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Chapter 26

Change of Variable in the Integral

I suppose most courses on basic calculus still devote a substantial amount of time to practice in the techniques of integrating standard functions. Surely the most powerful single technique is that of substitution: replacing $\int g(y)dy$ by $\int g(\phi(x))\phi'(x)dx$ for an appropriate function ϕ. At this level one usually concentrates on the skills of guessing at appropriate ϕ and getting the formulae right. I will not address such questions here, except for rare special cases; in this book I am concerned rather with validating the process. For functions of one variable, it can usually be justified by an appeal to the Fundamental Theorem of Calculus, and for any particular case I would normally go first to §225 in the hope that the results there would cover it. But for functions of two or more variables some much deeper ideas are necessary.

I have already treated the general problem of integration-by-substitution in abstract measure spaces in §235. There I described conditions under which $\int g(y)dy = \int g(\phi(x))J(x)dx$ for an appropriate function J. The context there gave very little scope for suggestions as to how to compute J; at best, it could be presented as a Radon-Nikodým derivative (235M). In this chapter I give a form of the fundamental theorem for the case of Lebesgue measure, in which ϕ is a more or less differentiable function between Euclidean spaces, and J is a 'Jacobian', the modulus of the determinant of the derivative of ϕ (263D). This necessarily depends on a serious investigation of the relationship between Lebesgue measure and geometry. The first step is to establish a form of Vitali's theorem for r-dimensional space, together with r-dimensional density theorems; I do this in §261, following closely the scheme of §§221 and 223 above. We need to know quite a lot about differentiable functions between Euclidean spaces, and it turns out that the theory is intertwined with that of 'Lipschitz' functions; I treat these in §262.

In the next two sections of the chapter, I turn to a separate problem for which some of the same techniques turn out to be appropriate: the description of surface measure on (smooth) surfaces in Euclidean space, like the surface of a cone or sphere. I suppose there is no difficulty in forming a robust intuition as to what is meant by the 'area' of such a surface and of suitably simple regions within it, and there is a very strong presumption that there ought to be an expression for this intuition in terms of measure theory as presented in this book; but the details are not I think straightforward. The first point to note is that for any calculation of the area of a region G in a surface S, one would always turn at once to a parametrization of the region, that is, a bijection $\phi : D \to G$ from some subset D of Euclidean space. But obviously one needs to be sure that the result of the calculation is independent of the parametrization chosen, and while it would be possible to base the theory on results showing such independence directly, that does not seem to me to be a true reflection of the underlying intuition, which is that the area of simple surfaces, at least, is something intrinsic to their geometry. I therefore see no acceptable alternative to a theory of 'r-dimensional measure' which can be described in purely geometric terms. This is the burden of §264, in which I give the definition and most fundamental properties of Hausdorff r-dimensional measure in Euclidean spaces. With this established, we find that the techniques of §§261-263 are sufficient to relate it to calculations through parametrizations, which is what I do in §265.

The chapter ends with a brief account of the Brunn-Minkowski inequality (266C), which is an essential tool for the geometric measure theory of convex sets.

Version of 11.12.12

261 Vitali's theorem in $\mathbb{R}^r$

The main aim of this section is to give r-dimensional versions of Vitali's theorem and Lebesgue's Density Theorem, following ideas already presented in §§221 and 223. I end with a proof that Lebesgue outer measure can be defined in terms of coverings by balls instead of by intervals (261F).

261B Vitali's theorem in $\mathbb{R}^r$ Let $A \subseteq \mathbb{R}^r$ be any set, and $\mathcal{I}$ a family of closed non-trivial balls in $\mathbb{R}^r$ such that every point of A is contained in arbitrarily small members of $\mathcal{I}$. Then there is a countable disjoint set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\mu(A \setminus \bigcup \mathcal{I}_0) = 0$.

261C Density Theorem in $\mathbb{R}^r$: integral form Let D be a subset of $\mathbb{R}^r$, and f a real-valued function which is integrable over D. Then

$$f(x) = \lim_{\delta \downarrow 0} \frac{1}{\mu B(x,\delta)} \int_{D \cap B(x,\delta)} f d\mu$$

for almost every $x \in D$.

261D Corollary (a) If $D \subseteq \mathbb{R}^r$ is any set, then

$$\lim_{\delta \downarrow 0} \frac{\mu^*(D \cap B(x,\delta))}{\mu B(x,\delta)} = 1$$

for almost every $x \in D$.

(b) If $E \subseteq \mathbb{R}^r$ is a measurable set, then

$$\lim_{\delta \downarrow 0} \frac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)} = \chi E(x)$$

for almost every $x \in \mathbb{R}^r$.

(c) If $D \subseteq \mathbb{R}^r$ and $f : D \to \mathbb{R}$ is any function, then for almost every $x \in D$,

$$\lim_{\delta \downarrow 0} \frac{\mu^*(\{y : y \in D, |f(y) - f(x)| \le \epsilon\} \cap B(x,\delta))}{\mu B(x,\delta)} = 1$$

for every $\epsilon > 0$.

(d) If $D \subseteq \mathbb{R}^r$ and $f : D \to \mathbb{R}$ is measurable, then for almost every $x \in D$,

$$\lim_{\delta \downarrow 0} \frac{\mu^*(\{y : y \in D, |f(y) - f(x)| \ge \epsilon\} \cap B(x,\delta))}{\mu B(x,\delta)} = 0$$

for every $\epsilon > 0$.

261E Theorem Let f be a locally integrable function defined on a conegligible subset of $\mathbb{R}^r$. Then

$$\lim_{\delta \downarrow 0} \frac{1}{\mu B(x,\delta)} \int_{B(x,\delta)} |f(y) - f(x)| dy = 0$$

for almost every $x \in \mathbb{R}^r$.

Remark The set

$$\{x : x \in \operatorname{dom} f, \lim_{\delta \downarrow 0} \frac{1}{\mu B(x,\delta)} \int_{B(x,\delta)} |f(y) - f(x)| dy = 0\}$$

is sometimes called the **Lebesgue set** of f.

261F Proposition Let $A \subseteq \mathbb{R}^r$ be any set, and $\epsilon > 0$. Then there is a sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ of closed balls in $\mathbb{R}^r$, all of radius at most ϵ, such that $A \subseteq \bigcup_{n \in \mathbb{N}} B_n$ and $\sum_{n=0}^{\infty} \mu B_n \le \mu^* A + \epsilon$. Moreover, we may suppose that the balls in the sequence whose centres do not lie in A have measures summing to at most ϵ.

Version of 11.8.15

262 Lipschitz and differentiable functions

In preparation for the main work of this chapter in §263, I devote a section to two important classes of functions between Euclidean spaces. What we really need is the essentially elementary material down to 262I, together with the technical lemma 262M and its corollaries. Theorem 262Q is not relied on in this volume, though I believe that it makes the patterns which will develop more natural and comprehensible.

As in §261, r (and here also s) will be a strictly positive integer, and 'measurable', 'negligible', 'integrable' will refer to Lebesgue measure unless otherwise stated.

262A Lipschitz functions Suppose that $\phi : D \to \mathbb{R}^s$ is a function, where $D \subseteq \mathbb{R}^r$. ϕ is **γ-Lipschitz**, where $\gamma \in [0, \infty[$, if

$$\|\phi(x) - \phi(y)\| \le \gamma\|x - y\|$$

for all x, $y \in D$, writing $\|x\| = \sqrt{\xi_1^2 + \ldots + \xi_r^2}$ if $x = (\xi_1, \ldots, \xi_r) \in \mathbb{R}^r$, $\|z\| = \sqrt{\zeta_1^2 + \ldots + \zeta_s^2}$ if $z = (\zeta_1, \ldots, \zeta_s) \in \mathbb{R}^s$. In this case, γ is **a Lipschitz constant for** ϕ.

A **Lipschitz function** is a function ϕ which is γ-Lipschitz for some $\gamma \ge 0$. Evidently a Lipschitz function is (uniformly) continuous.

262B Lemma Let $D \subseteq \mathbb{R}^r$ be a set and $\phi : D \to \mathbb{R}^s$ a function.

(a) ϕ is Lipschitz iff $\phi_i : D \to \mathbb{R}$ is Lipschitz for every i, writing $\phi(x) = (\phi_1(x), \ldots, \phi_s(x))$ for every $x \in D = \operatorname{dom}\phi \subseteq \mathbb{R}^r$.

(b) In this case, there is a Lipschitz function $\tilde\phi : \mathbb{R}^r \to \mathbb{R}^s$ extending ϕ.

(c) If $r = s = 1$ and $D = [a, b]$ is an interval, then ϕ is Lipschitz iff it is absolutely continuous and has a bounded derivative.

262D Proposition If $\phi : D \to \mathbb{R}^r$ is a γ-Lipschitz function, where $D \subseteq \mathbb{R}^r$, then $\mu^*\phi[A] \le \gamma^r\mu^*A$ for every $A \subseteq D$, where μ is Lebesgue measure on $\mathbb{R}^r$. In particular, $\phi[D \cap A]$ is negligible for every negligible set $A \subseteq \mathbb{R}^r$.

262E Corollary Let $\phi : D \to \mathbb{R}^r$ be an injective Lipschitz function, where $D \subseteq \mathbb{R}^r$, and f a measurable function from a subset of $\mathbb{R}^r$ to $\mathbb{R}$.

(a) If ϕ^{-1} is defined almost everywhere in a subset H of $\mathbb{R}^r$ and f is defined almost everywhere in $\mathbb{R}^r$, then $f\phi^{-1}$ is defined almost everywhere in H.

(b) If $E \subseteq D$ is Lebesgue measurable then $\phi[E]$ is measurable.

(c) If D is measurable then $f\phi^{-1}$ is measurable.

262F Differentiability: Definitions Suppose that ϕ is a function from a subset $D = \operatorname{dom}\phi$ of $\mathbb{R}^r$ to $\mathbb{R}^s$.

(a) ϕ is **differentiable** at $x \in D$ if there is a real $s \times r$ matrix T such that

$$\lim_{y\to x} \frac{\|\phi(y)-\phi(x)-T(y-x)\|}{\|y-x\|} = 0;$$

in this case we may write $T = \phi'(x)$.

(b) I will say that ϕ is **differentiable relative to its domain** at x, and that T is **a** derivative of ϕ at x, if $x \in D$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(y) - \phi(x) - T(y - x)\| \le \epsilon\|y - x\|$ for every $y \in B(x, \delta) \cap D$.

262H The norm of a matrix Some of the calculations below will rely on the notion of 'norm' of a matrix. The one I will use is the 'operator norm', defined by saying

$$\|T\| = \sup\{\|Tx\| : x \in \mathbb{R}^r,\ \|x\| \le 1\}$$

for any $s \times r$ matrix T.

(a) If $T = \langle\tau_{ij}\rangle_{i\le s, j\le r}$ then $\|T\| \le r\sqrt{s}\max_{i\le s, j\le r}|\tau_{ij}|$.

(b) $|\tau_{ij}| \le \|T\|$ for all i, j.

262I Lemma Let $\phi : D \to \mathbb{R}^s$ be a function, where $D \subseteq \mathbb{R}^r$. For $i \le s$ let $\phi_i : D \to \mathbb{R}$ be its ith coordinate, so that $\phi(x) = (\phi_1(x), \dots , \phi_s(x))$ for $x \in D$.

(a) If ϕ is differentiable relative to its domain at $x \in D$, then ϕ is continuous at x.

(b) If $x \in D$, then ϕ is differentiable relative to its domain at x iff each ϕ_i is differentiable relative to its domain at x.

(c) If ϕ is differentiable at $x \in D$, then all the partial derivatives $\frac{\partial\phi_i}{\partial\xi_j}$ of ϕ are defined at x, and the derivative of ϕ at x is the matrix $\langle\frac{\partial\phi_i}{\partial\xi_j}(x)\rangle_{i\le s, j\le r}$.

(d) If all the partial derivatives $\frac{\partial\phi_i}{\partial\xi_j}$, for $i \le s$ and $j \le r$, are defined in a neighbourhood of $x \in D$ and are continuous at x, then ϕ is differentiable at x.

262L Lemma Suppose that $D \subseteq \mathbb{R}^r$ and $x \in \mathbb{R}^r$ are such that $\lim_{\delta\downarrow 0}\frac{\mu^*(D\cap B(x,\delta))}{\mu B(x,\delta)} = 1$. Then $\lim_{z\to\mathbf{0}}\frac{\rho(x+z,D)}{\|z\|} = 0$, where $\rho(x + z, D) = \inf_{y\in D}\|x + z - y\|$.

262M Lemma Let ϕ be a function from a subset D of $\mathbb{R}^r$ to $\mathbb{R}^s$ which is differentiable at each point of its domain. For each $x \in D$ let $T(x)$ be a derivative of ϕ. Let M_{sr} be the set of $s \times r$ matrices and $\zeta : A \to \left]0, \infty\right[$ a strictly positive function, where $A \subseteq M_{sr}$ is a non-empty set containing $T(x)$ for every $x \in D$. Then we can find sequences $\langle D_n\rangle_{n\in\mathbb{N}}$, $\langle T_n\rangle_{n\in\mathbb{N}}$ such that

(i) $\langle D_n\rangle_{n\in\mathbb{N}}$ is a partition of D into sets which are relatively measurable in D;

(ii) $T_n \in A$ for every n;

(iii) $\|\phi(x) - \phi(y) - T_n(x - y)\| \le \zeta(T_n)\|x - y\|$ for every $n \in \mathbb{N}$ and x, $y \in D_n$;

(iv) $\|T(x) - T_n\| \le \zeta(T_n)$ for every $x \in D_n$.

262N Corollary Let ϕ be a function from a subset D of $\mathbb{R}^r$ to $\mathbb{R}^s$, and suppose that ϕ is differentiable relative to its domain at each point of D. Then D can be expressed as the union of a disjoint sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of relatively measurable subsets of D such that $\phi\restriction D_n$ is Lipschitz for each $n \in \mathbb{N}$.

262O Corollary Suppose that ϕ is an injective function from a measurable subset D of $\mathbb{R}^r$ to $\mathbb{R}^r$, and that ϕ is differentiable relative to its domain at every point of D.

(a) If $A \subseteq D$ is negligible, $\phi[A]$ is negligible.

(b) If $E \subseteq D$ is measurable, then $\phi[E]$ is measurable.

(c) If D is measurable and f is a measurable function defined on a subset of $\mathbb{R}^r$, then $f\phi^{-1}$ is measurable.

(d) If $H \subseteq \mathbb{R}^r$ and ϕ^{-1} is defined almost everywhere in H, and if f is a function defined almost everywhere in $\mathbb{R}^r$, then $f\phi^{-1}$ is defined almost everywhere in H.

262P Corollary Let ϕ be a function from a a subset D of $\mathbb{R}^r$ to $\mathbb{R}^s$, and suppose that ϕ is differentiable relative to its domain, with a derivative $T(x)$, at each point $x \in D$. Then the function $x \mapsto T(x)$ is measurable in the sense that $\tau_{ij} : D \to \mathbb{R}$ is measurable for all $i \le s$ and $j \le r$, where $\tau_{ij}(x)$ is the (i, j)th coefficient of the matrix $T(x)$ for all i, j and x.

***262Q Rademacher's theorem** Let ϕ be a Lipschitz function from a subset of $\mathbb{R}^r$ to $\mathbb{R}^s$. Then ϕ is differentiable relative to its domain almost everywhere in its domain.

Version of 4.4.13

263 Differentiable transformations in $\mathbb{R}^r$

This section is devoted to the proof of a single major theorem (263D) concerning differentiable transformations between subsets of $\mathbb{R}^r$. There will be a generalization of this result in §265, and those with some familiarity with the topic, or sufficient hardihood, may wish to read §264 before taking this section and §265 together. I end with a few simple corollaries and an extension of the main result which can be made in the one-dimensional case (263J).

Throughout this section, as in the rest of the chapter, μ will denote Lebesgue measure on $\mathbb{R}^r$.

263A Linear transformations: Theorem Let T be a real $r \times r$ matrix; regard T as a linear operator from $\mathbb{R}^r$ to itself. Let $J = |\det T|$ be the modulus of its determinant. Then

$$\mu T[E] = J\mu E$$

for every measurable set $E \subseteq \mathbb{R}^r$. If T is a permutation (that is, if $J \neq 0$), then

$$\mu F = J\mu(T^{-1}[F])$$

for every measurable $F \subseteq \mathbb{R}^r$, and

$$\int_F g\,d\mu = J\int_{T^{-1}[F]} gT\,d\mu$$

for every integrable function g and measurable set F.

263C Lemma Let T be a real $r \times r$ matrix; set $J = |\det T|$. Then for any $\epsilon > 0$ there is a $\zeta = \zeta(T,\epsilon) > 0$ such that

(i) $|\det S - \det T| \le \epsilon$ whenever S is an $r \times r$ matrix and $\|S - T\| \le \zeta$;

(ii) whenever $D \subseteq \mathbb{R}^r$ is a bounded set and $\phi : D \to \mathbb{R}^r$ is a function such that $\|\phi(x) - \phi(y) - T(x-y)\| \le \zeta\|x - y\|$ for all x, $y \in D$, then $|\mu^*\phi[D] - J\mu^*D| \le \epsilon\mu^*D$.

263D Theorem Let $D \subseteq \mathbb{R}^r$ be any set, and $\phi : D \to \mathbb{R}^r$ a function differentiable relative to its domain at each point of D. For each $x \in D$ let $T(x)$ be a derivative of ϕ relative to D at x, and set $J(x) = |\det T(x)|$. Then

(i) $J : D \to [0,\infty[$ is a measurable function,

(ii) $\mu^*\phi[D] \le \int_D J\,d\mu$,

allowing ∞ as the value of the integral. If D is measurable, then

(iii) $\phi[D]$ is measurable.

If D is measurable and ϕ is injective, then

(iv) $\mu\phi[D] = \int_D J\,d\mu$,

(v) for every real-valued function g defined on a subset of $\phi[D]$,

$$\int_{\phi[D]} g\,d\mu = \int_D J \times g\phi\,d\mu$$

if either integral is defined in $[-\infty,\infty]$, provided we interpret $J(x)g(\phi(x))$ as zero when $J(x) = 0$ and $g(\phi(x))$ is undefined.

***263F Corollary** Let $D \subseteq \mathbb{R}^r$ be any set and $\phi : D \to \mathbb{R}^r$ a Lipschitz function. Let D_1 be the set of points at which ϕ has a derivative relative to D, and for each $x \in D_1$ let $T(x)$ be such a derivative, with $J(x) = |\det T(x)|$. Then

(i) $D \setminus D_1$ is negligible;

(ii) $J : D_1 \to [0,\infty[$ is measurable;

(iii) $\mu^*\phi[D] \le \int_D J(x)dx$.

If D is measurable, then

(iv) $\phi[D]$ is measurable.

If D is measurable and ϕ is injective, then

(v) $\mu\phi[D] = \int_D J\,d\mu$,

(vi) for every real-valued function g defined on a subset of $\phi[D]$,

$$\int_{\phi[D]} g\,d\mu = \int_D J \times g\phi\,d\mu$$

if either integral is defined in $[-\infty,\infty]$, provided we interpret $J(x)g(\phi(x))$ as zero when $J(x) = 0$ and $g(\phi(x))$ is undefined.

263G Proposition $\int_{-\infty}^{\infty} e^{-t^2/2}dt = \sqrt{2\pi}$.

263H Corollary If $k \in \mathbb{N}$ is odd,

$$\int_{-\infty}^{\infty} x^k e^{-x^2/2}dx = 0;$$

if $k = 2l \in \mathbb{N}$ is even, then

$$\int_{-\infty}^{\infty} x^k e^{-x^2/2}dx = \frac{(2l)!}{2^l l!}\sqrt{2\pi}.$$

263I Theorem Let $D \subseteq \mathbb{R}^r$ be a measurable set, and $\phi : D \to \mathbb{R}^r$ a function differentiable relative to its domain at each point of D. For each $x \in D$ let $T(x)$ be a derivative of ϕ relative to D at x, and set $J(x) = |\det T(x)|$.

(a) Let ν be counting measure on $\mathbb{R}^r$. Then $\int_{\mathbb{R}^r} \nu(\phi^{-1}[\{y\}])dy$ and $\int_D J\,d\mu$ are defined in $[0,\infty]$ and equal.

(b) Let g be a real-valued function defined on a subset of $\phi[D]$ such that $\int_D g(\phi(x))\det T(x)dx$ is defined in $\mathbb{R}$, interpreting $g(\phi(x))\det T(x)$ as zero when $\det T(x) = 0$ and $g(\phi(x))$ is undefined. Set

$$C = \{y : y \in \phi[D],\ \phi^{-1}[\{y\}] \text{ is finite}\}, \quad R(y) = \textstyle\sum_{x\in\phi^{-1}[\{y\}]} \operatorname{sgn}\det T(x)$$

for $y \in C$, where $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(\alpha) = \frac{\alpha}{|\alpha|}$ for non-zero α. If we interpret $g(y)R(y)$ as zero when $g(y) = 0$ and $R(y)$ is undefined, then $\int_{\phi[D]} g \times R\,d\mu$ is defined and equal to $\int_D g(\phi(x))\det T(x)dx$.

263J The one-dimensional case: Proposition Let $I \subseteq \mathbb{R}$ be an interval with more than one point, and $\phi : I \to \mathbb{R}$ a function which is absolutely continuous on any closed bounded subinterval of I. Write $u = \inf I$, $u' = \sup I$ in $[-\infty,\infty]$, and suppose that $v = \lim_{x\downarrow u}\phi(x)$ and $v' = \lim_{x\uparrow u'}\phi(x)$ are defined in $[-\infty,\infty]$. Let g be a real function such that $\int_I g(\phi(x))\phi'(x)dx$ is defined, on the understanding that we interpret $g(\phi(x))\phi'(x)$ as 0 when $\phi'(x) = 0$ and $g(\phi(x))$ is undefined. Then $\int_v^{v'} g$ is defined and equal to $\int_I g(\phi(x))\phi'(x)dx$, where here we interpret $\int_v^{v'} g$ as $-\int_{v'}^{v} g$ if $v' < v$.

Version of 12.5.03

264 Hausdorff measures

The next topic I wish to approach is the question of 'surface measure'; a useful example to bear in mind throughout this section and the next is the notion of area for regions on the sphere, but any other smoothly curved two-dimensional surface in three-dimensional space will serve equally well. It is I think more than plausible that our intuitive concepts of 'area' for such surfaces should correspond to appropriate measures. But formalizing this intuition is non-trivial, especially if we seek the generality that simple geometric ideas lead us to; I mean, not contenting ourselves with arguments that depend on the special nature of the sphere, for instance, to describe spherical surface area. I divide the problem into two parts. In this section I will describe a construction which enables us to define the r-dimensional measure of an r-dimensional surface – among other things – in s-dimensional space. In the next section I will set out the basic theorems making it possible to calculate these measures effectively in the leading cases.

264A Definitions Let $s \geq 1$ be an integer, and $r > 0$. For any $A \subseteq \mathbb{R}^s$, $\delta > 0$ set

$$\theta_{r\delta}A = \inf\{\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r : \langle A_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } \mathbb{R}^s \text{ covering } A,$$
$$\operatorname{diam} A_n \leq \delta \text{ for every } n \in \mathbb{N}\}.$$

It is convenient in this context to say that $\operatorname{diam}\emptyset = 0$. Now set

$$\theta_r A = \sup_{\delta>0}\theta_{r\delta}A;$$

θ_r is **r-dimensional Hausdorff outer measure** on $\mathbb{R}^s$.

264B Lemma θ_r, as defined in 264A, is always an outer measure.

264C Definition If $s \geq 1$ is an integer, and $r > 0$, then **Hausdorff r-dimensional measure** on $\mathbb{R}^s$ is the measure μ_{Hr} on $\mathbb{R}^s$ defined by Carathéodory's method from the outer measure θ_r of 264A-264B.

264D Remarks (b) In the definitions above I require $r > 0$. It is sometimes appropriate to take μ_{H0} to be counting measure.

(c) All Hausdorff measures must be complete. For $r > 0$, they are atomless.

264E Theorem Let $s \geq 1$ be an integer, and $r \geq 0$; let μ_{Hr} be Hausdorff r-dimensional measure on $\mathbb{R}^s$, and Σ_{Hr} its domain. Then every Borel subset of $\mathbb{R}^s$ belongs to Σ_{Hr}.

264F Proposition Let $s \geq 1$ be an integer, and $r > 0$; let θ_r be r-dimensional Hausdorff outer measure on $\mathbb{R}^s$, and write μ_{Hr} for r-dimensional Hausdorff measure on $\mathbb{R}^s$, Σ_{Hr} for its domain. Then

(a) for every $A \subseteq \mathbb{R}^s$ there is a Borel set $E \supseteq A$ such that $\mu_{Hr}E = \theta_r A$;

(b) $\theta_r = \mu_{Hr}^*$, the outer measure defined from μ_{Hr};

(c) if $E \in \Sigma_{Hr}$ is expressible as a countable union of sets of finite measure, there are Borel sets E', E'' such that $E' \subseteq E \subseteq E''$ and $\mu_{Hr}(E'' \setminus E') = 0$.

264G Lipschitz functions: Proposition Let m, $s \geq 1$ be integers, and $\phi : D \to \mathbb{R}^s$ a γ-Lipschitz function, where D is a subset of $\mathbb{R}^m$. Then for any $A \subseteq D$ and $r \geq 0$,

$$\mu_{Hr}^*(\phi[A]) \leq \gamma^r \mu_{Hr}^* A$$

for every $A \subseteq D$, writing μ_{Hr} for r-dimensional Hausdorff outer measure on either $\mathbb{R}^m$ or $\mathbb{R}^s$.

264H Theorem Let $r \geq 1$ be an integer, and A a bounded subset of $\mathbb{R}^r$; write μ_r for Lebesgue measure on $\mathbb{R}^r$ and $d = \operatorname{diam} A$. Then

$$\mu_r^*(A) \leq \mu_r B(\mathbf{0}, \tfrac{d}{2}) = 2^{-r}\beta_r d^r,$$

where $B(\mathbf{0}, \frac{d}{2})$ is the ball with centre $\mathbf{0}$ and diameter d, so that $B(\mathbf{0}, 1)$ is the unit ball in $\mathbb{R}^r$, and has measure

$$\begin{aligned}\beta_r &= \frac{1}{k!}\pi^k \text{ if } r = 2k \text{ is even,}\\ &= \frac{2^{2k+1}k!}{(2k+1)!}\pi^k \text{ if } r = 2k+1 \text{ is odd.}\end{aligned}$$

264I Theorem Let $r \geq 1$ be an integer; let μ be Lebesgue measure on $\mathbb{R}^r$, and let μ_{Hr} be r-dimensional Hausdorff measure on $\mathbb{R}^r$. Then μ and μ_{Hr} have the same measurable sets and

$$\mu E = 2^{-r}\beta_r \mu_{Hr} E$$

for every measurable set $E \subseteq \mathbb{R}^r$, where $\beta_r = \mu B(\mathbf{0}, 1)$, so that the normalizing factor is

$$\begin{aligned}2^{-r}\beta_r &= \frac{1}{2^{2k}k!}\pi^k \text{ if } r = 2k \text{ is even,}\\ &= \frac{k!}{(2k+1)!}\pi^k \text{ if } r = 2k+1 \text{ is odd.}\end{aligned}$$

***264J The Cantor set: Proposition** Let C be the Cantor set in $[0, 1]$. Set $r = \ln 2/\ln 3$. Then the r-dimensional Hausdorff measure of C is 1.

***264K General metric spaces** Let (X,ρ) be a metric space, and $r>0$. For any $A\subseteq X$, $\delta>0$ set

$$\theta_{r\delta}A=\inf\{\sum_{n=0}^{\infty}(\operatorname{diam}A_n)^r:\langle A_n\rangle_{n\in\mathbb{N}}\text{ is a sequence of subsets of }X\text{ covering }A,$$

$$\operatorname{diam}A_n\le\delta\text{ for every }n\in\mathbb{N}\},$$

interpreting the diameter of the empty set as 0, and $\inf\emptyset$ as ∞, so that $\theta_{r\delta}A=\infty$ if A cannot be covered by a sequence of sets of diameter at most δ. Say that $\theta_rA=\sup_{\delta>0}\theta_{r\delta}A$ is the **r-dimensional Hausdorff outer measure** of A, and take the measure μ_{Hr} defined by Carathéodory's method from this outer measure to be **r-dimensional Hausdorff measure** on X.

Version of 3.9.13

265 Surface measures

In this section I offer a new version of the arguments of §263, this time not with the intention of justifying integration-by-substitution, but instead to give a practically effective method of computing the Hausdorff r-dimensional measure of a smooth r-dimensional surface in an s-dimensional space. The basic case to bear in mind is $r=2$, $s=3$, though any other combination which you can easily visualize will also be a valuable aid to intuition. I give a fundamental theorem (265E) providing a formula from which we can hope to calculate the r-dimensional measure of a surface in s-dimensional space which is parametrized by a differentiable function, and work through some of the calculations in the case of the r-sphere (265F-265H).

265A Normalized Hausdorff measure In this section I will use **normalized Hausdorff measure**, meaning $\nu_r=2^{-r}\beta_r\mu_{Hr}$, where μ_{Hr} is r-dimensional Hausdorff measure and $\beta_r=\mu_rB(\mathbf{0},1)$ is the Lebesgue measure of any ball of radius 1 in $\mathbb{R}^r$. This normalization makes ν_r on $\mathbb{R}^r$ agree with Lebesgue measure μ_r. $\nu_r^*=2^{-r}\beta_r\mu_{Hr}^*$.

265B Linear subspaces: Theorem Suppose that r, s are integers with $1\le r\le s$, and that T is a real $s\times r$ matrix; regard T as a linear operator from $\mathbb{R}^r$ to $\mathbb{R}^s$. Set $J=\sqrt{\det T^\top T}$, where $T^\top$ is the transpose of T. Write ν_r for normalized r-dimensional Hausdorff measure on $\mathbb{R}^s$, T_r for its domain, and μ_r for Lebesgue measure on $\mathbb{R}^r$. Then

$$\nu_rT[E]=J\mu_rE$$

for every measurable set $E\subseteq\mathbb{R}^r$. If T is injective (that is, if $J\ne0$), then

$$\nu_rF=J\mu_rT^{-1}[F]$$

whenever $F\in\mathrm{T}_r$ and $F\subseteq T[\mathbb{R}^r]$.

265C Corollary Under the conditions of 265B,

$$\nu_r^*T[A]=J\mu_r^*A$$

for every $A\subseteq\mathbb{R}^r$.

265D Lemma Suppose that $1\le r\le s$ and that T is an $s\times r$ matrix; set $J=\sqrt{\det T^\top T}$, and suppose that $J\ne0$. Then for any $\epsilon>0$ there is a $\zeta=\zeta(T,\epsilon)>0$ such that

(i) $|\sqrt{\det S^\top S}-J|\le\epsilon$ whenever S is an $s\times r$ matrix and $\|S-T\|\le\zeta$;

(ii) whenever $D\subseteq\mathbb{R}^r$ is a bounded set and $\phi:D\to\mathbb{R}^s$ is a function such that $\|\phi(x)-\phi(y)-T(x-y)\|\le\zeta\|x-y\|$ for all x, $y\in D$, then $|\nu_r^*\phi[D]-J\mu_r^*D|\le\epsilon\mu_r^*D$.

265E Theorem Suppose that $1\le r\le s$; write μ_r for Lebesgue measure on $\mathbb{R}^r$, ν_r for normalized Hausdorff measure on $\mathbb{R}^s$, and T_r for the domain of ν_r. Let $D\subseteq\mathbb{R}^r$ be any set, and $\phi:D\to\mathbb{R}^s$ a function differentiable relative to its domain at each point of D. For each $x \in D$ let $T(x)$ be a derivative of ϕ at x relative to D, and set $J(x) = \sqrt{\det T(x)^\top T(x)}$. Set $D' = \{x : x \in D,\ J(x) > 0\}$. Then

(i) $J : D \to [0, \infty[$ is a measurable function;

(ii) $\nu_r^* \phi[D] \le \int_D J(x)\mu_r(dx)$,

allowing ∞ as the value of the integral;

(iii) $\nu_r^* \phi[D \setminus D'] = 0$.

If D is Lebesgue measurable, then

(iv) $\phi[D] \in \mathrm{T}_r$.

If D is measurable and ϕ is injective, then

(v) $\nu_r \phi[D] = \int_D J\, d\mu_r$;

(vi) for any set $E \subseteq \phi[D]$, $E \in \mathrm{T}_r$ iff $\phi^{-1}[E] \cap D'$ is Lebesgue measurable, and in this case

$$\nu_r E = \int_{\phi^{-1}[E]} J(x)\mu_r(dx) = \int_D J \times \chi(\phi^{-1}[E]) d\mu_r;$$

(vii) for every real-valued function g defined on a subset of $\phi[D]$,

$$\int_{\phi[D]} g\, d\nu_r = \int_D J \times g\phi\, d\mu_r$$

if either integral is defined in $[-\infty, \infty]$, provided we interpret $J(x)g(\phi(x))$ as zero when $J(x) = 0$ and $g(\phi(x))$ is undefined.

265F The surface of a sphere Write S_r for $\{z : z \in \mathbb{R}^{r+1}, \|z\| = 1\}$, the r-sphere. Then the normalized r-dimensional Hausdorff measure of S_r is $2\pi\beta_{r-1}$, where β_{r-1} is the volume of the unit ball of $\mathbb{R}^{r-1}$ (interpreting β_0 as 1.)

265G Theorem Let μ_{r+1} be Lebesgue measure on $\mathbb{R}^{r+1}$, and ν_r normalized r-dimensional Hausdorff measure on $\mathbb{R}^{r+1}$. If f is a locally μ_{r+1}-integrable real-valued function, $y \in \mathbb{R}^{r+1}$ and $\delta > 0$,

$$\int_{B(y,\delta)} f d\mu_{r+1} = \int_0^\delta \int_{\partial B(y,t)} f d\nu_r dt,$$

where I write $\partial B(y,t)$ for the sphere $\{x : \|x - y\| = t\}$.

265H Corollary If ν_r is normalized r-dimensional Hausdorff measure on $\mathbb{R}^{r+1}$, then $\nu_r S_r = (r+1)\beta_{r+1}$.

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*266 The Brunn-Minkowski inequality

We now have most of the essential ingredients for a proof of the Brunn-Minkowski inequality (266C) in a strong form. I do not at present expect to use it in this treatise, but it is one of the basic results of geometric measure theory and from where we now stand is not difficult, so I include it here. The preliminary results on arithmetic and geometric means (266A) and essential closures (266B) are of great importance for other reasons.

266A Arithmetic and geometric means: Proposition If $u_0,\dots,u_n,p_0,\dots,p_n\in[0,\infty[$ and $\sum_{i=0}^n p_i=1$, then $\prod_{i=0}^n u_i^{p_i}\le\sum_{i=0}^n p_iu_i$.

266B Proposition For any set $D\subseteq\mathbb{R}^r$ set

$$\mathrm{cl}^*D=\{x:\limsup_{\delta\downarrow 0}\frac{\mu^*(D\cap B(x,\delta))}{\mu B(x,\delta)}>0\},$$

where μ is Lebesgue measure on $\mathbb{R}^r$.

(a) $D\setminus\mathrm{cl}^*D$ is negligible.

(b) $\mathrm{cl}^*D\subseteq\overline{D}$.

(c) cl^*D is a Borel set.

(d) $\mu(\mathrm{cl}^*D)=\mu^*D$.

(e) If $C\subseteq\mathbb{R}$ then $\overline{C}+\mathrm{cl}^*D\subseteq\mathrm{cl}^*(C+D)$, writing $C+D$ for $\{x+y:x\in C,\ y\in D\}$.

266C Theorem Let A, $B\subseteq\mathbb{R}^r$ be non-empty sets, where $r\ge 1$ is an integer. If μ is Lebesgue measure on $\mathbb{R}^r$, and $A+B=\{x+y:x\in A,\ y\in B\}$, then $\mu^*(A+B)^{1/r}\ge(\mu^*A)^{1/r}+(\mu^*B)^{1/r}$.