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Chapter 28

Fourier analysis

For the last chapter of this volume, I attempt a brief account of one of the most important topics in analysis. This is a bold enterprise, and I cannot hope to satisfy the reasonable demands of anyone who knows and loves the subject as it deserves. But I also cannot pass it by without being false to my own subject, since problems contributed by the study of Fourier series and transforms have led measure theory throughout its history. What I will try to do, therefore, is to give versions of those results which everyone ought to know in language unifying them with the rest of this treatise, aiming to open up a channel for the transfer of intuitions and techniques between the abstract general study of measure spaces, which is the centre of our work, and this particular family of applications of the theory of integration.

I have divided the material of this chapter, conventionally enough, into three parts: Fourier series, Fourier transforms and the characteristic functions of probability theory. While it will be obvious that many ideas are common to all three, I do not think it useful, at this stage, to try to formulate an explicit generalization to unify them; that belongs to a more general theory of harmonic analysis on groups, which must wait until Volume 4. I begin however with a section on the Stone-Weierstrass theorem (§281), which is one of the basic tools of functional analysis, as well as being useful for this chapter. The final section (§286), a proof of Carleson's theorem, is at a rather different level from the rest.

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281 The Stone-Weierstrass theorem

Before we begin work on the real subject of this chapter, it will be helpful to have a reasonably general statement of a fundamental theorem on the approximation of continuous functions. In fact I give a variety of forms (281A, 281E, 281F and 281G, together with 281Ya, 281Yd and 281Yg), all of which are sometimes useful. I end the section with a version of Weyl's Equidistribution Theorem (281M-281N).

281A Stone-Weierstrass theorem: first form Let X be a topological space and K a compact subset of X. Write $C_b(X)$ for the space of all bounded continuous real-valued functions on X, so that $C_b(X)$ is a linear space over $\mathbb{R}$. Let $A \subseteq C_b(X)$ be such that

A is a linear subspace of $C_b(X)$;

$|f| \in A$ for every $f \in A$;

$\chi X \in A$;

whenever x, y are distinct points of K there is an $f \in A$ such that $f(x) \neq f(y)$.

Then for every continuous $h : K \to \mathbb{R}$ and $\epsilon > 0$ there is an $f \in A$ such that

$|f(x) - h(x)| \leq \epsilon$ for every $x \in K$,

if $K \neq \emptyset$, $\inf_{x\in X} f(x) \geq \inf_{x\in K} h(x)$ and $\sup_{x\in X} f(x) \leq \sup_{x\in K} h(x)$.

281B Lemma Let X be any set. Write $\ell^\infty(X)$ for the set of bounded functions from X to $\mathbb{R}$. For $f \in \ell^\infty(X)$, set

$$\|f\|_\infty = \sup_{x\in X} |f(x)|,$$

counting the supremum as 0 if X is empty. Then

(a) $\ell^\infty(X)$ is a normed space.

(b) Let $A \subseteq \ell^\infty(X)$ be a subset and $\overline{A}$ its closure.

(i) If A is a linear subspace of $\ell^\infty(X)$, so is $\overline{A}$.
(ii) If $f \times g \in A$ whenever f, $g \in A$, then $f \times g \in \overline{A}$ whenever f, $g \in \overline{A}$.
(iii) If $|f| \in A$ whenever $f \in A$, then $|f| \in \overline{A}$ whenever $f \in \overline{A}$.

281C Lemma There is a sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ of real polynomials such that $\lim_{n\to\infty} p_n(x) = |x|$ uniformly for $x \in [-1, 1]$.

281D Corollary Let X be a set, and A a norm-closed linear subspace of $\ell^\infty(X)$ containing χX and such that $f \times g \in A$ whenever f, $g \in A$. Then $|f| \in A$ for every $f \in A$.

281E Stone-Weierstrass theorem: second form Let X be a topological space and K a compact subset of X. Write $C_b(X)$ for the space of all bounded continuous real-valued functions on X. Let $A \subseteq C_b(X)$ be such that

A is a linear subspace of $C_b(X)$;

$f \times g \in A$ for every f, $g \in A$;

$\chi X \in A$;

whenever x, y are distinct points of K there is an $f \in A$ such that $f(x) \neq f(y)$.

Then for every continuous $h : K \to \mathbb{R}$ and $\epsilon > 0$ there is an $f \in A$ such that

$|f(x) - h(x)| \leq \epsilon$ for every $x \in K$,

if $K \neq \emptyset$, $\inf_{x\in X} f(x) \geq \inf_{x\in K} h(x)$ and $\sup_{x\in X} f(x) \leq \sup_{x\in K} h(x)$.

281F Corollary: Weierstrass' theorem Let K be any closed bounded subset of $\mathbb{R}$. Then every continuous $h : K \to \mathbb{R}$ can be uniformly approximated on K by polynomials.

281G Stone-Weierstrass theorem: third form Let X be a topological space and K a compact subset of X. Write $C_b(X;\mathbb{C})$ for the space of all bounded continuous complex-valued functions on X, so that $C_b(X;\mathbb{C})$ is a linear space over $\mathbb{C}$. Let $A \subseteq C_b(X;\mathbb{C})$ be such that

A is a linear subspace of $C_b(X;\mathbb{C})$;

$f \times g \in A$ for every f, $g \in A$;

$\chi X \in A$;

the complex conjugate $\bar{f}$ of f belongs to A for every $f \in A$;

whenever x, y are distinct points of K there is an $f \in A$ such that $f(x) \neq f(y)$.

Then for every continuous $h : K \to \mathbb{C}$ and $\epsilon > 0$ there is an $f \in A$ such that

$|f(x) - h(x)| \leq \epsilon$ for every $x \in K$,

if $K \neq \emptyset$, $\sup_{x\in X} |f(x)| \leq \sup_{x\in K} |h(x)|$.

281H Corollary Let $[a, b] \subseteq \mathbb{R}$ be a non-empty bounded closed interval and $h : [a, b] \to \mathbb{C}$ a continuous function. Then for any $\epsilon > 0$ there are $y_0, \dots, y_n \in \mathbb{R}$ and $c_0, \dots, c_n \in \mathbb{C}$ such that

$$|h(x) - \textstyle\sum_{k=0}^n c_k e^{iy_k x}| \leq \epsilon \text{ for every } x \in [a, b],$$

$$\sup_{x\in\mathbb{R}} |\textstyle\sum_{k=0}^n c_k e^{iy_k x}| \leq \sup_{x\in[a,b]} |h(x)|.$$

281I Corollary Let S^1 be the unit circle $\{z : |z| = 1\} \subseteq \mathbb{C}$. Then for any continuous function $h : S^1 \to \mathbb{C}$ and $\epsilon > 0$, there are $n \in \mathbb{N}$ and $c_{-n}, c_{-n+1}, \dots, c_0, \dots, c_n \in \mathbb{C}$ such that $|h(z) - \sum_{k=-n}^n c_k z^k| \leq \epsilon$ for every $z \in S^1$.

281J Corollary Let $h : [-\pi, \pi] \to \mathbb{C}$ be a continuous function such that $h(\pi) = h(-\pi)$. Then for any $\epsilon > 0$ there are $n \in \mathbb{N}$, $c_{-n}, \dots, c_n \in \mathbb{C}$ such that $|h(x) - \sum_{k=-n}^n c_k e^{ikx}| \leq \epsilon$ for every $x \in [-\pi, \pi]$.

281K Corollary Suppose that $r \ge 1$ and that $K \subseteq \mathbb{R}^r$ is a non-empty closed bounded set. Let $h : K \to \mathbb{C}$ be a continuous function, and $\epsilon > 0$. Then there are $y_0, \dots, y_n \in \mathbb{Q}^r$ and $c_0, \dots, c_n \in \mathbb{C}$ such that

$$|h(x) - \sum_{k=0}^n c_k e^{iy_k \centerdot x}| \le \epsilon \text{ for every } x \in K,$$

$$\sup_{x\in\mathbb{R}^r} |\sum_{k=0}^n c_k e^{iy_k \centerdot x}| \le \sup_{x\in K} |h(x)|,$$

writing $y \centerdot x = \sum_{j=1}^r \eta_j \xi_j$ when $y = (\eta_1, \dots, \eta_r)$ and $x = (\xi_1, \dots, \xi_r)$ belong to $\mathbb{R}^r$.

281L Corollary Suppose that $r \ge 1$ and that $K \subseteq \mathbb{R}^r$ is a non-empty closed bounded set. Let $h : K \to \mathbb{R}$ be a continuous function, and $\epsilon > 0$. Then there are $y_0, \dots, y_n \in \mathbb{R}^r$ and $c_0, \dots, c_n \in \mathbb{C}$ such that, writing $g(x) = \sum_{k=0}^n c_k e^{iy_k \centerdot x}$, g is real-valued and

$$|h(x) - g(x)| \le \epsilon \text{ for every } x \in K,$$

$$\inf_{y\in K} h(y) \le g(x) \le \sup_{y\in K} h(y) \text{ for every } x \in \mathbb{R}^r.$$

281M Weyl's Equidistribution Theorem For any real number x, write $<x>$ for that number in $[0, 1[$ such that $x - <x>$ is an integer.

281N Theorem Let $\eta_1, \dots, \eta_r$ be real numbers such that $1, \eta_1, \dots, \eta_r$ are linearly independent over $\mathbb{Q}$. Then whenever $0 \le \alpha_j \le \beta_j \le 1$ for each $j \le r$,

$$\lim_{n\to\infty} \frac{1}{n+1} \#(\{m : m \le n, <m\eta_j> \in [\alpha_j, \beta_j] \text{ for every } j \le r\}) = \prod_{j=1}^r (\beta_j - \alpha_j).$$

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282 Fourier series

Out of the enormous theory of Fourier series, I extract a few results which may at least provide a basis for further study. I give the definitions of Fourier and Fejér sums (282A), with five of the most important results concerning their convergence (282G, 282H, 282J, 282L, 282O). On the way I include the Riemann-Lebesgue lemma (282E). I end by mentioning convolutions (282Q).

282A Definition Let f be an integrable complex-valued function defined almost everywhere in $]-\pi, \pi]$.

(a) The **Fourier coefficients** of f are the complex numbers

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

for $k \in \mathbb{Z}$.

(b) The **Fourier sums** of f are the functions

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

for $x \in]-\pi, \pi]$, $n \in \mathbb{N}$.

(c) The **Fourier series** of f is the series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$, or the series $c_0 + \sum_{k=1}^{\infty} (c_k e^{ikx} + c_{-k} e^{-ikx})$.

(d) The **Fejér sums** of f are the functions

$$\sigma_m = \frac{1}{m+1} \sum_{n=0}^{m} s_n$$

for $m \in \mathbb{N}$.

(e) If f is any function with $\operatorname{dom} f \subseteq\ ]-\pi,\pi]$, its **periodic extension** is the function $\tilde f$, with domain $\bigcup_{k\in\mathbb{Z}}(\operatorname{dom} f + 2k\pi)$, such that $\tilde f(x) = f(x-2k\pi)$ whenever $k\in\mathbb{Z}$ and $x\in\operatorname{dom} f + 2k\pi$.

282C The problems (a) Under what conditions, and in what senses, do the Fourier and Fejér sums s_n and σ_m of a function f converge to f?

(b) How do the properties of the double-ended sequence $\langle c_k\rangle_{k\in\mathbb{Z}}$ reflect the properties of f, and vice versa?

282D Lemma Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and

$$c_k = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx,\quad s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx},\quad \sigma_m(x) = \tfrac{1}{m+1}\sum_{n=0}^{m} s_n(x)$$

its Fourier coefficients, Fourier sums and Fejér sums. Write $\tilde f$ for the periodic extension of f. For $m\in\mathbb{N}$, write

$$\psi_m(t) = \tfrac{1-\cos(m+1)t}{2\pi(m+1)(1-\cos t)}$$

for $0<|t|\le\pi$.

(a) For each $n\in\mathbb{N}$, $x\in\ ]-\pi,\pi]$,

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(n+\frac12)(x-t)}{\sin\frac12(x-t)}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde f(x+t)\frac{\sin(n+\frac12)t}{\sin\frac12 t}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x -_{2\pi} t)\frac{\sin(n+\frac12)t}{\sin\frac12 t}dt, \end{aligned}$$

writing $x -_{2\pi} t$ for whichever of $x-t$, $x-t-2\pi$, $x-t+2\pi$ belongs to $]-\pi,\pi]$.

(b) For each $m\in\mathbb{N}$, $x\in\ ]-\pi,\pi]$,

$$\begin{aligned} \sigma_m(x) &= \int_{-\pi}^{\pi}\tilde f(x+t)\psi_m(t)dt \\ &= \int_0^{\pi}(\tilde f(x+t)+\tilde f(x-t))\psi_m(t)dt \\ &= \int_{-\pi}^{\pi} f(x -_{2\pi} t)\psi_m(t)dt. \end{aligned}$$

(c) For any $n\in\mathbb{N}$,

$$\frac{1}{2\pi}\int_{-\pi}^{0}\frac{\sin(n+\frac12)t}{\sin\frac12 t}dt = \frac{1}{2\pi}\int_{0}^{\pi}\frac{\sin(n+\frac12)t}{\sin\frac12 t}dt = \frac12,\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin(n+\frac12)t}{\sin\frac12 t}dt = 1.$$

(d) For any $m\in\mathbb{N}$,

(i) $0\le\psi_m(t)\le\dfrac{m+1}{2\pi}$ for every t;

(ii) for any $\delta>0$, $\lim_{m\to\infty}\psi_m(t)=0$ uniformly on $\{t:\delta\le|t|\le\pi\}$;

(iii) $\int_{-\pi}^{0}\psi_m = \int_0^{\pi}\psi_m = \dfrac12$, $\quad\int_{-\pi}^{\pi}\psi_m = 1$.

282E The Riemann-Lebesgue lemma Let f be a complex-valued function which is integrable over $\mathbb{R}$. Then

$$\lim_{y\to\infty}\int f(x)e^{-iyx}dx = \lim_{y\to-\infty}\int f(x)e^{-iyx}dx = 0.$$

282F Corollary (a) Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and $\langle c_k\rangle_{k\in\mathbb{Z}}$ its sequence of Fourier coefficients. Then $\lim_{k\to\infty} c_k = \lim_{k\to-\infty} c_k = 0$.

(b) Let f be a complex-valued function which is integrable over $\mathbb{R}$. Then $\lim_{y\to\infty}\int f(x)\sin yx\,dx = 0$.

282G Theorem Let $f :]-\pi,\pi] \to \mathbb{C}$ be a continuous function such that $\lim_{t\downarrow-\pi} f(t) = f(\pi)$. Then its sequence $\langle\sigma_m\rangle_{m\in\mathbb{N}}$ of Fejér sums converges uniformly to f on $]-\pi,\pi]$.

282H Theorem Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and $\langle\sigma_m\rangle_{m\in\mathbb{N}}$ its sequence of Fejér sums. Suppose that $x\in\,]-\pi,\pi]$ and $c\in\mathbb{C}$ are such that

$$\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^\delta |\tilde f(x+t)+\tilde f(x-t)-2c|dt = 0,$$

writing $\tilde f$ for the periodic extension of f; then $\lim_{m\to\infty}\sigma_m(x) = c$.

282I Corollary Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and $\langle\sigma_m\rangle_{m\in\mathbb{N}}$ its sequence of Fejér sums.

(a) $f(x) = \lim_{m\to\infty}\sigma_m(x)$ for almost every $x\in\,]-\pi,\pi]$.

(b) $\lim_{m\to\infty}\int_{-\pi}^{\pi}|f-\sigma_m| = 0$.

(c) If g is another integrable function with the same Fourier coefficients, then $f =_{\text{a.e.}} g$.

(d) If $x\in\,]-\pi,\pi[$ is such that $a = \lim_{t\in\operatorname{dom} f, t\uparrow x} f(t)$ and $b = \lim_{t\in\operatorname{dom} f, t\downarrow x} f(t)$ are both defined in $\mathbb{C}$, then

$$\lim_{m\to\infty}\sigma_m(x) = \frac{1}{2}(a+b).$$

(e) If $a = \lim_{t\in\operatorname{dom} f, t\uparrow\pi} f(t)$ and $b = \lim_{t\in\operatorname{dom} f, t\downarrow-\pi} f(t)$ are both defined in $\mathbb{C}$, then

$$\lim_{m\to\infty}\sigma_m(\pi) = \frac{1}{2}(a+b).$$

(f) If f is defined and continuous at $x\in\,]-\pi,\pi[$, then

$$\lim_{m\to\infty}\sigma_m(x) = f(x).$$

(g) If $\tilde f$, the periodic extension of f, is defined and continuous at π, then

$$\lim_{m\to\infty}\sigma_m(\pi) = f(\pi).$$

282J Theorem Let f be a complex-valued function which is square-integrable over $]-\pi,\pi]$. Let $\langle c_k\rangle_{k\in\mathbb{Z}}$ be its Fourier coefficients and $\langle s_n\rangle_{n\in\mathbb{N}}$ its Fourier sums. Then

(i) $\sum_{k=-\infty}^{\infty}|c_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2$,

(ii) $\lim_{n\to\infty}\int_{-\pi}^{\pi}|f-s_n|^2 = 0$.

282K Corollary Let $L^2_{\mathbb{C}}$ be the Hilbert space of equivalence classes of square-integrable complex-valued functions on $]-\pi,\pi]$, with the inner product

$$(f^\bullet|g^\bullet) = \int_{-\pi}^{\pi} f\times\bar g$$

and norm

$$\|f^\bullet\|_2 = \Bigl(\int_{-\pi}^{\pi}|f|^2\Bigr)^{1/2},$$

writing $f^\bullet\in L^2_{\mathbb{C}}$ for the equivalence class of a square-integrable function f. Let $\ell^2_{\mathbb{C}}(\mathbb{Z})$ be the Hilbert space of square-summable double-ended complex sequences, with the inner product

$$(\boldsymbol{c}|\boldsymbol{d}) = \sum_{k=-\infty}^{\infty} c_k \bar{d}_k$$

and norm

$$\|\boldsymbol{c}\|_2 = \Bigl(\sum_{k=-\infty}^{\infty} |c_k|^2\Bigr)^{1/2}$$

for $\boldsymbol{c} = \langle c_k\rangle_{k\in\mathbb{Z}}$, $\boldsymbol{d} = \langle d_k\rangle_{k\in\mathbb{Z}}$ in $\ell^2_{\mathbb{C}}(\mathbb{Z})$. Then we have an inner-product-space isomorphism $S : L^2_{\mathbb{C}} \to \ell^2_{\mathbb{C}}(\mathbb{Z})$ defined by saying that

$$S(f^\bullet)(k) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

for every square-integrable function f and every $k \in \mathbb{Z}$.

282L Theorem Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and $\langle s_n\rangle_{n\in\mathbb{N}}$ its sequence of Fourier sums.

(i) If f is differentiable at $x \in\,]-\pi,\pi[$, then $f(x) = \lim_{n\to\infty} s_n(x)$.

(ii) If the periodic extension $\tilde f$ of f is differentiable at π, then $f(\pi) = \lim_{n\to\infty} s_n(\pi)$.

282M Lemma Suppose that f is a complex-valued function, defined almost everywhere and of bounded variation on $]-\pi,\pi]$. Then $\sup_{k\in\mathbb{Z}} |kc_k| < \infty$, where c_k is the kth Fourier coefficient of f, as in 282A.

282N Lemma Let $\langle d_k\rangle_{k\in\mathbb{N}}$ be a complex sequence, and set $t_n = \sum_{k=0}^{n} d_k$, $\tau_m = \frac{1}{m+1}\sum_{n=0}^{m} t_n$ for n, $m \in \mathbb{N}$. Suppose that $\sup_{k\in\mathbb{N}} |kd_k| = M < \infty$. Then for any $j \ge 1$ and any $c \in \mathbb{C}$,

$$|t_n - c| \le \frac{M}{j} + (2j+3)\sup_{m\ge n-n/j} |\tau_m - c|$$

for every $n \ge j^2$.

282O Theorem Let f be a complex-valued function of bounded variation, defined almost everywhere in $]-\pi,\pi]$, and let $\langle s_n\rangle_{n\in\mathbb{N}}$ be its sequence of Fourier sums.

(i) If $x \in\,]-\pi,\pi[$, then

$$\lim_{n\to\infty} s_n(x) = \frac{1}{2}(\lim_{t\in\operatorname{dom} f, t\uparrow x} f(t) + \lim_{t\in\operatorname{dom} f, t\downarrow x} f(t)).$$

(ii) $\lim_{n\to\infty} s_n(\pi) = \frac{1}{2}(\lim_{t\in\operatorname{dom} f, t\uparrow\pi} f(t) + \lim_{t\in\operatorname{dom} f, t\downarrow -\pi} f(t))$.

(iii) If f is defined throughout $]-\pi,\pi]$, is continuous, and $\lim_{t\downarrow -\pi} f(t) = f(\pi)$, then $s_n(x) \to f(x)$ uniformly on $]-\pi,\pi]$.

282P Corollary Let f be a complex-valued function which is integrable over $]-\pi,\pi]$, and $\langle s_n\rangle_{n\in\mathbb{N}}$ its sequence of Fourier sums.

(i) Suppose that $x \in\,]-\pi,\pi[$ is such that f is of bounded variation on some neighbourhood of x. Then

$$\lim_{n\to\infty} s_n(x) = \frac{1}{2}(\lim_{t\in\operatorname{dom} f, t\uparrow x} f(t) + \lim_{t\in\operatorname{dom} f, t\downarrow x} f(t)).$$

(ii) If there is a $\delta > 0$ such that f is of bounded variation on both $]-\pi,-\pi+\delta]$ and $[\pi-\delta,\pi]$, then

$$\lim_{n\to\infty} s_n(\pi) = \frac{1}{2}(\lim_{t\in\operatorname{dom} f, t\uparrow\pi} f(t) + \lim_{t\in\operatorname{dom} f, t\downarrow -\pi} f(t)).$$

282Q Theorem Let f and g be complex-valued functions which are integrable over $]-\pi,\pi]$, and $\langle c_k\rangle_{k\in\mathbb{N}}$, $\langle d_k\rangle_{k\in\mathbb{N}}$ their Fourier coefficients. Let $f * g$ be their convolution, defined by the formula

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x -_{2\pi} t)g(t)dt = \int_{-\pi}^{\pi} \tilde{f}(x - t)g(t)dt,$$

writing $\tilde{f}$ for the periodic extension of f. Then the Fourier coefficients of $f * g$ are $\langle 2\pi c_k d_k \rangle_{k\in\mathbb{Z}}$.

***282R Proposition** (a) Let $f : [-\pi, \pi] \to \mathbb{C}$ be an absolutely continuous function such that $f(-\pi) = f(\pi)$, and $\langle c_k \rangle_{k\in\mathbb{Z}}$ its sequence of Fourier coefficients. Then the Fourier coefficients of f' are $\langle ikc_k \rangle_{k\in\mathbb{Z}}$.

(b) Let $f : \mathbb{R} \to \mathbb{C}$ be a differentiable function such that f' is absolutely continuous on $[-\pi, \pi]$, and $f(\pi) = f(-\pi)$. If $\langle c_k \rangle_{k\in\mathbb{Z}}$ are the Fourier coefficients of $f\restriction]-\pi, \pi]$, then $\sum_{k=-\infty}^{\infty} |c_k|$ is finite.

Version of 31.3.13

283 Fourier transforms I

I turn now to the theory of Fourier transforms on $\mathbb{R}$. In the first of two sections on the subject, I present those parts of the elementary theory which can be dealt with using the methods of the previous section on Fourier series. I find no way of making sense of the theory, however, without introducing a fragment of L.Schwartz' theory of distributions, which I present in §284. As in §282, of course, this treatment also is nothing but a start in the topic.

The whole theory can also be done in $\mathbb{R}^r$. I leave this extension to the exercises, however, since there are few new ideas, the formulae are significantly more complicated, and I shall not, in this volume at least, have any use for the multidimensional versions of these particular theorems, though some of the same ideas will appear, in multidimensional form, in §285.

283A Definitions Let f be a complex-valued function which is integrable over $\mathbb{R}$.

(a) The **Fourier transform** of f is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by setting

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} f(x)dx$$

for every $y \in \mathbb{R}$.

(b) The **inverse Fourier transform** of f is the function $\check{f} : \mathbb{R} \to \mathbb{C}$ defined by setting

$$\check{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x)dx$$

for every $y \in \mathbb{R}$.

283C Proposition Let f and g be complex-valued functions which are integrable over $\mathbb{R}$.

(a) $(f + g)^\wedge = \hat{f} + \hat{g}$.

(b) $(cf)^\wedge = c\hat{f}$ for every $c \in \mathbb{C}$.

(c) If $c \in \mathbb{R}$ and $h(x) = f(x + c)$ whenever this is defined, then $\hat{h}(y) = e^{icy}\hat{f}(y)$ for every $y \in \mathbb{R}$.

(d) If $c \in \mathbb{R}$ and $h(x) = e^{icx} f(x)$ for every $x \in \operatorname{dom} f$, then $\hat{h}(y) = \hat{f}(y - c)$ for every $y \in \mathbb{R}$.

(e) If $c > 0$ and $h(x) = f(cx)$ whenever this is defined, then $\hat{h}(y) = \frac{1}{c}\hat{f}(\frac{y}{c})$ for every $y \in \mathbb{R}$.

(f) $\hat{f} : \mathbb{R} \to \mathbb{C}$ is continuous.

(g) $\lim_{y\to\infty} \hat{f}(y) = \lim_{y\to-\infty} \hat{f}(y) = 0$.

(h) If $\int_{-\infty}^{\infty} |xf(x)|dx < \infty$, then $\hat{f}$ is differentiable, and its derivative is

$$\hat{f}'(y) = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} xf(x)dx$$

for every $y \in \mathbb{R}$.

(i) If f is absolutely continuous on every bounded interval and f' is integrable, then $(f')^\wedge(y) = iy\hat{f}(y)$ for every $y \in \mathbb{R}$.

283D Lemma (a) $\lim_{a\to\infty}\int_0^a \frac{\sin x}{x}dx = \frac{\pi}{2}$, $\lim_{a\to\infty}\int_{-a}^a \frac{\sin x}{x}dx = \pi$.
(b) There is a $K<\infty$ such that $|\int_a^b \frac{\sin cx}{x}dx| \le K$ whenever $a \le b$ and $c \in \mathbb{R}$.

283E Lemma Whenever $c<d$ in $\mathbb{R}$,

$$\begin{aligned}\lim_{a\to\infty}\int_{-a}^a e^{-iyx}\frac{e^{idy}-e^{icy}}{y}dy &= 2\pi i \text{ if } c<x<d,\\ &= \pi i \text{ if } x=c \text{ or } x=d,\\ &= 0 \text{ if } x<c \text{ or } x>d.\end{aligned}$$

283F Theorem Let f be a complex-valued function which is integrable over $\mathbb{R}$, and $\hat{f}$ its Fourier transform. Then whenever $c \le d$ in $\mathbb{R}$,

$$\int_c^d f = \frac{i}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^a \frac{e^{icy}-e^{idy}}{y}\hat{f}(y)dy.$$

283G Corollary If f and g are complex-valued functions which are integrable over $\mathbb{R}$, then $\hat{f}=\hat{g}$ iff $f =_{\text{a.e.}} g$.

283H Lemma Let f be a complex-valued function which is integrable over $\mathbb{R}$, and $\hat{f}$ its Fourier transform. Then

$$\frac{1}{\sqrt{2\pi}}\int_{-a}^a e^{ixy}\hat{f}(y)dy = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin a(x-t)}{x-t}f(t)dt = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin at}{t}f(x-t)dt$$

whenever $a>0$ and $x \in \mathbb{R}$.

283I Theorem Let f be a complex-valued function which is integrable over $\mathbb{R}$, and suppose that f is differentiable at $x \in \mathbb{R}$. Then

$$f(x) = \frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^a e^{ixy}\hat{f}(y)dy = \frac{1}{\sqrt{2\pi}}\lim_{a\to\infty}\int_{-a}^a e^{-ixy}\check{f}(y)dy.$$

283J Corollary Let $f:\mathbb{R}\to\mathbb{C}$ be an integrable function such that f is differentiable and $\hat{f}$ is integrable. Then $f = (\hat{f})^\vee = (\check{f})^\wedge$.

283K Proposition Suppose that f is a twice-differentiable function from $\mathbb{R}$ to $\mathbb{C}$ such that f, f' and f'' are all integrable. Then $\hat{f}$ is integrable.

283L Theorem Let f be a complex-valued function which is integrable over $\mathbb{R}$, with Fourier transform $\hat{f}$ and inverse Fourier transform $\check{f}$, and suppose that f is of bounded variation on some neighbourhood of $x \in \mathbb{R}$. Set $a = \lim_{t\in\operatorname{dom} f, t\uparrow x} f(t)$, $b = \lim_{t\in\operatorname{dom} f, t\downarrow x} f(t)$. Then

$$\frac{1}{\sqrt{2\pi}}\lim_{\gamma\to\infty}\int_{-\gamma}^{\gamma} e^{ixy}\hat{f}(y)dy = \frac{1}{\sqrt{2\pi}}\lim_{\gamma\to\infty}\int_{-\gamma}^{\gamma} e^{-ixy}\check{f}(y)dy = \frac{1}{2}(a+b).$$

283M Theorem Let f and g be complex-valued functions which are integrable over $\mathbb{R}$, and $f*g$ their convolution product, defined by setting

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

whenever this is defined. Then

$$(f*g)^\wedge(y) = \sqrt{2\pi}\hat{f}(y)\hat{g}(y), \quad (f*g)^\vee(y) = \sqrt{2\pi}\check{f}(y)\check{g}(y)$$

for every $y \in \mathbb{R}$.

283N Lemma For $\sigma > 0$, set $\psi_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ for $x \in \mathbb{R}$. Then its Fourier transform and inverse Fourier transform are

$$\hat{\psi}_\sigma = \check{\psi}_\sigma = \frac{1}{\sigma}\psi_{1/\sigma}.$$

In particular, $\hat{\psi}_1 = \psi_1$.

283O Proposition Let f and g be two complex-valued functions which are integrable over $\mathbb{R}$. Then $\int_{-\infty}^{\infty} f \times \hat{g} = \int_{-\infty}^{\infty} \hat{f} \times g$ and $\int_{-\infty}^{\infty} f \times \check{g} = \int_{-\infty}^{\infty} \check{f} \times g$.

Version of 30.8.13

284 Fourier transforms II

The basic paradox of Fourier transforms is the fact that while for certain functions (see 283J-283K) we have $(\hat{f})^\vee = f$, 'ordinary' integrable functions f (for instance, the indicator functions of non-trivial intervals) give rise to non-integrable Fourier transforms $\hat{f}$ for which there is no direct definition available for $\check{\hat{f}}$, making it a puzzle to decide in what sense the formula $f = \check{\hat{f}}$ might be true. What now seems by far the most natural resolution of the problem lies in declaring the Fourier transform to be an operation on *distributions* rather than on *functions*. I shall not attempt to describe this theory properly (almost any book on 'Distributions' will cover the ground better than I can possibly do here), but will try to convey the fundamental ideas, so far as they are relevant to the questions dealt with here, in language which will make the transition to a fuller treatment straightforward. At the same time, these methods make it easy to prove strong versions of the 'classical' theorems concerning Fourier transforms.

284A Test functions: Definition Throughout this section, a **rapidly decreasing test function** or **Schwartz function** will be a function $h : \mathbb{R} \to \mathbb{C}$ such that h is **smooth**, that is, differentiable everywhere any finite number of times, and moreover

$$\sup_{x\in\mathbb{R}} |x|^k |h^{(m)}(x)| < \infty$$

for all k, $m \in \mathbb{N}$, writing $h^{(m)}$ for the mth derivative of h.

284B Lemma (a) If g and h are rapidly decreasing test functions, so are $g + h$ and ch, for any $c \in \mathbb{C}$.

(b) If h is a rapidly decreasing test function and $y \in \mathbb{R}$, then $x \mapsto h(y - x)$ is a rapidly decreasing test function.

(c) If h is any rapidly decreasing test function, then h and h^2 are integrable.

(d) If h is a rapidly decreasing test function, so is its derivative h'.

(e) If h is a rapidly decreasing test function, so is the function $x \mapsto xh(x)$.

(f) For any $\epsilon > 0$, the function $x \mapsto e^{-\epsilon x^2}$ is a rapidly decreasing test function.

284C Proposition Let $h : \mathbb{R} \to \mathbb{C}$ be a rapidly decreasing test function. Then $\hat{h} : \mathbb{R} \to \mathbb{C}$ and $\check{h} : \mathbb{R} \to \mathbb{C}$ are rapidly decreasing test functions, and $\check{\hat{h}} = \hat{\check{h}} = h$.

284D Definition I will use the phrase **tempered function** on $\mathbb{R}$ to mean a measurable complex-valued function f, defined almost everywhere in $\mathbb{R}$, such that

$$\int_{-\infty}^{\infty} \frac{1}{1+|x|^k} |f(x)| dx < \infty$$

for some $k \in \mathbb{N}$.

284E Lemma (a) If f and g are tempered functions, so are $|f|$, $f + g$ and cf, for any $c \in \mathbb{C}$.

(b) If f is a tempered function then it is integrable over any bounded interval.

(c) If f is a tempered function and $x \in \mathbb{R}$, then $t \mapsto f(x+t)$ and $t \mapsto f(x-t)$ are both tempered functions.

284F Lemma Let f be a tempered function on $\mathbb{R}$ and h a rapidly decreasing test function. Then $f \times h$ is integrable.

284G Lemma Suppose that f_1 and f_2 are tempered functions and that $\int f_1 \times h = \int f_2 \times h$ for every rapidly decreasing test function h. Then $f_1 =_{\text{a.e.}} f_2$.

284H Definition Let f and g be tempered functions. Then I will say that g **represents the Fourier transform of** f if

$$\int_{-\infty}^{\infty} g \times h = \int_{-\infty}^{\infty} f \times \hat{h}$$

for every rapidly decreasing test function h.

284I Remarks (a) If f is an integrable complex-valued function on $\mathbb{R}$ and $\hat{f}$ is its Fourier transform, then $\hat{f}$ 'represents the Fourier transform of f'.

(b) Note also that if g_1, g_2 are two tempered functions both representing the Fourier transform of f, then $g_1 =_{\text{a.e.}} g_2$

(c) It is I suppose obvious that if f_1, f_2, g_1 and g_2 are tempered functions and g_i represents the Fourier transform of f_i for both i, then $cg_1 + g_2$ represents the Fourier transform of $cf_1 + f_2$ for every $c \in \mathbb{C}$.

(e) g **represents the inverse Fourier transform of** f when $\int f \times h = \int g \times \hat{h}$ for every rapidly decreasing test function h.

(f) If f, g are tempered functions and we write $\overleftrightarrow{g}(x) = g(-x)$ whenever this is defined, then
g represents the Fourier transform of f
$\iff$ $\overleftrightarrow{g}$ represents the inverse Fourier transform of f.

284J Lemma Let f be any tempered function and h a rapidly decreasing test function. Then $f * h$, defined by the formula

$$(f * h)(y) = \int_{-\infty}^{\infty} f(t)h(y-t)dt,$$

is defined everywhere.

284K Proposition Let f and g be tempered functions such that g represents the Fourier transform of f, and h a rapidly decreasing test function.

(a) The Fourier transform of the integrable function $f \times h$ is $\frac{1}{\sqrt{2\pi}} g * \hat{h}$.

(b) The Fourier transform of the continuous function $f * h$ is represented by the product $\sqrt{2\pi} g \times \hat{h}$.

284L Proposition Let f be a tempered function. Writing $\psi_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ for $x \in \mathbb{R}$ and $\sigma > 0$, then

$$\lim_{\sigma \downarrow 0} (f * \psi_\sigma)(x) = c$$

whenever $x \in \mathbb{R}$ and $c \in \mathbb{C}$ are such that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_0^\delta |f(x+t) + f(x-t) - 2c| dt = 0.$$

284M Theorem Let f and g be tempered functions such that g represents the Fourier transform of f. Then

(a)(i) $g(y) = \lim_{\epsilon \downarrow 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} e^{-\epsilon x^2} f(x) dx$ for almost every $y \in \mathbb{R}$.

(ii) If $y \in \mathbb{R}$ is such that $a = \lim_{t \in \operatorname{dom} g, t \uparrow y} g(t)$ and $b = \lim_{t \in \operatorname{dom} g, t \downarrow y} g(t)$ are both defined in $\mathbb{C}$, then

$$\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}e^{-\epsilon x^2}f(x)dx = \frac{1}{2}(a+b).$$

(b)(i) $f(x) = \lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}e^{-\epsilon y^2}g(y)dy$ for almost every $x \in \mathbb{R}$.

(ii) If $x \in \mathbb{R}$ is such that $a = \lim_{t\in\operatorname{dom} f, t\uparrow x} f(t)$ and $b = \lim_{t\in\operatorname{dom} f, t\downarrow x} f(t)$ are both defined in $\mathbb{C}$, then

$$\lim_{\epsilon\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}e^{-\epsilon y^2}g(y)dy = \frac{1}{2}(a+b).$$

284N L^2 spaces: Lemma Let $\mathcal{L}^2_{\mathbb{C}}$ be the space of square-integrable complex-valued functions on $\mathbb{R}$, and $\mathcal{S}$ the space of rapidly decreasing test functions. Then for every $f \in \mathcal{L}^2_{\mathbb{C}}$ and $\epsilon > 0$ there is an $h \in \mathcal{S}$ such that $\|f - h\|_2 \le \epsilon$.

284O Theorem (a) Let f be any complex-valued function which is square-integrable over $\mathbb{R}$. Then f is a tempered function and its Fourier transform is represented by another square-integrable function g, and $\|g\|_2 = \|f\|_2$.

(b) If f_1 and f_2 are complex-valued functions, square-integrable over $\mathbb{R}$, with Fourier transforms represented by functions g_1, g_2, then

$$\int_{-\infty}^{\infty} f_1 \times \bar{f}_2 = \int_{-\infty}^{\infty} g_1 \times \bar{g}_2.$$

(c) If f_1 and f_2 are complex-valued functions, square-integrable over $\mathbb{R}$, with Fourier transforms represented by functions g_1, g_2, then the integrable function $f_1 \times f_2$ has Fourier transform $\frac{1}{\sqrt{2\pi}}g_1 * g_2$.

(d) If f_1 and f_2 are complex-valued functions, square-integrable over $\mathbb{R}$, with Fourier transforms represented by functions g_1, g_2, then $\sqrt{2\pi}g_1 \times g_2$ represents the Fourier transform of the continuous function $f_1 * f_2$.

284P Corollary Writing $L^2_{\mathbb{C}}$ for the Hilbert space of equivalence classes of square-integrable complex-valued functions on $\mathbb{R}$, we have a linear isometry $T : L^2_{\mathbb{C}} \to L^2_{\mathbb{C}}$ given by saying that $T(f^\bullet) = g^\bullet$ whenever f, $g \in \mathcal{L}^2_{\mathbb{C}}$ and g represents the Fourier transform of f.

Version of 18.9.14

285 Characteristic functions

I come now to one of the most effective applications of Fourier transforms, the use of 'characteristic functions' to analyse probability distributions. It turns out not only that the Fourier transform of a probability distribution determines the distribution (285M) but that many of the things we want to know about a distribution are easily calculated from its transform. Even more strikingly, pointwise convergence of Fourier transforms corresponds (for sequences) to convergence for the vague topology in the space of distributions, so they provide a new and extremely powerful method for proving such results as the Central Limit Theorem and Poisson's theorem (285Q).

As the applications of the ideas here mostly belong to probability theory, I return to probabilists' terminology, as in Chapter 27. There will nevertheless be many points at which it is appropriate to speak of integrals, and there will often be more than one measure in play; so I should say directly that an integral $\int f(x)dx$ will be with respect to Lebesgue measure (usually, but not always, one-dimensional), as in the rest of this chapter, while integrals with respect to other measures will be expressed in the forms $\int f d\nu$ or $\int f(x)\nu(dx)$.

285A Definition (a) Let ν be a Radon probability measure on $\mathbb{R}^r$. Then the **characteristic function** of ν is the function $\varphi_\nu : \mathbb{R}^r \to \mathbb{C}$ given by the formula

$$\varphi_\nu(y) = \int e^{iy\,\boldsymbol{.}\,x}\nu(dx)$$

for every $y \in \mathbb{R}^r$, writing $y\boldsymbol{.}x = \eta_1\xi_1 + \ldots + \eta_r\xi_r$ if $y = (\eta_1, \ldots, \eta_r)$ and $x = (\xi_1, \ldots, \xi_r)$.

(b) Let $X_1,\dots,X_r$ be real-valued random variables on the same probability space. The **characteristic function** of $\boldsymbol{X}=(X_1,\dots,X_r)$ is the characteristic function $\varphi_{\boldsymbol{X}}=\varphi_{\nu_{\boldsymbol{X}}}$ of their joint probability distribution $\nu_{\boldsymbol{X}}$ as defined in 271C.

285C Proposition Let $X_1,\dots,X_r$ be real-valued random variables on the same probability space, and $\nu_{\boldsymbol{X}}$ their joint distribution. Then their characteristic function $\varphi_{\nu_{\boldsymbol{X}}}$ is given by

$$\varphi_{\nu_{\boldsymbol{X}}}(y)=\mathbb{E}(e^{iy\,.\boldsymbol{X}})=\mathbb{E}(e^{i\eta_1X_1}e^{i\eta_2X_2}\dots e^{i\eta_rX_r})$$

for every $y=(\eta_1,\dots,\eta_r)\in\mathbb{R}^r$.

285D Proposition Let ν be a Radon probability measure on $\mathbb{R}$. Write

$$\hat{\nu}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-iyx}\nu(dx)$$

for every $y\in\mathbb{R}$, and φ_ν for the characteristic function of ν.

(a) $\hat{\nu}(y)=\frac{1}{\sqrt{2\pi}}\varphi_\nu(-y)$ for every $y\in\mathbb{R}$.

(b) For any Lebesgue integrable complex-valued function h defined almost everywhere in $\mathbb{R}$,

$$\int_{-\infty}^{\infty}\hat{\nu}(y)h(y)dy=\int_{-\infty}^{\infty}\hat{h}(x)\nu(dx).$$

(c) For any rapidly decreasing test function h on $\mathbb{R}$,

$$\int_{-\infty}^{\infty}h(x)\nu(dx)=\int_{-\infty}^{\infty}\check{h}(y)\hat{\nu}(y)dy.$$

(d) If ν is an indefinite-integral measure over Lebesgue measure, with Radon-Nikodým derivative f, then $\hat{\nu}$ is the Fourier transform of f.

285E Lemma Let X be a normal random variable with expectation a and variance σ^2, where $\sigma>0$. Then the characteristic function of X is given by the formula

$$\varphi(y)=e^{iya}e^{-\sigma^2y^2/2}.$$

285F Proposition Let ν be a Radon probability measure on $\mathbb{R}^r$, and φ its characteristic function.

(a) $\varphi(0)=1$.

(b) $\varphi:\mathbb{R}^r\to\mathbb{C}$ is uniformly continuous.

(c) $\varphi(-y)=\overline{\varphi(y)}$, $|\varphi(y)|\le 1$ for every $y\in\mathbb{R}^r$.

(d) If $r=1$ and $\int|x|\nu(dx)<\infty$, then $\varphi'(y)$ exists and is equal to $i\int xe^{ixy}\nu(dx)$ for every $y\in\mathbb{R}$.

(e) If $r=1$ and $\int x^2\nu(dx)<\infty$, then $\varphi''(y)$ exists and is equal to $-\int x^2e^{ixy}\nu(dx)$ for every $y\in\mathbb{R}$.

285G Corollary (a) Let X be a real-valued random variable with finite expectation, and φ its characteristic function. Then $\varphi'(0)=i\mathbb{E}(X)$.

(b) Let X be a real-valued random variable with finite variance, and φ its characteristic function. Then $\varphi''(0)=-\mathbb{E}(X^2)$.

285I Proposition Let $X_1,\dots,X_n$ be independent real-valued random variables, with characteristic functions $\varphi_1,\dots,\varphi_n$. Let φ be the characteristic function of their sum $X=X_1+\dots+X_n$. Then

$$\varphi(y)=\prod_{j=1}^n\varphi_j(y)$$

for every $y\in\mathbb{R}$.

285J Lemma Let ν be a Radon probability measure on $\mathbb{R}^r$, and φ its characteristic function. Then for $1\le j\le r$ and $a>0$,

$$\nu\{x:|\xi_j|\ge a\}\le 7a\int_0^{1/a}(1-\mathcal{R}\mathrm{e}\,\varphi(te_j))dt,$$

where $e_j\in\mathbb{R}^r$ is the jth unit vector.

285L Theorem Let ν, $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be Radon probability measures on $\mathbb{R}^r$, with characteristic functions φ, $\langle\varphi_n\rangle_{n\in\mathbb{N}}$. Then the following are equiveridical:
(i) $\nu = \lim_{n\to\infty}\nu_n$ for the vague topology;
(ii) $\int h\,d\nu = \lim_{n\to\infty}\int h\,d\nu_n$ for every bounded continuous $h : \mathbb{R}^r \to \mathbb{R}$;
(iii) $\lim_{n\to\infty}\varphi_n(y) = \varphi(y)$ for every $y \in \mathbb{R}^r$.

285M Corollary (a) Let ν, ν' be two Radon probability measures on $\mathbb{R}^r$ with the same characteristic functions. Then they are equal.
(b) Let $(X_1, \dots, X_r)$ and $(Y_1, \dots, Y_r)$ be two families of real-valued random variables. If

$$\mathbb{E}(e^{i\eta_1 X_1+\ldots+i\eta_r X_r}) = \mathbb{E}(e^{i\eta_1 Y_1+\ldots+i\eta_r Y_r})$$

for all $\eta_1, \dots, \eta_r \in \mathbb{R}$, then $(X_1, \dots, X_r)$ has the same joint distribution as $(Y_1, \dots, Y_r)$.

285O Lemma Let $c_0, \dots, c_n, d_0, \dots, d_n$ be complex numbers of modulus at most 1. Then

$$|\prod_{k=0}^n c_k - \prod_{k=0}^n d_k| \le \sum_{k=0}^n |c_k - d_k|.$$

285P Lemma Suppose that $M \ge 0$ and $\epsilon > 0$. Then there are $\eta > 0$ and $y_0, \dots, y_n \in \mathbb{R}$ such that whenever X, Z are two real-valued random variables with $\mathbb{E}(|X|) \le M$, $\mathbb{E}(|Z|) \le M$ and $|\varphi_X(y_j) - \varphi_Z(y_j)| \le \eta$ for every $j \le n$, then $F_X(a) \le F_Z(a+\epsilon) + \epsilon$ for every $a \in \mathbb{R}$, where I write φ_X for the characteristic function of X and F_X for the distribution function of X.

285Q Law of Rare Events: Theorem For any $M \ge 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that whenever $X_0, \dots, X_n$ are independent $\{0,1\}$-valued random variables with $\Pr(X_k = 1) = p_k \le \delta$ for every $k \le n$ and $\sum_{k=0}^n p_k = \lambda \le M$, and $X = X_0 + \ldots + X_n$, then

$$|\Pr(X = m) - \frac{\lambda^m}{m!}e^{-\lambda}| \le \epsilon$$

for every $m \in \mathbb{N}$.

285R Convolutions If ν, $\tilde\nu$ are Radon probability measures on $\mathbb{R}^r$ then $\varphi_{\nu*\tilde\nu}(y) = \varphi_\nu(y)\varphi_{\tilde\nu}(y)$ for every $y \in \mathbb{R}^r$.

285V Proposition Let ν be a Radon probability measure on $\mathbb{R}^r$ such that $\nu * \nu = \nu$. Then ν is the Dirac measure δ_0 concentrated at 0.

285S The vague topology and pointwise convergence of characteristic functions Write

$$\rho'_y(\nu, \nu') = |\int e^{iy\cdot x}\nu(dx) - \int e^{iy\cdot x}\nu'(dx)|$$

for Radon probability measures ν, ν' on $\mathbb{R}^r$ and $y \in \mathbb{R}^r$. Write $\mathfrak{T}$ for the vague topology and $\mathfrak{S}$ for the topology defined by $\{\rho'_y : y \in \mathbb{R}^r\}$

285T Proposition Suppose that $y_0, \dots, y_n \in \mathbb{R}$ and $\eta > 0$. Then there are infinitely many $m \in \mathbb{N}$ such that $|1 - e^{iy_k m}| \le \eta$ for every $k \le n$.

285U Corollary The topologies $\mathfrak{S}$ and $\mathfrak{T}$ on the space of Radon probability measures on $\mathbb{R}$, as described in 285S, are different.

Version of 30.3.16

286 Carleson's theorem

Carleson's theorem (Carleson 66) was the (unexpected) solution to a long-standing problem. Remarkably, it can be proved by 'elementary' methods. The hardest part of the work below, in 286J-286L, demands only the laborious verification of inequalities. How the inequalities were chosen is a different matter; for once, some of the ideas of the proof are embodied in the statements of the lemmas. The argument here is a greatly expanded version of Lacey & Thiele 00.

The Hardy-Littlewood Maximal Theorem (286A) is important, and worth learning even if you leave the rest of the section as an unexamined monument. I bring 286B-286D forward to the beginning of the section, even though they are little more than worked exercises, because they also have potential uses in other contexts.

In this section all integrals are with respect to Lebesgue measure μ on $\mathbb{R}$ unless otherwise stated.

286A The Maximal Theorem Suppose that $1 < p < \infty$ and that $f \in \mathcal{L}^p_{\mathbb{C}}(\mu)$. Set

$$f^*(x) = \sup\{\frac{1}{b-a}\int_a^b |f| : a \le x \le b,\, a < b\}$$

for $x \in \mathbb{R}$. Then $\|f^*\|_p \le \frac{2^{1/p}p}{p-1}\|f\|_p$.

286B Lemma Let $g : \mathbb{R} \to [0,\infty[$ be a function which is non-decreasing on $]-\infty,\alpha]$, non-increasing on $[\beta,\infty[$ and constant on $[\alpha,\beta]$, where $\alpha \le \beta$. Then for any measurable function $f : \mathbb{R} \to [0,\infty]$, $\int_{-\infty}^{\infty} f \times g \le \int_{-\infty}^{\infty} g \cdot \sup_{a\le\alpha,b\ge\beta,a<b} \frac{1}{b-a}\int_a^b f$.

286C Shift, modulation and dilation For any function f with domain included in $\mathbb{R}$, and $\alpha \in \mathbb{R}$, we can define

$$(S_\alpha f)(x) = f(x+\alpha), \quad (M_\alpha f)(x) = e^{i\alpha x}f(x), \quad (D_\alpha f)(x) = f(\alpha x)$$

whenever the right-hand sides are defined.

(a) $S_{-\alpha}S_\alpha f = f$, $D_{1/\alpha}D_\alpha f = f$ if $\alpha \ne 0$.

(b) $S_\alpha(f \times g) = S_\alpha f \times S_\alpha g$, $D_\alpha(f \times g) = D_\alpha f \times D_\alpha g$.

(c) $D_\alpha|f| = |D_\alpha f|$.

(d) If f is integrable, then

$$(M_\alpha f)^\wedge = S_{-\alpha}\hat{f}, \quad (S_\alpha f)^\wedge = M_\alpha \hat{f}, \quad (S_\alpha f)^\vee = M_{-\alpha}\check{f};$$

if moreover $\alpha > 0$, then

$$\alpha(D_\alpha f)^\wedge = D_{1/\alpha}\hat{f}, \quad \alpha(D_\alpha f)^\vee = D_{1/\alpha}\check{f}.$$

(e) If f belongs to $\mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$, so do $S_\alpha f$, $M_\alpha f$ and (if $\alpha \ne 0$) $D_\alpha f$, and in this case

$$\|S_\alpha f\|_1 = \|M_\alpha f\|_1 = \|f\|_1, \quad \|D_\alpha f\|_1 = \frac{1}{|\alpha|}\|f\|_1.$$

(f) If f belongs to $\mathcal{L}^2_{\mathbb{C}}$ so do $S_\alpha f$, $M_\alpha f$ and (if $\alpha \ne 0$) $D_\alpha f$, and in this case

$$\|S_\alpha f\|_2 = \|M_\alpha f\|_2 = \|f\|_2, \quad \|D_\alpha f\|_2 = \frac{1}{\sqrt{|\alpha|}}\|f\|_2.$$

(g) If h is a rapidly decreasing test function, so are $M_\alpha h$ and $S_\alpha h$ and (if $\alpha \ne 0$) $D_\alpha h$.

286D Lemma Suppose that $g : \mathbb{R} \to [0,\infty]$ is a measurable function such that, for some constant $C \ge 0$, $\int_E g \le C\sqrt{\mu E}$ whenever $\mu E < \infty$. Then g is finite almost everywhere and $\int_{-\infty}^{\infty} \frac{1}{1+|x|}g(x)dx$ is finite.

286E The Lacey-Thiele construction (a) Let $\mathcal{I}$ be the family of all **dyadic intervals** of the form $\left[2^k n, 2^k(n+1)\right[$ where k, $n \in \mathbb{Z}$. The essential geometric property of $\mathcal{I}$ is that if I, $J \in \mathcal{I}$ then either $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$. Let Q be the set of all pairs $\sigma = (I_\sigma, J_\sigma) \in \mathcal{I}^2$ such that $\mu I_\sigma \cdot \mu J_\sigma = 1$. For $\sigma \in Q$, let $k_\sigma \in \mathbb{Z}$ be such that $\mu J_\sigma = 2^{k_\sigma}$ and $\mu I_\sigma = 2^{-k_\sigma}$; let x_σ be the midpoint of I_σ, y_σ the midpoint of J_σ, $J^l_\sigma \in \mathcal{I}$ the left-hand half-interval of J_σ, $J^r_\sigma \in \mathcal{I}$ the right-hand half-interval of J_σ, and y^l_σ the lower quartile of J_σ.

(b) There is a rapidly decreasing test function ϕ such that $\hat{\phi}$ is real-valued and $\chi[-\frac{1}{6}, \frac{1}{6}] \le \hat{\phi} \le \chi[-\frac{1}{5}, \frac{1}{5}]$. For $\sigma \in Q$, set

$$\phi_\sigma(x) = \sqrt{\mu J_\sigma} e^{iy^l_\sigma x} \phi((x - x_\sigma)\mu J_\sigma).$$

ϕ_σ is a rapidly decreasing test function.

$$\hat{\phi}_\sigma(y) = \sqrt{\mu I_\sigma} e^{-ix_\sigma(y - y^l_\sigma)} \hat{\phi}((y - y^l_\sigma)\mu I_\sigma),$$

which is zero unless $y \in J^l_\sigma$.

(i) $\|\phi_\sigma\|_2 = \|\phi\|_2$ for every $\sigma \in Q$.

(ii) $\|\hat{\phi}_\sigma\|_1 = \sqrt{\mu J_\sigma}\|\hat{\phi}\|_1$ for every $\sigma \in Q$.

(iii) If σ, $\tau \in Q$ and $J^l_\sigma \cap J^l_\tau = \emptyset$ then

$$(\phi_\sigma|\phi_\tau) = (\hat{\phi}_\sigma|\hat{\phi}_\tau) = 0.$$

(For f, $g \in \mathcal{L}^2_{\mathbb{C}}$, I write $(f|g)$ for $\int_{-\infty}^{\infty} f \times \bar{g}$.)

(iv) If σ, $\tau \in Q$ and $J_\sigma \ne J_\tau$ and $J^r_\sigma \cap J^r_\tau$ is non-empty, then $(\phi_\sigma|\phi_\tau) = 0$.

(c) Set $w(x) = \frac{1}{(1+|x|)^3}$ for $x \in \mathbb{R}$. For $\sigma \in Q$, set

$$w_\sigma(x) = w((x - x_\sigma)\mu J_\sigma)\mu J_\sigma \le \mu J_\sigma = 2^{k_\sigma}$$

for every x. Note that $w_\sigma = w_\tau$ whenever $I_\sigma = I_\tau$.

286F A partial order (a) For σ, $\tau \in Q$ say that $\tau \le \sigma$ if $J_\tau \subseteq J_\sigma$ and $I_\sigma \subseteq I_\tau$. Then $\le$ is a partial order on Q.

(i) If $\tau \le \sigma$, then $k_\tau \le k_\sigma$.

(ii) If σ and τ are incomparable, then $(I_\sigma \times J_\sigma) \cap (I_\tau \times J_\tau)$ is empty.

(iii) If σ, σ' are incomparable and both greater than or equal to τ, then $I_\sigma \cap I_{\sigma'} = \emptyset$.

(iv) If $\tau \le \sigma$ and $k_\tau \le k \le k_\sigma$, then there is a (unique) υ such that $\tau \le \upsilon \le \sigma$ and $k_\upsilon = k$.

(b) If $R \subseteq Q$, say that

$$R^+ = \bigcup_{\tau \in R} \{\sigma : \tau \le \sigma \in Q\}.$$

(c) For $\tau \in Q$ set

$$T_\tau = \{\sigma : \sigma \in Q,\, \tau \le \sigma,\, J^r_\tau \subseteq J^r_\sigma\}.$$

Note that if σ, $\sigma' \in T_\tau$ and $k_\sigma \ne k_{\sigma'}$ then $(\phi_\sigma|\phi_{\sigma'}) = 0$.

286G Lemma (a) $\int_{-\infty}^{\infty} w_\sigma = \int_{-\infty}^{\infty} w = 1$ for every $\sigma \in Q$.

(b) For any $m \in \mathbb{N}$, $\sum_{n=m}^{\infty} w(n + \frac{1}{2}) \le \frac{1}{2(1+m)^2}$.

(c) Suppose that $\sigma \in Q$ and that I is an interval not containing x_σ in its interior. Then $\int_I w_\sigma \ge w_\sigma(x)\mu I$, where x is the midpoint of I.

(d) For any $x \in \mathbb{R}$, $\sum_{n=-\infty}^{\infty} w(x - n) \le 2$.

(e) There is a constant $C_1 \ge 0$ such that $|\phi(x)| \le C_1 \min(w(3), w(x)^2)$ for every $x \in \mathbb{R}$ and

$$|\phi_\sigma(x)| \le C_1 \sqrt{\mu I_\sigma} w_\sigma(x) \min(1, w_\sigma(x)\mu I_\sigma)$$

for every $x \in \mathbb{R}$ and $\sigma \in Q$.

(f) There is a constant $C_2 \ge 0$ such that $\int_{-\infty}^{\infty} w(x)w(\alpha x+\beta)dx \le C_2 w(\beta)$ whenever $0 \le \alpha \le 1$ and $\beta \in \mathbb{R}$.

(g) There is a constant $C_3 \ge 0$ such that $|(\phi_\sigma|\phi_\tau)| \le C_3\sqrt{\mu I_\sigma}\sqrt{\mu J_\tau}\int_{I_\tau} w_\sigma$ whenever σ, $\tau \in Q$ and $k_\sigma \le k_\tau$.

(h) There is a constant $C_4 \ge 0$ such that

$$\sum_{\sigma\in Q,\sigma\ge\tau,k_\sigma=k}\int_{\mathbb{R}\setminus I_\tau} w_\sigma \le C_4$$

whenever $\tau \in Q$ and $k \in \mathbb{Z}$.

286H ‘Mass’ and ‘energy’ If P is a subset of Q, $E \subseteq \mathbb{R}$ is measurable, $g:\mathbb{R}\to\mathbb{R}$ is measurable, and $f \in \mathcal{L}^2_{\mathbb{C}}$, set

$$\operatorname{mass}_{Eg}(P) = \sup_{\sigma\in P,\tau\in Q,\tau\le\sigma}\int_{E\cap g^{-1}[J_\tau]} w_\tau \le \sup_{\tau\in Q}\int_{-\infty}^{\infty} w_\tau = 1,$$

$$\Delta_f(P) = \sum_{\sigma\in P}|(f|\phi_\sigma)|^2,$$

$$\operatorname{energy}_f(P) = \sup_{\tau\in Q}\sqrt{\mu J_\tau}\sqrt{\Delta_f(P\cap T_\tau)}.$$

If $P' \subseteq P$ then $\operatorname{mass}_{Eg}(P') \le \operatorname{mass}_{Eg}(P)$ and $\operatorname{energy}_f(P') \le \operatorname{energy}_f(P)$. $\operatorname{energy}_f(\{\sigma\}) = \sqrt{\mu J_\sigma}|(f|\phi_\sigma)|$ for any $\sigma \in Q$.

286I Lemma If $P \subseteq Q$ is finite and $f \in \mathcal{L}^2_{\mathbb{C}}$, then

(a) $\Delta_f(P) \le \|\sum_{\sigma\in P}(f|\phi_\sigma)\phi_\sigma\|_2\|f\|_2$,

(b) $\sum_{\sigma,\tau\in P,J_\sigma=J_\tau}\bigl|(f|\phi_\sigma)(\phi_\sigma|\phi_\tau)(\phi_\tau|f)\bigr| \le C_3\Delta_f(P)$.

286J Lemma Set $C_5 = 2^{12}$. If $P \subseteq Q$ is finite, $E \subseteq \mathbb{R}$ is measurable, $g:\mathbb{R}\to\mathbb{R}$ is measurable, and $\gamma \ge \operatorname{mass}_{Eg}(P)$, then we can find $R \subseteq Q$ such that $\gamma\sum_{\tau\in R}\mu I_\tau \le C_5\mu E$ and $\operatorname{mass}_{Eg}(P\setminus R^+) \le \frac{1}{4}\gamma$.

286K Lemma Set $C_6 = 4(C_3 + 4C_3\sqrt{2C_4})$. Suppose that $P \subseteq Q$ is a finite set, $f \in \mathcal{L}^2_{\mathbb{C}}$, $\|f\|_2 = 1$ and $\gamma \ge \operatorname{energy}_f(P)$. Then we can find $R \subseteq Q$ such that $\gamma^2\sum_{\tau\in R}\mu I_\tau \le C_6$ and $\operatorname{energy}_f(P\setminus R^+) \le \frac{1}{2}\gamma$.

286L Lemma Set

$$C_7 = C_1\bigl(\frac{7}{2} + \frac{8}{7} + \frac{28}{w(3/2)} + \frac{4\sqrt{14C_3}}{w(3/2)}\bigr).$$

Suppose that P is a finite subset of Q with a lower bound τ in Q for the ordering $\le$, $E \subseteq \mathbb{R}$ is measurable, $g:\mathbb{R}\to\mathbb{R}$ is measurable and $f \in \mathcal{L}^2_{\mathbb{C}}$. Then

$$\sum_{\sigma\in P}|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]}\phi_\sigma| \le C_7\operatorname{energy}_f(P)\operatorname{mass}_{Eg}(P)\mu I_\tau.$$

286M The Lacey-Thiele lemma Set $C_8 = 3C_7(C_5+C_6)$. Then

$$\sum_{\sigma\in Q}|(f|\phi_\sigma)\int_{E\cap g^{-1}[J^r_\sigma]}\phi_\sigma| \le C_8$$

whenever $f \in \mathcal{L}^2_{\mathbb{C}}$, $\|f\|_2 = 1$, $\mu E \le 1$ and $g:\mathbb{R}\to\mathbb{R}$ is measurable.

286N Lemma Set $C_9 = C_8\sqrt{2}$. Suppose that $f \in \mathcal{L}^2_{\mathbb{C}}$, $g:\mathbb{R}\to\mathbb{R}$ is measurable and $\mu F < \infty$. Then

$$\sum_{\sigma\in Q}|(f|\phi_\sigma)\int_{F\cap g^{-1}[J^r_\sigma]}\phi_\sigma| \le C_9\|f\|_2\sqrt{\mu F}.$$

286O Lemma (a) For $z \in \mathbb{R}$, define $\theta_z:\mathbb{R}\to[0,1]$ by setting

$$\theta_z(y) = \hat{\phi}(2^{-k}(y-\hat{y}))^2$$

whenever there is a dyadic interval $J \in \mathcal{I}$ of length 2^k such that z belongs to the right-hand half of J and y belongs to the left-hand half of J and $\hat{y}$ is the lower quartile of J, and zero if there is no such J. Then $(y, z) \mapsto \theta_z(y)$ is Borel measurable, $0 \le \theta_z(y) \le 1$ for all y, $z \in \mathbb{R}$, and $\theta_z(y) = 0$ if $y \ge z$.

(b) For $k \in \mathbb{Z}$, set $Q_k = \{\sigma : \sigma \in Q,\ k_\sigma = k\}$. Let $[Q]^{<\omega}$ be the set of finite subsets of Q, $[\mathbb{Z}]^{<\omega}$ the set of finite subsets of $\mathbb{Z}$ and $\mathcal{L}$ the set of subsets L of Q such that $L \cap Q_k$ is finite for every k. If $K \in [\mathbb{Z}]^{<\omega}$ and $L \in \mathcal{L}$, set

$$\begin{aligned}\mathcal{P}_{KL} = \{P : P \in [Q]^{<\omega},\ P \cap Q_k \supseteq L \cap Q_k &\text{ whenever } k \in \mathbb{Z} \\ &\text{and either } k \in K \text{ or } P \cap Q_k \ne \emptyset\};\end{aligned}$$

set

$$\mathcal{F} = \{\mathcal{P} : \mathcal{P} \subseteq [Q]^{<\omega} \text{ and there are } K \in [\mathbb{Z}]^{<\omega},\ L \in \mathcal{L} \text{ such that } \mathcal{P} \supseteq \mathcal{P}_{KL}\}.$$

Then $\mathcal{F}$ is a filter on $[Q]^{<\omega}$ and

$$2\pi \int_F (\hat{h} \times \theta_z)^\vee = \lim_{P \to \mathcal{F}} \sum_{\sigma \in P, z \in J_\sigma^r} (h|\phi_\sigma) \int_F \phi_\sigma$$

for every $z \in \mathbb{R}$ and rapidly decreasing test function h and measurable set $F \subseteq \mathbb{R}$ of finite measure.

286P Lemma Suppose that h is a rapidly decreasing test function. For $x \in \mathbb{R}$, set

$$Ah(x) = \sup_{z \in \mathbb{R}} |2\pi(\hat{h} \times \theta_z)^\vee(x)|.$$

Then $Ah : \mathbb{R} \to [0, \infty]$ is Borel measurable, and $\int_F Ah \le 4C_9 \|h\|_2 \sqrt{\mu F}$ whenever $\mu F < \infty$.

286Q Lemma For $\alpha > 0$ and y, z, $\beta \in \mathbb{R}$, set $\theta'_{z\alpha\beta}(y) = \theta_{\alpha z + \beta}(\alpha y + \beta)$. Then
(a) the function $(\alpha, \beta, y, z) \mapsto \theta'_{z\alpha\beta}(y) :]0, \infty[\times \mathbb{R}^3 \to [0, 1]$ is Borel measurable;
(b) $\theta'_{z\alpha\beta}(y) = 0$ whenever $y \ge z$;
(c) for any rapidly decreasing test function h, and any $z \in \mathbb{R}$,

$$2\pi |(\hat{h} \times \theta'_{z\alpha\beta})^\vee| \le D_{1/\alpha} A M_\beta D_\alpha h$$

at every point.

286R Lemma For any y, $z \in \mathbb{R}$,

$$\tilde{\theta}_z(y) = \int_1^2 \frac{1}{\alpha} \Bigl(\lim_{n \to \infty} \frac{1}{n} \int_0^n \theta'_{z\alpha\beta}(y) d\beta\Bigr) d\alpha$$

is defined, and

$$\begin{aligned}\tilde{\theta}_z(y) &= \tilde{\theta}_1(0) > 0 \text{ if } y < z, \\ &= 0 \text{ if } y \ge z.\end{aligned}$$

286S Lemma Suppose that h is a rapidly decreasing test function.
(a) For every $x \in \mathbb{R}$,

$$(\tilde{A}h)(x) = \liminf_{n \to \infty} \frac{1}{n} \int_1^2 \frac{1}{\alpha} \int_0^n (D_{1/\alpha} A M_\beta D_\alpha h)(x) d\beta d\alpha$$

is defined in $[0, \infty]$, and $\tilde{A}h : \mathbb{R} \to [0, \infty]$ is Borel measurable.
(b) $\int_F \tilde{A}h \le 3C_9 \|h\|_2 \sqrt{\mu F}$ whenever $\mu F < \infty$.
(c) If $z \in \mathbb{R}$, $2\pi |(\hat{h} \times \tilde{\theta}_z)^\vee| \le \tilde{A}h$ at every point.

286T Lemma Set $C_{10} = 3C_9 / \pi \tilde{\theta}_1(0)$. For $f \in \mathcal{L}^2_{\mathbb{C}}$, define $\hat{A}f : \mathbb{R} \to [0, \infty]$ by setting

$$(\hat{A}f)(y) = \sup_{a \le b} \frac{1}{\sqrt{2\pi}} \Bigl|\int_a^b e^{-ixy} f(x) dx\Bigr|$$

for each $y \in \mathbb{R}$. Then $\int_F \hat{A}f \le C_{10} \|f\|_2 \sqrt{\mu F}$ whenever $\mu F < \infty$.

286U Theorem If $f \in \mathcal{L}^2_{\mathbb{C}}$ then

$$g(y) = \lim_{a \to -\infty, b \to \infty} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-ixy} f(x) dx$$

is defined in $\mathbb{C}$ for almost every $y \in \mathbb{R}$, and g represents the Fourier transform of f.

286V Theorem For any square-integrable complex-valued function on $]-\pi, \pi]$, its sequence of Fourier sums converges to it almost everywhere.

Version of 6.1.10

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

285Xm Cauchy distribution The exercise introducing the Cauchy distribution, referred to in the 2002, 2004 and 2012 printings of Volume 3, is now 285Xp.

285Xo Poisson distribution The exercise naming the Poisson distribution, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 285Xr.

285Xr Bochner's theorem The exercise on a special case of Bochner's theorem, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 285Xu.

286U Carleson's theorem The sequential form, referred to in Bogachev 07, is now in 286V.

References

Bogachev V.I. [07] *Measure theory.* Springer, 2007.