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Chapter 25

Product Measures

I come now to another chapter on 'pure' measure theory, discussing a fundamental construction – or, as you may prefer to consider it, two constructions, since the problems involved in forming the product of two arbitrary measure spaces (§251) are rather different from those arising in the product of arbitrarily many probability spaces (§254). This work is going to stretch our technique to the utmost, for while the fundamental theorems to which we are moving are natural aims, the proofs are lengthy and there are many pitfalls beside the true paths.

The central idea is that of 'repeated integration'. You have probably already seen formulae of the type '$\iint f(x,y)dxdy$' used to calculate the integral of a function of two real variables over a region in the plane. One of the basic techniques of advanced calculus is reversing the order of integration; for instance, we expect $\int_0^1(\int_y^1 f(x,y)dx)dy$ to be equal to $\int_0^1(\int_0^x f(x,y)dy)dx$. As I have developed the subject, we already have a third calculation to compare with these two: $\int_D f$, where $D=\{(x,y):0\le y\le x\le 1\}$ and the integral is taken with respect to Lebesgue measure on the plane. The first two sections of this chapter are devoted to an analysis of the relationship between one- and two-dimensional Lebesgue measure which makes these operations valid – some of the time; part of the work has to be devoted to a careful description of the exact conditions which must be imposed on f and D if we are to be safe.

Repeated integration, in one form or another, appears everywhere in measure theory, and it is therefore necessary sooner or later to develop the most general possible expression of the idea. The standard method is through the theory of products of general measure spaces. Given measure spaces (X,Σ,μ) and (Y,T,ν), the aim is to find a measure λ on $X\times Y$ which will, at least, give the right measure $\mu E\cdot\nu F$ to a 'rectangle' $E\times F$ where $E\in\Sigma$ and $F\in\mathrm{T}$. It turns out that there are already difficulties in deciding what 'the' product measure is, and to do the job properly I find I need, even at this stage, to describe two related but distinguishable constructions. These constructions and their elementary properties take up the whole of §251. In §252 I turn to integration over the product, with Fubini's and Tonelli's theorems relating $\int fd\lambda$ with $\iint f(x,y)\mu(dx)\nu(dy)$. Because the construction of λ is symmetric between the two factors, this automatically provides theorems relating $\iint f(x,y)\mu(dx)\nu(dy)$ with $\iint f(x,y)\nu(dy)\mu(dx)$. §253 looks at the space $L^1(\lambda)$ and its relationship with $L^1(\mu)$ and $L^1(\nu)$.

For general measure spaces, there are obstacles in the way of forming an infinite product; to start with, if $\langle(X_n,\mu_n)\rangle_{n\in\mathbb{N}}$ is a sequence of measure spaces, then a product measure λ on $X=\prod_{n\in\mathbb{N}}X_n$ ought to set $\lambda X=\prod_{n=0}^\infty\mu_nX_n$, and there is no guarantee that the product will converge, or behave well when it does. But for probability spaces, when $\mu_nX_n=1$ for every n, this problem at least evaporates. It is possible to define the product of any family of probability spaces; this is the burden of §254.

I end the chapter with three sections which are a preparation for Chapters 27 and 28, but are also important in their own right as an investigation of the way in which the group structure of $\mathbb{R}^r$ interacts with Lebesgue and other measures. §255 deals with the 'convolution' $f*g$ of two functions, where $(f*g)(x)=\int f(y)g(x-y)dy$ (the integration being with respect to Lebesgue measure). In §257 I show that some of the same ideas, suitably transformed, can be used to describe a convolution $\nu_1*\nu_2$ of two measures on $\mathbb{R}^r$; in preparation for this I include a section on Radon measures on $\mathbb{R}^r$ (§256).

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251 Finite products

The first construction to set up is the product of a pair of measure spaces. It turns out that there are already substantial technical difficulties in the way of finding a canonical universally applicable method. I find myself therefore describing two related, but distinct, constructions, the 'primitive' and 'c.l.d.' product measures (251C, 251F). After listing the fundamental properties of the c.l.d product measure (251I-251J), I work through the identification of the product of Lebesgue measure with itself (251N) and a fairly thorough discussion of subspaces (251O-251S).

251A Definition Let (X,Σ,μ) and (Y,T,ν) be two measure spaces. For $A\subseteq X\times Y$ set

$$\theta A=\inf\{\sum_{n=0}^{\infty}\mu E_n\cdot\nu F_n : E_n\in\Sigma,\ F_n\in\mathrm{T}\ \forall\ n\in\mathbb{N},\ A\subseteq\bigcup_{n\in\mathbb{N}}E_n\times F_n\}.$$

251B Lemma In the context of 251A, θ is an outer measure on $X\times Y$.

251C Definition Let (X,Σ,μ) and (Y,T,ν) be measure spaces. By the **primitive product measure** on $X\times Y$ I shall mean the measure λ_0 derived by Carathéodory's method from the outer measure θ defined in 251A.

251D Definition If X and Y are sets with σ-algebras $\Sigma\subseteq\mathcal{P}X$ and $\mathrm{T}\subseteq\mathcal{P}Y$, I will write $\Sigma\widehat{\otimes}\mathrm{T}$ for the σ-algebra of subsets of $X\times Y$ generated by $\{E\times F : E\in\Sigma,\ F\in\mathrm{T}\}$.

251E Proposition Let (X,Σ,μ) and (Y,T,ν) be measure spaces; let λ_0 be the primitive product measure on $X\times Y$, and Λ its domain. Then $\Sigma\widehat{\otimes}\mathrm{T}\subseteq\Lambda$ and $\lambda_0(E\times F)=\mu E\cdot\nu F$ for all $E\in\Sigma$ and $F\in\mathrm{T}$.

251F Definition Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and λ_0 the primitive product measure. By the **c.l.d. product measure** on $X\times Y$ I shall mean the function $\lambda:\operatorname{dom}\lambda_0\to[0,\infty]$ defined by setting

$$\lambda W=\sup\{\lambda_0(W\cap(E\times F)) : E\in\Sigma,\ F\in\mathrm{T},\ \mu E<\infty,\ \nu F<\infty\}$$

for $W\in\operatorname{dom}\lambda_0$.

251G Remark λ is a measure.

251H Lemma Let (X,Σ,μ) and (Y,T,ν) be two measure spaces; let λ_0 be the primitive product measure on $X\times Y$, and Λ its domain. If $H\subseteq X\times Y$ and $H\cap(E\times F)\in\Lambda$ whenever $\mu E<\infty$ and $\nu F<\infty$, then $H\in\Lambda$.

251I Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces; let λ be the c.l.d. product measure on $X\times Y$, and Λ its domain. Then

(a) $\Sigma\widehat{\otimes}\mathrm{T}\subseteq\Lambda$ and $\lambda(E\times F)=\mu E\cdot\nu F$ whenever $E\in\Sigma$, $F\in\mathrm{T}$ and $\mu E\cdot\nu F<\infty$;

(b) for every $W\in\Lambda$ there is a $V\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $V\subseteq W$ and $\lambda V=\lambda W$;

(c) $(X\times Y,\Lambda,\lambda)$ is complete and locally determined, and in fact is the c.l.d. version of $(X\times Y,\Lambda,\lambda_0)$; in particular, $\lambda W=\lambda_0 W$ whenever $\lambda_0 W<\infty$;

(d) if $W\in\Lambda$ and $\lambda W>0$ then there are $E\in\Sigma$, $F\in\mathrm{T}$ such that $\mu E<\infty$, $\nu F<\infty$ and $\lambda(W\cap(E\times F))>0$;

(e) if $W\in\Lambda$ and $\lambda W<\infty$, then for every $\epsilon>0$ there are $E_0,\dots,E_n\in\Sigma$, $F_0,\dots,F_n\in\mathrm{T}$, all of finite measure, such that $\lambda(W\triangle\bigcup_{i\le n}(E_i\times F_i))\le\epsilon$.

251J Proposition If (X,Σ,μ) and (Y,T,ν) are semi-finite measure spaces and λ is the c.l.d. product measure on $X\times Y$, then $\lambda(E\times F)=\mu E\cdot\nu F$ for all $E\in\Sigma$, $F\in\mathrm{T}$.

251K σ-finite spaces: Proposition Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces. Then the c.l.d. product measure on $X\times Y$ is equal to the primitive product measure, and is the completion of its restriction to $\Sigma\widehat{\otimes}\mathrm{T}$; moreover, this common product measure is σ-finite.

***251L Proposition** Let (X_1,Σ_1,μ_1), (X_2,Σ_2,μ_2), (Y_1,T_1,ν_1) and (Y_2,T_2,ν_2) be σ-finite measure spaces; let λ_1, λ_2 be the product measures on $X_1\times Y_1$, $X_2\times Y_2$ respectively. Suppose that $f:X_1\to X_2$ and $g:Y_1\to Y_2$ are inverse-measure-preserving functions, and that $h(x,y)=(f(x),g(y))$ for $x\in X_1$, $y\in Y_1$. Then h is inverse-measure-preserving.

251M Proposition Let r, $s \geq 1$ be integers. Then we have a natural bijection $\phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^{r+s}$, defined by setting

$$\phi((\xi_1, \dots, \xi_r), (\eta_1, \dots, \eta_s)) = (\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_s)$$

for $\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_s \in \mathbb{R}$. If we write $\mathcal{B}_r$, $\mathcal{B}_s$ and $\mathcal{B}_{r+s}$ for the Borel σ-algebras of $\mathbb{R}^r$, $\mathbb{R}^s$ and $\mathbb{R}^{r+s}$ respectively, then ϕ identifies $\mathcal{B}_{r+s}$ with $\mathcal{B}_r \widehat{\otimes} \mathcal{B}_s$.

251N Theorem Let r, $s \geq 1$ be integers. Then the bijection $\phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^{r+s}$ described in 251M identifies Lebesgue measure on $\mathbb{R}^{r+s}$ with the c.l.d. product λ of Lebesgue measure on $\mathbb{R}^r$ and Lebesgue measure on $\mathbb{R}^s$.

251O Proposition Let (X, Σ, μ) and (Y, T, ν) be strictly localizable measure spaces. Then the c.l.d. product measure on $X \times Y$ is strictly localizable; moreover, if $\langle X_i \rangle_{i \in I}$ and $\langle Y_j \rangle_{j \in J}$ are decompositions of X and Y respectively, $\langle X_i \times Y_j \rangle_{(i,j) \in I \times J}$ is a decomposition of $X \times Y$.

251P Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and λ the c.l.d. product measure on $X \times Y$. Let λ^* be the corresponding outer measure. Then

$$\lambda^* C = \sup\{\theta(C \cap (E \times F)) : E \in \Sigma,\ F \in \mathrm{T},\ \mu E < \infty,\ \nu F < \infty\}$$

for every $C \subseteq X \times Y$, where θ is the outer measure of 251A.

251Q Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $A \subseteq X$, $B \subseteq Y$ subsets; write μ_A, ν_B for the subspace measures on A, B respectively. Let λ be the c.l.d. product measure on $X \times Y$, and $\lambda^\#$ the subspace measure it induces on $A \times B$. Let $\tilde{\lambda}$ be the c.l.d. product measure of μ_A and ν_B on $A \times B$. Then

(i) $\tilde{\lambda}$ extends $\lambda^\#$.

(ii) If

either (α) $A \in \Sigma$ and $B \in \mathrm{T}$

or (β) A and B can both be covered by sequences of sets of finite measure

or (γ) μ and ν are both strictly localizable,

then $\tilde{\lambda} = \lambda^\#$.

251R Corollary Let r, $s \geq 1$ be integers, and $\phi : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^{r+s}$ the natural bijection. If $A \subseteq \mathbb{R}^r$ and $B \subseteq \mathbb{R}^s$, then the restriction of ϕ to $A \times B$ identifies the product of Lebesgue measure on A and Lebesgue measure on B with Lebesgue measure on $\phi[A \times B] \subseteq \mathbb{R}^{r+s}$.

251S Corollary Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and λ the c.l.d. product measure on $X \times Y$. If $A \subseteq X$ and $B \subseteq Y$ can be covered by sequences of sets of finite measure, then $\lambda^*(A \times B) = \mu^* A \cdot \nu^* B$.

251T Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces. Write $(X, \hat{\Sigma}, \hat{\mu})$ and $(X, \tilde{\Sigma}, \tilde{\mu})$ for the completion and c.l.d. version of (X, Σ, μ). Let λ, $\hat{\lambda}$ and $\tilde{\lambda}$ be the three c.l.d. product measures on $X \times Y$ obtained from the pairs (μ, ν), $(\hat{\mu}, \nu)$ and $(\tilde{\mu}, \nu)$ of factor measures. Then $\lambda = \hat{\lambda} = \tilde{\lambda}$.

251U Proposition Let (X, Σ, μ) be an atomless measure space, and let λ be the c.l.d. measure on $X \times X$. Then $\Delta = \{(x, x) : x \in X\}$ is λ-negligible.

***251W Products of more than two spaces** Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a finite family of measure spaces, and set $X = \prod_{i \in I} X_i$. Write $\Sigma_i^f = \{E : E \in \Sigma_i,\ \mu_i E < \infty\}$ for each $i \in I$.

(a) For $A \subseteq X$ set

$$\theta A = \inf\{\sum_{n=0}^{\infty} \prod_{i \in I} \mu_i E_{ni} : E_{ni} \in \Sigma_i\ \forall\ i \in I,\ n \in \mathbb{N},\ A \subseteq \bigcup_{n \in \mathbb{N}} \prod_{i \in I} E_{ni}\}.$$

Then θ is an outer measure on X. Let λ_0 be the measure on X derived by Carathéodory's method from θ, and Λ its domain.

(b) If $\langle X_i\rangle_{i\in I}$ is a finite family of sets and Σ_i is a σ-algebra of subsets of X_i for each $i\in I$, then $\widehat{\bigotimes}_{i\in I}\Sigma_i$ is the σ-algebra of subsets of $X=\prod_{i\in I}X_i$ generated by $\{\prod_{i\in I}E_i:E_i\in\Sigma_i$ for every $i\in I\}$.

(c) $\lambda_0(\prod_{i\in I}E_i)$ is defined and equal to $\prod_{i\in I}\mu_iE_i$ whenever $E_i\in\Sigma_i$ for each $i\in I$.

(d) The **c.l.d. product measure** on X is the measure λ defined by setting

$$\lambda W=\sup\{\lambda_0(W\cap\prod_{i\in I}E_i):E_i\in\Sigma_i^f\text{ for each }i\in I\}$$

for $W\in\Lambda$. If I is empty, set $\lambda X=1$.

(e) If $H\subseteq X$, then $H\in\Lambda$ iff $H\cap\prod_{i\in I}E_i\in\Lambda$ whenever $E_i\in\Sigma_i^f$ for each $i\in I$.

(f)(i) $\widehat{\bigotimes}_{i\in I}\Sigma_i\subseteq\Lambda$ and $\lambda(\prod_{i\in I}E_i)=\prod_{i\in I}\mu_iE_i$ whenever $E_i\in\Sigma_i^f$ for each i.

(ii) For every $W\in\Lambda$ there is a $V\in\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $V\subseteq W$ and $\lambda V=\lambda W$.

(iii) λ is complete and locally determined, and is the c.l.d. version of λ_0.

(iv) If $W\in\Lambda$ and $\lambda W>0$ then there are $E_i\in\Sigma_i^f$, for $i\in I$, such that $\lambda(W\cap\prod_{i\in I}E_i)>0$.

(v) If $W\in\Lambda$ and $\lambda W<\infty$, then for every $\epsilon>0$ there are $n\in\mathbb{N}$ and $E_{0i},\dots,E_{ni}\in\Sigma_i^f$, for each $i\in I$, such that $\lambda(W\triangle\bigcup_{k\le n}\prod_{i\in I}E_{ki})\le\epsilon$.

(g) If each μ_i is σ-finite, so is λ, and $\lambda=\lambda_0$ is the completion of its restriction to $\widehat{\bigotimes}_{i\in I}\Sigma_i$.

(h) If $\langle I_j\rangle_{j\in J}$ is any partition of I, then λ can be identified with the c.l.d. product of $\langle\lambda_j\rangle_{j\in J}$, where λ_j is the c.l.d. product of $\langle\mu_i\rangle_{i\in I_j}$.

(i) If $I=\{1,\dots,n\}$ and each μ_i is Lebesgue measure on $\mathbb{R}$, then λ can be identified with Lebesgue measure on $\mathbb{R}^n$.

(j) If, for each $i\in I$, we have a decomposition $\langle X_{ij}\rangle_{j\in J_i}$ of X_i, then $\langle\prod_{i\in I}X_{i,f(i)}\rangle_{f\in\prod_{i\in I}J_i}$ is a decomposition of X.

(k) For any $C\subseteq X$,

$$\lambda^*C=\sup\{\theta(C\cap\prod_{i\in I}E_i):E_i\in\Sigma_i^f\text{ for every }i\in I\}.$$

(l) Suppose that $A_i\subseteq X_i$ for each $i\in I$. Write $\lambda^\#$ for the subspace measure on $A=\prod_{i\in I}A_i$, and $\tilde\lambda$ for the c.l.d. product of the subspace measures on the A_i. Then $\tilde\lambda$ extends $\lambda^\#$, and if

either $A_i\in\Sigma_i$ for every i

or every A_i can be covered by a sequence of sets of finite measure

or every μ_i is strictly localizable,

then $\tilde\lambda=\lambda^\#$.

(m) If $A_i\subseteq X_i$ can be covered by a sequence of sets of finite measure for each $i\in I$, then $\lambda^*(\prod_{i\in I}A_i)=\prod_{i\in I}\mu_i^*A_i$.

(n) Writing $\hat\mu_i$, $\tilde\mu_i$ for the completion and c.l.d. version of each μ_i, λ is the c.l.d. product of $\langle\hat\mu_i\rangle_{i\in I}$ and also of $\langle\tilde\mu_i\rangle_{i\in I}$.

(o) If all the (X_i,Σ_i,μ_i) are the same atomless measure space (X,Σ,μ), then $\{x:x\in X,\ i\mapsto x(i)$ is injective$\}$ is λ-conegligible.

(p) Now suppose that we have another family $\langle(Y_i,\mathrm{T}_i,\nu_i)\rangle_{i\in I}$ of measure spaces, with product (Y,Λ',λ'), and inverse-measure-preserving functions $f_i:X_i\to Y_i$ for each i; define $f:X\to Y$ by saying that $f(x)(i)=f_i(x(i))$ for $x\in X$ and $i\in I$. If all the ν_i are σ-finite, then f is inverse-measure-preserving for λ and λ'.

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252 Fubini's theorem

Perhaps the most important feature of the concept of 'product measure' is the fact that we can use it to discuss repeated integrals. In this section I give versions of Fubini's theorem and Tonelli's theorem (252B, 252G) with a variety of corollaries, the most useful ones being versions for σ-finite spaces (252C, 252H). As applications I describe the relationship between integration and measuring ordinate sets (252N) and calculate the r-dimensional volume of a ball in $\mathbb{R}^r$ (252Q). I mention counter-examples showing the difficulties which can arise with non-σ-finite measures and non-integrable functions (252K-252L).

252A Repeated integrals Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and f a real-valued function defined on a set $\operatorname{dom} f\subseteq X\times Y$. We can seek to form the **repeated integral**

$$\iint f(x,y)\nu(dy)\mu(dx)=\int\Bigl(\int f(x,y)\nu(dy)\Bigr)\mu(dx),$$

which should be interpreted as follows: set

$$D=\{x:x\in X,\ \int f(x,y)\nu(dy)\text{ is defined in }[-\infty,\infty]\},$$

$$g(x)=\int f(x,y)\nu(dy)\text{ for }x\in D,$$

and then write $\iint f(x,y)\nu(dy)\mu(dx)=\int g(x)\mu(dx)$ if this is defined.

252B Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Suppose that ν is σ-finite and that μ is *either* strictly localizable *or* complete and locally determined. Let f be a $[-\infty,\infty]$-valued function such that $\int fd\lambda$ is defined in $[-\infty,\infty]$. Then $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and is equal to $\int fd\lambda$.

252C Corollary Let (X,Σ,μ) and (Y,T,ν) be two σ-finite measure spaces, and λ the c.l.d. product measure on $X\times Y$. If f is λ-integrable, then $\iint f(x,y)\nu(dy)\mu(dx)$ and $\iint f(x,y)\mu(dx)\nu(dy)$ are defined, finite and equal.

252D Corollary Let (X,Σ,μ) and (Y,T,ν) be measure spaces and λ the c.l.d. product measure on $X\times Y$. Suppose that ν is σ-finite and that μ is either strictly localizable or complete and locally determined.

(i) If $W\in\operatorname{dom}\lambda$, then $\int\nu^*W[\{x\}]\mu(dx)$ is defined in $[0,\infty]$ and equal to λW.

(ii) If ν is complete, we can write $\int\nu W[\{x\}]\mu(dx)$ in place of $\int\nu^*W[\{x\}]\mu(dx)$.

252E Corollary Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Suppose that ν is σ-finite and that μ has locally determined negligible sets. Then if f is a Λ-measurable real-valued function defined on a subset of $X\times Y$, $y\mapsto f(x,y)$ is ν-virtually measurable for μ-almost every $x\in X$.

252F Corollary Let (X,Σ,μ) and (Y,T,ν) be two measure spaces, λ the c.l.d. product measure on $X\times Y$, and Λ its domain. Let $W\in\Lambda$ be such that the vertical section $W[\{x\}]$ is ν-negligible for μ-almost every $x\in X$. Then $\lambda W=0$.

252G Tonelli's theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $(X\times Y,\Lambda,\lambda)$ their c.l.d. product. Let f be a Λ-measurable $[-\infty,\infty]$-valued function defined on a member of Λ, and suppose that either $\iint|f(x,y)|\mu(dx)\nu(dy)$ or $\iint|f(x,y)|\nu(dy)\mu(dx)$ exists in $\mathbb{R}$. Then f is λ-integrable.

252H Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces, λ the c.l.d. product measure on $X\times Y$, and Λ its domain.

(a) Let f be a Λ-measurable $[-\infty,\infty]$-valued function defined on a member of Λ. Then if one of

$$\int_{X\times Y}|f(x,y)|\lambda(d(x,y)),\quad\int_Y\int_X|f(x,y)|\mu(dx)\nu(dy),\quad\int_X\int_Y|f(x,y)|\nu(dy)\mu(dx)$$

exists in $\mathbb{R}$, so do the other two, and in this case

$$\int_{X\times Y} f(x,y)\lambda(d(x,y)) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx).$$

(b) Let f be a Λ-measurable $[0,\infty]$-valued function defined on a member of Λ. Then

$$\int_{X\times Y} f(x,y)\lambda(d(x,y)) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx)$$

in the sense that if one of the integrals is defined in $[0,\infty]$ so are the other two, and all three are then equal.

252I Corollary Let (X,Σ,μ) and (Y,T,ν) be measure spaces, λ the c.l.d. product measure on $X\times Y$, and Λ its domain. Take $W\in\Lambda$. If either of the integrals

$$\int \mu^*W^{-1}[\{y\}]\nu(dy), \quad \int \nu^*W[\{x\}]\mu(dx)$$

exists and is finite, then $\lambda W<\infty$.

252K Example Let (X,Σ,μ) be $[0,1]$ with Lebesgue measure, and let (Y,T,ν) be $[0,1]$ with counting measure. Consider the set

$$W = \{(t,t) : t\in[0,1]\} \subseteq X\times Y.$$

$$\iint \chi W(x,y)\mu(dx)\nu(dy) = 0,$$

$$\iint \chi W(x,y)\nu(dy)\mu(dx) = 1,$$

so the two repeated integrals differ.

252N Integration through ordinate sets I: Proposition Let (X,Σ,μ) be a complete locally determined measure space, and λ the c.l.d. product measure on $X\times\mathbb{R}$, where $\mathbb{R}$ is given Lebesgue measure; write Λ for the domain of λ. For any $[0,\infty]$-valued function f defined on a conegligible subset of X, write Ω_f, Ω'_f for the **ordinate sets**

$$\Omega_f = \{(x,a) : x\in\operatorname{dom} f,\ 0\le a\le f(x)\} \subseteq X\times\mathbb{R},$$

$$\Omega'_f = \{(x,a) : x\in\operatorname{dom} f,\ 0\le a< f(x)\} \subseteq X\times\mathbb{R}.$$

Then

$$\lambda\Omega_f = \lambda\Omega'_f = \int f d\mu$$

in the sense that if one of these is defined in $[0,\infty]$, so are the other two, and they are equal.

252O Integration through ordinate sets II: Proposition Let (X,Σ,μ) be a measure space, and f a non-negative μ-virtually measurable function defined on a conegligible subset of X. Then

$$\int f d\mu = \int_0^\infty \mu^*\{x : x\in\operatorname{dom} f,\ f(x)\ge t\}dt = \int_0^\infty \mu^*\{x : x\in\operatorname{dom} f,\ f(x)> t\}dt$$

in $[0,\infty]$, where the integrals $\int\ldots dt$ are taken with respect to Lebesgue measure.

***252P Proposition** Let (X,Σ,μ) be a measure space, and (Y,T,ν) a σ-finite measure space. Then for any $\Sigma\widehat{\otimes}\mathrm{T}$-measurable function $f : X\times Y\to[0,\infty]$, $x\mapsto\int f(x,y)\nu(dy) : X\to[0,\infty]$ is Σ-measurable. If μ is semi-finite, $\iint f(x,y)\nu(dy)\mu(dx) = \int f d\lambda$, where λ is the c.l.d. product measure on $X\times Y$.

252Q The volume of a ball The Lebesgue measure of the unit ball in $\mathbb{R}^r$ is

$$\begin{aligned}\beta_r &= \frac{1}{k!}\pi^k \text{ if } r=2k \text{ is even,}\\ &= \frac{2^{2k+1}k!}{(2k+1)!}\pi^k \text{ if } r=2k+1 \text{ is odd.}\end{aligned}$$

252R Complex-valued functions (a) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and λ the c.l.d. product measure on $X \times Y$. Suppose that ν is σ-finite and that μ is either strictly localizable or complete and locally determined. Let f be a λ-integrable complex-valued function. Then $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\lambda$.

(b) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, λ the c.l.d. product measure on $X \times Y$, and Λ its domain. Let f be a Λ-measurable complex-valued function defined on a member of Λ, and suppose that either $\iint |f(x,y)|\mu(dx)\nu(dy)$ or $\iint |f(x,y)|\nu(dy)\mu(dx)$ is defined and finite. Then f is λ-integrable.

(c) Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces, λ the c.l.d. product measure on $X \times Y$, and Λ its domain. Let f be a Λ-measurable complex-valued function defined on a member of Λ. Then if one of

$$\int_{X\times Y} |f(x,y)|\lambda(d(x,y)), \quad \int_Y \int_X |f(x,y)|\mu(dx)\nu(dy), \quad \int_X \int_Y |f(x,y)|\nu(dy)\mu(dx)$$

exists in $\mathbb{R}$, so do the other two, and in this case

$$\int_{X\times Y} f(x,y)\lambda(d(x,y)) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx).$$

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253 Tensor products

The theorems of the last section show that the integrable functions on a product of two measure spaces can be effectively studied in terms of integration on each factor space separately. In this section I present a very striking relationship between the L^1 space of a product measure and the L^1 spaces of its factors, which actually determines the product L^1 up to isomorphism as Banach lattice. I start with a brief note on bilinear operators (253A) and a description of the canonical bilinear operator from $L^1(\mu)\times L^1(\nu)$ to $L^1(\mu\times\nu)$ (253B-253E). The main theorem of the section is 253F, showing that this canonical map is universal for continuous bilinear operators from $L^1(\mu)\times L^1(\nu)$ to Banach spaces; it also determines the ordering of $L^1(\mu\times\nu)$ (253G). I end with a description of a fundamental type of conditional expectation operator (253H) and notes on products of indefinite-integral measures (253I) and upper integrals of special kinds of function (253J, 253K).

253A Bilinear operators (a) Let U, V and W be linear spaces over $\mathbb{R}$. A map $\phi : U \times V \to W$ is **bilinear** if it is linear in each variable separately, that is,

$$\phi(u_1 + u_2, v) = \phi(u_1, v) + \phi(u_2, v),$$

$$\phi(u, v_1 + v_2) = \phi(u, v_1) + \phi(u, v_2),$$

$$\phi(\alpha u, v) = \alpha\phi(u, v) = \phi(u, \alpha v)$$

for all u, u_1, $u_2 \in U$, v, v_1, $v_2 \in V$ and scalars α. Observe that such a ϕ gives rise to, and in turn can be defined by, a linear operator $T : U \to \mathsf{L}(V; W)$, where

$$(Tu)(v) = \phi(u, v)$$

for all $u \in U$, $v \in V$. $\phi(0, v) = \phi(u, 0) = 0$ whenever $u \in U$ and $v \in V$.

If W' is another linear space and $S : W \to W'$ is a linear operator, then $S\phi : U \times V \to W'$ is bilinear.

(b) Now suppose that U, V and W are normed spaces, and $\phi : U \times V \to W$ a bilinear operator. ϕ is **bounded** if $\sup\{\|\phi(u, v)\| : \|u\| \le 1, \|v\| \le 1\}$ is finite, and in this case we call this supremum the norm $\|\phi\|$ of ϕ. $\|\phi(u, v)\| \le \|\phi\|\|u\|\|v\|$ for all $u \in U$, $v \in V$.

If W' is another normed space and $S : W \to W'$ is a bounded linear operator, then $S\phi : U \times V \to W'$ is a bounded bilinear operator, and $\|S\phi\| \le \|S\|\|\phi\|$.

253B Definition Let f and g be real-valued functions. I will write $f \otimes g$ for the function $(x, y) \mapsto f(x)g(y) : \operatorname{dom} f \times \operatorname{dom} g \to \mathbb{R}$.

253C Proposition (a) Let X and Y be sets, and Σ, T σ-algebras of subsets of X, Y respectively. If f is a Σ-measurable real-valued function defined on a subset of X, and g is a T-measurable real-valued function defined on a subset of Y, then $f \otimes g$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable.

(b) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and λ the c.l.d. product measure on $X \times Y$. If $f \in \mathcal{L}^0(\mu)$ and $g \in \mathcal{L}^0(\nu)$, then $f \otimes g \in \mathcal{L}^0(\lambda)$.

253D Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and write λ for the c.l.d. product measure on $X \times Y$. If $f \in \mathcal{L}^1(\mu)$ and $g \in \mathcal{L}^1(\nu)$, then $f \otimes g \in \mathcal{L}^1(\lambda)$ and $\int f \otimes g \, d\lambda = \int f \, d\mu \int g \, d\nu$.

253E The canonical map $L^1 \times L^1 \to L^1$ We may define $u \otimes v \in L^1(\lambda)$, for $u \in L^1(\mu)$ and $v \in L^1(\nu)$, by saying that $u \otimes v = (f \otimes g)^\bullet$ whenever $u = f^\bullet$ and $v = g^\bullet$. The map $(u, v) \mapsto u \otimes v$ is bilinear.

$$\|u \otimes v\|_1 = \|u\|_1 \|v\|_1$$

for all $u \in L^1(\mu)$, $v \in L^1(\nu)$. In particular, the bilinear operator $\otimes$ is bounded, with norm 1 (except in the trivial case in which one of $L^1(\mu)$, $L^1(\nu)$ is 0-dimensional).

253F Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and let λ be the c.l.d. product measure on $X \times Y$. Let W be any Banach space and $\phi : L^1(\mu) \times L^1(\nu) \to W$ a bounded bilinear operator. Then there is a unique bounded linear operator $T : L^1(\lambda) \to W$ such that $T(u \otimes v) = \phi(u, v)$ for all $u \in L^1(\mu)$ and $v \in L^1(\nu)$, and $\|T\| = \|\phi\|$.

253G The order structure of L^1: Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and λ the c.l.d. product measure on $X \times Y$. Then

(a) $u \otimes v \ge 0$ in $L^1(\lambda)$ whenever $u \ge 0$ in $L^1(\mu)$ and $v \ge 0$ in $L^1(\nu)$.

(b) The positive cone $\{w : w \ge 0\}$ of $L^1(\lambda)$ is precisely the closed convex hull C of $\{u \otimes v : u \ge 0, \, v \ge 0\}$ in $L^1(\lambda)$.

*(c) Let W be any Banach lattice, and $T : L^1(\lambda) \to W$ a bounded linear operator. Then the following are equiveridical:

(i) $Tw \ge 0$ in W whenever $w \ge 0$ in $L^1(\lambda)$;

(ii) $T(u \otimes v) \ge 0$ in W whenever $u \ge 0$ in $L^1(\mu)$ and $v \ge 0$ in $L^1(\nu)$.

253H Conditional expectations: Theorem Let (X, Σ, μ) and (Y, T, ν) be complete probability spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Set $\Lambda_1 = \{E \times Y : E \in \Sigma\}$. Then Λ_1 is a σ-subalgebra of Λ. Given a λ-integrable real-valued function f, set

$$g(x, y) = \int f(x, z) \nu(dz)$$

whenever $x \in X$, $y \in Y$ and the integral is defined in $\mathbb{R}$. Then g is a conditional expectation of f on Λ_1.

253I Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $f \in \mathcal{L}^0(\mu)$, $g \in \mathcal{L}^0(\nu)$ non-negative functions. Let μ', ν' be the corresponding indefinite-integral measures. Let λ be the c.l.d. product of μ and ν, and λ' the indefinite-integral measure defined from λ and $f \otimes g \in \mathcal{L}^0(\lambda)$. Then λ' is the c.l.d. product of μ' and ν'.

***253J Upper integrals: Proposition** Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces, with c.l.d. product measure λ. Then for any functions f and g, defined on conegligible subsets of X and Y respectively, and taking values in $[0, \infty]$,

$$\overline{\int} f \otimes g \, d\lambda = \overline{\int} f d\mu \cdot \overline{\int} g \, d\nu.$$

***253K Proposition** Let (X, Σ, μ) and (Y, T, ν) be probability spaces, with c.l.d. product measure λ. Then for any real-valued functions f, g defined on conegligible subsets of X, Y respectively,

$$\overline{\int} f(x) + g(y) \, \lambda(d(x, y)) = \overline{\int} f(x) \mu(dx) + \overline{\int} g(y) \nu(dy),$$

at least when the right-hand side is defined in $[-\infty, \infty]$.

253L Complex spaces As usual, the ideas apply essentially unchanged to complex L^1 spaces. Writing $L^1_{\mathbb{C}}(\mu)$, etc., for the complex L^1 spaces involved, we have the following. Throughout, let (X,Σ,μ) and (Y,T,ν) be measure spaces, and λ the c.l.d. product measure on $X\times Y$.

(a) If $f\in\mathcal{L}^0_{\mathbb{C}}(\mu)$ and $g\in\mathcal{L}^0_{\mathbb{C}}(\nu)$ then $f\otimes g$, defined by the formula $(f\otimes g)(x,y)=f(x)g(y)$ for $x\in\operatorname{dom}f$ and $y\in\operatorname{dom}g$, belongs to $\mathcal{L}^0_{\mathbb{C}}(\lambda)$.

(b) If $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)$ and $g\in\mathcal{L}^1_{\mathbb{C}}(\nu)$ then $f\otimes g\in\mathcal{L}^1_{\mathbb{C}}(\lambda)$ and $\int f\otimes g\,d\lambda=\int f d\mu\int g\,d\nu$.

(c) We have a bilinear operator $(u,v)\mapsto u\otimes v:L^1_{\mathbb{C}}(\mu)\times L^1_{\mathbb{C}}(\nu)\to L^1_{\mathbb{C}}(\lambda)$ defined by writing $f^\bullet\otimes g^\bullet=(f\otimes g)^\bullet$ for all $f\in\mathcal{L}^1_{\mathbb{C}}(\mu)$, $g\in\mathcal{L}^1_{\mathbb{C}}(\nu)$.

(d) If W is any complex Banach space and $\phi:L^1_{\mathbb{C}}(\mu)\times L^1_{\mathbb{C}}(\nu)\to W$ is any bounded bilinear operator, then there is a unique bounded linear operator $T:L^1_{\mathbb{C}}(\lambda)\to W$ such that $T(u\otimes v)=\phi(u,v)$ for every $u\in L^1_{\mathbb{C}}(\mu)$ and $v\in L^1_{\mathbb{C}}(\nu)$, and $\|T\|=\|\phi\|$.

(e) If μ and ν are complete probability measures, and $\Lambda_1=\{E\times Y:E\in\Sigma\}$, then for any $f\in\mathcal{L}^1_{\mathbb{C}}(\lambda)$ we have a conditional expectation g of f on Λ_1 given by setting $g(x,y)=\int f(x,z)\nu(dz)$ whenever this is defined.

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254 Infinite products

I come now to the second basic idea of this chapter: the description of a product measure on the product of a (possibly large) family of probability spaces. The section begins with a construction on similar lines to that of §251 (254A-254F) and its defining property in terms of inverse-measure-preserving functions (254G). I discuss the usual measure on $\{0,1\}^I$ (254J-254K), subspace measures (254L) and various properties of subproducts (254M-254T), including a study of the associated conditional expectation operators (254R-254T).

254A Definitions (a) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces. Set $X=\prod_{i\in I}X_i$, the family of functions x with domain I such that $x(i)\in X_i$ for every $i\in I$. In this context, I will say that a **measurable cylinder** is a subset of X expressible in the form

$$C=\prod_{i\in I}C_i,$$

where $C_i\in\Sigma_i$ for every $i\in I$ and $\{i:C_i\ne X_i\}$ is finite. Note that for a non-empty $C\subseteq X$ this expression is unique.

(b) We can therefore define a functional $\theta_0:\mathcal{C}\to[0,1]$, where $\mathcal{C}$ is the set of measurable cylinders, by setting

$$\theta_0 C=\prod_{i\in I}\mu_iC_i$$

whenever $C_i\in\Sigma_i$ for every $i\in I$ and $\{i:C_i\ne X_i\}$ is finite.

(c) Now define $\theta:\mathcal{P}X\to[0,1]$ by setting

$$\theta A=\inf\{\sum_{n=0}^\infty\theta_0C_n:C_n\in\mathcal{C}\text{ for every }n\in\mathbb{N},\,A\subseteq\bigcup_{n\in\mathbb{N}}C_n\}.$$

254B Lemma The functional θ defined in 254Ac is always an outer measure on X.

254C Definition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any indexed family of probability spaces, and X the Cartesian product $\prod_{i\in I}X_i$. The **product measure** on X is the measure defined by Carathéodory's method from the outer measure θ defined in 254A.

254E Definition Let $\langle X_i\rangle_{i\in I}$ be any family of sets, and $X=\prod_{i\in I}X_i$. If Σ_i is a σ-subalgebra of subsets of X_i for each $i\in I$, I write $\widehat{\bigotimes}_{i\in I}\Sigma_i$ for the σ-algebra of subsets of X generated by

$$\{\{x:x\in X,\,x(i)\in E\}:i\in I,\,E\in\Sigma_i\}.$$

254F Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, and let λ be the product measure on $X=\prod_{i\in I}X_i$; let Λ be its domain.

(a) $\lambda X=1$.

(b) If $E_i\in\Sigma_i$ for every $i\in I$, and $\{i:E_i\ne X_i\}$ is countable, then $\prod_{i\in I}E_i\in\Lambda$, and $\lambda(\prod_{i\in I}E_i)=\prod_{i\in I}\mu_iE_i$. In particular, $\lambda C=\theta_0C$ for every measurable cylinder C, and if $j\in I$ then $x\mapsto x(j):X\to X_j$ is inverse-measure-preserving.

(c) $\widehat{\bigotimes}_{i\in I}\Sigma_i\subseteq\Lambda$.

(d) λ is complete.

(e) For every $W\in\Lambda$ and $\epsilon>0$ there is a finite family $C_0,\dots,C_n$ of measurable cylinders such that $\lambda(W\triangle\bigcup_{k\le n}C_k)\le\epsilon$.

(f) For every $W\in\Lambda$ there are $W_1,\,W_2\in\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $W_1\subseteq W\subseteq W_2$ and $\lambda(W_2\setminus W_1)=0$.

254G Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces with product (X,Λ,λ). Let (Y,T,ν) be a complete probability space and $\phi:Y\to X$ a function. Suppose that $\nu^*\phi^{-1}[C]\le\lambda C$ for every measurable cylinder $C\subseteq X$. Then ϕ is inverse-measure-preserving. In particular, ϕ is inverse-measure-preserving iff $\phi^{-1}[C]\in\mathrm{T}$ and $\nu\phi^{-1}[C]=\lambda C$ for every measurable cylinder $C\subseteq X$.

254H Corollary Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle(Y_i,\mathrm{T}_i,\nu_i)\rangle_{i\in I}$ be two families of probability spaces, with products (X,Λ,λ) and (Y,Λ',λ'). Suppose that for each $i\in I$ we are given an inverse-measure-preserving function $\phi_i:X_i\to Y_i$. Set $\phi(x)=\langle\phi_i(x(i))\rangle_{i\in I}$ for $x\in X$. Then $\phi:X\to Y$ is inverse-measure-preserving.

254I Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, λ the product measure on $X=\prod_{i\in I}X_i$, and Λ its domain. Then λ is also the product of the completions $\hat\mu_i$ of the μ_i.

254J The product measure on $\{0,1\}^I$ (a) Perhaps the most important of all examples of infinite product measures is the case in which each factor X_i is just $\{0,1\}$ and each μ_i is the 'fair-coin' probability measure, setting

$$\mu_i\{0\}=\mu_i\{1\}=\frac{1}{2}.$$

In this case, the product $X=\{0,1\}^I$ has a family $\langle E_i\rangle_{i\in I}$ of measurable sets such that, writing λ for the product measure on X,

$$\lambda(\textstyle\bigcap_{i\in J}E_i)=2^{-\#(J)}\text{ if }J\subseteq I\text{ is finite.}$$

I will call this λ the **usual measure** on $\{0,1\}^I$. Observe that if I is finite then $\lambda\{x\}=2^{-\#(I)}$ for each $x\in X$. On the other hand, if I is infinite, then $\lambda\{x\}=0$ for every $x\in X$.

(b) There is a natural bijection between $\{0,1\}^I$ and $\mathcal{P}I$, matching $x\in\{0,1\}^I$ with $\{i:i\in I,\,x(i)=1\}$. So we get a standard measure $\tilde\lambda$ on $\mathcal{P}I$, which I will call the **usual measure on $\mathcal{P}I$**. Note that for any finite $b\subseteq I$ and any $c\subseteq b$ we have

$$\tilde\lambda\{a:a\cap b=c\}=\lambda\{x:x(i)=1\text{ for }i\in c,\,x(i)=0\text{ for }i\in b\setminus c\}=2^{-\#(b)}.$$

(c) Of course we can apply 254G to these measures; if (Y,T,ν) is a complete probability space, a function $\phi:Y\to\{0,1\}^I$ is inverse-measure-preserving iff

$$\nu\{y:y\in Y,\,\phi(y)\restriction J=z\}=2^{-\#(J)}$$

whenever $J\subseteq I$ is finite and $z\in\{0,1\}^J$.

(d) Define addition on X by setting $(x+y)(i)=x(i)+_2 y(i)$ for every $i\in I$, x, $y\in X$, where $0+_2 0=1+_2 1=0$, $0+_2 1=1+_2 0=1$. If $y\in X$, the map $x\mapsto x+y:X\to X$ is a measure space automorphism of (X,λ).

***(e)** Just because all the factors (X_i,μ_i) are the same, we have another class of automorphisms of (X,λ), corresponding to permutations of I. If $\pi:I\to I$ is any permutation, then we have a corresponding function $x\mapsto x\pi:X\to X$. $x\mapsto x\pi$ is a measure space automorphism.

254K Proposition Let λ be the usual measure on $X=\{0,1\}^{\mathbb{N}}$, and let μ be Lebesgue measure on $[0,1]$; write Λ for the domain of λ and Σ for the domain of μ.

(i) For $x\in X$ set $\phi(x)=\sum_{i=0}^{\infty}2^{-i-1}x(i)$. Then

$\phi^{-1}[E]\in\Lambda$ and $\lambda\phi^{-1}[E]=\mu E$ for every $E\in\Sigma$;

$\phi[F]\in\Sigma$ and $\mu\phi[F]=\lambda F$ for every $F\in\Lambda$.

(ii) There is a bijection $\tilde\phi:X\to[0,1]$ which is equal to ϕ at all but countably many points, and any such bijection is an isomorphism between (X,Λ,λ) and $([0,1],\Sigma,\mu)$.

254L Subspaces: Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, and (X,Λ,λ) their product.

(a) For each $i\in I$, let $A_i\subseteq X_i$ be a set of full outer measure, and write $\tilde\mu_i$ for the subspace measure on A_i. Let $\tilde\lambda$ be the product measure on $A=\prod_{i\in I}A_i$. Then $\tilde\lambda$ is the subspace measure on A induced by λ.

(b) $\lambda^*(\prod_{i\in I}A_i)=\prod_{i\in I}\mu_i^*A_i$ whenever $A_i\subseteq X_i$ for every i.

254M Let $\langle X_i\rangle_{i\in I}$ be a family of sets, with product X.

(a) For $J\subseteq I$, write X_J for $\prod_{i\in J}X_i$. We have a canonical bijection $x\mapsto(x\restriction J,x\restriction I\setminus J):X\to X_I\times X_{I\setminus J}$. Associated with this we have the map $x\mapsto\pi_J(x)=x\restriction J:X\to X_J$. Now I will say that a set $W\subseteq X$ is **determined by coordinates in** J if there is a $V\subseteq X_J$ such that $W=\pi_J^{-1}[V]$; that is, W corresponds to $V\times X_{I\setminus J}\subseteq X_J\times X_{I\setminus J}$.

$$
\begin{aligned}
&W \text{ is determined by coordinates in } J\\
&\qquad\iff x'\in W \text{ whenever } x\in W,\ x'\in X \text{ and } x'\restriction J=x\restriction J\\
&\qquad\iff W=\pi_J^{-1}[\pi_J[W]].
\end{aligned}
$$

It follows that if W is determined by coordinates in J, and $J\subseteq K\subseteq I$, W is also determined by coordinates in K. The family $\mathcal{W}_J$ of subsets of X determined by coordinates in J is closed under complementation and arbitrary unions and intersections.

(b)

$$\mathcal{W}=\bigcup\{\mathcal{W}_J:J\subseteq I\text{ is countable}\}$$

is a σ-algebra of subsets of X.

(c) If $i\in I$ and $E\subseteq X_i$ then $\{x:x\in X,\ x(i)\in E\}$ is determined by the single coordinate i, so surely belongs to $\mathcal{W}$; accordingly $\mathcal{W}$ must include $\widehat{\bigotimes}_{i\in I}\mathcal{P}X_i$.

254N Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces and $\langle K_j\rangle_{j\in J}$ a partition of I. For each $j\in J$ let λ_j be the product measure on $Z_j=\prod_{i\in K_j}X_i$, and write λ for the product measure on $X=\prod_{i\in I}X_i$. Then the natural bijection

$$x\mapsto\phi(x)=\langle x\restriction K_j\rangle_{j\in J}:X\to\prod_{j\in J}Z_j$$

identifies λ with the product of the family $\langle\lambda_j\rangle_{j\in J}$.

In particular, if $K\subseteq I$ is any set, then λ can be identified with the c.l.d. product of the product measures on $\prod_{i\in K}X_i$ and $\prod_{i\in I\setminus K}X_i$.

254O Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces. For each $J\subseteq I$ let λ_J be the product probability measure on $X_J=\prod_{i\in J}X_i$, and Λ_J its domain; write $X=X_I$, $\lambda=\lambda_I$ and $\Lambda=\Lambda_I$. For $x\in X$ and $J\subseteq I$ set $\pi_J(x)=x\restriction J\in X_J$.

(a) For every $J\subseteq I$, λ_J is the image measure $\lambda\pi_J^{-1}$; in particular, $\pi_J:X\to X_J$ is inverse-measure-preserving for λ and λ_J.

(b) If $J\subseteq I$ and $W\in\Lambda$ is determined by coordinates in J, then $\lambda_J\pi_J[W]$ is defined and equal to λW. Consequently there are W_1, W_2 belonging to the σ-algebra of subsets of X generated by

$$\{\{x:x(i)\in E\}:i\in J,\,E\in\Sigma_i\}$$

such that $W_1\subseteq W\subseteq W_2$ and $\lambda(W_2\setminus W_1)=0$.

(c) For every $W\in\Lambda$, we can find a countable set J and W_1, $W_2\in\Lambda$, both determined by coordinates in J, such that $W_1\subseteq W\subseteq W_2$ and $\lambda(W_2\setminus W_1)=0$.

(d) For every $W\in\Lambda$, there is a countable set $J\subseteq I$ such that $\pi_J[W]\in\Lambda_J$ and $\lambda_J\pi_J[W]=\lambda W$; so that $W'=\pi_J^{-1}[\pi_J[W]]$ belongs to Λ, and $\lambda(W'\setminus W)=0$.

254P Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, and for each $J\subseteq I$ let λ_J be the product probability measure on $X_J=\prod_{i\in J}X_i$, and Λ_J its domain; write $X=X_I$, $\Lambda=\Lambda_I$ and $\lambda=\lambda_I$. For $x\in X$ and $J\subseteq I$ set $\pi_J(x)=x\restriction J\in X_J$.

(a) If $J\subseteq I$ and g is a real-valued function defined on a subset of X_J, then g is Λ_J-measurable iff $g\pi_J$ is Λ-measurable.

(b) Whenever f is a Λ-measurable real-valued function defined on a λ-conegligible subset of X, we can find a countable set $J\subseteq I$ and a Λ_J-measurable function g defined on a λ_J-conegligible subset of X_J such that f extends $g\pi_J$.

254Q Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, and for each $J\subseteq I$ let λ_J be the product probability measure on $X_J=\prod_{i\in J}X_i$; write $X=X_I$, $\lambda=\lambda_I$. For $x\in X$, $J\subseteq I$ set $\pi_J(x)=x\restriction J\in X_J$.

(a) Let $\mathcal{S}$ be the linear subspace of $\mathbb{R}^X$ spanned by $\{\chi C:C\subseteq X$ is a measurable cylinder$\}$. Then for every λ-integrable real-valued function f and every $\epsilon>0$ there is a $g\in\mathcal{S}$ such that $\int|f-g|d\lambda\le\epsilon$.

(b) Whenever $J\subseteq I$ and g is a real-valued function defined on a subset of X_J, then $\int g\,d\lambda_J=\int g\pi_J d\lambda$ if either integral is defined in $[-\infty,\infty]$.

(c) Whenever f is a λ-integrable real-valued function, we can find a countable set $J\subseteq X$ and a λ_J-integrable function g such that f extends $g\pi_J$.

254R Conditional expectations again: Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces with product (X,Λ,λ). For $J\subseteq I$ let $\Lambda_J\subseteq\Lambda$ be the σ-subalgebra of sets determined by coordinates in J. Then we may regard $L^0(\lambda\restriction\Lambda_J)$ as a subspace of $L^0(\lambda)$. Let $P_J:L^1(\lambda)\to L^1(\lambda\restriction\Lambda_J)\subseteq L^1(\lambda)$ be the corresponding conditional expectation operator. Then

(a) for any J, $K\subseteq I$, $P_{K\cap J}=P_KP_J$;

(b) for any $u\in L^1(\lambda)$, there is a countable set $J^*\subseteq I$ such that $P_Ju=u$ iff $J\supseteq J^*$;

(c) for any $u\in L^0(\lambda)$, there is a unique smallest set $J^*\subseteq I$ such that $u\in L^0(\lambda\restriction\Lambda_{J^*})$, and this J^* is countable;

(d) for any $W\in\Lambda$ there is a unique smallest set $J^*\subseteq I$ such that $W\triangle W'$ is negligible for some $W'\in\Lambda_{J^*}$, and this J^* is countable;

(e) for any Λ-measurable real-valued function $f:X\to\mathbb{R}$ there is a unique smallest set $J^*\subseteq I$ such that f is equal almost everywhere to a Λ_J^*-measurable function, and this J^* is countable.

254S Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product (X,Λ,λ).

(a) If $A\subseteq X$ is determined by coordinates in $I\setminus\{j\}$ for every $j\in I$, then its outer measure λ^*A must be either 0 or 1.

(b) If $W\in\Lambda$ and $\lambda W>0$, then for every $\epsilon>0$ there are a $W'\in\Lambda$ and a finite set $J\subseteq I$ such that $\lambda W'\ge1-\epsilon$ and for every $x\in W'$ there is a $y\in W$ such that $x\restriction I\setminus J=y\restriction I\setminus J$.

254T Remarks (a) Let $\langle X_i\rangle_{i\in I}$ be a family of sets with Cartesian product X. For each $J\subseteq I$ let $\mathcal{W}_J$ be the set of subsets of X determined by coordinates in J. Then $\mathcal{W}_J\cap\mathcal{W}_K=\mathcal{W}_{J\cap K}$ for all J, $K\subseteq I$. Accordingly, for any $W\subseteq X$, $\mathcal{F}=\{J:W\in\mathcal{W}_J\}$ is a filter on I (unless $W=X$ or $W=\emptyset$, in which case $\mathcal{F}=\mathcal{P}X$).

(b) Set $X=\{0,1\}^{\mathbb{N}}$,

$$W=\{x:x\in X,\,\lim_{i\to\infty}x(i)=0\}.$$

Then for every $n\in\mathbb{N}$ W is determined by coordinates in $J_n=\{i:i\ge n\}$. But W is not determined by coordinates in $\bigcap_{n\in\mathbb{N}}J_n=\emptyset$.

***254U Example** There are a localizable measure space (X,Σ,μ) and a probability space (Y,T,ν) such that the c.l.d. product measure λ on $X\times Y$ is not localizable.

***254V Proposition** Let (X,Σ,μ) be an atomless probability space and I a countable set. Let λ be the product probability measure on X^I. Then $\{x:x\in X^I,\,x \text{ is injective}\}$ is λ-conegligible.

Version of 3.7.08

255 Convolutions of functions

I devote a section to a construction which is of great importance – and will in particular be very useful in Chapters 27 and 28 – and may also be regarded as a series of exercises on the work so far.

I find it difficult to know how much repetition to indulge in in this section, because the natural unified expression of the ideas is in the theory of topological groups, and I do not think we are yet ready for the general theory (I will come to it in Chapter 44 in Volume 4). The groups we need for this volume are

$\mathbb{R}$;

$\mathbb{R}^r$, for $r\ge 2$;

$S^1=\{z:z\in\mathbb{C},\,|z|=1\}$, the 'circle group';

$\mathbb{Z}$, the group of integers.

All the ideas already appear in the theory of convolutions on $\mathbb{R}$, and I will therefore present this material in relatively detailed form, before sketching the forms appropriate to the groups $\mathbb{R}^r$ and S^1 (or $]-\pi,\pi]$); $\mathbb{Z}$ can I think be safely left to the exercises.

255A Theorem Let μ be Lebesgue measure on $\mathbb{R}$ and μ_2 Lebesgue measure on $\mathbb{R}^2$; write Σ, Σ_2 for their domains.

(a) For any $a\in\mathbb{R}$, the map $x\mapsto a+x:\mathbb{R}\to\mathbb{R}$ is a measure space automorphism of $(\mathbb{R},\Sigma,\mu)$.

(b) The map $x\mapsto -x:\mathbb{R}\to\mathbb{R}$ is a measure space automorphism of $(\mathbb{R},\Sigma,\mu)$.

(c) For any $a\in\mathbb{R}$, the map $x\mapsto a-x:\mathbb{R}\to\mathbb{R}$ is a measure space automorphism of $(\mathbb{R},\Sigma,\mu)$.

(d) The map $(x,y)\mapsto(x+y,y):\mathbb{R}^2\to\mathbb{R}^2$ is a measure space automorphism of $(\mathbb{R}^2,\Sigma_2,\mu_2)$.

(e) The map $(x,y)\mapsto(x-y,y):\mathbb{R}^2\to\mathbb{R}^2$ is a measure space automorphism of $(\mathbb{R}^2,\Sigma_2,\mu_2)$.

255B Corollary (a) If $a\in\mathbb{R}$, then for any complex-valued function f defined on a subset of $\mathbb{R}$

$$\int f(x)dx=\int f(a+x)dx=\int f(-x)dx=\int f(a-x)dx$$

in the sense that if one of the integrals exists so do the others, and they are then all equal.

(b) If f is a complex-valued function defined on a subset of $\mathbb{R}^2$, then

$$\int f(x+y,y)d(x,y)=\int f(x-y,y)d(x,y)=\int f(x,y)d(x,y)$$

in the sense that if one of the integrals exists and is finite so does the other, and they are then equal.

255D Corollary Let f be a complex-valued function defined on a subset of $\mathbb{R}$.

(a) If f is measurable, then the functions $(x,y)\mapsto f(x+y)$, $(x,y)\mapsto f(x-y)$ are measurable.

(b) If f is defined almost everywhere in $\mathbb{R}$, then the functions $(x,y)\mapsto f(x+y)$, $(x,y)\mapsto f(x-y)$ are defined almost everywhere in $\mathbb{R}^2$.

255E The basic formula Let f and g be measurable complex-valued functions defined almost everywhere in $\mathbb{R}$. Write $f * g$ for the function defined by the formula

$$(f * g)(x) = \int f(x-y)g(y)dy$$

whenever the integral exists as a complex number. Then $f * g$ is the **convolution** of the functions f and g. $\operatorname{dom}(|f| * |g|) = \operatorname{dom}(f * g)$, and $|f * g| \le |f| * |g|$ everywhere on their common domain, for all f and g.

255F Elementary properties (a) Because integration is linear, we surely have

$$((f_1 + f_2) * g)(x) = (f_1 * g)(x) + (f_2 * g)(x),$$

$$(f * (g_1 + g_2))(x) = (f * g_1)(x) + (f * g_2)(x),$$

$$(cf * g)(x) = (f * cg)(x) = c(f * g)(x)$$

whenever the right-hand sides of the formulae are defined.

(b) If f and g are measurable complex-valued functions defined almost everywhere in $\mathbb{R}$, then $f*g = g*f$.

(c) If f_1, f_2, g_1, g_2 are measurable complex-valued functions defined almost everywhere in $\mathbb{R}$, $f_1 =_{\text{a.e.}} f_2$ and $g_1 =_{\text{a.e.}} g_2$, then $f_1 * g_1 = f_2 * g_2$.

It follows that if u, $v \in L^0_{\mathbb{C}}$, then we have a function $\theta(u,v)$ which is equal to $f * g$ whenever f, $g \in \mathcal{L}^0_{\mathbb{C}}$ are such that $f^\bullet = u$ and $g^\bullet = u$.

255G Theorem Let f, g and h be measurable complex-valued functions defined almost everywhere in $\mathbb{R}$.

(a) Suppose that $\int h(x+y)f(x)g(y)d(x,y)$ exists in $\mathbb{C}$. Then

$$\begin{aligned}\int h(x)(f * g)(x)dx &= \int h(x+y)f(x)g(y)d(x,y)\\ &= \iint h(x+y)f(x)g(y)dxdy = \iint h(x+y)f(x)g(y)dydx\end{aligned}$$

provided that in the expression $h(x)(f * g)(x)$ we interpret the product as 0 if $h(x) = 0$ and $(f * g)(x)$ is undefined.

(b) If, on a similar interpretation of $|h(x)|(|f| * |g|)(x)$, the integral $\int |h(x)|(|f| * |g|)(x)dx$ is finite, then $\int h(x+y)f(x)g(y)d(x,y)$ exists in $\mathbb{C}$.

255H Corollary If f, g are complex-valued functions which are integrable over $\mathbb{R}$, then $f*g$ is integrable, with

$$\int f * g = \int f \int g, \quad \int |f * g| \le \int |f| \int |g|.$$

255I Corollary For any measurable complex-valued functions f, g defined almost everywhere in $\mathbb{R}$, $f * g$ is measurable and has measurable domain.

255J Theorem Let f, g and h be complex-valued measurable functions, defined almost everywhere in $\mathbb{R}$, such that $f * g$ and $g * h$ are defined a.e. Suppose that $x \in \mathbb{R}$ is such that one of $(|f| * (|g| * |h|))(x)$, $((|f| * |g|) * |h|)(x)$ is defined in $\mathbb{R}$. Then $f * (g * h)$ and $(f * g) * h$ are defined and equal at x.

255K Proposition Suppose that f, g are measurable complex-valued functions defined almost everywhere in $\mathbb{R}$, and that $f \in \mathcal{L}^p_{\mathbb{C}}$, $g \in \mathcal{L}^q_{\mathbb{C}}$ where p, $q \in [1,\infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $f * g$ is defined everywhere in $\mathbb{R}$, is uniformly continuous, and

$$\begin{aligned}\sup_{x\in\mathbb{R}} |(f * g)(x)| &\le \|f\|_p \|g\|_q \text{ if } 1 < p < \infty,\ 1 < q < \infty,\\ &\le \|f\|_1 \operatorname{ess\,sup} |g| \text{ if } p = 1,\ q = \infty,\\ &\le \operatorname{ess\,sup} |f| \cdot \|g\|_1 \text{ if } p = \infty,\ q = 1.\end{aligned}$$

255L The r-dimensional case I have written 255A-255K out as theorems about Lebesgue measure on $\mathbb{R}$. However they all apply to Lebesgue measure on $\mathbb{R}^r$ for any $r \ge 1$.

255M The case of $]-\pi,\pi]$ **(a)** If we think of S^1 as the set $\{z : z \in \mathbb{C}, |z| = 1\}$, then the group operation is complex multiplication, and in the formulae above $x + y$ must be rendered as xy, while $x - y$ must be rendered as xy^{-1}. On the interval $]-\pi,\pi]$, the group operation is $+_{2\pi}$, where for x, $y \in \,]-\pi,\pi]$ I write $x +_{2\pi} y$ for whichever of $x + y$, $x + y + 2\pi$, $x + y - 2\pi$ belongs to $]-\pi,\pi]$.

(b) As for the measure, the measure to use on $]-\pi,\pi]$ is just Lebesgue measure.
On S^1, we need the corresponding measure induced by the canonical bijection between S^1 and $]-\pi,\pi]$.

255N Theorem Let μ be Lebesgue measure on $]-\pi,\pi]$ and μ_2 Lebesgue measure on $]-\pi,\pi] \times]-\pi,\pi]$; write Σ, Σ_2 for their domains.

(a) For any $a \in \,]-\pi,\pi]$, the map $x \mapsto a +_{2\pi} x :]-\pi,\pi] \to]-\pi,\pi]$ is a measure space automorphism of $(]-\pi,\pi]\,, \Sigma, \mu)$.

(b) The map $x \mapsto -_{2\pi}x :]-\pi,\pi] \to]-\pi,\pi]$ is a measure space automorphism of $(]-\pi,\pi]\,, \Sigma, \mu)$.

(c) For any $a \in \,]-\pi,\pi]$, the map $x \mapsto a -_{2\pi} x :]-\pi,\pi] \to]-\pi,\pi]$ is a measure space automorphism of $(]-\pi,\pi]\,, \Sigma, \mu)$.

(d) The map $(x, y) \mapsto (x +_{2\pi} y, y) :]-\pi,\pi]^2 \to]-\pi,\pi]^2$ is a measure space automorphism of $(]-\pi,\pi]^2,$ $\Sigma_2, \mu_2)$.

(e) The map $(x, y) \mapsto (x -_{2\pi} y, y) :]-\pi,\pi]^2 \to]-\pi,\pi]^2$ is a measure space automorphism of $(]-\pi,\pi]^2,$ $\Sigma_2, \mu_2)$.

255O Convolutions on $]-\pi,\pi]$ Write μ for Lebesgue measure on $]-\pi,\pi]$.

(a) Let f and g be measurable complex-valued functions defined almost everywhere in $]-\pi,\pi]$. Write $f * g$ for the function defined by the formula

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x -_{2\pi} y)g(y)dy$$

whenever $x \in \,]-\pi,\pi]$ and the integral exists as a complex number. Then $f * g$ is the **convolution** of the functions f and g.

(b) If f and g are measurable complex-valued functions defined almost everywhere in $]-\pi,\pi]$, then $f * g = g * f$.

(c) Let f, g and h be measurable complex-valued functions defined almost everywhere in $]-\pi,\pi]$. Then
(i)

$$\int_{-\pi}^{\pi} h(x)(f * g)(x)dx = \int_{]-\pi,\pi]^2} h(x +_{2\pi} y)f(x)g(y)d(x, y)$$

whenever the right-hand side exists and is finite, provided that in the expression $h(x)(f * g)(x)$ we interpret the product as 0 if $h(x) = 0$ and $(f * g)(x)$ is undefined.

(ii) If, on the same interpretation of $|h(x)|(|f| * |g|)(x)$, the integral $\int_{-\pi}^{\pi} |h(x)|(|f| * |g|)(x)dx$ is finite, then $\int_{]-\pi,\pi]^2} h(x +_{2\pi} y)f(x)g(y)d(x, y)$ exists in $\mathbb{C}$, so again we shall have

$$\int_{-\pi}^{\pi} h(x)(f * g)(x)dx = \int_{]-\pi,\pi]^2} h(x +_{2\pi} y)f(x)g(y)d(x, y).$$

(d) If f, g are complex-valued functions which are integrable over $]-\pi,\pi]$, then $f * g$ is integrable, with

$$\int_{-\pi}^{\pi} f * g = \int_{-\pi}^{\pi} f \int_{-\pi}^{\pi} g, \qquad \int_{-\pi}^{\pi} |f * g| \le \int_{-\pi}^{\pi} |f| \int_{-\pi}^{\pi} |g|.$$

(e) Let f, g, h be complex-valued measurable functions defined almost everywhere in $]-\pi,\pi]$, such that $f * g$ and $g * h$ are also defined almost everywhere. Suppose that $x \in \,]-\pi,\pi]$ is such that one of $(|f| * (|g| * |h|))(x)$, $((|f| * |g|) * |h|)(x)$ is defined in $\mathbb{R}$. Then $f * (g * h)$ and $(f * g) * h$ are defined and equal at x.

(f) Suppose that $f \in \mathcal{L}^p_{\mathbb{C}}(\mu)$, $g \in \mathcal{L}^q_{\mathbb{C}}(\mu)$ where p, $q \in [1,\infty]$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $f*g$ is defined everywhere in $]-\pi,\pi]$, and $\sup_{x\in]-\pi,\pi]}|(f*g)(x)| \le \|f\|_p\|g\|_q$, interpreting $\|\;\|_\infty$ as ess sup$|\;|$.

Version of 6.8.15

256 Radon measures on $\mathbb{R}^r$

In the next section, and again in Chapters 27 and 28, we need to consider the principal class of measures on Euclidean spaces. For a proper discussion of this class, and the interrelationships between the measures and the topologies involved, we must wait until Volume 4. For the moment, therefore, I present definitions adapted to the case in hand, warning you that the correct generalizations are not quite obvious. I give the definition (256A) and a characterization (256C) of Radon measures on Euclidean spaces, and theorems on the construction of Radon measures as indefinite integrals (256E, 256J), as image measures (256G) and as product measures (256K). In passing I give a version of Lusin's theorem concerning measurable functions on Radon measure spaces (256F).

Throughout this section, r and s will be integers greater than or equal to 1.

256A Definitions Let ν be a measure on $\mathbb{R}^r$ and Σ its domain.

(a) ν is a **topological measure** if every open set belongs to Σ.

(b) ν is **locally finite** if every bounded set has finite outer measure.

(c) If ν is a topological measure, it is **inner regular with respect to the compact sets** if

$$\nu E = \sup\{\nu K : K \subseteq E \text{ is compact}\}$$

for every $E \in \Sigma$.

(d) ν is a **Radon measure** if it is a complete locally finite topological measure which is inner regular with respect to the compact sets.

256B Lemma Let ν be a Radon measure on $\mathbb{R}^r$, and Σ its domain.

(a) ν is σ-finite.

(b) For any $E \in \Sigma$ and any $\epsilon > 0$ there are a closed set $F \subseteq E$ and an open set $G \supseteq E$ such that $\nu(G\setminus F) \le \epsilon$.

(c) For every $E \in \Sigma$ there is a set $H \subseteq E$, expressible as the union of a sequence of compact sets, such that $\nu(E\setminus H) = 0$.

(d) Every continuous real-valued function on $\mathbb{R}^r$ is Σ-measurable.

(e) If $h : \mathbb{R}^r \to \mathbb{R}$ is continuous and has bounded support, then h is ν-integrable.

256C Theorem A measure ν on $\mathbb{R}^r$ is a Radon measure iff it is the completion of a locally finite measure defined on the σ-algebra of Borel subsets of $\mathbb{R}^r$.

256D Proposition If ν and ν' are two Radon measures on $\mathbb{R}^r$, the following are equiveridical:

(i) $\nu = \nu'$;

(ii) $\nu K = \nu' K$ for every compact set $K \subseteq \mathbb{R}^r$;

(iii) $\nu G = \nu' G$ for every open set $G \subseteq \mathbb{R}^r$;

(iv) $\int h\,d\nu = \int h\,d\nu'$ for every continuous function $h : \mathbb{R}^r \to \mathbb{R}$ with bounded support.

256E Theorem Let ν be a Radon measure on $\mathbb{R}^r$, with domain Σ, and f a non-negative Σ-measurable function defined on a ν-conegligible subset of $\mathbb{R}^r$. Suppose that f is **locally integrable** in the sense that $\int_E f d\nu < \infty$ for every bounded set E. Then the indefinite-integral measure ν' on $\mathbb{R}^r$ defined by saying that

$$\nu' E = \int_E f d\nu \text{ whenever } E \cap \{x : x \in \operatorname{dom} f,\ f(x) > 0\} \in \Sigma$$

is a Radon measure on $\mathbb{R}^r$.

256F Theorem Let ν be a Radon measure on $\mathbb{R}^r$, and Σ its domain. Let $f : D \to \mathbb{R}$ be a Σ-measurable function, where $D \subseteq \mathbb{R}^r$. Then for every $\epsilon > 0$ there is a closed set $F \subseteq \mathbb{R}^r$ such that $\nu(\mathbb{R}^r \setminus F) \le \epsilon$ and $f{\restriction}F$ is continuous.

256G Theorem Let ν be a Radon measure on $\mathbb{R}^r$, with domain Σ, and suppose that $\phi : \mathbb{R}^r \to \mathbb{R}^s$ is measurable in the sense that all its coordinates are Σ-measurable. If the image measure $\nu' = \nu\phi^{-1}$ is locally finite, it is a Radon measure.

256H Examples (a) Lebesgue measure on $\mathbb{R}^r$ is a Radon measure.

(b) A point-supported measure on $\mathbb{R}^r$ is a Radon measure iff it is locally finite.

(c) Recall that the Cantor set C is a closed Lebesgue negligible subset of $[0,1]$, and that the Cantor function is a non-decreasing continuous function $f : [0,1] \to [0,1]$ such that $f(0) = 0$, $f(1) = 1$ and f is constant on each of the intervals composing $[0,1] \setminus C$. It follows that if we set $g(x) = \frac{1}{2}(x + f(x))$ for $x \in [0,1]$, then $g : [0,1] \to [0,1]$ is a continuous permutation. Now extend g to a permutation $h : \mathbb{R} \to \mathbb{R}$ by setting $h(x) = x$ for $x \in \mathbb{R} \setminus [0,1]$.

Let ν_1 be the indefinite-integral measure defined from Lebesgue measure μ on $\mathbb{R}$ and the function $2\chi(h[C])$. ν_1 is a Radon measure, and $\nu_1 h[C] = \nu_1\mathbb{R} = 1$. Let ν be the measure $\nu_1(h^{-1})^{-1}$, that is, $\nu E = \nu_1 h[E]$ for just those $E \subseteq \mathbb{R}$ such that $h[E] \in \operatorname{dom}\nu_1$. Then ν is a Radon probability measure on $\mathbb{R}$, and $\nu C = 1$, $\nu(\mathbb{R} \setminus C) = \mu C = 0$.

256J Absolutely continuous Radon measures: Proposition Let ν be a Radon measure on $\mathbb{R}^r$, and write μ for Lebesgue measure on $\mathbb{R}^r$. Then the following are equiveridical:

(i) ν is an indefinite-integral measure over μ;

(ii) $\nu E = 0$ whenever E is a Borel subset of $\mathbb{R}^r$ and $\mu E = 0$.

In this case, if $g \in \mathcal{L}^0(\mu)$ and $\int_E g\,d\mu = \nu E$ for every Borel set $E \subseteq \mathbb{R}^r$, then g is a Radon-Nikodým derivative of ν with respect to μ.

256K Products: Theorem Let ν_1, ν_2 be Radon measures on $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively. Let λ be their c.l.d. product measure on $\mathbb{R}^r \times \mathbb{R}^s$. Then λ is a Radon measure.

***256M Proposition** Let ν be a Radon measure on $\mathbb{R}^r$, and D any subset of $\mathbb{R}^r$. Let Φ be a non-empty upwards-directed family of non-negative continuous functions from D to $\mathbb{R}$. For $x \in D$ set $g(x) = \sup_{f\in\Phi} f(x)$ in $[0,\infty]$. Then

(a) $g : D \to [0,\infty]$ is lower semi-continuous, therefore Borel measurable;

(b) $\int_D g\,d\nu = \sup_{f\in\Phi} \int_D f d\nu$.

Version of 14.8.13

257 Convolutions of measures

The ideas of this chapter can be brought together in a satisfying way in the theory of convolutions of Radon measures, which will be useful in §272 and again in §285. I give just the definition (257A) and the central property (257B) of the convolution of totally finite Radon measures, with a few corollaries and a note on the relation between convolution of functions and convolution of measures (257F).

257A Definition Let $r \ge 1$ be an integer and ν_1, ν_2 two totally finite Radon measures on $\mathbb{R}^r$. Let λ be the product measure on $\mathbb{R}^r \times \mathbb{R}^r$. Define $\phi : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^r$ by setting $\phi(x,y) = x + y$. The **convolution** of ν_1 and ν_2, $\nu_1 * \nu_2$, is the image measure $\lambda\phi^{-1}$; this is a Radon measure.

Note that if ν_1 and ν_2 are Radon probability measures, then λ and $\nu_1 * \nu_2$ are also probability measures.

257B Theorem Let $r \geq 1$ be an integer, and ν_1 and ν_2 two totally finite Radon measures on $\mathbb{R}^r$; let $\nu = \nu_1 * \nu_2$ be their convolution, and λ their product on $\mathbb{R}^r \times \mathbb{R}^r$. Then for any real-valued function h defined on a subset of $\mathbb{R}^r$,

$$\int h(x+y)\lambda(d(x,y)) \text{ exists} = \int h(x)\nu(dx)$$

if either integral is defined in $[-\infty, \infty]$.

257C Corollary Let $r \geq 1$ be an integer, and ν_1, ν_2 two totally finite Radon measures on $\mathbb{R}^r$; let $\nu = \nu_1 * \nu_2$ be their convolution, and λ their product on $\mathbb{R}^r \times \mathbb{R}^r$; write Λ for the domain of λ. Let h be a Λ-measurable function defined λ-almost everywhere in $\mathbb{R}^r$. Suppose that any one of the integrals

$$\iint |h(x+y)|\nu_1(dx)\nu_2(dy), \quad \iint |h(x+y)|\nu_2(dy)\nu_1(dx), \quad \int h(x+y)\lambda(d(x,y))$$

exists and is finite. Then h is ν-integrable and

$$\int h(x)\nu(dx) = \iint h(x+y)\nu_1(dx)\nu_2(dy) = \iint h(x+y)\nu_2(dy)\nu_1(dx).$$

257D Corollary If ν_1 and ν_2 are totally finite Radon measures on $\mathbb{R}^r$, then $\nu_1 * \nu_2 = \nu_2 * \nu_1$.

257E Corollary If ν_1, ν_2 and ν_3 are totally finite Radon measures on $\mathbb{R}^r$, then $(\nu_1 * \nu_2) * \nu_3 = \nu_1 * (\nu_2 * \nu_3)$.

257F Theorem Suppose that ν_1 and ν_2 are totally finite Radon measures on $\mathbb{R}^r$ which are indefinite-integral measures over Lebesgue measure μ. Then $\nu_1 * \nu_2$ also is an indefinite-integral measure over μ; if f_1 and f_2 are Radon-Nikodým derivatives of ν_1, ν_2 respectively, then $f_1 * f_2$ is a Radon-Nikodým derivative of $\nu_1 * \nu_2$.

Version of 10.11.06

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

251N Paragraph numbers in the second half of §251, referred to in editions of Volumes 3 and 4 up to and including 2006, and in Bogachev 07, have been changed, so that 251M-251S are now 251N-251T.

252Yf Exercise This exercise, referred to in the first edition of Volume 1, has been moved to 252Ym.

254Yh Exercise This exercise, referred to in the 2013 edition of Volume 4, has been moved to 254Ye.

References

Bogachev V.I. [07] *Measure theory.* Springer, 2007.