mathfrak{A},\bar\mu)$ is a **probability algebra** if $\bar\mu 1=1$.

(b) $(\mathfrak{A},\bar\mu)$ is **totally finite** if $\bar\mu 1<\infty$.

(c) $(\mathfrak{A}, \bar{\mu})$ is **σ-finite** if there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\bar{\mu}a_n < \infty$ for every $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} a_n = 1$. Note that in this case $\langle a_n \rangle_{n\in\mathbb{N}}$ can be taken *either* to be non-decreasing *or* to be disjoint.

(d) $(\mathfrak{A}, \bar{\mu})$ is **semi-finite** if whenever $a \in \mathfrak{A}$ and $\bar{\mu}a = \infty$ there is a non-zero $b \subseteq a$ such that $\bar{\mu}b < \infty$.

(e) $(\mathfrak{A}, \bar{\mu})$ is **localizable** if it is semi-finite and the Boolean algebra $\mathfrak{A}$ is Dedekind complete.

322B Theorem Let (X, Σ, μ) be a measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Then
(a) (X, Σ, μ) is a probability space iff $(\mathfrak{A}, \bar{\mu})$ is a probability algebra;
(b) (X, Σ, μ) is totally finite iff $(\mathfrak{A}, \bar{\mu})$ is;
(c) (X, Σ, μ) is σ-finite iff $(\mathfrak{A}, \bar{\mu})$ is;
(d) (X, Σ, μ) is semi-finite iff $(\mathfrak{A}, \bar{\mu})$ is;
(e) (X, Σ, μ) is localizable iff $(\mathfrak{A}, \bar{\mu})$ is;
(f) if $E \in \Sigma$, then E is an atom for μ iff $E^\bullet$ is an atom in $\mathfrak{A}$;
(g) (X, Σ, μ) is atomless iff $\mathfrak{A}$ is;
(h) (X, Σ, μ) is purely atomic iff $\mathfrak{A}$ is.

322C Theorem (a) A probability algebra is totally finite.
(b) A totally finite measure algebra is σ-finite.
(c) A σ-finite measure algebra is localizable.
(d) A localizable measure algebra is semi-finite.

322D Proposition Let (X, Σ, μ) be a measure space, with completion $(X, \hat{\Sigma}, \hat{\mu})$ and c.l.d. version $(X, \tilde{\Sigma}, \tilde{\mu})$. Write $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ for the measure algebras of μ, $\hat{\mu}$ and $\tilde{\mu}$ respectively.

(a) The embedding $\Sigma \subsetneqq \hat{\Sigma}$ corresponds to an isomorphism between $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{A}_1, \bar{\mu}_1)$.

(b)(i) The embedding $\Sigma \subsetneqq \tilde{\Sigma}$ defines an order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_2$. Setting $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$, $\pi\restriction\mathfrak{A}^f$ is a measure-preserving bijection between $\mathfrak{A}^f$ and $\mathfrak{A}_2^f = \{c : c \in \mathfrak{A}_2,\ \bar{\mu}_2 c < \infty\}$.

(ii) π is injective iff μ is semi-finite, and in this case $\bar{\mu}_2(\pi a) = \bar{\mu}a$ for every $a \in \mathfrak{A}$.

(iii) If μ is localizable, π is a bijection.

322E Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.
(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff it has a partition of unity consisting of elements of finite measure.
(b) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite, $a = \sup\{b : b \subseteq a,\ \bar{\mu}b < \infty\}$ and $\bar{\mu}a = \sup\{\bar{\mu}b : b \subseteq a,\ \bar{\mu}b < \infty\}$ for every $a \in \mathfrak{A}$.

322F Proposition If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, then $\mathfrak{A}$ is a weakly (σ, ∞)-distributive Boolean algebra.

322G Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Then the following are equiveridical:
(i) $(\mathfrak{A}, \bar{\mu})$ is σ-finite;
(ii) $\mathfrak{A}$ is ccc;
(iii) *either* $\mathfrak{A} = \{0\}$ *or* there is a functional $\bar{\nu} : \mathfrak{A} \to [0, 1]$ such that $(\mathfrak{A}, \bar{\nu})$ is a probability algebra.

322H Principal ideals If $(\mathfrak{A}, \bar{\mu})$ is a measure algebra and $a \in \mathfrak{A}$, then $(\mathfrak{A}_a, \bar{\mu}\restriction\mathfrak{A}_a)$ is a measure algebra, where $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a.

322I Subspace measures: Proposition Let (X, Σ, μ) be a measure space, and $A \subseteq X$ a set with a measurable envelope E. Let μ_A be the subspace measure on A, and Σ_A its domain; let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of (X, Σ, μ) and $(\mathfrak{A}_A, \bar{\mu}_A)$ the measure algebra of (A, Σ_A, μ_A). Set $a = E^\bullet$ and let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Then we have an isomorphism between $(\mathfrak{A}_a, \bar{\mu}\restriction\mathfrak{A}_a)$ and $(\mathfrak{A}_A, \bar{\mu}_A)$ given by the formula

$$F^\bullet \mapsto (F \cap A)^\circ$$

whenever $F \in \Sigma$ and $F \subseteq E$, writing $F^\bullet$ for the equivalence class of F in $\mathfrak{A}$ and $(F \cap A)^\circ$ for the equivalence class of $F \cap A$ in $\mathfrak{A}_A$.

322J Corollary Let (X,Σ,μ) be a measure space, with measure algebra $(\mathfrak{A},\bar{\mu})$.

(a) If $E\in\Sigma$, then the measure algebra of the subspace measure μ_E can be identified with the principal ideal $\mathfrak{A}_{E^\bullet}$ of $\mathfrak{A}$.

(b) If $A\subseteq X$ is a set of full outer measure (in particular, if $\mu^*A=\mu X<\infty$), then the measure algebra of the subspace measure μ_A can be identified with $\mathfrak{A}$.

322K Indefinite-integral measures: Proposition Let (X,Σ,μ) be a measure space and ν an indefinite-integral measure over μ. Then the measure algebra of ν can be identified, as Boolean algebra, with a principal ideal of the measure algebra of μ.

322L Simple products (a) Let $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$ be an indexed family of measure algebras. Let $\mathfrak{A}$ be the simple product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$, and for $a\in\mathfrak{A}$ set $\bar{\mu}a=\sum_{i\in I}\bar{\mu}_i a(i)$. Then $(\mathfrak{A},\bar{\mu})$ is a measure algebra; I will call it the **simple product** of the family $\langle(\mathfrak{A}_i,\bar{\mu}_i)\rangle_{i\in I}$. Each of the $\mathfrak{A}_i$ corresponds to a principal ideal $\mathfrak{A}_{e_i}$ say in $\mathfrak{A}$, where $e_i\in\mathfrak{A}$ corresponds to $1_{\mathfrak{A}_i}\in\mathfrak{A}_i$, and the Boolean isomorphism between $\mathfrak{A}_i$ and $\mathfrak{A}_{e_i}$ is a measure algebra isomorphism between $(\mathfrak{A}_i,\bar{\mu}_i)$ and $(\mathfrak{A}_{e_i},\bar{\mu}\restriction\mathfrak{A}_{e_i})$.

(b) If $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is a family of measure spaces, with direct sum (X,Σ,μ), then the measure algebra $(\mathfrak{A},\bar{\mu})$ of (X,Σ,μ) can be identified with the simple product of the measure algebras $(\mathfrak{A}_i,\bar{\mu}_i)$ of the (X_i,Σ_i,μ_i).

(c) A simple product of measure algebras is semi-finite, or localizable, or atomless, or purely atomic, iff every factor is.

(d) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\langle e_i\rangle_{i\in I}$ a countable partition of unity in $\mathfrak{A}$. Then $(\mathfrak{A},\bar{\mu})$ is isomorphic to the product $\prod_{i\in I}(\mathfrak{A}_{e_i},\bar{\mu}\restriction\mathfrak{A}_{e_i})$ of the corresponding principal ideals.

(e) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra.

(i) If $\langle e_i\rangle_{i\in I}$ is any partition of unity in $\mathfrak{A}$, then $(\mathfrak{A},\bar{\mu})$ is isomorphic to the product $\prod_{i\in I}(\mathfrak{A}_{e_i},\bar{\mu}\restriction\mathfrak{A}_{e_i})$ of the corresponding principal ideals.

(ii) $(\mathfrak{A},\bar{\mu})$ is isomorphic to the measure algebra of a direct sum of totally finite measure spaces, which is strictly localizable.

***322M Strictly localizable spaces: Proposition** Let (X,Σ,μ) be a strictly localizable measure space with $\mu X>0$, and $(\mathfrak{A},\bar{\mu})$ its measure algebra. If $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, there is a partition $\langle X_i\rangle_{i\in I}$ of X into members of Σ such that $X_i^\bullet=a_i$ for every $i\in I$ and

$$\Sigma=\{E:E\subseteq X,\ E\cap X_i\in\Sigma\ \forall\ i\in I\},$$

$$\mu E=\textstyle\sum_{i\in I}\mu(E\cap X_i)\text{ for every }E\in\Sigma;$$

that is, the isomorphism between $\mathfrak{A}$ and the simple product $\prod_{i\in I}\mathfrak{A}_{a_i}$ of its principal ideals corresponds to an isomorphism between (X,Σ,μ) and the direct sum of the subspace measures on X_i.

322N Subalgebras: Proposition Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Set $\bar{\nu}=\bar{\mu}\restriction\mathfrak{B}$.

(a) $(\mathfrak{B},\bar{\nu})$ is a measure algebra.

(b) If $(\mathfrak{A},\bar{\mu})$ is totally finite, or a probability algebra, so is $(\mathfrak{B},\bar{\nu})$.

(c) If $(\mathfrak{A},\bar{\mu})$ is σ-finite and $(\mathfrak{B},\bar{\nu})$ is semi-finite, then $(\mathfrak{B},\bar{\nu})$ is σ-finite.

(d) If $(\mathfrak{A},\bar{\mu})$ is localizable and $\mathfrak{B}$ is order-closed and $(\mathfrak{B},\bar{\nu})$ is semi-finite, then $(\mathfrak{B},\bar{\nu})$ is localizable.

(e) If $(\mathfrak{B},\bar{\nu})$ is a probability algebra, or totally finite, or σ-finite, so is $(\mathfrak{A},\bar{\mu})$.

322O The Stone space of a localizable measure algebra: Theorem Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, Z the Stone space of $\mathfrak{A}$, and ν the standard measure on Z. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar{\mu})$ is localizable;

(ii) ν is localizable;

(iii) ν is locally determined;

(iv) ν is strictly localizable.

322P Theorem Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$. Then there is a unique extension of $\bar{\mu}$ to a functional $\tilde{\mu}$ on $\widehat{\mathfrak{A}}$ such that $(\widehat{\mathfrak{A}},\tilde{\mu})$ is a localizable measure algebra. The embedding $\mathfrak{A} \subsetneqq \widehat{\mathfrak{A}}$ identifies the ideals $\{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$ and $\{a : a \in \widehat{\mathfrak{A}},\ \tilde{\mu}a < \infty\}$.

322Q Definition Let $(\mathfrak{A},\bar{\mu})$ be any semi-finite measure algebra. I will call $(\widehat{\mathfrak{A}},\tilde{\mu})$, as constructed above, the **localization** of $(\mathfrak{A},\bar{\mu})$.

322R Further properties of Stone spaces: Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and (Z,Σ,ν) its Stone space.

(a) Meager sets in Z are nowhere dense; every $E \in \Sigma$ is uniquely expressible as $G \triangle M$ where $G \subseteq Z$ is open-and-closed and M is nowhere dense, and $\nu E = \sup\{\nu H : H \subseteq E \text{ is open-and-closed}\}$.

(b) The c.l.d. version $\tilde{\nu}$ of ν is strictly localizable, and has the same negligible sets as ν.

(c) If $(\mathfrak{A},\bar{\mu})$ is totally finite then $\nu E = \inf\{\nu H : H \supseteq E \text{ is open-and-closed}\}$ for every $E \in \Sigma$.

Version of 20.7.06

323 The topology of a measure algebra

I take a short section to discuss one of the fundamental tools for studying totally finite measure algebras, the natural metric that each carries. The same ideas, suitably adapted, can be applied to an arbitrary measure algebra, where we have a topology corresponding closely to the topology of convergence in measure on the function space L^0. Most of the section consists of an analysis of the relations between this topology and the order structure of the measure algebra.

323A The pseudometrics ρ_a (a) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. Write $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$. For $a \in \mathfrak{A}^f$ and b, $c \in \mathfrak{A}$, write $\rho_a(b,c) = \bar{\mu}(a \cap (b \triangle c))$. ρ_a is a pseudometric on $\mathfrak{A}$.

(b) Now the **measure-algebra topology** of the measure algebra $(\mathfrak{A},\bar{\mu})$ is that generated by the family $\mathrm{P} = \{\rho_a : a \in \mathfrak{A}^f\}$ of pseudometrics on $\mathfrak{A}$. Similarly the **measure-algebra uniformity** on $\mathfrak{A}$ is that generated by P.

(c) P is upwards-directed.

(d) On the ideal $\mathfrak{A}^f$ we have an actual metric ρ defined by saying that $\rho(a,b) = \bar{\mu}(a \triangle b)$ for a, $b \in \mathfrak{A}^f$; this is the **measure metric**. I will call the topology it generates the **strong measure-algebra topology** on $\mathfrak{A}^f$.

When $\bar{\mu}$ is totally finite, $\rho = \rho_1$ defines the measure-algebra topology and uniformity of $\mathfrak{A}$.

323B Proposition Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra, and give $\mathfrak{A}$ its measure-algebra topology.

(a) The operations $\cup$, $\cap$, $\setminus$ and $\triangle$ are all uniformly continuous.

(b) $\mathfrak{A}^f$ is dense in $\mathfrak{A}$.

323C Proposition (a) Let $(\mathfrak{A},\bar{\mu})$ be a totally finite measure algebra. Then $\bar{\mu} : \mathfrak{A} \to [0,\infty[$ is uniformly continuous.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra. Then $\bar{\mu} : \mathfrak{A} \to [0,\infty]$ is lower semi-continuous.

(c) Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra. If $a \in \mathfrak{A}$ and $\bar{\mu}a < \infty$, then $b \mapsto \bar{\mu}(b \cap a) : \mathfrak{A} \to \mathbb{R}$ is uniformly continuous.

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323D Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) Let $B\subseteq\mathfrak{A}$ be a non-empty upwards-directed set. For $b\in B$ set $F_b=\{c:b\subseteq c\in B\}$.

(i) $\{F_b:b\in B\}$ generates a Cauchy filter $\mathcal{F}(B\uparrow)$ on $\mathfrak{A}$.

(ii) If $\sup B$ is defined in $\mathfrak{A}$, then it is a topological limit of $\mathcal{F}(B\uparrow)$; it belongs to the topological closure of B.

(b) Let $B\subseteq\mathfrak{A}$ be a non-empty downwards-directed set. For $b\in B$ set $F'_b=\{c:b\supseteq c\in B\}$.

(i) $\{F'_b:b\in B\}$ generates a Cauchy filter $\mathcal{F}(B\downarrow)$ on $\mathfrak{A}$.

(ii) If $\inf B$ is defined in $\mathfrak{A}$, then it is a topological limit of $\mathcal{F}(B\downarrow)$; it belongs to the topological closure of B.

(c)(i) Closed subsets of $\mathfrak{A}$ are order-closed.

(ii) An order-dense subalgebra of $\mathfrak{A}$ must be dense in the topological sense.

(d) Now suppose that $(\mathfrak{A},\bar\mu)$ is semi-finite.

(i) The sets $\{b:b\subseteq c\}$, $\{b:b\supseteq c\}$ are closed for every $c\in\mathfrak{A}$.

(ii) If $B\subseteq\mathfrak{A}$ is non-empty and upwards-directed and e is a cluster point of $\mathcal{F}(B\uparrow)$, then $e=\sup B$.

(iii) If $B\subseteq\mathfrak{A}$ is non-empty and downwards-directed and e is a cluster point of $\mathcal{F}(B\downarrow)$, then $e=\inf B$.

323E Corollary Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) If $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum b, then $\langle b_n\rangle_{n\in\mathbb{N}}$ converges topologically to b.

(b) If $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum b, then $\langle b_n\rangle_{n\in\mathbb{N}}$ converges topologically to b.

323F Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $\langle c_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\sum_{n=0}^\infty\bar\mu(c_n\mathbin{\triangle}c_{n+1})$ is finite. Set $d_0=\sup_{n\in\mathbb{N}}\inf_{m\ge n}c_m$, $d_1=\inf_{n\in\mathbb{N}}\sup_{m\ge n}c_m$. Then $d_0=d_1$ and, writing d for their common value, $\lim_{n\to\infty}\bar\mu(c_n\mathbin{\triangle}d)=0$.

323G The classification of measure algebras: Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, $\mathfrak{T}$ its measure-algebra topology and $\mathcal{U}$ its measure-algebra uniformity.

(a) $(\mathfrak{A},\bar\mu)$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(b) $(\mathfrak{A},\bar\mu)$ is σ-finite iff $\mathfrak{T}$ is metrizable, and in this case $\mathcal{U}$ also is metrizable.

(c) $(\mathfrak{A},\bar\mu)$ is localizable iff $\mathfrak{T}$ is Hausdorff and $\mathfrak{A}$ is complete under $\mathcal{U}$.

323H Closed subalgebras: Theorem Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then it is topologically closed iff it is order-closed.

323I Notation In the context of 323H, I will say that $\mathfrak{B}$ is a **closed subalgebra** of $\mathfrak{A}$.

323J Proposition If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then the topological closure $\overline{\mathfrak{B}}$ of $\mathfrak{B}$ in $\mathfrak{A}$ is precisely the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$.

323K Lemma If $(\mathfrak{A},\bar\mu)$ is a localizable measure algebra and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then for any $a\in\mathfrak{A}$ the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}\cup\{a\}$ is closed.

323L Proposition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of measure algebras with simple product $(\mathfrak{A},\bar\mu)$. Then the measure-algebra topology on $\mathfrak{A}=\prod_{i\in I}\mathfrak{A}_i$ defined by $\bar\mu$ is the product of the measure-algebra topologies of the $\mathfrak{A}_i$.

***323M Proposition** Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and give $\mathfrak{A}^f$ its measure metric.

(a) The Boolean operations $\triangle$, $\cap$, $\cup$ and $\setminus$ on $\mathfrak{A}^f$ are uniformly continuous.

(b) $\bar\mu\restriction\mathfrak{A}^f:\mathfrak{A}^f\to[0,\infty[$ is 1-Lipschitz, therefore uniformly continuous.

(c) $\mathfrak{A}^f$ is complete.

Version of 29.11.17

324 Homomorphisms

In the course of Volume 2, I had occasion to remark that elementary measure theory is unusual among abstract topics in pure mathematics in not being dominated by any particular class of structure-preserving operators. We now come to what I think is one of the reasons for the gap: the most important operators of the theory are not between measure spaces at all, but between their measure algebras. In this section I run through the most elementary facts about Boolean homomorphisms between measure algebras. I start with results on the construction of such homomorphisms from functions between measure spaces (324A-324E), then investigate continuity and order-continuity of homomorphisms (324F-324H) before turning to measure-preserving homomorphisms (324I-324P).

324A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ their measure algebras. Write $\hat\Sigma$ for the domain of the completion $\hat\mu$ of μ. Let $D\subseteq X$ be a set of full outer measure, and $\hat\Sigma_D$ the subspace σ-algebra on D induced by $\hat\Sigma$. Let $\phi:D\to Y$ be a function such that $\phi^{-1}[F]\in\hat\Sigma_D$ for every $F\in\mathrm{T}$ and $\phi^{-1}[F]$ is μ-negligible whenever $\nu F=0$. Then there is a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by the formula

$$\pi F^\bullet=E^\bullet \text{ whenever } F\in\mathrm{T},\ E\in\Sigma \text{ and } (E\cap D)\triangle\phi^{-1}[F] \text{ is negligible.}$$

324B Corollary Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ their measure algebras. Let $\phi:X\to Y$ be a function such that $\phi^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$ and $\mu\phi^{-1}[F]=0$ whenever $\nu F=0$. Then there is a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by the formula

$$\pi F^\bullet=(\phi^{-1}[F])^\bullet \text{ for every } F\in\mathrm{T}.$$

324D Proposition Let (X,Σ,μ), (Y,T,ν) and (Z,Λ,λ) be measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$, $(\mathfrak{C},\bar\lambda)$. Suppose that $\phi:X\to Y$ and $\psi:Y\to Z$ satisfy the conditions of 324B, that is,

$$\phi^{-1}[F]\in\Sigma \text{ if } F\in\mathrm{T},\quad \mu\phi^{-1}[F]=0 \text{ if } \nu F=0,$$

$$\psi^{-1}[G]\in\mathrm{T} \text{ if } G\in\Lambda,\quad \mu\psi^{-1}[G]=0 \text{ if } \lambda G=0.$$

Let $\pi_\phi:\mathfrak{B}\to\mathfrak{A}$, $\pi_\psi:\mathfrak{C}\to\mathfrak{B}$ be the corresponding homomorphisms. Then $\psi\phi:X\to Z$ is another map of the same type, and $\pi_{\psi\phi}=\pi_\phi\pi_\psi:\mathfrak{C}\to\mathfrak{A}$.

324E Stone spaces: Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, with Stone spaces Z and W; let μ, ν be the corresponding measures on Z and W, and Σ, T their domains. If $\pi:\mathfrak{B}\to\mathfrak{A}$ is any order-continuous Boolean homomorphism, let $\phi:Z\to W$ be the corresponding continuous function. Then $\phi^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$, $\mu\phi^{-1}[F]=0$ whenever $\nu F=0$, and (writing E^* for the member of $\mathfrak{A}$ corresponding to $E\in\Sigma$) $\pi F^*=(\phi^{-1}[F])^*$ for every $F\in\mathrm{T}$.

324F Theorem Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism. Give $\mathfrak{A}$ and $\mathfrak{B}$ their measure-algebra topologies and uniformities.

(a) π is continuous iff it is continuous at 0 iff it is uniformly continuous.
(b) If $(\mathfrak{B},\bar\nu)$ is semi-finite and π is continuous, then it is order-continuous.
(c) If $(\mathfrak{A},\bar\mu)$ is semi-finite and π is order-continuous, then it is continuous.

324G Corollary If $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are semi-finite measure algebras, a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ is continuous iff it is order-continuous.

324H Corollary If $\mathfrak{A}$ is a Boolean algebra and $\bar\mu$, $\bar\nu$ are two measures both rendering $\mathfrak{A}$ a semi-finite measure algebra, then they endow $\mathfrak{A}$ with the same uniformity (and the same topology).

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324I Definition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras. A Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is **measure-preserving** if $\bar{\nu}(\pi a) = \bar{\mu} a$ for every $a \in \mathfrak{A}$.

324J Proposition Let $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{B}, \bar{\nu})$ and $(\mathfrak{C}, \bar{\lambda})$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$, $\theta : \mathfrak{B} \to \mathfrak{C}$ measure-preserving Boolean homomorphisms. Then $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism.

324K Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving Boolean homomorphism.

(a) π is injective.

(b) $(\mathfrak{A}, \bar{\mu})$ is totally finite iff $(\mathfrak{B}, \bar{\nu})$ is, and in this case π is order-continuous, therefore continuous, and $\pi[\mathfrak{A}]$ is a closed subalgebra of $\mathfrak{B}$.

(c) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite and $(\mathfrak{B}, \bar{\nu})$ is σ-finite, then $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(d) If $(\mathfrak{A}, \bar{\mu})$ is σ-finite and π is sequentially order-continuous, then $(\mathfrak{B}, \bar{\nu})$ is σ-finite.

(e) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite and π is order-continuous, then $(\mathfrak{B}, \bar{\nu})$ is semi-finite.

(f) If $(\mathfrak{A}, \bar{\mu})$ is atomless and semi-finite, and π is order-continuous, then $\mathfrak{B}$ is atomless.

(g) If $\mathfrak{B}$ is purely atomic and $(\mathfrak{A}, \bar{\mu})$ is semi-finite, then $\mathfrak{A}$ is purely atomic.

324L Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $(\mathfrak{B}, \bar{\nu})$ a measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the closed subalgebra of $\mathfrak{A}$ generated by C, then $\pi[\mathfrak{C}]$ is the closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

324M Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$. Let $\phi : X \to Y$ be inverse-measure-preserving. Then we have a sequentially order-continuous measure-preserving Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by setting $\pi F^\bullet = \phi^{-1}[F]^\bullet$ for every $F \in \mathrm{T}$.

324N Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, with Stone spaces Z and W; let μ, ν be the corresponding measures on Z and W. If $\pi : \mathfrak{B} \to \mathfrak{A}$ is an order-continuous measure-preserving Boolean homomorphism, and $\phi : Z \to W$ the corresponding continuous function, then ϕ is inverse-measure-preserving.

324O Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras, $\mathfrak{A}_0$ a topologically dense subalgebra of $\mathfrak{A}$, and $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism such that $\bar{\nu}\pi a = \bar{\mu} a$ for every $a \in \mathfrak{A}_0$. Then π has a unique extension to a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

***324P Proposition** Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras such that $\bar{\mu}1 = \bar{\nu}1$. Suppose that $A \subseteq \mathfrak{A}$ and $\phi : A \to \mathfrak{B}$ are such that $\bar{\nu}(\inf_{i \le n} \phi a_i) = \bar{\mu}(\inf_{i \le n} a_i)$ for all $a_0, \dots, a_n \in A$. Let $\mathfrak{C}$ be the smallest closed subalgebra of $\mathfrak{A}$ including A. Then ϕ has a unique extension to a measure-preserving Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$.

Version of 30.8.06

325 Free products and product measures

In this section I aim to describe the measure algebras of product measures as defined in Chapter 25. This will involve the concept of 'free product' set out in §315. It turns out that we cannot determine the measure algebra of a product measure from the measure algebras of the factors (325B), unless we are told that the product measure is localizable; but that there is nevertheless a general construction of 'localizable measure algebra free product', applicable to any pair of semi-finite measure algebras (325D), which represents the measure algebra of the product measure in the most important cases (325Eb). In the second part of the section (325I-325M) I deal with measure algebra free products of probability algebras, corresponding to the products of probability spaces treated in §254.

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325A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$. Let λ be the c.l.d. product measure on $X\times Y$, and Λ its domain; let $(\mathfrak{C},\bar\lambda)$ be the corresponding measure algebra.

(a)(i) The map $E\mapsto E\times Y:\Sigma\to\Lambda$ induces an order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.

(ii) The map $F\mapsto X\times F:\mathrm{T}\to\Lambda$ induces an order-continuous Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{C}$.

(b) The map $(E,F)\mapsto E\times F:\Sigma\times\mathrm{T}\to\Lambda$ induces a Boolean homomorphism $\psi:\mathfrak{A}\otimes\mathfrak{B}\to\mathfrak{C}$.

(c) $\psi[\mathfrak{A}\otimes\mathfrak{B}]$ is topologically dense in $\mathfrak{C}$ for the measure-algebra topology of $\mathfrak{C}$.

(d) For every $c\in\mathfrak{C}$,

$$\bar\lambda c=\sup\{\bar\lambda(c\cap\psi(a\otimes b)):a\in\mathfrak{A},\,b\in\mathfrak{B},\,\bar\mu a<\infty,\,\bar\nu b<\infty\}.$$

(e) If μ and ν are semi-finite, ψ is injective and $\bar\lambda\psi(a\otimes b)=\bar\mu a\cdot\bar\mu b$ for every $a\in\mathfrak{A}$, $b\in\mathfrak{B}$.

325B Characterizing the measure algebra of a product space: Example There are complete locally determined localizable measure spaces (X,μ), (X',μ'), with isomorphic measure algebras, and a probability space (Y,ν) such that the measure algebras of the c.l.d. product measures on $X\times Y$, $X'\times Y$ are not isomorphic.

325C Theorem Let (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2) be semi-finite measure spaces, with measure algebras $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$. Let λ be the c.l.d. product measure on $X_1\times X_2$, and $(\mathfrak{C},\bar\lambda)$ the corresponding measure algebra. Let $(\mathfrak{B},\bar\nu)$ be a localizable measure algebra, and $\phi_1:\mathfrak{A}_1\to\mathfrak{B}$, $\phi_2:\mathfrak{A}_2\to\mathfrak{B}$ order-continuous Boolean homomorphisms such that $\bar\nu(\phi_1(a_1)\cap\phi_2(a_2))=\bar\mu_1a_1\cdot\bar\mu_2a_2$ for all $a_1\in\mathfrak{A}_1$, $a_2\in\mathfrak{A}_2$. Then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi(\psi(a_1\otimes a_2))=\phi_1(a_1)\cap\phi_2(a_2)$ for all $a_1\in\mathfrak{A}_1$, $a_2\in\mathfrak{A}_2$, writing $\psi:\mathfrak{A}_1\otimes\mathfrak{A}_2\to\mathfrak{C}$ for the canonical map.

325D Theorem Let $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$ be semi-finite measure algebras.

(a) There is a localizable measure algebra $(\mathfrak{C},\bar\lambda)$, together with order-continuous Boolean homomorphisms $\varepsilon_1:\mathfrak{A}_1\to\mathfrak{C}$ and $\varepsilon_2:\mathfrak{A}_2\to\mathfrak{C}$, such that whenever $(\mathfrak{B},\bar\nu)$ is a localizable measure algebra, and $\phi_1:\mathfrak{A}_1\to\mathfrak{B}$, $\phi_2:\mathfrak{A}_2\to\mathfrak{B}$ are order-continuous Boolean homomorphisms and $\bar\nu(\phi_1(a_1)\cap\phi_2(a_2))=\bar\mu_1a_1\cdot\bar\mu_2a_2$ for all $a_1\in\mathfrak{A}_1$, $a_2\in\mathfrak{A}_2$, then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\varepsilon_j=\phi_j$ for both j.

(b) The structure $(\mathfrak{C},\bar\lambda,\varepsilon_1,\varepsilon_2)$ is determined up to isomorphism by this property.

(c)(i) The Boolean homomorphism $\psi:\mathfrak{A}_1\otimes\mathfrak{A}_2\to\mathfrak{C}$ defined from ε_1 and ε_2 is injective, and $\psi[\mathfrak{A}_1\otimes\mathfrak{A}_2]$ is topologically dense in $\mathfrak{C}$.

(ii) The closed subalgebra of $\mathfrak{C}$ generated by $\psi[\mathfrak{A}_1\otimes\mathfrak{A}_2]$ is the whole of $\mathfrak{C}$.

(d) If $j\in\{1,2\}$ and $(\mathfrak{A}_j,\bar\mu_j)$ is localizable, then $\varepsilon_j[\mathfrak{A}_j]$ is a closed subalgebra of $(\mathfrak{C},\bar\lambda)$.

325E Remarks We could say that a measure algebra $(\mathfrak{C},\bar\lambda)$, together with embeddings ε_1 and ε_2, as described in 325D, is a **localizable measure algebra free product** of $(\mathfrak{A}_1,\bar\mu_1)$ and $(\mathfrak{A}_2,\bar\mu_2)$.

325F Example Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure μ on $[0,1]$, and $(\mathfrak{C},\bar\lambda)$ the measure algebra of Lebesgue measure λ on $[0,1]^2$. Then $(\mathfrak{C},\bar\lambda)$ can be regarded as the localizable measure algebra free product of $(\mathfrak{A},\bar\mu)$ with itself. Let $\psi:\mathfrak{A}\otimes\mathfrak{A}\to\mathfrak{C}$ be the canonical map. Then $\psi[\mathfrak{A}\otimes\mathfrak{A}]$ is not order-dense in $\mathfrak{C}$, and ψ is not order-continuous.

325G Example Again, let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$, and $(\mathfrak{C},\bar\lambda)$ the measure algebra of Lebesgue measure on $[0,1]^2$. Then there is no order-continuous Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{A}$ such that $\phi(a\otimes b)=a\cap b$ for all a, $b\in\mathfrak{A}$.

***325H Products of more than two factors (a)** Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty finite family of semi-finite measure algebras. Then there is a localizable measure algebra $(\mathfrak{C},\bar\lambda)$, together with order-continuous Boolean homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ for $i\in I$, such that whenever $(\mathfrak{B},\bar\nu)$ is a localizable measure algebra, and $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ are order-continuous Boolean homomorphisms such that $\bar\nu(\inf_{i\in I}\phi_i(a_i))=\prod_{i\in I}\bar\mu_ia_i$ whenever $a_i\in\mathfrak{A}_i$ for each i, then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\phi_i$ for every i.

(b) The structure $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ is determined up to isomorphism by this property.

(c) The Boolean homomorphism $\psi:\bigotimes_{i\in I}\mathfrak{A}_i\to\mathfrak{C}$ defined from the ε_i is injective, and $\psi[\bigotimes_{i\in I}\mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$.

(d) Write $\widehat{\bigotimes}^{\mathrm{loc}}_{i\in I}(\mathfrak{A}_i,\bar\mu_i)$ for (a particular version of) the localizable measure algebra free product described in (a). If $\langle(A_i,\bar\mu_i)\rangle_{i\in I}$ is a finite family of semi-finite measure algebras and $\langle I_k\rangle_{k\in K}$ is a partition of I into non-empty sets, then $\widehat{\bigotimes}^{\mathrm{loc}}_{i\in I}(\mathfrak{A}_i,\bar\mu_i)$ is isomorphic, in a canonical way, to $\widehat{\bigotimes}^{\mathrm{loc}}_{k\in K}\bigl(\widehat{\bigotimes}^{\mathrm{loc}}_{i\in I_k}(\mathfrak{A}_i,\bar\mu_i)\bigr)$.

(e) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a finite family of semi-finite measure spaces, and write $(\mathfrak{A}_i,\bar\mu_i)$ for the measure algebra of (X_i,Σ_i,μ_i). Let λ be the c.l.d. product measure on $\prod_{i\in I}X_i$, and $(\mathfrak{C},\bar\lambda)$ the corresponding measure algebra. Then there is a canonical order-continuous measure-preserving embedding of $(\mathfrak{C},\bar\lambda)$ into the localizable measure algebra free product of the $(\mathfrak{A}_i,\bar\mu_i)$. If each μ_i is strictly localizable, this embedding is an isomorphism.

325I Infinite products: Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of probability spaces, with measure algebras $(\mathfrak{A}_i,\bar\mu_i)$. Let λ be the product measure on $X=\prod_{i\in I}X_i$, and $(\mathfrak{C},\bar\lambda)$ the corresponding measure algebra. For each $i\in I$, we have a measure-preserving homomorphism $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ corresponding to the inverse-measure-preserving function $x\mapsto x(i):X\to X_i$. Let $(\mathfrak{B},\bar\nu)$ be a probability algebra, and $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ Boolean homomorphisms such that $\bar\nu(\inf_{i\in J}\phi_i(a_i))=\prod_{i\in J}\bar\mu_i a_i$ whenever $J\subseteq I$ is a finite set and $a_i\in\mathfrak{A}_i$ for every i. Then there is a unique measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\phi_i$ for every $i\in I$.

325J Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras.

(a) There is a probability algebra $(\mathfrak{C},\bar\lambda)$, together with measure-preserving Boolean homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ for $i\in I$, such that whenever $(\mathfrak{B},\bar\nu)$ is a probability algebra, and $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ are Boolean homomorphisms such that $\bar\nu(\inf_{i\in J}\phi_i(a_i))=\prod_{i\in J}\bar\mu_i a_i$ whenever $J\subseteq I$ is finite and $a_i\in\mathfrak{A}_i$ for each $i\in J$, then there is a unique measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\phi_i$ for every $i\in I$.

(b) The structure $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ is determined up to isomorphism by this property.

(c) The Boolean homomorphism $\psi:\bigotimes_{i\in I}\mathfrak{A}_i\to\mathfrak{C}$ defined from the ε_i is injective, and $\psi[\bigotimes_{i\in I}\mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$.

325K Definition As in 325Ea, we can say that $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ is the **probability algebra free product** of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$.

325L Independent subalgebras If $(\mathfrak{A},\bar\mu)$ is a probability algebra, we say that a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **stochastically independent** if $\bar\mu(\inf_{i\in J}b_i)=\prod_{i\in J}\bar\mu b_i$ whenever $J\subseteq I$ is finite and $b_i\in\mathfrak{B}_i$ for each i. If every $\mathfrak{B}_i$ is closed, so that $(\mathfrak{B}_i,\bar\mu\restriction\mathfrak{B}_i)$ is a probability algebra, the identity maps $\iota_i:\mathfrak{B}_i\to\mathfrak{A}$ satisfy the conditions of the universal mapping theorem 325Ja, so we have a probability algebra free product $(\mathfrak{C},\bar\mu\restriction\mathfrak{C},\langle\iota_i\rangle_{i\in I})$ of $\langle(\mathfrak{B}_i,\bar\mu\restriction\mathfrak{B}_i)\rangle_{i\in I}$, where $\mathfrak{C}=\bigvee_{i\in I}\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$.

Conversely, if $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ is any family of probability algebras with probability algebra free product $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$, then $\langle\varepsilon_i[\mathfrak{A}_i]\rangle_{i\in I}$ is an independent family of closed subalgebras of $\mathfrak{C}$.

325M Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ their probability algebra free product. For $J\subseteq I$ let $\mathfrak{C}_J=\bigvee_{i\in J}\varepsilon_i[\mathfrak{A}_i]$ be the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i\in J}\varepsilon_i[\mathfrak{A}_i]$.

(a) For any $J\subseteq I$, $(\mathfrak{C}_J,\bar\lambda\restriction\mathfrak{C}_J,\langle\varepsilon_i\rangle_{i\in J})$ is a probability algebra free product of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in J}$.

(b)(i) For any $c\in\mathfrak{C}$, there is a unique smallest $J_c\subseteq I$ such that $c\in\mathfrak{C}_{J_c}$, and this J_c is countable.

(ii) If c, $d\in\mathfrak{C}$ and $c\subseteq d$, then there is an $e\in\mathfrak{C}_{J_c\cap J_d}$ such that $c\subseteq e\subseteq d$.

(c) For any non-empty family $\mathcal{J}\subseteq\mathcal{P}I$, $\bigcap_{J\in\mathcal{J}}\mathfrak{C}_J=\mathfrak{C}_{\bigcap\mathcal{J}}$.

***325N Notation** In this context, I will say that an element c of $\mathfrak{C}$ is **determined by coordinates in** J if $c\in\mathfrak{C}_J$.

Version of 21.5.11

326 Additive functionals on Boolean algebras

I devote two sections to the general theory of additive functionals on measure algebras. As many readers will rightly be in a hurry to get on to the next two chapters, I remark that the only significant result needed for §§331-332 is the Hahn decomposition of a countably additive functional (326M), and that this is no more than a translation into the language of measure algebras of a theorem already given in Chapter 23. The concept of 'standard extension' of a countably additive functional from a subalgebra (327F-327G) will be used for a theorem in §333, and as preparation for Chapter 36.

I begin with notes on the space of additive functionals on an arbitrary Boolean algebra (326A-326D), corresponding to 231A-231B, but adding a more general form of the Jordan decomposition of a bounded additive functional into positive and negative parts (326D). The next four paragraphs are starred, because they will not be needed in this volume; 326E is essential if you want to look at additive functionals on free products, 326F is a basic classification criterion, and 326H is an important extension of a fundamental fact about atomless measures noted in 215D, but all can be passed over on first reading. The next subsection (326I-326M) deals with countably additive functionals, corresponding to 231C-231F. In 326N-326T I develop a new idea, that of 'completely additive' functional, which does not match anything in the previous treatment.

326A Additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **finitely additive**, or just **additive**, if $\nu(a \cup b) = \nu a + \nu b$ whenever $a, b \in \mathfrak{A}$ and $a \cap b = 0$.

326B Elementary facts Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional.

(a) $\nu 0 = 0$.

(b) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is additive.

(c) $\alpha\nu$ is an additive functional for any $\alpha \in \mathbb{R}$. If ν' is another finitely additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is additive.

(d) If $\langle \nu_i \rangle_{i \in I}$ is any family of finitely additive functionals such that $\nu' a = \sum_{i \in I} \nu_i a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$, then ν' is additive.

(e) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism, then $\nu\pi : \mathfrak{B} \to \mathbb{R}$ is additive. In particular, if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then $\nu \restriction \mathfrak{B} : \mathfrak{B} \to \mathbb{R}$ is additive.

(f) ν is non-negative iff it is order-preserving – that is,

$$\nu a \geq 0 \text{ for every } a \in \mathfrak{A} \iff \nu b \leq \nu c \text{ whenever } b \subseteq c.$$

326C The space of additive functionals Let $\mathfrak{A}$ be any Boolean algebra. From 326Bc we see that the set M of all finitely additive real-valued functionals on $\mathfrak{A}$ is a linear space. We give it the ordering induced by that of $\mathbb{R}^{\mathfrak{A}}$. This renders it a partially ordered linear space.

326D The Jordan decomposition (I): Proposition Let $\mathfrak{A}$ be a Boolean algebra, and ν a finitely additive real-valued functional on $\mathfrak{A}$. Then the following are equiveridical:
(i) ν is bounded;
(ii) $\sup_{n \in \mathbb{N}} |\nu a_n| < \infty$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;
(iii) $\lim_{n \to \infty} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;
(iv) $\sum_{n=0}^{\infty} |\nu a_n| < \infty$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;
(v) ν is expressible as the difference of two non-negative additive functionals.

***326E Additive functionals on free products: Theorem** Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras, with free product $\mathfrak{A}$; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical maps, and

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$$C = \{\inf_{j\in J} \varepsilon_j(a_j) : J \subseteq I \text{ is finite, } a_j \in \mathfrak{A}_j \text{ for every } j \in J\}.$$

Suppose that $\theta : C \to \mathbb{R}$ is such that

$$\theta c = \theta(c \cap \varepsilon_i(a)) + \theta(c \cap \varepsilon_i(1 \setminus a))$$

whenever $c \in C$, $i \in I$ and $a \in \mathfrak{A}_i$. Then there is a unique finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ extending θ.

***326F Definition** Let $\mathfrak{A}$ be a Boolean algebra, and ν a finitely additive functional on $\mathfrak{A}$. ν is **properly atomless** if for every $\epsilon > 0$ there is a finite partition $\langle a_i\rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le \epsilon$ whenever $i \in I$ and $a \subseteq a_i$.

***326G Lemma** Let $\mathfrak{A}$ be a Boolean algebra.

(a)(i) If ν, $\nu' : \mathfrak{A} \to \mathbb{R}$ are properly atomless finitely additive functionals and $\alpha \in \mathbb{R}$, then $\alpha\nu$ and $\nu + \nu'$ are properly atomless additive functionals.

(ii) If $\nu : \mathfrak{A} \to \mathbb{R}$ is a properly atomless finitely additive functional, then ν is bounded and ν can be expressed as the difference of two non-negative properly atomless additive functionals.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and that $\langle \nu_i\rangle_{i\in I}$ is a family of non-negative additive functionals on $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$ there are an $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ and an $a' \subseteq a$ such that $\nu_i a' = \alpha\nu_i a$ for every $i \in I$. Then for any $a \in \mathfrak{A}$ there is a non-decreasing family $\langle a_t\rangle_{t\in[0,1]}$ in $\mathfrak{A}$ such that $a_0 = 0$, $a_1 = a$ and $\nu_i a_t = t\nu_i a$ for every $t \in [0, 1]$ and $i \in I$.

(c) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and that $\nu_0, \dots , \nu_n : \mathfrak{A} \to [0, \infty[$ are properly atomless additive functionals such that $\nu_i a \le \nu_0 a$ for every $i \le n$ and $a \in \mathfrak{A}$. Then for any $a \in \mathfrak{A}$ there is a non-decreasing family $\langle a_t\rangle_{t\in[0,1]}$ in $\mathfrak{A}$ such that $a_0 = 0$, $a_1 = a$ and $\nu_i a_t = t\nu_i a$ for every $t \in [0, 1]$ and $i \le n$.

***326H Liapounoff's convexity theorem** Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $r \ge 1$ an integer. Suppose that $\nu : \mathfrak{A} \to \mathbb{R}^r$ is additive in the sense that $\nu(a \cup b) = \nu a + \nu b$ whenever $a \cap b = 0$, and properly atomless in the sense that for every $\epsilon > 0$ there is a finite partition $\langle a_j\rangle_{j\in J}$ of unity in $\mathfrak{A}$ such that $\|\nu a\| \le \epsilon$ whenever $j \in J$ and $a \subseteq a_j$. Then $\{\nu a : a \in \mathfrak{A}\}$ is a convex set in $\mathbb{R}^r$.

326I Countably additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **countably additive** or **σ-additive** if $\sum_{n=0}^{\infty} \nu a_n$ is defined and equal to $\nu(\sup_{n\in\mathbb{N}} a_n)$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and $\sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{A}$.

326J Elementary facts Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional.

(a) ν is finitely additive.

(b) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with a supremum $a \in \mathfrak{A}$, then

$$\nu a = \lim_{n\to\infty} \nu a_n.$$

(c) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with an infimum $a \in \mathfrak{A}$, then

$$\nu a = \lim_{n\to\infty} \nu a_n.$$

(d) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is countably additive.

(e) $\alpha\nu$ is a countably additive functional for any $\alpha \in \mathbb{R}$. If ν' is another countably additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is countably additive.

(f) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then $\nu\pi$ is a countably additive functional on $\mathfrak{B}$.

(g) If $\mathfrak{A}$ is Dedekind σ-complete and $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B} : \mathfrak{B} \to \mathbb{R}$ is countably additive.

326K Corollary Let $\mathfrak{A}$ be a Boolean algebra and ν a finitely additive real-valued functional on $\mathfrak{A}$.

(a) ν is countably additive iff $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0 in $\mathfrak{A}$.

(b) If ν' is an additive functional on $\mathfrak{A}$ and $|\nu' a| \le \nu a$ for every $a \in \mathfrak{A}$, and ν is countably additive, then ν' is countably additive.

(c) If ν is non-negative, then ν is countably additive iff it is sequentially order-continuous.

326L The Jordan decomposition (II): Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a bounded countably additive real-valued functional on $\mathfrak{A}$. Then ν is expressible as the difference of two non-negative countably additive functionals.

326M The Hahn decomposition: Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional. Then ν is bounded and there is a $c \in \mathfrak{A}$ such that $\nu a \ge 0$ whenever $a \subseteq c$, while $\nu a \le 0$ whenever $a \cap c = 0$.

326N Completely additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **completely additive** or **τ-additive** if it is finitely additive and $\inf_{a\in A} |\nu a| = 0$ whenever A is a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0.

326O Basic facts Let $\mathfrak{A}$ be a Boolean algebra and ν a completely additive real-valued functional on $\mathfrak{A}$.

(a) ν is countably additive.

(b) Let A be a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0. Then for every $\epsilon > 0$ there is an $a \in A$ such that $|\nu b| \le \epsilon$ whenever $b \subseteq a$.

(c) If ν is non-negative, it is order-continuous.

(d) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is completely additive.

(e) $\alpha\nu$ is a completely additive functional for any $\alpha \in \mathbb{R}$. If ν' is another completely additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is completely additive.

(f) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is an order-continuous Boolean homomorphism, then $\nu\pi$ is a completely additive functional on $\mathfrak{B}$. In particular, if $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B}$ is completely additive.

(g) If ν' is another additive functional on $\mathfrak{A}$ and $|\nu' a| \le \nu a$ for every $a \in \mathfrak{A}$, then ν' is completely additive.

326P Proposition If $\mathfrak{A}$ is a ccc Boolean algebra, a functional $\nu : \mathfrak{A} \to \mathbb{R}$ is countably additive iff it is completely additive.

326Q The Jordan decomposition (III): Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a completely additive real-valued functional on $\mathfrak{A}$. Then ν is bounded and expressible as the difference of two non-negative completely additive functionals.

326R Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\nu : \mathfrak{A} \to \mathbb{R}$ a function. Then the following are equiveridical:

(i) ν is completely additive;

(ii) $\nu 1 = \sum_{i\in I} \nu a_i$ whenever $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$;

(iii) $\nu a = \sum_{i\in I} \nu a_i$ whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a.

326S Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a completely additive functional. Then there is a unique element of $\mathfrak{A}$, which I will denote $[\![\nu > 0]\!]$, such that $\nu a > 0$ whenever $0 \ne a \subseteq [\![\nu > 0]\!]$, while $\nu a \le 0$ whenever $a \cap [\![\nu > 0]\!] = 0$.

326T Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and μ, ν two completely additive functionals on $\mathfrak{A}$. Then there is a unique element of $\mathfrak{A}$, which I will denote $[\![\mu > \nu]\!]$, such that

$$\mu a > \nu a \text{ whenever } 0 \neq a \subseteq [\![\mu > \nu]\!],$$

$$\mu a \leq \nu a \text{ whenever } a \cap [\![\mu > \nu]\!] = 0.$$

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327 Additive functionals on measure algebras

When we turn to measure algebras, we have a simplification, relative to the general context of §326, because the algebras are always Dedekind σ-complete; but there are also elaborations, because we can ask how the additive functionals we examine are related to the measure. In 327A-327C I work through the relationships between the concepts of 'absolute continuity', '(true) continuity' and 'countable additivity', following §232, and adding 'complete additivity' from §326. These ideas provide a new interpretation of the Radon-Nikodým theorem (327D). I then use this theorem to develop some machinery (the 'standard extension' of an additive functional from a closed subalgebra to the whole algebra, 327F-327G) which will be used in §333.

327A Definition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Then ν is **absolutely continuous** with respect to $\bar{\mu}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \leq \epsilon$ whenever $\bar{\mu} a \leq \delta$.

327B Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Give $\mathfrak{A}$ its measure-algebra topology and uniformity.

(a) If ν is continuous at 0, it is completely additive.

(b) If ν is countably additive, it is absolutely continuous with respect to $\bar{\mu}$.

(c) The following are equiveridical:

(i) ν is continuous at 0;

(ii) ν is countably additive and whenever $a \in \mathfrak{A}$ and $\nu a \neq 0$ there is a $b \in \mathfrak{A}$ such that $\bar{\mu} b < \infty$ and $\nu(a \cap b) \neq 0$;

(iii) ν is continuous everywhere on $\mathfrak{A}$;

(iv) ν is uniformly continuous.

(d) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite, then ν is continuous iff it is completely additive.

(e) If $(\mathfrak{A}, \bar{\mu})$ is σ-finite, then ν is continuous iff it is countably additive iff it is completely additive.

(f) If $(\mathfrak{A}, \bar{\mu})$ is totally finite, then ν is continuous iff it is absolutely continuous with respect to $\bar{\mu}$ iff it is countably additive iff it is completely additive.

327C Proposition Let (X, Σ, μ) be a measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra.

(a) There is a one-to-one correspondence between finitely additive functionals $\bar{\nu}$ on $\mathfrak{A}$ and finitely additive functionals ν on Σ such that $\nu E = 0$ whenever $\mu E = 0$, given by the formula $\bar{\nu} E^\bullet = \nu E$ for every $E \in \Sigma$.

(b) In (a), $\bar{\nu}$ is absolutely continuous with respect to $\bar{\mu}$ iff ν is absolutely continuous with respect to μ.

(c) In (a), $\bar{\nu}$ is countably additive iff ν is countably additive; so that we have a one-to-one correspondence between the countably additive functionals on $\mathfrak{A}$ and the absolutely continuous countably additive functionals on Σ.

(d) In (a), $\bar{\nu}$ is continuous for the measure-algebra topology on $\mathfrak{A}$ iff ν is truly continuous in the sense of 232Ab.

(e) Suppose that μ is semi-finite. Then, in (a), $\bar{\nu}$ is completely additive iff ν is truly continuous.

327D The Radon-Nikodým theorem Let (X, Σ, μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Let L^1 be the space of equivalence classes of real-valued integrable functions on X, and write

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M_τ for the set of completely additive real-valued functionals on $\mathfrak{A}$. Then there is an ordered linear space bijection between M_τ and L^1 defined by saying that $\bar{\nu} \in M_\tau$ corresponds to $u \in L^1$ if

$$\bar{\nu}a = \int_E f \text{ whenever } a = E^\bullet \text{ in } \mathfrak{A} \text{ and } f^\bullet = u \text{ in } L^1.$$

327E Proposition If $(\mathfrak{A}, \bar{\mu})$ is a measure algebra, then the functional $a \mapsto \mu_c a = \bar{\mu}(a \cap c)$ is completely additive whenever $c \in \mathfrak{A}$ and $\bar{\mu}c < \infty$.

327F Standard extensions: Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C} \subseteq \mathfrak{A}$ a closed subalgebra. Write $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ for the spaces of countably additive real-valued functionals on $\mathfrak{A}$, $\mathfrak{C}$ respectively.

(a) There is an operator $R : M_\sigma(\mathfrak{C}) \to M_\sigma(\mathfrak{A})$ defined by saying that, for every $\nu \in M_\sigma(\mathfrak{C})$, $R\nu$ is the unique member of $M_\sigma(\mathfrak{A})$ such that $[\![R\nu > \alpha\bar{\mu}]\!] = [\![\nu > \alpha\bar{\mu}\restriction\mathfrak{C}]\!]$ for every $\alpha \in \mathbb{R}$.

(b)(i) $R\nu$ extends ν for every $\nu \in M_\sigma(\mathfrak{C})$.

(ii) R is linear and order-preserving.

(iii) $R(\bar{\mu}\restriction\mathfrak{C}) = \bar{\mu}$.

(iv) If $\langle \nu_n \rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functionals in $M_\sigma(\mathfrak{C})$ such that $\sum_{n=0}^\infty \nu_n c = \bar{\mu}c$ for every $c \in \mathfrak{C}$, then $\sum_{n=0}^\infty (R\nu_n)(a) = \bar{\mu}a$ for every $a \in \mathfrak{A}$.

327G Definition In the context of 327F, I will call $R\nu$ the **standard extension** of ν to $\mathfrak{A}$.

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*328 Reduced products and other constructions

I devote a section to some related constructions. At the end of §315 I mentioned projective and inductive limits of systems of Boolean algebras with linking homomorphisms. In the context of the present chapter, we naturally ask whether similar constructions can be found for probability algebras. For projective limits there is no difficulty (328I). For inductive limits the situation is more complex (328H). Some ideas in Volume 5 will depend on what I call 'reduced products' (328A-328F), which also provide a route to 328H. The same methods give a route to a useful result relating measure-preserving Boolean homomorphisms on a probability algebra to measure-preserving automorphisms on a larger probability algebra (328J).

328A Construction Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $\mathcal{F}$ an ultrafilter on I.

(a) Set

$$\mathcal{J}=\{\langle a_i\rangle_{i\in I}:\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i,\ \lim_{i\to\mathcal{F}}\bar\mu_i a_i=0\}.$$

Then $\mathcal{J}$ is an ideal in the simple product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$.

(b) Let $\mathfrak{A}$ be the quotient Boolean algebra $\prod_{i\in I}\mathfrak{A}_i/\mathcal{J}$. Then we have a functional $\bar\mu:\mathfrak{A}\to[0,1]$ defined by saying that

$$\bar\mu(\langle a_i\rangle^\bullet_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu_i a_i$$

whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$.

328B Proposition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras and $\mathcal{F}$ an ultrafilter on I, and construct $\mathfrak{A}$ and $\bar\mu$ as in 328A. Then $(\mathfrak{A},\bar\mu)$ is a probability algebra.

328C Definition In the context of 328A/328B, I will call $(\mathfrak{A},\bar\mu)$ the **probability algebra reduced product** of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ modulo $\mathcal{F}$; I will sometimes write it as $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$.

If all the $(\mathfrak{A}_i,\bar\mu_i)$ are the same, with common value $(\mathfrak{B},\bar\nu)$, I will write $(\mathfrak{B},\bar\nu)^I|\mathcal{F}$ for $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$, and call it the **probability algebra reduced power**.

328D Proposition Let I be a set, $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$, $\langle(\mathfrak{B}_i,\bar\nu_i)\rangle_{i\in I}$ and $\langle(\mathfrak{C}_i,\bar\lambda_i)\rangle_{i\in I}$ three families of probability algebras, and $\mathcal{F}$ an ultrafilter on I; let $(\mathfrak{A},\bar\mu)=\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$, $(\mathfrak{B},\bar\nu)=\prod_{i\in I}(\mathfrak{B}_i,\bar\nu_i)|\mathcal{F}$ and $(\mathfrak{C},\bar\lambda)=\prod_{i\in I}(\mathfrak{C}_i,\bar\lambda_i)|\mathcal{F}$ be the corresponding reduced products.

(a) If $\pi_i:\mathfrak{A}_i\to\mathfrak{B}_i$ is a measure-preserving Boolean homomorphism for each $i\in I$, we have a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ given by saying that

$$\pi(\langle a_i\rangle^\bullet_{i\in I})=\langle\pi_i a_i\rangle^\bullet_{i\in I}$$

whenever $a_i\in\mathfrak{A}_i$ for every $i\in I$.

(b) If, in addition, $\phi_i:\mathfrak{B}_i\to\mathfrak{C}_i$ is a measure-preserving Boolean homomorphism for each $i\in I$, and $\phi:\mathfrak{B}\to\mathfrak{C}$ is constructed as in (a), then $\phi\pi:\mathfrak{A}\to\mathfrak{C}$ corresponds to the family $\langle\phi_i\pi_i\rangle_{i\in I}$.

328E Proposition Let I be a non-empty set, $\le$ a reflexive transitive relation on I, and $\mathcal{F}$ an ultrafilter on I such that $\{j:j\in I,\ j\ge i\}$ belongs to $\mathcal{F}$ for every $i\in I$. Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, and suppose that we are given a family $\langle\pi_{ji}\rangle_{i\le j}$ such that

π_{ji} is a measure-preserving Boolean homomorphism from $\mathfrak{A}_i$ to $\mathfrak{A}_j$ whenever $i\le j$ in I,

$\pi_{ki}=\pi_{kj}\pi_{ji}$ whenever $i\le j\le k$ in I.

Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced product $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$.

(a) For each $i\in I$ we have a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}$ defined by saying that $\pi_i a=\langle a_j\rangle^\bullet_{j\in I}$ whenever $a_j=\pi_{ji}a$ for every $j\ge i$, and $\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$ in I.

(b) $\langle a_i\rangle^\bullet_{i\in I}\subseteq\sup_{j\in A}\pi_j a_j$ whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $A\in\mathcal{F}$.

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328F Corollary Suppose that $\langle(\mathfrak{A}_n,\bar\mu_n)\rangle_{n\in\mathbb{N}}$ is a sequence of probability algebras, $\phi_n:\mathfrak{A}_n\to\mathfrak{A}_{n+1}$ is a measure-preserving Boolean homomorphism for each n and $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$. Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced product $\prod_{n\in\mathbb{N}}(\mathfrak{A}_n,\bar\mu_n)|\mathcal{F}$. Then we have canonical measure-preserving Boolean homomorphisms $\pi_n:\mathfrak{A}_n\to\mathfrak{A}$ such that $\langle a_n\rangle^\bullet_{n\in\mathbb{N}}\subseteq\sup_{n\in A}\pi_n a_n$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}\mathfrak{A}_n$ and $A\in\mathcal{F}$, and $\pi_{n+1}\phi_n=\pi_n$ for every $n\in\mathbb{N}$.

328G Corollary Let $(\mathfrak{B},\bar\nu)$ be a probability algebra, I a non-empty set, and $\mathcal{F}$ an ultrafilter on I. Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced power $(\mathfrak{B},\bar\nu)^I|\mathcal{F}$.

(a) We have a measure-preserving Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by saying that $\pi b=\langle b\rangle^\bullet_{i\in I}$ for $b\in\mathfrak{B}$.

(b)

$$\langle b_i\rangle^\bullet_{i\in I}\subseteq\sup_{j\in A}\pi b_j=\pi(\sup_{j\in A}b_j)$$

whenever $A\in\mathcal{F}$ and $\langle b_i\rangle_{i\in I}\in\mathfrak{B}^I$.

328H Proposition Let $(I,\le)$ be an upwards-directed partially ordered set, and $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ a family of probability algebras; suppose that $\pi_{ji}:\mathfrak{A}_i\to\mathfrak{A}_j$ is a measure-preserving Boolean homomorphism whenever $i\le j$, and that $\pi_{ki}=\pi_{kj}\pi_{ji}$ whenever $i\le j\le k$. Then there are a probability algebra $(\mathfrak{C},\bar\lambda)$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i:\mathfrak{A}_i\to\mathfrak{C}$ is a measure-preserving Boolean homomorphism for each $i\in I$,

$\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$,

$\{0,1\}\cup\bigcup_{i\in I}\pi_i[\mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$,

and whenever $(\mathfrak{B},\bar\nu)$, $\langle\phi_i\rangle_{i\in I}$ are such that

$(\mathfrak{B},\bar\nu)$ is a probability algebra,

$\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ is a measure-preserving Boolean homomorphism for each $i\in I$,

$\phi_i=\phi_j\pi_{ji}$ whenever $i\le j$,

then there is a unique measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\pi_i=\phi_i$ for every $i\in I$.

328I Proposition Let $(I,\le)$ be a non-empty upwards-directed set, and $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ a family of probability algebras; suppose that $\pi_{ij}:\mathfrak{A}_j\to\mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for $i\le j$ in I, and that $\pi_{ij}\pi_{jk}=\pi_{ik}$ whenever $i\le j\le k$. Then there are a probability algebra $(\mathfrak{C},\bar\lambda)$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i\in I$,

$\pi_i=\pi_{ij}\pi_j$ whenever $i\le j$,

and whenever $(\mathfrak{B},\bar\nu)$, $\langle\phi_i\rangle_{i\in I}$ are such that

$(\mathfrak{B},\bar\nu)$ is a probability algebra,

$\phi_i:\mathfrak{B}\to\mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i\in I$,

$\phi_i=\pi_{ij}\phi_j$ whenever $i\le j$,

then there is a unique measure-preserving Boolean homomorphism $\phi:\mathfrak{B}\to\mathfrak{C}$ such that $\pi_i\phi=\phi_i$ for every $i\in I$.

328J Theorem Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and Φ a family of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself such that $\phi\psi=\psi\phi$ for all ϕ, $\psi\in\Phi$. Then there are a probability algebra $(\mathfrak{C},\bar\lambda)$, a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{C}$ and a family $\langle\tilde\phi\rangle_{\phi\in\Phi}$ such that

(i) $\tilde\phi:\mathfrak{C}\to\mathfrak{C}$ is a measure-preserving Boolean automorphism and $\tilde\phi\pi=\pi\phi$ for every $\phi\in\Phi$;

(ii) $(\phi\psi)^\sim=\tilde\phi\tilde\psi$ for all ϕ, $\psi\in\Phi$.

Version of 24.4.06

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

322K Paragraphs 322K (simple products of measure algebras), 322N (the Stone space of a measure algebra) and 322Q (further properties of Stone spaces), referred to in the 2003 and 2006 editions of Volume 4, are now 322L, 322O and 322R.

326E Countably additive functionals Definition 326E, referred to in the 2003 and 2006 editions of Volume 4 and the 2008 edition of Volume 5, is now 326I.

326G Corollary 326G, referred to in the 2008 edition of Volume 5, is now 326K.

326I Hahn decomposition Theorem 326I, referred to in the 2003 and 2006 editions of Volume 4, is now 326M.

326K Completely additive functionals The notes in 326K, referred to in the 2003 and 2006 editions of Volume 4, have been moved to 326O.

326Q Finitely additive functionals on free products Theorem 326Q, referred to in the 2003 and 2006 editions of Volume 4 and the 2008 edition of Volume 5, is now 326E.

328D Reduced products of probability algebras Paragraph 328D, referred to in the 2008 edition of Volume 5, is now 328E.

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