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# Chapter 24

## Function spaces

The extraordinary power of Lebesgue's theory of integration is perhaps best demonstrated by its ability to provide structures relevant to questions quite different from those to which it was at first addressed. In this chapter I give the constructions, and elementary properties, of some of the fundamental spaces of functional analysis.

I do not feel called on here to justify the study of normed spaces; if you have not met them before, I hope that the introduction here will show at least that they offer a basis for a remarkable fusion of algebra and analysis. The fragments of the theory of metric spaces, normed spaces and general topology which we shall need are sketched in §§2A2-2A5. The principal 'function spaces' described in this chapter in fact combine three structural elements: they are (infinite-dimensional) linear spaces, they are metric spaces, with associated concepts of continuity and convergence, and they are ordered spaces, with corresponding notions of supremum and infimum. The interactions between these three types of structure provide an inexhaustible wealth of ideas. Furthermore, many of these ideas are directly applicable to a wide variety of problems in more or less applied mathematics, particularly in differential and integral equations, but more generally in any system with infinitely many degrees of freedom.

I have laid out the chapter with sections on $L^0$ (the space of equivalence classes of all real-valued measurable functions, in which all the other spaces of the chapter are embedded), $L^1$ (equivalence classes of integrable functions), $L^\infty$ (equivalence classes of bounded measurable functions) and $L^p$ (equivalence classes of $p$th-power-integrable functions). While ordinary functional analysis gives much more attention to the Banach spaces $L^p$ for $1 \le p \le \infty$ than to $L^0$, from the special point of view of this book the space $L^0$ is at least as important and interesting as any of the others. Following these four sections, I return to a study of the standard topology on $L^0$, the topology of 'convergence in measure' (§245), and then to two linked sections on uniform integrability and weak compactness in $L^1$ (§§246-247).

There is a technical point here which must never be lost sight of. While it is customary and natural to call $L^1$, $L^2$ and the others 'function spaces', their elements are not in fact functions, but equivalence classes of functions. As you see from the language of the preceding paragraph, my practice is to scrupulously maintain the distinction; I give my reasons in the notes to §241.

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### 241 $\mathcal{L}^0$ and $L^0$

The chief aim of this chapter is to discuss the spaces $L^1$, $L^\infty$ and $L^p$ of the following three sections. However it will be convenient to regard all these as subspaces of a larger space $L^0$ of equivalence classes of (virtually) measurable functions, and I have collected in this section the basic facts concerning the ordered linear space $L^0$.

It is almost the first principle of measure theory that sets of measure zero can often be ignored; the phrase 'negligible set' itself asserts this principle. Accordingly, two functions which agree almost everywhere may often (not always!) be treated as identical. A suitable expression of this idea is to form the space of equivalence classes of functions, saying that two functions are equivalent if they agree on a conegligible set. This is the basis of all the constructions of this chapter. It is a remarkable fact that the spaces of equivalence classes so constructed are actually better adapted to certain problems than the spaces of functions from which they are derived, so that once the technique has been mastered it is easier to do one's thinking in the more abstract spaces.

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**241A The space $\mathcal{L}^0$: Definition** It is time to give a name to a set of functions which has already been used more than once. Let $(X,\Sigma,\mu)$ be a measure space. I write $\mathcal{L}^0$, or $\mathcal{L}^0(\mu)$, for the space of real-valued functions $f$ defined on conegligible subsets of $X$ which are virtually measurable, that is, such that $f{\restriction}E$ is measurable for some conegligible set $E\subseteq X$. Recall that $f$ is $\mu$-virtually measurable iff it is $\hat\Sigma$-measurable, where $\hat\Sigma$ is the completion of $\Sigma$ (212Fa).

**241B Basic properties** If $(X,\Sigma,\mu)$ is any measure space, then we have the following facts, corresponding to the fundamental properties of measurable functions listed in §121 of Volume 1. I work through them in order, so that if you have Volume 1 to hand you can see what has to be missed out.

**(a)** A constant real-valued function defined almost everywhere in $X$ belongs to $\mathcal{L}^0$ (121Ea).

**(b)** $f+g\in\mathcal{L}^0$ for all $f$, $g\in\mathcal{L}^0$ (for if $f{\restriction}F$ and $g{\restriction}G$ are measurable, then $(f+g){\restriction}(F\cap G)=(f{\restriction}F)+(g{\restriction}G)$ is measurable)(121Eb).

**(c)** $cf\in\mathcal{L}^0$ for all $f\in\mathcal{L}^0$, $c\in\mathbb{R}$ (121Ec).

**(d)** $f\times g\in\mathcal{L}^0$ for all $f$, $g\in\mathcal{L}^0$ (121Ed).

**(e)** If $f\in\mathcal{L}^0$ and $h:\mathbb{R}\to\mathbb{R}$ is Borel measurable, then $hf\in\mathcal{L}^0$ (121Eg).

**(f)** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f=\lim_{n\to\infty}f_n$ is defined (as a real-valued function) almost everywhere in $X$, then $f\in\mathcal{L}^0$ (121Fa).

**(g)** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f=\sup_{n\in\mathbb{N}}f_n$ is defined (as a real-valued function) almost everywhere in $X$, then $f\in\mathcal{L}^0$ (121Fb).

**(h)** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f=\inf_{n\in\mathbb{N}}f_n$ is defined (as a real-valued function) almost everywhere in $X$, then $f\in\mathcal{L}^0$ (121Fc).

**(i)** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f=\limsup_{n\to\infty}f_n$ is defined (as a real-valued function) almost everywhere in $X$, then $f\in\mathcal{L}^0$ (121Fd).

**(j)** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ and $f=\liminf_{n\to\infty}f_n$ is defined (as a real-valued function) almost everywhere in $X$, then $f\in\mathcal{L}^0$ (121Fe).

**(k)** $\mathcal{L}^0$ is just the set of real-valued functions, defined on subsets of $X$, which are equal almost everywhere to some $\Sigma$-measurable function from $X$ to $\mathbb{R}$. **P** (i) If $g:X\to\mathbb{R}$ is $\Sigma$-measurable and $f=_{\text{a.e.}}g$, then $F=\{x:x\in\operatorname{dom}f,\ f(x)=g(x)\}$ is conegligible and $f{\restriction}F=g{\restriction}F$ is measurable (121Eh), so $f\in\mathcal{L}^0$. (ii) If $f\in\mathcal{L}^0$, let $E\subseteq X$ be a conegligible set such that $f{\restriction}E$ is measurable. Then $D=E\cap\operatorname{dom}f$ is conegligible and $f{\restriction}D$ is measurable, so there is a measurable $h:X\to\mathbb{R}$ agreeing with $f$ on $D$ (121I); and $h=_{\text{a.e.}}f$. **Q**

**241C The space $L^0$: Definition** Let $(X,\Sigma,\mu)$ be any measure space. Then $=_{\text{a.e.}}$ is an equivalence relation on $\mathcal{L}^0$. Write $L^0$, or $L^0(\mu)$, for the set of equivalence classes in $\mathcal{L}^0$ under $=_{\text{a.e.}}$. For $f\in\mathcal{L}^0$, write $f^\bullet$ for its equivalence class in $L^0$.

**241D The linear structure of $L^0$** Let $(X,\Sigma,\mu)$ be any measure space, and set $\mathcal{L}^0=\mathcal{L}^0(\mu)$, $L^0=L^0(\mu)$.

**(a)** If $f_1$, $f_2$, $g_1$, $g_2\in\mathcal{L}^0$, $f_1=_{\text{a.e.}}f_2$ and $g_1=_{\text{a.e.}}g_2$ then $f_1+g_1=_{\text{a.e.}}f_2+g_2$. Accordingly we may define addition on $L^0$ by setting $f^\bullet+g^\bullet=(f+g)^\bullet$ for all $f$, $g\in\mathcal{L}^0$.

**(b)** If $f_1$, $f_2\in\mathcal{L}^0$ and $f_1=_{\text{a.e.}}f_2$, then $cf_1=_{\text{a.e.}}cf_2$ for every $c\in\mathbb{R}$. Accordingly we may define scalar multiplication on $L^0$ by setting $c\cdot f^\bullet=(cf)^\bullet$ for all $f\in\mathcal{L}^0$ and $c\in\mathbb{R}$.

**(c)** Now $L^0$ is a linear space over $\mathbb{R}$, with zero $\mathbf{0}^\bullet$, where $\mathbf{0}$ is the function with domain $X$ and constant value 0, and negatives $-(f^\bullet)=(-f)^\bullet$. **P** (i)

$$f+(g+h)=(f+g)+h \text{ for all } f,\ g,\ h\in\mathcal{L}^0,$$

so

$$u+(v+w)=(u+v)+w \text{ for all } u,\, v,\, w\in L^0.$$

(ii)

$$f+\mathbf{0}=\mathbf{0}+f=f \text{ for every } f\in\mathcal{L}^0,$$

so

$$u+\mathbf{0}^\bullet=\mathbf{0}^\bullet+u=u \text{ for every } u\in L^0.$$

(iii)

$$f+(-f)=_{\text{a.e.}}\mathbf{0} \text{ for every } f\in\mathcal{L}^0,$$

so

$$f^\bullet+(-f)^\bullet=\mathbf{0}^\bullet \text{ for every } f\in\mathcal{L}^0.$$

(iv)

$$f+g=g+f \text{ for all } f,\, g\in\mathcal{L}^0,$$

so

$$u+v=v+u \text{ for all } u,\, v\in L^0.$$

(v)

$$c(f+g)=cf+cg \text{ for all } f,\, g\in\mathcal{L}^0 \text{ and } c\in\mathbb{R},$$

so

$$c(u+v)=cu+cv \text{ for all } u,\, v\in L^0 \text{ and } c\in\mathbb{R}.$$

(vi)

$$(a+b)f=af+bf \text{ for all } f\in\mathcal{L}^0 \text{ and } a,\, b\in\mathbb{R},$$

so

$$(a+b)u=au+bu \text{ for all } u\in L^0 \text{ and } a,\, b\in\mathbb{R}.$$

(vii)

$$(ab)f=a(bf) \text{ for all } f\in\mathcal{L}^0 \text{ and } a,\, b\in\mathbb{R},$$

so

$$(ab)u=a(bu) \text{ for all } u\in L^0 \text{ and } a,\, b\in\mathbb{R}.$$

(viii)

$$1f=f \text{ for all } f\in\mathcal{L}^0,$$

so

$$1u=u \text{ for all } u\in L^0. \quad \mathbf{Q}$$

**241E The order structure of** $L^0$ Let $(X,\Sigma,\mu)$ be any measure space and set $\mathcal{L}^0=\mathcal{L}^0(\mu)$, $L^0=L^0(\mu)$.

**(a)** If $f_1,\, f_2,\, g_1,\, g_2\in\mathcal{L}^0$, $f_1=_{\text{a.e.}}f_2$, $g_1=_{\text{a.e.}}g_2$ and $f_1\le_{\text{a.e.}}g_1$, then $f_2\le_{\text{a.e.}}g_2$. Accordingly we may define a relation $\le$ on $L^0$ by saying that $f^\bullet\le g^\bullet$ iff $f\le_{\text{a.e.}}g$.

**(b)** Now $\le$ is a partial order on $L^0$. **P** (i) If $f,\, g,\, h\in\mathcal{L}^0$ and $f\le_{\text{a.e.}}g$ and $g\le_{\text{a.e.}}h$, then $f\le_{\text{a.e.}}h$. Accordingly $u\le w$ whenever $u,\, v,\, w\in L^0$, $u\le v$ and $v\le w$. (ii) If $f\in\mathcal{L}^0$ then $f\le_{\text{a.e.}}f$; so $u\le u$ for every $u\in L^0$. (iii) If $f,\, g\in\mathcal{L}^0$ and $f\le_{\text{a.e.}}g$ and $g\le_{\text{a.e.}}f$, then $f=_{\text{a.e.}}g$, so if $u\le v$ and $v\le u$ then $u=v$. **Q**

**(c)** In fact $L^0$, with $\le$, is a **partially ordered linear space**, that is, a (real) linear space with a partial order $\le$ such that

if $u \le v$ then $u + w \le v + w$ for every $w$,
if $0 \le u$ then $0 \le cu$ for every $c \ge 0$.

**P** (i) If $f$, $g$, $h \in \mathcal{L}^0$ and $f \le_{\text{a.e.}} g$, then $f + h \le_{\text{a.e.}} g + h$. (ii) If $f \in \mathcal{L}^0$ and $f \ge 0$ a.e., then $cf \ge 0$ a.e. for every $c \ge 0$. **Q**

**(d)** More: $L^0$ is a **Riesz space** or **vector lattice**, that is, a partially ordered linear space such that $u \vee v = \sup\{u, v\}$ and $u \wedge v = \inf\{u, v\}$ are defined for all $u$, $v \in L^0$. **P** Take $f$, $g \in \mathcal{L}^0$ such that $f^\bullet = u$ and $g^\bullet = v$. Then $f \vee g$, $f \wedge g \in \mathcal{L}^0$, writing

$$(f \vee g)(x) = \max(f(x), g(x)), \quad (f \wedge g)(x) = \min(f(x), g(x))$$

for $x \in \operatorname{dom} f \cap \operatorname{dom} g$. (Compare 241Bg-h.) Now, for any $h \in \mathcal{L}^0$, we have

$$f \vee g \le_{\text{a.e.}} h \iff f \le_{\text{a.e.}} h \text{ and } g \le_{\text{a.e.}} h,$$

$$h \le_{\text{a.e.}} f \wedge g \iff h \le_{\text{a.e.}} f \text{ and } h \le_{\text{a.e.}} g,$$

so for any $w \in L^0$ we have

$$(f \vee g)^\bullet \le w \iff u \le w \text{ and } v \le w,$$

$$w \le (f \wedge g)^\bullet \iff w \le u \text{ and } w \le v.$$

Thus we have

$$(f \vee g)^\bullet = \sup\{u, v\} = u \vee v, \quad (f \wedge g)^\bullet = \inf\{u, v\} = u \wedge v$$

in $L^0$. **Q**

**(e)** In particular, for any $u \in L^0$ we can speak of $|u| = u \vee (-u)$; if $f \in \mathcal{L}^0$ then $|f^\bullet| = |f|^\bullet$.
If $f$, $g \in \mathcal{L}^0$, $c \in \mathbb{R}$ then

$$|cf| = |c||f|, \quad f \vee g = \frac{1}{2}(f + g + |f - g|),$$

$$f \wedge g = \frac{1}{2}(f + g - |f - g|), \quad |f + g| \le_{\text{a.e.}} |f| + |g|,$$

so

$$|cu| = |c||u|, \quad u \vee v = \frac{1}{2}(u + v + |u - v|),$$

$$u \wedge v = \frac{1}{2}(u + v - |u - v|), \quad |u + v| \le |u| + |v|$$

for all $u$, $v \in L^0$.

**(f)** A special notation is often useful. If $f$ is a real-valued function, set $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$ for $x \in \operatorname{dom} f$, so that

$$f = f^+ - f^-, \quad |f| = f^+ + f^- = f^+ \vee f^-,$$

all these functions being defined on $\operatorname{dom} f$. In $L^0$, the corresponding operations are $u^+ = u \vee 0$, $u^- = (-u) \vee 0$, and we have

$$u = u^+ - u^-, \quad |u| = u^+ + u^- = u^+ \vee u^-, \quad u^+ \wedge u^- = 0.$$

**(g)** It is perhaps obvious, but I say it anyway: if $u \ge 0$ in $L^0$, then there is an $f \ge 0$ in $\mathcal{L}^0$ such that $f^\bullet = u$. **P** Take any $g \in \mathcal{L}^0$ such that $u = g^\bullet$, and set $f = g \vee \mathbf{0}$. **Q**

**241F Riesz spaces** There is an extensive abstract theory of Riesz spaces, which I think it best to leave aside for the moment; a general account may be found in Luxemburg & Zaanen 71 and Zaanen 83; my own book Fremlin 74 covers the elementary material, and Chapter 35 in the next volume repeats the most essential ideas. For our purposes here we need only a few definitions and some simple results which are most easily proved for the special cases in which we need them, without reference to the general theory.

**(a)** A Riesz space $U$ is **Archimedean** if whenever $u \in U$, $u > 0$ (that is, $u \geq 0$ and $u \neq 0$), and $v \in U$, there is an $n \in \mathbb{N}$ such that $nu \not\leq v$.

**(b)** A Riesz space $U$ is **Dedekind $\sigma$-complete** (or **$\sigma$-order-complete**, or **$\sigma$-complete**) if every non-empty countable set $A \subseteq U$ which is bounded above has a least upper bound in $U$.

**(c)** A Riesz space $U$ is **Dedekind complete** (or **order complete**, or **complete**) if every non-empty set $A \subseteq U$ which is bounded above in $U$ has a least upper bound in $U$.

**241G** Now we have the following important properties of $L^0$.

**Theorem** Let $(X, \Sigma, \mu)$ be a measure space. Set $L^0 = L^0(\mu)$.

(a) $L^0$ is Archimedean and Dedekind $\sigma$-complete.

(b) If $(X, \Sigma, \mu)$ is semi-finite, then $L^0$ is Dedekind complete iff $(X, \Sigma, \mu)$ is localizable.

**proof** Set $\mathcal{L}^0 = \mathcal{L}^0(\mu)$.

**(a)(i)** If $u$, $v \in L^0$ and $u > 0$, express $u$ as $f^\bullet$ and $v$ as $g^\bullet$ where $f$, $g \in \mathcal{L}^0$. Then $E = \{x : x \in \operatorname{dom} f,\ f(x) > 0\}$ is not negligible. So there is an $n \in \mathbb{N}$ such that

$$E_n = \{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ nf(x) > g(x)\}$$

is not negligible, since $E \cap \operatorname{dom} g \subseteq \bigcup_{n\in\mathbb{N}} E_n$. But now $nu \not\leq v$. As $u$ and $v$ are arbitrary, $L^0$ is Archimedean.

**(ii)** Now let $A \subseteq L^0$ be a non-empty countable set with an upper bound $w$ in $L^0$. Express $A$ as $\{f_n^\bullet : n \in \mathbb{N}\}$ where $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$, and $w$ as $h^\bullet$ where $h \in \mathcal{L}^0$. Set $f = \sup_{n\in\mathbb{N}} f_n$. Then we have $f(x)$ defined in $\mathbb{R}$ at any point $x \in \operatorname{dom} h \cap \bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n$ such that $f_n(x) \leq h(x)$ for every $n \in \mathbb{N}$, that is, for almost every $x \in X$; so $f \in \mathcal{L}^0$ (241Bg). Set $u = f^\bullet \in L^0$. If $v \in L^0$, say $v = g^\bullet$ where $g \in \mathcal{L}^0$, then

$$\begin{aligned}
&u_n \leq v \text{ for every } n \in \mathbb{N}\\
&\qquad\iff \text{for every } n \in \mathbb{N},\ f_n \leq_{\text{a.e.}} g\\
&\qquad\iff \text{for almost every } x \in X,\ f_n(x) \leq g(x) \text{ for every } n \in \mathbb{N}\\
&\qquad\iff f \leq_{\text{a.e.}} g \iff u \leq v.
\end{aligned}$$

Thus $u = \sup_{n\in\mathbb{N}} u_n$ in $L^0$. As $A$ is arbitrary, $L^0$ is Dedekind $\sigma$-complete.

**(b)(i)** Suppose that $(X, \Sigma, \mu)$ is localizable. Let $A \subseteq L^0$ be any non-empty set with an upper bound $w_0 \in L^0$. Set

$$\mathcal{A} = \{f : f \text{ is a measurable function from } X \text{ to } \mathbb{R},\ f^\bullet \in A\};$$

then every member of $A$ is of the form $f^\bullet$ for some $f \in \mathcal{A}$ (241Bk). For each $q \in \mathbb{Q}$, let $\mathcal{E}_q$ be the family of subsets of $X$ expressible in the form $\{x : f(x) \geq q\}$ for some $f \in \mathcal{A}$; then $\mathcal{E}_q \subseteq \Sigma$. Because $(X, \Sigma, \mu)$ is localizable, there is a set $F_q \in \Sigma$ which is an essential supremum for $\mathcal{E}_q$. For $x \in X$, set

$$g^*(x) = \sup\{q : q \in \mathbb{Q},\ x \in F_q\},$$

allowing $\infty$ as the supremum of a set which is not bounded above, and $-\infty$ as $\sup\emptyset$. Then

$$\{x : g^*(x) > a\} = \textstyle\bigcup_{q\in\mathbb{Q}, q>a} F_q \in \Sigma$$

for every $a \in \mathbb{R}$.

If $f \in \mathcal{A}$, then $f \leq_{\text{a.e.}} g^*$. **P** For each $q \in \mathbb{Q}$, set

$$E_q = \{x : f(x) \geq q\} \in \mathcal{E}_q;$$

then $E_q \setminus F_q$ is negligible. Set $H = \bigcup_{q\in\mathbb{Q}}(E_q \setminus F_q)$. If $x \in X \setminus H$, then

$$f(x) \geq q \implies g^*(x) \geq q,$$

so $f(x) \leq g^*(x)$; thus $f \leq_{\text{a.e.}} g^*$. **Q**

If $h : X \to \mathbb{R}$ is measurable and $u \le h^\bullet$ for every $u \in A$, then $g^* \le_{\text{a.e.}} h$. **P** Set $G_q = \{x : h(x) \ge q\}$ for each $q \in \mathbb{Q}$. If $E \in \mathcal{E}_q$, there is an $f \in \mathcal{A}$ such that $E = \{x : f(x) \ge q\}$; now $f \le_{\text{a.e.}} h$, so $E \setminus G_q \subseteq \{x : f(x) > h(x)\}$ is negligible. Because $F_q$ is an essential supremum for $\mathcal{E}_q$, $F_q \setminus G_q$ is negligible; and this is true for every $q \in \mathbb{Q}$. Consequently

$$\{x : h(x) < g^*(x)\} \subseteq \bigcup_{q\in\mathbb{Q}} F_q \setminus G_q$$

is negligible, and $g^* \le_{\text{a.e.}} h$. **Q**

Now recall that we are assuming that $A \ne \emptyset$ and that $A$ has an upper bound $w_0 \in L^0$. Take any $f_0 \in \mathcal{A}$ and a measurable $h_0 : X \to \mathbb{R}$ such that $h_0^\bullet = w_0$; then $f \le_{\text{a.e.}} h_0$ for every $f \in \mathcal{A}$, so $f_0 \le_{\text{a.e.}} g^* \le_{\text{a.e.}} h_0$, and $g^*$ must be finite a.e. Setting $g(x) = g^*(x)$ when $g^*(x) \in \mathbb{R}$, we have $g \in \mathcal{L}^0$ and $g =_{\text{a.e.}} g^*$, so that

$$f \le_{\text{a.e.}} g \le_{\text{a.e.}} h$$

whenever $f$, $h$ are measurable functions from $X$ to $\mathbb{R}$, $f^\bullet \in A$ and $h^\bullet$ is an upper bound for $A$; that is,

$$u \le g^\bullet \le w$$

whenever $u \in A$ and $w$ is an upper bound for $A$. But this means that $g^\bullet$ is a least upper bound for $A$ in $L^0$. As $A$ is arbitrary, $L^0$ is Dedekind complete.

**(ii)** Suppose that $L^0$ is Dedekind complete. We are assuming that $(X, \Sigma, \mu)$ is semi-finite. Let $\mathcal{E}$ be any subset of $\Sigma$. Set

$$A = \{0\} \cup \{(\chi E)^\bullet : E \in \mathcal{E}\} \subseteq L^0.$$

Then $A$ is bounded above by $(\chi X)^\bullet$ so has a least upper bound $w \in L^0$. Express $w$ as $h^\bullet$ where $h : X \to \mathbb{R}$ is measurable, and set $F = \{x : h(x) > 0\}$. Then $F$ is an essential supremum for $\mathcal{E}$ in $\Sigma$. **P** $(\alpha)$ If $E \in \mathcal{E}$, then $(\chi E)^\bullet \le w$ so $\chi E \le_{\text{a.e.}} h$, that is, $h(x) \ge 1$ for almost every $x \in E$, and $E \setminus F \subseteq \{x : x \in E,\ h(x) < 1\}$ is negligible. $(\beta)$ If $G \in \Sigma$ and $E \setminus G$ is negligible for every $E \in \mathcal{E}$, then $\chi E \le_{\text{a.e.}} \chi G$ for every $E \in \mathcal{E}$, that is, $(\chi E)^\bullet \le (\chi G)^\bullet$ for every $E \in \mathcal{E}$; so $w \le (\chi G)^\bullet$, that is, $h \le_{\text{a.e.}} \chi G$. Accordingly $F \setminus G \subseteq \{x : h(x) > (\chi G)(x)\}$ is negligible. **Q**

As $\mathcal{E}$ is arbitrary, $(X, \Sigma, \mu)$ is localizable.

**241H The multiplicative structure of $L^0$** Let $(X, \Sigma, \mu)$ be any measure space; write $L^0 = L^0(\mu)$, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$.

**(a)** If $f_1$, $f_2$, $g_1$, $g_2 \in \mathcal{L}^0$, $f_1 =_{\text{a.e.}} f_2$ and $g_1 =_{\text{a.e.}} g_2$ then $f_1 \times g_1 =_{\text{a.e.}} f_2 \times g_2$. Accordingly we may define multiplication on $L^0$ by setting $f^\bullet \times g^\bullet = (f \times g)^\bullet$ for all $f$, $g \in \mathcal{L}^0$.

**(b)** It is now easy to check that, for all $u$, $v$, $w \in L^0$ and $c \in \mathbb{R}$,

$u \times (v \times w) = (u \times v) \times w$,

$u \times e = e \times u = u$,

where $e = \chi X^\bullet$ is the equivalence class of the function with constant value 1,

$c(u \times v) = cu \times v = u \times cv$,

$u \times (v + w) = (u \times v) + (u \times w)$,

$(u + v) \times w = (u \times w) + (v \times w)$,

$u \times v = v \times u$,

$|u \times v| = |u| \times |v|$,

$u \times v = 0$ iff $|u| \wedge |v| = 0$,

$|u| \le |v|$ iff there is a $w$ such that $|w| \le e$ and $u = v \times w$.

**241I The action of Borel functions on $L^0$** Let $(X, \Sigma, \mu)$ be a measure space and $h : \mathbb{R} \to \mathbb{R}$ a Borel measurable function. Then $hf \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$ for every $f \in \mathcal{L}^0$ (241Be) and $hf =_{\text{a.e.}} hg$ whenever $f =_{\text{a.e.}} g$. So we have a function $\bar{h} : L^0 \to L^0$ defined by setting $\bar{h}(f^\bullet) = (hf)^\bullet$ for every $f \in \mathcal{L}^0$. For instance, if $u \in L^0$ and $p \ge 1$, we can consider $|u|^p = \bar{h}(u)$ where $h(x) = |x|^p$ for $x \in \mathbb{R}$.

**241J Complex $L^0$** The ideas of this chapter, like those of Chapters 22-23, are often applied to spaces based on complex-valued functions instead of real-valued functions. Let $(X, \Sigma, \mu)$ be a measure space.

**(a)** We may write $\mathcal{L}^0_{\mathbb{C}} = \mathcal{L}^0_{\mathbb{C}}(\mu)$ for the space of complex-valued functions $f$ such that $\operatorname{dom} f$ is a conegligible subset of $X$ and there is a conegligible subset $E \subseteq X$ such that $f\restriction E$ is measurable; that is, such that the real and imaginary parts of $f$ both belong to $\mathcal{L}^0(\mu)$. Next, $L^0_{\mathbb{C}} = L^0_{\mathbb{C}}(\mu)$ will be the space of equivalence classes in $\mathcal{L}^0_{\mathbb{C}}$ under the equivalence relation $=_{\text{a.e.}}$.

**(b)** Using just the same formulae as in 241D, it is easy to describe addition and scalar multiplication rendering $L^0_{\mathbb{C}}$ a linear space over $\mathbb{C}$. We no longer have quite the same kind of order structure, but we can identify a 'real part', being

$$\{f^\bullet : f \in \mathcal{L}^0_{\mathbb{C}} \text{ is real a.e.}\},$$

obviously identifiable with the real linear space $L^0$, and corresponding maps $u \mapsto \mathcal{R}\text{e}(u)$, $u \mapsto \mathcal{I}\text{m}(u) : L^0_{\mathbb{C}} \to L^0$ such that $u = \mathcal{R}\text{e}(u) + i\,\mathcal{I}\text{m}(u)$ for every $u$. Moreover, we have a notion of 'modulus', writing

$$|f^\bullet| = |f|^\bullet \in L^0 \text{ for every } f \in \mathcal{L}^0_{\mathbb{C}},$$

satisfying the basic relations $|cu| = |c||u|$, $|u + v| \le |u| + |v|$ for $u$, $v \in L^0_{\mathbb{C}}$ and $c \in \mathbb{C}$, as in 241Ef. We do of course still have a multiplication on $L^0_{\mathbb{C}}$, for which all the formulae in 241H are still valid.

**(c)** The following fact is useful. For any $u \in L^0_{\mathbb{C}}$, $|u|$ is the supremum in $L^0$ of $\{\mathcal{R}\text{e}(\zeta u) : \zeta \in \mathbb{C}, |\zeta| = 1\}$. **P** (i) If $|\zeta| = 1$, then $\mathcal{R}\text{e}(\zeta u) \le |\zeta u| = |u|$. So $|u|$ is an upper bound of $\{\mathcal{R}\text{e}(\zeta u) : |\zeta| = 1\}$. (ii) If $v \in L^0$ and $\mathcal{R}\text{e}(\zeta u) \le v$ whenever $|\zeta| = 1$, then express $u$, $v$ as $f^\bullet$, $g^\bullet$ where $f : X \to \mathbb{C}$ and $g : X \to \mathbb{R}$ are measurable. For any $q \in \mathbb{Q}$, $x \in X$ set $f_q(x) = \mathcal{R}\text{e}(e^{iqx} f(x))$. Then $f_q \le_{\text{a.e.}} g$. Accordingly $H = \{x : f_q(x) \le g(x)$ for every $q \in \mathbb{Q}\}$ is conegligible. But of course $H = \{x : |f(x)| \le g(x)\}$, so $|f| \le_{\text{a.e.}} g$ and $|u| \le v$. As $v$ is arbitrary, $|u|$ is the least upper bound of $\{\mathcal{R}\text{e}(\zeta u) : |\zeta| = 1\}$. **Q**

**241X Basic exercises >(a)** Let $X$ be a set, and let $\mu$ be counting measure on $X$ (112Bd). Show that $L^0(\mu)$ can be identified with $\mathcal{L}^0(\mu) = \mathbb{R}^X$.

**>(b)** Let $(X, \Sigma, \mu)$ be a measure space and $\hat\mu$ the completion of $\mu$. Show that $\mathcal{L}^0(\mu) = \mathcal{L}^0(\hat\mu)$ and $L^0(\mu) = L^0(\hat\mu)$.

**(c)** Let $(X, \Sigma, \mu)$ be a measure space. (i) Show that for every $u \in L^0(\mu)$ we may define an outer measure $\theta_u : \mathcal{P}\mathbb{R} \to [0, \infty]$ by writing $\theta_u(A) = \mu^* f^{-1}[A]$ whenever $A \subseteq \mathbb{R}$ and $f \in \mathcal{L}^0(\mu)$ is such that $f^\bullet = u$. (ii) Show that the measure defined from $\theta_u$ by Carathéodory's method measures every Borel subset of $\mathbb{R}$.

**(d)** Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum $(X, \Sigma, \mu)$ (214L). (i) Writing $\phi_i : X_i \to X$ for the canonical maps (in the construction of 214L, $\phi_i(x) = (x, i)$ for $x \in X_i$), show that $f \mapsto \langle f\phi_i\rangle_{i\in I}$ is a bijection between $\mathcal{L}^0(\mu)$ and $\prod_{i\in I} \mathcal{L}^0(\mu_i)$. (ii) Show that this corresponds to a bijection between $L^0(\mu)$ and $\prod_{i\in I} L^0(\mu_i)$.

**(e)** Let $U$ be a Dedekind $\sigma$-complete Riesz space and $A \subseteq U$ a non-empty countable set which is bounded below in $U$. Show that $\inf A$ is defined in $U$.

**(f)** Let $U$ be a Dedekind complete Riesz space and $A \subseteq U$ a non-empty set which is bounded below in $U$. Show that $\inf A$ is defined in $U$.

**(g)** Let $(X, \Sigma, \mu)$ and $(Y, \mathrm{T}, \nu)$ be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. (i) Show that we have a map $T : L^0(\nu) \to L^0(\mu)$ defined by setting $Tg^\bullet = (g\phi)^\bullet$ for every $g \in \mathcal{L}^0(\nu)$. (ii) Show that $T$ is linear, that $T(v \times w) = Tv \times Tw$ for all $v$, $w \in L^0(\nu)$, and that $T(\sup_{n\in\mathbb{N}} v_n) = \sup_{n\in\mathbb{N}} Tv_n$ whenever $\langle v_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^0(\nu)$ with an upper bound in $L^0(\nu)$.

**>(h)** Let $(X, \Sigma, \mu)$ be a measure space. Suppose that $r \ge 1$ and that $h : \mathbb{R}^r \to \mathbb{R}$ is a Borel measurable function. Show that there is a function $\bar h : L^0(\mu)^r \to L^0(\mu)$ defined by writing

$$\bar h(f_1^\bullet, \dots , f_r^\bullet) = (h(f_1, \dots , f_r))^\bullet$$

for $f_1, \dots , f_r \in \mathcal{L}^0(\mu)$.

**(i)** Let $(X,\Sigma,\mu)$ be a measure space and $g$, $h$, $\langle g_n\rangle_{n\in\mathbb{N}}$ Borel measurable functions from $\mathbb{R}$ to itself; write $\bar g$, $\bar h$, $\bar g_n$ for the corresponding functions from $L^0=L^0(\mu)$ to itself (241I). (i) Show that

$$\bar g(u)+\bar h(u)=\overline{g+h}(u),\quad \bar g(u)\times\bar h(u)=\overline{g\times h}(u),\quad \bar g(\bar h(u))=\overline{gh}(u)$$

for every $u\in L^0$. (ii) Show that if $g(t)\le h(t)$ for every $t\in\mathbb{R}$, then $\bar g(u)\le\bar h(u)$ for every $u\in L^0$. (iii) Show that if $g$ is non-decreasing, then $\bar g(u)\le\bar g(v)$ whenever $u\le v$ in $L^0$. (iv) Show that if $h(t)=\sup_{n\in\mathbb{N}}g_n(t)$ for every $t\in\mathbb{R}$, then $\bar h(u)=\sup_{n\in\mathbb{N}}\bar g_n(u)$ in $L^0$ for every $u\in L^0$.

**241Y Further exercises (a)** Let $U$ be any Riesz space. For $u\in U$ write $|u|=u\vee(-u)$, $u^+=u\vee 0$, $u^-=(-u)\vee 0$. Show that, for any $u$, $v\in U$,

$$u=u^+-u^-,\quad |u|=u^++u^-=u^+\vee u^-,\quad u^+\wedge u^-=0,$$

$$u\vee v=\frac{1}{2}(u+v+|u-v|)=u+(v-u)^+,$$

$$u\wedge v=\frac{1}{2}(u+v-|u-v|)=u-(u-v)^+,$$

$$|u+v|\le|u|+|v|.$$

**(b)** Let $U$ be a partially ordered linear space and $N$ a linear subspace of $U$ such that whenever $u$, $u'\in N$ and $u'\le v\le u$ then $v\in N$. (i) Show that the linear space quotient $U/N$ is a partially ordered linear space if we say that $u^\bullet\le v^\bullet$ in $U/N$ iff there is a $w\in N$ such that $u\le v+w$ in $U$. (ii) Show that in this case $U/N$ is a Riesz space if $U$ is a Riesz space and $|u|\in N$ for every $u\in N$.

**(c)** Let $(X,\Sigma,\mu)$ be a measure space. Write $\mathcal{L}^0_\Sigma$ for the space of all measurable functions from $X$ to $\mathbb{R}$, and $\mathcal{N}$ for the subspace of $\mathcal{L}^0_\Sigma$ consisting of measurable functions which are zero almost everywhere. (i) Show that $\mathcal{L}^0_\Sigma$ is a Dedekind $\sigma$-complete Riesz space. (ii) Show that $L^0(\mu)$ can be identified, as ordered linear space, with the quotient $\mathcal{L}^0_\Sigma/\mathcal{N}$ as defined in 241Yb above.

**(d)** Show that any Dedekind $\sigma$-complete Riesz space is Archimedean.

**(e)** A Riesz space $U$ is said to have the **countable sup property** if for every $A\subseteq U$ with a least upper bound in $U$, there is a countable $B\subseteq A$ such that $\sup B=\sup A$. Show that if $(X,\Sigma,\mu)$ is a semi-finite measure space, then it is $\sigma$-finite iff $L^0(\mu)$ has the countable sup property.

**(f)** Let $(X,\Sigma,\mu)$ be a measure space and $\tilde\mu$ the c.l.d. version of $\mu$ (213E). (i) Show that $\mathcal{L}^0(\mu)\subseteq\mathcal{L}^0(\tilde\mu)$. (ii) Show that this inclusion defines a linear operator $T:L^0(\mu)\to L^0(\tilde\mu)$ such that $T(u\times v)=Tu\times Tv$ for all $u$, $v\in L^0(\mu)$. (iii) Show that whenever $v>0$ in $L^0(\tilde\mu)$ there is a $u\ge 0$ in $L^0(\mu)$ such that $0<Tu\le v$. (iv) Show that $T(\sup A)=\sup T[A]$ whenever $A\subseteq L^0(\mu)$ is a non-empty set with a least upper bound in $L^0(\mu)$. (v) Show that $T$ is injective iff $\mu$ is semi-finite. (vi) Show that if $\mu$ is localizable, then $T$ is an isomorphism for the linear and order structures of $L^0(\mu)$ and $L^0(\tilde\mu)$. (*Hint*: 213Hb.)

**(g)** Let $(X,\Sigma,\mu)$ be a measure space and $Y$ any subset of $X$; let $\mu_Y$ be the subspace measure on $Y$. (i) Show that $\mathcal{L}^0(\mu_Y)=\{f{\restriction}Y:f\in\mathcal{L}^0(\mu)\}$. (ii) Show that there is a canonical surjection $T:L^0(\mu)\to L^0(\mu_Y)$ defined by setting $T(f^\bullet)=(f{\restriction}Y)^\bullet$ for every $f\in\mathcal{L}^0(\mu)$, which is linear and multiplicative and preserves finite suprema and infima, so that (in particular) $T(|u|)=|Tu|$ for every $u\in L^0(\mu)$. (iii) Show that $T$ is injective iff $Y$ has full outer measure.

**(h)** Suppose, in 241Yg, that $Y\in\Sigma$. Explain how $L^0(\mu_Y)$ may be identified (as ordered linear space) with the subspace $\{u:u\times\chi(X\setminus Y)^\bullet=0\}$ of $L^0(\mu)$.

**(i)** Let $(X,\Sigma,\mu)$ be a measure space, and $h:\mathbb{R}\to\mathbb{R}$ a non-decreasing function which is continuous on the left. Show that if $A\subseteq L^0=L^0(\mu)$ is a non-empty set with a supremum $v\in L^0$, then $\bar h(v)=\sup_{u\in A}\bar h(u)$, where $\bar h:L^0\to L^0$ is the function described in 241I.

**241 Notes and comments** As hinted in 241Ya and 241Yd, the elementary properties of the space $L^0$ which take up most of this section are strongly interdependent; it is not difficult to develop a theory of 'Riesz algebras' to incorporate the ideas of 241H into the rest. (Indeed, I sketch such a theory in §§352-353 in the next volume, under the name '$f$-algebra'.)

If we write $\mathcal{L}^0_\Sigma$ for the space of measurable functions from $X$ to $\mathbb{R}$, then $\mathcal{L}^0_\Sigma$ is also a Dedekind $\sigma$-complete Riesz space, and $L^0$ can be identified with the quotient $\mathcal{L}^0_\Sigma/\mathcal{N}$, writing $\mathcal{N}$ for the set of functions in $\mathcal{L}^0_\Sigma$ which are zero almost everywhere. (To do this properly, we need a theory of quotients of ordered linear spaces; see 241Yb-241Yc above.) Of course $\mathcal{L}^0$, as I define it, is not quite a linear space. I choose the slightly more awkward description of $L^0$ as a space of equivalence classes in $\mathcal{L}^0$ rather than in $\mathcal{L}^0_\Sigma$ because it frequently happens in practice that a member of $L^0$ arises from a member of $\mathcal{L}^0$ which is either not defined at every point of the underlying space, or not quite measurable; and to adjust such a function so that it becomes a member of $\mathcal{L}^0_\Sigma$, while trivial, is an arbitrary process which to my mind is liable to distort the true nature of such a construction. Of course the same argument could be used in favour of a slightly larger space, the space $\mathcal{L}^0_\infty$ of $\mu$-virtually measurable $[-\infty,\infty]$-valued functions defined and finite almost everywhere, relying on 135E rather than on 121E-121F. But I maintain that the operation of restricting a function in $\mathcal{L}^0_\infty$ to the set on which it is finite is *not* arbitrary, but canonical and entirely natural.

Reading the exposition above – or, for that matter, scanning the rest of this chapter – you are sure to notice a plethora of $^\bullet$s, adding a distinctive character to the pages which, I expect you will feel, is disagreeable to the eye and daunting, or at any rate wearisome, to the spirit. Many, perhaps most, authors prefer to simplify the typography by using the same symbol for a function in $\mathcal{L}^0$ or $\mathcal{L}^0_\Sigma$ and for its equivalence class in $L^0$; and indeed it is common to use syntax which does not distinguish between them either, so that an object which has been defined as a member of $L^0$ will suddenly become a function with actual values at points of the underlying measure space. I prefer to maintain a rigid distinction; you must choose for yourself whether to follow me. Since I have chosen the more cumbersome form, I suppose the burden of proof is on me, to justify my decision. (i) Anyone would agree that there is at least a formal difference between a function and a set of functions. This by itself does not justify insisting on the difference in every sentence; mathematical exposition would be impossible if we always insisted on consistency in such questions as whether (for instance) the number 3 belonging to the set $\mathbb{N}$ of natural numbers is exactly the same object as the number 3 belonging to the set $\mathbb{C}$ of complex numbers, or the ordinal 3. But the difference between an object and a set to which it belongs is a sufficient difference in kind to make any confusion extremely dangerous, and while I agree that you can study this topic without using different symbols for $f$ and $f^\bullet$, I do not think you can ever safely escape a mental distinction for more than a few lines of argument. (ii) As a teacher, I have to say that quite a few students, encountering this material for the first time, are misled by any failure to make the distinction between $f$ and $f^\bullet$ into believing that no distinction need be made; and – as a teacher – I always insist on a student convincing me, by correctly writing out the more pedantic forms of the arguments for a few weeks, that he understands the manipulations necessary, before I allow him to go his own way. (iii) The reason why it *is* possible to evade the distinction in certain types of argument is just that the Dedekind $\sigma$-complete Riesz space $\mathcal{L}^0_\Sigma$ parallels the Dedekind $\sigma$-complete Riesz space $L^0$ so closely that any proposition involving only countably many members of these spaces is likely to be valid in one if and only if it is valid in the other. In my view, the implications of this correspondence are at the very heart of measure theory. I prefer therefore to keep it constantly conspicuous, reminding myself through symbolism that every theorem has a Siamese twin, and rising to each challenge to express the twin theorem in an appropriate language. (iv) There are ways in which $\mathcal{L}^0_\Sigma$ and $L^0$ are actually very different, and many interesting ideas can be expressed only in a language which keeps them clearly separated.

For more than half my life now I have felt that these points between them are sufficient reason for being consistent in maintaining the formal distinction between $f$ and $f^\bullet$. You may feel that in (iii) and (iv) of the last paragraph I am trying to have things both ways; I am arguing that both the similarities and the differences between $L^0$ and $\mathcal{L}^0$ support my case. Indeed that is exactly my position. If they were totally different, using the same language for both would not give rise to confusion; if they were essentially the same, it would not matter if we were sometimes unclear which we were talking about.

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## 242 $L^1$

While the space $L^0$ treated in the previous section is of very great intrinsic interest, its chief use in the elementary theory is as a space in which some of the most important spaces of functional analysis are embedded. In the next few sections I introduce these one at a time.

The first is the space $L^1$ of equivalence classes of integrable functions. The importance of this space is not only that it offers a language in which to express those many theorems about integrable functions which do not depend on the differences between two functions which are equal almost everywhere. It can also appear as the natural space in which to seek solutions to a wide variety of integral equations, and as the completion of a space of continuous functions.

**242A The space** $L^1$ Let $(X, \Sigma, \mu)$ be any measure space.

**(a)** Let $\mathcal{L}^1 = \mathcal{L}^1(\mu)$ be the set of real-valued functions, defined on subsets of $X$, which are integrable over $X$. Then $\mathcal{L}^1 \subseteq \mathcal{L}^0 = \mathcal{L}^0(\mu)$, as defined in §241, and, for $f \in \mathcal{L}^0$, we have $f \in \mathcal{L}^1$ iff there is a $g \in \mathcal{L}^1$ such that $|f| \leq_{\text{a.e.}} g$; if $f \in \mathcal{L}^1$, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then $g \in \mathcal{L}^1$. (See 122P-122R.)

**(b)** Let $L^1 = L^1(\mu) \subseteq L^0 = L^0(\mu)$ be the set of equivalence classes of members of $\mathcal{L}^1$. If $f$, $g \in \mathcal{L}^1$ and $f =_{\text{a.e.}} g$ then $\int f = \int g$ (122Rb). Accordingly we may define a functional $\int$ on $L^1$ by writing $\int f^\bullet = \int f$ for every $f \in \mathcal{L}^1$.

**(c)** It will be convenient to be able to write $\int_A u$ for $u \in L^1$, $A \subseteq X$; this may be defined by saying that $\int_A f^\bullet = \int_A f$ for every $f \in \mathcal{L}^1$, where the integral is defined in 214D. **P** I have only to check that if $f =_{\text{a.e.}} g$ then $\int_A f = \int_A g$; and this is because $f \restriction A = g \restriction A$ almost everywhere in $A$. **Q**

If $E \in \Sigma$ and $u \in L^1$ then $\int_E u = \int u \times (\chi E)^\bullet$; this is because $\int_E f = \int f \times \chi E$ for every integrable function $f$ (131Fa).

**(d)** If $u \in L^1$, there is a $\Sigma$-measurable, $\mu$-integrable function $f : X \to \mathbb{R}$ such that $f^\bullet = u$. **P** As noted in 241Bk, there is a measurable $f : X \to \mathbb{R}$ such that $f^\bullet = u$; but of course $f$ is integrable because it is equal almost everywhere to some integrable function. **Q**

**242B Theorem** Let $(X, \Sigma, \mu)$ be any measure space. Then $L^1(\mu)$ is a linear subspace of $L^0(\mu)$ and $\int : L^1 \to \mathbb{R}$ is a linear functional.

**proof** If $u$, $v \in L^1 = L^1(\mu)$ and $c \in \mathbb{R}$ let $f$, $g$ be integrable functions such that $u = f^\bullet$ and $v = g^\bullet$; then $f + g$ and $cf$ are integrable, so $u + v = (f + g)^\bullet$ and $cu = (cf)^\bullet$ belong to $L^1$. Also

$$\int u + v = \int f + g = \int f + \int g = \int u + \int v$$

and

$$\int cu = \int cf = c \int f = c \int u.$$

**242C The order structure of** $L^1$ Let $(X, \Sigma, \mu)$ be any measure space.

**(a)** $L^1 = L^1(\mu)$ has an order structure derived from that of $L^0 = L^0(\mu)$ (241E); that is, $f^\bullet \leq g^\bullet$ iff $f \leq g$ a.e. Being a linear subspace of $L^0$, $L^1$ must be a partially ordered linear space; the two conditions of 241Ec are obviously inherited by linear subspaces.

Note also that if $u$, $v \in L^1$ and $u \leq v$ then $\int u \leq \int v$, because if $f$, $g$ are integrable functions and $f \leq_{\text{a.e.}} g$ then $\int f \leq \int g$ (122Od).

**(b)** If $u \in L^0$, $v \in L^1$ and $|u| \leq |v|$ then $u \in L^1$. **P** Let $f \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$, $g \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ be such that $u = f^\bullet$ and $v = g^\bullet$; then $g$ is integrable and $|f| \leq_{\text{a.e.}} |g|$, so $f$ is integrable and $u \in L^1$. **Q**

**(c)** In particular, $|u| \in L^1$ whenever $u \in L^1$, and

$$|\int u| = \max(\int u, \int(-u)) \le \int |u|,$$

because $u$, $-u \le |u|$.

**(d)** Because $|u| \in L^1$ for every $u \in L^1$,

$$u \vee v = \frac{1}{2}(u + v + |u - v|), \quad u \wedge v = \frac{1}{2}(u + v - |u - v|)$$

belong to $L^1$ for all $u$, $v \in L^1$. But if $w \in L^1$ we surely have

$$w \le u \;\&\; w \le v \iff w \le u \wedge v,$$

$$w \ge u \;\&\; w \ge v \iff w \ge u \vee v$$

because these are true for all $w \in L^0$, so $u \vee v = \sup\{u, v\}$ and $u \wedge v = \inf\{u, v\}$ in $L^1$. Thus $L^1$ is, in itself, a Riesz space.

**(e)** Note that if $u \in L^1$, then $u \ge 0$ iff $\int_E u \ge 0$ for every $E \in \Sigma$; this is because if $f$ is an integrable function on $X$ and $\int_E f \ge 0$ for every $E \in \Sigma$, then $f \ge 0$ a.e. (131Fb). More generally, if $u$, $v \in L^1$ and $\int_E u \le \int_E v$ for every $E \in \Sigma$, then $u \le v$. It follows at once that if $u$, $v \in L^1$ and $\int_E u = \int_E v$ for every $E \in \Sigma$, then $u = v$ (cf. 131Fc).

**(f)** If $u \ge 0$ in $L^1$, there is a non-negative $f \in \mathcal{L}^1$ such that $f^\bullet = u$ (compare 241Eg).

**242D The norm of** $L^1$ Let $(X, \Sigma, \mu)$ be any measure space.

**(a)** For $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ I write $\|f\|_1 = \int |f| \in [0, \infty[$. For $u \in L^1 = L^1(\mu)$ set $\|u\|_1 = \int |u|$, so that $\|f^\bullet\|_1 = \|f\|_1$ for every $f \in \mathcal{L}^1$. Then $\|\;\|_1$ is a norm on $L^1$. **P** (i) If $u$, $v \in L^1$ then $|u + v| \le |u| + |v|$, by 241Ee, so

$$\|u + v\|_1 = \int |u + v| \le \int |u| + |v| = \int |u| + \int |v| = \|u\|_1 + \|v\|_1.$$

(ii) If $u \in L^1$ and $c \in \mathbb{R}$ then

$$\|cu\|_1 = \int |cu| = \int |c||u| = |c| \int |u| = |c|\|u\|_1.$$

(iii) If $u \in L^1$ and $\|u\|_1 = 0$, express $u$ as $f^\bullet$, where $f \in \mathcal{L}^1$; then $\int |f| = \int |u| = 0$. Because $|f|$ is non-negative, it must be zero almost everywhere (122Rc), so $f = 0$ a.e. and $u = 0$ in $L^1$. **Q**

**(b)** Thus $L^1$, with $\|\;\|_1$, is a normed space and $\int : L^1 \to \mathbb{R}$ is a linear operator; observe that $\|\int\| \le 1$, because

$$|\int u| \le \int |u| = \|u\|_1$$

for every $u \in L^1$.

**(c)** If $u$, $v \in L^1$ and $|u| \le |v|$, then

$$\|u\|_1 = \int |u| \le \int |v| = \|v\|_1.$$

In particular, $\|u\|_1 = \||u|\|_1$ for every $u \in L^1$.

**(d)** Note the following property of the normed Riesz space $L^1$: if $u$, $v \in L^1$ and $u$, $v \ge 0$, then

$$\|u + v\|_1 = \int u + v = \int u + \int v = \|u\|_1 + \|v\|_1.$$

**(e)** The set $(L^1)^+ = \{u : u \ge 0\}$ is closed in $L^1$. **P** If $v \in L^1$, $u \in (L^1)^+$ then $\|u - v\|_1 \ge \|v \wedge 0\|_1$; this is because if $f$, $g \in \mathcal{L}^1$ and $f \ge 0$ a.e., $|f(x) - g(x)| \ge |\min(g(x), 0)|$ whenever $f(x)$ and $g(x)$ are both defined and $f(x) \ge 0$, which is almost everywhere, so

$$\|u - v\|_1 = \int |f - g| \ge \int |g \wedge \mathbf{0}| = \|v \wedge 0\|_1.$$

Now this means that if $v \in L^1$ and $v \not\ge 0$, the ball $\{w : \|w - v\|_1 < \delta\}$ does not meet $(L^1)^+$, where $\delta = \|v \wedge 0\|_1 > 0$ because $v \wedge 0 \ne 0$. Thus $L^1 \setminus (L^1)^+$ is open and $(L^1)^+$ is closed. **Q**

**242E** For the next result we need a variant of B.Levi's theorem.

**Lemma** Let $(X, \Sigma, \mu)$ be a measure space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of $\mu$-integrable real-valued functions such that $\sum_{n=0}^\infty \int |f_n| < \infty$. Then $f = \sum_{n=0}^\infty f_n$ is integrable and

$$\int f = \sum_{n=0}^\infty \int f_n, \quad \int |f| \le \sum_{n=0}^\infty \int |f_n|.$$

**proof (a)** Suppose first that every $f_n$ is non-negative. Set $g_n = \sum_{k=0}^n f_k$ for each $n$; then $\langle g_n \rangle_{n\in\mathbb{N}}$ is increasing a.e. and

$$\lim_{n\to\infty} \int g_n = \sum_{k=0}^\infty \int f_k$$

is finite, so by B.Levi's theorem (123A) $f = \lim_{n\to\infty} g_n$ is integrable and

$$\int f = \lim_{n\to\infty} \int g_n = \sum_{k=0}^\infty \int f_k.$$

In this case, of course,

$$\int |f| = \int f = \sum_{n=0}^\infty \int f_n = \sum_{n=0}^\infty \int |f_n|.$$

**(b)** For the general case, set $f_n^+ = \frac{1}{2}(|f_n| + f_n)$, $f_n^- = \frac{1}{2}(|f_n| - f_n)$, as in 241Ef; then $f_n^+$ and $f_n^-$ are non-negative integrable functions, and

$$\sum_{n=0}^\infty \int f_n^+ + \sum_{n=0}^\infty \int f_n^- = \sum_{n=0}^\infty \int |f_n| < \infty.$$

So $h_1 = \sum_{n=0}^\infty f_n^+$ and $h_2 = \sum_{n=0}^\infty f_n^-$ are both integrable. Now $f =_{\text{a.e.}} h_1 - h_2$, so

$$\int f = \int h_1 - \int h_2 = \sum_{n=0}^\infty \int f_n^+ - \sum_{n=0}^\infty \int f_n^- = \sum_{n=0}^\infty \int f_n.$$

Finally

$$\int |f| \le \int |h_1| + \int |h_2| = \sum_{n=0}^\infty \int f_n^+ + \sum_{n=0}^\infty \int f_n^- = \sum_{n=0}^\infty \int |f_n|.$$

**242F Theorem** For any measure space $(X, \Sigma, \mu)$, $L^1(\mu)$ is complete under its norm $\| \ \|_1$.

**proof** Let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a sequence in $L^1$ such that $\|u_{n+1} - u_n\|_1 \le 4^{-n}$ for every $n \in \mathbb{N}$. Choose integrable functions $f_n$ such that $f_0^\bullet = u_0$, $f_{n+1}^\bullet = u_{n+1} - u_n$ for each $n \in \mathbb{N}$. Then

$$\sum_{n=0}^\infty \int |f_n| = \|u_0\|_1 + \sum_{n=0}^\infty \|u_{n+1} - u_n\|_1 < \infty.$$

So $f = \sum_{n=0}^\infty f_n$ is integrable, by 242E, and $u = f^\bullet \in L^1$. Set $g_n = \sum_{j=0}^n f_j$ for each $n$; then $g_n^\bullet = u_n$, so

$$\|u - u_n\|_1 = \int |f - g_n| \le \int \sum_{j=n+1}^\infty |f_j| \le \sum_{j=n+1}^\infty 4^{-j} = 4^{-n}/3$$

for each $n$. Thus $u = \lim_{n\to\infty} u_n$ in $L^1$. As $\langle u_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $L^1$ is complete (2A4E).

**242G Definition** It will be convenient, for later reference, to introduce the following phrase. A **Banach lattice** is a Riesz space $U$ together with a norm $\| \ \|$ on $U$ such that (i) $\|u\| \le \|v\|$ whenever $u$, $v \in U$ and $|u| \le |v|$, writing $|u|$ for $u \vee (-u)$, as in 241Ee (ii) $U$ is complete under $\| \ \|$. Thus 242Dc and 242F amount to saying that the normed Riesz space $(L^1, \| \ \|_1)$ is a Banach lattice.

**242H $L^1$ as a Riesz space** We can discuss the ordered linear space $L^1$ in the language already used in 241E-241G for $L^0$.

**Theorem** Let $(X, \Sigma, \mu)$ be any measure space. Then $L^1 = L^1(\mu)$ is Dedekind complete.

**proof (a)** Let $A \subseteq L^1$ be any non-empty set which is bounded above in $L^1$. Set

$$A' = \{u_0 \vee \ldots \vee u_n : u_0, \ldots, u_n \in A\}.$$

Then $A \subseteq A'$, $A'$ has the same upper bounds as $A$ and $u \vee v \in A'$ for all $u$, $v \in A'$. Taking $w_0$ to be any upper bound of $A$ and $A'$, we have $\int u \le \int w_0$ for every $u \in A'$, so $\gamma = \sup_{u\in A'} \int u$ is defined in $\mathbb{R}$. For each $n \in \mathbb{N}$, choose $u_n \in A'$ such that $\int u_n \ge \gamma - 2^{-n}$. Because $L^0 = L^0(\mu)$ is Dedekind $\sigma$-complete (241Ga), $u^* = \sup_{n\in\mathbb{N}} u_n$ is defined in $L^0$, and $u_0 \le u^* \le w_0$ in $L^0$. Consequently

$$0 \le u^* - u_0 \le w_0 - u_0$$

in $L^0$. But $w_0 - u_0 \in L^1$, so $u^* - u_0 \in L^1$ (242Cb) and $u^* \in L^1$.

**(b)** The point is that $u^*$ is an upper bound for $A$. **P** If $u \in A$, then $u \vee u_n \in A'$ for every $n$, so

$$\|u - u \wedge u^*\|_1 = \int u - u \wedge u^* \le \int u - u \wedge u_n$$

(because $u \wedge u_n \le u_n \le u^*$, so $u \wedge u_n \le u \wedge u^*$)

$$= \int u \vee u_n - u_n$$

(because $u \vee u_n + u \wedge u_n = u + u_n$ – see the formulae in 242Cd)

$$= \int u \vee u_n - \int u_n \le \gamma - (\gamma - 2^{-n}) = 2^{-n}$$

for every $n$; so $\|u - u \wedge u^*\|_1 = 0$. But this means that $u = u \wedge u^*$, that is, that $u \le u^*$. As $u$ is arbitrary, $u^*$ is an upper bound for $A$. **Q**

**(c)** On the other hand, any upper bound for $A$ is surely an upper bound for $\{u_n : n \in \mathbb{N}\}$, so is greater than or equal to $u^*$. Thus $u^* = \sup A$ in $L^1$. As $A$ is arbitrary, $L^1$ is Dedekind complete.

**Remark** Note that the order-completeness of $L^1$, unlike that of $L^0$, does not depend on any particular property of the measure space $(X, \Sigma, \mu)$.

**242I The Radon-Nikodým theorem** I think it is worth re-writing the Radon-Nikodým theorem (232E) in the language of this chapter.

**Theorem** Let $(X, \Sigma, \mu)$ be a measure space. Then there is a canonical bijection between $L^1 = L^1(\mu)$ and the set of truly continuous additive functionals $\nu : \Sigma \to \mathbb{R}$, given by the formula

$$\nu F = \int_F u \text{ for } F \in \Sigma,\ u \in L^1.$$

**Remark** Recall that if $\mu$ is $\sigma$-finite, then the truly continuous additive functionals are just the absolutely continuous countably additive functionals; and that if $\mu$ is totally finite, then all absolutely continuous (finitely) additive functionals are truly continuous (232Bd).

**proof** For $u \in L^1$, $F \in \Sigma$ set $\nu_u F = \int_F u$. If $u \in L^1$, there is an integrable function $f$ such that $f^\bullet = u$, in which case

$$F \mapsto \nu_u F = \int_F f : \Sigma \to \mathbb{R}$$

is additive and truly continuous, by 232D. If $\nu : \Sigma \to \mathbb{R}$ is additive and truly continuous, then by 232E there is an integrable function $f$ such that $\nu F = \int_F f$ for every $F \in \Sigma$; setting $u = f^\bullet$ in $L^1$, $\nu = \nu_u$. Finally, if $u$, $v$ are distinct members of $L^1$, there is an $F \in \Sigma$ such that $\int_F u \ne \int_F v$ (242Ce), so that $\nu_u \ne \nu_v$; thus $u \mapsto \nu_u$ is injective as well as surjective.

**242J Conditional expectations revisited** We now have the machinery necessary for a new interpretation of some of the ideas of §233.

**(a)** Let $(X, \Sigma, \mu)$ be a measure space, and $\mathrm{T}$ a $\sigma$-subalgebra of $\Sigma$, as in 233A. Then $(X, \mathrm{T}, \mu\restriction\mathrm{T})$ is a measure space, and $\mathcal{L}^0(\mu\restriction\mathrm{T}) \subseteq \mathcal{L}^0(\mu)$; moreover, if $f$, $g \in \mathcal{L}^0(\mu\restriction\mathrm{T})$, then $f = g$ $(\mu\restriction\mathrm{T})$-a.e. iff $f = g$ $\mu$-a.e. **P** There are $\mu\restriction\mathrm{T}$-conegligible sets $F$, $G \in \mathrm{T}$ such that $f\restriction F$ and $g\restriction G$ are $\mathrm{T}$-measurable; set

$$E = \{x : x \in F \cap G,\ f(x) \ne g(x)\} \in \mathrm{T};$$

then

$$f = g \ (\mu\restriction\mathrm{T})\text{-a.e.} \iff (\mu\restriction\mathrm{T})(E) = 0 \iff \mu E = 0 \iff f = g\ \mu\text{-a.e.} \ \mathbf{Q}$$

Accordingly we have a canonical map $S : L^0(\mu\restriction\mathrm{T}) \to L^0(\mu)$ defined by saying that if $u \in L^0(\mu\restriction\mathrm{T})$ is the equivalence class of $f \in \mathcal{L}^0(\mu\restriction\mathrm{T})$, then $Su$ is the equivalence class of $f$ in $L^0(\mu)$. It is easy to check, working through the operations described in 241D, 241E and 241H, that $S$ is linear, injective and order-preserving, and that $|Su| = S|u|$, $S(u \vee v) = Su \vee Sv$ and $S(u \times v) = Su \times Sv$ for $u$, $v \in L^0(\mu\restriction\mathrm{T})$.

**(b)** Next, if $f \in \mathcal{L}^1(\mu\restriction\mathrm{T})$, then $f \in \mathcal{L}^1(\mu)$ and $\int f d\mu = \int f d(\mu\restriction\mathrm{T})$ (233B); so $Su \in L^1(\mu)$ and $\|Su\|_1 = \|u\|_1$ for every $u \in L^1(\mu\restriction\mathrm{T})$.

Observe also that every member of $L^1(\mu) \cap S[L^0(\mu\restriction\mathrm{T})]$ is actually in $S[L^1(\mu\restriction\mathrm{T})]$. **P** Take $u \in L^1(\mu) \cap S[L^0(\mu\restriction\mathrm{T})]$. Then $u$ is expressible both as $f^\bullet$ where $f \in \mathcal{L}^1(\mu)$, and as $g^\bullet$ where $g \in \mathcal{L}^0(\mu\restriction\mathrm{T})$. So $g =_{\text{a.e.}} f$, and $g$ is $\mu$-integrable, therefore $(\mu\restriction\mathrm{T})$-integrable (233B again). **Q**

This means that $S : L^1(\mu\restriction\mathrm{T}) \to L^1(\mu) \cap S[L^0(\mu\restriction\mathrm{T})]$ is a bijection.

**(c)** Now suppose that $\mu X = 1$, so that $(X, \Sigma, \mu)$ is a probability space. Recall that $g$ is a conditional expectation of $f$ on T if $g$ is $\mu\restriction\mathrm{T}$-integrable, $f$ is $\mu$-integrable and $\int_F g = \int_F f$ for every $F \in \mathrm{T}$; and that every $\mu$-integrable function has such a conditional expectation (233D). If $g$ is a conditional expectation of $f$ and $f_1 = f$ $\mu$-a.e. then $g$ is a conditional expectation of $f_1$, because $\int_F f_1 = \int_F f$ for every $F$; and I have already remarked in 233Dc that if $g$, $g_1$ are conditional expectations of $f$ on T then $g = g_1$ $\mu\restriction\mathrm{T}$-a.e.

**(d)** This means that we have an operator $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$ defined by saying that $P(f^\bullet) = g^\bullet$ whenever $g \in \mathcal{L}^1(\mu\restriction\mathrm{T})$ is a conditional expectation of $f \in \mathcal{L}^1(\mu)$ on T; that is, that $\int_F Pu = \int_F u$ whenever $u \in L^1(\mu)$ and $F \in \mathrm{T}$. If we identify $L^1(\mu)$, $L^1(\mu\restriction\mathrm{T})$ with the sets of absolutely continuous additive functionals defined on $\Sigma$ and T, as in 242I, then $P$ corresponds to the operation $\nu \mapsto \nu\restriction\mathrm{T}$.

**(e)** Because $Pu$ is uniquely defined in $L^1(\mu\restriction\mathrm{T})$ by the requirement $\int_F Pu = \int_F u$ for every $F \in \mathrm{T}$ (242Ce), we see that $P$ must be linear. **P** If $u$, $v \in L^1(\mu)$ and $c \in \mathbb{R}$, then

$$\int_F Pu + Pv = \int_F Pu + \int_F Pv = \int_F u + \int_F v = \int_F u + v = \int_F P(u + v),$$

$$\int_F P(cu) = \int_F cu = c\int_F u = c\int_F Pu = \int_F cPu$$

for every $F \in \mathrm{T}$. **Q** Also, if $u \ge 0$, then $\int_F Pu = \int_F u \ge 0$ for every $F \in \mathrm{T}$, so $Pu \ge 0$ (242Ce again).

It follows at once that $P$ is order-preserving, that is, that $Pu \le Pv$ whenever $u \le v$. Consequently

$$|Pu| = Pu \vee (-Pu) = Pu \vee P(-u) \le P|u|$$

for every $u \in L^1(\mu)$, because $u \le |u|$ and $-u \le |u|$. Finally, $P$ is a bounded linear operator, with norm 1. **P** The last formula tells us that

$$\|Pu\|_1 \le \|P|u|\|_1 = \int P|u| = \int |u| = \|u\|_1$$

for every $u \in L^1(\mu)$, so $\|P\| \le 1$. On the other hand, $P(\chi X^\bullet) = \chi X^\bullet \ne 0$, so $\|P\| = 1$. **Q**

**(f)** We may legitimately regard $Pu \in L^1(\mu\restriction\mathrm{T})$ as 'the' conditional expectation of $u \in L^1(\mu)$ on T; $P$ is the **conditional expectation operator**.

**(g)** If $u \in L^1(\mu\restriction\mathrm{T})$, then we have a corresponding $Su \in L^1(\mu)$, as in (b); now $PSu = u$. **P** $\int_F PSu = \int_F Su = \int_F u$ for every $F \in \mathrm{T}$. **Q** Consequently $SPSP = SP : L^1(\mu) \to L^1(\mu)$.

**(h)** The distinction drawn above between $u = f^\bullet \in L^0(\mu\restriction\mathrm{T})$ and $Su = f^\bullet \in L^0(\mu)$ is of course pedantic. I believe it is necessary to be aware of such distinctions, even though for nearly all purposes it is safe as well as convenient to regard $L^0(\mu\restriction\mathrm{T})$ as actually a subset of $L^0(\mu)$. If we do so, then (b) tells us that we can identify $L^1(\mu\restriction\mathrm{T})$ with $L^1(\mu) \cap L^0(\mu\restriction\mathrm{T})$, while (g) becomes '$P^2 = P$'.

**242K** The language just introduced allows the following re-formulations of 233J-233K.

**Theorem** Let $(X, \Sigma, \mu)$ be a probability space and T a $\sigma$-subalgebra of $\Sigma$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and $\bar\phi : L^0(\mu) \to L^0(\mu)$ the corresponding operator defined by setting $\bar\phi(f^\bullet) = (\phi f)^\bullet$ (241I). If $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$ is the conditional expectation operator, then $\bar\phi(Pu) \le P(\bar\phi u)$ whenever $u \in L^1(\mu)$ is such that $\bar\phi(u) \in L^1(\mu)$.

**proof** This is just a restatement of 233J.

**242L Proposition** Let $(X, \Sigma, \mu)$ be a probability space, and T a $\sigma$-subalgebra of $\Sigma$. Let $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$ be the corresponding conditional expectation operator. If $u \in L^1 = L^1(\mu)$ and $v \in L^0(\mu\restriction\mathrm{T})$, then $u \times v \in L^1$ iff $P|u| \times v \in L^1$, and in this case $P(u \times v) = Pu \times v$; in particular, $\int u \times v = \int Pu \times v$.

**proof** (I am here using the identification of $L^0(\mu\restriction\mathrm{T})$ as a subspace of $L^0(\mu)$, as suggested in 242Jh.) Express $u$ as $f^\bullet$ and $v$ as $h^\bullet$, where $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $h \in \mathcal{L}^0(\mu\restriction\mathrm{T})$. Let $g$, $g_0 \in \mathcal{L}^1(\mu\restriction\mathrm{T})$ be conditional expectations of $f$, $|f|$ respectively, so that $Pu = g^\bullet$ and $P|u| = g_0^\bullet$. Then, using 233K,

$$u \times v \in L^1 \iff f \times h \in \mathcal{L}^1 \iff g_0 \times h \in \mathcal{L}^1 \iff P|u| \times v \in L^1,$$

and in this case $g \times h$ is a conditional expectation of $f \times h$, that is, $Pu \times v = P(u \times v)$.

**242M $L^1$ as a completion** I mentioned in the introduction to this section that $L^1$ appears in functional analysis as a completion of some important spaces; put another way, some dense subspaces of $L^1$ are significant. The first is elementary.

**Proposition** Let $(X, \Sigma, \mu)$ be any measure space, and write $\mathcal{S}$ for the space of $\mu$-simple functions on $X$. Then

(a) whenever $f$ is a $\mu$-integrable real-valued function and $\epsilon > 0$, there is an $h \in \mathcal{S}$ such that $\int |f - h| \le \epsilon$;

(b) $S = \{f^\bullet : f \in \mathcal{S}\}$ is a dense linear subspace of $L^1 = L^1(\mu)$.

**proof (a)**(i) If $f$ is non-negative, then there is a simple function $h$ such that $h \le_{\text{a.e.}} f$ and $\int h \ge \int f - \frac{1}{2}\epsilon$ (122K), in which case

$$\int |f - h| = \int f - h = \int f - \int h \le \frac{1}{2}\epsilon.$$

(ii) In the general case, $f$ is expressible as a difference $f_1 - f_2$ of non-negative integrable functions. Now there are $h_1$, $h_2 \in \mathcal{S}$ such that $\int |f_j - h_j| \le \frac{1}{2}\epsilon$ for both $j$ and

$$\int |f - h| \le \int |f_1 - h_1| + \int |f_2 - h_2| \le \epsilon.$$

**(b)** Because $\mathcal{S}$ is a linear subspace of $\mathbb{R}^X$ included in $\mathcal{L}^1 = \mathcal{L}^1(\mu)$, $S$ is a linear subspace of $L^1$. If $u \in L^1$ and $\epsilon > 0$, there are an $f \in \mathcal{L}^1$ such that $f^\bullet = u$ and an $h \in \mathcal{S}$ such that $\int |f - h| \le \epsilon$; now $v = h^\bullet \in S$ and

$$\|u - v\|_1 = \int |f - h| \le \epsilon.$$

As $u$ and $\epsilon$ are arbitrary, $S$ is dense in $L^1$.

**242N** As always, Lebesgue measure on $\mathbb{R}^r$ and its subsets is by far the most important example; and in this case we have further classes of dense subspace of $L^1$. If you have reached this point without yet troubling to master multi-dimensional Lebesgue measure, just take $r = 1$. If you feel uncomfortable with general subspace measures, take $X$ to be $\mathbb{R}^r$ or $[0, 1] \subseteq \mathbb{R}$ or some other particular subset which you find interesting. The following term will be useful.

**Definition** If $f$ is a real- or complex-valued function defined on a subset of $\mathbb{R}^r$, say that the **support** of $f$ is $\overline{\{x : x \in \operatorname{dom} f,\ f(x) \ne 0\}}$.

**242O Theorem** Let $X$ be any subset of $\mathbb{R}^r$, where $r \ge 1$, and let $\mu$ be Lebesgue measure on $X$, that is, the subspace measure on $X$ induced by Lebesgue measure on $\mathbb{R}^r$. Write $C_k$ for the space of bounded continuous functions $f : \mathbb{R}^r \to \mathbb{R}$ which have bounded support, and $S_0$ for the space of linear combinations of functions of the form $\chi I$ where $I \subseteq \mathbb{R}^r$ is a bounded half-open interval. Then

(a) whenever $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $\epsilon > 0$, there are $g \in C_k$, $h \in S_0$ such that $\int_X |f - g| \le \epsilon$ and $\int_X |f - h| \le \epsilon$;

(b) $\{(g\restriction X)^\bullet : g \in C_k\}$ and $\{(h\restriction X)^\bullet : h \in S_0\}$ are dense linear subspaces of $L^1 = L^1(\mu)$.

**Remark** Of course there is a redundant 'bounded' in the description of $C_k$; see 242Xh.

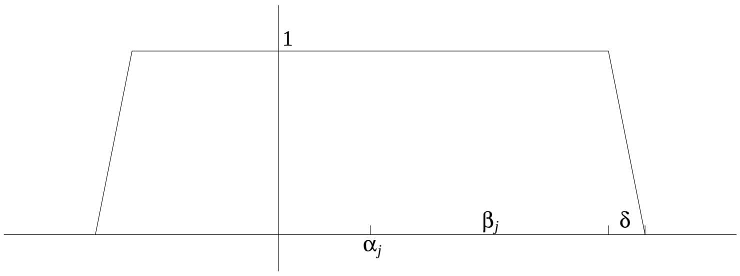
**proof (a)** I argue in turn that the result is valid for each of an increasing number of members $f$ of $\mathcal{L}^1 = \mathcal{L}^1(\mu)$. Write $\mu_r$ for Lebesgue measure on $\mathbb{R}^r$, so that $\mu$ is the subspace measure $(\mu_r)_X$.

**(i)** Suppose first that $f = \chi I\restriction X$ where $I \subseteq \mathbb{R}^r$ is a bounded half-open interval. Of course $\chi I$ is already in $S_0$, so I have only to show that it is approximated by members of $C_k$. If $I = \emptyset$ the result is trivial; we can take $g = 0$. Otherwise, express $I$ as $[a - b, a + b[$ where $a = (\alpha_1, \dots, \alpha_r)$, $b = (\beta_1, \dots, \beta_r)$ and $\beta_j > 0$ for each $j$. Let $\delta > 0$ be such that

$$2^r\prod_{j=1}^r(\beta_j+\delta)\le\epsilon+2^r\prod_{j=1}^r\beta_j.$$

For $\xi\in\mathbb{R}$ set

$$\begin{aligned}g_j(\xi)&=1 \text{ if } |\xi-\alpha_j|\le\beta_j,\\ &=(\beta_j+\delta-|\xi-\alpha_j|)/\delta \text{ if } \beta_j\le|\xi-\alpha_j|\le\beta_j+\delta,\\ &=0 \text{ if } |\xi-\alpha_j|\ge\beta_j+\delta.\end{aligned}$$




The function $g_j$

For $x=(\xi_1,\dots,\xi_r)\in\mathbb{R}^r$ set

$$g(x)=\prod_{j=1}^r g_j(\xi_j).$$

Then $g\in C_k$ and $\chi I\le g\le\chi J$, where $J=[a-b-\delta\mathbf{1},a+b+\delta\mathbf{1}]$ (writing $\mathbf{1}=(1,\dots,1)$), so that (by the choice of $\delta$) $\mu_rJ\le\mu_rI+\epsilon$, and

$$\begin{aligned}\int_X|g-f|&\le\int(\chi(J\cap X)-\chi(I\cap X))d\mu=\mu((J\setminus I)\cap X)\\ &\le\mu_r(J\setminus I)=\mu_rJ-\mu_rI\le\epsilon,\end{aligned}$$

as required.

**(ii)** Now suppose that $f=\chi(X\cap E)$ where $E\subseteq\mathbb{R}^r$ is a set of finite measure. Then there is a disjoint family $I_0,\dots,I_n$ of half-open intervals such that $\mu_r(E\triangle\bigcup_{j\le n}I_j)\le\frac{1}{2}\epsilon$. **P** There is an open set $G\supseteq E$ such that $\mu_r(G\setminus E)\le\frac{1}{4}\epsilon$ (134Fa). For each $m\in\mathbb{N}$, let $\mathcal{I}_m$ be the family of half-open intervals in $\mathbb{R}^r$ of the form $[a,b[$ where $a=(2^{-m}k_1,\dots,2^{-m}k_r)$, $k_1,\dots,k_r$ being integers, and $b=a+2^{-m}\mathbf{1}$; then $\mathcal{I}_m$ is a disjoint family. Set $H_m=\bigcup\{I:I\in\mathcal{I}_m,\ I\subseteq G\}$; then $\langle H_m\rangle_{m\in\mathbb{N}}$ is a non-decreasing family with union $G$, so that there is an $m$ such that $\mu_r(G\setminus H_m)\le\frac{1}{4}\epsilon$ and $\mu_r(E\triangle H_m)\le\frac{1}{2}\epsilon$. But now $H_m$ is expressible as a disjoint union $\bigcup_{j\le n}I_j$ where $I_0,\dots,I_n$ enumerate the members of $\mathcal{I}_m$ included in $H_m$. (The last sentence derails if $H_m$ is empty. But if $H_m=\emptyset$ then we can take $n=0$, $I_0=\emptyset$.) **Q**

Accordingly $h=\sum_{j=0}^n\chi I_j\in S_0$ and

$$\int_X|f-h|=\mu(X\cap(E\triangle\bigcup_{j\le n}I_j))\le\frac{1}{2}\epsilon.$$

As for $C_k$, (i) tells us that there is for each $j\le n$ a $g_j\in C_k$ such that $\int_X|g_j-\chi I_j|\le\epsilon/2(n+1)$, so that $g=\sum_{j=0}^n g_j\in C_k$ and

$$\int_X|f-g|\le\int_X|f-h|+\int_X|h-g|\le\frac{\epsilon}{2}+\sum_{j=0}^n\int_X|g_j-\chi I_j|\le\epsilon.$$

**(iii)** If $f$ is a simple function, express $f$ as $\sum_{k=0}^n a_k\chi E_k$ where each $E_k$ is of finite measure for $\mu$. Each $E_k$ is expressible as $X\cap F_k$ where $\mu_rF_k=\mu E_k$ (214Ca). By (ii), we can find $g_k\in C_k$, $h_k\in S_0$ such that

$$|a_k|\int_X|g_k-\chi F_k|\le\frac{\epsilon}{n+1},\quad |a_k|\int_X|h_k-\chi F_k|\le\frac{\epsilon}{n+1}$$

for each $k$. Set $g=\sum_{k=0}^n a_kg_k$ and $h=\sum_{k=0}^n a_kh_k$; then $g\in C_k$, $h\in S_0$ and

$$\int_X|f-g|\le\int_X\sum_{k=0}^n|a_k||\chi F_k-g_k|=\sum_{k=0}^n|a_k|\int_X|\chi F_k-g_k|\le\epsilon,$$

$$\int_X |f-h| \le \sum_{k=0}^n |a_k| \int_X |\chi F_k - h_k| \le \epsilon,$$

as required.

**(iv)** If $f$ is any integrable function on $X$, then by 242Ma we can find a simple function $f_0$ such that $\int |f-f_0| \le \frac{1}{2}\epsilon$, and now by (iii) there are $g \in C_k$, $h \in S_0$ such that $\int_X |f_0 - g| \le \frac{1}{2}\epsilon$, $\int_X |f_0 - h| \le \frac{1}{2}\epsilon$; so that

$$\int_X |f-g| \le \int_X |f-f_0| + \int_X |f_0-g| \le \epsilon,$$

$$\int_X |f-h| \le \int_X |f-f_0| + \int_X |f_0-h| \le \epsilon.$$

**(b)(i)** We must check first that if $g \in C_k$ then $g\restriction X$ is actually $\mu$-integrable. The point here is that if $g \in C_k$ and $a \in \mathbb{R}$ then

$$\{x : x \in X,\, g(x) > a\}$$

is the intersection of $X$ with an open subset of $\mathbb{R}^r$, and is therefore measured by $\mu$, because all open sets are measured by $\mu_r$ (115G). Next, $g$ is bounded and the set $E = \{x : x \in X,\, g(x) \ne 0\}$ is bounded in $\mathbb{R}^r$, therefore of finite outer measure for $\mu_r$ and of finite measure for $\mu$. Thus there is an $M \ge 0$ such that $|g| \le M\chi E$, which is $\mu$-integrable. Accordingly $g$ is $\mu$-integrable.

Of course $h\restriction X$ is $\mu$-integrable for every $h \in S_0$ because (by the definition of subspace measure) $\mu(I \cap X)$ is defined and finite for every bounded half-open interval $I$.

**(ii)** Now the rest follows by just the same arguments as in 242Mb. Because $\{g\restriction X : g \in C_k\}$ and $\{h\restriction X : h \in S_0\}$ are linear subspaces of $\mathbb{R}^X$ included in $\mathcal{L}^1(\mu)$, their images $C_k^\#$ and $S_0^\#$ are linear subspaces of $L^1$. If $u \in L^1$ and $\epsilon > 0$, there are an $f \in \mathcal{L}^1$ such that $f^\bullet = u$, and $g \in C_k$, $h \in S_0$ such that $\int_X |f-g|$, $\int_X |f-h| \le \epsilon$; now $v = (g\restriction X)^\bullet \in C_k^\#$ and $w = (h\restriction X)^\bullet \in S_0^\#$ and

$$\|u-v\|_1 = \int_X |f-g| \le \epsilon, \quad \|u-w\|_1 = \int_X |f-h| \le \epsilon.$$

As $u$ and $\epsilon$ are arbitrary, $C_k^\#$ and $S_0^\#$ are dense in $L^1$.

**242P Complex $L^1$** As you would, I hope, expect, we can repeat the work above with $\mathcal{L}^1_{\mathbb{C}}$, the space of complex-valued integrable functions, in place of $\mathcal{L}^1$, to construct a complex Banach space $L^1_{\mathbb{C}}$. The required changes, based on the ideas of 241J, are minor.

**(a)** In 242Aa, it is perhaps helpful to remark that, for $f \in \mathcal{L}^0_{\mathbb{C}}$,

$$f \in \mathcal{L}^1_{\mathbb{C}} \iff |f| \in \mathcal{L}^1 \iff \mathcal{R}\mathrm{e}(f), \mathcal{I}\mathrm{m}(f) \in \mathcal{L}^1.$$

Consequently, for $u \in L^0_{\mathbb{C}}$,

$$u \in L^1_{\mathbb{C}} \iff |u| \in L^1 \iff \mathcal{R}\mathrm{e}(u), \mathcal{I}\mathrm{m}(u) \in L^1.$$

**(b)** To prove a complex version of 242E, observe that if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^1_{\mathbb{C}}$ such that $\sum_{n=0}^\infty \int |f_n| < \infty$, then $\sum_{n=0}^\infty \int |\mathcal{R}\mathrm{e}(f_n)|$ and $\sum_{n=0}^\infty \int |\mathcal{I}\mathrm{m}(f_n)|$ are both finite, so we may apply 242E twice and see that

$$\int(\sum_{n=0}^\infty f_n) = \int(\sum_{n=0}^\infty \mathcal{R}\mathrm{e}(f_n)) + \int(\sum_{n=0}^\infty \mathcal{I}\mathrm{m}(f_n)) = \sum_{n=0}^\infty \int f_n.$$

Accordingly we can prove that $L^1_{\mathbb{C}}$ is complete under $\| \|_1$ by the argument of 242F.

**(c)** Similarly, little change is needed to adapt 242J to give a description of a conditional expectation operator $P : L^1_{\mathbb{C}}(\mu) \to L^1_{\mathbb{C}}(\mu\restriction\mathrm{T})$ when $(X, \Sigma, \mu)$ is a probability space and $\mathrm{T}$ is a $\sigma$-subalgebra of $\Sigma$. In the formula

$$|Pu| \le P|u|$$

of 242Je, we need to know that

$$|Pu| = \sup_{|\zeta|=1} \mathcal{R}\mathrm{e}(\zeta Pu)$$

in $L^0(\mu\restriction\mathrm{T})$ (241Jc), while

$$\mathcal{R}\mathrm{e}(\zeta Pu) = \mathcal{R}\mathrm{e}(P(\zeta u)) = P(\mathcal{R}\mathrm{e}(\zeta u)) \le P|u|$$

whenever $|\zeta| = 1$.

**(d)** In 242M, we need to replace $\mathcal{S}$ by $\mathcal{S}_{\mathbb{C}}$, the space of 'complex-valued simple functions' of the form $\sum_{k=0}^n a_k \chi E_k$ where each $a_k$ is a complex number and each $E_k$ is a measurable set of finite measure; then we get a dense linear subspace $S_{\mathbb{C}} = \{f^\bullet : f \in \mathcal{S}_{\mathbb{C}}\}$ of $L^1_{\mathbb{C}}$. In 242O, we must replace $C_k$ by $C_k(\mathbb{R}^r; \mathbb{C})$, the space of bounded continuous complex-valued functions of bounded support, and $S_0$ by the linear span over $\mathbb{C}$ of $\{\chi I : I$ is a bounded half-open interval$\}$.

**242X Basic exercises >(a)** Let $X$ be a set, and let $\mu$ be counting measure on $X$. Show that $L^1(\mu)$ can be identified with the space $\ell^1(X)$ of absolutely summable real-valued functions on $X$ (see 226A). In particular, the space $\ell^1 = \ell^1(\mathbb{N})$ of absolutely summable real-valued sequences is an $L^1$ space. Write out proofs of 242F adapted to these special cases.

**>(b)** Let $(X, \Sigma, \mu)$ be any measure space, and $\hat\mu$ the completion of $\mu$. Show that $\mathcal{L}^1(\hat\mu) = \mathcal{L}^1(\mu)$ and $L^1(\hat\mu) = L^1(\mu)$ (cf. 241Xb).

**(c)** Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, and $(X, \Sigma, \mu)$ their direct sum. Show that the isomorphism between $L^0(\mu)$ and $\prod_{i\in I} L^0(\mu_i)$ (241Xd) induces an identification between $L^1(\mu)$ and

$$\{u : u \in \textstyle\prod_{i\in I} L^1(\mu_i),\ \|u\| = \sum_{i\in I} \|u(i)\|_1 < \infty\} \subseteq \prod_{i\in I} L^1(\mu_i).$$

**(d)** Let $(X, \Sigma, \mu)$ and $(Y, \mathrm{T}, \nu)$ be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Show that $g \mapsto g\phi : \mathcal{L}^1(\nu) \to \mathcal{L}^1(\mu)$ (235G) induces a linear operator $T : L^1(\nu) \to L^1(\mu)$ such that $\|Tv\|_1 = \|v\|_1$ for every $v \in L^1(\nu)$.

**(e)** Let $U$ be a Riesz space (definition: 241Ed). A **Riesz norm** on $U$ is a norm $\|\ \|$ such that $\|u\| \le \|v\|$ whenever $|u| \le |v|$. Show that if $U$ is given its norm topology (2A4Bb) for such a norm, then (i) $u \mapsto |u| : U \to U$, $(u, v) \mapsto u \vee v : U \times U \to U$ are continuous (ii) $\{u : u \ge 0\}$ is closed.

**(f)** Show that any Banach lattice must be an Archimedean Riesz space (241Fa).

**(g)** Let $(X, \Sigma, \mu)$ be a probability space, and $\mathrm{T}$ a $\sigma$-subalgebra of $\Sigma$, $\Upsilon$ a $\sigma$-subalgebra of $\mathrm{T}$. Let $P_1 : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$, $P_2 : L_1(\mu\restriction\mathrm{T}) \to L^1(\mu\restriction\Upsilon)$ and $P : L^1(\mu) \to L^1(\mu\restriction\Upsilon)$ be the corresponding conditional expectation operators. Show that $P = P_2 P_1$.

**(h)** Show that if $g : \mathbb{R}^r \to \mathbb{R}$ is continuous and has bounded support it is bounded and attains its bounds. (*Hint*: 2A2F-2A2G.)

**(i)** Let $\mu$ be Lebesgue measure on $\mathbb{R}$. (i) Take $\delta > 0$. Show that if $\phi_\delta(x) = \exp(-\frac{1}{\delta^2 - x^2})$ for $|x| < \delta$, 0 for $|x| \ge \delta$ then $\phi$ is **smooth**, that is, differentiable arbitrarily often. (ii) Show that if $F_\delta(x) = \int_{-\infty}^x \phi_\delta d\mu$ for $x \in \mathbb{R}$ then $F_\delta$ is smooth. (iii) Show that if $a < b < c < d$ in $\mathbb{R}$ there is a smooth function $h$ such that $\chi[b, c] \le h \le \chi[a, d]$. (iv) Write $\mathcal{D}$ for the space of smooth functions $h : \mathbb{R} \to \mathbb{R}$ such that $\{x : h(x) \ne 0\}$ is bounded. Show that $\{h^\bullet : h \in \mathcal{D}\}$ is dense in $L^1(\mu)$. (v) Let $f$ be a real-valued function which is integrable over every bounded subset of $\mathbb{R}$. Show that $f \times h$ is integrable for every $h \in \mathcal{D}$, and that if $\int f \times h = 0$ for every $h \in \mathcal{D}$ then $f = 0$ a.e. (*Hint*: 222D.)

**(j)** Let $(X, \Sigma, \mu)$ be a probability space, $\mathrm{T}$ a $\sigma$-subalgebra of $\Sigma$ and $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T}) \subseteq L^1(\mu)$ the corresponding conditional expectation operator. Show that if $u$, $v \in L^1(\mu)$ are such that $P|u| \times P|v| \in L^1(\mu)$, then $\int Pu \times v = \int Pu \times Pv = \int u \times Pv$.

**242Y Further exercises (a)** Let $(X, \Sigma, \mu)$ be a measure space. Let $A \subseteq L^1 = L^1(\mu)$ be a non-empty downwards-directed set, and suppose that $\inf A = 0$ in $L^1$. (i) Show that $\inf_{u\in A} \|u\|_1 = 0$. (*Hint*: set $\gamma = \inf_{u\in A} \|u\|_1$; find a non-increasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $A$ such that $\lim_{n\to\infty} \|u_n\|_1 = \gamma$; set $v = \inf_{n\in\mathbb{N}} u_n$ and show that $u \wedge v = v$ for every $u \in A$, so that $v = 0$.) (ii) Show that if $U$ is any open set containing 0, there is a $u \in A$ such that $v \in U$ whenever $0 \le v \le u$.

**(b)** Let $(X,\Sigma,\mu)$ be a measure space and $Y$ any subset of $X$; let $\mu_Y$ be the subspace measure on $Y$ and $T:L^0(\mu)\to L^0(\mu_Y)$ the canonical map described in 241Yg. (i) Show that $Tu\in L^1(\mu_Y)$ and $\|Tu\|_1\le\|u\|_1$ for every $u\in L^1(\mu)$. (ii) Show that if $u\in L^1(\mu)$ then $\|Tu\|_1=\|u\|_1$ iff $\int_E u=\int_{Y\cap E}Tu$ for every $E\in\Sigma$. (iii) Show that $T$ is surjective and that $\|v\|_1=\min\{\|u\|_1:u\in L^1(\mu),\,Tu=v\}$ for every $v\in L^1(\mu_Y)$. (*Hint*: 214Eb.) (See also 244Yd below.)

**(c)** Let $(X,\Sigma,\mu)$ be a measure space. Write $\mathcal{L}^1_\Sigma$ for the space of all integrable $\Sigma$-measurable functions from $X$ to $\mathbb{R}$, and $\mathcal{N}$ for the subspace of $\mathcal{L}^1_\Sigma$ consisting of measurable functions which are zero almost everywhere. (i) Show that $\mathcal{L}^1_\Sigma$ is a Dedekind $\sigma$-complete Riesz space. (ii) Show that $L^1(\mu)$ can be identified, as ordered linear space, with the quotient $\mathcal{L}^1_\Sigma/\mathcal{N}$ as defined in 241Yb. (iii) Show that $\|\,\|_1$ is a seminorm on $\mathcal{L}^1_\Sigma$. (iv) Show that $f\mapsto|f|:\mathcal{L}^1_\Sigma\to\mathcal{L}^1_\Sigma$ is continuous if $\mathcal{L}^1_\Sigma$ is given the topology defined from $\|\,\|_1$. (v) Show that $\{f:f=0\text{ a.e.}\}$ is closed in $\mathcal{L}^1_\Sigma$, but that $\{f:f\ge0\}$ need not be.

**(d)** Let $(X,\Sigma,\mu)$ be a measure space, and $\tilde\mu$ the c.l.d. version of $\mu$ (213E). Show that the inclusion $\mathcal{L}^1(\mu)\subseteq\mathcal{L}^1(\tilde\mu)$ induces an isomorphism, as ordered normed linear spaces, between $L^1(\tilde\mu)$ and $L^1(\mu)$.

**(e)** Let $(X,\Sigma,\mu)$ be a measure space and $u_0,\dots,u_n\in L^1(\mu)$. (i) Suppose $k_0,\dots,k_n\in\mathbb{Z}$ are such that $\sum_{i=0}^n k_i=1$. Show that $\sum_{i=0}^n\sum_{j=0}^n k_ik_j\|u_i-u_j\|_1\le0$. (*Hint*: $\sum_{i=0}^n\sum_{j=0}^n k_ik_j|\alpha_i-\alpha_j|\le0$ for all $\alpha_0,\dots,\alpha_n\in\mathbb{R}$.) (ii) Suppose $\gamma_0,\dots,\gamma_n\in\mathbb{R}$ are such that $\sum_{i=0}^n\gamma_i=0$. Show that $\sum_{i=0}^n\sum_{j=0}^n\gamma_i\gamma_j\|u_i-u_j\|_1\le0$.

**(f)** Let $(X,\Sigma,\mu)$ be a measure space, and $A\subseteq L^1=L^1(\mu)$ a non-empty upwards-directed set. Suppose that $A$ is bounded for the norm $\|\,\|_1$. (i) Show that there is a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $A$ such that $\lim_{n\to\infty}\int u_n=\sup_{u\in A}\int u$, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is Cauchy. (ii) Show that $w=\sup A$ is defined in $L^1$ and belongs to the norm-closure of $A$ in $L^1$, so that, in particular, $\|w\|_1\le\sup_{u\in A}\|u\|_1$.

**(g)** A Riesz norm (definition: 242Xe) on a Riesz space $U$ is **order-continuous** if $\inf_{u\in A}\|u\|=0$ whenever $A\subseteq U$ is a non-empty downwards-directed set with infimum 0. (Thus 242Ya tells us that the norms $\|\,\|_1$ are all order-continuous.) Show that in this case (i) any non-decreasing sequence in $U$ which has an upper bound in $U$ must be Cauchy (ii) if $U$ is a Banach lattice, it is $U$ is Dedekind complete. (*Hint for (i)*: if $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with an upper bound in $U$, let $B$ be the set of upper bounds of $\{u_n:n\in\mathbb{N}\}$ and show that $A=\{v-u_n:v\in B,\,n\in\mathbb{N}\}$ has infimum 0 because $U$ is Archimedean.)

**(h)** Let $(X,\Sigma,\mu)$ be any measure space. Show that $L^1(\mu)$ has the countable sup property (241Ye).

**(i)** More generally, show that any Riesz space with an order-continuous Riesz norm has the countable sup property.

**(j)** Let $(X,\Sigma,\mu)$ and $(Y,\mathrm{T},\nu)$ be measure spaces and $U\subseteq L^0(\mu)$ a linear subspace. Let $T:U\to L^0(\nu)$ be a linear operator such that $Tu\ge0$ in $L^0(\nu)$ whenever $u\in U$ and $u\ge0$ in $L^0(\mu)$. Suppose that $w\in U$ is such that $w\ge0$ and $Tw=(\chi Y)^\bullet$. Show that whenever $\phi:\mathbb{R}\to\mathbb{R}$ is a convex function and $u\in L^0(\mu)$ is such that $w\times u$ and $w\times\bar\phi(u)\in U$, defining $\bar\phi:L^0(\mu)\to L^0(\mu)$ as in 241I, then $\bar\phi T(w\times u)\le T(w\times\bar\phi u)$. Explain how this result may be regarded as a common generalization of Jensen's inequality, as stated in 233I, and 242K above. See also 244M below.

**(k)**(i) A function $\phi:\mathbb{C}\to\mathbb{R}$ is **convex** if $\phi(ab+(1-a)c)\le a\phi(b)+(1-a)\phi(c)$ for all $b$, $c\in\mathbb{C}$ and $a\in[0,1]$. (ii) Show that such a function must be bounded on any bounded subset of $\mathbb{C}$. (iii) If $\phi:\mathbb{C}\to\mathbb{R}$ is convex and $c\in\mathbb{C}$, show that there is a $b\in\mathbb{C}$ such that $\phi(x)\ge\phi(c)+\mathcal{R}\mathrm{e}(b(x-c))$ for every $x\in\mathbb{C}$. (iv) If $\langle b_c\rangle_{c\in\mathbb{C}}$ is such that $\phi(x)\ge\phi_c(x)=\phi(c)+\mathcal{R}\mathrm{e}(b_c(x-c))$ for all $x$, $c\in\mathbb{C}$, show that $\{b_c:c\in I\}$ is bounded for any bounded $I\subseteq\mathbb{C}$. (v) Show that if $D\subseteq\mathbb{C}$ is any dense set, $\phi(x)=\sup_{c\in D}\phi_c(x)$ for every $x\in\mathbb{C}$.

**(l)** Let $(X,\Sigma,\mu)$ be a probability space and $\mathrm{T}$ a $\sigma$-subalgebra of $\Sigma$. Let $P:L^1_{\mathbb{C}}(\mu)\to L^1_{\mathbb{C}}(\mu\restriction\mathrm{T})$ be the conditional expectation operator. Show that if $\phi:\mathbb{C}\to\mathbb{R}$ is any convex function, and we define $\bar\phi(f^\bullet)=(\phi f)^\bullet$ for every $f\in\mathcal{L}^0_{\mathbb{C}}(\mu)$, then $\bar\phi(Pu)\le P(\bar\phi(u))$ whenever $u\in L^1_{\mathbb{C}}(\mu)$ is such that $\bar\phi(u)\in L^1(\mu)$.

**242 Notes and comments** Of course $L^1$-spaces compose one of the most important classes of Riesz space, and accordingly their properties have great prominence in the general theory; 242Xe, 242Xf, 242Ya and 242Yf-242Yi outline some of the interrelations between these properties. I will return to these questions in Chapter 35 in the next volume. I have mentioned in passing (242Dd) the additivity of the norm of $L^1$ on the positive elements. This elementary fact actually characterizes $L^1$ spaces among Banach lattices; see 369E in the next volume.

Just as $L^0(\mu)$ can be regarded as a quotient of a linear space $\mathcal{L}^0_\Sigma$, so can $L^1(\mu)$ be regarded as a quotient of a linear space $\mathcal{L}^1_\Sigma$ (242Yc). I have discussed this question in the notes to §241; all I try to do here is to be consistent.

We now have a language in which we can speak of 'the' conditional expectation of a function $f$, the equivalence class in $L^1(\mu\restriction\mathrm{T})$ consisting precisely of all the conditional expections of $f$ on T. If we think of $L^1(\mu\restriction\mathrm{T})$ as identified with its image in $L^1(\mu)$, then the conditional expectation operator $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$ becomes a projection (242Jh). We therefore have re-statements of 233J-233K, as in 242K, 242L and 242Yj.

I give 242O in a fairly general form; but its importance already appears if we take $X$ to be $[0,1]$ with one-dimensional Lebesgue measure. In this case, we have a natural norm on $C([0,1])$, the space of all continuous real-valued functions on $[0,1]$, given by setting

$$\|f\|_1 = \int_0^1 |f(x)|dx$$

for every $f \in C([0,1])$. The integral here can, of course, be taken to be the Riemann integral; we do not need the Lebesgue theory to show that $\|\;\|_1$ is a norm on $C([0,1])$. It is easy to check that $C([0,1])$ is not complete for this norm (if we set $f_n(x) = \min(1, 2^n x^n)$ for $x \in [0,1]$, then $\langle f_n\rangle_{n\in\mathbb{N}}$ is a $\|\;\|_1$-Cauchy sequence with no $\|\;\|_1$-limit in $C([0,1])$). We can use the abstract theory of normed spaces to construct a completion of $C([0,1])$; but it is much more satisfactory if this completion can be given a relatively concrete form, and this is what the identification of $L^1$ with the completion of $C([0,1])$ can do. (Note that the remark that $\|\;\|_1$ is a norm on $C([0,1])$, that is, that $\|f\|_1 \neq 0$ for every non-zero $f \in C([0,1])$, means just that the map $f \mapsto f^\bullet : C([0,1]) \to L^1$ is injective, so that $C([0,1])$ can be identified, as ordered normed space, with its image in $L^1$.) It would be even better if we could find a realization of the completion of $C([0,1])$ as a space of functions on some set $Z$, rather than as a space of equivalence classes of functions on $[0,1]$. Unfortunately this is not practical; such realizations do exist, but necessarily involve either a thoroughly unfamiliar base set $Z$, or an intolerably arbitrary embedding map from $C([0,1])$ into $\mathbb{R}^Z$.

You can get an idea of the obstacle to realizing the completion of $C([0,1])$ as a space of functions on $[0,1]$ itself by considering $f_n(x) = \frac{1}{n}x^n$ for $n \ge 1$. An easy calculation shows that $\sum_{n=1}^\infty \|f_n\|_1 < \infty$, so that $\sum_{n=1}^\infty f_n$ must exist in the completion of $C([0,1])$; but there is no natural value to assign to it at the point 1. Adaptations of this idea can give rise to indefinitely complicated phenomena – indeed, 242O shows that every integrable function is associated with some appropriate sequence from $C([0,1])$. In §245 I shall have more to say about what $\|\;\|_1$-convergent sequences look like.

From the point of view of measure theory, narrowly conceived, most of the interesting ideas appear most clearly with real functions and real linearspaces. But some of the most important applications of measure theory – important not only as mathematics in general, but also for the measure-theoretic questions they inspire – deal with complex functions and complex linear spaces. I therefore continue to offer sketches of the complex theory, as in 242P. I note that at irregular intervals we need ideas not already spelt out in the real theory, as in 242Pb and 242Yl.

Version of 30.4.04

## 243 $L^\infty$

The second of the classical Banach spaces of measure theory which I treat is the space $L^\infty$. As will appear below, $L^\infty$ is the polar companion of $L^1$, the linked opposite; for 'ordinary' measure spaces it is actually the dual of $L^1$ (243F-243G).

**243A Definitions** Let $(X, \Sigma, \mu)$ be any measure space. Let $\mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$ be the set of functions $f \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$ which are **essentially bounded**, that is, such that there is some $M \ge 0$ such that $\{x : x \in \operatorname{dom} f, |f(x)| \le M\}$ is conegligible, and write

$$L^\infty = L^\infty(\mu) = \{f^\bullet : f \in \mathcal{L}^\infty(\mu)\} \subseteq L^0(\mu).$$

Note that if $f \in \mathcal{L}^\infty$, $g \in \mathcal{L}^0$ and $g =_{\text{a.e.}} f$, then $g \in \mathcal{L}^\infty$; thus $\mathcal{L}^\infty = \{f : f \in \mathcal{L}^0,\ f^\bullet \in L^\infty\}$.

**243B Theorem** Let $(X, \Sigma, \mu)$ be any measure space. Then

(a) $L^\infty = L^\infty(\mu)$ is a linear subspace of $L^0 = L^0(\mu)$.

(b) If $u \in L^\infty$, $v \in L^0$ and $|v| \le |u|$ then $v \in L^\infty$. Consequently $|u|$, $u \vee v$, $u \wedge v$, $u^+ = u \vee 0$ and $u^- = (-u) \vee 0$ belong to $L^\infty$ for all $u$, $v \in L^\infty$.

(c) Writing $e = \chi X^\bullet$, the equivalence class in $L^0$ of the constant function with value 1, then an element $u$ of $L^0$ belongs to $L^\infty$ iff there is an $M \ge 0$ such that $|u| \le Me$.

(d) If $u$, $v \in L^\infty$ then $u \times v \in L^\infty$.

(e) If $u \in L^\infty$ and $v \in L^1 = L^1(\mu)$ then $u \times v \in L^1$.

**proof (a)** If $f$, $g \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$ and $c \in \mathbb{R}$, then $f + g$, $cf \in \mathcal{L}^\infty$. **P** We have $M_1$, $M_2 \ge 0$ such that $|f| \le M_1$ a.e. and $|g| \le M_2$ a.e. Now

$$|f + g| \le |f| + |g| \le M_1 + M_2 \text{ a.e.}, \quad |cf| \le |c||M_1| \text{ a.e.},$$

so $f + g$, $cf \in \mathcal{L}^\infty$. **Q** It follows at once that $u + v$, $cu \in L^\infty$ whenever $u$, $v \in L^\infty$ and $c \in \mathbb{R}$.

**(b)(i)** Take $f \in \mathcal{L}^\infty$, $g \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$ such that $u = f^\bullet$ and $v = g^\bullet$. Then $|g| \le_{\text{a.e.}} |f|$. Let $M \ge 0$ be such that $|f| \le M$ a.e.; then $|g| \le M$ a.e., so $g \in \mathcal{L}^\infty$ and $v \in L^\infty$.

**(ii)** Now $|\,|u|\,| = |u|$ so $|u| \in L^\infty$ whenever $u \in L^\infty$. Also $u \vee v = \frac{1}{2}(u+v+|u-v|)$, $u \wedge v = \frac{1}{2}(u+v-|u-v|)$ belong to $L^\infty$ for all $u$, $v \in L^\infty$.

**(c)**(i) If $u \in L^\infty$, take $f \in \mathcal{L}^\infty$ such that $f^\bullet = u$. Then there is an $M \ge 0$ such that $|f| \le M$ a.e., so that $|f| \le_{\text{a.e.}} M\chi X$ and $|u| \le Me$. (ii) Of course $\chi X \in \mathcal{L}^\infty$, so $e \in L^\infty$, and if $u \in L^0$ and $|u| \le Me$ then $u \in L^\infty$ by (b).

**(d)** $f \times g \in \mathcal{L}^\infty$ whenever $f$, $g \in \mathcal{L}^\infty$. **P** If $|f| \le M_1$ a.e. and $|g| \le M_2$ a.e., then

$$|f \times g| = |f| \times |g| \le M_1 M_2 \text{ a.e. } \mathbf{Q}$$

So $u \times v \in L^\infty$ for all $u$, $v \in L^\infty$.

**(e)** If $f \in \mathcal{L}^\infty$ and $g \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$, then there is an $M \ge 0$ such that $|f| \le M$ a.e., so $|f \times g| \le_{\text{a.e.}} M|g|$; because $M|g|$ is integrable and $f \times g$ is virtually measurable, $f \times g$ is integrable and $u \times v \in L^1$.

**243C The order structure of** $L^\infty$ Let $(X, \Sigma, \mu)$ be any measure space. Then $L^\infty = L^\infty(\mu)$, being a linear subspace of $L^0 = L^0(\mu)$, inherits a partial order which renders it a partially ordered linear space (compare 242Ca). Because $|u| \in L^\infty$ whenever $u \in L^\infty$ (243Bb), $u \wedge v$ and $u \vee v$ belong to $L^\infty$ whenever $u$, $v \in L^\infty$, and $L^\infty$ is a Riesz space (compare 242Cd).

The behaviour of $L^\infty$ as a Riesz space is dominated by the fact that it has an **order unit** $e$ with the property that

$$\text{for every } u \in L^\infty \text{ there is an } M \ge 0 \text{ such that } |u| \le Me$$

(243Bc).

**243D The norm of** $L^\infty$ Let $(X, \Sigma, \mu)$ be any measure space.

**(a)** For $f \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$, say that the **essential supremum** of $|f|$ is

$$\text{ess sup}\,|f| = \inf\{M : M \ge 0,\ \{x : x \in \operatorname{dom} f,\ |f(x)| \le M\} \text{ is conegligible}\}.$$

Then $|f| \le \text{ess sup}\,|f|$ a.e. **P** Set $M = \text{ess sup}\,|f|$. For each $n \in \mathbb{N}$, there is an $M_n \le M + 2^{-n}$ such that $|f| \le M_n$ a.e. Now

$$\{x : |f(x)| \le M\} = \bigcap_{n \in \mathbb{N}} \{x : |f(x)| \le M_n\}$$

is conegligible, so $|f| \le M$ a.e. **Q**

**(b)** If $f$, $g \in \mathcal{L}^\infty$ and $f =_{\text{a.e.}} g$, then $\operatorname{ess\,sup}|f| = \operatorname{ess\,sup}|g|$. Accordingly we may define a functional $\| \|_\infty$ on $L^\infty = L^\infty(\mu)$ by setting $\|u\|_\infty = \operatorname{ess\,sup}|f|$ whenever $u = f^\bullet$.

**(c)** From (a), we see that, for any $u \in L^\infty$, $\|u\|_\infty = \min\{\gamma : |u| \le \gamma e\}$, where, as before, $e = \chi X^\bullet \in L^\infty$. Consequently $\| \|_\infty$ is a norm on $L^\infty$. **P**(i) If $u$, $v \in L^\infty$ then

$$|u + v| \le |u| + |v| \le (\|u\|_\infty + \|v\|_\infty)e$$

so $\|u + v\|_\infty \le \|u\|_\infty + \|v\|_\infty$. (ii) If $u \in L^\infty$ and $c \in \mathbb{R}$ then

$$|cu| = |c||u| \le |c|\|u\|_\infty e,$$

so $\|cu\|_\infty \le |c|\|u\|_\infty$. (iii) If $\|u\|_\infty = 0$, there is an $f \in \mathcal{L}^\infty$ such that $f^\bullet = u$ and $|f| \le \|u\|_\infty$ a.e.; now $f = 0$ a.e. so $u = 0$. **Q**

**(d)** Note also that if $u \in L^0$, $v \in L^\infty$ and $|u| \le |v|$ then $|u| \le \|v\|_\infty e$ so $u \in L^\infty$ and $\|u\|_\infty \le \|v\|_\infty$; similarly,

$$\|u \times v\|_\infty \le \|u\|_\infty \|v\|_\infty, \quad \|u \vee v\|_\infty \le \max(\|u\|_\infty, \|v\|_\infty)$$

for all $u$, $v \in L^\infty$. Thus $L^\infty$ is a commutative Banach algebra (2A4J).

**(e)** Moreover,

$$|\int u \times v| \le \int |u \times v| = \|u \times v\|_1 \le \|u\|_1 \|v\|_\infty$$

whenever $u \in L^1$ and $v \in L^\infty$, because

$$|u \times v| = |u| \times |v| \le |u| \times \|v\|_\infty e = \|v\|_\infty |u|.$$

**(f)** Observe that if $u$, $v$ are non-negative members of $L^\infty$ then

$$\|u \vee v\|_\infty = \max(\|u\|_\infty, \|v\|_\infty);$$

this is because, for any $\gamma \ge 0$,

$$u \vee v \le \gamma e \iff u \le \gamma e \text{ and } v \le \gamma e.$$

**243E Theorem** For any measure space $(X, \Sigma, \mu)$, $L^\infty = L^\infty(\mu)$ is a Banach lattice under $\| \|_\infty$.

**proof (a)** We already know that $\|u\|_\infty \le \|v\|_\infty$ whenever $|u| \le |v|$ (243Dd); so we have just to check that $L^\infty$ is complete under $\| \|_\infty$. Let $\langle u_n \rangle_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^\infty$. For each $n \in \mathbb{N}$ choose $f_n \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$ such that $f_n^\bullet = u_n$ in $L^\infty$. For all $m$, $n \in \mathbb{N}$, $(f_m - f_n)^\bullet = u_m - u_n$. Consequently

$$E_{mn} = \{x : |f_m(x) - f_n(x)| > \|u_m - u_n\|_\infty\}$$

is negligible, by 243Da. This means that

$$E = \bigcap_{n\in\mathbb{N}} \{x : x \in \operatorname{dom} f_n,\, |f_n(x)| \le \|u_n\|_\infty\} \setminus \bigcup_{m,n\in\mathbb{N}} E_{mn}$$

is conegligible. But for every $x \in E$, $|f_m(x) - f_n(x)| \le \|u_m - u_n\|_\infty$ for all $m$, $n \in \mathbb{N}$, so that $\langle f_n(x) \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence, with a limit in $\mathbb{R}$. Thus $f = \lim_{n\to\infty} f_n$ is defined almost everywhere. Also, at least for $x \in E$,

$$|f(x)| \le \sup_{n\in\mathbb{N}} \|u_n\|_\infty < \infty,$$

so $f \in \mathcal{L}^\infty$ and $u = f^\bullet \in L^\infty$. If $m \in \mathbb{N}$, then, for every $x \in E$,

$$|f(x) - f_m(x)| \le \sup_{n\ge m} |f_n(x) - f_m(x)| \le \sup_{n\ge m} \|u_n - u_m\|_\infty,$$

so

$$\|u - u_m\|_\infty \le \sup_{n\ge m} \|u_n - u_m\|_\infty \to 0$$

as $m \to \infty$, and $u = \lim_{m\to\infty} u_m$ in $L^\infty$. As $\langle u_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $L^\infty$ is complete.

**243F The duality between $L^\infty$ and $L^1$** Let $(X,\Sigma,\mu)$ be any measure space.

**(a)** I have already remarked that if $u\in L^1=L^1(\mu)$ and $v\in L^\infty=L^\infty(\mu)$, then $u\times v\in L^1$ and $|\int u\times v|\le\|u\|_1\|v\|_\infty$ (243Bd, 243De).

**(b)** Consequently we have a bounded linear operator $T$ from $L^\infty$ to the normed space dual $(L^1)^*$ of $L^1$, given by writing

$$(Tv)(u)=\int u\times v \text{ for all } u\in L^1,\ v\in L^\infty.$$

**P** (i) By (a), $(Tv)(u)$ is well-defined for $u\in L^1$, $v\in L^\infty$. (ii) If $v\in L^\infty$, $u$, $u_1$, $u_2\in L^1$ and $c\in\mathbb{R}$, then

$$\begin{aligned}(Tv)(u_1+u_2)&=\int(u_1+u_2)\times v=\int(u_1\times v)+(u_2\times v)\\&=\int u_1\times v+\int u_2\times v=(Tv)(u_1)+(Tv)(u_2),\end{aligned}$$

$$(Tv)(cu)=\int cu\times v=\int c(u\times v)=c\int u\times v=c(Tv)(u).$$

This shows that $Tv:L^1\to\mathbb{R}$ is a linear functional for each $v\in L^\infty$. (iii) Next, for any $u\in L^1$ and $v\in L^\infty$,

$$|(Tv)(u)|=|\int u\times v|\le\|u\times v\|_1\le\|u\|_1\|v\|_\infty,$$

as remarked in (a). This means that $Tv\in(L^1)^*$ and $\|Tv\|\le\|v\|_\infty$ for every $v\in L^\infty$. (iv) If $v$, $v_1$, $v_2\in L^\infty$, $u\in L^1$ and $c\in\mathbb{R}$, then

$$\begin{aligned}T(v_1+v_2)(u)&=\int(v_1+v_2)\times u=\int(v_1\times u)+(v_2\times u)\\&=\int v_1\times u+\int v_2\times u=(Tv_1)(u)+(Tv_2)(u)\\&=(Tv_1+Tv_2)(u),\end{aligned}$$

$$T(cv)(u)=\int cv\times u=c\int v\times u=c(Tv)(u)=(cTv)(u).$$

As $u$ is arbitrary, $T(v_1+v_2)=Tv_1+Tv_2$ and $T(cv)=c(Tv)$; thus $T:L^\infty\to(L^1)^*$ is linear. (v) Recalling from (iii) that $\|Tv\|\le\|v\|_\infty$ for every $v\in L^\infty$, we see that $\|T\|\le 1$. **Q**

**(c)** Exactly the same arguments show that we have a linear operator $T':L^1\to(L^\infty)^*$, given by writing

$$(T'u)(v)=\int u\times v \text{ for all } u\in L^1,\ v\in L^\infty,$$

and that $\|T'\|$ also is at most 1.

**243G Theorem** Let $(X,\Sigma,\mu)$ be a measure space, and $T:L^\infty(\mu)\to(L^1(\mu))^*$ the canonical map described in 243F. Then

(a) $T$ is injective iff $(X,\Sigma,\mu)$ is semi-finite, and in this case is norm-preserving;

(b) $T$ is bijective iff $(X,\Sigma,\mu)$ is localizable, and in this case is a normed space isomorphism.

**proof (a)(i)** Suppose that $T$ is injective, and that $E\in\Sigma$ has $\mu E=\infty$. Then $\chi E$ is not equal almost everywhere to $\mathbf{0}$, so $(\chi E)^\bullet\ne 0$ in $L^\infty$, and $T(\chi E)^\bullet\ne 0$; let $u\in L^1$ be such that $T(\chi E)^\bullet(u)\ne 0$, that is, $\int u\times(\chi E)^\bullet\ne 0$. Express $u$ as $f^\bullet$ where $f$ is integrable; then $\int_E f\ne 0$ so $\int_E|f|\ne 0$. Let $g$ be a simple function such that $0\le g\le_{\text{a.e.}}|f|$ and $\int g>\int|f|-\int_E|f|$; then $\int_E g\ne 0$. Express $g$ as $\sum_{i=0}^n a_i\chi E_i$ where $\mu E_i<\infty$ for each $i$; then $0\ne\sum_{i=0}^n a_i\mu(E_i\cap E)$, so there is an $i\le n$ such that $\mu(E\cap E_i)\ne 0$, and now $E\cap E_i$ is a measurable subset of $E$ of non-zero finite measure.

As $E$ is arbitrary, this shows that $(X,\Sigma,\mu)$ must be semi-finite if $T$ is injective.

**(ii)** Now suppose that $(X,\Sigma,\mu)$ is semi-finite, and that $v\in L^\infty$ is non-zero. Express $v$ as $g^\bullet$ where $g:X\to\mathbb{R}$ is measurable; then $g\in\mathcal{L}^\infty$. Take any $a\in\,]0,\|v\|_\infty[$; then $E=\{x:|g(x)|\ge a\}$ has non-zero measure. Let $F\subseteq E$ be a measurable set of non-zero finite measure, and set $f(x)=|g(x)|/g(x)$ if $x\in F$, 0 otherwise; then $f\in\mathcal{L}^1$ and $(f\times g)(x)\ge a$ for $x\in F$, so, setting $u=f^\bullet\in L^1$, we have

$$(Tv)(u) = \int u \times v = \int f \times g \geq a\mu F = a\int |f| = a\|u\|_1 > 0.$$

This shows that $\|Tv\| \geq a$; as $a$ is arbitrary, $\|Tv\| \geq \|v\|_\infty$. We know already from 243F that $\|Tv\| \leq \|v\|_\infty$, so $\|Tv\| = \|v\|_\infty$ for every non-zero $v \in L^\infty$; the same is surely true for $v = 0$, so $T$ is norm-preserving and injective.

**(b)(i)** Using (a) and the definition of 'localizable', we see that under either of the conditions proposed $(X, \Sigma, \mu)$ is semi-finite and $T$ is injective and norm-preserving. I therefore have to show just that it is surjective iff $(X, \Sigma, \mu)$ is localizable.

**(ii)** Suppose that $T$ is surjective and that $\mathcal{E} \subseteq \Sigma$. Let $\mathcal{F}$ be the family of finite unions of members of $\mathcal{E}$, counting $\emptyset$ as the union of no members of $\mathcal{E}$, so that $\mathcal{F}$ is closed under finite unions and, for any $G \in \Sigma$, $E \setminus G$ is negligible for every $E \in \mathcal{E}$ iff $E \setminus G$ is negligible for every $E \in \mathcal{F}$.

If $u \in L^1$, then $h(u) = \lim_{E\in\mathcal{F},E\uparrow} \int_E u$ exists in $\mathbb{R}$. **P** If $u$ is non-negative, then

$$h(u) = \sup\{\int_E u : E \in \mathcal{F}\} \leq \int u < \infty.$$

For other $u$, we can express $u$ as $u_1 - u_2$, where $u_1$ and $u_2$ are non-negative, and now $h(u) = h(u_1) - h(u_2)$. **Q**

Evidently $h : L^1 \to \mathbb{R}$ is linear, being a limit of the linear functionals $u \mapsto \int_E u$, and also

$$|h(u)| \leq \sup_{E\in\mathcal{F}} |\int_E u| \leq \int |u|$$

for every $u$, so $h \in (L^1)^*$. Since we are supposing that $T$ is surjective, there is a $v \in L^\infty$ such that $Tv = h$. Express $v$ as $g^\bullet$ where $g : X \to \mathbb{R}$ is measurable and essentially bounded. Set $G = \{x : g(x) > 0\} \in \Sigma$.

If $F \in \Sigma$ and $\mu F < \infty$, then

$$\int_F g = \int (\chi F)^\bullet \times g^\bullet = (Tv)(\chi F)^\bullet = h(\chi F)^\bullet = \sup_{E\in\mathcal{F}} \mu(E \cap F).$$

**?** If $E \in \mathcal{E}$ and $E \setminus G$ is not negligible, then there is a set $F \subseteq E \setminus G$ such that $0 < \mu F < \infty$; now

$$\mu F = \mu(E \cap F) \leq \int_F g \leq 0,$$

as $g(x) \leq 0$ for $x \in F$. **X** Thus $E \setminus G$ is negligible for every $E \in \mathcal{E}$.

Let $H \in \Sigma$ be such that $E \setminus H$ is negligible for every $E \in \mathcal{E}$. **?** If $G \setminus H$ is not negligible, there is a set $F \subseteq G \setminus H$ of non-zero finite measure. Now

$$\mu(E \cap F) \leq \mu(H \cap F) = 0$$

for every $E \in \mathcal{E}$, so $\mu(E \cap F) = 0$ for every $E \in \mathcal{F}$, and $\int_F g = 0$; but $g(x) > 0$ for every $x \in F$, so $\mu F = 0$, which is impossible. **X** Thus $G \setminus H$ is negligible.

Accordingly $G$ is an essential supremum of $\mathcal{E}$ in $\Sigma$. As $\mathcal{E}$ is arbitrary, $(X, \Sigma, \mu)$ is localizable.

**(iii)** For the rest of this proof, I will suppose that $(X, \Sigma, \mu)$ is localizable and seek to show that $T$ is surjective.

Take $h \in (L^1)^*$ such that $\|h\| = 1$. Write $\Sigma^f = \{F : F \in \Sigma,\ \mu F < \infty\}$, and for $F \in \Sigma^f$ define $\nu_F : \Sigma \to \mathbb{R}$ by setting

$$\nu_F E = h(\chi(E \cap F)^\bullet)$$

for every $E \in \Sigma$. Then $\nu_F\emptyset = h(0) = 0$, and if $E$, $E' \in \Sigma$ are disjoint

$$\begin{aligned}\nu_F E + \nu_F E' &= h(\chi(E \cap F)^\bullet) + h(\chi(E' \cap F)^\bullet) = h((\chi(E \cap F) + \chi(E' \cap F))^\bullet)\\ &= h(\chi((E \cup E') \cap F)^\bullet) = \nu_F(E \cup E').\end{aligned}$$

Thus $\nu_F$ is additive. Also

$$|\nu_F E| \leq \|\chi(E \cap F)^\bullet\|_1 = \mu(E \cap F)$$

for every $E \in \Sigma$, so $\nu_F$ is truly continuous in the sense of 232Ab. By the Radon-Nikodým theorem (232E), there is an integrable function $g_F$ such that $\int_E g_F = \nu_F E$ for every $E \in \Sigma$; we may take it that $g_F$ is measurable and has domain $X$ (232He).

**(iv)** It is worth noting that $|g_F| \leq 1$ a.e. **P** If $G = \{x : g_F(x) > 1\}$, then

$$\int_G g_F = \nu_F G \le \mu(F\cap G) \le \mu G;$$

but this is possible only if $\mu G = 0$. Similarly, if $G' = \{x : g_F(x) < -1\}$, then

$$\int_{G'} g_F = \nu_F G' \ge -\mu G',$$

so again $\mu G' = 0$. **Q**

**(v)** If $F$, $F' \in \Sigma^f$, then $g_F = g_{F'}$ almost everywhere in $F\cap F'$. **P** If $E\in\Sigma$ and $E\subseteq F\cap F'$, then

$$\int_E g_F = h(\chi(E\cap F)^\bullet) = h(\chi(E\cap F')^\bullet) = \int_E g_{F'}.$$

So 131Hb gives the result. **Q** 213N (applied to $\{g_F{\restriction}F : F\in\Sigma^f\}$) now tells us that, because $\mu$ is localizable, there is a measurable function $g : X\to\mathbb{R}$ such that $g = g_F$ almost everywhere in $F$, for every $F\in\Sigma^f$.

**(vi)** For any $F\in\Sigma^f$, the set

$$\{x : x\in F,\ |g(x)|>1\} \subseteq \{x : |g_F(x)|>1\} \cup \{x : x\in F,\ g(x)\ne g_F(x)\}$$

is negligible; because $\mu$ is semi-finite, $\{x : |g(x)|>1\}$ is negligible, and $g\in\mathcal{L}^\infty$, with $\operatorname{ess\,sup}|g| \le 1$. Accordingly $v = g^\bullet \in L^\infty$, and we may speak of $Tv\in(L^1)^*$.

**(vii)** If $F\in\Sigma^f$, then

$$\int_F g = \int_F g_F = \nu_F X = h(\chi F^\bullet).$$

It follows at once that

$$(Tv)(f^\bullet) = \int f\times g = h(f^\bullet)$$

for every simple function $f : X\to\mathbb{R}$. Consequently $Tv = h$, because both $Tv$ and $h$ are continuous and the equivalence classes of simple functions form a dense subset of $L^1$ (242Mb, 2A3Uc). Thus $h = Tv$ is a value of $T$.

**(viii)** The argument as written above has assumed that $\|h\| = 1$. But of course any non-zero member of $(L^1)^*$ is a scalar multiple of an element of norm 1, so is a value of $T$. So $T : L^\infty\to(L^1)^*$ is indeed surjective, and is therefore an isometric isomorphism, as claimed.

**243H** Recall that $L^0$ is always Dedekind $\sigma$-complete and sometimes Dedekind complete (241G), while $L^1$ is always Dedekind complete (242H). In this respect $L^\infty$ follows $L^0$.

**Theorem** Let $(X,\Sigma,\mu)$ be a measure space.

(a) $L^\infty(\mu)$ is Dedekind $\sigma$-complete.

(b) If $\mu$ is localizable, $L^\infty(\mu)$ is Dedekind complete.

**proof** These are both consequences of 241G. If $A\subseteq L^\infty = L^\infty(\mu)$ is bounded above in $L^\infty$, fix $u_0\in A$ and an upper bound $w_0$ of $A$ in $L^\infty$. If $B$ is the set of upper bounds for $A$ in $L^0 = L^0(\mu)$, then $B\cap L^\infty$ is the set of upper bounds for $A$ in $L^\infty$. Moreover, if $B$ has a least member $v_0$, then we must have $u_0\le v_0\le w_0$, so that

$$0 \le v_0 - u_0 \le w_0 - u_0 \in L^\infty$$

and $v_0 - u_0$, $v_0$ belong to $L^\infty$. (Compare part (a) of the proof of 242H.) Thus $v_0 = \sup A$ in $L^\infty$.

Now we know that $L^0$ is Dedekind $\sigma$-complete; if $A\subseteq L^\infty$ is a non-empty countable set which is bounded above in $L^\infty$, it is surely bounded above in $L^0$, so has a supremum in $L^0$ which is also its supremum in $L^\infty$. As $A$ is arbitrary, $L^\infty$ is Dedekind $\sigma$-complete. While if $\mu$ is localizable, we can argue in the same way with arbitrary non-empty subsets of $L^\infty$ to see that $L^\infty$ is Dedekind complete because $L^0$ is.

**243I A dense subspace of $L^\infty$** In 242M and 242O I described a couple of important dense linear subspaces of $L^1$ spaces. The position concerning $L^\infty$ is a little different. However I can describe one important dense subspace.

**Proposition** Let $(X,\Sigma,\mu)$ be a measure space.

(a) Write $\mathcal{S}$ for the space of 'Σ-simple' functions on $X$, that is, the space of functions from $X$ to $\mathbb{R}$ expressible as $\sum_{k=0}^{n} a_k\chi E_k$ where $a_k \in \mathbb{R}$ and $E_k \in \Sigma$ for every $k \le n$. Then for every $f \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$ and every $\epsilon > 0$, there is a $g \in \mathcal{S}$ such that ess sup $|f - g| \le \epsilon$.

(b) $S = \{f^\bullet : f \in \mathcal{S}\}$ is a $\| \|_\infty$-dense linear subspace of $L^\infty = L^\infty(\mu)$.

(c) If $(X, \Sigma, \mu)$ is totally finite, then $\mathcal{S}$ is the space of $\mu$-simple functions, so $S$ becomes just the space of equivalence classes of simple functions, as in 242Mb.

**proof (a)** Let $\tilde f : X \to \mathbb{R}$ be a bounded measurable function such that $f =_{\text{a.e.}} \tilde f$. Let $n \in \mathbb{N}$ be such that $|f(x)| \le n\epsilon$ for every $x \in X$. For $-n \le k \le n$ set

$$E_k = \{x : k\epsilon \le \tilde f(x) < k + 1)\epsilon.$$

Set

$$g = \sum_{k=-n}^{n} k\epsilon\chi E_k \in \mathcal{S};$$

then $0 \le \tilde f(x) - g(x) \le \epsilon$ for every $x \in X$, so

$$\text{ess sup}\,|f - g| = \text{ess sup}\,|\tilde f - g| \le \epsilon.$$

**(b)** This follows immediately, as in 242Mb.

**(c)** also is elementary.

**243J Conditional expectations** Conditional expectations are so important that it is worth considering their interaction with every new concept.

**(a)** If $(X, \Sigma, \mu)$ is any measure space, and T is a $\sigma$-subalgebra of Σ, then the canonical embedding $S : L^0(\mu\restriction \mathrm{T}) \to L^0(\mu)$ (242Ja) embeds $L^\infty(\mu\restriction \mathrm{T})$ as a subspace of $L^\infty(\mu)$, and $\|Su\|_\infty = \|u\|_\infty$ for every $u \in L^\infty(\mu\restriction \mathrm{T})$. As in 242Jb, we can identify $L^\infty(\mu\restriction \mathrm{T})$ with its image in $L^\infty(\mu)$.

**(b)** Now suppose that $\mu X = 1$, and let $P : L^1(\mu) \to L^1(\mu\restriction \mathrm{T})$ be the conditional expectation operator (242Jd). Then $L^\infty(\mu)$ is actually a linear subspace of $L^1(\mu)$. Setting $e = \chi X^\bullet \in L^\infty(\mu)$, we see that $\int_F e = (\mu\restriction \mathrm{T})(F)$ for every $F \in \mathrm{T}$, so

$$Pe = \chi X^\bullet \in L^\infty(\mu\restriction \mathrm{T}).$$

If $u \in L^\infty(\mu)$, then setting $M = \|u\|_\infty$ we have $-Me \le u \le Me$, so $-MPe \le Pu \le MPe$, because $P$ is order-preserving (242Je); accordingly $\|Pu\|_\infty \le M = \|u\|_\infty$. Thus $P\restriction L^\infty(\mu) : L^\infty(\mu) \to L^\infty(\mu\restriction \mathrm{T})$ is an operator of norm 1.

If $u \in L^\infty(\mu\restriction \mathrm{T})$, then $Pu = u$; so $P[L^\infty]$ is the whole of $L^\infty(\mu\restriction \mathrm{T})$.

**243K Complex $L^\infty$** All the ideas needed to adapt the work above to complex $L^\infty$ spaces have already appeared in 241J and 242P. Let $\mathcal{L}^\infty_{\mathbb{C}}$ be

$$\{f : f \in \mathcal{L}^0_{\mathbb{C}}, \text{ ess sup}\,|f| < \infty\} = \{f : \mathcal{R}\mathrm{e}(f) \in \mathcal{L}^\infty, \mathcal{I}\mathrm{m}(f) \in \mathcal{L}^\infty\}.$$

Then

$$L^\infty_{\mathbb{C}} = \{f^\bullet : f \in \mathcal{L}^\infty_{\mathbb{C}}\} = \{u : u \in L^0_{\mathbb{C}}, \mathcal{R}\mathrm{e}(u) \in L^\infty, \mathcal{I}\mathrm{m}(u) \in L^\infty\}.$$

Setting

$$\|u\|_\infty = \||u|\|_\infty = \text{ess sup}\,|f| \text{ whenever } f^\bullet = u,$$

we have a norm on $L^\infty_{\mathbb{C}}$ rendering it a Banach space. We still have $u \times v \in L^\infty_{\mathbb{C}}$ and $\|u \times v\|_\infty \le \|u\|_\infty\|v\|_\infty$ for all $u, v \in L^\infty_{\mathbb{C}}$.

We now have a duality between $L^1_{\mathbb{C}}$ and $L^\infty_{\mathbb{C}}$ giving rise to a linear operator $T : L^\infty_{\mathbb{C}} \to (L^1_{\mathbb{C}})^*$ of norm at most 1, defined by the formula

$$(Tv)(u) = \int u \times v \text{ for every } u \in L^1, v \in L^\infty.$$

$T$ is injective iff the underlying measure space is semi-finite, and is a bijection iff the underlying measure space is localizable. (This can of course be proved by re-working the arguments of 243G; but it is perhaps easier to note that $T(\mathcal{R}\mathrm{e}(v)) = \mathcal{R}\mathrm{e}(Tv)$, $T(\mathcal{I}\mathrm{m}(v)) = \mathcal{I}\mathrm{m}(Tv)$ for every $v$, so that the result for complex spaces can be deduced from the result for real spaces.) To check that $T$ is norm-preserving when it is injective, the quickest route seems to be to imitate the argument of (a-ii) of the proof of 243G.

**243X Basic exercises (a)** Let $(X,\Sigma,\mu)$ be any measure space, and $\hat\mu$ the completion of $\mu$ (212C, 241Xb). Show that $\mathcal{L}^\infty(\hat\mu) = \mathcal{L}^\infty(\mu)$ and $L^\infty(\hat\mu) = L^\infty(\mu)$.

**>(b)** Let $(X,\Sigma,\mu)$ be a non-empty measure space. Write $\mathcal{L}^\infty_\Sigma$ for the space of bounded $\Sigma$-measurable real-valued functions with domain $X$. (i) Show that $L^\infty(\mu) = \{f^\bullet : f \in \mathcal{L}^\infty_\Sigma\} \subseteq L^0 = L^0(\mu)$. (ii) Show that $\mathcal{L}^\infty_\Sigma$ is a Dedekind $\sigma$-complete Banach lattice if we give it the norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \text{ for every } f \in \mathcal{L}^\infty_\Sigma.$$

(iii) Show that for every $u \in L^\infty = L^\infty(\mu)$, $\|u\|_\infty = \min\{\|f\|_\infty : f \in \mathcal{L}^\infty_\Sigma,\ f^\bullet = u\}$.

**>(c)** Let $(X,\Sigma,\mu)$ be any measure space, and $A$ a subset of $L^\infty(\mu)$. Show that $A$ is bounded for the norm $\| \, \|_\infty$ iff it is bounded above and below for the ordering of $L^\infty$.

**(d)** Let $(X,\Sigma,\mu)$ be any measure space, and $A \subseteq L^\infty(\mu)$ a non-empty set with a least upper bound $w$ in $L^\infty(\mu)$. Show that $\|w\|_\infty \le \sup_{u\in A} \|u\|_\infty$.

**(e)** Let $\langle (X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, and $(X,\Sigma,\mu)$ their direct sum (214L). Show that the canonical isomorphism between $L^0(\mu)$ and $\prod_{i\in I} L^0(\mu_i)$ (241Xd) induces an isomorphism between $L^\infty(\mu)$ and the subspace

$$\{u : u \in \textstyle\prod_{i\in I} L^\infty(\mu_i),\ \|u\| = \sup_{i\in I} \|u(i)\|_\infty < \infty\}$$

of $\prod_{i\in I} L^\infty(\mu_i)$.

**(f)** Let $(X,\Sigma,\mu)$ be any measure space, and $u \in L^1(\mu)$. Show that there is a $v \in L^\infty(\mu)$ such that $\|v\|_\infty \le 1$ and $\int u \times v = \|u\|_1$.

**(g)** Let $(X,\Sigma,\mu)$ be a semi-finite measure space and $v \in L^\infty(\mu)$. Show that

$$\|v\|_\infty = \sup\{\textstyle\int u \times v : u \in L^1,\ \|u\|_1 \le 1\} = \sup\{\|u \times v\|_1 : u \in L^1,\ \|u\|_1 \le 1\}.$$

**(h)** Give an example of a probability space $(X,\Sigma,\mu)$ and a $v \in L^\infty(\mu)$ such that $\|u \times v\|_1 < \|v\|_\infty$ whenever $u \in L^1(\mu)$ and $\|u\|_1 \le 1$.

**(i)** Write out proofs of 243G adapted to the special cases (i) $\mu X = 1$ (ii) $(X,\Sigma,\mu)$ is $\sigma$-finite.

**(j)** Let $(X,\Sigma,\mu)$ be any measure space. Show that $L^0(\mu)$ is Dedekind complete iff $L^\infty(\mu)$ is Dedekind complete.

**(k)** Let $(X,\Sigma,\mu)$ be a totally finite measure space and $\nu : \Sigma \to \mathbb{R}$ a functional. Show that the following are equiveridical: (i) there is a continuous linear functional $h : L^1(\mu) \to \mathbb{R}$ such that $h((\chi E)^\bullet) = \nu E$ for every $E \in \Sigma$ (ii) $\nu$ is additive and there is an $M \ge 0$ such that $|\nu E| \le M\mu E$ for every $E \in \Sigma$.

**>(l)** Let $X$ be any set, and let $\mu$ be counting measure on $X$. In this case it is customary to write $\ell^\infty(X)$ for $\mathcal{L}^\infty(\mu)$, and to identify it with $L^\infty(\mu)$. Write out statements and proofs of the results of this chapter adapted to this special case – if you like, with $X = \mathbb{N}$. In particular, write out a direct proof that $(\ell^1)^*$ can be identified with $\ell^\infty$. What happens when $X$ has just two members? or three?

**(m)** Show that if $(X,\Sigma,\mu)$ is any measure space and $u \in L^\infty_{\mathbb{C}}(\mu)$, then

$$\|u\|_\infty = \sup\{\| \mathcal{R}\mathrm{e}(\zeta u)\|_\infty : \zeta \in \mathbb{C},\ |\zeta| = 1\}.$$

**(n)** Let $(X,\Sigma,\mu)$ and $(Y,\mathrm{T},\nu)$ be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Show that $g\phi \in \mathcal{L}^\infty(\mu)$ for every $g \in \mathcal{L}^\infty(\nu)$, and that the map $g \mapsto g\phi$ induces a linear operator $T : L^\infty(\nu) \to L^\infty(\mu)$ defined by setting $T(g^\bullet) = (g\phi)^\bullet$ for every $g \in \mathcal{L}^\infty(\nu)$. (Compare 241Xg.) Show that $\|Tv\|_\infty = \|v\|_\infty$ for every $v \in L^\infty(\nu)$.

**(o)** For $f$, $g \in C = C([0,1])$, the space of continuous real-valued functions on the unit interval $[0,1]$, say

$$f \le g \text{ iff } f(x) \le g(x) \text{ for every } x \in [0,1],$$

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Show that $C$ is a Banach lattice, and that moreover

$$\|f \vee g\|_\infty = \max(\|f\|_\infty, \|g\|_\infty) \text{ whenever } f,\, g \ge 0,$$

$$\|f \times g\|_\infty \le \|f\|_\infty \|g\|_\infty \text{ for all } f,\, g \in C,$$

$$\|f\|_\infty = \min\{\gamma : |f| \le \gamma\chi X\} \text{ for every } f \in C.$$

**243Y Further exercises (a)** Let $(X, \Sigma, \mu)$ be a measure space, and $Y$ a subset of $X$; write $\mu_Y$ for the subspace measure on $Y$. Show that the canonical map from $L^0(\mu)$ onto $L^0(\mu_Y)$ (241Yg) induces a canonical map from $L^\infty(\mu)$ onto $L^\infty(\mu_Y)$, which is norm-preserving iff it is injective iff $Y$ has full outer measure.

**243 Notes and comments** I mention the formula

$$\|u \vee v\|_\infty = \max(\|u\|_\infty, \|v\|_\infty) \text{ for } u,\, v \ge 0$$

(243Df) because while it does not characterize $L^\infty$ spaces among Banach lattices (see 243Xo), it is in a sense dual to the characteristic property

$$\|u + v\|_1 = \|u\|_1 + \|v\|_1 \text{ for } u,\, v \ge 0$$

of the norm of $L^1$. (I will return to this in Chapter 35 in the next volume.)

The particular set $\mathcal{L}^\infty$ I have chosen (243A) is somewhat arbitrary. The space $L^\infty$ can very well be described entirely as a subspace of $L^0$, without going back to functions at all; see 243Bc, 243Dc. Just as with $\mathcal{L}^0$ and $\mathcal{L}^1$, there are occasions when it would be simpler to work with the linear space of essentially bounded measurable functions from $X$ to $\mathbb{R}$; and we now have a third obvious candidate, the linear space $\mathcal{L}^\infty_\Sigma$ of measurable functions from $X$ to $\mathbb{R}$ which are literally, rather than essentially, bounded, which is itself a Banach lattice (243Xb).

I suppose the most important theorem of this section is 243G, identifying $L^\infty$ with $(L^1)^*$. This identification is the chief reason for setting 'localizable' measure spaces apart. The proof of 243Gb is long because it depends on two separate ideas. The Radon-Nikodým theorem deals, in effect, with the totally finite case, and then in parts (b-v) and (b-vi) of the proof localizability is used to link the partial solutions $g_F$ together. Exercise 243Xi is supposed to help you to distinguish the two operations. The map $T' : L^1 \to (L^\infty)^*$ (243Fc) is also very interesting in its way, but I shall leave it for Chapter 36.

243G gives another way of looking at conditional expectation operators. If $(X, \Sigma, \mu)$ is a probability space and $\mathrm{T}$ is a $\sigma$-subalgebra of $\Sigma$, of course both $\mu$ and $\mu\restriction\mathrm{T}$ are localizable, so $L^\infty(\mu)$ can be identified with $(L^1(\mu))^*$ and $L^\infty(\mu\restriction\mathrm{T})$ can be identified with $(L^1(\mu\restriction\mathrm{T}))^*$. Now we have the canonical embedding $S : L^1(\mu\restriction\mathrm{T}) \to L^1(\mu)$ (242Jb) which is a norm-preserving linear operator, so gives rise to an adjoint operator $S' : L^1(\mu)^* \to L^1(\mu\restriction\mathrm{T})^*$ defined by the formula

$$(S'h)(v) = h(Sv) \text{ for all } v \in L^1(\mu\restriction\mathrm{T}),\, h \in L^1(\mu)^*.$$

Writing $T_\mu : L^\infty(\mu) \to L^1(\mu)^*$ and $T_{\mu\restriction\mathrm{T}} : L^\infty(\mu\restriction\mathrm{T}) \to L^1(\mu\restriction\mathrm{T})^*$ for the canonical maps, we get a map $Q = T^{-1}_{\mu\restriction\mathrm{T}} S' T_\mu : L^\infty(\mu) \to L^\infty(\mu\restriction\mathrm{T})$, defined by saying that

$$\int Qu \times v = (T_{\mu\restriction\mathrm{T}}Qu)(v) = (S'T_\mu u)(v) = (T_\mu v)(Su) = \int Su \times v = \int u \times v$$

whenever $v \in L^1(\mu\restriction\mathrm{T})$ and $u \in L^\infty(\mu)$. But this agrees with the formula of 242L: we have

$$\int Qu \times v = \int u \times v = \int P(u \times v) = \int Pu \times v.$$

Because $v$ is arbitrary, we must have $Qu = Pu$ for every $u \in L^\infty(\mu)$. Thus a conditional expectation operator is, in a sense, the adjoint of the appropriate embedding operator.

The discussion in the last paragraph applies, of course, only to the restriction $P\restriction L^\infty(\mu)$ of the conditional expectation operator to the $L^\infty$ space. Because $\mu$ is totally finite, $L^\infty(\mu)$ is a subspace of $L^1(\mu)$, and the real qualities of the operator $P$ are related to its behaviour on the whole space $L^1$. $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$ can also be expressed as an adjoint operator, but the expression needs more of the theory of Riesz spaces than I have space for here. I will return to this topic in Chapter 36.

Version of 6.3.09

## 244 $L^p$

Continuing with our tour of the classical Banach spaces, we come to the $L^p$ spaces for $1 < p < \infty$. The case $p = 2$ is more important than all the others put together, and it would be reasonable, perhaps even advisable, to read this section first with this case alone in mind. But the other spaces provide instructive examples and remain a basic part of the education of any functional analyst.

**244A Definitions** Let $(X, \Sigma, \mu)$ be any measure space, and $p \in ]1, \infty[$. Write $\mathcal{L}^p = \mathcal{L}^p(\mu)$ for the set of functions $f \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$ such that $|f|^p$ is integrable, and $L^p = L^p(\mu)$ for $\{f^\bullet : f \in \mathcal{L}^p\} \subseteq L^0 = L^0(\mu)$.

Note that if $f \in \mathcal{L}^p$, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then $|f|^p =_{\text{a.e.}} |g|^p$ so $|g|^p$ is integrable and $g \in \mathcal{L}^p$; thus $\mathcal{L}^p = \{f : f \in \mathcal{L}^0,\ f^\bullet \in L^p\}$.

Alternatively, we can define $u^p$ whenever $u \in L^0$, $u \ge 0$ by writing $(f^\bullet)^p = (f^p)^\bullet$ for every $f \in \mathcal{L}^0$ such that $f(x) \ge 0$ for every $x \in \operatorname{dom} f$ (compare 241I), and say that $L^p = \{u : u \in L^0,\ |u|^p \in L^1(\mu)\}$.

**244B Theorem** Let $(X, \Sigma, \mu)$ be any measure space, and $p \in [1, \infty]$.

(a) $L^p = L^p(\mu)$ is a linear subspace of $L^0 = L^0(\mu)$.

(b) If $u \in L^p$, $v \in L^0$ and $|v| \le |u|$, then $v \in L^p$. Consequently $|u|$, $u \vee v$ and $u \wedge v$ belong to $L^p$ for all $u$, $v \in L^p$.

**proof** The cases $p = 1$, $p = \infty$ are covered by 242B, 242C and 243B; so I suppose that $1 < p < \infty$.

**(a)(i)** Suppose that $f$, $g \in \mathcal{L}^p = \mathcal{L}^p(\mu)$. If $a$, $b \in \mathbb{R}$ then $|a + b|^p \le 2^p \max(|a|^p, |b|^p)$, so $|f + g|^p \le_{\text{a.e.}} 2^p(|f|^p \vee |g|^p)$; now $|f+g|^p \in \mathcal{L}^0$ and $2^p(|f|^p \vee |g|^p) \in \mathcal{L}^1$ so $|f+g|^p \in \mathcal{L}^1$. Thus $f+g \in \mathcal{L}^p$ for all $f$, $g \in \mathcal{L}^p$; it follows at once that $u + v \in L^p$ for all $u$, $v \in L^p$.

**(ii)** If $f \in \mathcal{L}^p$ and $c \in \mathbb{R}$ then $|cf|^p = |c|^p|f|^p \in \mathcal{L}^1$, so $cf \in \mathcal{L}^p$. Accordingly $cu \in L^p$ whenever $u \in L^p$ and $c \in \mathbb{R}$.

**(b)(i)** Express $u$ as $f^\bullet$ and $v$ as $g^\bullet$, where $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^0$. Then $|g| \le_{\text{a.e.}} |f|$, so $|g|^p \le_{\text{a.e.}} |f|^p$ and $|g|^p$ is integrable; accordingly $g \in \mathcal{L}^p$ and $v \in L^p$.

**(ii)** Now $|\,|u|\,| = |u|$ so $|u| \in L^p$ whenever $u \in L^p$. Finally $u \vee v = \frac{1}{2}(u + v + |u - v|)$ and $u \wedge v = \frac{1}{2}(u + v - |u - v|)$ belong to $L^p$ for all $u$, $v \in L^p$.

**244C The order structure of $L^p$** Let $(X, \Sigma, \mu)$ be any measure space, and $p \in [1, \infty]$. Then 244B is enough to ensure that the partial order inherited from $L^0(\mu)$ makes $L^p(\mu)$ a Riesz space (compare 242C, 243C).

**244D The norm of $L^p$** Let $(X, \Sigma, \mu)$ be a measure space, and $p \in ]1, \infty[$.

**(a)** For $f \in \mathcal{L}^p = \mathcal{L}^p(\mu)$, set $\|f\|_p = (\int |f|^p)^{1/p}$. If $f$, $g \in \mathcal{L}^p$ and $f =_{\text{a.e.}} g$ then $|f|^p =_{\text{a.e.}} |g|^p$ so $\|f\|_p = \|g\|_p$. Accordingly we may define $\|\,\|_p : L^p = L^p(\mu) \to [0, \infty[$ by writing $\|f^\bullet\|_p = \|f\|_p$ for every $f \in \mathcal{L}^p$.

Alternatively, we can say just that $\|u\|_p = (\int |u|^p)^{1/p}$ for every $u \in L^p = L^p(\mu)$.

**(b)** The notation $\|\,\|_p$ carries a promise that it is a norm on $L^p$; this is indeed so, but I hold the proof over to 244F below. For the moment, however, let us note just that $\|cu\|_p = |c|\|u\|_p$ for all $u \in L^p$ and $c \in \mathbb{R}$, and that if $\|u\|_p = 0$ then $\int |u|^p = 0$ so $|u|^p = 0$ and $u = 0$.

**(c)** If $|u| \le |v|$ in $L^p$ then $\|u\|_p \le \|v\|_p$; this is because $|u|^p \le |v|^p$.

**244E** I now work through the lemmas required to show that $\| \,\|_p$ is a norm on $L^p$ and, eventually, that the normed space dual of $L^p$ may be identified with a suitable $L^q$.

**Lemma** Suppose $(X, \Sigma, \mu)$ is a measure space, and that $p$, $q \in ]1, \infty[$ are such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) $ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$ for all real $a$, $b \ge 0$.

(b)(i) $f \times g$ is integrable and

$$|\int f \times g| \le \int |f \times g| \le \|f\|_p \|g\|_q$$

for all $f \in \mathcal{L}^p = \mathcal{L}^p(\mu)$, $g \in \mathcal{L}^q = \mathcal{L}^q(\mu)$;

(ii) $u \times v \in L^1 = L^1(\mu)$ and

$$|\int u \times v| \le \|u \times v\|_1 \le \|u\|_p \|v\|_q$$

for all $u \in L^p = L^p(\mu)$, $v \in L^q = L^q(\mu)$.

**proof (a)** If either $a$ or $b$ is 0, this is trivial. If both are non-zero, we may argue as follows. The function $x \mapsto x^{1/p} : [0, \infty[ \to \mathbb{R}$ is concave, with second derivative strictly less than 0, so lies entirely below any of its tangents; in particular, below its tangent at the point $(1, 1)$, which has equation $y = 1 + \frac{1}{p}(x - 1)$. Thus we have

$$x^{1/p} \le \frac{1}{p}x + 1 - \frac{1}{p} = \frac{1}{p}x + \frac{1}{q}$$

for every $x \in [0, \infty[$. So if $c$, $d > 0$, then

$$(\frac{c}{d})^{1/p} \le \frac{1}{p}\frac{c}{d} + \frac{1}{q};$$

multiplying both sides by $d$,

$$c^{1/p}d^{1/q} \le \frac{1}{p}c + \frac{1}{q}d;$$

setting $c = a^p$ and $d = b^q$, we get

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q,$$

as claimed.

**(b)(i)**($\alpha$) Suppose first that $\|f\|_p = \|g\|_q = 1$. For every $x \in \operatorname{dom} f \cap \operatorname{dom} g$ we have

$$|f(x)g(x)| \le \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

by (a). So

$$|f \times g| \le_{\text{a.e.}} \frac{1}{p}|f|^p + \frac{1}{q}|g|^q \in \mathcal{L}^1(\mu)$$

and $f \times g$ is integrable; also

$$\int |f \times g| \le \frac{1}{p}\int |f|^p + \frac{1}{q}\int |g|^q = \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q = \frac{1}{p} + \frac{1}{q} = 1.$$

($\beta$) If $\|f\|_p = 0$, then $\int |f|^p = 0$ so $|f|^p =_{\text{a.e.}} \mathbf{0}$, $f =_{\text{a.e.}} \mathbf{0}$, $f \times g =_{\text{a.e.}} \mathbf{0}$ and

$$\int |f \times g| = 0 = \|f\|_p \|g\|_q.$$

Similarly, if $\|g\|_q = 0$, then $g =_{\text{a.e.}} \mathbf{0}$ and again

$$\int |f \times g| = 0 = \|f\|_p \|g\|_q.$$

($\gamma$) Finally, for general $f \in \mathcal{L}^p$, $g \in \mathcal{L}^q$ such that $c = \|f\|_p$ and $d = \|g\|_q$ are both non-zero, we have $\|\frac{1}{c}f\|_p = \|\frac{1}{d}g\|_q = 1$ so

$$f \times g = cd(\frac{1}{c}f \times \frac{1}{d}g)$$

is integrable, and

$$\int |f \times g| = cd\int |\frac{1}{c}f \times \frac{1}{d}g| \le cd,$$

as required.

**(ii)** Now if $u \in L^p$, $v \in L^q$ take $f \in \mathcal{L}^p$, $g \in \mathcal{L}^q$ such that $u = f^\bullet$ and $v = g^\bullet$; $f \times g$ is integrable, so $u \times v \in L^1$, and

$$|\int u \times v| \le \|u \times v\|_1 = \int |f \times g| \le \|f\|_p \|g\|_q = \|u\|_p \|v\|_q.$$

**Remark** Part (b) is '**Hölder's inequality**'. In the case $p = q = 2$ it is '**Cauchy's inequality**'.

**244F Proposition** Let $(X, \Sigma, \mu)$ be a measure space and $p \in ]1, \infty[$. Set $q = p/(p-1)$, so that $\frac{1}{p} + \frac{1}{q} = 1$.
(a) For every $u \in L^p = L^p(\mu)$, $\|u\|_p = \max\{\int u \times v : v \in L^q(\mu),\ \|v\|_q \le 1\}$.
(b) $\|\ \|_p$ is a norm on $L^p$.

**proof (a)** For $u \in L^p$, set

$$\tau(u) = \sup\{\int u \times v : v \in L^q(\mu),\ \|v\|_q \le 1\}.$$

By 244E(b-ii), $\|u\|_p \ge \tau(u)$. If $\|u\|_p = 0$ then surely

$$0 = \|u\|_p = \tau(u) = \max\{\int u \times v : v \in L^q(\mu),\ \|v\|_q \le 1\}.$$

If $\|u\|_p = c > 0$, consider

$$v = c^{-p/q} \operatorname{sgn} u \times |u|^{p/q},$$

where for $a \in \mathbb{R}$ I write $\operatorname{sgn} a = |a|/a$ if $a \ne 0$, $0$ if $a = 0$, so that $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function; for $f \in \mathcal{L}^0$ I write $(\operatorname{sgn} f)(x) = \operatorname{sgn}(f(x))$ for $x \in \operatorname{dom} f$, so that $\operatorname{sgn} f \in \mathcal{L}^0$; and for $f \in \mathcal{L}^0$ I write $\operatorname{sgn}(f^\bullet) = (\operatorname{sgn} f)^\bullet$ to define a function $\operatorname{sgn} : L^0 \to L^0$ (cf. 241I). Then $v \in L^q = L^q(\mu)$ and

$$\|v\|_q = (\int |v|^q)^{1/q} = c^{-p/q}(\int |u|^p)^{1/q} = c^{-p/q}c^{p/q} = 1.$$

So

$$\begin{aligned}\tau(u) \ge \int u \times v &= c^{-p/q} \int \operatorname{sgn} u \times |u| \times \operatorname{sgn} u \times |u|^{p/q} \\ &= c^{-p/q} \int |u|^{1+\frac{p}{q}} = c^{-p/q} \int |u|^p = c^{p-\frac{p}{q}} = c,\end{aligned}$$

recalling that $1 + \frac{p}{q} = p$, $p - \frac{p}{q} = 1$. Thus $\tau(u) \ge \|u\|_p$ and

$$\tau(u) = \|u\|_p = \int u \times v.$$

**(b)** In view of the remarks in 244Db, I have only to check that $\|u + v\|_p \le \|u\|_p + \|v\|_p$ for all $u$, $v \in L^p$. But given $u$ and $v$, let $w \in L^q$ be such that $\|w\|_q = 1$ and $\int (u + v) \times w = \|u + v\|_p$. Then

$$\|u + v\|_p = \int (u + v) \times w = \int u \times w + \int v \times w \le \|u\|_p + \|v\|_p,$$

as required.

**Remark** The triangle inequality '$\|u + v\|_p \le \|u\|_p + \|v\|_p$' is **Minkowski's inequality**.

**244G Theorem** Let $(X, \Sigma, \mu)$ be any measure space, and $p \in [1, \infty]$. Then $L^p = L^p(\mu)$ is a Banach lattice under its norm $\|\ \|_p$.

**proof** The cases $p = 1$, $p = \infty$ are covered by 242F and 243E, so let us suppose that $1 < p < \infty$. We know already that $\|u\|_p \le \|v\|_p$ whenever $|u| \le |v|$, so it remains only to show that $L^p$ is complete.

Let $\langle u_n \rangle_{n \in \mathbb{N}}$ be a sequence in $L^p$ such that $\|u_{n+1} - u_n\|_p \le 4^{-n}$ for every $n \in \mathbb{N}$. Note that

$$\|u_n\|_p \le \|u_0\|_p + \sum_{k=0}^{n-1} \|u_{k+1} - u_k\|_p \le \|u_0\|_p + \sum_{k=0}^{\infty} 4^{-k} \le \|u_0\|_p + 2$$

for every $n$. For each $n \in \mathbb{N}$, choose $f_n \in \mathcal{L}^p$ such that $f_0^\bullet = u_0$, $f_n^\bullet = u_n - u_{n-1}$ for $n \geq 1$; do this in such a way that $\operatorname{dom} f_n = X$ and $f_n$ is $\Sigma$-measurable (241Bk). Then $\|f_n\|_p \leq 4^{-n+1}$ for $n \geq 1$.

For $m$, $n \in \mathbb{N}$, set

$$E_{mn} = \{x : |f_m(x)| \geq 2^{-n}\} \in \Sigma.$$

Then $|f_m(x)|^p \geq 2^{-np}$ for $x \in E_{mn}$, so

$$2^{-np}\mu E_{mn} \leq \int |f_m|^p < \infty$$

and $\mu E_{mn} < \infty$. So $\chi E_{mn} \in \mathcal{L}^q = \mathcal{L}^q(\mu)$ and

$$\int_{E_{mn}} |f_k| = \int |f_k| \times \chi E_{mn} \leq \|f_k\|_p \|\chi E_{mn}\|_q$$

for each $k$, by 244E(b-i). Accordingly

$$\sum_{k=0}^\infty \int_{E_{mn}} |f_k| \leq \|\chi E_{mn}\|_q \sum_{k=0}^\infty \|f_k\|_p < \infty,$$

and $\sum_{k=0}^\infty f_k(x)$ exists for almost every $x \in E_{mn}$, by 242E. This is true for all $m$, $n \in \mathbb{N}$. But if $x \in X \setminus \bigcup_{m,n\in\mathbb{N}} E_{mn}$, $f_n(x) = 0$ for every $n$, so $\sum_{k=0}^\infty f_k(x)$ certainly exists. Thus $g(x) = \sum_{k=0}^\infty f_k(x)$ is defined in $\mathbb{R}$ for almost every $x \in X$.

Set $g_n = \sum_{k=0}^n f_k$; then $g_n^\bullet = u_n \in L^p$ for each $n$, and $g(x) = \lim_{n\to\infty} g_n(x)$ is defined for almost every $x$. Now consider $|g|^p =_{\text{a.e.}} \lim_{n\to\infty} |g_n|^p$. We know that

$$\liminf_{n\to\infty} \int |g_n|^p = \liminf_{n\to\infty} \|u_n\|_p^p \leq (2 + \|u_0\|_p)^p < \infty,$$

so by Fatou's Lemma

$$\int |g|^p \leq \liminf_{k\to\infty} \int |g_k|^p < \infty.$$

Thus $u = g^\bullet \in L^p$. Moreover, for any $m \in \mathbb{N}$,

$$\begin{aligned}\int |g - g_m|^p &\leq \liminf_{n\to\infty} \int |g_n - g_m|^p = \liminf_{n\to\infty} \|u_n - u_m\|_p^p \\ &\leq \liminf_{n\to\infty} \sum_{k=m}^{n-1} 4^{-kp} = \sum_{k=m}^\infty 4^{-kp} = 4^{-mp}/(1 - 4^{-p}).\end{aligned}$$

So

$$\|u - u_m\|_p = (\int |g - g_m|^p)^{1/p} \leq 4^{-m}/(1 - 4^{-p})^{1/p} \to 0$$

as $m \to \infty$. Thus $u = \lim_{m\to\infty} u_m$ in $L^p$. As $\langle u_n \rangle_{n\in\mathbb{N}}$ is arbitrary, $L^p$ is complete.

**244H** Following 242M and 242O, I note that $L^p$ behaves like $L^1$ in respect of certain dense subspaces.

**Proposition** (a) Let $(X, \Sigma, \mu)$ be any measure space, and $p \in [1, \infty[$. Then the space $S$ of equivalence classes of $\mu$-simple functions is a dense linear subspace of $L^p = L^p(\mu)$.

(b) Let $X$ be any subset of $\mathbb{R}^r$, where $r \geq 1$, and let $\mu$ be the subspace measure on $X$ induced by Lebesgue measure on $\mathbb{R}^r$. Write $C_k$ for the set of (bounded) continuous functions $g : \mathbb{R}^r \to \mathbb{R}$ such that $\{x : g(x) \neq 0\}$ is bounded, and $S_0$ for the space of linear combinations of functions of the form $\chi I$, where $I \subseteq \mathbb{R}^r$ is a bounded half-open interval. Then $\{(g{\restriction}X)^\bullet : g \in C_k\}$ and $\{(h{\restriction}X)^\bullet : h \in S_0\}$ are dense in $L^p(\mu)$.

**proof (a)** I repeat the argument of 242M with a tiny modification.

**(i)** Suppose that $u \in L^p(\mu)$, $u \geq 0$ and $\epsilon > 0$. Express $u$ as $f^\bullet$ where $f : X \to [0, \infty[$ is a measurable function. Let $g : X \to \mathbb{R}$ be a simple function such that $0 \leq g \leq f^p$ and $\int g \geq \int f^p - \epsilon^p$. Set $h = g^{1/p}$. Then $h$ is a simple function and $h \leq f$. Because $p > 1$, $(f - h)^p + h^p \leq f^p$ and

$$\int (f - h)^p \leq \int f^p - g \leq \epsilon^p,$$

so

$$\|u - h^\bullet\|_p = (\int |f - h|^p)^{1/p} \leq \epsilon,$$

while $h^\bullet \in S$.

**(ii)** For general $u \in L^p$, $\epsilon > 0$, $u$ can be expressed as $u^+ - u^-$ where $u^+ = u \vee 0$, $u^- = (-u) \vee 0$ belong to $L^p$ and are non-negative. By (i), we can find $v_1$, $v_2 \in S$ such that $\|u^+ - v_1\|_p \le \frac{1}{2}\epsilon$ and $\|u^- - v_2\|_p \le \frac{1}{2}\epsilon$, so that $v = v_1 - v_2 \in S$ and $\|u - v\|_p \le \epsilon$. As $u$ and $\epsilon$ are arbitrary, $S$ is dense.

**(b)** Again, all the ideas are to be found in 242O; the changes needed are in the formulae, not in the method. They will go more easily if I note at the outset that whenever $f_1$, $f_2 \in \mathcal{L}^p(\mu)$ and $\int |f_1|^p \le \epsilon^p$, $\int |f_2|^p \le \delta^p$ (where $\epsilon$, $\delta \ge 0$), then $\int |f_1 + f_2|^p \le (\epsilon+\delta)^p$; this is just the triangle inequality for $\| \; \|_p$ (244Fb). Also I will regularly express the target relationships in the form '$\int_X |f - g|^p \le \epsilon^p$', '$\int_X |f - g|^p \le \epsilon^p$'. Now let me run through the argument of 242Oa, rather more briskly than before.

**(i)** Suppose first that $f = \chi I \restriction X$ where $I \subseteq \mathbb{R}^r$ is a bounded half-open interval. As before, we can set $h = \chi I$ and get $\int_X |f - h|^p = 0$. This time, use the same construction to find an interval $J$ and a function $g \in C_k$ such that $\chi I \le g \le \chi J$ and $\mu_r(J \setminus I) \le \epsilon^p$; this will ensure that $\int_X |f - g|^p \le \epsilon^p$.

**(ii)** Now suppose that $f = \chi(X \cap E)$ where $E \subseteq \mathbb{R}^r$ is a set of finite measure. Then, for the same reasons as before, there is a disjoint family $I_0, \dots, I_n$ of half-open intervals such that $\mu_r(E \triangle \bigcup_{j\le n} I_j) \le (\frac{1}{2}\epsilon)^p$. Accordingly $h = \sum_{j=0}^n \chi I_j \in S_0$ and $\int_X |f - h|^p \le (\frac{1}{2}\epsilon)^p$. And (i) tells us that there is for each $j \le n$ a $g_j \in C_k$ such that $\int_X |g_j - \chi I_j|^p \le (\epsilon/2(n+1))^p$, so that $g = \sum_{j=0}^n g_j \in C_k$ and $\int_X |f - g|^p \le \epsilon^p$.

**(iii)** The move to simple functions, and thence to arbitrary members of $\mathcal{L}^p(\mu)$, is just as before, but using $\|f\|_p$ in place of $\int_X |f|$. Finally, the translation from $\mathcal{L}^p$ to $L^p$ is again direct – remembering, as before, to check that $g \restriction X$, $h \restriction X$ belong to $\mathcal{L}^p(\mu)$ whenever $g \in C_k$ and $h \in S_0$.

***244I Corollary** In the context of 244Hb, $L^p(\mu)$ is separable.

**proof** Let $A$ be the set

$$\{(\sum_{j=0}^n q_j \chi([a_j, b_j[ \cap X))^\bullet : n \in \mathbb{N},\ q_0, \dots, q_n \in \mathbb{Q},\ a_0, \dots, a_n, b_0, \dots, b_n \in \mathbb{Q}^r\}.$$

Then $A$ is a countable subset of $L^p(\mu)$, and its closure must contain $(\sum_{j=0}^n c_j \chi([a_j, b_j[ \cap X))^\bullet$ whenever $c_0, \dots, c_n \in \mathbb{R}$ and $a_0, \dots, a_n, b_0, \dots, b_n \in \mathbb{R}^r$; that is, $\overline{A}$ is a closed set including $\{(h \restriction X)^\bullet : h \in S_0\}$, and is the whole of $L^p(\mu)$, by 244Hb.

**244J Duality in $L^p$ spaces** Let $(X, \Sigma, \mu)$ be any measure space, and $p \in ]1, \infty[$. Set $q = p/(p-1)$; note that $\frac{1}{p} + \frac{1}{q} = 1$ and that $p = q/(q-1)$; the relation between $p$ and $q$ is symmetric. Now $u \times v \in L^1(\mu)$ and $\|u \times v\|_1 \le \|u\|_p \|v\|_q$ whenever $u \in L^p = L^p(\mu)$ and $v \in L^q = L^q(\mu)$ (244E). Consequently we have a bounded linear operator $T$ from $L^q$ to the normed space dual $(L^p)^*$ of $L^p$, given by writing

$$(Tv)(u) = \int u \times v$$

for all $u \in L^p$ and $v \in L^q$, exactly as in 243F.

**244K Theorem** Let $(X, \Sigma, \mu)$ be a measure space, and $p \in ]1, \infty[$; set $q = p/(p-1)$. Then the canonical map $T : L^q(\mu) \to L^p(\mu)^*$, described in 244J, is a normed space isomorphism.

**Remark** I should perhaps remind anyone who is reading this section to learn about $L^2$ that the basic theory of Hilbert spaces yields this theorem in the case $p = q = 2$ without any need for the more generally applicable argument given below (see 244N, 244Yk).

**proof** We know that $T$ is a bounded linear operator of norm at most 1; I need to show (i) that $T$ is actually an isometry (that is, that $\|Tv\| = \|v\|_q$ for every $v \in L^q$), which will show incidentally that $T$ is injective (ii) that $T$ is surjective, which is the really substantial part of the theorem.

**(a)** If $v \in L^q$, then by 244Fa (recalling that $p = q/(q-1)$) there is a $u \in L^p$ such that $\|u\|_p \le 1$ and $\int u \times v = \|v\|_q$; thus $\|Tv\| \ge (Tv)(u) = \|v\|_q$. As we know already that $\|Tv\| \le \|v\|_q$, we have $\|Tv\| = \|v\|_q$ for every $v$, and $T$ is an isometry.

**(b)** The rest of the proof, therefore, will be devoted to showing that $T : L^q \to (L^p)^*$ is surjective. Fix $h \in (L^p)^*$ with $\|h\| = 1$.

I need to show that $h$ 'lives on' a countable union of sets of finite measure in $X$, in the following sense: there is a non-decreasing sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of sets of finite measure such that $h(f^\bullet) = 0$ whenever $f \in \mathcal{L}^p$ and $f(x) = 0$ for $x \in \bigcup_{n\in\mathbb{N}} E_n$. **P** Choose a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $L^p$ such that $\|u_n\|_p \le 1$ for every $n$ and $\lim_{n\to\infty} h(u_n) = \|h\| = 1$. For each $n$, express $u_n$ as $f_n^\bullet$, where $f_n : X \to \mathbb{R}$ is a measurable function. Set

$$E_n = \{x : \sum_{k=0}^n |f_k(x)|^p \ge 2^{-n}\}$$

for $n \in \mathbb{N}$; because $|f_k|^p$ is measurable and integrable and has domain $X$ for every $k$, $E_n \in \Sigma$ and $\mu E_n < \infty$ for each $n$.

Now suppose that $f \in \mathcal{L}^p(X)$ and that $f(x) = 0$ for $x \in \bigcup_{n\in\mathbb{N}} E_n$; set $u = f^\bullet \in L^p$. **?** Suppose, if possible, that $h(u) \ne 0$, and consider $h(cu)$, where

$$\operatorname{sgn} c = \operatorname{sgn} h(u), \quad 0 < |c| < (p\,|h(u)|\,\|u\|_p^{-p})^{1/(p-1)}.$$

(Of course $\|u\|_p \ne 0$ if $h(u) \ne 0$.) For each $n$, we have

$$\{x : f_n(x) \ne 0\} \subseteq \bigcup_{m\in\mathbb{N}} E_m \subseteq \{x : f(x) = 0\},$$

so $|f_n + cf|^p = |f_n|^p + |cf|^p$ and

$$h(u_n + cu) \le \|u_n + cu\|_p = (\|u_n\|_p^p + \|cu\|_p^p)^{1/p} \le (1 + |c|^p\|u\|_p^p)^{1/p}.$$

Letting $n \to \infty$,

$$1 + ch(u) \le (1 + |c|^p\|u\|_p^p)^{1/p}.$$

Because $\operatorname{sgn} c = \operatorname{sgn} h(u)$, $ch(u) = |c||h(u)|$ and we have

$$1 + p|c||h(u)| \le (1 + ch(u))^p \le 1 + |c|^p\|u\|_p^p,$$

so that

$$p|h(u)| \le |c|^{p-1}\|u\|_p^p < p|h(u)|$$

by the choice of $c$; which is impossible. **X**

This means that $h(f^\bullet) = 0$ whenever $f : X \to \mathbb{R}$ is measurable, belongs to $\mathcal{L}^q$, and is zero on $\bigcup_{n\in\mathbb{N}} E_n$. **Q**

**(c)** Set $H_n = E_n \setminus \bigcup_{k<n} E_k$ for each $n \in \mathbb{N}$; then $\langle H_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence of sets of finite measure. Now $h(u) = \sum_{n=0}^\infty h(u \times (\chi H_n)^\bullet)$ for every $u \in L^p$. **P** Express $u$ as $f^\bullet$, where $f : X \to \mathbb{R}$ is measurable. Set $f_n = f \times \chi H_n$ for each $n$, $g = f \times \chi(X \setminus \bigcup_{n\in\mathbb{N}} H_n)$; then $h(g^\bullet) = 0$, by (a), because $\bigcup_{n\in\mathbb{N}} H_n = \bigcup_{n\in\mathbb{N}} E_n$. Consider

$$g_n = g + \sum_{k=0}^n f_k \in \mathcal{L}^p$$

for each $n$. Then $\lim_{n\to\infty} f - g_n = \mathbf{0}$, and

$$|f - g_n|^p \le |f|^p \in \mathcal{L}^1$$

for every $n$, so by either Fatou's Lemma or Lebesgue's Dominated Convergence Theorem

$$\lim_{n\to\infty} \int |f - g_n|^p = 0,$$

and

$$\begin{aligned}\lim_{n\to\infty} \|u - g^\bullet - \sum_{k=0}^n u \times (\chi H_k)^\bullet\|_p &= \lim_{n\to\infty} \|u - g_n^\bullet\|_p \\ &= \lim_{n\to\infty} \Bigl(\int |f - g_n|^p\Bigr)^{1/p} = 0,\end{aligned}$$

that is,

$$u = g^\bullet + \sum_{k=0}^\infty u \times \chi H_k^\bullet$$

in $L^p$. Because $h : L^p \to \mathbb{R}$ is linear and continuous, it follows that

$$h(u) = h(g^\bullet) + \sum_{k=0}^\infty h(u \times \chi H_k^\bullet) = \sum_{k=0}^\infty h(u \times \chi H_k^\bullet),$$

as claimed. **Q**

**(d)** For each $n \in \mathbb{N}$, define $\nu_n : \Sigma \to \mathbb{R}$ by setting

$$\nu_n F = h(\chi(F \cap H_n)^\bullet)$$

for every $F \in \Sigma$. (Note that $\nu_n F$ is always defined because $\mu(F \cap H_n) < \infty$, so that

$$\|\chi(F \cap H_n)\|_p = \mu(F \cap H_n)^{1/p} < \infty.)$$

Then $\nu_n \emptyset = h(0) = 0$, and if $\langle F_k \rangle_{k\in\mathbb{N}}$ is a disjoint sequence in $\Sigma$,

$$\|\chi(\bigcup_{k\in\mathbb{N}} H_n \cap F_k) - \sum_{k=0}^m \chi(H_n \cap F_k)\|_p = \mu(H_n \cap \bigcup_{k=m+1}^\infty F_k)^{1/p} \to 0$$

as $m \to \infty$, so

$$\nu_n(\bigcup_{k\in\mathbb{N}} F_k) = \sum_{k=0}^\infty \nu_n F_k.$$

So $\nu_n$ is countably additive. Further, $|\nu_n F| \le \mu(H_n \cap F)^{1/p}$ for every $F \in \Sigma$, so $\nu_n$ is truly continuous in the sense of 232Ab.

There is therefore an integrable function $g_n$ such that $\nu_n F = \int_F g_n$ for every $F \in \Sigma$; let us suppose that $g_n$ is measurable and defined on the whole of $X$. Set $g(x) = g_n(x)$ whenever $n \in \mathbb{N}$ and $x \in H_n$, $g(x) = 0$ for $x \in X \setminus \bigcup_{n\in\mathbb{N}} H_n$.

**(e)** $g = \sum_{n=0}^\infty g_n \times \chi H_n$ is measurable and has the property that $\int_F g = h(\chi F^\bullet)$ whenever $n \in \mathbb{N}$ and $F$ is a measurable subset of $H_n$; consequently $\int_F g = h(\chi F^\bullet)$ whenever $n \in \mathbb{N}$ and $F$ is a measurable subset of $E_n = \bigcup_{k\le n} H_k$. Set $G = \{x : g(x) > 0\} \subseteq \bigcup_{n\in\mathbb{N}} E_n$. If $F \subseteq G$ and $\mu F < \infty$, then

$$\lim_{n\to\infty} \int g \times \chi(F \cap E_n) \le \sup_{n\in\mathbb{N}} h(\chi(F \cap E_n)^\bullet) \le \sup_{n\in\mathbb{N}} \|\chi(F \cap E_n)\|_p = (\mu F)^{1/p},$$

so by B.Levi's theorem

$$\int_F g = \int g \times \chi F = \lim_{n\to\infty} \int g \times \chi(F \cap E_n)$$

exists. Similarly, $\int_F g$ exists if $F \subseteq \{x : g(x) < 0\}$ has finite measure; while obviously $\int_F g$ exists if $F \subseteq \{x : g(x) = 0\}$. Accordingly $\int_F g$ exists for every set $F$ of finite measure. Moreover, by Lebesgue's Dominated Convergence Theorem,

$$\int_F g = \lim_{n\to\infty} \int_{F\cap E_n} g = \lim_{n\to\infty} h(\chi(F \cap E_n)^\bullet) = \sum_{n=0}^\infty h(\chi(F \cap H_n)^\bullet) = h(\chi F^\bullet)$$

for such $F$, by (c) above. It follows at once that

$$\int g \times f = h(f^\bullet)$$

for every simple function $f : X \to \mathbb{R}$.

**(f)** Now $g \in L^q$. **P** (i) We already know that $|g|^q : X \to \mathbb{R}$ is measurable, because $g$ is measurable and $a \mapsto |a|^q$ is continuous. (ii) Suppose that $f$ is a non-negative simple function and $f \le_{\text{a.e.}} |g|^q$. Then $f^{1/p}$ is a simple function, and $\operatorname{sgn} g$ is measurable and takes only the values 0, 1 and $-1$, so $f_1 = f^{1/p} \times \operatorname{sgn} g$ is simple. We see that $\int |f_1|^p = \int f$, so $\|f_1\|_p = (\int f)^{1/p}$. Accordingly

$$(\int f)^{1/p} \ge h(f_1^\bullet) = \int g \times f_1 = \int |g \times f^{1/p}|$$

$$\ge \int f^{1/q} \times f^{1/p}$$

(because $0 \le f^{1/q} \le_{\text{a.e.}} |g|$)

$$= \int f,$$

and we must have $\int f \le 1$. (iii) Thus

$$\sup\{\int f : f \text{ is a non-negative simple function, } f \le_{\text{a.e.}} |g|^q\} \le 1 < \infty.$$

But now observe that if $\epsilon > 0$ then

$$\{x : |g(x)|^q \ge \epsilon\} = \bigcup_{n\in\mathbb{N}}\{x : x \in E_n,\ |g(x)|^q \ge \epsilon\},$$

and for each $n \in \mathbb{N}$

$$\mu\{x : x \in E_n,\ |g(x)|^q \ge \epsilon\} \le \tfrac{1}{\epsilon},$$

because $f = \epsilon\chi\{x : x \in E_n,\ |g(x)|^q \ge \epsilon\}$ is a simple function less than or equal to $|g|^q$, so has integral at most 1. Accordingly

$$\mu\{x : |g(x)|^q \ge \epsilon\} = \sup_{n\in\mathbb{N}}\mu\{x : x \in E_n,\ |g(x)|^q \ge \epsilon\} \le \tfrac{1}{\epsilon} < \infty.$$

Thus $|g|^q$ is integrable, by the criterion in 122Ja. **Q**

**(g)** We may therefore speak of $h_1 = T(g^\bullet) \in (L^p)^*$, and we know that it agrees with $h$ on members of $L^p$ of the form $f^\bullet$ where $f$ is a simple function. But these form a dense subset of $L^p$, by 244Ha, and both $h$ and $h_1$ are continuous, so $h = h_1$, by 2A3Uc, and $h$ is a value of $T$. The argument as written so far has assumed that $\|h\| = 1$. But of course any non-zero member of $(L^p)^*$ is a scalar multiple of an element of norm 1, so is a value of $T$. So $T : L^q \to (L^p)^*$ is indeed surjective, and is therefore an isometric isomorphism, as claimed.

**244L** Continuing with the same topics as in §§242 and 243, I turn to the order-completeness of $L^p$.

**Theorem** Let $(X,\Sigma,\mu)$ be any measure space, and $p \in [1,\infty[$. Then $L^p = L^p(\mu)$ is Dedekind complete.

**proof** I use 242H. Let $A \subseteq L^p$ be a non-empty set which is bounded above in $L^p$. Fix $u_0 \in A$ and set

$$A' = \{u_0 \vee u : u \in A\},$$

so that $A'$ has the same upper bounds as $A$ and is bounded below by $u_0$. Fixing an upper bound $w_0$ of $A$ in $L^p$, then $u_0 \le u \le w_0$ for every $u \in A'$. Set

$$B = \{(u - u_0)^p : u \in A'\}.$$

Then

$$0 \le v \le (w_0 - u_0)^p \in L^1 = L^1(\mu)$$

for every $v \in B$, so $B$ is a non-empty subset of $L^1$ which is bounded above in $L^1$, and therefore has a least upper bound $v_1$ in $L^1$. Now $v_1^{1/p} \in L^p$; consider $w_1 = u_0 + v_1^{1/p}$. If $u \in A'$ then $(u - u_0)^p \le v_1$ so $u - u_0 \le v_1^{1/p}$ and $u \le w_1$; thus $w_1$ is an upper bound for $A'$. If $w \in L^p$ is an upper bound for $A'$, then $u - u_0 \le w - u_0$ and $(u - u_0)^p \le (w - u_0)^p$ for every $u \in A'$, so $(w - u_0)^p$ is an upper bound for $B$ and $v_1 \le (w - u_0)^p$, $v_1^{1/p} \le w - u_0$ and $w_1 \le w$. Thus $w = \sup A' = \sup A$ in $L^p$. As $A$ is arbitrary, $L^p$ is Dedekind complete.

**244M** As in the last two sections, the theory of conditional expectations is worth revisiting.

**Theorem** Let $(X,\Sigma,\mu)$ be a probability space, and $\mathrm{T}$ a $\sigma$-subalgebra of $\Sigma$. Take $p \in [1,\infty]$. Regard $L^0(\mu\restriction\mathrm{T})$ as a subspace of $L^0 = L^0(\mu)$, as in 242Jh, so that $L^p(\mu\restriction\mathrm{T})$ becomes $L^p(\mu) \cap L^0(\mu\restriction\mathrm{T})$. Let $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T})$ be the conditional expectation operator, as described in 242Jd. Then whenever $u \in L^p = L^p(\mu)$, $|Pu|^p \le P(|u|^p)$, so $Pu \in L^p(\mu\restriction\mathrm{T})$ and $\|Pu\|_p \le \|u\|_p$. Moreover, $P[L^p] = L^p(\mu\restriction\mathrm{T})$.

**proof** For $p = \infty$, this is 243Jb, so I assume henceforth that $p < \infty$. Concerning the identification of $L^p(\mu\restriction\mathrm{T})$ with $L^p \cap L^0(\mu\restriction\mathrm{T})$, if $S : L^0(\mu\restriction\mathrm{T}) \to L^0$ is the canonical embedding described in 242J, we have $|Su|^p = S(|u|^p)$ for every $u \in L^0(\mu\restriction\mathrm{T})$, so that $Su \in L^p$ iff $|u|^p \in L^1(\mu\restriction\mathrm{T})$ iff $u \in L^p(\mu\restriction\mathrm{T})$.

Set $\phi(t) = |t|^p$ for $t \in \mathbb{R}$; then $\phi$ is a convex function (because it is absolutely continuous on any bounded interval, and its derivative is non-decreasing), and $|u|^p = \bar\phi(u)$ for every $u \in L^0 = L^0(\mu)$, where $\bar\phi$ is defined as in 241I. Now if $u \in L^p = L^p(\mu)$, we surely have $u \in L^1$ (because $|u| \le |u|^p \vee (\chi X)^\bullet$, or otherwise); so 242K tells us that $|Pu|^p \le P|u|^p$. But this means that $Pu \in L^p \cap L^1(\mu\restriction\mathrm{T}) = L^p(\mu\restriction\mathrm{T})$, and

$$\|Pu\|_p = (\textstyle\int |Pu|^p)^{1/p} \le (\int P|u|^p)^{1/p} = (\int |u|^p)^{1/p} = \|u\|_p,$$

as claimed. If $u \in L^p(\mu\restriction\mathrm{T})$, then $Pu = u$, so $P[L^p]$ is the whole of $L^p(\mu\restriction\mathrm{T})$.

**244N The space $L^2$** **(a)** As I have already remarked, the really important function spaces are $L^0$, $L^1$, $L^2$ and $L^\infty$. $L^2$ has the special property of being an inner product space; if $(X,\Sigma,\mu)$ is any measure space and $u,\ v\in L^2=L^2(\mu)$ then $u\times v\in L^1(\mu)$, by 244Eb, and we may write $(u|v)=\int u\times v$. This makes $L^2$ a real inner product space (because

$$(u_1+u_2|v)=(u_1|v)+(u_2|v),\quad (cu|v)=c(u|v),\quad (u|v)=(v|u),$$

$$(u|u)\ge 0,\quad u=0 \text{ whenever } (u|u)=0$$

for all $u,\ u_1,\ u_2,\ v\in L^2$ and $c\in\mathbb{R}$) and its norm $\|\,\|_2$ is the associated norm (because $\|u\|_2=\sqrt{(u|u)}$ whenever $u\in L^2$). Because $L^2$ is complete (244G), it is a real Hilbert space. The fact that it may be identified with its own dual (244K) can of course be deduced from this.

I will use the phrase '**square-integrable**' to describe functions in $\mathcal{L}^2(\mu)$.

**(b)** Conditional expectations take a special form in the case of $L^2$. Let $(X,\Sigma,\mu)$ be a probability space, T a $\sigma$-subalgebra of $\Sigma$, and $P:L^1=L^1(\mu)\to L^1(\mu\restriction\mathrm{T})\subseteq L^1$ the corresponding conditional expectation operator. Then $P[L^2]\subseteq L^2$, where $L^2=L^2(\mu)$ (244M), so we have an operator $P_2=P\restriction L^2$ from $L^2$ to itself. Now $P_2$ is an orthogonal projection and its kernel is $\{u:u\in L^2,\ \int_F u=0$ for every $F\in\mathrm{T}\}$. **P** (i) If $u\in L^1$ then $Pu=0$ iff $\int_F u=0$ for every $F\in\mathrm{T}$ (cf. 242Je); so surely the kernel of $P_2$ is the set described. (ii) Since $P^2=P$, $P_2$ also is a projection; because $P_2$ has norm at most 1 (244M), and is therefore continuous,

$$U=P_2[L^2]=L^2(\mu\restriction\mathrm{T})=\{u:u\in L^2,\ P_2u=u\},\quad V=\{u:P_2u=0\}$$

are closed linear subspaces of $L^2$ such that $U\oplus V=L^2$. (iii) Now suppose that $u\in U$ and $v\in V$. Then $P|v|\in L^2$, so $u\times P|v|\in L^1$ and $P(u\times v)=u\times Pv$, by 242L. Accordingly

$$(u|v)=\int u\times v=\int P(u\times v)=\int u\times Pv=0.$$

Thus $U$ and $V$ are orthogonal subspaces of $L^2$, which is what we mean by saying that $P_2$ is an orthogonal projection. (Some readers will know that every projection of norm at most 1 on an inner product space is orthogonal.) **Q**

***244O** This is not the place for a detailed discussion of the geometry of $L^p$ spaces. However there is a particularly important fact about the shape of the unit ball which is accessible by the methods available to us here.

**Theorem** (Clarkson 1936) Suppose that $p\in\,]1,\infty[$ and $(X,\Sigma,\mu)$ is a measure space. Then $L^p=L^p(\mu)$ is uniformly convex (definition: 2A4K).

**proof** (Hanner 56, Naor 04)

**(a)(i)** For $0<t\le 1$ and $a,\ b\in\mathbb{R}$, set

$$\phi_0(t)=(1+t)^{p-1}+(1-t)^{p-1},$$

$$\phi_1(t)=\tfrac{(1+t)^{p-1}-(1-t)^{p-1}}{t^{p-1}}=(\frac{1}{t}+1)^{p-1}-(\frac{1}{t}-1)^{p-1},$$

$$\psi_{ab}(t)=|a|^p\phi_0(t)+|b|^p\phi_1(t),$$

$$\phi_2(b)=(1+b)^p+|1-b|^p.$$

**(ii)** We have

$$\phi_0'(t)=(p-1)((1+t)^{p-2}-(1-t)^{p-2}),\ \text{which has the same sign as } p-2,$$

(of course it is zero if $p=2$),

$$\begin{aligned}\phi_1'(t)&=-\tfrac{p-1}{t^2}((\frac{1}{t}-1)^{p-2}-(\frac{1}{t}-1)^{p-2})\\&=-\tfrac{p-1}{t^p}((1+t)^{p-2}-(1-t)^{p-2})=-\tfrac{1}{t^p}\phi_0'(t)\end{aligned}$$

for every $t\in\,]0,1[$. Accordingly $\phi_0'-\phi_1'$ has the same sign as $p-2$ everywhere on $]0,1[$. Also

$$\phi_0(1) = 2^{p-1} = \phi_1(1),$$

so $\phi_0 - \phi_1$ has the same sign as $2-p$ everywhere on $]0,1]$.

**(iii)** $\phi_2$ is strictly increasing on $[0,\infty[$. **P** For $b>0$,

$$\begin{aligned}\phi_2'(b) &= p((1+b)^{p-1} - (1-b)^{p-1}) > 0 \text{ if } b \le 1,\\ &= p((1+b)^{p-1} + (b-1)^{p-1}) > 0 \text{ if } b \ge 1. \ \mathbf{Q}\end{aligned}$$

**(iv)** If $0 < b \le a$, then

$$\begin{aligned}\psi_{ab}(\frac{b}{a}) &= a^p\phi_0(\frac{b}{a}) + b^p\phi_1(\frac{b}{a})\\ &= a^p(1+\frac{b}{a})^{p-1} + a^p(1-\frac{b}{a})^{p-1} + b^p(\frac{a}{b}+1)^{p-1} - b^p(\frac{a}{b}-1)^{p-1}\\ &= a(a+b)^{p-1} + a(a-b)^{p-1} + b(a+b)^{p-1} - b(a-b)^{p-1}\\ &= (a+b)^p + (a-b)^p = (a+b)^p + |a-b|^p. \qquad (\dagger)\end{aligned}$$

Also $\psi'_{ab}(t) = (a^p - \frac{b^p}{t^p})\phi'_0(t)$ has the sign of $2-p$ if $0 < t < \frac{b}{a}$ and the sign of $p-2$ if $\frac{b}{a} < t < 1$. Accordingly

— if $1 < p \le 2$, $\psi_{ab}(t) \le \psi_{ab}(\frac{b}{a}) = (a+b)^p + |a-b|^p$ for every $t \in ]0,1]$,

— if $p \ge 2$, $\psi_{ab}(t) \ge \psi_{ab}(\frac{b}{a}) = (a+b)^p + |a-b|^p$ for every $t \in ]0,1]$.

**(v)** Now consider the case $0 < a \le b$. If $1 < p \le 2$,

$$\psi_{ab}(t) = a^p\phi_0(t) + b^p\phi_1(t) \le a^p\phi_0(t) + b^p\phi_1(t) + (b^p - a^p)(\phi_0(t) - \phi_1(t))$$

(by (ii))

$$= b^p\phi_0(t) + a^p\phi_1(t) \le (b+a)^p + (b-a)^p = (a+b)^p + |a-b|^p$$

for every $t \in ]0,1]$. If $p \ge 2$, on the other hand,

$$\begin{aligned}\psi_{ab}(t) &= a^p\phi_0(t) + b^p\phi_1(t) \ge a^p\phi_0(t) + b^p\phi_1(t) + (b^p - a^p)(\phi_0(t) - \phi_1(t))\\ &= b^p\phi_0(t) + a^p\phi_1(t) \ge (a+b)^p + |a-b|^p\end{aligned}$$

for every $t$.

**(vi)** Thus we have the inequalities

$$\begin{aligned}\psi_{ab}(t) &\le |a+b|^p + |a-b|^p \text{ if } p \in ]1,2]\,,\\ &\ge |a+b|^p + |a-b|^p \text{ if } p \in [2,\infty[\end{aligned} \qquad (*)$$

whenever $t \in ]0,1]$ and $a$, $b \in ]0,\infty[$. Since $(a,b) \mapsto \psi_{ab}(t)$ is continuous for every $t$, the same inequalities are valid for all $a$, $b \in [0,\infty[$. And since

$$\psi_{ab}(t) = \psi_{|a|,|b|}(t), \quad |a+b|^p + |a-b|^p = \bigl||a|+|b|\bigr|^p + \bigl||a|-|b|\bigr|^p$$

for all $a$, $b \in \mathbb{R}$ and $t \in ]0,1]$, the inequalities (*) are valid for all $a$, $b \in \mathbb{R}$ and $t \in ]0,1]$.

**(b)** Suppose that $p \ge 2$.

**(i)**

$$\|u+v\|_p^p + \|u-v\|_p^p \le (\|u\|_p + \|v\|_p)^p + |\|u\|_p - \|v\|_p|^p$$

for all $u$, $v \in L^p$. **P** First consider the case $0 < \|v\|_p \le \|u\|_p$. Let $f$, $g : X \to \mathbb{R}$ be $\Sigma$-measurable functions such that $f^\bullet = u$ and $g^\bullet = v$. Then for any $t \in ]0,1]$,

$$\begin{aligned}\|u+v\|_p^p + \|u-v\|_p^p &= \int |f(x)+g(x)|^p + |f(x)-g(x)|^p \mu(dx)\\ &\le \int \psi_{f(x),g(x)}(t)\mu(dx)\end{aligned}$$

(by the second inequality in (*))

$$= \int |f(x)|^p\phi_0(t) + |g(x)|^p\phi_1(t)\mu(dx) = \|u\|_p^p\phi_0(t) + \|v\|_p^p\phi_1(t).$$

In particular, taking $t = \|v\|_p/\|u\|_p$, and applying (†) from (a-iv),

$$\|u+v\|_p^p + \|u-v\|_p^p \le (\|u\|_p + \|v\|_p)^p + |\|u\|_p - \|v\|_p|^p.$$

Of course the result will also be true if $0 < \|u\|_p \le \|v\|_p$, and the case in which either $u$ or $v$ is zero is trivial. **Q**

**(ii)** Let $\epsilon \in ]0,2]$. Set $\delta = 2 - (2^p - \epsilon^p)^{1/p} > 0$. If $u$, $v \in L^p$, $\|u\|_p = \|v\|_p = 1$ and $\|u-v\|_p \ge \epsilon$, then

$$\|u+v\|_p^p + \epsilon^p \le \|u+v\|_p^p + \|u-v\|_p^p \le (\|u\|_p + \|v\|_p)^p + |\|u\|_p - \|v\|_p|^p = 2^p,$$

so $\|u+v\|_p \le (2^p - \epsilon^p)^{1/p} = 2 - \delta$. As $u$, $v$ and $\epsilon$ are arbitrary, $L^p$ is uniformly convex.

**(c)** Next suppose that $p \in ]1,2]$.

**(i)**

$$(\|u\|_p + \|v\|_p)^p + |\|u\|_p - \|v\|_p|^p \le \|u+v\|_p^p + \|u-v\|_p^p$$

for all $u$, $v \in L^p$. **P** We can repeat all the ideas, and most of the formulae, of (b-i). As before, start with the case $0 < \|v\|_p \le \|u\|_p$. Let $f$, $g : X \to \mathbb{R}$ be $\Sigma$-measurable functions such that $f^\bullet = u$ and $g^\bullet = v$. Taking $t = \|v\|_p/\|u\|_p$,

$$\begin{aligned}\|u+v\|_p^p + \|u-v\|_p^p &= \int |f(x)+g(x)|^p + |f(x)-g(x)|^p \mu(dx)\\ &\ge \int \psi_{f(x),g(x)}(t)\mu(dx)\end{aligned}$$

(by the first inequality in (*))

$$= \|u\|_p^p\phi_0(t) + \|v\|_p^p\phi_1(t) = (\|u\|_p + \|v\|_p)^p + |\|u\|_p - \|v\|_p|^p.$$

Similarly if $0 < \|u\|_p \le \|v\|_p$, and the case in which either $u$ or $v$ is zero is trivial. **Q**

**(ii)** Let $\epsilon > 0$. Set $\gamma = \phi_2(\frac{\epsilon}{2}) > 2$ (see (a-iii) above) and $\delta = 2\bigl(1 - (\frac{2}{\gamma})^{1/p}\bigr) > 0$. Now suppose that $\|u\|_p = \|v\|_p = 1$ and $\|u-v\|_p \ge \epsilon$. Then $\|u+v\|_p \le 2 - \delta$. **P** If $u + v = 0$ this is trivial. Otherwise, set $a = \|u+v\|_p$ and $b = \|u-v\|_p$. Then $a \le 2$ and $b \ge \epsilon$, so

$$a^p\gamma = a^p\phi_2(\frac{\epsilon}{2}) \le a^p\phi_2(\frac{b}{a})$$

(by (a-iii) again)

$$\begin{aligned}&= (a+b)^p + |a-b|^p = (\|u+v\|_p + \|u-v\|_p)^p + |\|u+v\|_p - \|u-v\|_p|^p\\ &\le \|2u\|_p^p + \|2v\|_p^p\end{aligned}$$

(by (i) here)

$$= 2^{p+1}$$

and $a \le 2\bigl(\frac{2}{\gamma}\bigr)^{1/p} = 2 - \delta$. **Q** As $u$, $v$ and $\epsilon$ are arbitrary, $L^p$ is uniformly convex.

**Remark** The inequalities in (b-i) and (c-i) of the proof are called **Hanner's inequalities**.

**244P Complex $L^p$** Let $(X,\Sigma,\mu)$ be any measure space.

**(a)** For any $p\in\,]1,\infty[$, set

$$\mathcal{L}^p_{\mathbb{C}}=\mathcal{L}^p_{\mathbb{C}}(\mu)=\{f:f\in\mathcal{L}^0_{\mathbb{C}}(\mu),\,|f|^p\text{ is integrable}\},$$

$$\begin{aligned}L^p_{\mathbb{C}}=L^p_{\mathbb{C}}(\mu)&=\{f^\bullet:f\in\mathcal{L}^p_{\mathbb{C}}\}\\&=\{u:u\in L^0_{\mathbb{C}}(\mu),\,\mathcal{R}\mathrm{e}(u)\in L^p(\mu)\text{ and }\mathcal{I}\mathrm{m}(u)\in L^p(\mu)\}\\&=\{u:u\in L^0_{\mathbb{C}}(\mu),\,|u|\in L^p(\mu)\}.\end{aligned}$$

Then $L^p_{\mathbb{C}}$ is a linear subspace of $L^0_{\mathbb{C}}(\mu)$. Set $\|u\|_p=\||u|\|_p=(\int|u|^p)^{1/p}$ for $u\in L^p_{\mathbb{C}}$.

**(b)** The proof of 244E(b-i) applies unchanged to complex-valued functions, so taking $q=p/(p-1)$ we get

$$\|u\times v\|_1\le\|u\|_p\|v\|_q$$

for all $u\in L^p_{\mathbb{C}}$, $v\in L^q_{\mathbb{C}}$. 244Fa becomes

for every $u\in L^p_{\mathbb{C}}$ there is a $v\in L^q_{\mathbb{C}}$ such that $\|v\|_q\le1$ and

$$\int u\times v=|\int u\times v|=\|u\|_p;$$

the same proof works, if you omit all mention of the functional $\tau$, and allow me to write $\operatorname{sgn}a=|a|/a$ for all non-zero complex numbers – it would perhaps be more natural to write $\overline{\operatorname{sgn}}a$ in place of $\operatorname{sgn}a$. So, just as before, we find that $\|\,\|_p$ is a norm. We can use the argument of 244G to show that $L^p_{\mathbb{C}}$ is complete. (Alternatively, note that a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $L^0_{\mathbb{C}}$ is Cauchy, or convergent, iff its real and imaginary parts are.) The space $S_{\mathbb{C}}$ of equivalence classes of 'complex-valued simple functions' is dense in $L^p_{\mathbb{C}}$. If $X$ is a subset of $\mathbb{R}^r$ and $\mu$ is Lebesgue measure on $X$, then the space of equivalence classes of continuous complex-valued functions on $X$ with bounded support is dense in $L^p_{\mathbb{C}}$.

**(c)** The canonical map $T:L^q_{\mathbb{C}}\to(L^p_{\mathbb{C}})^*$, defined by writing $(Tv)(u)=\int u\times v$, is surjective because $T\restriction L^q:L^q\to(L^p)^*$ is surjective; and it is an isometry by the remarks in (b) just above. Thus we can still identify $L^q_{\mathbb{C}}$ with $(L^p_{\mathbb{C}})^*$.

**(d)** When we come to the complex form of Jensen's inequality, it seems that a new idea is needed. I have relegated this to 242Yk-242Yl. But for the complex form of 244M a simpler argument will suffice. If $(X,\Sigma,\mu)$ is a probability space, T is a $\sigma$-subalgebra of $\Sigma$ and $P:L^1_{\mathbb{C}}(\mu)\to L^1_{\mathbb{C}}(\mu\restriction\mathrm{T})$ is the corresponding conditional expectation operator, then for any $u\in L^p_{\mathbb{C}}$ we shall have

$$|Pu|^p\le(P|u|)^p\le P(|u|^p),$$

applying 242Pc and 244M. So $\|Pu\|_p\le\|u\|_p$, as before.

**(e)** There is a special point arising with $L^2_{\mathbb{C}}$. We now have to define

$$(u|v)=\int u\times\bar v$$

for $u$, $v\in L^2_{\mathbb{C}}$, so that $(u|u)=\int|u|^2=\|u\|_2^2$ for every $u$; this means that $(v|u)$ is the complex conjugate of $(u|v)$.

**244X Basic exercises >(a)** Let $(X,\Sigma,\mu)$ be a measure space, and $(X,\hat\Sigma,\hat\mu)$ its completion. Show that $\mathcal{L}^p(\hat\mu)=\mathcal{L}^p(\mu)$ and $L^p(\hat\mu)=L^p(\mu)$ for every $p\in[1,\infty]$.

**(b)** Let $(X,\Sigma,\mu)$ be a measure space, and $1\le p\le q\le r\le\infty$. Show that $L^p(\mu)\cap L^r(\mu)\subseteq L^q(\mu)\subseteq L^p(\mu)+L^r(\mu)\subseteq L^0(\mu)$. (See also 244Yh.)

**(c)** Let $(X,\Sigma,\mu)$ be a measure space. Suppose that $p$, $q$, $r\in[1,\infty]$ and that $\frac1p+\frac1q=\frac1r$, setting $\frac1\infty=0$ as usual. Show that $u\times v\in L^r(\mu)$ and $\|u\times v\|_r\le\|u\|_p\|v\|_q$ whenever $u\in L^p(\mu)$ and $v\in L^q(\mu)$. (*Hint*: if $r<\infty$ apply Hölder's inequality to $|u|^r\in L^{p/r}$, $|v|^r\in L^{q/r}$.)

**>(d)**(i) Let $(X,\Sigma,\mu)$ be a probability space. Show that if $1\le p\le r\le\infty$ then $\|f\|_p\le\|f\|_r$ for every $f\in\mathcal{L}^r(\mu)$. (*Hint*: use Hölder's inequality to show that $\int|f|^p\le\||f|^p\|_{r/p}$.) In particular, $\mathcal{L}^p(\mu)\supseteq\mathcal{L}^r(\mu)$. (ii) Let $(X,\Sigma,\mu)$ be a measure space such that $\mu E\ge 1$ whenever $E\in\Sigma$ and $\mu E>0$. (This happens, for instance, when $\mu$ is 'counting measure' on $X$.) Show that if $1\le p\le r\le\infty$ then $L^p(\mu)\subseteq L^r(\mu)$ and $\|u\|_p\ge\|u\|_r$ for every $u\in L^p(\mu)$. (*Hint*: look first at the case $\|u\|_p=1$.)

**>(e)** Let $(X,\Sigma,\mu)$ be a semi-finite measure space, and $p$, $q\in[1,\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. Show that if $u\in L^0(\mu)\setminus L^p(\mu)$ then there is a $v\in L^q(\mu)$ such that $u\times v\notin L^1(\mu)$. (*Hint*: reduce to the case $u\ge 0$. Show that in this case there is for each $n\in\mathbb{N}$ a $u_n\le u$ such that $4^n\le\|u_n\|_p<\infty$; take $v_n\in L^q$ such that $\|v_n\|_q\le 2^{-n}$ and $\int u_n\times v_n\ge 2^n$, and set $v=\sum_{n=0}^\infty v_n$.)

**(f)** Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, and $(X,\Sigma,\mu)$ their direct sum (214L). Take any $p\in[1,\infty[$. Show that the canonical isomorphism between $L^0(\mu)$ and $\prod_{i\in I}L^0(\mu_i)$ (241Xd) induces an isomorphism between $L^p(\mu)$ and the subspace

$$\{u:u\in\prod_{i\in I}L^p(\mu_i),\ \|u\|=\bigl(\sum_{i\in I}\|u(i)\|_p^p\bigr)^{1/p}<\infty\}$$

of $\prod_{i\in I}L^p(\mu_i)$.

**(g)** Let $(X,\Sigma,\mu)$ be a measure space. Set $M^{\infty,1}=L^1(\mu)\cap L^\infty(\mu)$. Show that for $u\in M^{\infty,1}$ the function $p\mapsto\|u\|_p:[1,\infty[\,\to[0,\infty[$ is continuous, and that $\|u\|_\infty=\lim_{p\to\infty}\|u\|_p$. (*Hint*: consider first the case in which $u$ is the equivalence class of a simple function.)

**(h)** Let $\mu$ be counting measure on $X=\{1,2\}$, so that $\mathcal{L}^0(\mu)=\mathbb{R}^2$ and $L^p(\mu)=L^0(\mu)$ can be identified with $\mathbb{R}^2$ for every $p\in[1,\infty]$. Sketch the unit balls $\{u:\|u\|_p\le 1\}$ in $\mathbb{R}^2$ for $p=1$, $\frac{3}{2}$, 2, 3 and $\infty$.

**(i)** Let $\mu$ be counting measure on $X=\{1,2,3\}$, so that $\mathcal{L}^0(\mu)=\mathbb{R}^3$ and $L^p(\mu)=L^0(\mu)$ can be identified with $\mathbb{R}^3$ for every $p\in[1,\infty]$. Describe the unit balls $\{u:\|u\|_p\le 1\}$ in $\mathbb{R}^3$ for $p=1$, 2 and $\infty$.

**(j)** At which points does the argument of 244Hb break down if we try to apply it to $L^\infty$ with $\|\ \|_\infty$?

**(k)** Let $p\in[1,\infty[$. (i) Show that $|a^p-b^p|\ge|a-b|^p$ for all $a$, $b\ge 0$. (*Hint*: for $a>b$, differentiate both sides with respect to $a$.) (ii) Let $(X,\Sigma,\mu)$ be a measure space and $U$ a linear subspace of $L^0(\mu)$ such that $(\alpha)$ $|u|\in U$ for every $u\in U$ $(\beta)$ $u^{1/p}\in U$ for every $u\in U$ $(\gamma)$ $U\cap L^1$ is dense in $L^1=L^1(\mu)$. Show that $U\cap L^p$ is dense in $L^p=L^p(\mu)$. (*Hint*: check first that $\{u:u\in U\cap L^1,\ u\ge 0\}$ is dense in $\{u:u\in L^1,\ u\ge 0\}$.) (iii) Use this to prove 244H from 242M and 242O.

**(l)** For any measure space $(X,\Sigma,\mu)$ write $M^{1,\infty}=M^{1,\infty}(\mu)$ for $\{v+w:v\in L^1(\mu),\ w\in L^\infty(\mu)\}\subseteq L^0(\mu)$. Show that $M^{1,\infty}$ is a linear subspace of $L^0$ including $L^p$ for every $p\in[1,\infty]$, and that if $u\in L^0$, $v\in M^{1,\infty}$ and $|u|\le|v|$ then $u\in M^{1,\infty}$. (*Hint*: $u=v\times w$ where $|w|\le\chi X^\bullet$.)

**(m)** Let $(X,\Sigma,\mu)$ and $(Y,\mathrm{T},\nu)$ be two measure spaces, and let $\mathcal{T}^+$ be the set of linear operators $T:M^{1,\infty}(\mu)\to M^{1,\infty}(\nu)$ such that $(\alpha)$ $Tu\ge 0$ whenever $u\ge 0$ in $M^{1,\infty}(\mu)$ $(\beta)$ $Tu\in L^1(\nu)$ and $\|Tu\|_1\le\|u\|_1$ whenever $u\in L^1(\mu)$ $(\gamma)$ $Tu\in L^\infty(\nu)$ and $\|Tu\|_\infty\le\|u\|_\infty$ whenever $u\in L^\infty(\mu)$. (i) Show that if $\phi:\mathbb{R}\to\mathbb{R}$ is a convex function such that $\phi(0)=0$, and $u\in M^{1,\infty}(\mu)$ is such that $\bar\phi(u)\in M^{1,\infty}(\mu)$ (interpreting $\bar\phi:L^0(\mu)\to L^0(\mu)$ as in 241I), then $\bar\phi(Tu)\in M^{1,\infty}(\nu)$ and $\bar\phi(Tu)\le T(\bar\phi(u))$ for every $T\in\mathcal{T}^+$. (ii) Hence show that if $p\in[1,\infty]$ and $u\in L^p(\mu)$, $Tu\in L^p(\nu)$ and $\|Tu\|_p\le\|u\|_p$ for every $T\in\mathcal{T}^+$.

**>(n)** Let $X$ be any set, and let $\mu$ be counting measure on $X$. In this case it is customary to write $\ell^p(X)$ for $\mathcal{L}^p(\mu)$, and to identify it with $L^p(\mu)$. In particular, $L^2(\mu)$ becomes identified with $\ell^2(X)$, the space of square-summable functions on $X$. Write out statements and proofs of the results of this section adapted to this special case.

**(o)** Let $(X,\Sigma,\mu)$ and $(Y,\mathrm{T},\nu)$ be measure spaces and $\phi:X\to Y$ an inverse-measure-preserving function. Show that the map $g\mapsto g\phi:\mathcal{L}^0(\nu)\to\mathcal{L}^0(\mu)$ (241Xg) induces a norm-preserving map from $L^p(\nu)$ to $L^p(\mu)$ for every $p\in[1,\infty]$, and also a map from $M^{1,\infty}(\nu)$ to $M^{1,\infty}(\mu)$ which belongs to the class $\mathcal{T}^+$ of 244Xm.

**244Y Further exercises (a)** Let $(X,\Sigma,\mu)$ be a measure space, and $(X,\tilde{\Sigma},\tilde{\mu})$ its c.l.d. version. Show that $\mathcal{L}^p(\mu)\subseteq\mathcal{L}^p(\tilde{\mu})$ and that this embedding induces a Banach lattice isomorphism between $L^p(\mu)$ and $L^p(\tilde{\mu})$, for every $p\in[1,\infty[$.

**(b)** Let $(X,\Sigma,\mu)$ be any measure space, and $p\in[1,\infty[$. Show that $L^p(\mu)$ has the countable sup property in the sense of 241Ye. (*Hint*: 242Yh.)

**(c)** Suppose that $(X,\Sigma,\mu)$ is a measure space, and that $p\in\,]0,1[$, $q<0$ are such that $\frac{1}{p}+\frac{1}{q}=1$. (i) Show that $ab\ge\frac{1}{p}a^p+\frac{1}{q}b^q$ for all real $a\ge 0$, $b>0$. (*Hint*: set $p'=\frac{1}{p}$, $q'=\frac{p'}{p'-1}$, $c=(ab)^p$, $d=b^{-p}$ and apply 244Ea.) (ii) Show that if $f$, $g\in\mathcal{L}^0(\mu)$ are non-negative and $E=\{x:x\in\operatorname{dom}g,\ g(x)>0\}$, then

$$(\int_E f^p)^{1/p}(\int_E g^q)^{1/q}\le\int f\times g.$$

(iii) Show that if $f$, $g\in\mathcal{L}^0(\mu)$ are non-negative, then

$$(\int f^p)^{1/p}+(\int g^p)^{1/p}\le(\int(f+g)^p)^{1/p}.$$

**(d)** Let $(X,\Sigma,\mu)$ be a measure space, and $Y$ a subset of $X$; write $\mu_Y$ for the subspace measure on $Y$. Show that the canonical map $T$ from $L^0(\mu)$ onto $L^0(\mu_Y)$ (241Yg) includes a surjection from $L^p(\mu)$ onto $L^p(\mu_Y)$ for every $p\in[1,\infty]$, and also a map from $M^{1,\infty}(\mu)$ to $M^{1,\infty}(\mu_Y)$ which belongs to the class $\mathcal{T}^+$ of 244Xm. Show that the following are equiveridical: (i) there is some $p\in[1,\infty[$ such that $T\restriction L^p(\mu)$ is injective; (ii) $T:L^p(\mu)\to L^p(\mu_Y)$ is norm-preserving for every $p\in[1,\infty[$; (iii) $F\cap Y\ne\emptyset$ whenever $F\in\Sigma$ and $0<\mu F<\infty$.

**(e)** Let $(X,\Sigma,\mu)$ be any measure space, and $p\in[1,\infty[$. Show that the norm $\|\,\|_p$ on $L^p(\mu)$ is order-continuous in the sense of 242Yg.

**(f)** Let $(X,\Sigma,\mu)$ be any measure space, and $p\in[1,\infty]$. Show that if $A\subseteq L^p(\mu)$ is upwards-directed and norm-bounded, then it is bounded above. (*Hint*: 242Yf.)

**(g)** Let $(X,\Sigma,\mu)$ be any measure space, and $p\in[1,\infty]$. Show that if a non-empty set $A\subseteq L^p(\mu)$ is upwards-directed and has a supremum in $L^p(\mu)$, then $\|\sup A\|_p\le\sup_{u\in A}\|u\|_p$. (*Hint*: consider first the case $0\in A$.)

**(h)** Let $(X,\Sigma,\mu)$ be a measure space and $u\in L^0(\mu)$. (i) Show that $I=\{p:p\in[1,\infty[\,,\ u\in L^p(\mu)\}$ is an interval. Give examples to show that it may be open, closed or half-open. (ii) Show that $p\mapsto p\ln\|u\|_p:I\to\mathbb{R}$ is convex. (*Hint*: if $p<q$ and $t\in\,]0,1[$, observe that $\int|u|^{tp+(1-t)q}\le\||u|^{pt}\|_{1/t}\||u|^{q(1-t)}\|_{1/(1-t)}$.) (iii) Show that if $p\le q\le r$ in $I$, then $\|u\|_q\le\max(\|u\|_p,\|u\|_r)$.

**(i)** Let $[a,b]$ be a non-trivial closed interval in $\mathbb{R}$ and $F:[a,b]\to\mathbb{R}$ a function; take $p\in\,]1,\infty[$. Show that the following are equiveridical: (i) $F$ is absolutely continuous and its derivative $F'$ belongs to $\mathcal{L}^p(\mu)$, where $\mu$ is Lebesgue measure on $[a,b]$ (ii)

$$c=\sup\{\sum_{i=1}^n\frac{|F(a_i)-F(a_{i-1})|^p}{(a_i-a_{i-1})^{p-1}}:a\le a_0<a_1<\ldots<a_n\le b\}$$

is finite, and that in this case $c=\|F'\|_p$. (*Hint*: (i) if $F$ is absolutely continuous and $F'\in\mathcal{L}^p$, use Hölder's inequality to show that $|F(b')-F(a')|^p\le(b'-a')^{p-1}\int_{a'}^{b'}|F'|^p$ whenever $a\le a'\le b'\le b$. (ii) If $F$ satisfies the condition, show that $(\sum_{i=0}^n|F(b_i)-F(a_i)|)^p\le c(\sum_{i=0}^n(b_i-a_i))^{p-1}$ whenever $a\le a_0\le b_0\le a_1\le\ldots\le b_n\le b$, so that $F$ is absolutely continuous. Take a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of polygonal functions approximating $F$; use 223Xj to show that $F_n'\to F'$ a.e., so that $\int|F'|^p\le\sup_{n\in\mathbb{N}}\int|F_n'|^p\le c^p$.)

**(j)** Let $G$ be an open set in $\mathbb{R}^r$ and write $\mu$ for Lebesgue measure on $G$. Let $C_k(G)$ be the set of continuous functions $f:G\to\mathbb{R}$ such that $\inf\{\|x-y\|:x\in G,\ f(x)\ne 0,\ y\in\mathbb{R}^r\setminus G\}>0$ (counting $\inf\emptyset$ as $\infty$). Show that for any $p\in[1,\infty[$ the set $\{f^\bullet:f\in C_k(G)\}$ is a dense linear subspace of $L^p(\mu)$.

**(k)** Let $U$ be any Hilbert space. (i) Show that if $C\subseteq U$ is convex (that is, $tu+(1-t)v\in C$ whenever $u$, $v\in C$ and $t\in[0,1]$; see 233Xd), closed and not empty, and $u\in U$, then there is a unique $v\in C$ such that $\|u-v\| = \inf_{w\in C}\|u-w\|$, and $(u-v|v-w) \ge 0$ for every $w \in C$. (ii) Show that if $h \in U^*$ there is a unique $v \in U$ such that $h(w) = (w|v)$ for every $w \in U$. (*Hint*: apply (i) with $C = \{w : h(w) = 1\}$, $u = 0$.) (iii) Show that if $V \subseteq U$ is a closed linear subspace then there is a unique linear projection $P$ on $U$ such that $P[U] = V$ and $(u - Pu|v) = 0$ for all $u \in U$, $v \in V$ ($P$ is 'orthogonal'). (*Hint*: take $Pu$ to be the point of $V$ nearest to $u$.)

**(l)** Let $(X, \Sigma, \mu)$ be a probability space, and $\mathrm{T}$ a $\sigma$-subalgebra of $\Sigma$. Use part (iii) of 244Yk to show that there is an orthogonal projection $P : L^2(\mu) \to L^2(\mu\restriction\mathrm{T})$ such that $\int_F Pu = \int_F u$ for every $u \in L^2(\mu)$ and $F \in \mathrm{T}$. Show that $Pu \ge 0$ whenever $u \ge 0$ and that $\int Pu = \int u$ for every $u$, so that $P$ has a unique extension to a continuous operator from $L^1(\mu)$ onto $L^1(\mu\restriction\mathrm{T})$. Use this to develop the theory of conditional expectations without using the Radon-Nikodým theorem.

**(m)** (Roselli & Willem 02) (i) Set $C = [0,\infty[^2 \subseteq \mathbb{R}^2$. Let $\phi : C \to \mathbb{R}$ be a continuous function such that $\phi(tz) = t\phi(z)$ for all $z \in C$. Show that $\phi$ is convex (definition: 233Xd) iff $t \mapsto \phi(1,t) : [0,\infty[ \to \mathbb{R}$ is convex. (ii) Show that if $p \in ]1,\infty[$ and $q = \frac{p}{p-1}$ then $(s,t) \mapsto -s^{1/p}t^{1/q}$, $(s,t) \mapsto -(s^{1/p}+t^{1/p})^p : C \to \mathbb{R}$ are convex. (iii) Show that if $p \in [1,2]$ then $(s,t) \mapsto |s^{1/p}+t^{1/p}|^p + |s^{1/p}-t^{1/p}|^p$ is convex. (iv) Show that if $p \in [2,\infty[$ then $(s,t) \mapsto -|s^{1/p}+t^{1/p}|^p - |s^{1/p}-t^{1/p}|^p$ is convex. (v) Use (ii) and 233Yj to prove Hölder's and Minkowski's inequalities. (vi) Use (iii) and (iv) to prove Hanner's inequalities. (vii) Adapt the method to answer (ii) and (iii) of 244Yc.

**(n)**(i) Show that any inner product space is uniformly convex. (ii) Let $U$ be a uniformly convex Banach space, $C \subseteq U$ a non-empty closed convex set, and $u \in U$. Show that there is a unique $v_0 \in C$ such that $\|u-v_0\| = \inf_{v\in C}\|u-v\|$. (iii) Let $U$ be a uniformly convex Banach space, and $A \subseteq U$ a non-empty bounded set. Set $\delta_0 = \inf\{\delta :$ there is some $u \in U$ such that $A \subseteq B(u,\delta) = \{v : \|v-u\| \le \delta\}\}$. Show that there is a unique $u_0 \in U$ such that $A \subseteq B(u_0,\delta_0)$.

**(o)** Let $(X, \Sigma, \mu)$ be a measure space, and $u \in L^0(\mu)$. Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is a sequence in $[1,\infty]$ with limit $p \in [1,\infty]$. Show that if $\limsup_{n\to\infty}\|u\|_{p_n}$ is finite then $\lim_{n\to\infty}\|u\|_{p_n}$ is defined and is equal to $\|u\|_p$.

**244 Notes and comments** At this point I feel we must leave the investigation of further function spaces. The next stage would have to be a systematic abstract analysis of general Banach lattices. The $L^p$ spaces give a solid foundation for such an analysis, since they introduce the basic themes of norm-completeness, order-completeness and identification of dual spaces. I have tried in the exercises to suggest the importance of the next layer of concepts: order-continuity of norms and the relationship between norm-boundedness and order-boundedness. What I have not had space to discuss is the subject of order-preserving linear operators between Riesz spaces, which is the key to understanding the order structure of the dual spaces here. (But you can make a start by re-reading the theory of conditional expectation operators in 242J-242L, 243J and 244M.) All these topics are treated in Fremlin 74 and in Chapters 35 and 36 of the next volume.

I remember that one of my early teachers of analysis said that the $L^p$ spaces (for $p \ne 1, 2, \infty$) had somehow got into the syllabus and had never been got out again. I would myself call them classics, in the sense that they have been part of the common experience of all functional analysts since functional analysis began; and while you are at liberty to dislike them, you can no more ignore them than you can ignore Milton if you are studying English poetry. Hölder's inequality, in particular, has a wealth of applications; not only 244F and 244K, but also 244Xc-244Xd and 244Yh-244Yi, for instance.

The $L^p$ spaces, for $1 \le p \le \infty$, form a kind of continuum. In terms of the concepts dealt with here, there is no distinction to be drawn between different $L^p$ spaces for $1 < p < \infty$ except the observation that the norm of $L^2$ is an inner product norm, corresponding to a Euclidean geometry on its finite-dimensional subspaces. To discriminate between the other $L^p$ spaces we need much more refined concepts in the geometry of normed spaces.

In terms of the theorems given here, $L^1$ seems closer to the middle range of $L^p$ for $1 < p < \infty$ than $L^\infty$ does; thus, for all $1 \le p < \infty$, we have $L^p$ Dedekind complete (independent of the measure space involved), the space $S$ of equivalence classes of simple functions is dense in $L^p$ (again, for every measure space), and the dual $(L^p)^*$ is (almost) identifiable as another function space. All of these should be regarded as consequences in one way or another of the order-continuity of the norm of $L^p$ for $p < \infty$. The chief obstacle to the universal identification of $(L^1)^*$ with $L^\infty$ is that for non-$\sigma$-finite measure spaces the space $L^\infty$ can be inadequate, rather than any pathology in the $L^1$ space itself. (This point, at least, I mean to return to in Volume 3.) There is also the point that for a non-semi-finite measure space the purely infinite sets can contribute to $L^\infty$ without any corresponding contribution to $L^1$. For $1 < p < \infty$, neither of these problems can arise. Any member of any such $L^p$ is supported entirely by a $\sigma$-finite part of the measure space, and the same applies to the dual – see part (c) of the proof of 244K.

Of course $L^1$ does have a markedly different geometry from the other $L^p$ spaces. The first sign of this is that it is not reflexive as a Banach space (except when it is finite-dimensional), whereas for $1 < p < \infty$ the identifications of $(L^p)^*$ with $L^q$ and of $(L^q)^*$ with $L^p$, where $q = p/(p-1)$, show that the canonical embedding of $L^p$ in $(L^p)^{**}$ is surjective, that is, that $L^p$ is reflexive. But even when $L^1$ is finite-dimensional the unit balls of $L^1$ and $L^\infty$ are clearly different in kind from the unit balls of $L^p$ for $1 < p < \infty$; they have corners instead of being smoothly rounded (244Xh-244Xi). A direct expression of the difference is in 244O. As usual, the case $p = 2$ is both much more important than the general case and enormously easier (244Yn(i)); and note how Hanner's inequalities reverse at $p = 2$. (See 244Yc for the reversal of Hölder's and Minkowski's inequalities at $p = 1$.) There are occasions on which it is useful to know that $\| \|_1$ and $\| \|_\infty$ can be approximated, in an exactly describable way, by uniformly convex norms (244Yo). I have written out a proof of 244O based on ingenuity and advanced calculus, like that of 244E. With a bit more about convex sets and functions, sketched in 233Yf-233Yj, there is a striking alternative proof (244Ym). Of course the proof of 244Ea also uses convexity, upside down.

The proof of 244K, identifying $(L^p)^*$, is a fairly long haul, and it is natural to ask whether we really have to work so hard, especially since in the case of $L^2$ we have a much easier argument (244Yk). Of course we can go faster if we know a bit more about Banach lattices (§369 in Volume 3 has the relevant facts), though this route uses some theorems quite as hard as 244K as given. There are alternative routes using the geometry of the $L^p$ spaces, following the ideas of 244Yk, but I do not think they are any easier, and the argument I have presented here at least has the virtue of using some of the same ideas as the identification of $(L^1)^*$ in 243G. The difference is that whereas in 243G we may have to piece together a large family of functions $g_F$ (part (b-v) of the proof), in 244K there are only countably many $g_n$; consequently we can make the argument work for arbitrary measure spaces, not just localizable ones.

The geometry of Hilbert space gives us an approach to conditional expectations which does not depend on the Radon-Nikodým theorem (244Yl). To turn these ideas into a proof of the Radon-Nikodým theorem itself, however, requires qualities of determination and ingenuity which can be better employed elsewhere.

The convexity arguments of 233J/242K can be used on many operators besides conditional expectations (see 244Xm). The class '$\mathcal{T}^+$' described there is not in fact the largest for which these arguments work; I take the ideas farther in Chapter 37. There is also a great deal more to be said if you put an arbitrary pair of $L^p$ spaces in place of $L^1$ and $L^\infty$ in 244Xl. 244Yh is a start, but for the real thing (the 'Riesz convexity theorem') I refer you to Zygmund 59, XII.1.11 or Dunford & Schwartz 57, VI.10.11.

Version of 25.3.06

## 245 Convergence in measure

I come now to an important and interesting topology on the spaces $\mathcal{L}^0$ and $L^0$. I start with the definition (245A) and with properties which echo those of the $L^p$ spaces for $p \geq 1$ (245D-245E). In 245G-245J I describe the most useful relationships between this topology and the norm topologies of the $L^p$ spaces. For $\sigma$-finite spaces, it is metrizable (245Eb) and sequential convergence can be described in terms of pointwise convergence of sequences of functions (245K-245L).

**245A Definitions** Let $(X, \Sigma, \mu)$ be a measure space.

**(a)** For any measurable set $F \subseteq X$ of finite measure, we have a functional $\tau_F$ on $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ defined by setting

$$\tau_F(f) = \int |f| \wedge \chi F$$

for every $f \in \mathcal{L}^0$. (The integral exists in $\mathbb{R}$ because $|f| \wedge \chi F$ belongs to $\mathcal{L}^0$ and is dominated by the integrable function $\chi F$). Now $\tau_F(f + g) \leq \tau_F(f) + \tau_F(g)$ whenever $f$, $g \in \mathcal{L}^0$. **P** We need only observe that

$$\min(|(f+g)(x)|, \chi F(x)) \leq \min(|f(x)|, \chi F(x)) + \min(|g(x)|, \chi F(x))$$

for every $x \in \operatorname{dom} f \cap \operatorname{dom} g$, which is almost every $x \in X$. **Q** Consequently, setting $\rho_F(f,g) = \tau_F(f-g)$, we have

$$\rho_F(f,h) = \tau_F((f-g)+(g-h)) \leq \tau_F(f-g) + \tau_F(g-h) = \rho_F(f,g) + \rho_F(g,h),$$

$$\rho_F(f,g) = \tau_F(f-g) \geq 0,$$

$$\rho_F(f,g) = \tau_F(f-g) = \tau_F(g-f) = \rho_F(g,f)$$

for all $f$, $g$, $h \in \mathcal{L}^0$; that is, $\rho_F$ is a pseudometric on $\mathcal{L}^0$.

**(b)** The family

$$\{\rho_F : F \in \Sigma,\ \mu F < \infty\}$$

now defines a topology on $\mathcal{L}^0$ (2A3F); I will call it the topology of **convergence in measure** on $\mathcal{L}^0$.

**(c)** If $f$, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then $|f| \wedge \chi F =_{\text{a.e.}} |g| \wedge \chi F$ and $\tau_F(f) = \tau_F(g)$, for every set $F$ of finite measure. Consequently we have functionals $\bar{\tau}_F$ on $L^0 = L^0(\mu)$ defined by writing

$$\bar{\tau}_F(f^\bullet) = \tau_F(f)$$

whenever $f \in \mathcal{L}^0$, $F \in \Sigma$ and $\mu F < \infty$. Corresponding to these we have pseudometrics $\bar{\rho}_F$ defined by either of the formulae

$$\bar{\rho}_F(u,v) = \bar{\tau}_F(u-v), \quad \bar{\rho}_F(f^\bullet, g^\bullet) = \rho_F(f,g)$$

for $u$, $v \in L^0$, $f$, $g \in \mathcal{L}^0$ and $F$ of finite measure. The family of these pseudometrics defines the **topology of convergence in measure** on $L^0$.

**(d)** I shall allow myself to say that a sequence (in $\mathcal{L}^0$ or $L^0$) **converges in measure** if it converges for the topology of convergence in measure (in the sense of 2A3M).

**245B Remarks (a)** Of course the topologies of $\mathcal{L}^0$, $L^0$ are about as closely related as it is possible for them to be. Not only is the topology of $L^0$ the quotient of the topology on $\mathcal{L}^0$ (that is, a set $G \subseteq L^0$ is open iff $\{f : f^\bullet \in G\}$ is open in $\mathcal{L}^0$), but every open set in $\mathcal{L}^0$ is the inverse image under the quotient map of an open set in $L^0$.

**(b)** It is convenient to note that if $F_0, \dots, F_n$ are measurable sets of finite measure with union $F$, then, in the notation of 245A, $\tau_{F_i} \leq \tau_F$ for every $i$; this means that a set $G \subseteq \mathcal{L}^0$ is open for the topology of convergence in measure iff for every $f \in G$ we can find a single set $F$ of finite measure and a $\delta > 0$ such that

$$\rho_F(g,f) \leq \delta \Longrightarrow g \in G.$$

Similarly, a set $G \subseteq L^0$ is open for the topology of convergence in measure iff for every $u \in G$ we can find a set $F$ of finite measure and a $\delta > 0$ such that

$$\bar{\rho}_F(v,u) \leq \delta \Longrightarrow v \in G.$$

**(c)** The phrase 'topology of convergence in measure' agrees well enough with standard usage when $(X, \Sigma, \mu)$ is totally finite. But a **warning!** the phrase 'topology of convergence in measure' is also used for the topology defined by the metric of 245Ye below, even when $\mu X = \infty$. I have seen the phrase **local convergence in measure** used for the topology of 245A. Most authors ignore non-$\sigma$-finite spaces in this context. However I hold that 245D-245E below are of sufficient interest to make the extension worth while.

**245C Pointwise convergence** The topology of convergence in measure is almost definable in terms of 'pointwise convergence', which is one of the roots of measure theory. The correspondence is closest in $\sigma$-finite measure spaces (see 245K), but there is still a very important relationship in the general case, as follows. Let $(X, \Sigma, \mu)$ be a measure space, and write $\mathcal{L}^0 = \mathcal{L}^0(\mu)$, $L^0 = L^0(\mu)$.

**(a)** If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ converging almost everywhere to $f\in\mathcal{L}^0$, then $\langle f_n\rangle_{n\in\mathbb{N}}\to f$ in measure. **P** By 2A3Mc, I have only to show that $\lim_{n\to\infty}\rho_F(f_n,f)=0$ whenever $\mu F<\infty$. But $\langle |f_n-f|\wedge\chi F\rangle_{n\in\mathbb{N}}$ converges to 0 a.e. and is dominated by the integrable function $\chi F$, so by Lebesgue's Dominated Convergence Theorem

$$\lim_{n\to\infty}\rho_F(f_n,f)=\lim_{n\to\infty}\int|f_n-f|\wedge\chi F=0. \ \mathbf{Q}$$

**(b)** To formulate a corresponding result applicable to $L^0$, we need the following concept. If $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ are sequences in $\mathcal{L}^0$ such that $f_n^\bullet=g_n^\bullet$ for every $n$, and $f$, $g\in\mathcal{L}^0$ are such that $f^\bullet=g^\bullet$, and $\langle f_n\rangle_{n\in\mathbb{N}}\to f$ a.e., then $\langle g_n\rangle_{n\in\mathbb{N}}\to g$ a.e., because

$$\{x: x\in\operatorname{dom}f\cap\operatorname{dom}g\cap\bigcap_{n\in\mathbb{N}}\operatorname{dom}f_n\cap\bigcap_{n\in\mathbb{N}}g_n,$$
$$g(x)=f(x)=\lim_{n\to\infty}f_n(x),\ f_n(x)=g_n(x)\ \forall\ n\in\mathbb{N}\}$$

is conegligible. Consequently we have a definition applicable to sequences in $L^0$; we can say that, for $f$, $f_n\in\mathcal{L}^0$, $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}$ is **order*-convergent**, or **order*-converges**, to $f^\bullet$ iff $f=_{\text{a.e.}}\lim_{n\to\infty}f_n$. In this case, of course, $\langle f_n\rangle_{n\in\mathbb{N}}\to f$ in measure. Thus, in $L^0$, a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ which order*-converges to $u\in L^0$ also converges to $u$ in measure.

**Remark** I suggest alternative descriptions of order-convergence in 245Xc; the conditions (iii)-(vi) there are in forms adapted to more general structures.

**(c)** For a typical example of a sequence which is convergent in measure without being order-convergent, consider the following. Take $\mu$ to be Lebesgue measure on $[0,1]$, and set $f_n(x)=2^m$ if $x\in[2^{-m}k,2^{-m}(k+1)]$, 0 otherwise, where $k=k(n)\in\mathbb{N}$, $m=m(n)\in\mathbb{N}$ are defined by saying that $n+1=2^m+k$ and $0\le k<2^m$. Then $\langle f_n\rangle_{n\in\mathbb{N}}\to 0$ for the topology of convergence in measure (since $\rho_F(f_n,0)\le 2^{-m}$ if $F\subseteq[0,1]$ is measurable and $2^m-1\le n$), though $\langle f_n\rangle_{n\in\mathbb{N}}$ is not convergent to 0 almost everywhere (indeed, $\limsup_{n\to\infty}f_n=\infty$ everywhere).

**245D Proposition** Let $(X,\Sigma,\mu)$ be any measure space.
(a) The topology of convergence in measure is a linear space topology on $L^0=L^0(\mu)$.
(b) The maps $\vee$, $\wedge: L^0\times L^0\to L^0$, and $u\mapsto|u|$, $u\mapsto u^+$, $u\mapsto u^-: L^0\to L^0$ are all continuous.
(c) The map $\times: L^0\times L^0\to L^0$ is continuous.
(d) For any continuous function $h:\mathbb{R}\to\mathbb{R}$, the corresponding function $\bar h: L^0\to L^0$ (241I) is continuous.

**proof (a)** The point is that the functionals $\bar\tau_F$, as defined in 245Ac, are F-seminorms in the sense of 2A5B. **P** Fix a set $F\in\Sigma$ of finite measure. I noted in 245Aa that

$$\tau_F(f+g)\le\tau_F(f)+\tau_F(g) \text{ for all } f,\ g\in\mathcal{L}^0,$$

so

$$\bar\tau_F(u+v)\le\bar\tau_F(u)+\bar\tau_F(v) \text{ for all } u,\ v\in L^0.$$

Next,

$$\bar\tau_F(cu)\le\bar\tau_F(u) \text{ whenever } u\in L^0 \text{ and } |c|\le 1 \qquad (*)$$

because $|cf|\wedge\chi F\le_{\text{a.e.}}|f|\wedge\chi F$ whenever $f\in\mathcal{L}^0$ and $|c|\le 1$. Finally, given $u\in L^0$ and $\epsilon>0$, let $f\in\mathcal{L}^0$ be such that $f^\bullet=u$. Then

$$\lim_{n\to\infty}|2^{-n}f|\wedge\chi F=0 \text{ a.e.},$$

so by Lebesgue's Dominated Convergence Theorem

$$\lim_{n\to\infty}\bar\tau_F(2^{-n}u)=\lim_{n\to\infty}\int|2^{-n}f|\wedge\chi F=0,$$

and there is an $n$ such that $\bar\tau_F(2^{-n}u)\le\epsilon$. It follows (by (*) just above) that $\bar\tau_F(cu)\le\epsilon$ whenever $|c|\le 2^{-n}$. As $\epsilon$ is arbitrary, $\lim_{c\to 0}\bar\tau_F(u)=0$ for every $u\in L^0$; which is the third condition in 2A5B. **Q**

Now 2A5B tells us that the topology defined by the $\bar\tau_F$ is a linear space topology.

**(b)** For any $u$, $v \in L^0$, $||u|-|v|| \le |u-v|$, so $\bar\rho_F(|u|,|v|) \le \bar\rho_F(u,v)$ for every set $F$ of finite measure. By 2A3H, $|\,| : L^0 \to L^0$ is continuous. Now

$$u \vee v = \frac{1}{2}(u+v+|u-v|), \quad u \wedge v = \frac{1}{2}(u+v-|u-v|),$$

$$u^+ = u \wedge 0, \quad u^- = (-u) \wedge 0.$$

As addition and subtraction are continuous, so are $\vee$, $\wedge$, $^+$ and $^-$.

**(c)** Take $u_0$, $v_0 \in L^0$, $F \in \Sigma$ a set of finite measure and $\epsilon > 0$. Represent $u_0$ and $v_0$ as $f_0^\bullet$, $g_0^\bullet$ respectively, where $f_0$, $g_0 : X \to \mathbb{R}$ are $\Sigma$-measurable (241Bk). If we set

$$F_m = \{x : x \in F,\, |f_0(x)| + |g_0(x)| \le m\},$$

then $\langle F_m\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence of sets with union $F$, so there is an $m \in \mathbb{N}$ such that $\mu(F\setminus F_m) \le \frac{1}{2}\epsilon$. Let $\delta > 0$ be such that $(2m+\mu F)\delta^2 + 2\delta \le \frac{1}{2}\epsilon$.

Now suppose that $u$, $v \in L^0$ are such that $\bar\rho_F(u,u_0) \le \delta^2$ and $\bar\rho_F(v,v_0) \le \delta^2$. Let $f$, $g : X \to \mathbb{R}$ be measurable functions such that $f^\bullet = u$ and $v^\bullet = v$. Then

$$\mu\{x : x \in F,\, |f(x)-f_0(x)| \ge \delta\} \le \delta, \quad \mu\{x : x \in F,\, |g(x)-g_0(x)| \ge \delta\} \le \delta,$$

so that

$$\mu\{x : x \in F,\, |f(x)-f_0(x)||g(x)-g_0(x)| \ge \delta^2\} \le 2\delta$$

and

$$\int_F \min(1, |f-f_0| \times |g-g_0|) \le \delta^2\mu F + 2\delta.$$

Also

$$|f \times g - f_0 \times g_0| \le |f-f_0| \times |g-g_0| + |f_0| \times |g-g_0| + |f-f_0| \times |g_0|,$$

so that

$$\begin{aligned}
\bar\rho_F(u \times v, u_0 \times v_0) &= \int_F \min(1, |f \times g - f_0 \times g_0|) \\
&\le \frac{1}{2}\epsilon + \int_{F_m} \min(1, |f \times g - f_0 \times g_0|) \\
&\le \frac{1}{2}\epsilon + \int_{F_m} \min(1, |f-f_0| \times |g-g_0| + m|g-g_0| + m|f-f_0|) \\
&\le \frac{1}{2}\epsilon + \int_F \min(1, |f-f_0| \times |g-g_0|) \\
&\qquad\qquad + m\int_F \min(1, |g-g_0|) + m\int_F \min(1, |f-f_0|) \\
&\le \frac{1}{2}\epsilon + \delta^2\mu F + 2\delta + 2m\delta^2 \le \epsilon.
\end{aligned}$$

As $F$ and $\epsilon$ are arbitrary, $\times$ is continuous at $(u_0, v_0)$; as $u_0$ and $v_0$ are arbitrary, $\times$ is continuous.

**(d)** Take $u \in L^0$, $F \in \Sigma$ of finite measure and $\epsilon > 0$. Then there is a $\delta > 0$ such that $\rho_F(\bar h(v), \bar h(u)) \le \epsilon$ whenever $\rho_F(v,u) \le \delta$. **P?** Otherwise, we can find, for each $n \in \mathbb{N}$, a $v_n$ such that $\bar\rho_F(v_n,u) \le 4^{-n}$ but $\bar\rho_F(\bar h(v_n), \bar h(u)) > \epsilon$. Express $u$ as $f^\bullet$ and $v_n$ as $g_n^\bullet$ where $f$, $g_n : X \to \mathbb{R}$ are measurable. Set

$$E_n = \{x : x \in F,\, |g_n(x) - f(x)| \ge 2^{-n}\}$$

for each $n$. Then $\bar\rho_F(v_n,u) \ge 2^{-n}\mu E_n$, so $\mu E_n \le 2^{-n}$ for each $n$, and $E = \bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n} E_m$ is negligible. But $\lim_{n\to\infty} g_n(x) = f(x)$ for every $x \in F \setminus E$, so (because $h$ is continuous) $\lim_{n\to\infty} h(g_n(x)) = h(f(x))$ for every $x \in F \setminus E$. Consequently (by Lebesgue's Dominated Convergence Theorem, as always)

$$\lim_{n\to\infty}\bar\rho_F(\bar h(v_n),\bar h(u))=\lim_{n\to\infty}\int_F\min(1,|h(g_n(x))-h(f(x))|\mu(dx)=0,$$

which is impossible. **XQ**

By 2A3H, $\bar h$ is continuous.

**Remark** I cannot say that the topology of convergence in measure on $\mathcal{L}^0$ is a linear space topology solely because (on the definitions I have chosen) $\mathcal{L}^0$ is not in general a linear space.

**245E** I turn now to the principal theorem relating the properties of the topological linear space $L^0(\mu)$ to the classification of measure spaces in Chapter 21.

**Theorem** Let $(X,\Sigma,\mu)$ be a measure space. Let $\mathfrak{T}$ be the topology of convergence in measure on $L^0=L^0(\mu)$, as described in 245A.

(a) $(X,\Sigma,\mu)$ is semi-finite iff $\mathfrak{T}$ is Hausdorff.

(b) $(X,\Sigma,\mu)$ is $\sigma$-finite iff $\mathfrak{T}$ is metrizable.

(c) $(X,\Sigma,\mu)$ is localizable iff $\mathfrak{T}$ is Hausdorff and $L^0$ is complete under $\mathfrak{T}$.

**proof** I use the pseudometrics $\rho_F$ on $\mathcal{L}^0=\mathcal{L}^0(\mu)$, $\bar\rho_F$ on $L^0$ described in 245A.

**(a)(i)** Suppose that $(X,\Sigma,\mu)$ is semi-finite and that $u$, $v$ are distinct members of $L^0$. Express them as $f^\bullet$ and $g^\bullet$ where $f$ and $g$ are measurable functions from $X$ to $\mathbb{R}$. Then $\mu\{x:f(x)\ne g(x)\}>0$ so, because $(X,\Sigma,\mu)$ is semi-finite, there is a set $F\in\Sigma$ of finite measure such that $\mu\{x:x\in F,\ f(x)\ne g(x)\}>0$. Now

$$\bar\rho_F(u,v)=\int_F\min(|f(x)-g(x)|,1)dx>0$$

(see 122Rc). As $u$ and $v$ are arbitrary, $\mathfrak{T}$ is Hausdorff (2A3L).

**(ii)** Suppose that $\mathfrak{T}$ is Hausdorff and that $E\in\Sigma$, $\mu E>0$. Then $u=\chi E^\bullet\ne0$ so there is an $F\in\Sigma$ such that $\mu F<\infty$ and $\bar\rho_F(u,0)\ne0$, that is, $\mu(E\cap F)>0$. Now $E\cap F$ is a non-negligible set of finite measure included in $E$. As $E$ is arbitrary, $(X,\Sigma,\mu)$ is semi-finite.

**(b)(i)** Suppose that $(X,\Sigma,\mu)$ is $\sigma$-finite. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering $X$. Set

$$\bar\rho(u,v)=\sum_{n=0}^\infty\frac{\bar\rho_{E_n}(u,v)}{1+2^n\mu E_n}$$

for $u$, $v\in L^0$. Then $\bar\rho$ is a metric on $L^0$. **P** Because every $\bar\rho_{E_n}$ is a pseudometric, so is $\bar\rho$. If $\bar\rho(u,v)=0$, express $u$ as $f^\bullet$, $v$ as $g^\bullet$ where $f$, $g\in\mathcal{L}^0(\mu)$; then

$$\int|f-g|\wedge\chi E_n=\bar\rho_{E_n}(u,v)=0,$$

so $f=g$ almost everywhere in $E_n$, for every $n$. Because $X=\bigcup_{n\in\mathbb{N}}E_n$, $f=_{\text{a.e.}}g$ and $u=v$. **Q**

If $F\in\Sigma$ and $\mu F<\infty$ and $\epsilon>0$, take $n$ such that $\mu(F\setminus E_n)\le\frac12\epsilon$. If $u$, $v\in L^0$ and $\bar\rho(u,v)\le\epsilon/2(1+2^n\mu E_n)$, then $\bar\rho_F(u,v)\le\epsilon$. **P** Express $u$ as $f^\bullet$, $v=g^\bullet$ where $f$, $g\in\mathcal{L}^0$. Then

$$\int|u-v|\wedge\chi E_n=\bar\rho_{E_n}(u,v)\le(1+2^n\mu E_n)\bar\rho(u,v)\le\frac{\epsilon}{2},$$

while

$$\int|f-g|\wedge\chi(F\setminus E_n)\le\mu(F\setminus E_n)\le\frac{\epsilon}{2},$$

so

$$\bar\rho_F(u,v)=\int|f-g|\wedge\chi F\le\int|f-g|\wedge\chi E_n+\int|f-g|\wedge\chi(F\setminus E_n)\le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.\ \textbf{Q}$$

In the other direction, given $\epsilon>0$, take $n\in\mathbb{N}$ such that $2^{-n}\le\frac12\epsilon$; then $\bar\rho(u,v)\le\epsilon$ whenever $\bar\rho_{E_n}(u,v)\le\epsilon/2(n+1)$.

These show that $\bar\rho$ defines the same topology as the $\bar\rho_F$ (2A3Ib), so that $\mathfrak{T}$, the topology defined by the $\bar\rho_F$, is metrizable.

**(ii)** Suppose that $\mathfrak{T}$ is metrizable. Let $\bar\rho$ be a metric defining $\mathfrak{T}$. For each $n \in \mathbb{N}$ there must be a measurable set $F_n$ of finite measure and a $\delta_n > 0$ such that

$$\bar\rho_{F_n}(u,0) \le \delta_n \Longrightarrow \bar\rho(u,0) \le 2^{-n}.$$

Set $E = X \setminus \bigcup_{n\in\mathbb{N}} F_n$. **?** If $E$ is not negligible, then $u = \chi E^\bullet \ne 0$; because $\bar\rho$ is a metric, there is an $n \in \mathbb{N}$ such that $\bar\rho(u,0) > 2^{-n}$; now

$$\mu(E \cap F_n) = \bar\rho_{F_n}(u,0) > \delta_n.$$

But $E \cap F_n = \emptyset$. **X**

So $\mu E = 0 < \infty$. Now $X = E \cup \bigcup_{n\in\mathbb{N}} F_n$ is a countable union of sets of finite measure, so $\mu$ is $\sigma$-finite.

**(c)** By (a), either hypothesis ensures that $\mu$ is semi-finite and that $\mathfrak{T}$ is Hausdorff.

**(i)** Suppose that $(X,\Sigma,\mu)$ is localizable. Let $\mathcal{F}$ be a Cauchy filter on $L^0$ (2A5F). For each measurable set $F$ of finite measure, choose a sequence $\langle A_n(F)\rangle_{n\in\mathbb{N}}$ of members of $\mathcal{F}$ such that $\bar\rho_F(u,v) \le 4^{-n}$ for every $u$, $v \in A_n(F)$ and every $n$ (2A5G). Choose $u_{Fn} \in \bigcap_{k\le n} A_n(F)$ for each $n$; then $\bar\rho_F(u_{F,n+1}, u_{Fn}) \le 2^{-n}$ for each $n$. Express each $u_{Fn}$ as $f_{Fn}^\bullet$ where $f_{Fn}$ is a measurable function from $X$ to $\mathbb{R}$. Then

$$\mu\{x : x \in F,\ |f_{F,n+1}(x) - f_{Fn}(x)| \ge 2^{-n}\} \le 2^n \bar\rho_F(u_{F,n+1}, u_{Fn}) \le 2^{-n}$$

for each $n$. It follows that $\langle f_{Fn}\rangle_{n\in\mathbb{N}}$ must converge almost everywhere in $F$. **P** Set

$$H_n = \{x : x \in F,\ |f_{F,n+1}(x) - f_{Fn}(x)| \ge 2^{-n}\}.$$

Then $\mu H_n \le 2^{-n}$ for each $n$, so

$$\mu(\bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} H_m) \le \inf_{n\in\mathbb{N}} \sum_{m=n}^\infty 2^{-m} = 0.$$

If $x \in F \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} H_m$, then there is some $k$ such that $x \in F \setminus \bigcup_{m\ge k} H_m$, so that $|f_{F,m+1}(x) - f_{Fm}(x)| \le 2^{-m}$ for every $m \ge k$, and $\langle f_{Fn}(x)\rangle_{n\in\mathbb{N}}$ is Cauchy, therefore convergent. **Q**

Set $f_F(x) = \lim_{n\to\infty} f_{Fn}(x)$ for every $x \in F$ such that the limit is defined in $\mathbb{R}$, so that $f_F$ is measurable and defined almost everywhere in $F$.

If $E$, $F$ are two sets of finite measure and $E \subseteq F$, then $\bar\rho_E(u_{En}, u_{Fn}) \le 2 \cdot 4^{-n}$ for each $n$. **P** $A_n(E)$ and $A_n(F)$ both belong to $\mathcal{F}$, so must have a point $w$ in common; now

$$\begin{aligned}\bar\rho_E(u_{En}, u_{Fn}) &\le \bar\rho_E(u_{En}, w) + \bar\rho_E(w, u_{Fn})\\ &\le \bar\rho_E(u_{En}, w) + \bar\rho_F(w, u_{Fn}) \le 4^{-n} + 4^{-n}. \ \mathbf{Q}\end{aligned}$$

Consequently

$$\mu\{x : x \in E,\ |f_{Fn}(x) - f_{En}(x)| \ge 2^{-n}\} \le 2^n \bar\rho_E(u_{Fn}, u_{En}) \le 2^{-n+1}$$

for each $n$, and $\lim_{n\to\infty} f_{Fn} - f_{En} = 0$ almost everywhere in $E$; so that $f_E = f_F$ a.e. on $E$.

Consequently, if $E$ and $F$ are any two sets of finite measure, $f_E = f_F$ a.e. on $E \cap F$, because both are equal almost everywhere on $E \cap F$ to $f_{E\cup F}$.

Because $\mu$ is localizable, it follows that there is an $f \in \mathcal{L}^0$ such that $f = f_E$ a.e. on $E$ for every measurable set $E$ of finite measure (213N). Consider $u = f^\bullet \in L^0$. For any set $E$ of finite measure,

$$\bar\rho_E(u, u_{En}) = \int_E \min(1, |f(x) - f_{En}(x)|)dx = \int_E \min(1, |f_E(x) - f_{En}(x)|)dx \to 0$$

as $n \to \infty$, using Lebesgue's Dominated Convergence Theorem. Now

$$\begin{aligned}\inf_{A\in\mathcal{F}} \sup_{v\in A} \bar\rho_E(v,u) &\le \inf_{n\in\mathbb{N}} \sup_{v\in A_{En}} \bar\rho_E(v,u)\\ &\le \inf_{n\in\mathbb{N}} \sup_{v\in A_{En}} (\bar\rho_E(v, u_{En}) + \bar\rho_E(u, u_{En}))\\ &\le \inf_{n\in\mathbb{N}} (4^{-n} + \bar\rho_E(u, u_{En})) = 0.\end{aligned}$$

As $E$ is arbitrary, $\mathcal{F} \to u$. As $\mathcal{F}$ is arbitrary, $L^0$ is complete under $\mathfrak{T}$.

**(ii)** Now suppose that $L^0$ is complete under $\mathfrak{T}$ and let $\mathcal{E}$ be any family of sets in $\Sigma$. Let $\mathcal{E}'$ be

$$\{\bigcup \mathcal{E}_0 : \mathcal{E}_0 \text{ is a finite subset of } \mathcal{E}\}.$$

Then the union of any two members of $\mathcal{E}'$ belongs to $\mathcal{E}'$. Let $\mathcal{F}$ be the set

$$\{A : A \subseteq L^0,\, A \supseteq A_E \text{ for some } E \in \mathcal{E}'\},$$

where for $E \in \mathcal{E}'$ I write

$$A_E = \{\chi F^\bullet : F \in \mathcal{E}',\, F \supseteq E\}.$$

Then $\mathcal{F}$ is a filter on $L^0$, because $A_E \cap A_F = A_{E\cup F}$ for all $E$, $F \in \mathcal{E}'$.

In fact $\mathcal{F}$ is a Cauchy filter. **P** Let $H$ be any set of finite measure and $\epsilon > 0$. Set $\gamma = \sup_{E\in\mathcal{E}'} \mu(H \cap E)$ and take $E \in \mathcal{E}'$ such that $\mu(H \cap E) \geq \gamma - \epsilon$. Consider $A_E \in \mathcal{F}$. If $F$, $G \in \mathcal{E}'$ and $F \supseteq E$, $G \supseteq E$ then

$$\begin{aligned}\bar\rho_H(\chi F^\bullet, \chi G^\bullet) &= \mu(H \cap (F \triangle G)) = \mu(H \cap (F \cup G)) - \mu(H \cap F \cap G)\\ &\leq \gamma - \mu(H \cap E) \leq \epsilon.\end{aligned}$$

Thus $\bar\rho_H(u, v) \leq \epsilon$ for all $u$, $v \in A_E$. As $H$ and $\epsilon$ are arbitrary, $\mathcal{F}$ is Cauchy. **Q**

Because $L^0$ is complete under $\mathfrak{T}$, $\mathcal{F}$ has a limit $w$ say. Express $w$ as $h^\bullet$, where $h : X \to \mathbb{R}$ is measurable, and consider $G = \{x : h(x) > \frac{1}{2}\}$.

**?** If $E \in \mathcal{E}$ and $\mu(E\setminus G) > 0$, let $F \subseteq E\setminus G$ be a set of non-zero finite measure. Then $\{u : \bar\rho_F(u, w) < \frac{1}{2}\mu F\}$ belongs to $\mathcal{F}$, so meets $A_E$; let $H \in \mathcal{E}'$ be such that $E \subseteq H$ and $\bar\rho_F(\chi H^\bullet, w) < \frac{1}{2}\mu F$. Then

$$\int_F \min(1, |1 - h(x)|) = \bar\rho_F(\chi H^\bullet, w) < \frac{1}{2}\mu F;$$

but because $F \cap G = \emptyset$, $1 - h(x) \geq \frac{1}{2}$ for every $x \in F$, so this is impossible. **X**

Thus $E \setminus G$ is negligible for every $E \in \mathcal{E}$.

Now suppose that $H \in \Sigma$ and $\mu(G \setminus H) > 0$. Then there is an $E \in \mathcal{E}$ such that $\mu(E \setminus H) > 0$. **P** Let $F \subseteq G \setminus H$ be a set of non-zero finite measure. Let $u \in A_\emptyset$ be such that $\bar\rho_F(u, w) < \frac{1}{2}\mu F$. Then $u$ is of the form $\chi C^\bullet$ for some $C \in \mathcal{E}'$, and

$$\int_F \min(1, |\chi C(x) - h(x)|) < \frac{1}{2}\mu F.$$

As $h(x) \geq \frac{1}{2}$ for every $x \in F$, $\mu(C \cap F) > 0$. But $C$ is a finite union of members of $\mathcal{E}$, so there is an $E \in \mathcal{E}$ such that $\mu(E \cap F) > 0$, and now $\mu(E \setminus H) > 0$. **Q**

As $H$ is arbitrary, $G$ is an essential supremum of $\mathcal{E}$ in $\Sigma$. As $\mathcal{E}$ is arbitrary, $(X, \Sigma, \mu)$ is localizable.

**245F Alternative description of the topology of convergence in measure** Let us return to arbitrary measure spaces $(X, \Sigma, \mu)$.

**(a)** For any $F \in \Sigma$ of finite measure and $\epsilon > 0$ define $\tau_{F\epsilon} : \mathcal{L}^0 \to [0, \infty[$ by

$$\tau_{F\epsilon}(f) = \mu^*\{x : x \in F \cap \operatorname{dom} f,\, |f(x)| > \epsilon\}$$

for $f \in \mathcal{L}^0$, taking $\mu^*$ to be the outer measure defined from $\mu$ (132B). If $f$, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then

$$\{x : x \in F \cap \operatorname{dom} f,\, |f(x)| > \epsilon\} \triangle \{x : x \in F \cap \operatorname{dom} g,\, |g(x)| > \epsilon\}$$

is negligible, so $\tau_{F\epsilon}(f) = \tau_{F\epsilon}(g)$; accordingly we have a functional from $L^0$ to $[0, \infty[$, given by

$$\bar\tau_{F\epsilon}(u) = \tau_{F\epsilon}(f)$$

whenever $f \in \mathcal{L}^0$ and $u = f^\bullet \in L^0$.

**(b)** Now $\tau_{F\epsilon}$ is not (except in trivial cases) subadditive, so does not define a pseudometric on $\mathcal{L}^0$ or $L^0$. But we can say that, for $f \in \mathcal{L}^0$,

$$\tau_F(f) \leq \epsilon \min(1, \epsilon) \Longrightarrow \tau_{F\epsilon}(f) \leq \epsilon \Longrightarrow \tau_F(f) \leq \epsilon(1 + \mu F).$$

(The point is that if $E \subseteq \operatorname{dom} f$ is a measurable conegligible set such that $f{\restriction}E$ is measurable, then

$$\tau_F(f) = \int_{E\cap F} \min(f(x), 1)dx, \quad \tau_{F\epsilon}(f) = \mu\{x : x \in E \cap F,\, f(x) > \epsilon\}.)$$

This means that a set $G \subseteq \mathcal{L}^0$ is open for the topology of convergence in measure iff for every $f \in G$ we can find a set $F$ of finite measure and $\epsilon$, $\delta > 0$ such that

$$\tau_{F\epsilon}(g-f) \le \delta \Longrightarrow g \in G.$$

Of course $\tau_{F\delta}(f) \ge \tau_{F\epsilon}(f)$ whenever $\delta \le \epsilon$, so we can equally say: $G \subseteq \mathcal{L}^0$ is open for the topology of convergence in measure iff for every $f \in G$ we can find a set $F$ of finite measure and $\epsilon > 0$ such that

$$\tau_{F\epsilon}(g-f) \le \epsilon \Longrightarrow g \in G.$$

Similarly, $G \subseteq L^0$ is open for the topology of convergence in measure on $L^0$ iff for every $u \in G$ we can find a set $F$ of finite measure and $\epsilon > 0$ such that

$$\bar{\tau}_{F\epsilon}(v-u) \le \epsilon \Longrightarrow v \in G.$$

**(c)** It follows at once that a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ converges in measure to $f \in \mathcal{L}^0$ iff

$$\lim_{n\to\infty} \mu^*\{x : x \in F \cap \operatorname{dom} f \cap \operatorname{dom} f_n,\, |f_n(x) - f(x)| > \epsilon\} = 0$$

whenever $F \in \Sigma$, $\mu F < \infty$ and $\epsilon > 0$. Similarly, a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $L^0$ converges in measure to $u$ iff $\lim_{n\to\infty} \bar{\tau}_{F\epsilon}(u - u_n) = 0$ whenever $\mu F < \infty$ and $\epsilon > 0$.

**(d)** In particular, if $(X, \Sigma, \mu)$ is totally finite, $\langle f_n \rangle_{n\in\mathbb{N}} \to f$ in $\mathcal{L}^0$ iff

$$\lim_{n\to\infty} \mu^*\{x : x \in \operatorname{dom} f \cap \operatorname{dom} f_n,\, |f(x) - f_n(x)| > \epsilon\} = 0$$

for every $\epsilon > 0$, and $\langle u_n \rangle_{n\in\mathbb{N}} \to u$ in $L^0$ iff

$$\lim_{n\to\infty} \bar{\tau}_{X\epsilon}(u - u_n) = 0$$

for every $\epsilon > 0$.

**245G Embedding $L^p$ in $L^0$: Proposition** Let $(X, \Sigma, \mu)$ be any measure space. Then for any $p \in [1, \infty]$, the embedding of $L^p = L^p(\mu)$ in $L^0 = L^0(\mu)$ is continuous for the norm topology of $L^p$ and the topology of convergence in measure on $L^0$.

**proof** Suppose that $u$, $v \in L^p$ and that $\mu F < \infty$. Then $(\chi F)^\bullet$ belongs to $L^q$, where $q = p/(p-1)$ (taking $q = 1$ if $p = \infty$, $q = \infty$ if $p = 1$ as usual), and

$$\bar{\rho}_F(u, v) \le \int |u - v| \times (\chi F)^\bullet \le \|u - v\|_p \|\chi F^\bullet\|_q$$

(244Eb). By 2A3H, this is enough to ensure that the embedding $L^p \subsetneqq L^0$ is continuous.

**245H** The case of $L^1$ is so important that I go farther with it.

**Proposition** Let $(X, \Sigma, \mu)$ be a measure space.

(a)(i) If $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $\epsilon > 0$, there are a $\delta > 0$ and a set $F \in \Sigma$ of finite measure such that $\int |f - g| \le \epsilon$ whenever $g \in \mathcal{L}^1$, $\int |g| \le \int |f| + \delta$ and $\rho_F(f, g) \le \delta$.

(ii) For any sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^1$ and any $f \in \mathcal{L}^1$, $\lim_{n\to\infty} \int |f - f_n| = 0$ iff $\langle f_n \rangle_{n\in\mathbb{N}} \to f$ in measure and $\limsup_{n\to\infty} \int |f_n| \le \int |f|$.

(b)(i) If $u \in L^1 = L^1(\mu)$ and $\epsilon > 0$, there are a $\delta > 0$ and a set $F \in \Sigma$ of finite measure such that $\|u - v\|_1 \le \epsilon$ whenever $v \in L^1$, $\|v\|_1 \le \|u\|_1 + \delta$ and $\bar{\rho}_F(u, v) \le \delta$.

(ii) For any sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $L^1$ and any $u \in L^1$, $\langle u_n \rangle_{n\in\mathbb{N}} \to u$ for $\| \,\|_1$ iff $\langle u_n \rangle_{n\in\mathbb{N}} \to u$ in measure and $\limsup_{n\to\infty} \|u_n\|_1 \le \|u\|_1$.

**proof (a)(i)** We know that there are a set $F$ of finite measure and an $\eta > 0$ such that $\int_E |f| \le \frac{1}{5}\epsilon$ whenever $\mu(E \cap F) \le \eta$ (225A). Take $\delta > 0$ such that $\delta(\epsilon + 5\mu F) \le \epsilon\eta$ and $\delta \le \frac{1}{5}\epsilon$. Then if $\int |g| \le \int |f| + \delta$ and $\rho_F(f, g) \le \delta$, let $G \subseteq \operatorname{dom} f \cap \operatorname{dom} g$ be a conegligible measurable set such that $f \restriction G$ and $g \restriction G$ are both measurable. Set

$$E = \{x : x \in F \cap G,\, |f(x) - g(x)| \ge \tfrac{\epsilon}{\epsilon + 5\mu F}\};$$

then

$$\delta \ge \rho_F(f, g) \ge \tfrac{\epsilon}{\epsilon + 5\mu F}\mu E,$$

so $\mu E\le\eta$. Set $H=F\setminus E$, so that $\mu(F\setminus H)\le\eta$ and $\int_{X\setminus H}|f|\le\frac{1}{5}\epsilon$. On the other hand, for almost every $x\in H$, $|f(x)-g(x)|\le\frac{\epsilon}{\epsilon+5\mu F}$, so $\int_H|f-g|\le\frac{1}{5}\epsilon$ and

$$\int_H|g|\ge\int_H|f|-\frac{1}{5}\epsilon\ge\int|f|-\int_{X\setminus H}|f|-\frac{1}{5}\epsilon\ge\int|f|-\frac{2}{5}\epsilon.$$

Since $\int|g|\le\int|f|+\frac{1}{5}\epsilon$, $\int_{X\setminus H}|g|\le\frac{3}{5}\epsilon$. Now this means that

$$\int|g-f|\le\int_{X\setminus H}|g|+\int_{X\setminus H}|f|+\int_H|g-f|\le\frac{3}{5}\epsilon+\frac{1}{5}\epsilon+\frac{1}{5}\epsilon=\epsilon,$$

as required.

**(ii)** If $\lim_{n\to\infty}\int|f-f_n|=0$, that is, $\lim_{n\to\infty}f_n^\bullet=f^\bullet$ in $L^1$, then by 245G we must have $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}\to f^\bullet$ in $L^0$, that is, $\langle f_n\rangle_{n\in\mathbb{N}}\to f$ for the topology of convergence in measure. Also, of course, $\lim_{n\to\infty}\int|f_n|=\int|f|$.

In the other direction, if $\limsup_{n\to\infty}\int|f_n|\le\int|f|$ and $\langle f_n\rangle_{n\in\mathbb{N}}\to f$ for the topology of convergence in measure, then whenever $\delta>0$ and $\mu F<\infty$ there must be an $m\in\mathbb{N}$ such that $\int|f_n|\le\int|f|+\delta$, $\rho_F(f,f_n)\le\delta$ for every $n\ge m$; so (i) tells us that $\lim_{n\to\infty}\int|f_n-f|=0$.

**(b)** This now follows immediately if we express $u$ as $f^\bullet$, $v$ as $g^\bullet$ and $u_n$ as $f_n^\bullet$.

**245I Remarks (a)** I think the phenomenon here is so important that it is worth looking at some elementary examples.

(i) If $\mu$ is counting measure on $\mathbb{N}$, and we set $f_n(n)=1$, $f_n(i)=0$ for $i\ne n$, then $\langle f_n\rangle_{n\in\mathbb{N}}\to0$ in measure, while $\int|f_n|=1$ for every $n$.

(ii) If $\mu$ is Lebesgue measure on $[0,1]$, and we set $f_n(x)=2^n$ for $0<x\le2^{-n}$, 0 for other $x$, then again $\langle f_n\rangle_{n\in\mathbb{N}}\to0$ in measure, while $\int|f_n|=1$ for every $n$.

(iii) In 245Cc we have another sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ converging to 0 in measure, while $\int|f_n|=1$ for every $n$. In all these cases (as required by Fatou's Lemma, at least in (i) and (ii)) we have $\int|f|\le\liminf_{n\to\infty}\int|f_n|$. (The next proposition shows that this applies to any sequence which is convergent in measure.)

The common feature of these examples is the way in which somehow the $f_n$ escape to infinity, either laterally (in (i)) or vertically (in (iii)) or both (in (ii)). Note that in all three examples we can set $f'_n=2^nf_n$ to obtain a sequence still converging to 0 in measure, but with $\lim_{n\to\infty}\int|f'_n|=\infty$.

**(b)** In 245H, I have used the explicit formulations '$\lim_{n\to\infty}\int|f_n-f|=0$' (for sequences of functions), '$\langle u_n\rangle_{n\in\mathbb{N}}\to u$ for $\| \|_1$' (for sequences in $L^1$). These are often expressed by saying that $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle u_n\rangle_{n\in\mathbb{N}}$ are **convergent in mean** to $f$, $u$ respectively.

**245J** For semi-finite spaces we have a further relationship.

**Proposition** Let $(X,\Sigma,\mu)$ be a semi-finite measure space. Write $\mathcal{L}^0=\mathcal{L}^0(\mu)$, etc.

(a)(i) For any $a\ge0$, the set $\{f:f\in\mathcal{L}^1,\int|f|\le a\}$ is closed in $\mathcal{L}^0$ for the topology of convergence in measure.

(ii) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^1$ which is convergent in measure to $f\in\mathcal{L}^0$, and $\liminf_{n\to\infty}\int|f_n|<\infty$, then $f$ is integrable and $\int|f|\le\liminf_{n\to\infty}\int|f_n|$.

(b)(i) For any $a\ge0$, the set $\{u:u\in L^1,\|u\|_1\le a\}$ is closed in $L^0$ for the topology of convergence in measure.

(ii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in $L^1$ which is convergent in measure to $u\in L^0$, and $\liminf_{n\to\infty}\|u_n\|_1<\infty$, then $u\in L^1$ and $\|u\|_1\le\liminf_{n\to\infty}\|u_n\|_1$.

**proof (a)(i)** Set $A=\{f:f\in\mathcal{L}^1,\int|f|\le a\}$, and let $g$ be any member of the closure of $A$ in $\mathcal{L}^0$. Let $h$ be any simple function such that $0\le h\le_{\text{a.e.}}|g|$, and $\epsilon>0$. If $h=0$ then of course $\int h\le a$. Otherwise, setting $F=\{x:h(x)>0\}$ and $M=\sup_{x\in X}h(x)$, there is an $f\in A$ such that $\mu^*\{x:x\in F\cap\operatorname{dom}f\cap\operatorname{dom}g,\,|f(x)-g(x)|\ge\epsilon\}\le\epsilon$ (245F); let $E\supseteq\{x:x\in F\cap\operatorname{dom}f\cap\operatorname{dom}g,\,|f(x)-g(x)|\ge\epsilon\}$ be a measurable set of measure at most $\epsilon$. Then $h\le_{\text{a.e.}}|f|+\epsilon\chi F+M\chi E$, so $\int h\le a+\epsilon(M+\mu F)$. As $\epsilon$ is arbitrary, $\int h\le a$. But we are supposing that $\mu$ is semi-finite, so this is enough to ensure that $g$ is integrable and that $\int|g|\le a$ (213B), that is, that $g\in A$. As $g$ is arbitrary, $A$ is closed.

**(ii)** Now if $\langle f_n\rangle_{n\in\mathbb{N}}$ is convergent in measure to $f$, and $\liminf_{n\to\infty}\int|f_n| = a$, then for any $\epsilon>0$ there is a subsequence $\langle f_{n(k)}\rangle_{k\in\mathbb{N}}$ such that $\int|f_{n(k)}|\le a+\epsilon$ for every $k$; since $\langle f_{n(k)}\rangle_{k\in\mathbb{N}}$ still converges in measure to $f$, $\int|f|\le a+\epsilon$. As $\epsilon$ is arbitrary, $\int|f|\le a$.

**(b)** As in 245H, this is just a translation of part (a) into the language of $L^1$ and $L^0$.

**245K** For $\sigma$-finite measure spaces, the topology of convergence in measure on $L^0$ is metrizable, so can be described effectively in terms of convergent sequences; it is therefore important that we have, in this case, a sharp characterization of sequential convergence in measure.

**Proposition** Let $(X,\Sigma,\mu)$ be a $\sigma$-finite measure space. Then

(a) a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0$ converges in measure to $f\in\mathcal{L}^0$ iff every subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence converging to $f$ almost everywhere;

(b) a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $L^0$ converges in measure to $u\in L^0$ iff every subsequence of $\langle u_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence which order*-converges to $u$.

**proof (a)(i)** Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}\to f$, that is, that $\lim_{n\to\infty}\int|f-f_n|\wedge\chi F=0$ for every set $F$ of finite measure. Let $\langle E_k\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering $X$, and let $\langle n(k)\rangle_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $\int|f-f_{n(k)}|\wedge\chi E_k\le 4^{-k}$ for every $k\in\mathbb{N}$. Then $\sum_{k=0}^{\infty}|f-f_{n(k)}|\wedge\chi E_k$ is finite almost everywhere (242E); but $\lim_{k\to\infty}f_{n(k)}(x)=f(x)$ whenever $\sum_{k=0}^{\infty}\min(1,|f(x)-f_{n(k)}(x)|)<\infty$, so $\langle f_{n(k)}\rangle_{k\in\mathbb{N}}\to f$ a.e.

**(ii)** The same applies to every subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$, so that every subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$ has a sub-subsequence converging to $f$ almost everywhere.

**(iii)** Now suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ does not converge to $f$. Then there is an open set $U$ containing $f$ such that $\{n:f_n\notin U\}$ is infinite, that is, $\langle f_n\rangle_{n\in\mathbb{N}}$ has a subsequence $\langle f'_n\rangle_{n\in\mathbb{N}}$ with $f'_n\notin U$ for every $n$. But now no sub-subsequence of $\langle f'_n\rangle_{n\in\mathbb{N}}$ converges to $f$ in measure, so no such sub-subsequence can converge almost everywhere, by 245Ca.

**(b)** This follows immediately from (a) if we express $u$ as $f^\bullet$, $u_n$ as $f_n^\bullet$.

**245L Corollary** Let $(X,\Sigma,\mu)$ be a $\sigma$-finite measure space.

(a) A subset $A$ of $\mathcal{L}^0=\mathcal{L}^0(\mu)$ is closed for the topology of convergence in measure iff $f\in A$ whenever $f\in\mathcal{L}^0$ and there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $A$ such that $f=_{\text{a.e.}}\lim_{n\to\infty}f_n$.

(b) A subset $A$ of $L^0=L^0(\mu)$ is closed for the topology of convergence in measure iff $u\in A$ whenever $u\in L^0$ and there is a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $A$ order*-converging to $u$.

**proof (a)(i)** If $A$ is closed for the topology of convergence in measure, and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $A$ converging to $f$ almost everywhere, then $\langle f_n\rangle_{n\in\mathbb{N}}$ converges to $f$ in measure, so surely $f\in A$ (since otherwise all but finitely many of the $f_n$ would have to belong to the open set $\mathcal{L}^0\setminus A$).

**(ii)** If $A$ is not closed, there is an $f\in\overline{A}\setminus A$. The topology can be defined by a metric $\rho$ (245Eb), and we can choose a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $A$ such that $\rho(f_n,f)\le 2^{-n}$ for every $n$, so that $\langle f_n\rangle_{n\in\mathbb{N}}\to f$ in measure. By 245K, $\langle f_n\rangle_{n\in\mathbb{N}}$ has a subsequence $\langle f'_n\rangle_{n\in\mathbb{N}}$ converging a.e. to $f$, and this witnesses that $A$ fails to satisfy the condition.

**(b)** This follows immediately, because $A\subseteq L^0$ is closed iff $\{f:f^\bullet\in A\}$ is closed in $\mathcal{L}^0$.

**245M Complex $L^0$** In 241J I briefly discussed the adaptations needed to construct the complex linear space $L^0_{\mathbb{C}}$. The formulae of 245A may be used unchanged to define topologies of convergence in measure on $\mathcal{L}^0_{\mathbb{C}}$ and $L^0_{\mathbb{C}}$. I think that every word of 245B-245L still applies if we replace each $L^0$ or $\mathcal{L}^0$ with $L^0_{\mathbb{C}}$ or $\mathcal{L}^0_{\mathbb{C}}$. Alternatively, to relate the 'real' and 'complex' forms of 245E, for instance, we can observe that because

$$\begin{aligned}\max(\rho_F(\mathcal{R}\mathrm{e}(u),\mathcal{R}\mathrm{e}(v)),\rho_F(\mathcal{I}\mathrm{m}(u),\mathcal{I}\mathrm{m}(v)))&\le\rho_F(u,v)\\&\le\rho_F(\mathcal{R}\mathrm{e}(u),\mathcal{R}\mathrm{e}(v))+\rho_F(\mathcal{I}\mathrm{m}(u),\mathcal{I}\mathrm{m}(v))\end{aligned}$$

for all $u$, $v\in L^0$ and all sets $F$ of finite measure, $L^0_{\mathbb{C}}$ can be identified, as uniform space, with $L^0\times L^0$, so is Hausdorff, or metrizable, or complete iff $L^0$ is.

**245X Basic exercises >(a)** Let $X$ be any set, and $\mu$ counting measure on $X$. Show that the topology of convergence in measure on $\mathcal{L}^0(\mu) = \mathbb{R}^X$ is just the product topology on $\mathbb{R}^X$ regarded as a product of copies of $\mathbb{R}$.

**>(b)** Let $(X, \Sigma, \mu)$ be any measure space, and $(X, \hat{\Sigma}, \hat{\mu})$ its completion. Show that the topologies of convergence in measure on $\mathcal{L}^0(\mu) = \mathcal{L}^0(\hat{\mu})$ (241Xb), corresponding to the families $\{\rho_F : F \in \Sigma,\, \mu F < \infty\}$, $\{\rho_F : F \in \hat{\Sigma},\, \hat{\mu} F < \infty\}$ are the same.

**>(c)** Let $(X, \Sigma, \mu)$ be any measure space; set $L^0 = L^0(\mu)$. Let $u$, $u_n \in L^0$ for $n \in \mathbb{N}$. Show that the following are equiveridical:

(i) $\langle u_n \rangle_{n\in\mathbb{N}}$ order*-converges to $u$ in the sense of 245C;

(ii) there are measurable functions $f$, $f_n : X \to \mathbb{R}$ such that $f^\bullet = u$, $f_n^\bullet = u_n$ for every $n \in \mathbb{N}$, and $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in X$;

(iii) $u = \inf_{n\in\mathbb{N}} \sup_{m\ge n} u_m = \sup_{n\in\mathbb{N}} \inf_{m\ge n} u_m$, the infima and suprema being taken in $L^0$;

(iv) $\inf_{n\in\mathbb{N}} \sup_{m\ge n} |u - u_m| = 0$ in $L^0$;

(v) there is a non-increasing sequence $\langle v_n \rangle_{n\in\mathbb{N}}$ in $L^0$ such that $\inf_{n\in\mathbb{N}} v_n = 0$ in $L^0$ and $u - v_n \le u_n \le u + v_n$ for every $n \in \mathbb{N}$;

(vi) there are sequences $\langle v_n \rangle_{n\in\mathbb{N}}$, $\langle w_n \rangle_{n\in\mathbb{N}}$ in $L^0$ such that $\langle v_n \rangle_{n\in\mathbb{N}}$ is non-decreasing, $\langle w_n \rangle_{n\in\mathbb{N}}$ is non-increasing, $\sup_{n\in\mathbb{N}} v_n = u = \inf_{n\in\mathbb{N}} w_n$ and $v_n \le u_n \le w_n$ for every $n \in \mathbb{N}$.

**(d)** Let $(X, \Sigma, \mu)$ be a semi-finite measure space. Show that a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $L^0 = L^0(\mu)$ is order*-convergent to $u \in L^0$ iff $\{|u_n| : n \in \mathbb{N}\}$ is bounded above in $L^0$ and $\langle \sup_{m\ge n} |u_m - u| \rangle_{n\in\mathbb{N}} \to 0$ for the topology of convergence in measure.

**(e)** Write out proofs that $L^0(\mu)$ is complete (as linear topological space) adapted to the special cases (i) $\mu X = 1$ (ii) $\mu$ is $\sigma$-finite, taking advantage of any simplifications you can find.

**(f)** Let $(X, \Sigma, \mu)$ be a measure space and $r \ge 1$; let $h : \mathbb{R}^r \to \mathbb{R}$ be a continuous function. (i) Suppose that for $1 \le k \le r$ we are given a sequence $\langle f_{kn} \rangle_{n\in\mathbb{N}}$ in $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ converging in measure to $f_k \in \mathcal{L}^0$. Show that $\langle h(f_{1n}, \dots, f_{kn}) \rangle_{n\in\mathbb{N}}$ converges in measure to $h(f_1, \dots, f_k)$. (ii) Generally, show that $(f_1, \dots, f_r) \mapsto h(f_1, \dots, f_r) : (\mathcal{L}^0)^r \to \mathcal{L}^0$ is continuous for the topology of convergence in measure. (iii) Show that the corresponding function $\bar{h} : (L^0)^r \to L^0$ (241Xh) is continuous for the topology of convergence in measure.

**(g)** Let $(X, \Sigma, \mu)$ be a measure space and $u \in L^1(\mu)$. Show that $v \mapsto \int u \times v : L^\infty \to \mathbb{R}$ is continuous for the topology of convergence in measure on the unit ball of $L^\infty$, but not, as a rule, on the whole of $L^\infty$.

**(h)** Let $(X, \Sigma, \mu)$ be a measure space and $v$ a non-negative member of $L^1 = L^1(\mu)$. Show that on the set $A = \{u : u \in L^1,\, |u| \le v\}$ the subspace topologies (2A3C) induced by the norm topology of $L^1$ and the topology of convergence in measure are the same. (*Hint*: given $\epsilon > 0$, take $F \in \Sigma$ of finite measure and $M \ge 0$ such that $\int (|v| - M\chi F^\bullet)^+ \le \epsilon$. Show that $\|u - u'\|_1 \le \epsilon + M\bar{\rho}_F(u, u')$ for all $u$, $u' \in A$.)

**(i)** Let $(X, \Sigma, \mu)$ be a measure space and $\mathcal{F}$ a filter on $L^1 = L^1(\mu)$ which is convergent, for the topology of convergence in measure, to $u \in L^1$. Show that $\mathcal{F} \to u$ for the norm topology of $L^1$ iff $\inf_{A\in\mathcal{F}} \sup_{v\in A} \|v\|_1 \le \|u\|_1$.

**(j)** Let $(X, \Sigma, \mu)$ be a measure space and $p \in [1, \infty[$. Suppose that $\langle u_n \rangle_{n\in\mathbb{N}}$ is a sequence in $L^p(\mu)$ which converges for $\|\,\|_p$ to $u \in L^p(\mu)$. Show that $\langle |u_n|^p \rangle_{n\in\mathbb{N}} \to |u|^p$ for $\|\,\|_1$. (*Hint*: 245G, 245Dd, 245H.)

**>(k)** Let $(X, \Sigma, \mu)$ be a semi-finite measure space and $p \in [1, \infty]$, $a \ge 0$. Show that $\{u : u \in L^p(\mu),\, \|u\|_p \le a\}$ is closed in $L^0(\mu)$ for the topology of convergence in measure.

**(l)** Let $(X, \Sigma, \mu)$ be a measure space, and $\langle u_n \rangle_{n\in\mathbb{N}}$ a sequence in $L^p = L^p(\mu)$, where $1 \le p < \infty$. Let $u \in L^p$. Show that the following are equiveridical: (i) $u = \lim_{n\to\infty} u_n$ for the norm topology of $L^p$ (ii) $\langle u_n \rangle_{n\in\mathbb{N}} \to u$ for the topology of convergence in measure and $\lim_{n\to\infty} \|u_n\|_p = \|u\|_p$ (iii) $\langle u_n \rangle_{n\in\mathbb{N}} \to u$ for the topology of convergence in measure and $\limsup_{n\to\infty} \|u_n\|_p \le \|u\|_p$.

**(m)** Let $X$ be a set and $\mu$, $\nu$ two measures on $X$ with the same measurable sets and the same negligible sets. (i) Show that $\mathcal{L}^0(\mu) = \mathcal{L}^0(\nu)$ and $L^0(\mu) = L^0(\nu)$. (ii) Show that if both $\mu$ and $\nu$ are semi-finite, then they define the same topology of convergence in measure on $\mathcal{L}^0$ and $L^0$. (*Hint*: use 215A to show that if $\mu E < \infty$ then $\mu E = \sup\{\mu F : F \subseteq E,\ \nu F < \infty\}$.)

**245Y Further exercises (a)** Let $(X, \Sigma, \mu)$ be a measure space and give $\Sigma$ the topology of convergence in measure (232Ya). Show that $\chi : \Sigma \to \mathcal{L}^0(\mu)$ is a homeomorphism between $\Sigma$ and its image $\chi[\Sigma]$ in $\mathcal{L}^0$, if $\mathcal{L}^0$ is given the topology of convergence in measure and $\chi[\Sigma]$ the subspace topology.

**(b)** Let $(X, \Sigma, \mu)$ be a measure space and $Y$ any subset of $X$; let $\mu_Y$ be the subspace measure on $Y$. Let $T : L^0(\mu) \to L^0(\mu_Y)$ be the canonical map defined by setting $T(f^\bullet) = (f{\restriction}Y)^\bullet$ for every $f \in \mathcal{L}^0(\mu)$ (241Yg). Show that $T$ is continuous for the topologies of convergence in measure on $L^0(\mu)$ and $L^0(\mu_Y)$.

**(c)** Let $(X, \Sigma, \mu)$ be a measure space, and $\tilde{\mu}$ the c.l.d. version of $\mu$. Show that the map $T : L^0(\mu) \to L^0(\tilde{\mu})$ induced by the inclusion $\mathcal{L}^0(\mu) \subseteq \mathcal{L}^0(\tilde{\mu})$ (241Yf) is continuous for the topologies of convergence in measure.

**(d)** Let $(X, \Sigma, \mu)$ be a measure space, and give $L^0 = L^0(\mu)$ the topology of convergence in measure. Let $A \subseteq L^0$ be a non-empty downwards-directed set, and suppose that $\inf A = 0$ in $L^0$. (i) Let $F \in \Sigma$ be any set of finite measure, and define $\bar{\tau}_F$ as in 245A; show that $\inf_{u \in A} \bar{\tau}_F(u) = 0$. (*Hint*: set $\gamma = \inf_{u \in A} \bar{\tau}_F(u)$; find a non-increasing sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $A$ such that $\lim_{n \to \infty} \bar{\tau}_F(u_n) = \gamma$; set $v = (\chi F)^\bullet \wedge \inf_{n \in \mathbb{N}} u_n$ and show that $u \wedge v = v$ for every $u \in A$, so that $v = 0$.) (ii) Show that if $U$ is any open set containing 0, there is a $u \in A$ such that $v \in U$ whenever $0 \le v \le u$.

**(e)** Let $(X, \Sigma, \mu)$ be a measure space. (i) Show that for $u \in L^0 = L^0(\mu)$ we may define $\psi_a(u)$, for $a \ge 0$, by setting $\psi_a(u) = \mu\{x : |f(x)| \ge a\}$ whenever $f : X \to \mathbb{R}$ is a measurable function and $f^\bullet = u$. (ii) Define $\rho : L^0 \times L^0 \to [0, 1]$ by setting $\rho(u, v) = \min(\{1\} \cup \{a : a \ge 0,\ \psi_a(u - v) \le a\}$. Show that $\rho$ is a metric on $L^0$, that $L^0$ is complete under $\rho$, and that $+, -, \wedge, \vee : L^0 \times L^0 \to L^0$ are continuous for $\rho$. (iii) Show that $c \mapsto cu : \mathbb{R} \to L^0$ is continuous for every $u \in L^0$ iff $(X, \Sigma, \mu)$ is totally finite, and that in this case $\rho$ defines the topology of convergence in measure on $L^0$.

**(f)** Let $(X, \Sigma, \mu)$ be a localizable measure space and $A \subseteq L^0 = L^0(\mu)$ a non-empty upwards-directed set which is bounded in the linear topological space sense (i.e., such that for every neighbourhood $U$ of 0 in $L^0$ there is a $k \in \mathbb{N}$ such that $A \subseteq kU$). Show that $A$ is bounded above in $L^0$, and that its supremum belongs to its closure.

**(g)** Let $(X, \Sigma, \mu)$ be a measure space, $p \in [1, \infty[$ and $v$ a non-negative member of $L^p = L^p(\mu)$. Show that on the set $A = \{u : u \in L^p,\ |u| \le v\}$ the subspace topologies induced by the norm topology of $L^p$ and the topology of convergence in measure are the same.

**(h)** Let $S$ be the set of all sequences $s : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n \to \infty} s(n) = \infty$. For every $s \in S$, let $(X_s, \Sigma_s, \mu_s)$ be $[0, 1]$ with Lebesgue measure, and let $(X, \Sigma, \mu)$ be the direct sum of $\langle (X_s, \Sigma_s, \mu_s) \rangle_{s \in S}$ (214L). For $s \in S$, $t \in [0, 1]$, $n \in \mathbb{N}$ set $h_n(s, t) = f_{s(n)}(t)$, where $\langle f_n \rangle_{n \in \mathbb{N}}$ is the sequence of 245Cc. Show that $\langle h_n \rangle_{n \in \mathbb{N}} \to 0$ for the topology of convergence in measure on $\mathcal{L}^0(\mu)$, but that $\langle h_n \rangle_{n \in \mathbb{N}}$ has no subsequence which is convergent to 0 almost everywhere.

**(i)** Let $X$ be a set, and suppose we are given a relation $\rightharpoonup$ between sequences in $X$ and members of $X$ such that ($\alpha$) if $x_n = x$ for every $n$ then $\langle x_n \rangle_{n \in \mathbb{N}} \rightharpoonup x$ ($\beta$) $\langle x'_n \rangle_{n \in \mathbb{N}} \rightharpoonup x$ whenever $\langle x_n \rangle_{n \in \mathbb{N}} \rightharpoonup x$ and $\langle x'_n \rangle_{n \in \mathbb{N}}$ is a subsequence of $\langle x_n \rangle_{n \in \mathbb{N}}$. Show that we have a topology $\mathfrak{T}$ on $X$ defined by saying that a subset $G$ of $X$ belongs to $\mathfrak{T}$ iff whenever $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in $X$ and $\langle x_n \rangle_{n \in \mathbb{N}} \rightharpoonup x \in G$ then some $x_n$ belongs to $G$. Show that a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in $X$ is $\mathfrak{T}$-convergent to $x$ iff every subsequence of $\langle x_n \rangle_{n \in \mathbb{N}}$ has a sub-subsequence $\langle x''_n \rangle_{n \in \mathbb{N}}$ such that $\langle x''_n \rangle_{n \in \mathbb{N}} \rightharpoonup x$.

**(j)** Let $\mu$ be Lebesgue measure on $\mathbb{R}^r$. Show that $L^0(\mu)$ is separable for the topology of convergence in measure. (*Hint*: 244I.)

**245 Notes and comments** In this section I am inviting you to regard the topology of (local) convergence in measure as the standard topology on $L^0$, just as the norms define the standard topologies on $L^p$ spaces for $p \geq 1$. The definition I have chosen is designed to make addition and scalar multiplication and the operations $\vee$, $\wedge$ and $\times$ continuous (245D); see also 245Xf. From the point of view of functional analysis these properties are more important than metrizability or even completeness.

Just as the algebraic and order structure of $L^0$ can be described in terms of the general theory of Riesz spaces, the more advanced results 241G and 245E also have interpretations in the general theory. It is not an accident that (for semi-finite measure spaces) $L^0$ is Dedekind complete iff it is complete as uniform space; you may find the relevant generalizations in 23K and 24E of Fremlin 74. Of course it is exactly because the two kinds of completeness are interrelated that I feel it necessary to use the phrase 'Dedekind completeness' to distinguish this particular kind of order-completeness from the more familiar uniformity-completeness described in 2A5F.

The usefulness of the topology of convergence in measure derives in large part from 245G-245J and the $L^p$ versions 245Xk and 245Xl. Some of the ideas here can be related to a question arising out of the basic convergence theorems. If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of integrable functions converging (pointwise) to a function $f$, in what ways can $\int f$ fail to be $\lim_{n\to\infty} \int f_n$? In the language of this section, this translates into: if we have a sequence (or filter) in $L^1$ converging for the topology of convergence in measure, in what ways can it fail to converge for the norm topology of $L^1$? The first answer is Lebesgue's Dominated Convergence Theorem: this cannot happen if the sequence is dominated, that is, lies within some set of the form $\{u : |u| \leq v\}$ where $v \in L^1$. (See 245Xh and 245Yg.) I will return to this in the next section. For the moment, though, 245H tells us that if $\langle u_n \rangle_{n\in\mathbb{N}}$ converges in measure to $u \in L^1$, but not for the topology of $L^1$, it is because $\limsup_{n\to\infty} \|u_n\|_1$ is too big; some of its weight is being lost at infinity, as in the examples of 245I. If $\langle u_n \rangle_{n\in\mathbb{N}}$ actually order*-converges to $u$, then Fatou's Lemma tells us that $\liminf_{n\to\infty} \|u_n\|_1 \geq \|u\|_1$, that is, that the limit cannot have greater weight (as measured by $\| \; \|_1$) than the sequence provides. 245J and 245Xk are generalizations of this to convergence in measure. If you want a generalization of B.Levi's theorem, then 242Yf remains the best expression in the language of this chapter; but 245Yf is a version in terms of the concepts of the present section.

In the case of $\sigma$-finite spaces, we have an alternative description of the topology of convergence in measure (245L) which makes no use of any of the functionals or pseudo-metrics in 245A. This can be expressed, at least in the context of $L^0$, in terms of a standard result from general topology (245Yi). You will see that that result gives a recipe for a topology on $L^0$ which could be applied in any measure space. What is remarkable is that for $\sigma$-finite spaces we get a linear space topology.

Version of 17.11.06

## 246 Uniform integrability

The next topic is a fairly specialized one, but it is of great importance, for different reasons, in both probability theory and functional analysis, and it therefore seems worth while giving a proper treatment straight away.

**246A Definition** Let $(X, \Sigma, \mu)$ be a measure space.

(a) A set $A \subseteq \mathcal{L}^1(\mu)$ is **uniformly integrable** if for every $\epsilon > 0$ we can find a set $E \in \Sigma$, of finite measure, and an $M \geq 0$ such that

$$\int (|f| - M\chi E)^+ \leq \epsilon \text{ for every } f \in A.$$

(b) A set $A \subseteq L^1(\mu)$ is **uniformly integrable** if for every $\epsilon > 0$ we can find a set $E \in \Sigma$, of finite measure, and an $M \geq 0$ such that

$$\int (|u| - M\chi E^\bullet)^+ \leq \epsilon \text{ for every } u \in A.$$

**246B Remarks (a)** Recall the formulae from 241Ef: $u^+ = u \vee 0$, so $(u - v)^+ = u - u \wedge v$.

**(b)** The phrase 'uniformly integrable' is not particularly helpful. But of course we can observe that for any particular integrable function $f$, there are simple functions approximating $f$ for $\| \|_1$ (242M), and such functions will be bounded (in modulus) by functions of the form $M\chi E$, with $\mu E < \infty$; thus singleton subsets of $\mathcal{L}^1$ and $L^1$ are uniformly integrable. A general uniformly integrable set of functions is one in which $M$ and $E$ can be chosen uniformly over the set.

**(c)** It will I hope be clear from the definitions that $A \subseteq \mathcal{L}^1$ is uniformly integrable iff $\{f^\bullet : f \in A\} \subseteq L^1$ is uniformly integrable.

**(d)** There is a useful simplification in the definition if $\mu X < \infty$ (in particular, if $(X, \Sigma, \mu)$ is a probability space). In this case a set $A \subseteq L^1(\mu)$ is uniformly integrable iff

$$\inf_{M \ge 0} \sup_{u \in A} \int (|u| - Me)^+ = 0$$

iff

$$\lim_{M \to \infty} \sup_{u \in A} \int (|u| - Me)^+ = 0,$$

writing $e = \chi X^\bullet \in L^1(\mu)$. (For if $\sup_{u \in A} \int (|u| - M\chi E^\bullet)^+ \le \epsilon$, then $\int (|u| - M'e)^+ \le \epsilon$ for every $M' \ge M$.) Similarly, $A \subseteq \mathcal{L}^1(\mu)$ is uniformly integrable iff

$$\lim_{M \to \infty} \sup_{f \in A} \int (|f| - M\chi X)^+ = 0$$

iff

$$\inf_{M \ge 0} \sup_{f \in A} \int (|f| - M\chi X)^+ = 0.$$

**Warning!** Some authors use the phrase 'uniformly integrable' for sets satisfying the conditions in (d) even when $\mu$ is not totally finite.

**246C** We have the following wide-ranging stability properties of the class of uniformly integrable sets in $L^1$ or $\mathcal{L}^1$.

**Proposition** Let $(X, \Sigma, \mu)$ be a measure space and $A$ a uniformly integrable subset of $L^1(\mu)$.

(a) $A$ is bounded for the norm $\| \|_1$.

(b) Any subset of $A$ is uniformly integrable.

(c) For any $a \in \mathbb{R}$, $aA = \{au : u \in A\}$ is uniformly integrable.

(d) There is a uniformly integrable $C \supseteq A$ such that $C$ is convex and $\| \|_1$-closed and $v \in C$ whenever $u \in C$ and $|v| \le |u|$.

(e) If $B$ is another uniformly integrable subset of $L^1$, then $A \cup B$ and $A + B = \{u + v : u \in A, v \in B\}$ are uniformly integrable.

**proof** Write $\Sigma^f$ for $\{E : E \in \Sigma, \mu E < \infty\}$.

**(a)** There must be $E \in \Sigma^f$, $M \ge 0$ such that $\int (|u| - M\chi E^\bullet)^+ \le 1$ for every $u \in A$; now

$$\|u\|_1 \le \int (|u| - M\chi E^\bullet)^+ + \int M\chi E^\bullet \le 1 + M\mu E$$

for every $u \in A$, so $A$ is bounded.

**(b)** This is immediate from the definition 246Ab.

**(c)** Given $\epsilon > 0$, we can find $E \in \Sigma^f$, $M \ge 0$ such that $|a| \int_E (|u| - M\chi E^\bullet)^+ \le \epsilon$ for every $u \in A$; now $\int_E (|v| - |a|M\chi E^\bullet)^+ \le \epsilon$ for every $v \in aA$.

**(d)** If $A$ is empty, take $C = A$. Otherwise, try

$$C = \{v : v \in L^1, \int (|v| - w)^+ \le \sup_{u \in A} \int (|u| - w)^+ \text{ for every } w \in L^1(\mu)\}.$$

Evidently $A \subseteq C$, and $C$ satisfies the definition 246Ab because $A$ does, considering $w$ of the form $M\chi E^\bullet$ where $E \in \Sigma^f$ and $M \ge 0$. The functionals

$$v \mapsto \int (|v| - w)^+ : L^1(\mu) \to \mathbb{R}$$

are all continuous for $\| \|_1$ (because the operators $v \mapsto |v|$, $v \mapsto v-w$, $v \mapsto v^+$, $v \mapsto \int v$ are continuous), so $C$ is closed. If $|v'| \le |v|$ and $v \in C$, then

$$\int(|v'|-w)^+ \le \int(|v|-w)^+ \le \sup_{u\in A}\int(|u|-w)^+$$

for every $w$, and $v' \in C$. If $v = av_1 + bv_2$ where $v_1$, $v_2 \in C$, $a \in [0,1]$ and $b = 1-a$, then $|v| \le a|v_1| + b|v_2|$, so

$$|v| - w \le (a|v_1| - aw) + (b|v_2| - bw) \le (a|v_1| - aw)^+ + (b|v_2| - bw)^+$$

and

$$(|v|-w)^+ \le a(|v_1|-w)^+ + b(|v_2|-w)^+$$

for every $w$; accordingly

$$\begin{aligned}\int(|v|-w)^+ &\le a\int(|v_1|-w)^+ + b\int(|v_2|-w)^+\\ &\le (a+b)\sup_{u\in A}\int(|u|-w)^+ = \sup_{u\in A}\int(|u|-w)^+\end{aligned}$$

for every $w$, and $v \in C$.

Thus $C$ has all the required properties.

**(e)** I show first that $A \cup B$ is uniformly integrable. **P** Given $\epsilon > 0$, let $M_1$, $M_2 \ge 0$ and $E_1$, $E_2 \in \Sigma^f$ be such that

$$\int(|u| - M_1\chi E_1^\bullet)^+ \le \epsilon \text{ for every } u \in A,$$

$$\int(|u| - M_2\chi E_2^\bullet)^+ \le \epsilon \text{ for every } u \in B.$$

Set $M = \max(M_1, M_2)$, $E = E_1 \cup E_2$; then $\mu E < \infty$ and

$$\int(|u| - M\chi E^\bullet)^+ \le \epsilon \text{ for every } u \in A \cup B.$$

As $\epsilon$ is arbitrary, $A \cup B$ is uniformly integrable. **Q**

Now (d) tells us that there is a convex uniformly integrable set $C$ including $A \cup B$, and in this case $A + B \subseteq 2C$, so $A + B$ is also uniformly integrable, using (b) and (c).

**246D Proposition** Let $(X, \Sigma, \mu)$ be a probability space and $A \subseteq L^1(\mu)$ a uniformly integrable set. Then there is a convex, $\| \|_1$-closed uniformly integrable set $C \subseteq L^1$ such that $A \subseteq C$, $w \in C$ whenever $v \in C$ and $|w| \le |v|$, and $Pv \in C$ whenever $v \in C$ and $P$ is the conditional expectation operator associated with a $\sigma$-subalgebra of $\Sigma$.

**proof** Set

$$C = \{v : v \in L^1(\mu),\ \int(|v| - Me)^+ \le \sup_{u\in A}\int(|u| - Me)^+ \text{ for every } M \ge 0\},$$

writing $e = \chi X^\bullet$ as usual. The arguments in the proof of 246Cd make it plain that $C \supseteq A$ is uniformly integrable, convex and closed, and that $w \in C$ whenever $v \in C$ and $|w| \le |v|$. As for the conditional expectation operators, if $v \in C$, T is a $\sigma$-subalgebra of $\Sigma$, $P$ is the associated conditional expectation operator, and $M \ge 0$, then

$$|Pv| \le P|v| = P((|v| \wedge Me) + (|v| - Me)^+) \le Me + P((|v| - Me)^+),$$

so

$$(|Pv| - Me)^+ \le P((|v| - Me)^+)$$

and

$$\int(|Pv| - Me)^+ \le \int P(|v| - Me)^+ = \int(|v| - Me)^+ \le \sup_{u\in A}\int(|u| - Me)^+;$$

as $M$ is arbitrary, $Pv \in C$.

**246E Remarks (a)** Of course 246D has an expression in terms of $\mathcal{L}^1$ rather than $L^1$: if $(X,\Sigma,\mu)$ is a probability space and $A\subseteq\mathcal{L}^1(\mu)$ is uniformly integrable, then there is a uniformly integrable set $C\supseteq A$ such that (i) $af+(1-a)g\in C$ whenever $f$, $g\in C$ and $a\in[0,1]$ (ii) $g\in C$ whenever $f\in C$, $g\in\mathcal{L}^0(\mu)$ and $|g|\le_{\text{a.e.}}|f|$ (iii) $f\in C$ whenever there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $C$ such that $\lim_{n\to\infty}\int|f-f_n|=0$ (iv) $g\in C$ whenever there is an $f\in C$ such that $g$ is a conditional expectation of $f$ with respect to some $\sigma$-subalgebra of $\Sigma$.

**(b)** In fact, there are obvious extensions of 246D; the proof there already shows that $T[C]\subseteq C$ whenever $T:L^1(\mu)\to L^1(\mu)$ is an order-preserving linear operator such that $\|Tu\|_1\le\|u\|_1$ for every $u\in L^1(\mu)$ and $\|Tu\|_\infty\le\|u\|_\infty$ for every $u\in L^1(\mu)\cap L^\infty(\mu)$ (246Yc). If we had done a bit more of the theory of operators on Riesz spaces I should be able to take you a good deal farther along this road; for instance, it is not in fact necessary to assume that the operators $T$ of the last sentence are order-preserving. I will return to this in Chapter 37 in the next volume.

**(c)** Moreover, the main theorem of the next section will show that for any measure spaces $(X,\Sigma,\mu)$, $(Y,\mathrm{T},\nu)$, $T[A]$ will be uniformly integrable in $L^1(\nu)$ whenever $A\subseteq L^1(\mu)$ is uniformly integrable and $T:L^1(\mu)\to L^1(\nu)$ is a continuous linear operator (247D).

**246F** We shall need an elementary lemma which I have not so far spelt out.

**Lemma** Let $(X,\Sigma,\mu)$ be a measure space. Then for any $u\in L^1(\mu)$,

$$\|u\|_1\le 2\sup_{E\in\Sigma}|\int_E u|.$$

**proof** Express $u$ as $f^\bullet$ where $f:X\to\mathbb{R}$ is measurable. Set $F=\{x:f(x)\ge0\}$. Then

$$\|u\|_1=\int|f|=|\int_F f|+|\int_{X\setminus F}f|\le 2\sup_{E\in\Sigma}|\int_E f|=2\sup_{E\in\Sigma}|\int_E u|.$$

**246G** Now we come to some of the remarkable alternative descriptions of uniform integrability.

**Theorem** Let $(X,\Sigma,\mu)$ be any measure space and $A$ a non-empty subset of $L^1(\mu)$. Then the following are equiveridical:

(i) $A$ is uniformly integrable;

(ii) $\sup_{u\in A}|\int_F u|<\infty$ for every $\mu$-atom $F\in\Sigma$, and for every $\epsilon>0$ there are $E\in\Sigma$, $\delta>0$ such that $\mu E<\infty$ and $|\int_F u|\le\epsilon$ whenever $u\in A$, $F\in\Sigma$ and $\mu(F\cap E)\le\delta$;

(iii) $\sup_{u\in A}|\int_F u|<\infty$ for every $\mu$-atom $F\in\Sigma$, and $\lim_{n\to\infty}\sup_{u\in A}|\int_{F_n}u|=0$ whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\Sigma$;

(iv) $\sup_{u\in A}|\int_F u|<\infty$ for every $\mu$-atom $F\in\Sigma$, and $\lim_{n\to\infty}\sup_{u\in A}|\int_{F_n}u|=0$ whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\Sigma$ with empty intersection.

**Remark** I use the phrase '$\mu$-atom' to emphasize that I mean an atom in the measure space sense (211I).

**proof (a)(i)⇒(iv)** Suppose that $A$ is uniformly integrable. Then surely if $F\in\Sigma$ is a $\mu$-atom,

$$\sup_{u\in A}|\int_F u|\le\sup_{u\in A}\|u\|_1<\infty,$$

by 246Ca. Now suppose that $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\Sigma$ with empty intersection, and that $\epsilon>0$. Take $E\in\Sigma$, $M\ge0$ such that $\mu E<\infty$ and $\int(|u|-M\chi E^\bullet)^+\le\frac12\epsilon$ whenever $u\in A$. Then for all $n$ large enough, $M\mu(F_n\cap E)\le\frac12\epsilon$, so that

$$|\int_{F_n}u|\le\int_{F_n}|u|\le\int(|u|-M\chi E^\bullet)^++\int_{F_n}M\chi E^\bullet\le\frac{\epsilon}{2}+M\mu(F_n\cap E)\le\epsilon$$

for every $u\in A$. As $\epsilon$ is arbitrary, $\lim_{n\to\infty}\sup_{u\in A}|\int_{F_n}u|=0$, and (iv) is true.

**(b)(iv)⇒(iii)** Suppose that (iv) is true. Then of course $\sup_{u\in A}|\int_F u|<\infty$ for every $\mu$-atom $F\in\Sigma$. **?** Suppose, if possible, that $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\Sigma$ such that

$$\epsilon=\limsup_{n\to\infty}\sup_{u\in A}\min(1,\tfrac13|\int_{F_n}u|)>0.$$

Set $H_n = \bigcup_{i \ge n} F_i$ for each $n$, so that $\langle H_n \rangle_{n\in\mathbb{N}}$ is non-increasing and has empty intersection, and $\int_{H_n} u \to 0$ as $n \to \infty$ for every $u \in L^1(\mu)$. Choose $\langle n_i \rangle_{i\in\mathbb{N}}$, $\langle m_i \rangle_{i\in\mathbb{N}}$, $\langle u_i \rangle_{i\in\mathbb{N}}$ inductively, as follows. $n_0 = 0$. Given $n_i \in \mathbb{N}$, take $m_i \ge n_i$, $u_i \in A$ such that $|\int_{F_{m_i}} u_i| \ge 2\epsilon$. Take $n_{i+1} > m_i$ such that $\int_{H_{n_{i+1}}} |u_i| \le \epsilon$. Continue.

Set $G_k = \bigcup_{i \ge k} F_{m_i}$ for each $k$. Then $\langle G_k \rangle_{k\in\mathbb{N}}$ is a non-increasing sequence in $\Sigma$ with empty intersection. But $F_{m_i} \subseteq G_i \subseteq F_{m_i} \cup H_{n_{i+1}}$, so

$$|\int_{G_i} u_i| \ge |\int_{F_{m_i}} u_i| - |\int_{G_i \setminus F_{m_i}} u_i| \ge 2\epsilon - \int_{H_{n_{i+1}}} |u_i| \ge \epsilon$$

for every $i$, contradicting the hypothesis (iv). **X**

This means that $\lim_{n\to\infty} \sup_{u\in A} |\int_{F_n} u|$ must be zero, and (iii) is true.

**(c)(iii)⇒(ii)** We still have $\sup_{u\in A} |\int_F u| < \infty$ for every $\mu$-atom $F$. **?** Suppose, if possible, that there is an $\epsilon > 0$ such that for every measurable set $E$ of finite measure and every $\delta > 0$ there are $u \in A$, $F \in \Sigma$ such that $\mu(F \cap E) \le \delta$ and $|\int_F u| \ge \epsilon$. Choose a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of sets of finite measure, a sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ in $\Sigma$, a sequence $\langle \delta_n \rangle_{n\in\mathbb{N}}$ of strictly positive real numbers and a sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in $A$ as follows. Given $u_k$, $E_k$, $\delta_k$ for $k < n$, choose $u_n \in A$ and $G_n \in \Sigma$ such that $\mu(G_n \cap \bigcup_{k<n} E_k) \le 2^{-n} \min(\{1\} \cup \{\delta_k : k < n\})$ and $|\int_{G_n} u_n| \ge \epsilon$; then choose a set $E_n$ of finite measure and a $\delta_n > 0$ such that $\int_F |u_n| \le \frac{1}{2}\epsilon$ whenever $F \in \Sigma$ and $\mu(F \cap E_n) \le \delta_n$ (see 225A). Continue.

On completing the induction, set $F_n = E_n \cap G_n \setminus \bigcup_{k>n} G_k$ for each $n$; then $\langle F_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\Sigma$. By the choice of $G_k$,

$$\mu(E_n \cap \bigcup_{k>n} G_k) \le \sum_{k=n+1}^{\infty} 2^{-k}\delta_n \le \delta_n,$$

so $\mu(E_n \cap (G_n \setminus F_n)) \le \delta_n$ and $\int_{G_n \setminus F_n} |u_n| \le \frac{1}{2}\epsilon$. This means that $|\int_{F_n} u_n| \ge |\int_{G_n} u_n| - \frac{1}{2}\epsilon \ge \frac{1}{2}\epsilon$. But this is contrary to the hypothesis (iii). **X**

**(d)(ii)⇒(i)($\alpha$)** Assume (ii). Let $\epsilon > 0$. Then there are $E \in \Sigma$, $\delta > 0$ such that $\mu E < \infty$ and $|\int_F u| \le \epsilon$ whenever $u \in A$, $F \in \Sigma$ and $\mu(F \cap E) \le \delta$. Now $\sup_{u\in A} \int_E |u| < \infty$. **P** Write $\mathcal{I}$ for the family of those $F \in \Sigma$ such that $F \subseteq E$ and $\sup_{u\in A} \int_F |u|$ is finite. If $F \subseteq E$ is an atom for $\mu$, then $\sup_{u\in A} \int_F |u| = \sup_{u\in A} |\int_F u| < \infty$, so $F \in \mathcal{I}$. (The point is that if $f : X \to \mathbb{R}$ is a measurable function such that $f^\bullet = u$, then one of $F' = \{x : x \in F, f(x) \ge 0\}$, $F'' = \{x : x \in F, f(x) < 0\}$ must be negligible, so that $\int_F |u|$ is either $\int_{F'} u = \int_F u$ or $-\int_{F''} u = -\int_F u$.) If $F \in \Sigma$, $F \subseteq E$ and $\mu F \le \delta$ then

$$\sup_{u\in A} \int_F |u| \le 2 \sup_{u\in A, G\in\Sigma, G\subseteq F} |\int_G u| \le 2\epsilon$$

(by 246F), so $F \in \mathcal{I}$. Next, if $F$, $G \in \mathcal{I}$ then $\sup_{u\in A} \int_{F\cup G} |u| \le \sup_{u\in A} \int_F |u| + \sup_{u\in A} \int_G |u|$ is finite, so $F \cup G \in \mathcal{I}$. Finally, if $\langle F_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{I}$, and $F = \bigcup_{n\in\mathbb{N}} F_n$, there is some $n \in \mathbb{N}$ such that $\mu(F \setminus \bigcup_{i\le n} F_i) \le \delta$; now $\bigcup_{i\le n} F_i$ and $F \setminus \bigcup_{i\le n} F_i$ both belong to $\mathcal{I}$, so $F \in \mathcal{I}$.

By 215Ab, there is an $F \in \mathcal{I}$ such that $H \setminus F$ is negligible for every $H \in \mathcal{I}$. Now observe that $E \setminus F$ cannot include any non-negligible member of $\mathcal{I}$; in particular, cannot include either an atom or a non-negligible set of measure less than $\delta$. But this means that the subspace measure on $E \setminus F$ is atomless, totally finite and has no non-negligible measurable sets of measure less than $\delta$; by 215D, $\mu(E \setminus F) = 0$ and $E \setminus F$ and $E$ belong to $\mathcal{I}$, as required. **Q**

Since $\int_{X\setminus E} |u| \le \delta$ for every $u \in A$, $\gamma = \sup_{u\in A} \int |u|$ is finite.

**($\beta$)** Set $M = \gamma/\delta$. If $u \in A$, express $u$ as $f^\bullet$, where $f : X \to \mathbb{R}$ is measurable, and consider

$$F = \{x : f(x) \ge M\chi E(x)\}.$$

Then

$$M\mu(F \cap E) \le \int_F f = \int_F u \le \gamma,$$

so $\mu(F \cap E) \le \gamma/M = \delta$. Accordingly $\int_F u \le \epsilon$. Similarly, $\int_{F'} (-u) \le \epsilon$, writing $F' = \{x : -f(x) \ge M\chi E(x)\}$. But this means that

$$\int (|u| - M\chi E^\bullet)^+ = \int (|f| - M\chi E)^+ \le \int_{F\cup F'} |f| = \int_{F\cup F'} |u| \le 2\epsilon$$

for every $u \in A$. As $\epsilon$ is arbitrary, $A$ is uniformly integrable.

**246H Remarks (a)** Of course conditions (ii)-(iv) of this theorem, like (i), have direct translations in terms of members of $\mathcal{L}^1$. Thus a non-empty set $A \subseteq \mathcal{L}^1$ is uniformly integrable iff $\sup_{f\in A} |\int_F f|$ is finite for every atom $F \in \Sigma$ and

*either* for every $\epsilon > 0$ we can find $E \in \Sigma$, $\delta > 0$ such that $\mu E < \infty$ and $|\int_F f| \le \epsilon$ whenever $f \in A$, $F \in \Sigma$ and $\mu(F \cap E) \le \delta$

*or* $\lim_{n\to\infty} \sup_{f\in A} |\int_{F_n} f| = 0$ for every disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\Sigma$

*or* $\lim_{n\to\infty} \sup_{f\in A} |\int_{F_n} f| = 0$ for every non-increasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\Sigma$ with empty intersection.

**(b)** There are innumerable further equivalent expressions characterizing uniform integrability; every author has his own favourite. Many of them are variants on (i)-(iv) of this theorem, as in 246I and 246Yd-246Yf. For a condition of a quite different kind, see Theorem 247C.

**246I Corollary** Let $(X,\Sigma,\mu)$ be a probability space. For $f \in \mathcal{L}^0(\mu)$, $M \ge 0$ set $F(f,M) = \{x : x \in \operatorname{dom} f, |f(x)| \ge M\}$. Then a non-empty set $A \subseteq \mathcal{L}^1(\mu)$ is uniformly integrable iff

$$\lim_{M\to\infty} \sup_{f\in A} \int_{F(f,M)} |f| = 0.$$

**proof (a)** If $A$ satisfies the condition, then

$$\inf_{M\ge 0} \sup_{f\in A} \int (|f| - M\chi X)^+ \le \inf_{M\ge 0} \sup_{f\in A} \int_{F(f,M)} |f| = 0,$$

so $A$ is uniformly integrable.

**(b)** If $A$ is uniformly integrable, and $\epsilon > 0$, there is an $M_0 \ge 0$ such that $\int (|f| - M_0\chi X)^+ \le \epsilon$ for every $f \in A$; also, $\gamma = \sup_{f\in A} \int |f|$ is finite (246Ca). Take any $M \ge M_0 \max(1, (1+\gamma)/\epsilon)$. If $f \in A$, then

$$|f| \times \chi F(f,M) \le (|f| - M_0\chi X)^+ + M_0\chi F(f,M) \le (|f| - M_0\chi X)^+ + \tfrac{\epsilon}{\gamma+1}|f|$$

everywhere on $\operatorname{dom} f$, so

$$\int_{F(f,M)} |f| \le \int (|f| - M_0\chi X)^+ + \tfrac{\epsilon}{\gamma+1}\int |f| \le 2\epsilon.$$

As $\epsilon$ is arbitrary, $\lim_{M\to\infty} \sup_{f\in A} \int_{F(f,M)} |f| = 0$.

**246J** The next step is to set out some remarkable connexions between uniform integrability and the topology of convergence in measure discussed in the last section.

**Theorem** Let $(X,\Sigma,\mu)$ be a measure space.

(a) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly integrable sequence of real-valued functions on $X$, and $f(x) = \lim_{n\to\infty} f_n(x)$ for almost every $x \in X$, then $f$ is integrable and $\lim_{n\to\infty} \int |f_n - f| = 0$; consequently $\int f = \lim_{n\to\infty} \int f_n$.

(b) If $A \subseteq L^1 = L^1(\mu)$ is uniformly integrable, then the norm topology of $L^1$ and the topology of convergence in measure of $L^0 = L^0(\mu)$ agree on $A$.

(c) For any $u \in L^1$ and any sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $L^1$, the following are equiveridical:

(i) $u = \lim_{n\to\infty} u_n$ for $\| \|_1$;

(ii) $\{u_n : n \in \mathbb{N}\}$ is uniformly integrable and $\langle u_n\rangle_{n\in\mathbb{N}}$ converges to $u$ in measure.

(d) If $(X,\Sigma,\mu)$ is semi-finite, and $A \subseteq L^1$ is uniformly integrable, then the closure $\overline{A}$ of $A$ in $L^0$ for the topology of convergence in measure is still a uniformly integrable subset of $L^1$.

**proof (a)** Note first that because $\sup_{n\in\mathbb{N}} \int |f_n| < \infty$ (246Ca) and $|f| = \liminf_{n\to\infty} |f_n|$, Fatou's Lemma assures us that $|f|$ is integrable, with $\int |f| \le \limsup_{n\to\infty} \int |f_n|$. It follows immediately that $\{f_n - f : n \in \mathbb{N}\}$ is uniformly integrable, being the sum of two uniformly integrable sets (246Cc, 246Ce).

Given $\epsilon > 0$, there are $M \ge 0$, $E \in \Sigma$ such that $\mu E < \infty$ and $\int (|f_n - f| - M\chi E)^+ \le \epsilon$ for every $n \in \mathbb{N}$. Also $|f_n - f| \wedge M\chi E \to 0$ a.e., so

$$\limsup_{n\to\infty}\int|f_n-f|\le\limsup_{n\to\infty}\int(|f_n-f|-M\chi E)^+ + \limsup_{n\to\infty}\int|f_n-f|\wedge M\chi E \le\epsilon,$$

by Lebesgue's Dominated Convergence Theorem. As $\epsilon$ is arbitrary, $\lim_{n\to\infty}\int|f_n-f|=0$ and $\lim_{n\to\infty}\int f_n-f=0$.

**(b)** Let $\mathfrak{T}_A$, $\mathfrak{S}_A$ be the topologies on $A$ induced by the norm topology of $L^1$ and the topology of convergence in measure on $L^0$ respectively.

**(i)** Given $\epsilon>0$, let $F\in\Sigma$, $M\ge 0$ be such that $\mu F<\infty$ and $\int(|v|-M\chi F^\bullet)^+\le\epsilon$ for every $v\in A$, and consider $\bar\rho_F$, defined as in 245A. Then for any $f$, $g\in\mathcal{L}^0$,

$$|f-g|\le(|f|-M\chi F)^+ + (|g|-M\chi F)^+ + M(|f-g|\wedge\chi F)$$

everywhere on $\operatorname{dom}f\cap\operatorname{dom}g$, so

$$|u-v|\le(|u|-M\chi F^\bullet)^+ + (|v|-M\chi F^\bullet)^+ + M(|u-v|\wedge\chi F^\bullet)$$

for all $u$, $v\in L^0$. Consequently

$$\|u-v\|_1\le 2\epsilon+M\bar\rho_F(u,v)$$

for all $u$, $v\in A$.

This means that, given $\epsilon>0$, we can find $F$, $M$ such that, for $u$, $v\in A$,

$$\bar\rho_F(u,v)\le\tfrac{\epsilon}{1+M}\implies\|u-v\|_1\le 3\epsilon.$$

It follows that every subset of $A$ which is open for $\mathfrak{T}_A$ is open for $\mathfrak{S}_A$ (2A3Ib).

**(ii)** In the other direction, we have $\bar\rho_F(u,v)\le\|u-v\|_1$ for every $u\in L^1$ and every set $F$ of finite measure, so every subset of $A$ which is open for $\mathfrak{S}_A$ is open for $\mathfrak{T}_A$.

**(c)** If $\langle u_n\rangle_{n\in\mathbb{N}}\to u$ for $\|\;\|_1$, $A=\{u_n:n\in\mathbb{N}\}$ is uniformly integrable. **P** Given $\epsilon>0$, let $m$ be such that $\|u_n-u\|_1\le\epsilon$ whenever $n\ge m$. Set $v=|u|+\sum_{i\le m}|u_i|\in L^1$, and let $M\ge 0$, $E\in\Sigma$ be such that $\mu E$ is finite and $\int_E(v-M\chi E^\bullet)^+\le\epsilon$. Then, for $w\in A$,

$$(|w|-M\chi E^\bullet)^+\le(|w|-v)^+ + (v-M\chi E^\bullet)^+,$$

so

$$\int_E(|w|-M\chi E^\bullet)^+\le\|(|w|-v)^+\|_1+\int_E(v-M\chi E^\bullet)^+\le 2\epsilon.\ \mathbf{Q}$$

Thus on either hypothesis we can be sure that $\{u_n:n\in\mathbb{N}\}$ and $A=\{u\}\cup\{u_n:n\in\mathbb{N}\}$ are uniformly integrable, so that the two topologies agree on $A$ (by (b)) and $\langle u_n\rangle_{n\in\mathbb{N}}$ converges to $u$ in one topology iff it converges to $u$ in the other.

**(d)** Because $A$ is $\|\;\|_1$-bounded (246Ca) and $\mu$ is semi-finite, $\overline{A}\subseteq L^1$ (245J(b-i)). Given $\epsilon>0$, let $M\ge 0$, $E\in\Sigma$ be such that $\mu E<\infty$ and $\int(|u|-M\chi E^\bullet)^+\le\epsilon$ for every $u\in A$. Now the maps $u\mapsto|u|$, $u\mapsto u-M\chi E^\bullet$, $u\mapsto u^+:L^0\to L^0$ are all continuous for the topology of convergence in measure (245D), while $\{u:\|u\|_1\le\epsilon\}$ is closed for the same topology (245J again), so $\{u:u\in L^0,\ \int(|u|-M\chi E^\bullet)^+\le\epsilon\}$ is closed and must include $\overline{A}$. Thus $\int(|u|-M\chi E^\bullet)^+\le\epsilon$ for every $u\in\overline{A}$. As $\epsilon$ is arbitrary, $\overline{A}$ is uniformly integrable.

**246K Complex $\mathcal{L}^1$ and $L^1$** The definitions and theorems above can be repeated without difficulty for spaces of (equivalence classes of) complex-valued functions, with just one variation: in the complex equivalent of 246F, the constant must be changed. It is easy to see that, for $u\in L^1_{\mathbb{C}}(\mu)$,

$$\|u\|_1\le\|\mathcal{R}\mathrm{e}(u)\|_1+\|\mathcal{I}\mathrm{m}(u)\|_1 \le 2\sup_{F\in\Sigma}|\int_F\mathcal{R}\mathrm{e}(u)|+2\sup_{F\in\Sigma}|\int_F\mathcal{I}\mathrm{m}(u)|\le 4\sup_{F\in\Sigma}|\int_F u|.$$

(In fact, $\|u\|_1 \le \pi \sup_{F\in\Sigma} |\int_F u|$; see 246Yl and 252Yt.) Consequently some of the arguments of 246G need to be written out with different constants, but the results, as stated, are unaffected.

**246X Basic exercises (a)** Let $(X,\Sigma,\mu)$ be a measure space and $A$ a subset of $L^1 = L^1(\mu)$. Show that the following are equiveridical: (i) $A$ is uniformly integrable; (ii) for every $\epsilon > 0$ there is a $w \ge 0$ in $L^1$ such that $\int(|u| - w)^+ \le \epsilon$ for every $u \in A$; (iii) $\langle(|u_{n+1}| - \sup_{i\le n} |u_i|)^+\rangle_{n\in\mathbb{N}} \to 0$ in $L^1$ for every sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $A$. (*Hint*: for (ii)⇒(iii), set $v_n = \sup_{i\le n} |u_i|$ and note that $\langle v_n \wedge w\rangle_{n\in\mathbb{N}}$ is convergent in $L^1$ for every $w \ge 0$.)

**>(b)** Let $(X,\Sigma,\mu)$ be a totally finite measure space. Show that for any $p > 1$ and $M \ge 0$ the set $\{f : f \in \mathcal{L}^p(\mu), \|f\|_p \le M\}$ is uniformly integrable. (*Hint*: $\int(|f| - M\chi X)^+ \le M^{1-p} \int |f|^p$.)

**>(c)** Let $\mu$ be counting measure on $\mathbb{N}$. Show that a set $A \subseteq \mathcal{L}^1(\mu) = \ell^1$ is uniformly integrable iff (i) $\sup_{f\in A} |f(n)| < \infty$ for every $n \in \mathbb{N}$ (ii) for every $\epsilon > 0$ there is an $m \in \mathbb{N}$ such that $\sum_{n=m}^{\infty} |f(n)| \le \epsilon$ for every $f \in A$.

**(d)** Let $X$ be a set, and let $\mu$ be counting measure on $X$. Show that a set $A \subseteq \mathcal{L}^1(\mu) = \ell^1(X)$ is uniformly integrable iff (i) $\sup_{f\in A} |f(x)| < \infty$ for every $x \in X$ (ii) for every $\epsilon > 0$ there is a finite set $I \subseteq X$ such that $\sum_{x\in X\setminus I} |f(x)| \le \epsilon$ for every $f \in A$. Show that in this case $A$ is relatively compact for the norm topology of $\ell^1(X)$.

**(e)** Let $(X,\Sigma,\mu)$ be a measure space, $\delta > 0$, and $\mathcal{I} \subseteq \Sigma$ a family such that (i) every atom belongs to $\mathcal{I}$ (ii) $E \in \mathcal{I}$ whenever $E \in \Sigma$ and $\mu E \le \delta$ (iii) $E \cup F \in \mathcal{I}$ whenever $E$, $F \in \mathcal{I}$ and $E \cap F = \emptyset$. Show that every set of finite measure belongs to $\mathcal{I}$.

**(f)** Let $(X,\Sigma,\mu)$ and $(Y,\mathrm{T},\nu)$ be measure spaces and $\phi : X \to Y$ an inverse-measure-preserving function. Show that a set $A \subseteq \mathcal{L}^1(\nu)$ is uniformly integrable iff $\{g\phi : g \in A\}$ is uniformly integrable in $\mathcal{L}^1(\mu)$. (*Hint*: use 246G for 'if', 246A for 'only if'.)

**>(g)** Let $(X,\Sigma,\mu)$ be a measure space and $p \in [1,\infty[$. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^p = \mathcal{L}^p(\mu)$ such that $\{|f_n|^p : n \in \mathbb{N}\}$ is uniformly integrable and $f_n \to f$ a.e. Show that $f \in \mathcal{L}^p$ and $\lim_{n\to\infty} \int |f_n - f|^p = 0$.

**(h)** Let $(X,\Sigma,\mu)$ be a semi-finite measure space and $p \in [1,\infty[$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a sequence in $L^p = L^p(\mu)$ and $u \in L^0(\mu)$. Show that the following are equiveridical: (i) $u \in L^p$ and $\langle u_n\rangle_{n\in\mathbb{N}}$ converges to $u$ for $\| \|_p$ (ii) $\langle u_n\rangle_{n\in\mathbb{N}}$ converges in measure to $u$ and $\{|u_n|^p : n \in \mathbb{N}\}$ is uniformly integrable. (*Hint*: 245Xl.)

**(i)** Let $(X,\Sigma,\mu)$ be a totally finite measure space, and $1 \le p < r \le \infty$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$ be a $\| \|_r$-bounded sequence in $L^r(\mu)$ which converges in measure to $u \in L^0(\mu)$. Show that $\langle u_n\rangle_{n\in\mathbb{N}}$ converges to $u$ for $\| \|_p$. (*Hint*: show that $\{|u_n|^p : n \in \mathbb{N}\}$ is uniformly integrable.)

**246Y Further exercises (a)** Let $(X,\Sigma,\mu)$ be a totally finite measure space. Show that $A \subseteq \mathcal{L}^1(\mu)$ is uniformly integrable iff there is a convex function $\phi : [0,\infty[ \to \mathbb{R}$ such that $\lim_{a\to\infty} \phi(a)/a = \infty$ and $\sup_{f\in A} \int \phi(|f|) < \infty$.

**(b)** For any metric space $(Z,\rho)$, let $\mathcal{C}_Z$ be the family of closed subsets of $Z$, and for $F$, $F' \in \mathcal{C}_Z \setminus \{\emptyset\}$ set $\tilde\rho(F,F') = \min(1, \max(\sup_{z\in F} \inf_{z'\in F'} \rho(z,z'), \sup_{z'\in F'} \inf_{z\in F} \rho(z,z')))$. Show that $\tilde\rho$ is a metric on $\mathcal{C}_Z \setminus \{\emptyset\}$ (it is the **Hausdorff metric**). Show that if $(Z,\rho)$ is complete then the family $\mathcal{K}_Z \setminus \{\emptyset\}$ of non-empty compact subsets of $Z$ is closed for $\tilde\rho$. Now let $(X,\Sigma,\mu)$ be any measure space and take $Z = L^1 = L^1(\mu)$, $\rho(z,z') = \|z - z'\|_1$ for $z$, $z' \in Z$. Show that the family of non-empty closed uniformly integrable subsets of $L^1$ is a closed subset of $\mathcal{C}_Z \setminus \{\emptyset\}$ including $\mathcal{K}_Z \setminus \{\emptyset\}$.

**(c)** Let $(X,\Sigma,\mu)$ be a totally finite measure space and $A \subseteq L^1(\mu)$ a uniformly integrable set. Show that there is a uniformly integrable set $C \supseteq A$ such that (i) $C$ is convex and closed in $L^0(\mu)$ for the topology of convergence in measure (ii) if $u \in C$ and $|v| \le |u|$ then $v \in C$ (iii) if $T$ belongs to the set $\mathcal{T}^+$ of operators from $L^1(\mu) = M^{1,\infty}(\mu)$ to itself, as described in 244Xm, then $T[C] \subseteq C$.

**(d)** Let $\mu$ be Lebesgue measure on $\mathbb{R}$. Show that a set $A \subseteq \mathcal{L}^1(\mu)$ is uniformly integrable iff $\lim_{n\to\infty} \int_{F_n} f_n = 0$ for every disjoint sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ of compact sets in $\mathbb{R}$ and every sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $A$.

**(e)** Let $\mu$ be Lebesgue measure on $\mathbb{R}$. Show that a set $A \subseteq \mathcal{L}^1(\mu)$ is uniformly integrable iff $\lim_{n\to\infty} \int_{G_n} f_n = 0$ for every disjoint sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ of open sets in $\mathbb{R}$ and every sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $A$.

**(f)** Repeat 246Yd and 246Ye for Lebesgue measure on arbitrary subsets of $\mathbb{R}^r$.

**(g)** Let $X$ be a set and $\Sigma$ a $\sigma$-algebra of subsets of $X$. Let $\langle \nu_n \rangle_{n\in\mathbb{N}}$ be a sequence of countably additive functionals on $\Sigma$ such that $\nu E = \lim_{n\to\infty} \nu_n E$ is defined for every $E \in \Sigma$. Show that $\lim_{n\to\infty} \nu_n F_n = 0$ whenever $\langle F_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\Sigma$. (*Hint*: suppose otherwise. By taking suitable subsequences reduce to the case in which $|\nu_n F_i - \nu F_i| \le 2^{-n}\epsilon$ for $i < n$, $|\nu_n F_n| \ge 3\epsilon$, $|\nu_n F_i| \le 2^{-i}\epsilon$ for $i > n$. Set $F = \bigcup_{i\in\mathbb{N}} F_{2i+1}$ and show that $|\nu_{2n+1} F - \nu_{2n} F| \ge \epsilon$ for every $n$.)

**(h)** Let $(X, \Sigma, \mu)$ be a measure space and $\langle u_n \rangle_{n\in\mathbb{N}}$ a sequence in $L^1 = L^1(\mu)$ such that $\lim_{n\to\infty} \int_F u_n$ is defined for every $F \in \Sigma$. Show that $\{u_n : n \in \mathbb{N}\}$ is uniformly integrable. (*Hint*: suppose not. Then there are a disjoint sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ in $\Sigma$ and a subsequence $\langle u'_n \rangle_{n\in\mathbb{N}}$ of $\langle u_n \rangle_{n\in\mathbb{N}}$ such that $\inf_{n\in\mathbb{N}} |\int_{F_n} u'_n| = \epsilon > 0$. But this contradicts 246Yg.)

**(i)** In 246Yg, show that $\nu$ is countably additive. (*Hint*: Set $\mu = \sum_{n=0}^{\infty} a_n \nu_n$ for a suitable sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ of strictly positive numbers. For each $n$ choose a Radon-Nikodým derivative $f_n$ of $\nu_n$ with respect to $\mu$. Show that $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable, so that $\nu$ is truly continuous.)(This is the **Vitali-Hahn-Saks theorem**.)

**(j)** Let $(X, \Sigma, \mu)$ be any measure space, and $A \subseteq L^1(\mu)$. Show that the following are equiveridical: (i) $A$ is $\| \|_1$-bounded; (ii) $\sup_{u\in A} |\int_F u| < \infty$ for every $\mu$-atom $F \in \Sigma$ and $\limsup_{n\to\infty} \sup_{u\in A} |\int_{F_n} u| < \infty$ for every disjoint sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure; (iii) $\sup_{u\in A} |\int_E u| < \infty$ for every $E \in \Sigma$. (*Hint*: show that $\langle a_n u_n \rangle_{n\in\mathbb{N}}$ is uniformly integrable whenever $\lim_{n\to\infty} a_n = 0$ in $\mathbb{R}$ and $\langle u_n \rangle_{n\in\mathbb{N}}$ is a sequence in $A$.)

**(k)** Let $(X, \Sigma, \mu)$ be a measure space and $A \subseteq L^1(\mu)$ a non-empty set. Show that the following are equiveridical: (i) $A$ is uniformly integrable; (ii) whenever $B \subseteq L^\infty(\mu)$ is non-empty and downwards-directed and has infimum 0 in $L^\infty(\mu)$ then $\inf_{v\in B} \sup_{u\in A} |\int u \times v| = 0$. (*Hint*: for (i)$\Rightarrow$(ii), note that $\inf_{v\in B} w \times v = 0$ for every $w \ge 0$ in $L^0$. For (ii)$\Rightarrow$(i), use 246G(iv).)

**(l)** Set $f(x) = e^{ix}$ for $x \in [-\pi, \pi]$. Show that $|\int_E f| \le 2$ for every $E \subseteq [-\pi, \pi]$.

**246 Notes and comments** I am holding over to the next section the most striking property of uniformly integrable sets (they are the relatively weakly compact sets in $L^1$) because this demands some non-trivial ideas from functional analysis and general topology. In this section I give the results which can be regarded as essentially measure-theoretic in inspiration. The most important new concept, or technique, is that of 'disjoint-sequence theorem'. A typical example is in condition (iii) of 246G, relating uniform integrability to the behaviour of functionals on disjoint sequences of sets. I give variants of this in 246Yd-246Yf, and 246Yg-246Yj are further results in which similar methods can be used. The central result of the next section (247C) will also use disjoint sequences in the proof, and they will appear more than once in Chapter 35 in the next volume.

The phrase 'uniformly integrable' ought to mean something like 'uniformly approximable by simple functions', and the definition 246A can be forced into such a form, but I do not think it very useful to do so. However condition (ii) of 246G amounts to something like 'uniformly truly continuous', if we think of members of $L^1$ as truly continuous functionals on $\Sigma$, as in 242I. (See 246Yi.) Note that in each of the statements (ii)-(iv) of 246G we need to take special note of any atoms for the measure, since they are not controlled by the main condition imposed. In an atomless measure space, of course, we have a simplification here, as in 246Yd-246Yf.

Another way of justifying the 'uniformly' in 'uniformly integrable' is by considering functionals $\theta_w$ where $w \ge 0$ in $L^1$, setting $\theta_w(u) = \int (|u| - w)^+$ for $u \in L^1$; then $A \subseteq L^1$ is uniformly integrable iff $\theta_w \to 0$

uniformly on $A$ as $w$ rises in $L^1$ (246Xa). It is sometimes useful to know that if this is true at all then it is necessarily witnessed by elements $w$ which can be built directly from materials at hand (see (iii) of 246Xa). Furthermore, the sets $A_{w\epsilon} = \{u : \theta_w(u) \le \epsilon\}$ are always convex, $\| \|_1$-closed and 'solid' (if $u \in A_{w\epsilon}$ and $|v| \le |u|$ then $v \in A_{w\epsilon}$)(246Cd); they are closed under pointwise convergence of sequences (246Ja) and in semi-finite measure spaces they are closed for the topology of convergence in measure (246Jd); in probability spaces, for level $w$, they are closed under conditional expectations (246D) and similar operators (246Yc). Consequently we can expect that any uniformly integrable set will be included in a uniformly integrable set which is closed under operations of several different types.

Yet another 'uniform' property of uniformly integrable sets is in 246Yk. The norm $\| \|_\infty$ is never (in interesting cases) order-continuous in the way that other $\| \|_p$ are (244Ye); but the uniformly integrable subsets of $L^1$ provide interesting order-continuous seminorms on $L^\infty$.

246J supplements results from §245. In the notes to that section I mentioned the question: if $\langle f_n\rangle_{n\in\mathbb{N}} \to f$ a.e., in what ways can $\langle\int f_n\rangle_{n\in\mathbb{N}}$ fail to converge to $\int f$? Here we find that $\langle\int|f_n-f|\rangle_{n\in\mathbb{N}} \to 0$ iff $\{f_n : n \in \mathbb{N}\}$ is uniformly integrable; this is a way of making precise the expression 'none of the weight of the sequence is lost at infinity'. Generally, for sequences, convergence in $\| \|_p$, for $p \in [1,\infty[$, is convergence in measure for $p$th-power-uniformly-integrable sequences (246Xh).

Version of 26.8.13

## 247 Weak compactness in $L^1$

I now come to the most striking feature of uniform integrability: it provides a description of the relatively weakly compact subsets of $L^1$ (247C). I have put this into a separate section because it demands some knowledge of functional analysis – in particular, of course, of weak topologies on Banach spaces. I will try to give an account in terms which are accessible to novices in the theory of normed spaces because the result is essentially measure-theoretic, as well as being of vital importance to applications in probability theory. I have written out the essential definitions in §§2A3-2A5.

**247A** Part of the argument of the main theorem below will run more smoothly if I separate out an idea which is, in effect, a simple special case of a theme which has been running through the exercises of this chapter (241Yg, 242Yb, 243Ya, 244Yd).

**Lemma** Let $(X, \Sigma, \mu)$ be a measure space, and $G$ any member of $\Sigma$. Let $\mu_G$ be the subspace measure on $G$, so that $\mu_G E = \mu E$ when $E \subseteq G$ and $E \in \Sigma$. Set

$$U = \{u : u \in L^1(\mu),\ u \times \chi G^\bullet = u\} \subseteq L^1(\mu).$$

Then we have an isomorphism $S$ between the ordered normed spaces $U$ and $L^1(\mu_G)$, given by writing

$$S(f^\bullet) = (f\restriction G)^\bullet$$

for every $f \in \mathcal{L}^1(\mu)$ such that $f^\bullet \in U$.

**proof** Of course I should remark explicitly that $U$ is a linear subspace of $L^1(\mu)$. I have discussed integration over subspaces in §§131 and 214; in particular, I noted that $f\restriction G$ is integrable, and that

$$\int |f\restriction G| d\mu_G = \int |f| \times \chi G\, d\mu \le \int |f| d\mu$$

for every $f \in \mathcal{L}^1(\mu)$ (131Fa). If $f$, $g \in \mathcal{L}^1(\mu)$ and $f = g$ $\mu$-a.e., then $f\restriction G = g\restriction G$ $\mu_G$-a.e.; so the proposed formula for $S$ does indeed define a map from $U$ to $L^1(\mu_G)$.

Because

$$(f + g)\restriction G = (f\restriction G) + (g\restriction G), \quad (cf)\restriction G = c(f\restriction G)$$

for all $f$, $g \in \mathcal{L}^1(\mu)$ and all $c \in \mathbb{R}$, $S$ is linear. Because

$$f \le g \ \mu\text{-a.e.} \Longrightarrow f\restriction G \le g\restriction G \ \mu_G\text{-a.e.},$$

$S$ is order-preserving. Because $\int |f\restriction G| d\mu_G \le \int |f| d\mu$ for every $f \in \mathcal{L}^1(\mu)$, $\|Su\|_1 \le \|u\|_1$ for every $u \in U$.

To see that $S$ is surjective, take any $v \in L^1(\mu_G)$. Express $v$ as $g^\bullet$ where $g \in \mathcal{L}^1(\mu_G)$. By 131E, $f \in \mathcal{L}^1(\mu)$, where $f(x) = g(x)$ for $x \in \operatorname{dom} g$, 0 for $x \in X \setminus G$; so that $f^\bullet \in U$ and $f\restriction G = g$ and $v = S(f^\bullet) \in S[U]$.

To see that $S$ is norm-preserving, note that, for any $f \in \mathcal{L}^1(\mu)$,

$$\int |f{\restriction}G| d\mu_G = \int |f| \times \chi G \, d\mu,$$

so that if $u = f^\bullet \in U$ we shall have

$$\|Su\|_1 = \int |f{\restriction}G| d\mu_G = \int |f| \times \chi G \, d\mu = \|u \times \chi G^\bullet\|_1 = \|u\|_1.$$

**247B Corollary** Let $(X, \Sigma, \mu)$ be any measure space, and let $G \in \Sigma$ be a measurable set expressible as a countable union of sets of finite measure. Define $U$ as in 247A, and let $h : L^1(\mu) \to \mathbb{R}$ be any continuous linear functional. Then there is a $v \in L^\infty(\mu)$ such that $h(u) = \int u \times v \, d\mu$ for every $u \in U$.

**proof** Let $S : U \to L^1(\mu_G)$ be the isomorphism described in 247A. Then $S^{-1} : L^1(\mu_G) \to U$ is linear and continuous, so $h_1 = hS^{-1}$ belongs to the normed space dual $(L^1(\mu_G))^*$ of $L^1(\mu_G)$. Now of course $\mu_G$ is $\sigma$-finite, therefore localizable (211L), so 243Gb tells us that there is a $v_1 \in L^\infty(\mu_G)$ such that

$$h_1(u) = \int u \times v_1 d\mu_G$$

for every $u \in L^1(\mu_G)$.

Express $v_1$ as $g_1^\bullet$ where $g_1 : G \to \mathbb{R}$ is a bounded measurable function. Set $g(x) = g_1(x)$ for $x \in G$, 0 for $x \in X \setminus G$; then $g : X \to \mathbb{R}$ is a bounded measurable function, and $v = g^\bullet \in L^\infty(\mu)$. If $u \in U$, express $u$ as $f^\bullet$ where $f \in \mathcal{L}^1(\mu)$; then

$$h(u) = h(S^{-1}Su) = h_1((f{\restriction}G)^\bullet) = \int (f{\restriction}G) \times g_1 d\mu_G$$
$$= \int (f \times g){\restriction}G \, d\mu_G = \int f \times g \times \chi G \, d\mu = \int f \times g \, d\mu = \int u \times v.$$

As $u$ is arbitrary, this proves the result.

**247C Theorem** Let $(X, \Sigma, \mu)$ be any measure space and $A$ a subset of $L^1 = L^1(\mu)$. Then $A$ is uniformly integrable iff it is relatively compact in $L^1$ for the weak topology of $L^1$.

**proof (a)** Suppose that $A$ is relatively compact for the weak topology. I seek to show that it satisfies the condition (iii) of 246G.

**(i)** If $F \in \Sigma$, then surely $\sup_{u \in A} |\int_F u| < \infty$, because $u \mapsto \int_F u$ belongs to $(L^1)^*$, and if $h \in (L^1)^*$ then the image of any relatively weakly compact set under $h$ must be bounded (2A5Ie).

**(ii)** Now suppose that $\langle F_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\Sigma$. **?** Suppose, if possible, that

$$\langle \sup_{u \in A} |\int_{F_n} u| \rangle_{n \in \mathbb{N}}$$

does not converge to 0. Then there is a strictly increasing sequence $\langle n(k) \rangle_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that

$$\gamma = \frac{1}{2} \inf_{k \in \mathbb{N}} \sup_{u \in A} |\int_{F_{n(k)}} u| > 0.$$

For each $k$, choose $u_k \in A$ such that $|\int_{F_{n(k)}} u_k| \ge \gamma$. Because $A$ is relatively compact for the weak topology, there is a cluster point $u$ of $\langle u_k \rangle_{k \in \mathbb{N}}$ in $L^1$ for the weak topology (2A3Ob). Set $\eta_j = 2^{-j}\gamma/6 > 0$ for each $j \in \mathbb{N}$.

We can now choose a strictly increasing sequence $\langle k(j) \rangle_{j \in \mathbb{N}}$ inductively so that, for each $j$,

$$\int_{F_{n(k(j))}} (|u| + \sum_{i=0}^{j-1} |u_{k(i)}|) \le \eta_j$$

$$\sum_{i=0}^{j-1} |\int_{F_{n(k(i))}} u - \int_{F_{n(k(i))}} u_{k(j)}| \le \eta_j$$

for every $j$, interpreting $\sum_{i=0}^{-1}$ as 0. **P** Given $\langle k(i) \rangle_{i<j}$, set $v^* = |u| + \sum_{i=0}^{j-1} |u_{k(i)}|$; then $\lim_{k \to \infty} \int_{F_{n(k)}} v^* = 0$, by Lebesgue's Dominated Convergence Theorem or otherwise, so there is a $k^*$ such that $k^* > k(i)$ for every $i < j$ and $\int_{F_{n(k)}} v^* \le \eta_j$ for every $k \ge k^*$. Next,

$$w \mapsto \sum_{i=0}^{j-1} |\int_{F_{n(k(i))}} u - \int_{F_{n(k(i))}} w| : L^1 \to \mathbb{R}$$

is continuous for the weak topology of $L^1$ and zero at $u$, and $u$ belongs to every weakly open set containing $\{u_k : k \ge k^*\}$, so there is a $k(j) \ge k^*$ such that $\sum_{i=0}^{j-1} |\int_{F_{n(k(i))}} u - \int_{F_{n(k(i))}} u_{k(j)}| < \eta_j$, which continues the construction. **Q**

Let $v$ be any cluster point in $L^1$, for the weak topology, of $\langle u_{k(j)}\rangle_{j\in\mathbb{N}}$. Setting $G_i = F_{n(k(i))}$, we have $|\int_{G_i} u - \int_{G_i} u_{k(j)}| \le \eta_j$ whenever $i < j$, so $\lim_{j\to\infty} \int_{G_i} u_{k(j)}$ exists $= \int_{G_i} u$ for each $i$, and $\int_{G_i} v = \int_{G_i} u$ for every $i$; setting $G = \bigcup_{i\in\mathbb{N}} G_i$,

$$\int_G v = \sum_{i=0}^{\infty} \int_{G_i} v = \sum_{i=0}^{\infty} \int_{G_i} u = \int_G u,$$

by 232D, because $\langle G_i\rangle_{i\in\mathbb{N}}$ is disjoint.

For each $j \in \mathbb{N}$,

$$\begin{aligned}\sum_{i=0}^{j-1} |\int_{G_i} u_{k(j)}| &+ \sum_{i=j+1}^{\infty} |\int_{G_i} u_{k(j)}| \\ &\le \sum_{i=0}^{j-1} \int_{G_i} |u| + \sum_{i=0}^{j-1} |\int_{G_i} u - \int_{G_i} u_{k(j)}| + \sum_{i=j+1}^{\infty} \int_{G_i} |u_{k(j)}| \\ &\le \sum_{i=0}^{j-1} \eta_i + \eta_j + \sum_{i=j+1}^{\infty} \eta_i = \sum_{i=0}^{\infty} \eta_i = \frac{\gamma}{3}.\end{aligned}$$

On the other hand, $|\int_{G_j} u_{k(j)}| \ge \gamma$. So

$$|\int_G u_{k(j)}| = |\sum_{i=0}^{\infty} \int_{G_i} u_{k(j)}| \ge \frac{2}{3}\gamma.$$

This is true for every $j$; because every weakly open set containing $v$ meets $\{u_{k(j)} : j \in \mathbb{N}\}$, $|\int_G v| \ge \frac{2}{3}\gamma$ and $|\int_G u| \ge \frac{2}{3}\gamma$. On the other hand,

$$|\int_G u| = |\sum_{i=0}^{\infty} \int_{G_i} u| \le \sum_{i=0}^{\infty} \int_{G_i} |u| \le \sum_{i=0}^{\infty} \eta_i = \frac{\gamma}{3},$$

which is absurd. **X**

This contradiction shows that $\lim_{n\to\infty} \sup_{u\in A} |\int_{F_n} u| = 0$. As $\langle F_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $A$ satisfies the condition 246G(iii) and is uniformly integrable.

**(b)** Now assume that $A$ is uniformly integrable. I seek a weakly compact set $C \supseteq A$.

**(i)** For each $n \in \mathbb{N}$, choose $E_n \in \Sigma$, $M_n \ge 0$ such that $\mu E_n < \infty$ and $\int (|u| - M_n \chi E_n^\bullet)^+ \le 2^{-n}$ for every $u \in A$. Set

$$C = \{v : v \in L^1, |\int_F v| \le M_n \mu(F \cap E_n) + 2^{-n} \ \forall\, n \in \mathbb{N},\, F \in \Sigma\},$$

and note that $A \subseteq C$, because if $u \in A$ and $F \in \Sigma$,

$$|\int_F u| \le \int_F (|u| - M_n \chi E_n^\bullet)^+ + \int_F M_n \chi E_n^\bullet \le 2^{-n} + M_n \mu(F \cap E_n)$$

for every $n$. Observe also that $C$ is $\| \|_1$-bounded, because

$$\|u\|_1 \le 2 \sup_{F\in\Sigma} |\int_F u| \le 2 \sup_{F\in\Sigma} (1 + M_0 \mu(F \cap E_0)) \le 2(1 + M_0 \mu E_0)$$

for every $u \in C$ (using 246F).

**(ii)** Because I am seeking to prove this theorem for arbitrary measure spaces $(X, \Sigma, \mu)$, I cannot use 243G to identify the dual of $L^1$. Nevertheless, 247B above shows that 243Gb it is 'nearly' valid, in the following sense: if $h \in (L^1)^*$, there is a $v \in L^\infty$ such that $h(u) = \int u \times v$ for every $u \in C$. **P** Set $G = \bigcup_{n\in\mathbb{N}} E_n \in \Sigma$, and define $U \subseteq L^1$ as in 247A-247B. By 247B, there is a $v \in L^\infty$ such that $h(u) = \int u \times v$ for every $u \in U$. But if $u \in C$, we can express $u$ as $f^\bullet$ where $f : X \to \mathbb{R}$ is measurable. If $F \in \Sigma$ and $F \cap G = \emptyset$, then

$$|\int_F f| = |\int_F u| \le 2^{-n} + M_n\mu(F\cap E_n) = 2^{-n}$$

for every $n\in\mathbb{N}$, so $\int_F f = 0$; it follows that $f=0$ a.e. on $X\setminus G$ (131Fc), so that $f\times\chi G =_{\text{a.e.}} f$ and $u = u\times\chi G^\bullet$, that is, $u\in U$, and $h(u)=\int u\times v$, as required. **Q**

**(iii)** So we may proceed, having an adequate description, not of $(L^1(\mu))^*$ itself, but of its action on $C$. Let $\mathcal{F}$ be any ultrafilter on $L^1$ containing $C$ (see 2A3R). For each $F\in\Sigma$, set

$$\nu F = \lim_{u\to\mathcal{F}}\int_F u;$$

because

$$\sup_{u\in C}|\int_F u| \le \sup_{u\in C}\|u\|_1 < \infty,$$

this is well-defined in $\mathbb{R}$ (2A3S(e-ii)). If $E$, $F$ are disjoint members of $\Sigma$, then $\int_{E\cup F} u = \int_E u + \int_F u$ for every $u\in C$, so

$$\nu(E\cup F) = \lim_{u\to\mathcal{F}}\int_{E\cup F} u = \lim_{u\to\mathcal{F}}\int_E u + \lim_{u\to\mathcal{F}}\int_F u = \nu E + \nu F$$

(2A3Sf). Thus $\nu:\Sigma\to\mathbb{R}$ is additive. Next, it is truly continuous with respect to $\mu$. **P** Given $\epsilon>0$, take $n\in\mathbb{N}$ such that $2^{-n}\le\frac{1}{2}\epsilon$, set $\delta=\epsilon/2(M_n+1)>0$ and observe that

$$|\nu F| \le \sup_{u\in C}|\int_F u| \le 2^{-n} + M_n\mu(F\cap E_n) \le \epsilon$$

whenever $\mu(F\cap E_n)\le\delta$. **Q** By the Radon-Nikodým theorem (232E), there is an $f_0\in\mathcal{L}^1$ such that $\int_F f_0 = \nu F$ for every $F\in\Sigma$. Set $u_0 = f_0^\bullet\in L^1$. If $n\in\mathbb{N}$, $F\in\Sigma$ then

$$|\int_F u_0| = |\nu F| \le \sup_{u\in C}|\int_F u| \le 2^{-n} + M_n\mu(F\cap E_n),$$

so $u_0\in C$.

**(iv)** Of course the point is that $\mathcal{F}$ converges to $u_0$. **P** Let $h\in(L^1)^*$. Then there is a $v\in L^\infty$ such that $h(u)=\int u\times v$ for every $u\in C$. Express $v$ as $g^\bullet$, where $g:X\to\mathbb{R}$ is bounded and $\Sigma$-measurable. Let $\epsilon>0$. Take $a_0\le a_1\le\ldots\le a_n$ such that $a_{i+1}-a_i\le\epsilon$ for each $i$ while $a_0\le g(x)<a_n$ for each $x\in X$. Set $F_i=\{x: a_{i-1}\le g(x)<a_i\}$ for $1\le i\le n$, and set $\tilde g=\sum_{i=1}^n a_i\chi F_i$, $\tilde v=\tilde g^\bullet$; then $\|\tilde v - v\|_\infty\le\epsilon$. We have

$$\begin{aligned}\int u_0\times\tilde v &= \sum_{i=1}^n a_i\int_{F_i} u = \sum_{i=1}^n a_i\nu F_i\\ &= \sum_{i=1}^n a_i\lim_{u\to\mathcal{F}}\int_{F_i} u = \lim_{u\to\mathcal{F}}\sum_{i=1}^n a_i\int_{F_i} u = \lim_{u\to\mathcal{F}}\int u\times\tilde v.\end{aligned}$$

Consequently

$$\begin{aligned}\limsup_{u\to\mathcal{F}}|\int u\times v - \int u_0\times v| &\le |\int u_0\times v - \int u_0\times\tilde v| + \sup_{u\in C}|\int u\times v - \int u\times\tilde v|\\ &\le \|u_0\|_1\|v-\tilde v\|_\infty + \sup_{u\in C}\|u\|_1\|v-\tilde v\|_\infty\\ &\le 2\epsilon\sup_{u\in C}\|u\|_1.\end{aligned}$$

As $\epsilon$ is arbitrary,

$$\limsup_{u\to\mathcal{F}}|h(u)-h(u_0)| = \limsup_{u\to\mathcal{F}}|\int u\times v - \int u_0\times v| = 0.$$

As $h$ is arbitrary, $u_0$ is a limit of $\mathcal{F}$ in $C$ for the weak topology of $L^1$. **Q**

As $\mathcal{F}$ is arbitrary, $C$ is weakly compact in $L^1$, and the proof is complete.

**247D Corollary** Let $(X,\Sigma,\mu)$ and $(Y,\mathrm{T},\nu)$ be any two measure spaces, and $T: L^1(\mu)\to L^1(\nu)$ a continuous linear operator. Then $T[A]$ is a uniformly integrable subset of $L^1(\nu)$ whenever $A$ is a uniformly integrable subset of $L^1(\mu)$.

**proof** The point is that $T$ is continuous for the respective weak topologies (2A5If). If $A \subseteq L^1(\mu)$ is uniformly integrable, then there is a weakly compact $C \supseteq A$, by 247C; $T[C]$, being the image of a compact set under a continuous map, must be weakly compact (2A3N(b-ii)); so $T[C]$ and $T[A]$ are uniformly integrable by the other half of 247C.

**247E Complex $L^1$** There are no difficulties, and no surprises, in proving 247C for $L^1_{\mathbb{C}}$. If we follow the same proof, everything works, but of course we must remember to change the constant when applying 246F, or rather 246K, in part (b-i) of the proof.

**247X Basic exercises >(a)** Let $(X, \Sigma, \mu)$ be any measure space. Show that if $A \subseteq L^1 = L^1(\mu)$ is relatively weakly compact, then $\{v : v \in L^1, |v| \le |u| \text{ for some } u \in A\}$ is relatively weakly compact.

**(b)** Let $(X, \Sigma, \mu)$ be a measure space. On $L^1 = L^1(\mu)$ define pseudometrics $\rho_F$, $\rho'_w$ for $F \in \Sigma$, $w \in L^\infty(\mu)$ by setting $\rho_F(u, v) = |\int_F u - \int_F v|$, $\rho'_w(u, v) = |\int u \times w - \int v \times w|$ for $u, v \in L^1$. Show that on any $\| \|_1$-bounded subset of $L^1$, the topology defined by $\{\rho_F : F \in \Sigma\}$ agrees with the topology generated by $\{\rho'_w : w \in L^\infty\}$.

**>(c)** Show that for any set $X$ a subset of $\ell^1 = \ell^1(X)$ is compact for the weak topology of $\ell^1$ iff it is compact for the norm topology of $\ell^1$. (*Hint*: 246Xd.)

**(d)** Use the argument of (a-ii) in the proof of 247C to show directly that if $A \subseteq \ell^1(\mathbb{N})$ is weakly compact then $\inf_{n \in \mathbb{N}} |u_n(n)| = 0$ for any sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $A$.

**(e)** Let $(X, \Sigma, \mu)$ and $(Y, \mathrm{T}, \nu)$ be measure spaces, and $T : L^2(\nu) \to L^1(\mu)$ any bounded linear operator. Show that $\{Tu : u \in L^2(\nu), \|u\|_2 \le 1\}$ is uniformly integrable in $L^1(\mu)$. (*Hint*: use 244K to see that $\{u : \|u\|_2 \le 1\}$ is weakly compact in $L^2(\nu)$.)

**247Y Further exercises (a)** Let $(X, \Sigma, \mu)$ be a measure space. Take $1 < p < \infty$ and $M \ge 0$ and set $A = \{u : u \in L^p = L^p(\mu), \|u\|_p \le M\}$. Write $\mathfrak{S}_A$ for the topology of convergence in measure on $A$, that is, the subspace topology induced by the topology of convergence in measure on $L^0(\mu)$. Show that if $h \in (L^p)^*$ then $h \restriction A$ is continuous for $\mathfrak{S}_A$; so that if $\mathfrak{T}$ is the weak topology on $L^p$, then the subspace topology $\mathfrak{T}_A$ is included in $\mathfrak{S}_A$.

**(b)** Let $(X, \Sigma, \mu)$ be a measure space and $\langle u_n \rangle_{n \in \mathbb{N}}$ a sequence in $L^1 = L^1(\mu)$ such that $\lim_{n \to \infty} \int_F u_n$ is defined for every $F \in \Sigma$. Show that $\{u_n : n \in \mathbb{N}\}$ is weakly convergent. (*Hint*: 246Yh.)

**247 Notes and comments** In 247D and 247Xa I try to suggest the power of the identification between weak compactness and uniform integrability. That a continuous image of a weakly compact set should be weakly compact is a commonplace of functional analysis; that the solid hull of a uniformly integrable set should be uniformly integrable is immediate from the definition. But I see no simple arguments to show that a continuous image of a uniformly integrable set should be uniformly integrable, or that the solid hull of a weakly compact set should be relatively weakly compact. (Concerning the former, an alternative route does exist; see 371Xf in the next volume.)

I can distinguish two important ideas in the proof of 247C. The first, in (a-ii) of the proof, is a careful manipulation of sequences; it is the argument needed to show that a weakly compact subset of $\ell^1$ is norm-compact. (You may find it helpful to write out a solution to 247Xd.) The $F_{n(k)}$ and $u_k$ are chosen to mimic the situation in which we have a sequence in $\ell^1$ such that $u_k(k) = 1$ for each $k$. The $k(i)$ are chosen so that the 'hump' moves sufficiently rapidly along for $u_{k(j)}(k(i))$ to be very small whenever $i \ne j$. But this means that $\sum_{i=0}^\infty u_{k(j)}(k(i))$ (corresponding to $\int_G u_{k(j)}$ in the proof) is always substantial, while $\sum_{i=0}^\infty v(k(i))$ will be small for any proposed cluster point $v$ of $\langle u_{k(j)} \rangle_{j \in \mathbb{N}}$. I used similar techniques in §246; compare 246Yg.

In the other half of the proof of 247C, the strategy is clearer. Members of $L^1$ correspond to truly continuous functionals on $\Sigma$; the uniform integrability of $C$ makes the corresponding set of functionals 'uniformly truly continuous', so that any limit functional will also be truly continuous and will give us a member of $L^1$ via the Radon-Nikodým theorem. A straightforward approximation argument ((b-iv) in the proof, and 247Xb) shows that $\lim_{u \in \mathcal{F}} \int u \times w = \int v \times w$ for every $w \in L^\infty$. For localizable measures $\mu$, this would complete the proof. For the general case, we need another step, here done in 247A-247B; a uniformly integrable subset of $L^1$ effectively lives on a $\sigma$-finite part of the measure space, so that we can ignore the rest of the measure and suppose that we have a localizable measure space.

The conditions (ii)-(iv) of 246G make it plain that weak compactness in $L^1$ can be effectively discussed in terms of sequences; see also 246Yh. I should remark that this is a general feature of weak compactness in Banach spaces (2A5J). Of course the disjoint-sequence formulations in 246G are characteristic of $L^1$ – I mean that while there are similar results applicable elsewhere (see Fremlin 74, chap. 8), the ideas are clearest and most dramatically expressed in their application to $L^1$.

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## Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

**241Yd Countable sup property** This exercise, referred to in the 2002 edition of Volume 3, has been moved to 241Ye.

**241Yh Quotient Riesz spaces** This exercise, referred to in the 2002 edition of Volume 3, has been moved to 241Yc.

**242Xf Inverse-measure-preserving functions** This exercise, referred to in the 2002 edition of Volume 3, has been moved to 242Xd.

**242Yc Order-continuous norms** This exercise, referred to in the 2002 edition of Volume 3, has been moved to 242Yg.

**244O Complex** $L^p$ This paragraph, referred to in the 2002 and 2004 editions of Volume 3, and the 2003 and 2006 editions of Volume 4, is now 244P.

**244Xf** $L^p$ **and** $L^q$ This exercise, referred to in the 2003 edition of Volume 4, has been moved to 244Xe.

**244Yd-244Yf** $L^p$ **as Banach lattice** These exercises, referred to in the 2002 and 2004 editions of Volume 3, are now 244Ye-244Yg.

Version of 17.12.12

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