Version of 15.11.13

Chapter 24

Function spaces

The extraordinary power of Lebesgue's theory of integration is perhaps best demonstrated by its ability to provide structures relevant to questions quite different from those to which it was at first addressed. In this chapter I give the constructions, and elementary properties, of some of the fundamental spaces of functional analysis.

I do not feel called on here to justify the study of normed spaces; if you have not met them before, I hope that the introduction here will show at least that they offer a basis for a remarkable fusion of algebra and analysis. The fragments of the theory of metric spaces, normed spaces and general topology which we shall need are sketched in §§2A2-2A5. The principal 'function spaces' described in this chapter in fact combine three structural elements: they are (infinite-dimensional) linear spaces, they are metric spaces, with associated concepts of continuity and convergence, and they are ordered spaces, with corresponding notions of supremum and infimum. The interactions between these three types of structure provide an inexhaustible wealth of ideas. Furthermore, many of these ideas are directly applicable to a wide variety of problems in more or less applied mathematics, particularly in differential and integral equations, but more generally in any system with infinitely many degrees of freedom.

I have laid out the chapter with sections on L^0 (the space of equivalence classes of all real-valued measurable functions, in which all the other spaces of the chapter are embedded), L^1 (equivalence classes of integrable functions), L^∞ (equivalence classes of bounded measurable functions) and L^p (equivalence classes of pth-power-integrable functions). While ordinary functional analysis gives much more attention to the Banach spaces L^p for $1 \le p \le \infty$ than to L^0, from the special point of view of this book the space L^0 is at least as important and interesting as any of the others. Following these four sections, I return to a study of the standard topology on L^0, the topology of 'convergence in measure' (§245), and then to two linked sections on uniform integrability and weak compactness in L^1 (§§246-247).

There is a technical point here which must never be lost sight of. While it is customary and natural to call L^1, L^2 and the others 'function spaces', their elements are not in fact functions, but equivalence classes of functions. As you see from the language of the preceding paragraph, my practice is to scrupulously maintain the distinction; I give my reasons in the notes to §241.

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241 $\mathcal{L}^0$ and L^0

The chief aim of this chapter is to discuss the spaces L^1, L^∞ and L^p of the following three sections. However it will be convenient to regard all these as subspaces of a larger space L^0 of equivalence classes of (virtually) measurable functions, and I have collected in this section the basic facts concerning the ordered linear space L^0.

It is almost the first principle of measure theory that sets of measure zero can often be ignored; the phrase 'negligible set' itself asserts this principle. Accordingly, two functions which agree almost everywhere may often (not always!) be treated as identical. A suitable expression of this idea is to form the space of equivalence classes of functions, saying that two functions are equivalent if they agree on a conegligible set. This is the basis of all the constructions of this chapter. It is a remarkable fact that the spaces of equivalence classes so constructed are actually better adapted to certain problems than the spaces of functions from which they are derived, so that once the technique has been mastered it is easier to do one's thinking in the more abstract spaces.

241A The space $\mathcal{L}^0$: Definition Let (X, Σ, μ) be a measure space. I write $\mathcal{L}^0$, or $\mathcal{L}^0(\mu)$, for the space of real-valued functions f defined on conegligible subsets of X which are virtually measurable.

241B Basic properties(a) A constant real-valued function defined almost everywhere in X belongs to $\mathcal{L}^0$.

(b) $f + g \in \mathcal{L}^0$ for all f, $g \in \mathcal{L}^0$.

(c) $cf \in \mathcal{L}^0$ for all $f \in \mathcal{L}^0$, $c \in \mathbb{R}$.

(d) $f \times g \in \mathcal{L}^0$ for all f, $g \in \mathcal{L}^0$.

(e) If $f \in \mathcal{L}^0$ and $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $hf \in \mathcal{L}^0$.

(f) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f = \lim_{n\to\infty} f_n$ is defined almost everywhere in X, then $f \in \mathcal{L}^0$.

(g) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f = \sup_{n\in\mathbb{N}} f_n$ is defined almost everywhere in X, then $f \in \mathcal{L}^0$.

(h) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f = \inf_{n\in\mathbb{N}} f_n$ is defined almost everywhere in X, then $f \in \mathcal{L}^0$.

(i) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f = \limsup_{n\to\infty} f_n$ is defined almost everywhere in X, then $f \in \mathcal{L}^0$.

(j) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f = \liminf_{n\to\infty} f_n$ is defined almost everywhere in X, then $f \in \mathcal{L}^0$.

(k) $\mathcal{L}^0$ is just the set of real-valued functions, defined on subsets of X, which are equal almost everywhere to some Σ-measurable function from X to $\mathbb{R}$.

241C The space L^0: Definition Let (X, Σ, μ) be any measure space. Then $=_{\text{a.e.}}$ is an equivalence relation on $\mathcal{L}^0$. Write L^0, or $L^0(\mu)$, for the set of equivalence classes in $\mathcal{L}^0$ under $=_{\text{a.e.}}$. For $f \in \mathcal{L}^0$, write $f^\bullet$ for its equivalence class in L^0.

241D The linear structure of L^0 Let (X, Σ, μ) be any measure space, and set $\mathcal{L}^0 = \mathcal{L}^0(\mu)$, $L^0 = L^0(\mu)$.

(a) If f_1, f_2, g_1, $g_2 \in \mathcal{L}^0$, $f_1 =_{\text{a.e.}} f_2$ and $g_1 =_{\text{a.e.}} g_2$ then $f_1 + g_1 =_{\text{a.e.}} f_2 + g_2$. Accordingly we may define addition on L^0 by setting $f^\bullet + g^\bullet = (f + g)^\bullet$ for all f, $g \in \mathcal{L}^0$.

(b) If f_1, $f_2 \in \mathcal{L}^0$ and $f_1 =_{\text{a.e.}} f_2$, then $cf_1 =_{\text{a.e.}} cf_2$ for every $c \in \mathbb{R}$. Accordingly we may define scalar multiplication on L^0 by setting $c \cdot f^\bullet = (cf)^\bullet$ for all $f \in \mathcal{L}^0$ and $c \in \mathbb{R}$.

(c) L^0 is a linear space over $\mathbb{R}$, with zero $\mathbf{0}^\bullet$, where $\mathbf{0}$ is the function with domain X and constant value 0, and negatives $-(f^\bullet) = (-f)^\bullet$.

241E The order structure of L^0 Let (X, Σ, μ) be any measure space and set $\mathcal{L}^0 = \mathcal{L}^0(\mu)$, $L^0 = L^0(\mu)$.

(a) If f_1, f_2, g_1, $g_2 \in \mathcal{L}^0$, $f_1 =_{\text{a.e.}} f_2$, $g_1 =_{\text{a.e.}} g_2$ and $f_1 \le_{\text{a.e.}} g_1$, then $f_2 \le_{\text{a.e.}} g_2$. Accordingly we may define a relation $\le$ on L^0 by saying that $f^\bullet \le g^\bullet$ iff $f \le_{\text{a.e.}} g$.

(b) $\le$ is a partial order on L^0.

(c) L^0, with $\le$, is a **partially ordered linear space**, that is, a (real) linear space with a partial order $\le$ such that

if $u \le v$ then $u + w \le v + w$ for every w,
if $0 \le u$ then $0 \le cu$ for every $c \ge 0$.

(d) L^0 is a **Riesz space** or **vector lattice**, that is, a partially ordered linear space such that $u \vee v = \sup\{u, v\}$ and $u \wedge v = \inf\{u, v\}$ are defined for all u, $v \in L^0$.

(e) In particular, for any $u \in L^0$ we can speak of $|u| = u \vee (-u)$; if $f \in \mathcal{L}^0$ then $|f^\bullet| = |f|^\bullet$.
If $c \in \mathbb{R}$ then

$$|cu| = |c||u|, \quad u \vee v = \frac{1}{2}(u + v + |u - v|),$$

$$u \wedge v = \frac{1}{2}(u + v - |u - v|), \quad |u + v| \le |u| + |v|$$

for all u, $v \in L^0$.

(f) If f is a real-valued function, set $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$ for $x \in \operatorname{dom} f$, so that

$$f = f^+ - f^-, \quad |f| = f^+ + f^- = f^+ \vee f^-,$$

all these functions being defined on $\operatorname{dom} f$. In L^0, the corresponding operations are $u^+ = u \vee 0$, $u^- = (-u) \vee 0$, and we have

$$u = u^+ - u^-, \quad |u| = u^+ + u^- = u^+ \vee u^-, \quad u^+ \wedge u^- = 0.$$

(g) If $u \ge 0$ in L^0, then there is an $f \ge 0$ in $\mathcal{L}^0$ such that $f^\bullet = u$.

241F Riesz spaces (a) A Riesz space U is **Archimedean** if whenever $u \in U$, $u > 0$ and $v \in U$, there is an $n \in \mathbb{N}$ such that $nu \not\le v$.

(b) A Riesz space U is **Dedekind σ-complete** if every non-empty countable set $A \subseteq U$ which is bounded above has a least upper bound in U.

(c) A Riesz space U is **Dedekind complete** (or **order complete**, or **complete**) if every non-empty set $A \subseteq U$ which is bounded above in U has a least upper bound in U.

241G Theorem Let (X, Σ, μ) be a measure space. Set $L^0 = L^0(\mu)$.
(a) L^0 is Archimedean and Dedekind σ-complete.
(b) If (X, Σ, μ) is semi-finite, then L^0 is Dedekind complete iff (X, Σ, μ) is localizable.

241H The multiplicative structure of L^0 Let (X, Σ, μ) be any measure space; write $L^0 = L^0(\mu)$, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$.

(a) If f_1, f_2, g_1, $g_2 \in \mathcal{L}^0$, $f_1 =_{\text{a.e.}} f_2$ and $g_1 =_{\text{a.e.}} g_2$ then $f_1 \times g_1 =_{\text{a.e.}} f_2 \times g_2$. Accordingly we may define multiplication on L^0 by setting $f^\bullet \times g^\bullet = (f \times g)^\bullet$ for all f, $g \in \mathcal{L}^0$.

(b) For all u, v, $w \in L^0$ and $c \in \mathbb{R}$,
$u \times (v \times w) = (u \times v) \times w$,
$u \times e = e \times u = u$,
where $e = \chi X^\bullet$ is the equivalence class of the function with constant value 1,
$c(u \times v) = cu \times v = u \times cv$,
$u \times (v + w) = (u \times v) + (u \times w)$,
$(u + v) \times w = (u \times w) + (v \times w)$,
$u \times v = v \times u$,
$|u \times v| = |u| \times |v|$,
$u \times v = 0$ iff $|u| \wedge |v| = 0$,
$|u| \le |v|$ iff there is a w such that $|w| \le e$ and $u = v \times w$.

241I The action of Borel functions on L^0 Let (X, Σ, μ) be a measure space and $h : \mathbb{R} \to \mathbb{R}$ a Borel measurable function. Then we have a function $\bar{h} : L^0 \to L^0$ defined by setting $\bar{h}(f^\bullet) = (hf)^\bullet$ for every $f \in \mathcal{L}^0$.

241J Complex L^0 Let (X,Σ,μ) be a measure space.

(a) We may write $\mathcal{L}^0_{\mathbb{C}}=\mathcal{L}^0_{\mathbb{C}}(\mu)$ for the space of complex-valued functions f such that $\operatorname{dom} f$ is a conegligible subset of X and there is a conegligible subset $E\subseteq X$ such that $f\restriction E$ is measurable; that is, such that the real and imaginary parts of f both belong to $\mathcal{L}^0(\mu)$. $L^0_{\mathbb{C}}=L^0_{\mathbb{C}}(\mu)$ will be the space of equivalence classes in $\mathcal{L}^0_{\mathbb{C}}$ under the equivalence relation $=_{\text{a.e.}}$.

(b) Using just the same formulae as in 241D, it is easy to describe addition and scalar multiplication rendering $L^0_{\mathbb{C}}$ a linear space over $\mathbb{C}$. We can identify a 'real part', being

$$\{f^\bullet : f\in\mathcal{L}^0_{\mathbb{C}} \text{ is real a.e.}\},$$

identifiable with the real linear space L^0, and corresponding maps $u\mapsto\mathcal{R}\mathrm{e}(u)$, $u\mapsto\mathcal{I}\mathrm{m}(u):L^0_{\mathbb{C}}\to L^0$ such that $u=\mathcal{R}\mathrm{e}(u)+i\,\mathcal{I}\mathrm{m}(u)$ for every u. Moreover, we have a notion of 'modulus', writing

$$|f^\bullet|=|f|^\bullet\in L^0 \text{ for every } f\in\mathcal{L}^0_{\mathbb{C}},$$

satisfying the basic relations $|cu|=|c||u|$, $|u+v|\le|u|+|v|$ for u, $v\in L^0_{\mathbb{C}}$ and $c\in\mathbb{C}$. We do still have a multiplication on $L^0_{\mathbb{C}}$, for which all the formulae in 241H are still valid.

(c) For any $u\in L^0_{\mathbb{C}}$, $|u|$ is the supremum in L^0 of $\{\mathcal{R}\mathrm{e}(\zeta u):\zeta\in\mathbb{C},\ |\zeta|=1\}$.

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242 L^1

While the space L^0 treated in the previous section is of very great intrinsic interest, its chief use in the elementary theory is as a space in which some of the most important spaces of functional analysis are embedded. In the next few sections I introduce these one at a time.

The first is the space L^1 of equivalence classes of integrable functions. The importance of this space is not only that it offers a language in which to express those many theorems about integrable functions which do not depend on the differences between two functions which are equal almost everywhere. It can also appear as the natural space in which to seek solutions to a wide variety of integral equations, and as the completion of a space of continuous functions.

242A The space L^1 Let (X,Σ,μ) be any measure space.

(a) Let $\mathcal{L}^1=\mathcal{L}^1(\mu)$ be the set of real-valued functions, defined on subsets of X, which are integrable over X. Then $\mathcal{L}^1\subseteq\mathcal{L}^0=\mathcal{L}^0(\mu)$, and, for $f\in\mathcal{L}^0$, we have $f\in\mathcal{L}^1$ iff there is a $g\in\mathcal{L}^1$ such that $|f|\le_{\text{a.e.}} g$; if $f\in\mathcal{L}^1$, $g\in\mathcal{L}^0$ and $f=_{\text{a.e.}} g$, then $g\in\mathcal{L}^1$.

(b) Let $L^1=L^1(\mu)\subseteq L^0=L^0(\mu)$ be the set of equivalence classes of members of $\mathcal{L}^1$. If f, $g\in\mathcal{L}^1$ and $f=_{\text{a.e.}} g$ then $\int f=\int g$. Accordingly we may define a functional $\int$ on L^1 by writing $\int f^\bullet=\int f$ for every $f\in\mathcal{L}^1$.

(c) $\int_A u$ for $u\in L^1$, $A\subseteq X$ is defined by saying that $\int_A f^\bullet=\int_A f$ for every $f\in\mathcal{L}^1$.
If $E\in\Sigma$ and $u\in L^1$ then $\int_E u=\int u\times(\chi E)^\bullet$.

(d) If $u\in L^1$, there is a Σ-measurable, μ-integrable function $f:X\to\mathbb{R}$ such that $f^\bullet=u$.

242B Theorem Let (X,Σ,μ) be any measure space. Then $L^1(\mu)$ is a linear subspace of $L^0(\mu)$ and $\int:L^1\to\mathbb{R}$ is a linear functional.

242C The order structure of L^1 Let (X,Σ,μ) be any measure space.

(a) $L^1 = L^1(\mu)$ has an order structure derived from that of $L^0 = L^0(\mu)$. L^1 is a partially ordered linear space.

If u, $v \in L^1$ and $u \le v$ then $\int u \le \int v$.

(b) If $u \in L^0$, $v \in L^1$ and $|u| \le |v|$ then $u \in L^1$.

(c) In particular, $|u| \in L^1$ whenever $u \in L^1$, and

$$|\int u| = \max(\int u, \int(-u)) \le \int |u|.$$

(d) L^1 is a Riesz space.

(e) Note that if $u \in L^1$, then $u \ge 0$ iff $\int_E u \ge 0$ for every $E \in \Sigma$. If u, $v \in L^1$ and $\int_E u \le \int_E v$ for every $E \in \Sigma$, then $u \le v$. If u, $v \in L^1$ and $\int_E u = \int_E v$ for every $E \in \Sigma$, then $u = v$.

(f) If $u \ge 0$ in L^1, there is a non-negative $f \in \mathcal{L}^1$ such that $f^\bullet = u$.

242D The norm of L^1 Let (X, Σ, μ) be any measure space.

(a) For $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ I write $\|f\|_1 = \int |f| \in [0, \infty[$. For $u \in L^1 = L^1(\mu)$ set $\|u\|_1 = \int |u|$, so that $\|f^\bullet\|_1 = \|f\|_1$ for every $f \in \mathcal{L}^1$. Then $\| \|_1$ is a norm on L^1.

(b) L^1, with $\| \|_1$, is a normed space and $\int : L^1 \to \mathbb{R}$ is a linear operator; $\| \int \| \le 1$.

(c) If u, $v \in L^1$ and $|u| \le |v|$, then

$$\|u\|_1 = \int |u| \le \int |v| = \|v\|_1.$$

In particular, $\|u\|_1 = \||u|\|_1$ for every $u \in L^1$.

(d) If u, $v \in L^1$ and u, $v \ge 0$, then

$$\|u + v\|_1 = \|u\|_1 + \|v\|_1.$$

(e) The set $(L^1)^+ = \{u : u \ge 0\}$ is closed in L^1.

242E Lemma Let (X, Σ, μ) be a measure space and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of μ-integrable real-valued functions such that $\sum_{n=0}^\infty \int |f_n| < \infty$. Then $f = \sum_{n=0}^\infty f_n$ is integrable and

$$\int f = \sum_{n=0}^\infty \int f_n, \quad \int |f| \le \sum_{n=0}^\infty \int |f_n|.$$

242F Theorem For any measure space (X, Σ, μ), $L^1(\mu)$ is complete under its norm $\| \|_1$.

242G Definition A **Banach lattice** is a Riesz space U together with a norm $\| \|$ on U such that (i) $\|u\| \le \|v\|$ whenever u, $v \in U$ and $|u| \le |v|$ (ii) U is complete under $\| \|$.

242H L^1 as a Riesz space: Theorem Let (X, Σ, μ) be any measure space. Then $L^1 = L^1(\mu)$ is Dedekind complete.

242I The Radon-Nikodým Theorem Let (X, Σ, μ) be a measure space. Then there is a canonical bijection between $L^1 = L^1(\mu)$ and the set of truly continuous additive functionals $\nu : \Sigma \to \mathbb{R}$, given by the formula

$$\nu F = \int_F u \text{ for } F \in \Sigma,\ u \in L^1.$$

242J Conditional expectations revisited(a) Let (X,Σ,μ) be a measure space, and T a σ-subalgebra of Σ. Then $(X,\mathrm{T},\mu\restriction\mathrm{T})$ is a measure space, and $\mathcal{L}^0(\mu\restriction\mathrm{T})\subseteq\mathcal{L}^0(\mu)$; if f, $g\in\mathcal{L}^0(\mu\restriction\mathrm{T})$, then $f=g$ $(\mu\restriction\mathrm{T})$-a.e. iff $f=g$ μ-a.e.

Accordingly we have a canonical map $S:L^0(\mu\restriction\mathrm{T})\to L^0(\mu)$ defined by saying that if $u\in L^0(\mu\restriction\mathrm{T})$ is the equivalence class of $f\in\mathcal{L}^0(\mu\restriction\mathrm{T})$, then Su is the equivalence class of f in $L^0(\mu)$. S is linear, injective and order-preserving, and $|Su|=S|u|$, $S(u\vee v)=Su\vee Sv$ and $S(u\times v)=Su\times Sv$ for u, $v\in L^0(\mu\restriction\mathrm{T})$.

(b) Next, if $f\in\mathcal{L}^1(\mu\restriction\mathrm{T})$, then $f\in\mathcal{L}^1(\mu)$ and $\int f d\mu=\int f d(\mu\restriction\mathrm{T})$; so $Su\in L^1(\mu)$ and $\|Su\|_1=\|u\|_1$ for every $u\in L^1(\mu\restriction\mathrm{T})$.

Observe also that every member of $L^1(\mu)\cap S[L^0(\mu\restriction\mathrm{T})]$ is actually in $S[L^1(\mu\restriction\mathrm{T})]$.

This means that $S:L^1(\mu\restriction\mathrm{T})\to L^1(\mu)\cap S[L^0(\mu\restriction\mathrm{T})]$ is a bijection.

(c) Now suppose that $\mu X=1$, so that (X,Σ,μ) is a probability space. If g is a conditional expectation of f and $f_1=f$ μ-a.e. then g is a conditional expectation of f_1; and if g, g_1 are conditional expectations of f on T then $g=g_1$ $\mu\restriction\mathrm{T}$-a.e.

(d) This means that we have an operator $P:L^1(\mu)\to L^1(\mu\restriction\mathrm{T})$ defined by saying that $P(f^\bullet)=g^\bullet$ whenever $g\in\mathcal{L}^1(\mu\restriction\mathrm{T})$ is a conditional expectation of $f\in\mathcal{L}^1(\mu)$ on T.

(e) P is linear and order-preserving. Consequently

$$|Pu|=Pu\vee(-Pu)=Pu\vee P(-u)\le P|u|$$

for every $u\in L^1(\mu)$. Finally, P is a bounded linear operator, with norm 1.

(f) We may legitimately regard $Pu\in L^1(\mu\restriction\mathrm{T})$ as 'the' conditional expectation of $u\in L^1(\mu)$ on T; P is the **conditional expectation operator**.

(g) If $u\in L^1(\mu\restriction\mathrm{T})$, then $Su\in L^1(\mu)$, as in (b); now $PSu=u$. Consequently $SPSP=SP:L^1(\mu)\to L^1(\mu)$.

242K Theorem Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\phi:\mathbb{R}\to\mathbb{R}$ be a convex function and $\bar\phi:L^0(\mu)\to L^0(\mu)$ the corresponding operator defined by setting $\bar\phi(f^\bullet)=(\phi f)^\bullet$. If $P:L^1(\mu)\to L^1(\mu\restriction\mathrm{T})$ is the conditional expectation operator, then $\bar\phi(Pu)\le P(\bar\phi u)$ whenever $u\in L^1(\mu)$ is such that $\bar\phi(u)\in L^1(\mu)$.

242L Proposition Let (X,Σ,μ) be a probability space, and T a σ-subalgebra of Σ. Let $P:L^1(\mu)\to L^1(\mu\restriction\mathrm{T})$ be the corresponding conditional expectation operator. If $u\in L^1=L^1(\mu)$ and $v\in L^0(\mu\restriction\mathrm{T})$, then $u\times v\in L^1$ iff $P|u|\times v\in L^1$, and in this case $P(u\times v)=Pu\times v$; in particular, $\int u\times v=\int Pu\times v$.

242M L^1 as a completion:Proposition Let (X,Σ,μ) be any measure space, and write $\mathcal{S}$ for the space of μ-simple functions on X. Then

(a) whenever f is a μ-integrable real-valued function and $\epsilon>0$, there is an $h\in\mathcal{S}$ such that $\int|f-h|\le\epsilon$;

(b) $S=\{f^\bullet:f\in\mathcal{S}\}$ is a dense linear subspace of $L^1=L^1(\mu)$.

242N Definition If f is a real- or complex-valued function defined on a subset of $\mathbb{R}^r$, say that the **support** of f is $\overline{\{x:x\in\operatorname{dom}f,\ f(x)\ne0\}}$.

242O Theorem Let X be any subset of $\mathbb{R}^r$, where $r\ge1$, and let μ be Lebesgue measure on X. Write C_k for the space of bounded continuous functions $f:\mathbb{R}^r\to\mathbb{R}$ which have bounded support, and S_0 for the space of linear combinations of functions of the form χI where $I\subseteq\mathbb{R}^r$ is a bounded half-open interval. Then

(a) whenever $f\in\mathcal{L}^1=\mathcal{L}^1(\mu)$ and $\epsilon>0$, there are $g\in C_k$, $h\in S_0$ such that $\int_X|f-g|\le\epsilon$ and $\int_X|f-h|\le\epsilon$;

(b) $\{(g\restriction X)^\bullet:g\in C_k\}$ and $\{(h\restriction X)^\bullet:h\in S_0\}$ are dense linear subspaces of $L^1=L^1(\mu)$.

242P Complex L^1(a) For $f \in \mathcal{L}^0_{\mathbb{C}}$,

$$f \in \mathcal{L}^1_{\mathbb{C}} \iff |f| \in \mathcal{L}^1 \iff \mathcal{R}\mathrm{e}(f),\, \mathcal{I}\mathrm{m}(f) \in \mathcal{L}^1.$$

Consequently, for $u \in L^0_{\mathbb{C}}$,

$$u \in L^1_{\mathbb{C}} \iff |u| \in L^1 \iff \mathcal{R}\mathrm{e}(u),\, \mathcal{I}\mathrm{m}(u) \in L^1.$$

(b) $L^1_{\mathbb{C}}$ is complete under $\| \|_1$.

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243 L^∞

The second of the classical Banach spaces of measure theory which I treat is the space L^∞. As will appear below, L^∞ is the polar companion of L^1, the linked opposite; for 'ordinary' measure spaces it is actually the dual of L^1 (243F-243G).

243A Definitions Let (X,Σ,μ) be any measure space. Let $\mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$ be the set of functions $f \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$ which are **essentially bounded**, that is, such that there is some $M \ge 0$ such that $\{x : x \in \operatorname{dom} f,\, |f(x)| \le M\}$ is conegligible, and write

$$L^\infty = L^\infty(\mu) = \{f^\bullet : f \in \mathcal{L}^\infty(\mu)\} \subseteq L^0(\mu).$$

Note that $\mathcal{L}^\infty = \{f : f \in \mathcal{L}^0,\, f^\bullet \in L^\infty\}$.

243B Theorem Let (X,Σ,μ) be any measure space. Then

(a) $L^\infty = L^\infty(\mu)$ is a linear subspace of $L^0 = L^0(\mu)$.

(b) If $u \in L^\infty$, $v \in L^0$ and $|v| \le |u|$ then $v \in L^\infty$. Consequently $|u|$, $u \vee v$, $u \wedge v$, u^+ and u^- belong to L^∞ for all u, $v \in L^\infty$.

(c) Writing $e = \chi X^\bullet$, the equivalence class in L^0 of the constant function with value 1, then an element u of L^0 belongs to L^∞ iff there is an $M \ge 0$ such that $|u| \le Me$.

(d) If u, $v \in L^\infty$ then $u \times v \in L^\infty$.

(e) If $u \in L^\infty$ and $v \in L^1 = L^1(\mu)$ then $u \times v \in L^1$.

243C The order structure of L^∞ Let (X,Σ,μ) be any measure space. Then $L^\infty = L^\infty(\mu)$, being a linear subspace of $L^0 = L^0(\mu)$, inherits a partial order which renders it a partially ordered linear space. Because $|u| \in L^\infty$ whenever $u \in L^\infty$, $u \wedge v$ and $u \vee v$ belong to L^∞ whenever u, $v \in L^\infty$, and L^∞ is a Riesz space.

L^∞ has an **order unit** e with the property that

$$\text{for every } u \in L^\infty \text{ there is an } M \ge 0 \text{ such that } |u| \le Me.$$

243D The norm of L^∞ Let (X,Σ,μ) be any measure space.

(a) For $f \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$, the **essential supremum** of $|f|$ is

$$\operatorname{ess\,sup}|f| = \inf\{M : M \ge 0,\, \{x : x \in \operatorname{dom} f,\, |f(x)| \le M\} \text{ is conegligible}\}.$$

Then $|f| \le \operatorname{ess\,sup}|f|$ a.e.

(b) If f, $g \in \mathcal{L}^\infty$ and $f =_{\text{a.e.}} g$, then $\operatorname{ess\,sup}|f| = \operatorname{ess\,sup}|g|$. Accordingly we may define a functional $\| \|_\infty$ on $L^\infty = L^\infty(\mu)$ by setting $\|u\|_\infty = \operatorname{ess\,sup}|f|$ whenever $u = f^\bullet$.

(c) For any $u \in L^\infty$, $\|u\|_\infty = \min\{\gamma : |u| \le \gamma e\}$, where $e = \chi X^\bullet \in L^\infty$. $\| \|_\infty$ is a norm on L^∞.

(d) Note also that if $u \in L^0$, $v \in L^\infty$ and $|u| \le |v|$ then $|u| \le \|v\|_\infty e$ so $u \in L^\infty$ and $\|u\|_\infty \le \|v\|_\infty$; similarly,

$$\|u \times v\|_\infty \le \|u\|_\infty \|v\|_\infty, \quad \|u \vee v\|_\infty \le \max(\|u\|_\infty, \|v\|_\infty)$$

for all u, $v \in L^\infty$. L^∞ is a commutative Banach algebra.

(e)

$$|\int u\times v|\le\int|u\times v|=\|u\times v\|_1\le\|u\|_1\|v\|_\infty$$

whenever $u\in L^1$ and $v\in L^\infty$.

(f) Observe that if u, v are non-negative members of L^∞ then

$$\|u\vee v\|_\infty=\max(\|u\|_\infty,\|v\|_\infty).$$

243E Theorem For any measure space (X,Σ,μ), $L^\infty=L^\infty(\mu)$ is a Banach lattice under $\|\;\|_\infty$.

243F The duality between L^∞ and L^1 Let (X,Σ,μ) be any measure space.

(b) We have a bounded linear operator T from L^∞ to the normed space dual $(L^1)^*$ of L^1, given by writing

$$(Tv)(u)=\int u\times v\text{ for all }u\in L^1,\,v\in L^\infty.$$

(c) We have a linear operator $T':L^1\to(L^\infty)^*$, given by writing

$$(T'u)(v)=\int u\times v\text{ for all }u\in L^1,\,v\in L^\infty,$$

and $\|T'\|$ also is at most 1.

243G Theorem Let (X,Σ,μ) be a measure space, and $T:L^\infty(\mu)\to(L^1(\mu))^*$ the canonical map described in 243F. Then

(a) T is injective iff (X,Σ,μ) is semi-finite, and in this case is norm-preserving;

(b) T is bijective iff (X,Σ,μ) is localizable, and in this case is a normed space isomorphism.

243H Theorem Let (X,Σ,μ) be a measure space.

(a) $L^\infty(\mu)$ is Dedekind σ-complete.

(b) If μ is localizable, $L^\infty(\mu)$ is Dedekind complete.

243I A dense subspace of L^∞: Proposition Let (X,Σ,μ) be a measure space.

(a) Write $\mathcal{S}$ for the space of 'Σ-simple' functions on X, that is, the space of functions from X to $\mathbb{R}$ expressible as $\sum_{k=0}^n a_k\chi E_k$ where $a_k\in\mathbb{R}$ and $E_k\in\Sigma$ for every $k\le n$. Then for every $f\in\mathcal{L}^\infty=\mathcal{L}^\infty(\mu)$ and every $\epsilon>0$, there is a $g\in\mathcal{S}$ such that $\operatorname{ess\,sup}|f-g|\le\epsilon$.

(b) $S=\{f^\bullet:f\in\mathcal{S}\}$ is a $\|\;\|_\infty$-dense linear subspace of $L^\infty=L^\infty(\mu)$.

(c) If (X,Σ,μ) is totally finite, then $\mathcal{S}$ is the space of μ-simple functions, so S becomes just the space of equivalence classes of simple functions.

243J Conditional expectations(a) If (X,Σ,μ) is any measure space, and T is a σ-subalgebra of Σ, then the canonical embedding $S:L^0(\mu\restriction\mathrm{T})\to L^0(\mu)$ embeds $L^\infty(\mu\restriction\mathrm{T})$ as a subspace of $L^\infty(\mu)$, and $\|Su\|_\infty=\|u\|_\infty$ for every $u\in L^\infty(\mu\restriction\mathrm{T})$. We can identify $L^\infty(\mu\restriction\mathrm{T})$ with its image in $L^\infty(\mu)$.

(b) Now suppose that $\mu X=1$, and let $P:L^1(\mu)\to L^1(\mu\restriction\mathrm{T})$ be the conditional expectation operator. Then $L^\infty(\mu)$ is a linear subspace of $L^1(\mu)$. Setting $e=\chi X^\bullet\in L^\infty(\mu)$,

$$Pe=\chi X^\bullet\in L^\infty(\mu\restriction\mathrm{T}).$$

$P\restriction L^\infty(\mu):L^\infty(\mu)\to L^\infty(\mu\restriction\mathrm{T})$ is an operator of norm 1.

If $u\in L^\infty(\mu\restriction\mathrm{T})$, then $Pu=u$; so $P[L^\infty]$ is the whole of $L^\infty(\mu\restriction\mathrm{T})$.

243K Complex L^∞ Let $\mathcal{L}^\infty_{\mathbb{C}}$ be

$$\{f:f\in\mathcal{L}^0_{\mathbb{C}},\ \operatorname{ess\,sup}|f|<\infty\}=\{f:\mathcal{R}\mathrm{e}(f)\in\mathcal{L}^\infty,\,\mathcal{I}\mathrm{m}(f)\in\mathcal{L}^\infty\}.$$

Then

$$L^\infty_{\mathbb{C}} = \{f^\bullet : f \in \mathcal{L}^\infty_{\mathbb{C}}\} = \{u : u \in L^0_{\mathbb{C}},\, \mathcal{R}\mathrm{e}(u) \in L^\infty,\, \mathcal{I}\mathrm{m}(u) \in L^\infty\}.$$

Setting

$$\|u\|_\infty = \||u|\|_\infty,$$

we have a norm on $L^\infty_{\mathbb{C}}$ rendering it a Banach space. $u \times v \in L^\infty_{\mathbb{C}}$ and $\|u \times v\|_\infty \le \|u\|_\infty \|v\|_\infty$ for all u, $v \in L^\infty_{\mathbb{C}}$.

We now have a duality between $L^1_{\mathbb{C}}$ and $L^\infty_{\mathbb{C}}$ giving rise to a linear operator $T : L^\infty_{\mathbb{C}} \to (L^1_{\mathbb{C}})^*$ of norm at most 1, defined by the formula

$$(Tv)(u) = \int u \times v \text{ for every } u \in L^1,\, v \in L^\infty.$$

T is injective iff the underlying measure space is semi-finite, and is a bijection iff the underlying measure space is localizable. T is norm-preserving when it is injective.

Version of 6.3.09

244 L^p

Continuing with our tour of the classical Banach spaces, we come to the L^p spaces for $1 < p < \infty$. The case $p = 2$ is more important than all the others put together, and it would be reasonable, perhaps even advisable, to read this section first with this case alone in mind. But the other spaces provide instructive examples and remain a basic part of the education of any functional analyst.

244A Definitions Let (X, Σ, μ) be any measure space, and $p \in]1, \infty[$. Write $\mathcal{L}^p = \mathcal{L}^p(\mu)$ for the set of functions $f \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$ such that $|f|^p$ is integrable, and $L^p = L^p(\mu)$ for $\{f^\bullet : f \in \mathcal{L}^p\} \subseteq L^0 = L^0(\mu)$.

$\mathcal{L}^p = \{f : f \in \mathcal{L}^0,\, f^\bullet \in L^p\}$.

244B Theorem Let (X, Σ, μ) be any measure space, and $p \in [1, \infty]$.

(a) $L^p = L^p(\mu)$ is a linear subspace of $L^0 = L^0(\mu)$.

(b) If $u \in L^p$, $v \in L^0$ and $|v| \le |u|$, then $v \in L^p$. Consequently $|u|$, $u \vee v$ and $u \wedge v$ belong to L^p for all u, $v \in L^p$.

244C The order structure of L^p Let (X, Σ, μ) be any measure space, and $p \in [1, \infty]$. Then the partial order inherited from $L^0(\mu)$ makes $L^p(\mu)$ a Riesz space.

244D The norm of L^p Let (X, Σ, μ) be a measure space, and $p \in]1, \infty[$.

(a) For $f \in \mathcal{L}^p = \mathcal{L}^p(\mu)$, set $\|f\|_p = (\int |f|^p)^{1/p}$. If f, $g \in \mathcal{L}^p$ and $f =_{\text{a.e.}} g$ then $|f|^p =_{\text{a.e.}} |g|^p$ so $\|f\|_p = \|g\|_p$. Accordingly we may define $\| \,\|_p : L^p = L^p(\mu) \to [0, \infty[$ by writing $\|f^\bullet\|_p = \|f\|_p$ for every $f \in \mathcal{L}^p$.

(b) $\|cu\|_p = |c|\|u\|_p$ for all $u \in L^p$ and $c \in \mathbb{R}$, and if $\|u\|_p = 0$ then $u = 0$.

(c) If $|u| \le |v|$ in L^p then $\|u\|_p \le \|v\|_p$.

244E Lemma Suppose (X, Σ, μ) is a measure space, and that p, $q \in]1, \infty[$ are such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) $ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$ for all real a, $b \ge 0$.

(b)(i) $f \times g$ is integrable and

$$|\int f \times g| \le \int |f \times g| \le \|f\|_p \|g\|_q$$

for all $f \in \mathcal{L}^p = \mathcal{L}^p(\mu)$, $g \in \mathcal{L}^q = \mathcal{L}^q(\mu)$;

(ii) $u \times v \in L^1 = L^1(\mu)$ and

$$|\int u \times v| \le \|u \times v\|_1 \le \|u\|_p \|v\|_q$$

for all $u \in L^p = L^p(\mu)$, $v \in L^q = L^q(\mu)$.

244F Proposition Let (X,Σ,μ) be a measure space and $p\in\,]1,\infty[$. Set $q=p/(p-1)$, so that $\frac{1}{p}+\frac{1}{q}=1$.

(a) For every $u\in L^p=L^p(\mu)$, $\|u\|_p=\max\{\int u\times v: v\in L^q(\mu),\,\|v\|_q\le 1\}$.

(b) $\|\,\|_p$ is a norm on L^p.

244G Theorem Let (X,Σ,μ) be any measure space, and $p\in[1,\infty]$. Then $L^p=L^p(\mu)$ is a Banach lattice under its norm $\|\,\|_p$.

244H Proposition (a) Let (X,Σ,μ) be any measure space, and $p\in[1,\infty[$. Then the space S of equivalence classes of μ-simple functions is a dense linear subspace of $L^p=L^p(\mu)$.

(b) Let X be any subset of $\mathbb{R}^r$, where $r\ge 1$, and let μ be the subspace measure on X induced by Lebesgue measure on $\mathbb{R}^r$. Write C_k for the set of (bounded) continuous functions $g:\mathbb{R}^r\to\mathbb{R}$ such that $\{x:g(x)\ne 0\}$ is bounded, and S_0 for the space of linear combinations of functions of the form χI, where $I\subseteq\mathbb{R}^r$ is a bounded half-open interval. Then $\{(g\restriction X)^\bullet: g\in C_k\}$ and $\{(h\restriction X)^\bullet: h\in S_0\}$ are dense in $L^p(\mu)$.

***244I Corollary** In the context of 244Hb, $L^p(\mu)$ is separable.

244J Duality in L^p spaces Let (X,Σ,μ) be any measure space, and $p\in\,]1,\infty[$. Set $q=p/(p-1)$. Now $u\times v\in L^1(\mu)$ and $\|u\times v\|_1\le\|u\|_p\|v\|_q$ whenever $u\in L^p=L^p(\mu)$ and $v\in L^q=L^q(\mu)$. Consequently we have a bounded linear operator T from L^q to the normed space dual $(L^p)^*$ of L^p, given by writing

$$(Tv)(u)=\int u\times v$$

for all $u\in L^p$ and $v\in L^q$.

244K Theorem Let (X,Σ,μ) be a measure space, and $p\in\,]1,\infty[$; set $q=p/(p-1)$. Then the canonical map $T:L^q(\mu)\to L^p(\mu)^*$ is a normed space isomorphism.

244L Theorem Let (X,Σ,μ) be any measure space, and $p\in[1,\infty[$. Then $L^p=L^p(\mu)$ is Dedekind complete.

244M Theorem Let (X,Σ,μ) be a probability space, and T a σ-subalgebra of Σ. Take $p\in[1,\infty]$. Regard $L^0(\mu\restriction\mathrm{T})$ as a subspace of $L^0=L^0(\mu)$, so that $L^p(\mu\restriction\mathrm{T})$ becomes $L^p(\mu)\cap L^0(\mu\restriction\mathrm{T})$. Let $P:L^1(\mu)\to L^1(\mu\restriction\mathrm{T})$ be the conditional expectation operator. Then whenever $u\in L^p=L^p(\mu)$, $|Pu|^p\le P(|u|^p)$, so $Pu\in L^p(\mu\restriction\mathrm{T})$ and $\|Pu\|_p\le\|u\|_p$. Moreover, $P[L^p]=L^p(\mu\restriction\mathrm{T})$.

244N The space L^2 (a) L^2 has the special property of being an inner product space; if (X,Σ,μ) is any measure space and u, $v\in L^2=L^2(\mu)$ then $u\times v\in L^1(\mu)$, and we may write $(u|v)=\int u\times v$. This makes L^2 a real inner product space and its norm $\|\,\|_2$ is the associated norm. L^2 is a real Hilbert space.

I will use the phrase '**square-integrable**' to describe functions in $\mathcal{L}^2(\mu)$.

(b) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $P:L^1=L^1(\mu)\to L^1(\mu\restriction\mathrm{T})\subseteq L^1$ the corresponding conditional expectation operator. Then $P[L^2]\subseteq L^2$, where $L^2=L^2(\mu)$, so we have an operator $P_2=P\restriction L^2$ from L^2 to itself. Now P_2 is an orthogonal projection and its kernel is $\{u:u\in L^2,\,\int_F u=0$ for every $F\in\mathrm{T}\}$.

***244O Theorem** Suppose that $p\in\,]1,\infty[$ and (X,Σ,μ) is a measure space. Then $L^p=L^p(\mu)$ is uniformly convex.

244P Complex L^p Let (X,Σ,μ) be any measure space.

(a) For any $p\in\,]1,\infty[$, set

$$\mathcal{L}^p_{\mathbb{C}}=\mathcal{L}^p_{\mathbb{C}}(\mu)=\{f:f\in\mathcal{L}^0_{\mathbb{C}}(\mu),\,|f|^p\text{ is integrable}\},$$

$$\begin{aligned}L^p_{\mathbb{C}}=L^p_{\mathbb{C}}(\mu)&=\{f^\bullet:f\in\mathcal{L}^p_{\mathbb{C}}\}\\&=\{u:u\in L^0_{\mathbb{C}}(\mu),\,\mathcal{R}\mathrm{e}(u)\in L^p(\mu)\text{ and }\mathcal{I}\mathrm{m}(u)\in L^p(\mu)\}\\&=\{u:u\in L^0_{\mathbb{C}}(\mu),\,|u|\in L^p(\mu)\}.\end{aligned}$$

Then $L^p_{\mathbb{C}}$ is a linear subspace of $L^0_{\mathbb{C}}(\mu)$. Set $\|u\|_p=\||u|\|_p=(\int|u|^p)^{1/p}$ for $u\in L^p_{\mathbb{C}}$.

(b) The proof of 244E(b-i) applies unchanged to complex-valued functions, so taking $q = p/(p-1)$ we get

$$\|u \times v\|_1 \le \|u\|_p \|v\|_q$$

for all $u \in L^p_{\mathbb{C}}$, $v \in L^q_{\mathbb{C}}$. 244Fa becomes

for every $u \in L^p_{\mathbb{C}}$ there is a $v \in L^q_{\mathbb{C}}$ such that $\|v\|_q \le 1$ and

$$\int u \times v = |\int u \times v| = \|u\|_p.$$

$\| \|_p$ is a norm. $L^p_{\mathbb{C}}$ is complete. The space $S_{\mathbb{C}}$ of equivalence classes of 'complex-valued simple functions' is dense in $L^p_{\mathbb{C}}$. If X is a subset of $\mathbb{R}^r$ and μ is Lebesgue measure on X, then the space of equivalence classes of continuous complex-valued functions on X with bounded support is dense in $L^p_{\mathbb{C}}$.

(c) The canonical map $T : L^q_{\mathbb{C}} \to (L^p_{\mathbb{C}})^*$, defined by writing $(Tv)(u) = \int u \times v$, is surjective and an isometry. Thus we can still identify $L^q_{\mathbb{C}}$ with $(L^p_{\mathbb{C}})^*$.

(d) If (X, Σ, μ) is a probability space, T is a σ-subalgebra of Σ and $P : L^1_{\mathbb{C}}(\mu) \to L^1_{\mathbb{C}}(\mu \restriction \mathrm{T})$ is the corresponding conditional expectation operator, then for any $u \in L^p_{\mathbb{C}}$ $\|Pu\|_p \le \|u\|_p$.

(e) We now have to define

$$(u|v) = \int u \times \bar{v}$$

for u, $v \in L^2_{\mathbb{C}}$; $(v|u)$ is the complex conjugate of $(u|v)$.

Version of 25.3.06

245 Convergence in measure

I come now to an important and interesting topology on the spaces $\mathcal{L}^0$ and L^0. I start with the definition (245A) and with properties which echo those of the L^p spaces for $p \ge 1$ (245D-245E). In 245G-245J I describe the most useful relationships between this topology and the norm topologies of the L^p spaces. For σ-finite spaces, it is metrizable (245Eb) and sequential convergence can be described in terms of pointwise convergence of sequences of functions (245K-245L).

245A Definitions Let (X, Σ, μ) be a measure space.

(a) For any measurable set $F \subseteq X$ of finite measure, we have a functional τ_F on $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ defined by setting

$$\tau_F(f) = \int |f| \wedge \chi F$$

for every $f \in \mathcal{L}^0$. Now $\tau_F(f+g) \le \tau_F(f) + \tau_F(g)$ whenever f, $g \in \mathcal{L}^0$. Consequently, setting $\rho_F(f,g) = \tau_F(f-g)$, ρ_F is a pseudometric on $\mathcal{L}^0$.

(b) The family

$$\{\rho_F : F \in \Sigma, \, \mu F < \infty\}$$

now defines a topology on $\mathcal{L}^0$; I will call it the topology of **convergence in measure** on $\mathcal{L}^0$.

(c) If f, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then $\tau_F(f) = \tau_F(g)$, for every set F of finite measure. Consequently we have functionals $\bar{\tau}_F$ on $L^0 = L^0(\mu)$ defined by writing

$$\bar{\tau}_F(f^\bullet) = \tau_F(f)$$

whenever $f \in \mathcal{L}^0$, $F \in \Sigma$ and $\mu F < \infty$. Corresponding to these we have pseudometrics $\bar{\rho}_F$ defined by either of the formulae

$$\bar{\rho}_F(u,v) = \bar{\tau}_F(u-v), \quad \bar{\rho}_F(f^\bullet, g^\bullet) = \rho_F(f,g)$$

for u, $v \in L^0$, f, $g \in \mathcal{L}^0$ and F of finite measure. The family of these pseudometrics defines the **topology of convergence in measure** on L^0.

(d) I shall allow myself to say that a sequence (in $\mathcal{L}^0$ or L^0) **converges in measure** if it converges for the topology of convergence in measure.

245C Pointwise convergence Let (X, Σ, μ) be a measure space, and write $\mathcal{L}^0 = \mathcal{L}^0(\mu)$, $L^0 = L^0(\mu)$.

(a) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ converging almost everywhere to $f \in \mathcal{L}^0$, then $\langle f_n \rangle_{n\in\mathbb{N}} \to f$ in measure.

(b) For f, $f_n \in \mathcal{L}^0$, $\langle f_n^\bullet \rangle_{n\in\mathbb{N}}$ is **order*-convergent**, or **order*-converges**, to $f^\bullet$ iff $f =_{\text{a.e.}} \lim_{n\to\infty} f_n$. In L^0, a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ which order*-converges to $u \in L^0$ also converges to u in measure.

(c) Take μ to be Lebesgue measure on $[0, 1]$, and set $f_n(x) = 2^m$ if $x \in [2^{-m}k, 2^{-m}(k + 1)]$, 0 otherwise, where $k = k(n) \in \mathbb{N}$, $m = m(n) \in \mathbb{N}$ are defined by saying that $n + 1 = 2^m + k$ and $0 \le k < 2^m$. Then $\langle f_n \rangle_{n\in\mathbb{N}} \to 0$ for the topology of convergence in measure, though $\langle f_n \rangle_{n\in\mathbb{N}}$ is not convergent to 0 almost everywhere.

245D Proposition Let (X, Σ, μ) be any measure space.
(a) The topology of convergence in measure is a linear space topology on $L^0 = L^0(\mu)$.
(b) The maps $\vee, \wedge : L^0 \times L^0 \to L^0$, and $u \mapsto |u|$, $u \mapsto u^+$, $u \mapsto u^- : L^0 \to L^0$ are all continuous.
(c) The map $\times : L^0 \times L^0 \to L^0$ is continuous.
(d) For any continuous function $h : \mathbb{R} \to \mathbb{R}$, the corresponding function $\bar{h} : L^0 \to L^0$ is continuous.

245E Theorem Let (X, Σ, μ) be a measure space. Let $\mathfrak{T}$ be the topology of convergence in measure on $L^0 = L^0(\mu)$.
(a) (X, Σ, μ) is semi-finite iff $\mathfrak{T}$ is Hausdorff.
(b) (X, Σ, μ) is σ-finite iff $\mathfrak{T}$ is metrizable.
(c) (X, Σ, μ) is localizable iff $\mathfrak{T}$ is Hausdorff and L^0 is complete under $\mathfrak{T}$.

245F Alternative description of the topology of convergence in measure Let us return to arbitrary measure spaces (X, Σ, μ).

(a) For any $F \in \Sigma$ of finite measure and $\epsilon > 0$ define $\tau_{F\epsilon} : \mathcal{L}^0 \to [0, \infty[$ by

$$\tau_{F\epsilon}(f) = \mu^*\{x : x \in F \cap \operatorname{dom} f,\, |f(x)| > \epsilon\}$$

for $f \in \mathcal{L}^0$. If f, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then $\tau_{F\epsilon}(f) = \tau_{F\epsilon}(g)$; accordingly we have a functional from L^0 to $[0, \infty[$, given by

$$\bar{\tau}_{F\epsilon}(u) = \tau_{F\epsilon}(f)$$

whenever $f \in \mathcal{L}^0$ and $u = f^\bullet \in L^0$.

(b) $G \subseteq \mathcal{L}^0$ is open for the topology of convergence in measure iff for every $f \in G$ we can find a set F of finite measure and $\epsilon > 0$ such that

$$\tau_{F\epsilon}(g - f) \le \epsilon \Longrightarrow g \in G.$$

$G \subseteq L^0$ is open for the topology of convergence in measure on L^0 iff for every $u \in G$ we can find a set F of finite measure and $\epsilon > 0$ such that

$$\bar{\tau}_{F\epsilon}(v - u) \le \epsilon \Longrightarrow v \in G.$$

(c) It follows at once that a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ converges in measure to $f \in \mathcal{L}^0$ iff

$$\lim_{n\to\infty} \mu^*\{x : x \in F \cap \operatorname{dom} f \cap \operatorname{dom} f_n,\, |f_n(x) - f(x)| > \epsilon\} = 0$$

whenever $F \in \Sigma$, $\mu F < \infty$ and $\epsilon > 0$. Similarly, a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in L^0 converges in measure to u iff $\lim_{n\to\infty} \bar{\tau}_{F\epsilon}(u - u_n) = 0$ whenever $\mu F < \infty$ and $\epsilon > 0$.

(d) In particular, if (X, Σ, μ) is totally finite, $\langle f_n \rangle_{n\in\mathbb{N}} \to f$ in $\mathcal{L}^0$ iff

$$\lim_{n\to\infty} \mu^*\{x : x \in \operatorname{dom} f \cap \operatorname{dom} f_n,\ |f(x) - f_n(x)| > \epsilon\} = 0$$

for every $\epsilon > 0$, and $\langle u_n\rangle_{n\in\mathbb{N}} \to u$ in L^0 iff

$$\lim_{n\to\infty} \bar\tau_{X\epsilon}(u - u_n) = 0$$

for every $\epsilon > 0$.

245G Embedding L^p in L^0: Proposition Let (X, Σ, μ) be any measure space. Then for any $p \in [1, \infty]$, the embedding of $L^p = L^p(\mu)$ in $L^0 = L^0(\mu)$ is continuous for the norm topology of L^p and the topology of convergence in measure on L^0.

245H Proposition Let (X, Σ, μ) be a measure space.

(a)(i) If $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $\epsilon > 0$, there are a $\delta > 0$ and a set $F \in \Sigma$ of finite measure such that $\int |f - g| \le \epsilon$ whenever $g \in \mathcal{L}^1$, $\int |g| \le \int |f| + \delta$ and $\rho_F(f, g) \le \delta$.

(ii) For any sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^1$ and any $f \in \mathcal{L}^1$, $\lim_{n\to\infty} \int |f - f_n| = 0$ iff $\langle f_n\rangle_{n\in\mathbb{N}} \to f$ in measure and $\limsup_{n\to\infty} \int |f_n| \le \int |f|$.

(b)(i) If $u \in L^1 = L^1(\mu)$ and $\epsilon > 0$, there are a $\delta > 0$ and a set $F \in \Sigma$ of finite measure such that $\|u - v\|_1 \le \epsilon$ whenever $v \in L^1$, $\|v\|_1 \le \|u\|_1 + \delta$ and $\bar\rho_F(u, v) \le \delta$.

(ii) For any sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in L^1 and any $u \in L^1$, $\langle u_n\rangle_{n\in\mathbb{N}} \to u$ for $\| \|_1$ iff $\langle u_n\rangle_{n\in\mathbb{N}} \to u$ in measure and $\limsup_{n\to\infty} \|u_n\|_1 \le \|u\|_1$.

245J Proposition Let (X, Σ, μ) be a semi-finite measure space. Write $\mathcal{L}^0 = \mathcal{L}^0(\mu)$, etc.

(a)(i) For any $a \ge 0$, the set $\{f : f \in \mathcal{L}^1, \int |f| \le a\}$ is closed in $\mathcal{L}^0$ for the topology of convergence in measure.

(ii) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^1$ which is convergent in measure to $f \in \mathcal{L}^0$, and $\liminf_{n\to\infty} \int |f_n| < \infty$, then f is integrable and $\int |f| \le \liminf_{n\to\infty} \int |f_n|$.

(b)(i) For any $a \ge 0$, the set $\{u : u \in L^1, \|u\|_1 \le a\}$ is closed in L^0 for the topology of convergence in measure.

(ii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in L^1 which is convergent in measure to $u \in L^0$, and $\liminf_{n\to\infty} \|u_n\|_1 < \infty$, then $u \in L^1$ and $\|u\|_1 \le \liminf_{n\to\infty} \|u_n\|_1$.

245K Proposition Let (X, Σ, μ) be a σ-finite measure space. Then

(a) a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0$ converges in measure to $f \in \mathcal{L}^0$ iff every subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence converging to f almost everywhere;

(b) a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in L^0 converges in measure to $u \in L^0$ iff every subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence which order*-converges to u.

245L Corollary Let (X, Σ, μ) be a σ-finite measure space.

(a) A subset A of $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ is closed for the topology of convergence in measure iff $f \in A$ whenever $f \in \mathcal{L}^0$ and there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in A such that $f =_{\text{a.e.}} \lim_{n\to\infty} f_n$.

(b) A subset A of $L^0 = L^0(\mu)$ is closed for the topology of convergence in measure iff $u \in A$ whenever $u \in L^0$ and there is a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in A order*-converging to u.

245M Complex L^0 In 241J I briefly discussed the adaptations needed to construct the complex linear space $L^0_{\mathbb{C}}$. The formulae of 245A may be used unchanged to define topologies of convergence in measure on $\mathcal{L}^0_{\mathbb{C}}$ and $L^0_{\mathbb{C}}$. I think that every word of 245B-245L still applies if we replace each L^0 or $\mathcal{L}^0$ with $L^0_{\mathbb{C}}$ or $\mathcal{L}^0_{\mathbb{C}}$.

Version of 17.11.06

246 Uniform integrability

The next topic is a fairly specialized one, but it is of great importance, for different reasons, in both probability theory and functional analysis, and it therefore seems worth while giving a proper treatment straight away.

246A Definition Let (X, Σ, μ) be a measure space.

(a) A set $A \subseteq \mathcal{L}^1(\mu)$ is **uniformly integrable** if for every $\epsilon > 0$ we can find a set $E \in \Sigma$, of finite measure, and an $M \geq 0$ such that

$$\int (|f| - M\chi E)^+ \leq \epsilon \text{ for every } f \in A.$$

(b) A set $A \subseteq L^1(\mu)$ is **uniformly integrable** if for every $\epsilon > 0$ we can find a set $E \in \Sigma$, of finite measure, and an $M \geq 0$ such that

$$\int (|u| - M\chi E^\bullet)^+ \leq \epsilon \text{ for every } u \in A.$$

246B Remarks (c) $A \subseteq \mathcal{L}^1$ is uniformly integrable iff $\{f^\bullet : f \in A\} \subseteq L^1$ is uniformly integrable.

(d) If $\mu X < \infty$ a set $A \subseteq L^1(\mu)$ is uniformly integrable iff

$$\inf_{M \geq 0} \sup_{u \in A} \int (|u| - Me)^+ = 0$$

iff

$$\lim_{M \to \infty} \sup_{u \in A} \int (|u| - Me)^+ = 0,$$

writing $e = \chi X^\bullet \in L^1(\mu)$. $A \subseteq \mathcal{L}^1(\mu)$ is uniformly integrable iff

$$\lim_{M \to \infty} \sup_{f \in A} \int (|f| - M\chi X)^+ = 0$$

iff

$$\inf_{M \geq 0} \sup_{f \in A} \int (|f| - M\chi X)^+ = 0.$$

246C Proposition Let (X, Σ, μ) be a measure space and A a uniformly integrable subset of $L^1(\mu)$.
(a) A is bounded for the norm $\| \|_1$.
(b) Any subset of A is uniformly integrable.
(c) For any $a \in \mathbb{R}$, $aA = \{au : u \in A\}$ is uniformly integrable.
(d) There is a uniformly integrable $C \supseteq A$ such that C is convex and $\| \|_1$-closed and $v \in C$ whenever $u \in C$ and $|v| \leq |u|$.
(e) If B is another uniformly integrable subset of L^1, then $A \cup B$ and $A + B = \{u + v : u \in A, v \in B\}$ are uniformly integrable.

246D Proposition Let (X, Σ, μ) be a probability space and $A \subseteq L^1(\mu)$ a uniformly integrable set. Then there is a convex, $\| \|_1$-closed uniformly integrable set $C \subseteq L^1$ such that $A \subseteq C$, $w \in C$ whenever $v \in C$ and $|w| \leq |v|$, and $Pv \in C$ whenever $v \in C$ and P is the conditional expectation operator associated with a σ-subalgebra of Σ.

246F Lemma Let (X, Σ, μ) be a measure space. Then for any $u \in L^1(\mu)$,

$$\|u\|_1 \leq 2 \sup_{E \in \Sigma} |\int_E u|.$$

246G Theorem Let (X, Σ, μ) be any measure space and A a non-empty subset of $L^1(\mu)$. Then the following are equiveridical:

(i) A is uniformly integrable;

(ii) $\sup_{u \in A} |\int_F u| < \infty$ for every μ-atom $F \in \Sigma$, and for every $\epsilon > 0$ there are $E \in \Sigma$, $\delta > 0$ such that $\mu E < \infty$ and $|\int_F u| \leq \epsilon$ whenever $u \in A$, $F \in \Sigma$ and $\mu(F \cap E) \leq \delta$;

(iii) $\sup_{u \in A} |\int_F u| < \infty$ for every μ-atom $F \in \Sigma$, and $\lim_{n \to \infty} \sup_{u \in A} |\int_{F_n} u| = 0$ whenever $\langle F_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in Σ;

(iv) $\sup_{u \in A} |\int_F u| < \infty$ for every μ-atom $F \in \Sigma$, and $\lim_{n \to \infty} \sup_{u \in A} |\int_{F_n} u| = 0$ whenever $\langle F_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection.

246I Corollary Let (X, Σ, μ) be a probability space. For $f \in \mathcal{L}^0(\mu)$, $M \geq 0$ set $F(f, M) = \{x : x \in \operatorname{dom} f, |f(x)| \geq M\}$. Then a non-empty set $A \subseteq \mathcal{L}^1(\mu)$ is uniformly integrable iff

$$\lim_{M\to\infty} \sup_{f\in A} \int_{F(f,M)} |f| = 0.$$

246J Theorem Let (X, Σ, μ) be a measure space.

(a) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a uniformly integrable sequence of real-valued functions on X, and $f(x) = \lim_{n\to\infty} f_n(x)$ for almost every $x \in X$, then f is integrable and $\lim_{n\to\infty} \int |f_n - f| = 0$; consequently $\int f = \lim_{n\to\infty} \int f_n$.

(b) If $A \subseteq L^1 = L^1(\mu)$ is uniformly integrable, then the norm topology of L^1 and the topology of convergence in measure of $L^0 = L^0(\mu)$ agree on A.

(c) For any $u \in L^1$ and any sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in L^1, the following are equiveridical:

(i) $u = \lim_{n\to\infty} u_n$ for $\| \|_1$;

(ii) $\{u_n : n \in \mathbb{N}\}$ is uniformly integrable and $\langle u_n \rangle_{n\in\mathbb{N}}$ converges to u in measure.

(d) If (X, Σ, μ) is semi-finite, and $A \subseteq L^1$ is uniformly integrable, then the closure $\overline{A}$ of A in L^0 for the topology of convergence in measure is still a uniformly integrable subset of L^1.

246K Complex $\mathcal{L}^1$ and L^1 For $u \in L^1_{\mathbb{C}}(\mu)$, $\|u\|_1 \leq 4 \sup_{F\in\Sigma} |\int_F u|$.

Version of 26.8.13

247 Weak compactness in L^1

I now come to the most striking feature of uniform integrability: it provides a description of the relatively weakly compact subsets of L^1 (247C). I have put this into a separate section because it demands some knowledge of functional analysis – in particular, of course, of weak topologies on Banach spaces. I will try to give an account in terms which are accessible to novices in the theory of normed spaces because the result is essentially measure-theoretic, as well as being of vital importance to applications in probability theory. I have written out the essential definitions in §§2A3-2A5.

247A Lemma Let (X, Σ, μ) be a measure space, and G any member of Σ. Let μ_G be the subspace measure on G. Set

$$U = \{u : u \in L^1(\mu),\ u \times \chi G^\bullet = u\} \subseteq L^1(\mu).$$

Then we have an isomorphism S between the ordered normed spaces U and $L^1(\mu_G)$, given by writing

$$S(f^\bullet) = (f{\restriction}G)^\bullet$$

for every $f \in \mathcal{L}^1(\mu)$ such that $f^\bullet \in U$.

247B Corollary Let (X, Σ, μ) be any measure space, and let $G \in \Sigma$ be a measurable set expressible as a countable union of sets of finite measure. Define U as in 247A, and let $h : L^1(\mu) \to \mathbb{R}$ be any continuous linear functional. Then there is a $v \in L^\infty(\mu)$ such that $h(u) = \int u \times v\, d\mu$ for every $u \in U$.

247C Theorem Let (X, Σ, μ) be any measure space and A a subset of $L^1 = L^1(\mu)$. Then A is uniformly integrable iff it is relatively compact in L^1 for the weak topology of L^1.

247D Corollary Let (X, Σ, μ) and (Y, T, ν) be any two measure spaces, and $T : L^1(\mu) \to L^1(\nu)$ a continuous linear operator. Then $T[A]$ is a uniformly integrable subset of $L^1(\nu)$ whenever A is a uniformly integrable subset of $L^1(\mu)$.