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Chapter 32

Measure algebras

I now come to the real work of this volume, the study of the Boolean algebras of equivalence classes of measurable sets. In this chapter I work through the 'elementary' theory, defining this to consist of the parts which do not depend on Maharam's theorem or the lifting theorem or non-trivial set theory.

§321 gives the definition of 'measure algebra', and relates this idea to its origin as the quotient of a σ-algebra of measurable sets by a σ-ideal of negligible sets, both in its elementary properties (following those of measure spaces treated in §112) and in an appropriate version of the Stone representation. §322 deals with the classification of measure algebras according to the scheme already developed in §211 for measure spaces. §323 discusses the standard topology and uniformity of a measure algebra. §324 contains results concerning Boolean homomorphisms between measure algebras, with the relationships between topological continuity, order-continuity and preservation of measure. §325 is devoted to the measure algebras of product measures, and their abstract characterization as completed free products. §§326-327 address the properties of additive functionals on Boolean algebras, generalizing the ideas of Chapter 23. Finally, §328 looks at 'reduced products' of probability algebras and some related constructions, including inductive limits.

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321 Measure algebras

I begin by defining 'measure algebra' and relating this concept to the work of Chapter 31 and to the elementary properties of measure spaces.

321A Definition A **measure algebra** is a pair $(\mathfrak{A},\bar\mu)$, where $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\bar\mu:\mathfrak{A}\to[0,\infty]$ is a function such that

$\bar\mu 0=0$;

whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, $\bar\mu(\sup_{n\in\mathbb{N}}a_n)=\sum_{n=0}^\infty\bar\mu a_n$;

$\bar\mu a>0$ whenever $a\in\mathfrak{A}$ and $a\ne 0$.

321B Elementary properties of measure algebras Corresponding to the most elementary properties of measure spaces (112C in Volume 1), we have the following basic properties of measure algebras. Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) If a, $b\in\mathfrak{A}$ and $a\cap b=0$ then $\bar\mu(a\cup b)=\bar\mu a+\bar\mu b$. **P** Set $a_0=a$, $a_1=b$, $a_n=0$ for $n\ge 2$; then

$$\bar\mu(a\cup b)=\bar\mu(\sup_{n\in\mathbb{N}}a_n)=\sum_{n=0}^\infty\bar\mu a_n=\bar\mu a+\bar\mu b. \textbf{ Q}$$

(b) If a, $b\in\mathfrak{A}$ and $a\subseteq b$ then $\bar\mu a\le\bar\mu b$. **P**

$$\bar\mu a\le\bar\mu a+\bar\mu(b\setminus a)=\bar\mu b. \textbf{ Q}$$

(c) For any a, $b\in\mathfrak{A}$, $\bar\mu(a\cup b)\le\bar\mu a+\bar\mu b$. **P**

$$\bar\mu(a\cup b)=\bar\mu a+\bar\mu(b\setminus a)\le\bar\mu a+\bar\mu b. \textbf{ Q}$$

(d) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{A}$, then $\bar\mu(\sup_{n\in\mathbb{N}}a_n)\le\sum_{n=0}^\infty\bar\mu a_n$. **P** For each n, set $b_n=a_n\setminus\sup_{i<n}a_i$. Inducing on n, we see that $\sup_{i\le n}a_i=\sup_{i\le n}b_i$ for each n, so $\sup_{n\in\mathbb{N}}a_n=\sup_{n\in\mathbb{N}}b_n$ and

$$\bar\mu(\sup_{n\in\mathbb{N}} a_n) = \bar\mu(\sup_{n\in\mathbb{N}} b_n) = \sum_{n=0}^{\infty} \bar\mu b_n \le \sum_{n=0}^{\infty} \bar\mu a_n$$

because $\langle b_n\rangle_{n\in\mathbb{N}}$ is disjoint. **Q**

(e) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$, then $\bar\mu(\sup_{n\in\mathbb{N}} a_n) = \lim_{n\to\infty} \bar\mu a_n$. **P** Set $b_0 = a_0$, $b_n = a_n \setminus a_{n-1}$ for $n \ge 1$. Then

$$\bar\mu(\sup_{n\in\mathbb{N}} a_n) = \bar\mu(\sup_{n\in\mathbb{N}} b_n) = \sum_{n=0}^{\infty} \bar\mu b_n$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n} \bar\mu b_i = \lim_{n\to\infty} \bar\mu(\sup_{i\le n} b_i) = \lim_{n\to\infty} \bar\mu a_n. \ \textbf{Q}$$

(f) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} \bar\mu a_n < \infty$, then $\bar\mu(\inf_{n\in\mathbb{N}} a_n) = \lim_{n\to\infty} \bar\mu a_n$. **P** (Cf. 112Cf.) Set $a = \inf_{n\in\mathbb{N}} a_n$. Take $k \in \mathbb{N}$ such that $\bar\mu a_k < \infty$. Set $b_n = a_k \setminus a_n$ for $n \in \mathbb{N}$; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and $\sup_{n\in\mathbb{N}} b_n = a_k \setminus a$ (313Ab). Because $\bar\mu a_k$ is finite,

$$\bar\mu a = \bar\mu a_k - \bar\mu(a_k \setminus a) = \bar\mu a_k - \lim_{n\to\infty} \bar\mu b_n$$

(by (e) above)

$$= \lim_{n\to\infty} \bar\mu(a_k \setminus b_n) = \lim_{n\to\infty} \bar\mu a_n. \ \textbf{Q}$$

321C Proposition Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, and $A \subseteq \mathfrak{A}$ a non-empty upwards-directed set. If $\sup_{a\in A} \bar\mu a < \infty$, then $\sup A$ is defined in $\mathfrak{A}$ and $\bar\mu(\sup A) = \sup_{a\in A} \bar\mu a$.

proof (Compare 215A.) Set $\gamma = \sup_{a\in A} \bar\mu a$, and for each $n \in \mathbb{N}$ choose $a_n \in A$ such that $\bar\mu a_n \ge \gamma - 2^{-n}$. Next, choose $\langle b_n\rangle_{n\in\mathbb{N}}$ in A such that $b_{n+1} \supseteq b_n \cup a_n$ for each n, and set $b = \sup_{n\in\mathbb{N}} b_n$. Then

$$\bar\mu b = \lim_{n\to\infty} \bar\mu b_n \le \gamma, \quad \bar\mu a_n \le \bar\mu b \text{ for every } n \in \mathbb{N},$$

so $\bar\mu b = \gamma$.

If $a \in A$, then for every $n \in \mathbb{N}$ there is an $a'_n \in A$ such that $a \cup a_n \subseteq a'_n$, so that

$$\bar\mu(a \setminus b) \le \bar\mu(a \setminus a_n) \le \bar\mu(a'_n \setminus a_n) = \bar\mu a'_n - \bar\mu a_n \le \gamma - \bar\mu a_n \le 2^{-n}.$$

This means that $\bar\mu(a \setminus b) = 0$, so $a \setminus b = 0$ and $a \subseteq b$. Accordingly b is an upper bound of A, and is therefore $\sup A$; since we already know that $\bar\mu b = \gamma$, the proof is complete.

321D Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $A \subseteq \mathfrak{A}$ a non-empty upwards-directed set. If $\sup A$ is defined in $\mathfrak{A}$, then $\bar\mu(\sup A) = \sup_{a\in A} \bar\mu a$.

proof If $\sup_{a\in A} \bar\mu a = \infty$, this is trivial; otherwise it follows from 321C.

321E Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $A \subseteq \mathfrak{A}$ a disjoint set. If $\sup A$ is defined in $\mathfrak{A}$, then $\bar\mu(\sup A) = \sum_{a\in A} \bar\mu a$.

proof If $A = \emptyset$ then $\sup A = 0$ and the result is trivial. Otherwise, set $B = \{a_0 \cup \ldots \cup a_n : a_0, \ldots, a_n \in A$ are distinct$\}$. Then B is upwards-directed, and $\sup_{b\in B} \bar\mu b = \sum_{a\in A} \bar\mu a$ because A is disjoint. Also B has the same upper bounds as A, so $\sup B = \sup A$ and

$$\bar\mu(\sup A) = \bar\mu(\sup B) = \sup_{b\in B} \bar\mu b = \sum_{a\in A} \bar\mu a.$$

321F Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $A \subseteq \mathfrak{A}$ a non-empty downwards-directed set. If $\inf_{a\in A} \bar\mu a < \infty$, then $\inf A$ is defined in $\mathfrak{A}$ and $\bar\mu(\inf A) = \inf_{a\in A} \bar\mu a$.

proof Take $a_0 \in A$ with $\bar{\mu}a_0 < \infty$, and set $B = \{a_0 \setminus a : a \in A\}$. Then B is upwards-directed, and $\sup_{b\in B} \bar{\mu}b \le \bar{\mu}a_0 < \infty$, so $\sup B$ is defined. Accordingly $\inf A = a_0 \setminus \sup B$ is defined (313Aa), and

$$\begin{aligned}\bar{\mu}(\inf A) &= \bar{\mu}a_0 - \bar{\mu}(\sup B) = \bar{\mu}a_0 - \sup_{b\in B} \bar{\mu}b \\ &= \inf_{b\in B} \bar{\mu}(a_0 \setminus b) = \inf_{a\in A} \bar{\mu}(a_0 \cap a) = \inf_{a\in A} \bar{\mu}a.\end{aligned}$$

321G Subalgebras If $(\mathfrak{A}, \bar{\mu})$ is a measure algebra, and $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, then $(\mathfrak{B}, \bar{\mu}\restriction\mathfrak{B})$ is a measure algebra. **P** As remarked in 314Eb, $\mathfrak{B}$ is Dedekind σ-complete. If $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$, then the supremum $b = \sup_{n\in\mathbb{N}} b_n$ is the same whether taken in $\mathfrak{B}$ or $\mathfrak{A}$, so that we have $\bar{\mu}b = \sum_{n=0}^{\infty} \bar{\mu}b_n$. **Q**

321H The measure algebra of a measure space I introduce the abstract notion of 'measure algebra' because I believe that this is the right language in which to formulate the questions addressed in this volume. However it is very directly linked with the idea of 'measure space', as the next two results show.

Theorem Let (X, Σ, μ) be a measure space, and $\mathcal{N}$ the null ideal of μ. Let $\mathfrak{A}$ be the Boolean algebra quotient $\Sigma/\Sigma \cap \mathcal{N}$. Then we have a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ defined by setting

$$\bar{\mu}E^\bullet = \mu E \text{ for every } E \in \Sigma,$$

and $(\mathfrak{A}, \bar{\mu})$ is a measure algebra. The canonical map $E \mapsto E^\bullet : \Sigma \to \mathfrak{A}$ is sequentially order-continuous.

proof (a) By 314C, $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra. By 313Qb, $E \mapsto E^\bullet$ is sequentially order-continuous, because $\Sigma \cap \mathcal{N}$ is a σ-ideal of Σ.

(b) If E, $F \in \Sigma$ and $E^\bullet = F^\bullet$ in $\mathfrak{A}$, then $E \triangle F \in \mathcal{N}$, so

$$\mu E \le \mu F + \mu(E \setminus F) = \mu F \le \mu E + \mu(F \setminus E) = \mu E$$

and $\mu E = \mu F$. Accordingly the given formula does indeed define a function $\bar{\mu} : \mathfrak{A} \to [0, \infty]$.

(c) Now

$$\bar{\mu}0 = \bar{\mu}\emptyset^\bullet = \mu\emptyset = 0.$$

If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, choose for each $n \in \mathbb{N}$ an $E_n \in \Sigma$ such that $E_n^\bullet = a_n$. Set $F_n = E_n \setminus \bigcup_{i<n} E_i$; then

$$F_n^\bullet = E_n^\bullet \setminus \sup_{i<n} E_i^\bullet = a_n \setminus \sup_{i<n} a_i = a_n$$

for each n, so $\bar{\mu}a_n = \mu F_n$ for each n. Now set $E = \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n$; then $E^\bullet = \sup_{n\in\mathbb{N}} F_n^\bullet = \sup_{n\in\mathbb{N}} a_n$. So

$$\bar{\mu}(\sup_{n\in\mathbb{N}} a_n) = \mu E = \sum_{n=0}^{\infty} \mu F_n = \sum_{n=0}^{\infty} \bar{\mu}a_n.$$

Finally, if $a \ne 0$, then there is an $E \in \Sigma$ such that $E^\bullet = a$, and $E \notin \mathcal{N}$, so $\bar{\mu}a = \mu E > 0$. Thus $(\mathfrak{A}, \bar{\mu})$ is a measure algebra.

321I Definition For any measure space (X, Σ, μ) I will call $(\mathfrak{A}, \bar{\mu})$, as constructed above, the **measure algebra of** (X, Σ, μ).

321J The Stone representation of a measure algebra Just as with Dedekind σ-complete Boolean algebras (314N), every measure algebra is obtainable from the construction above.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be any measure algebra. Then it is isomorphic, as measure algebra, to the measure algebra of some measure space.

proof (a) We know from 314M that $\mathfrak{A}$ is isomorphic, as Boolean algebra, to a quotient algebra $\Sigma/\mathcal{M}$ where Σ is a σ-algebra of subsets of the Stone space Z of $\mathfrak{A}$, and $\mathcal{M}$ is the ideal of meager subsets of Z. Let $\pi : \Sigma/\mathcal{M} \to \mathfrak{A}$ be the canonical isomorphism, and set $\theta E = \pi E^\bullet$ for each $E \in \Sigma$; then $\theta : \Sigma \to \mathfrak{A}$ is a sequentially order-continuous surjective Boolean homomorphism with kernel $\mathcal{M}$.

(b) For $E \in \Sigma$, set

$$\nu E = \bar{\mu}(\theta E).$$

Then (Z, Σ, ν) is a measure space. **P** (i) We know already that Σ is a σ-algebra of subsets of Z. (ii)

$$\nu\emptyset = \bar{\mu}(\theta\emptyset) = \bar{\mu}0 = 0.$$

(iii) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, then (because θ is a Boolean homomorphism) $\langle \theta E_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and (because θ is sequentially order-continuous) $\theta(\bigcup_{n\in\mathbb{N}} E_n) = \sup_{n\in\mathbb{N}} \theta E_n$; so

$$\nu(\bigcup_{n\in\mathbb{N}} E_n) = \bar{\mu}(\sup_{n\in\mathbb{N}} \theta E_n) = \sum_{n=0}^{\infty} \bar{\mu}(\theta E_n) = \sum_{n=0}^{\infty} \nu E_n. \text{ } \mathbf{Q}$$

(c) For $E \in \Sigma$,

$$\nu E = 0 \iff \bar{\mu}(\theta E) = 0 \iff \theta E = 0 \iff E \in \mathcal{M}.$$

So the measure algebra of (Z, Σ, ν) is just $\Sigma/\mathcal{M}$, with

$$\bar{\nu}E^\bullet = \nu E = \bar{\mu}(\theta E) = \bar{\mu}(\pi E^\bullet)$$

for every $E \in \Sigma$. Thus the Boolean algebra isomorphism π is also an isomorphism between the measure algebras $(\Sigma/\mathcal{M}, \bar{\nu})$ and $(\mathfrak{A}, \bar{\mu})$, and $(\mathfrak{A}, \bar{\mu})$ is represented in the required form.

321K Definition I will call the measure space (Z, Σ, ν) constructed in the proof of 321J the **Stone space** of the measure algebra $(\mathfrak{A}, \bar{\mu})$.

For later reference, I repeat the description of this space as developed in 311E, 311I, 314M and 321J. Z is a compact Hausdorff space, being the Stone space of $\mathfrak{A}$. $\mathfrak{A}$ can be identified with the algebra of open-and-closed sets in Z. The null ideal of ν coincides with the ideal of meager subsets of Z; in particular, ν is complete. The measurable sets are precisely those expressible in the form $E = \widehat{a} \triangle M$ where $a \in \mathfrak{A}$, $\widehat{a} \subseteq Z$ is the corresponding open-and-closed set, and M is meager; in this case $\nu E = \bar{\mu}a$ and $a = \theta E$ is the member of $\mathfrak{A}$ corresponding to E.

For the most important classes of measure algebras, more can be said; see 322O *et seq.* below.

321X Basic exercises >(a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $a \in \mathfrak{A}$; write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a. Show that $(\mathfrak{A}_a, \bar{\mu}\restriction\mathfrak{A}_a)$ is a measure algebra.

(b) Let $(X, \Sigma, \bar{\mu})$ be a measure space, and $\mathfrak{A}$ its measure algebra. (i) Show that if T is a σ-subalgebra of Σ, then $\{E^\bullet : E \in \mathrm{T}\}$ is a σ-subalgebra of $\mathfrak{A}$. (ii) Show that if $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$ then $\{E : E \in \Sigma,\ E^\bullet \in \mathfrak{B}\}$ is a σ-subalgebra of Σ.

321Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $I \lhd \mathfrak{A}$ a σ-ideal. For $u \in \mathfrak{A}/I$ set $\bar{\nu}u = \inf\{\bar{\mu}a : a \in \mathfrak{A},\ a^\bullet = u\}$. (i) Show that the infimum is always attained. (ii) Show that $(\mathfrak{A}/I, \bar{\nu})$ is a measure algebra.

321 Notes and comments The idea behind taking the quotient $\Sigma/\mathcal{N}$, where Σ is the algebra of measurable sets and $\mathcal{N}$ is the null ideal, is just that if negligible sets can be ignored – as is the case for a very large proportion of the results of measure theory – then two measurable sets can be counted as virtually the same if they differ by a negligible set, that is, if they represent the same member of the measure algebra. The definition in 321A is designed to be an exact characterization of these quotient algebras, taking into account the measures with which they are endowed. In the course of the present chapter I will work through many of the basic ideas dealt with in Volumes 1 and 2 to show how they can be translated into theorems about measure algebras, as I have done in 321B-321F. It is worth checking these correspondences carefully, because some of the ideas mutate significantly in translation. In measure algebras, it becomes sensible to take seriously the suprema and infima of uncountable sets (see 321C-321F).

I should perhaps remark that while the Stone representation (321J-321K) is significant, it is not the most important method of representing measure algebras, which is surely Maharam's theorem, to be dealt with in the next chapter. Nevertheless, the Stone representation is a canonical one, and will appear at each point that we meet a new construction involving measure algebras, just as the ordinary Stone representation of Boolean algebras can be expected to throw light on any aspect of Boolean algebra.

Version of 24.4.06

322 Taxonomy of measure algebras

Before going farther with the general theory of measure algebras, I run through those parts of the classification of measure spaces in §211 which have expressions in terms of measure algebras. The most important concepts at this stage are those of 'semi-finite', 'localizable' and 'σ-finite' measure algebra (322Ac-322Ae); these correspond exactly to the same terms applied to measure spaces (322B). I briefly investigate the Boolean-algebra properties of semi-finite and σ-finite measure algebras (322F, 322G), with mentions of completions and c.l.d. versions (322D), subspace measures (322I-322J), indefinite-integral measures (322K), direct sums of measure spaces (322L, 322M) and subalgebras of measure algebras (322N). It turns out that localizability of a measure algebra is connected in striking ways to the properties of the canonical measure on its Stone space (322O). I end the section with a description of the 'localization' of a semi-finite measure algebra (322P-322Q) and with some further properties of Stone spaces (322R).

322A Definitions Let $(\mathfrak{A},\bar\mu)$ be a measure algebra.

(a) I will say that $(\mathfrak{A},\bar\mu)$ is a **probability algebra** if $\bar\mu 1=1$.

(b) $(\mathfrak{A},\bar\mu)$ is **totally finite** if $\bar\mu 1<\infty$.

(c) $(\mathfrak{A},\bar\mu)$ is **σ-finite** if there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\bar\mu a_n<\infty$ for every $n\in\mathbb{N}$ and $\sup_{n\in\mathbb{N}}a_n=1$. Note that in this case $\langle a_n\rangle_{n\in\mathbb{N}}$ can be taken *either* to be non-decreasing (consider $a'_n=\sup_{i<n}a_i$) *or* to be disjoint (consider $a''_n=a_n\setminus a'_n$).

(d) $(\mathfrak{A},\bar\mu)$ is **semi-finite** if whenever $a\in\mathfrak{A}$ and $\bar\mu a=\infty$ there is a non-zero $b\subseteq a$ such that $\bar\mu b<\infty$.

(e) $(\mathfrak{A},\bar\mu)$ is **localizable** if it is semi-finite and the Boolean algebra $\mathfrak{A}$ is Dedekind complete.

322B The first step is to relate these concepts to the corresponding ones for measure spaces.

Theorem Let (X,Σ,μ) be a measure space, and $(\mathfrak{A},\bar\mu)$ its measure algebra. Then

(a) (X,Σ,μ) is a probability space iff $(\mathfrak{A},\bar\mu)$ is a probability algebra;
(b) (X,Σ,μ) is totally finite iff $(\mathfrak{A},\bar\mu)$ is;
(c) (X,Σ,μ) is σ-finite iff $(\mathfrak{A},\bar\mu)$ is;
(d) (X,Σ,μ) is semi-finite iff $(\mathfrak{A},\bar\mu)$ is;
(e) (X,Σ,μ) is localizable iff $(\mathfrak{A},\bar\mu)$ is;
(f) if $E\in\Sigma$, then E is an atom for μ iff $E^\bullet$ is an atom in $\mathfrak{A}$;
(g) (X,Σ,μ) is atomless iff $\mathfrak{A}$ is;
(h) (X,Σ,μ) is purely atomic iff $\mathfrak{A}$ is.

proof (a), (b) are trivial, since $\bar\mu 1=\mu X$.

(c)(i) If μ is σ-finite, let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of finite measure covering X; then $\bar\mu E_n^\bullet<\infty$ for every n, and

$$\sup_{n\in\mathbb{N}}E_n^\bullet=(\bigcup_{n\in\mathbb{N}}E_n)^\bullet=1,$$

so $(\mathfrak{A},\bar\mu)$ is σ-finite.

(ii) If $(\mathfrak{A},\bar\mu)$ is σ-finite, let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\bar\mu a_n<\infty$ for every n and $\sup_{n\in\mathbb{N}}a_n=1$. For each n, choose $E_n\in\Sigma$ such that $E_n^\bullet=a_n$. Set $E=\bigcup_{n\in\mathbb{N}}E_n$; then $E^\bullet=\sup_{n\in\mathbb{N}}a_n=1$, so E is conegligible. Now $(X\setminus E,E_0,E_1,\dots)$ is a sequence of sets of finite measure covering X, so μ is σ-finite.

(d)(i) Suppose that μ is semi-finite and that $a\in\mathfrak{A}$, $\bar\mu a=\infty$. Then there is an $E\in\Sigma$ such that $E^\bullet=a$, so that $\mu E=\bar\mu a=\infty$. As μ is semi-finite, there is an $F\in\Sigma$ such that $F\subseteq E$ and $0<\mu F<\infty$. Set $b=F^\bullet$; then $b\subseteq a$ and $0<\bar\mu b<\infty$.

(ii) Suppose that $(\mathfrak{A},\bar\mu)$ is semi-finite and that $E\in\Sigma$, $\mu E=\infty$. Then $\bar\mu E^\bullet=\infty$, so there is a $b\subseteq E^\bullet$ such that $0<\bar\mu b<\infty$. Let $F\in\Sigma$ be such that $F^\bullet=b$. Then $F\cap E\in\Sigma$, $F\cap E\subseteq E$ and $(F\cap E)^\bullet=E^\bullet\cap b=b$, so that $\mu(F\cap E)=\bar\mu b\in\,]0,\infty[$.

(e)(i) Note first that if $\mathcal{E} \subseteq \Sigma$ and $F \in \Sigma$, then

$$\begin{aligned} &E \setminus F \text{ is negligible for every } E \in \mathcal{E} \\ &\iff E^\bullet \setminus F^\bullet = 0 \text{ for every } E \in \mathcal{E} \\ &\iff F^\bullet \text{ is an upper bound for } \{E^\bullet : E \in \mathcal{E}\}. \end{aligned}$$

So if $\mathcal{E} \subseteq \Sigma$ and $H \in \Sigma$, then H is an essential supremum of $\mathcal{E}$ in Σ, in the sense of 211G, iff $H^\bullet$ is the supremum of $A = \{E^\bullet : E \in \mathcal{E}\}$ in $\mathfrak{A}$. **P** Writing $\mathcal{F}$ for

$$\{F : F \in \Sigma,\ E \setminus F \text{ is negligible for every } E \in \mathcal{E}\},$$

we see that $B = \{F^\bullet : F \in \mathcal{F}\}$ is just the set of upper bounds of A, and that H is an essential supremum of $\mathcal{E}$ iff $H \in \mathcal{F}$ and $H^\bullet$ is a lower bound for B; that is, iff $H^\bullet = \sup A$. **Q**

(ii) Thus $\mathfrak{A}$ is Dedekind complete iff every family in Σ has an essential supremum in Σ. Since we already know that $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff μ is, we see that $(\mathfrak{A}, \bar{\mu})$ is localizable iff μ is.

(f) This is immediate from the definitions in 211I and 316K, if we remember always that $\{b : b \subseteq E^\bullet\} = \{F^\bullet : F \in \Sigma,\ F \subseteq E\}$ (312Lb).

(g), (h) follow at once from (f).

322C I copy out the relevant parts of Theorem 211L in the new context.

Theorem (a) A probability algebra is totally finite.

(b) A totally finite measure algebra is σ-finite.

(c) A σ-finite measure algebra is localizable.

(d) A localizable measure algebra is semi-finite.

proof All except (c) are trivial; and (c) may be deduced from 211Lc-211Ld, 322Bc, 322Be and 321J, or from 316Fa and 322G below.

322D Of course not all the definitions in §211 are directly relevant to measure algebras. The concepts of 'complete', 'locally determined' and 'strictly localizable' measure space do not correspond in any direct way to properties of the measure algebras. Indeed, completeness is just irrelevant, as the next proposition shows.

Proposition Let (X, Σ, μ) be a measure space, with completion $(X, \hat{\Sigma}, \hat{\mu})$ and c.l.d. version $(X, \tilde{\Sigma}, \tilde{\mu})$ (213E). Write $(\mathfrak{A}, \bar{\mu})$, $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ for the measure algebras of μ, $\hat{\mu}$ and $\tilde{\mu}$ respectively.

(a) The embedding $\Sigma \subsetneqq \hat{\Sigma}$ corresponds to an isomorphism between $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{A}_1, \bar{\mu}_1)$.

(b)(i) The embedding $\Sigma \subsetneqq \tilde{\Sigma}$ defines an order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_2$. Setting $\mathfrak{A}^f = \{a : a \in \mathfrak{A},\ \bar{\mu}a < \infty\}$, $\pi \restriction \mathfrak{A}^f$ is a measure-preserving bijection between $\mathfrak{A}^f$ and $\mathfrak{A}_2^f = \{c : c \in \mathfrak{A}_2,\ \bar{\mu}_2 c < \infty\}$.

(ii) π is injective iff μ is semi-finite, and in this case $\bar{\mu}_2(\pi a) = \bar{\mu}a$ for every $a \in \mathfrak{A}$.

(iii) If μ is localizable, π is a bijection.

proof For $E \in \Sigma$, I write E° for its image in $\mathfrak{A}$; for $F \in \hat{\Sigma}$, I write F^* for its image in $\mathfrak{A}_1$; and for $G \in \tilde{\Sigma}$, I write $G^\bullet$ for its image in $\mathfrak{A}_2$.

(a) This is nearly trivial. The map $E \mapsto E^* : \Sigma \to \mathfrak{A}_1$ is a Boolean homomorphism, being the composition of the Boolean homomorphisms $E \mapsto E : \Sigma \to \hat{\Sigma}$ and $F \mapsto F^* : \hat{\Sigma} \to \mathfrak{A}_1$. Its kernel is $\{E : E \in \Sigma,\ \hat{\mu}E = 0\} = \{E : E \in \Sigma,\ \mu E = 0\}$, so it induces an injective Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{A}_1$ given by the formula $\phi(E^\circ) = E^*$ for every $E \in \Sigma$ (312F, 3A2G). To see that ϕ is surjective, take any $b \in \mathfrak{A}_1$. There is an $F \in \hat{\Sigma}$ such that $F^* = b$, and there is an $E \in \Sigma$ such that $E \subseteq F$ and $\hat{\mu}(F \setminus E) = 0$, so that

$$\pi(E^\circ) = E^* = F^* = b.$$

Thus π is a Boolean algebra isomorphism. It is a measure algebra isomorphism because for any $E \in \Sigma$

$$\bar{\mu}_1 \phi(E^\circ) = \bar{\mu}_1 E^* = \hat{\mu}E = \mu E = \bar{\mu}E^\circ.$$

(b)(i) The map $E \mapsto E^\bullet : \Sigma \to \mathfrak{A}_2$ is a Boolean homomorphism with kernel $\{E : E \in \Sigma,\, \tilde{\mu}E = 0\} \supseteq \{E : E \in \Sigma,\, \mu E = 0\}$, so induces a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}_2$, defined by saying that $\pi E^\circ = E^\bullet$ for every $E \in \Sigma$.

If $a \in \mathfrak{A}^f$, it is expressible as E° where $\mu E < \infty$. Then $\tilde{\mu}E = \mu E$ (213Fa), so $\pi a = E^\bullet$ belongs to $\mathfrak{A}_2^f$, and $\bar{\mu}_2(\pi a) = \bar{\mu}a$. If a, a' are distinct members of $\mathfrak{A}^f$, then

$$\bar{\mu}_2(\pi a \mathbin{\triangle} \pi a') = \bar{\mu}_2 \pi(a \mathbin{\triangle} a') = \bar{\mu}(a \mathbin{\triangle} a') > 0,$$

so $\pi a \neq \pi a'$; thus $\pi\restriction\mathfrak{A}^f$ is an injective map from $\mathfrak{A}^f$ to $\mathfrak{A}_2^f$. If $c \in \mathfrak{A}_2^f$, then $c = G^\bullet$ where $\tilde{\mu}G < \infty$; by 213Fc, there is an $E \in \Sigma$ such that $E \subseteq G$, $\mu E = \tilde{\mu}G$ and $\tilde{\mu}(G \setminus E) = 0$, so that $E^\circ \in \mathfrak{A}^f$ and

$$\pi E^\circ = E^\bullet = G^\bullet = c.$$

As c is arbitrary, $\phi[\mathfrak{A}^f] = \mathfrak{A}_2^f$.

Finally, π is order-continuous. **P** Let $A \subseteq \mathfrak{A}$ be a non-empty downwards-directed set with infimum 0, and $b \in \mathfrak{A}_2$ a lower bound for $\pi[A]$. **?** If $b \neq 0$, then (because $(\mathfrak{A}_2, \bar{\mu}_2)$ is semi-finite) there is a $b_0 \in \mathfrak{A}_2^f$ such that $0 \neq b_0 \subseteq b$. Let $a_0 \in \mathfrak{A}$ be such that $\pi a_0 = b_0$. Then $a_0 \neq 0$, so there is an $a \in A$ such that $a \not\supseteq a_0$, that is, $a \cap a_0 \neq a_0$. But now, because $\pi\restriction\mathfrak{A}^f$ is injective,

$$b_0 = \pi a_0 \neq \pi(a \cap a_0) = \pi a \cap \pi a_0 = \pi a \cap b_0,$$

and $b_0 \not\subseteq \pi a$, which is impossible. **X** Thus $b = 0$, and 0 is the only lower bound of $\pi[A]$. As A is arbitrary, π is order-continuous (313L(b-ii)). **Q**

(ii) (α) If μ is semi-finite, then $\tilde{\mu}E = \mu E$ for every $E \in \Sigma$ (213Hc), so

$$\bar{\mu}_2(\pi E^\circ) = \bar{\mu}_2 E^\bullet = \tilde{\mu}E = \mu E = \bar{\mu}E^\circ$$

for every $E \in \Sigma$. In particular,

$$\pi a = 0 \Longrightarrow 0 = \bar{\mu}_2(\pi a) = \bar{\mu}a \Longrightarrow a = 0,$$

so π is injective. (β) If μ is not semi-finite, there is an $E \in \Sigma$ such that $\mu E = \infty$ but $\mu H = 0$ whenever $H \in \Sigma$, $H \subseteq E$ and $\mu H < \infty$; so that $\tilde{\mu}E = 0$ and

$$E^\circ \neq 0, \quad \pi E^\circ = E^\bullet = 0.$$

So in this case π is not injective.

(iii) Now suppose that μ is localizable. Then for every $G \in \tilde{\Sigma}$ there is an $E \in \Sigma$ such that $\tilde{\mu}(E \triangle G) = 0$, by 213Hb; accordingly $\pi E^\circ = E^\bullet = G^\bullet$. As G is arbitrary, π is surjective; and we know from (ii) that π is injective, so it is a bijection, as claimed.

322E Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra.

(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff it has a partition of unity consisting of elements of finite measure.

(b) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite, $a = \sup\{b : b \subseteq a,\, \bar{\mu}b < \infty\}$ and $\bar{\mu}a = \sup\{\bar{\mu}b : b \subseteq a,\, \bar{\mu}b < \infty\}$ for every $a \in \mathfrak{A}$.

proof Set $\mathfrak{A}^f = \{b : b \in \mathfrak{A},\, \bar{\mu}b < \infty\}$.

(a)(i) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite, then $\mathfrak{A}^f$ is order-dense in $\mathfrak{A}$, so there is a partition of unity consisting of members of $\mathfrak{A}^f$ (313K).

(ii) If there is a partition of unity $C \subseteq \mathfrak{A}^f$, and $\bar{\mu}a = \infty$, then there is a $c \in C$ such that $a \cap c \neq 0$, and now $a \cap c \subseteq a$ and $0 < \bar{\mu}(a \cap c) < \infty$; as a is arbitrary, $(\mathfrak{A}, \bar{\mu})$ is semi-finite.

(b) Of course $\mathfrak{A}^f$ is upwards-directed, by 321Bc, and we are supposing that its supremum is 1. If $a \in \mathfrak{A}$, then

$$B = \{b : b \in \mathfrak{A}^f,\, b \subseteq a\} = \{a \cap b : b \in \mathfrak{A}^f\}$$

is upwards-directed and has supremum a (313Ba), so $\bar{\mu}a = \sup_{b \in B} \bar{\mu}b$, by 321D.

Remark Compare 213A.

322F Proposition If $(\mathfrak{A}, \bar{\mu})$ is a semi-finite measure algebra, then $\mathfrak{A}$ is a weakly (σ, ∞)-distributive Boolean algebra.

proof Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-empty downwards-directed subsets of $\mathfrak{A}$, all with infimum 0. Set

$$B = \{b : \text{for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \supseteq a\}.$$

If $c \in \mathfrak{A} \setminus \{0\}$, let $c' \subseteq c$ be such that $0 < \bar{\mu}c' < \infty$. For each $n \in \mathbb{N}$, $\inf_{a\in A_n} \bar{\mu}(c' \cap a) = 0$, by 321F; so we may choose $a_n \in A_n$ such that $\bar{\mu}(c' \cap a_n) \le 2^{-n-2}\bar{\mu}b$. Set $b = \sup_{n\in\mathbb{N}} a_n \in B$. Then

$$\bar{\mu}(c' \cap b) \le \sum_{n=0}^{\infty} \bar{\mu}(c' \cap a_n) < \bar{\mu}c',$$

so $c' \not\subseteq b$ and $c \not\subseteq b$. As c is arbitrary, $\inf B = 0$; as $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive (316G).

322G Corresponding to 215B, we have the following description of σ-finite algebras.

Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra. Then the following are equiveridical:

(i) $(\mathfrak{A},\bar{\mu})$ is σ-finite;

(ii) $\mathfrak{A}$ is ccc;

(iii) *either* $\mathfrak{A} = \{0\}$ *or* there is a functional $\bar{\nu} : \mathfrak{A} \to [0,1]$ such that $(\mathfrak{A},\bar{\nu})$ is a probability algebra.

proof (i)$\Leftrightarrow$(ii) By 321J, it is enough to consider the case in which $(\mathfrak{A},\bar{\mu})$ is the measure algebra of a measure space (X,Σ,μ), and μ is semi-finite, by 322Bd. We know that $\mathfrak{A}$ is ccc iff there is no uncountable disjoint set in $\Sigma \setminus \mathcal{N}$, where $\mathcal{N}$ is the null ideal of μ (316D). But 215B(iii) shows that this is equivalent to μ being σ-finite, which is equivalent to $(\mathfrak{A},\bar{\mu})$ being σ-finite, by 322Bc.

(i)$\Rightarrow$(iii) If $(\mathfrak{A},\bar{\mu})$ is σ-finite, and $\mathfrak{A} \ne \{0\}$, let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$ such that $\bar{\mu}a_n < \infty$ for every n and $\sup_{n\in\mathbb{N}} a_n = 1$. Then $\bar{\mu}a_n > 0$ for some n, so there are $\gamma_n > 0$ such that $\sum_{n=0}^{\infty} \gamma_n\bar{\mu}a_n = 1$. (Set $\gamma'_n = 2^{-n}/(1 + \bar{\mu}a_n)$, $\gamma_n = \gamma'_n/(\sum_{i=0}^{\infty} \gamma'_i\bar{\mu}a_i)$.) Set $\bar{\nu}a = \sum_{n=0}^{\infty} \gamma_n\bar{\mu}(a \cap a_n)$ for every $a \in \mathfrak{A}$; it is easy to check that $(\mathfrak{A},\bar{\nu})$ is a probability algebra.

(iii)$\Rightarrow$(i) is a consequence of (i)$\Leftrightarrow$(ii).

322H Principal ideals If $(\mathfrak{A},\bar{\mu})$ is a measure algebra and $a \in \mathfrak{A}$, then it is easy to see (using 314Eb) that $(\mathfrak{A}_a, \bar{\mu}\restriction\mathfrak{A}_a)$ is a measure algebra, where $\mathfrak{A}_a$ is the principal ideal of $\mathfrak{A}$ generated by a.

322I Subspace measures General subspace measures give rise to complications in the measure algebra (see 322Xf, 322Yd). But subspaces with measurable envelopes (132D, 213L) are manageable.

Proposition Let (X,Σ,μ) be a measure space, and $A \subseteq X$ a set with a measurable envelope E. Let μ_A be the subspace measure on A, and Σ_A its domain; let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of (X,Σ,μ) and $(\mathfrak{A}_A,\bar{\mu}_A)$ the measure algebra of (A,Σ_A,μ_A). Set $a = E^\bullet$ and let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Then we have an isomorphism between $(\mathfrak{A}_a, \bar{\mu}\restriction\mathfrak{A}_a)$ and $(\mathfrak{A}_A,\bar{\mu}_A)$ given by the formula

$$F^\bullet \mapsto (F \cap A)^\circ$$

whenever $F \in \Sigma$ and $F \subseteq E$, writing $F^\bullet$ for the equivalence class of F in $\mathfrak{A}$ and $(F \cap A)^\circ$ for the equivalence class of $F \cap A$ in $\mathfrak{A}_A$.

proof Set $\Sigma_E = \{E \cap F : F \in \Sigma\}$. For F, $G \in \Sigma_E$,

$$F^\bullet = G^\bullet \iff \mu(F \triangle G) = 0 \iff \mu_A(A \cap (F \triangle G)) = 0 \iff (F \cap A)^\circ = (G \cap A)^\circ,$$

because E is a measurable envelope of A. Accordingly the given formula defines an injective function from the image $\{F^\bullet : F \in \Sigma_E\}$ of Σ_E in $\mathfrak{A}$ to $\mathfrak{A}_A$; but this image is just the principal ideal $\mathfrak{A}_a$. It is easy to check that the map is a Boolean homomorphism from $\mathfrak{A}_a$ to $\mathfrak{A}_A$, and it is a Boolean isomorphism because $\Sigma_A = \{F \cap A : F \in \Sigma_E\}$. Finally, it is measure-preserving because

$$\bar{\mu}F^\bullet = \mu F = \mu^*(F \cap A) = \mu_A(F \cap A) = \bar{\mu}_A(F \cap A)^\circ$$

for every $F \in \Sigma_E$, again using the fact that E is a measurable envelope of A.

322J Corollary Let (X,Σ,μ) be a measure space, with measure algebra $(\mathfrak{A},\bar{\mu})$.

(a) If $E \in \Sigma$, then the measure algebra of the subspace measure μ_E can be identified with the principal ideal $\mathfrak{A}_{E^\bullet}$ of $\mathfrak{A}$.

(b) If $A \subseteq X$ is a set of full outer measure (in particular, if $\mu^*A = \mu X < \infty$), then the measure algebra of the subspace measure μ_A can be identified with $\mathfrak{A}$.

322K Indefinite-integral measures: Proposition Let (X,Σ,μ) be a measure space and ν an indefinite-integral measure over μ (234J). Then the measure algebra of ν can be identified, as Boolean algebra, with a principal ideal of the measure algebra of μ.

proof Taking $(X,\hat\Sigma,\hat\mu)$ to be the completion of (X,Σ,μ), then we can identify the measure algebras of μ and $\hat\mu$, by 322Da; and ν is still an indefinite-integral measure over $\hat\mu$, just because μ and $\hat\mu$ give rise to the same theory of integration (212Fb). Now there is a $G\in\hat\Sigma$ such that the domain T of ν is $\{E:E\subseteq X,$ $E\cap G\in\hat\Sigma\}$ and the null ideal $\mathcal{N}_\nu$ of ν is $\{A:A\subseteq X,\ A\cap G\in\mathcal{N}_\mu\}$, where $\mathcal{N}_\mu$ is the null ideal of μ or $\hat\mu$ (234Lc[1], 212Eb). Writing $\mathfrak{A}$ for the measure algebra of $\hat\mu$, $c=G^\bullet\in\mathfrak{A}$, and $\mathfrak{A}_c$ for the principal ideal of $\mathfrak{A}$ generated by c, we have a Boolean homomorphism $E\mapsto(E\cap G)^\bullet:\mathrm{T}\to\mathfrak{A}_c$ with kernel $\mathcal{N}_\nu$. So, writing $E^\circ\in\mathfrak{B}$ for the equivalence class of $E\in\mathrm{T}$, we have an injective Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}_c$ defined by setting $\pi E^\circ=(E\cap G)^\bullet$ for every $E\in\mathrm{T}$. Of course

$$\pi[\mathfrak{B}]\supseteq\{(E\cap G)^\bullet:E\in\hat\Sigma\}=\{a\cap c:a\in\mathfrak{A}\}=\mathfrak{A}_c,$$

so π is actually an isomorphism, as required.

322L Simple products (a) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be an indexed family of measure algebras. Let $\mathfrak{A}$ be the simple product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$ (315A), and for $a\in\mathfrak{A}$ set $\bar\mu a=\sum_{i\in I}\bar\mu_i a(i)$. Then it is easy to check (using 315D(e-ii)) that $(\mathfrak{A},\bar\mu)$ is a measure algebra; I will call it the **simple product** of the family $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$. Each of the $\mathfrak{A}_i$ corresponds to a principal ideal $\mathfrak{A}_{e_i}$ say in $\mathfrak{A}$, where $e_i\in\mathfrak{A}$ corresponds to $1_{\mathfrak{A}_i}\in\mathfrak{A}_i$ (315E), and the Boolean isomorphism between $\mathfrak{A}_i$ and $\mathfrak{A}_{e_i}$ is a measure algebra isomorphism between $(\mathfrak{A}_i,\bar\mu_i)$ and $(\mathfrak{A}_{e_i},\bar\mu\restriction\mathfrak{A}_{e_i})$.

(b) If $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ is a family of measure spaces, with direct sum (X,Σ,μ) (214L), then the measure algebra $(\mathfrak{A},\bar\mu)$ of (X,Σ,μ) can be identified with the simple product of the measure algebras $(\mathfrak{A}_i,\bar\mu_i)$ of the (X_i,Σ_i,μ_i). **P** If, as in 214L, we set $X=\{(x,i):i\in I,\ x\in X_i\}$, and for $E\subseteq X$, $i\in I$ we set $E_i=\{x:(x,i)\in E\}$, then the Boolean isomorphism $E\mapsto\langle E_i\rangle_{i\in I}:\Sigma\to\prod_{i\in I}\Sigma_i$ induces a Boolean isomorphism from $\mathfrak{A}$ to $\prod_{i\in I}\mathfrak{A}_i$, which is also a measure algebra isomorphism, because

$$\bar\mu E^\bullet=\mu E=\sum_{i\in I}\mu_i E_i=\sum_{i\in I}\bar\mu_i E_i^\bullet$$

for every $E\in\Sigma$. **Q**

(c) A simple product of measure algebras is semi-finite, or localizable, or atomless, or purely atomic, iff every factor is. (Compare 214Kb.)

(d) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\langle e_i\rangle_{i\in I}$ a countable partition of unity in $\mathfrak{A}$. Then $(\mathfrak{A},\bar\mu)$ is isomorphic to the product $\prod_{i\in I}(\mathfrak{A}_{e_i},\bar\mu\restriction\mathfrak{A}_{e_i})$ of the corresponding principal ideals. **P** By 315F(ii), the map $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a Boolean isomorphism between $\mathfrak{A}$ and $\prod_{i\in I}\mathfrak{A}_i$. Because $\langle e_i\rangle_{i\in I}$ is disjoint and $a=\sup_{i\in I}a\cap e_i$, $\bar\mu a=\sum_{i\in I}\bar\mu(a\cap e_i)$ for every $a\in\mathfrak{A}$ (321E, or otherwise). So $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a measure algebra isomorphism between $(\mathfrak{A},\bar\mu)$ and $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu\restriction\mathfrak{A}_{e_i})$. **Q**

(e) Let $(\mathfrak{A},\bar\mu)$ be a localizable measure algebra.

(i) If $\langle e_i\rangle_{i\in I}$ is any partition of unity in $\mathfrak{A}$, then $(\mathfrak{A},\bar\mu)$ is isomorphic to the product $\prod_{i\in I}(\mathfrak{A}_{e_i},\bar\mu\restriction\mathfrak{A}_{e_i})$ of the corresponding principal ideals. **P** By 315F(iii), the map $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a Boolean isomorphism between $\mathfrak{A}$ and $\prod_{i\in I}\mathfrak{A}_i$. Because $\langle e_i\rangle_{i\in I}$ is disjoint and $a=\sup_{i\in I}a\cap e_i$, $\bar\mu a=\sum_{i\in I}\bar\mu(a\cap e_i)$ (321E, in its full strength), for every $a\in\mathfrak{A}$. So $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a measure algebra isomorphism between $(\mathfrak{A},\bar\mu)$ and $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu\restriction\mathfrak{A}_{e_i})$. **Q**

(ii) In particular, since $\mathfrak{A}$ has a partition of unity consisting of elements of finite measure (322Ea), $(\mathfrak{A},\bar\mu)$ is isomorphic to a simple product of totally finite measure algebras. Each of these is isomorphic to the measure algebra of a totally finite measure space, so $(\mathfrak{A},\bar\mu)$ is isomorphic to the measure algebra of a direct sum of totally finite measure spaces, which is strictly localizable.

Thus every localizable measure algebra is isomorphic to the measure algebra of a strictly localizable measure space. (See also 322O below.)

[1]Formerly 234D.

***322M Strictly localizable spaces** The following fact is occasionally useful.

Proposition Let (X,Σ,μ) be a strictly localizable measure space with $\mu X > 0$, and $(\mathfrak{A},\bar\mu)$ its measure algebra. If $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, there is a partition $\langle X_i\rangle_{i\in I}$ of X into members of Σ such that $X_i^\bullet = a_i$ for every $i\in I$ and

$$\Sigma = \{E : E\subseteq X,\, E\cap X_i\in\Sigma\ \forall\, i\in I\},$$

$$\mu E = \sum_{i\in I}\mu(E\cap X_i) \text{ for every } E\in\Sigma;$$

that is, the isomorphism between $\mathfrak{A}$ and the simple product $\prod_{i\in I}\mathfrak{A}_{a_i}$ of its principal ideals (315F) corresponds to an isomorphism between (X,Σ,μ) and the direct sum of the subspace measures on X_i.

proof (a) Suppose to begin with that $\mu X<\infty$. In this case $J=\{i : a_i\ne 0\}$ must be countable (322G). For each $i\in J$, choose $E_i\in\Sigma$ such that $E_i^\bullet = a_i$, and set $F_i = E_i\setminus\bigcup_{j\in J,j\ne i}E_j$; then $F_i^\bullet = a_i$ for each $i\in J$, and $\langle F_i\rangle_{i\in J}$ is disjoint. Because $\mu X>0$, J is non-empty; fix some $j_0\in J$ and set

$$\begin{aligned} X_i &= F_{j_0}\cup(X\setminus\bigcup_{j\in J}F_j) \text{ if } i=j_0,\\ &= F_i \text{ for } i\in J\setminus\{j_0\},\\ &= \emptyset \text{ for } i\in I\setminus J. \end{aligned}$$

Then $\langle X_i\rangle_{i\in I}$ is a disjoint family in Σ, $\bigcup_{i\in I}X_i = X$ and $X_i^\bullet = a_i$ for every i. Moreover, because only countably many of the X_i are non-empty, we certainly have

$$\Sigma = \{E : E\subseteq X,\, E\cap X_i\in\Sigma\ \forall\, i\in I\},$$

$$\mu E = \sum_{i\in I}\mu(E\cap X_i) \text{ for every } E\in\Sigma.$$

(b) For the general case, start by taking a decomposition $\langle Y_j\rangle_{j\in J}$ of X. We can suppose that no Y_j is negligible, because there is certainly some j_0 such that $\mu Y_{j_0}>0$, and we can if necessary replace Y_{j_0} by $Y_{j_0}\cup\bigcup\{Y_j : \mu Y_j = 0\}$. For each j, we can identify the measure algebra of the subspace measure on Y_j with the principal ideal $\mathfrak{A}_{b_j}$ generated by $b_j = Y_j^\bullet$ (322I). Now $\langle a_i\cap b_j\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}_{b_j}$, so by (a) just above we can find a disjoint family $\langle X_{ji}\rangle_{i\in I}$ in Σ such that $\bigcup_{i\in I}X_{ji} = Y_j$, $X_{ji}^\bullet = a_i\cap b_j$ for every i and

$$\Sigma\cap\mathcal{P}Y_j = \{E : E\subseteq Y_j,\, E\cap X_{ji}\in\Sigma\ \forall\, i\in I\},$$

$$\mu E = \sum_{i\in I}\mu(E\cap X_{ji}) \text{ for every } E\in\Sigma\cap\mathcal{P}Y_j.$$

Set $X_i = \bigcup_{j\in I}X_{ji}$ for every $i\in I$. Then $\langle X_i\rangle_{i\in I}$ is a partition of X. Because $X_i\cap Y_j = X_{ji}$ is measurable for every j, $X_i\in\Sigma$. Because $X_i^\bullet\supseteq a_i\cap b_j$ for every j, and $\langle b_j\rangle_{j\in J}$ is a partition of unity in $\mathfrak{A}$ (322Lb), $X_i^\bullet\supseteq a_i$ for each i; because $\langle X_i^\bullet\rangle_{i\in I}$ is disjoint and $\sup_{i\in I}a_i = 1$, $X_i^\bullet = a_i$ for every i. If $E\subseteq X$ is such that $E\cap X_i\in\Sigma$ for every i, then $E\cap X_{ji}\in\Sigma$ for all $i\in I$ and $j\in J$, so $E\cap Y_j\in\Sigma$ for every $j\in J$ and $E\in\Sigma$. If $E\in\Sigma$, then

$$\begin{aligned} \mu E &= \sum_{j\in J}\mu(E\cap Y_j) = \sum_{j\in J}\sum_{i\in I}\mu(E\cap X_{ji})\\ &= \sum_{i\in I}\sum_{j\in J}\mu(E\cap X_i\cap Y_j) = \sum_{i\in I}\mu(E\cap X_i). \end{aligned}$$

Thus $\langle X_i\rangle_{i\in I}$ is a suitable family.

322N Subalgebras: Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\mathfrak{B}$ a σ-subalgebra of $\mathfrak{A}$. Set $\bar\nu = \bar\mu\restriction\mathfrak{B}$.

(a) $(\mathfrak{B},\bar\nu)$ is a measure algebra.

(b) If $(\mathfrak{A},\bar\mu)$ is totally finite, or a probability algebra, so is $(\mathfrak{B},\bar\nu)$.

(c) If $(\mathfrak{A},\bar\mu)$ is σ-finite and $(\mathfrak{B},\bar\nu)$ is semi-finite, then $(\mathfrak{B},\bar\nu)$ is σ-finite.

(d) If $(\mathfrak{A},\bar\mu)$ is localizable and $\mathfrak{B}$ is order-closed and $(\mathfrak{B},\bar\nu)$ is semi-finite, then $(\mathfrak{B},\bar\nu)$ is localizable.

(e) If $(\mathfrak{B},\bar\nu)$ is a probability algebra, or totally finite, or σ-finite, so is $(\mathfrak{A},\bar\mu)$.

proof (a) By 314Eb, $\mathfrak{B}$ is Dedekind σ-complete, and the identity map $\pi:\mathfrak{B}\to\mathfrak{A}$ is sequentially order-continuous; so that $\bar\nu=\bar\mu\pi$ will be countably additive and $(\mathfrak{B},\bar\nu)$ will be a measure algebra.

(b) This is trivial.

(c) Use 322G. Every disjoint subset of $\mathfrak{B}$ is disjoint in $\mathfrak{A}$, therefore countable, because $\mathfrak{A}$ is ccc; so $\mathfrak{B}$ also is ccc and $(\mathfrak{B},\bar\nu)$ (being semi-finite) is σ-finite.

(d) By 314Ea, $\mathfrak{B}$ is Dedekind complete; we are supposing that $(\mathfrak{B},\bar\nu)$ is semi-finite, so it is localizable.

(e) This is elementary.

322O The Stone space of a localizable measure algebra I said above that the concepts 'strictly localizable' and 'locally determined' measure space have no equivalents in the theory of measure algebras. But when we look at the canonical measure on the Stone space of a measure algebra, we can of course hope that properties of the measure algebra will be reflected in the properties of this measure, as happens in the next theorem.

Theorem Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, Z the Stone space of $\mathfrak{A}$, and ν the standard measure on Z constructed by the method of 321J-321K. Then the following are equiveridical:
(i) $(\mathfrak{A},\bar\mu)$ is localizable;
(ii) ν is localizable;
(iii) ν is locally determined;
(iv) ν is strictly localizable.

proof Write Σ for the domain of ν, that is,

$$\{E\triangle A: E\subseteq Z\text{ is open-and-closed, }A\subseteq Z\text{ is meager}\},$$

and $\mathcal{M}$ for the ideal of meager subsets of Z, that is, the null ideal of ν (314M, 321K). Then $a\mapsto\widehat{a}^\bullet:\mathfrak{A}\to\Sigma/\mathcal{M}$ is an isomorphism between $(\mathfrak{A},\bar\mu)$ and the measure algebra of (Z,Σ,ν) (314M). Note that because any subset of a meager set is meager, ν is surely complete.

(a)(i)$\Leftrightarrow$(ii) is a consequence of 322Be.

(b)(ii)$\Rightarrow$(iii) Suppose that ν is localizable. Of course it is semi-finite. Let $V\subseteq Z$ be a set such that $V\cap E\in\Sigma$ whenever $E\in\Sigma$ and $\nu E<\infty$. Because ν is localizable, there is a $W\in\Sigma$ which is an essential supremum in Σ of $\{V\cap E: E\in\Sigma,\,\nu E<\infty\}$, that is, $W^\bullet=\sup\{(V\cap E)^\bullet:\nu E<\infty\}$ in $\Sigma/\mathcal{M}$. I claim that $W\triangle V$ is nowhere dense. **P** Let $G\subseteq Z$ be a non-empty open set. Then there is a non-zero $a\in\mathfrak{A}$ such that $\widehat{a}\subseteq G$. Because $(\mathfrak{A},\bar\mu)$ is semi-finite, we may suppose that $\bar\mu a<\infty$. Now

$$(W\cap\widehat{a})^\bullet=W^\bullet\cap\widehat{a}^\bullet=\sup\nolimits_{\nu E<\infty}(V\cap E)^\bullet\cap\widehat{a}^\bullet=\sup\nolimits_{\nu E<\infty}(V\cap E\cap\widehat{a})^\bullet=(V\cap\widehat{a})^\bullet,$$

so $(W\triangle V)\cap\widehat{a}$ is negligible, therefore meager. But we know that $\mathfrak{A}$ is weakly (σ,∞)-distributive (322F), so that meager sets in Z are nowhere dense (316I), and there is a non-empty open set $H\subseteq\widehat{a}\setminus(W\triangle V)$. Now $H\subseteq G\setminus\overline{W\triangle V}$. As G is arbitrary, $\operatorname{int}\overline{W\triangle V}=\emptyset$ and $W\triangle V$ is nowhere dense. **Q**

But this means that $W\triangle V\in\mathcal{M}\subseteq\Sigma$ and $V=W\triangle(W\triangle V)\in\Sigma$. As V is arbitrary, ν is locally determined.

(c)(iii)$\Rightarrow$(iv) Assume that ν is locally determined. Because $(\mathfrak{A},\bar\mu)$ is semi-finite, there is a partition of unity $C\subseteq\mathfrak{A}$ consisting of elements of finite measure (322Ea). Set $\mathcal{C}=\{\widehat{c}:c\in C\}$. This is a disjoint family of sets of finite measure for ν. Now suppose that $F\in\Sigma$ and $\nu F>0$. Then there is an open-and-closed set $E\subseteq Z$ such that $F\triangle E$ is meager, and E is of the form $\widehat{a}$ for some $a\in\mathfrak{A}$. Since

$$\bar\mu a=\nu\widehat{a}=\nu F>0,$$

there is some $c\in C$ such that $a\cap c\neq0$, and now

$$\nu(F\cap\widehat{c})=\bar\mu(a\cap c)>0.$$

This means that ν satisfies the conditions of 213Oa and must be strictly localizable.

(d)(iv)$\Rightarrow$(ii) This is just 211Ld.

322P Theorem Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (314U). Then there is a unique extension of $\bar{\mu}$ to a functional $\tilde{\mu}$ on $\widehat{\mathfrak{A}}$ such that $(\widehat{\mathfrak{A}},\tilde{\mu})$ is a localizable measure algebra. The embedding $\mathfrak{A} \subseteq\!\!\!\!\!\rightarrow \widehat{\mathfrak{A}}$ identifies the ideals $\{a : a \in \mathfrak{A},\, \bar{\mu}a < \infty\}$ and $\{a : a \in \widehat{\mathfrak{A}},\, \tilde{\mu}a < \infty\}$.

proof (I write the argument out as if $\mathfrak{A}$ were actually a subalgebra of $\widehat{\mathfrak{A}}$.) For $c \in \widehat{\mathfrak{A}}$, set

$$\tilde{\mu}c = \sup\{\bar{\mu}a : a \in \mathfrak{A},\, a \subseteq c\}.$$

Evidently $\tilde{\mu}$ is a function from $\widehat{\mathfrak{A}}$ to $[0,\infty]$ extending $\bar{\mu}$, so $\tilde{\mu}0 = 0$. Because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $\tilde{\mu}c > 0$ whenever $c \neq 0$, because any such c includes a non-zero member of $\mathfrak{A}$. If $\langle c_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\widehat{\mathfrak{A}}$ with supremum c, then $\tilde{\mu}c = \sum_{n=0}^{\infty}\tilde{\mu}c_n$. **P** Let A be the set of all members of $\mathfrak{A}$ expressible as $a = \sup_{n\in\mathbb{N}} a_n$ where $a_n \in \mathfrak{A}$ and $a_n \subseteq c_n$ for every $n \in \mathbb{N}$. Now

$$\begin{aligned}\sup_{a\in A}\bar{\mu}a &= \sup\{\sum_{n=0}^{\infty}\bar{\mu}a_n : a_n \in \mathfrak{A},\, a_n \subseteq c_n \text{ for every } n \in \mathbb{N}\}\\ &= \sum_{n=0}^{\infty}\sup\{\bar{\mu}a_n : a_n \subseteq c_n\} = \sum_{n=0}^{\infty}\tilde{\mu}c_n.\end{aligned}$$

Also, because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $c_n = \sup\{a : a \in \mathfrak{A},\, a \subseteq c_n\}$ for each n, and $\sup A$, taken in $\widehat{\mathfrak{A}}$, must be c. But this means that if $a' \in \mathfrak{A}$ and $a' \subseteq c$ then $a' = \sup_{a\in A} a' \cap a$ in $\widehat{\mathfrak{A}}$ and therefore also in $\mathfrak{A}$; so that

$$\bar{\mu}a' = \sup_{a\in A}\bar{\mu}(a' \cap a) \le \sup_{a\in A}\bar{\mu}a.$$

Accordingly

$$\tilde{\mu}c = \sup_{a\in A}\bar{\mu}a = \sum_{n=0}^{\infty}\tilde{\mu}c_n. \quad \mathbf{Q}$$

This shows that $(\widehat{\mathfrak{A}},\tilde{\mu})$ is a measure algebra. It is semi-finite because $(\mathfrak{A},\bar{\mu})$ is and every non-zero element of $\widehat{\mathfrak{A}}$ includes a non-zero element of $\mathfrak{A}$, which in turn includes a non-zero element of finite measure. Since $\widehat{\mathfrak{A}}$ is Dedekind complete, $(\widehat{\mathfrak{A}},\bar{\mu})$ is localizable.

If $\bar{\mu}a$ is finite, then surely $\tilde{\mu}a = \bar{\mu}a$ is finite. If $\tilde{\mu}c$ is finite, then $A = \{a : a \in \mathfrak{A},\, a \subseteq c\}$ is upwards-directed and $\sup_{a\in A}\bar{\mu}a = \tilde{\mu}c$ is finite, so $b = \sup A$ is defined in $\mathfrak{A}$ and $\bar{\mu}b = \tilde{\mu}c$. Because $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$, $b = c$ (313K, 313O) and $c \in \mathfrak{A}$, with $\bar{\mu}c = \tilde{\mu}c$.

322Q Definition Let $(\mathfrak{A},\bar{\mu})$ be any semi-finite measure algebra. I will call $(\widehat{\mathfrak{A}},\tilde{\mu})$, as constructed above, the **localization** of $(\mathfrak{A},\bar{\mu})$. Of course it is unique just in so far as the Dedekind completion of $\mathfrak{A}$ is.

322R Further properties of Stone spaces: Proposition Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra and (Z,Σ,ν) its Stone space.

(a) Meager sets in Z are nowhere dense; every $E \in \Sigma$ is uniquely expressible as $G\triangle M$ where $G \subseteq Z$ is open-and-closed and M is nowhere dense, and $\nu E = \sup\{\nu H : H \subseteq E \text{ is open-and-closed}\}$.

(b) The c.l.d. version $\tilde{\nu}$ of ν is strictly localizable, and has the same negligible sets as ν.

(c) If $(\mathfrak{A},\bar{\mu})$ is totally finite then $\nu E = \inf\{\nu H : H \supseteq E \text{ is open-and-closed}\}$ for every $E \in \Sigma$.

proof (a) I have already remarked (in the proof of 322O) that $\mathfrak{A}$ is weakly (σ,∞)-distributive, so that meager sets in Z are nowhere dense. But we know that every member of Σ is expressible as $G\triangle M$ where G is open-and-closed and M is meager, therefore nowhere dense. Moreover, the expression is unique, because if $G\triangle M = G'\triangle M'$ then $G\triangle G' \subseteq M \cup M'$ is open and nowhere dense, therefore empty, so $G = G'$ and $M = M'$.

Now let $a \in \mathfrak{A}$ be such that $\widehat{a} = G$, and consider $B = \{b : b \in \mathfrak{A},\, \widehat{b} \subseteq E\}$. Then $\sup B = a$ in $\mathfrak{A}$. **P** If $b \in B$, then $\widehat{b} \setminus \widehat{a} \subseteq M$ is nowhere dense, therefore empty; so a is an upper bound for B. **?** If a is not the supremum of B, then there is a non-zero $c \subseteq a$ such that $b \subseteq a \setminus c$ for every $b \in B$. But now $\widehat{c}$ cannot be empty, so $\widehat{c} \setminus \overline{M}$ is non-empty, and there is a non-zero $d \in \mathfrak{A}$ such that $\widehat{d} \subseteq \widehat{c} \setminus \overline{M}$. In this case $d \in B$ and $d \not\subseteq a \setminus c$. **X** Thus $a = \sup B$. **Q**

It follows that

$$\nu E=\nu G=\bar\mu a=\sup_{b\in B}\bar\mu b$$
$$=\sup_{b\in B}\nu\widehat{b}\le\sup\{\nu H:H\subseteq E\text{ is open-and-closed}\}\le\nu E$$

and $\nu E=\sup\{\nu H:H\subseteq E$ is open-and-closed$\}$.

(b) This is the same as part (c) of the proof of 322O. We have a disjoint family $\mathcal{C}$ of sets of finite measure for ν such that whenever $E\in\Sigma$ and $\nu E>0$ there is a $C\in\mathcal{C}$ such that $\mu(C\cap E)>0$. Now if $\tilde\nu F$ is defined and not 0, there is an $E\in\Sigma$ such that $E\subseteq F$ and $\nu E>0$ (213Fc), so that there is a $C\in\mathcal{C}$ such that $\nu(E\cap C)>0$; since $\nu C<\infty$, we have

$$\tilde\nu(F\cap C)\ge\tilde\nu(E\cap C)=\nu(E\cap C)>0.$$

And of course $\tilde\nu C<\infty$ for every $C\in\mathcal{C}$. This means that $\mathcal{C}$ witnesses that $\tilde\nu$ satisfies the conditions of 213Oa, so that $\tilde\nu$ is strictly localizable.

Any ν-negligible set is surely $\tilde\nu$-negligible. If M is $\tilde\nu$-negligible then it is nowhere dense. **P** If $G\subseteq Z$ is open and not empty then there is a non-empty open-and-closed set $H_1\subseteq G$, and now $H_1\in\Sigma$, so there is a non-empty open-and-closed set $H\subseteq H_1$ such that νH is finite (because ν is semi-finite). In this case $H\cap M$ is ν-negligible, therefore nowhere dense, and $H\not\subseteq\overline{M}$. But this means that $G\not\subseteq\overline{M}$; as G is arbitrary, M is nowhere dense. **Q** Accordingly $M\in\mathcal{M}$ and is ν-negligible.

Thus ν and $\tilde\nu$ have the same negligible sets.

(c) Because $\nu Z<\infty$,

$$\nu E=\nu Z-\nu(Z\setminus E)=\nu Z-\sup\{\nu H:H\subseteq Z\setminus E\text{ is open-and-closed}\}$$
$$=\inf\{\nu(Z\setminus H):H\subseteq Z\setminus E\text{ is open-and-closed}\}$$
$$=\inf\{\nu H:H\supseteq E\text{ is open-and-closed}\}.$$

322X Basic exercises >(a) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Let I_∞ be the set of those $a\in\mathfrak{A}$ which are either 0 or 'purely infinite', that is, $\bar\mu b=\infty$ for every non-zero $b\subseteq a$. Show that I_∞ is a σ-ideal of $\mathfrak{A}$. Show that there is a function $\bar\mu_{\mathrm{sf}}:\mathfrak{A}/I_\infty\to[0,\infty]$ defined by setting $\bar\mu_{\mathrm{sf}}a^\bullet=\sup\{\bar\mu b:b\subseteq a,\,\bar\mu b<\infty\}$ for every $a\in\mathfrak{A}$. Show that $(\mathfrak{A}/I_\infty,\bar\mu_{\mathrm{sf}})$ is a semi-finite measure algebra.

(b) Let (X,Σ,μ) be a measure space and let μ_{sf} be the 'semi-finite version' of μ, as defined in 213Xc. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of (X,Σ,μ). Show that the measure algebra of $(X,\Sigma,\mu_{\mathrm{sf}})$ is isomorphic to the measure algebra $(\mathfrak{A}/I_\infty,\bar\mu_{\mathrm{sf}})$ of (a) above.

(c) Let (X,Σ,μ) be a measure space and $(X,\tilde\Sigma,\tilde\mu)$ its c.l.d. version. Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A}_2,\bar\mu_2)$ be the corresponding measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{A}_2$ the canonical homomorphism, as in 322Db. Show that the kernel of π is the ideal I_∞, as described in 322Xa, so that $\mathfrak{A}/I_\infty$ is isomorphic, as Boolean algebra, to $\pi[\mathfrak{A}]\subseteq\mathfrak{A}_2$. Show that this isomorphism identifies $\bar\mu_{\mathrm{sf}}$, as described in 322Xa, with $\bar\mu_2\restriction\pi[\mathfrak{A}]$.

(d) Give a direct proof of 322G, not relying on 215B and 321J.

>(e) Let $(\mathfrak{A},\bar\mu)$ be any measure algebra, A a non-empty subset of $\mathfrak{A}$, and $c\in\mathfrak{A}$ such that $\bar\mu c<\infty$. Show that (i) $c_0=\sup\{a\cap c:a\in A\}$ is defined in $\mathfrak{A}$ (ii) there is a countable set $B\subseteq A$ such that $c_0=\sup\{a\cap c:a\in B\}$.

(f) Let (X,Σ,μ) be a measure space and A any subset of X; let μ_A be the subspace measure on A and Σ_A its domain. Write $(\mathfrak{A},\bar\mu)$ for the measure algebra of (X,Σ,μ) and $(\mathfrak{A}_A,\bar\mu_A)$ for the measure algebra of (A,Σ_A,μ_A). Show that the formula $F^\bullet\mapsto(F\cap A)^\bullet$ defines a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{A}_A$ which has kernel $I=\{F^\bullet:F\in\Sigma,\,F\cap A=\emptyset\}$. Show that for any $a\in\mathfrak{A}$, $\bar\mu_A(\pi a)=\min\{\bar\mu b:b\in\mathfrak{A},\,a\setminus b\in I\}$.

(g) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $\mathfrak{B}$ a regularly embedded σ-subalgebra of $\mathfrak{A}$. Suppose that $(\mathfrak{B},\bar\mu\restriction\mathfrak{B})$ is semi-finite. Show that $(\mathfrak{A},\bar\mu)$ is semi-finite.

(h) Let $(\mathfrak{A},\bar{\mu})$ be any measure algebra and (Z,Σ,ν) its Stone space. Show that the c.l.d. version of ν is strictly localizable.

322Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ. Set $\mathcal{N}=\{N:\exists\,F\in\mathcal{I},\,N\subseteq F\}$. Show that $\mathcal{N}$ is a σ-ideal of subsets of X. Set $\hat{\Sigma}=\{E\triangle N:E\in\Sigma,\,N\in\mathcal{N}\}$. Show that $\hat{\Sigma}$ is a σ-algebra of subsets of X and that $\hat{\Sigma}/\mathcal{N}$ is isomorphic to $\Sigma/\mathcal{I}$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a semi-finite measure algebra, and (Z,Σ,ν) its Stone space. Let $\tilde{\nu}$ be the c.l.d. version of ν, and $\tilde{\Sigma}$ its domain. Show that $\tilde{\Sigma}$ is precisely the Baire-property algebra $\{G\triangle A:G\subseteq Z$ is open, $A\subseteq Z$ is meager$\}$, so that $\tilde{\Sigma}/\mathcal{M}$ can be identified with the regular open algebra of Z (314Yd) and the measure algebra of $\tilde{\nu}$ can be identified with the localization of $\mathfrak{A}$.

(c) Give an example of a localizable measure algebra $(\mathfrak{A},\bar{\mu})$ with a σ-subalgebra $\mathfrak{B}$ such that $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B})$ is semi-finite and atomless, but $\mathfrak{A}$ has an atom.

(d) Let (X,Σ,μ) be a measure space and $A\subseteq X$ a subset; let μ_A be the subspace measure on A, $\mathfrak{A}$ and $\mathfrak{A}_A$ the measure algebras of μ and μ_A, and $\pi:\mathfrak{A}\to\mathfrak{A}_A$ the canonical homomorphism, as described in 322Xf. (i) Show that if μ_A is semi-finite, then π is order-continuous. (ii) Show that if μ is semi-finite but μ_A is not, then π is not order-continuous.

(e) Show that if $(\mathfrak{A},\bar{\mu})$ is a semi-finite measure algebra, with Stone space (Z,Σ,ν), then ν has locally determined negligible sets in the sense of 213I.

(f) Let $(\mathfrak{A},\bar{\mu})$ be a localizable measure algebra and (Z,Σ,ν) its Stone space. (i) Show that a function $f:Z\to\mathbb{R}$ is Σ-measurable iff there is a conegligible set $G\subseteq X$ such that $f\restriction G$ is continuous. (*Hint*: 316Yi.) (ii) Show that $f:Z\to[0,1]$ is Σ-measurable iff there is a continuous function $g:Z\to[0,1]$ such that $f=g$ ν-a.e.

322 Notes and comments I have taken this leisurely tour through the concepts of Chapter 21 partly to recall them (or persuade you to look them up) and partly to give you practice in the elementary manipulations of measure algebras. The really vital result here is the correspondence between 'localizability' in measure spaces and measure algebras. Part of the object of this volume (particularly in Chapter 36) is to try to make sense of the properties of localizable measure spaces, as discussed in Chapter 24 and elsewhere, in terms of their measure algebras. I hope that 322Be has already persuaded you that the concept really belongs to measure algebras, and that the formulation in terms of 'essential suprema' is a dispensable expedient.

I have given proofs of 322C and 322G depending on the realization of an arbitrary measure algebra as the measure algebra of a measure space, and the corresponding theorems for measure spaces, because this seems the natural approach from where we presently stand; but I am sympathetic to the view that such proofs must be inappropriate, and that it is in some sense better style to look for arguments which speak only of measure algebras (322Xd).

For any measure algebra $(\mathfrak{A},\bar{\mu})$, the set $\mathfrak{A}^f$ of elements of finite measure is an ideal of $\mathfrak{A}$; consequently it is order-dense iff it includes a partition of unity (322E). In 322F we have something deeper: any semi-finite measure algebra must be weakly (σ,∞)-distributive when regarded as a Boolean algebra, and this has significant consequences in its Stone space, which are used in the proofs of 322O and 322R. Of course a result of this kind must depend on the semi-finiteness of the measure algebra, since any Dedekind σ-complete Boolean algebra becomes a measure algebra if we give every non-zero element the measure ∞. It is natural to look for algebraic conditions on a Boolean algebra sufficient to make it 'measurable', in the sense that it should carry a semi-finite measure; this is an unresolved problem to which I will return in Chapter 39.

Subspace measures, indefinite-integral measures, simple products, direct sums, principal ideals and order-closed subalgebras give no real surprises; I spell out the details in 322H-322N and 322Xf-322Xg. It is worth noting that completing a measure space has no effect on its measure algebra (322D, 322Ya). We see also that from the point of view of measure algebras there is no distinction to be made between 'localizable' and 'strictly localizable', since every localizable measure algebra is representable as the measure algebra of a strictly localizable measure space (322Le). (But strict localizability does have implications for some processes starting in the measure algebra; see 322M.) It is nevertheless remarkable that the canonical measure on the Stone space of a semi-finite measure algebra is localizable iff it is strictly localizable (322O). This canonical measure has many other interesting properties, which I skim over in 322R, 322Xh, 322Yb and 322Yf. In Chapter 21 I discussed a number of methods of improving measure spaces, notably 'completions' (212C) and 'c.l.d. versions' (213E). Neither of these is applicable in any general way to measure algebras. But in fact we have a more effective construction, at least for semi-finite measure algebras, that of 'localization' (322P-322Q); I say that it is more effective just because localizability is more important than completeness or local determinedness, being of vital importance in the behaviour of function spaces (241Gb, 243Gb, 245Ec, 363M, 364M, 365L, 367M, 369A, 369C). Note that the localization of a semi-finite measure algebra does in fact correspond to the c.l.d. version of a certain measure (322Yb). But of course $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ do *not* have the same Stone spaces, even when $\widehat{\mathfrak{A}}$ can be effectively represented as the measure algebra of a measure on the Stone space of $\mathfrak{A}$. What is happening in 322Yb is that we are using all the open sets of Z to represent members of $\widehat{\mathfrak{A}}$, not just the open-and-closed sets, which correspond to members of $\mathfrak{A}$.

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323 The topology of a measure algebra

I take a short section to discuss one of the fundamental tools for studying totally finite measure algebras, the natural metric that each carries. The same ideas, suitably adapted, can be applied to an arbitrary measure algebra, where we have a topology corresponding closely to the topology of convergence in measure on the function space L^0. Most of the section consists of an analysis of the relations between this topology and the order structure of the measure algebra.

323A The pseudometrics ρ_a (a) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. Write $\mathfrak{A}^f=\{a:a\in\mathfrak{A},\,\bar\mu a<\infty\}$. For $a\in\mathfrak{A}^f$ and b, $c\in\mathfrak{A}$, write $\rho_a(b,c)=\bar\mu(a\cap(b\mathbin{\triangle}c))$. Then ρ_a is a pseudometric on $\mathfrak{A}$. **P** (i) Because $\bar\mu a<\infty$, ρ_a takes values in $[0,\infty[$. (ii) If b, c, $d\in\mathfrak{A}$ then $b\mathbin{\triangle}d\subseteq(b\mathbin{\triangle}c)\cup(c\mathbin{\triangle}d)$, so

$$\begin{aligned}\rho_a(b,d)=\bar\mu(a\cap(b\mathbin{\triangle}d))&\le\bar\mu((a\cap(b\mathbin{\triangle}c))\cup(a\cap(c\mathbin{\triangle}d)))\\&\le\bar\mu(a\cap(b\mathbin{\triangle}c))+\bar\mu(a\cap(c\mathbin{\triangle}d))=\rho_a(b,c)+\rho_a(c,d).\end{aligned}$$

(iii) If b, $c\in\mathfrak{A}$ then

$$\rho_a(b,c)=\bar\mu(a\cap(b\mathbin{\triangle}c))=\bar\mu(a\cap(c\mathbin{\triangle}b))=\rho_a(c,b).\ \mathbf{Q}$$

(b) Now the **measure-algebra topology** of the measure algebra $(\mathfrak{A},\bar\mu)$ is that generated by the family $\mathrm{P}=\{\rho_a:a\in\mathfrak{A}^f\}$ of pseudometrics on $\mathfrak{A}$. Similarly the **measure-algebra uniformity** on $\mathfrak{A}$ is that generated by P. For the rest of this section I will take it that every measure algebra is endowed with its measure-algebra topology and uniformity.

(For a general discussion of topologies defined by pseudometrics, see 2A3F *et seq.* For the associated uniformities see §3A4.)

(c) Note that P is upwards-directed, since $\rho_{a\cup a'}\ge\max(\rho_a,\rho_{a'})$ for all a, $a'\in\mathfrak{A}^f$.

(d) On the ideal $\mathfrak{A}^f$ we have an actual metric ρ defined by saying that $\rho(a,b)=\bar\mu(a\mathbin{\triangle}b)$ for a, $b\in\mathfrak{A}^f$ (to see that ρ is a metric, repeat the formulae of (a) above); this is the **measure metric** or **Fréchet-Nikodým metric**. I will call the topology it generates the **strong measure-algebra topology** on $\mathfrak{A}^f$.

When $\bar\mu$ is totally finite, that is, $\mathfrak{A}^f=\mathfrak{A}$, $\rho=\rho_1$ defines the measure-algebra topology and uniformity of $\mathfrak{A}$.

323B Proposition Let $(\mathfrak{A},\bar\mu)$ be any measure algebra, and give $\mathfrak{A}$ its measure-algebra topology.

(a) The operations $\cup$, $\cap$, $\setminus$ and $\triangle$ are all uniformly continuous.

(b) $\mathfrak{A}^f$ is dense in $\mathfrak{A}$.

proof (a) The point is that for any b, c, b', $c' \in \mathfrak{A}$ we have

$$(b * c) \triangle (b' * c') \subseteq (b \triangle b') \cup (c \triangle c')$$

for any of the operations $* = \cup$, $\cap$ etc.; so that if $a \in \mathfrak{A}^f$ then

$$\rho_a(b * c, b' * c') \le \rho_a(b, b') + \rho_a(c, c').$$

Consequently the operation $*$ must be uniformly continuous.

(b) Given $b \in \mathfrak{A}$, $a \in \mathfrak{A}^f$ and $\epsilon > 0$, then $a \cap b \in \mathfrak{A}^f$ and $\rho_a(b, a \cap b) = 0$. Because the family $\{\rho_a : a \in \mathfrak{A}^f\}$ is upwards-directed, this is enough to show that every neighbourhood of b meets $\mathfrak{A}^f$; as b is arbitrary, $\mathfrak{A}^f$ is dense.

323C Proposition (a) Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra. Then $\bar\mu : \mathfrak{A} \to [0, \infty[$ is uniformly continuous.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra. Then $\bar\mu : \mathfrak{A} \to [0, \infty]$ is lower semi-continuous.

(c) Let $(\mathfrak{A}, \bar\mu)$ be any measure algebra. If $a \in \mathfrak{A}$ and $\bar\mu a < \infty$, then $b \mapsto \bar\mu(b \cap a) : \mathfrak{A} \to \mathbb{R}$ is uniformly continuous.

proof (a) For any a, $b \in \mathfrak{A}$,

$$|\bar\mu a - \bar\mu b| \le \bar\mu(a \triangle b) = \rho_1(a, b).$$

(b) Suppose that $b \in \mathfrak{A}$ and $\bar\mu b > \alpha \in \mathbb{R}$. Then there is an $a \subseteq b$ such that $\alpha < \bar\mu a < \infty$ (322Eb). If $c \in \mathfrak{A}$ is such that $\rho_a(b, c) < \bar\mu a - \alpha$, then

$$\bar\mu c \ge \bar\mu(a \cap c) = \bar\mu a - \bar\mu(a \cap (b \setminus c)) > \alpha.$$

Thus $\{b : \bar\mu b > \alpha\}$ is open; as α is arbitrary, $\bar\mu$ is lower semi-continuous.

(c) $|\bar\mu(a \cap b) - \bar\mu(a \cap c)| \le \rho_a(b, c)$ for all b, $c \in \mathfrak{A}$.

323D The following facts are basic to any understanding of the relationship between the order structure and topology of a measure algebra.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) Let $B \subseteq \mathfrak{A}$ be a non-empty upwards-directed set. For $b \in B$ set $F_b = \{c : b \subseteq c \in B\}$.

(i) $\{F_b : b \in B\}$ generates a Cauchy filter $\mathcal{F}(B\uparrow)$ on $\mathfrak{A}$.

(ii) If $\sup B$ is defined in $\mathfrak{A}$, then it is a topological limit of $\mathcal{F}(B\uparrow)$; in particular, it belongs to the topological closure of B.

(b) Let $B \subseteq \mathfrak{A}$ be a non-empty downwards-directed set. For $b \in B$ set $F'_b = \{c : b \supseteq c \in B\}$.

(i) $\{F'_b : b \in B\}$ generates a Cauchy filter $\mathcal{F}(B\downarrow)$ on $\mathfrak{A}$.

(ii) If $\inf B$ is defined in $\mathfrak{A}$, then it is a topological limit of $\mathcal{F}(B\downarrow)$; in particular, it belongs to the topological closure of B.

(c)(i) Closed subsets of $\mathfrak{A}$ are order-closed in the sense of 313Da.

(ii) An order-dense subalgebra of $\mathfrak{A}$ must be dense in the topological sense.

(d) Now suppose that $(\mathfrak{A}, \bar\mu)$ is semi-finite.

(i) The sets $\{b : b \subseteq c\}$, $\{b : b \supseteq c\}$ are closed for every $c \in \mathfrak{A}$.

(ii) If $B \subseteq \mathfrak{A}$ is non-empty and upwards-directed and e is a cluster point of $\mathcal{F}(B\uparrow)$, then $e = \sup B$.

(iii) If $B \subseteq \mathfrak{A}$ is non-empty and downwards-directed and e is a cluster point of $\mathcal{F}(B\downarrow)$, then $e = \inf B$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a)(i) (α) If b, $c \in B$ then there is a $d \in B$ such that $b \cup c \subseteq d$, so that $F_d \subseteq F_b \cap F_c$; consequently

$$\mathcal{F}(B\uparrow) = \{F : F \subseteq \mathfrak{A}, \exists\, b \in B, F_b \subseteq F\}$$

is a filter on $\mathfrak{A}$. (β) Let $a \in \mathfrak{A}^f$, $\epsilon > 0$. Then there is a $b \in B$ such that $\bar\mu(a \cap c) \le \bar\mu(a \cap b) + \frac{1}{2}\epsilon$ for every $c \in B$, and $F_b \in \mathcal{F}(B\uparrow)$. If now c, $c' \in F_b$, $c \triangle c' \subseteq (c \setminus b) \cup (c' \setminus b)$, so

$$\rho_a(c, c') \le \bar\mu(a \cap c \setminus b) + \bar\mu(a \cap c' \setminus b) = \bar\mu(a \cap c) + \bar\mu(a \cap c') - 2\bar\mu(a \cap b) \le \epsilon.$$

As a and ϵ are arbitrary, $\mathcal{F}(B\uparrow)$ is Cauchy.

(ii) Suppose that $e = \sup B$ is defined in $\mathfrak{A}$. Let $a \in \mathfrak{A}^f$, $\epsilon > 0$. By 313Ba, $a \cap e = \sup_{b \in B} a \cap b$; but $\{a \cap b : b \in B\}$ is upwards-directed, so $\bar\mu(a \cap e) = \sup_{b \in B} \bar\mu(a \cap b)$, by 321D. Let $b \in B$ be such that $\bar\mu(a \cap b) \ge \bar\mu(a \cap e) - \epsilon$. Then for any $c \in F_b$, $e \triangle c \subseteq e \setminus b$, so

$$\rho_a(e, c) = \bar\mu(a \cap (e \triangle c)) \le \bar\mu(a \cap (e \setminus b)) = \bar\mu(a \cap e) - \bar\mu(a \cap b) \le \epsilon.$$

As a and ϵ are arbitrary, $\mathcal{F}(B\uparrow) \to e$.

Because $B \in \mathcal{F}(B\uparrow)$, e surely belongs to the topological closure of B.

(b) Either repeat the arguments above, with appropriate inversions, using 321F in place of 321D, or apply (a) to the set $\{1 \setminus b : b \in B\}$.

(c)(i) This follows at once from (a) and (b) and the definition in 313Da.

(ii) If $\mathfrak{B} \subseteq \mathfrak{A}$ is an order-dense subalgebra and $a \in \mathfrak{A}$, then $B = \{b : b \in \mathfrak{B}, b \subseteq a\}$ is upwards-directed and has supremum a (313K); by (a-ii), $a \in \overline{B} \subseteq \overline{\mathfrak{B}}$. As a is arbitrary, $\mathfrak{B}$ is topologically dense.

(d)(i) Set $F = \{b : b \subseteq c\}$. If $d \in \mathfrak{A} \setminus F$, then (because $(\mathfrak{A}, \bar\mu)$ is semi-finite) there is an $a \in \mathfrak{A}^f$ such that $\delta = \bar\mu(a \cap d \setminus c) > 0$; now if $b \in F$,

$$\rho_a(d, b) \ge \bar\mu(a \cap d \setminus b) \ge \delta,$$

so that d cannot belong to the closure of F. As d is arbitrary, F is closed. Similarly, $\{b : b \supseteq c\}$ is closed.

(ii) (α) If $b \in B$, then $e \in \overline{F_b}$, because $F_b \in \mathcal{F}(B\uparrow)$; but $\{c : b \subseteq c\}$ is a closed set including F_b, so contains e, and $b \subseteq e$. As b is arbitrary, e is an upper bound for B. (β) If d is an upper bound of B, then $\{c : c \subseteq d\}$ is a closed set belonging to $\mathcal{F}(B\uparrow)$, so contains e. As d is arbitrary, this shows that e is the supremum of B, as claimed.

(iii) Use the same arguments as in (ii), but inverted.

323E Corollary Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra.

(a) If $\langle b_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum b, then $\langle b_n \rangle_{n \in \mathbb{N}}$ converges topologically to b.

(b) If $\langle b_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum b, then $\langle b_n \rangle_{n \in \mathbb{N}}$ converges topologically to b.

proof I call this a 'corollary' because it is the special case of 323Da-323Db in which B is the set of terms of a monotonic sequence; but it is probably easier to work directly from the definition in 323A, and use 321Be or 321Bf to see that $\lim_{n\to\infty} \rho_a(b_n, b) = 0$ whenever $\bar\mu a < \infty$.

323F The following is a useful calculation.

Lemma Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra and $\langle c_n \rangle_{n \in \mathbb{N}}$ a sequence in $\mathfrak{A}$ such that the sum $\sum_{n=0}^{\infty} \bar\mu(c_n \triangle c_{n+1})$ is finite. Set $d_0 = \sup_{n \in \mathbb{N}} \inf_{m \ge n} c_m$, $d_1 = \inf_{n \in \mathbb{N}} \sup_{m \ge n} c_m$. Then $d_0 = d_1$ and, writing d for their common value, $\lim_{n\to\infty} \bar\mu(c_n \triangle d) = 0$.

proof Write $\alpha_n = \bar\mu(c_n \triangle c_{n+1})$, $\beta_n = \sum_{k=n}^{\infty} \alpha_k$ for $n \in \mathbb{N}$; we are supposing that $\lim_{n\to\infty} \beta_n = 0$. Set $b_n = \sup_{m \ge n} c_m \triangle c_{m+1}$; then

$$\bar\mu b_n \le \textstyle\sum_{m=n}^{\infty} \bar\mu(c_m \triangle c_{m+1}) = \beta_n$$

for each n. If $m \ge n$, then

$$c_m \triangle c_n \subseteq \sup_{n \le k < m} c_k \triangle c_{k+1} \subseteq b_n,$$

so

$$c_n \setminus b_n \subseteq c_m \subseteq c_n \cup b_n.$$

Consequently

$$c_n \setminus b_n \subseteq \inf_{k \ge m} c_k \subseteq \sup_{k \ge m} c_k \subseteq c_n \cup b_n$$

for every $m \ge n$, and

$$c_n \setminus b_n \subseteq d_0 \subseteq d_1 \subseteq c_n \cup b_n,$$

so that

$$c_n \triangle d_0 \subseteq b_n, \quad c_n \triangle d_1 \subseteq b_n, \quad d_1 \setminus d_0 \subseteq b_n.$$

As this is true for every n,

$$\lim_{n\to\infty} \bar{\mu}(c_n \triangle d_i) \le \lim_{n\to\infty} \bar{\mu}b_n = 0$$

for both i, and

$$\bar{\mu}(d_1 \triangle d_0) \le \inf_{n\in\mathbb{N}} \bar{\mu}b_n = 0,$$

so that $d_1 = d_0$.

323G The classification of measure algebras: Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, $\mathfrak{T}$ its measure-algebra topology and $\mathcal{U}$ its measure-algebra uniformity.

(a) $(\mathfrak{A}, \bar{\mu})$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(b) $(\mathfrak{A}, \bar{\mu})$ is σ-finite iff $\mathfrak{T}$ is metrizable, and in this case $\mathcal{U}$ also is metrizable.

(c) $(\mathfrak{A}, \bar{\mu})$ is localizable iff $\mathfrak{T}$ is Hausdorff and $\mathfrak{A}$ is complete under $\mathcal{U}$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a)(i) Suppose that $(\mathfrak{A}, \bar{\mu})$ is semi-finite and that b, c are distinct members of $\mathfrak{A}$. Then there is an $a \subseteq b \triangle c$ such that $0 < \bar{\mu}a < \infty$, and now $\rho_a(b, c) > 0$. As b and c are arbitrary, $\mathfrak{T}$ is Hausdorff (2A3L).

(ii) Suppose that $\mathfrak{T}$ is Hausdorff and that $b \in \mathfrak{A}$ has $\bar{\mu}b = \infty$. Then $b \ne 0$ so there must be an $a \in \mathfrak{A}^f$ such that $\bar{\mu}(a \cap b) = \rho_a(0, b) > 0$; in which case $a \cap b \subseteq b$ and $0 < \bar{\mu}(a \cap b) < \infty$. As b is arbitrary, $\bar{\mu}$ is semi-finite.

(b)(i) Suppose that $\bar{\mu}$ is σ-finite. Let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{A}^f$ with supremum 1. Set

$$\rho(b, c) = \sum_{n=0}^{\infty} \frac{\rho_{a_n}(b, c)}{1 + 2^n \bar{\mu}a_n}$$

for b, $c \in \mathfrak{A}$. Then ρ is a metric on $\mathfrak{A}$, because if $\rho(b, c) = 0$ then $a_n \cap (b \triangle c) = 0$ for every n, so $b \triangle c = 0$ and $b = c$.

If $a \in \mathfrak{A}^f$ and $\epsilon > 0$, take n such that $\bar{\mu}(a \setminus a_n) \le \frac{1}{2}\epsilon$. If b, $c \in \mathfrak{A}$ and $\rho(b, c) \le \epsilon/2(1 + 2^n \bar{\mu}a_n)$, then

$$\begin{aligned} \rho_a(b, c) &= \rho_{a\setminus a_n}(b, c) + \rho_{a\cap a_n}(b, c) \le \bar{\mu}(a \setminus a_n) + \rho_{a_n}(b, c) \\ &\le \frac{1}{2}\epsilon + (1 + 2^n \bar{\mu}a_n)\rho(b, c) \le \epsilon. \end{aligned}$$

In the other direction, given $\epsilon > 0$, take $n \in \mathbb{N}$ such that $2^{-n} \le \frac{1}{2}\epsilon$; then $\rho(b, c) \le \epsilon$ whenever $\rho_{a_n}(b, c) \le \epsilon/2(n + 1)$.

This shows that $\mathcal{U}$ is the same as the metrizable uniformity defined by $\{\rho\}$; accordingly $\mathfrak{T}$ also is defined by ρ.

(ii) Now suppose that $\mathfrak{T}$ is metrizable, and let ρ be a metric defining $\mathfrak{T}$. For each $n \in \mathbb{N}$ there must be $a_{n0}, \dots, a_{nk_n} \in \mathfrak{A}^f$ and $\delta_n > 0$ such that

$$\rho_{a_{ni}}(b, 1) \le \delta_n \text{ for every } i \le k_n \implies \rho(b, 1) \le 2^{-n}.$$

Set $d = \sup_{n\in\mathbb{N}, i\le k_n} a_{ni}$. Then $\rho_{a_{ni}}(d, 1) = 0$ for every n and i, so $\rho(d, 1) \le 2^{-n}$ for every n and $d = 1$. Thus 1 is the supremum of countably many elements of finite measure and $(\mathfrak{A}, \bar{\mu})$ is σ-finite.

(c)(i) Suppose that $(\mathfrak{A}, \bar{\mu})$ is localizable. Then $\mathfrak{T}$ is Hausdorff, by (a). Let $\mathcal{F}$ be a Cauchy filter on $\mathfrak{A}$. For each $a \in \mathfrak{A}^f$, choose a sequence $\langle F_n(a) \rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\rho_a(b, c) \le 2^{-n}$ whenever b, $c \in F_n(a)$ and $n \in \mathbb{N}$. Choose $c_{an} \in \bigcap_{k\le n} F_k(a)$ for each n; then $\rho_a(c_{an}, c_{a,n+1}) \le 2^{-n}$ for each n. Set $d_a = \sup_{n\in\mathbb{N}} \inf_{k\ge n} a \cap c_{ak}$. Then

$$\lim_{n\to\infty} \rho_a(d_a, c_{an}) = \lim_{n\to\infty} \bar{\mu}(d_a \triangle (a \cap c_{an})) = 0,$$

by 323F.

If a, $b \in \mathfrak{A}^f$ and $a \subseteq b$, then $d_a = a \cap d_b$. **P** For each $n \in \mathbb{N}$, $F_n(a)$ and $F_n(b)$ both belong to $\mathcal{F}$, so must have a point e in common; now

$$\begin{aligned}\rho_a(d_a, d_b) &\le \rho_a(d_a, c_{an}) + \rho_a(c_{an}, e) + \rho_a(e, c_{bn}) + \rho_a(c_{bn}, d_b)\\ &\le \rho_a(d_a, c_{an}) + \rho_a(c_{an}, e) + \rho_b(e, c_{bn}) + \rho_b(c_{bn}, d_b)\\ &\le \rho_a(d_a, c_{an}) + 2^{-n} + 2^{-n} + \rho_b(c_{bn}, d_b)\\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Consequently $\rho_a(d_a, d_b) = 0$, that is,

$$d_a = a \cap d_a = a \cap d_b. \text{ **Q**}$$

Set $d = \sup\{d_b : b \in \mathfrak{A}^f\}$; this is defined because $\mathfrak{A}$ is Dedekind complete. Then $\mathcal{F} \to d$. **P** If $a \in \mathfrak{A}^f$ and $\epsilon > 0$, then

$$a \cap d = \sup_{b \in \mathfrak{A}^f} a \cap d_b = \sup_{b \in \mathfrak{A}^f} a \cap b \cap d_{a \cup b} = \sup_{b \in \mathfrak{A}^f} a \cap b \cap d_a = a \cap d_a.$$

So if we choose $n \in \mathbb{N}$ such that $2^{-n} + \rho_a(c_{an}, d_a) \le \epsilon$, then for any $e \in F_n(a)$ we shall have

$$\rho_a(e, d) \le \rho_a(e, c_{an}) + \rho_a(c_{an}, d) \le 2^{-n} + \rho_a(c_{an}, d_a) \le \epsilon.$$

Thus

$$\{e : \rho_a(d, e) \le \epsilon\} \supseteq F_n(a) \in \mathcal{F}.$$

As a, ϵ are arbitrary, $\mathcal{F}$ converges to d. **Q** As $\mathcal{F}$ is arbitrary, $\mathfrak{A}$ is complete.

(ii) Now suppose that $\mathfrak{T}$ is Hausdorff and that $\mathfrak{A}$ is complete under $\mathcal{U}$. By (a), $(\mathfrak{A}, \bar\mu)$ is semi-finite. Let B be any non-empty subset of $\mathfrak{A}$, and set $B' = \{b_0 \cup \ldots \cup b_n : b_0, \ldots, b_n \in B\}$, so that B' is upwards-directed and has the same upper bounds as B. By 323Da, we have a Cauchy filter $\mathcal{F}(B'\uparrow)$; because $\mathfrak{A}$ is complete, this is convergent; and because $(\mathfrak{A}, \bar\mu)$ is semi-finite, its limit must be $\sup B' = \sup B$, by 323Dd. As B is arbitrary, $\mathfrak{A}$ is Dedekind complete, so $(\mathfrak{A}, \bar\mu)$ is localizable.

323H Closed subalgebras The ideas used in the proof of (c) above have many other applications, of which one of the most important is the following. You may find it helpful to read the next theorem first on the assumption that $(\mathfrak{A}, \bar\mu)$ is a probability algebra.

Theorem Let $(\mathfrak{A}, \bar\mu)$ be a localizable measure algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Then it is topologically closed iff it is order-closed.

proof (a) If $\mathfrak{B}$ is closed, it must be order-closed, by 323Dc.

(b) Now suppose that $\mathfrak{B}$ is order-closed. I repeat the ideas of part (c-i) of the proof of 323G. Let e be any member of the closure of $\mathfrak{B}$ in $\mathfrak{A}$. For each $a \in \mathfrak{A}^f$ and $n \in \mathbb{N}$ choose $c_{an} \in \mathfrak{B}$ such that $\rho_a(c_{an}, e) \le 2^{-n}$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} \bar\mu((a \cap c_{an}) \triangle (a \cap c_{a,n+1})) &= \sum_{n=0}^{\infty} \rho_a(c_{an}, c_{a,n+1})\\ &\le \sum_{n=0}^{\infty} \rho_a(c_{an}, e) + \rho_a(e, c_{a,n+1}) < \infty.\end{aligned}$$

So if we set $e_a = \sup_{n \in \mathbb{N}} \inf_{k \ge n} c_{ak}$, then

$$\rho_a(e_a, c_{an}) = \rho_a(a \cap e_a, a \cap c_{an}) \to 0$$

as $n \to \infty$, by 323F, and $\rho_a(e, e_a) = 0$, that is, $a \cap e_a = a \cap e$. Also, because $\mathfrak{B}$ is order-closed, $\inf_{k \ge n} c_{ak} \in \mathfrak{B}$ for every n, and $e_a \in \mathfrak{B}$.

Because $\mathfrak{A}$ is Dedekind complete, we can set

$$e'_a = \inf\{e_b : b \in \mathfrak{A}^f,\ a \subseteq b\};$$

then $e'_a \in \mathfrak{B}$ and

$$e'_a \cap a = \inf_{b \supseteq a} e_b \cap a = \inf_{b \supseteq a} e_b \cap b \cap a = \inf_{b \supseteq a} e \cap b \cap a = e \cap a.$$

Now $e'_a \subseteq e'_b$ whenever $a \subseteq b$, so $B = \{e'_a : a \in \mathfrak{A}^f\}$ is upwards-directed, and

$$\sup B = \sup\{e'_a \cap a : a \in \mathfrak{A}^f\} = \sup\{e \cap a : a \in \mathfrak{A}^f\} = e$$

because $(\mathfrak{A}, \bar{\mu})$ is semi-finite. Accordingly $e \in \mathfrak{B}$. As e is arbitrary, $\mathfrak{B}$ is closed, as claimed.

323I Notation In the context of 323H, I will say simply that $\mathfrak{B}$ is a **closed subalgebra** of $\mathfrak{A}$.

323J Proposition If $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then the topological closure $\overline{\mathfrak{B}}$ of $\mathfrak{B}$ in $\mathfrak{A}$ is precisely the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$.

proof Write $\mathfrak{B}_\tau$ for the smallest order-closed subset of $\mathfrak{A}$ including $\mathfrak{B}$. By 313Gc, $\mathfrak{B}_\tau$ is a subalgebra of $\mathfrak{A}$, and is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$. Being an order-closed subalgebra of $\mathfrak{A}$, it is topologically closed, by 323H, and must include $\overline{\mathfrak{B}}$. On the other hand, $\overline{\mathfrak{B}}$, being topologically closed, is order-closed (323D(c-i)), so includes $\mathfrak{B}_\tau$. Thus $\overline{\mathfrak{B}} = \mathfrak{B}_\tau$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$.

323K I note some simple results for future reference.

Lemma If $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then for any $a \in \mathfrak{A}$ the subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{a\}$ is closed.

proof By 314Ja, $\mathfrak{C}$ is order-closed.

323L Proposition Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a family of measure algebras with simple product $(\mathfrak{A}, \bar{\mu})$ (322K). Then the measure-algebra topology on $\mathfrak{A} = \prod_{i\in I} \mathfrak{A}_i$ defined by $\bar{\mu}$ is just the product of the measure-algebra topologies of the $\mathfrak{A}_i$.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A. Write $\mathfrak{T}$ for the topology of $\mathfrak{A}$ and $\mathfrak{S}$ for the product topology. For $i \in I$ and $d \in \mathfrak{A}_i^f$ define a pseudometric $\tilde{\rho}_{di}$ on $\mathfrak{A}$ by setting

$$\tilde{\rho}_{di}(b, c) = \rho_d(b(i), c(i))$$

whenever b, $c \in \mathfrak{A}$; then $\mathfrak{S}$ is defined by $\mathrm{P} = \{\tilde{\rho}_{di} : i \in I,\ a \in \mathfrak{A}_i^f\}$ (3A3Ig). Now each $\tilde{\rho}_{di}$ is one of the defining pseudometrics for $\mathfrak{T}$, since

$$\tilde{\rho}_{di}(b, c) = \bar{\mu}(\tilde{d} \cap (b \triangle c))$$

where $\tilde{d}(i) = d$, $\tilde{d}(j) = 0$ for $j \neq i$. So $\mathfrak{S} \subseteq \mathfrak{T}$.

Now suppose that $a \in \mathfrak{A}^f$ and $\epsilon > 0$. Then $\sum_{i\in I} \bar{\mu}_i a(i) = \bar{\mu}a$ is finite, so there is a finite set $J \subseteq I$ such that $\sum_{i\in I\setminus J} \bar{\mu}_i a(i) \le \frac{1}{2}\epsilon$. For each $j \in J$, $\tau_j = \tilde{\rho}_{a(j),j}$ belongs to P, and

$$\begin{aligned}\rho_a(b, c) &= \sum_{i\in I} \bar{\mu}_i(a(i) \cap (b(i) \triangle c(i))) \\ &\le \sum_{j\in J} \bar{\mu}_j(a(j) \cap (b(j) \triangle c(j))) + \frac{1}{2}\epsilon = \sum_{j\in J} \tau_j(b, c) + \frac{1}{2}\epsilon \le \epsilon\end{aligned}$$

whenever b, c are such that $\tau_j(b, c) \le \epsilon/(1 + 2\#(J))$ for every $j \in J$. By 2A3H, the identity map from $(\mathfrak{A}, \mathfrak{S})$ to $(\mathfrak{A}, \mathfrak{T})$ is continuous, that is, $\mathfrak{T} \subseteq \mathfrak{S}$.

Putting these together, we see that $\mathfrak{S} = \mathfrak{T}$, as claimed.

***323M** In this volume we shall have little need to consider the measure metric on $\mathfrak{A}^f$, but the following facts are sometimes useful.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and give $\mathfrak{A}^f$ its measure metric.

(a) The Boolean operations $\triangle$, $\cap$, $\cup$ and $\setminus$ on $\mathfrak{A}^f$ are uniformly continuous.

(b) $\bar{\mu}\restriction\mathfrak{A}^f : \mathfrak{A}^f \to [0, \infty[$ is 1-Lipschitz, therefore uniformly continuous.

(c) $\mathfrak{A}^f$ is complete.

proof (a) Writing ρ for the measure metric on $\mathfrak{A}^f$, then, just as in the proof of 323Ba,

$$\rho(b * c, b' * c') \le \rho(b, b') + \rho(c, c')$$

for all b, c, b', $c' \in \mathfrak{A}^f$ and any of the Boolean operations $* = \triangle, \cap, \cup$ and $\setminus$.

(b) If a, $b \in \mathfrak{A}^f$ then

$$|\bar{\mu}a - \bar{\mu}b| \le |\bar{\mu}a - \bar{\mu}(a \cap b)| + |\bar{\mu}b - \bar{\mu}(a \cap b)| = \bar{\mu}(a \setminus b) + \bar{\mu}(b \setminus a) = \rho(a, b).$$

(c) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}^f$ such that $\sum_{n=0}^{\infty} \rho(a_n, a_{n+1}) < \infty$, set $d = \sup_{n\in\mathbb{N}} \inf_{m\ge n} a_m$. By 323F, $\lim_{n\to\infty} \bar{\mu}(d \triangle a_n) = 0$. In particular, there is some $n \in \mathbb{N}$ such that $\bar{\mu}(d \setminus a_n)$ is finite, so $d \in \mathfrak{A}^f$ and $\lim_{n\to\infty} \rho(d, a_n) = 0$. As in 2A4E, this is enough to show that $\mathfrak{A}^f$ is complete.

323X Basic exercises >(a) Let (X, Σ, μ) be a measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. (i) Show that we have an injection $\chi : \mathfrak{A} \to L^0(\mu)$ (see §241) given by setting $\chi(E^\bullet) = (\chi E)^\bullet$ for every $E \in \Sigma$. (ii) Show that χ is a homeomorphism between $\mathfrak{A}$ and its image if $\mathfrak{A}$ is given its measure-algebra topology and $L^0(\mu)$ is given its topology of convergence in measure (245A).

>(b) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and ρ the measure metric on the ideal $\mathfrak{A}^f$ of elements of finite measure. (i) Show that the embedding $\mathfrak{A}^f \subseteq \mathfrak{A}$ is uniformly continuous for the measure-algebra uniformity on $\mathfrak{A}$. (ii) In the context of 323Xa, show that $\chi : \mathfrak{A}^f \to L^0(\mu)$ is an isometry between $\mathfrak{A}^f$ and a subset of $L^1(\mu)$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ be a semi-finite measure algebra. Show that the set $\{(a, b) : a \subseteq b\}$ is a closed set in $\mathfrak{A} \times \mathfrak{A}$.

>(d) Let (X, Σ, μ) be a σ-finite measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. (i) Show that if T is a σ-subalgebra of Σ, then $\{F^\bullet : F \in \mathrm{T}\}$ is a closed subalgebra of $\mathfrak{A}$. (ii) Show that if $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$, then $\{F : F \in \Sigma, F^\bullet \in \mathfrak{B}\}$ is a σ-subalgebra of Σ.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra, and $C \subseteq \mathfrak{A}$ a set such that $\sup A$, $\inf A$ belong to C for all non-empty subsets A of C. Show that C is topologically closed.

(f) Show that if $(\mathfrak{A}, \bar{\mu})$ is any measure algebra and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then its topological closure $\overline{\mathfrak{B}}$ is again a subalgebra.

(g) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $e \in \mathfrak{A}$; let $\mathfrak{A}_e$ be the principal ideal of $\mathfrak{A}$ generated by e, and $\bar{\mu}_e$ its measure (322H). (i) Show that the measure-algebra topology on $\mathfrak{A}_e$ defined by $\bar{\mu}_e$ is just the subspace topology induced by the measure-algebra topology of $\mathfrak{A}$. (ii) Show that the measure-algebra uniformity on $\mathfrak{A}_e$ is the subspace uniformity induced by the measure-algebra uniformity of $\mathfrak{A}$. (iii) Show that the strong measure-algebra topology on $\mathfrak{A}_e^f$ is the subspace topology induced by the strong measure-algebra topology of $\mathfrak{A}^f$.

(h) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra. Show that its localization (322P) can be identified with its completion under its measure-algebra uniformity.

323Y Further exercises (a) Let $(\mathfrak{A}, \bar{\mu})$ be a σ-finite measure algebra. Show that a set $F \subseteq \mathfrak{A}$ is topologically closed iff $e \in F$ whenever there are non-empty sets B, $C \subseteq \mathfrak{A}$ such that B is upwards-directed, C is downwards-directed, $\sup B = \inf C = e$ and $[b, c] \cap F \ne \emptyset$ for every $b \in B$, $c \in C$, writing $[b, c] = \{d : b \subseteq d \subseteq c\}$.

(b) Give an example to show that (a) is false for general localizable measure algebras.

(c) Give an example of a semi-finite measure algebra $(\mathfrak{A}, \bar{\mu})$ with an order-closed subalgebra which is not topologically closed.

(d) Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra and write $\mathbb{B}$ for the family of closed subalgebras of $\mathfrak{A}$. For $\mathfrak{B}$, $\mathfrak{C}\in\mathbb{B}$ set $\rho(\mathfrak{B},\mathfrak{C})=\max(\sup_{b\in\mathfrak{B}}\inf_{c\in\mathfrak{C}}\bar{\mu}(b\,\triangle\,c),\sup_{c\in\mathfrak{C}}\inf_{b\in\mathfrak{B}}\bar{\mu}(b\,\triangle\,c))$. Show that $(\mathbb{B},\rho)$ is a complete metric space. (Cf. 246Yb, 4A2T.)

(e) Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of Lebesgue measure on $\mathbb{R}$. Show that it is separable in its measure-algebra topology. (*Hint*: 245Yj.)

323 Notes and comments The message of this section is that the topology of a measure algebra is essentially defined by its order and algebraic structure; see also 324F-324H below. Of course the results are really about semi-finite measure algebras, and indeed this whole volume, like the rest of measure theory, has little of interest to say about others; they are included only because they arise occasionally and it is not absolutely essential to exclude them. We therefore expect to be able to describe such things as closed subalgebras and continuous homomorphisms in terms of the ordering, as in 323H and 324G. For σ-finite algebras, indeed, there is an easy description of the topology in terms of the order (323Ya). I think the result of this section on which I shall most often depend is 323H: in most contexts, there is no need to distinguish between 'topologically closed subalgebra' and 'order-closed subalgebra'. However a σ-subalgebra of a localizable measure algebra need not be topologically sequentially closed; there is an example in Fremlin Pagter & Ricker 05.

It is also the case that the topology of a measure algebra corresponds very closely indeed to the topology of convergence in measure. A description of this correspondence is in 323Xa. Indeed all the results of this section have analogues in the theory of topological Riesz spaces. I will enlarge on the idea here in §367. For the moment, however, if you look back to Chapter 24, you will see that 323B and 323G are closely paralleled by 245D and 245E, while 323Ya is related to 245L.

It is I think natural to ask whether there are any other topological Boolean algebras with the properties 323B-323D. In fact there are; see 393G and 393Xf below.

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324 Homomorphisms

In the course of Volume 2, I had occasion to remark that elementary measure theory is unusual among abstract topics in pure mathematics in not being dominated by any particular class of structure-preserving operators. We now come to what I think is one of the reasons for the gap: the most important operators of the theory are not between measure spaces at all, but between their measure algebras. In this section I run through the most elementary facts about Boolean homomorphisms between measure algebras. I start with results on the construction of such homomorphisms from functions between measure spaces (324A-324E), then investigate continuity and order-continuity of homomorphisms (324F-324H) before turning to measure-preserving homomorphisms (324I-324P).

324A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $(\mathfrak{A},\bar{\mu})$, $(\mathfrak{B},\bar{\nu})$ their measure algebras. Write $\hat{\Sigma}$ for the domain of the completion $\hat{\mu}$ of μ. Let $D\subseteq X$ be a set of full outer measure (definition: 132F), and $\hat{\Sigma}_D$ the subspace σ-algebra on D induced by $\hat{\Sigma}$ (121A). Let $\phi:D\to Y$ be a function such that $\phi^{-1}[F]\in\hat{\Sigma}_D$ for every $F\in\mathrm{T}$ and $\phi^{-1}[F]$ is μ-negligible whenever $\nu F=0$. Then there is a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by the formula

$$\pi F^\bullet=E^\bullet \text{ whenever } F\in\mathrm{T},\ E\in\Sigma \text{ and } (E\cap D)\triangle\phi^{-1}[F] \text{ is negligible.}$$

proof Let $F\in\mathrm{T}$. Then there is an $H\in\hat{\Sigma}$ such that $H\cap D=\phi^{-1}[F]$; now there is an $E\in\Sigma$ such that $E\triangle H$ is negligible, so that $(E\cap D)\triangle\phi^{-1}[F]$ is negligible. If E_1 is another member of Σ such that $(E_1\cap D)\triangle\phi^{-1}[F]$ is negligible, then $(E\triangle E_1)\cap D$ is negligible, so is included in a negligible member G of Σ. Since $(E\triangle E_1)\setminus G$ belongs to Σ and is disjoint from D, it is negligible; accordingly $E\triangle E_1$ is negligible and $E^\bullet=E_1^\bullet$ in $\mathfrak{A}$.

What this means is that the formula offered defines a map $\pi:\mathfrak{B}\to\mathfrak{A}$. It is now easy to check that π is a Boolean homomorphism, because if

$$(E\cap D)\triangle\phi^{-1}[F],\quad (E'\cap D)\triangle\phi^{-1}[F']$$

are negligible, so are

$$((E\cap E')\cap D)\triangle\phi^{-1}[F\cap F'],\quad ((X\setminus E)\cap D)\triangle\phi^{-1}[Y\setminus F],$$

and we can apply 312H.

To see that π is sequentially order-continuous, let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}$. For each n we may choose an $F_n\in\mathrm{T}$ such that $F_n^\bullet=b_n$, and $E_n\in\Sigma$ such that $(E_n\cap D)\triangle\phi^{-1}[F_n]$ is negligible; now, setting $F=\bigcup_{n\in\mathbb{N}}F_n$, $E=\bigcup_{n\in\mathbb{N}}E_n$,

$$(E\cap D)\triangle\phi^{-1}[F]\subseteq\bigcup_{n\in\mathbb{N}}(E_n\cap D)\triangle\phi^{-1}[F_n]$$

is negligible, so

$$\pi(\sup_{n\in\mathbb{N}}b_n)=\pi(F^\bullet)=E^\bullet=\sup_{n\in\mathbb{N}}E_n^\bullet=\sup_{n\in\mathbb{N}}\pi b_n.$$

(Recall that the maps $E\mapsto E^\bullet$, $F\mapsto F^\bullet$ are sequentially order-continuous, by 321H.) So π is sequentially order-continuous (313L(c-iii)).

324B Corollary Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ their measure algebras. Let $\phi:X\to Y$ be a function such that $\phi^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$ and $\mu\phi^{-1}[F]=0$ whenever $\nu F=0$. Then there is a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by the formula

$$\pi F^\bullet=(\phi^{-1}[F])^\bullet \text{ for every } F\in\mathrm{T}.$$

324C Remarks (a) In §235 and elsewhere in Volume 2 I spent a good deal of time on functions between measure spaces which satisfy the conditions of 324A. Indeed, I take the trouble to spell 324A out in such generality just in order to catch these applications. Some of the results of the present chapter (322D, 322Jb) can also be regarded as special cases of 324A.

(b) The question of which homomorphisms between the measure algebras of measure spaces (X,Σ,μ), (Y,T,ν) can be realized by functions between X and Y is important and deep; I will return to it in §§343-344.

(c) In the simplified context of 324B, I have actually defined a contravariant functor; the relevant facts are the following.

324D Proposition Let (X,Σ,μ), (Y,T,ν) and (Z,Λ,λ) be measure spaces, with measure algebras $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$, $(\mathfrak{C},\bar\lambda)$. Suppose that $\phi:X\to Y$ and $\psi:Y\to Z$ satisfy the conditions of 324B, that is,

$$\phi^{-1}[F]\in\Sigma \text{ if } F\in\mathrm{T},\quad \mu\phi^{-1}[F]=0 \text{ if } \nu F=0,$$

$$\psi^{-1}[G]\in\mathrm{T} \text{ if } G\in\Lambda,\quad \mu\psi^{-1}[G]=0 \text{ if } \lambda G=0.$$

Let $\pi_\phi:\mathfrak{B}\to\mathfrak{A}$, $\pi_\psi:\mathfrak{C}\to\mathfrak{B}$ be the corresponding homomorphisms. Then $\psi\phi:X\to Z$ is another map of the same type, and $\pi_{\psi\phi}=\pi_\phi\pi_\psi:\mathfrak{C}\to\mathfrak{A}$.

proof The necessary checks are all elementary.

324E Stone spaces While in the context of general measure spaces the question of realizing homomorphisms is difficult, in the case of the Stone representation it is relatively straightforward.

Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, with Stone spaces Z and W; let μ, ν be the corresponding measures on Z and W, as described in 321J-321K, and Σ, T their domains. If $\pi:\mathfrak{B}\to\mathfrak{A}$ is any order-continuous Boolean homomorphism, let $\phi:Z\to W$ be the corresponding continuous function, as described in 312Q. Then $\phi^{-1}[F]\in\Sigma$ for every $F\in\mathrm{T}$, $\mu\phi^{-1}[F]=0$ whenever $\nu F=0$, and (writing E^* for the member of $\mathfrak{A}$ corresponding to $E\in\Sigma$) $\pi F^*=(\phi^{-1}[F])^*$ for every $F\in\mathrm{T}$.

proof Recall that $E^* = a$ iff $E\triangle\widehat{a}$ is meager, where $\widehat{a}$ is the open-and-closed subset of Z corresponding to $a \in \mathfrak{A}$. In particular, $\mu E = 0$ iff E is meager. Now the point is that $\phi^{-1}[F]$ is nowhere dense in Z whenever F is a nowhere dense subset of W, by 313R. Consequently $\phi^{-1}[F]$ is meager whenever F is meager in W, since F is then just a countable union of nowhere dense sets. Thus we see already that $\mu\phi^{-1}[F] = 0$ whenever $\nu F = 0$. If F is any member of T, there is an open-and-closed set F_0 such that $F\triangle F_0$ is meager; now $\phi^{-1}[F_0]$ is open-and-closed, so $\phi^{-1}[F] = \phi^{-1}[F_0]\triangle\phi^{-1}[F\triangle F_0]$ belongs to Σ. Moreover, if $b \in \mathfrak{B}$ is such that $\widehat{b} = F_0$, and $a = \pi b$, then $\widehat{a} = \phi^{-1}[F_0]$, so

$$\pi F^* = \pi b = a = (\phi^{-1}[F_0])^* = (\phi^{-1}[F])^*,$$

as required.

324F I turn now to the behaviour of order-continuous homomorphisms between measure algebras.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism. Give $\mathfrak{A}$ and $\mathfrak{B}$ their measure-algebra topologies and uniformities (323Ab).

(a) π is continuous iff it is continuous at 0 iff it is uniformly continuous.

(b) If $(\mathfrak{B}, \bar{\nu})$ is semi-finite and π is continuous, then it is order-continuous.

(c) If $(\mathfrak{A}, \bar{\mu})$ is semi-finite and π is order-continuous, then it is continuous.

proof I use the notations $\mathfrak{A}^f$, ρ_a from 323A.

(a) Suppose that π is continuous at 0; I seek to show that it is uniformly continuous. Take $b \in \mathfrak{B}^f$ and $\epsilon > 0$. Then there are $a_0, \dots , a_n \in \mathfrak{A}^f$ and $\delta > 0$ such that

$$\bar{\nu}(b \cap \pi c) = \rho_b(\pi c, 0) \le \epsilon \text{ whenever } \max_{i\le n} \rho_{a_i}(c, 0) \le \delta;$$

setting $a = \sup_{i\le n} a_i$,

$$\bar{\nu}(b \cap \pi c) \le \epsilon \text{ whenever } \bar{\mu}(a \cap c) \le \delta.$$

Now suppose that $\rho_a(c, c') \le \delta$. Then $\bar{\mu}(a \cap (c \triangle c')) \le \delta$, so

$$\rho_b(\pi c, \pi c') = \bar{\nu}(b \cap (\pi c \triangle \pi c')) = \bar{\nu}(b \cap \pi(c \triangle c')) \le \epsilon.$$

As b and ϵ are arbitrary, π is uniformly continuous. The rest of the implications are elementary.

(b) Let A be a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0. Then $0 \in \overline{A}$ (323D(b-ii)); because π is continuous, $0 \in \overline{\pi[A]}$. **?** If b is a non-zero lower bound for $\pi[A]$ in $\mathfrak{B}$, then (because $(\mathfrak{B}, \bar{\nu})$ is semi-finite) there is a $c \subseteq b$ with $0 < \bar{\nu}c < \infty$; now

$$\rho_c(\pi a, 0) = \bar{\nu}(c \cap \pi a) = \bar{\nu}c > 0$$

for every $a \in A$, so $0 \notin \overline{\pi[A]}$. **X**

Thus $\inf \pi[A] = 0$ in $\mathfrak{B}$; as A is arbitrary, π is order-continuous (313L(b-ii)).

(c) By (a), it will be enough to show that π is continuous at 0. Take $b \in \mathfrak{B}^f$ and $\epsilon > 0$. **?** Suppose, if possible, that whenever $a \in \mathfrak{A}^f$ and $\delta > 0$ there is a $c \in \mathfrak{A}$ such that $\bar{\mu}(a \cap c) \le \delta$ but $\bar{\nu}(b \cap \pi c) \ge \epsilon$. For each $a \in \mathfrak{A}^f$, $n \in \mathbb{N}$ choose c_{an} such that $\bar{\mu}(a \cap c_{an}) \le 2^{-n}$ but $\bar{\nu}(b \cap \pi c_{an}) \ge \epsilon$. Set $c_a = \inf_{n\in\mathbb{N}} \sup_{m\ge n} c_{am}$; then

$$\bar{\mu}(a \cap c_a) \le \inf_{n\in\mathbb{N}} \sum_{m=n}^{\infty} \bar{\mu}(a \cap c_{an}) = 0,$$

so $c_a \cap a = 0$. On the other hand, because π is order-continuous, $\pi c_a = \inf_{n\in\mathbb{N}} \sup_{m\ge n} \pi c_{am}$, so that

$$\bar{\nu}(b \cap \pi c_a) = \lim_{n\to\infty} \bar{\nu}(b \cap \sup_{m\ge n} \pi c_{am}) \ge \epsilon.$$

This shows that

$$\rho_b(\pi(1 \setminus a), 0) = \bar{\nu}(b \cap \pi(1 \setminus a)) \ge \bar{\nu}(b \cap \pi c_a) \ge \epsilon.$$

But now observe that $A = \{1 \setminus a : a \in \mathfrak{A}^f\}$ is a downwards-directed subset of $\mathfrak{A}$ with infimum 0, because $(\mathfrak{A}, \bar{\mu})$ is semi-finite. So $\pi[A]$ is downwards-directed and has infimum 0, and 0 must be in the closure of $\pi[A]$, by 323D(b-ii) again; while we have just seen that $\rho_b(d, 0) \ge \epsilon$ for every $d \in \pi[A]$. **X**

Thus there must be $a \in \mathfrak{A}^f$, $\delta > 0$ such that

$$\rho_b(\pi c, 0) = \bar{\nu}(b \cap \pi c) \le \epsilon$$

whenever

$$\rho_a(c,0)=\bar\mu(a\cap c)\le\delta.$$

As b and ϵ are arbitrary, π is continuous at 0 and therefore continuous.

324G Corollary If $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are semi-finite measure algebras, a Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ is continuous iff it is order-continuous.

324H Corollary If $\mathfrak{A}$ is a Boolean algebra and $\bar\mu$, $\bar\nu$ are two measures both rendering $\mathfrak{A}$ a semi-finite measure algebra, then they endow $\mathfrak{A}$ with the same uniformity (and, of course, the same topology).

proof By 324G, the identity map from $\mathfrak{A}$ to itself is continuous whichever of the topologies we place on $\mathfrak{A}$; and by 324Fa it is therefore uniformly continuous.

324I Definition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras. A Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}$ is **measure-preserving** if $\bar\nu(\pi a)=\bar\mu a$ for every $a\in\mathfrak{A}$.

324J Proposition Let $(\mathfrak{A},\bar\mu)$, $(\mathfrak{B},\bar\nu)$ and $(\mathfrak{C},\bar\lambda)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$, $\theta:\mathfrak{B}\to\mathfrak{C}$ measure-preserving Boolean homomorphisms. Then $\theta\pi:\mathfrak{A}\to\mathfrak{C}$ is a measure-preserving Boolean homomorphism.

proof Elementary.

324K Proposition Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a measure-preserving Boolean homomorphism.

(a) π is injective.

(b) $(\mathfrak{A},\bar\mu)$ is totally finite iff $(\mathfrak{B},\bar\nu)$ is, and in this case π is order-continuous, therefore continuous, and $\pi[\mathfrak{A}]$ is a closed subalgebra of $\mathfrak{B}$.

(c) If $(\mathfrak{A},\bar\mu)$ is semi-finite and $(\mathfrak{B},\bar\nu)$ is σ-finite, then $(\mathfrak{A},\bar\mu)$ is σ-finite.

(d) If $(\mathfrak{A},\bar\mu)$ is σ-finite and π is sequentially order-continuous, then $(\mathfrak{B},\bar\nu)$ is σ-finite.

(e) If $(\mathfrak{A},\bar\mu)$ is semi-finite and π is order-continuous, then $(\mathfrak{B},\bar\nu)$ is semi-finite.

(f) If $(\mathfrak{A},\bar\mu)$ is atomless and semi-finite, and π is order-continuous, then $\mathfrak{B}$ is atomless.

(g) If $\mathfrak{B}$ is purely atomic and $(\mathfrak{A},\bar\mu)$ is semi-finite, then $\mathfrak{A}$ is purely atomic.

proof (a) If $a\ne0$ in $\mathfrak{A}$, then $\bar\nu\pi a=\bar\mu a>0$ so $\pi a\ne0$. By 3A2Db, π is injective.

(b) Because

$$\bar\nu 1_{\mathfrak{B}}=\bar\nu\pi 1_{\mathfrak{A}}=\bar\mu 1_{\mathfrak{A}},$$

$(\mathfrak{A},\bar\mu)$ is totally finite iff $(\mathfrak{B},\bar\nu)$ is. Now suppose that $A\subseteq\mathfrak{A}$ is downwards-directed and non-empty and that $\inf A=0$. Then

$$\inf_{a\in A}\bar\nu\pi a=\inf_{a\in A}\bar\mu a=0$$

by 321F. So $\bar\nu b=0$ for any lower bound b of $\pi[A]$, and $\inf\pi[A]=0$. As A is arbitrary, π is order-continuous, by 313Lb again.

By 324Fc, π is continuous. By 314Fa, $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$, that is, 'closed' in the sense of 323I.

(c) I appeal to 322G. If C is a disjoint family in $\mathfrak{A}\setminus\{0\}$, then $\langle\pi c\rangle_{c\in C}$ is a disjoint family in $\mathfrak{B}\setminus\{0\}$, so is countable, and C must be countable, because π is injective. Thus $\mathfrak{A}$ is ccc and (being semi-finite) $(\mathfrak{A},\bar\mu)$ is σ-finite.

(d) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\bar\mu a_n<\infty$ for every n and $\sup_{n\in\mathbb{N}}a_n=1$. Then $\bar\nu\pi a_n<\infty$ for every n and (because π is sequentially order-continuous) $\sup_{n\in\mathbb{N}}\pi a_n=1$, so $(\mathfrak{B},\bar\nu)$ is σ-finite.

(e) Setting $\mathfrak{A}^f=\{a:\bar\mu a<\infty\}$, $\sup\mathfrak{A}^f=1$; because π is order-continuous, $\sup\pi[\mathfrak{A}^f]=1$ in $\mathfrak{B}$. So if $\bar\nu b=\infty$, there is an $a\in\mathfrak{A}^f$ such that $\pi a\cap b\ne0$, and now $0<\bar\nu(b\cap\pi a)<\infty$.

(f) Take any non-zero $b\in\mathfrak{B}$. As in (e), there is an $a\in\mathfrak{A}$ such that $\bar\mu a<\infty$ and $\pi a\cap b\ne0$. If $\pi a\cap b\ne b$, then surely b is not an atom. Otherwise, set

$$C = \{c : c \in \mathfrak{A},\, c \subseteq a,\, b \subseteq \pi c\}.$$

Then C is downwards-directed and contains a, so $c_0 = \inf C$ is defined in $\mathfrak{A}$ (321F again), and

$$\bar{\mu}c_0 = \inf_{c\in C} \bar{\mu}c \ge \bar{\nu}b > 0,$$

so $c_0 \ne 0$. Because $\mathfrak{A}$ is atomless, there is a $d \subseteq c_0$ such that neither d nor $c_0 \setminus d$ is zero, so that neither $c_0 \setminus d$ nor d can belong to C. But this means that $b \cap \pi d$ and $b \cap \pi(c_0 \setminus d)$ are both non-zero, so that again b is not an atom. As b is arbitrary, $\mathfrak{B}$ is atomless.

(g) Take any non-zero $a \in \mathfrak{A}$. Then there is an $a' \subseteq a$ such that $0 < \bar{\mu}a' < \infty$. Because $\mathfrak{B}$ is purely atomic, there is an atom b of $\mathfrak{B}$ with $b \subseteq \pi a'$. Set

$$C = \{c : c \in \mathfrak{A},\, c \subseteq a',\, b \subseteq \pi c\}.$$

Then C is downwards-directed and contains a', so $c_0 = \inf C$ is defined in $\mathfrak{A}$, and

$$\bar{\mu}c_0 = \inf_{c\in C} \bar{\mu}c \ge \bar{\nu}b > 0,$$

so $c_0 \ne 0$. If $d \subseteq c_0$, then $b \cap \pi d$ must be either b or 0. If $b \cap \pi d = b$, then $d \in C$ and $d = c_0$. If $b \cap \pi d = 0$, then $c_0 \setminus d \in C$ and $d = 0$. Thus c_0 is an atom in $\mathfrak{A}$. As a is arbitrary, $\mathfrak{A}$ is purely atomic.

324L Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra, $(\mathfrak{B}, \bar{\nu})$ a measure algebra, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the closed subalgebra of $\mathfrak{A}$ generated by C, then $\pi[\mathfrak{C}]$ is the closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

proof By 324Ka, π is order-continuous, so we can apply 314H.

324M Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with measure algebras $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$. Let $\phi : X \to Y$ be inverse-measure-preserving. Then we have a sequentially order-continuous measure-preserving Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{A}$ defined by setting $\pi F^\bullet = \phi^{-1}[F]^\bullet$ for every $F \in \mathrm{T}$.

proof This is immediate from 324B.

324N Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, with Stone spaces Z and W; let μ, ν be the corresponding measures on Z and W. If $\pi : \mathfrak{B} \to \mathfrak{A}$ is an order-continuous measure-preserving Boolean homomorphism, and $\phi : Z \to W$ the corresponding continuous function, then ϕ is inverse-measure-preserving.

proof Use 324E. In the notation there, if $F \in \mathrm{T}$, then

$$\nu F = \bar{\nu}F^* = \bar{\mu}\pi F^* = \bar{\mu}\phi^{-1}[F]^* = \mu\phi^{-1}[F].$$

324O Proposition Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be totally finite measure algebras, $\mathfrak{A}_0$ a topologically dense subalgebra of $\mathfrak{A}$, and $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism such that $\bar{\nu}\pi a = \bar{\mu}a$ for every $a \in \mathfrak{A}_0$. Then π has a unique extension to a measure-preserving homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

proof Let ρ, σ be the measure metrics on $\mathfrak{A}$, $\mathfrak{B}$ respectively, as in 323Ad. Then for any a, $a' \in \mathfrak{A}_0$

$$\sigma(\pi a, \pi a') = \bar{\nu}(\pi a \triangle \pi a') = \bar{\nu}\pi(a \triangle a') = \bar{\mu}(a \triangle a') = \rho(a, a');$$

that is, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ is an isometry. Because $\mathfrak{A}_0$ is dense in the metric space $(\mathfrak{A}, \rho)$, while $\mathfrak{B}$ is complete under σ (323Gc), there is a unique continuous function $\hat{\pi} : \mathfrak{A} \to \mathfrak{B}$ extending π (3A4G). Now the operations

$$(a, a') \mapsto \hat{\pi}(a \cap a'), \quad (a, a') \mapsto \hat{\pi}a \cap \hat{\pi}a' : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{B},$$

are continuous and agree on the dense subset $\mathfrak{A}_0 \times \mathfrak{A}_0$ of $\mathfrak{A} \times \mathfrak{A}$; because the topology of $\mathfrak{B}$ is Hausdorff, they agree on $\mathfrak{A} \times \mathfrak{A}$, that is, $\hat{\pi}(a \cap a') = \hat{\pi}a \cap \hat{\pi}a'$ for all a, $a' \in \mathfrak{A}$ (2A3Uc). Similarly, the operations

$$a \mapsto \hat{\pi}(1 \setminus a), \quad a \mapsto 1 \setminus \hat{\pi}a : \mathfrak{A} \to \mathfrak{B}$$

are continuous and agree on the dense subset $\mathfrak{A}_0$ of $\mathfrak{A}$, so they agree on $\mathfrak{A}$, that is, $\hat{\pi}(1 \setminus a) = 1 \setminus a$ for every $a \in \mathfrak{A}$. Thus $\hat{\pi}$ is a Boolean homomorphism (312H again). To see that it is measure-preserving, observe that

$$a \mapsto \bar{\mu}a = \rho(a, 0), \quad a \mapsto \bar{\nu}(\hat{\pi}a) = \sigma(\hat{\pi}a, 0) : \mathfrak{A} \to \mathbb{R}$$

are continuous and agree on $\mathfrak{A}_0$, so agree on $\mathfrak{A}$. Finally, $\hat{\pi}$ is the only measure-preserving Boolean homomorphism extending π, because any such map must be continuous (324Kb), and $\hat{\pi}$ is the only continuous extension of π.

***324P** The following fact will be applied in §387, by which time it will seem perfectly elementary; for the moment, it may be a useful exercise.

Proposition Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be totally finite measure algebras such that $\bar{\mu}1=\bar{\nu}1$. Suppose that $A\subseteq\mathfrak{A}$ and $\phi:A\to\mathfrak{B}$ are such that $\bar{\nu}(\inf_{i\le n}\phi a_i)=\bar{\mu}(\inf_{i\le n}a_i)$ for all $a_0,\dots,a_n\in A$. Let $\mathfrak{C}$ be the smallest closed subalgebra of $\mathfrak{A}$ including A. Then ϕ has a unique extension to a measure-preserving Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$.

proof (a) Let Ψ be the family of all functions ψ extending ϕ and having the same properties; that is, ψ is a function from a subset of $\mathfrak{A}$ to $\mathfrak{B}$, and $\bar{\nu}(\inf_{i\le n}\psi a_i)=\bar{\mu}(\inf_{i\le n}a_i)$ for all $a_0,\dots,a_n\in\operatorname{dom}\psi$. By Zorn's Lemma, Ψ has a maximal member θ. Write D for the domain of θ.

(b)(i) If c, $d\in D$ then $c\cap d\in D$. **P?** Otherwise, set $D'=D\cup\{c\cap d\}$ and extend θ to $\theta':D'\to\mathfrak{B}$ by writing $\theta'(c\cap d)=\theta c\cap\theta d$. It is easy to check that $\theta'\in\Psi$, which is supposed to be impossible. **XQ**

Now

$$\bar{\nu}(\theta c\cap\theta d\cap\theta(c\cap d))=\bar{\mu}(c\cap d)=\bar{\nu}(\theta c\cap\theta d)=\bar{\nu}\theta(c\cap d),$$

so $\theta(c\cap d)=\theta c\cap\theta d$.

(ii) If $d\in D$ then $1\setminus d\in D$. **P?** Otherwise, set $D'=D\cup\{1\setminus d\}$ and extend θ to D' by writing $\theta'(1\setminus d)=1\setminus\theta d$. Once again, it is easy to check that $\theta'\in\Psi$, which is impossible. **XQ**

Consequently (since D is certainly not empty, even if A is), D is a subalgebra of $\mathfrak{A}$ (312B(iii)).

(iii) Since

$$\bar{\nu}\theta 1=\bar{\mu}1=\bar{\nu}1,$$

$\theta 1=1$. If $d\in D$ then

$$\bar{\nu}\theta(1\setminus d)=\bar{\mu}(1\setminus d)=\bar{\mu}1-\bar{\mu}d=\bar{\nu}1-\bar{\nu}\theta d=\bar{\nu}(1\setminus\theta d),$$

while

$$\bar{\nu}(\theta d\cap\theta(1\setminus d))=\bar{\mu}(d\cap(1\setminus d))=0,$$

so $\theta d\cap\theta(1\setminus d))=0$, $\theta(1\setminus d)\subseteq 1\setminus\theta d$ and $\theta(1\setminus d)$ must be equal to $1\setminus\theta d$.

By 312H(ii), $\theta:D\to\mathfrak{B}$ is a Boolean homomorphism.

(iv) Let $\mathfrak{D}$ be the topological closure of D in $\mathfrak{A}$. Then it is an order-closed subalgebra of $\mathfrak{A}$ (323J), so, with $\bar{\mu}\restriction\mathfrak{D}$, is a totally finite measure algebra in which D is a topologically dense subalgebra. By 324O, there is an extension of θ to a measure-preserving Boolean homomorphism from $\mathfrak{D}$ to $\mathfrak{B}$; of course this extension belongs to Ψ, so in fact $D=\mathfrak{D}$ is a closed subalgebra of $\mathfrak{A}$.

(c) Since $A\subseteq D$, $\mathfrak{C}\subseteq\mathfrak{D}$ and $\phi_1=\theta\restriction\mathfrak{C}$ is a suitable extension of ϕ.

To see that ϕ_1 is unique, let $\phi_2:\mathfrak{C}\to\mathfrak{B}$ be any other measure-preserving Boolean homomorphism extending ϕ. Set $C=\{a:\phi_1 a=\phi_2 a\}$; then C is a topologically closed subalgebra of $\mathfrak{A}$ including A, so is the whole of $\mathfrak{C}$, and $\phi_2=\phi_1$.

324X Basic exercises (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, of which $\mathfrak{A}$ is Dedekind σ-complete, and $\phi:\mathfrak{A}\to\mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. Let I be an ideal of $\mathfrak{A}$ included in the kernel of ϕ. Show that we have a sequentially order-continuous Boolean homomorphism $\pi:\mathfrak{A}/I\to\mathfrak{B}$ given by setting $\pi(a^\bullet)=\phi a$ for every $a\in\mathfrak{A}$.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra, and $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$ such that $(\mathfrak{B},\bar{\mu}\restriction\mathfrak{B})$ is semi-finite. Show that the topology on $\mathfrak{B}$ induced by $\bar{\mu}\restriction\mathfrak{B}$ is just the subspace topology induced by the topology of $\mathfrak{A}$.

(c) Let (X,Σ,μ) be a measure space and $(X,\tilde{\Sigma},\tilde{\mu})$ its c.l.d. version. Let $\mathfrak{A}$, $\mathfrak{A}_2$ be the corresponding measure algebras and $\pi:\mathfrak{A}\to\mathfrak{A}_2$ the canonical homomorphism (see 322Db). Show that π is topologically continuous.

(d) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be measure algebras, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a bijective measure-preserving Boolean homomorphism. Show that $\pi^{-1}:\mathfrak{B}\to\mathfrak{A}$ is a measure-preserving homomorphism.

(e) Let $\bar{\mu}$ be counting measure on $\mathcal{P}\mathbb{N}$. Show that $(\mathcal{P}\mathbb{N},\bar{\mu})$ is a σ-finite measure algebra. Find a measure-preserving Boolean homomorphism from $\mathcal{P}\mathbb{N}$ to itself which is not sequentially order-continuous.

324Y Further exercises (a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, of which $\mathfrak{A}$ is Dedekind complete, and $\phi:\mathfrak{A}\to\mathfrak{B}$ an order-continuous Boolean homomorphism. Let I be an ideal of $\mathfrak{A}$ included in the kernel of ϕ. Show that we have an order-continuous Boolean homomorphism $\pi:\mathfrak{A}/I\to\mathfrak{B}$ given by setting $\pi(a^\bullet)=\phi a$ for every $a\in\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and Z its Stone space. Write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z, and $\mathcal{Z}$ for the family of nowhere dense zero sets of Z; let $\mathcal{Z}_\sigma$ be the σ-ideal of subsets of Z generated by $\mathcal{Z}$. Show that $\Sigma=\{E\triangle U: E\in\mathcal{E},\, U\in\mathcal{Z}_\sigma\}$ is a σ-algebra of subsets of Z, and describe a canonical isomorphism between $\Sigma/\mathcal{Z}_\sigma$ and $\mathfrak{A}$.

(c) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Dedekind σ-complete Boolean algebras, with Stone spaces Z and W. Construct $\mathcal{Z}_\sigma\subseteq\Sigma\subseteq\mathcal{P}Z$ as in 324Yb, and let $\mathcal{W}_\sigma\subseteq\mathrm{T}\subseteq\mathcal{P}W$ be the corresponding structure defined from $\mathfrak{B}$. Let $\pi:\mathfrak{B}\to\mathfrak{A}$ be a sequentially order-continuous Boolean homomorphism, and $\phi:Z\to W$ the corresponding continuous map. Show that if $E^*\in\mathfrak{A}$ corresponds to $E\in\Sigma$, then $\pi F^*=\phi^{-1}[F]^*$ for every $F\in\mathrm{T}$.

(d) Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a ccc Boolean algebra and $\pi:\mathfrak{A}\to\mathfrak{B}$ an injective Boolean homomorphism. Show that $\mathfrak{A}$ is ccc.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra, and $\pi:\mathfrak{A}\to\mathfrak{B}$ an order-continuous Boolean homomorphism. Show that for every atom $b\in\mathfrak{B}$ there is an atom $a\in\mathfrak{A}$ such that $\pi a\supseteq b$. Hence show that if $\mathfrak{A}$ is atomless so is $\mathfrak{B}$, and that if $\mathfrak{B}$ is purely atomic and π is injective then $\mathfrak{A}$ is purely atomic.

(f) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be localizable measure algebras and $\mathfrak{A}_0$ an order-dense subalgebra of $\mathfrak{A}$. Suppose that $\pi:\mathfrak{A}_0\to\mathfrak{B}$ is an order-continuous Boolean homomorphism such that $\bar{\nu}\pi a=\bar{\mu}a$ for every $a\in\mathfrak{A}_0$. Show that π has a unique extension to a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(g) Let $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$ be probability algebras, and $f:\mathfrak{A}\to\mathfrak{B}$ an isometry for the measure metrics. Show that $a\mapsto f(a)\triangle f(0)$ is a measure-preserving Boolean homomorphism.

324 Notes and comments If you examine the arguments of this section carefully, you will see that rather little depends on the measures named. Really this material deals with structures $(X,\Sigma,\mathcal{I})$ where X is a set, Σ is a σ-ideal of subsets of X, and $\mathcal{I}$ is a σ-ideal of Σ, corresponding to the family of measurable negligible sets. In this abstract form it is natural to think in terms of sequentially order-continuous homomorphisms, as in 324Yc. I have stated 324E in terms of order-continuous homomorphisms just for a slight gain in simplicity. But in fact, when there is a difference, it is likely that order-continuity, rather than sequential order-continuity, will be the more significant condition. Note that when the domain algebra is σ-finite, it is ccc (322G), so the two concepts coincide (316Fd).

Of course I need to refer to measures when looking at such concepts as σ-finite measure algebra or measure-preserving homomorphism, but even here the real ideas involved are such notions as order-continuity and the countable chain condition, as you will see if you work through 324K. It is instructive to look at the translations of these facts into the context of inverse-measure-preserving functions; see 234B.

324H shows that we may speak of 'the' topology and uniformity of a Dedekind σ-complete Boolean algebra which carries any semi-finite measure; the topology of such an algebra is determined by its algebraic structure. Contrast this with the theory of normed spaces: two Banach spaces (e.g., ℓ^1 and ℓ^2) can be isomorphic as linear spaces, both being of algebraic dimension $\mathfrak{c}$, while they are not isomorphic as topological linear spaces. When we come to the theory of ordered linear topological spaces, however, we shall again find ourselves with operators whose algebraic properties guarantee continuity (355C, 367O).

Version of 30.8.06

325 Free products and product measures

In this section I aim to describe the measure algebras of product measures as defined in Chapter 25. This will involve the concept of 'free product' set out in §315. It turns out that we cannot determine the measure algebra of a product measure from the measure algebras of the factors (325B), unless we are told that the product measure is localizable; but that there is nevertheless a general construction of 'localizable measure algebra free product', applicable to any pair of semi-finite measure algebras (325D), which represents the measure algebra of the product measure in the most important cases (325Eb). In the second part of the section (325I-325M) I deal with measure algebra free products of probability algebras, corresponding to the products of probability spaces treated in §254.

325A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with measure algebras $(\mathfrak{A},\bar{\mu})$ and $(\mathfrak{B},\bar{\nu})$. Let λ be the c.l.d. product measure on $X\times Y$, and Λ its domain; let $(\mathfrak{C},\bar{\lambda})$ be the corresponding measure algebra.

(a)(i) The map $E\mapsto E\times Y:\Sigma\to\Lambda$ induces an order-continuous Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.

(ii) The map $F\mapsto X\times F:\mathrm{T}\to\Lambda$ induces an order-continuous Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{C}$.

(b) The map $(E,F)\mapsto E\times F:\Sigma\times\mathrm{T}\to\Lambda$ induces a Boolean homomorphism $\psi:\mathfrak{A}\otimes\mathfrak{B}\to\mathfrak{C}$.

(c) $\psi[\mathfrak{A}\otimes\mathfrak{B}]$ is topologically dense in $\mathfrak{C}$ for the measure-algebra topology of $\mathfrak{C}$.

(d) For every $c\in\mathfrak{C}$,

$$\bar{\lambda}c=\sup\{\bar{\lambda}(c\cap\psi(a\otimes b)):a\in\mathfrak{A},\ b\in\mathfrak{B},\ \bar{\mu}a<\infty,\ \bar{\nu}b<\infty\}.$$

(e) If μ and ν are semi-finite, ψ is injective and $\bar{\lambda}\psi(a\otimes b)=\bar{\mu}a\cdot\bar{\mu}b$ for every $a\in\mathfrak{A}$, $b\in\mathfrak{B}$.

proof (a) $E\times Y\in\Lambda$ for every $E\in\Sigma$ (251E), and $\lambda(E\times Y)=0$ whenever $\mu E=0$ (251Ia). Thus $E\mapsto(E\times Y)^\bullet:\Sigma\to\mathfrak{C}$ is a Boolean homomorphism with kernel including $\{E:\mu E=0\}$, so descends to a Boolean homomorphism $\varepsilon_1:\mathfrak{A}\to\mathfrak{C}$.

To see that ε_1 is order-continuous, let $A\subseteq\mathfrak{A}$ be a non-empty downwards-directed set with infimum 0. **?** If there is a non-zero lower bound c of $\varepsilon_1[A]$, express c as $W^\bullet$ where $W\in\Lambda$. We have $\lambda(W)>0$; by the definition of λ (251F), there are $G\in\Sigma$, $H\in\mathrm{T}$ such that $\mu G<\infty$, $\nu H<\infty$ and $\lambda(W\cap(G\times H))>0$. Of course $\inf_{a\in A}a\cap G^\bullet=0$ in $\mathfrak{A}$, so $\inf_{a\in A}\bar{\mu}(a\cap G^\bullet)=0$, by 321F; let $a\in A$ be such that $\bar{\mu}(a\cap G^\bullet)\cdot\nu H<\lambda(W\cap(G\times H))$. Express a as $E^\bullet$, where $E\in\Sigma$. Then $\lambda(W\setminus(E\times Y))=0$. But this means that

$$\lambda(W\cap(G\times H))\le\lambda((E\cap G)\times H)=\mu(E\cap G)\cdot\nu H=\bar{\mu}(a\cap G^\bullet)\cdot\nu H,$$

contradicting the choice of a. **X** Thus $\inf\varepsilon_1[A]=0$ in $\mathfrak{C}$; as A is arbitrary, ε_1 is order-continuous.

Similarly $\varepsilon_2:\mathfrak{B}\to\mathfrak{C}$, induced by $F\mapsto X\times F:\mathrm{T}\to\Lambda$, is order-continuous.

(b) Now there must be a corresponding Boolean homomorphism $\psi:\mathfrak{A}\otimes\mathfrak{B}\to\mathfrak{C}$ such that $\psi(a\otimes b)=\varepsilon_1a\cap\varepsilon_2b$ for every $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$, that is,

$$\psi(E^\bullet\otimes F^\bullet)=(E\times Y)^\bullet\cap(X\times F)^\bullet=(E\times F)^\bullet$$

for every $E\in\Sigma$, $F\in\mathrm{T}$ (315Jb).

(c) Suppose that c, $e\in\mathfrak{C}$, $\bar{\lambda}e<\infty$ and $\epsilon>0$. Express c, e as $U^\bullet$, $W^\bullet$ where U, $W\in\Lambda$. By 251Ie, there are $E_0,\dots,E_n\in\Sigma$, $F_0,\dots,F_n\in\mathrm{T}$, all of finite measure, such that $\lambda((U\cap W)\triangle\bigcup_{i\le n}E_i\times F_i)\le\epsilon$. Set

$$c_1=(\bigcup_{i\le n}E_i\times F_i)^\bullet\in\psi[\mathfrak{A}\otimes\mathfrak{B}];$$

then

$$\bar{\lambda}(e\cap(c\,\triangle\,c_1))=\lambda(W\cap(U\triangle\bigcup_{i\le n}E_i\times F_i))\le\epsilon.$$

As c, e and ϵ are arbitrary, $\psi[\mathfrak{A}\otimes\mathfrak{B}]$ is topologically dense in $\mathfrak{C}$.

(d) By the definition of λ, we have

$$\lambda W=\sup\{\lambda(W\cap(E\times F)):E\in\Sigma,\ F\in\mathrm{T},\ \mu E<\infty,\ \nu F<\infty\}$$

for every $W \in \Lambda$; so all we have to do is express c as $W^\bullet$.

(e) Now suppose that μ and ν are semi-finite. Then $\lambda(E \times F) = \mu E \cdot \nu F$ for any $E \in \Sigma$, $F \in \mathrm{T}$ (251J), so $\bar\lambda\psi(a \otimes b) = \bar\mu a \cdot \bar\nu b$ for every $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$.

To see that ψ is injective, take any non-zero $c \in \mathfrak{A} \otimes \mathfrak{B}$; then there must be non-zero $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ such that $a \otimes b \subseteq c$ (315Kb), so that

$$\bar\lambda\psi c \ge \bar\lambda\psi(a \otimes b) = \bar\mu a \cdot \bar\nu b > 0$$

and $\psi c \ne 0$.

325B Characterizing the measure algebra of a product space A very natural question to ask is, whether it is possible to define a 'measure algebra free product' of two abstract measure algebras in a way which will correspond to one of the constructions above. I give an example to show the difficulties involved.

Example There are complete locally determined localizable measure spaces (X, μ), (X', μ'), with isomorphic measure algebras, and a probability space (Y, ν) such that the measure algebras of the c.l.d. product measures on $X \times Y$, $X' \times Y$ are not isomorphic.

proof Let (X, Σ, μ) be the complete locally determined localizable not-strictly-localizable measure space described in 216E. Recall that, for $E \in \Sigma$, $\mu E = \#(\{\gamma : \gamma \in C,\ f_\gamma \in E\})$ if this is finite, ∞ otherwise (216Eb), where C is a set with cardinal greater than $\mathfrak{c}$. The map $E \mapsto \{\gamma : f_\gamma \in E\} : \Sigma \to \mathcal{P}C$ is surjective (216Ec), so descends to an isomorphism between $\mathfrak{A}$, the measure algebra of μ, and $\mathcal{P}C$. Let (X', Σ', μ') be C with counting measure, so that its measure algebra $(\mathfrak{A}', \bar\mu')$ is isomorphic to $(\mathfrak{A}, \bar\mu)$, while μ' is of course strictly localizable.

Let (Y, T, ν) be $\{0,1\}^C$ with its usual measure. Let λ, λ' be the c.l.d. product measures on $X \times Y$, $X' \times Y$ respectively, and $(\mathfrak{C}, \bar\lambda)$, $(\mathfrak{C}', \bar\lambda')$ the corresponding measure algebras. Then λ is not localizable (254U), so $(\mathfrak{C}, \bar\lambda)$ is not localizable (322Be). On the other hand, λ', being the c.l.d. product of strictly localizable measures, is strictly localizable (251O), therefore localizable, so $(\mathfrak{C}', \bar\lambda')$ is localizable, and is not isomorphic to $(\mathfrak{C}, \bar\lambda)$.

325C Thus there can be no universally applicable method of identifying the measure algebra of a product measure from the measure algebras of the factors. However, you have no doubt observed that the example above involves non-σ-finite spaces, and conjectured that this is not an accident. In contexts in which we know that the algebras involved are localizable, there are positive results available, such as the following.

Theorem Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be semi-finite measure spaces, with measure algebras $(\mathfrak{A}_1, \bar\mu_1)$ and $(\mathfrak{A}_2, \bar\mu_2)$. Let λ be the c.l.d. product measure on $X_1 \times X_2$, and $(\mathfrak{C}, \bar\lambda)$ the corresponding measure algebra. Let $(\mathfrak{B}, \bar\nu)$ be a localizable measure algebra, and $\phi_1 : \mathfrak{A}_1 \to \mathfrak{B}$, $\phi_2 : \mathfrak{A}_2 \to \mathfrak{B}$ order-continuous Boolean homomorphisms such that $\bar\nu(\phi_1(a_1) \cap \phi_2(a_2)) = \bar\mu_1 a_1 \cdot \bar\mu_2 a_2$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$. Then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi(\psi(a_1 \otimes a_2)) = \phi_1(a_1) \cap \phi_2(a_2)$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$, writing $\psi : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{C}$ for the canonical map described in 325A.

proof (a) Because ψ is injective, it is an isomorphism between $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ and its image in $\mathfrak{C}$. I trust it will cause no confusion if I abuse notation slightly and treat $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ as actually a subalgebra of $\mathfrak{C}$. Now the Boolean homomorphisms ϕ_1, ϕ_2 correspond to a Boolean homomorphism $\theta : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{B}$. The point is that $\bar\nu\theta c = \bar\lambda c$ for every $c \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$. **P** By 315Kb, every member of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is expressible as $\sup_{i \le n} a_i \otimes a'_i$, where $a_i \in \mathfrak{A}_1$, $a'_i \in \mathfrak{A}_2$ for each i and $\langle a_i \otimes a'_i \rangle_{i \le n}$ is disjoint. Now for each i we have

$$\bar\nu\theta(a_i \otimes a'_i) = \bar\nu(\phi_1(a_i) \cap \phi_2(a'_i)) = \bar\mu_1 a_i \cdot \bar\mu_2 a'_i = \bar\lambda(a_i \otimes a'_i),$$

by 325Ae. So

$$\bar\nu\theta(c) = \sum_{i=0}^n \bar\nu\theta(a_i \otimes a'_i) = \sum_{i=0}^n \bar\lambda(a_i \otimes a'_i) = \bar\lambda c. \ \mathbf{Q}$$

(b) The following fact will underlie many of the arguments below. If $e \in \mathfrak{B}$, $\bar\nu e < \infty$ and $\epsilon > 0$, there are $e_1 \in \mathfrak{A}_1^f$, $e_2 \in \mathfrak{A}_2^f$ such that $\bar\nu(e \setminus \theta(e_1 \otimes e_2)) \le \epsilon$, writing $\mathfrak{A}_i^f$ for $\{a : \bar\mu_i a < \infty\}$. **P** Because $(\mathfrak{A}_1, \bar\mu_1)$ is semi-finite, $\mathfrak{A}_1^f$ has supremum 1 in $\mathfrak{A}_1$; because ϕ_1 is order-continuous, $\sup\{\phi_1(a) : a \in \mathfrak{A}_1^f\} = 1$ in $\mathfrak{B}$, and

$\inf\{e \setminus \phi_1(a) : a \in \mathfrak{A}_1^f\} = 0$ (313Aa). Because $\mathfrak{A}_1^f$ is upwards-directed, $\{e \setminus \phi_1(a) : a \in \mathfrak{A}_1^f\}$ is downwards-directed, so $\inf\{\bar\nu(e \setminus \phi(a)) : a \in \mathfrak{A}_1^f\} = 0$ (321F again). Let $e_1 \in \mathfrak{A}_1^f$ be such that $\bar\nu(e \setminus \phi_1(e_1)) \le \frac{1}{2}\epsilon$.

In the same way, there is an $e_2 \in \mathfrak{A}_2^f$ such that $\bar\nu(e \setminus \phi_2(e_2)) \le \frac{1}{2}\epsilon$. Consider $e' = e_1 \otimes e_2 \in \mathfrak{C}$. Then

$$\bar\nu(e \setminus \theta e') = \bar\nu(e \setminus (\phi_1(e_1) \cap \phi_2(e_2))) \le \bar\nu(e \setminus \phi_1(e_1)) + \bar\nu(e \setminus \phi_2(e_2)) \le \epsilon. \text{ **Q**}$$

(c) The next step is to check that θ is uniformly continuous for the measure-algebra uniformities defined by $\bar\nu$ and $\bar\lambda$. **P** Take any $e \in \mathfrak{B}^f$ and $\epsilon > 0$. By (b), there are e_1, e_2 such that $\bar\lambda(e_1 \otimes e_2) < \infty$ and $\bar\nu(e \setminus \theta(e_1 \otimes e_2)) \le \frac{1}{2}\epsilon$. Set $e' = e_1 \otimes e_2$. Now suppose that c, $c' \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$ and $\bar\lambda((c \triangle c') \cap e') \le \frac{1}{2}\epsilon$. Then

$$\bar\nu((\theta(c) \triangle \theta(c')) \cap e) \le \bar\nu\theta((c \triangle c') \cap e') + \bar\nu(e \setminus \theta e') \le \bar\lambda((c \triangle c') \cap e') + \frac{1}{2}\epsilon \le \epsilon.$$

By 3A4Cc, θ is uniformly continuous for the subspace uniformity on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$. **Q**

(d) Recall that $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is topologically dense in $\mathfrak{C}$ (325Ac), while $\mathfrak{B}$ is complete for its uniformity (323Gc). So there is a uniformly continuous function $\phi : \mathfrak{C} \to \mathfrak{B}$ extending θ (3A4G).

(e) Because θ is a Boolean homomorphism, so is ϕ. **P** (i) The functions $c \mapsto \phi(1 \setminus c)$, $c \mapsto 1 \setminus \phi(c)$ are continuous and the topology of $\mathfrak{B}$ is Hausdorff, so $\{c : \phi(1 \setminus c) = 1 \setminus \phi(c)\}$ is closed; as it includes $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, it must be the whole of $\mathfrak{C}$. (ii) The functions $(c, c') \mapsto \phi(c \cup c')$, $(c, c') \mapsto \phi(c) \cup \phi(c')$ are continuous, so $\{(c, c') : \phi(c \cup c') = \phi(c) \cup \phi(c')\}$ is closed in $\mathfrak{C} \times \mathfrak{C}$; as it includes $(\mathfrak{A}_1 \otimes \mathfrak{A}_2) \times (\mathfrak{A}_1 \otimes \mathfrak{A}_2)$, it must be the whole of $\mathfrak{C} \times \mathfrak{C}$. **Q**

(f) Because θ is measure-preserving, so is ϕ. **P** Take any $e_1 \in \mathfrak{A}_1^f$, $e_2 \in \mathfrak{A}_2^f$. Then the functions $c \mapsto \bar\lambda(c \cap (e_1 \otimes e_2))$, $c \mapsto \bar\nu\phi(c \cap (e_1 \otimes e_2))$ are continuous and equal on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, so are equal on $\mathfrak{C}$. The argument of (b) shows that for any $b \in \mathfrak{B}$,

$$\begin{aligned}\bar\nu b &= \sup\{\bar\nu(b \cap e) : e \in \mathfrak{B}^f\} \\ &= \sup\{\bar\nu(b \cap \phi(e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\},\end{aligned}$$

so that

$$\begin{aligned}\bar\nu\phi(c) &= \sup\{\bar\nu\phi(c \cap (e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\} \\ &= \sup\{\bar\lambda(c \cap (e_1 \otimes e_2)) : e_1 \in \mathfrak{A}_1^f,\ e_2 \in \mathfrak{A}_2^f\} = \bar\lambda c\end{aligned}$$

for every $c \in \mathfrak{C}$. **Q**

(g) To see that ϕ is order-continuous, take any non-empty downwards-directed set $C \subseteq \mathfrak{C}$ with infimum 0. **?** If $\phi[C]$ has a non-zero lower bound b in $\mathfrak{B}$, let $e \subseteq b$ be such that $0 < \bar\nu e < \infty$. Let $e' \in \mathfrak{C}$ be such that $\bar\lambda e' < \infty$ and $\bar\nu(e \setminus \phi(e')) < \bar\nu e$, as in (b) above, so that $\bar\nu(e \cap \phi(e')) > 0$. Now, because $\inf C = 0$, there is a $c \in C$ such that $\bar\lambda(c \cap e') < \bar\nu(e \cap \phi(e'))$. But this means that

$$\bar\nu(b \cap \phi(e')) \le \bar\nu\phi(c \cap e') = \bar\lambda(c \cap e') < \bar\nu(e \cap \phi(e')) \le \bar\nu(b \cap \phi(e')),$$

which is absurd. **X** Thus $\inf \phi[C] = 0$ in $\mathfrak{B}$. As C is arbitrary, ϕ is order-continuous.

(h) Finally, to see that ϕ is unique, observe that any order-continuous Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$ must be continuous (324Fc); so that if it agrees with ϕ on $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ it must agree with ϕ on $\mathfrak{C}$.

325D Theorem Let $(\mathfrak{A}_1, \bar\mu_1)$ and $(\mathfrak{A}_2, \bar\mu_2)$ be semi-finite measure algebras.

(a) There is a localizable measure algebra $(\mathfrak{C}, \bar\lambda)$, together with order-continuous Boolean homomorphisms $\varepsilon_1 : \mathfrak{A}_1 \to \mathfrak{C}$ and $\varepsilon_2 : \mathfrak{A}_2 \to \mathfrak{C}$, such that whenever $(\mathfrak{B}, \bar\nu)$ is a localizable measure algebra, and $\phi_1 : \mathfrak{A}_1 \to \mathfrak{B}$, $\phi_2 : \mathfrak{A}_2 \to \mathfrak{B}$ are order-continuous Boolean homomorphisms and $\bar\nu(\phi_1(a_1) \cap \phi_2(a_2)) = \bar\mu_1 a_1 \cdot \bar\mu_2 a_2$ for all $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$, then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\varepsilon_j = \phi_j$ for both j.

(b) The structure $(\mathfrak{C}, \bar\lambda, \varepsilon_1, \varepsilon_2)$ is determined up to isomorphism by this property.

(c)(i) The Boolean homomorphism $\psi : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{C}$ defined from ε_1 and ε_2 is injective, and $\psi[\mathfrak{A}_1 \otimes \mathfrak{A}_2]$ is topologically dense in $\mathfrak{C}$.

(ii) The closed subalgebra of $\mathfrak{C}$ generated by $\psi[\mathfrak{A}_1 \otimes \mathfrak{A}_2]$ is the whole of $\mathfrak{C}$.

(d) If $j \in \{1,2\}$ and $(\mathfrak{A}_j, \bar{\mu}_j)$ is localizable, then $\varepsilon_j[\mathfrak{A}_j]$ is a closed subalgebra of $(\mathfrak{C}, \bar{\lambda})$.

proof (a)(i) We may regard $(\mathfrak{A}_1, \bar{\mu}_1)$ as the measure algebra of (Z_1, Σ_1, μ_1) where Z_1 is the Stone space of $\mathfrak{A}_1$, Σ_1 is the algebra of subsets of Z_1 differing from an open-and-closed set by a meager set, and μ_1 is an appropriate measure (321K). Note that in this representation, each $a \in \mathfrak{A}_1$ becomes identified with $\widehat{a}^\bullet$, where $\widehat{a}$ is the open-and-closed subset of Z_1 corresponding to a. Similarly, we may think of $(\mathfrak{A}_2, \bar{\mu}_2)$ as the measure algebra of (Z_2, Σ_2, μ_2), where Z_2 is the Stone space of $\mathfrak{A}_2$.

(ii) Let λ be the c.l.d. product measure on $Z_1 \times Z_2$. The point is that λ is strictly localizable. **P** By 322Ea, both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have partitions of unity consisting of elements of finite measure; let $\langle c_i \rangle_{i\in I}$, $\langle d_j \rangle_{j\in J}$ be such partitions. Then $\langle \widehat{c}_i \times \widehat{d}_j \rangle_{i\in I, j\in J}$ is a disjoint family of sets of finite measure in $Z_1 \times Z_2$. If $W \subseteq Z_1 \times Z_2$ is such that $\lambda W > 0$, there must be sets E_1, E_2 of finite measure such that $\lambda(W \cap (E_1 \times E_2)) > 0$. Because $E_1^\bullet = \sup_{i\in I} E_1^\bullet \cap c_i$, we must have

$$\mu_1 E_1 = \bar{\mu}_1 E_1^\bullet = \textstyle\sum_{i\in I} \bar{\mu}_1(E_1^\bullet \cap c_i) = \sum_{i\in I} \mu_1(E_1 \cap \widehat{c}_i).$$

Similarly, $\mu_2 E_2 = \sum_{i\in J} \mu_2(E_2 \cap \widehat{d}_j)$. But this means that there must be finite $I' \subseteq I$, $J' \subseteq J$ such that

$$\textstyle\sum_{i\in I', j\in J'} \mu_1(E_1 \cap \widehat{c}_i)\mu_2(E_2 \cap \widehat{d}_j) > \mu_1 E_1 \cdot \mu_2 E_2 - \lambda(W \cap (E_1 \times E_2)),$$

so that there have to be $i \in I'$, $j \in J'$ such that $\lambda(W \cap (\widehat{c}_i \times \widehat{d}_j)) > 0$.

Now this means that $\langle \widehat{c}_i \times \widehat{d}_j \rangle_{i\in I, j\in J}$ satisfies the conditions of 213O. Because λ is surely complete and locally determined, it is strictly localizable. **Q**

(iii) We may therefore take $(\mathfrak{C}, \bar{\lambda})$ to be just the measure algebra of λ. The maps ε_1, ε_2 will be the canonical maps described in 325Aa, inducing the map $\psi : \mathfrak{A}_1 \otimes \mathfrak{A}_2 \to \mathfrak{C}$ referred to in 325C; and 325C now gives the result.

(b) This is nearly obvious. Suppose we had an alternative structure $(\mathfrak{C}', \bar{\lambda}', \varepsilon_1', \varepsilon_2')$ with the same property. Then we must have an order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{C}'$ such that $\phi\varepsilon_j = \varepsilon_j'$ for both j; and similarly we have an order-continuous measure-preserving Boolean homomorphism $\phi' : \mathfrak{C}' \to \mathfrak{C}$ such that $\phi'\varepsilon_j' = \varepsilon_j$ for both j. Now $\phi'\phi : \mathfrak{C} \to \mathfrak{C}$ is an order-continuous measure-preserving Boolean homomorphism such that $\phi'\varepsilon_j = \varepsilon_j$ for both j. By the uniqueness assertion in (a), applied with $\mathfrak{B} = \mathfrak{C}$, $\phi'\phi$ must be the identity on $\mathfrak{C}$. In the same way, $\phi\phi'$ is the identity on $\mathfrak{C}'$. So ϕ and ϕ' are the two halves of the required isomorphism.

(c) In view of the construction for $\mathfrak{C}$ offered in part (a) of the proof, (i) is just a consequence of 325Ac and 325Ae. Now (ii) follows by 323J.

(d) If $\mathfrak{A}_j$ is Dedekind complete then $\varepsilon_j[\mathfrak{A}_j]$ is order-closed in $\mathfrak{C}$ because ε_j is order-continuous (314F(a-i)).

325E Remarks (a) We could say that a measure algebra $(\mathfrak{C}, \bar{\lambda})$, together with embeddings ε_1 and ε_2, as described in 325D, is a **localizable measure algebra free product** of $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$; and its uniqueness up to isomorphism makes it safe, most of the time, to call it 'the' localizable measure algebra free product. Observe that it can equally well be regarded as the uniform space completion of the algebraic free product; see 325Yc.

(b) As the example in 325B shows, the localizable measure algebra free product of the measure algebras of given measure spaces need not appear directly as the measure algebra of their product. But there is one context in which it must so appear: if the product measure is localizable, 325C tells us at once that it has the right measure algebra. For σ-finite measure algebras, of course, any corresponding measure spaces have to be strictly localizable, so again we can use the product measure directly.

325F I ought not to proceed to the next topic without giving another pair of examples to show the subtlety of the concept of 'measure algebra free product'.

Example Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure μ on $[0,1]$, and $(\mathfrak{C}, \bar{\lambda})$ the measure algebra of Lebesgue measure λ on $[0,1]^2$. Then $(\mathfrak{C}, \bar{\lambda})$ can be regarded as the localizable measure algebra free product of $(\mathfrak{A}, \bar{\mu})$ with itself, by 251N and 325Eb. Let $\psi : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{C}$ be the canonical map, as described in 325A. Then $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ is not order-dense in $\mathfrak{C}$, and ψ is not order-continuous.

proof (a) Let $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \epsilon_n = \infty$, but $\sum_{n=0}^{\infty} \epsilon_n^2 < 1$; for instance, we could take $\epsilon_n = \frac{1}{n+2}$. Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a stochastically independent sequence of measurable subsets of $[0,1]$ such that $\mu E_n = \epsilon_n$ for each n. In $\mathfrak{A}$ set $a_n = E_n^\bullet$, and consider $c_n = \sup_{i \le n} a_i \otimes a_i \in \mathfrak{A} \otimes \mathfrak{A}$ for each n.

(b) We have $\sup_{n \in \mathbb{N}} c_n = 1$ in $\mathfrak{A} \widehat{\otimes} \mathfrak{A}$. **P?** Otherwise, there is a non-zero $a \in \mathfrak{A} \widehat{\otimes} \mathfrak{A}$ such that $a \cap (a_n \otimes a_n) = 0$ for every n, and now there are non-zero b, $b' \in \mathfrak{A}$ such that $b \otimes b' \subseteq a$. Set $I = \{n : a_n \cap b = 0\}$, $J = \{n : a_n \cap b'\} = 0$. Then $\langle E_n \rangle_{n \in I}$ is an independent family and $\mu(\bigcup_{n \in I} E_i) \le 1 - \bar{\mu} b < 1$, so $\sum_{n \in I} \mu E_n < \infty$, by the Borel-Cantelli lemma (273K). Similarly $\sum_{n \in J} \mu E_n < \infty$. Because $\sum_{n \in \mathbb{N}} \mu E_n = \infty$, there must be some $n \in \mathbb{N} \setminus (I \cup J)$. Now $a_n \cap b$ and $a_n \cap b'$ are both non-zero, so

$$0 \ne (a_n \cap b) \otimes (a_n \cap b') = (a_n \otimes a_n) \cap (b \otimes b') = 0,$$

which is absurd. **XQ**

(c) On the other hand,

$$\sum_{n=0}^{\infty} \bar{\lambda} \psi(c_n) \le \sum_{n=0}^{\infty} (\bar{\mu} a_n)^2 = \sum_{n=0}^{\infty} \epsilon_n^2 < 1,$$

by the choice of the ϵ_n. So $\sup_{n \in \mathbb{N}} \psi(c_n)$ cannot be 1 in $\mathfrak{C}$.

Thus ψ is not order-continuous.

(d) By 313P(a-ii) and 313O, $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ cannot be order-dense in $\mathfrak{C}$; alternatively, (b) shows that there can be no non-zero member of $\psi[\mathfrak{A} \otimes \mathfrak{A}]$ included in $1 \setminus \sup_{n \in \mathbb{N}} \psi(c_n)$. (Both these arguments rely tacitly on the fact that ψ is injective, as noted in 325Ae.)

325G Since 325F shows that the free product and the localizable measure algebra free product are very different constructions, I had better repeat an idea from §315 in the new context.

Example Again, let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$, and $(\mathfrak{C}, \bar{\lambda})$ the measure algebra of Lebesgue measure on $[0,1]^2$. Then there is no order-continuous Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{A}$ such that $\phi(a \otimes b) = a \cap b$ for all a, $b \in \mathfrak{A}$. **P** Let $\phi : \mathfrak{C} \to \mathfrak{A}$ be a Boolean homomorphism such that $\phi(a \otimes b) = a \cap b$ for all a, $b \in \mathfrak{A}$. For $i < 2^n$ let a_{ni} be the equivalence class in $\mathfrak{A}$ of the interval $[2^{-n} i, 2^{-n}(i+1)]$, and set $c_n = \sup_{i < 2^n} a_{ni} \otimes a_{ni}$. Then $\phi c_n = 1$ for every n, but $\bar{\lambda} c_n = 2^{-n}$ for each n, so $\inf_{n \in \mathbb{N}} c_n = 0$ in $\mathfrak{C}$; thus ϕ cannot be order-continuous. **Q** (Compare 315Q.)

***325H Products of more than two factors** We can of course extend the ideas of 325A, 325C and 325D to products of any finite number of factors. No new ideas are needed, so I spell the results out without proofs.

(a) Let $\langle (\mathfrak{A}_i, \bar{\mu}_i) \rangle_{i \in I}$ be a non-empty finite family of semi-finite measure algebras. Then there is a localizable measure algebra $(\mathfrak{C}, \bar{\lambda})$, together with order-continuous Boolean homomorphisms $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{C}$ for $i \in I$, such that whenever $(\mathfrak{B}, \bar{\nu})$ is a localizable measure algebra, and $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ are order-continuous Boolean homomorphisms such that $\bar{\nu}(\inf_{i \in I} \phi_i(a_i)) = \prod_{i \in I} \bar{\mu}_i a_i$ whenever $a_i \in \mathfrak{A}_i$ for each i, then there is a unique order-continuous measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi \varepsilon_i = \phi_i$ for every i.

(b) The structure $(\mathfrak{C}, \bar{\lambda}, \langle \varepsilon_i \rangle_{i \in I})$ is determined up to isomorphism by this property.

(c) The Boolean homomorphism $\psi : \bigotimes_{i \in I} \mathfrak{A}_i \to \mathfrak{C}$ defined from the ε_i is injective, and $\psi[\bigotimes_{i \in I} \mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$.

(d) Write $\widehat{\bigotimes}^{\text{loc}}_{i \in I} (\mathfrak{A}_i, \bar{\mu}_i)$ for (a particular version of) the localizable measure algebra free product described in (a). If $\langle (A_i, \bar{\mu}_i) \rangle_{i \in I}$ is a finite family of semi-finite measure algebras and $\langle I_k \rangle_{k \in K}$ is a partition of I into non-empty sets, then $\widehat{\bigotimes}^{\text{loc}}_{i \in I} (\mathfrak{A}_i, \bar{\mu}_i)$ is isomorphic, in a canonical way, to $\widehat{\bigotimes}^{\text{loc}}_{k \in K} \bigl(\widehat{\bigotimes}^{\text{loc}}_{i \in I_k} (\mathfrak{A}_i, \bar{\mu}_i) \bigr)$.

(e) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a finite family of semi-finite measure spaces, and write $(\mathfrak{A}_i, \bar{\mu}_i)$ for the measure algebra of (X_i, Σ_i, μ_i). Let λ be the c.l.d. product measure on $\prod_{i \in I} X_i$ (251W), and $(\mathfrak{C}, \bar{\lambda})$ the corresponding measure algebra. Then there is a canonical order-continuous measure-preserving embedding of $(\mathfrak{C},\bar{\lambda})$ into the localizable measure algebra free product of the $(\mathfrak{A}_i,\bar{\mu}_i)$. If each μ_i is strictly localizable, this embedding is an isomorphism.

325I Infinite products Just as in §254, we can now turn to products of infinite families of probability algebras.

Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of probability spaces, with measure algebras $(\mathfrak{A}_i,\bar{\mu}_i)$. Let λ be the product measure on $X=\prod_{i\in I}X_i$, and $(\mathfrak{C},\bar{\lambda})$ the corresponding measure algebra. For each $i\in I$, we have a measure-preserving homomorphism $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ corresponding to the inverse-measure-preserving function $x\mapsto x(i):X\to X_i$. Let $(\mathfrak{B},\bar{\nu})$ be a probability algebra, and $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ Boolean homomorphisms such that $\bar{\nu}(\inf_{i\in J}\phi_i(a_i))=\prod_{i\in J}\bar{\mu}_i a_i$ whenever $J\subseteq I$ is a finite set and $a_i\in\mathfrak{A}_i$ for every i. Then there is a unique measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\phi_i$ for every $i\in I$.

proof (a) As remarked in 254Fb, all the maps $x\mapsto x(i)$ are inverse-measure-preserving, so correspond to measure-preserving homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ (324M). It will be helpful to use some notation from §254. Write $\mathcal{C}$ for the family of measurable cylinders in X expressible in the form

$$E=\{x:x\in X,\,x(i)\in E_i\text{ for every }i\in J\},$$

where $J\subseteq I$ is finite and $E_i\in\Sigma_i$ for every $i\in J$. Note that in this case

$$E^\bullet=\inf_{i\in J}\varepsilon_i(E_i^\bullet).$$

Set

$$C=\{E^\bullet:E\in\mathcal{C}\}\subseteq\mathfrak{C},$$

so that C is precisely the family of elements of $\mathfrak{C}$ expressible in the form $\inf_{i\in J}\phi_i(a_i)$ where $J\subseteq I$ is finite and $a_i\in\mathfrak{A}_i$ for each i.

The homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ define a Boolean homomorphism $\psi:\bigotimes_{i\in I}\mathfrak{A}_i\to\mathfrak{C}$ (315J), which is injective. **P** If $c\in\bigotimes_{i\in I}\mathfrak{A}_i$ is non-zero, there must be a finite set $J\subseteq I$ and a family $\langle a_i\rangle_{i\in J}$ such that $a_i\in\mathfrak{A}_i\setminus\{0\}$ for each i and $c\supseteq\inf_{i\in J}\tilde{\varepsilon}_i(a_i)$, where for the moment I write $\tilde{\varepsilon}_i$ for the canonical map from $\mathfrak{A}_i$ to $\bigotimes_{i\in I}\mathfrak{A}_i$ (315Kb). Express each a_i as $E_i^\bullet$, where $E_i\in\Sigma_i$. Then

$$E=\{x:x\in X,\,x(i)\in E_i\text{ for each }i\in J\}$$

has measure

$$\lambda E=\prod_{i\in J}\mu E_i=\prod_{i\in J}\bar{\mu}a_i\neq0,$$

while

$$E^\bullet=\psi(\inf_{i\in J}\tilde{\varepsilon}_i(a_i))\subseteq\psi(c),$$

so $\psi(c)\neq0$. As c is arbitrary, ψ is injective. **Q**

(b) Because ψ is injective, it is an isomorphism between $\bigotimes_{i\in I}\mathfrak{A}_i$ and its image in $\mathfrak{C}$. I trust it will cause no confusion if I abuse notation slightly and treat $\bigotimes_{i\in I}\mathfrak{A}_i$ as actually a subalgebra of $\mathfrak{C}$, so that $\varepsilon_j:\mathfrak{A}_j\to\mathfrak{C}$ becomes identified with $\tilde{\varepsilon}_j:\mathfrak{A}_j\to\bigotimes_{i\in I}\mathfrak{A}_i$. Now the Boolean homomorphisms $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ correspond to a Boolean homomorphism $\theta:\bigotimes_{i\in I}\mathfrak{A}_i\to\mathfrak{B}$. The point is that $\bar{\nu}\theta(c)=\bar{\lambda}c$ for every $c\in\bigotimes_{i\in I}\mathfrak{A}_i$. **P** Suppose to begin with that $c\in C$. Then we have $c=E^\bullet$, where $E=\{x:x(i)\in E_i\ \forall\ i\in J\}$ and $E_i\in\Sigma_i$ for each $i\in J$. So

$$\begin{aligned}\bar{\lambda}c=\lambda E=\prod_{i\in J}\mu E_i=\prod_{i\in J}\bar{\mu}_iE_i^\bullet=\bar{\nu}(\inf_{i\in J}\phi a_i)\\=\bar{\nu}(\inf_{i\in J}\theta\varepsilon_i(a_i))=\bar{\nu}\theta(\inf_{i\in J}\varepsilon_i(a_i))=\bar{\nu}\theta(c).\end{aligned}$$

Next, any $c\in\bigotimes_{i\in I}\mathfrak{A}_i$ is expressible as the supremum of a finite disjoint family $\langle c_k\rangle_{k\in K}$ in C (315Kb), so

$$\bar{\nu}\theta(c)=\sum_{k\in K}\bar{\nu}\theta(c_k)=\sum_{k\in K}\bar{\lambda}(c_k)=\bar{\lambda}c.\ \mathbf{Q}$$

(c) It follows that θ is uniformly continuous for the measure metrics defined by $\bar{\nu}$ and $\bar{\lambda}$, since

$$\bar\nu(\theta(c)\mathbin{\triangle}\theta(c'))=\bar\nu\theta(c\mathbin{\triangle}c')=\bar\lambda(c\mathbin{\triangle}c')$$

for all c, $c'\in\bigotimes_{i\in I}\mathfrak{A}_i$.

(d) Next, $\bigotimes_{i\in I}\mathfrak{A}_i$ is topologically dense in $\mathfrak{C}$. **P** Let $c\in\mathfrak{C}$, $\epsilon>0$. Express c as $W^\bullet$. Then by 254Fe there are $H_0,\dots,H_k\in\mathcal{C}$ such that $\lambda(W\triangle\bigcup_{j\le k}H_j)\le\epsilon$. Now $c_j=H_j^\bullet\in C$ for each j, so

$$c'=\sup_{j\le k}c_j=(\bigcup_{j\le k}H_j)^\bullet\in\bigotimes_{i\in I}\mathfrak{A}_i,$$

and $\bar\lambda(c\mathbin{\triangle}c')\le\epsilon$. **Q**

Since $\mathfrak{B}$ is complete for its uniformity (323Gc), there is a uniformly continuous function $\phi:\mathfrak{C}\to\mathfrak{B}$ extending θ (3A4G).

(e) Because θ is a Boolean homomorphism, so is ϕ. **P** (i) The functions $c\mapsto\phi(1\setminus c)$, $1\setminus\phi(c)$ are continuous and the topology of $\mathfrak{B}$ is Hausdorff, so $\{c:\phi(1\setminus c)=1\setminus\phi(c)\}$ is closed; as it includes $\bigotimes_{i\in I}\mathfrak{A}_i$, it must be the whole of $\mathfrak{C}$. (ii) The functions $(c,c')\mapsto\phi(c\cup c')$, $(c,c')\mapsto\phi(c)\cup\phi(c')$ are continuous, so $\{(c,c'):\phi(c\cup c')=\phi(c)\cup\phi(c')\}$ is closed in $\mathfrak{C}\times\mathfrak{C}$; as it includes $\bigotimes_{i\in I}\mathfrak{A}_I\times\bigotimes_{i\in I}\mathfrak{A}_i$, it must be the whole of $\mathfrak{C}\times\mathfrak{C}$. **Q**

(f) Because θ is measure-preserving, so is ϕ. **P** The functions $c\mapsto\bar\lambda c$, $c\mapsto\bar\nu\phi(c)$ are continuous and equal on $\bigotimes_{i\in I}\mathfrak{A}_i$, so are equal on $\mathfrak{C}$. **Q**

(g) Finally, to see that ϕ is unique, observe that any measure-preserving Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$ must be continuous, so that if it agrees with ϕ on $\bigotimes_{i\in I}\mathfrak{A}_i$ it must agree with ϕ on $\mathfrak{C}$.

325J Of course this leads at once to a result corresponding to 325D.

Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras.

(a) There is a probability algebra $(\mathfrak{C},\bar\lambda)$, together with measure-preserving Boolean homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{C}$ for $i\in I$, such that whenever $(\mathfrak{B},\bar\nu)$ is a probability algebra, and $\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ are Boolean homomorphisms such that $\bar\nu(\inf_{i\in J}\phi_i(a_i))=\prod_{i\in J}\bar\mu_ia_i$ whenever $J\subseteq I$ is finite and $a_i\in\mathfrak{A}_i$ for each $i\in J$, then there is a unique measure-preserving Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\phi_i$ for every $i\in I$.

(b) The structure $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ is determined up to isomorphism by this property.

(c) The Boolean homomorphism $\psi:\bigotimes_{i\in I}\mathfrak{A}_i\to\mathfrak{C}$ defined from the ε_i is injective, and $\psi[\bigotimes_{i\in I}\mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$.

proof For (a) and (c), all we have to do is represent each $(\mathfrak{A}_i,\bar\mu_i)$ as the measure algebra of a probability space, and apply 325I. The uniqueness of $\mathfrak{C}$ and the ε_i follows from the uniqueness of the homomorphisms ϕ, as in 325Db.

325K Definition As in 325Ea, we can say that $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ is a, or the, **probability algebra free product** of $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$.

325L Independent subalgebras If $(\mathfrak{A},\bar\mu)$ is a probability algebra, we say that a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **stochastically independent** if $\bar\mu(\inf_{i\in J}b_i)=\prod_{i\in J}\bar\mu b_i$ whenever $J\subseteq I$ is finite and $b_i\in\mathfrak{B}_i$ for each i. (Compare 272Ab.) If every $\mathfrak{B}_i$ is closed, so that $(\mathfrak{B}_i,\bar\mu\restriction\mathfrak{B}_i)$ is a probability algebra, the identity maps $\iota_i:\mathfrak{B}_i\to\mathfrak{A}$ satisfy the conditions of the universal mapping theorem 325Ja, so we have a probability algebra free product $(\mathfrak{C},\bar\mu\restriction\mathfrak{C},\langle\iota_i\rangle_{i\in I})$ of $\langle(\mathfrak{B}_i,\bar\mu\restriction\mathfrak{B}_i)\rangle_{i\in I}$, where $\mathfrak{C}=\bigvee_{i\in I}\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$.

Conversely, if $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ is any family of probability algebras with probability algebra free product $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$, then $\langle\varepsilon_i[\mathfrak{A}_i]\rangle_{i\in I}$ is an independent family of closed subalgebras of $\mathfrak{C}$. (Compare 272J, 315Xp.)

325M We can now make a general trawl through Chapters 25 and 27 seeking results which can be expressed in the language of this section. I give some in 325Xf-325Xi. Some ideas from §254 which are thrown into sharper relief by a reformulation are in the following theorem.

Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$ their probability algebra free product. For $J\subseteq I$ let $\mathfrak{C}_J=\bigvee_{i\in J}\varepsilon_i[\mathfrak{A}_i]$ be the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i\in J}\varepsilon_i[\mathfrak{A}_i]$.

(a) For any $J \subseteq I$, $(\mathfrak{C}_J, \bar{\lambda}\restriction\mathfrak{C}_J, \langle\varepsilon_i\rangle_{i\in J})$ is a probability algebra free product of $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in J}$.

(b)(i) For any $c \in \mathfrak{C}$, there is a unique smallest $J_c \subseteq I$ such that $c \in \mathfrak{C}_{J_c}$, and this J_c is countable.

(ii) If c, $d \in \mathfrak{C}$ and $c \subseteq d$, then there is an $e \in \mathfrak{C}_{J_c\cap J_d}$ such that $c \subseteq e \subseteq d$.

(c) For any non-empty family $\mathcal{J} \subseteq \mathcal{P}I$, $\bigcap_{J\in\mathcal{J}} \mathfrak{C}_J = \mathfrak{C}_{\bigcap\mathcal{J}}$.

proof (a) If $(\mathfrak{B}, \bar{\nu}, \langle\phi_i\rangle_{i\in J})$ is any probability algebra free product of $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in J}$, then we have a measure-preserving homomorphism $\psi : \mathfrak{B} \to \mathfrak{C}$ such that $\psi\phi_i = \varepsilon_i$ for every $i \in J$. Because the subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ generated by $\bigcup_{i\in J} \phi_i[\mathfrak{A}_i]$ is topologically dense in $\mathfrak{B}$ (325Jc), and ψ is continuous (324Kb), $\bigcup_{i\in J} \varepsilon_i[\mathfrak{A}_i]$ is topologically dense in $\psi[\mathfrak{B}]$; also $\psi[\mathfrak{B}]$ is closed in $\mathfrak{C}$ (324Kb again). But this means that $\psi[\mathfrak{B}]$ is just the topological closure of $\bigcup_{i\in I} \varepsilon_i[\mathfrak{A}_i]$ and must be $\mathfrak{C}_J$. Thus ψ is an isomorphism, and

$$(\mathfrak{C}_J, \bar{\lambda}\restriction\mathfrak{C}_J, \langle\varepsilon_i\rangle_{i\in J}) = (\psi[\mathfrak{B}], \bar{\nu}\psi^{-1}, \langle\psi\phi_i\rangle_{i\in J})$$

also is a probability algebra free product of $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in J}$.

(b) As in 325J, we may suppose that each $(\mathfrak{A}_i, \bar{\mu}_i)$ is the measure algebra of a probability space (X_i, Σ_i, μ_i), and that $\mathfrak{C}$ is the measure algebra of their product (X, Λ, λ). For $J \subseteq I$ let Λ_J be the set of members of Λ which are determined by coordinates in J. Then $\{x : x(i) \in E\} \in \Lambda_J$ for every $i \in J$ and $E \in \Sigma_i$; so $\{U^\bullet : U \in \Lambda_J\}$ is a closed subalgebra of $\mathfrak{C}$ including $\varepsilon_i[\mathfrak{A}_i]$ for every $i \in J$, and therefore including $\mathfrak{C}_J$. On the other hand, as observed in 254Ob, any member of Λ_J is approximated, in measure, by sets in the σ-algebra T_J generated by sets of the form $\{x : x(i) \in E\}$ where $i \in J$ and $E \in \Sigma_i$. Of course $\mathrm{T}_J \subseteq \Lambda_J$, so $\{W^\bullet : W \in \Lambda_J\} = \{W^\bullet : W \in \mathrm{T}_J\}$ is the closed subalgebra of $\mathfrak{C}$ generated by $\bigcup_{i\in J} \varepsilon_i[\mathfrak{A}_i]$, which is $\mathfrak{C}_J$.

(i) Let $W \in \Lambda$ be such that $c = W^\bullet$. By 254Rd, there is a smallest $J_c \subseteq I$ such that $W\triangle U$ is negligible for some $U \in \Lambda_{J_c}$, and J_c is countable. By the remarks above, J_c is also the unique smallest subset of I such that $c \in \mathfrak{C}_{J_c}$.

(ii) Let $U \in \Lambda_{J_c}$, $V \in \Lambda_{J_d}$ be such that $c = U^\bullet$ and $d = V^\bullet$. We can think of λ as a product $\lambda' \times \lambda''$ where λ' is the product measure on $X' = \prod_{i\in J_d} X_i$ and λ'' is the product measure on $X'' = \prod_{i\in I\setminus J_d} X_i$ (254N). Express V as $V_0 \times X''$ where $V_0 \subseteq X'$ belongs to the domain of λ' (254Ob). Consider

$$W_0 = \{z : z \in X',\ \{w : w \in X'',\ (z, w) \in U\} \text{ is not } \lambda''\text{-negligible}\};$$

then W_0 is measured by λ', by Fubini's theorem (252B or 252D). Because $c \subseteq d$, $U \setminus V$ is λ-negligible and $W_0 \setminus V_0$ is λ'-negligible, while W_0 is determined by coordinates in $J_c\cap J_d$. So $W = W_0 \times X''$ also is determined by coordinates in $J_c \cap J_d$, while $U \setminus W$ and $W \setminus V$ are λ-negligible. We can therefore take $e = W^\bullet$.

(c) Of course $\mathfrak{C}_K \subseteq \mathfrak{C}_J$ whenever $K \subseteq J \subseteq I$, so $\bigcap_{J\in\mathcal{J}} \mathfrak{C}_J \supseteq \mathfrak{C}_{\bigcap\mathcal{J}}$. On the other hand, suppose that $c \in \bigcap_{J\in\mathcal{J}} \mathfrak{C}_J$; then by (b-i) there is some $K \subseteq \bigcap\mathcal{J}$ such that $c \in \mathfrak{C}_K \subseteq \mathfrak{C}_{\bigcap\mathcal{J}}$. As c is arbitrary, $\bigcap_{J\in\mathcal{J}} \mathfrak{C}_J = \mathfrak{C}_{\bigcap\mathcal{J}}$.

***325N Notation** In this context, I will say that an element c of $\mathfrak{C}$ is **determined by coordinates in** J if $c \in \mathfrak{C}_J$.

325X Basic exercises (a) Let $(\mathfrak{A}_1, \bar{\mu}_1)$, $(\mathfrak{A}_2, \bar{\mu}_2)$ be two semi-finite measure algebras, and suppose that for each j we are given a closed subalgebra $\mathfrak{B}_j$ of $\mathfrak{A}_j$ such that $(\mathfrak{B}_j, \bar{\nu}_j)$ also is semi-finite, where $\bar{\nu}_j = \bar{\mu}_j\restriction\mathfrak{B}_j$. Show that the localizable measure algebra free product $(\mathfrak{B}_1, \bar{\nu}_1)\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{B}_2, \bar{\nu}_2)$ can be thought of as a closed subalgebra of $(\mathfrak{A}_1, \bar{\mu}_1)\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{A}_2, \bar{\mu}_2)$.

(b) Let $(\mathfrak{A}_1, \bar{\mu}_1)$ and $(\mathfrak{A}_2, \bar{\mu}_2)$ be two semi-finite measure algebras, and suppose that for each j we are given a principal ideal $\mathfrak{B}_j$ of $\mathfrak{A}_j$. Set $\bar{\nu}_j = \bar{\mu}_j\restriction\mathfrak{B}_j$. Show that the localizable measure algebra free product $(\mathfrak{B}_1, \bar{\nu}_1)\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{B}_2, \bar{\nu}_2)$ can be thought of as a principal ideal of $(\mathfrak{A}_1, \bar{\mu}_1)\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{A}_2, \bar{\mu}_2)$.

(c) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be semi-finite measure algebras with localizations $(\widehat{\mathfrak{A}}, \tilde{\mu})$ and $(\widehat{\mathfrak{B}}, \tilde{\nu})$. Show that the localizable measure algebra free products $(\mathfrak{A}, \bar{\mu})\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{B}, \bar{\nu})$ and $(\widehat{\mathfrak{A}}, \tilde{\mu})\widehat{\otimes}_{\mathrm{loc}}(\widehat{\mathfrak{B}}, \tilde{\nu})$ are isomorphic.

>(d) Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ and $\langle(\mathfrak{B}_j, \bar{\nu}_j)\rangle_{j\in J}$ be families of semi-finite measure algebras, with simple products $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ (322L). Show that the localizable measure algebra free product $(\mathfrak{A}, \bar{\mu})\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{B}, \bar{\nu})$ can be identified with the simple product of the family $\langle(\mathfrak{A}_i, \bar{\mu}_i)\widehat{\otimes}_{\mathrm{loc}}(\mathfrak{B}_j, \bar{\nu}_j)\rangle_{i\in I,j\in J}$.

>(e) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ and $\langle(\mathfrak{A}'_i,\bar\mu'_i)\rangle_{i\in I}$ be two families of probability algebras, and $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$, $(\mathfrak{C}',\bar\lambda',\langle\varepsilon'_i\rangle_{i\in I})$ their probability algebra free products. Suppose that for each $i\in I$ we are given a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}'_i$. Show that there is a unique measure-preserving Boolean homomorphism $\pi:\mathfrak{C}\to\mathfrak{C}'$ such that $\pi\varepsilon_i=\varepsilon'_i\pi_i$ for every $i\in I$.

>(f) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. We say that a family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ is **stochastically independent** if $\bar\mu(\inf_{i\in J}a_i)=\prod_{i\in J}\bar\mu a_i$ for every non-empty finite $J\subseteq I$. Show that this is so iff $\langle\mathfrak{A}_i\rangle_{i\in I}$ is stochastically independent, where $\mathfrak{A}_i=\{0,a_i,1\setminus a_i,1\}$ for each i. (Compare 272F.)

>(g) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\langle\mathfrak{A}_i\rangle_{i\in I}$ a stochastically independent family of closed subalgebras of $\mathfrak{A}$. Let $\langle J(k)\rangle_{k\in K}$ be a disjoint family of subsets of I, and for each $k\in K$ let $\mathfrak{B}_k=\bigvee_{i\in J(k)}\mathfrak{A}_i$ be the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in J(k)}\mathfrak{A}_i$. Show that $\langle\mathfrak{B}_k\rangle_{k\in K}$ is stochastically independent. (Compare 272K.)

(h) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra, and $\langle\mathfrak{A}_i\rangle_{i\in I}$ a stochastically independent family of closed subalgebras of $\mathfrak{A}$. For $J\subseteq I$ set $\mathfrak{B}_J=\bigvee_{i\in J}\mathfrak{A}_i$. Show that $\bigcap\{\mathfrak{B}_{I\setminus J}:J$ is a finite subset of $I\}=\{0,1\}$. (*Hint*: For $J\subseteq I$, show that $\bar\mu(b\cap c)=\bar\mu b\cdot\bar\mu c$ for every $b\in\mathfrak{B}_{I\setminus J}$ and $c\in\mathfrak{B}_J$. Compare 272O, 325M.)

(i) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras with probability algebra free product $(\mathfrak{C},\bar\lambda,\langle\varepsilon_i\rangle_{i\in I})$. For $J\subseteq I$ set $\mathfrak{C}_J=\bigvee_{i\in J}\varepsilon_i[\mathfrak{A}_i]$. Show that for any J, $K\subseteq I$ and $c\in\mathfrak{C}$, $\mathfrak{C}_J\cap\mathfrak{C}_K=\mathfrak{C}_{J\cap K}$ and the upper envelope $\operatorname{upr}(c,\mathfrak{C}_{J\cap K})$ is equal to $\operatorname{upr}(\operatorname{upr}(c,\mathfrak{C}_J),\mathfrak{C}_K)$.

325Y Further exercises (a) Let μ be counting measure on $X=\{0\}$, μ' the countable-cocountable measure on $X'=\omega_1$, and ν counting measure on $Y=\omega_1$. Show that the measure algebras of the primitive product measures on $X\times Y$, $X'\times Y$ are not isomorphic.

(b) Let $(\mathfrak{A}_1,\bar\mu_1)$, $(\mathfrak{A}_2,\bar\mu_2)$, $(\mathfrak{A}'_1,\bar\mu'_1)$ and $(\mathfrak{A}'_2,\bar\mu'_2)$ be semi-finite measure algebras with localizable measure algebra free products $(\mathfrak{C},\bar\lambda,\varepsilon_1,\varepsilon_2)$ and $(\mathfrak{C}',\bar\lambda',\varepsilon'_1,\varepsilon'_2)$. Suppose that $\pi_1:\mathfrak{A}_1\to\mathfrak{A}'_1$ and $\pi_2:\mathfrak{A}_2\to\mathfrak{A}'_2$ are measure-preserving Boolean homomorphisms. Show that there is a measure-preserving Boolean homomorphism $\pi:\mathfrak{C}\to\mathfrak{C}'$ such that $\pi\varepsilon_i=\varepsilon'_i\pi_i$ for both i, but that π is not necessarily unique.

(c) Let $\mathfrak{A}$ be a Boolean algebra, and $\mu:\mathfrak{A}\to[0,\infty]$ a functional such that $\mu0=0$, $\mu a>0$ for every $a\ne0$, and $\mu(a\cup b)=\mu a+\mu b$ whenever a, $b\in\mathfrak{A}$ and $a\cap b=0$; suppose that $\mathfrak{A}^f=\{a:\mu a<\infty\}$ is order-dense in $\mathfrak{A}$. For $e\in\mathfrak{A}^f$, a, $b\in\mathfrak{A}$ set $\rho_e(a,b)=\mu(e\cap(a\triangle b))$. Give $\mathfrak{A}$ the uniformity defined by $\{\rho_e:\mu e<\infty\}$. (i) Show that the completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ under this uniformity has a measure $\hat\mu$, extending μ, under which it is a localizable measure algebra. (ii) Show that if $a\in\widehat{\mathfrak{A}}$, $\hat\mu a<\infty$ and $\epsilon>0$, there is a $b\in\mathfrak{A}$ such that $\hat\mu(a\triangle b)\le\epsilon$. (iii) Show that for every $a\in\widehat{\mathfrak{A}}$ there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $a\supseteq\sup_{n\in\mathbb{N}}\inf_{m\ge n}a_m$ and $\hat\mu a=\hat\mu(\sup_{n\in\mathbb{N}}\inf_{m\ge n}a_m)$. (iv) In particular, the set of infima in $\widehat{\mathfrak{A}}$ of sequences in $\mathfrak{A}$ is order-dense in $\widehat{\mathfrak{A}}$. (v) Explain the relevance of this construction to the embedding $\mathfrak{A}_1\otimes\mathfrak{A}_2\subsetneqq\mathfrak{C}$ in 325D.

(d) In 325F, set $W=\bigcup_{n\in\mathbb{N}}E_n\times E_n$. Show that if A, B are any non-negligible subsets of $[0,1]$, then $W\cap(A\times B)$ is not negligible.

(e) Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of Lebesgue measure on $[0,1]$. Show that $\mathfrak{A}\otimes\mathfrak{A}$ is ccc but not weakly (σ,∞)-distributive. (*Hint*: (i) $\mathfrak{A}\otimes\mathfrak{A}$ is embeddable as a subalgebra of a probability algebra (ii) in the notation of 325F, look at $c_{mn}=\sup_{m\le i\le n}e_i\otimes e_i$.)

(f) Repeat 325F-325G and 325Yd-325Ye with an arbitrary atomless probability space in place of $[0,1]$.

(g) Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and $\langle a_i\rangle_{i\in I}$ a stochastically independent family in $\mathfrak{A}$. Show that for any $a\in\mathfrak{A}$ and $\epsilon>0$ the set $\{i:i\in I,\ |\bar\mu(a\cap a_i)-\bar\mu a\cdot\bar\mu a_i|\ge\epsilon\}$ is finite, so that $\{i:\bar\mu(a\cap a_i)\ne\bar\mu a\cdot\bar\mu a_i\}$ is countable. (*Hint*: 272Ye[2].)

[2]Formerly 272Yd.

325 Notes and comments 325B shows that the measure algebra of a product measure may be irregular if we have factor measures which are not strictly localizable. But two facts lead the way to the 'localizable measure algebra free product' in 325D-325E. The first is that every semi-finite measure algebra is embeddable, in a canonical way, in a localizable measure algebra (322P); and the second is that the Stone representation of a localizable measure algebra is strictly localizable (322O). It is a happy coincidence that we can collapse these two facts together in the construction of 325D. Another way of looking at the localizable measure algebra free product of two localizable measure algebras is to express it as the simple product of measure algebra free products of totally finite measure algebras, using 325Xd and the fact that for σ-finite measure algebras there is only one reasonable measure algebra free product, being that provided by any representation of them as measure algebras of measure spaces (325Eb).

Yet a third way of approaching measure algebra free products is as the uniform space completions of algebraic free products, using 325Yc. This gives the same result as the construction of 325D because the algebraic free product appears as a topologically dense subalgebra of the localizable measure algebra free product, which is complete as uniform space (325Dc). (I have to repeat such phrases as 'topologically dense' because the algebraic free product is emphatically *not* order-dense in the measure algebra free product (325F).) The results in 251I on approximating measurable sets for a c.l.d. product measure by combinations of measurable rectangles correspond to general facts about completions of finitely-additive measures (325Yc(ii), 325Yc(iii)). It is worth noting that the completion process can be regarded as made up of two steps; first take infima of sequences of sets of finite measure, and then take arbitrary suprema (325Yc(iv)).

The idea of 325F appears in many guises, and this is only the first time that I shall wish to call on it. The point of the set $W = \bigcup_{n\in\mathbb{N}} E_n \times E_n$ is that it is a measurable subset of the square (indeed, by taking the E_n to be open sets we can arrange that W should be open), of measure strictly less than 1 (in fact, as small as we wish), such that its complement does not include any non-negligible 'measurable rectangle' $G \times H$; indeed, $W \cap (A \times B)$ is non-negligible for any non-negligible sets A, $B \subseteq [0,1]$ (325Yd). I believe that the first published example of such a set was by Erdős & Oxtoby 55 (a version of which is in 532N in Volume 5); I learnt the method of 325F from R.O.Davies.

I include 325G as a kind of guard-rail. The relationship between preservation of measure and order-continuity is a subtle one, as I have already tried to show in 324K, and it is often worth considering the possibility that a result involving order-continuous measure-preserving homomorphisms has a form applying to all order-continuous homomorphisms. However, there is no simple expression of such an idea in the present context.

In the context of infinite free products of probability algebras, there is a degree of simplification, since there is only one algebra which can plausibly be called the probability algebra free product, and this is produced by any realization of the algebras as measure algebras of probability spaces (325I-325K). The examples 325F-325G apply equally, of course, to this context. At this point I mention the concept of 'stochastically independent' family (325L, 325Xf) because we have the machinery to translate several results from §272 into the language of measure algebras (325Xf-325Xh). I feel that I have to use the phrase 'stochastically independent' here because there is the much weaker alternative concept of 'Boolean independence' (315Xp) also present. But I leave most of this as exercises, because the language of measure algebras offers few ideas to the probability theory already covered in Chapter 27. All it can do is formalise the ever-present principle that negligible sets often can and should be ignored.

Version of 21.5.11

326 Additive functionals on Boolean algebras

I devote two sections to the general theory of additive functionals on measure algebras. As many readers will rightly be in a hurry to get on to the next two chapters, I remark that the only significant result needed for §§331-332 is the Hahn decomposition of a countably additive functional (326M), and that this is no more than a translation into the language of measure algebras of a theorem already given in Chapter 23. The concept of 'standard extension' of a countably additive functional from a subalgebra (327F-327G) will be used for a theorem in §333, and as preparation for Chapter 36.

I begin with notes on the space of additive functionals on an arbitrary Boolean algebra (326A-326D), corresponding to 231A-231B, but adding a more general form of the Jordan decomposition of a bounded additive functional into positive and negative parts (326D). The next four paragraphs are starred, because they will not be needed in this volume; 326E is essential if you want to look at additive functionals on free products, 326F is a basic classification criterion, and 326H is an important extension of a fundamental fact about atomless measures noted in 215D, but all can be passed over on first reading. The next subsection (326I-326M) deals with countably additive functionals, corresponding to 231C-231F. In 326N-326T I develop a new idea, that of 'completely additive' functional, which does not match anything in the previous treatment.

326A Additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **finitely additive**, or just **additive**, if $\nu(a \cup b) = \nu a + \nu b$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0$.

A non-negative additive functional is sometimes called a **finitely additive measure** or **charge**.

326B Elementary facts Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. The following will I hope be obvious.

(a) $\nu 0 = 0$ (because $\nu 0 = \nu 0 + \nu 0$).

(b) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is additive (because $(a \cap c) \cup (b \cap c) = (a \cup b) \cap c$).

(c) $\alpha\nu$ is an additive functional for any $\alpha \in \mathbb{R}$. If ν' is another finitely additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is additive.

(d) If $\langle \nu_i \rangle_{i \in I}$ is any family of finitely additive functionals such that $\nu' a = \sum_{i \in I} \nu_i a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$, then ν' is additive.

(e) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism, then $\nu\pi : \mathfrak{B} \to \mathbb{R}$ is additive. In particular, if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then $\nu \restriction \mathfrak{B} : \mathfrak{B} \to \mathbb{R}$ is additive.

(f) ν is non-negative iff it is order-preserving – that is,

$$\nu a \ge 0 \text{ for every } a \in \mathfrak{A} \iff \nu b \le \nu c \text{ whenever } b \subseteq c$$

(because $\nu c = \nu b + \nu(c \setminus b)$ if $b \subseteq c$).

326C The space of additive functionals Let $\mathfrak{A}$ be any Boolean algebra. From 326Bc we see that the set M of all finitely additive real-valued functionals on $\mathfrak{A}$ is a linear space (a linear subspace of $\mathbb{R}^{\mathfrak{A}}$). We give it the ordering induced by that of $\mathbb{R}^{\mathfrak{A}}$, so that $\nu \le \nu'$ iff $\nu a \le \nu' a$ for every $a \in \mathfrak{A}$. This renders it a partially ordered linear space (because $\mathbb{R}^{\mathfrak{A}}$ is).

326D The Jordan decomposition (I): Proposition Let $\mathfrak{A}$ be a Boolean algebra, and ν a finitely additive real-valued functional on $\mathfrak{A}$. Then the following are equiveridical:

(i) ν is bounded;

(ii) $\sup_{n \in \mathbb{N}} |\nu a_n| < \infty$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;

(iii) $\lim_{n \to \infty} |\nu a_n| = 0$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;

(iv) $\sum_{n=0}^{\infty} |\nu a_n| < \infty$ for every disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}$;

(v) ν is expressible as the difference of two non-negative additive functionals.

proof (a)(i)$\Rightarrow$(v) Assume that ν is bounded. For each $a \in \mathfrak{A}$, set

$$\nu^+ a = \sup\{\nu b : b \subseteq a\}.$$

Because ν is bounded, ν^+ is real-valued. Now ν^+ is additive. **P** If a, $b \in \mathfrak{A}$ and $a \cap b = 0$, then

$$\nu^+(a \cup b) = \sup_{c \subseteq a \cup b} \nu c = \sup_{d \subseteq a, e \subseteq b} \nu(d \cup e) = \sup_{d \subseteq a, e \subseteq b} \nu d + \nu e$$

(because $d \cap e \subseteq a \cap b = 0$ whenever $d \subseteq a$, $e \subseteq b$)

$$= \sup_{d \subseteq a} \nu d + \sup_{e \subseteq b} \nu e = \nu^+ a + \nu^+ b. \textbf{ Q}$$

Consequently $\nu^- = \nu^+ - \nu$ also is additive (326Bc).

Since

$$0 = \nu 0 \le \nu^+ a, \quad \nu a \le \nu^+ a$$

for every $a \in \mathfrak{A}$, $\nu^+ \ge 0$ and $\nu^- \ge 0$. Thus $\nu = \nu^+ - \nu^-$ is the difference of two non-negative additive functionals.

(b)(v)⇒(iv) If ν is expressible as $\nu_1 - \nu_2$, where ν_1 and ν_2 are non-negative additive functionals, and $\langle a_n \rangle_{n \in \mathbb{N}}$ is disjoint, then

$$\sum_{i=0}^{n} \nu_j a_i = \nu_j (\sup_{i \le n} a_i) \le \nu_j 1$$

for every n, both j, so that

$$\sum_{i=0}^{\infty} |\nu a_i| \le \sum_{i=0}^{\infty} \nu_1 a_i + \sum_{i=0}^{\infty} \nu_2 a_i \le \nu_1 1 + \nu_2 1 < \infty.$$

(c)(iv)⇒(iii)⇒(ii) are trivial.

(d) not-(i)⇒not-(ii) Suppose that ν is unbounded. Choose sequences $\langle a_n \rangle_{n \in \mathbb{N}}$, $\langle b_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $b_0 = 1$. Given that $\sup_{a \subseteq b_n} |\nu a| = \infty$, choose $c_n \subseteq b_n$ such that $|\nu c_n| \ge |\nu b_n| + n$; then $|\nu c_n| \ge n$ and

$$|\nu(b_n \setminus c_n)| = |\nu b_n - \nu c_n| \ge |\nu c_n| - |\nu b_n| \ge n.$$

We have

$$\begin{aligned} \infty = \sup_{a \subseteq b_n} |\nu a| &= \sup_{a \subseteq b_n} |\nu(a \cap c_n) + \nu(a \setminus c_n)| \\ &\le \sup_{a \subseteq b_n} |\nu(a \cap c_n)| + |\nu(a \setminus c_n)| \le \sup_{a \subseteq b_n \cap c_n} |\nu a| + \sup_{a \subseteq b_n \setminus c_n} |\nu a|, \end{aligned}$$

so at least one of $\sup_{a \subseteq b_n \cap c_n} |\nu a|$, $\sup_{a \subseteq b_n \setminus c_n} |\nu a|$ must be infinite; take b_{n+1} to be one of c_n, $b_n \setminus c_n$ such that $\sup_{a \subseteq b_{n+1}} |\nu a| = \infty$, and set $a_n = b_n \setminus b_{n+1}$, so that $|\nu a_n| \ge n$. Continue.

On completing the induction, we have a disjoint sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ such that $|\nu a_n| \ge n$ for every n, so that (ii) is false.

Remark I hope that this reminds you of the decomposition of a function of bounded variation as the difference of monotonic functions (224D).

***326E Additive functionals on free products** In Volume 4, when we return to the construction of measures on product spaces, the following fundamental fact will be useful.

Theorem Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a non-empty family of Boolean algebras, with free product $\mathfrak{A}$; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical maps, and

$$C = \{\inf_{j \in J} \varepsilon_j(a_j) : J \subseteq I \text{ is finite, } a_j \in \mathfrak{A}_j \text{ for every } j \in J\}.$$

Suppose that $\theta : C \to \mathbb{R}$ is such that

$$\theta c = \theta(c \cap \varepsilon_i(a)) + \theta(c \cap \varepsilon_i(1 \setminus a))$$

whenever $c \in C$, $i \in I$ and $a \in \mathfrak{A}_i$. Then there is a unique finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ extending θ.

proof (a) It will help if I note at once that $\theta 0 = 0$. **P**

$$\theta 0 = \theta(0 \cap \varepsilon_i(0)) + \theta(0 \cap \varepsilon_i(1)) = 2\theta 0$$

for any $i \in I$. **Q**

(b) The key is of course the following fact: if $\langle c_r \rangle_{r \le m}$ and $\langle d_s \rangle_{s \le n}$ are two disjoint families in C with the same supremum in $\mathfrak{A}$, then $\sum_{r=0}^{m} \theta c_r = \sum_{s=0}^{n} \theta d_s$. **P** Let $J \subseteq I$ be a finite set and $\mathfrak{B}_i \subseteq \mathfrak{A}_i$ a finite subalgebra, for each $i \in J$, such that every c_r and every d_s belongs to the subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ generated by $\{\varepsilon_j(b) : j \in J, b \in \mathfrak{B}_j\}$. Next, if $j \in J$ and $b \in \mathfrak{B}_j$, then

$$\sum_{r=0}^{m} \theta c_r = \sum_{r=0}^{m} \theta(c_r \cap \varepsilon_j(b)) + \sum_{r=0}^{m} \theta(c_r \setminus \varepsilon_j(b)).$$

We can therefore find a disjoint family $\langle c'_r\rangle_{r\le m'}$ in $C \cap \mathfrak{A}_0$ such that

$$\sup_{r\le m'} c'_r = \sup_{r\le m} c_r, \quad \sum_{r=0}^{m'} \theta c'_r = \sum_{r=0}^{m} \theta c_r,$$

and whenever $r \le m'$, $j \in J$ and $b \in \mathfrak{B}_j$ then either $c'_r \subseteq \varepsilon_j(b)$ or $c'_r \cap \varepsilon_j(b) = 0$; that is, every c'_r is either 0 or of the form $\inf_{j\in J} \varepsilon_j(b_j)$ where b_j is an atom of $\mathfrak{B}_j$ for every j. Similarly, we can find $\langle d'_s\rangle_{s\le n'}$ such that

$$\sup_{s\le n'} d'_s = \sup_{s\le n} d_s, \quad \sum_{s=0}^{n'} \theta d'_s = \sum_{s=0}^{n} \theta d_s,$$

and whenever $s \le n'$ and $j \in J$ then d'_s is either 0 or of the form $\inf_{j\in J} \varepsilon_j(b_j)$ where b_j is an atom of $\mathfrak{B}_j$ for every j. But we now have $\sup_{r\le m'} c'_r = \sup_{s\le n'} d'_s$ while for any $r \le m'$, $s \le n'$ either $c'_r = d'_s$ or $c'_r \cap d'_s = 0$. It follows that the non-zero terms in the finite sequence $\langle c'_r\rangle_{r\le m'}$ are just a rearrangement of the non-zero terms in $\langle d'_s\rangle_{s\le n'}$, so that

$$\sum_{r=0}^{m} \theta c_r = \sum_{r=0}^{m'} \theta c'_r = \sum_{s=0}^{n'} \theta d'_s = \sum_{s=0}^{n} \theta d_s,$$

as required. **Q**

(c) By 315Kb, this means that we have a functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu(\sup_{r\le m} c_r) = \sum_{r=0}^{m} \theta c_r$ whenever $\langle c_r\rangle_{r\le m}$ is a disjoint family in C. It is now elementary to check that ν is additive, and it is clearly the only additive functional on $\mathfrak{A}$ extending θ.

***326F** I give a couple of pages to an interesting property of additive functionals on Dedekind σ-complete Boolean algebras. I do not think it will be used in this book, and it really belongs to the theory of vector measures, which is hardly considered here, but the ideas are important, and the following definition has other uses.

Definition Let $\mathfrak{A}$ be a Boolean algebra, and ν a finitely additive functional on $\mathfrak{A}$. I will say that ν is **properly atomless** if for every $\epsilon > 0$ there is a finite partition $\langle a_i\rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le \epsilon$ whenever $i \in I$ and $a \subseteq a_i$.

***326G Lemma** Let $\mathfrak{A}$ be a Boolean algebra.

(a)(i) If ν, $\nu' : \mathfrak{A} \to \mathbb{R}$ are properly atomless finitely additive functionals and $\alpha \in \mathbb{R}$, then $\alpha\nu$ and $\nu + \nu'$ are properly atomless additive functionals.

(ii) If $\nu : \mathfrak{A} \to \mathbb{R}$ is a properly atomless finitely additive functional, then ν is bounded and ν can be expressed as the difference of two non-negative properly atomless additive functionals.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and that $\langle \nu_i\rangle_{i\in I}$ is a family of non-negative additive functionals on $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$ there are an $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ and an $a' \subseteq a$ such that $\nu_i a' = \alpha\nu_i a$ for every $i \in I$. Then for any $a \in \mathfrak{A}$ there is a non-decreasing family $\langle a_t\rangle_{t\in[0,1]}$ in $\mathfrak{A}$ such that $a_0 = 0$, $a_1 = a$ and $\nu_i a_t = t\nu_i a$ for every $t \in [0, 1]$ and $i \in I$.

(c) Suppose that $\mathfrak{A}$ is Dedekind σ-complete and that $\nu_0, \dots , \nu_n : \mathfrak{A} \to [0, \infty[$ are properly atomless additive functionals such that $\nu_i a \le \nu_0 a$ for every $i \le n$ and $a \in \mathfrak{A}$. Then for any $a \in \mathfrak{A}$ there is a non-decreasing family $\langle a_t\rangle_{t\in[0,1]}$ in $\mathfrak{A}$ such that $a_0 = 0$, $a_1 = a$ and $\nu_i a_t = t\nu_i a$ for every $t \in [0, 1]$ and $i \le n$.

proof (a)(i) Let $\epsilon > 0$. Then there are finite partitions $\langle a_i\rangle_{i\in I}$, $\langle b_j\rangle_{j\in J}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le \frac{\epsilon}{2+|\alpha}$ whenever $i \in I$ and $a \subseteq a_i$, while $|\nu' a| \le \frac{\epsilon}{2}$ whenever $j \in J$ and $a \subseteq b_j$. Now $|(\alpha\nu)(a)| \le \epsilon$ whenever $i \in I$ and $a \subseteq a_i$. Moreover, $\langle a_i \cap b_j\rangle_{(i,j)\in I\times J}$ is a finite partition of unity in $\mathfrak{A}$, and $|(\nu + \nu')(a)| \le \epsilon$ whenever $i \in I$, $j \in J$ and $a \subseteq a_i \cap b_j$.

(ii)(α) There is a finite partition $\langle c_j\rangle_{j\in J}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le 1$ whenever $i \in J$ and $a \subseteq c_j$; now $|\nu a| \le \sum_{j\in J} |\nu(a \cap c_j)| \le \#(J)$ for every $a \in \mathfrak{A}$, so ν is bounded.

(β) Define ν^+ as in part (a) of the proof of 326D, so that $\nu^+ : \mathfrak{A} \to [0, \infty[$ is additive. Now ν^+ is properly atomless. **P** Given $\epsilon > 0$, there is a finite partition $\langle a_i\rangle_{i\in I}$ of unity in $\mathfrak{A}$ such that $|\nu a| \le \epsilon$

whenever $i \in I$ and $a \subseteq a_i$; in which case $\nu^+ a = \sup_{b \subseteq a} \nu b \le \epsilon$ whenever $i \in I$ and $a \subseteq a_i$. **Q** As in 326D, $\nu^- = \nu^+ - \nu$ is non-negative, and by (i) just above (or otherwise) it is properly atomless, so $\nu = \nu^+ - \nu^-$ is the difference of non-negative properly atomless functionals.

(b) If $\nu_i a = 0$ for every $i \in I$, we can take $a_t = 0$ for $0 \le t < 1$ and $a_1 = a$. So suppose that $k \in I$ is such that $\nu_k a > 0$. For $i \in I$, set $\gamma_i = \dfrac{\nu_i a}{\nu_k a}$. Choose $\langle D_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $D_0 = \{0, a\}$. Given that D_n is a finite totally ordered subset of $\{b : b \subseteq a\}$ containing 0 and a and $\nu_i d = \gamma_i \nu_k d$ for every $d \in D_n$ and $i \in I$, then for each $d \in D_n \setminus \{a\}$ let d' be the next member of D_n strictly including d, and take $b_d \subseteq d' \setminus d$, $\alpha_d \in [\frac{1}{3}, \frac{2}{3}]$ such that $\nu_i b_d = \alpha_d \nu_i (d' \setminus d)$ for every $i \in I$. Then

$$\nu_i(d \cup b_d) = (1 - \alpha_d)\nu_i d + \alpha_d \nu_i d' = \gamma_i((1 - \alpha_d)\nu_k d + \alpha_d \nu_k d') = \gamma_i \nu_k (d \cup b_d)$$

for every i. Set $D_{n+1} = D_n \cup \{d \cup b_d : d \in D_n\}$; observe that D_{n+1} is still totally ordered, and continue. At the end of the induction, it is easy to see that $\nu_k(d' \setminus d) \le (\frac{2}{3})^n \nu_i a$ whenever $n \in \mathbb{N}$ and $d \subset d'$ are successive members of D_n.

Set $D = \bigcup_{n \in \mathbb{N}} D_n$. Then D is a countable totally ordered set with least element 0 and greatest element a, and $\{\nu_k d : d \in D\}$ is dense in $[0, \nu_k a]$. For $t \in \,]0, 1]$, set $a_t = \sup\{d : d \in D,\ \nu_k d \le t\nu_k a\}$; this is where we need to know that $\mathfrak{A}$ is Dedekind σ-complete. Set $a_0 = 0$. Then $\langle a_t \rangle_{t \in [0,1]}$ is a non-decreasing family with $a_0 = 0$ and $a_1 = a$. If $0 < t < 1$, $i \in I$ and $\epsilon > 0$, there are d, $d' \in D$ such that

$$t\nu_k a - \epsilon \le \nu_k d \le t\nu_k a < \nu_k d' \le t\nu_k a + \epsilon,$$

$$t\nu_i a - \gamma_i \epsilon \le \nu_i d \le t\nu_i a < \nu_i d' \le t\nu_i a + \gamma_i \epsilon;$$

in this case $d \subseteq a_t \subseteq d'$, so

$$t\nu_i a - \gamma_i \epsilon \le \nu_i a_t \le t\nu_i a + \gamma_i \epsilon;$$

as ϵ is arbitrary, $\nu_i a_t = t\nu_i a$. Thus we have a suitable family $\langle a_t \rangle_{t \ge 0}$.

(c) Induce on n.

(i) The induction starts with a single non-negative properly atomless functional ν_0. Now for any $a \in \mathfrak{A}$ there is an $a' \subseteq a$ such that $\frac{1}{3}\nu_0 a \le \nu_0 a' \le \frac{2}{3}\nu_0 a$. **P** This is trivial if $\nu_0 a = 0$. Otherwise, let C be a finite partition of unity in $\mathfrak{A}$ such that $\nu_0 c \le \frac{1}{3}\nu_0 a$ for every $c \in C$. Enumerate C as $\langle c_i \rangle_{i < m}$ and for $i \le m$ set $b_i = a \cap \sup_{j < i} c_j$. Then $b_0 = 0$, $b_m = a$ and $\nu_0 b_{i+1} - \nu_0 b_i \le \nu_0 c_i \le \frac{1}{3}\nu_0 a$ for each i. So there must be an $i \le m$ such that $\frac{1}{3}\nu_0 a \le \nu_0 b_i \le \frac{2}{3}\nu_0 a$, and we can set $a' = b_i$. **Q**

Now (b), with $I = \{0\}$, gives the result.

(ii) For the inductive step to $n \ge 1$, I show first that if $a \in \mathfrak{A}$ there is an $a' \subseteq a$ such that $\nu_i a' = \frac{1}{2}\nu_i a$ for every $i \le n$. **P** By the inductive hypothesis, we have a non-decreasing family $\langle a_t \rangle_{t \in [0,1]}$ such that $a_0 = 0$, $a_1 = a$ and $\nu_i a_t = t\nu_i a$ whenever $t \in [0, 1]$ and $i < n$. Now observe that for $0 \le s \le t \le 1$,

$$|\nu_n a_t - \nu_n a_s| = \nu_n(a_t \setminus a_s) \le \nu_0(a_t \setminus a_s) = (t - s)\nu_0 a.$$

So the functions $t \mapsto \nu_n a_t : [0, 1] \to [0, \infty[$ and $f : [0, \frac{1}{2}] \to [0, \infty[$ are continuous, where $f(t) = \nu_n a_{t+\frac{1}{2}} - \nu_n a_t$ for $0 \le t \le \frac{1}{2}$. However, $f(0) + f(\frac{1}{2}) = \nu_n a$, so $\frac{1}{2}\nu_n a$ lies between $f(0)$ and $f(\frac{1}{2})$ and there is a $t \in [0, \frac{1}{2}]$ such that $f(t) = \frac{1}{2}\nu_n a$. Set $a' = a_{t+\frac{1}{2}} \setminus a_t$; then $\nu_i a' = \frac{1}{2}\nu_i a$ for every $i \le n$, as required. **Q**

Once again (b), with $I = \{0, \dots, n\}$, shows that for any $a \in \mathfrak{A}$ we have a non-decreasing family $\langle a_t \rangle_{t \in [0,1]}$ such that $a_0 = 0$, $a_1 = 1$ and $\nu_i a_t = t\nu_i a$ whenever $t \in [0, 1]$ and $i \le n$.

***326H Liapounoff's convexity theorem** (Liapounoff 1940) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $r \ge 1$ an integer. Suppose that $\nu : \mathfrak{A} \to \mathbb{R}^r$ is additive in the sense that $\nu(a \cup b) = \nu a + \nu b$ whenever $a \cap b = 0$ (see 361B), and properly atomless in the sense that for every $\epsilon > 0$ there is a finite partition $\langle a_j \rangle_{j \in J}$ of unity in $\mathfrak{A}$ such that $\|\nu a\| \le \epsilon$ whenever $j \in J$ and $a \subseteq a_j$. Then $\{\nu a : a \in \mathfrak{A}\}$ is a convex set in $\mathbb{R}^r$.([3])

proof For $1 \le i \le r$, let ν_i be the ith component of ν, so that $\nu a = \langle \nu_i a \rangle_{1 \le i \le r}$ for each $a \in \mathfrak{A}$. Then every ν_i is additive. Moroever, it is properly atomless. **P** Given $\epsilon > 0$, there is a finite partition $\langle a_j \rangle_{j \in J}$ of unity

[3] I learnt this version of the theorem from K.P.S.Bhaskara Rao.

in $\mathfrak{A}$ such that $|\nu_i a| \le \|\nu a\| \le \epsilon$ whenever $j \in J$ and $a \subseteq a_j$. **Q** So we can express ν_i as $\nu_i^+ - \nu_i^-$ where ν_i^+ and ν_i^- are non-negative properly atomless non-negative functionals (326G(a-ii)). Set $\tilde{\nu}a = \sum_{i=1}^{r} \nu_i^+ a + \nu_i^- a$ for $a \in \mathfrak{A}$. Then $\tilde{\nu}$ is again properly atomless (326G(a-i)).

Suppose that x, $y \in \nu[\mathfrak{A}]$ and $\alpha \in [0,1]$. Let a, $b \in \mathfrak{A}$ be such that $\nu a = x$ and $\nu b = y$. By 326Gc, applied to $\tilde{\nu}, \nu_1^+, \nu_1^-, \dots, \nu_r^+, \nu_r^-$, there is an $c \subseteq a \setminus b$ such that

$$\nu_i^+ c = \alpha \nu_i^+ (a \setminus b), \quad \nu_i^- c = \alpha \nu_i^- (a \setminus b),$$

for every $i \le r$, so that $\nu_i c = \alpha \nu_i (a \setminus b)$ for every $i \le r$. Similarly, there is a $d \subseteq b \setminus a$ such that $\nu d = (1-\alpha)\nu(b \setminus d)$. Now $e = c \cup (a \cap b) \cup d$,

$$\begin{aligned} \alpha x + (1-\alpha) y &= \alpha \nu a + (1-\alpha)\nu b \\ &= \alpha \nu(a \setminus b) + \alpha \nu(a \cap b) + (1-\alpha)\nu(a \cap b) + (1-\alpha)\nu(b \setminus a) \\ &= \nu c + \nu(a \cap b) + \nu d = \nu(c \cup (a \cap b) \cup d) \in \nu[\mathfrak{A}]. \end{aligned}$$

As x, y and α are arbitrary, $\nu[\mathfrak{A}]$ is convex.

326I Countably additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **countably additive** or **σ-additive** if $\sum_{n=0}^{\infty} \nu a_n$ is defined and equal to $\nu(\sup_{n\in\mathbb{N}} a_n)$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ and $\sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{A}$.

A warning is perhaps in order. It can happen that $\mathfrak{A}$ is presented to us as a subalgebra of a larger algebra $\mathfrak{B}$; for instance, $\mathfrak{A}$ might be an algebra of sets, a subalgebra of some σ-algebra $\Sigma \subseteq \mathcal{P}X$. In this case, there may be sequences in $\mathfrak{A}$ which have a supremum in $\mathfrak{A}$ which is not a supremum in $\mathfrak{B}$ (indeed, this will happen just when the embedding is not sequentially order-continuous). So we can have a countably additive functional $\nu : \mathfrak{B} \to \mathbb{R}$ such that $\nu \restriction \mathfrak{A}$ is not countably additive in the sense used here. A similar phenomenon will arise when we come to the Daniell integral in Volume 4 (§436).

326J Elementary facts Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional.

(a) ν is finitely additive. (Setting $a_n = 0$ for every n, we see from the definition in 326I that $\nu 0 = 0$. Now, given $a \cap b = 0$, set $a_0 = a$, $a_1 = b$, $a_n = 0$ for $n \ge 2$ to see that $\nu(a \cup b) = \nu a + \nu b$.)

(b) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with a supremum $a \in \mathfrak{A}$, then

$$\nu a = \nu a_0 + \sum_{n=0}^{\infty} \nu(a_{n+1} \setminus a_n) = \lim_{n\to\infty} \nu a_n.$$

(c) If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with an infimum $a \in \mathfrak{A}$, then $\langle a_0 \setminus a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum $a_0 \setminus a$, so

$$\nu a = \nu a_0 - \nu(a_0 \setminus a) = \nu a_0 - \lim_{n\to\infty} \nu(a_0 \setminus a_n) = \lim_{n\to\infty} \nu a_n.$$

(d) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is countably additive. (For $\sup_{n\in\mathbb{N}} a_n \cap c = c \cap \sup_{n\in\mathbb{N}} a_n$ whenever the right-hand-side is defined, by 313Ba.)

(e) $\alpha\nu$ is a countably additive functional for any $\alpha \in \mathbb{R}$. If ν' is another countably additive functional on $\mathfrak{A}$, then $\nu + \nu'$ is countably additive.

(f) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{B} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism, then $\nu\pi$ is a countably additive functional on $\mathfrak{B}$. (For if $\langle b_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$ with supremum b, then $\langle \pi b_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence with supremum πb.)

(g) If $\mathfrak{A}$ is Dedekind σ-complete and $\mathfrak{B}$ is a σ-subalgebra of $\mathfrak{A}$, then $\nu \restriction \mathfrak{B} : \mathfrak{B} \to \mathbb{R}$ is countably additive. (For the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is sequentially order-continuous, by 314Gb.)

326K Corollary Let $\mathfrak{A}$ be a Boolean algebra and ν a finitely additive real-valued functional on $\mathfrak{A}$.

(a) ν is countably additive iff $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0 in $\mathfrak{A}$.

(b) If ν' is an additive functional on $\mathfrak{A}$ and $|\nu' a| \le \nu a$ for every $a \in \mathfrak{A}$, and ν is countably additive, then ν' is countably additive.

(c) If ν is non-negative, then ν is countably additive iff it is sequentially order-continuous.

proof (a)(i) If ν is countably additive and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\lim_{n\to\infty} \nu a_n = 0$ by 326Jc. **(ii)** If ν satisfies the condition, and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, set $b_n = a \setminus \sup_{i\le n} a_i$ for each $n \in \mathbb{N}$; then $\langle b_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so

$$\nu a - \sum_{i=0}^{n} \nu a_i = \nu a - \nu(\sup_{i\le n} a_i) = \nu b_n \to 0$$

as $n \to \infty$, and $\nu a = \sum_{n=0}^{\infty} \nu a_n$; thus ν is countably additive.

(b) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, set $b_n = \sup_{i\le n} a_i$ for each n; then $\nu a = \lim_{n\to\infty} \nu b_n$, so

$$\lim_{n\to\infty} |\nu' a - \nu' b_n| = \lim_{n\to\infty} |\nu'(a \setminus b_n)| \le \lim_{n\to\infty} \nu(a \setminus b_n) = 0,$$

and

$$\sum_{n=0}^{\infty} \nu' a_n = \lim_{n\to\infty} \nu' b_n = \nu' a.$$

(c) If ν is countably additive, then it is sequentially order-continuous by 326Jb-326Jc. If ν is sequentially order-continuous, then of course it satisfies the condition of (a), so is countably additive.

326L The Jordan decomposition (II): Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a bounded countably additive real-valued functional on $\mathfrak{A}$. Then ν is expressible as the difference of two non-negative countably additive functionals.

proof Consider the functional $\nu^+ a = \sup_{b\subseteq a} \nu b$ defined in the proof of 326D. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum a, and $b \subseteq a$, then

$$\nu b = \sum_{n=0}^{\infty} \nu(b \cap a_n) \le \sum_{n=0}^{\infty} \nu^+ a_n.$$

As b is arbitrary, $\nu^+ a \le \sum_{n=0}^{\infty} \nu^+ a_n$. But of course

$$\nu^+ a \ge \nu^+(\sup_{i\le n} a_i) = \sum_{i=0}^{n} \nu^+ a_i$$

for every $n \in \mathbb{N}$, so $\nu^+ a = \sum_{n=0}^{\infty} \nu^+ a_n$. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ν^+ is countably additive.

Now $\nu^- = \nu^+ - \nu$ also is countably additive, and $\nu = \nu^+ - \nu^-$ is the difference of non-negative countably additive functionals.

326M The Hahn decomposition: Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional. Then ν is bounded and there is a $c \in \mathfrak{A}$ such that $\nu a \ge 0$ whenever $a \subseteq c$, while $\nu a \le 0$ whenever $a \cap c = 0$.

first proof By 314M, there are a set X and a σ-algebra Σ of subsets of X and a sequentially order-continuous Boolean homomorphism π from Σ onto $\mathfrak{A}$. Set $\nu_1 = \nu\pi : \Sigma \to \mathbb{R}$. Then ν_1 is countably additive (326Jf). So ν_1 is bounded and there is a set $H \in \Sigma$ such that $\nu_1 F \ge 0$ whenever $F \in \Sigma$ and $F \subseteq H$ and $\nu_1 F \le 0$ whenever $F \in \Sigma$ and $F \cap H = \emptyset$ (231Eb). Set $c = \pi H \in \mathfrak{A}$. If $a \subseteq c$, then there is an $F \in \Sigma$ such that $\pi F = a$; now $\pi(F \cap H) = a \cap c = a$, so $\nu a = \nu_1(F \cap H) \ge 0$. If $a \cap c = 0$, then there is an $F \in \Sigma$ such that $\pi F = a$; now $\pi(F \setminus H) = a \setminus c = a$, so $\nu a = \nu_1(F \setminus H) \le 0$.

second proof (a) Note first that ν is bounded. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, then $\sum_{n=0}^{\infty} \nu a_n$ must exist and be equal to $\nu(\sup_{n\in\mathbb{N}} a_n)$; in particular, $\lim_{n\to\infty} \nu a_n = 0$. By 326D, ν is bounded. **Q**

(b)(i) We know that $\gamma = \sup\{\nu a : a \in \mathfrak{A}\} < \infty$. Choose a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\nu a_n \ge \gamma - 2^{-n}$ for every $n \in \mathbb{N}$. For $m \le n \in \mathbb{N}$, set $b_{mn} = \inf_{m\le i\le n} a_i$. Then $\nu b_{mn} \ge \gamma - 2 \cdot 2^{-m} + 2^{-n}$ for every $n \ge m$. **P** Induce on n. For $n = m$, this is due to the choice of $a_m = b_{mm}$. For the inductive step, we have $b_{m,n+1} = b_{mn} \cap a_{n+1}$, while surely $\gamma \ge \nu(a_{n+1} \cup b_{mn})$, so

$$\begin{aligned}\gamma + \nu b_{m,n+1} &\ge \nu(a_{n+1} \cup b_{mn}) + \nu(a_{n+1} \cap b_{mn})\\ &= \nu a_{n+1} + \nu b_{mn} \ge \gamma - 2^{-n-1} + \gamma - 2 \cdot 2^{-m} + 2^{-n}\end{aligned}$$

(by the choice of a_{n+1} and the inductive hypothesis)

$$= 2\gamma - 2 \cdot 2^{-m} + 2^{-n-1}.$$

Subtracting γ from both sides, $\nu b_{m,n+1} \ge \gamma - 2 \cdot 2^{-m} + 2^{-n-1}$ and the induction proceeds. **Q**

(ii) Set

$$b_m = \inf_{n \ge m} b_{mn} = \inf_{n \ge m} a_n.$$

Then

$$\nu b_m = \lim_{n\to\infty} \nu b_{mn} \ge \gamma - 2 \cdot 2^{-m},$$

by 326Jc. Next, $\langle b_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, so setting $c = \sup_{n\in\mathbb{N}} b_n$ we have

$$\nu c = \lim_{n\to\infty} \nu b_n \ge \gamma;$$

since νc is surely less than or equal to γ, $\nu c = \gamma$.

If $b \in \mathfrak{A}$ and $b \subseteq c$, then

$$\nu c - \nu b = \nu(c \setminus b) \le \gamma = \nu c,$$

so $\nu b \ge 0$. If $b \in \mathfrak{A}$ and $b \cap c = 0$ then

$$\nu c + \nu b = \nu(c \cup b) \le \gamma = \nu c$$

so $\nu b \le 0$. This completes the proof.

326N Completely additive functionals: Definition Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{R}$ is **completely additive** or **τ-additive** if it is finitely additive and $\inf_{a\in A} |\nu a| = 0$ whenever A is a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0.

326O Basic facts Let $\mathfrak{A}$ be a Boolean algebra and ν a completely additive real-valued functional on $\mathfrak{A}$.

(a) ν is countably additive. **P** If $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then for any infinite $I \subseteq \mathbb{N}$ the set $\{a_i : i \in I\}$ is downwards-directed and has infimum 0, so $\inf_{i\in I} |\nu a_i| = 0$; which means that $\lim_{n\to\infty} \nu a_n$ must be zero. By 326Ka, ν is countably additive. **Q**

(b) Let A be a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0. Then for every $\epsilon > 0$ there is an $a \in A$ such that $|\nu b| \le \epsilon$ whenever $b \subseteq a$. **P?** Suppose, if possible, otherwise. Set

$$B = \{b : |\nu b| \ge \epsilon, \exists\, a \in A,\ b \supseteq a\}.$$

If $a \in A$ there is a $b' \subseteq a$ such that $|\nu b'| > \epsilon$. Now $\{a' \setminus b' : a' \in A,\ a' \subseteq a\}$ is downwards-directed and has infimum 0, so there is an $a' \in A$ such that $a' \subseteq a$ and $|\nu(a' \setminus b')| \le |\nu b'| - \epsilon$. Set $b = b' \cup a'$; then $a' \subseteq b$ and

$$|\nu b| = |\nu b' + \nu(a' \setminus b')| \ge |\nu b'| - |\nu(a' \setminus b')| \ge \epsilon,$$

so $b \in B$. But also $b \subseteq a$. Thus every member of A includes some member of B. Since every member of B includes a member of A, B is downwards-directed and has infimum 0; but this is impossible, since $\inf_{b\in B} |\nu b| \ge \epsilon$. **XQ**

(c) If ν is non-negative, it is order-continuous. **P** (i) If A is a non-empty upwards-directed set with supremum a_0, then $\{a_0 \setminus a : a \in A\}$ is a non-empty downwards-directed set with infimum 0, so

$$\sup_{a\in A} \nu a = \nu a_0 - \inf_{a\in A} \nu(a_0 \setminus a) = \nu a_0.$$

(ii) If A is a non-empty downwards-directed set with infimum a_0, then $\{a \setminus a_0 : a \in A\}$ is a non-empty downwards-directed set with infimum 0, so

$$\inf_{a\in A} \nu a = \nu a_0 + \inf_{a\in A} \nu(a \setminus a_0) = \nu a_0.\ \mathbf{Q}$$

(d) If $c \in \mathfrak{A}$, then $a \mapsto \nu(a \cap c)$ is completely additive. **P** If A is a non-empty downwards-directed set with infimum 0, so is $\{a \cap c : a \in A\}$, and $\inf_{a\in A} |\nu(a \cap c)| = 0$. **Q**

(e) $\alpha\nu$ is a completely additive functional for any $\alpha\in\mathbb{R}$. If ν' is another completely additive functional on $\mathfrak{A}$, then $\nu+\nu'$ is completely additive. **P** We know from 326Bc that $\nu+\nu'$ is additive. Let A be a non-empty downwards-directed set with infimum 0. For any $\epsilon>0$, (b) tells us that there are a, $a'\in A$ such that $|\nu b|\le\epsilon$ whenever $b\subseteq a$ and $|\nu' b|\le\epsilon$ whenever $b\subseteq a'$. But now, because A is downwards-directed, there is a $b\in A$ such that $b\subseteq a\cap a'$, which means that $|\nu b+\nu' b|\le|\nu b|+|\nu' b|$ is at most 2ϵ. As ϵ is arbitrary, $\inf_{a\in A}|(\nu+\nu')(a)|=0$, and $\nu+\nu'$ is completely additive. **Q**

(f) If $\mathfrak{B}$ is another Boolean algebra and $\pi:\mathfrak{B}\to\mathfrak{A}$ is an order-continuous Boolean homomorphism, then $\nu\pi$ is a completely additive functional on $\mathfrak{B}$. **P** By 326Be, $\nu\pi$ is additive. If $B\subseteq\mathfrak{B}$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{B}$, then $\pi[B]$ is a non-empty downwards-directed set with infimum 0 in $\mathfrak{A}$, because π is order-continuous, so $\inf_{b\in B}|\nu\pi b|=0$. **Q** In particular, if $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then $\nu\restriction\mathfrak{B}$ is completely additive.

(g) If ν' is another additive functional on $\mathfrak{A}$ and $|\nu' a|\le\nu a$ for every $a\in\mathfrak{A}$, then ν' is completely additive. **P** If $A\subseteq\mathfrak{A}$ is non-empty and downwards-directed and $\inf A=0$, then $\inf_{a\in A}|\nu' a|\le\inf_{a\in A}\nu a=0$. **Q**

326P I squeeze a useful fact in here.

Proposition If $\mathfrak{A}$ is a ccc Boolean algebra, a functional $\nu:\mathfrak{A}\to\mathbb{R}$ is countably additive iff it is completely additive.

proof If ν is completely additive it is countably additive, by 326Oa. If ν is countably additive and A is a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0, then there is a (non-empty) countable subset B of A also with infimum 0 (316E). Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a sequence running over B, and choose $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $a_0=b_0$, $a_{n+1}\subseteq a_n\cap b_n$ for every $n\in\mathbb{N}$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0, so $\lim_{n\to\infty}\nu a_n=0$ (326Jc) and $\inf_{a\in A}|\nu a|=0$. As A is arbitrary, ν is completely additive.

326Q The Jordan decomposition (III): Proposition Let $\mathfrak{A}$ be a Boolean algebra and ν a completely additive real-valued functional on $\mathfrak{A}$. Then ν is bounded and expressible as the difference of two non-negative completely additive functionals.

proof (a) I must first check that ν is bounded. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Set

$$A=\{a:a\in\mathfrak{A}\text{, there is an }n\in\mathbb{N}\text{ such that }a_i\subseteq a\text{ for every }i\ge n\}.$$

Then A is closed under $\cap$, and if b is any lower bound for A then $b\subseteq 1\setminus a_n\in A$, so $b\cap a_n=0$, for every $n\in\mathbb{N}$; but this means that $1\setminus b\in A$, so that $b\subseteq 1\setminus b$ and $b=0$. Thus $\inf A=0$. By 326Ob, there is an $a\in A$ such that $|\nu b|\le 1$ whenever $b\subseteq a$. By the definition of A, there must be an $n\in\mathbb{N}$ such that $|\nu a_i|\le 1$ for every $i\ge n$. But this means that $\sup_{n\in\mathbb{N}}|\nu a_n|$ is finite. As $\langle a_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ν is bounded, by 326D(ii). **Q**

(b) As in 326D and 326L, set $\nu^+a=\sup_{b\subseteq a}\nu b$ for every $a\in\mathfrak{A}$. Then ν^+ is completely additive. **P** We know that ν^+ is additive. If A is a non-empty downwards-directed subset of $\mathfrak{A}$ with infimum 0, then for every $\epsilon>0$ there is an $a\in A$ such that $|\nu b|\le\epsilon$ whenever $b\subseteq a$; in particular, $\nu^+a\le\epsilon$. As ϵ is arbitrary, $\inf_{a\in A}\nu^+a=0$; as A is arbitrary, ν^+ is completely additive. **Q**

Consequently $\nu^-=\nu^+-\nu$ is completely additive (326Oe) and $\nu=\nu^+-\nu^-$ is the difference of non-negative completely additive functionals.

326R I give an alternative definition of 'completely additive' which you may feel clarifies the concept.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\nu:\mathfrak{A}\to\mathbb{R}$ a function. Then the following are equiveridical:

(i) ν is completely additive;

(ii) $\nu 1=\sum_{i\in I}\nu a_i$ whenever $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$;

(iii) $\nu a=\sum_{i\in I}\nu a_i$ whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a.

proof (For notes on sums $\sum_{i\in I}$, see 226A.)

(a)(i)⇒(ii) If ν is completely additive and $\langle a_i\rangle_{i\in I}$ is a partition of unity in A, then (inducing on $\#(J)$) $\nu(\sup_{i\in J}a_i)=\sum_{i\in J}\nu a_i$ for every finite $J\subseteq I$. Consider

$$A = \{1 \setminus \sup_{i\in J} a_i : J \subseteq I \text{ is finite}\}.$$

Then A is non-empty and downwards-directed and has infimum 0, so for every $\epsilon > 0$ there is an $a \in A$ such that $|\nu b| \le \epsilon$ whenever $b \subseteq a$ (326Ob again). Express a as $1 \setminus \sup_{i\in J} a_i$ where $J \subseteq I$ is finite. If now K is another finite subset of I including J,

$$|\nu 1 - \sum_{i\in K} a_i| = |\nu(1 \setminus \sup_{i\in K} a_i)| \le \epsilon.$$

As remarked in 226Ad, this means that $\nu 1 = \sum_{i\in I} \nu a_i$, as claimed.

(b)(ii)⇒(iii) Suppose that ν satisfies the condition (ii), and that $\langle a_i\rangle_{i\in I}$ is a disjoint family with supremum a. Take any $j \notin I$, set $J = I \cup \{j\}$ and $a_j = 1 \setminus a$; then $\langle a_i\rangle_{i\in J}$, $(a, 1 \setminus a)$ are both partitions of unity, so

$$\nu(1 \setminus a) + \nu a = \nu 1 = \sum_{i\in J} \nu a_i = \nu(1 \setminus a) + \sum_{i\in I} \nu a_i,$$

and $\nu a = \sum_{i\in I} \nu a_i$.

(c)(iii)⇒(i) Suppose that ν satisfies (iii). Then ν is additive.

(α) ν is bounded. **P** Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{A}$. Applying Zorn's Lemma to the set $\mathcal{C}$ of all disjoint families $C \subseteq \mathfrak{A}$ including $\{a_n : n \in \mathbb{N}\}$, we find a partition of unity $C \supseteq \{a_n : n \in \mathbb{N}\}$. Now $\sum_{c\in C} \nu c$ is defined in $\mathbb{R}$, so $\sup_{n\in\mathbb{N}} |\nu a_n| \le \sup_{c\in C} |\nu c|$ is finite. By 326D, ν is bounded. **Q**

(β) Define ν^+ from ν as in 326D. Then ν^+ satisfies the same condition as ν. **P** Let $\langle a_i\rangle_{i\in I}$ be a disjoint family in $\mathfrak{A}$ with supremum a. Then for any $b \subseteq a$, we have $b = \sup_{i\in I} b \cap a_i$, so

$$\nu b = \sum_{i\in I} \nu(b \cap a_i) \le \sum_{i\in I} \nu^+ a_i.$$

Thus $\nu^+ a \le \sum_{i\in I} \nu^+ a_i$. But of course

$$\begin{aligned}\sum_{i\in I} \nu^+ a_i &= \sup\{\sum_{i\in J} \nu^+ a_i : J \subseteq I \text{ is finite}\}\\ &= \sup\{\nu^+(\sup_{i\in J} a_i) : J \subseteq I \text{ is finite}\} \le \nu^+ a,\end{aligned}$$

so $\nu^+ a = \sum_{i\in I} \nu^+ a_i$. **Q**

(γ) It follows that ν^+ is completely additive. **P** If A is a non-empty downwards-directed set with infimum 0, then $B = \{b : \exists\, a \in A,\, b \cap a = 0\}$ is order-dense in $\mathfrak{A}$, so there is a partition of unity $\langle b_i\rangle_{i\in I}$ lying in B (313K). Now if $J \subseteq I$ is finite, there is an $a \in A$ such that $a \cap \sup_{i\in J} b_i = 0$ (because A is downwards-directed), and

$$\nu^+ a + \sum_{i\in J} \nu^+ b_i \le \nu^+ 1.$$

Since $\nu^+ 1 = \sup_{J\subseteq I \text{ is finite}} \sum_{i\in J} \nu^+ b_i$, $\inf_{a\in A} \nu^+ a = 0$. As A is arbitrary, ν^+ is completely additive. **Q**

(δ) Now consider $\nu^- = \nu^+ - \nu$. Of course

$$\nu^- a = \nu^+ a - \nu a = \sum_{i\in I} \nu^+ a_i - \sum_{i\in I} \nu a_i = \sum_{i\in I} \nu^- a_i$$

whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ with supremum a. Because ν^- is non-negative, the argument of (γ) shows that $\nu^- = (\nu^-)^+$ is completely additive. So $\nu = \nu^+ - \nu^-$ is completely additive, as required.

326S For completely additive functionals, we have a useful refinement of the Hahn decomposition. I give it in a form adapted to the applications I have in mind.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a completely additive functional. Then there is a unique element of $\mathfrak{A}$, which I will denote $[\![\nu > 0]\!]$, 'the region where $\nu > 0$', such that $\nu a > 0$ whenever $0 \ne a \subseteq [\![\nu > 0]\!]$, while $\nu a \le 0$ whenever $a \cap [\![\nu > 0]\!] = 0$.

proof Set

$$C_1 = \{c : c \in \mathfrak{A} \setminus \{0\},\, \nu a > 0 \text{ whenever } 0 \ne a \subseteq c\},$$

$$C_2 = \{c : c \in \mathfrak{A},\, \nu a \le 0 \text{ whenever } a \subseteq c\}.$$

Then $C_1 \cup C_2$ is order-dense in $\mathfrak{A}$. **P** There is a $c_0 \in \mathfrak{A}$ such that $\nu a \geq 0$ for every $a \subseteq c_0$ and $\nu a \leq 0$ whenever $a \cap c_0 = 0$ (326M). Given $b \in \mathfrak{A} \setminus \{0\}$, then $b \setminus c_0 \in C_2$, so if $b \setminus c_0 \neq 0$ we can stop. Otherwise, $b \subseteq c_0$. If $b \in C_1$ we can stop. Otherwise, there is a non-zero $c \subseteq b$ such that $\nu c \leq 0$; but in this case $\nu a \geq 0$ and $\nu(c \setminus a) \geq 0$ so $\nu a = 0$ for every $a \subseteq c$, and $c \in C_2$. **Q**

There is therefore a partition of unity $D \subseteq C_1 \cup C_2$. Now $D \cap C_1$ is countable. **P** If $d \in D \cap C_1$, $\nu d > 0$. Also

$$\#(\{d : d \in D,\, \nu d \geq 2^{-n}\}) \leq 2^n \sup_{a \in \mathfrak{A}} \nu a$$

is finite for each n, so $D \cap C_1$ is the union of a sequence of finite sets, and is countable. **Q**

Accordingly $D \cap C_1$ has a supremum e. If $0 \neq a \subseteq e$ then

$$\nu a = \sum_{c \in D} \nu(a \cap c) = \sum_{c \in D \cap C_1} \nu(a \cap c) \geq 0$$

by 326R. Also there must be some $c \in D \cap C_1$ such that $a \cap c \neq 0$, in which case $\nu(a \cap c) > 0$, so that $\nu a > 0$. If $a \cap e = 0$, then

$$\nu a = \sum_{c \in D} \nu(a \cap c) = \sum_{c \in D \cap C_2} \nu(a \cap c) \leq 0.$$

Thus e has the properties demanded of $[\![\nu > 0]\!]$. To see that e is unique, we need observe only that if e' has the same properties then $\nu(e \setminus e') \leq 0$ (because $(e \setminus e') \cap e' = 0$), so $e \setminus e' = 0$ (because $e \setminus e' \subseteq e$). Similarly, $e' \setminus e = 0$ and $e = e'$. Thus we may properly denote e by the formula $[\![\nu > 0]\!]$.

326T Corollary Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and μ, ν two completely additive functionals on $\mathfrak{A}$. Then there is a unique element of $\mathfrak{A}$, which I will denote $[\![\mu > \nu]\!]$, 'the region where $\mu > \nu$', such that

$$\mu a > \nu a \text{ whenever } 0 \neq a \subseteq [\![\mu > \nu]\!],$$

$$\mu a \leq \nu a \text{ whenever } a \cap [\![\mu > \nu]\!] = 0.$$

proof Apply 326S to the functional $\mu - \nu$, and set $[\![\mu > \nu]\!] = [\![\mu - \nu > 0]\!]$.

326X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that (i) $\nu(a \cup b) = \nu a + \nu b - \nu(a \cap b)$ (ii) $\nu(a \cup b \cup c) = \nu a + \nu b + \nu c - \nu(a \cap b) - \nu(a \cap c) - \nu(b \cap c) + \nu(a \cap b \cap c)$ for all a, b, $c \in \mathfrak{A}$. Generalize these results to longer sequences in $\mathfrak{A}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that a finitely additive functional ν is properly atomless iff there is a properly atomless additive functional ν' such that $|\nu a| \leq \nu' a$ for every $a \in \mathfrak{A}$. (ii) Show that a non-negative finitely additive functional ν on $\mathfrak{A}$ is properly atomless iff whenever ν' is a non-zero finitely additive functional such that $0 \leq \nu' a \leq \nu a$ for every $a \in \mathfrak{A}$ there is an $a \in \mathfrak{A}$ such that $\nu' a$ and $\nu'(1 \setminus a)$ are both non-zero.

(c)(i) Suppose that $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ is countably additive. Show that $\mathcal{I} = \{a : \nu b = 0 \text{ for every } b \subseteq a\}$ is an ideal of $\mathfrak{A}$. Show that the following are equiveridical: (α) ν is properly atomless; (β) whenever $\nu a \neq 0$ there is a $b \subseteq a$ such that $\nu b \notin \{0, \nu a\}$; ($\gamma$) the quotient algebra $\mathfrak{A}/\mathcal{I}$ is atomless. (ii) Find an atomless Dedekind complete Boolean algebra $\mathfrak{A}$ and a finitely additive $\nu : \mathfrak{A} \to [0, 1]$ such that $\nu a > 0$ for every non-zero $a \in \mathfrak{A}$ but ν is not properly atomless.

(d) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that the following are equiveridical: (i) ν is countably additive; (ii) $\lim_{n\to\infty} \nu a_n = \nu a$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a.

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Show that the following are equiveridical: (i) ν is countably additive; (ii) $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} \sup_{m\geq n} a_m = 0$; (iii) $\lim_{n\to\infty} \nu a_n = \nu a$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $a = \inf_{n\in\mathbb{N}} \sup_{m\geq n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\geq n} a_m$. (*Hint*: for (i)⇒(iii), consider non-negative ν first.)

(f) Let X be an uncountable set, and J an infinite subset of X. Let $\mathfrak{A}$ be the finite-cofinite algebra of X (316Yl), and for $a \in A$ set $\nu a = \#(a \cap J)$ if a is finite, $-\#(J \setminus a)$ if a is cofinite. Show that ν is countably additive and unbounded.

>(g) Let $\mathfrak{A}$ be the algebra of subsets of $[0,1]$ generated by the family of (closed) intervals. Show that there is a unique additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu[\alpha, \beta] = \beta - \alpha$ whenever $0 \le \alpha \le \beta \le 1$. Show that ν is countably additive but not completely additive.

(h)(i) Let (X, Σ, μ) be any atomless probability space. Show that $\mu : \Sigma \to \mathbb{R}$ is a countably additive functional which is not completely additive. (ii) Let X be any uncountable set and μ the countable-cocountable measure on X (211R). Show that μ is countably additive but not completely additive.

(i) Let $\mathfrak{A}$ be an atomless Boolean algebra. Show that every completely additive functional on $\mathfrak{A}$ is properly atomless.

(j) Let $\mathfrak{A}$ be a Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a function. (i) Show that ν is finitely additive iff $\sum_{i \in I} \nu a_i = \nu 1$ for every finite partition of unity $\langle a_i \rangle_{i \in I}$. (ii) Show that ν is countably additive iff $\sum_{i \in I} \nu a_i = \nu 1$ for every countable partition of unity $\langle a_i \rangle_{i \in I}$.

(k) Show that 326S can fail if ν is only countably additive, rather than completely additive. (*Hint*: 326Xh.)

(l) Let $\mathfrak{A}$ be a Boolean algebra and ν a finitely additive real-valued functional on $\mathfrak{A}$. Let us say that $a \in \mathfrak{A}$ is a **support** of ν if (α) $\nu b = 0$ whenever $b \cap a = 0$ (β) for every non-zero $b \subseteq a$ there is a $c \subseteq b$ such that $\nu c \ne 0$. (i) Check that ν can have at most one support. (ii) Show that if a is a support for ν and ν is bounded, then the principal ideal $\mathfrak{A}_a$ generated by a is ccc. (iii) Show that if $\mathfrak{A}$ is Dedekind σ-complete and ν is countably additive, then ν is completely additive iff it has a support, and that in the language of 326S this is $[\![\nu > 0]\!] \cup [\![-\nu > 0]\!]$. (iv) Taking $J = X$ in 326Xf, show that X is the support of the functional ν there.

326Y Further exercises (a) Show that there is a finitely additive functional $\nu : \mathcal{P}\mathbb{N} \to \mathbb{R}$ such that $\nu\{n\} = 1$ for every $n \in \mathbb{N}$, so that ν is not bounded. (*Hint*: Use Zorn's Lemma to construct a maximal linearly independent subset of ℓ^∞ including $\{\chi\{n\} : n \in \mathbb{N}\}$, and hence to construct a linear map $f : \ell^\infty \to \mathbb{R}$ such that $f(\chi\{n\}) = 1$ for every n.)

(b) Let $\mathfrak{A}$ be any infinite Boolean algebra. Show that there is an unbounded finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$. (*Hint*: let $\langle t_n \rangle_{n \in \mathbb{N}}$ be a sequence of distinct points in the Stone space of $\mathfrak{A}$, and set $\nu a = \nu'\{n : t_n \in \widehat{a}\}$ for a suitable ν'.)

(c) Let $\mathfrak{A}$ be a Boolean algebra, and give $\mathbb{R}^{\mathfrak{A}}$ its product topology. Show that the space of finitely additive functionals on $\mathfrak{A}$ is a closed subset of $\mathbb{R}^{\mathfrak{A}}$, but that the space of bounded finitely additive functionals is closed only when $\mathfrak{A}$ is finite.

(d) Let $\mathfrak{A}$ be a Boolean algebra, and M the linear space of all bounded finitely additive real-valued functionals on $\mathfrak{A}$. For ν, $\nu' \in M$ say that $\nu \le \nu'$ if $\nu a \le \nu' a$ for every $a \in \mathfrak{A}$. Show that

(i) ν^+, as defined in the proof of 326D, is just $\sup\{0, \nu\}$ in M;

(ii) M is a Dedekind complete Riesz space (241E-241F, 353H);

(iii) for ν, $\nu' \in M$, $|\nu| = \nu \vee (-\nu)$, $\nu \vee \nu'$ and $\nu \wedge \nu'$ are given by the formulae

$$|\nu|(a) = \sup_{b \subseteq a} \nu b - \nu(a \setminus b), \quad (\nu \vee \nu')(a) = \sup_{b \subseteq a} \nu b + \nu'(a \setminus b),$$

$$(\nu \wedge \nu')(a) = \inf_{b \subseteq a} \nu b + \nu'(a \setminus b);$$

(iv) for any non-empty $A \subseteq M$, A is bounded above in M iff

$$\sup\{\textstyle\sum_{i=0}^n \nu_i a_i : \nu_i \in A \text{ for each } i \le n,\ \langle a_i \rangle_{i \le n} \text{ is disjoint}\}$$

is finite, and then $\sup A$ is defined by the formula

$$(\sup A)(a) = \sup\{\sum_{i=0}^{n} \nu_i a_i : \nu_i \in A \text{ for each } i \le n,\ \langle a_i\rangle_{i\le n} \text{ is disjoint, } \sup_{i\le n} a_i = a\}$$

for every $a \in \mathfrak{A}$;

(v) setting $\|\nu\| = |\nu|(1)$, $\|\ \|$ is an order-continuous norm (definition: 354Dc) on M under which M is a Banach lattice.

(e) Let $\mathfrak{A}$ be a Boolean algebra. A functional $\nu : \mathfrak{A} \to \mathbb{C}$ is **finitely additive** if its real and imaginary parts are. Show that the space of bounded finitely additive functionals from $\mathfrak{A}$ to $\mathbb{C}$ is a Banach space under the total variation norm $\|\nu\| = \sup\{\sum_{i=0}^{n} |\nu a_i| : \langle a_i\rangle_{i\le n}$ is a partition of unity in $\mathfrak{A}\}$.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and μ, ν finitely additive functionals on $\mathfrak{A}$, $\mathfrak{B}$ respectively. Show that there is a unique finitely additive functional λ on the free product $\mathfrak{A}\otimes\mathfrak{B} \to \mathbb{R}$ such that $\lambda(a\otimes b) = \mu a\cdot\nu b$ for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

(g) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with free product $(\bigotimes_{i\in I}\mathfrak{A}_i, \langle\varepsilon_i\rangle_{i\in I})$, and for each $i \in I$ let ν_i be a finitely additive functional on $\mathfrak{A}_i$ such that $\nu_i 1 = 1$. Show that there is a unique finitely additive functional $\nu : \bigotimes_{i\in I}\mathfrak{A}_i \to \mathbb{R}$ such that $\nu(\inf_{i\in J}\varepsilon_i(a_i)) = \prod_{i\in J}\nu_i a_i$ whenever $J \subseteq I$ is non-empty and finite and $a_i \in \mathfrak{A}_i$ for each $i \in J$.

(h) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty[$ a countably additive functional. Show that ν is properly atomless iff whenever $a \in \mathfrak{A}$ and $\nu a \ne 0$ there is a $b \subseteq a$ such that $0 < \nu b < \nu a$.

(i) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional. Show that $\nu[\mathfrak{A}]$ is a compact subset of $\mathbb{R}$.

(j) Let $\mathfrak{G}$ be the regular open algebra of $\mathbb{R}$ (314P). Find a properly atomless finitely additive $\nu : \mathfrak{G} \to \mathbb{R}$ such that $\nu[\mathfrak{G}]$ is not compact.

(k) (Halmos 1948) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $r \ge 1$ an integer. (i) Let $C \subseteq \mathbb{R}^r$ be a non-empty bounded convex set, and for $z \in \mathbb{R}^r$ set $H_z = \{x : x\,.\,z = \sup_{y\in C} y\,.\,z\}$. Suppose that $H_z \cap \overline{C} \subseteq C$ for every $z \in \mathbb{R}^r \setminus \{0\}$. Show that C is closed. (ii) Suppose that $\nu : \mathfrak{A} \to \mathbb{R}^r$ is countably additive in the sense that all its coordinates are countably additive functionals. Show that $\nu[\mathfrak{A}]$ is compact.

(l) Let $\mathfrak{A}$ be a Boolean algebra, and give it the topology $\mathfrak{T}_\sigma$ for which the closed sets are the sequentially order-closed sets. Show that a finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is countably additive iff it is continuous for $\mathfrak{T}_\sigma$.

(m) Let $\mathfrak{A}$ be a Boolean algebra, and M_σ the set of all bounded countably additive real-valued functionals on $\mathfrak{A}$. Show that M_σ is a closed and order-closed linear subspace of the normed space M of all additive functionals on $\mathfrak{A}$ (326Yd), and that $|\nu| \in M_\sigma$ whenever $\nu \in M_\sigma$.

(n) Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative finitely additive functional on $\mathfrak{A}$. Set

$$\nu_\sigma a = \inf\{\sup_{n\in\mathbb{N}} \nu a_n : \langle a_n\rangle_{n\in\mathbb{N}} \text{ is a non-decreasing sequence with supremum } a\}$$

for every $a \in \mathfrak{A}$. Show that ν_σ is countably additive, and is $\sup\{\nu' : \nu' \le \nu$ is countably additive$\}$.

(o) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\langle\nu_n\rangle_{n\in\mathbb{N}}$ a sequence of countably additive real-valued functionals on $\mathfrak{A}$ such that $\nu a = \lim_{n\to\infty}\nu_n a$ is defined in $\mathbb{R}$ for every $a \in \mathfrak{A}$. Show that ν is countably additive.

(p) Let $\mathfrak{A}$ be a Boolean algebra, and M_τ the set of all completely additive real-valued functionals on $\mathfrak{A}$. Show that M_τ is a closed and order-closed linear subspace of the normed space M of all additive functionals, and that $|\nu| \in M_\tau$ whenever $\nu \in M_\tau$.

(q) Let $\mathfrak{A}$ be a Boolean algebra and ν a non-negative finitely additive functional on $\mathfrak{A}$. Set

$$\nu_\tau b = \inf\{\sup_{a\in A}\nu a : A \text{ is a non-empty upwards-directed set with supremum } b\}$$

for every $b \in \mathfrak{A}$. Show that ν_τ is completely additive, and is $\sup\{\nu' : \nu' \le \nu$ is completely additive$\}$.

(r) Let $\mathfrak{A}$ be a Boolean algebra, and give it the topology $\mathfrak{T}$ for which the closed sets are the order-closed sets (313Xb). Show that a finitely additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is completely additive iff it is continuous for $\mathfrak{T}$.

(s) Let X be a set, Σ any σ-algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a functional. Show that ν is completely additive iff there are sequences $\langle x_n \rangle_{n\in\mathbb{N}}$, $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ such that $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $\nu E = \sum_{n=0}^{\infty} \alpha_n \chi E(x_n)$ for every $E \in \Sigma$.

326 Notes and comments I have not mentioned the phrase 'measure algebra' anywhere in this section, and in principle this material could have been part of Chapter 31; but countably additive functionals are kissing cousins of measures, and most of the ideas here surely belong to 'measure theory' rather than to 'Boolean algebra', in so far as such divisions are meaningful at all. I have given as much as possible of the theory in a general form because the simplifications which are possible when we look only at measure algebras are seriously confusing if they are allowed too much prominence. In particular, it is important to understand that the principal properties of completely additive functionals do not depend on Dedekind completeness of the algebra, provided we take care over the definitions. Similarly, the definition of 'countably additive' functional for algebras which are not Dedekind σ-complete needs a moment's attention to the phrase 'and $\sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{A}$'. It can happen that a functional is countably additive mostly because there are too few such sequences (326Xf).

The formulations I have chosen as principal definitions (326A, 326I, 326N) are those which I find closest to my own intuitions of the concepts, but you may feel that 326K(i), 326Xe(iii) and 326R, or 326Yl and 326Yr, provide useful alternative patterns. The point is that countable additivity corresponds to sequential order-continuity (326Jb, 326Jc, 326Jf), while complete additivity corresponds to order-continuity (326Oc, 326Of); the difficulty is that we must consider functionals which are not order-preserving, so that the simple definitions in 313H cannot be applied directly. It is fair to say that all the additive functionals ν we need to understand are bounded, and therefore may be studied in terms of their positive and negative parts ν^+, ν^-, which are order-preserving (326Bf); but many of the most important applications of these ideas depend precisely on using facts about ν to deduce facts about ν^+ and ν^-.

It is in 326D that we seem to start getting more out of the theory than we have put in. The ideas here have vast ramifications. What it amounts to is that we can discover much more than we might expect by looking at disjoint sequences. To begin with, the conditions here lead directly to 326M and 326Q: every completely additive functional is bounded, and every countably additive functional on a Dedekind σ-complete Boolean algebra is bounded. (But note 326Ya-326Yb.)

I have expressed 326H in terms of an additive function from a Boolean algebra to a finite-dimensional space (it is already non-trivial in the two-dimensional case, which would correspond to an additive complex-valued functional, as in 326Ye). It is usually regarded as a theorem about countably additive functions, or 'vector measures' (see 394O below), but rather remarkably we do not in fact need countable additivity. Of course it can also be regarded as a kind of ham-sandwich theorem for measures; we can simultaneously bisect an element of a Dedekind σ-complete Boolean algebra with respect to finitely many additive functionals. If you like, the dimensionality requirement of the ordinary ham-sandwich theorems of topology is met by the requirement of atomlessness here. A companion result, also due to Liapounoff, which requires countable additivity but allows atoms, is in 362Yx.

Naturally enough, the theory of countably additive functionals on general Boolean algebras corresponds closely to the special case of countably additive functionals on σ-algebras of sets, already treated in §§231-232 for the sake of the Radon-Nikodým theorem. This should make 326I-326M very straightforward. When we come to completely additive functionals, however, there is room for many surprises. The natural map from a σ-algebra of measurable sets to the corresponding measure algebra is sequentially order-continuous but rarely order-continuous, so that there can be completely additive functionals on the measure algebra which do not correspond to completely additive functionals on the σ-algebra. Indeed there are very few completely additive functionals on σ-algebras of sets (326Ys). Of course these surprises can arise only when there is a difference between completely additive and countably additive functionals, that is, when the algebra involved is not ccc (326P). But I think that neither 326Q nor 326R is obvious.

I find myself generally using the phrase 'countably additive' in preference to 'completely additive' in the context of ccc algebras, where there is no difference between them. This is an attempt at user-friendliness;

the phrase 'countably additive' is the commoner one in ordinary use. But I must say that my personal inclination is to the other side. The reason why so many theorems apply to countably additive functionals in these contexts is just that they are completely additive.

I have given two proofs of 326M. I certainly assume that if you have got this far you are acquainted with the Radon-Nikodým theorem and the associated basic facts about countably additive functionals on σ-algebras of sets; so that the 'first proof' should be easy and natural. On the other hand, there are purist objections on two fronts. First, it relies on the Stone representation, which involves a much stronger form of the axiom of choice than is actually necessary. Second, the classical Hahn decomposition in 231E is evidently a special case of 326M, and if we need both (as we certainly do) then one expects the ideas to stand out more clearly if they are applied directly to the general case. In fact the two versions of the argument are so nearly identical that (as you will observe, if you have Volume 2 to hand) they can share nearly every word. You can take the 'second proof', therefore, as a worked example in the translation of ideas from the context of σ-algebras of sets to the context of Dedekind σ-complete Boolean algebras. What makes it possible is the fact that the only limit operations referred to involve countable families.

Arguments not involving limit operations can generally, of course, be applied to all Boolean algebras; I have lifted some exercises (326Yd, 326Yn) from §231 to give you some practice in such generalizations.

Almost any non-trivial measure provides an example of a countably additive functional on a Dedekind σ-complete algebra which is not completely additive (326Xh). The question of whether such a functional can exist on a Dedekind complete algebra is the 'Banach-Ulam problem', to which I will return in 363S.

In this section I have looked only at questions which can be adequately treated in terms of the underlying algebras $\mathfrak{A}$, without using any auxiliary structure. To go much farther we shall need to study the 'function spaces' $S(\mathfrak{A})$ and $L^\infty(\mathfrak{A})$ of Chapter 36. In particular, the ideas of 326Ya, 326Yd-326Ye and 326Ym-326Yq will make better sense when redeveloped in §362.

Version of 13.7.11

327 Additive functionals on measure algebras

When we turn to measure algebras, we have a simplification, relative to the general context of §326, because the algebras are always Dedekind σ-complete; but there are also elaborations, because we can ask how the additive functionals we examine are related to the measure. In 327A-327C I work through the relationships between the concepts of 'absolute continuity', '(true) continuity' and 'countable additivity', following §232, and adding 'complete additivity' from §326. These ideas provide a new interpretation of the Radon-Nikodým theorem (327D). I then use this theorem to develop some machinery (the 'standard extension' of an additive functional from a closed subalgebra to the whole algebra, 327F-327G) which will be used in §333.

327A I start with the following definition and theorem corresponding to 232A-232B.

Definition Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Then ν is **absolutely continuous** with respect to $\bar{\mu}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu a| \le \epsilon$ whenever $\bar{\mu} a \le \delta$.

327B Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and $\nu : \mathfrak{A} \to \mathbb{R}$ a finitely additive functional. Give $\mathfrak{A}$ its measure-algebra topology and uniformity (§323).

(a) If ν is continuous at 0, it is completely additive.

(b) If ν is countably additive, it is absolutely continuous with respect to $\bar{\mu}$.

(c) The following are equiveridical:

(i) ν is continuous at 0;

(ii) ν is countably additive and whenever $a \in \mathfrak{A}$ and $\nu a \neq 0$ there is a $b \in \mathfrak{A}$ such that $\bar{\mu} b < \infty$ and $\nu(a \cap b) \neq 0$;

(iii) ν is continuous everywhere on $\mathfrak{A}$;

(iv) ν is uniformly continuous.

(d) If $(\mathfrak{A},\bar\mu)$ is semi-finite, then ν is continuous iff it is completely additive.

(e) If $(\mathfrak{A},\bar\mu)$ is σ-finite, then ν is continuous iff it is countably additive iff it is completely additive.

(f) If $(\mathfrak{A},\bar\mu)$ is totally finite, then ν is continuous iff it is absolutely continuous with respect to $\bar\mu$ iff it is countably additive iff it is completely additive.

proof (a) If ν is continuous, and $A\subseteq\mathfrak{A}$ is non-empty, downwards-directed and has infimum 0, then $0\in\overline{A}$ (323D(b-ii)), so $\inf_{a\in A}|\nu a|=0$.

(b) ? Suppose, if possible, that ν is countably additive but not absolutely continuous. Then there is an $\epsilon>0$ such that for every $\delta>0$ there is an $a\in\mathfrak{A}$ such that $\bar\mu a\le\delta$ but $|\nu a|\ge\epsilon$. For each $n\in\mathbb{N}$ we may choose a $b_n\in\mathfrak{A}$ such that $\bar\mu b_n\le 2^{-n}$ and $|\nu b_n|\ge\epsilon$. Consider $b_n^*=\sup_{k\ge n}b_k$, $b=\inf_{n\in\mathbb{N}}b_n^*$. Then we have

$$\bar\mu b\le\inf_{n\in\mathbb{N}}\bar\mu(\sup_{k\ge n}b_k)\le\inf_{n\in\mathbb{N}}\sum_{k=n}^\infty 2^{-k}=0,$$

so $\bar\mu b=0$ and $b=0$. On the other hand, ν is expressible as a difference $\nu^+-\nu^-$ of non-negative countably additive functionals (326L), each of which is sequentially order-continuous (326Kc), and

$$0=\lim_{n\to\infty}(\nu^++\nu^-)b_n^*\ge\inf_{n\in\mathbb{N}}(\nu^++\nu^-)b_n\ge\inf_{n\in\mathbb{N}}|\nu b_n|\ge\epsilon,$$

which is absurd. **X**

(c)(i)⇒(ii) Suppose that ν is continuous at 0. Then it is completely additive, by (a), therefore countably additive. If $\nu a\ne0$, there must be a b of finite measure such that $|\nu d|<|\nu a|$ whenever $d\cap b=0$, so that $|\nu(a\setminus b)|<|\nu a|$ and $\nu(a\cap b)\ne0$. Thus the conditions are satisfied.

(ii)⇒(iv) Now suppose that ν satisfies the two conditions in (ii). Because $\mathfrak{A}$ is Dedekind σ-complete, ν must be bounded (326M), therefore expressible as the difference $\nu^+-\nu^-$ of countably additive functionals. Set $\nu_1=\nu^++\nu^-$. Set

$$\gamma=\sup\{\nu_1 b: b\in\mathfrak{A},\,\bar\mu b<\infty\},$$

and choose a sequence $\langle b_n\rangle_{n\in\mathbb{N}}$ of elements of $\mathfrak{A}$ of finite measure such that $\lim_{n\to\infty}\nu_1 b_n=\gamma$; set $b^*=\sup_{n\in\mathbb{N}}b_n$. If $d\in\mathfrak{A}$ and $d\cap b^*=0$ then $\nu d=0$. **P** If $b\in\mathfrak{A}$ and $\bar\mu b<\infty$, then

$$|\nu(d\cap b)|\le\nu_1(d\cap b)\le\nu_1(b\setminus b_n)=\nu_1(b\cup b_n)-\nu_1 b_n\le\gamma-\nu_1 b_n$$

for every $n\in\mathbb{N}$, so $\nu(d\cap b)=0$. As b is arbitrary, the second condition in (ii) tells us that $\nu d=0$. **Q**

Setting $b_n^*=\sup_{k\le n}b_k$ for each n, we have $\lim_{n\to\infty}\nu_1(b^*\setminus b_n^*)=0$. Take any $\epsilon>0$, and (using (b) above) let $\delta>0$ be such that $|\nu a|\le\epsilon$ whenever $\bar\mu a\le\delta$. Let n be such that $\nu_1(b^*\setminus b_n^*)\le\epsilon$. Then

$$\begin{aligned}|\nu a|&\le|\nu(a\cap b_n^*)|+|\nu(a\cap(b^*\setminus b_n^*))|+|\nu(a\setminus b^*)|\\&\le|\nu(a\cap b_n^*)|+\nu_1(b^*\setminus b_n^*)\le|\nu(a\cap b_n^*)|+\epsilon\end{aligned}$$

for any $a\in\mathfrak{A}$.

Now if b, $c\in\mathfrak{A}$ and $\bar\mu((b\,\triangle\,c)\cap b_n^*)\le\delta$ then

$$\begin{aligned}|\nu b-\nu c|&\le|\nu(b\setminus c)|+|\nu(c\setminus b)|\\&\le|\nu((b\setminus c)\cap b^*)|+|\nu((c\setminus b)\cap b^*)|+2\epsilon\le\epsilon+\epsilon+2\epsilon=4\epsilon\end{aligned}$$

because $\bar\mu((b\setminus c)\cap b_n^*)$, $\bar\mu((c\setminus b)\cap b_n^*)$ are both less than or equal to δ. As ϵ is arbitrary, ν is uniformly continuous.

(iv)⇒(iii)⇒(i) are trivial.

(d) One implication is covered by (a). For the other, suppose that ν is completely additive. Then it is countably additive. On the other hand, if $\nu a\ne0$, consider $B=\{b:b\subseteq a,\,\bar\mu b<\infty\}$. Then B is upwards-directed and $\sup B=a$, because $\bar\mu$ is semi-finite (322Eb), so $\{a\setminus b:b\in B\}$ is downwards-directed and has infimum 0. Accordingly $\inf_{b\in B}|\nu(a\setminus b)|=0$, and there must be a $b\in B$ such that $\nu b\ne0$. But this means that condition (ii) of (c) is satisfied, so that ν is continuous.

(e) Now suppose that $(\mathfrak{A},\bar\mu)$ is σ-finite. In this case $\mathfrak{A}$ is ccc (322G) so complete additivity and countable additivity are the same (326P) and we have a special case of (d).

(f) Finally, suppose that $\bar{\mu}1 < \infty$ and that ν is absolutely continuous with respect to $\bar{\mu}$. If $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and has infimum 0, then $\inf_{a \in A} \bar{\mu}a = 0$ (321F), so $\inf_{a \in A} |\nu a|$ must be 0; thus ν is completely additive. With (b) and (e) this shows that all four conditions are equiveridical.

327C Proposition Let (X, Σ, μ) be a measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra.

(a) There is a one-to-one correspondence between finitely additive functionals $\bar{\nu}$ on $\mathfrak{A}$ and finitely additive functionals ν on Σ such that $\nu E = 0$ whenever $\mu E = 0$, given by the formula $\bar{\nu}E^\bullet = \nu E$ for every $E \in \Sigma$.

(b) In (a), $\bar{\nu}$ is absolutely continuous with respect to $\bar{\mu}$ iff ν is absolutely continuous with respect to μ.

(c) In (a), $\bar{\nu}$ is countably additive iff ν is countably additive; so that we have a one-to-one correspondence between the countably additive functionals on $\mathfrak{A}$ and the absolutely continuous countably additive functionals on Σ.

(d) In (a), $\bar{\nu}$ is continuous for the measure-algebra topology on $\mathfrak{A}$ iff ν is truly continuous in the sense of 232Ab.

(e) Suppose that μ is semi-finite. Then, in (a), $\bar{\nu}$ is completely additive iff ν is truly continuous.

proof (a) This should be nearly obvious. If $\bar{\nu} : \mathfrak{A} \to \mathbb{R}$ is additive, then the formula defines a functional $\nu : \Sigma \to \mathbb{R}$ which is additive by 326Be. Also, of course,

$$\mu E = 0 \implies E^\bullet = 0 \implies \nu E = 0.$$

On the other hand, if ν is an additive functional on Σ which is zero on negligible sets, then, for E, $F \in \Sigma$,

$$\begin{aligned} E^\bullet = F^\bullet &\implies \mu(E \setminus F) = \mu(F \setminus E) = 0 \\ &\implies \nu(E \setminus F) = \nu(F \setminus E) = 0 \\ &\implies \nu F = \nu E - \nu(E \setminus F) + \nu(F \setminus E) = \nu E, \end{aligned}$$

so we have a function $\bar{\nu} : \mathfrak{A} \to \mathbb{R}$ defined by the given formula. If E, $F \in \Sigma$ and $E^\bullet \cap F^\bullet = 0$, then

$$\begin{aligned} \bar{\nu}(E^\bullet \cup F^\bullet) &= \bar{\nu}(E \cup F)^\bullet = \nu(E \cup F) \\ &= \nu(E \setminus F) + \nu F = \bar{\nu}E^\bullet + \bar{\nu}F^\bullet \end{aligned}$$

because $(E \setminus F)^\bullet = E^\bullet \setminus F^\bullet = E^\bullet$. Thus $\bar{\nu}$ is additive, and the correspondence is complete.

(b) This is immediate from the definitions.

(c)(i) If ν is countably additive, and $\langle a_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$, we can express it as $\langle E_n \rangle_{n \in \mathbb{N}}$ where $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in Σ. Setting $F_n = E_n \setminus \bigcup_{i<n} E_i$, $\langle F_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in Σ and

$$F_n^\bullet = a_n \setminus \sup_{i<n} a_i = a_n$$

for each n. So

$$\bar{\nu}(\sup_{n \in \mathbb{N}} a_n) = \nu(\bigcup_{n \in \mathbb{N}} F_n) = \sum_{n=0}^\infty \nu F_n = \sum_{n=0}^\infty \bar{\nu} a_n.$$

As $\langle a_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\bar{\nu}$ is countably additive.

(ii) If $\bar{\nu}$ is countably additive, then ν is countably additive by 326Jf.

(iii) For the last remark, note that by 232Ba a countably additive functional on Σ is absolutely continuous with respect to μ iff it is zero on the μ-negligible sets.

(d) The definition of 'truly continuous' functional translates directly to continuity at 0 in the measure algebra. But by 327Bc this is the same thing as continuity.

(e) Put (d) and 327Bd together.

327D The Radon-Nikodým theorem We are now ready for another look at this theorem.

Theorem Let (X, Σ, μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Let L^1 be the space of equivalence classes of real-valued integrable functions on X (§242), and write M_τ for the set of completely additive real-valued functionals on $\mathfrak{A}$. Then there is an ordered linear space bijection between M_τ and L^1 defined by saying that $\bar{\nu} \in M_\tau$ corresponds to $u \in L^1$ if

$$\bar{\nu}a = \int_E f \text{ whenever } a = E^\bullet \text{ in } \mathfrak{A} \text{ and } f^\bullet = u \text{ in } L^1.$$

proof (a) Given $\bar{\nu} \in M_\tau$, we have a truly continuous $\nu : \Sigma \to \mathbb{R}$ given by setting $\nu E = \bar{\nu}E^\bullet$ for every $E \in \Sigma$ (327Ce). Now there is an integrable function f such that $\nu E = \int_E f$ for every $E \in \Sigma$ (232E). There is likely to be more than one such function, but any two must be equal almost everywhere (232Hd), so the corresponding equivalence class $u_{\bar{\nu}} = f^\bullet$ is uniquely defined.

(b) Conversely, given $u \in L^1$, we have a well-defined functional ν_u on Σ given by setting

$$\nu_u E = \int_E u = \int_E f \text{ whenever } f^\bullet = u$$

for every $E \in \Sigma$ (242Ac). By 232D, ν_u is additive and truly continuous, and of course it is zero when μ is zero, so corresponds to a completely additive functional $\bar{\nu}_u$ on $\mathfrak{A}$ (327Ce).

(c) Clearly the maps $u \mapsto \bar{\nu}_u$ and $\bar{\nu} \mapsto u_{\bar{\nu}}$ are now the two halves of a one-to-one correspondence. To see that it is linear, we need note only that

$$(\bar{\nu}_u + \bar{\nu}_v)E^\bullet = \bar{\nu}_u E^\bullet + \bar{\nu}_v E^\bullet = \int_E u + \int_E v = \int_E u + v = \bar{\nu}_{u+v}E^\bullet$$

for every $E \in \Sigma$, so $\bar{\nu}_u + \bar{\nu}_v = \bar{\nu}_{u+v}$ for all u, $v \in L^1$; and similarly $\bar{\nu}_{\alpha u} = \alpha\bar{\nu}_u$ for $u \in L^1$ and $\alpha \in \mathbb{R}$. As for the ordering, given u and $v \in L^1$, take integrable f, g such that $u = f^\bullet$ and $v = g^\bullet$; then

$$\begin{aligned}
\bar{\nu}_u \le \bar{\nu}_v &\iff \bar{\nu}_u E^\bullet \le \bar{\nu}_v E^\bullet \text{ for every } E \in \Sigma \\
&\iff \int_E u \le \int_E v \text{ for every } E \in \Sigma \\
&\iff \int_E f \le \int_E g \text{ for every } E \in \Sigma \\
&\iff f \le_{\text{a.e.}} g \iff u \le v,
\end{aligned}$$

using 131Ha.

327E I slip in an elementary fact.

Proposition If $(\mathfrak{A}, \bar{\mu})$ is a measure algebra, then the functional $a \mapsto \mu_c a = \bar{\mu}(a \cap c)$ is completely additive whenever $c \in \mathfrak{A}$ and $\bar{\mu}c < \infty$.

proof μ_c is additive because $\bar{\mu}$ is additive, and by 321F again $\inf_{a \in A} \mu_c a = 0$ whenever A is non-empty, downwards-directed and has infimum 0.

327F Standard extensions The machinery of 327D provides the basis of a canonical method for extending countably additive functionals from closed subalgebras, which we shall need in §333.

Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a totally finite measure algebra and $\mathfrak{C} \subseteq \mathfrak{A}$ a closed subalgebra. Write $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ for the spaces of countably additive real-valued functionals on $\mathfrak{A}$, $\mathfrak{C}$ respectively.

(a) There is an operator $R : M_\sigma(\mathfrak{C}) \to M_\sigma(\mathfrak{A})$ defined by saying that, for every $\nu \in M_\sigma(\mathfrak{C})$, $R\nu$ is the unique member of $M_\sigma(\mathfrak{A})$ such that $[\![R\nu > \alpha\bar{\mu}]\!] = [\![\nu > \alpha\bar{\mu}\restriction\mathfrak{C}]\!]$ for every $\alpha \in \mathbb{R}$.

(b)(i) $R\nu$ extends ν for every $\nu \in M_\sigma(\mathfrak{C})$.

(ii) R is linear and order-preserving.

(iii) $R(\bar{\mu}\restriction\mathfrak{C}) = \bar{\mu}$.

(iv) If $\langle\nu_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-negative functionals in $M_\sigma(\mathfrak{C})$ such that $\sum_{n=0}^\infty \nu_n c = \bar{\mu}c$ for every $c \in \mathfrak{C}$, then $\sum_{n=0}^\infty (R\nu_n)(a) = \bar{\mu}a$ for every $a \in \mathfrak{A}$.

Remarks When saying that $\mathfrak{C}$ is 'closed', I mean, indifferently, 'topologically closed' or 'order-closed'; see 323H-323I.

For the notation '$[\![\nu > \alpha\bar{\mu}]\!]$' see 326S-326T.

proof (a)(i) By 321J-321K, we may represent $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a measure space (X, Σ, μ); write π for the canonical map from Σ to $\mathfrak{A}$. Write T for $\{E : E \in \Sigma, \pi E \in \mathfrak{C}\}$. Because $\mathfrak{C}$ is a σ-subalgebra of $\mathfrak{C}$ and π is a sequentially order-continuous Boolean homomorphism, T is a σ-subalgebra of Σ.

(ii) For each $\nu \in M_\sigma(\mathfrak{C})$, $\nu\pi : \mathrm{T} \to \mathbb{R}$ is countably additive and zero on $\{F : F \in \mathrm{T}, \mu F = 0\}$, so we can choose a T-measurable function $f_\nu : X \to \mathbb{R}$ such that $\int_F f_\nu d(\mu\restriction\mathrm{T}) = \nu\pi F$ for every $F \in \mathrm{T}$. Of course we can now think of f_ν as a μ-integrable function (233B), so we get a corresponding countably additive functional $R\nu : \mathfrak{A} \to \mathbb{R}$ defined by setting $(R\nu)(\pi E) = \int_E f_\nu$ for every $E \in \Sigma$ (327D). (In this context, of course, countably additive functionals are completely additive, by 327Bf.) Note that if $c \in \mathfrak{C}$ there is an $F \in \mathrm{T}$ such that $F^\bullet = c$, so that

$$(R\nu)(c) = \int_F f_\nu = \nu c.$$

For $\alpha \in \mathbb{R}$, set $H_\alpha = \{x : f_\nu(x) > \alpha\} \in \mathrm{T}$. Then for any $E \in \Sigma$,

$$E \subseteq H_\alpha,\ \mu E > 0 \Longrightarrow \int_E f_\nu > \alpha\mu E,$$

$$E \cap H_\alpha = \emptyset \Longrightarrow \int_E f_\nu \le \alpha\mu E.$$

Translating into terms of elements of $\mathfrak{A}$, and setting $c_\alpha = \pi H_\alpha \in \mathfrak{C}$, we have

$$0 \ne a \subseteq c_\alpha \Longrightarrow (R\nu)(a) > \alpha\bar\mu a,$$

$$a \cap c_\alpha = 0 \Longrightarrow (R\nu)(a) \le \alpha\bar\mu a.$$

So $[\![R\nu > \alpha\bar\mu]\!] = c_\alpha \in \mathfrak{C}$. Of course we now have

$$\nu c = (R\nu)(c) > \alpha\bar\mu c \text{ when } c \in \mathfrak{C},\ 0 \ne c \subseteq c_\alpha,$$

$$\nu c \le \alpha\bar\mu c \text{ when } c \in \mathfrak{C},\ c \cap c_\alpha = 0,$$

so that c_α is also equal to $[\![\nu > \alpha\bar\mu\restriction\mathfrak{C}]\!]$.

Thus the functional $R\nu$ satisfies the declared formula.

(iii) To see that $R\nu$ is uniquely defined, observe that if $\lambda \in M_\sigma(\mathfrak{A})$ and $[\![\lambda > \alpha\bar\mu]\!] = [\![R\nu > \alpha\bar\mu]\!]$ for every α, then there is a Σ-measurable function $g : X \to \mathbb{R}$ such that $\int_E g\,d\mu = \lambda\pi E$ for every $E \in \Sigma$; but in this case (just as in (ii)) $[\![\lambda > \alpha\bar\mu]\!] = \pi G_\alpha$, where $G_\alpha = \{x : g(x) > \alpha\}$, for each α. So we must have $\pi G_\alpha = \pi H_\alpha$, that is, $\mu(G_\alpha \triangle H_\alpha) = 0$, for every α. Accordingly

$$\{x : f_\nu(x) \ne g(x)\} = \bigcup_{q\in\mathbb{Q}} G_q \triangle H_q$$

is negligible; $f_\nu =_{\text{a.e.}} g$, $\int_E f_\nu d\mu = \int_E g\,d\mu$ for every $E \in \Sigma$ and $\lambda = R\nu$.

(b)(i) I have already noted that $(R\nu)c = \nu c$ for every $\nu \in M_\sigma(\mathfrak{C})$ and $c \in \mathfrak{C}$.

(ii) If $\nu = \nu_1 + \nu_2$, we must have, in the language of (a) above,

$$\int_F f_\nu = \nu\pi F = \nu_1\pi F + \nu_2\pi F = \int_F f_{\nu_1} + \int_F f_{\nu_2} = \int_F f_{\nu_1} + f_{\nu_2}$$

for every $F \in \mathrm{T}$, so $f_\nu =_{\text{a.e.}} f_{\nu_1} + f_{\nu_2}$, and we can repeat the formulae

$$(R\nu)(\pi E) = \int_E f_\nu = \int_E f_{\nu_1} + f_{\nu_2} = \int_E f_{\nu_1} + \int_E f_{\nu_2} = (R\nu_1)(\pi E) + (R\nu_2)(\pi E),$$

in a different order, for every $E \in \Sigma$, to see that $R\nu = R\nu_1 + R\nu_2$. Similarly, if $\nu \in M_\sigma(\mathfrak{C})$ and $\gamma \in \mathbb{R}$, $f_{\gamma\nu} =_{\text{a.e.}} \gamma f_\nu$ and $R(\gamma\nu) = \gamma R\nu$. If $\nu_1 \le \nu_2$ in $M_\sigma(\mathfrak{C})$, then

$$\int_F f_{\nu_1} = \nu_1\pi F \le \nu_2\pi F = \int_F f_{\nu_2}$$

for every $F \in \mathrm{T}$, so $f_{\nu_1} \le_{\text{a.e.}} f_{\nu_2}$ (131Ha again), and $R\nu_1 \le R\nu_2$.

Thus R is linear and order-preserving.

(iii) If $\nu = \bar\mu\restriction\mathfrak{C}$ then

$$\int_F f_\nu = \nu\pi F = \mu F = \int_F \chi X$$

for every $F \in \mathrm{T}$, so $f_\nu =_{\text{a.e.}} \chi X$ and $R\nu = \bar\mu$.

(iv) Now suppose that $\langle\nu_n\rangle_{n\in\mathbb{N}}$ is a sequence in $M_\sigma(\mathfrak{C})$ such that, for every $c \in \mathfrak{C}$, $\nu_n c \ge 0$ for every n and $\sum_{n=0}^\infty \nu_n c = \bar\mu c$. Set $g_n = \sum_{i=0}^n f_{\nu_i}$ for each n; then $0 \le_{\text{a.e.}} g_n \le_{\text{a.e.}} g_{n+1} \le_{\text{a.e.}} \chi X$ for every n, and

$$\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \sum_{i=0}^n \nu_i 1 = \bar\mu 1.$$

But this means that, setting $g = \lim_{n\to\infty} g_n$, $g \leq_{\text{a.e.}} \chi X$ and $\int g = \int \chi X$, so that $g =_{\text{a.e.}} \chi X$ and

$$\sum_{n=0}^{\infty}(R\nu_i)(\pi E) = \lim_{n\to\infty}\int_E g_n = \mu E$$

for every $E \in \Sigma$. Thus $\sum_{n=0}^{\infty}(R\nu_i)(a) = \bar\mu a$ for every $a \in \mathfrak{A}$.

327G Definition In the context of 327F, I will call $R\nu$ the **standard extension** of ν to $\mathfrak{A}$.

Remark The point of my insistence on the uniqueness of R, and on the formula in 327Fa, is that $R\nu$ really is defined by the abstract structure $(\mathfrak{A}, \bar\mu, \mathfrak{C}, \nu)$, even though I have used a proof which runs through the representation of $(\mathfrak{A}, \bar\mu)$ as the measure algebra of a measure space (X, Σ, μ).

327X Basic exercises (a) Let (X, Σ, μ) be a probability space, and T a σ-subalgebra of Σ. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of (X, Σ, μ). Show that $\mathfrak{C} = \{F^\bullet : F \in \mathrm{T}\}$ is a closed subalgebra of $\mathfrak{A}$. Identify the spaces $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ of countably additive functionals with $L^1(\mu)$, $L^1(\mu\restriction\mathrm{T})$, as in 327D. Show that the conditional expectation operator $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$ (242Jd) corresponds to the map $\nu \mapsto \nu\restriction\mathfrak{C} : M_\sigma(\mathfrak{A}) \to M_\sigma(\mathfrak{C})$.

(b) Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\nu : \mathfrak{A} \to \mathbb{R}$ a countably additive functional. Show that, for any $a \in \mathfrak{A}$,

$$\nu a = \int_0^\infty \bar\mu(a \cap [\![\nu > \alpha\bar\mu]\!])d\alpha - \int_{-\infty}^0 \bar\mu(a \setminus [\![\nu > \alpha\bar\mu]\!])d\alpha,$$

the integrals being taken with respect to Lebesgue measure. (*Hint*: take $(\mathfrak{A}, \bar\mu)$ to be the measure algebra of (X, Σ, μ); represent ν by a μ-integrable function f; apply Fubini's theorem to the sets $\{(x,t) : x \in E,\ 0 \leq t < f(x)\}$, $\{(x,t) : x \in E,\ f(x) \leq t \leq 0\}$ in $X \times \mathbb{R}$, where $a = E^\bullet$.)

(c) Let $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\mu')$ be totally finite measure algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a measure-preserving Boolean homomorphism. Let $\mathfrak{C}$ be a closed subalgebra of $\mathfrak{A}$, and ν a countably additive functional on the closed subalgebra $\pi[\mathfrak{C}]$ of $\mathfrak{B}$ (324L). (i) Show that $\nu\pi$ is a countably additive functional on $\mathfrak{C}$. (ii) Show that if $\tilde\nu$ is the standard extension of ν to $\mathfrak{B}$, then $\tilde\nu\pi$ is the standard extension of $\nu\pi$ to $\mathfrak{A}$. (*Hint*: take $\alpha \in \mathbb{R}$ and set $e_0 = [\![\tilde\nu > \alpha\bar\mu']\!] = [\![\nu > \alpha\bar\mu'\restriction\pi[\mathfrak{C}]]\!]$; there is a $c_0 \in \mathfrak{C}$ such that $\pi c_0 = e_0$; check that $c_0 = [\![\tilde\nu\pi > \alpha\bar\mu]\!] = [\![\nu\pi > \alpha\bar\mu\restriction\mathfrak{C}]\!]$.)

(d) Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$ and $\nu : \mathfrak{C} \to \mathbb{R}$ a countably additive functional with standard extension $\tilde\nu : \mathfrak{A} \to \mathbb{R}$. Show that, for any $a \in \mathfrak{A}$,

$$\tilde\nu a = \int_0^\infty \bar\mu(a \cap [\![\nu > \alpha\bar\mu\restriction\mathfrak{C}]\!])d\alpha - \int_{-\infty}^0 \bar\mu(a \setminus [\![\nu > \alpha\bar\mu\restriction\mathfrak{C}]\!])d\alpha.$$

(e) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\mathfrak{B}$, $\mathfrak{C}$ stochastically independent closed subalgebras of $\mathfrak{A}$ (definition: 325L). Let ν be a countably additive functional on $\mathfrak{C}$, and $\tilde\nu$ its standard extension to $\mathfrak{A}$. Show that $\tilde\nu(b \cap c) = \bar\mu b \cdot \nu c$ for every $b \in \mathfrak{B}$, $c \in \mathfrak{C}$.

(f) Let (X, Σ, μ) be a probability space, and T a σ-subalgebra of Σ. Let ν be a probability measure with domain T such that $\nu E = 0$ whenever $E \in \mathrm{T}$ and $\mu E = 0$. Show that there is a probability measure λ with domain Σ which extends ν.

327Y Further exercises (a) Let $(\mathfrak{A}_1, \bar\mu_1)$ and $(\mathfrak{A}_2, \bar\mu_2)$ be localizable measure algebras with localizable measure algebra free product $(\mathfrak{C}, \bar\lambda)$. Show that if ν_1, ν_2 are completely additive functionals on $\mathfrak{A}_1$, $\mathfrak{A}_2$ respectively, there is a unique completely additive functional $\nu : \mathfrak{C} \to \mathbb{R}$ such that $\nu(a_1 \otimes a_2) = \nu_1 a_1 \cdot \nu_2 a_2$ for every $a_1 \in \mathfrak{A}_1$, $a_2 \in \mathfrak{A}_2$. (*Hint*: 253D.)

(b) Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra; let $R : M_\sigma(\mathfrak{C}) \to M_\sigma(\mathfrak{A})$ be the standard extension operator (327G). Show (i) that R is order-continuous (ii) that $R(\nu^+) = (R\nu)^+$, $\|R\nu\| = \|\nu\|$ for every $\nu \in M_\sigma(\mathfrak{C})$, defining ν^+ and $\|\nu\|$ as in 326Yd.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. For a countably additive functional ν on $\mathfrak{C}$ write $\tilde\nu$ for its standard extension to $\mathfrak{A}$. Show that if ν, $\langle\nu_n\rangle_{n\in\mathbb{N}}$ are countably additive functionals on $\mathfrak{C}$ and $\lim_{n\to\infty}\nu_n c = \nu c$ for every $c \in \mathfrak{C}$, then $\lim_{n\to\infty}\tilde\nu_n a = \tilde\nu a$ for every $a \in \mathfrak{A}$. (*Hint*: use ideas from §§246-247, as well as from 327F and 326Yo.)

327 Notes and comments When we come to measure algebras, it is the completely additive functionals which fit most naturally into the topological theory (327Bd); they correspond to the 'truly continuous' functionals which I discussed in §232 (327Cd), and therefore to the Radon-Nikodým theorem (327D). I will return to some of these questions in Chapter 36. I myself regard the form here as the best expression of the essence of the Radon-Nikodým theorem, if not the one most commonly applied.

The concept of 'standard extension' of a countably additive functional (or, as we could equally well say, of a completely additive functional, since in the context of 327F the two coincide) is in a sense dual to the concept of 'conditional expectation'. If (X,Σ,μ) is a probability space and T is a σ-subalgebra of Σ, then we have a corresponding closed subalgebra $\mathfrak{C}$ of the measure algebra $(\mathfrak{A},\bar\mu)$ of μ, and identifications between the spaces $M_\sigma(\mathfrak{A})$, $M_\sigma(\mathfrak{C})$ of countably additive functionals and the spaces $L^1(\mu)$, $L^1(\mu\restriction\mathrm{T})$. Now we have a natural embedding S of $L^1(\mu\restriction\mathrm{T})$ as a subspace of $L^1(\mu)$ (242Jb), and a natural restriction map from $M_\sigma(\mathfrak{A})$ to $M_\sigma(\mathfrak{C})$. These give rise to corresponding operators between the opposite members of each pair; the standard extension operator R of 327F-327G, and the conditional expectation operator P of 242Jd. (See 327Xa.) The fundamental fact

$$PSv=v \text{ for every } v\in L^1(\mu\restriction\mathrm{T})$$

(242Jg) is matched by the fact that

$$R\nu\restriction\mathfrak{C}=\nu \text{ for every } \nu\in M_\sigma(\mathfrak{C}).$$

The further identification of $R\nu$ in terms of integrals $\int\bar\mu(a\cap[\![\nu>\alpha\bar\mu]\!])d\alpha$ (327Xd) is relatively inessential, but is striking, and perhaps makes it easier to believe that R is truly 'standard' in the abstract contexts which will arise in §333 below. It is also useful in such calculations as 327Xe.

The isomorphisms between M_τ spaces and L^1 spaces described here mean that any of the concepts involving L^1 spaces discussed in Chapter 24 can be applied to M_τ spaces, at least in the case of measure algebras. In fact, as I will show in Chapter 36, there is much more to be said here; the space of bounded additive functionals on a Boolean algebra is already an L^1 space in an abstract sense, and ideas such as 'uniform integrability' are relevant and significant there, as well as in the spaces of countably additive and completely additive functionals. I hope that 326Yd, 326Ym-326Yn, 326Yp-326Yq and 327Yb will provide some hints to be going on with for the moment.

Version of 2.6.09

*328 Reduced products and other constructions

I devote a section to some related constructions. At the end of §315 I mentioned projective and inductive limits of systems of Boolean algebras with linking homomorphisms. In the context of the present chapter, we naturally ask whether similar constructions can be found for probability algebras. For projective limits there is no difficulty (328I). For inductive limits the situation is more complex (328H). Some ideas in Volume 5 will depend on what I call 'reduced products' (328A-328F), which also provide a route to 328H. The same methods give a route to a useful result relating measure-preserving Boolean homomorphisms on a probability algebra to measure-preserving automorphisms on a larger probability algebra (328J).

328A Construction Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $\mathcal{F}$ an ultrafilter on I.

(a) Set

$$\mathcal{J}=\{\langle a_i\rangle_{i\in I}:\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i,\ \lim_{i\to\mathcal{F}}\bar\mu_i a_i=0\}.$$

Then $\mathcal{J}$ is an ideal in the simple product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$. **P** If $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ belong to $\mathcal{J}$, and $\langle c_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ is such that $\langle c_i\rangle_{i\in I}\subseteq\langle a_i\rangle_{i\in I}\cup\langle b_i\rangle_{i\in I}$, then $c_i\subseteq a_i\cup b_i$ for every i, so

$$\lim_{i\to\mathcal{F}}\bar\mu_i c_i\le\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\bar\mu_i b_i=\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\lim_{i\to\mathcal{F}}\bar\mu_i b_i=0$$

and $\langle c_i\rangle_{i\in I}\in\mathcal{J}$. Of course $\langle 0_{\mathfrak{A}_i}\rangle_{i\in I}$ belongs to $\mathcal{J}$, so $\mathcal{J}\triangleleft\prod_{i\in I}\mathfrak{A}_i$. **Q**

(b) Let $\mathfrak{A}$ be the quotient Boolean algebra $\prod_{i\in I}\mathfrak{A}_i/\mathcal{J}$. Then we have a functional $\bar\mu:\mathfrak{A}\to[0,1]$ defined by saying that

$$\bar\mu(\langle a_i\rangle^\bullet_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu_i a_i$$

whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$. **P** If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $\langle a_i\rangle^\bullet_{i\in I}=\langle b_i\rangle^\bullet_{i\in I}$, then $\langle a_i\triangle b_i\rangle_{i\in I}\in\mathcal{J}$, so

$$|\lim_{i\to\mathcal{F}}\bar\mu_i a_i-\lim_{i\to\mathcal{F}}\bar\mu_i b_i|=\lim_{i\to\mathcal{F}}|\bar\mu_i a_i-\bar\mu_i b_i|\le\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\triangle b_i)=0.\ \mathbf{Q}$$

328B Proposition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras and $\mathcal{F}$ an ultrafilter on I, and construct $\mathfrak{A}$ and $\bar\mu$ as in 328A. Then $(\mathfrak{A},\bar\mu)$ is a probability algebra.

proof (a) If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $\langle a_i\rangle^\bullet_{i\in I}\cap\langle b_i\rangle^\bullet_{i\in I}=0$, then $\langle a_i\cap b_i\rangle_{i\in I}\in\mathcal{J}$, so

$$\begin{aligned}\bar\mu(\langle a_i\rangle^\bullet_{i\in I}\cup\langle b_i\rangle^\bullet_{i\in I})&=\bar\mu(\langle a_i\cup b_i\rangle^\bullet_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\cup b_i)\\&=\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\bar\mu_i b_i-\bar\mu_i(a_i\cap b_i)\\&=\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\lim_{i\to\mathcal{F}}\bar\mu_i b_i-\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\cap b_i)\\&=\lim_{i\to\mathcal{F}}\bar\mu_i a_i+\lim_{i\to\mathcal{F}}\bar\mu_i b_i=\bar\mu(\langle a_i\rangle^\bullet_{i\in I})+\bar\mu(\langle b_i\rangle^\bullet_{i\in I}).\end{aligned}$$

So $\bar\mu$ is additive.

(b) $1_{\mathfrak{A}}=\langle 1_{\mathfrak{A}_i}\rangle^\bullet_{i\in I}$ so

$$\bar\mu 1_{\mathfrak{A}}=\lim_{i\to\mathcal{F}}\bar\mu_i 1_{\mathfrak{A}_i}=1.$$

(c) If $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $\bar\mu(\langle a_i\rangle^\bullet_{i\in I})=0$, then $\langle a_i\rangle_{i\in I}\in\mathcal{J}$ and $\langle a_i\rangle^\bullet_{i\in I}=0$; thus $\bar\mu$ is strictly positive.

(d) Suppose that $\langle\tilde a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$. Express each $\tilde a_n$ as $\langle a_{ni}\rangle^\bullet_{i\in I}$ where $a_{ni}\in\mathfrak{A}_i$ for each i. Set $b_{ni}=\sup_{m\le n}a_{mi}$ for $n\in\mathbb{N}$ and $i\in I$; then $\langle b_{ni}\rangle^\bullet_{i\in I}=\sup_{m\le n}\tilde a_m$ in $\mathfrak{A}$. Set

$$\gamma=\sum_{n=0}^\infty\bar\mu\tilde a_n=\sup_{n\in\mathbb{N}}\bar\mu(\langle b_{ni}\rangle^\bullet_{i\in I})=\sup_{n\in\mathbb{N}}\lim_{i\to\mathcal{F}}\bar\mu_i b_{ni}.$$

Set $A_n = \{i : i \in I,\ \bar{\mu}_i b_{ni} \le \gamma + 2^{-n}\}$ for each $n \in \mathbb{N}$; then $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{F}$, and $A_0 = I$. For $i \in I$ set

$$b_i = b_{ni} \text{ if } i \in A_n \setminus A_{n+1},$$
$$= \sup_{n\in\mathbb{N}} b_{ni} \text{ if } i \in \bigcap_{n\in\mathbb{N}} A_n.$$

Consider $\tilde{b} = \langle b_i\rangle^\bullet_{i\in I} \in \mathfrak{A}$. For each $n \in \mathbb{N}$, $\{i : a_{ni} \subseteq b_i,\ \bar{\mu} b_i \le \gamma + 2^{-n}\}$ includes $A_n \in \mathcal{F}$, so $\tilde{a}_n \subseteq \tilde{b}$ for every n and $\bar{\mu}\tilde{b} \le \gamma$.

If $\tilde{c} \in \mathfrak{A}$ is another upper bound for $\{\tilde{a}_n : n \in \mathbb{N}\}$, then, using (a),

$$\gamma = \sup_{n\in\mathbb{N}} \bar{\mu}(\sup_{m\le n} \tilde{a}_m) \le \bar{\mu}(\tilde{b} \cap \tilde{c}) \le \bar{\mu}\tilde{b} \le \gamma;$$

so $\bar{\mu}(\tilde{b} \setminus \tilde{c}) = 0$ and $\tilde{b} \setminus \tilde{c} = 0$, by (c). Thus $\tilde{b} = \sup_{n\in\mathbb{N}} \tilde{a}_n$ in $\mathfrak{A}$, while $\bar{\mu}\tilde{b} = \sum_{n=0}^\infty \bar{\mu}\tilde{a}_n$.

(e) If $\langle \tilde{a}_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathfrak{A}$, then (iv) tells us that $\{\tilde{a}_n \setminus \sup_{m<n} \tilde{a}_m : n \in \mathbb{N}\}$ has a supremum in $\mathfrak{A}$, which is also the supremum of $\{\tilde{a}_n : n \in \mathbb{N}\}$. So $\mathfrak{A}$ is Dedekind σ-complete. Now (d) tells us also that $\bar{\mu}$ is countably additive, so that $(\mathfrak{A}, \bar{\mu})$ is a probability algebra.

328C Definition In the context of 328A/328B, I will call $(\mathfrak{A}, \bar{\mu})$ the **probability algebra reduced product** of $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ modulo $\mathcal{F}$; I will sometimes write it as $\prod_{i\in I}(\mathfrak{A}_i, \bar{\mu}_i)|\mathcal{F}$. (There are dangers in this notation. In 351M I will speak of 'reduced powers' $\mathbb{R}^I|\mathcal{F}$, and the rules will be significantly different there.)

If all the $(\mathfrak{A}_i, \bar{\mu}_i)$ are the same, with common value $(\mathfrak{B}, \bar{\nu})$, I will write $(\mathfrak{B}, \bar{\nu})^I|\mathcal{F}$ for $\prod_{i\in I}(\mathfrak{A}_i, \bar{\mu}_i)|\mathcal{F}$, and call it the **probability algebra reduced power**.

328D Proposition Let I be a set, $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$, $\langle(\mathfrak{B}_i, \bar{\nu}_i)\rangle_{i\in I}$ and $\langle(\mathfrak{C}_i, \bar{\lambda}_i)\rangle_{i\in I}$ three families of probability algebras, and $\mathcal{F}$ an ultrafilter on I; let $(\mathfrak{A}, \bar{\mu}) = \prod_{i\in I}(\mathfrak{A}_i, \bar{\mu}_i)|\mathcal{F}$, $(\mathfrak{B}, \bar{\nu}) = \prod_{i\in I}(\mathfrak{B}_i, \bar{\nu}_i)|\mathcal{F}$ and $(\mathfrak{C}, \bar{\lambda}) = \prod_{i\in I}(\mathfrak{C}_i, \bar{\lambda}_i)|\mathcal{F}$ be the corresponding reduced products.

(a) If $\pi_i : \mathfrak{A}_i \to \mathfrak{B}_i$ is a measure-preserving Boolean homomorphism for each $i \in I$, we have a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ given by saying that

$$\pi(\langle a_i\rangle^\bullet_{i\in I}) = \langle \pi_i a_i\rangle^\bullet_{i\in I}$$

whenever $a_i \in \mathfrak{A}_i$ for every $i \in I$.

(b) If, in addition, $\phi_i : \mathfrak{B}_i \to \mathfrak{C}_i$ is a measure-preserving Boolean homomorphism for each $i \in I$, and $\phi : \mathfrak{B} \to \mathfrak{C}$ is constructed as in (a), then $\phi\pi : \mathfrak{A} \to \mathfrak{C}$ corresponds to the family $\langle \phi_i\pi_i\rangle_{i\in I}$.

proof (a) Following through the construction in 328A, we have ideals

$$\mathcal{J} = \{\langle a_i\rangle_{i\in I} : \lim_{i\to\mathcal{F}} \bar{\mu}_i a_i = 0\} \vartriangleleft \prod_{i\in I} \mathfrak{A}_i,$$

$$\mathcal{K} = \{\langle b_i\rangle_{i\in I} : \lim_{i\to\mathcal{F}} \bar{\nu}_i b_i = 0\} \vartriangleleft \prod_{i\in I} \mathfrak{B}_i,$$

and a Boolean homomorphism $\hat{\pi} : \prod_{i\in I} \mathfrak{A}_i \to \prod_{i\in I} \mathfrak{B}_i$ given by the formula $\hat{\pi}\langle a_i\rangle_{i\in I} = \langle \pi_i a_i\rangle_{i\in I}$ (use 315Bb). Because the homomorphisms π_i are measure-preserving, $\hat{\pi}\boldsymbol{a} \in \mathcal{K}$ whenever $\boldsymbol{a} \in \mathcal{J}$. Consequently we have a Boolean homomorphism $\pi : \prod_{i\in I} \mathfrak{A}_i/\mathcal{J} \to \prod_{i\in I} \mathfrak{B}_i/\mathcal{K}$ given by setting $\pi\boldsymbol{a}^\bullet = (\hat{\pi}\boldsymbol{a})^\bullet$ whenever $\boldsymbol{a} \in \prod_{i\in I} \mathfrak{A}_i$ (3A2G). And

$$\bar{\nu}\pi(\langle a_i\rangle^\bullet_{i\in I}) = \bar{\nu}(\langle \pi_i a_i\rangle^\bullet_{i\in I}) = \lim_{i\to\mathcal{F}} \bar{\nu}_i\pi_i a_i = \lim_{i\to\mathcal{F}} \bar{\mu}_i a_i = \bar{\mu}(\langle a_i\rangle^\bullet_{i\in I})$$

whenever $\langle a_i\rangle_{i\in I} \in \prod_{i\in I} \mathfrak{A}_i$, so π is measure-preserving.

(b) is now just a matter of writing the defining formulae out.

328E Proposition Let I be a non-empty set, $\le$ a reflexive transitive relation on I, and $\mathcal{F}$ an ultrafilter on I such that $\{j : j \in I,\ j \ge i\}$ belongs to $\mathcal{F}$ for every $i \in I$. Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras, and suppose that we are given a family $\langle \pi_{ji}\rangle_{i\le j}$ such that

π_{ji} is a measure-preserving Boolean homomorphism from $\mathfrak{A}_i$ to $\mathfrak{A}_j$ whenever $i \le j$ in I,

$\pi_{ki} = \pi_{kj}\pi_{ji}$ whenever $i \le j \le k$ in I.

Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced product $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$.

(a) For each $i\in I$ we have a measure-preserving Boolean homomorphism $\pi_i:\mathfrak{A}_i\to\mathfrak{A}$ defined by saying that $\pi_i a=\langle a_j\rangle^\bullet_{j\in I}$ whenever $a_j=\pi_{ji}a$ for every $j\ge i$, and $\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$ in I.

(b) $\langle a_i\rangle^\bullet_{i\in I}\subseteq\sup_{j\in A}\pi_j a_j$ whenever $\langle a_i\rangle_{i\in I}\in\prod_{i\in I}\mathfrak{A}_i$ and $A\in\mathcal{F}$.

proof (a) π_i is well-defined because $\{j:j\ge i\}\in\mathcal{F}$. It is a measure-preserving Boolean homomorphism because every π_{ji} is. If $i\le j$ in I, $a\in\mathfrak{A}_i$ and $a_k=\pi_{ki}a$ for every $k\ge i$, then $a_k=\pi_{kj}\pi_{ji}a$ for every $k\ge j$, so $\pi_j\pi_{ji}a=\langle a_k\rangle^\bullet_{k\in I}=\pi_i a$; as a is arbitrary, $\pi_j\pi_{ji}=\pi_i$.

(b) Set $c=\sup_{j\in A}\pi_j a_j$ in $\mathfrak{A}$. For any $\epsilon>0$, there is a finite $K\subseteq A$ such that $\bar\mu c\le\epsilon+\bar\mu(\sup_{j\in K}\pi_j a_j)$ (321C). The set $B=\{k:k\in I,\ j\le k$ for every $j\in K\}$ belongs to $\mathcal{F}$, so is not empty; fix $k\in B$, and set $b=\sup_{j\in K}\pi_{kj}a_j\in\mathfrak{A}_k$,

$$\begin{aligned} b_i&=\pi_{ik}b \text{ if } i\ge k,\\ &=0 \text{ for other } i\in I.\end{aligned}$$

Then

$$\langle b_i\rangle^\bullet_{i\in I}=\pi_k b=\pi_k(\sup\nolimits_{j\in K}\pi_{kj}a_j)=\sup\nolimits_{j\in K}\pi_k\pi_{kj}a_j=\sup\nolimits_{j\in K}\pi_j a_j\subseteq c.$$

If $i\in A$ and $i\ge k$, then

$$\begin{aligned}\bar\mu_i(a_i\setminus b_i)&=\bar\mu(\pi_i a_i\setminus\pi_i b_i)=\bar\mu(\pi_i a_i\setminus\pi_i\pi_{ik}b)\\ &=\bar\mu(\pi_i a_i\setminus\pi_k b)=\bar\mu(\pi_i a_i\setminus\sup_{j\in K}\pi_j a_j)\le\bar\mu(c\setminus\sup_{j\in K}\pi_j a_j)\le\epsilon\end{aligned}$$

by the choice of K. So

$$\begin{aligned}\bar\mu(\langle a_i\rangle^\bullet_{i\in I}\setminus c)&\le\bar\mu(\langle a_i\rangle^\bullet_{i\in I}\setminus\langle b_i\rangle^\bullet_{i\in I})=\bar\mu(\langle a_i\setminus b_i\rangle^\bullet_{i\in I})\\ &=\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\setminus b_i)\le\sup_{i\in A,i\ge k}\bar\mu_i(a_i\setminus b_i)\le\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\langle a_i\rangle^\bullet_{i\in I}\subseteq c$.

328F Corollary Suppose that $\langle(\mathfrak{A}_n,\bar\mu_n)\rangle_{n\in\mathbb{N}}$ is a sequence of probability algebras, $\phi_n:\mathfrak{A}_n\to\mathfrak{A}_{n+1}$ is a measure-preserving Boolean homomorphism for each n and $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$. Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced product $\prod_{n\in\mathbb{N}}(\mathfrak{A}_n,\bar\mu_n)|\mathcal{F}$. Then we have canonical measure-preserving Boolean homomorphisms $\pi_n:\mathfrak{A}_n\to\mathfrak{A}$ such that $\langle a_n\rangle^\bullet_{n\in\mathbb{N}}\subseteq\sup_{n\in A}\pi_n a_n$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}\mathfrak{A}_n$ and $A\in\mathcal{F}$, and $\pi_{n+1}\phi_n=\pi_n$ for every $n\in\mathbb{N}$.

proof Apply 328E with $\pi_{ji}=\phi_{j-1}\ldots\phi_{i+1}\phi_i$ whenever $i<j$.

328G Corollary Let $(\mathfrak{B},\bar\nu)$ be a probability algebra, I a non-empty set, and $\mathcal{F}$ an ultrafilter on I. Let $(\mathfrak{A},\bar\mu)$ be the probability algebra reduced power $(\mathfrak{B},\bar\nu)^I|\mathcal{F}$.

(a) We have a measure-preserving Boolean homomorphism $\pi:\mathfrak{B}\to\mathfrak{A}$ defined by saying that $\pi b=\langle b\rangle^\bullet_{i\in I}$ for $b\in\mathfrak{B}$.

(b)

$$\langle b_i\rangle^\bullet_{i\in I}\subseteq\sup\nolimits_{j\in A}\pi b_j=\pi(\sup\nolimits_{j\in A}b_j)$$

whenever $A\in\mathcal{F}$ and $\langle b_i\rangle_{i\in I}\in\mathfrak{B}^I$.

proof Apply 328E with $\le\,=I\times I$ and π_{ji} the identity operator on $\mathfrak{B}$ for all i, $j\in I$.

328H Proposition Let $(I,\le)$ be an upwards-directed partially ordered set, and $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ a family of probability algebras; suppose that $\pi_{ji}:\mathfrak{A}_i\to\mathfrak{A}_j$ is a measure-preserving Boolean homomorphism whenever $i\le j$, and that $\pi_{ki}=\pi_{kj}\pi_{ji}$ whenever $i\le j\le k$. Then there are a probability algebra $(\mathfrak{C},\bar\lambda)$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i:\mathfrak{A}_i\to\mathfrak{C}$ is a measure-preserving Boolean homomorphism for each $i\in I$,

$\pi_i=\pi_j\pi_{ji}$ whenever $i\le j$,

$\{0,1\} \cup \bigcup_{i \in I} \pi_i[\mathfrak{A}_i]$ is topologically dense in $\mathfrak{C}$,

and whenever $(\mathfrak{B}, \bar{\nu})$, $\langle \phi_i \rangle_{i \in I}$ are such that

$(\mathfrak{B}, \bar{\nu})$ is a probability algebra,

$\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism for each $i \in I$,

$\phi_i = \phi_j \pi_{ji}$ whenever $i \le j$,

then there is a unique measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\pi_i = \phi_i$ for every $i \in I$.

proof (a) If I is empty the result is trivial (take $\mathfrak{C} = \{0,1\}$); so let us suppose henceforth that $I \neq \emptyset$. In this case,

$$\{A : A \subseteq I, \text{ there is some } i \in I \text{ such that } j \in A \text{ whenever } i \le j\}$$

is a filter on I, and is included in an ultrafilter $\mathcal{F}$ say (2A1O). Let $(\mathfrak{A}, \bar{\mu})$ be the reduced product $\prod_{i \in I}(\mathfrak{A}_i, \bar{\mu}_i)|\mathcal{F}$. Then we have for each $i \in I$ a measure-preserving Boolean homomorphism $\pi_i : \mathfrak{A}_i \to \mathfrak{A}$ such that $\pi_i = \pi_j \pi_{ji}$ whenever $i \le j$ (328E). If $i \le j$ in I, then $\pi_i[\mathfrak{A}_i] \subseteq \pi_j[\mathfrak{A}_j]$; because $(I, \le)$ is upwards-directed, $\langle \pi_i[\mathfrak{A}_i] \rangle_{i \in I}$ is an upwards-directed family of subalgebras of $\mathfrak{A}$, and $\mathfrak{D} = \bigcup_{i \in I} \pi_i[\mathfrak{A}_i]$ is a subalgebra of $\mathfrak{A}$; let $\mathfrak{C}$ be its closure (323J). Set $\bar{\lambda} = \bar{\mu} \restriction \mathfrak{C}$, so that $(\mathfrak{C}, \bar{\lambda})$ is a probability algebra, and $\pi_i : \mathfrak{A}_i \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism for each $i \in I$, with $\pi_i = \pi_j \pi_{ji}$ whenever $i \le j$.

(b) Now suppose that $\mathfrak{B}$ and $\langle \phi_i \rangle_{i \in I}$ are as declared.

(i) Set

$$\phi' = \{(\pi_i a, \phi_i a) : i \in I,\ a \in \mathfrak{A}_i\} \subseteq \mathfrak{D} \times \mathfrak{B}.$$

Then ϕ' is (the graph of) a function from $\mathfrak{D}$ to $\mathfrak{B}$. **P** If $c \in \mathfrak{D}$, there is surely an $i \in I$ such that $c \in \pi_i[\mathfrak{A}_i]$, so that $(c, \phi_i a) \in \phi'$ for some $a \in \mathfrak{A}_i$. If (c, b) and (c, b') belong to ϕ', there are i, $j \in I$ and $a \in \mathfrak{A}_i$, $a' \in \mathfrak{A}_j$ such that

$$\pi_i a = \pi_j a' = c, \quad \phi_i a = b, \quad \phi_j a' = b'.$$

Let $k \in I$ be such that $i \le k$ and $j \le k$; then

$$\pi_k \pi_{ki} a = \pi_i a = c = \pi_j a' = \pi_k \pi_{kj} a'.$$

As π_k is measure-preserving, therefore injective, $\pi_{ki} a = \pi_{kj} a'$, and

$$b = \phi_i a = \phi_k \pi_{ki} a = \phi_k \pi_{kj} a' = \phi_j a' = b'.$$

So each element of $\mathfrak{D}$ is the first member of exactly one element of ϕ', and ϕ' is the graph of a function. **Q** Of course the defining formula for ϕ' guarantees that $\phi'\pi_i = \phi_i : \mathfrak{A}_i \to \mathfrak{B}$ for every $i \in I$.

(ii) Next, $\phi' : \mathfrak{D} \to \mathfrak{B}$ is a measure-preserving Boolean homomorphism. **P** If c, $c' \in \mathfrak{D}$ then there are i, $j \in I$ and $a \in \mathfrak{A}_i$, $a' \in \mathfrak{A}_j$ such that $c = \pi_i a$ and $c' = \pi_j a'$. Again take $k \in I$ such that $i \le k$ and $j \le k$; then

$$c = \pi_k \pi_{ki} a, \quad c' = \pi_k \pi_{kj} a', \quad \phi' c = \phi_k \pi_{ki} a, \quad \phi' c' = \phi_k \pi_{kj} a'.$$

In this case, for either of the Boolean operations $\star = \triangle$ or $\star = \cap$, we have

$$\begin{aligned}\phi' c \star \phi' c' &= \phi_k \pi_{ki} a \star \phi_k \pi_{kj} a' = \phi_k(\pi_{ki} a \star \pi_{kj} a') \\ &= \phi' \pi_k(\pi_{ki} a \star \pi_{kj} a') = \phi'(\pi_k \pi_{ki} a \star \pi_k \pi_{kj} a') = \phi'(c \star c').\end{aligned}$$

As c, c' and $\star$ are arbitrary, ϕ' is a ring homomorphism. Moreover, in the same context,

$$\bar{\nu}\phi' c = \bar{\nu}\phi_i a = \bar{\mu}_i a = \bar{\mu}\pi_i a = \bar{\lambda} c,$$

so ϕ' is measure-preserving. It follows that $\phi' 1_{\mathfrak{C}} = 1_{\mathfrak{B}}$, and ϕ' is a Boolean homomorphism. **Q**

(iii) By 324O, there is a unique extension of ϕ' to a measure-preserving Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$; and of course we still have $\phi\pi_i = \phi_i$ for every $i \in I$.

(iv) To see that ϕ is unique, take any measure-preserving Boolean homomorphism $\tilde{\phi} : \mathfrak{C} \to \mathfrak{B}$ such that $\tilde{\phi}\pi_i = \phi_i$ for every i. Then $\tilde{\phi}$ must agree with ϕ on $\pi_i[\mathfrak{A}_i]$ for every i, so $\tilde{\phi} \restriction \mathfrak{D} = \phi \restriction \mathfrak{D}$; as $\mathfrak{D}$ is topologically dense in $\mathfrak{C}$, $\tilde{\phi} = \phi$ (324O again).

328I For completeness, I spell out the relatively elementary construction for projective limits.

Proposition Let $(I, \leq)$ be a non-empty upwards-directed set, and $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i \in I}$ a family of probability algebras; suppose that $\pi_{ij} : \mathfrak{A}_j \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for $i \leq j$ in I, and that $\pi_{ij}\pi_{jk} = \pi_{ik}$ whenever $i \leq j \leq k$. Then there are a probability algebra $(\mathfrak{C}, \bar{\lambda})$ and a family $\langle \pi_i \rangle_{i \in I}$ such that

$\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i \in I$,

$\pi_i = \pi_{ij}\pi_j$ whenever $i \leq j$,

and whenever $(\mathfrak{B}, \bar{\nu})$, $\langle \phi_i \rangle_{i \in I}$ are such that

$(\mathfrak{B}, \bar{\nu})$ is a probability algebra,

$\phi_i : \mathfrak{B} \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism for each $i \in I$,

$\phi_i = \pi_{ij}\phi_j$ whenever $i \leq j$,

then there is a unique measure-preserving Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{C}$ such that $\pi_i \phi = \phi_i$ for every $i \in I$.

proof (a) Let $\mathfrak{C} \subseteq \prod_{i \in I} \mathfrak{A}_i$ be the set

$$\{\langle a_i \rangle_{i \in I} : \pi_{ij} a(j) = a(i) \text{ whenever } i \leq j \text{ in } I\}.$$

Because every π_{ij} is a Boolean homomorphism, $\mathfrak{C}$ is a subalgebra of $\prod_{i \in I} \mathfrak{A}_i$; taking $\pi_j(\langle a_i \rangle_{i \in I}) = a_j$ whenever $\langle a_i \rangle_{i \in I} \in \mathfrak{C}$, $\pi_j : \mathfrak{C} \to \mathfrak{A}_j$ is a Boolean homomorphism for every $j \in I$, and $\pi_i = \pi_{ij}\pi_j$ whenever $i \leq j$.

Because every π_{ij} is order-continuous, $\mathfrak{C}$ is an order-closed subalgebra of $\prod_{i \in I} \mathfrak{A}_i$, so is Dedekind complete.

(b) If $c = \langle a_i \rangle_{i \in I} \in \mathfrak{C}$, then

$$\bar{\mu}_i \pi_i c = \bar{\mu}_i a_i = \bar{\mu}_i \pi_{ij} a_j = \bar{\mu}_j a_j = \bar{\mu}_j \pi_j c$$

whenever $i \leq j$ in I; because I is upwards-directed, $\bar{\mu}_i \pi_i c = \bar{\mu}_j \pi_j c$ for all i, $j \in I$. So we have a functional $\bar{\lambda} : \mathfrak{C} \to [0,1]$ defined by setting $\bar{\lambda} c = \bar{\mu}_i \pi_i c$ whenever $c \in \mathfrak{C}$ and $i \in I$. Note that $1_{\mathfrak{C}} = \langle 1_{\mathfrak{A}_i} \rangle_{i \in I}$, so $\bar{\lambda} 1_{\mathfrak{C}} = \bar{\mu}_i 1_{\mathfrak{A}_i} = 1$, for any $i \in I$.

If $\langle c_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathfrak{C}$ with supremum c, then express each c_n as $\langle a_{ni} \rangle_{i \in I}$; we must have $c = \langle \sup_{n \in \mathbb{N}} a_{ni} \rangle_{i \in I}$, so

$$\bar{\lambda} c = \bar{\mu}_i(\sup_{n \in \mathbb{N}} a_{ni}) = \sum_{n=0}^{\infty} \bar{\mu}_i a_{ni} = \sum_{n=0}^{\infty} \bar{\lambda} c_n$$

for any $i \in I$. Thus $\bar{\lambda}$ is countably additive. If $c \in \mathfrak{C}$ is non-zero, express it as $\langle a_i \rangle_{i \in I}$; there must be an $i \in I$ such that $a_i \neq 0$, so that $\bar{\lambda} c = \bar{\mu}_i a_i > 0$. Thus $\bar{\lambda}$ is strictly positive, and $(\mathfrak{C}, \bar{\lambda})$ is a probability algebra.

(c) If $(\mathfrak{B}, \bar{\nu})$ is a probability algebra and $\langle \phi_i \rangle_{i \in I}$ is a family such that $\phi_i : \mathfrak{B} \to \mathfrak{A}_i$ is a measure-preserving Boolean homomorphism and $\phi_i = \pi_{ij}\phi_j$ whenever $i \leq j$ in I, set $\phi b = \langle \phi_i b \rangle_{i \in I}$ for $b \in \mathfrak{B}$. Then $\phi : \mathfrak{B} \to \prod_{i \in I} \mathfrak{A}_i$ is a Boolean homomorphism; also

$$\pi_{ij}(\phi b)(j) = \pi_{ij}\phi_j b = \phi_i b = (\phi b)(i)$$

whenever $i \leq j$ and $b \in \mathfrak{B}$, so $\phi[\mathfrak{B}] \subseteq \mathfrak{C}$, while $\pi_i \phi = \phi_i$ for every $i \in I$. And of course this uniquely determines ϕ. To see that ϕ is measure-preserving, we have only to check that

$$\bar{\lambda}\phi b = \bar{\mu}_i \pi_i \phi b = \bar{\mu}_i \phi_i b = \bar{\nu} b$$

whenever $b \in \mathfrak{B}$ and $i \in I$.

328J A different application of the method in 328A yields the following result on commuting families of Boolean homomorphisms.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and Φ a family of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself such that $\phi\psi = \psi\phi$ for all ϕ, $\psi \in \Phi$. Then there are a probability algebra $(\mathfrak{C}, \bar{\lambda})$, a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ and a family $\langle \tilde{\phi} \rangle_{\phi \in \Phi}$ such that

(i) $\tilde{\phi} : \mathfrak{C} \to \mathfrak{C}$ is a measure-preserving Boolean automorphism and $\tilde{\phi}\pi = \pi\phi$ for every $\phi \in \Phi$;

(ii) $(\phi\psi)^{\sim} = \tilde{\phi}\tilde{\psi}$ for all ϕ, $\psi \in \Phi$.

proof (a) Let Ψ be the set of all products $\phi_0\phi_1\ldots\phi_n$ where $\phi_i \in \Phi\cup\{\iota\}$ for every $i \le n$, ι here being the identity map from $\mathfrak{A}$ to itself. Then Ψ is a family of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself, and $\phi\psi = \psi\phi \in \Psi$ for all ϕ, $\psi \in \Psi$.

(b) For ϕ, $\psi \in \Psi$, say that $\phi \le \psi$ if there is a $\theta \in \Psi$ such that $\phi\theta = \psi$. Then $\le$ is a reflexive transitive relation on Ψ. Note that if $\phi \le \psi$ in Ψ then there is exactly one $\theta \in \Psi$ such that $\phi\theta = \psi$, because ϕ is injective. So we may define $\pi_{\psi,\phi} \in \Psi$ by saying that $\phi\pi_{\psi,\phi} = \psi$ whenever $\phi \le \psi$ in Ψ; that is, $\pi_{\phi\psi,\phi} = \psi$ whenever ϕ, $\psi \in \Psi$. Observe that if $\phi \le \psi \le \theta$ in Ψ, then

$$\phi\pi_{\psi,\phi}\pi_{\theta,\psi} = \psi\pi_{\theta,\psi} = \theta = \phi\pi_{\theta,\phi},$$

so

$$\pi_{\theta,\phi} = \pi_{\psi,\phi}\pi_{\theta,\psi} = \pi_{\theta,\psi}\pi_{\psi,\phi}.$$

Of course $\iota \le \phi$ for every $\phi \in \Psi$.

(c) If ϕ_1, $\phi_2 \in \Psi$ then $\phi_1 \le \phi_1\phi_2$ and $\phi_2 \le \phi_2\phi_1 = \phi_1\phi_2$; generally, if $D \subseteq \Psi$ is finite, there is a $\psi \in \Psi$ such that $\phi \le \psi$ for every $\phi \in D$. Consequently

$$\{A : A \subseteq \Psi, \text{ there is some } \phi \in \Psi \text{ such that } \psi \in A \text{ whenever } \phi \le \psi\}$$

is a filter on Ψ, and is included in an ultrafilter $\mathcal{F}$ say. Let $(\mathfrak{C}_0, \bar\lambda_0)$ be the probability algebra reduced power $(\mathfrak{A}, \bar\mu)^\Psi|\mathcal{F}$. By 328E, we have for each $\phi \in \Psi$ a measure-preserving Boolean homomorphism $\pi_\phi : \mathfrak{A} \to \mathfrak{C}_0$ defined by saying that $\pi_\phi a = \langle a_\psi\rangle^\bullet_{\psi\in\Psi}$ if $a_\psi = \pi_{\psi,\phi}a$ whenever $\phi \le \psi$ in Ψ, and $\pi_\phi = \pi_\psi\pi_{\psi,\phi}$ whenever $\phi \le \psi$. Re-interpreting this in terms of the definitions of $\le$ and $\pi_{\psi,\phi}$, we have $\pi_\phi = \pi_{\phi\psi}\psi$ whenever ϕ, $\psi \in \Psi$.

(d) If ϕ, ψ in Ψ, then

$$\pi_\phi[\mathfrak{A}] \cup \pi_\psi[\mathfrak{A}] = \pi_{\phi\psi}[\psi[\mathfrak{A}]] \cup \pi_{\psi\phi}[\phi[\mathfrak{A}]] \subseteq \pi_{\phi\psi}[\mathfrak{A}] \cup \pi_{\psi\phi}[\mathfrak{A}] = \pi_{\phi\psi}[\mathfrak{A}],$$

which is a subalgebra of $\mathfrak{C}_0$. So $\mathfrak{D} = \bigcup_{\phi\in\Psi}\pi_\phi[\mathfrak{A}]$ is a subalgebra of $\mathfrak{C}_0$, and its closure $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{C}_0$; set $\bar\lambda = \bar\lambda_0\restriction\mathfrak{C}$. Then $\pi = \pi_\iota : \mathfrak{A} \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism.

(e) If $\theta \in \Psi$, we have a measure-preserving Boolean homomorphism $\hat\theta : \mathfrak{C} \to \mathfrak{C}$ defined by the formula

$$\hat\theta(\langle a_\psi\rangle^\bullet_{\psi\in\Psi}) = \langle\theta a_\psi\rangle^\bullet_{\psi\in\Psi}$$

for every family $\langle a_\psi\rangle_{\psi\in\Psi}$ in $\mathfrak{A}$ (328Da); and $\widehat{\theta\phi} = \hat\theta\hat\phi$ for all θ, $\phi \in \Psi$ (328Db). Also $\hat\theta\pi_\phi = \pi_\phi\theta$ for every ϕ, $\theta \in \Psi$. **P** Let $a \in \mathfrak{A}$. Define $\langle a_\psi\rangle_{\psi\in\Psi}$, $\langle a'_\psi\rangle_{\psi\in\Psi}$ by setting

$$\begin{aligned} a_\psi &= \pi_{\psi,\phi}a \text{ when } \phi \le \psi, \\ &= 0 \text{ otherwise}, \\ a'_\psi &= \pi_{\psi,\phi}\theta a = \theta\pi_{\psi,\phi}a \text{ when } \phi \le \psi, \\ &= 0 \text{ otherwise}. \end{aligned}$$

Then

$$\pi_\phi a = \langle a_\psi\rangle^\bullet_{\psi\in\Psi},$$

$$\hat\theta\pi_\phi a = \langle\theta a_\psi\rangle^\bullet_{\psi\in\Psi} = \langle a'_\psi\rangle^\bullet_{\psi\in\Psi} = \pi_\phi\theta a. \text{ **Q**}$$

(f) It follows that, for $\theta \in \Psi$,

$$\hat\theta[\mathfrak{D}] = \bigcup_{\phi\in\Psi}\hat\theta[\pi_\phi[\mathfrak{A}]] = \bigcup_{\phi\in\Psi}\pi_\phi[\theta[\mathfrak{A}]] \subseteq \mathfrak{D}.$$

But in fact $\hat\theta[\mathfrak{D}] = \mathfrak{D}$. **P** If $d \in \mathfrak{D}$, there are $\phi \in \Psi$ and $a \in \mathfrak{A}$ such that $\pi_\phi a = d$. Now define

$$\begin{aligned} a_\psi &= \pi_{\psi,\phi}a \text{ if } \phi \le \psi, \\ &= 0 \text{ for other } \psi \in \Psi, \\ a'_\psi &= \pi_{\psi,\phi\theta}a \text{ if } \phi\theta \le \psi, \\ &= 0 \text{ for other } \psi \in \Psi, \\ d' &= \pi_{\phi\theta}a = \langle a'_\psi \rangle^\bullet_{\psi\in\Psi}. \end{aligned}$$

In this case, if $\phi\theta \le \psi$,

$$\phi\theta a'_\psi = \psi a, \quad \phi a_\psi = \psi a$$

so $\theta a'_\psi = a_\psi$. Consequently

$$\hat\theta d' = \hat\theta(\langle a'_\psi \rangle^\bullet_{\psi\in\Psi}) = \langle \theta a'_\psi \rangle^\bullet_{\psi\in\Psi} = \langle a_\psi \rangle^\bullet_{\psi\in\Psi}$$

(because $\{\psi : \phi\theta \le \psi\} \in \mathcal{F}$)

$$= d,$$

and $d = \hat\theta\pi_{\phi\theta}a \in \hat\theta[\mathfrak{D}]$. **Q**

(g) Since $\hat\theta[\mathfrak{C}]$ is a closed subalgebra of $\mathfrak{C}_0$ (324Kb) in which $\hat\theta[\mathfrak{D}] = \mathfrak{D}$ is topologically dense (3A3Eb), $\hat\theta[\mathfrak{C}] = \mathfrak{C}$. Setting $\tilde\theta = \hat\theta{\restriction}\mathfrak{C}$, we see that $\tilde\theta : \mathfrak{C} \to \mathfrak{C}$ is a surjective measure-preserving Boolean homomorphism, so is a Boolean automorphism. Since $\widehat{\phi\theta} = \hat\phi\hat\theta$, we have $(\phi\theta)^\sim = \tilde\phi\tilde\theta$ for all ϕ, $\theta \in \Psi$.

(h) Finally, as observed at the beginning of (e),

$$\tilde\theta\pi = \tilde\theta\pi_\iota = \hat\theta\pi_\iota = \pi_\iota\theta = \pi\theta$$

for every $\theta \in \Psi$. So $(\mathfrak{C}, \bar\lambda, \pi, \langle\tilde\theta\rangle_{\theta\in\Phi})$ has the required properties.

328X Basic exercises (a) Write out a version of the proof of 328J adapted to the case in which $\Phi = \{\phi\}$ is a singleton. (This is an abstract version of a construction known as the 'natural extension' of an inverse-measure-preserving function; see Petersen 83, 1.3G.)

(b) Let $\nu_\mathbb{N}$ be the usual measure on $X = \{0,1\}^\mathbb{N}$, and $(\mathfrak{B}_\mathbb{N}, \bar\nu_\mathbb{N})$ its measure algebra. (i) Find inverse-measure-preserving functions f, $g : X \to X$ such that $gf = g$ but $f(x) \ne x$ for every $x \in X$. (*Hint*: try $g(x)(n) = x(n+1)$.) (ii) Find measure-preserving Boolean homomorphisms ϕ, $\psi : \mathfrak{B}_\mathbb{N} \to \mathfrak{B}_\mathbb{N}$ such that $\phi\psi = \psi$ but ϕ is not the identity. (iii) In 328J, show that the hypothesis that members of Φ commute cannot be omitted.

(c) Let $(\mathfrak{A}, \bar\mu)$ be a purely atomic probability algebra, I a non-empty set and $\mathcal{F}$ an ultrafilter on I. Show that $(\mathfrak{A}, \bar\mu)^I|\mathcal{F}$ is isomorphic to $(\mathfrak{A}, \bar\mu)$.

328 Notes and comments I have starred this section because it is far from the main line of argument of the volume, and most readers should be moving on to Maharam's theorem and the Lifting Theorem. However the results here, while natural enough, have some features which demand a little attention, and it will be useful to be able to call on exact formulations of the ideas.

The proof of 328H begins by taking an ultrafilter on I. This ought to ring bells. It should be clear from the statement of the proposition that $(\mathfrak{C}, \bar\lambda, \langle\pi_i\rangle_{i\in I})$ is determined up to isomorphism by the properties declared here. It cannot therefore depend on which ultrafilter we pick, and there ought to be a construction not relying on this approach (and, we can hope, not demanding any application of the axiom of choice). This is indeed the case, and in 392Yd below I will sketch a method which can be adapted to give such a proof. Yet another proof of 328H is proposed in 418Yn[4] in Volume 4.

[4] Formerly 418Yp.

The same remarks apply to the proof of 328J. In the result as stated, I have not imposed conditions on the structure $(\mathfrak{C}, \bar{\lambda}, \pi, \langle\tilde{\phi}\rangle_{\phi\in\Phi})$ sufficient to define it uniquely, but once again it is not necessary to employ an ultrafilter, and in fact the filter

$$\{A : A \subseteq \Psi, \text{ there is some } \phi \in \Psi \text{ such that } \psi \in A \text{ whenever } \phi \leq \psi\}$$

is already enough, if we take the trouble to move to the right subalgebra of $\mathfrak{A}^\Psi$ before taking the quotient algebra.

Version of 24.4.06

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

322K Paragraphs 322K (simple products of measure algebras), 322N (the Stone space of a measure algebra) and 322Q (further properties of Stone spaces), referred to in the 2003 and 2006 editions of Volume 4, are now 322L, 322O and 322R.

326E Countably additive functionals Definition 326E, referred to in the 2003 and 2006 editions of Volume 4 and the 2008 edition of Volume 5, is now 326I.

326G Corollary 326G, referred to in the 2008 edition of Volume 5, is now 326K.

326I Hahn decomposition Theorem 326I, referred to in the 2003 and 2006 editions of Volume 4, is now 326M.

326K Completely additive functionals The notes in 326K, referred to in the 2003 and 2006 editions of Volume 4, have been moved to 326O.

326Q Finitely additive functionals on free products Theorem 326Q, referred to in the 2003 and 2006 editions of Volume 4 and the 2008 edition of Volume 5, is now 326E.

328D Reduced products of probability algebras Paragraph 328D, referred to in the 2008 edition of Volume 5, is now 328E.

Version of 25.8.17

References for Volume 3

Anderson I. [87] *Combinatorics of Finite Sets.* Oxford U.P., 1987. [332Xk, 3A1K.]

Anzai H. [51] 'On an example of a measure-preserving transformation which is not conjugate to its inverse', Proc. Japanese Acad. Sci. 27 (1951) 517-522. [§382 *notes*.]

Balcar B., Główczyński W. & Jech T. [98] 'The sequential topology on complete Boolean algebras', Fundamenta Math. 155 (1998) 59-78. [393L, 393Q.]

Balcar B., Jech T. & Pazák T. [05] 'Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann', Bull. London Math. Soc. 37 (2005) 885-898. [393L, 393Q.]

Becker H. & Kechris A.S. [96] *The descriptive set theory of Polish group actions.* Cambridge U.P., 1996 (London Math. Soc. Lecture Note Series 232). [§395 *notes*.]

Bekkali M. & Bonnet R. [89] 'Rigid Boolean algebras', pp. 637-678 in Monk 89. [384L.]

Bellow A. & Kölzow D. [76] *Measure Theory, Oberwolfach 1975.* Springer, 1976 (Lecture Notes in Mathematics 541).

Bhaskara Rao, K.P.S. & Bhaskara Rao, M. *Theory of Charges.* Academic, 1983.

Billingsley P. [65] *Ergodic Theory and Information.* Wiley, 1965. [§372 *notes*.]

Bollobás B. [79] *Graph Theory.* Springer, 1979. [332Xk, 3A1K.]

Bose R.C. & Manvel B. [84] *Introduction to Combinatorial Theory.* Wiley, 1984. [3A1K.]

Bourbaki N. [66] *General Topology.* Hermann/Addison-Wesley, 1968. [§3A3, §3A4.]

Bourbaki N. [68] *Theory of Sets.* Hermann/Addison-Wesley, 1968. [§315 *notes*.]

Bourbaki N. [87] *Topological Vector Spaces.* Springer, 1987. [§3A5.]

Bukhvalov A.V. [95] 'Optimization without compactness, and its applications', pp. 95-112 in Huijsmans Kaashoek Luxemburg & Pagter 95. [367U.]

Burke M.R. [93] 'Liftings for Lebesgue measure', pp. 119-150 in Judah 93. [341L, 345F.]

Burke M.R. [n95] 'Consistent liftings', privately circulated, 1995. [346Ya.]

Burnside W. [1911] *Theory of Groups of Finite Order.* Cambridge U.P., 1911 (reprinted by Dover, 1955). [§384 *notes*.]

Chacon R.V. [69] 'Weakly mixing transformations which are not strongly mixing', Proc. Amer. Math. Soc. 22 (1969) 559-562. [372R.]

Chacon R.V. & Krengel U. [64] 'Linear modulus of a linear operator', Proc. Amer. Math. Soc. 15 (1964) 553-559. [§371 *notes*.]

Choksi J.R. & Prasad V.S. [82] 'Ergodic theory of homogeneous measure algebras', pp. 367-408 of Kölzow & Maharam-Stone 82. [383I.]

Cohn H. [06] 'A short proof of the simple continued fraction expansion of e', Amer. Math. Monthly 113 (2006) 57-62; `arXiv:math.NT/0601660`. [372L.]

Coleman A.J. & Ribenboim P. [67] (eds.) *Proceedings of the Symposium in Analysis, Queen's University, June 1967.* Queen's University, Kingston, Ontario, 1967.

Comfort W.W. & Negrepontis S. [82] *Chain Conditions in Topology.* Cambridge U.P., 1982. [§391 *notes*.]

Cziszár I. [67] 'Information-type measures of difference of probability distributions and indirect observations', Studia Scientiarum Math. Hungarica 2 (1967) 299-318. [386G.]

Davey B.A. & Priestley H.A. [90] 'Introduction to Lattices and Order', Cambridge U.P., 1990. [3A6C.]

Dugundji J. [66] *Topology.* Allyn & Bacon, 1966. [§3A3, 3A4Bb.]

Dunford N. [1936] 'Integration and linear operators', Trans. Amer. Math. Soc. 40 (1936) 474-494. [§376 *notes*.]

Dunford N. & Schwartz J.T. [57] *Linear Operators I.* Wiley, 1957 (reprinted 1988). [§356 *notes*, §371 *notes*, §376 *notes*, §3A5.]

Dye H.A. [59] 'On groups of measure preserving transformations I', Amer. J. Math. 81 (1959) 119-159. [§388 *notes*.]

Eigen S.J. [82] 'The group of measure-preserving transformations of $[0,1]$ has no outer automorphisms', Math. Ann. 259 (1982) 259-270. [384D.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [§3A3, §3A4.]

Enderton H.B. [77] *Elements of Set Theory.* Academic, 1977. [§3A1.]

Erdős P. & Oxtoby J.C. [55] 'Partitions of the plane into sets having positive measure in every non-null product set', Trans. Amer. Math. Soc. 79 (1955) 91-102. [§325 *notes*.]

Fathi A. [78] 'Le groupe des transformations de $[0,1]$ qui préservent la mesure de Lebesgue est un groupe simple', Israel J. Math. 29 (1978) 302-308. [§382 *notes*, 383I.]

Fremlin D.H. [74a] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [Chap. 35 *intro*., 354Yb, §355 *notes*, §356 *notes*, §363 *notes*, §365 *notes*, §371 *notes*, §376 *notes*.]

Fremlin D.H. [74b] 'A characterization of L-spaces', Indag. Math. 36 (1974) 270-275. [§371 *notes*.]

Fremlin D.H., de Pagter B. & Ricker W.J. [05] 'Sequential closedness of Boolean algebras of projections in Banach spaces', Studia Math. 167 (2005) 45-62. [§323 *notes*.]

Frolík Z. [68] 'Fixed points of maps of extremally disconnected spaces and complete Boolean algebras', Bull. Acad. Polon. Sci. 16 (1968) 269-275. [382E.]

Gaal S.A. [64] *Point Set Topology.* Academic, 1964. [§3A3, §3A4.]

Gaifman H. [64] 'Concerning measures on Boolean algebras', Pacific J. Math. 14 (1964) 61-73. [§391 *notes*.]

Gale D. [60] *The theory of linear economic models.* McGraw-Hill, 1960. [3A5D.]

Gnedenko B.V. & Kolmogorov A.N. [54] *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, 1954. [§342 *notes*.]

Graf S. & Weizsäcker H.von [76] 'On the existence of lower densities in non-complete measure spaces', pp. 155-158 in Bellow & Kölzow 76. [341L.]

Hajian A. & Ito Y. [69] 'Weakly wandering sets and invariant measures for a group of transformations', J. of Math. and Mech. 18 (1969) 1203-1216. [396B.]

Hajian A., Ito Y. & Kakutani S. [75] 'Full groups and a theorem of Dye', Advances in Math. 17 (1975) 48-59. [§388 *notes*.]

Halmos P.R. [1948] 'The range of a vector measure', Bull. Amer. Math. Soc. 54 (1948) 416-421. [326Yk.]

Halmos P.R. [60] *Naive Set Theory.* Van Nostrand, 1960. [3A1D.]

Huijsmans C.B., Kaashoek M.A., Luxemburg W.A.J. & de Pagter B. [95] (eds.) *Operator Theory in Function Spaces and Banach Lattices.* Birkhäuser, 1995.

Ionescu Tulcea C. & Ionescu Tulcea A. [69] *Topics in the Theory of Lifting.* Springer, 1969. [§341 *notes*.]

James I.M. [87] *Topological and Uniform Spaces.* Springer, 1987. [§3A3, §3A4.]

Jech T. [03] *Set Theory, Millennium Edition.* Springer, 2002. [§3A1.]

Jech T. [08] 'Algebraic characterizations of measure algebras', Proc. Amer. Math. Soc. 136 (2008) 1285-1294. [393Xj.]

Johnson R.A. [80] 'Strong liftings which are not Borel liftings', Proc. Amer. Math. Soc. 80 (1980) 234-236. [345F.]

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Kakutani S. [1941] 'Concrete representation of abstract L-spaces and the mean ergodic theorem', Annals of Math. 42 (1941) 523-537. [369E.]

Kalton N.J., Peck N.T. & Roberts J.W. [84] 'An F-space sampler', Cambridge U.P., 1984. [§375 *notes*.]

Kalton N.J. & Roberts J.W. [83] 'Uniformly exhaustive submeasures and nearly additive set functions', Trans. Amer. Math. Soc. 278 (1983) 803-816. [392D, §392 *notes*.]

Kantorovich L.V., Vulikh B.Z. & Pinsker A.G. [50] *Functional Analysis in Partially Ordered Spaces*, Gostekhizdat, 1950. [391D.]

Kawada Y. [1944] 'Über die Existenz der invarianten Integrale', Jap. J. Math. 19 (1944) 81-95. [§395 *notes*.]

Kelley J.L. [59] 'Measures on Boolean algebras', Pacific J. Math. 9 (1959) 1165-1177. [§391 *notes*.]

Kolmogorov A.N. [58] 'New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces', Dokl. Akad. Nauk SSSR 119 (1958) 861-864. [385P.]

Kölzow D. & Maharam-Stone D. [82] (eds.) *Measure Theory Oberwolfach 1981.* Springer, 1982 (Lecture Notes in Math. 945).

Koppelberg S. [89] *General Theory of Boolean Algebras*, vol. 1 of Monk 89. [Chap. 31 *intro.*, §332 *notes.*]

Köthe G. [69] *Topological Vector Spaces I.* Springer, 1969. [§356 *notes*, 3A5N.]

Kranz P. & Labuda I. [93] (eds.) *Proceedings of the Orlicz Memorial Conference, 1991*, unpublished manuscript, available from first editor (`mmkranz@olemiss.edu`).

Krengel, U. [63] '"Uber den Absolutbetrag stetiger linearer Operatoren und seine Anwendung auf ergodische Zerlegungen', Math. Scand. 13 (1963) 151-187. [371Xb.]

Krieger W. [76] 'On ergodic flows and the isomorphism of factors', Math. Ann. 223 (1976) 19-70. [§388 *notes.*]

Krivine J.-L. [71] *Introduction to Axiomatic Set Theory.* D.Reidel, 1971. [§3A1.]

Kullback S. [67] 'A lower bound for discrimination information in terms of variation', IEEE Trans. on Information Theory 13 (1967) 126-127. [386G.]

Kunen K. [80] *Set Theory.* North-Holland, 1980. [§3A1.]

Kwapien S. [73] 'On the form of a linear operator on the space of all measurable functions', Bull. Acad. Polon. Sci. 21 (1973) 951-954. [§375 *notes.*]

Lang S. [93] *Real and Functional Analysis.* Springer, 1993. [§3A5.]

Liapounoff A.A. [1940] 'Sur les fonctions-vecteurs complètement additives', Bull. Acad. Sci. URSS (Izvestia Akad. Nauk SSSR) 4 (1940) 465-478. [326H.]

Lindenstrauss J. & Tzafriri L. [79] *Classical Banach Spaces II.* Springer, 1979, reprinted in Lindenstrauss & Tzafriri 96. [§354 *notes*, 374Xj.]

Lindenstrauss J. & Tzafriri L. [96] *Classical Banach Spaces I & II.* Springer, 1996.

Lipschutz S. [64] *Set Theory and Related Topics.* McGraw-Hill, 1964 (Schaum's Outline Series). [3A1D.]

Luxemburg W.A.J. [67a] 'Is every integral normal?', Bull. Amer. Math. Soc. 73 (1967) 685-688. [363S.]

Luxemburg W.A.J. [67b] 'Rearrangement-invariant Banach function spaces', pp. 83-144 in Coleman & Ribenboim 67. [§374 *notes.*]

Luxemburg W.A.J. & Zaanen A.C. [71] *Riesz Spaces I.* North-Holland, 1971. [Chap. 35 *intro.*]

Macheras N.D., Musiał K. & Strauss W. [99] 'On products of admissible liftings and densities', J. for Analysis and its Applications 18 (1999) 651-668. [346G.]

Macheras N.D. & Strauss W. [95] 'Products of lower densities', J. for Analysis and its Applications 14 (1995) 25-32. [346Xf.]

Macheras N.D. & Strauss W. [96a] 'On products of almost strong liftings', J. Australian Math. Soc. (A) 60 (1996) 1-23. [346Yc.]

Macheras N.D. & Strauss W. [96b] 'The product lifting for arbitrary products of complete probability spaces', Atti Sem. Math. Fis. Univ. Modena 44 (1996) 485-496. [346H, 346Yd.]

Maharam D. [1942] 'On homogeneous measure algebras', Proc. Nat. Acad. Sci. U.S.A. 28 (1942) 108-111. [331F, 332B.]

Maharam D. [1947] 'An algebraic characterization of measure algebras', Ann. Math. 48 (1947) 154-167. [393J.]

Maharam D. [58] 'On a theorem of von Neumann', Proc. Amer. Math. Soc. 9 (1958) 987-994. [§341 *notes*, §346 *notes.*]

Marczewski E. [53] 'On compact measures', Fund. Math. 40 (1953) 113-124. [342A.]

McCune W. [97] 'Solution of the Robbins problem', J. Automated Reasoning 19 (1997) 263-276. [311Yc.]

Miller B.D. [04] PhD Thesis, University of California, Berkeley, 2004. [382Xc, §382 *notes.*]

Monk J.D. [89] (ed.) *Handbook of Boolean Algebra.* North-Holland, 1989.

Nadkarni M.G. [90] 'On the existence of a finite invariant measure', Proc. Indian Acad. Sci., Math. Sci. 100 (1990) 203-220. [§395 *notes.*]

Ornstein D.S. [74] *Ergodic Theory, Randomness and Dynamical Systems.* Yale U.P., 1974. [§387 *notes.*]

Ornstein D.S. & Shields P.C. [73] 'An uncountable family of K-automorphisms', Advances in Math. 10 (1973) 63-88. [§382 *notes.*]

Perović Ž. & Veličković B. [18] 'Ranks of Maharam algebras', Advances in Math. 330 (2018) 253-279. [394A.]

Petersen K. [83] *Ergodic Theory.* Cambridge U.P., 1983. [328Xa, 385C, §385 *notes*, 386E.]

Roberts J.W. [93] 'Maharam's problem', in Kranz & Labuda 93. [§394 *notes*.]

Rotman J.J. [84] 'An Introduction to the Theory of Groups', Allyn & Bacon, 1984. [§384 *notes*, 3A6B.]

Rudin W. [91] *Functional Analysis.* McGraw-Hill, 1991. [§3A5.]

Ryzhikov V.V. [93] 'Factorization of an automorphism of a complete Boolean algebra into a product of three involutions', Mat. Zametki (=Math. Notes of Russian Acad. Sci.) 54 (1993) 79-84. [§382 *notes*.]

Sazonov V.V. [66] 'On perfect measures', A.M.S. Translations (2) 48 (1966) 229-254. [§342 *notes*.]

Schaefer H.H. [66] *Topological Vector Spaces.* MacMillan, 1966; reprinted with corrections Springer, 1971. [3A4A, 3A5J, 3A5N.]

Schaefer H.H. [74] *Banach Lattices and Positive Operators.* Springer, 1974. [Chap. 35 *intro*., §354 *notes*.]

Schubert H. [68] *Topology.* Allyn & Bacon, 1968. [§3A3, §3A4.]

Shelah S. [98] 'The lifting problem with the full ideal', J. Applied Analysis 4 (1998) 1-17. [341L.]

Sikorski R. [64] *Boolean Algebras.* Springer, 1964. [Chap. 31 *intro*.]

Sinaĭ Ya.G. [59] 'The notion of entropy of a dynamical system', Dokl. Akad. Nauk SSSR 125 (1959) 768-771. [385P.]

Sinaĭ Ya.G. [62] 'Weak isomorphism of transformations with an invariant measure', Soviet Math. 3 (1962) 1725-1729. [387E.]

Smorodinsky M. [71] *Ergodic Theory, Entropy.* Springer, 1971 (Lecture Notes in Math., 214). [§387 *notes*.]

Štěpánek P. & Rubin M. [89] 'Homogeneous Boolean algebras', pp. 679-715 in Monk 89. [382S, §382 *notes*.]

Talagrand M. [82a] 'Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations', Ann. Institut Fourier (Grenoble) 32 (1982) 39-69. [§346 *notes*.]

Talagrand M. [82b] 'La pathologie des relèvements invariants', Proc. Amer. Math. Soc. 84 (1982) 379-382. [345F.]

Talagrand M. [84] *Pettis integral and measure theory.* Mem. Amer. Math. Soc. 307 (1984). [§346 *notes*.]

Talagrand M. [08] 'Maharam's problem', Annals of Math. 168 (2008) 981-1009. [§394.]

Taylor A.E. [64] *Introduction to Functional Analysis.* Wiley, 1964. [§3A5.]

Todorčević S. [04] 'A problem of von Neumann and Maharam about algebras supporting continuous submeasures', Fund. Math. 183 (2004) 169-183. [393S.]

Truss J.K. [89] 'Infinite permutation groups I: products of conjugacy classes', J. Algebra 120 (1989) 454-493. [§382 *notes*.]

Vladimirov D.A. [02] *Boolean algebras in analysis.* Kluwer, 2002 (Math. and its Appl. 540). [367L.]

Vulikh B.C. [67] *Introduction to the Theory of Partially Ordered Vector Spaces.* Wolters-Noordhoff, 1967. [§364 *notes*.]

Wagon S. [85] *The Banach-Tarski Paradox.* Cambridge U.P., 1985. [§395 *notes*.]

Wilansky A. [64] *Functional Analysis.* Blaisdell, 1964. [§3A5.]

Zaanen A.C. [83] *Riesz Spaces II.* North-Holland, 1983. [Chap. 35 *intro*., 376K, §376 *notes*.]