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Chapter 23

The Radon-Nikodým Theorem

In Chapter 22, I discussed the indefinite integrals of integrable functions on $\mathbb{R}$, and gave what I hope you feel are satisfying descriptions both of the functions which are indefinite integrals (the absolutely continuous functions) and of how to find which functions they are indefinite integrals of (you differentiate them). For general measure spaces, we have no structure present which can give such simple formulations; but nevertheless the same questions can be asked and, up to a point, answered.

The first section of this chapter introduces the basic machinery needed, the concept of 'countably additive' functional and its decomposition into positive and negative parts. The main theorem takes up the second section: indefinite integrals are the 'truly continuous' additive functionals; on σ-finite spaces, these are the 'absolutely continuous' countably additive functionals. In §233 I discuss the most important single application of the theorem, its use in providing a concept of 'conditional expectation'. This is one of the central concepts of probability theory – as you very likely know; but the form here is a dramatic generalization of the elementary concept of the conditional probability of one event given another, and needs the whole strength of the general theory of measure and integration as developed in Volume 1 and this chapter. I include some notes on convex functions, up to and including versions of Jensen's inequality (233I-233J).

While we are in the area of 'pure' measure theory, I take the opportunity to discuss some further topics. I begin with some essentially elementary constructions, image measures, sums of measures and indefinite-integral measures; I think the details need a little attention, and I work through them in §234. Rather deeper ideas are needed to deal with 'measurable transformations'. In §235 I set out the techniques necessary to provide an abstract basis for a general method of integration-by-substitution, with a detailed account of sufficient conditions for a formula of the type

$$\int g(y)dy = \int g(\phi(x))J(x)dx$$

to be valid.

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231 Countably additive functionals

I begin with an abstract description of the objects which will, in appropriate circumstances, correspond to the indefinite integrals of general integrable functions. In this section I give those parts of the theory which do not involve a measure, but only a set with a distinguished σ-algebra of subsets. The basic concepts are those of 'finitely additive' and 'countably additive' functional, and there is one substantial theorem, the 'Hahn decomposition' (231E).

231A Definition Let X be a set and Σ an algebra of subsets of X. A functional $\nu : \Sigma \to \mathbb{R}$ is **finitely additive**, or just **additive**, if $\nu(E \cup F) = \nu E + \nu F$ whenever $E, F \in \Sigma$ and $E \cap F = \emptyset$.

231B Elementary facts Let X be a set, Σ an algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a finitely additive functional.

(a) $\nu\emptyset = 0$.

(b) If $E_0, \dots , E_n$ are disjoint members of Σ then $\nu(\bigcup_{i\le n} E_i) = \sum_{i=0}^n \nu E_i$.

(c) If E, $F\in\Sigma$ and $E\subseteq F$ then $\nu F=\nu E+\nu(F\setminus E)$. More generally, for any E, $F\in\Sigma$,

$$\nu F=\nu(F\cap E)+\nu(F\setminus E),$$

$$\nu E+\nu F=\nu(E\cup F)+\nu(E\cap F),$$

$$\nu E-\nu F=\nu(E\setminus F)-\nu(F\setminus E).$$

231C Definition Let X be a set and Σ an algebra of subsets of X. A function $\nu:\Sigma\to\mathbb{R}$ is **countably additive** or **σ-additive** if $\sum_{n=0}^{\infty}\nu E_n$ exists in $\mathbb{R}$ and is equal to $\nu(\bigcup_{n\in\mathbb{N}}E_n)$ for every disjoint sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ such that $\bigcup_{n\in\mathbb{N}}E_n\in\Sigma$.

231D Elementary facts Let X be a set, Σ a σ-algebra of subsets of X and $\nu:\Sigma\to\mathbb{R}$ a countably additive functional.

(a) ν is finitely additive.

(b) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ, with union $E\in\Sigma$, then

$$\nu E=\nu E_0+\sum_{n=0}^{\infty}\nu(E_{n+1}\setminus E_n)=\lim_{n\to\infty}\nu E_n.$$

(c) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with intersection $E\in\Sigma$, then

$$\nu E=\lim_{n\to\infty}\nu E_n.$$

(d) If $\nu':\Sigma\to\mathbb{R}$ is another countably additive functional, and $c\in\mathbb{R}$, then $\nu+\nu':\Sigma\to\mathbb{R}$ and $c\nu:\Sigma\to\mathbb{R}$ are countably additive.

(e) If $H\in\Sigma$, then $\nu_H:\Sigma\to\mathbb{R}$ is countably additive, where $\nu_H E=\nu(E\cap H)$ for every $E\in\Sigma$.

231E Theorem Let X be a set, Σ a σ-algebra of subsets of X, and $\nu:\Sigma\to\mathbb{R}$ a countably additive functional. Then
(a) ν is bounded;
(b) there is a set $H\in\Sigma$ such that

$$\nu F\ge 0 \text{ whenever } F\in\Sigma \text{ and } F\subseteq H,$$

$$\nu F\le 0 \text{ whenever } F\in\Sigma \text{ and } F\cap H=\emptyset.$$

231F Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\nu:\Sigma\to\mathbb{R}$ a countably additive functional. Then ν can be expressed as the difference of two totally finite measures with domain Σ.

Remark This is called the '**Jordan decomposition**' of ν. The expression of 231Eb is a '**Hahn decomposition**'.

Version of 19.5.17

232 The Radon-Nikodým theorem

I come now to the chief theorem of this chapter, one of the central results of measure theory, relating countably additive functionals to indefinite integrals. The objective is to give a complete description of the functionals which can arise as indefinite integrals of integrable functions (232E). These can be characterized as the 'truly continuous' additive functionals (232Ab). A more commonly used concept, and one adequate in many cases, is that of 'absolutely continuous' additive functional (232Aa); I spend the first few paragraphs (232B-232D) on elementary facts about truly continuous and absolutely continuous functionals. I end the section with a discussion of decompositions of general countably additive functionals (232I).

232A Absolutely continuous functionals Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a finitely additive functional.

(a) ν is **absolutely continuous** with respect to μ (sometimes written '$\nu \ll \mu$') if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu E| \le \epsilon$ whenever $E \in \Sigma$ and $\mu E \le \delta$.

(b) ν is **truly continuous** with respect to μ if for every $\epsilon > 0$ there are $E \in \Sigma$ and $\delta > 0$ such that $\mu E < \infty$ and $|\nu F| \le \epsilon$ whenever $F \in \Sigma$ and $\mu(E \cap F) \le \delta$.

(c) If ν is countably additive, it is **singular** with respect to μ if there is a set $F \in \Sigma$ such that $\mu F = 0$ and $\nu E = 0$ whenever $E \in \Sigma$ and $E \subseteq X \setminus F$.

232B Proposition Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a finitely additive functional.

(a) If ν is countably additive, it is absolutely continuous with respect to μ iff $\nu E = 0$ whenever $\mu E = 0$.

(b) ν is truly continuous with respect to μ iff (α) it is countably additive (β) it is absolutely continuous with respect to μ (γ) whenever $E \in \Sigma$ and $\nu E \ne 0$ there is an $F \in \Sigma$ such that $\mu F < \infty$ and $\nu(E \cap F) \ne 0$.

(c) If (X, Σ, μ) is σ-finite, then ν is truly continuous with respect to μ iff it is countably additive and absolutely continuous with respect to μ.

(d) If (X, Σ, μ) is totally finite, then ν is truly continuous with respect to μ iff it is absolutely continuous with respect to μ.

232C Lemma Let (X, Σ, μ) be a measure space and ν, ν' two countably additive functionals on Σ which are truly continuous with respect to μ. Take $c \in \mathbb{R}$ and $H \in \Sigma$, and set $\nu_H E = \nu(E \cap H)$ for $E \in \Sigma$. Then $\nu + \nu'$, $c\nu$ and ν_H are all truly continuous with respect to μ, and ν is expressible as the difference of non-negative countably additive functionals which are truly continuous with respect to μ.

232D Proposition Let (X, Σ, μ) be a measure space, and f a μ-integrable real-valued function. For $E \in \Sigma$ set $\nu E = \int_E f$. Then $\nu : \Sigma \to \mathbb{R}$ is a countably additive functional and is truly continuous with respect to μ, therefore absolutely continuous with respect to μ.

Remark The functional $E \mapsto \int_E f$ is called the **indefinite integral** of f.

232E The Radon-Nikodým theorem Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a function. Then the following are equiveridical:

(i) there is a μ-integrable function f such that $\nu E = \int_E f$ for every $E \in \Sigma$;

(ii) ν is finitely additive and truly continuous with respect to μ.

232F Corollary Let (X, Σ, μ) be a σ-finite measure space and $\nu : \Sigma \to \mathbb{R}$ a function. Then there is a μ-integrable function f such that $\nu E = \int_E f$ for every $E \in \Sigma$ iff ν is countably additive and absolutely continuous with respect to μ.

232G Corollary Let (X, Σ, μ) be a totally finite measure space and $\nu : \Sigma \to \mathbb{R}$ a function. Then there is a μ-integrable function f on X such that $\nu E = \int_E f$ for every $E \in \Sigma$ iff ν is finitely additive and absolutely continuous with respect to μ.

232H Remarks If (X, Σ, μ) is a measure space and ν is a $[-\infty, \infty]$-valued functional defined on a family of subsets of X, I will say that a $[-\infty, \infty]$-valued function f defined on a subset of X is a **Radon-Nikodým derivative** of ν with respect to μ if $\int_E f d\mu$ is defined and equal to νE for every $E \in \operatorname{dom} \nu$.

232I The Lebesgue decomposition of a countably additive functional: Proposition (a) Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a countably additive functional. Then ν has unique expressions as

$$\nu = \nu_s + \nu_{ac} = \nu_s + \nu_{tc} + \nu_e,$$

where ν_s is singular with respect to μ, ν_{ac} is absolutely continuous with respect to μ, ν_{tc} is truly continuous with respect to μ, and ν_e is absolutely continuous with respect to μ and zero on every set of finite measure.

(b) If $X = \mathbb{R}^r$, Σ is the algebra of Borel sets in $\mathbb{R}^r$ and μ is the restriction of Lebesgue measure to Σ, then ν is uniquely expressible as $\nu_p + \nu_{cs} + \nu_{ac}$ where ν_{ac} is absolutely continuous with respect to μ, ν_{cs} is singular with respect to μ and zero on singletons, and $\nu_p E = \sum_{x \in E} \nu_p\{x\}$ for every $E \in \Sigma$.

Remark The expression $\nu = \nu_p + \nu_{cs} + \nu_{ac}$ of (b) is the **Lebesgue decomposition** of ν.

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233 Conditional expectations

I devote a section to a first look at one of the principal applications of the Radon-Nikodým theorem. It is one of the most vital ideas of measure theory, and will appear repeatedly in one form or another. Here I give the definition and most basic properties of conditional expectations as they arise in abstract probability theory, with notes on convex functions and a version of Jensen's inequality (233I-233J).

233A σ-subalgebras Let X be a set and Σ a σ-algebra of subsets of X. A **σ-subalgebra** of Σ is a σ-algebra T of subsets of X such that $\mathrm{T} \subseteq \Sigma$. If (X, Σ, μ) is a measure space and T is a σ-subalgebra of Σ, then $(X, \mathrm{T}, \mu\restriction\mathrm{T})$ is again a measure space.

233B Lemma Let (X, Σ, μ) be a measure space and T a σ-subalgebra of Σ. A real-valued function f defined on a subset of X is $\mu\restriction\mathrm{T}$-integrable iff (i) it is μ-integrable (ii) $\operatorname{dom} f$ is $\mu\restriction\mathrm{T}$-conegligible (iii) f is $\mu\restriction\mathrm{T}$-virtually measurable; and in this case $\int f d(\mu\restriction\mathrm{T}) = \int f d\mu$.

233D Conditional expectations Let (X, Σ, μ) be a probability space. Let $\mathrm{T} \subseteq \Sigma$ be a σ-subalgebra. If f is a μ-integrable real-valued function, there is a $\mu\restriction\mathrm{T}$-integrable function g such that $\int_F g = \int_F f$ for every $F \in \mathrm{T}$; such a function is **a conditional expectation** of f on T.

233E Proposition Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence of μ-integrable real-valued functions, and for each n let g_n be a conditional expectation of f_n on T. Then

(a) $g_1 + g_2$ is a conditional expectation of $f_1 + f_2$ on T;

(b) for any $c \in \mathbb{R}$, cg_0 is a conditional expectation of cf_0 on T;

(c) if $f_1 \leq_{\text{a.e.}} f_2$ then $g_1 \leq_{\text{a.e.}} g_2$;

(d) if $\langle f_n \rangle_{n \in \mathbb{N}}$ is non-decreasing a.e. and $f = \lim_{n\to\infty} f_n$ is μ-integrable, then $\lim_{n\to\infty} g_n$ is a conditional expectation of f on T;

(e) if $f = \lim_{n\to\infty} f_n$ is defined a.e. and there is a μ-integrable function h such that $|f_n| \leq_{\text{a.e.}} h$ for every n, then $\lim_{n\to\infty} g_n$ is a conditional expectation of f on T;

(f) if $F \in \mathrm{T}$ then $g_0 \times \chi F$ is a conditional expectation of $f_0 \times \chi F$ on T;

(g) if h is a bounded, $\mu\restriction\mathrm{T}$-virtually measurable real-valued function defined $\mu\restriction\mathrm{T}$-almost everywhere in X, then $g_0 \times h$ is a conditional expectation of $f_0 \times h$ on T;

(h) if Υ is a σ-subalgebra of T, then a function h_0 is a conditional expectation of f_0 on Υ iff it is a conditional expectation of g_0 on Υ.

233G Convex functions Recall that a real-valued function ϕ defined on an interval $I \subseteq \mathbb{R}$ is **convex** if

$$\phi(tb + (1-t)c) \leq t\phi(b) + (1-t)\phi(c)$$

whenever b, $c \in I$ and $t \in [0, 1]$.

233H Lemma Let $I \subseteq \mathbb{R}$ be a non-empty open interval and $\phi : I \to \mathbb{R}$ a convex function.

(a) For every $a \in I$ there is a $b \in \mathbb{R}$ such that $\phi(x) \geq \phi(a) + b(x - a)$ for every $x \in I$.

(b) If we take, for each $q \in I \cap \mathbb{Q}$, a $b_q \in \mathbb{R}$ such that $\phi(x) \geq \phi(q) + b_q(x - q)$ for every $x \in I$, then

$$\phi(x) = \sup_{q \in I \cap \mathbb{Q}} \phi(q) + b_q(x - q)$$

for every $x \in I$.

(c) ϕ is Borel measurable.

233I Jensen's inequality Let (X, Σ, μ) be a measure space and $\phi : \mathbb{R} \to \mathbb{R}$ a convex function.

(a) Suppose that f and g are real-valued μ-virtually measurable functions defined almost everywhere in X and that $g \geq 0$ almost everywhere, $\int g = 1$ and $g \times f$ is integrable. Then $\phi(\int g \times f) \leq \int g \times \phi f$, where we may need to interpret the right-hand integral as ∞.

(b) In particular, if $\mu X = 1$ and f is a real-valued function which is integrable over X, then $\phi(\int f) \leq \int \phi f$.

233J Theorem Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and f a μ-integrable real-valued function defined almost everywhere in X such that the composition ϕf is also integrable. If g and h are conditional expectations on T of f, ϕf respectively, then $\phi g \leq_{\text{a.e.}} h$. Consequently $\int \phi g \leq \int \phi f$.

233K Proposition Let (X, Σ, μ) be a probability space, and T a σ-subalgebra of Σ. Suppose that f is a μ-integrable function and h is a $(\mu\restriction\mathrm{T})$-virtually measurable real-valued function defined $(\mu\restriction\mathrm{T})$-almost everywhere in X. Let g, g_0 be conditional expectations of f and $|f|$ on T. Then $f \times h$ is integrable iff $g_0 \times h$ is integrable, and in this case $g \times h$ is a conditional expectation of $f \times h$ on T.

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234 Operations on measures

I take a few pages to describe some standard constructions. The ideas are straightforward, but a number of details need to be worked out if they are to be securely integrated into the general framework I employ. The first step is to formally introduce inverse-measure-preserving functions (234A-234B), the most important class of transformations between measure spaces. For construction of new measures, we have the notions of image measure (234C-234E), sum of measures (234G-234H) and indefinite-integral measure (234I-234O). Finally I mention a way of ordering the measures on a given set (234P-234Q).

234A Inverse-measure-preserving functions If (X, Σ, μ) and (Y, T, ν) are measure spaces, a function $\phi : X \to Y$ is **inverse-measure-preserving** if $\phi^{-1}[F] \in \Sigma$ and $\mu(\phi^{-1}[F]) = \nu F$ for every $F \in \mathrm{T}$.

234B Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function.

(a) If $\hat{\mu}$, $\hat{\nu}$ are the completions of μ, ν respectively, ϕ is also inverse-measure-preserving for $\hat{\mu}$ and $\hat{\nu}$.

(b) μ is a probability measure iff ν is a probability measure.

(c) μ is totally finite iff ν is totally finite.

(d)(i) If ν is σ-finite, then μ is σ-finite.

(ii) If ν is semi-finite and μ is σ-finite, then ν is σ-finite.

(e)(i) If ν is σ-finite and atomless, then μ is atomless.

(ii) If ν is semi-finite and μ is purely atomic, then ν is purely atomic.

(f)(i) $\mu^*\phi^{-1}[B] \leq \nu^*B$ for every $B \subseteq Y$.

(ii) $\mu^*A \leq \nu^*\phi[A]$ for every $A \subseteq X$.

(g) If (Z, Λ, λ) is another measure space, and $\psi : Y \to Z$ is inverse-measure-preserving, then $\psi\phi : X \to Z$ is inverse-measure-preserving.

234C Image measures: Proposition Let (X, Σ, μ) be a measure space, Y any set, and $\phi : X \to Y$ a function. Set

$$\mathrm{T} = \{F : F \subseteq Y,\ \phi^{-1}[F] \in \Sigma\}, \quad \nu F = \mu(\phi^{-1}[F]) \text{ for every } F \in \mathrm{T}.$$

Then (Y, T, ν) is a measure space.

234D Definition In the context of 234C, ν is called the **image measure**; I will denote it $\mu\phi^{-1}$.

234E Proposition Let (X, Σ, μ) be a measure space, Y a set and $\phi : X \to Y$ a function; let $\mu\phi^{-1}$ be the image measure on Y.

(a) ϕ is inverse-measure-preserving for μ and $\mu\phi^{-1}$.

(b) If μ is complete, so is $\mu\phi^{-1}$.

(c) If Z is another set, and $\psi : Y \to Z$ a function, then the image measures $\mu(\psi\phi)^{-1}$ and $(\mu\phi^{-1})\psi^{-1}$ on Z are the same.

***234F Proposition** Let X be a set, (Y, T, ν) a measure space, and $\phi : X \to Y$ a function such that $\phi[X]$ has full outer measure in Y. Then there is a measure μ on X, with domain $\Sigma = \{\phi^{-1}[F] : F \in \mathrm{T}\}$, such that ϕ is inverse-measure-preserving for μ and ν.

234G Sums of measures: Proposition Let X be a set, and $\langle \mu_i \rangle_{i\in I}$ a family of measures on X. For each $i \in I$, let Σ_i be the domain of μ_i. Set $\Sigma = \mathcal{P}X \cap \bigcap_{i\in I} \Sigma_i$ and define $\mu : \Sigma \to [0, \infty]$ by setting $\mu E = \sum_{i\in I} \mu_i E$ for every $E \in \Sigma$. Then μ is a measure on X.

Remark In this context, I will call μ the **sum** of the family $\langle \mu_i \rangle_{i\in I}$.

234H Proposition Let X be a set and $\langle \mu_i \rangle_{i\in I}$ a family of complete measures on X with sum μ.

(a) μ is complete.

(b)(i) A subset of X is μ-negligible iff it is μ_i-negligible for every $i \in I$.

(ii) A subset of X is μ-conegligible iff it is μ_i-conegligible for every $i \in I$.

(c) Let f be a function from a subset of X to $[-\infty, \infty]$. Then $\int f d\mu$ is defined in $[-\infty, \infty]$ iff $\int f d\mu_i$ is defined in $[-\infty, \infty]$ for every i and one of $\sum_{i\in I} f^+ d\mu_i$, $\sum_{i\in I} f^- d\mu_i$ is finite, and in this case $\int f d\mu = \sum_{i\in I} \int f d\mu_i$.

234I Indefinite-integral measures: Theorem Let (X, Σ, μ) be a measure space, and f a non-negative μ-virtually measurable real-valued function defined on a conegligible subset of X. Write $\nu F = \int f \times \chi F \, d\mu$ whenever $F \subseteq X$ is such that the integral is defined in $[0, \infty]$. Then ν is a complete measure on X, and its domain includes Σ.

234J Definition Let (X, Σ, μ) be a measure space, and ν another measure on X with domain T. I will call ν an **indefinite-integral measure** over μ, or sometimes a **completed indefinite-integral measure**, if it can be obtained by the method of 234I from some non-negative virtually measurable function f defined almost everywhere on X. In this case, f is a Radon-Nikodým derivative of ν with respect to μ in the sense of 232Hf.

234K Remarks Let (X, Σ, μ) be a measure space, and f a μ-virtually measurable non-negative real-valued function defined almost everywhere on X; let ν be the associated indefinite-integral measure.

(a) ν has a Radon-Nikodým derivative which is Σ-measurable and defined everywhere.

(b) If E is μ-negligible, then $\nu E = 0$.

(d) $\nu E = \int_E f d\mu$ for every $E \in \operatorname{dom} \nu$.

(e) μ and its completion define the same indefinite-integral measures.

234L The domain of an indefinite-integral measure: Proposition Let (X, Σ, μ) be a measure space, f a non-negative μ-virtually measurable function defined almost everywhere in X, and ν the associated indefinite-integral measure. Set $G = \{x : x \in \operatorname{dom} f,\ f(x) > 0\}$, and let $\hat{\Sigma}$ be the domain of the completion $\hat{\mu}$ of μ.

(a) The domain T of ν is $\{E : E \subseteq X,\ E \cap G \in \hat{\Sigma}\}$; $\mathrm{T} \supseteq \hat{\Sigma} \supseteq \Sigma$.

(b) ν is the completion of its restriction to Σ.

(c) A set $A \subseteq X$ is ν-negligible iff $A \cap G$ is μ-negligible.

(d) In particular, if μ is complete, then $\mathrm{T} = \{E : E \subseteq X,\ E \cap G \in \Sigma\}$ and $\nu A = 0$ iff $\mu(A \cap G) = 0$.

234M Corollary If (X,Σ,μ) is a complete measure space and $G \in \Sigma$, then the indefinite-integral measure over μ defined by χG is just the measure $\mu \llcorner G$ defined by setting

$$(\mu \llcorner G)(F) = \mu(F \cap G) \text{ whenever } F \subseteq X \text{ and } F \cap G \in \Sigma.$$

***234N Proposition** Let (X,Σ,μ) be a measure space, and ν an indefinite-integral measure over μ.
(a) If μ is semi-finite, so is ν.
(b) If μ is complete and locally determined, so is ν.
(c) If μ is localizable, so is ν.
(d) If μ is strictly localizable, so is ν.
(e) If μ is σ-finite, so is ν.
(f) If μ is atomless, so is ν.

***234O Theorem** Let (X,Σ,μ) be a localizable measure space. Then a measure ν, with domain $\mathrm{T} \supseteq \Sigma$, is an indefinite-integral measure over μ iff (α) ν is semi-finite and zero on μ-negligible sets (β) ν is the completion of its restriction to Σ (γ) whenever $\nu E > 0$ there is an $F \subseteq E$ such that $F \in \Sigma$, $\mu F < \infty$ and $\nu F > 0$.

234P Ordering measures: Definition Let μ, ν be two measures on a set X. I will say that $\mu \le \nu$ if μE is defined, and $\mu E \le \nu E$, whenever ν measures E.

234Q Proposition Let X be a set, and write M for the set of all measures on X.
(a) Defining $\le$ as in 234P, $(\mathrm{M}, \le)$ is a partially ordered set.
(b) If μ, $\nu \in \mathrm{M}$, then $\mu \le \nu$ iff there is a $\lambda \in \mathrm{M}$ such that $\mu + \lambda = \nu$.
(c) If $\mu \le \nu$ in M and f is a $[-\infty,\infty]$-valued function, defined on a subset of X, such that $\int f d\nu$ is defined in $[-\infty,\infty]$, then $\int f d\mu$ is defined; if f is non-negative, $\int f d\mu \le \int f d\nu$.

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235 Measurable transformations

I turn now to a topic which is separate from the Radon-Nikodým theorem, but which seems to fit better here than in either of the next two chapters. I seek to give results which will generalize the basic formula of calculus

$$\int g(y)dy = \int g(\phi(x))\phi'(x)dx$$

in the context of a general transformation ϕ between measure spaces. The principal results are I suppose 235A/235E, which are very similar expressions of the basic idea, and 235J, which gives a general criterion for a stronger result. A formulation from a different direction is in 235R.

235A Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $\phi : D_\phi \to Y$, $J : D_J \to [0,\infty[$ functions defined on conegligible subsets D_ϕ, D_J of X such that

$$\int J \times \chi(\phi^{-1}[F])d\mu \text{ exists} = \nu F$$

whenever $F \in \mathrm{T}$ and $\nu F < \infty$. Then

$$\int_{\phi^{-1}[H]} J \times g\phi \, d\mu \text{ exists} = \int_H g \, d\nu$$

for every ν-integrable function g taking values in $[-\infty,\infty]$ and every $H \in \mathrm{T}$, provided that we interpret $(J \times g\phi)(x)$ as 0 when $J(x) = 0$ and $g(\phi(x))$ is undefined. Consequently, interpreting $J \times f\phi$ in the same way,

$$\underline{\int} f d\nu \le \underline{\int} J \times f\phi \, d\mu \le \overline{\int} J \times f\phi \, d\mu \le \overline{\int} f d\nu$$

for every $[-\infty,\infty]$-valued function f defined almost everywhere in Y.

235C Lemma Let Σ, T be σ-algebras of subsets of X and Y respectively. Suppose that $D \subseteq X$ and that $\phi : D \to Y$ is a function such that $\phi^{-1}[F] \in \Sigma_D$, the subspace σ-algebra, for every $F \in \mathrm{T}$. Then $g\phi$ is Σ-measurable for every $[-\infty, \infty]$-valued T-measurable function g defined on a subset of Y.

235D Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with completions $(X, \hat{\Sigma}, \hat{\mu})$ and $(Y, \hat{\mathrm{T}}, \hat{\nu})$. Let $\phi : D_\phi \to Y$, $J : D_J \to [0, \infty[$ be functions defined on conegligible subsets of X.

(a) If $\int J \times \chi(\phi^{-1}[F])d\mu = \nu F$ whenever $F \in \mathrm{T}$ and $\nu F < \infty$, then $\int J \times \chi(\phi^{-1}[F])d\hat{\mu} = \hat{\nu} F$ whenever $F \in \hat{\mathrm{T}}$ and $\hat{\nu} F < \infty$.

(b) If $\int J \times \chi(\phi^{-1}[F])d\mu = \nu F$ whenever $F \in \mathrm{T}$, then $\int J \times \chi(\phi^{-1}[F])d\hat{\mu} = \hat{\nu} F$ whenever $F \in \hat{\mathrm{T}}$.

235E Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $\phi : D_\phi \to Y$, $J : D_J \to [0, \infty[$ two functions defined on conegligible subsets of X such that

$$\int J \times \chi(\phi^{-1}[F])d\mu = \nu F$$

for every $F \in \mathrm{T}$, allowing ∞ as a value of the integral.

(a) $J \times g\phi$ is μ-virtually measurable for every ν-virtually measurable function g defined on a subset of Y.

(b) Let g be a ν-virtually measurable $[-\infty, \infty]$-valued function defined on a conegligible subset of Y. Then $\int J \times g\phi\, d\mu = \int g\, d\nu$ whenever either integral is defined in $[-\infty, \infty]$, if we interpret $(J \times g\phi)(x)$ as 0 when $J(x) = 0$ and $g(\phi(x))$ is undefined.

235G Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces and $\phi : X \to Y$ an inverse-measure-preserving function. Then

(a) if g is a ν-virtually measurable $[-\infty, \infty]$-valued function defined on a subset of Y, $g\phi$ is μ-virtually measurable;

(b) if g is a ν-virtually measurable $[-\infty, \infty]$-valued function defined on a conegligible subset of Y, $\int g\phi\, d\mu = \int g\, d\nu$ if either integral is defined in $[-\infty, \infty]$;

(c) if g is a ν-virtually measurable $[-\infty, \infty]$-valued function defined on a conegligible subset of Y, and $F \in \mathrm{T}$, then $\int_{\phi^{-1}[F]} g\phi\, d\mu = \int_F g\, d\nu$ if either integral is defined in $[-\infty, \infty]$.

235I Lemma Let Σ, T be σ-algebras of subsets of X, Y respectively, and ϕ a function from a subset D of X to Y. Suppose that $G \subseteq X$ and that

$$\mathrm{T} = \{F : F \subseteq Y,\ G \cap \phi^{-1}[F] \in \Sigma\}.$$

Then a real-valued function g, defined on a member of T, is T-measurable iff $\chi G \times g\phi$ is Σ-measurable.

235J Theorem Let (X, Σ, μ) and (Y, T, ν) be complete measure spaces. Let $\phi : D_\phi \to Y$, $J : D_J \to [0, \infty[$ be functions defined on conegligible subsets of X, and set $G = \{x : x \in D_J,\ J(x) > 0\}$. Suppose that

$$\mathrm{T} = \{F : F \subseteq Y,\ G \cap \phi^{-1}[F] \in \Sigma\},$$

$$\nu F = \int J \times \chi(\phi^{-1}[F])d\mu \text{ for every } F \in \mathrm{T}.$$

Then, for any real-valued function g defined on a subset of Y, $\int J \times g\phi\, d\mu = \int g\, d\nu$ whenever either integral is defined in $[-\infty, \infty]$, provided that we interpret $(J \times g\phi)(x)$ as 0 when $J(x) = 0$ and $g(\phi(x))$ is undefined.

235K Corollary Let (X, Σ, μ) be a complete measure space, and J a non-negative measurable function defined on a conegligible subset of X. Let ν be the associated indefinite-integral measure, and T its domain. Then for any real-valued function g defined on a subset of X, g is T-measurable iff $J \times g$ is Σ-measurable, and $\int g\, d\nu = \int J \times g\, d\mu$ if either integral is defined in $[-\infty, \infty]$, provided that we interpret $(J \times g)(x)$ as 0 when $J(x) = 0$ and $g(x)$ is undefined.

235M Theorem Let (X, Σ, μ) be a σ-finite measure space, (Y, T, ν) a semi-finite measure space, and $\phi : D \to Y$ a function such that

(i) D is a conegligible subset of X,

(ii) $\phi^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$;

(iii) $\mu\phi^{-1}[F] > 0$ whenever $F \in \mathrm{T}$ and $\nu F > 0$.

Then there is a Σ-measurable function $J : X \to [0, \infty[$ such that $\int J \times \chi\phi^{-1}[F]\, d\mu = \nu F$ for every $F \in \mathrm{T}$.

235N Remark Theorem 235M can fail if μ is only strictly localizable rather than σ-finite.

***235O Lemma** Let (X,Σ,μ) be a measure space and f a non-negative integrable function on X. If $A\subseteq X$ is such that $\int_A f+\int_{X\setminus A} f=\int f$, then $f\times\chi A$ is integrable.

***235P Proposition** Let (X,Σ,μ) be a complete measure space and (Y,T,ν) a complete σ-finite measure space. Suppose that $\phi:D_\phi\to Y$, $J:D_J\to[0,\infty[$ are functions defined on conegligible subsets D_ϕ, D_J of X such that $\int_{\phi^{-1}[F]}J\,d\mu$ exists and is equal to νF whenever $F\in\mathrm{T}$ and $\nu F<\infty$.

(a) $J\times g\phi$ is Σ-measurable for every T-measurable real-valued function g defined on a subset of Y.

(b) If g is a T-measurable real-valued function defined almost everywhere in Y, then $\int J\times g\phi\,d\mu=\int g\,d\nu$ whenever either integral is defined in $[-\infty,\infty]$, interpreting $(J\times g\phi)(x)$ as 0 when $J(x)=0$, $g(\phi(x))$ is undefined.

***235Q Example** Set $X=Y=[0,2]$. Write Σ_L for the algebra of Lebesgue measurable subsets of $\mathbb{R}$, and μ_L for Lebesgue measure; write μ_c for counting measure on $\mathbb{R}$. Set

$$\Sigma=\mathrm{T}=\{E:E\subseteq[0,2],\ E\cap[0,1[\ \in\Sigma_L\}.$$

For $E\in\Sigma=\mathrm{T}$, set

$$\mu E=\nu E=\mu_L(E\cap[0,1[)+\mu_c(E\cap[1,2]).$$

Let $A\subseteq[0,1[$ be a non-Lebesgue-measurable set such that $\mu_L^*(E\setminus A)=\mu_L E$ for every Lebesgue measurable $E\subseteq[0,1[$. Define $\phi:[0,2]\to[0,2]$ by setting $\phi(x)=x+1$ if $x\in A$, $\phi(x)=x$ if $x\in[0,2]\setminus A$.

If $F\in\Sigma$, then $\mu^*(\phi^{-1}[F])=\mu F$.

This means that if we set $J(x)=1$ for every $x\in[0,2]$, ϕ, J satisfy the amended hypotheses for 235A. But if we set $g=\chi\,[0,1[$, then g is μ-integrable, with $\int g\,d\mu=1$, while $J\times g\phi$ is not μ-integrable.

235R Reversing the burden: Theorem Let (X,Σ,μ), (Y,T,ν) be measure spaces and $\phi:X\to Y$, $J:Y\to[0,\infty[$ functions such that $\int_F J\,d\nu$ and $\mu\phi^{-1}[F]$ are defined in $[0,\infty]$ and equal for every $F\in\mathrm{T}$. Then $\int g\phi\,d\mu=\int J\times g\,d\nu$ whenever g is ν-virtually measurable and defined ν-almost everywhere and either integral is defined in $[-\infty,\infty]$.