Version of 10.9.13

One of the first things one learns, as a student of measure theory, is that sets of measure zero are frequently 'negligible' in the straightforward sense that they can safely be ignored. This is not quite a universal principle, and one of my purposes in writing this treatise is to call attention to the exceptional cases in which negligible sets are important. But very large parts of the theory, including some of the topics already treated in Volume 2, can be expressed in an appropriately abstract language in which negligible sets have been factored out. This is what the present volume is about. A 'measure algebra' is a quotient of an algebra of measurable sets by a null ideal; that is, the elements of the measure algebra are equivalence classes of measurable sets. At the cost of an extra layer of abstraction, we obtain a language which can give concise and elegant expression to a substantial proportion of the ideas of measure theory, and which offers insights almost everywhere in the subject.

It is here that I embark wholeheartedly on 'pure' measure theory. I think it is fair to say that the applications of measure theory to other branches of mathematics are more often through measure *spaces* rather than measure *algebras*. Certainly there will be in this volume many theorems of wide importance outside measure theory; but typically their usefulness will be in forms translated back into the language of the first two volumes. But it is also fair to say that the language of measure algebras is the only reasonable way to discuss large parts of a subject which, as pure mathematics, can bear comparison with any.

In the structure of this volume I can distinguish seven 'working' and two 'accessory' chapters. The 'accessory' chapters are 31 and 35. In these I develop the theories of Boolean algebras and Riesz spaces (= vector lattices) which are needed later. As in Volume 2 you have a certain amount of choice in the order in which you take the material. Everything except Chapter 35 depends on Chapter 31, and everything except Chapters 31 and 35 depends on Chapter 32. Chapters 33, 34 and 36 can be taken in any order, but Chapter 36 relies on Chapter 35. (I do not mean that Chapter 33 is never referred to in Chapter 34, nor even that the later chapters do not rely on results from Chapter 33. What I mean is that their most important ideas are accessible without learning the material of Chapter 33 properly.) Chapter 37 depends on Chapters 35 and 36. Chapter 38 would be difficult to make sense of without some notion of what has been done in Chapter 33. Chapter 39 uses fragments of Chapters 35 and 36.

The first third of the volume follows almost the only line permitted by the structure of the subject. If we are going to study measure algebras at all, we must know the relevant facts about Boolean algebras (Chapter 31) and how to translate what we know about measure spaces into the new language (Chapter 32). Then we must get a proper grip on the two most important theorems: Maharam's theorem on the classification of measure algebras (Chapter 33) and the von Neumann-Maharam lifting theorem (Chapter 34). Since I am now writing for readers who are committed – I hope, happily committed – to learning as much as they can about the subject, I take the space to push these ideas as far as they can easily go, giving a full classification of closed subalgebras of probability algebras, for instance (§333), and investigating special types of lifting (§§345-346). I mention here three sections interpolated into Chapter 34 (§§342-344) which attack a subtle and important question: when can we expect homomorphisms between measure algebras to be realizable in terms of transformations between measure spaces, as discussed briefly in §234 and elsewhere.

Chapters 36 and 37 are devoted to re-working the ideas of Chapter 24 on 'function spaces' in the more abstract context now available, and relating them to the general Riesz spaces of Chapter 35. I am concerned here not to develop new structures, nor even to prove striking new theorems, but rather to offer new ways of looking at the old ones. Only in the Ergodic Theorem (§372) do I come to a really important new result. Chapter 38 looks at two questions, both obvious ones to ask if you have been trained in twentieth-century pure mathematics: what does the automorphism group of a measure algebra look like, and inside such an automorphism group, what do the conjugacy classes look like? (The second question is a fancy way of asking how to decide, given two automorphisms of one of the structures considered in this volume, whether they are really different, or just copies of each other obtained by looking at the structure a different way up.)

Finally, in Chapter 39, I discuss what is known about the question of which Boolean algebras can appear as measure algebras.

Concerning the prerequisites for this volume, we certainly do not need everything in Volume 2. The important chapters there are 21, 23, 24, 25 and 27. If you are approaching this volume without having read the earlier parts of this treatise, you will need the Radon-Nikodým theorem and product measures (of arbitrary families of probability spaces), for Maharam's theorem; a simple version of the martingale theorem, for the lifting theorem; and an acquaintance with L^p spaces (particularly, with L^0 spaces) for Chapter 36. But I would recommend the results-only versions of Volumes 1 and 2 in case some reference is totally obscure. Outside measure theory, I call on quite a lot of terms from general topology, but none of the ideas needed are difficult (Baire's and Tychonoff's theorems are the deepest); they are sketched in §§3A3 and 3A4. We do need some functional analysis for Chapters 36 and 39, but very little more than was already used in Volume 2, except that I now call on versions of the Hahn-Banach theorem (§3A5).

In this volume I assume that readers have substantial experience in both real and abstract analysis, and I make few concessions which would not be appropriate when addressing active researchers, except that perhaps I am a little gentler when calling on ideas from set theory and general topology than I should be with my own colleagues, and I continue to include all the easiest exercises I can think of. I do maintain my practice of giving proofs in very full detail, not so much because I am trying to make them easier, but because one of my purposes here is to provide a complete account of the ideas of the subject. I hope that the result will be accessible to most doctoral students who are studying topics in, or depending on, measure theory.

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Chapter 31

Boolean algebras

The theory of measure algebras naturally depends on certain parts of the general theory of Boolean algebras. In this chapter I collect those results which will be useful later. Since many students encounter the formal notion of Boolean algebra for the first time in this context, I start at the beginning; and indeed I include in the Appendix (§3A2) a brief account of the necessary part of the theory of rings, as not everyone will have had time for this bit of abstract algebra in an undergraduate course. But unless you find the algebraic theory of Boolean algebras so interesting that you wish to study it for its own sake – in which case you should perhaps turn to Sikorski 64 or Koppelberg 89 – I do not think it would be very sensible to read the whole of this chapter before proceeding to the main work of the volume in Chapter 32. Probably §311 is necessary to get an idea of what a Boolean algebra looks like, and a glance at the statements of the theorems in §312 and 313A-313B would be useful, but the later sections can wait until you have need of them, on the understanding that apparently innocent formal manipulations may depend on concepts which take some time to master. I hope that the cross-references will be sufficiently well-targeted to make it possible to read this material in parallel with its applications.

As for the actual material covered, §311 introduces Boolean rings and algebras, with M.H.Stone's theorem on their representation as rings and algebras of sets. §312 is devoted to subalgebras, homomorphisms and quotients, following a path parallel to the corresponding ideas in group theory, ring theory and linear algebra. In §313 I come to the special properties of Boolean algebras associated with their lattice structures, with notions of order-preservation, order-continuity and order-closure. §314 continues this with a discussion of order-completeness, and the elaboration of the Stone representation of an arbitrary Boolean algebra into the Loomis-Sikorski representation of a σ-complete Boolean algebra; this brings us to regular open algebras. §315 deals with 'simple' and 'free' products of Boolean algebras, corresponding to 'products' and 'tensor products' of linear spaces, and to projective and inductive limits of families of Boolean algebras. Finally, §316 examines three special topics: the countable chain condition, weak distributivity and homogeneity.

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311 Boolean algebras

In this section I try to give a sufficient notion of the character of abstract Boolean algebras to make the calculations which will appear on almost every page of this volume seem both elementary and natural. The principal result is of course M.H.Stone's theorem: every Boolean algebra can be expressed as an algebra of sets (311E). So the section divides naturally into the first part, proving Stone's theorem, and the second, consisting of elementary consequences of the theorem and a little practice in using the insights it offers.

311A Definitions (a) A **Boolean ring** is a ring $(\mathfrak{A},+,.)$ in which $a^2=a$ for every $a\in\mathfrak{A}$.

(b) A **Boolean algebra** is a Boolean ring $\mathfrak{A}$ with a multiplicative identity $1=1_{\mathfrak{A}}$; I allow $1=0$ in this context.

311B Examples (a) For any set X, $(\mathcal{P}X,\triangle,\cap)$ is a Boolean algebra; its zero is $\emptyset$ and its multiplicative identity is X.

(b) Recall that an 'algebra of subsets of X' is a family $\Sigma\subseteq\mathcal{P}X$ such that $\emptyset\in\Sigma$, $X\setminus E\in\Sigma$ for every $E\in\Sigma$, and $E\cup F\in\Sigma$ for all E, $F\in\Sigma$. In this case $(\Sigma,\triangle,\cap)$ is a Boolean algebra with zero $\emptyset$ and identity X.

(c) Consider the ring $\mathbb{Z}_2=\{0,1\}$, with its ring operations $+_2$, $\cdot$ given by setting

$$0+_2 0=1+_2 1=0,\quad 0+_2 1=1+_2 0=1,$$

$$0\cdot 0=0\cdot 1=1\cdot 0=0,\quad 1\cdot 1=1.$$

Because $0\cdot 0=0$ and $1\cdot 1=1$, it is a Boolean algebra.

311C Proposition Let $\mathfrak{A}$ be a Boolean ring.
(a) $a + a = 0$ for every $a \in \mathfrak{A}$.
(b) $ab = ba$ for all $a, b \in \mathfrak{A}$.

311D Lemma Let $\mathfrak{A}$ be a Boolean ring, I an ideal of $\mathfrak{A}$, and $a \in \mathfrak{A} \setminus I$. Then there is a ring homomorphism $\phi : \mathfrak{A} \to \mathbb{Z}_2$ such that $\phi a = 1$ and $\phi d = 0$ for every $d \in I$.

311E M.H.Stone's theorem: first form Let $\mathfrak{A}$ be any Boolean ring, and let Z be the set of ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Then we have an injective ring homomorphism $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$, setting $\widehat{a} = \{z : z \in Z, z(a) = 1\}$. If $\mathfrak{A}$ is a Boolean algebra, then $\widehat{1_{\mathfrak{A}}} = Z$.

311F Remarks For any Boolean ring $\mathfrak{A}$, I will say that the **Stone space** of $\mathfrak{A}$ is the set Z of non-zero ring homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, and the canonical map $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$ is the **Stone representation**.

311G The operations $\cup$, $\setminus$, $\triangle$ on a Boolean ring Let $\mathfrak{A}$ be a Boolean ring.

(a) Set

$$a \cup b = a + b + ab, \quad a \cap b = ab, \quad a \setminus b = a + ab, \quad a \triangle b = a + b$$

for $a, b \in \mathfrak{A}$.

(b) I will say that a set $A \subseteq \mathfrak{A}$ is **disjoint** if $a \cap b = 0$, that is, $ab = 0$, for all distinct $a, b \in A$; and that an indexed family $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **disjoint** if $a_i \cap a_j = 0$ for all distinct $i, j \in I$. (I allow $0 \in A$ or $a_i = 0$.)

(c) A **partition of unity** in $\mathfrak{A}$ will be *either* a disjoint set $C \subseteq \mathfrak{A}$ such that there is no non-zero $a \in \mathfrak{A}$ such that $a \cap c = 0$ for every $c \in C$ *or* a disjoint family $\langle c_i \rangle_{i \in I}$ in $\mathfrak{A}$ such that there is no non-zero $a \in \mathfrak{A}$ such that $a \cap c_i = 0$ for every $i \in I$.

(d) Note that a set $C \subseteq \mathfrak{A}$ is a partition of unity iff $C \cup \{0\}$ is a maximal disjoint set.
If $A \subseteq \mathfrak{A}$ is any disjoint set, there is a partition of unity including A.

(e) If C and D are two partitions of unity, I say that C **refines** D if for every $c \in C$ there is a $d \in D$ such that $cd = c$. Note that if C refines D and D refines E then C refines E.

311H The order structure of a Boolean ring Again treating a Boolean ring $\mathfrak{A}$ as an algebra of sets, it has a natural ordering, setting $a \subseteq b$ if $ab = a$, so that $a \subseteq b$ iff $\widehat{a} \subseteq \widehat{b}$. This translation makes it obvious that $\subseteq$ is a partial order on $\mathfrak{A}$, with least element 0, and with greatest element 1 iff $\mathfrak{A}$ is a Boolean algebra. Moreover, $\mathfrak{A}$ is a lattice, with $a \cup b = \sup\{a, b\}$ and $a \cap b = \inf\{a, b\}$ for all $a, b \in \mathfrak{A}$. Generally, for $a_0, \ldots, a_n \in \mathfrak{A}$,

$$\sup_{i \le n} a_i = a_0 \cup \ldots \cup a_n, \quad \inf_{i \le n} a_i = a_0 \cap \ldots \cap a_n;$$

suprema and infima of finite subsets of $\mathfrak{A}$ correspond to unions and intersections of the corresponding families in the Stone space.

311I The topology of a Stone space: Theorem Let Z be the Stone space of a Boolean ring $\mathfrak{A}$, and let $\mathfrak{T}$ be

$$\{G : G \subseteq Z \text{ and for every } z \in G \text{ there is an } a \in \mathfrak{A} \text{ such that } z \in \widehat{a} \subseteq G\}.$$

Then $\mathfrak{T}$ is a topology on Z, under which Z is a locally compact zero-dimensional Hausdorff space, and $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$ is precisely the set of compact open subsets of Z. $\mathfrak{A}$ is a Boolean algebra iff Z is compact.

311J Proposition Let X be a locally compact zero-dimensional Hausdorff space. Then the set $\mathfrak{A}$ of open-and-compact subsets of X is a subring of $\mathcal{P}X$. If Z is the Stone space of $\mathfrak{A}$, there is a unique homeomorphism $\theta : Z \to X$ such that $\widehat{a} = \theta^{-1}[a]$ for every $a \in \mathfrak{A}$.

311L Complemented distributive lattices: Proposition Let $\mathfrak{A}$ be a lattice such that

(i) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, $c \in \mathfrak{A}$;

(ii) there is a permutation $a \mapsto a' : \mathfrak{A} \to \mathfrak{A}$ which is order-reversing, that is, $a \le b$ iff $b' \le a'$, and such that $a'' = a$ for every a;

(iii) $\mathfrak{A}$ has a least element 0 and $a \wedge a' = 0$ for every $a \in \mathfrak{A}$.

Then $\mathfrak{A}$ has a Boolean algebra structure for which $a \subseteq b$ iff $a \le b$.

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312 Homomorphisms

I continue the theory of Boolean algebras with a section on subalgebras, ideals and homomorphisms. From now on, I will relegate Boolean rings which are not algebras to the exercises; I think there is no need to set out descriptions of the (mostly trifling) modifications necessary to deal with the extra generality. The first part of the section (312A-312L) concerns the translation of the basic concepts of ring theory into the language which I propose to use for Boolean algebras. 312M shows that the order relation on a Boolean algebra defines the algebraic structure, and in 312N-312O I give a fundamental result on the extension of homomorphisms. I end the section with results relating the previous ideas to the Stone representation of a Boolean algebra (312P-312T).

312A Subalgebras Let $\mathfrak{A}$ be a Boolean algebra. I will use the phrase **subalgebra of** $\mathfrak{A}$ to mean a subring of $\mathfrak{A}$ containing its multiplicative identity.

312B Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Then the following are equiveridical, that is, if one is true so are the others:

(i) $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$;

(ii) $0 \in \mathfrak{B}$, $a \cup b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$, and $1 \setminus a \in \mathfrak{B}$ for all $a \in \mathfrak{B}$;

(iii) $\mathfrak{B} \ne \emptyset$, $a \cap b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$, and $1 \setminus a \in \mathfrak{B}$ for all $a \in \mathfrak{B}$.

312C Ideals in Boolean algebras: Proposition If $\mathfrak{A}$ is a Boolean algebra, a set $I \subseteq \mathfrak{A}$ is an ideal of $\mathfrak{A}$ iff $0 \in I$, $a \cup b \in I$ for all a, $b \in I$, and $a \in I$ whenever $b \in I$ and $a \subseteq b$.

312E Proposition Let $\mathfrak{A}$ be a Boolean algebra, and a any element of $\mathfrak{A}$. Then the principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ generated by a is just $\{b : b \in \mathfrak{A},\ b \subseteq a\}$, and (with the inherited operations $\cap{\restriction}\mathfrak{A}_a \times \mathfrak{A}_a$, $\triangle{\restriction}\mathfrak{A}_a \times \mathfrak{A}_a$) is a Boolean algebra in its own right, with multiplicative identity a.

312F Boolean homomorphisms Now suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras. I will use the phrase **Boolean homomorphism** to mean a function $\pi : \mathfrak{A} \to \mathfrak{B}$ which is a ring homomorphism and is uniferent, that is, $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

312G Proposition Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be Boolean algebras.

(a) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism, then $\pi[\mathfrak{A}]$ is a subalgebra of $\mathfrak{B}$.

(b) If $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ are Boolean homomorphisms, then $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ is a Boolean homomorphism.

(c) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a bijective Boolean homomorphism, then $\pi^{-1} : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism.

312H Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a function. Then the following are equiveridical:

(i) π is a Boolean homomorphism;

(ii) $\pi(a \cap b) = \pi a \cap \pi b$ and $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for all a, $b \in \mathfrak{A}$;

(iii) $\pi(a \cup b) = \pi a \cup \pi b$ and $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for all a, $b \in \mathfrak{A}$;

(iv) $\pi(a \cup b) = \pi a \cup \pi b$ and $\pi a \cap \pi b = 0_{\mathfrak{B}}$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0_{\mathfrak{A}}$, and $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

312I Proposition If $\mathfrak{A}$, $\mathfrak{B}$ are Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism, then $\pi a \subseteq \pi b$ whenever $a \subseteq b$ in $\mathfrak{A}$.

312J Proposition Let $\mathfrak{A}$ be a Boolean algebra, and a any member of $\mathfrak{A}$. Then the map $b \mapsto a \cap b$ is a surjective Boolean homomorphism from $\mathfrak{A}$ onto the principal ideal $\mathfrak{A}_a$ generated by a.

***312K Fixed-point subalgebras** If $\mathfrak{A}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, then $\{a : a \in \mathfrak{A},\ \pi a = a\}$ is a subalgebra of $\mathfrak{A}$; I will call it the **fixed-point subalgebra** of π.

312L Quotient algebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$. Then the quotient ring $\mathfrak{A}/I$ is a Boolean algebra, and the canonical map $a \mapsto a^\bullet : \mathfrak{A} \to \mathfrak{A}/I$ is a Boolean homomorphism, so that

$$(a \mathbin{\triangle} b)^\bullet = a^\bullet \mathbin{\triangle} b^\bullet, \quad (a \cup b)^\bullet = a^\bullet \cup b^\bullet, \quad (a \cap b)^\bullet = a^\bullet \cap b^\bullet, \quad (a \setminus b)^\bullet = a^\bullet \setminus b^\bullet$$

for all a, $b \in \mathfrak{A}$.

(b) The order relation on $\mathfrak{A}/I$ is defined by the formula

$$a^\bullet \subseteq b^\bullet \iff a \setminus b \in I.$$

For any $a \in \mathfrak{A}$,

$$\{u : u \subseteq a^\bullet\} = \{b^\bullet : b \subseteq a\}.$$

312M Proposition If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a bijection such that $\pi a \subseteq \pi b$ whenever $a \subseteq b$, then π is a Boolean algebra isomorphism.

312N Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$; let c be any member of $\mathfrak{A}$. Then

$$\mathfrak{A}_1 = \{(a \cap c) \cup (b \setminus c) : a,\ b \in \mathfrak{A}_0\}$$

is a subalgebra of $\mathfrak{A}$; it is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$.

312O Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism, and $c \in \mathfrak{A}$. If $v \in \mathfrak{B}$ is such that $\pi a \subseteq v \subseteq \pi b$ whenever a, $b \in \mathfrak{A}_0$ and $a \subseteq c \subseteq b$, then there is a unique Boolean homomorphism π_1 from the subalgebra $\mathfrak{A}_1$ of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ such that π_1 extends π and $\pi_1 c = v$.

312P Homomorphisms and Stone spaces: Proposition Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; write $\widehat{a} \subseteq Z$ for the open-and-closed set corresponding to $a \in \mathfrak{A}$. Then there is a one-to-one correspondence between ideals I of $\mathfrak{A}$ and open sets $G \subseteq Z$, given by the formulae

$$G = \bigcup_{a \in I} \widehat{a}, \quad I = \{a : \widehat{a} \subseteq G\}.$$

312Q Theorem Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z, W; write $\widehat{a} \subseteq Z$, $\widehat{b} \subseteq W$ for the open-and-closed sets corresponding to $a \in \mathfrak{A}$, $b \in \mathfrak{B}$. Then we have a one-to-one correspondence between Boolean homomorphisms $\pi : \mathfrak{A} \to \mathfrak{B}$ and continuous functions $\phi : W \to Z$, given by the formula

$$\pi a = b \iff \phi^{-1}[\widehat{a}] = \widehat{b},$$

that is, $\phi^{-1}[\widehat{a}] = \widehat{\pi a}$.

312R Theorem Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Boolean algebras, with Stone spaces Z, W and V. Let $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ be Boolean homomorphisms, with corresponding continuous functions $\phi : W \to Z$ and $\psi : V \to W$. Then the Boolean homomorphism $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ corresponds to the continuous function $\phi\psi : V \to Z$.

312S Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism, with associated continuous function $\phi : W \to Z$. Then

(a) π is injective iff ϕ is surjective;

(b) π is surjective iff ϕ is injective.

312T Principal ideals If $\mathfrak{A}$ is a Boolean algebra and $a \in \mathfrak{A}$, we have a natural surjective Boolean homomorphism $b \mapsto b \cap a : \mathfrak{A} \to \mathfrak{A}_a$, the principal ideal generated by a. Writing Z for the Stone space of $\mathfrak{A}$ and Z_a for the Stone space of $\mathfrak{A}_a$, this homomorphism must correspond to an injective continuous function $\phi : Z_a \to Z$. ϕ must be a homeomorphism between Z_a and its image $\phi[Z_a] \subseteq Z$. $\phi[Z_a] = \widehat{a}$.

Version of 8.6.11

313 Order-continuous homomorphisms

Because a Boolean algebra has a natural partial order, we have corresponding notions of upper bounds, lower bounds, suprema and infima. These are particularly important in the Boolean algebras arising in measure theory, and the infinitary operations 'sup' and 'inf' require rather more care than the basic binary operations '$\cup$', '$\cap$', because intuitions from elementary set theory are sometimes misleading. I therefore take a section to work through the most important properties of these operations, together with the homomorphisms which preserve them.

313A Relative complementation: Proposition Let $\mathfrak{A}$ be a Boolean algebra, e a member of $\mathfrak{A}$, and A a non-empty subset of $\mathfrak{A}$.

(a) If $\sup A$ is defined in $\mathfrak{A}$, then $\inf\{e \setminus a : a \in A\}$ is defined and equal to $e \setminus \sup A$.

(b) If $\inf A$ is defined in $\mathfrak{A}$, then $\sup\{e \setminus a : a \in A\}$ is defined and equal to $e \setminus \inf A$.

313B General distributive laws: Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $e \in \mathfrak{A}$ and $A \subseteq \mathfrak{A}$ is a non-empty set such that $\sup A$ is defined in $\mathfrak{A}$, then $\sup\{e \cap a : a \in A\}$ is defined and equal to $e \cap \sup A$.

(b) If $e \in \mathfrak{A}$ and $A \subseteq \mathfrak{A}$ is a non-empty set such that $\inf A$ is defined in $\mathfrak{A}$, then $\inf\{e \cup a : a \in A\}$ is defined and equal to $e \cup \inf A$.

(c) Suppose that A, $B \subseteq \mathfrak{A}$ are non-empty and $\sup A$, $\sup B$ are defined in $\mathfrak{A}$. Then $\sup\{a \cap b : a \in A,\ b \in B\}$ is defined and is equal to $\sup A \cap \sup B$.

(d) Suppose that A, $B \subseteq \mathfrak{A}$ are non-empty and $\inf A$, $\inf B$ are defined in $\mathfrak{A}$. Then $\inf\{a \cup b : a \in A,\ b \in B\}$ is defined and is equal to $\inf A \cup \inf B$.

313C Proposition Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; for $a \in \mathfrak{A}$ write $\widehat{a}$ for the corresponding open-and-closed subset of Z.

(a) If $A \subseteq \mathfrak{A}$ and $a_0 \in \mathfrak{A}$ then $a_0 = \sup A$ in $\mathfrak{A}$ iff $\widehat{a}_0 = \overline{\bigcup_{a \in A} \widehat{a}}$.

(b) If $A \subseteq \mathfrak{A}$ is non-empty and $a_0 \in \mathfrak{A}$ then $a_0 = \inf A$ in $\mathfrak{A}$ iff $\widehat{a}_0 = \operatorname{int} \bigcap_{a \in A} \widehat{a}$.

(c) If $A \subseteq \mathfrak{A}$ is non-empty then $\inf A = 0$ in $\mathfrak{A}$ iff $\bigcap_{a \in A} \widehat{a}$ is nowhere dense in Z.

313D Definitions Let P be a partially ordered set and C a subset of P.

(a) C is **order-closed** if $\sup A \in C$ whenever A is a non-empty upwards-directed subset of C such that $\sup A$ is defined in P, and $\inf A \in C$ whenever A is a non-empty downwards-directed subset of C such that $\inf A$ is defined in P.

(b) C is **sequentially order-closed** if $\sup_{n \in \mathbb{N}} p_n \in C$ whenever $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in C such that $\sup_{n \in \mathbb{N}} p_n$ is defined in P, and $\inf_{n \in \mathbb{N}} p_n \in C$ whenever $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in C such that $\inf_{n \in \mathbb{N}} p_n$ is defined in P.

313E Order-closed subalgebras and ideals(a) Let $\mathfrak{B}$ be a subalgebra of a Boolean algebra $\mathfrak{A}$.

(i) The following are equiveridical:

(α) $\mathfrak{B}$ is order-closed in $\mathfrak{A}$;

(β) $\sup B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ and $\sup B$ is defined in $\mathfrak{A}$;

(β') $\inf B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ and $\inf B$ is defined in $\mathfrak{A}$;

(γ) $\sup B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ is non-empty and upwards-directed and $\sup B$ is defined in $\mathfrak{A}$;

(γ') $\inf B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ is non-empty and downwards-directed and $\inf B$ is defined in $\mathfrak{A}$.

(ii) The following are equiveridical:

(α) $\mathfrak{B}$ is sequentially order-closed in $\mathfrak{A}$;

(β) $\sup_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $\sup_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(β') $\inf_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $\inf_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(γ) $\sup_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{B}$ and $\sup_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(γ') $\inf_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{B}$ and $\inf_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$.

(b) Now suppose that I is an ideal of $\mathfrak{A}$. Then

I is order-closed iff $\sup A \in I$ whenever $A \subseteq I$ is non-empty, upwards-directed and has a supremum in $\mathfrak{A}$;

I is sequentially order-closed iff $\sup_{n\in\mathbb{N}} a_n \in I$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in I with a supremum in $\mathfrak{A}$.

I is order-closed iff $\sup A \in I$ whenever $A \subseteq I$ has a supremum in $\mathfrak{A}$;

I is sequentially order-closed iff $\sup_{n\in\mathbb{N}} a_n \in I$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in I with a supremum in $\mathfrak{A}$.

(c) I will normally use the phrases **σ-subalgebra**, **σ-ideal** for sequentially order-closed subalgebras and ideals of Boolean algebras.

313F Order-closures and generated sets (a)(i) If $\mathcal{S}$ is any non-empty family of subalgebras of a Boolean algebra $\mathfrak{A}$, then $\bigcap\mathcal{S}$ is a subalgebra of $\mathfrak{A}$;

(ii) if $\mathcal{F}$ is any non-empty family of order-closed subsets of a partially ordered set P, then $\bigcap\mathcal{F}$ is an order-closed subset of P;

(iii) if $\mathcal{F}$ is any non-empty family of sequentially order-closed subsets of a partially ordered set P, then $\bigcap\mathcal{F}$ is a sequentially order-closed subset of P.

(b) Consequently, given any Boolean algebra $\mathfrak{A}$ and a subset B of $\mathfrak{A}$, we have a smallest subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ including B, being the intersection of all the subalgebras of $\mathfrak{A}$ which include B; a smallest σ-subalgebra $\mathfrak{B}_\sigma$ of $\mathfrak{A}$ including B, being the intersection of all the σ-subalgebras of $\mathfrak{A}$ which include B; and a smallest order-closed subalgebra $\mathfrak{B}_\tau$ of $\mathfrak{A}$ including B, being the intersection of all the order-closed subalgebras of $\mathfrak{A}$ which include B. We call $\mathfrak{B}$, $\mathfrak{B}_\sigma$ and $\mathfrak{B}_\tau$ the subalgebra, σ-subalgebra and order-closed subalgebra **generated** by B.

(c) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ any subalgebra of $\mathfrak{A}$, then the smallest order-closed subset $\overline{\mathfrak{B}}$ of $\mathfrak{A}$ which includes $\mathfrak{B}$ is again a subalgebra of $\mathfrak{A}$ (so is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$).

313G Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) Suppose that $1 \in I \subseteq A \subseteq \mathfrak{A}$ and that

$$a \cap b \in I \text{ for all } a,\, b \in I,$$

$$b \setminus a \in A \text{ whenever } a,\, b \in A \text{ and } a \subseteq b.$$

Then A includes the subalgebra of $\mathfrak{A}$ generated by I.

(b) If moreover $\sup_{n\in\mathbb{N}} a_n \in A$ for every non-decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A with a supremum in $\mathfrak{A}$, then A includes the σ-subalgebra of $\mathfrak{A}$ generated by I.

(c) And if $\sup C \in A$ whenever $C \subseteq A$ is an upwards-directed set with a supremum in $\mathfrak{A}$, then A includes the order-closed subalgebra of $\mathfrak{A}$ generated by I.

313H Definitions Let P and Q be two partially ordered sets, and $\phi : P \to Q$ an **order-preserving** function, that is, a function such that $\phi(p) \le \phi(q)$ in Q whenever $p \le q$ in P.

(a) I say that ϕ is **order-continuous** if (i) $\phi(\sup A) = \sup_{p\in A} \phi(p)$ whenever A is a non-empty upwards-directed subset of P and $\sup A$ is defined in P (ii) $\phi(\inf A) = \inf_{p\in A} \phi(p)$ whenever A is a non-empty downwards-directed subset of P and $\inf A$ is defined in P.

(b) I say that ϕ is **sequentially order-continuous** or **σ-order-continuous** if (i) $\phi(p) = \sup_{n\in\mathbb{N}} \phi(p_n)$ whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in P and $p = \sup_{n\in\mathbb{N}} p_n$ in P (ii) $\phi(p) = \inf_{n\in\mathbb{N}} \phi(p_n)$ whenever $\langle p_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in P and $p = \inf_{n\in\mathbb{N}} p_n$ in P.

313I Proposition Let P, Q and R be partially ordered sets, and $\phi : P \to Q$, $\psi : Q \to R$ order-preserving functions.

(a) $\psi\phi : P \to R$ is order-preserving.

(b) If ϕ and ψ are order-continuous, so is $\psi\phi$.

(c) If ϕ and ψ are sequentially order-continuous, so is $\psi\phi$.

(d) ϕ is order-continuous iff $\phi^{-1}[B]$ is order-closed for every order-closed $B \subseteq Q$.

313J Definition Let $\mathfrak{A}$ be a Boolean algebra. A set $D \subseteq \mathfrak{A}$ is **order-dense** if for every non-zero $a \in \mathfrak{A}$ there is a non-zero $d \in D$ such that $d \subseteq a$.

313K Lemma If $\mathfrak{A}$ is a Boolean algebra and $D \subseteq \mathfrak{A}$ is order-dense, then for any $a \in \mathfrak{A}$ there is a disjoint $C \subseteq D$ such that $\sup C = a$; in particular, $a = \sup\{d : d \in D, d \subseteq a\}$ and there is a partition of unity $C \subseteq D$.

313L Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) π is order-preserving.

(b) The following are equiveridical:

(i) π is order-continuous;

(ii) whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$ in $\mathfrak{A}$, then $\inf \pi[A] = 0$ in $\mathfrak{B}$;

(iii) whenever $A \subseteq \mathfrak{A}$ is non-empty and upwards-directed and $\sup A = 1$ in $\mathfrak{A}$, then $\sup \pi[A] = 1$ in $\mathfrak{B}$;

(iv) whenever $A \subseteq \mathfrak{A}$ and $\sup A$ is defined in $\mathfrak{A}$, then $\pi(\sup A) = \sup \pi[A]$ in $\mathfrak{B}$;

(v) whenever $A \subseteq \mathfrak{A}$ and $\inf A$ is defined in $\mathfrak{A}$, then $\pi(\inf A) = \inf \pi[A]$ in $\mathfrak{B}$;

(vi) whenever $C \subseteq \mathfrak{A}$ is a partition of unity, then $\pi[C]$ is a partition of unity in $\mathfrak{B}$.

(c) The following are equiveridical:

(i) π is sequentially order-continuous;

(ii) whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} a_n = 0$ in $\mathfrak{A}$, then $\inf_{n\in\mathbb{N}} \pi a_n = 0$ in $\mathfrak{B}$;

(iii) whenever $A \subseteq \mathfrak{A}$ is countable and $\sup A$ is defined in $\mathfrak{A}$, then $\pi(\sup A) = \sup \pi[A]$ in $\mathfrak{B}$;

(iv) whenever $A \subseteq \mathfrak{A}$ is countable and $\inf A$ is defined in $\mathfrak{A}$, then $\pi(\inf A) = \inf \pi[A]$ in $\mathfrak{B}$;

(v) whenever $C \subseteq \mathfrak{A}$ is a countable partition of unity, then $\pi[C]$ is a partition of unity in $\mathfrak{B}$.

(d) If π is bijective, it is order-continuous.

313M Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism.

(a) If $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{B}$, then $\pi^{-1}[\mathfrak{D}]$ is an order-closed subalgebra of $\mathfrak{A}$.

(b) If $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $C \subseteq \mathfrak{A}$, then the order-closed subalgebra $\mathfrak{D}$ of $\mathfrak{B}$ generated by $\pi[C]$ includes $\pi[\mathfrak{C}]$.

(c) Now suppose that π is surjective and that $C \subseteq \mathfrak{A}$ is such that the order-closed subalgebra of $\mathfrak{A}$ generated by C is $\mathfrak{A}$ itself. Then the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$ is $\mathfrak{B}$.

313N Definition The phrase **regular embedding** is sometimes used to mean an injective order-continuous Boolean homomorphism; a subalgebra $\mathfrak{B}$ of a Boolean algebra $\mathfrak{A}$ is said to be **regularly embedded** in $\mathfrak{A}$ if the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is order-continuous. .

It will be useful to be able to say '$\mathfrak{B}$ can be regularly embedded in $\mathfrak{A}$' to mean that there is an injective order-continuous Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$; that is, that $\mathfrak{B}$ is isomorphic to a regularly embedded subalgebra of $\mathfrak{A}$. In this form it is obvious that if $\mathfrak{C}$ can be regularly embedded in $\mathfrak{B}$, and $\mathfrak{B}$ can be regularly embedded in $\mathfrak{A}$, then $\mathfrak{C}$ can be regularly embedded in $\mathfrak{A}$.

313O Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Then $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$. In particular, if $B \subseteq \mathfrak{B}$ and $c \in \mathfrak{B}$ then $c = \sup B$ in $\mathfrak{B}$ iff $c = \sup B$ in $\mathfrak{A}$.

313P Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism with kernel I.

(a)(i) If π is order-continuous then I is order-closed.

(ii) If $\pi[\mathfrak{A}]$ is regularly embedded in $\mathfrak{B}$ and I is order-closed then π is order-continuous.

(b)(i) If π is sequentially order-continuous then I is a σ-ideal.

(ii) If $\pi[\mathfrak{A}]$ is regularly embedded in $\mathfrak{B}$ and I is a σ-ideal then π is sequentially order-continuous.

313Q Corollary Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$; write π for the canonical map from $\mathfrak{A}$ to $\mathfrak{A}/I$.

(a) π is order-continuous iff I is order-closed.

(b) π is sequentially order-continuous iff I is a σ-ideal.

313R Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism. Let Z and W be their Stone spaces, and $\phi : W \to Z$ the corresponding continuous function. Then the following are equiveridical:

(i) π is order-continuous;

(ii) $\phi^{-1}[M]$ is nowhere dense in W for every nowhere dense set $M \subseteq Z$;

(iii) $\operatorname{int} \phi[H] \neq \emptyset$ for every non-empty open set $H \subseteq W$.

313S Upper envelopes(a) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. For $a \in \mathfrak{A}$, the **upper envelope** of a in $\mathfrak{C}$ is

$$\operatorname{upr}(a, \mathfrak{C}) = \inf\{c : c \in \mathfrak{C},\ a \subseteq c\}$$

if the infimum is defined in $\mathfrak{C}$.

(b) If $A \subseteq \mathfrak{A}$ is such that $\operatorname{upr}(a, \mathfrak{C})$ is defined for every $a \in A$, $a_0 = \sup A$ is defined in $\mathfrak{A}$ and $c_0 = \sup_{a\in A} \operatorname{upr}(a, \mathfrak{C})$ is defined in $\mathfrak{C}$, then $c_0 = \operatorname{upr}(a_0, \mathfrak{C})$. $\operatorname{upr}(a \cup a', \mathfrak{C}) = \operatorname{upr}(a, \mathfrak{C}) \cup \operatorname{upr}(a', \mathfrak{C})$ whenever the right-hand side is defined.

(c) If $a \in \mathfrak{A}$ is such that $\operatorname{upr}(a, \mathfrak{C})$ is defined, then $\operatorname{upr}(a \cap c, \mathfrak{C}) = c \cap \operatorname{upr}(a, \mathfrak{C})$ for every $c \in \mathfrak{C}$.

Version of 26.7.07

314 Order-completeness

The results of §313 are valid in all Boolean algebras, but of course are of most value when many suprema and infima exist. I now set out the most useful definitions which guarantee the existence of suprema and infima (314A) and work through their elementary relationships with the concepts introduced so far (314C-314J). I then embark on the principal theorems concerning order-complete Boolean algebras: the extension theorem for homomorphisms to a Dedekind complete algebra (314K), the Loomis-Sikorski representation of a Dedekind σ-complete algebra as a quotient of a σ-algebra of sets (314M), the characterization of Dedekind complete algebras in terms of their Stone spaces (314S), and the idea of 'Dedekind completion' of a Boolean algebra (314T-314U). On the way I describe 'regular open algebras' (314O-314Q).

314A Definitions Let P be a partially ordered set.

(a) P is **Dedekind complete** if every non-empty subset of P with an upper bound has a least upper bound.

(b) P is **Dedekind σ-complete** if (i) every countable non-empty subset of P with an upper bound has a least upper bound (ii) every countable non-empty subset of P with a lower bound has a greatest lower bound.

314C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and I a σ-ideal of $\mathfrak{A}$. Then the quotient Boolean algebra $\mathfrak{A}/I$ is Dedekind σ-complete.

314D Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X. Then $\Sigma \cap \mathcal{I}$ is a σ-ideal of the Boolean algebra Σ, and $\Sigma/\Sigma \cap \mathcal{I}$ is Dedekind σ-complete.

314E Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{A}$ is Dedekind complete, then all its order-closed subalgebras and principal ideals are Dedekind complete.

(b) If $\mathfrak{A}$ is Dedekind σ-complete, then all its σ-subalgebras and principal ideals are Dedekind σ-complete.

314F Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a)(i) If $\mathfrak{A}$ is Dedekind complete and π is order-continuous, then $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$.

(ii) If $\mathfrak{B}$ is Dedekind complete and π is injective and $\pi[\mathfrak{A}]$ is order-closed then π is order-continuous.

(b)(i) If $\mathfrak{A}$ is Dedekind σ-complete and π is sequentially order-continuous, then $\pi[\mathfrak{A}]$ is a σ-subalgebra of $\mathfrak{B}$.

(ii) If $\mathfrak{B}$ is Dedekind σ-complete and π is injective and $\pi[\mathfrak{A}]$ is a σ-subalgebra of $\mathfrak{B}$ then π is sequentially order-continuous.

314G Corollary Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$.

(a) If $\mathfrak{A}$ is Dedekind complete, then $\mathfrak{B}$ is order-closed iff it is Dedekind complete in itself and is regularly embedded in $\mathfrak{A}$.

(b) If $\mathfrak{A}$ is Dedekind σ-complete, then $\mathfrak{B}$ is a σ-subalgebra iff it is Dedekind σ-complete in itself and the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is sequentially order-continuous.

314H Corollary Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by C, then $\pi[\mathfrak{C}]$ is the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

314I Corollary (a) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, $\mathfrak{B}$ is a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{B}$ is an injective Boolean homomorphism and $\pi[\mathfrak{A}]$ is order-dense in $\mathfrak{B}$, then π is an isomorphism.

(b) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ which is Dedekind complete in itself, then $\mathfrak{B} = \mathfrak{A}$.

314J Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$. Take any $c \in \mathfrak{A}$, and set

$$\mathfrak{A}_1 = \{(a \cap c) \cup (b \setminus c) : a,\, b \in \mathfrak{A}_0\},$$

the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$.

(a) Suppose that $\mathfrak{A}$ is Dedekind complete. If $\mathfrak{A}_0$ is order-closed in $\mathfrak{A}$, so is $\mathfrak{A}_1$.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. If $\mathfrak{A}_0$ is a σ-subalgebra of $\mathfrak{A}$, so is $\mathfrak{A}_1$.

314K Extension of homomorphisms: Theorem Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a Dedekind complete Boolean algebra. Let $\mathfrak{A}_0$ be a Boolean subalgebra of $\mathfrak{A}$ and $\pi_0 : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism. Then there is a Boolean homomorphism $\pi_1 : \mathfrak{A} \to \mathfrak{B}$ extending π_0.

314L Lemma Let X be any topological space, and write $\mathcal{M}$ for the family of meager subsets of X. Then $\mathcal{M}$ is a σ-ideal of subsets of X.

314M Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and Z its Stone space. Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z, and $\mathcal{M}$ the σ-ideal of meager subsets of Z. Then $\Sigma = \{E \triangle A : E \in \mathcal{E},\, A \in \mathcal{M}\}$ is a σ-algebra of subsets of Z, $\mathcal{M}$ is a σ-ideal of Σ, and $\mathfrak{A}$ is isomorphic, as Boolean algebra, to $\Sigma/\mathcal{M}$.

314N Corollary A Boolean algebra $\mathfrak{A}$ is Dedekind σ-complete iff it is isomorphic to a quotient $\Sigma/\mathcal{I}$ where Σ is a σ-algebra of sets and $\mathcal{I}$ is a σ-ideal of Σ.

314O Regular open algebras: Definition Let X be a topological space. A **regular open set** in X is an open set $G \subseteq X$ such that $G = \operatorname{int} \overline{G}$.

Note that if $F \subseteq X$ is any closed set, then $G = \operatorname{int} F$ is a regular open set.

314P Theorem Let X be any topological space, and write $\mathrm{RO}(X)$ for the set of regular open sets in X. Then $\mathrm{RO}(X)$ is a Dedekind complete Boolean algebra, with $1_{\mathrm{RO}(X)} = X$ and $0_{\mathrm{RO}(X)} = \emptyset$, and with Boolean operations given by

$$G \cap_{\mathrm{RO}} H = G \cap H, \quad G \triangle_{\mathrm{RO}} H = \operatorname{int} \overline{G \triangle H},$$

$$G \cup_{\mathrm{RO}} H = \operatorname{int} \overline{G \cup H}, \quad G \setminus_{\mathrm{RO}} H = G \setminus \overline{H},$$

with Boolean ordering given by

$$G \subseteq_{\mathrm{RO}} H \iff G \subseteq H,$$

and with suprema and infima given by

$$\sup \mathcal{H} = \operatorname{int} \overline{\bigcup \mathcal{H}}, \quad \inf \mathcal{H} = \operatorname{int} \bigcap \mathcal{H} = \operatorname{int} \overline{\bigcap \mathcal{H}}$$

for all non-empty $\mathcal{H} \subseteq \mathrm{RO}(X)$.

314Q Remark $\mathrm{RO}(X)$ is called the **regular open algebra** of the topological space X.

***314R Lemma** (a) Let X and Y be topological spaces, and $f : X \to Y$ a continuous function such that $f^{-1}[M]$ is nowhere dense in X for every nowhere dense $M \subseteq Y$. Then we have an order-continuous Boolean homomorphism π from the regular open algebra $\mathrm{RO}(Y)$ of Y to the regular open algebra $\mathrm{RO}(X)$ of X defined by setting $\pi H = \operatorname{int} \overline{f^{-1}[H]}$ for every $H \in \mathrm{RO}(Y)$.

(b) Let X be a topological space.

(i) If $U \subseteq X$ is open, then $G \mapsto G \cap U$ is a surjective order-continuous Boolean homomorphism from $\mathrm{RO}(X)$ onto $\mathrm{RO}(U)$.

(ii) If $U \in \mathrm{RO}(X)$ then $\mathrm{RO}(U)$ is the principal ideal of $\mathrm{RO}(X)$ generated by U.

314S Theorem Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z, and $\mathrm{RO}(Z)$ for the regular open algebra of Z. Then the following are equiveridical:

(i) $\mathfrak{A}$ is Dedekind complete;

(ii) Z is extremally disconnected;

(iii) $\mathcal{E} = \mathrm{RO}(Z)$.

314T Theorem Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z; for $a \in \mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z. Let $\widehat{\mathfrak{A}}$ be the regular open algebra of Z.

(a) The map $a \mapsto \widehat{a}$ is an injective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

(b) If $\mathfrak{B}$ is any Dedekind complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a unique order-continuous Boolean homomorphism $\pi_1 : \widehat{\mathfrak{A}} \to \mathfrak{B}$ such that $\pi_1 \widehat{a} = \pi a$ for every $a \in \mathfrak{A}$.

314U The Dedekind completion of a Boolean algebra (a) For any Boolean algebra $\mathfrak{A}$, I will say that the Boolean algebra $\widehat{\mathfrak{A}}$ constructed in 314T is the **Dedekind completion** of $\mathfrak{A}$.

(b) If $\mathfrak{C}$ is a Dedekind complete Boolean algebra and $\mathfrak{A}$ is an order-dense subalgebra of $\mathfrak{C}$, then the embedding $\mathfrak{A} \hookrightarrow \mathfrak{C}$ induces an isomorphism from $\widehat{\mathfrak{A}}$ to $\mathfrak{C}$.

(c) Suppose that Z is a zero-dimensional compact Hausdorff space, and $\mathcal{E}$ is the algebra of open-and-closed subsets of Z. Then $\mathcal{E}$ is order-dense in the regular open algebra $\mathrm{RO}(Z)$, so the Dedekind completion of $\mathcal{E}$ can be identified with $\mathrm{RO}(Z)$.

Version of 13.11.12

315 Products and free products

I describe here two algebraic constructions of fundamental importance. They are very different in character, indeed may be regarded as opposites, despite the common use of the word 'product'. The first part of the section (315A-315H) deals with the easier construction, the 'simple product'; the second part (315I-315Q) with the 'free product'. These constructions lead to descriptions of projective and inductive limits (315R-315S).

315A Products of Boolean algebras (a) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be any family of Boolean algebras. Set $\mathfrak{A} = \prod_{i \in I} \mathfrak{A}_i$, with the natural ring structure

$$a \triangle b = \langle a(i) \triangle b(i) \rangle_{i \in I},$$

$$a \cap b = \langle a(i) \cap b(i) \rangle_{i \in I}$$

for a, $b \in \mathfrak{A}$. Then $\mathfrak{A}$ is a Boolean algebra. I will call $\mathfrak{A}$ the **simple product** of the family $\langle \mathfrak{A}_i \rangle_{i \in I}$.

(b) The Boolean operations on $\mathfrak{A}$ are now defined by the formulae

$$a \cup b = \langle a(i) \cup b(i) \rangle_{i \in I}, \quad a \setminus b = \langle a(i) \setminus b(i) \rangle_{i \in I}$$

for all a, $b \in \mathfrak{A}$.

315B Theorem Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a family of Boolean algebras, and $\mathfrak{A}$ their simple product.

(a) The maps $a \mapsto \pi_i(a) = a(i) : \mathfrak{A} \to \mathfrak{A}_i$ are all Boolean homomorphisms.

(b) If $\mathfrak{B}$ is any other Boolean algebra, then a map $\phi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism iff $\pi_i \phi : \mathfrak{B} \to \mathfrak{A}_i$ is a Boolean homomorphism for every $i \in I$.

315C Products of partially ordered sets (a) If $\langle P_i \rangle_{i \in I}$ is any family of partially ordered sets, its **product** is the set $P = \prod_{i \in I} P_i$ ordered by saying that $p \le q$ iff $p(i) \le q(i)$ for every $i \in I$.

(b) The point is that if $\mathfrak{A}$ is the simple product of a family $\langle \mathfrak{A}_i \rangle_{i \in I}$ of Boolean algebras, then the ordering of $\mathfrak{A}$ is just the product partial order:

$$a \subseteq b \iff a(i) \subseteq b(i) \ \forall \ i \in I.$$

315D Proposition Let $\langle P_i \rangle_{i \in I}$ be a family of non-empty partially ordered sets with product P.

(a) For any non-empty set $A \subseteq P$ and $q \in P$,

(i) $\sup A = q$ in P iff $\sup_{p \in A} p(i) = q(i)$ in P_i for every $i \in I$,

(ii) $\inf A = q$ in P iff $\inf_{p \in A} p(i) = q(i)$ in P_i for every $i \in I$.

(b) The coordinate maps $p \mapsto \pi_i(p) = p(i) : P \to P_i$ are all order-preserving and order-continuous.

(c) For any partially ordered set Q and function $\phi : Q \to P$, ϕ is order-preserving iff $\pi_i \phi$ is order-preserving for every $i \in I$.

(d) For any partially ordered set Q and order-preserving function $\phi : Q \to P$,

(i) ϕ is order-continuous iff $\pi_i\phi$ is order-continuous for every i,

(ii) ϕ is sequentially order-continuous iff $\pi_i\phi$ is sequentially order-continuous for every i.

(e)(i) P is Dedekind complete iff every P_i is Dedekind complete.

(ii) P is Dedekind σ-complete iff every P_i is Dedekind σ-complete.

315E Factor algebras as principal ideals If $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of Boolean algebras with simple product $\mathfrak{A}$, define $\theta_i:\mathfrak{A}_i\to\mathfrak{A}$ by setting $(\theta_i a)(i)=a$, $(\theta_i a)(j)=0_{\mathfrak{A}_j}$ if $i\in I$, $a\in\mathfrak{A}_i$ and $j\in I\setminus\{i\}$. Each θ_i is a ring homomorphism, and is a Boolean isomorphism between $\mathfrak{A}_i$ and the principal ideal of $\mathfrak{A}$ generated by $\theta_i(1_{\mathfrak{A}_i})$. The family $\langle\theta_i(1_{\mathfrak{A}_i})\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$.

315F Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\langle e_i\rangle_{i\in I}$ a partition of unity in $\mathfrak{A}$. Suppose

either (i) that I is finite

or (ii) that I is countable and $\mathfrak{A}$ is Dedekind σ-complete

or (iii) that $\mathfrak{A}$ is Dedekind complete.

Then the map $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a Boolean isomorphism between $\mathfrak{A}$ and $\prod_{i\in I}\mathfrak{A}_{e_i}$, writing $\mathfrak{A}_{e_i}$ for the principal ideal of $\mathfrak{A}$ generated by e_i for each i.

315G Algebras of sets and their quotients: Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i an algebra of subsets of X_i for each i.

(a) The simple product $\prod_{i\in I}\Sigma_i$ may be identified with the algebra

$$\Sigma=\{E:E\subseteq X,\,\{x:(x,i)\in E\}\in\Sigma_i\text{ for every }i\in I\}$$

of subsets of $X=\{(x,i):i\in I,\,x\in X_i\}$, with the canonical homomorphisms $\pi_i:\Sigma\to\Sigma_i$ being given by

$$\pi_i E=\{x:(x,i)\in E\}$$

for each $E\in\Sigma$.

(b) Now suppose that $\mathcal{J}_i$ is an ideal of Σ_i for each i. Then $\prod_{i\in I}\Sigma_i/\mathcal{J}_i$ may be identified with $\Sigma/\mathcal{J}$, where

$$\mathcal{J}=\{E:E\in\Sigma,\,\{x:(x,i)\in E\}\in\mathcal{J}_i\text{ for every }i\in I\},$$

and the canonical homomorphisms $\tilde\pi_i:\Sigma/\mathcal{J}\to\Sigma_i/\mathcal{J}_i$ are given by the formula $\tilde\pi_i(E^\bullet)=(\pi_iE)^\bullet$ for every $E\in\Sigma$.

***315H Proposition** Let X be a topological space, and $\mathcal{U}$ a disjoint family of open subsets of X with union dense in X. Then the regular open algebra $\mathrm{RO}(X)$ is isomorphic to the simple product $\prod_{U\in\mathcal{U}}\mathrm{RO}(U)$.

315I Free products (a) Definition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras. For each $i\in I$, let Z_i be the Stone space of $\mathfrak{A}_i$. Set $Z=\prod_{i\in I}Z_i$, with the product topology. Then the **free product** of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is the algebra $\mathfrak{A}$ of open-and-closed sets in Z; I will denote it by $\bigotimes_{i\in I}\mathfrak{A}_i$.

(b) For $i\in I$ and $a\in\mathfrak{A}_i$, the set $\varepsilon_i(a)=\{z:z(i)\in\widehat{a}\}$ belongs to $\mathfrak{A}$. In this context I will call $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ the **canonical map**.

(c) The topological space Z may be identified with the Stone space of the Boolean algebra $\mathfrak{A}$.

315J Theorem Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with free product $\mathfrak{A}$.

(a) The canonical map $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ is a Boolean homomorphism for every $i\in I$.

(b) For any Boolean algebra $\mathfrak{B}$ and any family $\langle\phi_i\rangle_{i\in I}$ such that ϕ_i is a Boolean homomorphism from $\mathfrak{A}_i$ to $\mathfrak{B}$ for every i, there is a unique Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$ such that $\phi_i=\phi\varepsilon_i$ for each i.

315K Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and $\mathfrak{A}$ their free product; write $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ for the canonical homomorphisms.

(a) $\mathfrak{A}$ is the subalgebra of itself generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{A}_i]$.

(b) Write C for the set of those members of $\mathfrak{A}$ expressible in the form $\inf_{j\in J}\varepsilon_j(a_j)$, where $J\subseteq I$ is finite and $a_j\in\mathfrak{A}_j$ for every j. Then every member of $\mathfrak{A}$ is expressible as the supremum of a disjoint finite subset of C. In particular, C is order-dense in $\mathfrak{A}$.

(c) Every ε_i is order-continuous.

(d) $\mathfrak{A} = \{0_{\mathfrak{A}}\}$ iff there is some $i \in I$ such that $\mathfrak{A}_i = \{0_{\mathfrak{A}_i}\}$.

(e) Now suppose that $\mathfrak{A}_i \neq \{0_{\mathfrak{A}_i}\}$ for every $i \in I$.

(i) ε_i is injective for every $i \in I$.

(ii) If $J \subseteq I$ is finite and a_j is a non-zero member of $\mathfrak{A}_j$ for each $j \in J$, then $\inf_{j\in J} \varepsilon_j(a_j) \neq 0$.

(iii) If i, j are distinct members of I, $a \in \mathfrak{A}_i$ and $b \in \mathfrak{A}_j$, then $\varepsilon_i(a) = \varepsilon_j(b)$ iff either $a = 0_{\mathfrak{A}_i}$ and $b = 0_{\mathfrak{A}_j}$ or $a = 1_{\mathfrak{A}_i}$ and $b = 1_{\mathfrak{A}_j}$.

315L Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, and $\langle J_k\rangle_{k\in K}$ any partition of I. Then the free product $\mathfrak{A}$ of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is isomorphic to the free product $\mathfrak{B}$ of $\langle\mathfrak{B}_k\rangle_{k\in K}$, where each $\mathfrak{B}_k$ is the free product of $\langle\mathfrak{A}_i\rangle_{i\in J_k}$.

315M Algebras of sets and their quotients: Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i an algebra of subsets of X_i for each i.

(a) The free product $\bigotimes_{i\in I}\Sigma_i$ may be identified with the algebra Σ of subsets of $X = \prod_{i\in I} X_i$ generated by the set $\{\varepsilon_i(E) : i \in I, E \in \Sigma_i\}$, where $\varepsilon_i(E) = \{x : x \in X, x(i) \in E\}$.

(b) Now suppose that $\mathcal{J}_i$ is an ideal of Σ_i for each i. Then $\bigotimes_{i\in I}\Sigma_i/\mathcal{J}_i$ may be identified with $\Sigma/\mathcal{J}$, where $\mathcal{J}$ is the ideal of Σ generated by $\{\varepsilon_i(E) : i \in I, E \in \mathcal{J}_i\}$; the corresponding canonical maps $\tilde\varepsilon_i : \Sigma_i/\mathcal{J}_i \to \Sigma/\mathcal{J}$ being defined by the formula $\tilde\varepsilon_i(E^\bullet) = (\varepsilon_i(E))^\bullet$ for $i \in I$, $E \in \Sigma_i$.

315N Notation If $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras, I write $\mathfrak{A}\otimes\mathfrak{B}$ for their free product, and for $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ I write $a \otimes b$ for $\varepsilon_1(a) \cap \varepsilon_2(b)$, where $\varepsilon_1 : \mathfrak{A} \to \mathfrak{A}\otimes\mathfrak{B}$, $\varepsilon_2 : \mathfrak{B} \to \mathfrak{A}\otimes\mathfrak{B}$ are the canonical maps. Observe that $(a_1 \otimes b_1) \cap (a_2 \otimes b_2) = (a_1 \cap a_2) \otimes (b_1 \cap b_2)$, and that the maps $a \mapsto a \otimes b_0$, $b \mapsto a_0 \otimes b$ are always ring homomorphisms. Now $a \otimes b = 0$ only when one of a, b is 0. In the context of 315M, we can identify $E \otimes F$ with $E \times F$ for $E \in \Sigma_1$ and $F \in \Sigma_2$, and $E^\bullet \otimes F^\bullet$ with $(E \times F)^\bullet$.

315O Lemma Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras.

(a) Any element of $\mathfrak{A}\otimes\mathfrak{B}$ is expressible as $\sup_{i\in I} a_i \otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$.

(b) If $c \in \mathfrak{A}\otimes\mathfrak{B}$ is non-zero there are non-zero $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ such that $a \otimes b \subseteq c$.

315P Example $\mathfrak{A} = \mathcal{P}\mathbb{N}\otimes\mathcal{P}\mathbb{N}$ is not Dedekind σ-complete.

315Q Example Now let $\mathfrak{A}$ be any non-trivial atomless Boolean algebra, and $\mathfrak{B}$ the free product $\mathfrak{A}\otimes\mathfrak{A}$. Then the identity homomorphism from $\mathfrak{A}$ to itself induces a homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$ given by setting $\phi(a \otimes b) = a \cap b$ for every a, $b \in \mathfrak{A}$. ϕ is not order-continuous.

Thus the free product does not respect order-continuity.

***315R Projective and inductive limits: Proposition** Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and R a subset of $I \times I$; suppose that $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$ is a Boolean homomorphism for each $(i, j) \in R$.

(a) There are a Boolean algebra $\mathfrak{C}$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ is a Boolean homomorphism for each $i \in I$,

$\pi_j = \pi_{ji}\pi_i$ whenever $(i, j) \in R$,

and whenever $\mathfrak{B}$, $\langle\phi_i\rangle_{i\in I}$ are such that

$\mathfrak{B}$ is a Boolean algebra,

$\phi_i : \mathfrak{B} \to \mathfrak{A}_i$ is a Boolean homomorphism for each $i \in I$,

$\phi_j = \pi_{ji}\phi_i$ whenever $(i, j) \in R$,

then there is a unique Boolean homomorphism $\phi : \mathfrak{B} \to \mathfrak{C}$ such that $\pi_i\phi = \phi_i$ for every $i \in I$.

(b) There are a Boolean algebra $\mathfrak{C}$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i : \mathfrak{A}_i \to \mathfrak{C}$ is a Boolean homomorphism for each $i \in I$,

$\pi_i = \pi_j\pi_{ji}$ whenever $(i, j) \in R$,

and whenever $\mathfrak{B}$, $\langle\phi_i\rangle_{i\in I}$ are such that

$\mathfrak{B}$ is a Boolean algebra,

$\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ is a Boolean homomorphism for each $i \in I$,

$\phi_i = \phi_j\pi_{ji}$ whenever $(i, j) \in R$,

then there is a unique Boolean homomorphism $\phi : \mathfrak{C} \to \mathfrak{B}$ such that $\phi\pi_i = \phi_i$ for every $i \in I$.

***315S Definitions** In 315Ra, we call $\mathfrak{A}$, together with $\langle \pi_i \rangle_{i \in I}$, 'the' **projective limit** of $(\langle \mathfrak{A}_i \rangle_{i \in I}, \langle \pi_{ji} \rangle_{(i,j) \in R})$; in 315Rb, we call $\mathfrak{A}$, together with $\langle \pi_i \rangle_{i \in I}$, 'the' **inductive limit** of $(\langle \mathfrak{A}_i \rangle_{i \in I}, \langle \pi_{ji} \rangle_{(i,j) \in R})$.

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316 Further topics

I introduce three special properties of Boolean algebras which will be of great importance in the rest of this volume: the countable chain condition (316A-316F), weak (σ,∞)-distributivity (316G-316J) and homogeneity (316N-316Q). I add some brief notes on atoms in Boolean algebras (316K-316L), with a characterization of the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316M).

316A Definitions (a) A Boolean algebra $\mathfrak{A}$ is **ccc**, or satisfies the **countable chain condition**, if every disjoint subset of $\mathfrak{A}$ is countable.

(b) A topological space X is **ccc**, or satisfies the **countable chain condition**, or has **Souslin's property**, if every disjoint collection of open sets in X is countable.

316B Theorem A Boolean algebra $\mathfrak{A}$ is ccc iff its Stone space Z is ccc.

316C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathcal{I}$ a σ-ideal of $\mathfrak{A}$. Then the quotient algebra $\mathfrak{B} = \mathfrak{A}/\mathcal{I}$ is ccc iff every disjoint family in $\mathfrak{A} \setminus \mathcal{I}$ is countable.

316D Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ. Then the quotient algebra $\Sigma/\mathcal{I}$ is ccc iff every disjoint family in $\Sigma \setminus \mathcal{I}$ is countable.

316E Proposition Let $\mathfrak{A}$ be a ccc Boolean algebra. Then for any $A \subseteq \mathfrak{A}$ there is a countable $B \subseteq A$ such that B has the same upper and lower bounds as A.

316F Corollary Let $\mathfrak{A}$ be a ccc Boolean algebra.

(a) If $\mathfrak{A}$ is Dedekind σ-complete it is Dedekind complete.

(b) If $A \subseteq \mathfrak{A}$ is sequentially order-closed it is order-closed.

(c) If Q is any partially ordered set and $\phi : \mathfrak{A} \to Q$ is a sequentially order-continuous order-preserving function, it is order-continuous.

(d) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, it is order-continuous.

316G Definition Let $\mathfrak{A}$ be a Boolean algebra. I will say that $\mathfrak{A}$ is **weakly (σ,∞)-distributive** if whenever $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of downwards-directed subsets of $\mathfrak{A}$ and $\inf A_n = 0$ for every n, then $\inf B = 0$, where

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \supseteq a\}.$$

316H Proposition Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is weakly (σ,∞)-distributive;

(ii) whenever $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$, there is a partition of unity B in $\mathfrak{A}$ such that $\{a : a \in A_n,\ a \cap b \neq 0\}$ is finite for every $n \in \mathbb{N}$ and $b \in B$;

(iii) whenever $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of upwards-directed subsets of $\mathfrak{A}$, each with a supremum $c_n = \sup A_n$, and

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \subseteq a\},$$

then $\inf\{c_n \setminus b : n \in \mathbb{N},\ b \in B\} = 0$;

(iv) whenever $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of upwards-directed subsets of $\mathfrak{A}$, each with a supremum $c_n = \sup A_n$, and $\inf_{n \in \mathbb{N}} c_n = c$ is defined, then $c = \sup B$, where

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \subseteq a\}.$$

316I Theorem Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Then $\mathfrak{A}$ is weakly (σ,∞)-distributive iff every meager set in Z is nowhere dense.

316J The regular open algebra of $\mathbb{R}$: Proposition The algebra $\mathrm{RO}(\mathbb{R})$ of regular open subsets of $\mathbb{R}$ is not weakly (σ,∞)-distributive.

316K Atoms in Boolean algebras (a) If $\mathfrak{A}$ is a Boolean algebra, an **atom** in $\mathfrak{A}$ is a non-zero $a \in \mathfrak{A}$ such that the only elements included in a are 0 and a.

(b) A Boolean algebra is **atomless** if it has no atoms.

(c) A Boolean algebra is **purely atomic** if every non-zero element includes an atom.

316L Proposition Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z.

(a) There is a one-to-one correspondence between atoms a of $\mathfrak{A}$ and isolated points $z \in Z$, given by the formula $\widehat{a} = \{z\}$.

(b) $\mathfrak{A}$ is atomless iff Z has no isolated points.

(c) $\mathfrak{A}$ is purely atomic iff the isolated points of Z form a dense subset of Z.

316M Proposition Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. Then a Boolean algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ iff it is atomless, countable and not $\{0\}$.

316N Definition A Boolean algebra $\mathfrak{A}$ is **homogeneous** if every non-trivial principal ideal of $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$.

***316O Lemma** Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that

$$D = \{d : d \in \mathfrak{A},\ \mathfrak{A} \text{ is isomorphic to the principal ideal } \mathfrak{A}_d\}$$

is order-dense in $\mathfrak{A}$. Then $\mathfrak{A}$ is homogeneous.

***316P Proposition** Let $\mathfrak{A}$ be a homogeneous Boolean algebra. Then its Dedekind completion is homogeneous.

***316Q Proposition** The free product of any family of homogeneous Boolean algebras is homogeneous.

***316R Proposition** Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ which is regularly embedded in $\mathfrak{A}$.

(a) Every atom of $\mathfrak{A}$ is included in an atom of $\mathfrak{B}$.

(b) If $\mathfrak{B}$ is atomless, so is $\mathfrak{A}$.