One of the first things one learns, as a student of measure theory, is that sets of measure zero are frequently 'negligible' in the straightforward sense that they can safely be ignored. This is not quite a universal principle, and one of my purposes in writing this treatise is to call attention to the exceptional cases in which negligible sets are important. But very large parts of the theory, including some of the topics already treated in Volume 2, can be expressed in an appropriately abstract language in which negligible sets have been factored out. This is what the present volume is about. A 'measure algebra' is a quotient of an algebra of measurable sets by a null ideal; that is, the elements of the measure algebra are equivalence classes of measurable sets. At the cost of an extra layer of abstraction, we obtain a language which can give concise and elegant expression to a substantial proportion of the ideas of measure theory, and which offers insights almost everywhere in the subject.

It is here that I embark wholeheartedly on 'pure' measure theory. I think it is fair to say that the applications of measure theory to other branches of mathematics are more often through measure *spaces* rather than measure *algebras*. Certainly there will be in this volume many theorems of wide importance outside measure theory; but typically their usefulness will be in forms translated back into the language of the first two volumes. But it is also fair to say that the language of measure algebras is the only reasonable way to discuss large parts of a subject which, as pure mathematics, can bear comparison with any.

In the structure of this volume I can distinguish seven 'working' and two 'accessory' chapters. The 'accessory' chapters are 31 and 35. In these I develop the theories of Boolean algebras and Riesz spaces (= vector lattices) which are needed later. As in Volume 2 you have a certain amount of choice in the order in which you take the material. Everything except Chapter 35 depends on Chapter 31, and everything except Chapters 31 and 35 depends on Chapter 32. Chapters 33, 34 and 36 can be taken in any order, but Chapter 36 relies on Chapter 35. (I do not mean that Chapter 33 is never referred to in Chapter 34, nor even that the later chapters do not rely on results from Chapter 33. What I mean is that their most important ideas are accessible without learning the material of Chapter 33 properly.) Chapter 37 depends on Chapters 35 and 36. Chapter 38 would be difficult to make sense of without some notion of what has been done in Chapter 33. Chapter 39 uses fragments of Chapters 35 and 36.

The first third of the volume follows almost the only line permitted by the structure of the subject. If we are going to study measure algebras at all, we must know the relevant facts about Boolean algebras (Chapter 31) and how to translate what we know about measure spaces into the new language (Chapter 32). Then we must get a proper grip on the two most important theorems: Maharam's theorem on the classification of measure algebras (Chapter 33) and the von Neumann-Maharam lifting theorem (Chapter 34). Since I am now writing for readers who are committed – I hope, happily committed – to learning as much as they can about the subject, I take the space to push these ideas as far as they can easily go, giving a full classification of closed subalgebras of probability algebras, for instance (§333), and investigating special types of lifting (§§345-346). I mention here three sections interpolated into Chapter 34 (§§342-344) which attack a subtle and important question: when can we expect homomorphisms between measure algebras to be realizable in terms of transformations between measure spaces, as discussed briefly in §234 and elsewhere.

Chapters 36 and 37 are devoted to re-working the ideas of Chapter 24 on 'function spaces' in the more abstract context now available, and relating them to the general Riesz spaces of Chapter 35. I am concerned here not to develop new structures, nor even to prove striking new theorems, but rather to offer new ways of looking at the old ones. Only in the Ergodic Theorem (§372) do I come to a really important new result. Chapter 38 looks at two questions, both obvious ones to ask if you have been trained in twentieth-century pure mathematics: what does the automorphism group of a measure algebra look like, and inside such an automorphism group, what do the conjugacy classes look like? (The second question is a fancy way of asking how to decide, given two automorphisms of one of the structures considered in this volume, whether they are really different, or just copies of each other obtained by looking at the structure a different way up.)

Finally, in Chapter 39, I discuss what is known about the question of which Boolean algebras can appear as measure algebras.

Concerning the prerequisites for this volume, we certainly do not need everything in Volume 2. The important chapters there are 21, 23, 24, 25 and 27. If you are approaching this volume without having read the earlier parts of this treatise, you will need the Radon-Nikodým theorem and product measures (of arbitrary families of probability spaces), for Maharam's theorem; a simple version of the martingale theorem, for the lifting theorem; and an acquaintance with L^p spaces (particularly, with L^0 spaces) for Chapter 36. But I would recommend the results-only versions of Volumes 1 and 2 in case some reference is totally obscure. Outside measure theory, I call on quite a lot of terms from general topology, but none of the ideas needed are difficult (Baire's and Tychonoff's theorems are the deepest); they are sketched in §§3A3 and 3A4. We do need some functional analysis for Chapters 36 and 39, but very little more than was already used in Volume 2, except that I now call on versions of the Hahn-Banach theorem (§3A5).

In this volume I assume that readers have substantial experience in both real and abstract analysis, and I make few concessions which would not be appropriate when addressing active researchers, except that perhaps I am a little gentler when calling on ideas from set theory and general topology than I should be with my own colleagues, and I continue to include all the easiest exercises I can think of. I do maintain my practice of giving proofs in very full detail, not so much because I am trying to make them easier, but because one of my purposes here is to provide a complete account of the ideas of the subject. I hope that the result will be accessible to most doctoral students who are studying topics in, or depending on, measure theory.

Version of 29.10.12

Chapter 31

Boolean algebras

The theory of measure algebras naturally depends on certain parts of the general theory of Boolean algebras. In this chapter I collect those results which will be useful later. Since many students encounter the formal notion of Boolean algebra for the first time in this context, I start at the beginning; and indeed I include in the Appendix (§3A2) a brief account of the necessary part of the theory of rings, as not everyone will have had time for this bit of abstract algebra in an undergraduate course. But unless you find the algebraic theory of Boolean algebras so interesting that you wish to study it for its own sake – in which case you should perhaps turn to Sikorski 64 or Koppelberg 89 – I do not think it would be very sensible to read the whole of this chapter before proceeding to the main work of the volume in Chapter 32. Probably §311 is necessary to get an idea of what a Boolean algebra looks like, and a glance at the statements of the theorems in §312 and 313A-313B would be useful, but the later sections can wait until you have need of them, on the understanding that apparently innocent formal manipulations may depend on concepts which take some time to master. I hope that the cross-references will be sufficiently well-targeted to make it possible to read this material in parallel with its applications.

As for the actual material covered, §311 introduces Boolean rings and algebras, with M.H.Stone's theorem on their representation as rings and algebras of sets. §312 is devoted to subalgebras, homomorphisms and quotients, following a path parallel to the corresponding ideas in group theory, ring theory and linear algebra. In §313 I come to the special properties of Boolean algebras associated with their lattice structures, with notions of order-preservation, order-continuity and order-closure. §314 continues this with a discussion of order-completeness, and the elaboration of the Stone representation of an arbitrary Boolean algebra into the Loomis-Sikorski representation of a σ-complete Boolean algebra; this brings us to regular open algebras. §315 deals with 'simple' and 'free' products of Boolean algebras, corresponding to 'products' and 'tensor products' of linear spaces, and to projective and inductive limits of families of Boolean algebras. Finally, §316 examines three special topics: the countable chain condition, weak distributivity and homogeneity.

Version of 15.10.08

311 Boolean algebras

In this section I try to give a sufficient notion of the character of abstract Boolean algebras to make the calculations which will appear on almost every page of this volume seem both elementary and natural. The principal result is of course M.H.Stone's theorem: every Boolean algebra can be expressed as an algebra of sets (311E). So the section divides naturally into the first part, proving Stone's theorem, and the second, consisting of elementary consequences of the theorem and a little practice in using the insights it offers.

311A Definitions (a) A **Boolean ring** is a ring $(\mathfrak{A},+,.)$ in which $a^2=a$ for every $a\in\mathfrak{A}$.

(b) A **Boolean algebra** is a Boolean ring $\mathfrak{A}$ with a multiplicative identity $1=1_{\mathfrak{A}}$; I allow $1=0$ in this context.

Remark For notes on those parts of the elementary theory of rings which we shall need, see §3A2.

I hope that the rather arbitrary use of the word 'algebra' here will give no difficulties; it gives me the freedom to insist that the ring $\{0\}$ should be accepted as a Boolean algebra.

311B Examples (a) For any set X, $(\mathcal{P}X,\triangle,\cap)$ is a Boolean algebra; its zero is $\emptyset$ and its multiplicative identity is X. **P** We have to check the following, which are all easily established, using Venn diagrams or otherwise:

$A\triangle B\subseteq X$ for all A, $B\subseteq X$,

$(A\triangle B)\triangle C=A\triangle(B\triangle C)$ for all A, B, $C\subseteq X$,

so that $(\mathcal{P}X,\triangle)$ is a semigroup;

$A\triangle\emptyset=\emptyset\triangle A=A$ for every $A\subseteq X$,

so that $\emptyset$ is the identity in $(\mathcal{P}X,\triangle)$;

$A\triangle A=\emptyset$ for every $A\subseteq X$,

so that every element of $\mathcal{P}X$ is its own inverse in $(\mathcal{P}X,\triangle)$, and $(\mathcal{P}X,\triangle)$ is a group;

$A\triangle B=B\triangle A$ for all A, $B\subseteq X$,

so that $(\mathcal{P}X,\triangle)$ is an abelian group;

$A\cap B\subseteq X$ for all A, $B\subseteq X$,

$(A\cap B)\cap C=A\cap(B\cap C)$ for all A, B, $C\subseteq X$,

so that $(\mathcal{P}X,\cap)$ is a semigroup;

$A\cap(B\triangle C)=(A\cap B)\triangle(A\cap C)$, $(A\triangle B)\cap C=(A\cap C)\triangle(B\cap C)$ for all A, B, $C\subseteq X$,

so that $(\mathcal{P}X,\triangle,\cap)$ is a ring;

$A\cap A=A$ for every $A\subseteq X$,

so that $(\mathcal{P}X,\triangle,\cap)$ is a Boolean ring;

$A\cap X=X\cap A=A$ for every $A\subseteq X$,

so that $(\mathcal{P}X,\triangle,\cap)$ is a Boolean algebra and X is its identity. **Q**

(b) Recall that an 'algebra of subsets of X' (136E) is a family $\Sigma\subseteq\mathcal{P}X$ such that $\emptyset\in\Sigma$, $X\setminus E\in\Sigma$ for every $E\in\Sigma$, and $E\cup F\in\Sigma$ for all E, $F\in\Sigma$. In this case $(\Sigma,\triangle,\cap)$ is a Boolean algebra with zero $\emptyset$ and identity X. **P** If E, $F\in\Sigma$, then

$$E\cap F=X\setminus((X\setminus E)\cup(X\setminus F))\in\Sigma,$$

$$E\triangle F=(E\cap(X\setminus F))\cup(F\cap(X\setminus E))\in\Sigma.$$

Because $\emptyset$ and $X=X\setminus\emptyset$ both belong to Σ, we can work through the identities in (a) above to see that Σ, like $\mathcal{P}X$, is a Boolean algebra. **Q**

(c) Consider the ring $\mathbb{Z}_2=\{0,1\}$, with its ring operations $+_2$, $\cdot$ given by setting

$$0+_2 0=1+_2 1=0,\quad 0+_2 1=1+_2 0=1,$$

$$0\cdot 0=0\cdot 1=1\cdot 0=0,\quad 1\cdot 1=1.$$

I leave it to you to check, if you have not seen it before, that this is a ring. Because $0\cdot 0=0$ and $1\cdot 1=1$, it is a Boolean algebra.

311C Proposition Let $\mathfrak{A}$ be a Boolean ring.

(a) $a+a=0$, that is, $a=-a$, for every $a\in\mathfrak{A}$.

(b) $ab=ba$ for all a, $b\in\mathfrak{A}$.

proof (a) If $a\in\mathfrak{A}$, then

$$a+a=(a+a)(a+a)=a^2+a^2+a^2+a^2=a+a+a+a,$$

so we must have $0=a+a$.

(b) Now for any a, $b\in\mathfrak{A}$,

$$a+b=(a+b)(a+b)=a^2+ab+ba+b^2=a+ab+ba+b,$$

so

$$0=ab+ba=ab+ab$$

and $ab=ba$.

311D Lemma Let $\mathfrak{A}$ be a Boolean ring, I an ideal of $\mathfrak{A}$ (3A2E), and $a\in\mathfrak{A}\setminus I$. Then there is a ring homomorphism $\phi:\mathfrak{A}\to\mathbb{Z}_2$ such that $\phi a=1$ and $\phi d=0$ for every $d\in I$.

proof (a) Let $\mathcal{I}$ be the family of those ideals J of $\mathfrak{A}$ which include I and do not contain a. Then $\mathcal{I}$ has a maximal element K say. **P** Apply Zorn's lemma. Since $I\in\mathcal{I}$, $\mathcal{I}\neq\emptyset$. If $\mathcal{J}$ is a non-empty totally ordered subset of $\mathcal{I}$, then set $J^* = \bigcup \mathcal{J}$. If b, $c \in J^*$ and $d \in \mathfrak{A}$, then there are J_1, $J_2 \in \mathcal{J}$ such that $b \in J_1$ and $c \in J_2$; now $J = J_1 \cup J_2$ is equal to one of J_1, J_2, so belongs to $\mathcal{J}$, and 0, $b+c$, bd all belong to J, so all belong to J^*. Thus $J^* \lhd \mathfrak{A}$; of course $I \subseteq J^*$ and $a \notin J^*$, so $J^* \in \mathcal{I}$ and is an upper bound for $\mathcal{J}$ in $\mathcal{I}$. As $\mathcal{J}$ is arbitrary, the hypotheses of Zorn's lemma are satisfied and $\mathcal{I}$ has a maximal element. **Q**

(b) For $b \in \mathfrak{A}$ set $K_b = \{d : d \in \mathfrak{A},\, bd \in K\}$. The following are easy to check:

(i) $K \subseteq K_b$ for every $b \in \mathfrak{A}$, because K is an ideal.

(ii) $K_b \lhd \mathfrak{A}$ for every $b \in \mathfrak{A}$. **P** $0 \in K \subseteq K_b$. If d, $d' \in K_b$ and $c \in \mathfrak{A}$ then

$$b(d+d') = bd + bd', \quad b(dc) = (bd)c$$

belong to K, so $d+d'$, $dc \in K_b$. **Q**

(iii) If $b \in \mathfrak{A}$ and $a \notin K_b$, then $K_b \in \mathcal{I}$ so $K_b = K$.

(iv) Now $a^2 = a \notin K$, so $a \notin K_a$ and $K_a = K$.

(v) If $b \in \mathfrak{A} \setminus K$ then $b \notin K_a$, that is, $ba = ab \notin K$, and $a \notin K_b$; consequently $K_b = K$.

(vi) If b, $c \in \mathfrak{A} \setminus K$ then $c \notin K_b$ so $bc \notin K$.

(vii) If b, $c \in \mathfrak{A} \setminus K$ then

$$bc(b+c) = b^2c + bc^2 = bc + bc = 0 \in K,$$

so $b + c \in K_{bc}$. By (vi) and (v), $K_{bc} = K$ so $b + c \in K$.

(c) Now define $\phi : \mathfrak{A} \to \mathbb{Z}_2$ by setting $\phi d = 0$ if $d \in K$, $\phi d = 1$ if $d \in \mathfrak{A} \setminus K$. Then ϕ is a ring homomorphism. **P**

(i) If b, $c \in K$ then $b+c$, $bc \in K$ so

$$\phi(b+c) = 0 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c.$$

(ii) If $b \in K$, $c \in \mathfrak{A} \setminus K$ then

$$c = (b+b) + c = b + (b+c) \notin K$$

so $b + c \notin K$, while $bc \in K$, so

$$\phi(b+c) = 1 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c.$$

(iii) Similarly,

$$\phi(b+c) = 1 = \phi b +_2 \phi c, \quad \phi(bc) = 0 = \phi b\, \phi c$$

if $b \in \mathfrak{A} \setminus K$ and $c \in K$.

(iv) If b, $c \in \mathfrak{A} \setminus K$, then by (b-vi) and (b-vii) we have $b + c \in K$, $bc \notin K$ so

$$\phi(b+c) = 0 = \phi b +_2 \phi c, \quad \phi(bc) = 1 = \phi b\, \phi c.$$

Thus ϕ is a ring homomorphism. **Q**

(d) Finally, if $d \in I$ then $d \in K$ so $\phi d = 0$; and $\phi a = 1$ because $a \notin K$.

311E M.H.Stone's theorem: first form Let $\mathfrak{A}$ be any Boolean ring, and let Z be the set of ring homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Then we have an injective ring homomorphism $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$, setting $\widehat{a} = \{z : z \in Z,\, z(a) = 1\}$. If $\mathfrak{A}$ is a Boolean algebra, then $\widehat{1_{\mathfrak{A}}} = Z$.

proof (a) If a, $b \in \mathfrak{A}$, then

$$\widehat{a+b} = \{z : z(a+b) = 1\} = \{z : z(a) +_2 z(b) = 1\} = \{z : \{z(a), z(b)\} = \{0,1\}\} = \widehat{a} \triangle \widehat{b},$$

$$\widehat{ab} = \{z : z(ab) = 1\} = \{z : z(a)z(b) = 1\} = \{z : z(a) = z(b) = 1\} = \widehat{a} \cap \widehat{b}.$$

Thus $a \mapsto \widehat{a}$ is a ring homomorphism.

(b) If $a \in \mathfrak{A}$ and $a \neq 0$, then by 311D, with $I = \{0\}$, there is a $z \in Z$ such that $z(a) = 1$, that is, $z \in \widehat{a}$; so that $\widehat{a} \neq \emptyset$. This shows that the kernel of $a \mapsto \widehat{a}$ is $\{0\}$, so that the homomorphism is injective (3A2Db).

(c) If $\mathfrak{A}$ is a Boolean algebra, and $z \in Z$, then there is some $a \in \mathfrak{A}$ such that $z(a) = 1$, so that $z(1_{\mathfrak{A}})z(a) = z(1_{\mathfrak{A}}a) \neq 0$ and $z(1_{\mathfrak{A}}) \neq 0$; thus $\widehat{1_{\mathfrak{A}}} = Z$.

311F Remarks (a) For any Boolean ring $\mathfrak{A}$, I will say that the **Stone space** of $\mathfrak{A}$ is the set Z of non-zero ring homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, and the canonical map $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$ is the **Stone representation**.

(b) Because the map $a \mapsto \widehat{a} : \mathfrak{A} \to \mathcal{P}Z$ is an injective ring homomorphism, $\mathfrak{A}$ is isomorphic, as Boolean ring, to its image $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$, which is a subring of $\mathcal{P}Z$. Thus the Boolean rings $\mathcal{P}X$ of 311Ba are leading examples in a very strong sense.

(c) I have taken the set Z of the Stone representation to be actually the set of homomorphisms from $\mathfrak{A}$ onto $\mathbb{Z}_2$. Of course we could equally well take any set which is in a natural one-to-one correspondence with Z; a popular choice is the set of maximal proper ideals of $\mathfrak{A}$, since a subset of $\mathfrak{A}$ is a maximal ideal iff it is the kernel of a member of Z, which is then uniquely defined.

311G The operations $\cup$, $\setminus$, $\triangle$ on a Boolean ring Let $\mathfrak{A}$ be a Boolean ring.

(a) Using the Stone representation, we can see that the elementary operations $\cup$, $\cap$, $\setminus$, $\triangle$ of set theory all correspond to operations on $\mathfrak{A}$. If we set

$$a \cup b = a + b + ab, \quad a \cap b = ab, \quad a \setminus b = a + ab, \quad a \triangle b = a + b$$

for a, $b \in \mathfrak{A}$, then we see that

$$\widehat{a \cup b} = \widehat{a} \triangle \widehat{b} \triangle (\widehat{a} \cap \widehat{b}) = \widehat{a} \cup \widehat{b},$$

$$\widehat{a \cap b} = \widehat{a} \cap \widehat{b},$$

$$\widehat{a \setminus b} = \widehat{a} \setminus \widehat{b},$$

$$\widehat{a \triangle b} = \widehat{a} \triangle \widehat{b}.$$

Consequently all the familiar rules for manipulation of $\cap$, $\cup$, etc. will apply also to $\cap$, $\cup$, and we shall have, for instance,

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c), \quad a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$$

for any members a, b, c of any Boolean ring $\mathfrak{A}$.

(b) Still importing terminology from elementary set theory, I will say that a set $A \subseteq \mathfrak{A}$ is **disjoint** if $a \cap b = 0$, that is, $ab = 0$, for all distinct a, $b \in A$; and that an indexed family $\langle a_i \rangle_{i \in I}$ in $\mathfrak{A}$ is **disjoint** if $a_i \cap a_j = 0$ for all distinct i, $j \in I$. (Just as I allow $\emptyset$ to be a member of a disjoint family of sets, I allow $0 \in A$ or $a_i = 0$ in the present context.)

(c) A **partition of unity** in $\mathfrak{A}$ will be *either* a disjoint set $C \subseteq \mathfrak{A}$ such that there is no non-zero $a \in \mathfrak{A}$ such that $a \cap c = 0$ for every $c \in C$ *or* a disjoint family $\langle c_i \rangle_{i \in I}$ in $\mathfrak{A}$ such that there is no non-zero $a \in \mathfrak{A}$ such that $a \cap c_i = 0$ for every $i \in I$. (In the first case I allow $0 \in C$, and in the second I allow $c_i = 0$.)

(d) Note that a set $C \subseteq \mathfrak{A}$ is a partition of unity iff $C \cup \{0\}$ is a maximal disjoint set. **P** If C is a partition of unity and $a \in \mathfrak{A} \setminus (C \cup \{0\})$, then there must be a $c \in C$ such that $a \cap c \neq 0$, so that $C \cup \{0, a\}$ is not disjoint; thus $C \cup \{0\}$ is a maximal disjoint set. If $C \cup \{0\}$ is a maximal disjoint set, and $a \in \mathfrak{A} \setminus \{0\}$, then either $a \in C$ and $a \cap a \neq 0$, or $C \cup \{0, a\}$ is not disjoint, so there is a $c \in C$ such that $a \cap c \neq 0$; thus C is a partition of unity. **Q**

If $A \subseteq \mathfrak{A}$ is any disjoint set, there is a partition of unity including A. **P** Apply Zorn's Lemma to $\{C : C$ is a disjoint set including $A\}$. **Q**

(e) If C and D are two partitions of unity, I say that C **refines** D if for every $c \in C$ there is a $d \in D$ such that $cd = c$ (that is, $c \subseteq d$ in the language of 311H below). Note that if C refines D and D refines E then C refines E. **P** If $c \in C$, there is a $d \in D$ such that $cd = c$; now there is an $e \in E$ such that $de = d$; in this case,

$$ce = (cd)e = c(de) = cd = c;$$

as c is arbitrary, C refines E. **Q**

311H The order structure of a Boolean ring Again treating a Boolean ring $\mathfrak{A}$ as an algebra of sets, it has a natural ordering, setting $a \subseteq b$ if $ab = a$, so that $a \subseteq b$ iff $\widehat{a} \subseteq \widehat{b}$. This translation makes it obvious that $\subseteq$ is a partial order on $\mathfrak{A}$, with least element 0, and with greatest element 1 iff $\mathfrak{A}$ is a Boolean algebra. Moreover, $\mathfrak{A}$ is a lattice (definition: 2A1Ad), with $a \cup b = \sup\{a, b\}$ and $a \cap b = \inf\{a, b\}$ for all a, $b \in \mathfrak{A}$. Generally, for $a_0, \dots, a_n \in \mathfrak{A}$,

$$\sup_{i \le n} a_i = a_0 \cup \dots \cup a_n, \quad \inf_{i \le n} a_i = a_0 \cap \dots \cap a_n;$$

suprema and infima of finite subsets of $\mathfrak{A}$ correspond to unions and intersections of the corresponding families in the Stone space. (But suprema and infima of *infinite* subsets of $\mathfrak{A}$ are a very different matter; see §313 below.)

It may be obvious, but it is nevertheless vital to recognise that when $\mathfrak{A}$ is a ring of sets then $\subseteq$ agrees with $\subseteq$.

311I The topology of a Stone space: Theorem Let Z be the Stone space of a Boolean ring $\mathfrak{A}$, and let $\mathfrak{T}$ be

$$\{G : G \subseteq Z \text{ and for every } z \in G \text{ there is an } a \in \mathfrak{A} \text{ such that } z \in \widehat{a} \subseteq G\}.$$

Then $\mathfrak{T}$ is a topology on Z, under which Z is a locally compact zero-dimensional Hausdorff space, and $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$ is precisely the set of compact open subsets of Z. $\mathfrak{A}$ is a Boolean algebra iff Z is compact.

proof (a) Because $\mathcal{E}$ is closed under $\cap$, and $\bigcup \mathcal{E} = Z$ (recall that Z is the set of surjective homomorphisms from $\mathfrak{A}$ to $\mathbb{Z}_2$, so that every $z \in Z$ is somewhere non-zero and belongs to some $\widehat{a}$), $\mathcal{E}$ is a topology base, and $\mathfrak{T}$ is a topology.

(b) $\mathfrak{T}$ is Hausdorff. **P** Take any distinct z, $w \in Z$. Then there is an $a \in \mathfrak{A}$ such that $z(a) \ne w(a)$; let us take it that $z(a) = 1$, $w(a) = 0$. There is also a $b \in \mathfrak{A}$ such that $w(b) = 1$, so that $w(b + ab) = w(b) +_2 w(a)w(b) = 1$ and $w \in (b + ab)^\frown$; also

$$a(b + ab) = ab + a^2 b = ab + ab = 0,$$

so

$$\widehat{a} \cap (b + ab)^\frown = (a(b + ab))^\frown = \widehat{0} = \emptyset,$$

and $\widehat{a}$, $(b + ab)^\frown$ are disjoint members of $\mathfrak{T}$ containing z, w respectively. **Q**

(c) If $a \in \mathfrak{A}$ then $\widehat{a}$ is compact. **P** Let $\mathcal{F}$ be an ultrafilter on Z containing $\widehat{a}$. For each $b \in \mathfrak{A}$, $z_0(b) = \lim_{z \to \mathcal{F}} z(b)$ must be defined in $\mathbb{Z}_2$, since one of the sets $\{z : z(b) = 0\}$, $\{z : z(b) = 1\}$ must belong to $\mathcal{F}$. If b, $c \in \mathfrak{A}$, then the set

$$F = \{z : z(b) = z_0(b),\ z(c) = z_0(c),\ z(b + c) = z_0(b + c),\ z(bc) = z_0(bc)\}$$

belongs to $\mathcal{F}$, so is not empty; take any $z_1 \in F$; then

$$z_0(b + c) = z_1(b + c) = z_1(b) +_2 z_1(c) = z_0(b) +_2 z_0(c),$$

$$z_0(bc) = z_1(bc) = z_1(b) z_1(c) = z_0(b) z_0(c).$$

As b, c are arbitrary, $z_0 : \mathfrak{A} \to \mathbb{Z}_2$ is a ring homomorphism. Also $z_0(a) = 1$, because $\widehat{a} \in \mathcal{F}$, so $z_0 \in \widehat{a}$. Now let G be any open subset of Z containing z_0; then there is a $b \in \mathfrak{A}$ such that $z_0 \subseteq \widehat{b} \subseteq G$; since $\lim_{z \to \mathcal{F}} z(b) = z_0(b) = 1$, we must have $\widehat{b} = \{z : z(b) = 1\} \in \mathcal{F}$ and $G \in \mathcal{F}$. Thus $\mathcal{F}$ converges to z_0. As $\mathcal{F}$ is arbitrary, $\widehat{a}$ is compact (2A3R). **Q**

(d) This shows that $\widehat{a}$ is a compact open set for every $a \in \mathfrak{A}$. Moreover, since every point of Z belongs to some $\widehat{a}$, every point of Z has a compact neighbourhood, and Z is locally compact. Every $\widehat{a}$ is closed (because it is compact, or otherwise), so $\mathcal{E}$ is a base for $\mathfrak{T}$ consisting of open-and-closed sets, and $\mathfrak{T}$ is zero-dimensional.

(e) Now suppose that $E \subseteq Z$ is an open compact set. If $E = \emptyset$ then $E = \widehat{0}$. Otherwise, set

$$\mathcal{G} = \{\widehat{a} : a \in \mathfrak{A},\ \widehat{a} \subseteq E\}.$$

Then $\mathcal{G}$ is a family of open subsets of Z and $\bigcup \mathcal{G} = E$, because E is open. But E is also compact, so there is a finite $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $E = \bigcup \mathcal{G}_0$. Express $\mathcal{G}_0$ as $\{\widehat{a}_0, \dots, \widehat{a}_n\}$. Then

$$E = \widehat{a}_0 \cup \ldots \cup \widehat{a}_n = (a_0 \cup \ldots \cup a_n)\widehat{\ }.$$

This shows that every compact open subset of Z is of the form $\widehat{a}$ for some $a \in \mathfrak{A}$.

(f) Finally, if $\mathfrak{A}$ is a Boolean algebra then $Z = \widehat{1}$ is compact, by (c); while if Z is compact then (e) tells us that $Z = \widehat{a}$ for some $a \in \mathfrak{A}$, and of course this a must be a multiplicative identity for $\mathfrak{A}$, so that $\mathfrak{A}$ is a Boolean algebra.

311J We have a kind of converse of Stone's theorem.

Proposition Let X be a locally compact zero-dimensional Hausdorff space. Then the set $\mathfrak{A}$ of open-and-compact subsets of X is a subring of $\mathcal{P}X$. If Z is the Stone space of $\mathfrak{A}$, there is a unique homeomorphism $\theta : Z \to X$ such that $\widehat{a} = \theta^{-1}[a]$ for every $a \in \mathfrak{A}$.

proof (a) Because X is Hausdorff, all its compact sets are closed, so every member of $\mathfrak{A}$ is closed. Consequently $a \cup b$, $a \setminus b$, $a \cap b$ and $a \triangle b$ belong to $\mathfrak{A}$ for all a, $b \in \mathfrak{A}$, and $\mathfrak{A}$ is a subring of $\mathcal{P}X$.

It will be helpful to know that $\mathfrak{A}$ is a base for the topology of X. **P** If $G \subseteq X$ is open and $x \in G$, then (because X is locally compact) there is a compact set $K \subseteq X$ such that $x \in \operatorname{int} K$; now (because X is zero-dimensional) there is an open-and-closed set $a \subseteq X$ such that $x \in a \subseteq G \cap \operatorname{int} K$; because a is a closed subset of a compact subset of X, it is compact, and belongs to $\mathfrak{A}$, while $x \in a \subseteq G$. **Q**

(b) Let $R \subseteq Z \times X$ be the relation

$$\{(z, x)\text{: for every } a \in \mathfrak{A},\ x \in a \iff z(a) = 1\}.$$

Then R is the graph of a bijective function $\theta : Z \to X$.

P (i) If $z \in Z$ and x, $x' \in X$ are distinct, then, because X is Hausdorff, there is an open set $G \subseteq X$ containing x and not containing x'; because $\mathfrak{A}$ is a base for the topology of X, there is an $a \in \mathfrak{A}$ such that $x \in a \subseteq G$, so that $x' \notin a$. Now either $z(a) = 1$ and $(z, x') \notin R$, or $z(a) = 0$ and $(z, x) \notin R$. Thus R is the graph of a function θ with domain included in Z and taking values in X.

(ii) If $z \in Z$, there is an $a_0 \in \mathfrak{A}$ such that $z(a_0) = 1$. Consider $\mathcal{A} = \{a : z(a) = 1\}$. This is a family of closed subsets of X containing the compact set a_0, and $a \cap b \in \mathcal{A}$ for all a, $b \in \mathcal{A}$. So $\bigcap \mathcal{A}$ is not empty (3A3Db); take $x \in \bigcap \mathcal{A}$. Then $x \in a$ whenever $z(a) = 1$. On the other hand, if $z(a) = 0$, then

$$z(a_0 \setminus a) = z(a_0 \triangle (a \cap a_0)) = z(a_0) +_2 z(a_0)z(a) = 1,$$

so $x \in a_0 \setminus a$ and $x \notin a$. Thus $(z, x) \in R$ and $\theta(z) = x$ is defined. As z is arbitrary, the domain of θ is the whole of Z.

(iii) If $x \in X$, define $z : \mathfrak{A} \to \mathbb{Z}_2$ by setting $z(a) = 1$ if $x \in a$, 0 otherwise. It is elementary to check that z is a ring homomorphism form $\mathfrak{A}$ to $\mathbb{Z}_2$. To see that it takes the value 1, note that because $\mathfrak{A}$ is a base for the topology of X there is an $a \in \mathfrak{A}$ such that $x \in a$, so that $z(a) = 1$. So $z \in Z$, and of course $(z, x) \in R$. As x is arbitrary, θ is surjective.

(iv) If z, $z' \in Z$ and $\theta(z) = \theta(z')$, then, for any $a \in \mathfrak{A}$,

$$z(a) = 1 \iff \theta(z) \in a \iff \theta(z') \in a \iff z'(a) = 1,$$

so $z = z'$. Thus θ is injective. **Q**

(c) For any $a \in \mathfrak{A}$,

$$\theta^{-1}[a] = \{z : \theta(z) \in a\} = \{z : z(a) = 1\} = \widehat{a}.$$

It follows that θ is a homeomorphism. **P** (i) If $G \subseteq X$ is open, then (because $\mathfrak{A}$ is a base for the topology of X) $G = \bigcup\{a : a \in \mathfrak{A},\ a \subseteq G\}$ and

$$\theta^{-1}[G] = \bigcup\{\theta^{-1}[a] : a \in \mathfrak{A},\ a \subseteq G\} = \bigcup\{\widehat{a} : a \in \mathfrak{A},\ a \subseteq G\}$$

is an open subset of Z. As G is arbitrary, θ is continuous. (ii) On the other hand, if $G \subseteq X$ and $\theta^{-1}[G]$ is open, then $\theta^{-1}[G]$ is of the form $\bigcup_{a \in \mathcal{A}} \widehat{a}$ for some $\mathcal{A} \subseteq \mathfrak{A}$, so that $G = \bigcup \mathcal{A}$ is an open set in X. Accordingly θ is a homeomorphism. **Q**

(d) Finally, I must check the uniqueness of θ. But of course if $\tilde{\theta} : Z \to X$ is any function such that $\tilde{\theta}^{-1}[a] = \widehat{a}$ for every $a \in \mathfrak{A}$, then the graph of $\tilde{\theta}$ must be R, so $\tilde{\theta} = \theta$.

311K Remark Thus we have a correspondence between Boolean rings and zero-dimensional locally compact Hausdorff spaces which is (up to isomorphism, on the one hand, and homeomorphism, on the other) one-to-one. Every property of Boolean rings which we study will necessarily correspond to some property of zero-dimensional locally compact Hausdorff spaces.

311L Complemented distributive lattices I have introduced Boolean algebras through the theory of rings; this seems to be the quickest route to them from an ordinary undergraduate course in abstract algebra. However there are alternative approaches, taking the order structure rather than the algebraic operations as fundamental, and for the sake of an application in Chapter 35 I give the details of one of these.

Proposition Let $\mathfrak{A}$ be a lattice such that

(i) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, $c \in \mathfrak{A}$;

(ii) there is a permutation $a \mapsto a' : \mathfrak{A} \to \mathfrak{A}$ which is order-reversing, that is, $a \le b$ iff $b' \le a'$, and such that $a'' = a$ for every a;

(iii) $\mathfrak{A}$ has a least element 0 and $a \wedge a' = 0$ for every $a \in \mathfrak{A}$.

Then $\mathfrak{A}$ has a Boolean algebra structure for which $a \subseteq b$ iff $a \le b$.

proof (a) Write 1 for $0'$; if $a \in \mathfrak{A}$, then $a' \ge 0$ so $a = a'' \le 0' = 1$, and 1 is the greatest element of $\mathfrak{A}$.

If a, $b \in \mathfrak{A}$ then, because $'$ is an order-reversing permutation, $a' \vee b' = (a \wedge b)'$. **P** For $c \in \mathfrak{A}$,

$$\begin{aligned} a' \vee b' \le c &\iff a' \le c \ \&\ b' \le c \iff c' \le a \ \&\ c' \le b \\ &\iff c' \le a \wedge b \iff (a \wedge b)' \le c. \ \mathbf{Q} \end{aligned}$$

Similarly, $a' \wedge b' = (a \vee b)'$. If a, b, $c \in \mathfrak{A}$ then

$$(a \wedge b) \vee c = ((a' \vee b') \wedge c')' = ((a' \wedge c') \vee (b' \wedge c'))' = (a \vee c) \wedge (b \vee c).$$

(b) Define addition and multiplication on $\mathfrak{A}$ by setting

$$a + b = (a \wedge b') \vee (a' \wedge b), \quad ab = a \wedge b$$

for a, $b \in \mathfrak{A}$.

(c)(i) If a, $b \in \mathfrak{A}$ then

$$\begin{aligned} (a + b)' &= (a' \vee b) \wedge (a \vee b') = (a' \wedge a) \vee (a' \wedge b') \vee (b \wedge a) \vee (b \wedge b') \\ &= 0 \vee (a' \wedge b') \vee (b \wedge a) = (a' \wedge b') \vee (a \wedge b). \end{aligned}$$

So if a, b, $c \in \mathfrak{A}$ then

$$\begin{aligned} (a + b) + c &= ((a + b) \wedge c') \vee ((a + b)' \wedge c) \\ &= (((a \wedge b') \vee (a' \wedge b)) \wedge c') \vee (((a' \wedge b') \vee (a \wedge b)) \wedge c) \\ &= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee (a' \wedge b' \wedge c) \vee (a \wedge b \wedge c); \end{aligned}$$

as this last formula is symmetric in a, b and c, it is also equal to $a + (b + c)$. Thus addition is associative.

(ii) For any $a \in \mathfrak{A}$,

$$a + 0 = 0 + a = (a' \wedge 0) \vee (a \wedge 0') = 0 \vee (a \wedge 1) = a,$$

so 0 is the additive identity of $\mathfrak{A}$. Also

$$a + a = (a \wedge a') \vee (a' \wedge a) = 0 \vee 0 = 0$$

so each element of $\mathfrak{A}$ is its own additive inverse, and $(\mathfrak{A}, +)$ is a group. It is abelian because $\vee$ and $\wedge$ are commutative.

(d) Because $\wedge$ is associative and commutative, $(\mathfrak{A}, \cdot)$ is a commutative semigroup; also 1 is its identity, because $a \wedge 1 = a$ for every $a \in \mathfrak{A}$. As for the distributive law in $\mathfrak{A}$,

$$\begin{aligned} ab + ac &= (a \wedge b \wedge (a \wedge c)') \vee ((a \wedge b)' \wedge a \wedge c) \\ &= (a \wedge b \wedge (a' \vee c')) \vee ((a' \vee b') \wedge a \wedge c) \\ &= (a \wedge b \wedge a') \vee (a \wedge b \wedge c') \vee (a' \wedge a \wedge c) \vee (b' \wedge a \wedge c) \\ &= (a \wedge b \wedge c') \vee (b' \wedge a \wedge c) \\ &= a \wedge ((b \wedge c') \vee (b' \wedge c)) = a(b + c) \end{aligned}$$

for all a, b, $c \in \mathfrak{A}$. Thus $(\mathfrak{A}, +, \cdot)$ is a ring; because $a \wedge a = a$ for every a, it is a Boolean ring.

(e) For a, $b \in \mathfrak{A}$,

$$a \subseteq b \iff ab = a \iff a \wedge b = a \iff a \leq b,$$

so the order relations of $\mathfrak{A}$ coincide.

Remark It is the case that the Boolean algebra structure of $\mathfrak{A}$ is uniquely determined by its order structure, but I delay the proof to the next section (312M).

311X Basic exercises (a) Let $A_0, \dots, A_n$ be sets. Show that

$$A_0 \triangle \dots \triangle A_n = \{x : \#(\{i : i \leq n,\, x \in A_i\}) \text{ is odd}\}.$$

(b) Let X be a set, and $\Sigma \subseteq \mathcal{P}X$. Show that the following are equiveridical: (i) Σ is an algebra of subsets of X; (ii) Σ is a subring of $\mathcal{P}X$ (that is, contains $\emptyset$ and is closed under $\triangle$ and $\cap$) and contains X; (iii) $\emptyset \in \Sigma$, $X \setminus E \in \Sigma$ for every $E \in \Sigma$, and $E \cap F \in \Sigma$ for all E, $F \in \Sigma$.

(c) Let $\mathfrak{A}$ be any Boolean ring. Let $a \mapsto a'$ be any bijection between $\mathfrak{A}$ and a set B disjoint from $\mathfrak{A}$. Set $\mathfrak{B} = \mathfrak{A} \cup B$, and extend the addition and multiplication of $\mathfrak{A}$ to form binary operations on $\mathfrak{B}$ by using the formulae

$$a + b' = a' + b = (a + b)', \quad a' + b' = a + b,$$

$$a'b = b + ab, \quad ab' = a + ab,\ a'b' = (a + b + ab)'.$$

Show that $\mathfrak{B}$ is a Boolean algebra and that $\mathfrak{A}$ is an ideal in $\mathfrak{B}$.

>(d) Let $\mathfrak{A}$ be a Boolean ring, and K a finite subset of $\mathfrak{A}$. Show that the subring of $\mathfrak{A}$ generated by K has at most $2^{2^{\#(K)}-1}$ members. (*Hint*: count its minimal non-zero elements.)

>(e) Show that any finite Boolean ring is isomorphic to $\mathcal{P}X$ for some finite set X (and, in particular, is a Boolean algebra).

(f) Let $\mathfrak{A}$ be any Boolean ring. Show that

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c), \quad a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

for all a, b, $c \in \mathfrak{A}$ directly from the definitions in 311G, without using Stone's theorem.

>(g) Let $\mathfrak{A}$ be any Boolean ring. Show that if we regard the Stone space Z of $\mathfrak{A}$ as a subset of $\{0,1\}^{\mathfrak{A}}$, then the topology of Z (311I) is just the subspace topology induced by the ordinary product topology of $\{0,1\}^{\mathfrak{A}}$.

(h) Let I be any set, and set $X = \{0,1\}^I$ with its usual topology (3A3K). Show that for a subset E of X the following are equiveridical: (i) E is open-and-compact; (ii) E is determined by coordinates in a finite subset of I (definition: 254M); (iii) E belongs to the algebra of subsets of X generated by $\{E_i : i \in I\}$, where $E_i = \{x : x(i) = 1\}$ for each i.

(i) Let $(\mathfrak{A}, \leq)$ be a lattice such that (α) $\mathfrak{A}$ has a least element 0 and a greatest element 1 (β) for every a, b, $c \in \mathfrak{A}$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (γ) for every $a \in \mathfrak{A}$ there is an $a' \in \mathfrak{A}$ such that $a \vee a' = 1$ and $a \wedge a' = 0$. Show that there is a Boolean algebra structure on $\mathfrak{A}$ for which $\leq$ agrees with $\subseteq$.

311Y Further exercises (a) Let $\mathfrak{A}$ be a Boolean ring, and $\mathfrak{B}$ the Boolean algebra constructed by the method of 311Xc. Show that the Stone space of $\mathfrak{B}$ can be identified with the one-point compactification (3A3O) of the Stone space of $\mathfrak{A}$.

(b) Let $(\mathfrak{A}, \vee, \wedge, 0, 1)$ be such that (i) $(\mathfrak{A}, \vee)$ is a commutative semigroup with identity 0 (ii) $(\mathfrak{A}, \wedge)$ is a commutative semigroup with identity 1 (iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all a, b, $c \in \mathfrak{A}$ (iv) $a \vee a = a \wedge a = a$ for every $a \in \mathfrak{A}$ (v) for every $a \in \mathfrak{A}$ there is an $a' \in \mathfrak{A}$ such that $a \vee a' = 1$ and $a \wedge a' = 0$. Show that there is a Boolean algebra structure on $\mathfrak{A}$ for which $\vee = \cup$, $\wedge = \cap$.

(c) Let $(\mathfrak{A}, \vee, ')$ be such that (i) $(\mathfrak{A}, \vee)$ is a non-empty commutative semigroup (ii) $' : \mathfrak{A} \to \mathfrak{A}$ is a function (iii) $((a \vee b)' \vee (a \vee b')')' = a$ for all a, $b \in \mathfrak{A}$. Show that there is a Boolean algebra structure on $\mathfrak{A}$ for which $\vee = \cup$ and $'$ is complementation. (*Hint*: McCune 97.)

(d) Let P be a distributive lattice, and Z the set of surjective lattice homomorphisms from P to $\{0, 1\}$. Show that there is a sublattice of $\mathcal{P}Z$ isomorphic to P.

311 Notes and comments My aim in this section has been to get as quickly as possible to Stone's theorem, since this is surely the best route to a picture of general Boolean algebras; they are isomorphic to algebras of sets. This means that all their elementary algebraic properties – indeed, all their first-order properties – can be effectively studied in the context of elementary set theory. In 311G-311H I describe a few of the ways in which the Stone representation suggests algebraic properties of Boolean algebras.

You should not, however, come too readily to the conclusion that Boolean algebras will never be able to surprise you. In this book, in particular, we shall need to work a good deal with suprema and infima of infinite sets in Boolean algebras, for the ordering of 311H; and even though this corresponds to the ordering $\subseteq$ of ordinary sets, we find that $(\sup A)\widehat{\ }$ is sufficiently different from $\bigcup_{a \in A} \widehat{a}$ to need new kinds of intuition. (The point is that $\bigcup_{a \in A} \widehat{a}$ is an open set in the Stone space, but need not be compact if A is infinite, so may well be smaller than $(\sup A)\widehat{\ }$, even when $\sup A$ is defined in $\mathfrak{A}$.) There is also the fact that Stone's theorem depends crucially on a fairly strong form of the axiom of choice (employed through Zorn's Lemma in the argument of 311D). Of course I shall be using the axiom of choice without scruple throughout this volume. But it should be clear that such results as 312B-312C in the next section cannot possibly need the axiom of choice for their proofs, and that to use Stone's theorem in such a context is slightly misleading.

Nevertheless, it is so useful to be able to regard a Boolean algebra as an algebra of sets – especially when dealing with only finitely many elements of the algebra at a time – that henceforth I will almost always use the symbols $\triangle$, $\cap$ for the addition and multiplication of a Boolean ring, and will use $\cup$, $\setminus$, $\subseteq$ without further comment, just as if I were considering $\cup$, $\setminus$ and $\subseteq$ in the Stone space. (In 311Gb I have given a definition of 'disjointness' in a Boolean algebra based on the same idea.) Even without the axiom of choice this approach can be justified, once we have observed that finitely-generated Boolean algebras are finite (311Xd), since relatively elementary methods show that any finite Boolean algebra is isomorphic to $\mathcal{P}X$ for some finite set X.

I have taken a Boolean algebra to be a particular kind of commutative ring with identity. Of course there are other approaches. If we wish to think of the order relation as primary, then 311L and 311Xi are reasonably natural. Other descriptions can be based on a list of the properties of the binary operations $\cup$, $\cap$ and the complementation operation $a \mapsto a' = 1 \setminus a$, as in 311Yb. (The hardest I know of is in 311Yc.) I give extra space to 311L only because this is well adapted to an application in 352Q below.

Version of 29.5.07

312 Homomorphisms

I continue the theory of Boolean algebras with a section on subalgebras, ideals and homomorphisms. From now on, I will relegate Boolean rings which are not algebras to the exercises; I think there is no need to set out descriptions of the (mostly trifling) modifications necessary to deal with the extra generality. The first part of the section (312A-312L) concerns the translation of the basic concepts of ring theory into the language which I propose to use for Boolean algebras. 312M shows that the order relation on a Boolean algebra defines the algebraic structure, and in 312N-312O I give a fundamental result on the extension of homomorphisms. I end the section with results relating the previous ideas to the Stone representation of a Boolean algebra (312P-312T).

312A Subalgebras Let $\mathfrak{A}$ be a Boolean algebra. I will use the phrase **subalgebra of** $\mathfrak{A}$ to mean a subring of $\mathfrak{A}$ containing its multiplicative identity $1 = 1_{\mathfrak{A}}$.

312B Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Then the following are equiveridical, that is, if one is true so are the others:
(i) $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$;
(ii) $0 \in \mathfrak{B}$, $a \cup b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$, and $1 \setminus a \in \mathfrak{B}$ for all $a \in \mathfrak{B}$;
(iii) $\mathfrak{B} \neq \emptyset$, $a \cap b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$, and $1 \setminus a \in \mathfrak{B}$ for all $a \in \mathfrak{B}$.

proof (i)$\Rightarrow$(iii) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, and a, $b \in \mathfrak{B}$, then of course we shall have

$$0,\, 1 \in \mathfrak{B}, \text{ so } \mathfrak{B} \neq \emptyset,$$

$$a \cap b \in \mathfrak{B}, \quad 1 \setminus a = 1 \triangle a \in \mathfrak{B}.$$

(iii)$\Rightarrow$(ii) If (iii) is true, then there is some $b_0 \in \mathfrak{B}$; now $1 \setminus b_0 \in \mathfrak{B}$, so

$$0 = b_0 \cap (1 \setminus b_0) \in \mathfrak{B}.$$

If a, $b \in \mathfrak{B}$, then

$$a \cup b = 1 \setminus ((1 \setminus a) \cap (1 \setminus b)) \in \mathfrak{B}.$$

So (ii) is true.

(ii)$\Rightarrow$(i) If (ii) is true, then for any a, $b \in \mathfrak{B}$,

$$a \cap b = 1 \setminus ((1 \setminus a) \cup (1 \setminus b)) \in \mathfrak{B},$$

$$a \triangle b = (a \cap (1 \setminus b)) \cup (b \cap (1 \setminus a)) \in \mathfrak{B},$$

so (because also $0 \in \mathfrak{B}$) $\mathfrak{B}$ is a subring of $\mathfrak{A}$, and

$$1 = 1 \setminus 0 \in \mathfrak{B},$$

so $\mathfrak{B}$ is a subalgebra.

Remark Thus an algebra of subsets of a set X, as defined in 136E or 311Bb, is just a subalgebra of the Boolean algebra $\mathcal{P}X$.

312C Ideals in Boolean algebras: Proposition If $\mathfrak{A}$ is a Boolean algebra, a set $I \subseteq \mathfrak{A}$ is an ideal of $\mathfrak{A}$ iff $0 \in I$, $a \cup b \in I$ for all a, $b \in I$, and $a \in I$ whenever $b \in I$ and $a \subseteq b$.

proof (a) Suppose that I is an ideal. Then of course $0 \in I$. If a, $b \in I$ then $a \cap b \in I$ so $a \cup b = (a \triangle b) \triangle (a \cap b) \in I$. If $b \in I$ and $a \subseteq b$ then $a = a \cap b \in I$.

(b) Now suppose that I satisfies the conditions proposed. If a, $b \in I$ then

$$a \triangle b \subseteq a \cup b \in I$$

so $a \triangle b \in I$, while of course $-a = a \in I$, and also $0 \in I$, by hypothesis; thus I is a subgroup of $(\mathfrak{A}, \triangle)$. Finally, if $a \in I$ and $b \in \mathfrak{A}$ then

$$a \cap b \subseteq a \in I,$$

so $b \cap a = a \cap b \in I$; thus I is an ideal.

312D Principal ideals Of course, while an ideal I in a Boolean algebra $\mathfrak{A}$ is necessarily a subring, it is not as a rule a subalgebra, except in the special case $I = \mathfrak{A}$. But if we say that a **principal ideal** of $\mathfrak{A}$ is the ideal $\mathfrak{A}_a$ generated by a single element a of $\mathfrak{A}$, we have a special phenomenon.

312E Proposition Let $\mathfrak{A}$ be a Boolean algebra, and a any element of $\mathfrak{A}$. Then the principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ generated by a is just $\{b : b \in \mathfrak{A},\, b \subseteq a\}$, and (with the inherited operations $\cap\restriction\mathfrak{A}_a \times \mathfrak{A}_a$, $\triangle\restriction\mathfrak{A}_a \times \mathfrak{A}_a$) is a Boolean algebra in its own right, with multiplicative identity a.

proof $b \subseteq a$ iff $b \cap a = b$, so that

$$\mathfrak{A}_a = \{b : b \subseteq a\} = \{b \cap a : b \in \mathfrak{A}\}$$

is an ideal of $\mathfrak{A}$, and of course it is the smallest ideal of $\mathfrak{A}$ containing a. Being an ideal, it is a subring; the idempotent relation $b \cap b = b$ is inherited from $\mathfrak{A}$, so it is a Boolean ring; and a is plainly its multiplicative identity.

312F Boolean homomorphisms Now suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras. I will use the phrase **Boolean homomorphism** to mean a function $\pi : \mathfrak{A} \to \mathfrak{B}$ which is a ring homomorphism (that is, $\pi(a \triangle b) = \pi a \triangle \pi b$, $\pi(a \cap b) = \pi a \cap \pi b$ for all a, $b \in \mathfrak{A}$) and is uniferent, that is, $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

312G Proposition Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ be Boolean algebras.

(a) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism, then $\pi[\mathfrak{A}]$ is a subalgebra of $\mathfrak{B}$.

(b) If $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ are Boolean homomorphisms, then $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ is a Boolean homomorphism.

(c) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a bijective Boolean homomorphism, then $\pi^{-1} : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism.

proof These are all immediate consequences of the corresponding results for ring homomorphisms (3A2D).

312H Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a function. Then the following are equiveridical:

(i) π is a Boolean homomorphism;

(ii) $\pi(a \cap b) = \pi a \cap \pi b$ and $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for all a, $b \in \mathfrak{A}$;

(iii) $\pi(a \cup b) = \pi a \cup \pi b$ and $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for all a, $b \in \mathfrak{A}$;

(iv) $\pi(a \cup b) = \pi a \cup \pi b$ and $\pi a \cap \pi b = 0_{\mathfrak{B}}$ whenever a, $b \in \mathfrak{A}$ and $a \cap b = 0_{\mathfrak{A}}$, and $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

proof (i)⇒(iv) If π is a Boolean homomorphism then of course $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$; also, given that $a \cap b = 0$ in $\mathfrak{A}$,

$$\pi a \cap \pi b = \pi(a \cap b) = \pi(0_{\mathfrak{A}}) = 0_{\mathfrak{B}},$$

$$\pi(a \cup b) = \pi(a \triangle b) = \pi a \triangle \pi b = \pi a \cup \pi b.$$

(iv)⇒(iii) Assume (iv), and take a, $b \in \mathfrak{A}$. Then

$$\pi a = \pi(a \cap b) \cup \pi(a \setminus b), \quad \pi b = \pi(a \cap b) \cup \pi(b \setminus a),$$

so

$$\pi(a \cup b) = \pi a \cup \pi(b \setminus a) = \pi(a \cap b) \cup \pi(a \setminus b) \cup \pi(b \setminus a) = \pi a \cup \pi b.$$

Taking $b = 1 \setminus a$, we must have

$$1_{\mathfrak{B}} = \pi(1_{\mathfrak{A}}) = \pi a \cup \pi(1_{\mathfrak{A}} \setminus a), \quad 0_{\mathfrak{B}} = \pi a \cap \pi(1_{\mathfrak{A}} \setminus a),$$

so $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$. Thus (iii) is true.

(iii)⇒(ii) If (iii) is true and a, $b \in \mathfrak{A}$, then

$$\begin{aligned}\pi(a \cap b) &= \pi(1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus a) \cup (1_{\mathfrak{A}} \setminus b)))\\ &= 1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus \pi a) \cup (1_{\mathfrak{B}} \setminus \pi b))) = \pi a \cap \pi b.\end{aligned}$$

So (ii) is true.

(ii)⇒(i) If (ii) is true, then

$$\begin{aligned}\pi(a \triangle b) &= \pi((1_{\mathfrak{A}} \setminus ((1_{\mathfrak{A}} \setminus a) \cap (1_{\mathfrak{A}} \setminus b))) \cap (1_{\mathfrak{A}} \setminus (a \cap b)))\\ &= (1_{\mathfrak{B}} \setminus ((1_{\mathfrak{B}} \setminus \pi a) \cap (1_{\mathfrak{B}} \setminus \pi b)) \cap (1_{\mathfrak{B}} \setminus (\pi a \cap \pi b))) = \pi a \triangle \pi b\end{aligned}$$

for all a, $b \in \mathfrak{A}$, so π is a ring homomorphism; and now

$$\pi(1_{\mathfrak{A}}) = \pi(1_{\mathfrak{A}} \setminus 0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus \pi(0_{\mathfrak{A}}) = 1_{\mathfrak{B}} \setminus 0_{\mathfrak{B}} = 1_{\mathfrak{B}},$$

so that π is a Boolean homomorphism.

312I Proposition If $\mathfrak{A}$, $\mathfrak{B}$ are Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism, then $\pi a \subseteq \pi b$ whenever $a \subseteq b$ in $\mathfrak{A}$.

proof

$$a \subseteq b \Longrightarrow a \cap b = a \Longrightarrow \pi a \cap \pi b = \pi a \Longrightarrow \pi a \subseteq \pi b.$$

312J Proposition Let $\mathfrak{A}$ be a Boolean algebra, and a any member of $\mathfrak{A}$. Then the map $b \mapsto a \cap b$ is a surjective Boolean homomorphism from $\mathfrak{A}$ onto the principal ideal $\mathfrak{A}_a$ generated by a.

proof This is an elementary verification.

***312K Fixed-point subalgebras** For future reference I introduce the following idea. If $\mathfrak{A}$ is a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean homomorphism, then $\{a : a \in \mathfrak{A},\ \pi a = a\}$ is a subalgebra of $\mathfrak{A}$ (put the definitions 312A and 312F together); I will call it the **fixed-point subalgebra** of π.

312L Quotient algebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$. Then the quotient ring $\mathfrak{A}/I$ (3A2F) is a Boolean algebra, and the canonical map $a \mapsto a^\bullet : \mathfrak{A} \to \mathfrak{A}/I$ is a Boolean homomorphism, so that

$$(a \triangle b)^\bullet = a^\bullet \triangle b^\bullet, \quad (a \cup b)^\bullet = a^\bullet \cup b^\bullet, \quad (a \cap b)^\bullet = a^\bullet \cap b^\bullet, \quad (a \setminus b)^\bullet = a^\bullet \setminus b^\bullet$$

for all a, $b \in \mathfrak{A}$.

(b) The order relation on $\mathfrak{A}/I$ is defined by the formula

$$a^\bullet \subseteq b^\bullet \iff a \setminus b \in I.$$

For any $a \in \mathfrak{A}$,

$$\{u : u \subseteq a^\bullet\} = \{b^\bullet : b \subseteq a\}.$$

proof (a) Of course the map $a \mapsto a^\bullet = \{a \triangle b : b \in I\}$ is a ring homomorphism (3A2Fd). Because

$$(a^\bullet)^2 = (a^2)^\bullet = a^\bullet$$

for every $a \in \mathfrak{A}$, $\mathfrak{A}/I$ is a Boolean ring; because $1^\bullet$ is a multiplicative identity, it is a Boolean algebra, and $a \mapsto a^\bullet$ is a Boolean homomorphism. The formulae given are now elementary.

(b) We have

$$a^\bullet \subseteq b^\bullet \iff a^\bullet \setminus b^\bullet = 0 \iff a \setminus b \in I.$$

Now

$$\{u : u \subseteq a^\bullet\} = \{u \cap a^\bullet : u \in \mathfrak{A}/I\} = \{(b \cap a)^\bullet : b \in \mathfrak{A}\} = \{b^\bullet : b \subseteq a\}.$$

312M The above results are both repetitive and nearly trivial. Now I come to something with a little more meat to it.

Proposition If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a bijection such that $\pi a \subseteq \pi b$ whenever $a \subseteq b$, then π is a Boolean algebra isomorphism.

proof (a) Because π is surjective, there must be c_0, $c_1 \in \mathfrak{A}$ such that $\pi c_0 = 0_{\mathfrak{B}}$, $\pi c_1 = 1_{\mathfrak{B}}$; now $\pi(0_{\mathfrak{A}}) \subseteq \pi c_0$ and $\pi c_1 \subseteq \pi(1_{\mathfrak{A}})$, so we must have $\pi(0_{\mathfrak{A}}) = 0_{\mathfrak{B}}$ and $\pi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

(b) If $a \in \mathfrak{A}$, then $\pi a \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}}$. **P** There is a $c \in \mathfrak{A}$ such that $\pi c = 1_{\mathfrak{B}} \setminus (\pi a \cup \pi(1_{\mathfrak{A}} \setminus a))$. Now

$$\pi(c \cap a) \subseteq \pi c \cap \pi a = 0_{\mathfrak{B}}, \quad \pi(c \setminus a) \subseteq \pi c \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}};$$

as π is injective, $c \cap a = c \setminus a = 0_{\mathfrak{A}}$ and $c = 0_{\mathfrak{A}}$, $\pi c = 0_{\mathfrak{B}}$, $\pi a \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}}$. **Q**

(c) If $a \in \mathfrak{A}$, then $\pi a \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}}$. **P** It may be clear to you that this is just a dual form of (b). If not, I repeat the argument in the form now appropriate. There is a $c \in \mathfrak{A}$ such that $\pi c = 1_{\mathfrak{B}} \setminus (\pi a \cap \pi(1_{\mathfrak{A}} \setminus a))$. Now

$$\pi(c \cup a) \supseteq \pi c \cup \pi a = 1_{\mathfrak{B}}, \quad \pi(c \cup (1_{\mathfrak{A}} \setminus a)) \supseteq \pi c \cup \pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}};$$

as π is injective, $c \cup a = c \cup (1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{A}}$ and $c = 1_{\mathfrak{A}}$, $\pi c = 1_{\mathfrak{B}}$, $\pi a \cap \pi(1_{\mathfrak{A}} \setminus a) = 0_{\mathfrak{B}}$. **Q**

(d) Putting (b) and (c) together, we have $\pi(1_{\mathfrak{A}} \setminus a) = 1_{\mathfrak{B}} \setminus \pi a$ for every $a \in \mathfrak{A}$. Now $\pi(a \cup b) = \pi a \cup \pi b$ for every a, $b \in \mathfrak{A}$. **P** Surely $\pi a \cup \pi b \subseteq \pi(a \cup b)$. Let $c \in \mathfrak{A}$ be such that $\pi c = \pi(a \cup b) \setminus (\pi a \cup \pi b)$. Then

$$\pi(c \cap a) \subseteq \pi c \cap \pi a = 0_{\mathfrak{B}}, \quad \pi(c \cap b) \subseteq \pi c \cap \pi b = 0_{\mathfrak{B}},$$

so $c \cap a = c \cap b = 0$ and $c \subseteq 1_{\mathfrak{A}} \setminus (a \cup b)$; accordingly

$$\pi c \subseteq \pi(1_{\mathfrak{A}} \setminus (a \cup b)) = 1_{\mathfrak{B}} \setminus \pi(a \cup b);$$

as also $\pi c \subseteq \pi(a \cup b)$, $\pi c = 0_{\mathfrak{B}}$ and $\pi(a \cup b) = \pi a \cup \pi b$. **Q**

(e) So the conditions of 312H(iii) are satisfied and π is a Boolean homomorphism; being bijective, it is an isomorphism.

312N I turn next to a fundamental lemma on the construction of homomorphisms. We need to start with a proper description of a certain type of subalgebra.

Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$; let c be any member of $\mathfrak{A}$. Then

$$\mathfrak{A}_1 = \{(a \cap c) \cup (b \setminus c) : a,\, b \in \mathfrak{A}_0\}$$

is a subalgebra of $\mathfrak{A}$; it is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$.

proof We have to check the following:

$$a = (a \cap c) \cup (a \setminus c) \in \mathfrak{A}_1$$

for every $a \in \mathfrak{A}_0$, so $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$; in particular, $0 \in \mathfrak{A}_1$.

$$1 \setminus ((a \cap c) \cup (b \setminus c)) = ((1 \setminus a) \cap c) \cup ((1 \setminus b) \setminus c) \in \mathfrak{A}_1$$

for all a, $b \in \mathfrak{A}_0$, so $1 \setminus d \in \mathfrak{A}_1$ for every $d \in \mathfrak{A}_1$.

$$(a \cap c) \cup (b \setminus c) \cup (a' \cap c) \cup (b' \setminus c) = ((a \cup a') \cap c) \cup ((b \cup b') \setminus c) \in \mathfrak{A}_1$$

for all a, b, a', $b' \in \mathfrak{A}_0$, so $d \cup d' \in \mathfrak{A}_1$ for all d, $d' \in \mathfrak{A}_1$. Thus $\mathfrak{A}_1$ is a subalgebra of $\mathfrak{A}$ (using 312B).

$$c = (1 \cap c) \cup (0 \setminus c) \in \mathfrak{A}_1,$$

so $\mathfrak{A}_1$ includes $\mathfrak{A}_0 \cup \{c\}$; and finally it is clear that any subalgebra of $\mathfrak{A}$ including $\mathfrak{A}_0 \cup \{c\}$, being closed under $\cap$, $\cup$ and complementation, must include $\mathfrak{A}_1$, so that $\mathfrak{A}_1$ is the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$.

312O Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism, and $c \in \mathfrak{A}$. If $v \in \mathfrak{B}$ is such that $\pi a \subseteq v \subseteq \pi b$ whenever a, $b \in \mathfrak{A}_0$ and $a \subseteq c \subseteq b$, then there is a unique Boolean homomorphism π_1 from the subalgebra $\mathfrak{A}_1$ of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ such that π_1 extends π and $\pi_1 c = v$.

proof (a) The basic fact we need to know is that if a, a', b, $b' \in \mathfrak{A}_0$ and

$$(a \cap c) \cup (b \setminus c) = d = (a' \cap c) \cup (b' \setminus c),$$

then

$$(\pi a \cap v) \cup (\pi b \setminus v) = (\pi a' \cap v) \cup (\pi b' \setminus v).$$

P We have

$$a \cap c = d \cap c = a' \cap c.$$

Accordingly $(a \triangle a') \cap c = 0$ and $c \subseteq 1 \setminus (a \triangle a')$. Consequently (since $a \triangle a'$ surely belongs to $\mathfrak{A}_0$)

$$v \subseteq \pi(1 \setminus (a \triangle a')) = 1 \setminus (\pi a \triangle \pi a'),$$

and

$$\pi a \cap v = \pi a' \cap v.$$

Similarly,

$$b \setminus c = d \setminus c = b' \setminus c,$$

so

$$(b \triangle b') \setminus c = 0, \quad b \triangle b' \subseteq c, \quad \pi b \triangle \pi b' = \pi(b \triangle b') \subseteq v$$

and

$$\pi b \setminus v = \pi b' \setminus v.$$

Putting these together, we have the result. **Q**

(b) Consequently, we have a function π_1 defined by writing

$$\pi_1((a \cap c) \cup (b \setminus c)) = (\pi a \cap v) \cup (\pi b \setminus v)$$

for all a, $b \in \mathfrak{A}_0$; and 312N tells us that the domain of π_1 is just $\mathfrak{A}_1$. Now π_1 is a Boolean homomorphism. **P** This amounts to running through the proof of 312N again.

(i) If a, $b \in \mathfrak{A}_0$, then

$$\begin{aligned}\pi_1(1 \setminus ((a \cap c) \cup (b \setminus c))) &= \pi_1(((1 \setminus a) \cap c) \cup ((1 \setminus b) \setminus c))\\ &= (\pi(1 \setminus a) \cap v) \cup (\pi(1 \setminus b) \setminus v)\\ &= ((1 \setminus \pi a) \cap v) \cup ((1 \setminus \pi b) \setminus v)\\ &= 1 \setminus ((\pi a \cap v) \cup (\pi b \setminus v)) = 1 \setminus \pi_1((a \cap c) \cup (b \setminus c)).\end{aligned}$$

So $\pi_1(1 \setminus d) = 1 \setminus \pi_1 d$ for every $d \in \mathfrak{A}_1$.

(ii) If a, b, a', $b' \in \mathfrak{A}_0$, then

$$\begin{aligned}\pi_1((a \cap c) \cup (b \setminus c) \cup (a' \cap c) \cup (b' \setminus c)) &= \pi_1(((a \cup a') \cap c) \cup ((b \cup b') \setminus c))\\ &= (\pi(a \cup a') \cap v) \cup (\pi(b \cup b') \setminus v)\\ &= ((\pi a \cup \pi a') \cap v) \cup ((\pi b \cup \pi b') \setminus v)\\ &= (\pi a \cap v) \cup (\pi b \setminus v) \cup (\pi a' \cap v) \cup (\pi b' \setminus v)\\ &= \pi_1((a \cap c) \cup (b \setminus c)) \cup \pi_1((a' \cap c) \cup (b' \setminus c)).\end{aligned}$$

So $\pi_1(d \cup d') = \pi_1 d \cup \pi_1 d'$ for all d, $d' \in \mathfrak{A}_1$.

By 312H(iii), π_1 is a Boolean homomorphism. **Q**

(c) If $a \in \mathfrak{A}_0$, then

$$\pi_1 a = \pi_1((a \cap c) \cup (a \setminus c)) = (\pi a \cap v) \cup (\pi a \setminus v) = \pi a,$$

so π_1 extends π. As for the action of π_1 on c,

$$\pi_1 c = \pi_1((1 \cap c) \cup (0 \setminus c)) = (\pi 1 \cap v) \cup (\pi 0 \setminus v) = (1 \cap v) \cup (0 \setminus v) = v,$$

as required.

(d) Finally, the formula of (b) is the only possible definition for any Boolean homomorphism from $\mathfrak{A}_1$ to $\mathfrak{B}$ which will extend π and take c to v. So π_1 is unique.

312P Homomorphisms and Stone spaces Because the Stone space Z of a Boolean algebra $\mathfrak{A}$ (311E) can be constructed explicitly from the algebraic structure of $\mathfrak{A}$, it must in principle be possible to describe any feature of the Boolean structure of $\mathfrak{A}$ in terms of Z. In the next few paragraphs I work through the most important identifications.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; write $\widehat{a} \subseteq Z$ for the open-and-closed set corresponding to $a \in \mathfrak{A}$. Then there is a one-to-one correspondence between ideals I of $\mathfrak{A}$ and open sets $G \subseteq Z$, given by the formulae

$$G = \bigcup_{a \in I} \widehat{a}, \quad I = \{a : \widehat{a} \subseteq G\}.$$

proof (a) For any ideal $I \lhd \mathfrak{A}$, set $H(I) = \bigcup_{a \in I} \widehat{a}$; then $H(I)$ is a union of open subsets of Z, so is open. For any open set $G \subseteq Z$, set $J(G) = \{a : a \in \mathfrak{A}, \widehat{a} \subseteq G\}$; then $J(G)$ satisfies the conditions of 312C, so is an ideal of $\mathfrak{A}$.

(b) If $I \lhd \mathfrak{A}$, then $J(H(I)) = I$. **P** (i) If $a \in I$, then $\widehat{a} \subseteq H(I)$ so $a \in J(H(I))$. (ii) If $a \in J(H(I))$, then $\widehat{a} \subseteq H(I) = \bigcup_{b \in I} \widehat{b}$. Because $\widehat{a}$ is compact and all the $\widehat{b}$ are open, there must be finitely many $b_0, \dots, b_n \in I$ such that $\widehat{a} \subseteq \widehat{b}_0 \cup \ldots \cup \widehat{b}_n$. But now $a \subseteq b_0 \cup \ldots \cup b_n \in I$, so $a \in I$. **Q**

(c) If $G \subseteq Z$ is open, then $H(J(G)) = G$. **P** (i) If $z \in G$, then (because $\{\widehat{a} : a \in \mathfrak{A}\}$ is a base for the topology of Z) there is an $a \in \mathfrak{A}$ such that $z \in \widehat{a} \subseteq G$; now $a \in J(G)$ and $z \in H(J(G))$. (ii) If $z \in H(J(G))$, there is an $a \in J(G)$ such that $z \in \widehat{a}$; now $\widehat{a} \subseteq G$, so $z \in G$. **Q**

This shows that the maps $G \mapsto J(G)$, $I \mapsto H(I)$ are two halves of a one-to-one correspondence, as required.

312Q Theorem Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z, W; write $\widehat{a} \subseteq Z$, $\widehat{b} \subseteq W$ for the open-and-closed sets corresponding to $a \in \mathfrak{A}$, $b \in \mathfrak{B}$. Then we have a one-to-one correspondence between Boolean homomorphisms $\pi : \mathfrak{A} \to \mathfrak{B}$ and continuous functions $\phi : W \to Z$, given by the formula

$$\pi a = b \iff \phi^{-1}[\widehat{a}] = \widehat{b},$$

that is, $\phi^{-1}[\widehat{a}] = \widehat{\pi a}$.

proof (a) Recall that I have constructed Z and W as the sets of Boolean homomorphisms from $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathbb{Z}_2$ (311F). So if $\pi : \mathfrak{A} \to \mathfrak{B}$ is any Boolean homomorphism, and $w \in W$, $\psi_\pi(w) = w\pi$ is a Boolean homomorphism from $\mathfrak{A}$ to $\mathbb{Z}_2$ (312Gb), and belongs to Z. Now $\psi_\pi^{-1}[\widehat{a}] = \widehat{\pi a}$ for every $a \in \mathfrak{A}$. **P**

$$\psi_\pi^{-1}[\widehat{a}] = \{w : \psi_\pi(w) \in \widehat{a}\} = \{w : w\pi \in \widehat{a}\} = \{w : w\pi(a) = 1\} = \{w : w \in \widehat{\pi a}\}. \text{ **Q**}$$

Consequently ψ_π is continuous. **P** Let G be any open subset of Z. Then $G = \bigcup\{\widehat{a} : \widehat{a} \subseteq G\}$, so

$$\psi_\pi^{-1}[G] = \bigcup\{\psi_\pi^{-1}[\widehat{a}] : \widehat{a} \subseteq G\} = \bigcup\{\widehat{\pi a} : \widehat{a} \subseteq G\}$$

is open. As G is arbitrary, ψ_π is continuous. **Q**

(b) If $\phi : W \to Z$ is continuous, then for any $a \in \mathfrak{A}$ the set $\phi^{-1}[\widehat{a}]$ must be an open-and-closed set in W; consequently there is a unique member of $\mathfrak{B}$, call it $\theta_\phi a$, such that $\phi^{-1}[\widehat{a}] = \widehat{\theta_\phi a}$. Observe that, for any $w \in W$ and $a \in \mathfrak{A}$,

$$w(\theta_\phi a) = 1 \iff w \in \widehat{\theta_\phi a} \iff \phi(w) \in \widehat{a} \iff (\phi(w))(a) = 1,$$

so $\phi(w) = w\theta_\phi$.

Now θ_ϕ is a Boolean homomorphism. **P** (i) If $a, b \in \mathfrak{A}$ then

$$\theta_\phi(a \cup b)\widehat{\ } = \phi^{-1}[(a \cup b)\widehat{\ }] = \phi^{-1}[\widehat{a} \cup \widehat{b}] = \phi^{-1}[\widehat{a}] \cup \phi^{-1}[\widehat{b}] = \widehat{\theta_\phi a} \cup \widehat{\theta_\phi b} = (\theta_\phi a \cup \theta_\phi b)\widehat{\ },$$

so $\theta_\phi(a \cup b) = \theta_\phi a \cup \theta_\phi b$. (ii) If $a \in \mathfrak{A}$, then

$$\theta_\phi(1 \setminus a)\widehat{\ } = \phi^{-1}[(1 \setminus a)\widehat{\ }] = \phi^{-1}[Z \setminus \widehat{a}] = W \setminus \phi^{-1}[\widehat{a}] = W \setminus \widehat{\theta_\phi a} = (1 \setminus \theta_\phi a)\widehat{\ },$$

so $\theta_\phi(1 \setminus a) = 1 \setminus \theta_\phi a$. (iii) By 312H, θ_ϕ is a Boolean homomorphism. **Q**

(c) For any Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$, $\pi = \theta_{\psi_\pi}$. **P** For $a \in \mathfrak{A}$,

$$(\theta_{\psi_\pi} a)\widehat{\ } = \psi_\pi^{-1}[\widehat{a}] = \widehat{\pi a},$$

so $\theta_{\psi_\pi} a = a$. **Q**

(d) For any continuous function $\phi : W \to Z$, $\phi = \psi_{\theta_\phi}$. **P** For any $w \in W$,

$$\psi_{\theta_\phi}(w) = w\theta_\phi = \phi(w). \text{ **Q**}$$

(e) Thus $\pi \mapsto \psi_\pi$, $\phi \mapsto \theta_\phi$ are the two halves of a one-to-one correspondence, as required.

312R Theorem Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Boolean algebras, with Stone spaces Z, W and V. Let $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ be Boolean homomorphisms, with corresponding continuous functions $\phi : W \to Z$ and $\psi : V \to W$. Then the Boolean homomorphism $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ corresponds to the continuous function $\phi\psi : V \to Z$.

proof For any $a \in \mathfrak{A}$,

$$\widehat{\theta\pi a} = (\theta(\pi a))\widehat{\ } = \psi^{-1}[\widehat{\pi a}] = \psi^{-1}[\phi^{-1}[\widehat{a}]] = (\phi\psi)^{-1}[\widehat{a}].$$

312S Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism, with associated continuous function $\phi : W \to Z$. Then

(a) π is injective iff ϕ is surjective;

(b) π is surjective iff ϕ is injective.

proof (a) If $a \in \mathfrak{A}$, then

$$\widehat{a} \cap \phi[W] = \emptyset \iff \phi^{-1}[\widehat{a}] = \emptyset \iff \widehat{\pi a} = \emptyset \iff \pi a = 0.$$

Now W is compact, so $\phi[W]$ also is compact, therefore closed, and

$$\begin{aligned} \phi \text{ is not surjective} &\iff Z \setminus \phi[W] \neq \emptyset \\ &\iff \text{there is a non-zero } a \in \mathfrak{A} \text{ such that } \widehat{a} \subseteq Z \setminus \phi[W] \\ &\iff \text{there is a non-zero } a \in \mathfrak{A} \text{ such that } \pi a = 0 \\ &\iff \pi \text{ is not injective} \end{aligned}$$

(3A2Db).

(b)(i) If π is surjective and w, w' are distinct members of W, then there is a $b \in \mathfrak{B}$ such that $w \in \widehat{b}$ and $w' \notin \widehat{b}$. Now $b = \pi a$ for some $a \in \mathfrak{A}$, so $\phi(w) \in \widehat{a}$ and $\phi(w') \notin \widehat{a}$, and $\phi(w) \neq \phi(w')$. As w and w' are arbitrary, ϕ is injective.

(ii) If ϕ is injective and $b \in \mathfrak{B}$, then $K = \phi[\widehat{b}]$, $L = \phi[W \setminus \widehat{b}]$ are disjoint compact subsets of Z. Consider $I = \{a : a \in \mathfrak{A},\ L \cap \widehat{a} = \emptyset\}$. Then $\bigcup_{a \in I} \widehat{a} = Z \setminus L \supseteq K$. Because K is compact and every $\widehat{a}$ is open, there is a finite family $a_0, \dots, a_n \in I$ such that $K \subseteq \widehat{a}_0 \cup \ldots \cup \widehat{a}_n$. Set $a = a_0 \cup \ldots \cup a_n$. Then $\widehat{a} = \widehat{a}_0 \cup \ldots \cup \widehat{a}_n$ includes K and is disjoint from L. So $\widehat{\pi a} = \phi^{-1}[\widehat{a}]$ includes $\widehat{b}$ and is disjoint from $W \setminus \widehat{b}$; that is, $\widehat{\pi a} = \widehat{b}$ and $\pi a = b$. As b is arbitrary, π is surjective.

312T Principal ideals If $\mathfrak{A}$ is a Boolean algebra and $a \in \mathfrak{A}$, we have a natural surjective Boolean homomorphism $b \mapsto b \cap a : \mathfrak{A} \to \mathfrak{A}_a$, the principal ideal generated by a (312J). Writing Z for the Stone space of $\mathfrak{A}$ and Z_a for the Stone space of $\mathfrak{A}_a$, this homomorphism must correspond to an injective continuous function $\phi : Z_a \to Z$ (312Sb). Because Z_a is compact and Z is Hausdorff, ϕ must be a homeomorphism between Z_a and its image $\phi[Z_a] \subseteq Z$ (3A3Dd). To identify $\phi[Z_a]$, note that it is compact, therefore closed, and that

$$\begin{aligned} Z \setminus \phi[Z_a] &= \bigcup\{\widehat{b} : b \in \mathfrak{A},\ \widehat{b} \cap \phi[Z_a] = \emptyset\} \\ &= \bigcup\{\widehat{b} : \phi^{-1}[\widehat{b}] = \emptyset\} = \bigcup\{\widehat{b} : b \cap a = 0\} = Z \setminus \widehat{a}, \end{aligned}$$

so that $\phi[Z_a] = \widehat{a}$. It is therefore natural to identify Z_a with the open-and-closed set $\widehat{a} \subseteq Z$.

312X Basic exercises (a) Let $\mathfrak{A}$ be a Boolean ring, and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Show that $\mathfrak{B}$ is a subring of $\mathfrak{A}$ iff $0 \in \mathfrak{B}$ and $a \cup b$, $a \setminus b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$.

(b) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a subset of $\mathfrak{A}$. Show that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ iff $1 \in \mathfrak{B}$ and $a \setminus b \in \mathfrak{B}$ for all a, $b \in \mathfrak{B}$.

(c) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $I \subseteq A \subseteq \mathfrak{A}$ are such that $1 \in A$, $a \cap b \in I$ for all $a, b \in I$ and $a \setminus b \in A$ whenever $a, b \in A$ and $b \subseteq a$. Show that A includes the subalgebra of $\mathfrak{A}$ generated by I. (*Hint*: 136Xf.)

(d) Show that if $\mathfrak{A}$ is a Boolean ring, a set $I \subseteq \mathfrak{A}$ is an ideal of $\mathfrak{A}$ iff $0 \in I$, $a \cup b \in I$ for all $a, b \in I$, and $a \in I$ whenever $b \in I$ and $a \subseteq b$.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a function such that (i) $\pi(a) \subseteq \pi(b)$ whenever $a \subseteq b$ (ii) $\pi(a) \cap \pi(b) = 0_{\mathfrak{B}}$ whenever $a \cap b = 0_{\mathfrak{A}}$ (iii) $\pi(a) \cup \pi(b) \cup \pi(c) = 1_{\mathfrak{B}}$ whenever $a \cup b \cup c = 1_{\mathfrak{A}}$. Show that π is a Boolean homomorphism.

(f) Let $\mathfrak{A}$ be a Boolean ring, and a any member of $\mathfrak{A}$. Show that the map $b \mapsto a \cap b$ is a ring homomorphism from $\mathfrak{A}$ onto the principal ideal $\mathfrak{A}_a$ generated by a.

(g) Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be Boolean rings, and let $\mathfrak{B}_1$, $\mathfrak{B}_2$ be the Boolean algebras constructed from them by the method of 311Xc. Show that any ring homomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ has a unique extension to a Boolean homomorphism from $\mathfrak{B}_1$ to $\mathfrak{B}_2$.

(h) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings, $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$, $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a ring homomorphism, and $c \in \mathfrak{A}$. Show that if $v \in \mathfrak{B}$ is such that $\pi a \setminus v = \pi b \cap v = 0$ whenever $a, b \in \mathfrak{A}_0$ and $a \setminus c = b \cap c = 0$, then there is a unique ring homomorphism π_1 from the subring $\mathfrak{A}_1$ of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ such that π_1 extends π_0 and $\pi_1 c = v$.

(i) Let $\mathfrak{A}$ be a Boolean ring, and Z its Stone space. Show that there is a one-to-one correspondence between ideals I of $\mathfrak{A}$ and open sets $G \subseteq Z$, given by the formulae $G = \bigcup_{a \in I} \widehat{a}$, $I = \{a : \widehat{a} \subseteq G\}$.

(j) Let $\mathfrak{A}$ be a Boolean algebra, and suppose that $\mathfrak{A}$ is the subalgebra of itself generated by $\mathfrak{A}_0 \cup \{c\}$, where $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}$ and $c \in \mathfrak{A}$. Let Z be the Stone space of $\mathfrak{A}$ and Z_0 the Stone space of $\mathfrak{A}_0$. Let $\psi : Z \to Z_0$ be the continuous surjection corresponding to the embedding of $\mathfrak{A}_0$ in $\mathfrak{A}$. Show that $\psi \restriction \widehat{c}$ and $\psi \restriction Z \setminus \widehat{c}$ are injective.

Now let $\mathfrak{B}$ be another Boolean algebra, with Stone space W, and $\pi : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism, with corresponding function $\phi : W \to Z_0$. Show that there is a continuous function $\phi_1 : W \to Z$ such that $\psi\phi_1 = \phi$ iff there is an open-and-closed set $V \subseteq W$ such that $\phi[V] \subseteq \psi[\widehat{c}]$ and $\phi[W \setminus V] \subseteq \psi[Z \setminus \widehat{c}]$.

(k) Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z, and I an ideal of $\mathfrak{A}$, corresponding to an open set $G \subseteq Z$. Show that the Stone space of the quotient algebra $\mathfrak{A}/I$ may be identified with $Z \setminus G$.

(l) Let $\mathfrak{A}$ be a Boolean algebra, and $A \subseteq \mathfrak{A}$ a set, closed under $\cup$ and $\cap$, such that $0, 1 \in A$. Let B be the set of elements of $\mathfrak{A}$ expressible as $a \setminus a'$ where $a, a' \in A$, and C the set of elements of $\mathfrak{A}$ expressible as $b_0 \cup \ldots \cup b_n$ where $b_0, \ldots, b_n \in B$ are disjoint. Show that C is a subalgebra of $\mathfrak{A}$.

(m) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras, and $A \subseteq \mathfrak{A}$ a set, closed under $\cup$ and $\cap$, such that $0_{\mathfrak{A}}, 1_{\mathfrak{A}} \in A$; let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by A. Let $\pi : A \to \mathfrak{B}$ be such that $\pi 0_{\mathfrak{A}} = 0_{\mathfrak{B}}$ and $\pi 1_{\mathfrak{A}} = 1_{\mathfrak{B}}$, and $\pi(a \cup a') = \pi a \cup \pi a'$, $\pi(a \cap a') = \pi a \cap \pi a'$ for all $a, a' \in \mathfrak{A}$. Show that π has a unique extension to a Boolean homomorphism from $\mathfrak{C}$ to $\mathfrak{B}$.

312Y Further exercises (a) Find a function $\phi : \mathcal{P}\{0,1,2\} \to \mathbb{Z}_2$ such that $\phi(1 \setminus a) = 1 \setminus \phi a$ for every $a \in \mathcal{P}\{0,1,2\}$ and $\phi(a) \subseteq \phi(b)$ whenever $a \subseteq b$, but ϕ is not a Boolean homomorphism.

(b) Let $\mathfrak{A}$ be the Boolean ring of finite subsets of $\mathbb{N}$. Show that there is a permutation $\pi : \mathfrak{A} \to \mathfrak{A}$ such that $\pi a \subseteq \pi b$ whenever $a \subseteq b$ but π is not a ring homomorphism.

(c) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean rings, with Stone spaces Z, W. Show that we have a one-to-one correspondence between ring homomorphisms $\pi : \mathfrak{A} \to \mathfrak{B}$ and continuous functions $\phi : H \to Z$, where $H \subseteq W$ is an open set, such that $\phi^{-1}[K]$ is compact for every compact set $K \subseteq Z$, given by the formula $\pi a = b \iff \phi^{-1}[\widehat{a}] = \widehat{b}$.

(d) Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be Boolean rings, with Stone spaces Z, W and V. Let $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\theta : \mathfrak{B} \to \mathfrak{C}$ be ring homomorphisms, with corresponding continuous functions $\phi : H \to Z$ and $\psi : G \to W$. Show that the ring homomorphism $\theta\pi : \mathfrak{A} \to \mathfrak{C}$ corresponds to the continuous function $\phi\psi : \psi^{-1}[H] \to Z$.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean rings, with Stone spaces Z and W, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a ring homomorphism, with associated continuous function $\phi : H \to Z$. Show that π is injective iff $\phi[H]$ is dense in Z, and that π is surjective iff ϕ is injective and $H = W$.

(f) Let $\mathfrak{A}$ be a Boolean ring and $a \in \mathfrak{A}$. Show that the Stone space of the principal ideal $\mathfrak{A}_a$ of $\mathfrak{A}$ generated by a can be identified with the compact open set $\widehat{a}$ in the Stone space of $\mathfrak{A}$. Show that the identity map is a ring homomorphism from $\mathfrak{A}_a$ to $\mathfrak{A}$, and corresponds to the identity function on $\widehat{a}$.

312 Notes and comments The definitions of 'subalgebra' and 'Boolean homomorphism' (312A, 312F), like that of 'Boolean algebra', are a trifle arbitrary, but will be a convenient way of mandating appropriate treatment of multiplicative identities. I run through the work of 312A-312J essentially for completeness; once you are familiar with Boolean algebras, they should all seem obvious. 312M has a little bit more to it. It shows that the order structure of a Boolean algebra defines the ring structure, in a fairly strong sense.

I call 312O a 'lemma', but actually it is the most important result in this section; it is the basic tool we have for extending a homomorphism from a subalgebra to a slightly larger one, and with Zorn's Lemma (another 'lemma' which deserves a capital L) will provide us with general methods of constructing homomorphisms.

In 312P-312T I describe the basic relationships between the Boolean homomorphisms and continuous functions on Stone spaces. 312Q-312R show that, in the language of category theory, the Stone representation provides a 'contravariant functor' from the category of Boolean algebras with Boolean homomorphisms to the category of topological spaces with continuous functions. Using 311I-311J, we know exactly which topological spaces appear, the zero-dimensional compact Hausdorff spaces; and we know also that the functor is faithful, that is, that we can recover Boolean algebras and homomorphisms from the corresponding topological spaces and continuous functions. There is an agreeable duality in 312S. All of this can be done for Boolean rings, but there are some extra complications (312Yc-312Yf).

To my mind, the very essence of the theory of Boolean algebras is the fact that they are abstract rings, but at the same time can be thought of 'locally' as algebras of sets. Consequently we can bring two quite separate kinds of intuition to bear. 312O gives an example of a ring-theoretic problem, concerning the extension of homomorphisms, which has a resolution in terms of the order relation, a concept most naturally described in terms of algebras-of-sets. It is very much a matter of taste and habit, but I myself find that a Boolean homomorphism is easiest to think of in terms of its action on finite subalgebras, which are directly representable as $\mathcal{P}X$ for some finite X (311Xe); the corresponding continuous map between Stone spaces is less helpful. I offer 312Xj, the Stone-space version of 312O, for you to test your own intuitions on.

Version of 8.6.11

313 Order-continuous homomorphisms

Because a Boolean algebra has a natural partial order (311H), we have corresponding notions of upper bounds, lower bounds, suprema and infima. These are particularly important in the Boolean algebras arising in measure theory, and the infinitary operations 'sup' and 'inf' require rather more care than the basic binary operations '$\cup$', '$\cap$', because intuitions from elementary set theory are sometimes misleading. I therefore take a section to work through the most important properties of these operations, together with the homomorphisms which preserve them.

313A Relative complementation: Proposition Let $\mathfrak{A}$ be a Boolean algebra, e a member of $\mathfrak{A}$, and A a non-empty subset of $\mathfrak{A}$.

(a) If $\sup A$ is defined in $\mathfrak{A}$, then $\inf\{e \setminus a : a \in A\}$ is defined and equal to $e \setminus \sup A$.

(b) If $\inf A$ is defined in $\mathfrak{A}$, then $\sup\{e \setminus a : a \in A\}$ is defined and equal to $e \setminus \inf A$.

proof (a) Writing a_0 for $\sup A$, we have $e\setminus a_0\subseteq e\setminus a$ for every $a\in A$, so $e\setminus a_0$ is a lower bound for $C=\{e\setminus a:a\in A\}$. Now suppose that c is any lower bound for C. Then (because A is not empty) $c\subseteq e$, and

$$a=(a\setminus e)\cup(e\setminus(e\setminus a))\subseteq(a_0\setminus e)\cup(e\setminus c)$$

for every $a\in A$. Consequently $a_0\subseteq(a_0\setminus e)\cup(e\setminus c)$ is disjoint from c and

$$c=c\cap e\subseteq e\setminus a_0.$$

Accordingly $e\setminus a_0$ is the greatest lower bound of C, as claimed.

(b) This time set $a_0=\inf A$, $C=\{e\setminus a:a\in A\}$. As before, $e\setminus a_0$ is surely an upper bound for C. If c is any upper bound for C, then

$$e\setminus c\subseteq e\setminus(e\setminus a)=e\cap a\subseteq a$$

for every $a\in A$, so $e\setminus c\subseteq a_0$ and $e\setminus a_0\subseteq c$. As c is arbitrary, $e\setminus a_0$ is indeed the least upper bound of C.

Remark In the arguments above I repeatedly encourage you to treat $\cap$, $\cup$, $\setminus$, $\subseteq$ as if they were the corresponding operations and relation of basic set theory. This is perfectly safe so long as we take care that every manipulation so justified has only finitely many elements of the Boolean algebra in hand at once.

313B General distributive laws: Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $e\in\mathfrak{A}$ and $A\subseteq\mathfrak{A}$ is a non-empty set such that $\sup A$ is defined in $\mathfrak{A}$, then $\sup\{e\cap a:a\in A\}$ is defined and equal to $e\cap\sup A$.

(b) If $e\in\mathfrak{A}$ and $A\subseteq\mathfrak{A}$ is a non-empty set such that $\inf A$ is defined in $\mathfrak{A}$, then $\inf\{e\cup a:a\in A\}$ is defined and equal to $e\cup\inf A$.

(c) Suppose that A, $B\subseteq\mathfrak{A}$ are non-empty and $\sup A$, $\sup B$ are defined in $\mathfrak{A}$. Then $\sup\{a\cap b:a\in A,\,b\in B\}$ is defined and is equal to $\sup A\cap\sup B$.

(d) Suppose that A, $B\subseteq\mathfrak{A}$ are non-empty and $\inf A$, $\inf B$ are defined in $\mathfrak{A}$. Then $\inf\{a\cup b:a\in A,\,b\in B\}$ is defined and is equal to $\inf A\cup\inf B$.

proof (a) Set

$$B=\{e\setminus a:a\in A\},\quad C=\{e\setminus b:b\in B\}=\{e\cap a:a\in A\}.$$

Using 313A, we have

$$\inf B=e\setminus\sup A,\qquad \sup C=e\setminus\inf B=e\cap\sup A,$$

as required.

(b) Set $a_0=\inf A$, $B=\{e\cup a:a\in A\}$. Then $e\cup a_0\subseteq e\cup a$ for every $a\in A$, so $e\cup a_0$ is a lower bound for B. If c is any lower bound for B, then $c\setminus e\subseteq a$ for every $a\in A$, so $c\setminus e\subseteq a_0$ and $c\subseteq e\cup a_0$; thus $e\cup a_0$ is the greatest lower bound for B, as claimed.

(c) By (a), we have

$$a\cap\sup B=\sup_{b\in B}a\cap b$$

for every $a\in A$, so

$$\sup_{a\in A,b\in B}a\cap b=\sup_{a\in A}(a\cap\sup B)=\sup A\cap\sup B,$$

using (a) again.

(d) Similarly, using (b) twice,

$$\inf_{a\in A,b\in B}a\cup b=\inf_{a\in A}(a\cup\inf B)=\inf A\cup\inf B.$$

313C As always, it is worth developing a representation of the concepts of sup and inf in terms of Stone spaces.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; for $a\in\mathfrak{A}$ write $\widehat{a}$ for the corresponding open-and-closed subset of Z.

(a) If $A \subseteq \mathfrak{A}$ and $a_0 \in \mathfrak{A}$ then $a_0 = \sup A$ in $\mathfrak{A}$ iff $\widehat{a}_0 = \overline{\bigcup_{a \in A} \widehat{a}}$.
(b) If $A \subseteq \mathfrak{A}$ is non-empty and $a_0 \in \mathfrak{A}$ then $a_0 = \inf A$ in $\mathfrak{A}$ iff $\widehat{a}_0 = \operatorname{int} \bigcap_{a \in A} \widehat{a}$.
(c) If $A \subseteq \mathfrak{A}$ is non-empty then $\inf A = 0$ in $\mathfrak{A}$ iff $\bigcap_{a \in A} \widehat{a}$ is nowhere dense in Z.

proof (a) For any $b \in \mathfrak{A}$,

$$\begin{aligned} b \text{ is an upper bound for } A &\iff \widehat{a} \subseteq \widehat{b} \text{ for every } a \in A \\ &\iff \bigcup_{a \in A} \widehat{a} \subseteq \widehat{b} \iff \overline{\bigcup_{a \in A} \widehat{a}} \subseteq \widehat{b} \end{aligned}$$

because $\widehat{b}$ is certainly closed in Z. It follows at once that if $\widehat{a}_0$ is actually equal to $\overline{\bigcup_{a \in A} \widehat{a}}$ then a_0 must be the least upper bound of A in $\mathfrak{A}$. On the other hand, if $a_0 = \sup A$, then $\overline{\bigcup_{a \in A} \widehat{a}} \subseteq \widehat{a}_0$. **?** If $\widehat{a}_0 \neq \overline{\bigcup_{a \in A} \widehat{a}}$, then $\widehat{a}_0 \setminus \overline{\bigcup_{a \in A} \widehat{a}}$ is a non-empty open set in Z, so includes $\widehat{b}$ for some non-zero $b \in \mathfrak{A}$; now $\widehat{a} \subseteq \widehat{a}_0 \setminus \widehat{b}$, so $a \subseteq a_0 \setminus b$ for every $a \in A$, and $a_0 \setminus b$ is an upper bound for A strictly less than a_0. **X** Thus $\widehat{a}_0$ must be exactly $\overline{\bigcup_{a \in A} \widehat{a}}$.

(b) Take complements: setting $a_1 = 1 \setminus a_0$, we have

$$a_0 = \inf A \iff a_1 = \sup_{a \in A} 1 \setminus a$$

(by 313A)

$$\begin{aligned} &\iff \widehat{a}_1 = \overline{\bigcup_{a \in A} Z \setminus \widehat{a}} \\ &\iff \widehat{a}_0 = Z \setminus \overline{\bigcup_{a \in A} Z \setminus \widehat{a}} = \operatorname{int} \bigcap_{a \in A} \widehat{a}. \end{aligned}$$

(c) Since $\bigcap_{a \in A} \widehat{a}$ is surely a closed set, it is nowhere dense iff it has empty interior, that is, iff $0 = \inf A$.

313D I started the section with the results above because they are easily stated and of great importance. But I must now turn to some new definitions, and I think it may help to clarify the ideas involved if I give them in their own natural context, even though this is far more general than we have any immediate need for here.

Definitions Let P be a partially ordered set and C a subset of P.

(a) C is **order-closed** if $\sup A \in C$ whenever A is a non-empty upwards-directed subset of C such that $\sup A$ is defined in P, and $\inf A \in C$ whenever A is a non-empty downwards-directed subset of C such that $\inf A$ is defined in P.

(b) C is **sequentially order-closed** if $\sup_{n \in \mathbb{N}} p_n \in C$ whenever $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in C such that $\sup_{n \in \mathbb{N}} p_n$ is defined in P, and $\inf_{n \in \mathbb{N}} p_n \in C$ whenever $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in C such that $\inf_{n \in \mathbb{N}} p_n$ is defined in P.

Remark I hope it is obvious that an order-closed set is sequentially order-closed.

313E Order-closed subalgebras and ideals Of course, in the very special cases of a subalgebra or ideal of a Boolean algebra, the concepts 'order-closed' and 'sequentially order-closed' have expressions simpler than those in 313D. I spell them out.

(a) Let $\mathfrak{B}$ be a subalgebra of a Boolean algebra $\mathfrak{A}$.

(i) The following are equiveridical:
(α) $\mathfrak{B}$ is order-closed in $\mathfrak{A}$;
(β) $\sup B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ and $\sup B$ is defined in $\mathfrak{A}$;

(β') $\inf B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ and $\inf B$ is defined in $\mathfrak{A}$;

(γ) $\sup B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ is non-empty and upwards-directed and $\sup B$ is defined in $\mathfrak{A}$;

(γ') $\inf B \in \mathfrak{B}$ whenever $B \subseteq \mathfrak{B}$ is non-empty and downwards-directed and $\inf B$ is defined in $\mathfrak{A}$.

P Of course (β) $\Rightarrow$ (γ). If (γ) is true and $B \subseteq \mathfrak{B}$ is any set with a supremum in $\mathfrak{A}$, then $B' = \{0\} \cup \{b_0 \cup \ldots \cup b_n : b_0, \ldots, b_n \in B\}$ is a non-empty upwards-directed set with the same upper bounds as B, so $\sup B = \sup B' \in \mathfrak{B}$. Thus ($\gamma$) $\Rightarrow$ (β) and (β), (γ) are equiveridical. Next, if (β) is true and $B \subseteq \mathfrak{B}$ is a set with an infimum in $\mathfrak{A}$, then $B' = \{1 \setminus b : b \in \mathfrak{B}\} \subseteq \mathfrak{B}$ and $\sup B' = 1 \setminus \inf B$ is defined, so $\sup B'$ and $\inf B$ belong to $\mathfrak{B}$. Thus (β) $\Rightarrow$ (β'). In the same way, (γ') $\iff$ (β') $\Rightarrow$ (β) and (β), (β'), (γ), (γ') are all equiveridical. But since we also have (α) $\iff$ (γ)&(γ'), (α) is equiveridical with the others. **Q**

Replacing the sets B above by sequences, the same arguments provide conditions for $\mathfrak{B}$ to be sequentially order-closed, as follows.

(ii) The following are equiveridical:

(α) $\mathfrak{B}$ is sequentially order-closed in $\mathfrak{A}$;

(β) $\sup_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $\sup_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(β') $\inf_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{B}$ and $\inf_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(γ) $\sup_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{B}$ and $\sup_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$;

(γ') $\inf_{n\in\mathbb{N}} b_n \in \mathfrak{B}$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{B}$ and $\inf_{n\in\mathbb{N}} b_n$ is defined in $\mathfrak{A}$.

(b) Now suppose that I is an ideal of $\mathfrak{A}$. Then if $A \subseteq I$ is non-empty all lower bounds of A necessarily belong to I; so that

I is order-closed iff $\sup A \in I$ whenever $A \subseteq I$ is non-empty, upwards-directed and has a supremum in $\mathfrak{A}$;

I is sequentially order-closed iff $\sup_{n\in\mathbb{N}} a_n \in I$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in I with a supremum in $\mathfrak{A}$.

Moreover, because I is closed under $\cup$,

I is order-closed iff $\sup A \in I$ whenever $A \subseteq I$ has a supremum in $\mathfrak{A}$;

I is sequentially order-closed iff $\sup_{n\in\mathbb{N}} a_n \in I$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in I with a supremum in $\mathfrak{A}$.

(c) If $\mathfrak{A} = \mathcal{P}X$ is a power set, then a sequentially order-closed subalgebra of $\mathfrak{A}$ is just a σ-algebra of sets, while a sequentially order-closed ideal of $\mathfrak{A}$ is a what I have called a σ-ideal of sets (112Db). If $\mathfrak{A}$ is itself a σ-algebra of sets, then a sequentially order-closed subalgebra of $\mathfrak{A}$ is a 'σ-subalgebra' in the sense of 233A.

Accordingly I will normally use the phrases **σ-subalgebra**, **σ-ideal** for sequentially order-closed subalgebras and ideals of Boolean algebras.

313F Order-closures and generated sets (a) It is an immediate consequence of the definitions that

(i) if $\mathcal{S}$ is any non-empty family of subalgebras of a Boolean algebra $\mathfrak{A}$, then $\bigcap\mathcal{S}$ is a subalgebra of $\mathfrak{A}$;

(ii) if $\mathcal{F}$ is any non-empty family of order-closed subsets of a partially ordered set P, then $\bigcap\mathcal{F}$ is an order-closed subset of P;

(iii) if $\mathcal{F}$ is any non-empty family of sequentially order-closed subsets of a partially ordered set P, then $\bigcap\mathcal{F}$ is a sequentially order-closed subset of P.

(b) Consequently, given any Boolean algebra $\mathfrak{A}$ and a subset B of $\mathfrak{A}$, we have a smallest subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ including B, being the intersection of all the subalgebras of $\mathfrak{A}$ which include B; a smallest σ-subalgebra $\mathfrak{B}_\sigma$ of $\mathfrak{A}$ including B, being the intersection of all the σ-subalgebras of $\mathfrak{A}$ which include B; and a smallest order-closed subalgebra $\mathfrak{B}_\tau$ of $\mathfrak{A}$ including B, being the intersection of all the order-closed subalgebras of $\mathfrak{A}$ which include B. We call $\mathfrak{B}$, $\mathfrak{B}_\sigma$ and $\mathfrak{B}_\tau$ the subalgebra, σ-subalgebra and order-closed subalgebra **generated** by B. (I will return to this in 331E.)

(c) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ any subalgebra of $\mathfrak{A}$, then the smallest order-closed subset $\overline{\mathfrak{B}}$ of $\mathfrak{A}$ which includes $\mathfrak{B}$ is again a subalgebra of $\mathfrak{A}$ (so is the order-closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}$). **P** (i) $0 \in \mathfrak{B} \subseteq \overline{\mathfrak{B}}$. (ii) The set $\{b : 1 \setminus b \in \overline{\mathfrak{B}}\}$ is order-closed (use 313A) and includes $\mathfrak{B}$, so includes $\overline{\mathfrak{B}}$; thus

$1 \setminus b \in \overline{\mathfrak{B}}$ for every $b \in \overline{\mathfrak{B}}$. (iii) If $c \in \mathfrak{B}$, the set $\{b : b \cup c \in \overline{\mathfrak{B}}\}$ is order-closed (use 313Bb) and includes $\mathfrak{B}$, so includes $\overline{\mathfrak{B}}$; thus $b \cup c \in \overline{\mathfrak{B}}$ whenever $b \in \overline{\mathfrak{B}}$ and $c \in \mathfrak{B}$. (iv) If $c \in \overline{\mathfrak{B}}$, the set $\{b : b \cup c \in \overline{\mathfrak{B}}\}$ is order-closed and includes $\mathfrak{B}$ (by (iii)), so includes $\overline{\mathfrak{B}}$; thus $b \cup c \in \overline{\mathfrak{B}}$ whenever $b, c \in \overline{\mathfrak{B}}$. (v) By 312B(ii), $\overline{\mathfrak{B}}$ is a subalgebra of $\mathfrak{A}$. **Q**

313G This is a convenient moment at which to spell out an abstract version of the Monotone Class Theorem (136B).

Lemma Let $\mathfrak{A}$ be a Boolean algebra.

(a) Suppose that $1 \in I \subseteq A \subseteq \mathfrak{A}$ and that

$$a \cap b \in I \text{ for all } a,\, b \in I,$$

$$b \setminus a \in A \text{ whenever } a,\, b \in A \text{ and } a \subseteq b.$$

Then A includes the subalgebra of $\mathfrak{A}$ generated by I.

(b) If moreover $\sup_{n \in \mathbb{N}} a_n \in A$ for every non-decreasing sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in A with a supremum in $\mathfrak{A}$, then A includes the σ-subalgebra of $\mathfrak{A}$ generated by I.

(c) And if $\sup C \in A$ whenever $C \subseteq A$ is an upwards-directed set with a supremum in $\mathfrak{A}$, then A includes the order-closed subalgebra of $\mathfrak{A}$ generated by I.

proof (a)(i) Let $\mathfrak{P}$ be the family of all sets J such that $I \subseteq J \subseteq A$ and $a \cap b \in J$ for all $a, b \in J$. Then $I \in \mathfrak{P}$ and if $\mathfrak{Q} \subseteq \mathfrak{P}$ is upwards-directed and not empty, $\bigcup \mathfrak{Q} \in \mathfrak{P}$. By Zorn's Lemma, $\mathfrak{P}$ has a maximal element $\mathfrak{B}$.

(ii) Now

$$\mathfrak{B} = \{c : c \in \mathfrak{A},\, c \cap b \in A \text{ for every } b \in \mathfrak{B}\}.$$

P If $c \in \mathfrak{B}$, then of course $c \cap b \in \mathfrak{B} \subseteq A$ for every $b \in \mathfrak{B}$, because $\mathfrak{B} \in \mathfrak{P}$. If $c \in \mathfrak{A} \setminus \mathfrak{B}$, consider

$$J = \mathfrak{B} \cup \{c \cap b : b \in \mathfrak{B}\}.$$

Then $c = c \cap 1 \in J$ so J properly includes $\mathfrak{B}$ and cannot belong to $\mathfrak{P}$. On the other hand, if $b_1, b_2 \in \mathfrak{B}$,

$$b_1 \cap b_2 \in \mathfrak{B} \subseteq J, \quad (c \cap b_1) \cap b_2 = b_1 \cap (c \cap b_2) = (c \cap b_1) \cap (c \cap b_2) = c \cap (b_1 \cap b_2) \in J,$$

so $c_1 \cap c_2 \in J$ for all $c_1, c_2 \in J$; and of course $I \subseteq \mathfrak{B} \subseteq J$. So J cannot be a subset of A, and there must be a $b \in \mathfrak{B}$ such that $c \cap b \notin A$. **Q**

(iii) Consequently $c \setminus b \in \mathfrak{B}$ whenever $b, c \in \mathfrak{B}$ and $b \subseteq c$. **P** If $a \in \mathfrak{B}$, then $b \cap a$, $c \cap a \in \mathfrak{B} \subseteq A$ and $b \cap a \subseteq c \cap a$, so

$$(c \setminus b) \cap a = (c \cap a) \setminus (b \cap a) \in A$$

by the hypothesis on A. By (ii), $c \setminus b \in \mathfrak{B}$. **Q**

(iv) It follows that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. **P** If $b \in \mathfrak{B}$, then

$$b \subseteq 1 \in I \subseteq \mathfrak{B},$$

so $1 \setminus b \in \mathfrak{B}$. If $a, b \in \mathfrak{B}$, then

$$a \cup b = 1 \setminus ((1 \setminus a) \cap (1 \setminus b)) \in \mathfrak{B}.$$

$0 = 1 \setminus 1 \in \mathfrak{B}$, so that the conditions of 312B(ii) are satisfied. **Q**

Now the subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in A, as required.

(b) Now suppose that $\sup_{n \in \mathbb{N}} a_n$ belongs to A whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in A with a supremum in $\mathfrak{A}$. Then $\mathfrak{B}$, as defined in part (a) of the proof, is a σ-subalgebra of $\mathfrak{A}$. **P** Let $\langle b_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{B}$ with a supremum c in $\mathfrak{A}$. Then for any $b \in \mathfrak{B}$, $\langle b_n \cap b \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in A with a supremum $c \cap b$ in $\mathfrak{A}$ (313Ba). So $c \cap b \in A$. As b is arbitrary, $c \in \mathfrak{B}$, by the criterion in (a-ii) above. As $\langle b_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mathfrak{B}$ is a σ-subalgebra, by 313Ea. **Q**

Accordingly the σ-subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in $\mathfrak{A}$.

(c) Finally, if $\sup C \in A$ whenever C is a non-empty upwards-directed subset of A with a least upper bound in $\mathfrak{A}$, $\mathfrak{B}$ is order-closed. **P** Let $C \subseteq \mathfrak{B}$ be a non-empty upwards-directed set with a supremum c in $\mathfrak{A}$. Then for any $b \in \mathfrak{B}$, $\{c \cap b : c \in C\}$ is a non-empty upwards-directed set in A with supremum $c \cap b$ in $\mathfrak{A}$. So $c \cap b \in A$. As b is arbitrary, $c \in \mathfrak{B}$. As C is arbitrary, $\mathfrak{B}$ is order-closed in $\mathfrak{A}$ (313Ea(i-α)). **Q**

Accordingly the order-closed subalgebra of $\mathfrak{A}$ generated by I is included in $\mathfrak{B}$ and therefore in $\mathfrak{A}$.

313H Definitions It is worth distinguishing various types of supremum- and infimum-preserving function. Once again, I do this in almost the widest possible context. Let P and Q be two partially ordered sets, and $\phi : P \to Q$ an **order-preserving** function, that is, a function such that $\phi(p) \le \phi(q)$ in Q whenever $p \le q$ in P.

(a) I say that ϕ is **order-continuous** if (i) $\phi(\sup A) = \sup_{p \in A} \phi(p)$ whenever A is a non-empty upwards-directed subset of P and $\sup A$ is defined in P (ii) $\phi(\inf A) = \inf_{p \in A} \phi(p)$ whenever A is a non-empty downwards-directed subset of P and $\inf A$ is defined in P.

(b) I say that ϕ is **sequentially order-continuous** or **σ-order-continuous** if (i) $\phi(p) = \sup_{n \in \mathbb{N}} \phi(p_n)$ whenever $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in P and $p = \sup_{n \in \mathbb{N}} p_n$ in P (ii) $\phi(p) = \inf_{n \in \mathbb{N}} \phi(p_n)$ whenever $\langle p_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in P and $p = \inf_{n \in \mathbb{N}} p_n$ in P.

Remark You may feel that one of the equivalent formulations in Proposition 313Lb gives a clearer idea of what is really being demanded of ϕ in the ordinary cases we shall be looking at.

313I Proposition Let P, Q and R be partially ordered sets, and $\phi : P \to Q$, $\psi : Q \to R$ order-preserving functions.

(a) $\psi\phi : P \to R$ is order-preserving.

(b) If ϕ and ψ are order-continuous, so is $\psi\phi$.

(c) If ϕ and ψ are sequentially order-continuous, so is $\psi\phi$.

(d) ϕ is order-continuous iff $\phi^{-1}[B]$ is order-closed for every order-closed $B \subseteq Q$.

proof (a)-(c) I think the only point that needs remarking is that if $A \subseteq P$ is upwards-directed, then $\phi[A] \subseteq Q$ is upwards-directed, because ϕ is order-preserving. So if $\sup A$ is defined in P and ϕ, ψ are order-continuous, we shall have

$$\psi(\phi(\sup A)) = \psi(\sup \phi[A]) = \sup \psi[\phi[A]].$$

Similarly, if $A \subseteq P$ is downwards-directed and has an infimum, then $\phi[A]$ is downwards-directed, and if ϕ and ψ are order-continuous then

$$\psi(\phi(\inf A)) = \psi(\inf \phi[A]) = \inf \psi[\phi[A]].$$

For sequential order-continuity we argue in the same way but with sequences.

(d)(i) Suppose that ϕ is order-continuous and that $B \subseteq Q$ is order-closed. Let $A \subseteq \phi^{-1}[B]$ be a non-empty upwards-directed set with supremum $p \in P$. Then $\phi[A] \subseteq B$ is non-empty and upwards-directed, because ϕ is order-preserving, and $\phi(p) = \sup \phi[A]$ because ϕ is order-continuous. Because B is order-closed, $\phi(p) \in B$ and $p \in \phi^{-1}[B]$. Similarly, if $A \subseteq \phi^{-1}[B]$ is non-empty and downwards-directed, and $\inf A$ is defined in P, then $\phi(\inf A) = \inf \phi[A] \in B$ and $\inf A \in \phi^{-1}[B]$. Thus $\phi^{-1}[B]$ is order-closed; as B is arbitrary, ϕ satisfies the condition.

(ii) Now suppose that $\phi^{-1}[B]$ is order-closed in P whenever $B \subseteq Q$ is order-closed in Q. Let $A \subseteq P$ be a non-empty upwards-directed subset of P with a supremum $p \in P$. Then $\phi(p)$ is an upper bound of $\phi[A]$. Let q be any upper bound of $\phi[A]$ in Q. Consider $B = \{r : r \le q\}$; then $B \subseteq Q$ is upwards-directed and order-closed, so $\phi^{-1}[B]$ is order-closed. Also $A \subseteq \phi^{-1}[B]$ is non-empty and upwards-directed and has supremum p, so $p \in \phi^{-1}[B]$ and $\phi(p) \in B$, that is, $\phi(p) \le q$. As q is arbitrary, $\phi(p) = \sup \phi[A]$. Similarly, $\phi(\inf A) = \inf \phi[A]$ whenever $A \subseteq P$ is non-empty, downwards-directed and has an infimum in P; so ϕ is order-continuous.

313J It is useful to introduce here the following notion.

Definition Let $\mathfrak{A}$ be a Boolean algebra. A set $D \subseteq \mathfrak{A}$ is **order-dense** if for every non-zero $a \in \mathfrak{A}$ there is a non-zero $d \in D$ such that $d \subseteq a$.

Remark Many authors use the simple word 'dense' where I have insisted on the phrase 'order-dense'. In the work of this treatise it will be important to distinguish clearly between this concept of 'dense' set and the topological concept (2A3U).

313K Lemma If $\mathfrak{A}$ is a Boolean algebra and $D \subseteq \mathfrak{A}$ is order-dense, then for any $a \in \mathfrak{A}$ there is a disjoint $C \subseteq D$ such that $\sup C = a$; in particular, $a = \sup\{d : d \in D,\, d \subseteq a\}$ and there is a partition of unity $C \subseteq D$.

proof Set $D_a = \{d : d \in D,\, d \subseteq a\}$. Applying Zorn's lemma to the family $\mathcal{C}$ of disjoint sets $C \subseteq D_a$, we have a maximal $C \in \mathcal{C}$. Now if $b \in \mathfrak{A}$ and $b \not\supseteq a$, there is a $d \in D$ such that $0 \neq d \subseteq a \setminus b$. Because C is maximal, there must be a $c \in C$ such that $c \cap d \neq 0$, so that $c \not\subseteq b$. Turning this round, any upper bound of C must include a, so that $a = \sup C$. It follows at once that $a = \sup D_a$.

Taking $a = 1$ we obtain a partition of unity included in D.

313L Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a) π is order-preserving.

(b) The following are equiveridical:

(i) π is order-continuous;

(ii) whenever $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and $\inf A = 0$ in $\mathfrak{A}$, then $\inf \pi[A] = 0$ in $\mathfrak{B}$;

(iii) whenever $A \subseteq \mathfrak{A}$ is non-empty and upwards-directed and $\sup A = 1$ in $\mathfrak{A}$, then $\sup \pi[A] = 1$ in $\mathfrak{B}$;

(iv) whenever $A \subseteq \mathfrak{A}$ and $\sup A$ is defined in $\mathfrak{A}$, then $\pi(\sup A) = \sup \pi[A]$ in $\mathfrak{B}$;

(v) whenever $A \subseteq \mathfrak{A}$ and $\inf A$ is defined in $\mathfrak{A}$, then $\pi(\inf A) = \inf \pi[A]$ in $\mathfrak{B}$;

(vi) whenever $C \subseteq \mathfrak{A}$ is a partition of unity, then $\pi[C]$ is a partition of unity in $\mathfrak{B}$.

(c) The following are equiveridical:

(i) π is sequentially order-continuous;

(ii) whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ and $\inf_{n\in\mathbb{N}} a_n = 0$ in $\mathfrak{A}$, then $\inf_{n\in\mathbb{N}} \pi a_n = 0$ in $\mathfrak{B}$;

(iii) whenever $A \subseteq \mathfrak{A}$ is countable and $\sup A$ is defined in $\mathfrak{A}$, then $\pi(\sup A) = \sup \pi[A]$ in $\mathfrak{B}$;

(iv) whenever $A \subseteq \mathfrak{A}$ is countable and $\inf A$ is defined in $\mathfrak{A}$, then $\pi(\inf A) = \inf \pi[A]$ in $\mathfrak{B}$;

(v) whenever $C \subseteq \mathfrak{A}$ is a countable partition of unity, then $\pi[C]$ is a partition of unity in $\mathfrak{B}$.

(d) If π is bijective, it is order-continuous.

proof (a) This is 312I.

(b)(i)$\Rightarrow$(ii) is trivial, as $\pi 0 = 0$.

(ii)$\Rightarrow$(iv) Assume (ii), and let A be any subset of $\mathfrak{A}$ such that $c = \sup A$ is defined in $\mathfrak{A}$. If $A = \emptyset$, then $c = 0$ and $\sup \pi[A] = 0 = \pi c$. Otherwise, set

$$A' = \{a_0 \cup \ldots \cup a_n : a_0, \ldots, a_n \in A\}, \quad C = \{c \setminus a : a \in A'\}.$$

Then A' is upwards-directed and has the same upper bounds as A, so $c = \sup A'$ and $0 = \inf C$, by 313Aa. Also C is downwards-directed, so $\inf \pi[C] = 0$ in $\mathfrak{B}$. But now

$$\pi[C] = \{\pi c \setminus \pi a : a \in A'\} = \{\pi c \setminus b : b \in \pi[A']\},$$

$$\pi[A'] = \{\pi a_0 \cup \ldots \cup \pi a_n : a_0, \ldots, a_n \in A\} = \{b_0 \cup \ldots \cup b_n : b_0, \ldots, b_n \in \pi[A]\},$$

because π is a Boolean homomorphism. Again using 313Aa and the fact that $b \subseteq \pi c$ for every $b \in \pi[A']$, we get

$$\pi c = \sup \pi[A'] = \sup \pi[A].$$

As A is arbitrary, (iv) is satisfied.

(iv)$\Rightarrow$(v) If $A \subseteq \mathfrak{A}$ and $c = \inf A$ is defined in $\mathfrak{A}$, then $1 \setminus c = \sup_{a\in A} 1 \setminus a$, so

$$\pi c = 1 \setminus \pi(1 \setminus c) = 1 \setminus \sup_{a\in A} \pi(1 \setminus a) = \inf_{a\in A} 1 \setminus \pi(1 \setminus a) = \inf_{a\in A} \pi a.$$

(v)⇒(ii) is trivial, because $\pi 0 = 0$.

(iv)⇒(iii) is similarly trivial.

(iii)⇒(vi) Assume (iii), and let C be a partition of unity in $\mathfrak{A}$. Then $C' = \{c_0 \cup \ldots \cup c_n : c_0, \ldots , c_n \in C\}$ is upwards-directed and has supremum 1, so $\sup \pi[C'] = 1$. But (because π is a Boolean homomorphism) $\pi[C]$ and $\pi[C']$ have the same upper bounds, so $\sup \pi[C] = 1$, as required.

(vi)⇒(ii) Assume (vi), and let $A \subseteq \mathfrak{A}$ be a set with infimum 0. Set

$$D = \{d : d \in \mathfrak{A},\ \exists\, a \in A,\ d \cap a = 0\}.$$

Then D is order-dense in $\mathfrak{A}$. **P** If $e \in \mathfrak{A} \setminus \{0\}$, then there is an $a \in A$ such that $e \not\subseteq a$, so that $e \setminus a$ is a non-zero member of D included in e. **Q** Consequently there is a partition of unity $C \subseteq D$, by 313K. But now if b is any lower bound for $\pi[A]$ in $\mathfrak{B}$, we must have $b \cap \pi d = 0$ for every $d \in D$, so $\pi c \subseteq 1 \setminus b$ for every $c \in C$, and $1 \setminus b = 1$, $b = 0$. Thus $\inf \pi[A] = 0$. As A is arbitrary, (ii) is satisfied.

(v)&(iv)⇒(i) is trivial.

(c) We can use nearly identical arguments, remembering only to interpolate the word 'countable' from time to time. I spell out the new version of (ii)⇒(iv), even though it requires no more than an adaptation of the language. Assume (ii), and let A be a countable subset of $\mathfrak{A}$ with a supremum $c \in \mathfrak{A}$. If $A = \emptyset$, then $c = 0$ so $\pi c = 0 = \sup \pi[A]$. Otherwise, let $\langle a_n \rangle_{n\in\mathbb{N}}$ be a sequence running over A; set $a'_n = a_0 \cup \ldots \cup a_n$ and $c_n = c \setminus a'_n$ for each n. Then $\langle a'_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, with supremum c, and $\langle c_n \rangle_{n\in\mathbb{N}}$ is non-increasing, with infimum 0; so $\inf_{n\in\mathbb{N}} \pi c_n = 0$ and

$$\sup_{n\in\mathbb{N}} \pi a_n = \sup_{n\in\mathbb{N}} \pi a'_n = \pi c.$$

For (v)⇒(ii), however, a different idea is involved. Assume (v), and suppose that $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. Set $c_0 = 1 \setminus a_0$, $c_n = a_{n-1} \setminus a_n$ for $n \ge 1$; then $C = \{c_n : n \in \mathbb{N}\}$ is a partition of unity in $\mathfrak{A}$ (because if $c \cap c_n = 0$ for every n, then $c \subseteq a_n$ for every n), so $\pi[C]$ is a partition of unity in $\mathfrak{B}$. Now if $b \subseteq \pi a_n$ for every n, $b \cap \pi c_n$ for every n, so $b = 0$; thus $\inf_{n\in\mathbb{N}} \pi a_n = 0$. As $\langle a_n \rangle_{n\in\mathbb{N}}$ is arbitrary, (ii) is satisfied.

(d) Suppose that $A \subseteq \mathfrak{A}$ is non-empty and $\inf A = 0$ in $\mathfrak{A}$. Let $b \in \mathfrak{B}$ be a lower bound for $\pi[A]$. Because π is surjective, there is a $c \in \mathfrak{A}$ such that $\pi c = b$. If $a \in A$, then

$$\pi(a \cap c) = \pi a \cap \pi c = \pi a \cap b = b = \pi c;$$

because π is injective, $a \cap c = c$ and $c \subseteq a$. As a is arbitrary, c is a lower bound of A and must be 0; so $b_0 = \pi 0 = 0$. As b is arbitrary, $\inf \pi[A] = 0$; as A is arbitrary, π is order-continuous, by (b)(ii)⇒(i).

313M The following result is perfectly elementary, but it will save a moment later on to have it spelt out.

Lemma Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism.

(a) If $\mathfrak{D}$ is an order-closed subalgebra of $\mathfrak{B}$, then $\pi^{-1}[\mathfrak{D}]$ is an order-closed subalgebra of $\mathfrak{A}$.

(b) If $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by $C \subseteq \mathfrak{A}$, then the order-closed subalgebra $\mathfrak{D}$ of $\mathfrak{B}$ generated by $\pi[C]$ includes $\pi[\mathfrak{C}]$.

(c) Now suppose that π is surjective and that $C \subseteq \mathfrak{A}$ is such that the order-closed subalgebra of $\mathfrak{A}$ generated by C is $\mathfrak{A}$ itself. Then the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$ is $\mathfrak{B}$.

proof (a) Setting $\mathfrak{C} = \pi^{-1}[\mathfrak{D}]$: if a, $a' \in \mathfrak{C}$ then $\pi(a \cap b) = \pi a \cap \pi b$, $\pi(a \triangle a') = \pi a \triangle \pi a' \in \mathfrak{D}$, so $a \cap a'$, $a \triangle a' \in \mathfrak{C}$; $\pi 1 = 1 \in \mathfrak{D}$ so $1 \in \mathfrak{C}$; thus $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$. By 313Id, $\mathfrak{C}$ is order-closed.

(b) By (a), $\pi^{-1}[\mathfrak{D}]$ is an order-closed subalgebra of $\mathfrak{A}$. It includes C so includes $\mathfrak{C}$, and $\pi[\mathfrak{C}] \subseteq \mathfrak{D}$.

(c) In the language of (b), we have $\mathfrak{C} = \mathfrak{A}$, so $\mathfrak{D}$ must be $\mathfrak{B}$.

313N Definition The phrase **regular embedding** is sometimes used to mean an injective order-continuous Boolean homomorphism; a subalgebra $\mathfrak{B}$ of a Boolean algebra $\mathfrak{A}$ is said to be **regularly embedded** in $\mathfrak{A}$ if the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is order-continuous, that is, if whenever $b \in \mathfrak{B}$ is the supremum (in $\mathfrak{B}$) of $B \subseteq \mathfrak{B}$, then b is also the supremum in $\mathfrak{A}$ of B; and similarly for infima. One important case is when $\mathfrak{B}$ is order-dense (313O); another is in 314Ga below.

It will be useful to be able to say '$\mathfrak{B}$ can be regularly embedded in $\mathfrak{A}$' to mean that there is an injective order-continuous Boolean homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$; that is, that $\mathfrak{B}$ is isomorphic to a regularly embedded subalgebra of $\mathfrak{A}$. In this form it is obvious from 313Ib that if $\mathfrak{C}$ can be regularly embedded in $\mathfrak{B}$, and $\mathfrak{B}$ can be regularly embedded in $\mathfrak{A}$, then $\mathfrak{C}$ can be regularly embedded in $\mathfrak{A}$.

313O Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Then $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$. In particular, if $B \subseteq \mathfrak{B}$ and $c \in \mathfrak{B}$ then $c = \sup B$ in $\mathfrak{B}$ iff $c = \sup B$ in $\mathfrak{A}$.

proof I have to show that the identity homomorphism $\iota : \mathfrak{B} \to \mathfrak{A}$ is order-continuous. **?** Suppose, if possible, otherwise. By 313L(b-ii), there is a non-empty set $B \subseteq \mathfrak{B}$ such that $\inf B = 0$ in $\mathfrak{B}$ but $B = \iota[B]$ has a non-zero lower bound $a \in \mathfrak{A}$. In this case, however (because $\mathfrak{B}$ is order-dense) there is a non-zero $d \in \mathfrak{B}$ with $d \subseteq a$, in which case d is a non-zero lower bound for B in $\mathfrak{B}$. **X**

313P The most important use of these ideas to us concerns quotient algebras (313Q); I approach by means of a superficially more general result.

Theorem Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism with kernel I.

(a)(i) If π is order-continuous then I is order-closed.

(ii) If $\pi[\mathfrak{A}]$ is regularly embedded in $\mathfrak{B}$ and I is order-closed then π is order-continuous.

(b)(i) If π is sequentially order-continuous then I is a σ-ideal.

(ii) If $\pi[\mathfrak{A}]$ is regularly embedded in $\mathfrak{B}$ and I is a σ-ideal then π is sequentially order-continuous.

proof (a)(i) If $A \subseteq I$ is upwards-directed and has a supremum $c \in \mathfrak{A}$, then $\pi c = \sup \pi[A] = 0$, so $c \in I$. As remarked in 313Eb, this shows that I is order-closed.

(ii) We are supposing that the identity map from $\pi[\mathfrak{A}]$ to $\mathfrak{B}$ is order-continuous, so it will be enough to show that π is order-continuous when regarded as a map from $\mathfrak{A}$ to $\pi[\mathfrak{A}]$. Suppose that $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed and that $\inf A = 0$. **?** Suppose, if possible, that 0 is not the greatest lower bound of $\pi[A]$ in $\pi[\mathfrak{A}]$. Then there is a $c \in \mathfrak{A}$ such that $0 \neq \pi c \subseteq \pi a$ for every $a \in A$. Now

$$\pi(c \setminus a) = \pi c \setminus \pi a = 0$$

for every $a \in A$, so $c \setminus a \in I$ for every $a \in A$. The set $C = \{c \setminus a : a \in A\}$ is upwards-directed and has supremum c; because I is order-closed, $c = \sup C \in I$, and $\pi c = 0$, contradicting the specification of c. **X** Thus $\inf \pi[A] = 0$ in either $\pi[\mathfrak{A}]$ or $\mathfrak{B}$. As A is arbitrary, π is order-continuous, by the criterion (ii) of 313Lb.

(b) Argue in the same way, replacing each set A by a sequence.

313Q Corollary Let $\mathfrak{A}$ be a Boolean algebra and I an ideal of $\mathfrak{A}$; write π for the canonical map from $\mathfrak{A}$ to $\mathfrak{A}/I$.

(a) π is order-continuous iff I is order-closed.

(b) π is sequentially order-continuous iff I is a σ-ideal.

proof $\pi[\mathfrak{A}] = \mathfrak{A}/I$ is surely regularly embedded in $\mathfrak{A}/I$.

313R For order-continuous homomorphisms, at least, there is an elegant characterization in terms of Stone spaces.

Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism. Let Z and W be their Stone spaces, and $\phi : W \to Z$ the corresponding continuous function (312Q). Then the following are equiveridical:

(i) π is order-continuous;

(ii) $\phi^{-1}[M]$ is nowhere dense in W for every nowhere dense set $M \subseteq Z$;

(iii) $\operatorname{int} \phi[H] \neq \emptyset$ for every non-empty open set $H \subseteq W$.

proof (a)(i)⇒(iii) Suppose that π is order-continuous. **?** Suppose, if possible, that $H \subseteq W$ is a non-empty open set and $\operatorname{int} \phi[H] = \emptyset$. Let $b \in \mathfrak{B} \setminus \{0\}$ be such that $\widehat{b} \subseteq H$. Then $\phi[\widehat{b}]$ has empty interior; but also it is a closed set, so its complement is dense. Set $A=\{a:a\in\mathfrak{A},\,\widehat{a}\cap\phi[\widehat{b}]=\emptyset\}$. Then $\bigcup_{a\in A}\widehat{a}=Z\setminus\phi[\widehat{b}]$ is a dense open set, so $\sup A=1$ in $\mathfrak{A}$ (313Ca). Because π is order-continuous, $\sup\pi[A]=1$ in $\mathfrak{B}$ (313L(b-iii)), and there is an $a\in A$ such that $\pi a\cap b\ne 0$. But this means that $\widehat{b}\cap\phi^{-1}[\widehat{a}]\ne\emptyset$ and $\phi[\widehat{b}]\cap\widehat{a}\ne\emptyset$, contrary to the definition of A. **X**

Thus there is no such set H, and (iii) is true.

(b)(iii)⇒(ii) Now assume (iii). If $M\subseteq Z$ is nowhere dense, set $N=\phi^{-1}[\overline{M}]$, so that $N\subseteq W$ is a closed set. If $H=\operatorname{int}N$, then $\operatorname{int}\phi[H]\subseteq\operatorname{int}\overline{M}=\emptyset$, so (iii) tells us that H is empty; thus N and $\phi^{-1}[M]$ are nowhere dense, as required by (ii).

(c)(ii)⇒(i) Assume (ii), and let $A\subseteq\mathfrak{A}$ be a non-empty set such that $\inf A=0$ in $\mathfrak{A}$. Then $M=\bigcap_{a\in A}\widehat{a}$ has empty interior in Z (313Cb), so (being closed) is nowhere dense, and $\phi^{-1}[M]$ also is nowhere dense. If $b\in\mathfrak{B}\setminus\{0\}$, then

$$\widehat{b}\not\subseteq\phi^{-1}[M]=\bigcap_{a\in A}\phi^{-1}[\widehat{a}]=\bigcap_{a\in A}\widehat{\pi a},$$

so b is not a lower bound for $\pi[A]$. This shows that $\inf\pi[A]=0$ in $\mathfrak{B}$. As A is arbitrary, π is order-continuous (313L(b-ii)).

313S Upper envelopes(a) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$. For $a\in\mathfrak{A}$, the **upper envelope** of a in $\mathfrak{C}$, or **projection** of a on $\mathfrak{C}$, is

$$\operatorname{upr}(a,\mathfrak{C})=\inf\{c:c\in\mathfrak{C},\,a\subseteq c\}$$

if the infimum is defined in $\mathfrak{C}$.

Remark Note that the infima here are to be taken in the subalgebra, so that $\operatorname{upr}(a,\mathfrak{C})$ will always belong to $\mathfrak{C}$. In the great majority of elementary applications, $\mathfrak{C}$ will be order-closed in $\mathfrak{A}$, so that we do not need to distinguish between infima in $\mathfrak{C}$ and infima in $\mathfrak{A}$. But see 313Yh.

(b) If $A\subseteq\mathfrak{A}$ is such that $\operatorname{upr}(a,\mathfrak{C})$ is defined for every $a\in A$, $a_0=\sup A$ is defined in $\mathfrak{A}$ and $c_0=\sup_{a\in A}\operatorname{upr}(a,\mathfrak{C})$ is defined in $\mathfrak{C}$, then $c_0=\operatorname{upr}(a_0,\mathfrak{C})$. **P** If $c\in\mathfrak{C}$ then

$$\begin{aligned}c_0\subseteq c&\iff\operatorname{upr}(a,\mathfrak{C})\subseteq c\text{ for every }a\in A\\&\iff a\subseteq c\text{ for every }a\in A\iff a_0\subseteq c.\ \mathbf{Q}\end{aligned}$$

In particular, $\operatorname{upr}(a\cup a',\mathfrak{C})=\operatorname{upr}(a,\mathfrak{C})\cup\operatorname{upr}(a',\mathfrak{C})$ whenever the right-hand side is defined.

(c) If $a\in\mathfrak{A}$ is such that $\operatorname{upr}(a,\mathfrak{C})$ is defined, then $\operatorname{upr}(a\cap c,\mathfrak{C})=c\cap\operatorname{upr}(a,\mathfrak{C})$ for every $c\in\mathfrak{C}$. **P** For $c'\in\mathfrak{C}$,

$$\begin{aligned}a\cap c\subseteq c'&\iff a\subseteq c'\cup(1\setminus c)\\&\iff\operatorname{upr}(a,\mathfrak{C})\subseteq c'\cup(1\setminus c)\iff c\cap\operatorname{upr}(a,\mathfrak{C})\subseteq c'.\ \mathbf{Q}\end{aligned}$$

313X Basic exercises (a) Use 313C to give alternative proofs of 313A and 313B.

(b) Let P be a partially ordered set. Show that there is a topology on P for which the closed sets are just the order-closed sets.

(c) Let P be a partially ordered set, $Q\subseteq P$ an order-closed set, and R a subset of Q which is order-closed in Q when Q is given the partial ordering induced by that of P. Show that R is order-closed in P.

>(d) Let $\mathfrak{A}$ be a Boolean algebra. Suppose that $1\in I\subseteq\mathfrak{A}$ and that $a\cap b\in I$ for all a, $b\in I$. (i) Let $\mathfrak{B}$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I\subseteq A$ and $b\setminus a\in A$ whenever a, $b\in A$ and $a\subseteq b$. Show that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$. (ii) Let $\mathfrak{B}_\sigma$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I\subseteq A$, $b\setminus a\in A$ whenever a, $b\in A$ and $a\subseteq b$ and $\sup_{n\in\mathbb{N}}b_n\in A$ whenever $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in A with a supremum in $\mathfrak{A}$. Show that $\mathfrak{B}_\sigma$ is a σ-subalgebra of $\mathfrak{A}$. (iii) Let $\mathfrak{B}_\tau$ be the intersection of all those subsets A of $\mathfrak{A}$ such that $I\subseteq A$, $b\setminus a\in A$ whenever a, $b\in A$ and $a\subseteq b$ and $\sup B\in A$ whenever B is a non-empty upwards-directed subset of A with a supremum in $\mathfrak{A}$. Show that $\mathfrak{B}_\tau$ is an order-closed subalgebra of $\mathfrak{A}$. (iv) Hence give a proof of 313G not relying on Zorn's Lemma or any other use of the axiom of choice.

(e) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Let $\mathfrak{B}_\sigma$ be the smallest sequentially order-closed subset of $\mathfrak{A}$ including $\mathfrak{B}$. Show that $\mathfrak{B}_\sigma$ is a subalgebra of $\mathfrak{A}$.

>(f) Let X be a set, and $\mathcal{A}$ a subset of $\mathcal{P}X$. Show that $\mathcal{A}$ is an order-closed subalgebra of $\mathcal{P}X$ iff it is of the form $\{f^{-1}[F] : F \subseteq Y\}$ for some set Y and function $f : X \to Y$.

(g) Let P and Q be partially ordered sets, and $\phi : P \to Q$ an order-preserving function. Show that ϕ is sequentially order-continuous iff $\phi^{-1}[C]$ is sequentially order-closed in $\mathfrak{A}$ for every sequentially order-closed $C \subseteq \mathfrak{B}$.

(h) For partially ordered sets P and Q, let us call a function $\phi : P \to Q$ **monotonic** if it is *either* order-preserving *or* order-reversing. State and prove definitions and results corresponding to 313H, 313I and 313Xg for general monotonic functions.

>(i) Let $\mathfrak{A}$ be a Boolean algebra. Show that the operations $(a,b) \mapsto a \cup b$ and $(a,b) \mapsto a \cap b$ are order-continuous operations from $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{A}$, if we give $\mathfrak{A} \times \mathfrak{A}$ the product partial order, saying that $(a,b) \le (a',b')$ iff $a \subseteq a'$ and $b \subseteq b'$.

(j) Let $\mathfrak{A}$ be a Boolean algebra. Show that if a subalgebra of $\mathfrak{A}$ is order-dense then it is dense in the topology of 313Xb.

>(k) Let $\mathfrak{A}$ be a Boolean algebra and $A \subseteq \mathfrak{A}$ any disjoint set. Show that there is a partition of unity in $\mathfrak{A}$ including A.

>(l) Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras and π_1, $\pi_2 : \mathfrak{A} \to \mathfrak{B}$ two order-continuous Boolean homomorphisms. Show that $\{a : \pi_1 a = \pi_2 a\}$ is an order-closed subalgebra of $\mathfrak{A}$.

(m) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and π_1, $\pi_2 : \mathfrak{A} \to \mathfrak{B}$ two Boolean homomorphisms. Suppose that π_1 and π_2 agree on some order-dense subset of $\mathfrak{A}$, and that one of them is order-continuous. Show that they are equal. (*Hint*: if π_1 is order-continuous, $\pi_2 a \supseteq \pi_1 a$ for every a.)

(n) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\mathfrak{A}_0$ an order-dense subalgebra of $\mathfrak{A}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism. Show that π is order-continuous iff $\pi{\restriction}\mathfrak{A}_0 : \mathfrak{A}_0 \to \mathfrak{B}$ is order-continuous.

(o) Let $\mathfrak{A}$ be a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism with fixed-point subalgebra $\mathfrak{C}$ (312K). (i) Show that if π is sequentially order-continuous then $\mathfrak{C}$ is a σ-subalgebra of $\mathfrak{A}$. (ii) Show that if π is order-continuous then $\mathfrak{C}$ is order-closed.

>(p) Let $\mathfrak{A}$ be a Boolean algebra. For $A \subseteq \mathfrak{A}$ set $A^\perp = \{b : a \cap b = 0 \ \forall\, a \in A\}$. (i) Show that $A^\perp$ is an order-closed ideal of $\mathfrak{A}$. (ii) Show that a set $A \subseteq \mathfrak{A}$ is an order-closed ideal of $\mathfrak{A}$ iff $A = A^{\perp\perp}$. (iii) Show that if $I \subseteq \mathfrak{A}$ is an order-closed ideal then $\{a^\bullet : a \in I^\perp\}$ is an order-dense ideal in the quotient algebra $\mathfrak{A}/I$.

(q) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W; let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a Boolean homomorphism, and $\phi : W \to Z$ the corresponding continuous function. Show that the following are equiveridical: (i) π is order-continuous; (ii) $\operatorname{int} \phi^{-1}[F] = \phi^{-1}[\operatorname{int} F]$ for every closed $F \subseteq Z$ (iii) $\overline{\phi^{-1}[G]} = \phi^{-1}[\overline{G}]$ for every open $G \subseteq Z$.

(r) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\pi : \mathfrak{A} \to \mathfrak{B}$ an injective Boolean homomorphism and $\mathfrak{C}$ a Boolean subalgebra of $\mathfrak{A}$. Suppose that $a \in \mathfrak{A}$ is such that $c = \operatorname{upr}(a, \mathfrak{C})$ is defined. Show that $\operatorname{upr}(\pi a, \pi[\mathfrak{C}])$ is defined and equal to πc.

(s) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism and D an order-dense subset of $\mathfrak{A}$ containing 0. Show that π is injective iff $\pi{\restriction}D$ is injective.

(t) Let $\mathfrak{A}$ be a Boolean algebra and $A_0, \dots, A_n$ subsets of $\mathfrak{A}$ such that $\sup A_i$ is defined for each $i \le n$. Set $B = \{a_0 \cap \dots \cap a_n : a_i \in A_i \text{ for each } i\}$. Show that $\sup B$ is defined and equal to $\inf_{i<n} \sup A_i$.

313Y Further exercises (a) Prove 313A-313C for general Boolean rings.

(b) Let P be any partially ordered set, and let $\mathfrak{T}$ be the topology of 313Xb. (i) Show that a sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P is $\mathfrak{T}$-convergent to $p\in P$ iff every subsequence of $\langle p_n\rangle_{n\in\mathbb{N}}$ has a monotonic sub-subsequence with supremum or infimum equal to p. (ii) Show that a subset A of P is sequentially order-closed, in the sense of 313Db, iff the $\mathfrak{T}$-limit of any $\mathfrak{T}$-convergent sequence in A belongs to A. (iii) Suppose that A is an upwards-directed subset of P with supremum $p_0\in P$. For $a\in A$ set $F_a=\{p:a\le p\in A\}$, and let $\mathcal{F}$ be the filter on P generated by $\{F_a:a\in A\}$. Show that $\mathcal{F}\to p_0$ for $\mathfrak{T}$. (iv) Show that if Q is another partially ordered set, endowed with a topology $\mathfrak{S}$ in the same way, then a monotonic function $\phi:P\to Q$ is order-continuous iff it is continuous for the topologies $\mathfrak{T}$ and $\mathfrak{S}$, and is sequentially order-continuous iff it is sequentially continuous for these topologies.

(c) Let U be a Banach lattice (242G, 354Ab). Show that its norm is order-continuous in the sense of 242Yg and 354Dc iff its restriction to $\{u:u\ge 0\}$ is order-continuous in the sense of 313Ha.

(d) Let P and Q be lattices, and $f:P\to Q$ a bijective lattice homomorphism. Show that f is order-continuous.

(e) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, with Stone spaces Z and W, and $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism, with associated continuous function $\phi:W\to Z$. Show that π is sequentially order-continuous iff $\phi^{-1}[M]$ is nowhere dense for every nowhere dense zero set $M\subseteq Z$.

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with Stone spaces Z and W respectively, $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism and $\phi:W\to Z$ the corresponding continuous function. Show that $\pi[\mathfrak{A}]$ is order-dense in $\mathfrak{B}$ iff ϕ is **irreducible**, that is, $\phi[F]\ne\phi[W]$ for any proper closed subset F of W.

(g) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with Stone spaces Z and W respectively, $\pi:\mathfrak{A}\to\mathfrak{B}$ a Boolean homomorphism and $\phi:W\to Z$ the corresponding continuous function. Show that the following are equiveridical: (i) π is injective and order-continuous; (ii) for $M\subseteq Z$, M is nowhere dense iff $\phi^{-1}[M]$ is nowhere dense.

(h) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{C}$ a Boolean subalgebra of $\mathfrak{A}$. Let $\mathcal{I}$ be the set of those $a\in\mathfrak{A}$ such that the upper envelope $\operatorname{upr}(a,\mathfrak{C})$ is zero. (i) Show that $\mathcal{I}$ is an ideal in $\mathfrak{A}$. (ii) Show that $\mathfrak{C}$ is regularly embedded in $\mathfrak{A}$ iff $\mathcal{I}=\{0\}$. (iii) Let $\pi:\mathfrak{A}\to\mathfrak{A}/\mathcal{I}$ be the canonical map. Show that $\pi\restriction\mathfrak{C}$ is injective and order-continuous.

313 Notes and comments I give 'elementary' proofs of 313A-313B because I believe that they help to exhibit the relevant aspects of the structure of Boolean algebras; but various abbreviations are possible, notably if we allow ourselves to use the Stone representation (313Xa). 313A and 313Ba-b can be expressed by saying that the Boolean operations $\cup$, $\cap$ and $\setminus$ are (separately) order-continuous. Of course, $\setminus$ is order-reversing, rather than order-preserving, in the second variable; but the natural symmetry in the concept of partial order means that the ideas behind 313H-313I can be applied equally well to order-reversing functions (313Xh). In fact, $\cup$ and $\cap$ can be regarded as order-continuous functions on the product space (313Bc-d, 313Xi). Clearly 313Bc-d can be extended into forms valid for any finite sequence $A_0,\dots,A_n$ of subsets of $\mathfrak{A}$ in place of A, B. But if we seek to go to infinitely many subsets of $\mathfrak{A}$ we find ourselves saying something new; see 316G-316J below.

Proposition 313C, and its companions 313R, 313Xq and 313Ye, are worth studying not only as a useful technique, but also in order to understand the difference between $\sup A$, where A is a set in a Boolean algebra, and $\bigcup\mathcal{A}$, where $\mathcal{A}$ is a family of sets. Somehow $\sup A$ can be larger, and $\inf A$ smaller, than one's first intuition might suggest, corresponding to the fact that not every subset of the Stone space corresponds to an element of the Boolean algebra.

I should like to use the words 'order-closed' and 'sequentially order-closed' to mean closed, or sequentially closed, for some more or less canonical topology. The difficulty is that while a great many topologies can be defined from a partial order (one is described in 313Xb and 313Yb, and another in 367Yb and 393L), none of them has such pre-eminence that it can be called 'the' order-topology, except in the very special context of totally ordered spaces (see 4A2R in Volume 4). Accordingly there is a degree of arbitrariness in the language I use here. Nevertheless (sequentially) order-closed subalgebras and ideals are of such importance that they seem to deserve a concise denotation. The same remarks apply to (sequential) order-continuity. Concerning the term 'order-dense' in 313J, this has little to do with density in any topological sense, but the word 'dense', at least, is established in this context.

With all these definitions, there is a good deal of scope for possible interrelations. The most important to us is 313Q, which will be used repeatedly (typically, with $\mathfrak{A}$ an algebra of sets), but I think it is worth having the expanded version in 313P available.

I take the opportunity to present an abstract form of an important lemma on σ-algebras generated by families closed under $\cap$ (136B, 313Gb). This time round I use the Zorn's Lemma argument in the text and suggest the alternative, 'elementary' method in the exercises (313Xd). The two methods are opposing extremes in the sense that the Zorn's Lemma argument looks for maximal subalgebras included in A (which are not unique, and have to be picked out using the axiom of choice) and the other approach seeks minimal subalgebras including I (which are uniquely defined, and can be described without the axiom of choice).

Note that the concept of 'order-closed algebra of sets' is not particularly useful; there are too few order-closed subalgebras of $\mathcal{P}X$ and they are of too simple a form (313Xf). It is in abstract Boolean algebras that the idea becomes important. In many of the most important partially ordered sets of measure theory, the sequentially order-closed sets are the same as the order-closed sets (see, for instance, 316Fb below), and most of the important order-closed subalgebras dealt with in this chapter can be thought of as σ-subalgebras which are order-closed because they happen to lie in the right kind of algebra.

Version of 26.7.07

314 Order-completeness

The results of §313 are valid in all Boolean algebras, but of course are of most value when many suprema and infima exist. I now set out the most useful definitions which guarantee the existence of suprema and infima (314A) and work through their elementary relationships with the concepts introduced so far (314C-314J). I then embark on the principal theorems concerning order-complete Boolean algebras: the extension theorem for homomorphisms to a Dedekind complete algebra (314K), the Loomis-Sikorski representation of a Dedekind σ-complete algebra as a quotient of a σ-algebra of sets (314M), the characterization of Dedekind complete algebras in terms of their Stone spaces (314S), and the idea of 'Dedekind completion' of a Boolean algebra (314T-314U). On the way I describe 'regular open algebras' (314O-314Q).

314A Definitions Let P be a partially ordered set.

(a) P is **Dedekind complete**, or **order-complete**, or **conditionally complete** if every non-empty subset of P with an upper bound has a least upper bound.

(b) P is **Dedekind σ-complete**, or **σ-order-complete**, if (i) every countable non-empty subset of P with an upper bound has a least upper bound (ii) every countable non-empty subset of P with a lower bound has a greatest lower bound.

314B Remarks (a) I give these definitions in the widest possible generality because they are in fact of great interest for general partially ordered sets, even though for the moment we shall be concerned only with Boolean algebras. Indeed I have already presented the same idea in the context of Riesz spaces (241F).

(b) You will observe that the definition in (a) of 314A is asymmetric, unlike that in (b). This is because the inverted form of the definition is equivalent to that given; that is, P is Dedekind complete (on the definition 314Aa) iff every non-empty subset of P with a lower bound has a greatest lower bound. **P** (i) Suppose that P is Dedekind complete, and that $B \subseteq P$ is non-empty and bounded below. Let A be the set of lower bounds for B. Then A has at least one upper bound (since any member of B is an upper bound for A) and is not empty; so $a_0 = \sup A$ is defined. Now if $b \in B$, b is an upper bound for A, so $a_0 \le b$; thus $a_0 \in A$ and must be the greatest member of A, that is, the greatest lower bound of B. (ii) Similarly, if every non-empty subset of P with a lower bound has a greatest lower bound, P is Dedekind complete. **Q**

(c) In the special case of Boolean algebras, we do not need both halves of the definition 314Ab; in fact we have, for any Boolean algebra $\mathfrak{A}$,

$$\begin{aligned} &A \text{ is Dedekind } \sigma\text{-complete} \\ &\iff \text{every non-empty countable subset of } \mathfrak{A} \text{ has a least upper bound} \\ &\iff \text{every non-empty countable subset of } \mathfrak{A} \text{ has a greatest lower bound.} \end{aligned}$$

P Because $\mathfrak{A}$ has a least element 0 and a greatest element 1, every subset of $\mathfrak{A}$ has upper and lower bounds; so the two one-sided conditions together are equivalent to saying that $\mathfrak{A}$ is Dedekind σ-complete. I therefore have to show that they are equiveridical. Now if $A \subseteq \mathfrak{A}$ is a non-empty countable set, so is $B = \{1 \setminus a : a \in A\}$, and

$$\inf A = 1 \setminus \sup B, \quad \sup A = 1 \setminus \inf B$$

whenever the right-hand-sides are defined (313A). So if the existence of a supremum (resp. infimum) of B is guaranteed, so is the existence of an infimum (resp. supremum) of A. **Q**

The real point here is of course that $(\mathfrak{A}, \subseteq)$ is isomorphic to $(\mathfrak{A}, \supseteq)$.

(d) Most specialists in Boolean algebra speak of 'complete', or 'σ-complete', Boolean algebras. I prefer the longer phrases 'Dedekind complete' and 'Dedekind σ-complete' because we shall be studying metrics on Boolean algebras and shall need the notion of metric completeness as well as that of order-completeness.

(e) I have had to make some rather arbitrary choices in the definition here. The principal examples of partially ordered set to which we shall apply these definitions are Boolean algebras and Riesz spaces, which are all lattices. Consequently it is not possible to distinguish in these contexts between the property of Dedekind completeness, as defined above, and the weaker property, which we might call 'monotone order-completeness',

(i) whenever $A \subseteq P$ is non-empty, upwards-directed and bounded above then A has a least upper bound in P (ii) whenever $A \subseteq P$ is non-empty, downwards-directed and bounded below then A has a greatest lower bound in P.

(See 314Xa below. 'Monotone order-completeness' is the property involved in 314Ya, for instance.) Nevertheless I am prepared to say, on the basis of my own experience of working with other partially ordered sets, that 'Dedekind completeness', as I have defined it, is at least of sufficient importance to deserve a name. Note that it does not imply that P is a lattice, since it allows two elements of P to have no common upper bound.

(f) The phrase **complete lattice** is sometimes used to mean a Dedekind complete lattice with greatest and least elements; equivalently, a Dedekind complete partially ordered set with greatest and least elements. Thus a Dedekind complete Boolean algebra is a complete lattice in this sense, but $\mathbb{R}$ is not.

(g) The most important Dedekind complete Boolean algebras (at least from the point of view of measure theory) are the 'measure algebras' of the next chapter. I shall not pause here to give other examples, but will proceed directly with the general theory.

314C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and I a σ-ideal of $\mathfrak{A}$. Then the quotient Boolean algebra $\mathfrak{A}/I$ is Dedekind σ-complete.

proof I use the description in 314Bc. Let $B \subseteq \mathfrak{A}/I$ be a non-empty countable set. For each $u \in B$, choose an $a_u \in \mathfrak{A}$ such that $u = a_u^\bullet$. Then $c = \sup_{u \in B} a_u$ is defined in $\mathfrak{A}$; consider $v = c^\bullet$ in $\mathfrak{A}/I$. Because the map $a \mapsto a^\bullet$ is sequentially order-continuous (313Qb), $v = \sup B$. As B is arbitrary, $\mathfrak{A}/I$ is Dedekind σ-complete.

314D Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X. Then $\Sigma \cap \mathcal{I}$ is a σ-ideal of the Boolean algebra Σ, and $\Sigma/\Sigma \cap \mathcal{I}$ is Dedekind σ-complete.

proof Of course Σ is Dedekind σ-complete, because if $\langle E_n \rangle_{n \in \mathbb{N}}$ is any sequence in Σ then $\bigcup_{n \in \mathbb{N}} E_n$ is the least upper bound of $\{E_n : n \in \mathbb{N}\}$ in Σ. It is also easy to see that $\Sigma \cap \mathcal{I}$ is a σ-ideal of Σ, since $F \cap \bigcup_{n \in \mathbb{N}} E_n \in \mathcal{I}$ whenever $F \in \Sigma$ and $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\Sigma \cap \mathcal{I}$. So 314C gives the result.

314E Proposition Let $\mathfrak{A}$ be a Boolean algebra.

(a) If $\mathfrak{A}$ is Dedekind complete, then all its order-closed subalgebras and principal ideals are Dedekind complete.

(b) If $\mathfrak{A}$ is Dedekind σ-complete, then all its σ-subalgebras and principal ideals are Dedekind σ-complete.

proof All we need to note is that if $\mathfrak{C}$ is either an order-closed subalgebra or a principal ideal of $\mathfrak{A}$, and $B \subseteq \mathfrak{C}$ is such that $b = \sup B$ is defined in $\mathfrak{A}$, then $b \in \mathfrak{C}$ (see 313E(a-i-β)), so b is still the supremum of B in $\mathfrak{C}$; while the same is true if $\mathfrak{C}$ is a σ-subalgebra and $B \subseteq \mathfrak{C}$ is countable, using 313E(a-ii-β).

314F I spell out some further connexions between the concepts 'order-closed set', 'order-continuous function' and 'Dedekind complete Boolean algebra' which are elementary without being quite transparent.

Proposition Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a Boolean homomorphism.

(a)(i) If $\mathfrak{A}$ is Dedekind complete and π is order-continuous, then $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$.

(ii) If $\mathfrak{B}$ is Dedekind complete and π is injective and $\pi[\mathfrak{A}]$ is order-closed then π is order-continuous.

(b)(i) If $\mathfrak{A}$ is Dedekind σ-complete and π is sequentially order-continuous, then $\pi[\mathfrak{A}]$ is a σ-subalgebra of $\mathfrak{B}$.

(ii) If $\mathfrak{B}$ is Dedekind σ-complete and π is injective and $\pi[\mathfrak{A}]$ is a σ-subalgebra of $\mathfrak{B}$ then π is sequentially order-continuous.

proof (a)(i) If $B \subseteq \pi[\mathfrak{A}]$, then $a_0 = \sup(\pi^{-1}[B])$ is defined in $\mathfrak{A}$; now

$$\pi a_0 = \sup(\pi[\pi^{-1}[B]]) = \sup B$$

in $\mathfrak{B}$ (313L(b-iv)), and of course $\pi a_0 \in \pi[\mathfrak{A}]$. By 313E(a-i-$\beta$) again, this is enough to show that $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$.

(ii) Suppose that $A \subseteq \mathfrak{A}$ and $\inf A = 0$ in $\mathfrak{A}$. Then $\pi[A]$ has an infimum b_0 in $\mathfrak{B}$, which belongs to $\pi[\mathfrak{A}]$ because $\pi[\mathfrak{A}]$ is an order-closed subalgebra of $\mathfrak{B}$ (313E(a-i-β')). Now if $a_0 \in \mathfrak{A}$ is such that $\pi a_0 = b_0$, we have

$$\pi(a \cap a_0) = \pi a \cap \pi a_0 = \pi a_0$$

for every $a \in A$, so (because π is injective) $a \cap a_0 = a_0$ and $a_0 \subseteq a$ for every $a \in A$. But this means that $a_0 = 0$ and $b_0 = \pi 0 = 0$. As A is arbitrary, π is order-continuous (313L(b-ii)).

(b) Use the same arguments, but with sequences in place of the sets B, A above.

314G Corollary Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$.

(a) If $\mathfrak{A}$ is Dedekind complete, then $\mathfrak{B}$ is order-closed iff it is Dedekind complete in itself and is regularly embedded in $\mathfrak{A}$.

(b) If $\mathfrak{A}$ is Dedekind σ-complete, then $\mathfrak{B}$ is a σ-subalgebra iff it is Dedekind σ-complete in itself and the identity map from $\mathfrak{B}$ to $\mathfrak{A}$ is sequentially order-continuous.

proof (a) Let $\iota : \mathfrak{B} \to \mathfrak{A}$ be the identity map; then it is an injective Boolean homomorphism.

(i) If $\mathfrak{B}$ is order-closed, then it is Dedekind complete in itself by 314Ea. By 314F(a-ii), $\iota : \mathfrak{B} \to \mathfrak{A}$ is order-continuous, that is, $\mathfrak{B}$ is regularly embedded in $\mathfrak{A}$.

(ii) If $\mathfrak{B}$ is Dedekind complete in itself and ι is order-continuous, then $\mathfrak{B} = \iota[\mathfrak{B}]$ is order-closed in $\mathfrak{A}$ by 314F(a-i).

(b) Use the same arguments, but with 314Eb and 314Fb in place of 314Ea and 314Fa.

314H Corollary Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ an order-continuous Boolean homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the order-closed subalgebra of $\mathfrak{A}$ generated by C, then $\pi[\mathfrak{C}]$ is the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

proof Let $\mathfrak{D}$ be the order-closed subalgebra of $\mathfrak{B}$ generated by $\pi[C]$. By 313Mb, $\pi[\mathfrak{C}] \subseteq \mathfrak{D}$. On the other hand, the identity homomorphism $\iota : \mathfrak{C} \to \mathfrak{A}$ is order-continuous, by 314Ga, so $\pi\iota : \mathfrak{C} \to \mathfrak{B}$ is order-continuous, and $\pi[\mathfrak{C}] = \pi\iota[\mathfrak{C}]$ is order-closed in $\mathfrak{B}$, by 314F(a-i). But since $\pi[C]$ is surely included in $\pi[\mathfrak{C}]$, $\mathfrak{D}$ also is included in $\pi[\mathfrak{C}]$. Accordingly $\pi[\mathfrak{C}] = \mathfrak{D}$, as claimed.

314I Corollary (a) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra, $\mathfrak{B}$ is a Boolean algebra, $\pi : \mathfrak{A} \to \mathfrak{B}$ is an injective Boolean homomorphism and $\pi[\mathfrak{A}]$ is order-dense in $\mathfrak{B}$, then π is an isomorphism.

(b) If $\mathfrak{A}$ is a Boolean algebra and $\mathfrak{B}$ is an order-dense subalgebra of $\mathfrak{A}$ which is Dedekind complete in itself, then $\mathfrak{B} = \mathfrak{A}$.

proof (a) Because $\pi[\mathfrak{A}]$ is order-dense, it is regularly embedded in $\mathfrak{B}$ (313O); also, the kernel of π is $\{0\}$, which is surely order-closed in $\mathfrak{A}$, so 313P(a-ii) tells us that π is order-continuous. By 314F(a-i), $\pi[\mathfrak{A}]$ is order-closed in $\mathfrak{B}$; being order-dense, it must be the whole of $\mathfrak{B}$ (313K). Thus π is surjective; being injective, it is an isomorphism.

(b) Apply (a) to the identity map from $\mathfrak{B}$ to $\mathfrak{A}$.

314J When we come to applications of the extension procedure in 312O, the following will sometimes be needed.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{A}_0$ a subalgebra of $\mathfrak{A}$. Take any $c \in \mathfrak{A}$, and set

$$\mathfrak{A}_1 = \{(a \cap c) \cup (b \setminus c) : a,\, b \in \mathfrak{A}_0\},$$

the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_0 \cup \{c\}$ (312N).

(a) Suppose that $\mathfrak{A}$ is Dedekind complete. If $\mathfrak{A}_0$ is order-closed in $\mathfrak{A}$, so is $\mathfrak{A}_1$.

(b) Suppose that $\mathfrak{A}$ is Dedekind σ-complete. If $\mathfrak{A}_0$ is a σ-subalgebra of $\mathfrak{A}$, so is $\mathfrak{A}_1$.

proof (a) Let D be any subset of $\mathfrak{A}_1$. Set

$$E = \{e : e \in \mathfrak{A}, \text{ there is some } d \in D \text{ such that } e \subseteq d\},$$

$$A = \{a : a \in \mathfrak{A}_0,\, a \cap c \in E\}, \quad B = \{b : b \in \mathfrak{A}_0,\, b \setminus c \in E\}.$$

Because $\mathfrak{A}$ is Dedekind complete, $a^* = \sup A$ and $b^* = \sup B$ are defined in $\mathfrak{A}$; because $\mathfrak{A}_0$ is order-closed, both belong to $\mathfrak{A}_0$, so $d^* = (a^* \cap c) \cup (b^* \setminus c)$ belongs to $\mathfrak{A}_1$.

Now if $d \in D$, it is expressible as $(a \cap c) \cup (b \setminus c)$ for some $a,\, b \in \mathfrak{A}_0$; since $a \in A$ and $b \in B$, we have $a \subseteq a^*$ and $b \subseteq b^*$, so $d \subseteq d^*$. Thus d^* is an upper bound for D. On the other hand, if d' is any other upper bound for D in $\mathfrak{A}$, it is also an upper bound for E, so we must have

$$a^* \cap c = \sup_{a \in A} a \cap c \subseteq d', \quad b^* \setminus c = \sup_{b \in B} b \setminus c \subseteq d',$$

and $d^* \subseteq d'$. Thus $d^* = \sup D$. This shows that the supremum of any subset of $\mathfrak{A}_1$ belongs to $\mathfrak{A}_1$, so that $\mathfrak{A}_1$ is order-closed.

(b) The argument is the same, except that we replace D by a sequence $\langle d_n \rangle_{n\in\mathbb{N}}$, and A, B by sequences $\langle a_n \rangle_{n\in\mathbb{N}}$, $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_0$ such that $d_n = (a_n \cap c) \cup (b_n \setminus c)$ for every n.

314K Extension of homomorphisms The following is one of the most striking properties of Dedekind complete Boolean algebras.

Theorem Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ a Dedekind complete Boolean algebra. Let $\mathfrak{A}_0$ be a Boolean subalgebra of $\mathfrak{A}$ and $\pi_0 : \mathfrak{A}_0 \to \mathfrak{B}$ a Boolean homomorphism. Then there is a Boolean homomorphism $\pi_1 : \mathfrak{A} \to \mathfrak{B}$ extending π_0.

proof (a) Let P be the set of all Boolean homomorphisms π such that $\operatorname{dom}\pi$ is a Boolean subalgebra of $\mathfrak{A}$ including $\mathfrak{A}_0$ and π extends π_0. Identify each member of P with its graph, which is a subset of $\mathfrak{A} \times \mathfrak{B}$, and order P by inclusion, so that $\pi \subseteq \theta$ means just that θ extends π. Then any non-empty totally ordered subset Q of P has an upper bound in P. **P** Let π^* be the simple union of these graphs. (i) If (a, b) and (a, b') both belong to π^*, then there are $\pi,\, \pi' \in Q$ such that $\pi a = b$, $\pi' a = b'$; now either $\pi \subseteq \pi'$ or $\pi' \subseteq \pi$; in either case, $\theta = \pi \cup \pi' \in Q$, so that

$$b = \pi a = \theta a = \pi' a = b'.$$

This shows that π^* is a function. (ii) Because $Q \neq \emptyset$,

$$\operatorname{dom}\pi_0 \subseteq \operatorname{dom}\pi \subseteq \operatorname{dom}\pi^*$$

for some $\pi \in Q$; thus π^* extends π_0 (and, in particular, $0 \in \operatorname{dom}\pi^*$). (iii) Now suppose that a, $a' \in \operatorname{dom}(\pi^*)$. Then there are π, $\pi' \in Q$ such that $a \in \operatorname{dom}\pi$, $a' \in \operatorname{dom}\pi'$; once again, $\theta = \pi \cup \pi' \in Q$, so that a, $a' \in \operatorname{dom}\theta$, and

$$a \cap a' \in \operatorname{dom}\theta \subseteq \operatorname{dom}\pi^*, \quad 1 \setminus a \in \operatorname{dom}\theta \subseteq \operatorname{dom}\pi^*,$$

$$\pi^*(a \cap a') = \theta(a \cap a') = \theta a \cap \theta a' = \pi^* a \cap \pi^* a',$$

$$\pi^*(1 \setminus a) = \theta(1 \setminus a) = 1 \setminus \theta a = 1 \setminus \pi^* a.$$

(iv) This shows that $\operatorname{dom}\pi^*$ is a subalgebra of $\mathfrak{A}$ and that π^* is a Boolean homomorphism, that is, that $\pi^* \in P$; and of course π^* is an upper bound for Q in P. **Q**

(b) By Zorn's Lemma, P has a maximal element π_1 say.

? Suppose, if possible, that $\mathfrak{A}_1 = \operatorname{dom}\pi_1$ is not the whole of $\mathfrak{A}$; take $c \in \mathfrak{A} \setminus \mathfrak{A}_1$. Set $A = \{a : a \in \mathfrak{A}_1, a \subseteq c\}$. Because $\mathfrak{B}$ is Dedekind complete, $d = \sup \pi_1[A]$ is defined in $\mathfrak{B}$. If $a' \in \mathfrak{A}_1$ and $c \subseteq a'$, then of course $a \subseteq a'$ and $\pi_1 a \subseteq \pi_1 a'$ whenever $a \in A$, so that $\pi_1 a'$ is an upper bound for $\pi_1[A]$, and $d \subseteq \pi_1 a'$.

But this means that there is an extension of π_1 to a Boolean homomorphism π on the Boolean subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_1 \cup \{c\}$ (312O). And this π must be a member of P properly extending π_1, which is supposed to be maximal. **X**

Thus $\operatorname{dom}\pi_1 = \mathfrak{A}$ and π_1 is an extension of π_0 to $\mathfrak{A}$, as required.

314L The Loomis-Sikorski representation of a Dedekind σ-complete Boolean algebra The construction in 314D is not only the commonest way in which new Dedekind σ-complete Boolean algebras appear, but is adequate to describe them all. I start with an elementary general fact.

Lemma Let X be any topological space, and write $\mathcal{M}$ for the family of meager subsets of X. Then $\mathcal{M}$ is a σ-ideal of subsets of X.

proof The point is that if $A \subseteq X$ is nowhere dense, so is every subset of A; this is obvious, since if $B \subseteq A$ then $\overline{B} \subseteq \overline{A}$ so $\operatorname{int}\overline{B} \subseteq \operatorname{int}\overline{A} = \emptyset$. So if $B \subseteq A \in \mathcal{M}$, let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a sequence of nowhere dense sets with union A; then $\langle B \cap A_n \rangle_{n\in\mathbb{N}}$ is a sequence of nowhere dense sets with union B, so $B \in \mathcal{M}$. If $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}$ with union A, then for each n we may choose a sequence $\langle A_{nm} \rangle_{m\in\mathbb{N}}$ of nowhere dense sets with union A_n; then the countable family $\langle A_{nm} \rangle_{n,m\in\mathbb{N}}$ may be re-indexed as a sequence of nowhere dense sets with union A, so $A \in \mathcal{M}$. Finally, $\emptyset$ is nowhere dense, so belongs to $\mathcal{M}$.

314M Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and Z its Stone space. Let $\mathcal{E}$ be the algebra of open-and-closed subsets of Z, and $\mathcal{M}$ the σ-ideal of meager subsets of Z. Then $\Sigma = \{E \triangle A : E \in \mathcal{E}, A \in \mathcal{M}\}$ is a σ-algebra of subsets of Z, $\mathcal{M}$ is a σ-ideal of Σ, and $\mathfrak{A}$ is isomorphic, as Boolean algebra, to $\Sigma/\mathcal{M}$.

proof (a) I start by showing that Σ is a σ-algebra. **P** Of course $\emptyset = \emptyset \triangle \emptyset \in \Sigma$. If $F \in \Sigma$, express it as $E \triangle A$ where $E \in \mathcal{E}$, $A \in \mathcal{M}$; then $Z \setminus F = (Z \setminus E) \triangle A \in \Sigma$.

If $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in Σ, express each F_n as $E_n \triangle A_n$, where $E_n \in \mathcal{E}$ and $A_n \in \mathcal{M}$. Now each E_n is expressible as $\widehat{a}_n$, where $a_n \in \mathfrak{A}$. Because $\mathfrak{A}$ is Dedekind σ-complete, $a = \sup_{n\in\mathbb{N}} a_n$ is defined in $\mathfrak{A}$. Set $E = \widehat{a} \in \mathcal{E}$. By 313Ca, $E = \overline{\bigcup_{n\in\mathbb{N}} E_n}$, so the closed set $E \setminus \bigcup_{n\in\mathbb{N}} E_n$ has empty interior and is nowhere dense. Accordingly, setting $A = E \triangle \bigcup_{n\in\mathbb{N}} F_n$, we have

$$A \subseteq (E \setminus \textstyle\bigcup_{n\in\mathbb{N}} E_n) \cup \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{M},$$

so that $\bigcup_{n\in\mathbb{N}} F_n = E \triangle A \in \Sigma$. Thus Σ is closed under countable unions and is a σ-algebra. **Q**

Evidently $\mathcal{M} \subseteq \Sigma$, because $\emptyset \in \mathcal{E}$.

(b) For each $F \in \Sigma$, there is exactly one $E \in \mathcal{E}$ such that $F \triangle E \in \mathcal{M}$. **P** There is surely some $E \in \mathcal{E}$ such that F is expressible as $E \triangle A$ where $A \in \mathcal{M}$, so that $F \triangle E = A \in \mathcal{M}$. If E' is any other member of $\mathcal{E}$, then $E' \triangle E$ is a non-empty open set in X, while $E' \triangle E \subseteq A \cup (F \triangle E')$; by Baire's theorem for compact Hausdorff spaces (3A3G), $A \cup (F \triangle E') \notin \mathcal{M}$ and $F \triangle E' \notin \mathcal{M}$. Thus E is unique. **Q**

(c) Consequently the maps $E \mapsto E^\bullet : \mathcal{E} \to \Sigma/\mathcal{M}$ is a bijection. But since it is also a Boolean homomorphism, it is an isomorphism, and $\mathfrak{A} \cong \mathcal{E} \cong \Sigma/\mathcal{M}$, as claimed.

314N Corollary A Boolean algebra $\mathfrak{A}$ is Dedekind σ-complete iff it is isomorphic to a quotient $\Sigma/\mathcal{I}$ where Σ is a σ-algebra of sets and $\mathcal{I}$ is a σ-ideal of Σ.

proof Put 314D and 314M together.

314O Regular open algebras For Boolean algebras which are Dedekind complete in the full sense, there is another general method of representing them, which leads to further very interesting ideas.

Definition Let X be a topological space. A **regular open set** in X is an open set $G \subseteq X$ such that $G = \operatorname{int} \overline{G}$.

Note that if $F \subseteq X$ is any closed set, then $G = \operatorname{int} F$ is a regular open set, because $G \subseteq \overline{G} \subseteq F$ so

$$G \subseteq \operatorname{int} \overline{G} \subseteq \operatorname{int} F = G$$

and $G = \operatorname{int} \overline{G}$.

314P Theorem Let X be any topological space, and write $\mathrm{RO}(X)$ for the set of regular open sets in X. Then $\mathrm{RO}(X)$ is a Dedekind complete Boolean algebra, with $1_{\mathrm{RO}(X)} = X$ and $0_{\mathrm{RO}(X)} = \emptyset$, and with Boolean operations given by

$$G \cap_{\mathrm{RO}} H = G \cap H, \quad G \triangle_{\mathrm{RO}} H = \operatorname{int} \overline{G \triangle H},$$

$$G \cup_{\mathrm{RO}} H = \operatorname{int} \overline{G \cup H}, \quad G \setminus_{\mathrm{RO}} H = G \setminus \overline{H},$$

with Boolean ordering given by

$$G \subseteq_{\mathrm{RO}} H \iff G \subseteq H,$$

and with suprema and infima given by

$$\sup \mathcal{H} = \operatorname{int} \overline{\bigcup \mathcal{H}}, \quad \inf \mathcal{H} = \operatorname{int} \bigcap \mathcal{H} = \operatorname{int} \overline{\bigcap \mathcal{H}}$$

for all non-empty $\mathcal{H} \subseteq \mathrm{RO}(X)$.

Remark I use the expressions

$$\cap_{\mathrm{RO}} \quad \cup_{\mathrm{RO}} \quad \triangle_{\mathrm{RO}} \quad \setminus_{\mathrm{RO}} \quad \subseteq_{\mathrm{RO}}$$

in case the distinction between

$$\cap \quad \cup \quad \triangle \quad \setminus \quad \subseteq$$

and

$$\cap \quad \cup \quad \triangle \quad \setminus \quad \subseteq$$

is insufficiently marked.

proof I base the proof on the study of an auxiliary algebra of sets which involves some of the ideas already used in 314M.

(a) Let $\mathcal{I}$ be the family of nowhere dense subsets of X. Then $\mathcal{I}$ is an ideal of subsets of X. **P** Of course $\emptyset \in \mathcal{I}$. If $A \subseteq B \in \mathcal{I}$ then $\operatorname{int} \overline{A} \subseteq \operatorname{int} \overline{B} = \emptyset$. If A, $B \in \mathcal{I}$ and G is a non-empty open set, then $G \setminus \overline{A}$ is a non-empty open set and $(G \setminus \overline{A}) \setminus \overline{B}$ is non-empty; accordingly G cannot be a subset of $\overline{A} \cup \overline{B} = \overline{A \cup B}$. This shows that $\operatorname{int} \overline{A \cup B} = \emptyset$, so that $A \cup B \in \mathcal{I}$. **Q**

(b) For any set $A \subseteq X$, write ∂A for the boundary of A, that is, $\overline{A} \setminus \operatorname{int} A$. Set

$$\Sigma = \{E : E \subseteq X, \, \partial E \in \mathcal{I}\}.$$

The Σ is an algebra of subsets of X. **P** (i) $\partial\emptyset = \emptyset \in \mathcal{I}$ so $\emptyset \in \Sigma$. (ii) If A, $B \subseteq X$, then $\overline{A \cup B} = \overline{A} \cup \overline{B}$, while $\operatorname{int}(A \cup B) \supseteq \operatorname{int} A \cup \operatorname{int} B$; so $\partial(A \cup B) \subseteq \partial A \cup \partial B$. So if E, $F \in \Sigma$, $\partial(E \cup F) \subseteq \partial E \cup \partial F \in \mathcal{I}$ and $E \cup F \in \Sigma$. (iii) If $A \subseteq X$, then

$$\partial(X \setminus A) = \overline{X \setminus A} \setminus \operatorname{int}(X \setminus A) = (X \setminus \operatorname{int} A) \setminus (X \setminus \overline{A}) = \overline{A} \setminus \operatorname{int} A = \partial A.$$

So if $E \in \Sigma$, $\partial(X \setminus E) = \partial E \in \mathcal{I}$ and $X \setminus E \in \Sigma$. **Q**

If $A \in \mathcal{I}$, then of course $\partial A = \overline{A} \in \mathcal{I}$, so $A \in \Sigma$; accordingly $\mathcal{I}$ is an ideal in the Boolean algebra Σ, and we can form the quotient $\Sigma/\mathcal{I}$.

It will be helpful to note that every open set belongs to Σ, since if G is open then $\partial G = \overline{G} \setminus G$ cannot include any non-empty open set (since any open set meeting $\overline{G}$ must meet G).

(c) For each $E \in \Sigma$, set $V_E = \operatorname{int}\overline{E}$; then V_E is the unique member of $\mathrm{RO}(X)$ such that $E \triangle V_E \in \mathcal{I}$. **P** (i) Being the interior of a closed set, $V_E \in \mathrm{RO}(X)$. Since $\operatorname{int} E \subseteq V_E \subseteq \overline{E}$, $E \triangle V_E \subseteq \partial E \in \mathcal{I}$. (ii) If $G \in \mathrm{RO}(X)$ is such that $E \triangle G \in \mathcal{I}$, then

$$G \setminus \overline{V_E} \subseteq G \setminus V_E \subseteq (G \triangle E) \cup (V_E \triangle E) \in \mathcal{I},$$

so $G \setminus \overline{V_E}$, being open, must be actually empty, and $G \subseteq \overline{V_E}$; but this means that $G \subseteq \operatorname{int}\overline{V_E} = V_E$. Similarly, $V_E \subseteq G$ and $V_E = G$. This shows that V_E is unique. **Q**

(d) It follows that the map $G \mapsto G^\bullet : \mathrm{RO}(X) \to \Sigma/\mathcal{I}$ is a bijection, and we have a Boolean algebra structure on $\mathrm{RO}(X)$ defined by the Boolean algebra structure of $\Sigma/\mathcal{I}$. What this means is that for each of the binary Boolean operations $\cap_{\mathrm{RO}}$, $\triangle_{\mathrm{RO}}$, $\cup_{\mathrm{RO}}$, $\setminus_{\mathrm{RO}}$ and for G, $H \in \mathrm{RO}(X)$ we must have $G *_{\mathrm{RO}} H = \operatorname{int}\overline{G * H}$, writing $*_{\mathrm{RO}}$ for the operation on the algebra $\mathrm{RO}(X)$ and $*$ for the corresponding operation on Σ or $\mathcal{P}X$.

(e) Before working through the identifications, it will be helpful to observe that if $\mathcal{H}$ is any non-empty subset of $\mathrm{RO}(X)$, then $\operatorname{int}\bigcap\mathcal{H} = \operatorname{int}\overline{\bigcap\mathcal{H}}$. **P** Set $G = \operatorname{int}\overline{\bigcap\mathcal{H}}$. For every $H \in \mathcal{H}$, $G \subseteq \overline{H}$ so $G \subseteq \operatorname{int}\overline{H} = H$; thus

$$G \subseteq \operatorname{int}\bigcap\mathcal{H} \subseteq \operatorname{int}\overline{\bigcap\mathcal{H}} = G,$$

so $G = \operatorname{int}\bigcap\mathcal{H}$. **Q** Consequently $\operatorname{int}\bigcap\mathcal{H}$, being the interior of a closed set, belongs to $\mathrm{RO}(X)$.

(f)(i) If G, $H \in \mathrm{RO}(X)$ then their intersection in the algebra $\mathrm{RO}(X)$ is

$$G \cap_{\mathrm{RO}} H = \operatorname{int}\overline{G \cap H} = \operatorname{int}(G \cap H) = G \cap H,$$

using (d) for the first equality and (e) for the second.

(ii) Of course $X \in \mathrm{RO}(X)$ and $X^\bullet = 1_{\Sigma/\mathcal{I}}$, so $X = 1_{\mathrm{RO}(X)}$.

(iii) If $G \in \mathrm{RO}(X)$ then its complement $1_{\mathrm{RO}(X)} \setminus_{\mathrm{RO}} G$ in $\mathrm{RO}(X)$ is

$$\operatorname{int}\overline{X \setminus G} = \operatorname{int}(X \setminus G) = X \setminus \overline{G}.$$

(iv) If G, $H \in \mathrm{RO}(X)$, then the relative complement in $\mathrm{RO}(X)$ is

$$G \setminus_{\mathrm{RO}} H = G \cap_{\mathrm{RO}} (1_{\mathrm{RO}(X)} \setminus_{\mathrm{RO}} H) = G \cap (X \setminus \overline{H}) = G \setminus \overline{H} = \operatorname{int}(G \setminus H).$$

(v) If G, $H \in \mathrm{RO}(X)$, then $G \cup_{\mathrm{RO}} H = \operatorname{int}\overline{G \cup H}$ and $G \triangle_{\mathrm{RO}} H = \operatorname{int}\overline{G \triangle H}$, by the remarks in (d).

(g) We must note that for G, $H \in \mathrm{RO}(X)$,

$$G \subseteq_{\mathrm{RO}} H \iff G \cap_{\mathrm{RO}} H = G \iff G \cap H = G \iff G \subseteq H;$$

that is, the ordering of the Boolean algebra $\mathrm{RO}(X)$ is just the partial ordering induced on $\mathrm{RO}(X)$ by the Boolean ordering $\subseteq$ of $\mathcal{P}X$ or Σ.

(h) If $\mathcal{H}$ is any non-empty subset of $\mathrm{RO}(X)$, consider $G_0 = \operatorname{int}\bigcap\mathcal{H}$ and $G_1 = \operatorname{int}\overline{\bigcup\mathcal{H}}$.

$G_0 = \inf\mathcal{H}$ in $\mathrm{RO}(X)$. **P** By (e), $G_0 \in \mathrm{RO}(X)$. Of course $G_0 \subseteq H$ for every $H \in \mathcal{H}$, so G_0 is a lower bound for $\mathcal{H}$. If G is any lower bound for $\mathcal{H}$ in $\mathrm{RO}(X)$, then $G \subseteq H$ for every $H \in \mathcal{H}$, so $G \subseteq \bigcap\mathcal{H}$; but also G is open, so $G \subseteq \operatorname{int}\bigcap\mathcal{H} = G_0$. Thus G_0 is the greatest lower bound for $\mathcal{H}$. **Q**

$G_1 = \sup\mathcal{H}$ in $\mathrm{RO}(X)$. **P** Being the interior of a closed set, $G_1 \in \mathrm{RO}(X)$, and of course

$$H = \operatorname{int}\overline{H} \subseteq \operatorname{int}\overline{\bigcup\mathcal{H}} = G_1$$

for every $H \in \mathcal{H}$, so G_1 is an upper bound for $\mathcal{H}$ in $\mathrm{RO}(X)$. If G is any upper bound for $\mathcal{H}$ in $\mathrm{RO}(X)$, then

$$G = \operatorname{int}\overline{G} \supseteq \operatorname{int}\overline{\bigcup\mathcal{H}} = G_1;$$

thus G_1 is the least upper bound for $\mathcal{H}$ in $\mathrm{RO}(X)$. **Q**

This shows that every non-empty $\mathcal{H} \subseteq \mathrm{RO}(X)$ has a supremum and an infimum in $\mathrm{RO}(X)$; consequently $\mathrm{RO}(X)$ is Dedekind complete, and the proof is finished.

314Q Remarks (a) RO(X) is called the **regular open algebra** of the topological space X.

(b) Note that the map $E \mapsto V_E : \Sigma \to \mathrm{RO}(X)$ of part (c) of the proof above is a Boolean homomorphism, if RO(X) is given its Boolean algebra structure. Its kernel is of course $\mathcal{I}$; the induced map $E^\bullet \mapsto V_E : \Sigma/\mathcal{I} \to \mathrm{RO}(X)$ is just the inverse of the isomorphism $G \mapsto G^\bullet : \mathrm{RO}(X) \to \Sigma/\mathcal{I}$.

***314R** I interpolate a lemma corresponding to 313R, with a couple of other occasionally useful facts.

Lemma (a) Let X and Y be topological spaces, and $f : X \to Y$ a continuous function such that $f^{-1}[M]$ is nowhere dense in X for every nowhere dense $M \subseteq Y$. Then we have an order-continuous Boolean homomorphism π from the regular open algebra RO(Y) of Y to the regular open algebra RO(X) of X defined by setting $\pi H = \operatorname{int}\overline{f^{-1}[H]}$ for every $H \in \mathrm{RO}(Y)$.

(b) Let X be a topological space.

(i) If $U \subseteq X$ is open, then $G \mapsto G \cap U$ is a surjective order-continuous Boolean homomorphism from RO(X) onto RO(U).

(ii) If $U \in \mathrm{RO}(X)$ then RO(U) is the principal ideal of RO(X) generated by U.

proof (a)(i) By the remark in 314O, the formula for πH always defines a member of RO(X); and of course π is order-preserving.

Observe that if $H \in \mathrm{RO}(Y)$, then $f^{-1}[H]$ is open, so $f^{-1}[H] \subseteq \pi H$. It will be convenient to note straight away that if $V \subseteq Y$ is a dense open set then $f^{-1}[V]$ is dense in X. **P** $M = Y \setminus V$ is nowhere dense, so $f^{-1}[M]$ is nowhere dense and its complement $f^{-1}[V]$ is dense. **Q**

(ii) If H_1, $H_2 \in \mathrm{RO}(Y)$ then $\pi(H_1 \cap H_2) = \pi H_1 \cap \pi H_2$. **P** Because π is order-preserving, $\pi(H_1 \cap H_2) \subseteq \pi H_1 \cap \pi H_2$. **?** Suppose, if possible, that they are not equal. Then (because $\pi(H_1 \cap H_2)$ is a regular open set) $G = \pi H_1 \cap \pi H_2 \setminus \overline{\pi(H_1 \cap H_2)}$ is non-empty. Set $M = \overline{f[G]}$. Then $f^{-1}[M] \supseteq G$ is not nowhere dense, so $H = \operatorname{int} M$ must be non-empty. Now $G \subseteq \pi H_1 \subseteq \overline{f^{-1}[H_1]}$, so

$$f[G] \subseteq f[\overline{f^{-1}[H_1]}] \subseteq \overline{f[f^{-1}[H_1]]} \subseteq \overline{H}_1,$$

so $M \subseteq \overline{H}_1$ and $H \subseteq \operatorname{int}\overline{H}_1 = H_1$. Similarly, $H \subseteq H_2$, and $f^{-1}[H] \subseteq f^{-1}[H_1 \cap H_2] \subseteq \pi(H_1 \cap H_2)$. But also $H \cap f[G]$ is not empty, so

$$\emptyset \neq G \cap f^{-1}[H] \subseteq G \cap \pi(H_1 \cap H_2),$$

which is impossible. **XQ**

(iii) If $H \in \mathrm{RO}(Y)$ and $H' = Y \setminus \overline{H}$ is its complement in RO(Y) then $\pi H' = X \setminus \overline{\pi H}$ is the complement of πH in RO(X). **P** By (b), πH and $\pi H'$ are disjoint. Now $H \cup H'$ is a dense open subset of Y, so

$$\pi H \cup \pi H' \supseteq f^{-1}[H] \cup f^{-1}[H'] = f^{-1}[H \cup H']$$

is dense in X, and the regular open set $\pi H'$ must include the complement of πH in RO(X). **Q**

Putting this together with (b), we see that the conditions of 312H(ii) are satisfied, so that π is a Boolean homomorphism.

(iv) To see that it is order-continuous, let $\mathcal{H} \subseteq \mathrm{RO}(Y)$ be a non-empty set with supremum Y. Then $H_0 = \bigcup \mathcal{H}$ is a dense open subset of Y (see the formula in 314P). So

$$\bigcup_{H \in \mathcal{H}} \pi H \supseteq \bigcup_{H \in \mathcal{H}} f^{-1}[H] = f^{-1}[H_0]$$

is dense in X, and $\sup_{H \in \mathcal{H}} \pi H = X$ in RO(X). By 313L(b-iii), π is order-continuous.

(b)(i) The idea is to apply (a) to the identity function $f : U \to X$. If $M \subseteq X$ is nowhere dense, then any non-empty open subset of U has a non-empty open subset disjoint from M, so $f^{-1}[M] = M \cap U$ is nowhere dense in U; thus the condition is satisfied, and we have an order-continuous Boolean homomorphism $\pi : \mathrm{RO}(X) \to \mathrm{RO}(U)$ defined by setting $\pi H = \operatorname{int}_U \overline{H \cap U}^{(U)}$ for every $H \in \mathrm{RO}(X)$. (I write int_U, $\overline{}^{(U)}$ to indicate interior and closure in the subspace topology.) Now for any open set $G \subseteq X$,

$$U \cap \overline{G} = U \cap (\overline{G \cap U} \cup \overline{G \setminus U}) = U \cap \overline{G \cap U} = \overline{G \cap U}^{(U)}.$$

So if $H \in \mathrm{RO}(X)$, then

$$\pi H = \mathrm{int}_U \overline{H \cap U}^{(U)} = \mathrm{int}_U (U \cap \overline{H}) = U \cap \mathrm{int}\,\overline{H} = U \cap G.$$

So π takes the required form. To see that it is surjective, take any $V \in \mathrm{RO}(U)$. Then $\mathrm{int}\,\overline{V} \in \mathrm{RO}(X)$, and

$$V = \mathrm{int}_U \overline{V}^{(U)} = \mathrm{int}_U (U \cap \overline{V}) = U \cap \mathrm{int}\,\overline{V} = \pi(\mathrm{int}\,\overline{V})$$

is a value of π.

(ii) If $G \in \mathrm{RO}(X)$ and $G \subseteq U$, then $G = G \cap U \in \mathrm{RO}(U)$. Conversely, if $V \in \mathrm{RO}(U)$, there is a $G \in \mathrm{RO}(X)$ such that $V = G \cap U$; but $G \cap U \in \mathrm{RO}(X)$, by 314P, so $V \in \mathrm{RO}(X)$.

314S It is now easy to characterize the Stone spaces of Dedekind complete Boolean algebras.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space; write $\mathcal{E}$ for the algebra of open-and-closed subsets of Z, and $\mathrm{RO}(Z)$ for the regular open algebra of Z. Then the following are equiveridical:

(i) $\mathfrak{A}$ is Dedekind complete;

(ii) Z is extremally disconnected (definition: 3A3Af);

(iii) $\mathcal{E} = \mathrm{RO}(Z)$.

proof For $a \in \mathfrak{A}$, let $\widehat{a}$ be the corresponding member of $\mathcal{E}$.

(i)⇒(ii) If $\mathfrak{A}$ is Dedekind complete, let G be any open set in Z. Set $A = \{a : a \in \mathfrak{A}, \widehat{a} \subseteq G\}$, $a_0 = \sup A$. Then $G = \bigcup\{\widehat{a} : a \in A\}$, because $\mathcal{E}$ is a base for the topology of Z, so $\widehat{a}_0 = \overline{G}$, by 313Ca. Consequently $\overline{G}$ is open. As G is arbitrary, Z is extremally disconnected.

(ii)⇒(iii) If $E \in \mathcal{E}$, then of course $E = \overline{E} = \mathrm{int}\,\overline{E}$, so E is a regular open set. Thus $\mathcal{E} \subseteq \mathrm{RO}(Z)$. On the other hand, suppose that $G \subseteq Z$ is a regular open set. Because Z is extremally disconnected, $\overline{G}$ is open; so $G = \mathrm{int}\,\overline{G} = \overline{G}$ is open-and-closed, and belongs to $\mathcal{E}$. Thus $\mathcal{E} = \mathrm{RO}(Z)$.

(iii)⇒(i) Since $\mathrm{RO}(Z)$ is Dedekind complete (314P), $\mathcal{E}$ and $\mathfrak{A}$ are also Dedekind complete Boolean algebras.

Remark Note that if the conditions above are satisfied, either 312M or the formulae in 314P show that the Boolean structures of $\mathcal{E}$ and $\mathrm{RO}(Z)$ are identical.

314T I come now to a construction of great importance, both as a foundation for further constructions and as a source of insight into the nature of Dedekind completeness.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z; for $a \in \mathfrak{A}$ let $\widehat{a}$ be the corresponding open-and-closed subset of Z. Let $\widehat{\mathfrak{A}}$ be the regular open algebra of Z (314P).

(a) The map $a \mapsto \widehat{a}$ is an injective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

(b) If $\mathfrak{B}$ is any Dedekind complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a unique order-continuous Boolean homomorphism $\pi_1 : \widehat{\mathfrak{A}} \to \mathfrak{B}$ such that $\pi_1 \widehat{a} = \pi a$ for every $a \in \mathfrak{A}$.

proof (a)(i) Setting $\mathcal{E} = \{\widehat{a} : a \in \mathfrak{A}\}$, every member of $\mathcal{E}$ is open-and-closed, so is surely equal to the interior of its closure, and is a regular open set; thus $\widehat{a} \in \widehat{\mathfrak{A}}$ for every $a \in \mathfrak{A}$. The formulae in 314P tell us that if a, $b \in \mathfrak{A}$, then $\widehat{a} \cap \widehat{b}$, taken in $\widehat{\mathfrak{A}}$, is just the set-theoretic intersection $\widehat{a} \cap \widehat{b} = (a \cap b)\widehat{\ }$; while $1 \setminus \widehat{a}$, taken in $\widehat{\mathfrak{A}}$, is

$$Z \setminus \overline{\widehat{a}} = Z \setminus \widehat{a} = (1 \setminus a)\widehat{\ }.$$

And of course $\widehat{0} = \emptyset$ is the zero of $\widehat{\mathfrak{A}}$. Thus the map $a \mapsto \widehat{a} : \mathfrak{A} \to \widehat{\mathfrak{A}}$ preserves $\cap$ and complementation, so is a Boolean homomorphism (312H). Of course it is injective.

(ii) If $A \subseteq \mathfrak{A}$ is non-empty and $\inf A = 0$, then $\bigcap_{a \in A} \widehat{a}$ is nowhere dense in Z (313Cc), so

$$\inf\{\widehat{a} : a \in A\} = \mathrm{int}(\textstyle\bigcap_{a \in A} \widehat{a}) = \emptyset$$

(314P again). As A is arbitrary, the map $a \mapsto \widehat{a} : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is order-continuous.

(iii) If $G \in \widehat{\mathfrak{A}}$ is not empty, then there is a non-empty member of $\mathcal{E}$ included in it, by the definition of the topology of Z (311I). So $\mathcal{E}$ is an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

(b) Now suppose that $\mathfrak{B}$ is a Dedekind complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism. Write $\iota a = \widehat{a}$ for $a \in \mathfrak{A}$, so that $\iota : \mathfrak{A} \to \widehat{\mathfrak{A}}$ is an isomorphism between $\mathfrak{A}$ and the order-dense subalgebra $\mathcal{E}$ of $\widehat{\mathfrak{A}}$. Accordingly $\pi\iota^{-1} : \mathcal{E} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, being the composition of the order-continuous Boolean homomorphisms π and ι^{-1}. By 314K, it has an extension to a Boolean homomorphism $\pi_1 : \widehat{\mathfrak{A}} \to \mathfrak{B}$, and $\pi_1\iota = \pi$, that is, $\pi_1\widehat{a} = \pi a$ for every $a \in \mathfrak{A}$. Now π_1 is order-continuous. **P** Suppose that $\mathcal{H} \subseteq \widehat{\mathfrak{A}}$ has supremum 1 in $\widehat{\mathfrak{A}}$. Set

$$\mathcal{H}' = \{E : E \in \mathcal{E},\, E \subseteq H \text{ for some } H \in \mathcal{H}\}.$$

Because $\mathcal{E}$ is order-dense in $\widehat{\mathfrak{A}}$,

$$H = \sup_{E \in \mathcal{E}, E \subseteq H} E = \sup_{E \in \mathcal{H}', E \subseteq H} E$$

for every $H \in \mathcal{H}$ (313K), and $\sup \mathcal{H}' = 1$ in $\widehat{\mathfrak{A}}$. It follows at once that $\sup \mathcal{H}' = 1$ in $\mathcal{E}$, so $\sup \pi_1[\mathcal{H}'] = \sup(\pi\iota^{-1})[\mathcal{H}'] = 1$. Since any upper bound for $\pi_1[\mathcal{H}]$ must also be an upper bound for $\pi_1[\mathcal{H}']$, $\sup \pi_1[\mathcal{H}] = 1$ in $\mathfrak{B}$. As $\mathcal{H}$ is arbitrary, π_1 is order-continuous (313L(b-iii)). **Q**

If $\pi_1' : \widehat{\mathfrak{A}} \to \mathfrak{B}$ is any other Boolean homomorphism such that $\pi_1'\widehat{a} = \pi a$ for every $a \in \mathfrak{A}$, then π_1 and π_1' agree on $\mathcal{E}$, and the argument just above shows that π_1' is also order-continuous. But if $G \in \widehat{\mathfrak{A}}$, G is the supremum (in $\widehat{\mathfrak{A}}$) of $\mathcal{F} = \{E : E \in \mathcal{E},\, E \subseteq G\}$, so

$$\pi_1' G = \sup_{E \in \mathcal{F}} \pi_1' E = \sup_{E \in \mathcal{F}} \pi_1 E = \pi_1 G.$$

As G is arbitrary, $\pi_1' = \pi_1$. Thus π_1 is unique.

314U The Dedekind completion of a Boolean algebra (a) For any Boolean algebra $\mathfrak{A}$, I will say that the Boolean algebra $\widehat{\mathfrak{A}}$ constructed in 314T is the **Dedekind completion** of $\mathfrak{A}$.

When using this concept I shall frequently suppress the distinction between $a \in \mathfrak{A}$ and $\widehat{a} \in \widehat{\mathfrak{A}}$, and treat $\mathfrak{A}$ as itself an order-dense subalgebra of $\widehat{\mathfrak{A}}$.

(b) The universal mapping theorem in 314Tb assures us that the Dedekind completion is essentially unique. The commonest way in which this fact appears is the following. If $\mathfrak{C}$ is a Dedekind complete Boolean algebra and $\mathfrak{A}$ is an order-dense subalgebra of $\mathfrak{C}$, then the embedding $\mathfrak{A} \subsetneqq \mathfrak{C}$ induces an isomorphism from $\widehat{\mathfrak{A}}$ to $\mathfrak{C}$. **P** Write $\pi a = a$ for $a \in \mathfrak{A}$. Because $\mathfrak{A}$ is order-dense, π is order-continuous (313O), so extends to an order-continuous Boolean homomorphism $\pi_1 : \widehat{\mathfrak{A}} \to \mathfrak{C}$. If $b \in \widehat{\mathfrak{A}}$ is non-zero, there is a non-zero $a \in \mathfrak{A}$ such that $a \subseteq b$; now

$$0 \neq a = \pi a = \pi_1 a \subseteq \pi_1 b.$$

As b is arbitrary, π_1 is injective. Next, $\pi_1[\widehat{\mathfrak{A}}]$ must be order-closed in $\mathfrak{C}$, by 314F(a-i); since it includes $\mathfrak{A}$ and $\mathfrak{A}$ is order-dense in $\mathfrak{C}$, $\pi_1[\widehat{\mathfrak{A}}] = \mathfrak{C}$ and π_1 is an isomorphism. **Q**

(c) Looking at the construction in 314T from a different angle, we get the following. Suppose that Z is a zero-dimensional compact Hausdorff space, and $\mathcal{E}$ is the algebra of open-and-closed subsets of Z. Then $\mathcal{E}$ is order-dense in the regular open algebra $\mathrm{RO}(Z)$, so the Dedekind completion of $\mathcal{E}$ can be identified with $\mathrm{RO}(Z)$. (For by 311J we can identify Z with the Stone space of $\mathcal{E}$.)

314X Basic exercises >(a) Let $\mathfrak{A}$ be a Boolean algebra. (i) Show that the following are equiveridical: (α) $\mathfrak{A}$ is Dedekind complete (β) every upwards-directed subset of $\mathfrak{A}$ has a least upper bound (γ) every downwards-directed subset of $\mathfrak{A}$ has a greatest lower bound (δ) every disjoint subset of $\mathfrak{A}$ has a least upper bound. (ii) Show that the following are equiveridical: (α) $\mathfrak{A}$ is Dedekind σ-complete (β) every non-decreasing sequence in $\mathfrak{A}$ has a least upper bound (γ) every non-increasing sequence in $\mathfrak{A}$ has a greatest lower bound (δ) every disjoint sequence in $\mathfrak{A}$ has a least upper bound.

(b) Let $\mathfrak{A}$ be a Boolean algebra. Show that any principal ideal of $\mathfrak{A}$ is order-closed. Show that $\mathfrak{A}$ is Dedekind complete iff every order-closed ideal is principal.

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$, and $a \in \mathfrak{A}$; let $\mathfrak{A}_a$ be the principal ideal of $\mathfrak{A}$ generated by a. Show that $\{a \cap b : b \in \mathfrak{B}\}$ is an order-closed subalgebra of $\mathfrak{A}_a$.

>(d) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ a surjective order-continuous Boolean homomorphism. (i) Show that the kernel of π is a principal ideal in $\mathfrak{A}$. (ii) Show that $\mathfrak{B}$ is isomorphic to the complementary principal ideal in $\mathfrak{A}$, and in particular is Dedekind complete.

(e) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and $\mathfrak{C}$ an order-closed subalgebra of $\mathfrak{A}$. Show that an element a of $\mathfrak{A}$ belongs to $\mathfrak{C}$ iff $\operatorname{upr}(1 \setminus a, \mathfrak{C}) = 1 \setminus \operatorname{upr}(a, \mathfrak{C})$ iff $\operatorname{upr}(1 \setminus a, \mathfrak{C}) \cap \operatorname{upr}(a, \mathfrak{C}) = 0$, writing $\operatorname{upr}(a, \mathfrak{C})$ for the upper envelope of a in $\mathfrak{C}$, as in 313S.

>(f) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{C}$ an order-closed subalgebra of $\mathfrak{A}$, $a_0 \in \mathfrak{A}$ and $c_0 \in \mathfrak{C}$. Show that the following are equiveridical: (i) there is a Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi c = c$ for every $c \in \mathfrak{C}$ and $\pi a_0 = c_0$ (ii) $1 \setminus \operatorname{upr}(1 \setminus a_0, \mathfrak{C}) \subseteq c_0 \subseteq \operatorname{upr}(a_0, \mathfrak{C})$.

>(g) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, $\mathfrak{B}$ a Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ a sequentially order-continuous Boolean homomorphism. If $C \subseteq \mathfrak{A}$ and $\mathfrak{C}$ is the σ-subalgebra of $\mathfrak{A}$ generated by C, show that $\pi[\mathfrak{C}]$ is the σ-subalgebra of $\mathfrak{B}$ generated by $\pi[C]$.

(h) Let X and Y be extremally disconnected compact Hausdorff spaces, $\mathrm{RO}(X)$ and $\mathrm{RO}(Y)$ their regular open algebras, and $\phi : X \to Y$ a continuous surjection. Show that the following are equiveridical: (i) the Boolean homomorphism $V \mapsto \phi^{-1}[V]$ from $\mathrm{RO}(Y)$ to $\mathrm{RO}(X)$ (312Q, 314S) is order-continuous; (ii) $\phi[U]$ is open-and-closed in Y for every open-and-closed set $U \subseteq X$; (iii) $\phi[G]$ is open in Y for every open set $G \subseteq X$.

(i) Find a proof of 314Tb which does not appeal to 314K.

(j) Let $\mathfrak{B}$ be a Dedekind complete Boolean algebra, and $\mathfrak{A}$ a Boolean algebra which can be regularly embedded in $\mathfrak{B}$. Show that the Dedekind completion of $\mathfrak{A}$ can be regularly embedded in $\mathfrak{B}$.

(k) Let X be a topological space and Y a dense subset of X. Show that $G \mapsto G \cap Y$ is a Boolean isomorphism from $\mathrm{RO}(X)$ to $\mathrm{RO}(Y)$.

(l) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra, $\mathfrak{B}$ an order-closed subalgebra of $\mathfrak{A}$, c a member of $\mathfrak{A}$ and $\mathfrak{C}$ the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{c\}$. Show that $c \cap a = c \cap \operatorname{upr}(c \cap a, \mathfrak{B})$ for every $a \in \mathfrak{C}$.

314Y Further exercises (a) Let P be a Dedekind complete partially ordered set. Show that a set $Q \subseteq P$ is order-closed iff $\sup R$, $\inf R$ belong to Q whenever $R \subseteq Q$ is a totally ordered subset of Q with upper and lower bounds in P. (*Hint*: show by induction on κ that if $A \subseteq Q$ is upwards-directed and bounded above and $\#(A) \le \kappa$ then $\sup A \in Q$.)

(b) Let P be a lattice. Show that P is Dedekind complete iff every non-empty totally ordered subset of P with an upper bound in P has a least upper bound in P. (*Hint*: if $A \subseteq P$ is non-empty and bounded below in P, let B be the set of lower bounds of A and use Zorn's Lemma to find a maximal element of B.)

(c) Give an example of a Boolean algebra $\mathfrak{A}$ with an order-closed subalgebra $\mathfrak{A}_0$ and an element c such that the subalgebra generated by $\mathfrak{A}_0 \cup \{c\}$ is not order-closed.

(d) Let X be any topological space. Let $\mathcal{M}$ be the σ-ideal of meager subsets of X, and set

$$\widehat{\mathcal{B}} = \{G \triangle A : G \subseteq X \text{ is open, } A \in \mathcal{M}\}.$$

(i) Show that $\widehat{\mathcal{B}}$ is a σ-algebra of subsets of X, and that $\widehat{\mathcal{B}}/\mathcal{M}$ is Dedekind complete. (Members of $\widehat{\mathcal{B}}$ are said to be the subsets of X **with the Baire property**; $\widehat{\mathcal{B}}$ is the **Baire-property algebra** of X.) (ii) Show that if $A \subseteq X$ and $\bigcup\{G : G \subseteq X \text{ is open, } A \cap G \in \widehat{\mathcal{B}}\}$ is dense, then $A \in \widehat{\mathcal{B}}$. (iii) Show that there is a largest open set $V \in \mathcal{M}$. (iv) Let $\mathrm{RO}(X)$ be the regular open algebra of X. Show that the map $G \mapsto G^\bullet$ is an order-continuous Boolean homomorphism from $\mathrm{RO}(X)$ onto $\widehat{\mathcal{B}}/\mathcal{M}$, so induces a Boolean isomorphism between the principal ideal of $\mathrm{RO}(X)$ generated by $X \setminus \overline{V}$ and $\widehat{\mathcal{B}}/\mathcal{M}$. ($\widehat{\mathcal{B}}/\mathcal{M}$ is the **category algebra** of X; it is a Dedekind complete Boolean algebra. X is called a **Baire space** if $V = \emptyset$; in this case $\mathrm{RO}(X) \cong \widehat{\mathcal{B}}/\mathcal{M}$. See 4A3S in Volume 4.)

(e) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and $\langle a_n\rangle_{n\in\mathbb{N}}$ any sequence in $\mathfrak{A}$. For $n \in \mathbb{N}$ set $E_n = \{x : x \in \{0,1\}^{\mathbb{N}}, x(n) = 1\}$, and let $\mathcal{B}$ be the σ-algebra of subsets of $\{0,1\}^{\mathbb{N}}$ generated by $\{E_n : n \in \mathbb{N}\}$. ($\mathcal{B}$ is the 'Borel σ-algebra' of $\{0,1\}^{\mathbb{N}}$; see 4A3E in Volume 4.) Show that there is a unique sequentially order-continuous Boolean homomorphism $\theta : \mathcal{B} \to \mathfrak{A}$ such that $\theta(E_n) = a_n$ for every $n \in \mathbb{N}$. (*Hint*: define a suitable function ϕ from the Stone space Z of $\mathfrak{A}$ to $\{0,1\}^{\mathbb{N}}$, and consider $\{E : E \subseteq \{0,1\}^{\mathbb{N}}, \phi^{-1}[E]$ has the Baire property in $Z\}$.) Show that $\theta[\mathcal{B}]$ is the σ-subalgebra of $\mathfrak{A}$ generated by $\{a_n : n \in \mathbb{N}\}$.

(f) Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Show that $\mathfrak{A}$ is Dedekind σ-complete iff $\overline{G}$ is open whenever G is a cozero set in Z. (Such spaces are called **basically disconnected** or **quasi-Stonian**.)

(g) Let $\mathfrak{A}$, $\mathfrak{B}$ be Dedekind complete Boolean algebras and $D \subseteq \mathfrak{A}$ an order-dense set. Suppose that $\phi : D \to \mathfrak{B}$ is such that (i) $\phi[D]$ is order-dense in $\mathfrak{B}$ (ii) for all $d, d' \in D$, $d \cap d' = 0$ iff $\phi d \cap \phi d' = 0$. Show that ϕ has a unique extension to a Boolean isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

(h) Let $\mathfrak{A}$ be any Boolean algebra. Let $\mathcal{J}$ be the family of order-closed ideals in $\mathfrak{A}$. Show that (i) $\mathcal{J}$ is a Dedekind complete Boolean algebra with operations defined by the formulae $I \cap J = I \cap J$, $1 \setminus J = \{a : a \cap b = 0$ for every $b \in J\}$ (ii) the map $a \mapsto \mathfrak{A}_a$, the principal ideal generated by a, is an injective order-continuous Boolean homomorphism from $\mathfrak{A}$ onto an order-dense subalgebra of $\mathcal{J}$ (iii) $\mathcal{J}$ is isomorphic to the Dedekind completion of $\mathfrak{A}$.

314 Notes and comments At the risk of being tiresomely long-winded, I have taken the trouble to spell out a large proportion of the results in this section and the last in their 'sequential' as well as their 'unrestricted' forms. The point is that while (in my view) the underlying ideas are most clearly and dramatically expressed in terms of order-closed sets, order-continuous functions and Dedekind complete algebras, a large proportion of the applications in measure theory deal with sequentially order-closed sets, sequentially order-continuous functions and Dedekind σ-complete algebras. As a matter of simple technique, therefore, it is necessary to master both, and for the sake of later reference I generally give the statements of both versions in full. Perhaps the points to look at most keenly are just those where there is a difference in the ideas involved, as in 314Bb, or in which there is only one version given, as in 314M and 314T.

If you have seen the Hahn-Banach theorem (3A5A), it may have been recalled to your mind by Theorem 314K; in both cases we use an order relation and a bit of algebra to make a single step towards an extension of a function, and Zorn's lemma to turn this into the extension we seek. A good part of this section has turned out to be on the borderland between the theory of Boolean algebra and general topology; naturally enough, since (as always with the general theory of Boolean algebra) one of our first concerns is to establish connexions between algebras and their Stone spaces.

I think 314T is the first substantial 'universal mapping theorem' in this volume; it is by no means the last. The idea of the construction $\widehat{\mathfrak{A}}$ is not just that we obtain a Dedekind complete Boolean algebra in which $\mathfrak{A}$ is embedded as an order-dense subalgebra, but that we simultaneously obtain a theorem on the canonical extension to $\widehat{\mathfrak{A}}$ of order-continuous Boolean homomorphisms defined on $\mathfrak{A}$. This characterization is enough to define the pair $(\widehat{\mathfrak{A}}, a \mapsto \widehat{a})$ up to isomorphism, so the exact method of construction of $\widehat{\mathfrak{A}}$ becomes of secondary importance. The one used in 314T is very natural (at least, if we believe in Stone spaces), but there are others (see 314Yh), with different virtues.

314K and 314T both describe circumstances in which we can find extensions of Boolean homomorphisms. Clearly such results are fundamental in the theory of Boolean algebras, but I shall not attempt any systematic presentation here. 314Ye can also be regarded as belonging to this family of ideas.

Version of 13.11.12

315 Products and free products

I describe here two algebraic constructions of fundamental importance. They are very different in character, indeed may be regarded as opposites, despite the common use of the word 'product'. The first part of the section (315A-315H) deals with the easier construction, the 'simple product'; the second part (315I-315Q) with the 'free product'. These constructions lead to descriptions of projective and inductive limits (315R-315S).

315A Products of Boolean algebras (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras. Set $\mathfrak{A} = \prod_{i\in I}\mathfrak{A}_i$, with the natural ring structure

$$a \triangle b = \langle a(i) \triangle b(i)\rangle_{i\in I},$$

$$a \cap b = \langle a(i) \cap b(i)\rangle_{i\in I}$$

for a, $b \in \mathfrak{A}$. Then $\mathfrak{A}$ is a ring (3A2H); it is a Boolean ring because

$$a \cap a = \langle a(i) \cap a(i)\rangle_{i\in I} = a$$

for every $a \in \mathfrak{A}$; and it is a Boolean algebra because if we set $1_{\mathfrak{A}} = \langle 1_{\mathfrak{A}_i}\rangle_{i\in I}$, then $1_{\mathfrak{A}} \cap a = a$ for every $a \in \mathfrak{A}$. I will call $\mathfrak{A}$ the **simple product** of the family $\langle\mathfrak{A}_i\rangle_{i\in I}$.

I should perhaps remark that when $I = \emptyset$ then $\mathfrak{A}$ becomes $\{\emptyset\}$, to be interpreted as the singleton Boolean algebra.

(b) The Boolean operations on $\mathfrak{A}$ are now defined by the formulae

$$a \cup b = \langle a(i) \cup b(i)\rangle_{i\in I}, \quad a \setminus b = \langle a(i) \setminus b(i)\rangle_{i\in I}$$

for all a, $b \in \mathfrak{A}$.

315B Theorem Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and $\mathfrak{A}$ their simple product.

(a) The maps $a \mapsto \pi_i(a) = a(i) : \mathfrak{A} \to \mathfrak{A}_i$ are all Boolean homomorphisms.

(b) If $\mathfrak{B}$ is any other Boolean algebra, then a map $\phi : \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism iff $\pi_i\phi : \mathfrak{B} \to \mathfrak{A}_i$ is a Boolean homomorphism for every $i \in I$.

proof Verification of these facts amounts just to applying the definitions with attention.

315C Products of partially ordered sets (a) It is perhaps worth spelling out the following elementary definition. If $\langle P_i\rangle_{i\in I}$ is any family of partially ordered sets, its **product** is the set $P = \prod_{i\in I} P_i$ ordered by saying that $p \le q$ iff $p(i) \le q(i)$ for every $i \in I$; it is easy to check that P is now a partially ordered set.

(b) The point is that if $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, then the ordering of $\mathfrak{A}$ is just the product partial order:

$$a \subseteq b \iff a \cap b = a \iff a(i) \cap b(i) = a(i)\ \forall\, i \in I \iff a(i) \subseteq b(i)\ \forall\, i \in I.$$

Now we have the following elementary, but extremely useful, general facts about products of partially ordered sets.

315D Proposition Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets with product P.

(a) For any non-empty set $A \subseteq P$ and $q \in P$,

(i) $\sup A = q$ in P iff $\sup_{p\in A} p(i) = q(i)$ in P_i for every $i \in I$,

(ii) $\inf A = q$ in P iff $\inf_{p\in A} p(i) = q(i)$ in P_i for every $i \in I$.

(b) The coordinate maps $p \mapsto \pi_i(p) = p(i) : P \to P_i$ are all order-preserving and order-continuous.

(c) For any partially ordered set Q and function $\phi : Q \to P$, ϕ is order-preserving iff $\pi_i\phi$ is order-preserving for every $i \in I$.

(d) For any partially ordered set Q and order-preserving function $\phi : Q \to P$,

(i) ϕ is order-continuous iff $\pi_i\phi$ is order-continuous for every i,

(ii) ϕ is sequentially order-continuous iff $\pi_i\phi$ is sequentially order-continuous for every i.

(e)(i) P is Dedekind complete iff every P_i is Dedekind complete.

(ii) P is Dedekind σ-complete iff every P_i is Dedekind σ-complete.

proof All these are elementary verifications. Of course parts (b), (d) and (e) rely on (a).

315E Factor algebras as principal ideals Because Boolean algebras have least elements, we have a second type of canonical homomorphism associated with their products. If $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a family of Boolean algebras with simple product $\mathfrak{A}$, define $\theta_i : \mathfrak{A}_i \to \mathfrak{A}$ by setting $(\theta_i a)(i) = a$, $(\theta_i a)(j) = 0_{\mathfrak{A}_j}$ if $i \in I$, $a \in \mathfrak{A}_i$ and $j \in I \setminus \{i\}$. Each θ_i is a ring homomorphism, and is a Boolean isomorphism between $\mathfrak{A}_i$ and the principal ideal of $\mathfrak{A}$ generated by $\theta_i(1_{\mathfrak{A}_i})$. The family $\langle\theta_i(1_{\mathfrak{A}_i})\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$.

Associated with these embeddings is the following important result.

315F Proposition Let $\mathfrak{A}$ be a Boolean algebra and $\langle e_i\rangle_{i\in I}$ a partition of unity in $\mathfrak{A}$. Suppose
either (i) that I is finite
or (ii) that I is countable and $\mathfrak{A}$ is Dedekind σ-complete
or (iii) that $\mathfrak{A}$ is Dedekind complete.
Then the map $a\mapsto\langle a\cap e_i\rangle_{i\in I}$ is a Boolean isomorphism between $\mathfrak{A}$ and $\prod_{i\in I}\mathfrak{A}_{e_i}$, writing $\mathfrak{A}_{e_i}$ for the principal ideal of $\mathfrak{A}$ generated by e_i for each i.

proof The given map is a Boolean homomorphism because each of the maps $a\mapsto a\cap e_i:\mathfrak{A}\to\mathfrak{A}_{e_i}$ is (312J). It is injective because $\sup_{i\in I}e_i=1$, so if $a\in\mathfrak{A}\setminus\{0\}$ there is an i such that $a\cap e_i\ne 0$. It is surjective because $\langle e_i\rangle_{i\in I}$ is disjoint and if $c\in\prod_{i\in I}\mathfrak{A}_{e_i}$ then $a=\sup_{i\in I}c(i)$ is defined in $\mathfrak{A}$ and

$$a\cap e_j=\sup\nolimits_{i\in I}c(i)\cap e_j=c(j)$$

for every $j\in I$ (using 313Ba). The three alternative versions of the hypotheses of this proposition are designed to ensure that the supremum is always well-defined in $\mathfrak{A}$.

315G Algebras of sets and their quotients The Boolean algebras of measure theory are mostly presented as algebras of sets or quotients of algebras of sets, so it is perhaps worth spelling out the ways in which the product construction applies to such algebras.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i an algebra of subsets of X_i for each i.

(a) The simple product $\prod_{i\in I}\Sigma_i$ may be identified with the algebra

$$\Sigma=\{E:E\subseteq X,\,\{x:(x,i)\in E\}\in\Sigma_i\text{ for every }i\in I\}$$

of subsets of $X=\{(x,i):i\in I,\,x\in X_i\}$, with the canonical homomorphisms $\pi_i:\Sigma\to\Sigma_i$ being given by

$$\pi_iE=\{x:(x,i)\in E\}$$

for each $E\in\Sigma$.

(b) Now suppose that $\mathcal{J}_i$ is an ideal of Σ_i for each i. Then $\prod_{i\in I}\Sigma_i/\mathcal{J}_i$ may be identified with $\Sigma/\mathcal{J}$, where

$$\mathcal{J}=\{E:E\in\Sigma,\,\{x:(x,i)\in E\}\in\mathcal{J}_i\text{ for every }i\in I\},$$

and the canonical homomorphisms $\tilde\pi_i:\Sigma/\mathcal{J}\to\Sigma_i/\mathcal{J}_i$ are given by the formula $\tilde\pi_i(E^\bullet)=(\pi_iE)^\bullet$ for every $E\in\Sigma$.

proof (a) It is easy to check that Σ is a subalgebra of $\mathcal{P}X$, and that the map $E\mapsto\langle\pi_iE\rangle_{i\in I}:\Sigma\to\prod_{i\in I}\Sigma_i$ is a Boolean isomorphism.

(b) Again, it is easy to check that $\mathcal{J}$ is an ideal of Σ, that the proposed formula for $\tilde\pi_i$ does indeed define a map from $\Sigma/\mathcal{J}$ to $\Sigma_i/\mathcal{J}_i$, and that $E^\bullet\mapsto\langle\tilde\pi_iE^\bullet\rangle_{i\in I}$ is an isomorphism between $\Sigma/\mathcal{J}$ and $\prod_{i\in I}\Sigma_i/\mathcal{J}_i$.

***315H** There is a particular kind of simple product which arises naturally when we look at regular open algebras.

Proposition Let X be a topological space, and $\mathcal{U}$ a disjoint family of open subsets of X with union dense in X. Then the regular open algebra $\mathrm{RO}(X)$ is isomorphic to the simple product $\prod_{U\in\mathcal{U}}\mathrm{RO}(U)$.

proof By 314R(b-i), $G\mapsto G\cap U$ is a Boolean homomorphism from $\mathrm{RO}(X)$ onto $\mathrm{RO}(U)$, for any $U\in\mathcal{U}$. By 315B, we have a Boolean homomorphism $G\mapsto\pi G=\langle G\cap U\rangle_{U\in\mathcal{U}}:\mathrm{RO}(X)\to\prod_{U\in\mathcal{U}}\mathrm{RO}(U)$. If $G\in\mathrm{RO}(X)\setminus\{\emptyset\}$, then $G\cap\bigcup\mathcal{U}\ne\emptyset$, because $\bigcup\mathcal{U}$ is dense; now there is a $U\in\mathcal{U}$ such that $G\cap U\ne\emptyset$, so πG is non-zero in the Boolean algebra $\prod_{U\in\mathcal{U}}\mathrm{RO}(U)$. As G is arbitrary, π is injective (3A2Db).

To see that π is surjective, suppose that we are given a family $\langle V_U\rangle_{U\in\mathcal{U}}$ with $V_U\in\mathrm{RO}(U)$ for every $U\in\mathcal{U}$. Set $H=\bigcup_{U\in\mathcal{U}}V_U$, $G=\operatorname{int}\overline{H}\in\mathrm{RO}(X)$. Then, for any $U\in\mathcal{U}$, (writing int_U and $\overline{}^{(U)}$ for interior and closure in the subspace topology on U, as in part (b) of the proof of 314R)

$$G\cap U=U\cap\operatorname{int}\overline{H}=\operatorname{int}_U\overline{H\cap U}^{(U)}=\operatorname{int}_U\overline{V}_U^{(U)}=V_U,$$

so $\pi G=\langle V_U\rangle_{U\in\mathcal{U}}$. Thus π is bijective and is a Boolean isomorphism.

315I Free products I come now to the second construction of this section.

(a) Definition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras. For each $i \in I$, let Z_i be the Stone space of $\mathfrak{A}_i$. Set $Z = \prod_{i\in I} Z_i$, with the product topology. Then the **free product** of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is the algebra $\mathfrak{A}$ of open-and-closed sets in Z; I will denote it by $\bigotimes_{i\in I}\mathfrak{A}_i$.

(b) For $i \in I$ and $a \in \mathfrak{A}_i$, the set $\widehat{a} \subseteq Z_i$ representing a is an open-and-closed subset of Z_i; because $z \mapsto z(i) : Z \to Z_i$ is continuous, $\varepsilon_i(a) = \{z : z(i) \in \widehat{a}\}$ is open-and-closed, so belongs to $\mathfrak{A}$. In this context I will call $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ the **canonical map**.

(c) The topological space Z may be identified with the Stone space of the Boolean algebra $\mathfrak{A}$. **P** By Tychonoff's theorem (3A3J), Z is compact. If $z \in Z$ and G is an open subset of Z containing z, then there are J, $\langle G_j\rangle_{i\in J}$ such that J is a finite subset of I, G_j is an open subset of Z_j for each $j \in J$, and

$$z \in \{w : w \in Z,\ w(j) \in G_j \text{ for every } j \in J\} \subseteq G.$$

Because each Z_j is zero-dimensional, we can find an open-and-closed set $E_j \subseteq Z_j$ such that $z(j) \in E_j \subseteq G_j$. Now

$$H = Z \cap \bigcap_{j\in J}\{w : w(j) \in E_j\}$$

is a finite intersection of open-and-closed subsets of Z, so is open-and-closed; and $z \in H \subseteq G$. As z and G are arbitrary, Z is zero-dimensional. Finally, Z, being the product of Hausdorff spaces, is Hausdorff. So the result follows from 311J. **Q**

315J Theorem Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with free product $\mathfrak{A}$.

(a) The canonical map $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ is a Boolean homomorphism for every $i \in I$.

(b) For any Boolean algebra $\mathfrak{B}$ and any family $\langle\phi_i\rangle_{i\in I}$ such that ϕ_i is a Boolean homomorphism from $\mathfrak{A}_i$ to $\mathfrak{B}$ for every i, there is a unique Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ such that $\phi_i = \phi\varepsilon_i$ for each i.

proof These are both consequences of 312Q-312R. As in 315I, write Z_i for the Stone space of $\mathfrak{A}$, and Z for $\prod_{i\in I} Z_i$, identified with the Stone space of $\mathfrak{A}$, as observed in 315Ic. The maps $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ are defined as the homomorphisms corresponding to the continuous maps $z \mapsto \tilde{\varepsilon}_i(z) = z(i) : Z \to Z_i$, so (a) is surely true.

Now suppose that we are given a Boolean homomorphism $\phi_i : \mathfrak{A}_i \to \mathfrak{B}$ for each $i \in I$. Let W be the Stone space of $\mathfrak{B}$, and let $\tilde{\phi}_i : W \to Z_i$ be the continuous function corresponding to ϕ_i. By 3A3Ib, the map $w \mapsto \tilde{\phi}(w) = \langle\tilde{\phi}_i(w)\rangle_{i\in I} : W \to Z$ is continuous, and corresponds to a Boolean homomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$; because $\tilde{\phi}_i = \tilde{\varepsilon}_i\tilde{\phi}$, $\phi\varepsilon_i = \phi_i$ for each i. Moreover, ϕ is the only Boolean homomorphism with this property, because if $\psi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean homomorphism such that $\psi\varepsilon_i = \phi_i$ for every i, then ψ corresponds to a continuous function $\tilde{\psi} : W \to Z$, and we must have $\tilde{\varepsilon}_i\tilde{\psi} = \tilde{\phi}_i$ for each i, so that $\tilde{\psi} = \tilde{\phi}$ and $\psi = \phi$. This proves (b).

315K Of course 315J is the defining property of the free product (see 315Xi below). I list a few further basic facts.

Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and $\mathfrak{A}$ their free product; write $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ for the canonical homomorphisms.

(a) $\mathfrak{A}$ is the subalgebra of itself generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{A}_i]$.

(b) Write C for the set of those members of $\mathfrak{A}$ expressible in the form $\inf_{j\in J}\varepsilon_j(a_j)$, where $J \subseteq I$ is finite and $a_j \in \mathfrak{A}_j$ for every j. Then every member of $\mathfrak{A}$ is expressible as the supremum of a disjoint finite subset of C. In particular, C is order-dense in $\mathfrak{A}$.

(c) Every ε_i is order-continuous.

(d) $\mathfrak{A} = \{0_{\mathfrak{A}}\}$ iff there is some $i \in I$ such that $\mathfrak{A}_i = \{0_{\mathfrak{A}_i}\}$.

(e) Now suppose that $\mathfrak{A}_i \ne \{0_{\mathfrak{A}_i}\}$ for every $i \in I$.

(i) ε_i is injective for every $i \in I$.

(ii) If $J \subseteq I$ is finite and a_j is a non-zero member of $\mathfrak{A}_j$ for each $j \in J$, then $\inf_{j\in J}\varepsilon_j(a_j) \ne 0$.

(iii) If i, j are distinct members of I, $a \in \mathfrak{A}_i$ and $b \in \mathfrak{A}_j$, then $\varepsilon_i(a) = \varepsilon_j(b)$ iff either $a = 0_{\mathfrak{A}_i}$ and $b = 0_{\mathfrak{A}_j}$ or $a = 1_{\mathfrak{A}_i}$ and $b = 1_{\mathfrak{A}_j}$.

proof As usual, write Z_i for the Stone space of $\mathfrak{A}_i$, and $Z = \prod_{i\in I} Z_i$, identified with the Stone space of $\mathfrak{A}$ (315Ic).

(a) Write $\mathfrak{A}'$ for the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\varepsilon_i[\mathfrak{A}_i]$. Then $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}'$ is a Boolean homomorphism for each i, so by 315Jb there is a Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{A}'$ such that $\phi\varepsilon_i=\varepsilon_i$ for each i. Now, regarding ϕ as a Boolean homomorphism from $\mathfrak{A}$ to itself, the uniqueness assertion of 315Jb (with $\mathfrak{B}=\mathfrak{A}$) shows that ϕ must be the identity, so that $\mathfrak{A}'=\mathfrak{A}$.

(b) Write $\mathcal{D}$ for the set of finite partitions of unity in $\mathfrak{A}$ consisting of members of C, and A for the set of members of $\mathfrak{A}$ expressible in the form $\sup D'$ where D' is a subset of a member of $\mathcal{D}$. Then A is a subalgebra of $\mathfrak{A}$. **P** (i) $1_{\mathfrak{A}}\in C$ (set $J=\emptyset$ in the definition of members of C) so $\{1_{\mathfrak{A}}\}\in\mathcal{D}$ and $0_{\mathfrak{A}}$, $1_{\mathfrak{A}}\in A$. (ii) Note that if c, $d\in C$ then $c\cap d\in C$. (iii) If a, $b\in A$, express them as $\sup D'$, $\sup E'$ where $D'\subseteq D\in\mathcal{D}$, $E'\subseteq E\in\mathcal{D}$. Then

$$F=\{d\cap e:d\in D,\,e\in E\}\in\mathcal{D},$$

so

$$1_{\mathfrak{A}}\setminus a=\sup(D\setminus D')\in A,$$

$$a\cup b=\sup\{f:f\in F,\,f\subseteq a\cup b\}\in A.\ \mathbf{Q}$$

Also, $\varepsilon_i[\mathfrak{A}_i]\subseteq A$ for each $i\in I$. **P** If $a\in\mathfrak{A}_i$, then $\{\varepsilon_i(a),\varepsilon_i(1_{\mathfrak{A}_i}\setminus a)\}\in\mathcal{D}$, so $\varepsilon_i(a)\in A$. **Q**

So (a) tells us that $A=\mathfrak{A}$, and every member of $\mathfrak{A}$ is a finite disjoint union of members of C.

(c) If $i\in I$ and $A\subseteq\mathfrak{A}_i$ and $\inf A=0$ in $\mathfrak{A}_i$, take any non-zero $c\in\mathfrak{A}$. By (b), we can find a finite $J\subseteq I$ and a family $\langle a_j\rangle_{j\in J}$ such that $c'=\inf_{j\in J}\varepsilon_j(a_j)\subseteq c$ and $c'\ne 0$. Regarding c' as a subset of Z, we have a point $z\in c'$. Adding i to J and setting $a_i=1_{\mathfrak{A}_i}$ if necessary, we may suppose that $i\in J$. Now $c'\ne 0_{\mathfrak{A}}$ so $a_i\ne 0_{\mathfrak{A}_i}$ and there is an $a\in A$ such that $a_i\not\subseteq a$, so there is a $t\in\widehat{a}_i\setminus\widehat{a}$. In this case, setting $w(i)=t$, $w(j)=z(j)$ for $j\ne i$, we have $w\in c'\setminus\varepsilon_i(a)$, and c', c are not included in $\varepsilon_i(a)$. As c is arbitrary, this shows that $\inf\varepsilon_i[A]=0$. As A is arbitrary, ε_i is order-continuous.

(d) The point is that $\mathfrak{A}=\{0_{\mathfrak{A}}\}$ iff $Z=\emptyset$, which is so iff some Z_i is empty.

(e)(i) Because no Z_i is empty, all the coordinate maps from Z to Z_i are surjective, so the corresponding homomorphisms ε_i are injective (312Sa).

(ii) Because J is finite,

$$\inf_{j\in J}\varepsilon_j(a_j)=\{z:z\in Z,\,z(j)\in\widehat{a}_j\text{ for every }j\in J\}$$

is not empty.

(iii) If $\varepsilon_i(a)=\varepsilon_j(b)=0_{\mathfrak{A}}$ then (using (i)) $a=0_{\mathfrak{A}_i}$ and $b=0_{\mathfrak{A}_j}$; if $\varepsilon_i(a)=\varepsilon_j(b)=1_{\mathfrak{A}}$ then $a=1_{\mathfrak{A}_i}$ and $b=1_{\mathfrak{A}_j}$. **?** If $\varepsilon_i(a)=\varepsilon_j(b)\in\mathfrak{A}\setminus\{0_{\mathfrak{A}},1_{\mathfrak{A}}\}$, then there are $t\in\widehat{a}$ and $u\in Z_j\setminus\widehat{b}$. Now there is a $z\in Z$ such that $z(i)=t$ and $z(j)=u$, so that $z\in\varepsilon_i(a)\setminus\varepsilon_j(b)$. **X**

315L Proposition Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, and $\langle J_k\rangle_{k\in K}$ any partition (that is, disjoint cover) of I. Then the free product $\mathfrak{A}$ of $\langle\mathfrak{A}_i\rangle_{i\in I}$ is isomorphic to the free product $\mathfrak{B}$ of $\langle\mathfrak{B}_k\rangle_{k\in K}$, where each $\mathfrak{B}_k$ is the free product of $\langle\mathfrak{A}_i\rangle_{i\in J_k}$.

proof Write $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$, $\varepsilon'_i:\mathfrak{A}_i\to\mathfrak{B}_k$ and $\delta_k:\mathfrak{B}_k\to\mathfrak{B}$ for the canonical homomorphisms when $k\in K$ and $i\in J_k$. Then the homomorphisms $\delta_k\varepsilon'_i:\mathfrak{A}_i\to\mathfrak{B}$ correspond to a homomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$ such that $\phi\varepsilon_i=\delta_k\varepsilon'_i$ whenever $i\in J_k$. Next, for each k, the homomorphisms $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$, for $i\in J_k$, correspond to a homomorphism $\psi_k:\mathfrak{B}_k\to\mathfrak{A}$ such that $\psi_k\varepsilon'_i=\varepsilon_i$ for $i\in J_k$; and the family $\langle\psi_k\rangle_{k\in K}$ corresponds to a homomorphism $\psi:\mathfrak{B}\to\mathfrak{A}$ such that $\psi\delta_k=\psi_k$ for $k\in K$. Consequently

$$\psi\phi\varepsilon_i=\psi\delta_k\varepsilon'_i=\psi_k\varepsilon'_i=\varepsilon_i$$

whenever $k\in K$, $i\in J_k$. Once again using the uniqueness assertion in 315Jb, $\psi\phi$ is the identity homomorphism on $\mathfrak{A}$. On the other hand, if we look at $\phi\psi:\mathfrak{B}\to\mathfrak{B}$, then we see that

$$\phi\psi\delta_k\varepsilon'_i=\phi\psi_k\varepsilon'_i=\phi\varepsilon_i=\delta_k\varepsilon'_i$$

whenever $k\in K$, $i\in J_k$. Now, for given k, $\{b:b\in\mathfrak{B}_k,\,\phi\psi\delta_k b=\delta_k b\}$ is a subalgebra of $\mathfrak{B}_k$ including $\bigcup_{i\in J_k}\varepsilon'_i[\mathfrak{A}_i]$, and must be the whole of $\mathfrak{B}_k$, by 315Ka. So $\{b:b\in\mathfrak{B},\,\phi\psi b=b\}$ is a subalgebra of $\mathfrak{B}$ including $\bigcup_{k\in K}\delta_k[\mathfrak{B}_k]$, and is the whole of $\mathfrak{B}$. Thus $\phi\psi$ is the identity on $\mathfrak{B}$ and ϕ, ψ are the two halves of an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

315M Algebras of sets and their quotients Once again I devote a paragraph to spelling out the application of the construction to the algebras most important to us.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i an algebra of subsets of X_i for each i.

(a) The free product $\bigotimes_{i\in I}\Sigma_i$ may be identified with the algebra Σ of subsets of $X=\prod_{i\in I}X_i$ generated by the set $\{\varepsilon_i(E):i\in I,\,E\in\Sigma_i\}$, where $\varepsilon_i(E)=\{x:x\in X,\,x(i)\in E\}$.

(b) Now suppose that $\mathcal{J}_i$ is an ideal of Σ_i for each i. Then $\bigotimes_{i\in I}\Sigma_i/\mathcal{J}_i$ may be identified with $\Sigma/\mathcal{J}$, where $\mathcal{J}$ is the ideal of Σ generated by $\{\varepsilon_i(E):i\in I,\,E\in\mathcal{J}_i\}$; the corresponding canonical maps $\tilde\varepsilon_i:\Sigma_i/\mathcal{J}_i\to\Sigma/\mathcal{J}$ being defined by the formula $\tilde\varepsilon_i(E^\bullet)=(\varepsilon_i(E))^\bullet$ for $i\in I$, $E\in\Sigma_i$.

proof I start by proving (b) in detail; the argument for (a) is then easy to extract. Write $\mathfrak{A}_i=\Sigma_i/\mathcal{J}_i$, $\mathfrak{A}=\Sigma/\mathcal{J}$.

(i) Fix $i\in I$ for the moment. By the definition of Σ, $\varepsilon_i(E)\in\Sigma$ for $E\in\Sigma_i$, and it is easy to check that $\varepsilon_i:\Sigma_i\to\Sigma$ is a Boolean homomorphism. Again, because $\varepsilon_i(E)\in\mathcal{J}$ whenever $E\in\mathcal{J}_i$, the kernel of the homomorphism $E\mapsto(\varepsilon_i(E))^\bullet:\Sigma_i\to\mathfrak{A}$ includes $\mathcal{J}_i$, so the formula for $\tilde\varepsilon_i$ defines a homomorphism from $\mathfrak{A}_i$ to $\mathfrak{A}$.

Now let $\mathfrak{C}=\bigotimes_{i\in I}\mathfrak{A}_i$ be the free product, and write $\varepsilon'_i:\mathfrak{A}_i\to\mathfrak{C}$ for the canonical homomorphisms. By 315J, there is a Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{A}$ such that $\phi\varepsilon'_i=\tilde\varepsilon_i$ for each i. The set

$$\{H:H\in\Sigma,\,H^\bullet\in\phi[\mathfrak{C}]\}$$

is a subalgebra of Σ including $\varepsilon_i[\Sigma_i]$ for every i, so is Σ itself, and ϕ is surjective.

(ii) We need a simple description of the ideal $\mathcal{J}$, as follows: a set $H\in\Sigma$ belongs to $\mathcal{J}$ iff there are a finite $K\subseteq I$ and a family $\langle F_k\rangle_{k\in K}$ such that $F_k\in\mathcal{J}_k$ for each k and $H\subseteq\bigcup_{k\in K}\varepsilon_k(F_k)$. For evidently such sets have to belong to $\mathcal{J}$, since the $\varepsilon_k(F_k)$ will be in $\mathcal{J}$, while the family of all these sets is an ideal containing $\varepsilon_i(F)$ whenever $i\in I$ and $F\in\mathcal{J}_i$.

(iii) Now we can see that $\phi:\mathfrak{C}\to\mathfrak{A}$ is injective. **P** Take any non-zero $c\in\mathfrak{C}$. By 315Kb, we can find a finite $J\subseteq I$ and a family $\langle a_j\rangle_{j\in J}$ in $\prod_{j\in J}\mathfrak{A}_j$ such that $0\ne\inf_{j\in J}\varepsilon'_ja_j\subseteq c$. Express each a_j as $E_j^\bullet$, where $E_j\in\Sigma_j$, and consider $H=X\cap\bigcap_{j\in J}\varepsilon_j(E_j)\in\Sigma$. Then

$$H^\bullet=\inf_{j\in J}\tilde\varepsilon_ja_j=\phi(\inf_{j\in J}\varepsilon'_ja_j)\subseteq\phi(c).$$

Also, because $\varepsilon'_ja_j\ne0$, $E_j\notin\mathcal{J}_j$ for each j. But it follows that $H\notin\mathcal{J}$, because if $K\subseteq I$ is finite and $F_k\in\mathcal{J}_k$ for each $k\in K$, set $E_i=X_i$ for $i\in I\setminus J$, $F_i=\emptyset$ for $i\in I\setminus K$; then there is an $x\in X$ such that $x(i)\in E_i\setminus F_i$ for each $i\in I$, so that $x\in H\setminus\bigcup_{k\in K}\varepsilon_k(F_k)$. By the criterion of (ii), $H\notin\mathcal{J}$. So

$$0\ne E^\bullet\subseteq\phi(c).$$

As c is arbitrary, the kernel of ϕ is $\{0\}$, and ϕ is injective. **Q**

So $\phi:\mathfrak{C}\to\mathfrak{A}$ is the required isomorphism.

(iv) This proves (b). Reading through the arguments above, it is easy to see the simplifications which compose a proof of (a), reading Σ_i for $\mathfrak{A}_i$ and $\{\emptyset\}$ for $\mathcal{J}_i$.

315N Notation Free products are sufficiently surprising that I think it worth taking a moment to look at a pair of examples relevant to the kinds of application I wish to make of the concept in the next chapter. First let me introduce a somewhat more direct notation which seems appropriate for the free product of finitely many factors. If $\mathfrak{A}$ and $\mathfrak{B}$ are two Boolean algebras, I write $\mathfrak{A}\otimes\mathfrak{B}$ for their free product, and for $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ I write $a\otimes b$ for $\varepsilon_1(a)\cap\varepsilon_2(b)$, where $\varepsilon_1:\mathfrak{A}\to\mathfrak{A}\otimes\mathfrak{B}$, $\varepsilon_2:\mathfrak{B}\to\mathfrak{A}\otimes\mathfrak{B}$ are the canonical maps. Observe that $(a_1\otimes b_1)\cap(a_2\otimes b_2)=(a_1\cap a_2)\otimes(b_1\cap b_2)$, and that the maps $a\mapsto a\otimes b_0$, $b\mapsto a_0\otimes b$ are always ring homomorphisms. Now 315K(e-ii) tells us that $a\otimes b=0$ only when one of a, b is 0. In the context of 315M, we can identify $E\otimes F$ with $E\times F$ for $E\in\Sigma_1$ and $F\in\Sigma_2$, and $E^\bullet\otimes F^\bullet$ with $(E\times F)^\bullet$.

315O Lemma Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras.

(a) Any element of $\mathfrak{A}\otimes\mathfrak{B}$ is expressible as $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$.

(b) If $c\in\mathfrak{A}\otimes\mathfrak{B}$ is non-zero there are non-zero $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ such that $a\otimes b\subseteq c$.

proof (a) Let C be the set of elements of $\mathfrak{A}\otimes\mathfrak{B}$ representable in this form. Then C is a subalgebra of $\mathfrak{A}\otimes\mathfrak{B}$. **P** (i) If $\langle a_i\rangle_{i\in I}$, $\langle a'_j\rangle_{j\in J}$ are finite partitions of unity in $\mathfrak{A}$, and b_i, b'_j members of $\mathfrak{B}$ for $i\in I$ and $j\in J$, then $\langle a_i\cap a'_j\rangle_{i\in I,j\in J}$ is a partition of unity in $\mathfrak{A}$, and

$$\begin{aligned}(\sup_{i\in I}a_i\otimes b_i)\cap(\sup_{j\in J}a'_j\otimes b'_j)&=\sup_{i\in I,j\in J}(a_i\otimes b_i)\cap(a'_j\otimes b'_j)\\&=\sup_{i\in I,j\in J}(a_i\cap a'_j)\otimes(b_i\cap b'_j)\in C.\end{aligned}$$

So $c\cap c'\in C$ for all c, $c'\in C$. (ii) If $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$ and $b_i\in\mathfrak{B}$ for each i, then

$$1\setminus\sup_{i\in I}a_i\otimes b_i=(\sup_{i\in I}a_i\otimes 1)\setminus(\sup_{i\in I}a_i\otimes b_i)=\sup_{i\in I}a_i\otimes(1\setminus b_i)\in C.$$

Thus $1\setminus c\in C$ for every $c\in C$. **Q**

Since $a\otimes 1=(a\otimes 1)\cup((1\setminus a)\otimes 0)$ and $1\otimes b$ belong to C for every $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$, C must be the whole of $\mathfrak{A}\otimes\mathfrak{B}$, by 315Ka.

(b) Now this follows at once, as well as being a special case of 315Kb.

315P Example $\mathfrak{A}=\mathcal{P}\mathbb{N}\otimes\mathcal{P}\mathbb{N}$ is not Dedekind σ-complete. **P** Consider $A=\{\{n\}\otimes\{n\}:n\in\mathbb{N}\}\subseteq\mathfrak{A}$. **?** If A has a least upper bound c in $\mathfrak{A}$, then c is expressible as a supremum $\sup_{j\le k}a_j\otimes b_j$, by 315Kb. Because k is finite, there must be distinct m, n such that $\{j:m\in a_j\}=\{j:n\in a_j\}$. Now $\{n\}\times\{n\}\subseteq c$, so there is a $j\le k$ such that

$$(a_j\cap\{n\})\otimes(b_j\cap\{n\})=(\{n\}\otimes\{n\})\cap(a_j\otimes b_j)\ne 0,$$

so that neither $a_j\cap\{n\}$ nor $b_j\cap\{n\}$ is empty, that is, $n\in a_j\cap b_j$. But this means that $m\in a_j$, so that

$$(a_j\otimes b_j)\cap(\{m\}\otimes\{n\})=(a_j\cap\{m\})\otimes(b_j\cap\{n\})\ne 0,$$

and $c\cap(\{m\}\otimes\{n\})\ne 0$, even though $a\cap(\{m\}\otimes\{n\})=0$ for every $a\in A$. **X** Thus we have found a countable subset of $\mathfrak{A}$ with no supremum in $\mathfrak{A}$, and $\mathfrak{A}$ is not Dedekind σ-complete. **Q**

315Q Example Now let $\mathfrak{A}$ be any non-trivial atomless Boolean algebra, and $\mathfrak{B}$ the free product $\mathfrak{A}\otimes\mathfrak{A}$. Then the identity homomorphism from $\mathfrak{A}$ to itself induces a homomorphism $\phi:\mathfrak{B}\to\mathfrak{A}$ given by setting $\phi(a\otimes b)=a\cap b$ for every a, $b\in\mathfrak{A}$. The point I wish to make is that ϕ is not order-continuous. **P** Let C be the set $\{a\otimes b:a,\,b\in\mathfrak{A},\,a\cap b=0\}$. Then $\phi(c)=0_{\mathfrak{A}}$ for every $c\in C$. If $d\in\mathfrak{B}$ is non-zero, then by 315Ob there are non-zero a, $b\in\mathfrak{A}$ such that $a\otimes b\subseteq d$; now, because $\mathfrak{A}$ is atomless, there is a non-zero $a'\subseteq a$ such that $a\setminus a'\ne 0$. At least one of $b\setminus a'$, $b\setminus(a\setminus a')$ is non-zero; suppose the former. Then $a'\otimes(b\setminus a')$ is a non-zero member of C included in d. As d is arbitrary, this shows that $\sup C=1_{\mathfrak{B}}$. So

$$\sup_{c\in C}\phi(c)=0_{\mathfrak{A}}\ne 1_{\mathfrak{A}}=\phi(\sup C),$$

and ϕ is not order-continuous. **Q**

Thus the free product (unlike the product, see 315Dd) does not respect order-continuity.

***315R Projective and inductive limits: Proposition** Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and R a subset of $I\times I$; suppose that $\pi_{ji}:\mathfrak{A}_i\to\mathfrak{A}_j$ is a Boolean homomorphism for each $(i,j)\in R$.

(a) There are a Boolean algebra $\mathfrak{C}$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ is a Boolean homomorphism for each $i\in I$,

$\pi_j=\pi_{ji}\pi_i$ whenever $(i,j)\in R$,

and whenever $\mathfrak{B}$, $\langle\phi_i\rangle_{i\in I}$ are such that

$\mathfrak{B}$ is a Boolean algebra,

$\phi_i:\mathfrak{B}\to\mathfrak{A}_i$ is a Boolean homomorphism for each $i\in I$,

$\phi_j=\pi_{ji}\phi_i$ whenever $(i,j)\in R$,

then there is a unique Boolean homomorphism $\phi:\mathfrak{B}\to\mathfrak{C}$ such that $\pi_i\phi=\phi_i$ for every $i\in I$.

(b) There are a Boolean algebra $\mathfrak{C}$ and a family $\langle\pi_i\rangle_{i\in I}$ such that

$\pi_i:\mathfrak{A}_i\to\mathfrak{C}$ is a Boolean homomorphism for each $i\in I$,

$\pi_i=\pi_j\pi_{ji}$ whenever $(i,j)\in R$, and whenever $\mathfrak{B}$, $\langle\phi_i\rangle_{i\in I}$ are such that

$\mathfrak{B}$ is a Boolean algebra,

$\phi_i:\mathfrak{A}_i\to\mathfrak{B}$ is a Boolean homomorphism for each $i\in I$,

$\phi_i=\phi_j\pi_{ji}$ whenever $(i,j)\in R$,

then there is a unique Boolean homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ such that $\phi\pi_i=\phi_i$ for every $i\in I$.

proof (a) Let $\mathfrak{A}$ be the simple product $\prod_{i\in I}\mathfrak{A}_i$, and set

$$\mathfrak{C}=\{a:a\in\mathfrak{A},\,a(j)=\pi_{ji}a(i)\text{ whenever }(i,j)\in R\}.$$

Because every π_{ji} is a Boolean homomorphism, $\mathfrak{C}$ is a subalgebra of $\mathfrak{A}$. Set $\pi_i(a)=a(i)$ for $i\in I$ and $a\in\mathfrak{C}$; then $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ is a Boolean homomorphism for every i, and $\pi_j=\pi_{ji}\pi_i$ whenever $(i,j)\in R$.

Now suppose that $\mathfrak{B}$ and $\langle\phi_i\rangle_{i\in I}$ have the declared properties. For $b\in\mathfrak{B}$, set $\phi b=\langle\phi_i b\rangle_{i\in I}\in\mathfrak{A}$; because $\phi_j=\pi_{ji}\phi_i$ whenever $(i,j)\in R$, $\phi b\in\mathfrak{C}$. Of course ϕb is the unique member of $\mathfrak{C}$ such that $\pi_i\phi b=\phi_i b$ for every $i\in I$. And $\phi:\mathfrak{B}\to\mathfrak{A}$ is a Boolean homomorphism by 315Bb, so $\phi:\mathfrak{B}\to\mathfrak{C}$ is a Boolean homomorphism.

(b) This time, let $\mathfrak{A}=\bigotimes_{i\in I}\mathfrak{A}_i$ be the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$; for each $i\in I$, let $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ be the canonical map. Let J be the ideal of $\mathfrak{A}$ generated by elements of the form $\varepsilon_i a\,\triangle\,\varepsilon_j\pi_{ji}a$ where $(i,j)\in R$ and $a\in\mathfrak{A}_i$; let $\mathfrak{C}$ be the quotient algebra $\mathfrak{A}/J$, and set $\pi_i a=(\varepsilon_i a)^\bullet\in\mathfrak{C}$ for $i\in I$ and $a\in\mathfrak{A}_i$. Then every π_i is a Boolean homomorphism, and if $(i,j)\in R$ and $a\in\mathfrak{A}_i$, then

$$\pi_i a=(\varepsilon_i a)^\bullet=(\varepsilon_j\pi_{ji}a)^\bullet=\pi_j\pi_{ji}a$$

because $\varepsilon_i a\,\triangle\,\varepsilon_j\pi_{ji}a$ belongs to J.

Once again, suppose that $\mathfrak{B}$ and $\langle\phi_i\rangle_{i\in I}$ have the properties declared in this part of the proposition. By 315Jb, there is a Boolean homomorphism $\tilde\phi:\mathfrak{A}\to\mathfrak{B}$ such that $\tilde\phi\varepsilon_i=\phi_i$ for every $i\in I$. Now the kernel of $\tilde\phi$ includes J. **P** The kernel of $\tilde\phi$ is an ideal of $\mathfrak{A}$, so all we have to check is that it contains $\varepsilon_i a\,\triangle\,\varepsilon_j\pi_{ji}a$ whenever $(i,j)\in R$ and $a\in\mathfrak{A}_i$; but in this case

$$\tilde\phi(\varepsilon_i a\,\triangle\,\varepsilon_j\pi_{ji}a)=\tilde\phi\varepsilon_i a\,\triangle\,\tilde\phi\varepsilon_j\pi_{ji}a=\phi_i a\,\triangle\,\phi_j\pi_{ji}a=\phi_i a\,\triangle\,\phi_i a=0.\ \mathbf{Q}$$

Accordingly there is a unique ring homomorphism $\phi:\mathfrak{C}\to\mathfrak{B}$ defined by saying that $\phi c^\bullet=\tilde\phi c$ for every $c\in\mathfrak{A}$ (3A2G). As

$$\phi 1_{\mathfrak{C}}=\phi(1_{\mathfrak{A}}^\bullet)=\tilde\phi 1_{\mathfrak{A}}=1_{\mathfrak{B}},$$

ϕ is a Boolean homomorphism. Now, of course,

$$\phi\pi_i a=\phi(\varepsilon_i a)^\bullet=\tilde\phi\varepsilon_i a=\phi_i a$$

whenever $i\in I$ and $a\in\mathfrak{A}_i$.

To see that ϕ is unique, observe that if $\phi':\mathfrak{C}\to\mathfrak{B}$ has the same property, then we have a Boolean homomorphism $\tilde\phi':\mathfrak{A}\to\mathfrak{B}$ defined by setting $\tilde\phi' c=\phi' c^\bullet$ for every $c\in\mathfrak{A}$; in which case

$$\tilde\phi'\varepsilon_i a=\phi'(\varepsilon_i a)^\bullet=\phi'\pi_i a=\phi_i a$$

whenever $i\in I$ and $a\in\mathfrak{A}_i$, so that $\tilde\phi'=\tilde\phi$ and $\phi'=\phi$.

***315S Definitions** In 315Ra, we call $\mathfrak{A}$, together with $\langle\pi_i\rangle_{i\in I}$, 'the' **projective limit** of $(\langle\mathfrak{A}_i\rangle_{i\in I},\langle\pi_{ji}\rangle_{(i,j)\in R})$; in 315Rb, we call $\mathfrak{A}$, together with $\langle\pi_i\rangle_{i\in I}$, 'the' **inductive limit** of $(\langle\mathfrak{A}_i\rangle_{i\in I},\langle\pi_{ji}\rangle_{(i,j)\in R})$.

315X Basic exercises (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with simple product $\mathfrak{A}$, and $\pi_i:\mathfrak{A}\to\mathfrak{A}_i$ the coordinate homomorphisms. Suppose we have another Boolean algebra $\mathfrak{A}'$, with homomorphisms $\pi_i':\mathfrak{A}'\to\mathfrak{A}_i$, such that for every Boolean algebra $\mathfrak{B}$ and every family $\langle\phi_i\rangle_{i\in I}$ of homomorphisms from $\mathfrak{B}$ to the $\mathfrak{A}_i$ there is a unique homomorphism $\phi:\mathfrak{B}\to\mathfrak{A}'$ such that $\phi_i=\pi_i'\phi$ for every i. Show that there is a unique isomorphism $\psi:\mathfrak{A}\to\mathfrak{A}'$ such that $\pi_i'\psi=\pi_i$ for every $i\in I$.

(b) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with simple product $\mathfrak{A}=\prod_{i\in I}\mathfrak{A}_i$. (i) Show that $\mathfrak{A}$ is Dedekind complete iff every $\mathfrak{A}_i$ is Dedekind complete. (ii) Show that $\mathfrak{A}$ is Dedekind σ-complete iff every $\mathfrak{A}_i$ is Dedekind σ-complete.

(c) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with simple product $\mathfrak{A}=\prod_{i\in I}\mathfrak{A}_i$. Suppose that for every $i\in I$ we are given a subalgebra $\mathfrak{B}_i$ of $\mathfrak{A}_i$. (i) Show that the simple product $\mathfrak{B}=\prod_{i\in I}\mathfrak{B}_i$ is a subalgebra of $\mathfrak{A}$. (ii) Show that $\mathfrak{B}$ is order-closed in $\mathfrak{A}$ iff $\mathfrak{B}_i$ is order-closed in $\mathfrak{A}_i$ for every $i\in I$.

(d) Let $\langle P_i\rangle_{i\in I}$ be a family of non-empty partially ordered sets, with product partially ordered set P. Show that P is a lattice iff every P_i is a lattice, and that in this case it is the product lattice in the sense that $p\vee q=\langle p(i)\vee q(i)\rangle_{i\in I}$, $p\wedge q=\langle p(i)\wedge q(i)\rangle_{i\in I}$ for all p, $q\in P$.

(e) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras with simple product $\mathfrak{A}$. For each $i\in I$ let Z_i be the Stone space of $\mathfrak{A}_i$, and let Z be the Stone space of $\mathfrak{A}$. (i) Show that the coordinate maps from $\mathfrak{A}$ onto $\mathfrak{A}_i$ induce homeomorphisms between the Z_i and open-and-closed subsets Z_i^* of Z. (ii) Show that $\langle Z_i^*\rangle_{i\in I}$ is disjoint. (iii) Show that $\bigcup_{i\in I}Z_i^*$ is dense in Z, and is equal to Z iff $\{i:\mathfrak{A}_i\ne\{0\}\}$ is finite.

(f) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, with simple product $\mathfrak{A}$. Suppose that for each $i\in I$ we are given an ideal I_i of $\mathfrak{A}_i$. Show that $I=\prod_{i\in I}I_i$ is an ideal of $\mathfrak{A}$, and that $\mathfrak{A}/I$ may be identified, as Boolean algebra, with $\prod_{i\in I}\mathfrak{A}_i/I_i$.

(g) Let $\langle X_i\rangle_{i\in I}$ be any family of topological spaces. Let X be their disjoint union $\{(x,i):i\in I,\,x\in X_i\}$, with the disjoint union topology; that is, a set $G\subseteq X$ is open in X iff $\{x:(x,i)\in G\}$ is open in X_i for every $i\in I$. Show that the algebra of open-and-closed subsets of X can be identified, as Boolean algebra, with the simple product of the algebras of open-and-closed sets of the X_i.

(h) Show that the topological product of any family of zero-dimensional spaces is zero-dimensional.

(i) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with free product $\mathfrak{A}$, and $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ the canonical homomorphisms. Suppose we have another Boolean algebra $\mathfrak{A}'$, with homomorphisms $\varepsilon_i':\mathfrak{A}_i\to\mathfrak{A}'$, such that for every Boolean algebra $\mathfrak{B}$ and every family $\langle\phi_i\rangle_{i\in I}$ of homomorphisms from the $\mathfrak{A}_i$ to $\mathfrak{B}$ there is a unique homomorphism $\phi:\mathfrak{A}'\to\mathfrak{B}$ such that $\phi_i=\phi\varepsilon_i'$ for every i. Show that there is a unique isomorphism $\psi:\mathfrak{A}\to\mathfrak{A}'$ such that $\varepsilon_i'=\psi\varepsilon_i$ for every $i\in I$.

(j) Let I be any set, and let $\mathfrak{A}$ be the algebra of open-and-closed sets of $\{0,1\}^I$; for each $i\in I$ set $a_i=\{x:x\in\{0,1\}^I,\,x(i)=1\}\in\mathfrak{A}$. Show that for any Boolean algebra $\mathfrak{B}$ and any family $\langle b_i\rangle_{i\in I}$ in $\mathfrak{B}$ there is a unique Boolean homomorphism $\phi:\mathfrak{A}\to\mathfrak{B}$ such that $\phi(a_i)=b_i$ for every $i\in I$.

(k) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$, $\langle\mathfrak{B}_j\rangle_{j\in J}$ be two families of Boolean algebras. Show that there is a natural injective homomorphism $\phi:\prod_{i\in I}\mathfrak{A}_i\otimes\prod_{j\in J}\mathfrak{B}_j\to\prod_{i\in I,j\in J}\mathfrak{A}_i\otimes\mathfrak{B}_j$ defined by saying that

$$\phi(a\otimes b)=\langle a(i)\otimes b(j)\rangle_{i\in I,j\in J}$$

for $a\in\prod_{i\in I}\mathfrak{A}_i$, $b\in\prod_{j\in J}\mathfrak{B}_j$. Show that ϕ is surjective if I and J are finite.

(l) Let $\langle J(i)\rangle_{i\in I}$ be a family of sets, with product $Q=\prod_{i\in I}J(i)$. Let $\langle\mathfrak{A}_{ij}\rangle_{i\in I,j\in J(i)}$ be a family of Boolean algebras. Describe a natural injective homomorphism $\phi:\bigotimes_{i\in I}\prod_{j\in J(i)}\mathfrak{A}_{ij}\to\prod_{q\in Q}\bigotimes_{i\in I}\mathfrak{A}_{i,q(i)}$.

(m) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with partitions of unity $\langle a_i\rangle_{i\in I}$, $\langle b_j\rangle_{j\in J}$. Show that $\langle a_i\otimes b_j\rangle_{i\in I,j\in J}$ is a partition of unity in $\mathfrak{A}\otimes\mathfrak{B}$.

(n) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras and $a\in\mathfrak{A}$, $b\in\mathfrak{B}$. Write $\mathfrak{A}_a$, $\mathfrak{B}_b$ for the corresponding principal ideals. Show that there is a canonical isomorphism between $\mathfrak{A}_a\otimes\mathfrak{B}_b$ and the principal ideal of $\mathfrak{A}\otimes\mathfrak{B}$ generated by $a\otimes b$.

(o) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of Boolean algebras, with free product $\bigotimes_{i\in I}\mathfrak{A}_i$, and $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ the canonical maps. Show that $\varepsilon_i[\mathfrak{A}_i]$ is an order-closed subalgebra of $\mathfrak{A}$ for every i.

(p) Let $\mathfrak{A}$ be a Boolean algebra. Let us say that a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **Boolean-independent** if $\inf_{j\in J}a_j\ne0$ whenever $J\subseteq I$ is finite and $a_j\in\mathfrak{A}_j\setminus\{0\}$ for every $j\in J$. Show that in this case the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{A}_i$ is isomorphic to the free product $\bigotimes_{i\in I}\mathfrak{A}_i$.

(q) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ and $\langle\mathfrak{B}_i\rangle_{i\in I}$ be two families of Boolean algebras, and suppose that for each $i\in I$ we are given a Boolean homomorphism $\phi_i:\mathfrak{A}_i\to\mathfrak{B}_i$ with kernel $K_i\lhd\mathfrak{A}_i$. Show that the ϕ_i induce a Boolean homomorphism $\phi:\bigotimes_{i\in I}\mathfrak{A}_i\to\bigotimes_{i\in I}\mathfrak{B}_i$ with kernel generated by $\bigcup_{i\in I}\varepsilon[K_i]$, where $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$ is the canonical homomorphism. Show that if every ϕ_i is surjective, so is ϕ.

(r) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be any family of non-trivial Boolean algebras. Show that if $J\subseteq I$ and $\mathfrak{B}_j$ is a subalgebra of $\mathfrak{A}_j$ for each $j\in J$, then $\bigotimes_{j\in J}\mathfrak{B}_j$ is canonically embedded as a subalgebra of $\bigotimes_{i\in I}\mathfrak{A}_i$.

(s) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, neither $\{0\}$. Show that any element of $\mathfrak{A}\otimes\mathfrak{B}$ is uniquely expressible as $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}$, with no a_i equal to 0, and $b_i\ne b_j$ in $\mathfrak{B}$ for $i\ne j$.

315Y Further exercises (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ and $\langle\mathfrak{B}_i\rangle_{i\in I}$ be two families of Boolean algebras, and suppose that we are given Boolean homomorphisms $\phi_i:\mathfrak{A}_i\to\mathfrak{B}_i$ for each i; let $\phi:\bigotimes_{i\in I}\mathfrak{A}_i\to\bigotimes_{i\in I}\mathfrak{B}_i$ be the induced homomorphism. (i) Show that if every ϕ_i is order-continuous, so is ϕ. (ii) Show that if every ϕ_i is sequentially order-continuous, so is ϕ.

(b) Let $\langle Z_i\rangle_{i\in I}$ be any family of topological spaces with product Z. For $i\in I$, $z\in Z$ set $\tilde\varepsilon_i(z)=z(i)$. Show that if $M\subseteq Z_i$ is nowhere dense in Z_i then $\tilde\varepsilon_i^{-1}[M]$ is nowhere dense in Z. Use this to prove 315Kc.

(c) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and suppose that we are given subalgebras $\mathfrak{B}_i$ of $\mathfrak{A}_i$ for each i; set $\mathfrak{A}=\bigotimes_{i\in I}\mathfrak{A}_i$ and $\mathfrak{B}=\bigotimes_{i\in I}\mathfrak{B}_i$, and let $\phi:\mathfrak{B}\to\mathfrak{A}$ be the homomorphism induced by the embeddings $\mathfrak{B}_i\hookrightarrow\mathfrak{A}_i$. (i) Show that if every $\mathfrak{B}_i$ is order-closed in $\mathfrak{A}_i$, then $\phi[\mathfrak{B}]$ is order-closed in $\mathfrak{A}$. (ii) Show that if every $\mathfrak{B}_i$ is a σ-subalgebra of $\mathfrak{A}_i$, then $\phi[\mathfrak{B}]$ is a σ-subalgebra in $\mathfrak{A}$.

(d) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, with product X. Let $\mathrm{RO}(X_i)$, $\mathrm{RO}(X)$ be the corresponding regular open algebras. Show that $\mathrm{RO}(X)$ can be identified with the Dedekind completion of $\bigotimes_{i\in I}\mathrm{RO}(X_i)$.

(e) Use the ideas of 315Xj and 315M to give an alternative construction of 'free product', for which 315J and 315K(e-ii) are true, and which does not depend on the concept of Stone space nor on any other use of the axiom of choice. (*Hint*: show that for any Boolean algebra $\mathfrak{A}$ there is a canonical surjection from the algebra $\mathcal{E}_{\mathfrak{A}}$ onto $\mathfrak{A}$, where $\mathcal{E}_J$ is the algebra of subsets of $\{0,1\}^J$ generated by sets of the form $\{x:x(j)=1\}$; show that for such algebras $\mathcal{E}_J$, at least, the method of 315I-315J can be used; now apply the method of 315M to describe $\bigotimes_{i\in I}\mathfrak{A}_i$ as a quotient of $\mathcal{E}_J$ where $J=\{(a,i):i\in I,\ a\in\mathfrak{A}_i\}$. Finally check 315K(e-ii).)

(f) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras. Show that $\mathfrak{A}\otimes\mathfrak{B}$ is Dedekind complete iff *either* $\mathfrak{A}=\{0\}$ *or* $\mathfrak{B}=\{0\}$ *or* $\mathfrak{A}$ is finite and $\mathfrak{B}$ is Dedekind complete *or* $\mathfrak{B}$ is finite and $\mathfrak{A}$ is Dedekind complete.

(g) Let $\langle P_i\rangle_{i\in I}$ be any family of partially ordered spaces. (i) Give a construction of a partially ordered space P, together with a family of order-preserving maps $\varepsilon_i:P_i\to P$, such that whenever Q is a partially ordered set and $\phi_i:P_i\to Q$ is order-preserving for every $i\in I$, there is a unique order-preserving map $\phi:P\to Q$ such that $\phi_i=\phi\varepsilon_i$ for every i. (ii) Show that ϕ will be order-continuous iff every ϕ_i is. (iii) Show that P will be Dedekind complete iff every P_i is, but (except in trivial cases) is not a lattice.

(h) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, and R a subset of $I\times I$; suppose that $\pi_{ji}:\mathfrak{A}_i\to\mathfrak{A}_j$ is a Boolean homomorphism for each $(i,j)\in R$. For each $i\in I$, let Z_i be the Stone space of $\mathfrak{A}_i$; for $(i,j)\in R$, let $f_{ji}:Z_j\to Z_i$ be the continuous function corresponding to π_{ji}. Show that the Stone space of the inductive limit of the system $(\langle\mathfrak{A}_i\rangle_{i\in I},\langle\pi_{ji}\rangle_{(i,j)\in R})$ can be identified with $\{z:z\in\prod_{i\in I}Z_i,\ f_{ji}(z(j))=z(i)$ whenever $(i,j)\in R\}$.

315 Notes and comments In this section I find myself asking for slightly more sophisticated algebra than seems necessary elsewhere. The point is that simple products and free products are best regarded as defined by the properties described in 315B and 315J. That is, it is sometimes right to think of *a* simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras as being a structure $(\mathfrak{A},\langle\pi_i\rangle_{i\in I})$ where $\mathfrak{A}$ is a Boolean algebra, $\pi_i:\mathfrak{A}\to\mathfrak{A}_i$ is a homomorphism for every $i\in I$, and every family of homomorphisms from a Boolean algebra

$\mathfrak{B}$ to the $\mathfrak{A}_i$ can be uniquely represented by a single homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$. Similarly, reversing the direction of the homomorphisms, we can speak of *a* free product (it would be natural to say 'coproduct') $(\mathfrak{A}, \langle\varepsilon_i\rangle_{i\in I})$ of $\langle\mathfrak{A}_i\rangle_{i\in I}$. On such definitions, it is elementary that any two simple products, or free products, are isomorphic in the obvious sense (315Xa, 315Xi), and very general arguments from abstract algebra, not restricted to Boolean algebras (see BOURBAKI 68, IV.3.2), show that they exist. (But in order to prove such basic facts as that the π_i are surjective, or that the ε_i are, except when the construction collapses altogether, injective, we do of course have to look at the special properties of Boolean algebras.) Now in the case of simple products, the Cartesian product construction is so direct and so familiar that there seems no need to trouble our imaginations with any other. But in the case of free products, things are more complicated. I have given primacy to the construction in terms of Stone spaces because I believe that this is the fastest route to effective mental pictures. But in some ways this approach seems to be inappropriate. If you take what in my view is a tenable position, and say that a Boolean algebra is best regarded as the limit of its finite subalgebras, then you might prefer a construction of a free product as a limit of free products of finitely many finite subalgebras. Or you might feel that it is wrong to rely on the axiom of choice to prove a result which certainly does not need it (see 315Ye).

Because I believe that the universal mapping theorem 315J is the right basis for the study of free products, I am naturally led to use it as the starting point for proofs of theorems about free products, as in 315L. But 315K(e-ii) seems to lie deeper. (Note, for instance, that in 315M we *do* need the axiom of choice, in part (a-iii) of the proof, since without it the product $\prod_{i\in I} X_i$ could be empty.)

Both 'simple product' and 'free product' are essentially algebraic constructions involving the category of Boolean algebras and Boolean homomorphisms, and any relationships with such concepts as order-continuity can be regarded as accidental, in so far as there are accidents in mathematics. 315Cb and 315D show that simple products behave very straightforwardly when the homomorphisms involved are order-continuous. 315Q, 315Xo and 315Ya-315Yc show that free products are much more complex and subtle.

For finite products, we have a kind of distributivity; $(\mathfrak{A}\times\mathfrak{B})\otimes\mathfrak{C}$ can be identified with $(\mathfrak{A}\otimes\mathfrak{C})\times(\mathfrak{B}\otimes\mathfrak{C})$ (315Xk, 315Xl). There are contexts in which this makes it seem more natural to write $\mathfrak{A}\oplus\mathfrak{B}$ in place of $\mathfrak{A}\times\mathfrak{B}$, and indeed I have already spoken of a 'direct sum' of measure spaces (214L) in terms which correspond closely to the simple product of algebras of sets described in 315Ga. Generally, the simple product corresponds to disjoint unions of Stone spaces (315Xe) and the free product to products of Stone spaces. But the simple product is indeed the product Boolean algebra, in the ordinary category sense; the universal mapping theorem 315B is exactly of the type we expect from products of topological spaces (3A3Ib) or partially ordered sets (315Dc), etc. It is the 'free product' which is special to Boolean algebras. The nearest analogy that I know of elsewhere is with the concept of 'tensor product' of linear spaces (cf. §253).

It is perhaps worth noting that projective limits of systems of Boolean algebras have a straightforward description in terms of the algebras themselves (315Ra), while inductive limits have a similarly direct description in terms of Stone spaces (315Yh).

Version of 26.1.09

316 Further topics

I introduce three special properties of Boolean algebras which will be of great importance in the rest of this volume: the countable chain condition (316A-316F), weak (σ,∞)-distributivity (316G-316J) and homogeneity (316N-316Q). I add some brief notes on atoms in Boolean algebras (316K-316L), with a characterization of the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316M).

316A Definitions (a) A Boolean algebra $\mathfrak{A}$ is **ccc**, or satisfies the **countable chain condition**, if every disjoint subset of $\mathfrak{A}$ is countable.

(b) A topological space X is **ccc**, or satisfies the **countable chain condition**, or has **Souslin's property**, if every disjoint collection of open sets in X is countable.

316B Theorem A Boolean algebra $\mathfrak{A}$ is ccc iff its Stone space Z is ccc.

proof (a) If $\mathfrak{A}$ is ccc and $\mathcal{G}$ is a disjoint family of open sets in Z, then for each $G \in \mathcal{G}' = \mathcal{G} \setminus \{\emptyset\}$ we can find a non-zero $a_G \in \mathfrak{A}$ such that the corresponding open-and-closed set $\widehat{a}_G$ is included in G. Now $\{a_G : G \in \mathcal{G}'\}$ is a disjoint family in $\mathfrak{A}$, so is countable; since $a_G \neq a_H$ for distinct G, $H \in \mathcal{G}'$, $\mathcal{G}'$ and $\mathcal{G}$ must be countable. As $\mathcal{G}$ is arbitrary, Z is ccc.

(b) If Z is ccc and $A \subseteq \mathfrak{A}$ is disjoint, then $\{\widehat{a} : a \in A\}$ is a disjoint family of open subsets of Z, so must be countable, and A is countable. As A is arbitrary, $\mathfrak{A}$ is ccc.

316C Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\mathcal{I}$ a σ-ideal of $\mathfrak{A}$. Then the quotient algebra $\mathfrak{B} = \mathfrak{A}/\mathcal{I}$ is ccc iff every disjoint family in $\mathfrak{A} \setminus \mathcal{I}$ is countable.

proof (a) Suppose that $\mathfrak{B}$ is ccc and that A is a disjoint family in $\mathfrak{A} \setminus \mathcal{I}$. Then $\{a^\bullet : a \in A\}$ is a disjoint family in $\mathfrak{B}$, therefore countable, and $a^\bullet \neq b^\bullet$ when a, b are distinct members of A; so A is countable.

(b) Now suppose that $\mathfrak{B}$ is not ccc. Then there is an uncountable disjoint set $B \subseteq \mathfrak{B}$. Of course $B \setminus \{0\}$ is still uncountable, so may be enumerated as $\langle b_\xi \rangle_{\xi<\kappa}$, where κ is an uncountable cardinal (2A1K), so that $\omega_1 \leq \kappa$. For each $\xi < \omega_1$, choose $a_\xi \in \mathfrak{A}$ such that $a_\xi^\bullet = b_\xi$. Of course $a_\xi \notin \mathcal{I}$. If $\eta < \xi < \omega_1$, then $b_\eta \cap b_\xi = 0$, so $a_\xi \cap a_\eta \in \mathcal{I}$. Because $\xi < \omega_1$, it is countable; because $\mathcal{I}$ is a σ-ideal, and $\mathfrak{A}$ is Dedekind σ-complete,

$$d_\xi = \sup_{\eta<\xi} a_\xi \cap a_\eta,$$

belongs to $\mathcal{I}$, and

$$c_\xi = a_\xi \setminus d_\xi \in \mathfrak{A} \setminus \mathcal{I}.$$

But now of course

$$c_\xi \cap c_\eta \subseteq c_\xi \cap a_\eta \subseteq c_\xi \cap d_\xi = 0$$

whenever $\eta < \xi < \omega_1$, so $\{c_\xi : \xi < \omega_1\}$ is an uncountable disjoint family in $\mathfrak{A} \setminus \mathcal{I}$.

Remark An ideal $\mathcal{I}$ satisfying the conditions of this proposition is said to be ω_1**-saturated** in $\mathfrak{A}$.

316D Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of Σ. Then the quotient algebra $\Sigma/\mathcal{I}$ is ccc iff every disjoint family in $\Sigma \setminus \mathcal{I}$ is countable.

316E Proposition Let $\mathfrak{A}$ be a ccc Boolean algebra. Then for any $A \subseteq \mathfrak{A}$ there is a countable $B \subseteq A$ such that B has the same upper and lower bounds as A.

proof (a) Set

$$D = \bigcup_{a\in A} \{d : d \subseteq a\}.$$

Applying Zorn's lemma to the family $\mathcal{C}$ of disjoint subsets of D, we have a maximal $C_0 \in \mathcal{C}$. For each $c \in C_0$ choose a $b_c \in A$ such that $c \subseteq b_c$, and set $B_0 = \{b_c : c \in C_0\}$. Because $\mathfrak{A}$ is ccc, C_0 is countable, so B_0 also is countable. **?** If there is an upper bound e for B_0 which is not an upper bound for A, take $a \in A$ such that $c' = a \setminus e \neq 0$; then $c' \in D$ and $c' \cap c = c' \cap b_c = 0$ for every $c \in C_0$, so $C_0 \cup \{c'\} \in \mathcal{C}$; but C_0 was supposed to be maximal in $\mathcal{C}$. **X** Thus every upper bound for B_0 is also an upper bound for A.

(b) Similarly, there is a countable set $B_1' \subseteq A' = \{1 \setminus a : a \in A\}$ such that every upper bound for B_1' is an upper bound for A'. Set $B_1 = \{1 \setminus b : b \in B_1'\}$; then B_1 is a countable subset of A and every lower bound for B_1 is a lower bound for A. Try $B = B_0 \cup B_1$. Then B is a countable subset of A and every upper (resp. lower) bound for B is an upper (resp. lower) bound for A; so that B must have exactly the same upper and lower bounds as A has.

316F Corollary Let $\mathfrak{A}$ be a ccc Boolean algebra.

(a) If $\mathfrak{A}$ is Dedekind σ-complete it is Dedekind complete.

(b) If $A \subseteq \mathfrak{A}$ is sequentially order-closed it is order-closed.

(c) If Q is any partially ordered set and $\phi : \mathfrak{A} \to Q$ is a sequentially order-continuous order-preserving function, it is order-continuous.

(d) If $\mathfrak{B}$ is another Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, it is order-continuous.

proof (a) If A is any subset of $\mathfrak{A}$, let $B \subseteq A$ be a countable set with the same upper bounds as A; then $\sup B$ is defined in $\mathfrak{A}$ and must be $\sup A$.

(b) Suppose that $B \subseteq A$ is non-empty and upwards-directed and has a supremum a in $\mathfrak{A}$. Then there is a non-empty countable set $C \subseteq B$ with the same upper bounds as B. Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a sequence running over C. Because B is upwards-directed, we can choose $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively such that

$$b_0 = c_0, \quad b_{n+1} \in B,\, b_{n+1} \supseteq b_n \cup c_{n+1} \text{ for every } n \in \mathbb{N}.$$

Now any upper bound for $\{b_n : n \in \mathbb{N}\}$ must also be an upper bound for $\{c_n : n \in \mathbb{N}\} = C$, so is an upper bound for the whole set B. But this means that $a = \sup_{n\in\mathbb{N}} b_n$. As $\langle b_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in A, and A is sequentially order-closed, $a \in A$.

In the same way, if $B \subseteq A$ is downwards-directed and has an infimum in $\mathfrak{A}$, this is also the infimum of some non-increasing sequence in B, so must belong to A. Thus A is order-closed.

(c)(i) Suppose that $A \subseteq \mathfrak{A}$ is a non-empty upwards-directed set with supremum $a_0 \in \mathfrak{A}$. As in (b), there is a non-decreasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ with supremum a_0. Because ϕ is sequentially order-continuous, $\phi a_0 = \sup_{n\in\mathbb{N}} \phi c_n$ in Q. But this means that ϕa_0 must be the least upper bound of $\phi[A]$.

(ii) Similarly, if $A \subseteq \mathfrak{A}$ is a non-empty downwards-directed set with infimum a_0, there is a non-increasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in A with infimum a_0, so that

$$\inf \phi[A] = \inf_{n\in\mathbb{N}} \phi c_n = \phi a_0.$$

Putting this together with (i), we see that ϕ is order-continuous, as claimed.

(d) This is a special case of (c).

316G Definition Let $\mathfrak{A}$ be a Boolean algebra. I will say that $\mathfrak{A}$ is **weakly** (σ,∞)**-distributive** if whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of downwards-directed subsets of $\mathfrak{A}$ and $\inf A_n = 0$ for every n, then $\inf B = 0$, where

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \supseteq a\}.$$

316H Proposition Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) $\mathfrak{A}$ is weakly (σ,∞)-distributive;

(ii) whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$, there is a partition of unity B in $\mathfrak{A}$ such that $\{a : a \in A_n,\, a \cap b \neq 0\}$ is finite for every $n \in \mathbb{N}$ and $b \in B$;

(iii) whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of upwards-directed subsets of $\mathfrak{A}$, each with a supremum $c_n = \sup A_n$, and

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \subseteq a\},$$

then $\inf\{c_n \setminus b : n \in \mathbb{N},\, b \in B\} = 0$;

(iv) whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of upwards-directed subsets of $\mathfrak{A}$, each with a supremum $c_n = \sup A_n$, and $\inf_{n\in\mathbb{N}} c_n = c$ is defined, then $c = \sup B$, where

$$B = \{b : b \in \mathfrak{A}, \text{ for every } n \in \mathbb{N} \text{ there is an } a \in A_n \text{ such that } b \subseteq a\}.$$

proof (i)$\Rightarrow$(ii) Suppose that $\mathfrak{A}$ is weakly (σ,∞)-distributive and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$. For each $n \in \mathbb{N}$, set

$$C_n = \{1 \setminus \sup D : D \in A_n \text{ is finite}\},$$

so that C_n is downwards-directed and has infimum 0. Set $E = \{e :$ for every $n \in \mathbb{N}$ there is a $c \in C_n$ such that $c \subseteq e\}$; then $\inf E = 0$. So $B_0 = \{b : b \cap e = 0$ for some $e \in E\}$ is order-dense in $\mathfrak{A}$ and includes a partition B of unity. If $n \in \mathbb{N}$ and $b \in B$, take $e \in E$ such that $b \cap e = 0$, $c \in C_n$ such that $c \subseteq e$, and a finite set $D \subseteq A_n$ such that $c = 1 \setminus \sup D$; then

$$b \subseteq 1 \setminus e \subseteq 1 \setminus c \subseteq \sup D$$

and $\{a : a \in A_n,\, a \cap b \neq 0\} \subseteq D$ is finite. As $\langle A_n \rangle_{n\in\mathbb{N}}$ is arbitrary, (ii) is true.

(ii)$\Rightarrow$(iii) Suppose that (ii) is true, and that $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of upwards-directed subsets of $\mathfrak{A}$, each with a supremum $c_n = \sup A_n$. For each $n \in \mathbb{N}$,

$$D_n = \{d : d \subseteq 1 \setminus c_n\} \cup \bigcup_{a\in A_n} \{d : d \subseteq a\}$$

is order-dense in $\mathfrak{A}$, so there is a partition of unity $D'_n \subseteq D_n$ (313K). By (ii), there is a partition of unity E such that $\{d : d \in D'_n,\, d \cap e \neq 0\}$ is finite for every $n \in \mathbb{N}$ and $e \in E$. **?** Suppose, if possible, that $\{c_n \setminus b : n \in \mathbb{N},\, b \in B\}$ has a non-zero lower bound c. Let $e \in E$ be such that $c \cap e \neq 0$. For each $n \in \mathbb{N}$, set $D''_n = \{d : d \in D'_n,\, c \cap e \cap d \neq 0\}$. Then D''_n is finite so $d_n = \sup D''_n$ is defined and $c \cap e \subseteq d_n$. Also, because $c \subseteq c_n$, each element of D''_n is included in a member of A_n; as A_n is upwards-directed, so are d_n and $c \cap e$. As n is arbitrary, $c \cap e \in B$; and c was supposed to be disjoint from every member of B. **X**

Thus $\inf\{c_n \setminus b : n \in \mathbb{N},\, b \in B\} = 0$; as $\langle A_n \rangle_{n\in\mathbb{N}}$ is arbitrary, (iii) is true.

(iii)$\Rightarrow$(iv) Suppose that (iii) is true and that A_n, c_n, c and B are as in the statement of (iv). Then

$$\inf\{c \setminus b : b \in B\} = \inf\{c_n \setminus b : n \in \mathbb{N},\, b \in B\} = 0;$$

as $b \subseteq c_n$ whenever $b \in B$ and $n \in \mathbb{N}$, we have $b \subseteq c$ for every $b \in B$, and $\sup B = c$, by 313Ab. Thus (iv) is true.

(iv)$\Rightarrow$(i) Suppose that (iv) is true and that A_n and B are as in 316G. Set $A'_n = \{1 \setminus a : a \in A_n\}$, so that A'_n is an upwards-directed set with supremum 1 for each n, and

$$B' = \{b : \text{for every } n \in \mathbb{N} \text{ there is an } a \in A'_n \text{ such that } b \subseteq a\} = \{1 \setminus b : b \in B\};$$

then

$$\inf B = 1 \setminus \sup B' = 1 \setminus \inf_{n\in\mathbb{N}} \sup A'_n = 0,$$

as required.

316I As usual, a characterization of the property in terms of the Stone spaces is extremely valuable.

Theorem Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Then $\mathfrak{A}$ is weakly (σ,∞)-distributive iff every meager set in Z is nowhere dense.

proof (a) The point is that if $M \subseteq Z$ then M is nowhere dense iff there is a partition of unity A in $\mathfrak{A}$ such that $M \cap \widehat{a} = \emptyset$ for every $a \in A$. **P** If M is nowhere dense, then $\{a : M \cap \widehat{a} = \emptyset\}$ is order-dense in $\mathfrak{A}$, so includes a partition of unity. In the other direction, if A is a partition of unity such that M is disjoint from $\widehat{a}$ for every $a \in A$, then $\sup A = 1$ so $G = \bigcup_{a\in A} \widehat{a}$ is dense (313Ca); now G is a dense open set disjoint from M, so M is nowhere dense. **Q**

(b) Suppose that $\mathfrak{A}$ is weakly (σ,∞)-distributive and that M is a meager subset of Z. Then M can be expressed as $\bigcup_{n\in\mathbb{N}} M_n$ where each M_n is nowhere dense. For each $n \in \mathbb{N}$, let A_n be a partition of unity such that $M_n \cap \widehat{a} = \emptyset$ for every $a \in A_n$. By 316H(i)$\Rightarrow$(ii), there is a partition of unity B such that $\{a : a \in A_n,\, a \cap b \neq 0\}$ is finite for every $n \in \mathbb{N}$ and $b \in B$. Now $M_n \cap \widehat{b} = \emptyset$ for every $n \in \mathbb{N}$ and $b \in B$. **P** $C = \{a : a \in A_n,\, b \cap a \neq 0\}$ is finite. So $F = \bigcup_{a\in C} \widehat{a}$ is closed and $G = \widehat{b} \setminus F$ is open. But $G \cap \widehat{a} = \emptyset$ for every $a \in A$, so G is empty and $\widehat{b} \subseteq F \subseteq Z \setminus M_n$. **Q** Accordingly $M \cap \widehat{b} = \emptyset$ for every $b \in B$ and M is nowhere dense.

(c) Suppose that every meager set in Z is nowhere dense, and that $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$. Then $M_n = Z \setminus \bigcup_{a\in A_n} \widehat{a}$ is nowhere dense for each n (313Cc), so $M = \bigcup_{n\in\mathbb{N}} M_n$ is meager, therefore nowhere dense. Let B be a partition of unity in $\mathfrak{A}$ such that $M \cap \widehat{b} = \emptyset$ for every $b \in B$. If $n \in \mathbb{N}$ and $b \in B$, then

$$\widehat{b} \subseteq Z \setminus M \subseteq Z \setminus M_n = \bigcup_{a\in A_n} \widehat{a}.$$

As $\widehat{b}$ is compact, there is some finite $C \subseteq A$ such that $\widehat{b} \subseteq \bigcup_{a\in C} \widehat{a}$ and $b \subseteq \sup C$; but this means that $\{a : a \in A_n,\, a \cap b \neq 0\} \subseteq C$ is finite. As $\langle A_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive, by 316H(ii)$\Rightarrow$(i).

316J The regular open algebra of $\mathbb{R}$ For examples of weakly (σ,∞)-distributive algebras, I think we can wait for Chapter 32 (see also 393C). But the standard example of an algebra which is *not* weakly (σ,∞)-distributive is of such importance that (even though it has nothing to do with measure theory, narrowly defined) I think it right to describe it here.

Proposition The algebra $\mathrm{RO}(\mathbb{R})$ of regular open subsets of $\mathbb{R}$ (314O) is not weakly (σ,∞)-distributive.

proof Enumerate $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$. For each $n\in\mathbb{N}$, set

$$A_n=\{G:G\in\mathrm{RO}(\mathbb{R}),\,q_i\in G\text{ for every }i\le n\}.$$

Then A_n is downwards-directed, and

$$\inf A_n=\operatorname{int}\bigcap A_n=\operatorname{int}\{q_i:i\le n\}=\emptyset.$$

But if $A\subseteq\mathrm{RO}(\mathbb{R})$ is such that

for every $n\in\mathbb{N}$, $G\in A$ there is an $H\in A_n$ such that $H\subseteq G$,

then we must have $\mathbb{Q}\subseteq G$ for every $G\in A$, so that

$$\mathbb{R}=\operatorname{int}\overline{\mathbb{Q}}\subseteq\operatorname{int}\overline{G}=G$$

for every $G\in A$, and $A\subseteq\{\mathbb{R}\}$; which means that $\inf A\ne\emptyset$ in $\mathrm{RO}(\mathbb{R})$, and 316G cannot be satisfied.

316K Atoms in Boolean algebras (a) If $\mathfrak{A}$ is a Boolean algebra, an **atom** in $\mathfrak{A}$ is a non-zero $a\in\mathfrak{A}$ such that the only elements included in a are 0 and a.

(b) A Boolean algebra is **atomless** if it has no atoms.

(c) A Boolean algebra is **purely atomic** if every non-zero element includes an atom.

316L Proposition Let $\mathfrak{A}$ be a Boolean algebra, with Stone space Z.

(a) There is a one-to-one correspondence between atoms a of $\mathfrak{A}$ and isolated points $z\in Z$, given by the formula $\widehat{a}=\{z\}$.

(b) $\mathfrak{A}$ is atomless iff Z has no isolated points.

(c) $\mathfrak{A}$ is purely atomic iff the isolated points of Z form a dense subset of Z.

proof (a)(i) If z is an isolated point in Z, then $\{z\}$ is an open-and-closed subset of Z, so is of the form $\widehat{a}$ for some $a\in\mathfrak{A}$; now if $b\subseteq a$, $\widehat{b}$ must be either $\emptyset$ or $\{z\}$, so b must be either a or 0, and a is an atom.

(ii) If $a\in\mathfrak{A}$ and $\widehat{a}$ has two points z and w, then (because Z is Hausdorff, 311I) there is an open set G containing z but not w. Now there is a $c\in\mathfrak{A}$ such that $z\in\widehat{c}\subseteq G$, so that $a\cap c$ must be different from both 0 and a, and a is not an atom.

(b) This follows immediately from (a).

(c) From (a) we see that $\mathfrak{A}$ is purely atomic iff $\widehat{a}$ contains an isolated point for every non-zero $a\in\mathfrak{A}$; since every non-empty open set in Z includes a non-empty set of the form $\widehat{a}$, this happens iff every non-empty open set in Z contains an isolated point, that is, iff the set of isolated points is dense.

316M Proposition Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. Then a Boolean algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ iff it is atomless, countable and not $\{0\}$.

proof (a) I must check that $\mathfrak{B}$ has the declared properties. The point is that it is the subalgebra $\mathfrak{B}'$ of $\mathcal{P}X$ generated by $\{b_i:i\in\mathbb{N}\}$, where I write $X=\{0,1\}^{\mathbb{N}}$, $b_i=\{x:x\in X,\,x(i)=1\}$. **P** Of course b_i and its complement $\{x:x(i)=0\}$ are open, so $b_i\in\mathfrak{B}$ for each i, and $\mathfrak{B}'\subseteq\mathfrak{B}$. In the other direction, the open cylinder sets of X are all of the form $c_z=\{x:x(i)=z(i)\text{ for every }i\in J\}$, where $J\subseteq I$ is finite and $z\in\{0,1\}^J$; now

$$c_z=X\cap\bigcap\nolimits_{z(i)=1}b_i\setminus\bigcup\nolimits_{z(i)=0}b_i\in\mathfrak{B}'.$$

If $b\in\mathfrak{B}$ then b is expressible as a union of such cylinder sets, because it is open; but also it is compact, so is the union of finitely many of them, and must belong to $\mathfrak{B}'$. Thus $\mathfrak{B}=\mathfrak{B}'$, as claimed. **Q**

For each $n \in \mathbb{N}$ let $\mathfrak{B}_n$ be the finite subalgebra of $\mathfrak{B}$ generated by $\{b_i : i < n\}$ (so that $\mathfrak{B}_0 = \{0, 1\}$). Then $\langle \mathfrak{B}_n \rangle_{n\in\mathbb{N}}$ is an increasing sequence of subalgebras of $\mathfrak{B}$ with union $\mathfrak{B}$; so $\mathfrak{B}$ is countable. Also $b \cap b_n$, $b \setminus b_n$ are non-zero for every $n \in \mathbb{N}$ and non-zero $b \in \mathfrak{B}_n$, so no member of any $\mathfrak{B}_n$ can be an atom in $\mathfrak{B}$, and $\mathfrak{B}$ is atomless.

(b) Now suppose that $\mathfrak{A}$ is another algebra with the same properties. Enumerate $\mathfrak{A}$ as $\langle a_n \rangle_{n\in\mathbb{N}}$. Choose finite subalgebras $\mathfrak{A}_n \subseteq \mathfrak{A}$ and isomorphisms $\pi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ as follows. $\mathfrak{A}_0 = \{0, 1\}$, $\pi_0 0 = 0$, $\pi_0 1 = 1$. Given $\mathfrak{A}_n$ and π_n, let A_n be the set of atoms of $\mathfrak{A}_n$. For $a \in A_n$, choose $a' \in \mathfrak{A}$ such that

if $a_n \cap a$, $a_n \setminus a$ are both non-zero, then $a' = a_n \cap a$;

otherwise, $a' \subseteq a$ is any element such that a', $a \setminus a'$ are both non-zero.

(This is where I use the hypothesis that $\mathfrak{A}$ is atomless.) Set $a'_n = \sup_{a\in A_n} a'$. Then we see that $a \cap a'_n$, $a \setminus a'_n$ are non-zero for every $a \in A_n$ and therefore for every non-zero $a \in \mathfrak{A}_n$, that is, that

$$\sup\{a : a \in \mathfrak{A}_n,\ a \subseteq a'_n\} = 0, \quad \inf\{a : a \in \mathfrak{A}_n,\ a \supseteq a'_n\} = 1.$$

Let $\mathfrak{A}_{n+1}$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_n \cup \{a'_n\}$. Since we have

$$\sup\{b : b \in \mathfrak{B}_n,\ b \subseteq b_n\} = 0, \quad \inf\{b : b \in \mathfrak{B}_n,\ b \supseteq b_n\} = 1,$$

there is a (unique) extension of $\pi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ to a homomorphism $\pi_{n+1} : \mathfrak{A}_{n+1} \to \mathfrak{B}_{n+1}$ such that $\pi_{n+1}a'_n = b_n$ (312O). Since we similarly have an extension ϕ of π_n^{-1} to a homomorphism from $\mathfrak{B}_{n+1}$ to $\mathfrak{A}_{n+1}$ with $\phi b_n = a'_n$, and since $\phi\pi_{n+1}$, $\pi_{n+1}\phi$ must be the respective identity homomorphisms, π_{n+1} is an isomorphism, and the induction continues.

Since π_{n+1} extends π_n for each n, these isomorphisms join together to give us an isomorphism

$$\pi : \bigcup_{n\in\mathbb{N}} \mathfrak{A}_n \to \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n = \mathfrak{B}.$$

Observe next that the construction ensures that $a_n \in \mathfrak{A}_{n+1}$ for each n, since $a_n \cap a$ is either 0 or a or $a'_n \cap a$ for every $a \in A_n$, and in all cases belongs to $\mathfrak{A}_{n+1}$. So $\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n$ contains every a_n and (by the choice of $\langle a_n \rangle_{n\in\mathbb{N}}$) must be the whole of $\mathfrak{A}$. Thus $\pi : \mathfrak{A} \to \mathfrak{B}$ witnesses that $\mathfrak{A} \cong \mathfrak{B}$.

316N Definition A Boolean algebra $\mathfrak{A}$ is **homogeneous** if every non-trivial principal ideal of $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$.

***316O Lemma** Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra such that

$$D = \{d : d \in \mathfrak{A},\ \mathfrak{A} \text{ is isomorphic to the principal ideal } \mathfrak{A}_d\}$$

is order-dense in $\mathfrak{A}$. Then $\mathfrak{A}$ is homogeneous.

proof (a) If $\mathfrak{A} = \{0\}$ then it has no non-trivial principal ideals, so is homogeneous. If $\mathfrak{A}$ is not atomless, let $a \in \mathfrak{A}$ be an atom; then there is a non-zero $d \in D$ such that $d \subseteq a$ and $d = a$; so $\mathfrak{A} \cong \mathfrak{A}_d = \{0, d\}$ and $\mathfrak{A} = \{0, 1\}$ is homogeneous because its only non-trivial principal ideal is itself. So suppose henceforth that $\mathfrak{A}$ is atomless and not $\{0\}$.

(b) Take any $a \in \mathfrak{A} \setminus \{0\}$. Choose $\langle a_n \rangle_{n\in\mathbb{N}}$ inductively in $\mathfrak{A}$ in such a way that $a_0 = a$ and that $a_{n+1} \subseteq a_n$ is neither 0 nor a_n for any n. Let D' be

$$\{d : d \in D, \text{ either } d \subseteq \inf_{n\in\mathbb{N}} a_n \text{ or } d \subseteq a_n \setminus a_{n+1} \text{ for some } n \text{ or } d \subseteq 1 \setminus a_0\}.$$

Then $D' \subseteq D$ is still order-dense. Let $C \subseteq D'$ be a partition of unity (313K); then C is infinite. We have

$$\mathfrak{A} \cong \prod_{c\in C} \mathfrak{A}_c \cong \mathfrak{A}^C$$

(315F). Moreover, every member of C is either included in a or disjoint from it, so setting $C' = \{c : c \in C, c \subseteq a\}$ we see that C' is a partition of unity in $\mathfrak{A}_a$; as $\mathfrak{A}_a$ is Dedekind complete (314Ea),

$$\mathfrak{A}_a \cong \prod_{c\in C'} \mathfrak{A}_c \cong \mathfrak{A}^{C'} \cong (\mathfrak{A}^C)^{C'} \cong \mathfrak{A}^{C\times C'} \cong \mathfrak{A}^C \cong \mathfrak{A}$$

(because C is infinite and C' is not empty, so $\#(C \times C') = \#(C)$). As a is arbitrary, $\mathfrak{A}$ is homogeneous.

***316P Proposition** Let $\mathfrak{A}$ be a homogeneous Boolean algebra. Then its Dedekind completion $\widehat{\mathfrak{A}}$ is homogeneous.

proof Regarding $\mathfrak{A}$ as a subset of $\widehat{\mathfrak{A}}$, it is order-dense. Next, if $a \in \mathfrak{A}$, then the principal ideal $\widehat{\mathfrak{A}}_a$ which it generates in $\widehat{\mathfrak{A}}$ can be identified with the Dedekind completion of the principal ideal $\mathfrak{A}_a$ which it generates in $\mathfrak{A}$. **P** $\mathfrak{A}_a$ is order-dense in $\widehat{\mathfrak{A}}_a$ and $\widehat{\mathfrak{A}}_a$ is Dedekind complete, so we can use 314Ub. **Q** But this means that

$$\widehat{\mathfrak{A}}_a \cong \widehat{\mathfrak{A}_a} \cong \widehat{\mathfrak{A}}$$

for every $a \in \mathfrak{A} \setminus \{0\}$. As $\mathfrak{A} \setminus \{0\}$ is order-dense in $\widehat{\mathfrak{A}}$, 316O tells us that $\widehat{\mathfrak{A}}$ is homogeneous.

***316Q Proposition** The free product of any family of homogeneous Boolean algebras is homogeneous.

proof (a) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of homogeneous Boolean algebras and $\mathfrak{A} = \bigotimes_{i\in I}\mathfrak{A}_i$ their free product; let $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}$ be the canonical homomorphisms. If any of the $\mathfrak{A}_i$ is $\{0\}$, so is $\mathfrak{A}$ (315Kd), and the result is trivial; so let us suppose that every $\mathfrak{A}_i$ has at least two elements. If $\mathfrak{A}_i = \{0,1\}$ for every $i \in I$, then $\mathfrak{A} = \{0,1\}$ is homogeneous; so we may suppose that at least one $\mathfrak{A}_i$ is infinite.

(b) If we have a family $\langle a_i\rangle_{i\in I}$ such that $a_i \in \mathfrak{A}_i$ for every i and $J = \{i : a_i \ne 1\}$ is finite, consider $a = \inf_{i\in J}\varepsilon_i(a_i)$ in $\mathfrak{A}$. Then $\mathfrak{A}_a \cong \bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$. **P** For $j \in I$, let ε'_j be the canonical homomorphism from $(\mathfrak{A}_j)_{a_j}$ to $\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$. Set $\phi_i(c) = a \cap \varepsilon_i(c)$ for $i \in I$ and $c \in (\mathfrak{A}_i)_{a_i}$. Then $\phi_i : (\mathfrak{A}_i)_{a_i} \to \mathfrak{A}_a$ is always a Boolean homomorphism, so we have a Boolean homomorphism $\phi : \bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i} \to \mathfrak{A}_a$ such that $\phi_i = \phi\varepsilon'_i$ for each i (315J).

If $K \subseteq I$ is finite, $J \subseteq K$, $b_k \in (\mathfrak{A}_k)_{a_k}$ for each $k \in K$ and b is the infimum $\inf_{k\in K}\varepsilon'_k(b_k)$ taken in $\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$, then

$$\phi(b) = \inf_{k\in K}\phi\varepsilon'_k(b_k)$$

(here taking the infimum in $\mathfrak{A}_a$)

$$= \inf_{k\in K}\phi_k(b_k) = \inf_{k\in K} a \cap \varepsilon_k(b_k) = a \cap \inf_{k\in K}\varepsilon_k(b_k)$$

(here taking the infimum in $\mathfrak{A}$)

$$= \inf_{k\in K}\varepsilon_k(b_k)$$

because $K \supseteq J$.

Now suppose that $b \in \bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$ is non-zero. Then there are a finite $K \subseteq I$ and a family $\langle b_k\rangle_{k\in K}$ such that $b_k \in (\mathfrak{A}_k)_{a_k} \setminus \{0\}$ for each k and b includes $\inf_{k\in K}\varepsilon'_k b_k$ in $\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}$ (315Kb). Set $K' = J \cup K$ and $b_k = a_k$ for $k \in J \setminus K$. Then

$$\phi(b) \supseteq \phi(\inf_{k\in K}\varepsilon'_k(b_k)) \supseteq \inf_{k\in K'}\phi\varepsilon'_k(b_k) = \inf_{k\in K'}\varepsilon_k(b_k) \ne 0$$

(315K(e-ii)). As b is arbitrary, ϕ is injective.

To see that ϕ is surjective, use 315Kb; every element of $\mathfrak{A}_a$ is expressible as a finite union of elements of the form $c = \inf_{k\in K}\varepsilon_k(c_k)$ where $K \subseteq I$ is finite and $c_k \in \mathfrak{A}_k$ for each $k \in K$. Again set $K' = J \cup K$; this time, take $c_k = 1$ for any $k \in J \setminus K$. Then

$$\begin{aligned} c = c \cap a &= \inf_{k\in K'}\varepsilon_k(c_k) \cap \inf_{k\in K'}\varepsilon_k(a_k) \\ &= \inf_{k\in K'}(\varepsilon_k(c_k) \cap \varepsilon_k(a_k)) = \inf_{k\in K'}(\varepsilon_k(c_k \cap a_k)) \\ &= \phi(\inf_{k\in K'}\varepsilon'_k(c_k \cap a_k) \in \phi[\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}]. \end{aligned}$$

So $\mathfrak{A}_a \subseteq \phi[\bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i}]$. **Q**

(c) Let A be the set of those $a \in \mathfrak{A}$ expressible in the form considered in (b), with every a_i non-zero. If $a \in A$, then

$$\mathfrak{A}_a \cong \bigotimes_{i\in I}(\mathfrak{A}_i)_{a_i} \cong \bigotimes_{i\in I}\mathfrak{A}_i = \mathfrak{A}$$

because every $\mathfrak{A}_i$ is homogeneous.

(d) We need to know that $\mathfrak{A}$ is isomorphic to the simple power $\mathfrak{A}^n$ for every $n \geq 1$. **P** We are supposing that there is a $k \in I$ such that $\mathfrak{A}_k$ is infinite. In this case there must be a partition of unity $(d_1, \dots , d_n)$ in $\mathfrak{A}_k \setminus \{0\}$. (Induce on n, noting at the inductive step that if $\{d_1, \dots , d_n\}$ is a partition of unity then not all the d_j can be atoms, because $\#(\mathfrak{A}_k) > 2^n$.) Now, setting $a^{(j)} = \varepsilon_k(d_j)$ for each j, $(a^{(1)}, \dots , a^{(n)})$ is a partition of unity in $\mathfrak{A}$ and (by (c)) all the principal ideals $\mathfrak{A}_{a^{(j)}}$ are isomorphic to $\mathfrak{A}$, so

$$\mathfrak{A} \cong \prod_{j\leq n} \mathfrak{A}_{a^{(j)}} \cong \mathfrak{A}^n$$

by 315F(i). **Q**

(e) Now take any $a \in \mathfrak{A} \setminus \{0\}$. Then a is expressible as $\sup_{1\leq j\leq n} a^{(j)}$ where $a^{(1)}, \dots , a^{(n)}$ are disjoint members of A (315Kb). So, putting (c) and (d) together,

$$\mathfrak{A}_a \cong \prod_{1\leq j\leq n} \mathfrak{A}_{a^{(j)}} \cong \mathfrak{A}^n \cong \mathfrak{A}.$$

As a is arbitrary, $\mathfrak{A}$ is homogeneous.

***316R** It will be useful in later volumes to be able to quote a simple fact.

Proposition Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ which is regularly embedded in $\mathfrak{A}$.
(a) Every atom of $\mathfrak{A}$ is included in an atom of $\mathfrak{B}$.
(b) If $\mathfrak{B}$ is atomless, so is $\mathfrak{A}$.

proof (a) If $a \in \mathfrak{A}$ is an atom, consider $B = \{b : b \in \mathfrak{B},\ a \subseteq b\}$. Then B has a non-zero lower bound b_0 in $\mathfrak{B}$. **P?** Otherwise, $\inf B = 0$ in $\mathfrak{B}$; as $b \mapsto b : \mathfrak{B} \to \mathfrak{A}$ is order-continuous and B is downwards-directed, $\inf B = 0$ in $\mathfrak{A}$; but a is a non-zero lower bound for B in $\mathfrak{A}$. **XQ** If now $b \in \mathfrak{B}$, $b \subseteq b_0$ and $b \neq 0$, $b_0 \not\subseteq 1 \setminus b$ so $1 \setminus b \notin B$, $a \not\subseteq 1 \setminus b$ and $a \cap b \neq 0$; as a is an atom in $\mathfrak{A}$, $a \setminus b = 0$, $a \subseteq b$, $b \in B$ and $b = b_0$. Thus b_0 is an atom in $\mathfrak{B}$. Repeating the argument just above with b_0 in the place of b, we see that $a \subseteq b_0$. Thus a is included in an atom of $\mathfrak{B}$.

(b) follows at once.

316X Basic exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that it is ccc iff there is no family $\langle a_\xi\rangle_{\xi<\omega_1}$ in $\mathfrak{A}$ such that $a_\xi \subset a_\eta$ whenever $\xi < \eta < \omega_1$.

(b) Let $\mathfrak{A}$ be a ccc Boolean algebra. Show that if $\mathcal{I}$ is a σ-ideal of $\mathfrak{A}$, then it is order-closed, and $\mathfrak{A}/\mathcal{I}$ is ccc.

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\mathcal{I}$ an order-closed ideal of $\mathfrak{A}$. Show that $\mathfrak{A}/\mathcal{I}$ is ccc iff there is no uncountable disjoint family in $\mathfrak{A} \setminus \mathcal{I}$.

(d) Let $\mathfrak{A}$ be a Boolean algebra. Show that the following are equiveridical: (i) $\mathfrak{A}$ is ccc; (ii) every σ-ideal of $\mathfrak{A}$ is order-closed; (iii) every σ-subalgebra of $\mathfrak{A}$ is order-closed; (iv) every sequentially order-continuous Boolean homomorphism from $\mathfrak{A}$ to another Boolean algebra is order-continuous. (*Hint*: 313Q.)

(e) Show that any purely atomic Boolean algebra is weakly (σ, ∞)-distributive.

>(f) Let $\mathfrak{A}$ be a Dedekind complete purely atomic Boolean algebra. Show that it is isomorphic to $\mathcal{P}A$, where A is the set of atoms of $\mathfrak{A}$.

(g) Show that a homogeneous Boolean algebra is either atomless or $\{0, 1\}$.

(h) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$. Show that if $\mathfrak{A}$ is ccc, then $\mathfrak{B}$ is ccc.

(i) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ which is regularly embedded in $\mathfrak{A}$. (i) Show that if $\mathfrak{A}$ is purely atomic, so is $\mathfrak{B}$. (ii) Show that if $\mathfrak{A}$ is weakly (σ, ∞)-distributive, then $\mathfrak{B}$ is weakly (σ, ∞)-distributive.

(j) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. (i) Show that $\mathfrak{A}$ is ccc iff $\mathfrak{B}$ is ccc. (ii) Show that $\mathfrak{A}$ is weakly (σ, ∞)-distributive iff $\mathfrak{B}$ is weakly (σ, ∞)-distributive. (iii) Show that $\mathfrak{A}$ and $\mathfrak{B}$ have the same atoms, so that $\mathfrak{A}$ is atomless, or purely atomic, iff $\mathfrak{B}$ is.

(k) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and $\pi : \mathfrak{A} \to \mathfrak{B}$ a surjective order-continuous Boolean homomorphism. (i) Show that if $\mathfrak{A}$ is ccc, then $\mathfrak{B}$ is ccc. (ii) Show that if $\mathfrak{A}$ is weakly (σ, ∞)-distributive, then $\mathfrak{B}$ is weakly (σ, ∞)-distributive. (iii) Show that if $\mathfrak{A}$ is atomless, then $\mathfrak{B}$ is atomless. (iv) Show that if $\mathfrak{A}$ is purely atomic, then $\mathfrak{B}$ is purely atomic.

(l) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, neither $\{0\}$, and $\mathfrak{A} \otimes \mathfrak{B}$ their free product. (i) Show that if $\mathfrak{A} \otimes \mathfrak{B}$ is ccc, then $\mathfrak{A}$ and $\mathfrak{B}$ are both ccc. (ii) Show that if $\mathfrak{A} \otimes \mathfrak{B}$ is weakly (σ, ∞)-distributive, then $\mathfrak{A}$ and $\mathfrak{B}$ are both weakly (σ, ∞)-distributive. (iii) Show that $\mathfrak{A} \otimes \mathfrak{B}$ is atomless iff either $\mathfrak{A}$ or $\mathfrak{B}$ is atomless. (iv) Show that $\mathfrak{A} \otimes \mathfrak{B}$ is purely atomic iff $\mathfrak{A}$ and $\mathfrak{B}$ are both purely atomic.

(m) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{A}_a$ a principal ideal of $\mathfrak{A}$. (i) Show that of $\mathfrak{A}$ is ccc, then $\mathfrak{A}_a$ is ccc. (ii) Show that if $\mathfrak{A}$ is weakly (σ, ∞)-distributive, then $\mathfrak{A}_a$ is weakly (σ, ∞)-distributive. (iii) Show that if $\mathfrak{A}$ is atomless, then $\mathfrak{A}_a$ is atomless. (iv) Show that if $\mathfrak{A}$ is purely atomic, then $\mathfrak{A}_a$ is purely atomic. (v) Show that if $\mathfrak{A}$ is homogeneous, then $\mathfrak{A}_a$ is homogeneous.

(n) Let $\langle \mathfrak{A}_i \rangle_{i \in I}$ be a family of Boolean algebras, with simple product $\mathfrak{A}$. (i) Show that $\mathfrak{A}$ is ccc iff every $\mathfrak{A}_i$ is ccc and $\{i : \mathfrak{A}_i \neq \{0\}\}$ is countable. (ii) Show that $\mathfrak{A}$ is weakly (σ, ∞)-distributive iff every $\mathfrak{A}_i$ is weakly (σ, ∞)-distributive. (iii) Show that $\mathfrak{A}$ is atomless iff every $\mathfrak{A}_i$ is atomless. (iv) Show that $\mathfrak{A}$ is purely atomic iff every $\mathfrak{A}_i$ is purely atomic.

>(o) Let X be a separable topological space. Show that X is ccc.

(p) Let X be a topological space, and $\mathrm{RO}(X)$ its regular open algebra. Show that X is ccc iff $\mathrm{RO}(X)$ is ccc.

(q) Let X be a zero-dimensional compact Hausdorff space. Show that the regular open algebra of X is weakly (σ, ∞)-distributive iff the algebra of open-and-closed subsets of X is weakly (σ, ∞)-distributive.

(r) Show that the algebra of open-and-closed subsets of $\{0, 1\}^{\mathbb{N}}$, with its usual topology, is ccc, homogeneous and not weakly (σ, ∞)-distributive.

(s) Show that the regular open algebra $\mathrm{RO}(\mathbb{R})$ is ccc and homogeneous.

316Y Further exercises (a) Let I be any set. Show that $\{0, 1\}^I$, with its usual topology, is ccc. (*Hint*: show that if $E \subseteq \{0, 1\}^I$ is a non-empty open-and-closed set, then $\nu_I E > 0$, where ν_I is the usual measure on $\{0, 1\}^I$.)

(b) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. Show that $\mathfrak{A}$ is ccc iff every nowhere dense subset of Z is included in a nowhere dense zero set.

(c) Let X be a zero-dimensional topological space. Show that X is ccc iff the regular open algebra of X is ccc iff the algebra of open-and-closed subsets of X is ccc.

(d) Set $X = \{0, 1\}^{\omega_1}$, and for $\xi < \omega_1$ set $E_\xi = \{x : x \in X,\, x(\xi) = 1\}$. Let Σ be the algebra of subsets of X generated by $\{E_\xi : \xi < \omega_1\} \cup \{\{x\} : x \in X\}$, and $\mathcal{I}$ the σ-ideal of Σ generated by $\{E_\xi \cap E_\eta : \xi < \eta < \omega_1\} \cup \{\{x\} : x \in X\}$. Show that $\Sigma/\mathcal{I}$ is not ccc, but that there is no uncountable disjoint family in $\Sigma \setminus \mathcal{I}$.

(e) Let $\mathfrak{A}$ be a Boolean algebra. $\mathfrak{A}$ is **weakly σ-distributive** if whenever $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of countable partitions of unity in $\mathfrak{A}$ then there is a partition B of unity such that $\{a : a \in A_n,\, a \cap b \neq 0\}$ is finite for every $b \in B$ and $n \in \mathbb{N}$. (Dedekind complete weakly σ-distributive algebras are also called **ω^ω-bounding**.) $\mathfrak{A}$ has the **Egorov property** if whenever $\langle A_n \rangle_{n \in \mathbb{N}}$ is a sequence of countable partitions of unity in $\mathfrak{A}$ then there is a *countable* partition B of unity such that $\{a : a \in A_n,\, a \cap b \neq 0\}$ is finite for every $b \in B$ and $n \in \mathbb{N}$. (i) Show that if $\mathfrak{A}$ is ccc then it is weakly (σ, ∞)-distributive iff it has the Egorov property iff it is weakly σ-distributive. (ii) Show that $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ does not have the Egorov property, even though it is weakly (σ, ∞)-distributive. (*Hint*: try $a_{nm} = \{f : f(n) = m\}$.)

(f) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with the Egorov property and I a σ-ideal of $\mathfrak{A}$. Show that $\mathfrak{A}/I$ has the Egorov property.

(g) Let X be a regular topological space and $\mathrm{RO}(X)$ its regular open algebra. Show that $\mathrm{RO}(X)$ is weakly (σ,∞)-distributive iff every meager set in X is nowhere dense.

(h) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space. (i) Show that $\mathfrak{A}$ is weakly σ-distributive iff the union of any sequence of nowhere dense zero sets in Z is nowhere dense. (ii) Show that $\mathfrak{A}$ has the Egorov property iff the union of any sequence of nowhere dense zero sets in Z is included in a nowhere dense zero set.

(i) Let $\mathfrak{A}$ be a Dedekind σ-complete weakly (σ,∞)-distributive Boolean algebra, Z its Stone space, $\mathcal{E}$ the algebra of open-and-closed subsets of Z, $\mathcal{M}$ the σ-ideal of meager subsets of Z, and Σ the algebra $\{E\triangle M : E\in\mathcal{E},\, M\in\mathcal{M}\}$, as in 314M. (i) Let $f : Z\to\mathbb{R}$ be a function. Show that f is Σ-measurable iff there is a dense open set $G\subseteq Z$ such that $f{\restriction}G$ is continuous. (ii) Now suppose that $\mathfrak{A}$ is Dedekind complete and that $f : Z\to\mathbb{R}$ is a bounded function. Show that f is Σ-measurable iff there is a continuous function $g : Z\to\mathbb{R}$ such that $\{z : f(z)\ne g(z)\}$ is meager. (*Hint*: if G is a dense open set and $f{\restriction}G$ is continuous, the closure in $Z\times\mathbb{R}$ of the graph of $f{\restriction}G$ is a function, because Z is extremally disconnected.)

(j) Show that the Stone space of $\mathrm{RO}(\mathbb{R})$ is separable. More generally, show that if a topological space X is separable so is the Stone space of its regular open algebra.

(k)(i) Let X be a non-empty separable Hausdorff space without isolated points. Show that its regular open algebra is not weakly (σ,∞)-distributive. (ii) Let (X,ρ) be a non-empty metric space without isolated points. Show that its regular open algebra is not weakly (σ,∞)-distributive. (iii) Let I be any infinite set. Show that the algebra of open-and-closed subsets of $\{0,1\}^I$ is not weakly (σ,∞)-distributive. Show that the regular open algebra of $\{0,1\}^I$ is not weakly (σ,∞)-distributive.

(l) For any set X, write

$$\mathcal{C}_X = \{I : I\subseteq X \text{ is finite}\}\cup\{X\setminus I : I\subseteq X \text{ is finite}\}.$$

(i) Show that $\mathcal{C}_X$ is an algebra of subsets of X (the **finite-cofinite algebra**). (ii) Show that a Boolean algebra is purely atomic iff it has an order-dense subalgebra isomorphic to the finite-cofinite algebra of some set. (iii) Show that a Dedekind σ-complete Boolean algebra is purely atomic iff it has an order-dense subalgebra isomorphic to the countable-cocountable algebra of some set.

(m) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, none of them $\{0\}$, with free product $\mathfrak{A}$. (i) Show that $\mathfrak{A}$ is purely atomic iff every $\mathfrak{A}_i$ is purely atomic and $\{i : \mathfrak{A}_i\ne\{0,1\}\}$ is finite. (ii) Show that $\mathfrak{A}$ is atomless iff either some $\mathfrak{A}_i$ is atomless or $\{i : \mathfrak{A}_i\ne\{0,1\}\}$ is infinite.

(n) Let X be a Hausdorff space and $\mathrm{RO}(X)$ its regular open algebra. (i) Show that the atoms of $\mathrm{RO}(X)$ are precisely the sets $\{x\}$ where x is an isolated point in X. (ii) Show that $\mathrm{RO}(X)$ is atomless iff X has no isolated points. (iii) Show that $\mathrm{RO}(X)$ is purely atomic iff the set of isolated points in X is dense in X.

(o) Show that a Boolean algebra is isomorphic to $\mathrm{RO}(\mathbb{R})$ iff it is atomless, Dedekind complete, has a countable order-dense subalgebra and is not $\{0\}$.

(p) Let $\langle\mathfrak{A}_i\rangle_{i\in I}$ be a family of Boolean algebras, none of them $\{0\}$, and $\mathfrak{A}$ their simple product. Show that $\mathfrak{A}$ is homogeneous iff (i) $\mathfrak{A}_i$ is isomorphic to $\mathfrak{A}_j$ for all i, $j\in I$ (ii) for every $i\in I$ there is a partition of unity $A\subseteq\mathfrak{A}_i\setminus\{0\}$ with $\#(A)=\#(I)$.

(q) Let $\mathfrak{A}$ be a Boolean algebra such that $\{d : d\in\mathfrak{A},\, \mathfrak{A}_d\cong\mathfrak{A}\}$ is order-dense in $\mathfrak{A}$. Show that the Dedekind completion $\widehat{\mathfrak{A}}$ is homogeneous.

(r) Write $[\mathbb{N}]^{<\omega}$ for the ideal of $\mathcal{P}\mathbb{N}$ consisting of the finite subsets of $\mathbb{N}$. Show that $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is atomless, weakly (σ,∞)-distributive and not ccc, and that its Dedekind completion is homogeneous.

(s) Show that the regular open algebra of $\{0,1\}^I$ is homogeneous for any infinite set I.

(t) Suppose that $\mathfrak{A}$ is a weakly (σ,∞)-distributive Boolean algebra, and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty upwards-directed subsets of $\mathfrak{A}$. Set

$$B=\bigcap_{n\in\mathbb{N}}\{b:b\in\mathfrak{A},\ \{b\cap a:a\in A_n\}\text{ has a greatest member}\}.$$

Show that B is upwards-directed and $\sup B=1$.

316 Notes and comments The phrase 'countable chain condition' is perhaps unfortunate, since the disjoint sets to which the definition 316A refers could more naturally be called 'antichains'; but there is in fact a connexion between countable chains and countable antichains (316Xa). While some authors speak of the 'countable antichain condition' or 'cac', the term 'ccc' has become solidly established. In the Boolean algebra context, it could equally well be called the 'countable sup property' (316E).

The countable chain condition can be thought of as a restriction on the 'width' of a Boolean algebra; it means that the algebra cannot spread too far laterally (see 316Xn(i)), though it may be indefinitely complex in other ways. Generally it means that in a wide variety of contexts we need look only at countable families and monotonic sequences, rather than arbitrary families and directed sets (316E, 316F, 316Ye). Many of the ideas of 316C-316E have already appeared in 215B; see 322G below.

I remarked in the notes to §313 that the distributive laws described in 313B have important generalizations, of which 'weak (σ,∞)-distributivity' and its cousin 'weak σ-distributivity' (316Ye) are two. They are characteristic of the measure algebras which are the chief subject of this volume. The 'Egorov property' (316Ye again) is an alternative formulation applicable to ccc spaces.

Of course every property of Boolean algebras has a reflection in a topological property of their Stone spaces; happily, most of the concepts of this section correspond to reasonably natural topological expressions (316B, 316I, 316L, 316Yh). 'Homogeneity' is the odd one out. In fact only the definition of 'homogeneous' Boolean algebra is particularly worth noting at this stage. The homogeneous algebras we are primarily interested in will appear in §331, and they are too special for any general theory to be very helpful.

With five new properties (ccc, weakly (σ,∞)-distributive, atomless, purely atomic, homogeneous) to incorporate into the constructions of the last few sections, a very large number of questions can be asked; most are elementary. In 316Xh-316Xn I list the properties which are inherited by subalgebras, order-continuous homomorphic images, free products, principal ideals and simple products. The countable chain condition is so important that it is worth noting that a sequentially order-continuous image of a ccc algebra is ccc (316Xb), and that there is a useful necessary and sufficient condition for a sequentially order-continuous image of a σ-complete algebra to be ccc (316C, 316D, 316Xc; but see also 316Yd). To see that sequentially order-continuous images do not inherit weak (σ,∞)-distributivity, recall that the regular open algebra of $\mathbb{R}$ is isomorphic to the quotient of the Baire-property algebra $\widehat{\mathcal{B}}$ of $\mathbb{R}$ by the meager ideal $\mathcal{M}$ (314Yd); but that $\widehat{\mathcal{B}}$ is purely atomic (since it contains all singletons), therefore weakly (σ,∞)-distributive (316Xe). Similarly, $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ is a non-ccc image of a ccc algebra (316Yr). Free products of weakly (σ,∞)-distributive algebras need not be weakly (σ,∞)-distributive (325Ye). There are important cases in which the simple product of homogeneous algebras is homogeneous (316Yp).

The definitions here provide a language in which a remarkably interesting question can be asked: is the free product of ccc Boolean algebras always ccc? equivalently, is the product of ccc topological spaces always ccc? What is special about this question is that it cannot be answered within the ordinary rules of mathematics (even including the axiom of choice); it is undecidable, in the same way that the continuum hypothesis is. I will deal with a variety of undecidable questions in Volume 5; this particular one is mentioned in 516U, 517G and 553J.

I have taken the opportunity to mention three of the most important of all Boolean algebras: the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$ (316M, 316Xr), the regular open algebra of $\mathbb{R}$ (316J, 316Xs, 316Yo) and the quotient $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ (316Yr). A fourth algebra which belongs in this company is the Lebesgue measure algebra, which is atomless, ccc, weakly (σ,∞)-distributive and homogeneous (so that every countable subset of its Stone space Z is nowhere dense, and Z is a non-separable ccc space); but for this I wait for the next chapter.

Version of 30.9.07

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

314V Upper envelopes The note on upper envelopes, referred to in the 2003 and 2006 printings of Volume 4, has been moved to 313S.

315H Paragraphs 315H-315N, referred to in the 2003 and 2006 printings of Volume 4 and the 2008 printing of Volume 5, are now 315I-315O.

316J Weakly (σ,∞)-distributive algebras The reference to 316J in the 2003 printing of Volume 4 should be changed to 316H.

Version of 25.8.17

References for Volume 3

Anderson I. [87] *Combinatorics of Finite Sets.* Oxford U.P., 1987. [332Xk, 3A1K.]

Anzai H. [51] 'On an example of a measure-preserving transformation which is not conjugate to its inverse', Proc. Japanese Acad. Sci. 27 (1951) 517-522. [§382 *notes*.]

Balcar B., Główczyński W. & Jech T. [98] 'The sequential topology on complete Boolean algebras', Fundamenta Math. 155 (1998) 59-78. [393L, 393Q.]

Balcar B., Jech T. & Pazák T. [05] 'Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann', Bull. London Math. Soc. 37 (2005) 885-898. [393L, 393Q.]

Becker H. & Kechris A.S. [96] *The descriptive set theory of Polish group actions.* Cambridge U.P., 1996 (London Math. Soc. Lecture Note Series 232). [§395 *notes*.]

Bekkali M. & Bonnet R. [89] 'Rigid Boolean algebras', pp. 637-678 in Monk 89. [384L.]

Bellow A. & Kölzow D. [76] *Measure Theory, Oberwolfach 1975.* Springer, 1976 (Lecture Notes in Mathematics 541).

Bhaskara Rao, K.P.S. & Bhaskara Rao, M. *Theory of Charges.* Academic, 1983.

Billingsley P. [65] *Ergodic Theory and Information.* Wiley, 1965. [§372 *notes*.]

Bollobás B. [79] *Graph Theory.* Springer, 1979. [332Xk, 3A1K.]

Bose R.C. & Manvel B. [84] *Introduction to Combinatorial Theory.* Wiley, 1984. [3A1K.]

Bourbaki N. [66] *General Topology.* Hermann/Addison-Wesley, 1968. [§3A3, §3A4.]

Bourbaki N. [68] *Theory of Sets.* Hermann/Addison-Wesley, 1968. [§315 *notes*.]

Bourbaki N. [87] *Topological Vector Spaces.* Springer, 1987. [§3A5.]

Bukhvalov A.V. [95] 'Optimization without compactness, and its applications', pp. 95-112 in Huijsmans Kaashoek Luxemburg & Pagter 95. [367U.]

Burke M.R. [93] 'Liftings for Lebesgue measure', pp. 119-150 in Judah 93. [341L, 345F.]

Burke M.R. [n95] 'Consistent liftings', privately circulated, 1995. [346Ya.]

Burnside W. [1911] *Theory of Groups of Finite Order.* Cambridge U.P., 1911 (reprinted by Dover, 1955). [§384 *notes*.]

Chacon R.V. [69] 'Weakly mixing transformations which are not strongly mixing', Proc. Amer. Math. Soc. 22 (1969) 559-562. [372R.]

Chacon R.V. & Krengel U. [64] 'Linear modulus of a linear operator', Proc. Amer. Math. Soc. 15 (1964) 553-559. [§371 *notes*.]

Choksi J.R. & Prasad V.S. [82] 'Ergodic theory of homogeneous measure algebras', pp. 367-408 of Kölzow & Maharam-Stone 82. [383I.]

Cohn H. [06] 'A short proof of the simple continued fraction expansion of e', Amer. Math. Monthly 113 (2006) 57-62; `arXiv:math.NT/0601660`. [372L.]

Coleman A.J. & Ribenboim P. [67] (eds.) *Proceedings of the Symposium in Analysis, Queen's University, June 1967.* Queen's University, Kingston, Ontario, 1967.

Comfort W.W. & Negrepontis S. [82] *Chain Conditions in Topology.* Cambridge U.P., 1982. [§391 *notes*.]

Csiszár I. [67] 'Information-type measures of difference of probability distributions and indirect observations', Studia Scientiarum Math. Hungarica 2 (1967) 299-318. [386G.]

Davey B.A. & Priestley H.A. [90] 'Introduction to Lattices and Order', Cambridge U.P., 1990. [3A6C.]

Dugundji J. [66] *Topology.* Allyn & Bacon, 1966. [§3A3, 3A4Bb.]

Dunford N. [1936] 'Integration and linear operators', Trans. Amer. Math. Soc. 40 (1936) 474-494. [§376 *notes*.]

Dunford N. & Schwartz J.T. [57] *Linear Operators I.* Wiley, 1957 (reprinted 1988). [§356 *notes*, §371 *notes*, §376 *notes*, §3A5.]

Dye H.A. [59] 'On groups of measure preserving transformations I', Amer. J. Math. 81 (1959) 119-159. [§388 *notes*.]

Eigen S.J. [82] 'The group of measure-preserving transformations of $[0,1]$ has no outer automorphisms', Math. Ann. 259 (1982) 259-270. [384D.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [§3A3, §3A4.]

Enderton H.B. [77] *Elements of Set Theory.* Academic, 1977. [§3A1.]

Erdős P. & Oxtoby J.C. [55] 'Partitions of the plane into sets having positive measure in every non-null product set', Trans. Amer. Math. Soc. 79 (1955) 91-102. [§325 *notes*.]

Fathi A. [78] 'Le groupe des transformations de $[0,1]$ qui préservent la mesure de Lebesgue est un groupe simple', Israel J. Math. 29 (1978) 302-308. [§382 *notes*, 383I.]

Fremlin D.H. [74a] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [Chap. 35 *intro.*, 354Yb, §355 *notes*, §356 *notes*, §363 *notes*, §365 *notes*, §371 *notes*, §376 *notes*.]

Fremlin D.H. [74b] 'A characterization of L-spaces', Indag. Math. 36 (1974) 270-275. [§371 *notes*.]

Fremlin D.H., de Pagter B. & Ricker W.J. [05] 'Sequential closedness of Boolean algebras of projections in Banach spaces', Studia Math. 167 (2005) 45-62. [§323 *notes*.]

Frolík Z. [68] 'Fixed points of maps of extremally disconnected spaces and complete Boolean algebras', Bull. Acad. Polon. Sci. 16 (1968) 269-275. [382E.]

Gaal S.A. [64] *Point Set Topology.* Academic, 1964. [§3A3, §3A4.]

Gaifman H. [64] 'Concerning measures on Boolean algebras', Pacific J. Math. 14 (1964) 61-73. [§391 *notes*.]

Gale D. [60] *The theory of linear economic models.* McGraw-Hill, 1960. [3A5D.]

Gnedenko B.V. & Kolmogorov A.N. [54] *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, 1954. [§342 *notes*.]

Graf S. & Weizsäcker H.von [76] 'On the existence of lower densities in non-complete measure spaces', pp. 155-158 in Bellow & Kölzow 76. [341L.]

Hajian A. & Ito Y. [69] 'Weakly wandering sets and invariant measures for a group of transformations', J. of Math. and Mech. 18 (1969) 1203-1216. [396B.]

Hajian A., Ito Y. & Kakutani S. [75] 'Full groups and a theorem of Dye', Advances in Math. 17 (1975) 48-59. [§388 *notes*.]

Halmos P.R. [1948] 'The range of a vector measure', Bull. Amer. Math. Soc. 54 (1948) 416-421. [326Yk.]

Halmos P.R. [60] *Naive Set Theory.* Van Nostrand, 1960. [3A1D.]

Huijsmans C.B., Kaashoek M.A., Luxemburg W.A.J. & de Pagter B. [95] (eds.) *Operator Theory in Function Spaces and Banach Lattices.* Birkhäuser, 1995.

Ionescu Tulcea C. & Ionescu Tulcea A. [69] *Topics in the Theory of Lifting.* Springer, 1969. [§341 *notes*.]

James I.M. [87] *Topological and Uniform Spaces.* Springer, 1987. [§3A3, §3A4.]

Jech T. [03] *Set Theory, Millennium Edition.* Springer, 2002. [§3A1.]

Jech T. [08] 'Algebraic characterizations of measure algebras', Proc. Amer. Math. Soc. 136 (2008) 1285-1294. [393Xj.]

Johnson R.A. [80] 'Strong liftings which are not Borel liftings', Proc. Amer. Math. Soc. 80 (1980) 234-236. [345F.]

Judah H. [93] (ed.) *Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991.* Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Kakutani S. [1941] 'Concrete representation of abstract L-spaces and the mean ergodic theorem', Annals of Math. 42 (1941) 523-537. [369E.]

Kalton N.J., Peck N.T. & Roberts J.W. [84] 'An F-space sampler', Cambridge U.P., 1984. [§375 *notes*.]

Kalton N.J. & Roberts J.W. [83] 'Uniformly exhaustive submeasures and nearly additive set functions', Trans. Amer. Math. Soc. 278 (1983) 803-816. [392D, §392 *notes*.]

Kantorovich L.V., Vulikh B.Z. & Pinsker A.G. [50] *Functional Analysis in Partially Ordered Spaces*, Gostekhizdat, 1950. [391D.]

Kawada Y. [1944] 'Über die Existenz der invarianten Integrale', Jap. J. Math. 19 (1944) 81-95. [§395 *notes*.]

Kelley J.L. [59] 'Measures on Boolean algebras', Pacific J. Math. 9 (1959) 1165-1177. [§391 *notes*.]

Kolmogorov A.N. [58] 'New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces', Dokl. Akad. Nauk SSSR 119 (1958) 861-864. [385P.]

Kölzow D. & Maharam-Stone D. [82] (eds.) *Measure Theory Oberwolfach 1981.* Springer, 1982 (Lecture Notes in Math. 945).

Koppelberg S. [89] *General Theory of Boolean Algebras*, vol. 1 of MONK 89. [Chap. 31 *intro.*, §332 *notes.*]

Köthe G. [69] *Topological Vector Spaces I.* Springer, 1969. [§356 *notes*, 3A5N.]

Kranz P. & Labuda I. [93] (eds.) *Proceedings of the Orlicz Memorial Conference, 1991*, unpublished manuscript, available from first editor (`mmkranz@olemiss.edu`).

Krengel, U. [63] '"Uber den Absolutbetrag stetiger linearer Operatoren und seine Anwendung auf ergodische Zerlegungen', Math. Scand. 13 (1963) 151-187. [371Xb.]

Krieger W. [76] 'On ergodic flows and the isomorphism of factors', Math. Ann. 223 (1976) 19-70. [§388 *notes.*]

Krivine J.-L. [71] *Introduction to Axiomatic Set Theory.* D.Reidel, 1971. [§3A1.]

Kullback S. [67] 'A lower bound for discrimination information in terms of variation', IEEE Trans. on Information Theory 13 (1967) 126-127. [386G.]

Kunen K. [80] *Set Theory.* North-Holland, 1980. [§3A1.]

Kwapien S. [73] 'On the form of a linear operator on the space of all measurable functions', Bull. Acad. Polon. Sci. 21 (1973) 951-954. [§375 *notes.*]

Lang S. [93] *Real and Functional Analysis.* Springer, 1993. [§3A5.]

Liapounoff A.A. [1940] 'Sur les fonctions-vecteurs complètement additives', Bull. Acad. Sci. URSS (Izvestia Akad. Nauk SSSR) 4 (1940) 465-478. [326H.]

Lindenstrauss J. & Tzafriri L. [79] *Classical Banach Spaces II.* Springer, 1979, reprinted in LINDENSTRAUSS & TZAFRIRI 96. [§354 *notes*, 374Xj.]

Lindenstrauss J. & Tzafriri L. [96] *Classical Banach Spaces I & II.* Springer, 1996.

Lipschutz S. [64] *Set Theory and Related Topics.* McGraw-Hill, 1964 (Schaum's Outline Series). [3A1D.]

Luxemburg W.A.J. [67a] 'Is every integral normal?', Bull. Amer. Math. Soc. 73 (1967) 685-688. [363S.]

Luxemburg W.A.J. [67b] 'Rearrangement-invariant Banach function spaces', pp. 83-144 in COLEMAN & RIBENBOIM 67. [§374 *notes.*]

Luxemburg W.A.J. & Zaanen A.C. [71] *Riesz Spaces I.* North-Holland, 1971. [Chap. 35 *intro.*]

Macheras N.D., Musiał K. & Strauss W. [99] 'On products of admissible liftings and densities', J. for Analysis and its Applications 18 (1999) 651-668. [346G.]

Macheras N.D. & Strauss W. [95] 'Products of lower densities', J. for Analysis and its Applications 14 (1995) 25-32. [346Xf.]

Macheras N.D. & Strauss W. [96a] 'On products of almost strong liftings', J. Australian Math. Soc. (A) 60 (1996) 1-23. [346Yc.]

Macheras N.D. & Strauss W. [96b] 'The product lifting for arbitrary products of complete probability spaces', Atti Sem. Math. Fis. Univ. Modena 44 (1996) 485-496. [346H, 346Yd.]

Maharam D. [1942] 'On homogeneous measure algebras', Proc. Nat. Acad. Sci. U.S.A. 28 (1942) 108-111. [331F, 332B.]

Maharam D. [1947] 'An algebraic characterization of measure algebras', Ann. Math. 48 (1947) 154-167. [393J.]

Maharam D. [58] 'On a theorem of von Neumann', Proc. Amer. Math. Soc. 9 (1958) 987-994. [§341 *notes*, §346 *notes.*]

Marczewski E. [53] 'On compact measures', Fund. Math. 40 (1953) 113-124. [342A.]

McCune W. [97] 'Solution of the Robbins problem', J. Automated Reasoning 19 (1997) 263-276. [311Yc.]

Miller B.D. [04] PhD Thesis, University of California, Berkeley, 2004. [382Xc, §382 *notes.*]

Monk J.D. [89] (ed.) *Handbook of Boolean Algebra.* North-Holland, 1989.

Nadkarni M.G. [90] 'On the existence of a finite invariant measure', Proc. Indian Acad. Sci., Math. Sci. 100 (1990) 203-220. [§395 *notes.*]

Ornstein D.S. [74] *Ergodic Theory, Randomness and Dynamical Systems.* Yale U.P., 1974. [§387 *notes.*]

Ornstein D.S. & Shields P.C. [73] 'An uncountable family of K-automorphisms', Advances in Math. 10 (1973) 63-88. [§382 *notes.*]

Perović Ž. & Veličković B. [18] 'Ranks of Maharam algebras', Advances in Math. 330 (2018) 253-279. [394A.]

Petersen K. [83] *Ergodic Theory.* Cambridge U.P., 1983. [328Xa, 385C, §385 *notes*, 386E.]

Roberts J.W. [93] 'Maharam's problem', in Kranz & Labuda 93. [§394 *notes*.]

Rotman J.J. [84] 'An Introduction to the Theory of Groups', Allyn & Bacon, 1984. [§384 *notes*, 3A6B.]

Rudin W. [91] *Functional Analysis.* McGraw-Hill, 1991. [§3A5.]

Ryzhikov V.V. [93] 'Factorization of an automorphism of a complete Boolean algebra into a product of three involutions', Mat. Zametki (=Math. Notes of Russian Acad. Sci.) 54 (1993) 79-84. [§382 *notes*.]

Sazonov V.V. [66] 'On perfect measures', A.M.S. Translations (2) 48 (1966) 229-254. [§342 *notes*.]

Schaefer H.H. [66] *Topological Vector Spaces.* MacMillan, 1966; reprinted with corrections Springer, 1971. [3A4A, 3A5J, 3A5N.]

Schaefer H.H. [74] *Banach Lattices and Positive Operators.* Springer, 1974. [Chap. 35 *intro*., §354 *notes*.]

Schubert H. [68] *Topology.* Allyn & Bacon, 1968. [§3A3, §3A4.]

Shelah S. [98] 'The lifting problem with the full ideal', J. Applied Analysis 4 (1998) 1-17. [341L.]

Sikorski R. [64] *Boolean Algebras.* Springer, 1964. [Chap. 31 *intro*.]

Sinaĭ Ya.G. [59] 'The notion of entropy of a dynamical system', Dokl. Akad. Nauk SSSR 125 (1959) 768-771. [385P.]

Sinaĭ Ya.G. [62] 'Weak isomorphism of transformations with an invariant measure', Soviet Math. 3 (1962) 1725-1729. [387E.]

Smorodinsky M. [71] *Ergodic Theory, Entropy.* Springer, 1971 (Lecture Notes in Math., 214). [§387 *notes*.]

Štěpánek P. & Rubin M. [89] 'Homogeneous Boolean algebras', pp. 679-715 in Monk 89. [382S, §382 *notes*.]

Talagrand M. [82a] 'Closed convex hull of set of measurable functions, Riemann-measurable functions and measurability of translations', Ann. Institut Fourier (Grenoble) 32 (1982) 39-69. [§346 *notes*.]

Talagrand M. [82b] 'La pathologie des relèvements invariants', Proc. Amer. Math. Soc. 84 (1982) 379-382. [345F.]

Talagrand M. [84] *Pettis integral and measure theory.* Mem. Amer. Math. Soc. 307 (1984). [§346 *notes*.]

Talagrand M. [08] 'Maharam's problem', Annals of Math. 168 (2008) 981-1009. [§394.]

Taylor A.E. [64] *Introduction to Functional Analysis.* Wiley, 1964. [§3A5.]

Todorčević S. [04] 'A problem of von Neumann and Maharam about algebras supporting continuous submeasures', Fund. Math. 183 (2004) 169-183. [393S.]

Truss J.K. [89] 'Infinite permutation groups I: products of conjugacy classes', J. Algebra 120 (1989) 454-493. [§382 *notes*.]

Vladimirov D.A. [02] *Boolean algebras in analysis.* Kluwer, 2002 (Math. and its Appl. 540). [367L.]

Vulikh B.C. [67] *Introduction to the Theory of Partially Ordered Vector Spaces.* Wolters-Noordhoff, 1967. [§364 *notes*.]

Wagon S. [85] *The Banach-Tarski Paradox.* Cambridge U.P., 1985. [§395 *notes*.]

Wilansky A. [64] *Functional Analysis.* Blaisdell, 1964. [§3A5.]

Zaanen A.C. [83] *Riesz Spaces II.* North-Holland, 1983. [Chap. 35 *intro*., 376K, §376 *notes*.]