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Chapter 23

The Radon-Nikodým Theorem

In Chapter 22, I discussed the indefinite integrals of integrable functions on $\mathbb{R}$, and gave what I hope you feel are satisfying descriptions both of the functions which are indefinite integrals (the absolutely continuous functions) and of how to find which functions they are indefinite integrals of (you differentiate them). For general measure spaces, we have no structure present which can give such simple formulations; but nevertheless the same questions can be asked and, up to a point, answered.

The first section of this chapter introduces the basic machinery needed, the concept of 'countably additive' functional and its decomposition into positive and negative parts. The main theorem takes up the second section: indefinite integrals are the 'truly continuous' additive functionals; on σ-finite spaces, these are the 'absolutely continuous' countably additive functionals. In §233 I discuss the most important single application of the theorem, its use in providing a concept of 'conditional expectation'. This is one of the central concepts of probability theory – as you very likely know; but the form here is a dramatic generalization of the elementary concept of the conditional probability of one event given another, and needs the whole strength of the general theory of measure and integration as developed in Volume 1 and this chapter. I include some notes on convex functions, up to and including versions of Jensen's inequality (233I-233J).

While we are in the area of 'pure' measure theory, I take the opportunity to discuss some further topics. I begin with some essentially elementary constructions, image measures, sums of measures and indefinite-integral measures; I think the details need a little attention, and I work through them in §234. Rather deeper ideas are needed to deal with 'measurable transformations'. In §235 I set out the techniques necessary to provide an abstract basis for a general method of integration-by-substitution, with a detailed account of sufficient conditions for a formula of the type

$$\int g(y)dy = \int g(\phi(x))J(x)dx$$

to be valid.

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231 Countably additive functionals

I begin with an abstract description of the objects which will, in appropriate circumstances, correspond to the indefinite integrals of general integrable functions. In this section I give those parts of the theory which do not involve a measure, but only a set with a distinguished σ-algebra of subsets. The basic concepts are those of 'finitely additive' and 'countably additive' functional, and there is one substantial theorem, the 'Hahn decomposition' (231E).

231A Definition Let X be a set and Σ an algebra of subsets of X (136E). A functional $\nu : \Sigma \to \mathbb{R}$ is **finitely additive**, or just **additive**, if $\nu(E \cup F) = \nu E + \nu F$ whenever E, $F \in \Sigma$ and $E \cap F = \emptyset$.

231B Elementary facts Let X be a set, Σ an algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a finitely additive functional.

(a) $\nu\emptyset = 0$. (For $\nu\emptyset = \nu(\emptyset \cup \emptyset) = \nu\emptyset + \nu\emptyset$.)

(b) If $E_0, \dots , E_n$ are disjoint members of Σ then $\nu(\bigcup_{i\le n} E_i) = \sum_{i=0}^{n} \nu E_i$.

(c) If E, $F \in \Sigma$ and $E \subseteq F$ then $\nu F = \nu E + \nu(F \setminus E)$. More generally, for any E, $F \in \Sigma$,

$$\nu F = \nu(F \cap E) + \nu(F \setminus E),$$

$$\nu E + \nu F = \nu E + \nu(F \setminus E) + \nu(F \cap E) = \nu(E \cup F) + \nu(E \cap F),$$

$$\nu E - \nu F = \nu(E \setminus F) + \nu(E \cap F) - \nu(E \cap F) - \nu(F \setminus E) = \nu(E \setminus F) - \nu(F \setminus E).$$

231C Definition Let X be a set and Σ an algebra of subsets of X. A function $\nu : \Sigma \to \mathbb{R}$ is **countably additive** or **σ-additive** if $\sum_{n=0}^{\infty} \nu E_n$ exists in $\mathbb{R}$ and is equal to $\nu(\bigcup_{n\in\mathbb{N}} E_n)$ for every disjoint sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in Σ such that $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$.

Remark Note that when I use the phrase 'countably additive functional' I mean to exclude the possibility of ∞ as a value of the functional. Thus a measure is a countably additive functional iff it is totally finite (211C).

You will sometimes see the phrase '**signed measure**' used to mean what I call a countably additive functional.

231D Elementary facts Let X be a set, Σ a σ-algebra of subsets of X and $\nu : \Sigma \to \mathbb{R}$ a countably additive functional.

(a) ν is finitely additive. **P** (i) Setting $E_n = \emptyset$ for every $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \nu\emptyset$ must be defined in $\mathbb{R}$ so $\nu\emptyset$ must be 0. (ii) Now if E, $F \in \Sigma$ and $E \cap F = \emptyset$ we can set $E_0 = E$, $E_1 = F$, $E_n = \emptyset$ for $n \geq 2$ and get

$$\nu(E \cup F) = \nu(\textstyle\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \nu E_n = \nu E + \nu F. \ \mathbf{Q}$$

(b) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ, with union $E \in \Sigma$, then

$$\nu E = \nu E_0 + \textstyle\sum_{n=0}^{\infty} \nu(E_{n+1} \setminus E_n) = \lim_{n\to\infty} \nu E_n.$$

(c) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with intersection $E \in \Sigma$, then

$$\nu E = \nu E_0 - \lim_{n\to\infty} \nu(E_0 \setminus E_n) = \lim_{n\to\infty} \nu E_n.$$

(d) If $\nu' : \Sigma \to \mathbb{R}$ is another countably additive functional, and $c \in \mathbb{R}$, then $\nu + \nu' : \Sigma \to \mathbb{R}$ and $c\nu : \Sigma \to \mathbb{R}$ are countably additive.

(e) If $H \in \Sigma$, then $\nu_H : \Sigma \to \mathbb{R}$ is countably additive, where $\nu_H E = \nu(E \cap H)$ for every $E \in \Sigma$. **P** If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ with union $E \in \Sigma$ then $\langle E_n \cap H \rangle_{n\in\mathbb{N}}$ is disjoint, with union $E \cap H$, so

$$\nu_H E = \nu(H \cap E) = \nu(\textstyle\bigcup_{n\in\mathbb{N}}(H \cap E_n)) = \sum_{n=0}^{\infty} \nu(H \cap E_n) = \sum_{n=0}^{\infty} \nu_H E_n. \ \mathbf{Q}$$

Remark For the time being, we shall be using the notion of 'countably additive functional' only on σ-algebras Σ, in which case we can take it for granted that the unions and intersections above belong to Σ.

231E All the ideas above amount to minor modifications of ideas already needed at the very beginning of the theory of measure spaces. We come now to something more substantial.

Theorem Let X be a set, Σ a σ-algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a countably additive functional. Then

(a) ν is bounded;

(b) there is a set $H \in \Sigma$ such that

$$\nu F \geq 0 \text{ whenever } F \in \Sigma \text{ and } F \subseteq H,$$

$$\nu F \leq 0 \text{ whenever } F \in \Sigma \text{ and } F \cap H = \emptyset.$$

proof (a) ? Suppose, if possible, otherwise. For $E \in \Sigma$, set $M(E) = \sup\{|\nu F| : F \in \Sigma, F \subseteq E\}$; then $M(X) = \infty$. Moreover, whenever E_1, E_2, $F \in \Sigma$ and $F \subseteq E_1 \cup E_2$, then

$$|\nu F| = |\nu(F \cap E_1) + \nu(F \setminus E_1)| \le |\nu(F \cap E_1)| + |\nu(F \setminus E_1)| \le M(E_1) + M(E_2),$$

so $M(E_1 \cup E_2) \le M(E_1) + M(E_2)$. Choose a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ as follows. $E_0 = X$. Given that $M(E_n) = \infty$, where $n \in \mathbb{N}$, then surely there is an $F_n \subseteq E_n$ such that $|\nu F_n| \ge 1 + |\nu E_n|$, in which case $|\nu(E_n \setminus F_n)| \ge 1$. Now at least one of $M(F_n)$, $M(E_n \setminus F_n)$ is infinite; if $M(F_n) = \infty$, set $E_{n+1} = F_n$; otherwise, set $E_{n+1} = E_n \setminus F_n$; in either case, note that $|\nu(E_n \setminus E_{n+1})| \ge 1$ and $M(E_{n+1}) = \infty$, so that the induction will continue.

On completing this induction, set $G_n = E_n \setminus E_{n+1}$ for $n \in \mathbb{N}$. Then $\langle G_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, so $\sum_{n=0}^{\infty} \nu G_n$ is defined in $\mathbb{R}$ and $\lim_{n\to\infty} \nu G_n = 0$; but $|\nu G_n| \ge 1$ for every n. **X**

(b)(i) By (a), $\gamma = \sup\{\nu E : E \in \Sigma\} < \infty$. Choose a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ such that $\nu E_n \ge \gamma - 2^{-n}$ for every $n \in \mathbb{N}$. For $m \le n \in \mathbb{N}$, set $F_{mn} = \bigcap_{m\le i\le n} E_i$. Then $\nu F_{mn} \ge \gamma - 2 \cdot 2^{-m} + 2^{-n}$ for every $n \ge m$. **P** Induce on n. For $n = m$, this is due to the choice of $E_m = F_{mm}$. For the inductive step, we have $F_{m,n+1} = F_{mn} \cap E_{n+1}$, while surely $\gamma \ge \nu(E_{n+1} \cup F_{mn})$, so

$$\gamma + \nu F_{m,n+1} \ge \nu(E_{n+1} \cup F_{mn}) + \nu(E_{n+1} \cap F_{mn})$$
$$= \nu E_{n+1} + \nu F_{mn}$$

(231Bc)

$$\ge \gamma - 2^{-n-1} + \gamma - 2 \cdot 2^{-m} + 2^{-n}$$

(by the choice of E_{n+1} and the inductive hypothesis)

$$= 2\gamma - 2 \cdot 2^{-m} + 2^{-n-1}.$$

Subtracting γ from both sides, $\nu F_{m,n+1} \ge \gamma - 2 \cdot 2^{-m} + 2^{-n-1}$ and the induction proceeds. **Q**

(ii) For $m \in \mathbb{N}$, set

$$F_m = \bigcap_{n\ge m} F_{mn} = \bigcap_{n\ge m} E_n.$$

Then

$$\nu F_m = \lim_{n\to\infty} \nu F_{mn} \ge \gamma - 2 \cdot 2^{-m},$$

by 231Dc. Next, $\langle F_m\rangle_{m\in\mathbb{N}}$ is non-decreasing, so setting $H = \bigcup_{m\in\mathbb{N}} F_m$ we have

$$\nu H = \lim_{m\to\infty} \nu F_m \ge \gamma;$$

since νH is surely less than or equal to γ, $\nu H = \gamma$.

(iii) If $F \in \Sigma$ and $F \subseteq H$, then

$$\nu H - \nu F = \nu(H \setminus F) \le \gamma = \nu H,$$

so $\nu F \ge 0$. If $F \in \Sigma$ and $F \cap H = \emptyset$ then

$$\nu H + \nu F = \nu(H \cup F) \le \gamma = \nu H$$

so $\nu F \le 0$. This completes the proof.

231F Corollary Let X be a set, Σ a σ-algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a countably additive functional. Then ν can be expressed as the difference of two totally finite measures with domain Σ.

proof Take $H \in \Sigma$ as described in 231Eb. Set $\nu_1 E = \nu(E \cap H)$, $\nu_2 E = -\nu(E \setminus H)$ for $E \in \Sigma$. Then, as in 231Dd-e, both ν_1 and ν_2 are countably additive functionals on Σ, and of course $\nu = \nu_1 - \nu_2$. But also, by the choice of H, both ν_1 and ν_2 are non-negative, so are totally finite measures.

Remark This is called the '**Jordan decomposition**' of ν. The expression of 231Eb is a '**Hahn decomposition**'.

231X Basic exercises (a) Let Σ be the family of subsets A of $\mathbb{N}$ such that one of A, $\mathbb{N}\setminus A$ is finite. Show that Σ is an algebra of subsets of $\mathbb{N}$. (This is the **finite-cofinite algebra** of subsets of $\mathbb{N}$; compare 211Ra.)

(b) Let X be a set, Σ an algebra of subsets of X and $\nu:\Sigma\to\mathbb{R}$ a finitely additive functional. Show that $\nu(E\cup F\cup G)+\nu(E\cap F)+\nu(E\cap G)+\nu(F\cap G)=\nu E+\nu F+\nu G+\nu(E\cap F\cap G)$ for all E, F, $G\in\Sigma$. Generalize this result to longer sequences of sets.

>(c) Let Σ be the finite-cofinite algebra of subsets of $\mathbb{N}$, as in 231Xa. Define $\nu:\Sigma\to\mathbb{Z}$ by setting

$$\nu E=\lim_{n\to\infty}\bigl(\#(\{i:i\le n,\,2i\in E\})-\#(\{i:i\le n,\,2i+1\in E\})\bigr)$$

for every $E\in\Sigma$. Show that ν is well-defined and finitely additive and unbounded.

(d) Let X be a set and Σ an algebra of subsets of X. (i) Show that if $\nu:\Sigma\to\mathbb{R}$ and $\nu':\Sigma\to\mathbb{R}$ are finitely additive, so are $\nu+\nu'$ and $c\nu$ for any $c\in\mathbb{R}$. (ii) Show that if $\nu:\Sigma\to\mathbb{R}$ is finitely additive and $H\in\Sigma$, then ν_H is finitely additive, where $\nu_H(E)=\nu(H\cap E)$ for every $E\in\Sigma$.

(e) Let X be a set, Σ an algebra of subsets of X and $\nu:\Sigma\to\mathbb{R}$ a finitely additive functional. Let S be the linear space of those real-valued functions on X expressible in the form $\sum_{i=0}^n a_i\chi E_i$ where $E_i\in\Sigma$ for each i. (i) Show that we have a linear functional $\oint:S\to\mathbb{R}$ given by writing

$$\oint\sum_{i=0}^n a_i\chi E_i=\sum_{i=0}^n a_i\nu E_i$$

whenever $a_0,\dots,a_n\in\mathbb{R}$ and $E_0,\dots,E_n\in\Sigma$. (ii) Show that if $\nu E\ge 0$ for every $E\in\Sigma$ then $\oint f\ge 0$ whenever $f\in S$ and $f(x)\ge 0$ for every $x\in X$. (iii) Show that if ν is bounded and $X\ne\emptyset$ then

$$\sup\{|\oint f|:f\in S,\,\|f\|_\infty\le 1\}=\sup_{E,F\in\Sigma}|\nu E-\nu F|,$$

writing $\|f\|_\infty=\sup_{x\in X}|f(x)|$.

>(f) Let X be a set, Σ a σ-algebra of subsets of X and $\nu:\Sigma\to\mathbb{R}$ a finitely additive functional. Show that the following are equiveridical:

(i) ν is countably additive;

(ii) $\lim_{n\to\infty}\nu E_n=0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ and $\bigcap_{n\in\mathbb{N}}E_n=\emptyset$;

(iii) $\lim_{n\to\infty}\nu E_n=0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ and $\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}E_m=\emptyset$;

(iv) $\lim_{n\to\infty}\nu E_n=\nu E$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ and

$$E=\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}E_m=\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}E_m.$$

(*Hint*: for (i)$\Rightarrow$(iv), consider non-negative ν first.)

(g) Let X be a set and Σ a σ-algebra of subsets of X, and let $\nu:\Sigma\to[-\infty,\infty[$ be a function which is countably additive in the sense that $\nu\emptyset=0$ and whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, $\sum_{n=0}^\infty\nu E_n=\lim_{n\to\infty}\sum_{i=0}^n\nu E_i$ is defined in $[-\infty,\infty[$ and is equal to $\nu(\bigcup_{n\in\mathbb{N}}E_n)$. Show that ν is bounded above and attains its upper bound (that is, there is an $H\in\Sigma$ such that $\nu H=\sup_{F\in\Sigma}\nu F$). Hence, or otherwise, show that ν is expressible as the difference of a totally finite measure and a measure, both with domain Σ.

231Y Further exercises (a) Let X be a set, Σ an algebra of subsets of X, and $\nu:\Sigma\to\mathbb{R}$ a bounded finitely additive functional. Set

$$\nu^+E=\sup\{\nu F:F\in\Sigma,\,F\subseteq E\},$$

$$\nu^-E=-\inf\{\nu F:F\in\Sigma,\,F\subseteq E\},$$

$$|\nu|E=\sup\{\nu F_1-\nu F_2:F_1,\,F_2\in\Sigma,\,F_1,\,F_2\subseteq E\}.$$

Show that ν^+, ν^- and $|\nu|$ are all bounded finitely additive functionals on Σ and that $\nu=\nu^+-\nu^-$, $|\nu|=\nu^++\nu^-$. Show that if ν is countably additive so are ν^+, ν^- and $|\nu|$. ($|\nu|$ is sometimes called the **variation** of ν.)

(b) Let X be a set and Σ an algebra of subsets of X. Let ν_1, ν_2 be two bounded finitely additive functionals defined on Σ. Set

$$(\nu_1 \vee \nu_2)(E) = \sup\{\nu_1 F + \nu_2(E \setminus F) : F \in \Sigma,\, F \subseteq E\},$$

$$(\nu_1 \wedge \nu_2)(E) = \inf\{\nu_1 F + \nu_2(E \setminus F) : F \in \Sigma,\, F \subseteq E\}.$$

Show that $\nu_1 \vee \nu_2$ and $\nu_1 \wedge \nu_2$ are finitely additive functionals, and that $\nu_1 + \nu_2 = \nu_1 \vee \nu_2 + \nu_1 \wedge \nu_2$. Show that, in the language of 231Ya,

$$\nu^+ = \nu \vee 0, \quad \nu^- = (-\nu) \vee 0 = -(\nu \wedge 0), \quad |\nu| = \nu \vee (-\nu) = \nu^+ \vee \nu^- = \nu^+ + \nu^-,$$

$$\nu_1 \vee \nu_2 = \nu_1 + (\nu_2 - \nu_1)^+, \quad \nu_1 \wedge \nu_2 = \nu_1 - (\nu_1 - \nu_2)^+,$$

so that $\nu_1 \vee \nu_2$ and $\nu_1 \wedge \nu_2$ are countably additive if ν_1 and ν_2 are.

(c) Let X be a set and Σ an algebra of subsets of X. Let M be the set of all bounded finitely additive functionals from Σ to $\mathbb{R}$. Show that M is a linear space under the natural definitions of addition and scalar multiplication. Show that M has a partial order $\le$ defined by saying that

$$\nu \le \nu' \text{ iff } \nu E \le \nu' E \text{ for every } E \in \Sigma,$$

and that for this partial order $\nu_1 \vee \nu_2$, $\nu_1 \wedge \nu_2$, as defined in 231Yb, are $\sup\{\nu_1, \nu_2\}$, $\inf\{\nu_1, \nu_2\}$.

(d) Let X be a set and Σ an algebra of subsets of X. Let $\nu_0, \dots, \nu_n$ be bounded finitely additive functionals on Σ and set

$$\check\nu E = \sup\{\textstyle\sum_{i=0}^n \nu_i F_i : F_0, \dots, F_n \in \Sigma,\, \bigcup_{i\le n} F_i = E,\, F_i \cap F_j = \emptyset \text{ for } i \ne j\},$$

$$\hat\nu E = \inf\{\textstyle\sum_{i=0}^n \nu_i F_i : F_0, \dots, F_n \in \Sigma,\, \bigcup_{i\le n} F_i = E,\, F_i \cap F_j = \emptyset \text{ for } i \ne j\}$$

for $E \in \Sigma$. Show that $\check\nu$ and $\hat\nu$ are finitely additive and are, respectively, $\sup\{\nu_0, \dots, \nu_n\}$ and $\inf\{\nu_0, \dots, \nu_n\}$ in the partially ordered set of finitely additive functionals on Σ.

(e) Let X be a set and Σ a σ-algebra of subsets of X; let M be the partially ordered set of all bounded finitely additive functionals from Σ to $\mathbb{R}$. (i) Show that if $A \subseteq M$ is non-empty and bounded above in M, then A has a supremum $\check\nu$ in M, given by the formula

$$\check\nu E = \sup\{\sum_{i=0}^n \nu_i F_i : \nu_0, \dots, \nu_n \in A,\, F_0, \dots, F_n \in \Sigma,\, \bigcup_{i\le n} F_i = E,$$
$$F_i \cap F_j = \emptyset \text{ for } i \ne j\}.$$

(ii) Show that if $A \subseteq M$ is non-empty and bounded below in M then it has an infimum $\hat\nu \in M$, given by the formula

$$\hat\nu E = \inf\{\sum_{i=0}^n \nu_i F_i : \nu_0, \dots, \nu_n \in A,\, F_0, \dots, F_n \in \Sigma,\, \bigcup_{i\le n} F_i = E,$$
$$F_i \cap F_j = \emptyset \text{ for } i \ne j\}.$$

(f) Let X be a set, Σ an algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a non-negative finitely additive functional. For $E \in \Sigma$ set

$$\nu_{ca}(E) = \inf\{\sup_{n\in\mathbb{N}} \nu F_n : \langle F_n\rangle_{n\in\mathbb{N}} \text{ is a non-decreasing sequence in } \Sigma \text{ with union } E\}.$$

Show that ν_{ca} is a countably additive functional on Σ and that if ν' is any countably additive functional with $\nu' \le \nu$ then $\nu' \le \nu_{ca}$. Show that $\nu_{ca} \wedge (\nu - \nu_{ca}) = 0$.

(g) Let X be a set, Σ an algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a bounded finitely additive functional. Show that ν is uniquely expressible as $\nu_{ca} + \nu_{pfa}$, where ν_{ca} is countably additive, ν_{pfa} is finitely additive and if $0 \le \nu' \le |\nu_{pfa}|$ and ν' is countably additive then $\nu' = 0$.

(h) Let X be a set and Σ an algebra of subsets of X. Let M be the linear space of bounded finitely additive functionals on Σ, and for $\nu \in M$ set $\|\nu\| = |\nu|(X)$, defining $|\nu|$ as in 231Ya. ($\|\nu\|$ is the **total variation** of ν.) Show that $\| \ \|$ is a norm on M under which M is a Banach space. Show that the space of bounded countably additive functionals on Σ is a closed linear subspace of M.

(i) Repeat as many as possible of the results of this section for complex-valued functionals.

231 Notes and comments The real purpose of this section has been to describe the Hahn decomposition of a countably additive functional (231E). The leisurely exposition in 231A-231D is intended as a review of the most elementary properties of measures, in the slightly more general context of 'signed measures', with those properties corresponding to 'additivity' alone separated from those which depend on 'countable additivity'. In 231Xf I set out necessary and sufficient conditions for a finitely additive functional on a σ-algebra to be countably additive, designed to suggest that a finitely additive functional is countably additive iff it is 'sequentially order-continuous' in some sense. The fact that a countably additive functional can be expressed as the difference of non-negative countably additive functionals (231F) has an important counterpart in the theory of finitely additive functionals: a finitely additive functional can be expressed as the difference of non-negative finitely additive functionals if (and only if) it is bounded (231Ya). But I do not think that this, or the further properties of bounded finitely additive functionals described in 231Xe and 231Y, will be important to us before Volume 3.

Version of 19.5.17

232 The Radon-Nikodým theorem

I come now to the chief theorem of this chapter, one of the central results of measure theory, relating countably additive functionals to indefinite integrals. The objective is to give a complete description of the functionals which can arise as indefinite integrals of integrable functions (232E). These can be characterized as the 'truly continuous' additive functionals (232Ab). A more commonly used concept, and one adequate in many cases, is that of 'absolutely continuous' additive functional (232Aa); I spend the first few paragraphs (232B-232D) on elementary facts about truly continuous and absolutely continuous functionals. I end the section with a discussion of decompositions of general countably additive functionals (232I).

232A Absolutely continuous functionals Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a finitely additive functional.

(a) ν is **absolutely continuous** with respect to μ (sometimes written '$\nu \ll \mu$') if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|\nu E| \le \epsilon$ whenever $E \in \Sigma$ and $\mu E \le \delta$.

(b) ν is **truly continuous** with respect to μ if for every $\epsilon > 0$ there are $E \in \Sigma$ and $\delta > 0$ such that $\mu E < \infty$ and $|\nu F| \le \epsilon$ whenever $F \in \Sigma$ and $\mu(E \cap F) \le \delta$.

(c) For reference, I add another definition here. If ν is countably additive, it is **singular** with respect to μ if there is a set $F \in \Sigma$ such that $\mu F = 0$ and $\nu E = 0$ whenever $E \in \Sigma$ and $E \subseteq X \setminus F$.

232B Proposition Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a finitely additive functional.

(a) If ν is countably additive, it is absolutely continuous with respect to μ iff $\nu E = 0$ whenever $\mu E = 0$.

(b) ν is truly continuous with respect to μ iff (α) it is countably additive (β) it is absolutely continuous with respect to μ (γ) whenever $E \in \Sigma$ and $\nu E \neq 0$ there is an $F \in \Sigma$ such that $\mu F < \infty$ and $\nu(E \cap F) \neq 0$.

(c) If (X, Σ, μ) is σ-finite, then ν is truly continuous with respect to μ iff it is countably additive and absolutely continuous with respect to μ.

(d) If (X, Σ, μ) is totally finite, then ν is truly continuous with respect to μ iff it is absolutely continuous with respect to μ.

proof (a)(i) If ν is absolutely continuous with respect to μ and $\mu E = 0$, then $\mu E \le \delta$ for every $\delta > 0$, so $|\nu E| \le \epsilon$ for every $\epsilon > 0$ and $\nu E = 0$.

(ii) ? Suppose, if possible, that $\nu E = 0$ whenever $\mu E = 0$, but ν is not absolutely continuous. Then there is an $\epsilon > 0$ such that for every $\delta > 0$ there is an $E \in \Sigma$ such that $\mu E \le \delta$ but $|\nu E| \ge \epsilon$. For each $n \in \mathbb{N}$ we may choose an $F_n \in \Sigma$ such that $\mu F_n \le 2^{-n}$ and $|\nu F_n| \ge \epsilon$. Consider $F = \bigcap_{n\in\mathbb{N}} \bigcup_{k\ge n} F_k$. Then we have

$$\mu F \le \inf_{n\in\mathbb{N}} \mu(\bigcup_{k\ge n} F_k) \le \inf_{n\in\mathbb{N}} \sum_{k=n}^{\infty} 2^{-k} = 0,$$

so $\mu F = 0$.

Now recall that by 231Eb there is an $H \in \Sigma$ such that $\nu G \ge 0$ when $G \in \Sigma$ and $G \subseteq H$, while $\nu G \le 0$ when $G \in \Sigma$ and $G \cap H = \emptyset$. As in 231F, set $\nu_1 G = \nu(G \cap H)$, $\nu_2 G = -\nu(G \setminus H)$ for $G \in \Sigma$, so that ν_1 and ν_2 are totally finite measures, and $\nu_1 F = \nu_2 F = 0$ because $\mu(F \cap H) = \mu(F \setminus H) = 0$. Consequently

$$0 = \nu_i F = \lim_{n\to\infty} \nu_i(\bigcup_{m\ge n} F_m) \ge \limsup_{n\to\infty} \nu_i F_n \ge 0$$

for both i, and

$$0 = \lim_{n\to\infty}(\nu_1 F_n + \nu_2 F_n) \ge \liminf_{n\to\infty} |\nu F_n| \ge \epsilon > 0,$$

which is absurd. **X**

(b)(i) Suppose that ν is truly continuous with respect to μ. It is obvious from the definitions that ν is absolutely continuous with respect to μ. If $\nu E \ne 0$, there must be an F of finite measure such that $|\nu G| < |\nu E|$ whenever $G \cap F = \emptyset$, so that $|\nu(E \setminus F)| < |\nu E|$ and $\nu(E \cap F) \ne 0$. This deals with the conditions (β) and (γ).

To check that ν is countably additive, let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in Σ, with union E, and $\epsilon > 0$. Let $\delta > 0$ and $F \in \Sigma$ be such $\mu F < \infty$ and $|\nu G| \le \epsilon$ whenever $G \in \Sigma$ and $\mu(F \cap G) \le \delta$. Then

$$\sum_{n=0}^{\infty} \mu(E_n \cap F) \le \mu F < \infty,$$

so there is an $n \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} \mu(E_i \cap F) \le \delta$. Take any $m \ge n$ and consider $E_m^* = \bigcup_{i\le m} E_i$. We have

$$|\nu E - \sum_{i=0}^{m} \nu E_i| = |\nu E - \nu E_m^*| = |\nu(E \setminus E_m^*)| \le \epsilon,$$

because

$$\mu(F \cap E \setminus E_m^*) = \sum_{i=m+1}^{\infty} \mu(F \cap E_i) \le \delta.$$

As ϵ is arbitrary,

$$\nu E = \sum_{i=0}^{\infty} \nu E_i;$$

as $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, ν is countably additive.

(ii) Now suppose that ν satisfies the three conditions. By 231F, ν can be expressed as the difference of two non-negative countably additive functionals ν_1, ν_2; set $\nu' = \nu_1 + \nu_2$, so that ν' is a non-negative countably additive functional and $|\nu F| \le \nu' F$ for every $F \in \Sigma$. Set

$$\gamma = \sup\{\nu' F : F \in \Sigma,\ \mu F < \infty\} \le \nu' X < \infty,$$

and choose a sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of sets of finite measure such that $\lim_{n\to\infty} \nu' F_n = \gamma$; set $F^* = \bigcup_{n\in\mathbb{N}} F_n$. If $G \in \Sigma$ and $G \cap F^* = \emptyset$ then $\nu G = 0$. **P?** Otherwise, by condition (γ), there is an $F \in \Sigma$ such that $\mu F < \infty$ and $\nu(G \cap F) \ne 0$. It follows that

$$\nu'(F \setminus F^*) \ge \nu'(F \cap G) \ge |\nu(F \cap G)| > 0,$$

and there must be an $n \in \mathbb{N}$ such that

$$\gamma < \nu' F_n + \nu'(F \setminus F^*) = \nu'(F_n \cup (F \setminus F^*)) \le \nu'(F \cup F_n) \le \gamma$$

because $\mu(F \cup F_n) < \infty$; but this is impossible. **XQ**

Setting $F_n^* = \bigcup_{k\le n} F_k$ for each n, we have $\lim_{n\to\infty} \nu'(F^* \setminus F_n^*) = 0$. Take any $\epsilon > 0$, and (using condition (β)) let $\delta > 0$ be such that $|\nu E| \le \frac{1}{2}\epsilon$ whenever $\mu E \le \delta$. Let n be such that $\nu'(F^* \setminus F_n^*) \le \frac{1}{2}\epsilon$. Now if $F \in \Sigma$ and $\mu(F \cap F_n^*) \le \delta$ then

$$\begin{aligned}|\nu F| &\le |\nu(F\cap F_n^*)| + |\nu(F\cap F^*\setminus F_n^*)| + |\nu(F\setminus F^*)|\\ &\le \frac{1}{2}\epsilon + \nu'(F\cap F^*\setminus F_n^*) + 0\\ &\le \frac{1}{2}\epsilon + \nu'(F^*\setminus F_n^*) \le \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.\end{aligned}$$

And $\mu F_n^* < \infty$. As ϵ is arbitrary, ν is truly continuous.

(c) Now suppose that (X,Σ,μ) is σ-finite and that ν is countably additive and absolutely continuous with respect to μ. Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering X (211D). If $\nu E \ne 0$, then $\lim_{n\to\infty}\nu(E\cap X_n)\ne 0$, so $\nu(E\cap X_n)\ne 0$ for some n. This shows that ν satisfies condition (γ) of (b), so is truly continuous.

Of course the converse of this fact is already covered by (b).

(d) Finally, suppose that $\mu X < \infty$ and that ν is absolutely continuous with respect to μ. Then it must be truly continuous, because we can take $F = X$ in the definition 232Ab.

232C Lemma Let (X,Σ,μ) be a measure space and ν, ν' two countably additive functionals on Σ which are truly continuous with respect to μ. Take $c\in\mathbb{R}$ and $H\in\Sigma$, and set $\nu_H E = \nu(E\cap H)$ for $E\in\Sigma$, as in 231De. Then $\nu+\nu'$, $c\nu$ and ν_H are all truly continuous with respect to μ, and ν is expressible as the difference of non-negative countably additive functionals which are truly continuous with respect to μ.

proof Let $\epsilon > 0$. Set $\eta = \epsilon/(2+\epsilon+|c|) > 0$. Then there are δ, $\delta' > 0$ and E, $E'\in\Sigma$ such that $\mu E<\infty$, $\mu E'<\infty$ and $|\nu F|\le\eta$ whenever $\mu(F\cap E)\le\delta$, $|\nu' F|\le\eta$ whenever $\mu(F\cap E)\le\delta'$. Set $\delta^* = \min(\delta,\delta') > 0$, $E^* = E\cup E'\in\Sigma$; then

$$\mu E^* \le \mu E + \mu E' < \infty.$$

Suppose that $F\in\Sigma$ and $\mu(F\cap E^*)\le\delta^*$; then

$$\mu(F\cap H\cap E)\le\mu(F\cap E)\le\delta^*\le\delta,\quad \mu(F\cap E')\le\delta^*\le\delta'$$

so

$$|(\nu+\nu')F| \le |\nu F| + |\nu' F| \le \eta+\eta\le\epsilon,$$

$$|(c\nu)F| = |c||\nu F| \le |c|\eta \le \epsilon,$$

$$|\nu_H F| = |\nu(F\cap H)| \le \eta \le \epsilon.$$

As ϵ is arbitrary, $\nu+\nu'$, $c\nu$ and ν_H are all truly continuous.

Now, taking H from 231Eb, we see that $\nu_1 = \nu_H$ and $\nu_2 = -\nu_{X\setminus H}$ are truly continuous and non-negative, and $\nu = \nu_1 - \nu_2$ is the difference of truly continuous measures.

232D Proposition Let (X,Σ,μ) be a measure space, and f a μ-integrable real-valued function. For $E\in\Sigma$ set $\nu E = \int_E f$. Then $\nu:\Sigma\to\mathbb{R}$ is a countably additive functional and is truly continuous with respect to μ, therefore absolutely continuous with respect to μ.

proof Recall that $\int_E f = \int f\times\chi E$ is defined for every $E\in\Sigma$ (131Fa). So $\nu:\Sigma\to\mathbb{R}$ is well-defined. If E, $F\in\Sigma$ are disjoint then

$$\begin{aligned}\nu(E\cup F) = \int f\times\chi(E\cup F) &= \int (f\times\chi E) + (f\times\chi F)\\ &= \int f\times\chi E + \int f\times\chi F = \nu E + \nu F,\end{aligned}$$

so ν is finitely additive.

Now 225A, without using the phrase 'truly continuous', proved exactly that ν is truly continuous with respect to μ. It follows from 232Bb that ν is countably additive and absolutely continuous.

Remark The functional $E\mapsto\int_E f$ is called the **indefinite integral** of f.

232E We are now at last ready for the theorem.

The Radon-Nikodým theorem Let (X,Σ,μ) be a measure space and $\nu:\Sigma\to\mathbb{R}$ a function. Then the following are equiveridical:

(i) there is a μ-integrable function f such that $\nu E=\int_E f$ for every $E\in\Sigma$;

(ii) ν is finitely additive and truly continuous with respect to μ.

proof (a) If f is a μ-integrable real-valued function and $\nu E=\int_E f$ for every $E\in\Sigma$, then 232D tells us that ν is finitely additive and truly continuous.

(b) In the other direction, suppose that ν is finitely additive and truly continuous; note that (by 232B(a-b)) $\nu E=0$ whenever $\mu E=0$. To begin with, suppose that ν is non-negative and not zero.

In this case, there is a non-negative simple function f such that $\int f>0$ and $\int_E f\le\nu E$ for every $E\in\Sigma$. **P** Let $H\in\Sigma$ be such that $\nu H>0$; set $\epsilon=\frac{1}{3}\nu H>0$. Let $E\in\Sigma$, $\delta>0$ be such that $\mu E<\infty$ and $\nu F\le\epsilon$ whenever $F\in\Sigma$ and $\mu(F\cap E)\le\delta$; then $\nu(H\setminus E)\le\epsilon$ so $\nu E\ge\nu(H\cap E)\ge 2\epsilon$ and $\mu E\ge\mu(H\cap E)>0$. Set $\mu_E F=\mu(F\cap E)$ for every $F\in\Sigma$; then μ_E is a countably additive functional on Σ. Set $\nu'=\nu-\alpha\mu_E$, where $\alpha=\epsilon/\mu E$; then ν' is a countably additive functional and $\nu'E>0$. By 231Eb, as usual, there is a set $G\in\Sigma$ such that $\nu'F\ge 0$ if $F\in\Sigma$ and $F\subseteq G$, but $\nu'F\le 0$ if $F\in\Sigma$ and $F\cap G=\emptyset$. As $\nu'(E\setminus G)\le 0$,

$$0<\nu'E\le\nu'(E\cap G)\le\nu(E\cap G)$$

and $\mu(E\cap G)>0$. Set $f=\alpha\chi(E\cap G)$; then f is a non-negative simple function and $\int f=\alpha\mu(E\cap G)>0$.

If $F\in\Sigma$ then $\nu'(F\cap G)\ge 0$, that is,

$$\nu(F\cap G)\ge\alpha\mu_E(F\cap G)=\alpha\mu(F\cap E\cap G)=\int_F f.$$

So

$$\nu F\ge\nu(F\cap G)\ge\int_F f,$$

as required. **Q**

(c) Still supposing that ν is a non-negative, truly continuous additive functional, let Φ be the set of non-negative simple functions $f:X\to\mathbb{R}$ such that $\int_E f\le\nu E$ for every $E\in\Sigma$; then the constant function $\mathbf{0}$ belongs to Φ, so Φ is not empty.

If f, $g\in\Phi$ then $f\vee g\in\Phi$, where $(f\vee g)(x)=\max(f(x),g(x))$ for $x\in X$. **P** Set $H=\{x:(f-g)(x)\ge 0\}\in\Sigma$; then $f\vee g=(f\times\chi H)+(g\times\chi(X\setminus H))$ is a non-negative simple function, and for any $E\in\Sigma$,

$$\int_E f\vee g=\int_{E\cap H}f+\int_{E\setminus H}g\le\nu(E\cap H)+\nu(E\setminus H)=\nu E. \quad \mathbf{Q}$$

Set

$$\gamma=\sup\{\int f:f\in\Phi\}\le\nu X<\infty.$$

Choose a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in Φ such that $\lim_{n\to\infty}\int f_n=\gamma$. For each n, set $g_n=f_0\vee f_1\vee\ldots\vee f_n$; then $g_n\in\Phi$ and $\int f_n\le\int g_n\le\gamma$ for each n, so $\lim_{n\to\infty}\int g_n=\gamma$. By B.Levi's theorem, $f=\lim_{n\to\infty}g_n$ is integrable and $\int f=\gamma$. Note that if $E\in\Sigma$ then

$$\int_E f=\lim_{n\to\infty}\int_E f_n\le\nu E.$$

? Suppose, if possible, that there is an $H\in\Sigma$ such that $\int_H f\ne\nu H$. Set

$$\nu_1 F=\nu F-\int_F f\ge 0$$

for every $F\in\Sigma$; then by (a) of this proof and 232C, ν_1 is a truly continuous finitely additive functional, and we are supposing that $\nu_1\ne 0$. By (b) of this proof, there is a non-negative simple function g such that $\int_F g\le\nu_1 F$ for every $F\in\Sigma$ and $\int g>0$. Take $n\in\mathbb{N}$ such that $\int f_n+\int g>\gamma$. Then f_n+g is a non-negative simple function and

$$\int_F(f_n+g)=\int_F f_n+\int_F g\le\int_F f+\int_F g=\nu F-\nu_1 F+\int_F g\le\nu F$$

for any $F\in\Sigma$, so $f_n+g\in\Phi$, and

$$\gamma<\int f_n+\int g=\int f_n+g\le\gamma,$$

which is absurd. **X** Thus we have $\int_H f = \nu H$ for every $H \in \Sigma$.

(d) This proves the theorem for non-negative ν. For general ν, we need only observe that ν is expressible as $\nu_1 - \nu_2$, where ν_1 and ν_2 are non-negative truly continuous countably additive functionals, by 232C; so that there are integrable functions f_1, f_2 such that $\nu_i F = \int_F f_i$ for both i and every $F \in \Sigma$. Of course $f = f_1 - f_2$ is integrable and $\nu F = \int_F f$ for every $F \in \Sigma$. This completes the proof.

232F Corollary Let (X, Σ, μ) be a σ-finite measure space and $\nu : \Sigma \to \mathbb{R}$ a function. Then there is a μ-integrable function f such that $\nu E = \int_E f$ for every $E \in \Sigma$ iff ν is countably additive and absolutely continuous with respect to μ.

proof Put 232Bc and 232E together.

232G Corollary Let (X, Σ, μ) be a totally finite measure space and $\nu : \Sigma \to \mathbb{R}$ a function. Then there is a μ-integrable function f on X such that $\nu E = \int_E f$ for every $E \in \Sigma$ iff ν is finitely additive and absolutely continuous with respect to μ.

proof Put 232Bd and 232E together.

232H Remarks (a) Most authors are satisfied with 232F as the 'Radon-Nikodým theorem'. In my view the problem of identifying indefinite integrals is of sufficient importance to justify an analysis which applies to all measure spaces, even if it requires a new concept (the notion of 'truly continuous' functional).

(b) I ought to offer an example of an absolutely continuous functional which is not truly continuous. A simple one is the following. Let X be any uncountable set. Let Σ be the countable-cocountable σ-algebra of subsets of X and ν the countable-cocountable measure on X (211R). Let μ be the restriction to Σ of counting measure on X. If $\mu E = 0$ then $E = \emptyset$ and $\nu E = 0$, so ν is absolutely continuous. But for any E of finite measure we have $\nu(X \setminus E) = 1$, so ν is not truly continuous. See also 232Xf(i).

***(c)** The space (X, Σ, μ) of this example is, in terms of the classification developed in Chapter 21, somewhat irregular; for instance, it is neither locally determined nor localizable, and therefore not strictly localizable, though it is complete and semi-finite. Can this phenomenon occur in a strictly localizable measure space? We are led here into a fascinating question. Suppose, in (b), I used the same idea, but with $\Sigma = \mathcal{P}X$. No difficulty arises in constructing μ; but can there now be a ν with the required properties, that is, a non-zero countably additive functional from $\mathcal{P}X$ to $\mathbb{R}$ which is zero on all finite sets? This is the 'Banach-Ulam problem', on which I have written extensively elsewhere (Fremlin 93), and to which I will return in Chapter 54 in Volume 5. The present question is touched on again in 363S in Volume 3.

(d) Following the Radon-Nikodým theorem, the question immediately arises: for a given ν, how much possible variation is there in the corresponding f? The answer is straightforward enough: two integrable functions f and g give rise to the same indefinite integral iff they are equal almost everywhere (131Hb).

(e) I have stated the Radon-Nikodým theorem in terms of arbitrary integrable functions, meaning to interpret 'integrability' in a wide sense, according to the conventions of Volume 1. However, given a truly continuous additive functional ν, we can ask whether there is in any sense a canonical integrable function representing it. The answer is no. But we certainly do not need to take arbitrary integrable functions of the type considered in Chapter 12. If f is any integrable function, there is a conegligible set E such that $f{\restriction}E$ is measurable, and now we can find a conegligible measurable set $G \subseteq E \cap \operatorname{dom} f$; if we set $g(x) = f(x)$ for $x \in G$, 0 for $x \in X \setminus G$, then $f =_{\text{a.e.}} g$, so g has the same indefinite integral as f (as noted in (d) just above), while g is measurable and has domain X. Thus we can make a trivial, but sometimes convenient, refinement to the theorem: if (X, Σ, μ) is a measure space, and $\nu : \Sigma \to \mathbb{R}$ is finitely additive and truly continuous with respect to μ, then there is a Σ-measurable μ-integrable function $g : X \to \mathbb{R}$ such that $\int_E g = \nu E$ for every $E \in \Sigma$.

(f) It is convenient to introduce now a general definition. If (X, Σ, μ) is a measure space and ν is a $[-\infty, \infty]$-valued functional defined on a family of subsets of X, I will say that a $[-\infty, \infty]$-valued function

f defined on a subset of X is a **Radon-Nikodým derivative** of ν with respect to μ if $\int_E f d\mu$ is defined (in the sense of 214D) and equal to νE for every $E \in \operatorname{dom}\nu$. Thus the integrable functions called f in 232E-232G are all 'Radon-Nikodým derivatives'; later on we shall have less well-regulated examples.

When ν is a measure and f is non-negative, f may be called a **density function**.

(g) Throughout the work above I have taken it that ν is defined on the whole domain Σ of μ. In some of the most important applications, however, ν is defined only on some smaller σ-algebra T. In this case we commonly seek to apply the same results with $\mu\restriction\mathrm{T}$ in place of μ.

232I The Lebesgue decomposition of a countably additive functional: Proposition (a) Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a countably additive functional. Then ν has unique expressions as

$$\nu = \nu_s + \nu_{ac} = \nu_s + \nu_{tc} + \nu_e,$$

where ν_s is singular with respect to μ, ν_{ac} is absolutely continuous with respect to μ, ν_{tc} is truly continuous with respect to μ, and ν_e is absolutely continuous with respect to μ and zero on every set of finite measure.

(b) If $X = \mathbb{R}^r$, Σ is the algebra of Borel sets in $\mathbb{R}^r$ and μ is the restriction of Lebesgue measure to Σ, then ν is uniquely expressible as $\nu_p + \nu_{cs} + \nu_{ac}$ where ν_{ac} is absolutely continuous with respect to μ, ν_{cs} is singular with respect to μ and zero on singletons, and $\nu_p E = \sum_{x\in E} \nu_p\{x\}$ for every $E \in \Sigma$.

proof (a)(i) Suppose first that ν is non-negative. In this case, set

$$\nu_s E = \sup\{\nu(E \cap F) : F \in \Sigma,\ \mu F = 0\},$$

$$\nu_t E = \sup\{\nu(E \cap F) : F \in \Sigma,\ \mu F < \infty\}.$$

Then both ν_s and ν_t are countably additive. **P** Surely $\nu_s\emptyset = \nu_t\emptyset = 0$. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in Σ with union E. **(α)** If $F \in \Sigma$ and $\mu F = 0$, then

$$\nu(E \cap F) = \sum_{n=0}^{\infty} \nu(E_n \cap F) \le \sum_{n=0}^{\infty} \nu_s(E_n);$$

as F is arbitrary,

$$\nu_s E \le \sum_{n=0}^{\infty} \nu_s E_n.$$

(β) If $F \in \Sigma$ and $\mu F < \infty$, then

$$\nu(E \cap F) = \sum_{n=0}^{\infty} \nu(E_n \cap F) \le \sum_{n=0}^{\infty} \nu_t(E_n);$$

as F is arbitrary,

$$\nu_t E \le \sum_{n=0}^{\infty} \nu_t E_n.$$

(γ) If $\epsilon > 0$, then (because $\sum_{n=0}^{\infty} \nu E_n = \nu E < \infty$) there is an $n \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} \nu E_k \le \epsilon$. Now, for each $k \le n$, there is an $F_k \in \Sigma$ such that $\mu F_k = 0$ and $\nu(E_k \cap F_k) \ge \nu_s E_k - \frac{\epsilon}{n+1}$. In this case, $F = \bigcup_{k\le n} F_k \in \Sigma$, $\mu F = 0$ and

$$\nu_s E \ge \nu(E \cap F) \ge \sum_{k=0}^{n} \nu(E_k \cap F_k) \ge \sum_{k=0}^{n} \nu_s E_k - \epsilon \ge \sum_{k=0}^{\infty} \nu_s E_k - 2\epsilon,$$

because

$$\sum_{k=n+1}^{\infty} \nu_s E_k \le \sum_{k=n+1}^{\infty} \nu E_k \le \epsilon.$$

As ϵ is arbitrary,

$$\nu_s E \ge \sum_{k=0}^{\infty} \nu_s E_k.$$

(δ) Similarly, for each $k \le n$, there is an $F'_k \in \Sigma$ such that $\mu F'_k < \infty$ and $\nu(E_k \cap F'_k) \ge \nu_t E_k - \frac{\epsilon}{n+1}$. In this case, $F' = \bigcup_{k\le n} F'_k \in \Sigma$, $\mu F' < \infty$ and

$$\nu_t E \ge \nu(E \cap F') \ge \sum_{k=0}^{n} \nu(E_k \cap F'_k) \ge \sum_{k=0}^{n} \nu_t E_k - \epsilon \ge \sum_{k=0}^{\infty} \nu_t E_k - 2\epsilon,$$

because

$$\sum_{k=n+1}^{\infty} \nu_t E_k \le \sum_{k=n+1}^{\infty} \nu E_k \le \epsilon.$$

As ϵ is arbitrary,

$$\nu_t E \geq \sum_{k=0}^{\infty} \nu_t E_k.$$

(ϵ) Putting these together, $\nu_s E = \sum_{n=0}^{\infty} \nu_s E_n$ and $\nu_t E = \sum_{n=0}^{\infty} \nu_t E_n$. As $\langle E_n \rangle_{n\in\mathbb{N}}$ is arbitrary, ν_s and ν_t are countably additive. **Q**

(ii) Still supposing that ν is non-negative, if we choose a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ in Σ such that $\mu F_n = 0$ for each n and $\lim_{n\to\infty} \nu F_n = \nu_s X$, then $F^* = \bigcup_{n\in\mathbb{N}} F_n$ has $\mu F^* = 0$ and $\nu_s F^* = \nu F^* = \nu_s X$; so that $\nu_s(X \setminus F^*) = 0$, and ν_s is singular with respect to μ in the sense of 232Ac.

Note that $\nu_s F = \nu F$ whenever $\mu F = 0$. So if we write $\nu_{ac} = \nu - \nu_s$, then ν_{ac} is a countably additive functional and $\nu_{ac} F = 0$ whenever $\mu F = 0$; that is, ν_{ac} is absolutely continuous with respect to μ.

If we write $\nu_{tc} = \nu_t - \nu_s$, then ν_{tc} is a non-negative countably additive functional; $\nu_{tc} F = 0$ whenever $\mu F = 0$, and if $\nu_{tc} E > 0$ there is a set F with $\mu F < \infty$ and $\nu_{tc}(E \cap F) > 0$. So ν_{tc} is truly continuous with respect to μ, by 232Bb. Set $\nu_e = \nu - \nu_t = \nu_{ac} - \nu_{tc}$.

Thus for any non-negative countably additive functional ν, we have expressions

$$\nu = \nu_s + \nu_{ac}, \quad \nu_{ac} = \nu_{tc} + \nu_e$$

where ν_s, ν_{ac}, ν_{tc} and ν_e are all non-negative countably additive functionals, ν_s is singular with respect to μ, ν_{ac} and ν_e are absolutely continuous with respect to μ, ν_{tc} is truly continuous with respect to μ, and $\nu_e F = 0$ whenever $\mu F < \infty$.

(iii) For general countably additive functionals $\nu : \Sigma \to \mathbb{R}$, we can express ν as $\nu' - \nu''$, where ν' and ν'' are non-negative countably additive functionals. If we define ν'_s, $\nu''_s, \ldots, \nu''_e$ as in (i)-(ii), we get countably additive functionals

$$\nu_s = \nu'_s - \nu''_s, \quad \nu_{ac} = \nu'_{ac} - \nu''_{ac}, \quad \nu_{tc} = \nu'_{tc} - \nu''_{tc}, \quad \nu_e = \nu'_e - \nu''_e$$

such that ν_s is singular with respect to μ (if F', F'' are such that

$$\mu F = \mu F' = \nu'_s(X \setminus F) = \nu''_s(X \setminus F) = 0,$$

then $\mu(F' \cup F'') = 0$ and $\nu_s E = 0$ whenever $E \subseteq X \setminus (F' \cup F'')$), ν_{ac} is absolutely continuous with respect to μ, ν_{tc} is truly continuous with respect to μ, and $\nu_e F = 0$ whenever $\mu F < \infty$, while

$$\nu = \nu_s + \nu_{ac} = \nu_s + \nu_{tc} + \nu_e.$$

(iv) Moreover, these decompositions are unique. **P(α)** If, for instance, $\nu = \tilde{\nu}_s + \tilde{\nu}_{ac}$, where $\tilde{\nu}_s$ is singular and $\tilde{\nu}_{ac}$ is absolutely continuous with respect to μ, let F, $\tilde{F}$ be such that $\mu F = \mu\tilde{F} = 0$ and $\tilde{\nu}_s E = 0$ whenever $E \cap \tilde{F} = \emptyset$, $\nu_s E = 0$ whenever $E \cap F = \emptyset$. Set $F^* = F \cup \tilde{F}$. If $E \in \Sigma$, $\nu_{ac}(E \cap F^*) = 0$ because $\mu F^* = 0$ while $\nu_s(E \setminus F^*) = 0$ because $E \setminus F^*$ is disjoint from F, so

$$\nu_s E = \nu_s(E \cap F^*) + \nu_s(E \setminus F^*) = \nu(E \cap F^*) - \nu_{ac}(E \cap F^*) + 0 = \nu(E \cap F^*).$$

Similarly, $\tilde{\nu}_s E = \nu(E \cap F^*)$ and $\tilde{\nu}_s E = \nu_s E$; as E is arbitrary, $\tilde{\nu}_s = \nu_s$ and $\tilde{\nu}_{ac} = \nu_{ac}$.

(β) Similarly, if $\nu_{ac} = \tilde{\nu}_{tc} + \tilde{\nu}_e$ where $\tilde{\nu}_{tc}$ is truly continuous with respect to μ and $\tilde{\nu}_e F = 0$ whenever $\mu F < \infty$, then there are sequences $\langle F_n \rangle_{n\in\mathbb{N}}$, $\langle \tilde{F}_n \rangle_{n\in\mathbb{N}}$ of sets of finite measure such that $\nu_{tc} F = 0$ whenever $F \cap \bigcup_{n\in\mathbb{N}} F_n = \emptyset$ and $\tilde{\nu}_{tc} F = 0$ whenever $F \cap \bigcup_{n\in\mathbb{N}} \tilde{F}_n = \emptyset$. Write $F^* = \bigcup_{n\in\mathbb{N}}(F_n \cup \tilde{F}_n)$; then $\tilde{\nu}_e E = \nu_e E = 0$ whenever $E \subseteq F^*$ and $\tilde{\nu}_{tc} E = \nu_{tc} E = 0$ whenever $E \cap F^* = \emptyset$, so $\nu_{tc} E = \nu_{ac}(E \cap F^*) = \tilde{\nu}_{tc} E$ for every $E \in \Sigma$, $\nu_{tc} = \tilde{\nu}_{tc}$ and $\nu_e = \tilde{\nu}_e$. **Q**

(b) In this case, μ is σ-finite (cf. 211P), so every absolutely continuous countably additive functional is truly continuous (232Bc), and we shall always have $\nu_e = 0$, $\nu_{ac} = \nu_{tc}$. But in the other direction we know that singleton sets, and therefore countable sets, are all measurable. We therefore have a further decomposition $\nu_s = \nu_p + \nu_{cs}$, where there is a countable set $K \subseteq \mathbb{R}^r$ with $\nu_p E = 0$ whenever $E \in \Sigma$ and $E \cap K = \emptyset$, and ν_{cs} is singular with respect to μ and zero on countable sets. **P** (i) If $\nu \geq 0$, set

$$\nu_p E = \sup\{\nu(E \cap K) : K \subseteq \mathbb{R}^r \text{ is countable}\};$$

just as with ν_s, dealt with in (a) above, ν_p is countably additive and there is a countable $K \subseteq \mathbb{R}^r$ such that $\nu_p E = \nu(E \cap K)$ for every $E \in \Sigma$. (ii) For general ν, we can express ν as $\nu' - \nu''$ where ν' and ν'' are non-negative, and write $\nu_p = \nu'_p - \nu''_p$. (iii) ν_p is characterized by saying that there is a countable set K such that $\nu_p E = \nu(E \cap K)$ for every $E \in \Sigma$ and $\nu\{x\} = 0$ for every $x \in \mathbb{R}^r \setminus K$. (iv) So if we set $\nu_{cs} = \nu_s - \nu_p$, ν_{cs} will be singular with respect to μ and zero on countable sets. **Q**

Now, for any $E \in \Sigma$,

$$\nu_p E = \nu(E \cap K) = \sum_{x \in K \cap E} \nu\{x\} = \sum_{x \in E} \nu\{x\}.$$

Remark The expression $\nu = \nu_p + \nu_{cs} + \nu_{ac}$ of (b) is the **Lebesgue decomposition** of ν.

232X Basic exercises (a) Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a countably additive functional which is absolutely continuous with respect to μ. Show that the following are equiveridical: (i) ν is truly continuous with respect to μ; (ii) there is a sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in Σ such that $\mu E_n < \infty$ for every $n \in \mathbb{N}$ and $\nu F = 0$ whenever $F \in \Sigma$ and $F \cap \bigcup_{n \in \mathbb{N}} E_n = \emptyset$.

>(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded non-decreasing function and μ_g the associated Lebesgue-Stieltjes measure (114Xa). Show that μ_g is absolutely continuous (equivalently, truly continuous) with respect to Lebesgue measure iff the restriction of g to any closed bounded interval is absolutely continuous in the sense of 225B.

(c) Let (X, Σ, μ) be a measure space, and ν and λ additive functionals on Σ of which ν is positive and countably additive, so that (X, Σ, ν) also is a measure space. (i) Show that if ν is absolutely continuous with respect to μ and λ is absolutely continuous with respect to ν, then λ is absolutely continuous with respect to μ. (ii) Show that if ν is truly continuous with respect to μ and λ is absolutely continuous with respect to ν then λ is truly continuous with respect to μ.

>(d) Let X be a non-empty set and Σ a σ-algebra of subsets of X. Show that for any sequence $\langle \nu_n \rangle_{n \in \mathbb{N}}$ of countably additive functionals on Σ there is a probability measure μ on X, with domain Σ, such that every ν_n is absolutely continuous with respect to μ. (*Hint*: start with the case $\nu_n \ge 0$.)

(e) Let (X, Σ, μ) be a measure space and $(X, \hat{\Sigma}, \hat{\mu})$ its completion (212C). Let $\nu : \Sigma \to \mathbb{R}$ be an additive functional such that $\nu E = 0$ whenever $\mu E = 0$. Show that ν has a unique extension to an additive functional $\hat{\nu} : \hat{\Sigma} \to \mathbb{R}$ such that $\hat{\nu} E = 0$ whenever $\hat{\mu} E = 0$.

(f) Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$ including the filter $\{\mathbb{N} \setminus I : I \subseteq \mathbb{N}$ is finite$\}$ (2A1O). Define $\nu : \mathcal{P}\mathbb{N} \to \{0, 1\}$ by setting $\nu E = 1$ if $E \in \mathcal{F}$, 0 for $E \in \mathcal{P}\mathbb{N} \setminus \mathcal{F}$. (i) Let μ_1 be counting measure on $\mathcal{P}\mathbb{N}$. Show that ν is additive and absolutely continuous with respect to μ_1, but is not truly continuous. (ii) Define $\mu_2 : \mathcal{P}\mathbb{N} \to [0, 1]$ by setting $\mu_2 E = \sum_{n \in E} 2^{-n-1}$. Show that ν is zero on μ_2-negligible sets, but is not absolutely continuous with respect to μ_2.

(g) Let X be a set and Σ a σ-algebra of subsets of X; let $\nu : \Sigma \to \mathbb{R}$ be a countably additive functional. Let $\mathcal{I}$ be an **ideal** of Σ, that is, a subset of Σ such that (α) $\emptyset \in \mathcal{I}$ (β) $E \cup F \in \mathcal{I}$ for all E, $F \in \mathcal{I}$ (γ) if $E \in \Sigma$, $F \in \mathcal{I}$ and $E \subseteq F$ then $E \in \mathcal{I}$. Show that ν has a unique decomposition as $\nu = \nu_{\mathcal{I}} + \nu'_{\mathcal{I}}$, where $\nu_{\mathcal{I}}$ and $\nu'_{\mathcal{I}}$ are countably additive functionals, $\nu'_{\mathcal{I}} E = 0$ for every $E \in \mathcal{I}$, and whenever $E \in \Sigma$ and $\nu_{\mathcal{I}} E \ne 0$ there is an $F \in \mathcal{I}$ such that $\nu_{\mathcal{I}}(E \cap F) \ne 0$.

(h) Rewrite this section in terms of complex-valued additive functionals.

232Y Further exercises (a) Let (X, Σ, μ) be a measure space and $\nu : \Sigma \to \mathbb{R}$ a finitely additive functional. If E, F, $H \in \Sigma$ and $\mu H < \infty$ set $\rho_H(E, F) = \mu(H \cap (E \triangle F))$. (i) Show that ρ_H is a pseudometric on Σ (2A3Fa). (ii) Let $\mathfrak{T}$ be the topology on Σ generated by $\{\rho_H : H \in \Sigma, \mu H < \infty\}$ (2A3Fc). Show that ν is continuous for $\mathfrak{T}$ iff it is truly continuous in the sense of 232Ab. ($\mathfrak{T}$ is the topology of **convergence in measure** on Σ.)

(b) Let (X, Σ, μ) be a measure space and $(X, \tilde{\Sigma}, \tilde{\mu})$ its c.l.d. version (213E). Let $\nu : \Sigma \to \mathbb{R}$ be an additive functional which is truly continuous with respect to μ. Show that ν has a unique extension to a functional $\tilde{\nu} : \tilde{\Sigma} \to \mathbb{R}$ which is truly continuous with respect to $\tilde{\mu}$.

(c) (H.König) Let X be a set and μ, ν two measures on X with the same domain Σ. For $\alpha \ge 0$, $E \in \Sigma$ set $(\alpha\mu \wedge \nu)(E) = \inf\{\alpha\mu(E \cap F) + \nu(E \setminus F) : F \in \Sigma\}$ (cf. 112Ya). Show that the following are equiveridical: (i) $\nu E = 0$ whenever $\mu E = 0$; (ii) $\sup_{\alpha \ge 0}(\alpha\mu \wedge \nu)(E) = \nu E$ for every $E \in \Sigma$.

(d) Let (X, Σ, μ) be a measure space and f a μ-integrable real-valued function. Show that the indefinite integral of f is the unique additive functional $\nu : \Sigma \to \mathbb{R}$ such that whenever $E \in \Sigma$ and $f(x) \in [a, b]$ for almost every $x \in E$, then $a\mu E \le \nu E \le b\mu E$.

(e) Let (X, Σ, μ) be a measure space and f a non-negative real-valued function which is integrable over X; let ν be its indefinite integral. Show that for any function $g : X \to \mathbb{R}$, $\int g\,d\nu = \int f \times g\,d\mu$ in the sense that if one of these is defined in $[-\infty, \infty]$ so is the other, and they are then equal. (*Hint*: start with simple functions g.)

(f) Let (X, Σ, μ) be a measure space, f an integrable function, and $\nu : \Sigma \to \mathbb{R}$ the indefinite integral of f. Show that $|\nu|$, as defined in 231Ya, is the indefinite integral of $|f|$.

(g) Let X be a set, Σ a σ-algebra of subsets of X, and $\nu : \Sigma \to \mathbb{R}$ a countably additive functional. Show that ν has a Radon-Nikodým derivative with respect to $|\nu|$ as defined in 231Ya, and that any such derivative has modulus equal to 1 $|\nu|$-a.e.

(h) For a non-decreasing function $F : [a, b] \to \mathbb{R}$, where $a < b$, let ν_F be the corresponding Lebesgue-Stieltjes measure. Show that if we define $(\nu_F)_{ac}$, etc., with regard to Lebesgue measure on $[a, b]$, as in 232I, then

$$(\nu_F)_p = \nu_{F_p}, \quad (\nu_F)_{ac} = \nu_{F_{ac}}, \quad (\nu_F)_{cs} = \nu_{F_{cs}},$$

where F_p, F_{cs} and F_{ac} are defined as in 226C.

(i) Extend the idea of (h) to general functions F of bounded variation.

(j) Extend the ideas of (h) and (i) to open, half-open and unbounded intervals (cf. 226Yb).

(k) Say that two bounded additive functionals ν_1, ν_2 on an algebra Σ of sets are **mutually singular** if for any $\epsilon > 0$ there is an $H \in \Sigma$ such that

$$\sup\{|\nu_1 F| : F \in \Sigma,\ F \subseteq H\} \le \epsilon,$$

$$\sup\{|\nu_2 F| : F \in \Sigma,\ F \cap H = \emptyset\} \le \epsilon.$$

(i) Show that ν_1 and ν_2 are mutually singular iff, in the language of 231Ya-231Yb, $|\nu_1| \wedge |\nu_2| = 0$.

(ii) Show that if Σ is a σ-algebra and ν_1 and ν_2 are countably additive, then they are mutually singular iff there is an $H \in \Sigma$ such that $\nu_1 F = 0$ whenever $F \in \Sigma$ and $F \subseteq H$, while $\nu_2 F = 0$ whenever $F \in \Sigma$ and $F \cap H = \emptyset$.

(iii) Show that if ν_s, ν_{tc} and ν_e are defined from ν and μ as in 232I, then each pair of the three are mutually singular.

232 Notes and comments The Radon-Nikodým theorem must be on any list of the half-dozen most important theorems of measure theory, and not only the theorem itself, but the techniques necessary to prove it, are at the heart of the subject. In my book Fremlin 74 I discussed a variety of more or less abstract versions of the theorem and of the method, to some of which I will return in §§327 and 365 of the next volume.

As I have presented it here, the essence of the proof is split between 231E and 232E. I think we can distinguish the following elements. Let ν be a countably additive functional.

(i) ν is bounded (231Ea).

(ii) ν is expressible as the difference of non-negative functionals (231F).

(I gave this as a corollary of 231Eb, but it can also be proved by simpler methods, as in 231Ya.)

(iii) If $\nu > 0$, there is an integrable f such that $0 < \nu_f \le \nu$, writing ν_f for the indefinite integral of f. (This is the point at which we really do need the Hahn decomposition 231Eb.)

(iv) The set $\Psi = \{f : \nu_f \le \nu\}$ is closed under countable suprema, so there is an $f \in \Psi$ maximising $\int f$.

(In part (b) of the proof of 232E, I spoke of simple functions; but this was solely to simplify the technical details, and the same argument works if we apply it to Ψ instead of Φ. Note the use here of B.Levi's theorem.)

(v) Take f from (iv) and use (iii) to show that $\nu - \nu_f = 0$.

Each of the steps (i)-(iv) requires a non-trivial idea, and the importance of the theorem lies not only in its remarkable direct consequences in the rest of this chapter and elsewhere, but in the versatility and power of these ideas.

I introduce the idea of 'truly continuous' functional in order to give a reasonably straightforward account of the status of the Radon-Nikodým theorem in non-σ-finite measure spaces. Of course the whole point is that a truly continuous functional, like an indefinite integral, must be concentrated on a σ-finite part of the space (232Xa), so that 232E, as stated, can be deduced easily from the standard form 232F. I dare to use the word 'truly' in this context because this kind of continuity does indeed correspond to a topological notion (232Ya).

There is a possible trap in the definition I give of 'absolutely continuous' functional. Many authors use the condition of 232Ba as a definition, saying that ν is absolutely continuous with respect to μ if $\nu E = 0$ whenever $\mu E = 0$. For countably additive functionals this coincides with the ϵ-δ formulation in 232Aa; but for other additive functionals this need not be so (232Xf(ii)). Mostly the distinction is insignificant, but I note that in 232Bd it is critical, since ν there is not assumed to be countably additive.

In 232I I describe one of the many ways of decomposing a countably additive functional into mutually singular parts with special properties. In 231Yf-231Yg I have already suggested a method of decomposing an additive functional into the sum of a countably additive part and a 'purely finitely additive' part. All these results have natural expressions in terms of the ordered linear space of bounded additive functionals on an algebra (231Yc).

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233 Conditional expectations

I devote a section to a first look at one of the principal applications of the Radon-Nikodým theorem. It is one of the most vital ideas of measure theory, and will appear repeatedly in one form or another. Here I give the definition and most basic properties of conditional expectations as they arise in abstract probability theory, with notes on convex functions and a version of Jensen's inequality (233I-233J).

233A σ-subalgebras Let X be a set and Σ a σ-algebra of subsets of X. A **σ-subalgebra** of Σ is a σ-algebra T of subsets of X such that $\mathrm{T} \subseteq \Sigma$. If (X, Σ, μ) is a measure space and T is a σ-subalgebra of Σ, then $(X, \mathrm{T}, \mu\restriction\mathrm{T})$ is again a measure space; this is immediate from the definition (112A). Now we have the following straightforward lemma. It is a special case of 235G below, but I give a separate proof in case you do not wish as yet to embark on the general investigation pursued in §235.

233B Lemma Let (X, Σ, μ) be a measure space and T a σ-subalgebra of Σ. A real-valued function f defined on a subset of X is $\mu\restriction\mathrm{T}$-integrable iff (i) it is μ-integrable (ii) $\operatorname{dom} f$ is $\mu\restriction\mathrm{T}$-conegligible (iii) f is $\mu\restriction\mathrm{T}$-virtually measurable; and in this case $\int f d(\mu\restriction\mathrm{T}) = \int f d\mu$.

proof (a) Note first that if f is a $\mu\restriction\mathrm{T}$-simple function, that is, is expressible as $\sum_{i=0}^{n} a_i \chi E_i$ where $a_i \in \mathbb{R}$, $E_i \in \mathrm{T}$ and $(\mu\restriction\mathrm{T})E_i < \infty$ for each i, then f is μ-simple and

$$\int f d\mu = \sum_{i=0}^{n} a_i \mu E_i = \int f d(\mu\restriction\mathrm{T}).$$

(b) Let U_μ be the set of non-negative μ-integrable functions and $U_{\mu\restriction\mathrm{T}}$ the set of non-negative $\mu\restriction\mathrm{T}$-integrable functions.

Suppose $f \in U_{\mu\restriction\mathrm{T}}$. Then there is a non-decreasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of $\mu\restriction\mathrm{T}$-simple functions such that $f(x) = \lim_{n\to\infty} f_n$ $\mu\restriction\mathrm{T}$-a.e. and

$$\int f d(\mu\restriction\mathrm{T}) = \lim_{n\to\infty} \int f_n d(\mu\restriction\mathrm{T}).$$

But now every f_n is also μ-simple, and $\int f_n d\mu = \int f_n d(\mu\restriction\mathrm{T})$ for every n, and $f = \lim_{n\to\infty} f_n$ μ-a.e. So $f \in U_\mu$ and $\int f d\mu = \int f d(\mu\restriction\mathrm{T})$.

(c) Now suppose that f is $\mu\restriction\mathrm{T}$-integrable. Then it is the difference of two members of $U_{\mu\restriction\mathrm{T}}$, so is μ-integrable, and $\int f d\mu = \int f d(\mu\restriction\mathrm{T})$. Also conditions (ii) and (iii) are satisfied, according to the conventions established in Volume 1 (122Nc, 122P-122Q).

(d) Suppose that f satisfies conditions (i)-(iii). Then $|f| \in U_\mu$, and there is a conegligible set $E \subseteq \operatorname{dom} f$ such that $E \in \mathrm{T}$ and $f\restriction E$ is T-measurable. Accordingly $|f|\restriction E$ is T-measurable. Now, if $\epsilon > 0$, then

$$(\mu\restriction\mathrm{T})\{x : x \in E, |f|(x) \ge \epsilon\} = \mu\{x : x \in E, |f|(x) \ge \epsilon\} \le \tfrac{1}{\epsilon}\int |f| d\mu < \infty;$$

moreover,

$$\begin{aligned}\sup\{\int g\, d(\mu\restriction\mathrm{T}) : g \text{ is a } \mu\restriction\mathrm{T}\text{-simple function}, g \le |f|\ \mu\restriction\mathrm{T}\text{-a.e.}\}\\ = \sup\{\int g\, d\mu : g \text{ is a } \mu\restriction\mathrm{T}\text{-simple function}, g \le |f|\ \mu\restriction\mathrm{T}\text{-a.e.}\}\\ \le \sup\{\int g\, d\mu : g \text{ is a } \mu\text{-simple function}, g \le |f|\ \mu\text{-a.e.}\}\\ \le \int |f| d\mu < \infty.\end{aligned}$$

By the criterion of 122Ja, $|f| \in U_{\mu\restriction\mathrm{T}}$. Consequently f, being $\mu\restriction\mathrm{T}$-virtually T-measurable, is $\mu\restriction\mathrm{T}$-integrable, by 122P. This completes the proof.

233C Remarks (a) My argument just above is detailed to the point of pedantry. I think, however, that while I can be accused of wasting paper by writing everything down, every element of the argument is necessary to the result. To be sure, some of the details are needed only because I use such a wide notion of 'integrable function'; if you restrict the notion of 'integrability' to measurable functions defined on the whole measure space, there are simplifications at this stage, to be paid for later when you discover that many of the principal applications are to functions defined by formulae which do not apply on the whole underlying space.

The essential point which does have to be grasped is that while a $\mu\restriction\mathrm{T}$-negligible set is always μ-negligible, a μ-negligible set need not be $\mu\restriction\mathrm{T}$-negligible.

(b) As the simplest possible example of the problems which can arise, I offer the following. Let (X, Σ, μ) be $[0, 1]^2$ with Lebesgue measure. Let T be the set of those members of Σ expressible as $F \times [0, 1]$ for some $F \subseteq [0, 1]$; it is easy to see that T is a σ-subalgebra of Σ. Consider f, $g : X \to [0, 1]$ defined by saying that

$$f(t, u) = 1 \text{ if } u > 0,\ 0 \text{ otherwise},$$

$$g(t, u) = 1 \text{ if } t > 0,\ 0 \text{ otherwise}.$$

Then both f and g are μ-integrable, being constant μ-a.e. But only g is $\mu\restriction\mathrm{T}$-integrable, because any non-negligible $E \in \mathrm{T}$ includes a complete vertical section $\{t\} \times [0, 1]$, so that f takes both values 0 and 1 on E. If we set

$$h(t, u) = 1 \text{ if } u > 0, \text{ undefined otherwise},$$

then again (on the conventions I use) h is μ-integrable but not $\mu\restriction\mathrm{T}$-integrable, as there is no conegligible member of T included in the domain of h.

(c) If f is defined everywhere in X, and $\mu\restriction\mathrm{T}$ is complete, then of course f is $\mu\restriction\mathrm{T}$-integrable iff it is μ-integrable and T-measurable. But note that in the example just above, which is one of the archetypes for this topic, $\mu\restriction\mathrm{T}$ is not complete, as singleton sets are negligible but not measurable.

233D Conditional expectations Let (X,Σ,μ) be a probability space, that is, a measure space with $\mu X = 1$. (Nearly all the ideas here work perfectly well for any totally finite measure space, but there seems nothing to be gained from the extension, and the traditional phrase 'conditional expectation' demands a probability space.) Let $\mathrm{T} \subseteq \Sigma$ be a σ-subalgebra.

(a) For any μ-integrable real-valued function f defined on a conegligible subset of X, we have a corresponding indefinite integral $\nu_f : \Sigma \to \mathbb{R}$ given by the formula $\nu_f E = \int_E f$ for every $E \in \Sigma$. We know that ν_f is countably additive and truly continuous with respect to μ, which in the present context is the same as saying that it is absolutely continuous (232Bc-232Bd). Now consider the restrictions $\mu{\restriction}\mathrm{T}$, $\nu_f{\restriction}\mathrm{T}$ of μ and ν_f to the σ-algebra T. It follows directly from the definitions of 'countably additive' and 'absolutely continuous' that $\nu_f{\restriction}\mathrm{T}$ is countably additive and absolutely continuous with respect to $\mu{\restriction}\mathrm{T}$, therefore truly continuous with respect to $\mu{\restriction}\mathrm{T}$. Consequently, the Radon-Nikodým theorem (232E) tells us that there is a $\mu{\restriction}\mathrm{T}$-integrable function g such that $(\nu_f{\restriction}\mathrm{T})F = \int_F g\,d(\mu{\restriction}\mathrm{T})$ for every $F \in \mathrm{T}$.

(b) Let us define **a conditional expectation of** f **on** T to be such a function; that is, a $\mu{\restriction}\mathrm{T}$-integrable function g such that $\int_F g\,d(\mu{\restriction}\mathrm{T}) = \int_F f d\mu$ for every $F \in \mathrm{T}$. Looking back at 233B, we see that for such a g we have

$$\int_F g\,d(\mu{\restriction}\mathrm{T}) = \int g \times \chi F\,d(\mu{\restriction}\mathrm{T}) = \int g \times \chi F\,d\mu = \int_F g\,d\mu$$

for every $F \in \mathrm{T}$; also, that g is almost everywhere equal to a T-measurable function defined everywhere in X which is also a conditional expectation of f on T (232He).

(c) I set the word 'a' of the phrase 'a conditional expectation' in bold type to emphasize that there is nothing unique about the function g. In 242J I will return to this point, and describe an object which could properly be called 'the' conditional expectation of f on T. g is 'essentially unique' only in the sense that if g_1, g_2 are both conditional expectations of f on T then $g_1 = g_2$ $\mu{\restriction}\mathrm{T}$-a.e. (131Hb). This does of course mean that a very large number of its properties – for instance, the distribution function $G(a) = \hat{\mu}\{x : g(x) \le a\}$, where $\hat{\mu}$ is the completion of μ (212C) – are independent of which g we take.

(d) A word of explanation of the phrase 'conditional expectation' is in order. This derives from the standard identification of probability with measure, due to Kolmogorov, which I will discuss more fully in Chapter 27. A real-valued random variable may be regarded as a measurable, or virtually measurable, function f on a probability space (X,Σ,μ); its 'expectation' becomes identified with $\int f d\mu$, supposing that this exists. If $F \in \Sigma$ and $\mu F > 0$ then the 'conditional expectation of f given F' is $\frac{1}{\mu F}\int_F f$. If $F_0, \dots, F_n$ is a partition of X into measurable sets of non-zero measure, then the function g given by

$$g(x) = \frac{1}{\mu F_i}\int_{F_i} f \text{ if } x \in F_i$$

is a kind of anticipated conditional expectation; if we are one day told that $x \in F_i$, then $g(x)$ will be our subsequent estimate of the expectation of f. In the terms of the definition above, g is a conditional expectation of f on the finite algebra T generated by $\{F_0, \dots, F_n\}$. An appropriate intuition for general σ-algebras T is that they consist of the events which we shall be able to observe at some stated future time t_0, while the whole algebra Σ consists of all events, including those not observable until times later than t_0, if ever.

233E I list some of the elementary facts concerning conditional expectations.

Proposition Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of μ-integrable real-valued functions, and for each n let g_n be a conditional expectation of f_n on T. Then

(a) $g_1 + g_2$ is a conditional expectation of $f_1 + f_2$ on T;

(b) for any $c \in \mathbb{R}$, cg_0 is a conditional expectation of cf_0 on T;

(c) if $f_1 \le_{\text{a.e.}} f_2$ then $g_1 \le_{\text{a.e.}} g_2$;

(d) if $\langle f_n\rangle_{n\in\mathbb{N}}$ is non-decreasing a.e. and $f = \lim_{n\to\infty} f_n$ is μ-integrable, then $\lim_{n\to\infty} g_n$ is a conditional expectation of f on T;

(e) if $f = \lim_{n\to\infty} f_n$ is defined a.e. and there is a μ-integrable function h such that $|f_n| \le_{\text{a.e.}} h$ for every n, then $\lim_{n\to\infty} g_n$ is a conditional expectation of f on T;

(f) if $F \in \mathrm{T}$ then $g_0 \times \chi F$ is a conditional expectation of $f_0 \times \chi F$ on T;

(g) if h is a bounded, $\mu\restriction\mathrm{T}$-virtually measurable real-valued function defined $\mu\restriction\mathrm{T}$-almost everywhere in X, then $g_0 \times h$ is a conditional expectation of $f_0 \times h$ on T;

(h) if Υ is a σ-subalgebra of T, then a function h_0 is a conditional expectation of f_0 on Υ iff it is a conditional expectation of g_0 on Υ.

proof (a)-(b) We have only to observe that

$$\int_F g_1 + g_2 d(\mu\restriction\mathrm{T}) = \int_F g_1 d(\mu\restriction\mathrm{T}) + \int_F g_2 d(\mu\restriction\mathrm{T}) = \int_F f_1 d\mu + \int_F f_2 d\mu = \int_F f_1 + f_2 d\mu,$$

$$\int_F cg_0 d(\mu\restriction\mathrm{T}) = c\int_F g_0 d(\mu\restriction\mathrm{T}) = c\int_F f_0 d\mu = \int_F cf_0 d\mu$$

for every $F \in \mathrm{T}$.

(c) If $F \in \mathrm{T}$ then

$$\int_F g_1 d(\mu\restriction\mathrm{T}) = \int_F f_1 d\mu \le \int_F f_2 d\mu = \int_F g_2 d(\mu\restriction\mathrm{T})$$

for every $F \in \mathrm{T}$; consequently $g_1 \le g_2$ $\mu\restriction\mathrm{T}$-a.e. (131Ha).

(d) By (c), $\langle g_n\rangle_{n\in\mathbb{N}}$ is non-decreasing $\mu\restriction\mathrm{T}$-a.e.; moreover,

$$\sup_{n\in\mathbb{N}} \int g_n d(\mu\restriction\mathrm{T}) = \sup_{n\in\mathbb{N}} \int f_n d\mu = \int f d\mu < \infty.$$

By B.Levi's theorem, $g = \lim_{n\to\infty} g_n$ is defined $\mu\restriction\mathrm{T}$-almost everywhere, and

$$\int_F g\, d(\mu\restriction\mathrm{T}) = \lim_{n\to\infty} \int_F g_n d(\mu\restriction\mathrm{T}) = \lim_{n\to\infty} \int_F f_n d\mu = \int_F f d\mu$$

for every $F \in \mathrm{T}$, so g is a conditional expectation of f on T.

(e) Set $f'_n = \inf_{m\ge n} f_m$, $f''_n = \sup_{m\ge n} f_m$ for each $n \in \mathbb{N}$. Then we have

$$-h \le_{\text{a.e.}} f'_n \le f_n \le f''_n \le_{\text{a.e.}} h,$$

and $\langle f'_n\rangle_{n\in\mathbb{N}}$, $\langle f''_n\rangle_{n\in\mathbb{N}}$ are almost-everywhere-monotonic sequences of functions both converging almost everywhere to f. For each n, let g'_n, g''_n be conditional expectations of f'_n, f''_n on T. By (iii) and (iv), $\langle g'_n\rangle_{n\in\mathbb{N}}$ and $\langle g''_n\rangle_{n\in\mathbb{N}}$ are almost-everywhere-monotonic sequences converging almost everywhere to conditional expectations g', g'' of f. Of course $g' = g''$ $\mu\restriction\mathrm{T}$-a.e. (233Dc). Also, for each n, $g'_n \le_{\text{a.e.}} g_n \le_{\text{a.e.}} g''_n$, so $\langle g_n\rangle_{n\in\mathbb{N}}$ converges to g' $\mu\restriction\mathrm{T}$-a.e., and $g = \lim_{n\to\infty} g_n$ is defined almost everywhere and is a conditional expectation of f on T.

(f) For any $H \in \mathrm{T}$,

$$\int_H g_0 \times \chi F\, d(\mu\restriction\mathrm{T}) = \int_{H\cap F} g_0 d(\mu\restriction\mathrm{T}) = \int_{H\cap F} f_0 d\mu = \int_H f_0 \times \chi F\, d\mu.$$

(g)(i) If h is actually $(\mu\restriction\mathrm{T})$-simple, say $h = \sum_{i=0}^n a_i \chi F_i$ where $F_i \in \mathrm{T}$ for each i, then

$$\int_F g_0 \times h\, d(\mu\restriction\mathrm{T}) = \sum_{i=0}^n a_i \int_F g_0 \times \chi F_i d(\mu\restriction\mathrm{T}) = \sum_{i=0}^n a_i \int_F f \times \chi F_i\, d\mu = \int_F f \times h\, d\mu$$

for every $F \in \mathrm{T}$. (ii) For the general case, if h is $\mu\restriction\mathrm{T}$-virtually measurable and $|h(x)| \le M$ $\mu\restriction\mathrm{T}$-almost everywhere, then there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ of $\mu\restriction\mathrm{T}$-simple functions converging to h almost everywhere, and with $|h_n(x)| \le M$ for every x, n. Now $f_0 \times h_n \to f_0 \times h$ a.e. and $|f_0 \times h_n| \le_{\text{a.e.}} M|f_0|$ for each n, while $g_0 \times h_n$ is a conditional expectation of $f_0 \times h_n$ for every n, so by (e) we see that $\lim_{n\to\infty} g_0 \times h_n$ will be a conditional expectation of $f_0 \times h$; but this is equal almost everywhere to $g_0 \times h$.

(h) We need note only that $\int_H g_0 d(\mu\restriction\mathrm{T}) = \int_H f_0 d\mu$ for every $H \in \Upsilon$, so

$$\int_H h_0 d(\mu\restriction\Upsilon) = \int_H g_0 d(\mu\restriction\mathrm{T}) \text{ for every } H \in \Upsilon$$

$$\iff \int_H h_0 d(\mu\restriction\Upsilon) = \int_H f_0 d\mu \text{ for every } H \in \Upsilon.$$

233F Remarks Of course the results above are individually nearly trivial (though I think (e) and (g) might give you pause for thought if they were offered without previous preparation of the ground).

Cumulatively they amount to some quite strong properties. In §242 I will restate them in language which is syntactically more direct, but relies on a deeper level of abstraction.

As an illustration of the power of conditional expectations to surprise us, I offer the next proposition, which depends on the concept of 'convex' function.

233G Convex functions Recall that a real-valued function ϕ defined on an interval $I \subseteq \mathbb{R}$ is **convex** if

$$\phi(tb + (1-t)c) \le t\phi(b) + (1-t)\phi(c)$$

whenever b, $c \in I$ and $t \in [0,1]$.

Examples The formulae $|x|$, x^2, $e^{\pm x} \pm x$ define convex functions on $\mathbb{R}$; on $]-1,1[$ we have $1/(1-x^2)$; on $]0,\infty[$ we have $1/x$ and $x \ln x$; on $[0,1]$ we have the function which is zero on $]0,1[$ and 1 on $\{0,1\}$.

233H The general theory of convex functions is both extensive and important; I list a few of their more salient properties in 233Xe. For the moment the following lemma covers what we need.

Lemma Let $I \subseteq \mathbb{R}$ be a non-empty open interval (bounded or unbounded) and $\phi : I \to \mathbb{R}$ a convex function.

(a) For every $a \in I$ there is a $b \in \mathbb{R}$ such that $\phi(x) \ge \phi(a) + b(x-a)$ for every $x \in I$.

(b) If we take, for each $q \in I \cap \mathbb{Q}$, a $b_q \in \mathbb{R}$ such that $\phi(x) \ge \phi(q) + b_q(x-q)$ for every $x \in I$, then

$$\phi(x) = \sup_{q \in I \cap \mathbb{Q}} \phi(q) + b_q(x-q)$$

for every $x \in I$.

(c) ϕ is Borel measurable.

proof (a) If c, $c' \in I$ and $c < a < c'$, then a is expressible as $dc + (1-d)c'$ for some $d \in]0,1[$, so that $\phi(a) \le d\phi(c) + (1-d)\phi(c')$ and

$$\begin{aligned} \tfrac{\phi(a)-\phi(c)}{a-c} &\le \tfrac{d\phi(c)+(1-d)\phi(c')-\phi(c)}{dc+(1-d)c'-c} = \tfrac{(1-d)(\phi(c')-\phi(c))}{(1-d)(c'-c)} \\ &= \tfrac{d(\phi(c')-\phi(c))}{d(c'-c)} = \tfrac{\phi(c')-d\phi(c)-(1-d)\phi(c')}{c'-dc-(1-d)c'} \le \tfrac{\phi(c')-\phi(a)}{c'-a}. \end{aligned}$$

This means that

$$b = \sup_{c<a, c \in I} \tfrac{\phi(a)-\phi(c)}{a-c}$$

is finite, and $b \le \frac{\phi(c')-\phi(a)}{c'-a}$ whenever $a < c' \in I$; accordingly $\phi(x) \ge \phi(a) + b(x-a)$ for every $x \in I$.

(b) By the choice of the b_q, $\phi(x) \ge \sup_{q \in \mathbb{Q}} \phi_q(x)$. On the other hand, given $x \in I$, fix $y \in I$ such that $x < y$ and let $b \in \mathbb{R}$ be such that $\phi(z) \ge \phi(x) + b(z-x)$ for every $z \in I$. If $q \in \mathbb{Q}$ and $x < q < y$, we have $\phi(y) \ge \phi(q) + b_q(y-q)$, so that $b_q \le \frac{\phi(y)-\phi(q)}{y-q}$ and

$$\begin{aligned} \phi(q) + b_q(x-q) &= \phi(q) - b_q(q-x) \ge \phi(q) - \tfrac{\phi(y)-\phi(q)}{y-q}(q-x) \\ &= \tfrac{y-x}{y-q}\phi(q) - \tfrac{q-x}{y-q}\phi(y) \ge \tfrac{y-x}{y-q}(\phi(x)+b(q-x)) - \tfrac{q-x}{y-q}\phi(y). \end{aligned}$$

Now

$$\begin{aligned} \phi(x) &= \lim_{q \downarrow x} \tfrac{y-x}{y-q}(\phi(x)+b(q-x)) - \tfrac{q-x}{y-q}\phi(y) \\ &\le \sup_{q \in \mathbb{Q} \cap]x,y[} \tfrac{y-x}{y-q}(\phi(x)+b(q-x)) - \tfrac{q-x}{y-q}\phi(y) \\ &\le \sup_{q \in \mathbb{Q} \cap]x,y[} \phi(q) + b_q(x-q) \le \sup_{q \in \mathbb{Q} \cap I} \phi(q) + b_q(x-q). \end{aligned}$$

(c) Writing $\phi_q(x) = \phi(q) + b_q(x-q)$ for every $q \in \mathbb{Q} \cap I$, every ϕ_q is a Borel measurable function, and $\phi = \sup_{q \in I \cap \mathbb{Q}} \phi_q$ is the supremum of a countable family of Borel measurable functions, so is Borel measurable.

233I Jensen's inequality Let (X,Σ,μ) be a measure space and $\phi:\mathbb{R}\to\mathbb{R}$ a convex function.

(a) Suppose that f and g are real-valued μ-virtually measurable functions defined almost everywhere in X and that $g\ge 0$ almost everywhere, $\int g=1$ and $g\times f$ is integrable. Then $\phi(\int g\times f)\le\int g\times\phi f$, where we may need to interpret the right-hand integral as ∞.

(b) In particular, if $\mu X=1$ and f is a real-valued function which is integrable over X, then $\phi(\int f)\le\int\phi f$.

proof (a) For each $q\in\mathbb{Q}$ take b_q such that $\phi(t)\ge\phi_q(t)=\phi(q)+b_q(t-q)$ for every $t\in\mathbb{R}$ (233Ha). Because ϕ is Borel measurable (233Hc), ϕf is μ-virtually measurable (121H), so $g\times\phi f$ also is; since $g\times\phi f$ is defined almost everywhere and almost everywhere greater than or equal to the integrable function $g\times\phi_0 f$, $\int g\times\phi f$ is defined in $]-\infty,\infty]$. Now

$$\begin{aligned}\phi_q\Bigl(\int g\times f\Bigr)&=\phi(q)+b_q\int g\times f-b_q q\\&=\int g\times(b_q f+(\phi(q)-b_q q)\chi X)=\int g\times\phi_q f\le\int g\times\phi f,\end{aligned}$$

because $\int g=1$ and $g\ge 0$ a.e. By 233Hb,

$$\phi(\textstyle\int g\times f)=\sup_{q\in\mathbb{Q}}\phi_q(\int g\times f)\le\int g\times\phi f.$$

(b) Take g to be the constant function with value 1.

233J Even the special case 233Ib of Jensen's inequality is already very useful. It can be extended as follows.

Theorem Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\phi:\mathbb{R}\to\mathbb{R}$ be a convex function and f a μ-integrable real-valued function defined almost everywhere in X such that the composition ϕf is also integrable. If g and h are conditional expectations on T of f, ϕf respectively, then $\phi g\le_{\text{a.e.}}h$. Consequently $\int\phi g\le\int\phi f$.

proof We use the same ideas as in 233I. For each $q\in\mathbb{Q}$ take a $b_q\in\mathbb{R}$ such that $\phi(t)\ge\phi_q(t)=\phi(q)+b_q(t-q)$ for every $t\in\mathbb{R}$, so that $\phi(t)=\sup_{q\in\mathbb{Q}}\phi_q(t)$ for every $t\in\mathbb{R}$. Now setting

$$\psi_q(x)=\phi(q)+b_q(g(x)-q)$$

for $x\in\operatorname{dom}g$, we see that $\psi_q=\phi_q g$ is a conditional expectation of $\phi_q f$, and as $\phi_q f\le_{\text{a.e.}}\phi f$ we must have $\psi_q\le_{\text{a.e.}}h$. But also $\phi g=\sup_{q\in\mathbb{Q}}\psi_q$ wherever g is defined, so $\phi g\le_{\text{a.e.}}h$, as claimed.

It follows at once that $\int\phi g\le\int h=\int\phi f$.

233K I give the following proposition, an elaboration of 233Eg, in a very general form, as its applications can turn up anywhere.

Proposition Let (X,Σ,μ) be a probability space, and T a σ-subalgebra of Σ. Suppose that f is a μ-integrable function and h is a $(\mu\restriction\mathrm{T})$-virtually measurable real-valued function defined $(\mu\restriction\mathrm{T})$-almost everywhere in X. Let g, g_0 be conditional expectations of f and $|f|$ on T. Then $f\times h$ is integrable iff $g_0\times h$ is integrable, and in this case $g\times h$ is a conditional expectation of $f\times h$ on T.

proof (a) Suppose that h is a $\mu\restriction\mathrm{T}$-simple function. Then surely $f\times h$ and $g_0\times h$ are integrable, and $g\times h$ is a conditional expectation of $f\times h$ as in 233Eg.

(b) Now suppose that f, $h\ge 0$. Then $g=g_0\ge 0$ a.e. (233Ec). Let $\tilde h$ be a non-negative T-measurable function defined everywhere in X such that $h=_{\text{a.e.}}\tilde h$. For each $n\in\mathbb{N}$ set

$$\begin{aligned}h_n(x)&=2^{-n}k\text{ if }0\le k<4^n\text{ and }2^{-n}k\le\tilde h(x)<2^{-n}(k+1),\\&=2^n\text{ if }\tilde h(x)\ge 2^{-n}.\end{aligned}$$

Then h_n is a $(\mu\restriction\mathrm{T})$-simple function, so $g\times h_n$ is a conditional expectation of $f\times h_n$. Both $\langle f\times h_n\rangle_{n\in\mathbb{N}}$ and $\langle g\times h_n\rangle_{n\in\mathbb{N}}$ are almost everywhere non-decreasing sequences of integrable functions, with limits $f\times h$ and $g\times h$ respectively. By B.Levi's theorem,

$$\begin{aligned} f\times h \text{ is integrable} &\iff f\times\tilde h \text{ is integrable}\\ &\iff \sup_{n\in\mathbb{N}}\int f\times h_n<\infty \iff \sup_{n\in\mathbb{N}}\int g\times h_n<\infty \end{aligned}$$

(because $\int g\times h_n=\int f\times h_n$ for each n)

$$\iff g\times h \text{ is integrable} \iff g_0\times h \text{ is integrable.}$$

Moreover, in this case

$$\begin{aligned}\int_E f\times h&=\int_E f\times\tilde h=\lim_{n\to\infty}\int_E f\times h_n\\ &=\lim_{n\to\infty}\int_E g\times h_n=\int_E g\times\tilde h=\int_E g\times h\end{aligned}$$

for every $E\in\mathrm{T}$, while $g\times h$ is $(\mu\restriction\mathrm{T})$-virtually measurable, so $g\times h$ is a conditional expectation of $f\times h$.

(c) Finally, consider the general case of integrable f and virtually measurable h. Set $f^+=f\vee 0$, $f^-=(-f)\vee 0$, so that $f=f^+-f^-$ and $0\le f^+$, $f^-\le|f|$; similarly, set $h^+=h\vee 0$, $h^-=(-h)\vee 0$. Let g_1, g_2 be conditional expectations of f^+, f^- on T. Because $0\le f^+$, $f^-\le|f|$, $0\le g_1$, $g_2\le_{\text{a.e.}} g_0$, while $g=_{\text{a.e.}} g_1-g_2$.

We see that

$$\begin{aligned} f\times h \text{ is integrable} &\iff |f|\times|h|=|f\times h| \text{ is integrable}\\ &\iff g_0\times|h| \text{ is integrable}\\ &\iff g_0\times h \text{ is integrable.}\end{aligned}$$

And in this case all four of $f^+\times h^+,\dots,f^-\times h^-$ are integrable, so

$$(g_1-g_2)\times h=g_1\times h^+-g_2\times h^+-g_1\times h^-+g_2\times h^-$$

is a conditional expectation of

$$f^+\times h^+-f^-\times h^+-f^+\times h^-+f^-\times h^-=f\times h.$$

Since $g\times h=_{\text{a.e.}}(g_1-g_2)\times h$, this also is a conditional expectation of $f\times h$, and we're done.

233X Basic exercises (a) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-negative μ-integrable functions and suppose that g_n is a conditional expectation of f_n on T for each n. Suppose that $f=\liminf_{n\to\infty}f_n$ is integrable and has a conditional expectation g. Show that $g\le_{\text{a.e.}}\liminf_{n\to\infty}g_n$.

(b) Let $I\subseteq\mathbb{R}$ be an interval, and $\phi:I\to\mathbb{R}$ a function. Show that ϕ is convex iff $\{x:x\in I,\ \phi(x)+bx\le c\}$ is an interval for every b, $c\in\mathbb{R}$.

>(c) Let $I\subseteq\mathbb{R}$ be an open interval and $\phi:I\to\mathbb{R}$ a function. (i) Show that if ϕ is differentiable then it is convex iff ϕ' is non-decreasing. (ii) Show that if ϕ is absolutely continuous on every bounded closed subinterval of I then ϕ is convex iff ϕ' is non-decreasing on its domain.

(d) For any $r\ge 1$, a subset C of $\mathbb{R}^r$ is **convex** if $tx+(1-t)y\in C$ for all x, $y\in C$ and $t\in[0,1]$. If $C\subseteq\mathbb{R}^r$ is convex, then a function $\phi:C\to\mathbb{R}$ is **convex** if $\phi(tx+(1-t)y)\le t\phi(x)+(1-t)\phi(y)$ for all x, $y\in C$ and $t\in[0,1]$.

Let $C\subseteq\mathbb{R}^r$ be a convex set and $\phi:C\to\mathbb{R}$ a function. Show that the following are equiveridical: (i) the function ϕ is convex; (ii) the set $\{(x,t):x\in C,\ t\in\mathbb{R},\ t\ge\phi(x)\}$ is convex in $\mathbb{R}^{r+1}$; (iii) the set $\{x:x\in C,\ \phi(x)+b\centerdot x\le c\}$ is convex in $\mathbb{R}^r$ for every $b\in\mathbb{R}^r$ and $c\in\mathbb{R}$, writing $b\centerdot x=\sum_{i=1}^r\beta_i\xi_i$ if $b=(\beta_1,\dots,\beta_r)$ and $x=(\xi_1,\dots,\xi_r)$.

(e) Let $I \subseteq \mathbb{R}$ be an interval and $\phi : I \to \mathbb{R}$ a convex function.

(i) Show that if a, $d \in I$ and $a < b \le c < d$ then

$$\frac{\phi(b)-\phi(a)}{b-a} \le \frac{\phi(d)-\phi(c)}{d-c}.$$

(ii) Show that ϕ is continuous at every interior point of I.

(iii) Show that *either* ϕ is monotonic on I *or* there is a $c \in I$ such that $\phi(c) = \min_{x\in I}\phi(x)$ and ϕ is non-increasing on $I \cap \,]-\infty, c]$, monotonic non-decreasing on $I \cap [c, \infty[$.

(iv) Show that ϕ is differentiable at all but countably many points of I, and that its derivative is non-decreasing in the sense that $\phi'(x) \le \phi'(y)$ whenever x, $y \in \operatorname{dom}\phi'$ and $x \le y$.

(v) Show that if I is closed and bounded and ϕ is continuous then ϕ is absolutely continuous.

(vi) Show that if I is closed and bounded and $\psi : I \to \mathbb{R}$ is absolutely continuous with a non-decreasing derivative then ψ is convex.

(f) Show that if $I \subseteq \mathbb{R}$ is an interval and ϕ, $\psi : I \to \mathbb{R}$ are convex functions so is $a\phi + b\psi$ for any a, $b \ge 0$.

(g) In the context of 233K, give an example in which $g \times h$ is integrable but $f \times h$ is not. (*Hint*: take X, μ, T as in 233Cb, and arrange for g to be 0.)

(h) Let $I \subseteq \mathbb{R}$ be an interval and Φ a non-empty family of convex real-valued functions on I such that $\psi(x) = \sup_{\phi\in\Phi}\phi(x)$ is finite for every $x \in I$. Show that ψ is convex.

233Y Further exercises (a) If $I \subseteq \mathbb{R}$ is an interval, a function $\phi : I \to \mathbb{R}$ is **mid-convex** if $\phi(\frac{x+y}{2}) \le \frac{1}{2}(\phi(x) + \phi(y))$ for all x, $y \in I$. Show that a mid-convex function which is bounded on any non-trivial subinterval of I is convex.

(b) Generalize 233Xd to arbitrary normed spaces in place of $\mathbb{R}^r$.

(c) Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ. Let ϕ be a convex real-valued function with domain an interval $I \subseteq \mathbb{R}$, and f an integrable real-valued function on X such that $f(x) \in I$ for almost every $x \in X$ and ϕf is integrable. Let g, h be conditional expectations on T of f, ϕf respectively. Show that $g(x) \in I$ for almost every x and that $\phi g \le_{\text{a.e.}} h$.

(d)(i) Show that if $I \subseteq \mathbb{R}$ is a bounded interval, $E \subseteq I$ is Lebesgue measurable, and $\mu E > \frac{2}{3}\mu I$ where μ is Lebesgue measure, then for every $x \in I$ there are y, $z \in E$ such that $z = \frac{x+y}{2}$. (*Hint*: by 134Ya/263A, $\mu(x + E) + \mu(2E) > \mu(2I)$.) (ii) Show that if $f : [0, 1] \to \mathbb{R}$ is a mid-convex Lebesgue measurable function (definition: 233Ya), $a > 0$, and $E = \{x : x \in [0, 1],\ a \le f(x) < 2a\}$ is not negligible, then there is a non-trivial interval $I \subseteq [0, 1]$ such that $f(x) > 0$ for every $x \in I$. (*Hint*: 223B.) (iii) Suppose that $f : [0, 1] \to \mathbb{R}$ is a mid-convex function such that $f \le 0$ almost everywhere in $[0, 1]$. Show that $f \le 0$ everywhere in $]0, 1[$. (*Hint*: for every $x \in \,]0, 1[$, $\max(f(x - t), f(x + t)) \le 0$ for almost every $t \in [0, \min(x, 1 - x)]$.) (iv) Suppose that $f : [0, 1] \to \mathbb{R}$ is a mid-convex Lebesgue measurable function such that $f(0) = f(1) = 0$. Show that $f(x) \le 0$ for every $x \in [0, 1]$. (*Hint*: show that $\{x : f(x) \le 0\}$ is dense in $[0, 1]$, use (ii) to show that it is conegligible in $[0, 1]$ and apply (iii).) (v) Show that if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is a mid-convex Lebesgue measurable function then it is convex.

(e) Let (X, Σ, μ) be a probability space, T a σ-subalgebra of subsets of X, and $f : X \to [0, \infty]$ a Σ-measurable function. Show that (i) there is a T-measurable $g : X \to [0, \infty]$ such that $\int_F g = \int_F f$ for every $F \in \mathrm{T}$ (ii) any two such functions are equal a.e.

(f) Suppose that $r \ge 1$ and $C \subseteq \mathbb{R}^r \setminus \{0\}$ is a convex set. Show that there is a non-zero $b \in \mathbb{R}^r$ such that $b\,.\,z \ge 0$ for every $z \in C$. (*Hint*: if $r = 2$, identify $\mathbb{R}^2$ with $\mathbb{C}$; reduce to the case in which C contains no points which are real and negative; set $\theta = \sup\{\arg z : z \in C\}$ and $b = -ie^{i\theta}$. Now induce on r.)

(g) Suppose that $r \ge 1$, $C \subseteq \mathbb{R}^r$ is a convex set and $\phi : C \to \mathbb{R}$ is a convex function. Show that there is a function $h : \mathbb{R}^r \to [-\infty, \infty[$ such that $\phi(z) = \sup\{h(y) + z\,.\,y : y \in \mathbb{R}^r\}$ for every $z \in C$. (*Hint*: try $h(y) = \inf\{\phi(z) - z\,.\,y : z \in C\}$, and apply 233Yf to a translate of $\{(z, t) : \phi(z) \le t\}$.)

(h) Let (X,Σ,μ) be a probability space, $r\ge 1$ an integer and $C\subseteq\mathbb{R}^r$ a convex set. Let $f_1,\dots,f_r$ be μ-integrable real-valued functions and suppose that $\{x: x\in\bigcap_{j\le r}\operatorname{dom}f_j,\ (f_1(x),\dots,f_r(x))\in C\}$ is a conegligible subset of X. Show that $(\int f_1,\dots,\int f_r)\in C$. (*Hint*: induce on r.)

(i) Let (X,Σ,μ) be a probability space, $r\ge 1$ an integer, $C\subseteq\mathbb{R}^r$ a convex set and $\phi:C\to\mathbb{R}^r$ a convex function. Let $f_1,\dots,f_r$ be μ-integrable real-valued functions and suppose that $\{x: x\in\bigcap_{j\le r}\operatorname{dom}f_j$, $(f_1(x),\dots,f_r(x))\in C\}$ is a conegligible subset of X. Show that $\phi(\int f_1,\dots,\int f_r)\le\int\phi(f_1,\dots,f_r)$.

(j) Let (X,Σ,μ) be a measure space with $\mu X>0$, $r\ge 1$ an integer, $C\subseteq\mathbb{R}^r$ a convex set such that $tz\in C$ whenever $z\in C$ and $t>0$, and $\phi:C\to\mathbb{R}$ a convex function. Let $f_1,\dots,f_r$ be μ-integrable real-valued functions and suppose that $\{x: x\in\bigcap_{j\le r}\operatorname{dom}f_j,\ (f_1(x),\dots,f_r(x))\in C\}$ is a conegligible subset of X. Show that $(\int f_1,\dots,\int f_r)\in C$ and that $\phi(\int f_1,\dots,\int f_r)\le\int\phi(f_1,\dots,f_r)$. (*Hint*: putting 215B(viii) and 235K below together, show that there are a probability measure ν on X and a function $h:X\to[0,\infty[$ such that $\int f_j d\mu=\int f_j\times h\,d\nu$ for every j.)

233 Notes and comments The concept of 'conditional expectation' is fundamental in probability theory, and will reappear in Chapter 27 in its natural context. I hope that even as an exercise in technique, however, it will strike you as worth taking note of.

I introduced 233E as a 'list of elementary facts', and they are indeed straightforward. But below the surface there are some remarkable ideas waiting for expression. If you take T to be the trivial algebra $\{\emptyset,X\}$, so that the (unique) conditional expectation of an integrable function f is the constant function $(\int f)\chi X$, then 233Ed and 233Ee become versions of B.Levi's theorem and Lebesgue's Dominated Convergence Theorem. (Fatou's Lemma is in 233Xa.) Even 233Eg can be thought of as a generalization of the result that $\int cf=c\int f$, where the constant c has been replaced by a bounded T-measurable function. A recurrent theme in the later parts of this treatise will be the search for 'conditioned' versions of theorems. The proof of 233Ee is a typical example of an argument which has been translated from a proof of the original 'unconditioned' result.

I suggested that 233I-233J are surprising, and I think that most of us find them so, even applied to the list of convex functions given in 233G. But I should remark that in a way 233J has very little to do with conditional expectations. The only properties of conditional expectations used in the proof are (i) that if g is a conditional expectation of f, then $a\chi X+bg$ is a conditional expectation of $a\chi X+bf$ for all real a, b (ii) if g_1, g_2 are conditional expectations of f_1, f_2 and $f_1\le_{\text{a.e.}}f_2$, then $g_1\le_{\text{a.e.}}g_2$. See 244Xm below. Jensen's inequality has an interesting extension to the multidimensional case, explored in 233Yf-233Yj. If you have encountered 'geometric' forms of the Hahn-Banach theorem (see 3A5C in Volume 3) you will find 233Yf and 233Yg very natural, and you may notice that the finite-dimensional case is slightly different from the infinite-dimensional case you have probably been taught. I think that in fact the most delicate step is in 233Yh.

Note that 233Ib can be regarded as the special case of 233J in which $\mathrm{T}=\{\emptyset,X\}$. In fact 233Ia can be derived from 233Ib applied to the measure ν where $\nu E=\int_E g$ for every $E\in\Sigma$.

Like 233B, 233K seems to have rather a lot of technical detail in the argument. The point of this result is that we can deduce the integrability of $f\times h$ from that of $g_0\times h$ (but not from the integrability of $g\times h$; see 233Xg). Otherwise it should be routine.

Version of 11.4.09

234 Operations on measures

I take a few pages to describe some standard constructions. The ideas are straightforward, but a number of details need to be worked out if they are to be securely integrated into the general framework I employ. The first step is to formally introduce inverse-measure-preserving functions (234A-234B), the most important class of transformations between measure spaces. For construction of new measures, we have the notions of image measure (234C-234E), sum of measures (234G-234H) and indefinite-integral measure (234I-234O). Finally I mention a way of ordering the measures on a given set (234P-234Q).

234A Inverse-measure-preserving functions It is high time that I introduced the nearest thing in measure theory to a 'morphism'. If (X,Σ,μ) and (Y,T,ν) are measure spaces, a function $\phi:X\to Y$ is **inverse-measure-preserving** if $\phi^{-1}[F]\in\Sigma$ and $\mu(\phi^{-1}[F])=\nu F$ for every $F\in\mathrm{T}$.

234B Proposition Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $\phi:X\to Y$ an inverse-measure-preserving function.

(a) If $\hat\mu$, $\hat\nu$ are the completions of μ, ν respectively, ϕ is also inverse-measure-preserving for $\hat\mu$ and $\hat\nu$.

(b) μ is a probability measure iff ν is a probability measure.

(c) μ is totally finite iff ν is totally finite.

(d)(i) If ν is σ-finite, then μ is σ-finite.

(ii) If ν is semi-finite and μ is σ-finite, then ν is σ-finite.

(e)(i) If ν is σ-finite and atomless, then μ is atomless.

(ii) If ν is semi-finite and μ is purely atomic, then ν is purely atomic.

(f)(i) $\mu^*\phi^{-1}[B]\le\nu^*B$ for every $B\subseteq Y$.

(ii) $\mu^*A\le\nu^*\phi[A]$ for every $A\subseteq X$.

(g) If (Z,Λ,λ) is another measure space, and $\psi:Y\to Z$ is inverse-measure-preserving, then $\psi\phi:X\to Z$ is inverse-measure-preserving.

proof (a) If $\hat\nu$ measures F, there are F', $F''\in\mathrm{T}$ such that $F'\subseteq F\subseteq F''$ and $\nu(F''\setminus F')=0$. Now

$$\phi^{-1}[F']\subseteq\phi^{-1}[F]\subseteq\phi^{-1}[F''],\quad \mu(\phi^{-1}[F'']\setminus\phi^{-1}[F'])=\nu(F''\setminus F')=0,$$

so $\hat\mu$ measures $\phi^{-1}[F]$ and

$$\hat\mu(\phi^{-1}[F])=\mu\phi^{-1}[F']=\nu F'=\hat\nu F.$$

As F is arbitrary, ϕ is inverse-measure-preserving for $\hat\mu$ and $\hat\nu$.

(b)-(c) are surely obvious.

(d)(i) If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a cover of Y by sets of finite measure for ν, then $\langle\phi^{-1}[F_n]\rangle_{n\in\mathbb{N}}$ is a cover of X by sets of finite measure for μ.

(ii) Let $\mathcal{F}\subseteq\mathrm{T}$ be a disjoint family of non-ν-negligible sets. Then $\langle\phi^{-1}[F]\rangle_{F\in\mathcal{F}}$ is a disjoint family of non-negligible sets. By 215B(iii), $\mathcal{F}$ is countable. By 215B(iii) in the opposite direction, ν is σ-finite.

(e)(i) Suppose that $E\in\Sigma$ and $\mu E>0$. Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a cover of Y by sets of finite measure for ν. Because ν is atomless, we can find, for each n, a finite partition $\langle F_{ni}\rangle_{i\in I_n}$ of F_n such that $\nu F_{ni}<\mu E$ for every $i\in I_n$ (use 215D repeatedly). Now $X=\bigcup_{n\in\mathbb{N},i\in I_n}\phi^{-1}[F_{ni}]$, so there are $n\in\mathbb{N}$ and $i\in I_n$ with

$$0<\mu(E\cap\phi^{-1}[F_{ni})\le\mu\phi^{-1}[F_{ni}]=\nu F_{ni}<\mu E,$$

and E is not a μ-atom. As E is arbitrary, μ is atomless.

(ii) Suppose that $F\in\mathrm{T}$ and $\nu F>0$. Because ν is semi-finite, there is an $F_1\subseteq F$ such that $0<\nu F_1<\infty$. Now $\mu\phi^{-1}[F_1]>0$; because μ is purely atomic, there is a μ-atom $E\subseteq\phi^{-1}[F_1]$.

Let $\mathcal{G}$ be the set of those $G\in\mathrm{T}$ such that $G\subseteq F_1$ and $\mu(E\cap\phi^{-1}[G])=0$. Then the union of any sequence in $\mathcal{G}$ belongs to $\mathcal{G}$, so by 215Ac there is an $H\in\mathcal{G}$ such that $\nu(G\setminus H)=0$ whenever $G\in\mathcal{G}$. Consider $F_1\setminus H$. We have

$$\nu(F_1\setminus H)=\mu(\phi^{-1}[F_1]\setminus\phi^{-1}[H])\ge\mu(E\setminus\phi^{-1}[H])=\mu E>0.$$

If $G\in\mathrm{T}$ and $G\subseteq F_1\setminus H$, then one of $E\cap\phi^{-1}[G]$, $E\setminus\phi^{-1}[G]$ is μ-negligible. In the former case, $G\in\mathcal{G}$ and $G=G\setminus H$ is ν-negligible. In the latter case, $F_1\setminus G\in\mathcal{G}$ and $(F_1\setminus H)\setminus G$ is ν-negligible. As G is arbitrary, $F_1\setminus H$ is a ν-atom included in F; as F is arbitrary, ν is purely atomic.

(f)(i) Let $F\in\mathrm{T}$ be such that $B\subseteq F$ and $\nu^*B=\nu F$ (132Aa); then $\phi^{-1}[B]\subseteq\phi^{-1}[F]$ so

$$\mu^*\phi^{-1}[B]\le\mu\phi^{-1}[F]=\nu F=\nu^*B.$$

(ii) $\mu^*A\le\mu^*(\phi^{-1}[\phi[A]])\le\nu^*\phi[A]$ by (i).

(g) For any $W\in\Lambda$,

$$\mu(\psi\phi)^{-1}[W]=\mu\phi^{-1}[\psi^{-1}[W]]=\nu\psi^{-1}[W]=\lambda W.$$

234C Image measures The following construction is one of the commonest ways in which new measure spaces appear.

Proposition Let (X, Σ, μ) be a measure space, Y any set, and $\phi : X \to Y$ a function. Set

$$\mathrm{T} = \{F : F \subseteq Y, \phi^{-1}[F] \in \Sigma\}, \quad \nu F = \mu(\phi^{-1}[F]) \text{ for every } F \in \mathrm{T}.$$

Then (Y, T, ν) is a measure space.

proof (a) $\emptyset = \phi^{-1}[\emptyset] \in \Sigma$ so $\emptyset \in \mathrm{T}$.

(b) If $F \in \mathrm{T}$, then $\phi^{-1}[F] \in \Sigma$, so $X \setminus \phi^{-1}[F] \in \Sigma$; but $X \setminus \phi^{-1}[F] = \phi^{-1}[Y \setminus F]$, so $Y \setminus F \in \mathrm{T}$.

(c) If $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in T, then $\phi^{-1}[F_n] \in \Sigma$ for every n, so $\bigcup_{n\in\mathbb{N}} \phi^{-1}[F_n] \in \Sigma$; but $\phi^{-1}[\bigcup_{n\in\mathbb{N}} F_n] = \bigcup_{n\in\mathbb{N}} \phi^{-1}[F_n]$, so $\bigcup_{n\in\mathbb{N}} F_n \in \mathrm{T}$.

Thus T is a σ-algebra.

(d) $\nu\emptyset = \mu\phi^{-1}[\emptyset] = \mu\emptyset = 0$.

(e) If $\langle F_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in T, then $\langle \phi^{-1}[F_n] \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, so

$$\nu(\textstyle\bigcup_{n\in\mathbb{N}} F_n) = \mu\phi^{-1}[\bigcup_{n\in\mathbb{N}} F_n] = \mu(\bigcup_{n\in\mathbb{N}} \phi^{-1}[F_n]) = \sum_{n=0}^{\infty} \mu\phi^{-1}[F_n] = \sum_{n=0}^{\infty} \nu F_n.$$

So ν is a measure.

234D Definition In the context of 234C, ν is called the **image measure** or **push-forward measure**; I will denote it $\mu\phi^{-1}$.

Remark I ought perhaps to say that this construction does not always produce exactly the 'right' measure on Y; there are circumstances in which some modification of the measure $\mu\phi^{-1}$ described here is more useful. But I will note these explicitly when they occur; when I use the unadorned phrase 'image measure' I shall mean the measure constructed above.

234E Proposition Let (X, Σ, μ) be a measure space, Y a set and $\phi : X \to Y$ a function; let $\mu\phi^{-1}$ be the image measure on Y.

(a) ϕ is inverse-measure-preserving for μ and $\mu\phi^{-1}$.

(b) If μ is complete, so is $\mu\phi^{-1}$.

(c) If Z is another set, and $\psi : Y \to Z$ a function, then the image measures $\mu(\psi\phi)^{-1}$ and $(\mu\phi^{-1})\psi^{-1}$ on Z are the same.

proof (a) Immediate from the definitions.

(b) Write ν for $\mu\phi^{-1}$ and T for its domain. If $\nu^*B = 0$, then $\mu^*\phi^{-1}[B] = 0$, by 234B(f-i); as μ is complete, $\phi^{-1}[B] \in \Sigma$, so $B \in \mathrm{T}$. As B is arbitrary, ν is complete.

(c) For $G \subseteq Z$ and $u \in [0, \infty]$,

$$\begin{aligned}
&(\mu(\psi\phi)^{-1})(G) \text{ is defined and equal to } u\\
&\qquad \iff \mu((\psi\phi)^{-1}[G]) \text{ is defined and equal to } u\\
&\qquad \iff \mu(\phi^{-1}[\psi^{-1}[G]]) \text{ is defined and equal to } u\\
&\qquad \iff (\mu\phi^{-1})(\psi^{-1}[G])) \text{ is defined and equal to } u\\
&\qquad \iff ((\mu\phi^{-1})\psi^{-1})(G) \text{ is defined and equal to } u.
\end{aligned}$$

***234F** In the opposite direction, the following construction of a pull-back measure is sometimes useful.

Proposition Let X be a set, (Y, T, ν) a measure space, and $\phi : X \to Y$ a function such that $\phi[X]$ has full outer measure in Y. Then there is a measure μ on X, with domain $\Sigma = \{\phi^{-1}[F] : F \in \mathrm{T}\}$, such that ϕ is inverse-measure-preserving for μ and ν.

proof The check that Σ is a σ-algebra of subsets of X is straightforward; all we need to know is that $\phi^{-1}[\emptyset] = \emptyset$, $X \setminus \phi^{-1}[F] = \phi^{-1}[Y \setminus F]$ for every $F \subseteq Y$, and that $\phi^{-1}[\bigcup_{n\in\mathbb{N}} F_n] = \bigcup_{n\in\mathbb{N}} \phi^{-1}[F_n]$ for every sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ of subsets of Y. The key fact is that if F_1, $F_2 \in \mathrm{T}$ and $\phi^{-1}[F_1] = \phi^{-1}[F_2]$, then $\phi[X]$ does not meet $F_1\triangle F_2$; because $\phi[X]$ has full outer measure, $F_1\triangle F_2$ is ν-negligible and $\nu F_1 = \nu F_2$. Accordingly the formula $\mu\phi^{-1}[F] = \nu F$ does define a function $\mu : \Sigma \to [0, \infty]$. Now

$$\mu\emptyset = \mu\phi^{-1}[\emptyset] = \nu\emptyset = 0.$$

Next, if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ, choose $F_n \in \mathrm{T}$ such that $E_n = \phi^{-1}[F_n]$ for each $n \in \mathbb{N}$. The sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ need not be disjoint, but if we set $F'_n = F_n \setminus \bigcup_{i<n} F_i$ for each $n \in \mathbb{N}$, then $\langle F'_n\rangle_{n\in\mathbb{N}}$ is disjoint and

$$E_n = E_n \setminus \bigcup_{i<n} E_i = \phi^{-1}[F'_n]$$

for each n; so

$$\mu(\bigcup_{n\in\mathbb{N}} E_n) = \nu(\bigcup_{n\in\mathbb{N}} F'_n) = \sum_{n=0}^\infty \nu F'_n = \sum_{n=0}^\infty \mu E_n.$$

As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, μ is a measure on X, as required.

234G Sums of measures I come now to a quite different way of building measures. The idea is an obvious one, but the technical details, in the general case I wish to examine, need watching.

Proposition Let X be a set, and $\langle\mu_i\rangle_{i\in I}$ a family of measures on X. For each $i \in I$, let Σ_i be the domain of μ_i. Set $\Sigma = \mathcal{P}X \cap \bigcap_{i\in I}\Sigma_i$ and define $\mu : \Sigma \to [0, \infty]$ by setting $\mu E = \sum_{i\in I}\mu_i E$ for every $E \in \Sigma$. Then μ is a measure on X.

proof Σ is a σ-algebra of subsets of X because every Σ_i is. (Apply 111Ga with $\mathfrak{S} = \{\Sigma_i : i \in I\} \cup \{\mathcal{P}X\}$.) Of course μ takes values in $[0, \infty]$ (226A). $\mu\emptyset = 0$ because $\mu_i\emptyset = 0$ for every i. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ with union E, then

$$\mu E = \sum_{i\in I}\mu_i E = \sum_{i\in I}\sum_{n=0}^\infty \mu_i E_n = \sum_{n=0}^\infty\sum_{i\in I}\mu_i E_n$$

(226Af)

$$= \sum_{n=0}^\infty \mu E_n.$$

So μ is a measure.

Remark In this context, I will call μ the **sum** of the family $\langle\mu_i\rangle_{i\in I}$.

234H Proposition Let X be a set and $\langle\mu_i\rangle_{i\in I}$ a family of complete measures on X with sum μ.

(a) μ is complete.

(b)(i) A subset of X is μ-negligible iff it is μ_i-negligible for every $i \in I$.

(ii) A subset of X is μ-conegligible iff it is μ_i-conegligible for every $i \in I$.

(c) Let f be a function from a subset of X to $[-\infty, \infty]$. Then $\int f d\mu$ is defined in $[-\infty, \infty]$ iff $\int f d\mu_i$ is defined in $[-\infty, \infty]$ for every i and one of $\sum_{i\in I} f^+ d\mu_i$, $\sum_{i\in I} f^- d\mu_i$ is finite, and in this case $\int f d\mu = \sum_{i\in I}\int f d\mu_i$.

proof Write $\Sigma_i = \operatorname{dom}\mu_i$ for $i \in I$, $\Sigma = \mathcal{P}X \cap \bigcap_{i\in I}\Sigma_i = \operatorname{dom}\mu$.

(a) If $E \subseteq F \in \Sigma$ and $\mu F = 0$, then $\mu_i F = 0$ for every $i \in I$; because μ_i is complete, $E_i \in \Sigma_i$ for every $i \in I$, and $E \in \Sigma$.

(b) This now follows at once, since a set $A \subseteq X$ is μ-negligible iff $\mu A = 0$.

(c)(i) Note first that (b-ii) tells us that, under either hypothesis, $\operatorname{dom} f$ is conegligible both for μ and for every μ_i, so that if we extend f to X by giving it the value 0 on $X \setminus \operatorname{dom} f$ then neither $\int f d\mu$ nor $\sum_{i\in I}\int f d\mu_i$ is affected. So let us assume from now on that f is defined everywhere on X. Now it is plain that either hypothesis ensures that f is Σ-measurable, that is, is Σ_i-measurable for every $i \in I$.

(ii) Suppose that f is non-negative. For $n \in \mathbb{N}$ set $f_n(x) = \sum_{k=1}^{4^n} 2^{-n}\chi\{x : f(x) \geq 2^{-n}k\}$, so that $\langle f_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum f. We have

$$\int f_n d\mu = \sum_{k=1}^{4^n} 2^{-n}\mu\{x : f(x) \geq 2^{-n}k\} = \sum_{k=1}^{4^n}\sum_{i\in I} 2^{-n}\mu_i\{x : f(x) \geq 2^{-n}k\}$$
$$= \sum_{i\in I}\sum_{k=1}^{4^n} 2^{-n}\mu_i\{x : f(x) \geq 2^{-n}k\} = \sum_{i\in I}\int f_n d\mu_i$$

for every n, so

$$\int f d\mu = \sup_{n\in\mathbb{N}}\int f_n d\mu = \sup_{n\in\mathbb{N}}\sup_{J\subseteq I \text{ is finite}}\sum_{i\in J}\int f_n d\mu_i = \sup_{J\subseteq I \text{ is finite}}\sup_{n\in\mathbb{N}}\sum_{i\in J}\int f_n d\mu_i$$
$$= \sup_{J\subseteq I \text{ is finite}}\lim_{n\to\infty}\sum_{i\in J}\int f_n d\mu_i = \sup_{J\subseteq I \text{ is finite}}\sum_{i\in J}\lim_{n\to\infty}\int f_n d\mu_i = \sum_{i\in I}\int f d\mu_i.$$

(iii) Generally,

$$\int f d\mu \text{ is defined in } [\infty, \infty]$$
$$\iff \int f^+ d\mu \text{ and } \int f^- d\mu \text{ are defined and at most one is infinite}$$
$$\iff \sum_{i\in I}\int f^+ d\mu_i \text{ and } \sum_{i\in I}\int f^- d\mu_i \text{ are defined and at most one is infinite}$$
$$\iff \int f d\mu_i \text{ is defined for every } i \text{ and at most one of } \sum_{i\in I}\int f^+ d\mu_i,$$
$$\sum_{i\in I}\int f^- d\mu_i \text{ is infinite,}$$

and in this case

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu = \sum_{i\in I}\int f^+ d\mu_i - \sum_{i\in I}\int f^- d\mu_i = \sum_{i\in I}\int f d\mu_i.$$

234I Indefinite-integral measures Extending an idea already used in 232D, we are led to the following construction; once again, we need to take care over the formal details if we want to get full value from it.

Theorem Let (X, Σ, μ) be a measure space, and f a non-negative μ-virtually measurable real-valued function defined on a conegligible subset of X. Write $\nu F = \int f \times \chi F\, d\mu$ whenever $F \subseteq X$ is such that the integral is defined in $[0, \infty]$ according to the conventions of 133A. Then ν is a complete measure on X, and its domain includes Σ.

proof (a) Write T for the domain of ν, that is, the family of sets $F \subseteq X$ such that $\int f \times \chi F\, d\mu$ is defined in $[0, \infty]$, that is, $f \times \chi F$ is μ-virtually measurable (133A). Then T is a σ-algebra of subsets of X. **P** For each $F \in \mathrm{T}$ let $H_F \subseteq X$ be a μ-conegligible set such that $f \times \chi F \restriction H_F$ is Σ-measurable. Because f itself is μ-virtually measurable, $X \in \mathrm{T}$. If $F \in \mathrm{T}$, then

$$f \times \chi(X \setminus F)\restriction(H_X \cap H_F) = f\restriction(H_X \cap H_F) - (f \times \chi F)\restriction(H_X \cap H_F)$$

is Σ-measurable, while $H_X \cap H_F$ is μ-conegligible, so $X \setminus F \in \mathrm{T}$. If $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence in T with union F, set $H = \bigcap_{n\in\mathbb{N}} H_{F_n}$; then H is conegligible, $f \times \chi F_n\restriction H$ is Σ-measurable for every $n \in \mathbb{N}$, and $f \times \chi F = \sup_{n\in\mathbb{N}} f \times \chi F_n$, so $f \times \chi F\restriction H$ is Σ-measurable, and $F \in \mathrm{T}$. Thus T is a σ-algebra. If $F \in \Sigma$, then $f \times \chi F\restriction H_X$ is Σ-measurable, so $F \in \mathrm{T}$. **Q**

(b) Next, ν is a measure. **P** Of course $\nu F \in [0, \infty]$ for every $F \in \mathrm{T}$. $f \times \chi\emptyset = 0$ wherever it is defined, so $\nu\emptyset = 0$. If $\langle F_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in T with union F, then $f \times \chi F = \sum_{n=0}^{\infty} f \times \chi F_n$. If $\nu F_m = \infty$ for some m, then we surely have $\nu F = \infty = \sum_{n=0}^{\infty}\nu F_n$. If $\nu F_m < \infty$ for each m but $\sum_{n=0}^{\infty}\nu F_n = \infty$, then

$$\int f\times\chi(\bigcup_{n\le m}F_n)=\sum_{n=0}^{m}\int f\times\chi F_n\to\infty$$

as $m\to\infty$, so again $\nu F=\infty=\sum_{n=0}^{\infty}\nu F_n$. If $\sum_{n=0}^{\infty}\nu F_n<\infty$ then by B.Levi's theorem

$$\nu F=\int\sum_{n=0}^{\infty}f\times\chi F_n=\sum_{n=0}^{\infty}\int f\times\chi F_n=\sum_{n=0}^{\infty}\nu F_n.\ \mathbf{Q}$$

(c) Finally, ν is complete. **P** If $A\subseteq F\in\mathrm{T}$ and $\nu F=0$, then $f\times\chi F=0$ a.e., so $f\times\chi A=0$ a.e. and νA is defined and equal to zero. **Q**

234J Definition Let (X,Σ,μ) be a measure space, and ν another measure on X with domain T. I will call ν an **indefinite-integral measure** over μ, or sometimes a **completed indefinite-integral measure**, if it can be obtained by the method of 234I from some non-negative virtually measurable function f defined almost everywhere on X. In this case, f is a Radon-Nikodým derivative of ν with respect to μ in the sense of 232Hf. As in 232Hf, the phrase **density function** is also used in this context.

234K Remarks Let (X,Σ,μ) be a measure space, and f a μ-virtually measurable non-negative real-valued function defined almost everywhere on X; let ν be the associated indefinite-integral measure.

(a) There is a Σ-measurable function $g:X\to[0,\infty[$ such that $f=g$ μ-a.e. **P** Let $H\subseteq\operatorname{dom}f$ be a measurable conegligible set such that $f\restriction H$ is measurable, and set $g(x)=f(x)$ for $x\in H$, $g(x)=0$ for $x\in X\setminus H$. **Q** In this case, $\int f\times\chi E\,d\mu=\int g\times\chi E\,d\mu$ if either is defined. So g is a Radon-Nikodým derivative of ν, and ν has a Radon-Nikodým derivative which is Σ-measurable and defined everywhere.

(b) If E is μ-negligible, then $f\times\chi E=0$ μ-a.e., so $\nu E=0$. Many authors are prepared to say 'ν is absolutely continuous with respect to μ' in this context. But if ν is not totally finite, it need not be absolutely continuous in the ϵ-δ sense of 232Aa (234Xh), and further difficulties can arise if μ or ν is not σ-finite (see 234Yk, 234Ym).

(c) I have defined 'indefinite-integral measure' in such a way as to produce a complete measure. In my view this is what makes best sense in most applications. There are occasions on which it seems more appropriate to use the measure $\nu_0:\Sigma\to[0,\infty]$ defined by setting $\nu_0E=\int_E fd\mu=\int f\times\chi E\,d\mu$ for $E\in\Sigma$. I suppose I would call this the **uncompleted indefinite-integral measure** over μ defined by f. (ν is always the completion of ν_0; see 234Lb.)

(d) Note the way in which I formulated the definition of ν: '$\nu E=\int f\times\chi E\,d\mu$ if the integral is defined', rather than '$\nu E=\int_E fd\mu$'. The point is that the longer formula gives a rule for deciding what the domain of ν must be. Of course it is the case that $\nu E=\int_E fd\mu$ for every $E\in\operatorname{dom}\nu$ (apply 214F to $f\times\chi E$).

(e) Because μ and its completion define the same virtually measurable functions, the same null ideals and the same integrals (212Eb, 212F), they define the same indefinite-integral measures.

234L The domain of an indefinite-integral measure It is sometimes useful to have an explicit description of the domain of a measure constructed in this way.

Proposition Let (X,Σ,μ) be a measure space, f a non-negative μ-virtually measurable function defined almost everywhere in X, and ν the associated indefinite-integral measure. Set $G=\{x:x\in\operatorname{dom}f,\ f(x)>0\}$, and let $\hat\Sigma$ be the domain of the completion $\hat\mu$ of μ.

(a) The domain T of ν is $\{E:E\subseteq X,\ E\cap G\in\hat\Sigma\}$; in particular, $\mathrm{T}\supseteq\hat\Sigma\supseteq\Sigma$.

(b) ν is the completion of its restriction to Σ.

(c) A set $A\subseteq X$ is ν-negligible iff $A\cap G$ is μ-negligible.

(d) In particular, if μ itself is complete, then $\mathrm{T}=\{E:E\subseteq X,\ E\cap G\in\Sigma\}$ and $\nu A=0$ iff $\mu(A\cap G)=0$.

proof (a)(i) If $E\in\mathrm{T}$, then $f\times\chi E$ is virtually measurable, so there is a conegligible measurable set $H\subseteq\operatorname{dom}f$ such that $f\times\chi E\restriction H$ is measurable. Now $E\cap G\cap H=\{x:x\in H,\ (f\times\chi E)(x)>0\}$ must belong to Σ, while $E\cap G\setminus H$ is negligible, so belongs to $\hat\Sigma$, and $E\cap G\in\hat\Sigma$.

(ii) If $E\cap G\in\hat\Sigma$, let F_1, $F_2\in\Sigma$ be such that $F_1\subseteq E\cap G\subseteq F_2$ and $F_2\setminus F_1$ is negligible. Let $H\subseteq\operatorname{dom}f$ be a conegligible set such that $f\restriction H$ is measurable. Then $H'=H\setminus(F_2\setminus F_1)$ is conegligible and $f\times\chi E\restriction H'=f\times\chi F_1\restriction H'$ is measurable, so $f\times\chi E$ is virtually measurable and $E\in\mathrm{T}$.

(b) Thus the given formula does indeed describe T. If $E\in\mathrm{T}$, let F_1, $F_2\in\Sigma$ be such that $F_1\subseteq E\cap G\subseteq F_2$ and $\mu(F_2\setminus F_1)=0$. Because G itself also belongs to $\hat\Sigma$, there are G_1, $G_2\in\Sigma$ such that $G_1\subseteq G\subseteq G_2$ and $\mu(G_2\setminus G_1)=0$. Set $F_2'=F_2\cup(X\setminus G_1)$. Then $F_2'\in\Sigma$ and $F_1\subseteq E\subseteq F_2'$. But $(F_2'\setminus F_1)\cap G\subseteq(G_2\setminus G_1)\cup(F_2\setminus F_1)$ is μ-negligible, so $\nu(F_2'\setminus F_1)=0$.

This shows that if ν' is the completion of $\nu\restriction\Sigma$ and T' is its domain, then $\mathrm{T}\subseteq\mathrm{T}'$. But as ν is complete, it surely extends ν', so $\nu=\nu'$, as claimed.

(c) Now take any $A\subseteq X$. Because ν is complete,

$$\begin{aligned}A\text{ is }\nu\text{-negligible}&\iff\nu A=0\\&\iff\int f\times\chi A\,d\mu=0\\&\iff f\times\chi A=0\ \mu\text{-a.e.}\\&\iff A\cap G\text{ is }\mu\text{-negligible.}\end{aligned}$$

(d) This is just a restatement of (a) and (c) when $\mu=\hat\mu$.

234M Corollary If (X,Σ,μ) is a complete measure space and $G\in\Sigma$, then the indefinite-integral measure over μ defined by χG is just the measure $\mu\llcorner G$ defined by setting

$$(\mu\llcorner G)(F)=\mu(F\cap G)\text{ whenever }F\subseteq X\text{ and }F\cap G\in\Sigma.$$

proof 234Ld.

***234N** The next two results will not be relied on in this volume, but I include them for future reference, and to give an idea of the scope of these ideas.

Proposition Let (X,Σ,μ) be a measure space, and ν an indefinite-integral measure over μ.

(a) If μ is semi-finite, so is ν.
(b) If μ is complete and locally determined, so is ν.
(c) If μ is localizable, so is ν.
(d) If μ is strictly localizable, so is ν.
(e) If μ is σ-finite, so is ν.
(f) If μ is atomless, so is ν.

proof By 234Ka, we may express ν as the indefinite integral of a Σ-measurable function $f:X\to[0,\infty[$. Let T be the domain of ν, and $\hat\Sigma$ the domain of the completion $\hat\mu$ of μ; set $G=\{x:x\in X,\ f(x)>0\}\in\Sigma$.

(a) Suppose that $E\in\mathrm{T}$ and that $\nu E=\infty$. Then $E\cap G$ cannot be μ-negligible. Because μ is semi-finite, there is a non-negligible $F\in\Sigma$ such that $F\subseteq E\cap G$ and $\mu F<\infty$. Now $F=\bigcup_{n\in\mathbb{N}}\{x:x\in F,\ 2^{-n}\le f(x)\le n\}$, so there is an $n\in\mathbb{N}$ such that $F'=\{x:x\in F,\ 2^{-n}\le f(x)\le n\}$ is non-negligible. Because f is measurable, $F'\in\Sigma\subseteq\mathrm{T}$ and $2^{-n}\mu F'\le\nu F'\le n\mu F'$. Thus we have found an $F'\subseteq E$ such that $0<\nu F'<\infty$. As E is arbitrary, ν is semi-finite.

(b) We already know that ν is complete (234Lb) and semi-finite. Now suppose that $E\subseteq X$ is such that $E\cap F\in\mathrm{T}$, that is, $E\cap F\cap G\in\Sigma$ (234Ld), whenever $F\in\mathrm{T}$ and $\nu F<\infty$. Then $E\cap G\cap F\in\Sigma$ whenever $F\in\Sigma$ and $\mu F<\infty$. **P** Set $F_n=\{x:x\in F\cap G,\ f(x)\le n\}$. Then $\nu F_n\le n\mu F<\infty$, so $E\cap G\cap F_n\in\Sigma$ for every n. But this means that $E\cap G\cap F=\bigcup_{n\in\mathbb{N}}E\cap G\cap F_n\in\Sigma$. **Q** Because μ is locally determined, $E\cap G\in\Sigma$ and $E\in\mathrm{T}$. As E is arbitrary, ν is locally determined.

(c) Let $\mathcal{F}\subseteq\mathrm{T}$ be any set. Set $\mathcal{E}=\{F\cap G:F\in\mathcal{F}\}$, so that $\mathcal{E}\subseteq\hat\Sigma$. By 212Ga, $\hat\mu$ is localizable, so $\mathcal{E}$ has an essential supremum $H\in\hat\Sigma$. But now, for any $H'\in\mathrm{T}$, $H'\cup(X\setminus G)=(H'\cap G)\cup(X\setminus G)$ belongs to $\hat\Sigma$, so

$$\begin{aligned}\nu(F\setminus H')=0 \text{ for every } F\in\mathcal{F}\\ &\iff \hat\mu(F\cap G\setminus H')=0 \text{ for every } F\in\mathcal{F}\\ &\iff \hat\mu(E\setminus H')=0 \text{ for every } E\in\mathcal{E}\\ &\iff \hat\mu(E\setminus(H'\cup(X\setminus G)))=0 \text{ for every } E\in\mathcal{E}\\ &\iff \hat\mu(H\setminus((H'\cup(X\setminus G)))=0\\ &\iff \hat\mu(H\cap G\setminus H')=0\\ &\iff \nu(H\setminus H')=0.\end{aligned}$$

Thus H is also an essential supremum of $\mathcal{F}$ in T. As $\mathcal{F}$ is arbitrary, ν is localizable.

(d) Let $\langle X_i\rangle_{i\in I}$ be a decomposition of X for μ; then it is also a decomposition for $\hat\mu$ (212Gb). Set $F_0=X\setminus G$, $F_n=\{x:x\in G,\,n-1<f(x)\le n\}$ for $n\ge 1$. Then $\langle X_i\cap F_n\rangle_{i\in I,n\in\mathbb{N}}$ is a decomposition for ν. **P** (i) $\langle X_i\rangle_{i\in I}$ and $\langle F_n\rangle_{n\in\mathbb{N}}$ are partitions of X into members of $\Sigma\subseteq\mathrm{T}$, so $\langle X_i\cap F_n\rangle_{i\in I,n\in\mathbb{N}}$ also is. (ii) $\nu(X_i\cap F_0)=0$, $\nu(X_i\cap F_n)\le n\mu X_i<\infty$ for $i\in I$, $n\ge 1$. (iii) If $E\subseteq X$ and $E\cap X_i\cap F_n\in\mathrm{T}$ for every $i\in I$ and $n\in\mathbb{N}$ then $E\cap X_i\cap G=\bigcup_{n\in\mathbb{N}}E\cap X_i\cap F_n\cap G$ belongs to $\hat\Sigma$ for every i, so $E\cap G\in\hat\Sigma$ and $E\in\mathrm{T}$. (iv) If $E\in\mathrm{T}$, then of course

$$\textstyle\sum_{i\in I,n\in\mathbb{N}}\nu(E\cap X_i\cap F_n)=\sup_{J\subseteq I\times\mathbb{N}\text{ is finite}}\sum_{(i,n)\in J}\nu(E\cap X_i\cap F_n)\le\nu E.$$

So if $\sum_{i\in I,n\in\mathbb{N}}\nu(E\cap X_i\cap F_n)=\infty$ it is surely equal to νE. If the sum is finite, then $K=\{i:i\in I,\,\nu(E\cap X_i)>0\}$ must be countable. But for $i\in I\setminus K$, $\int_{E\cap X_i}fd\mu=0$, so $f=0$ μ-a.e. on $E\cap X_i$, that is, $\hat\mu(E\cap G\cap X_i)=0$. Because $\langle X_i\rangle_{i\in I}$ is a decomposition for $\hat\mu$, $\hat\mu(E\cap G\cap\bigcup_{i\in I\setminus K}X_i)=0$ and $\nu(E\cap\bigcup_{i\in I\setminus K}X_i)=0$. But this means that

$$\textstyle\nu E=\sum_{i\in K}\nu(E\cap X_i)=\sum_{i\in K,n\in\mathbb{N}}\nu(E\cap X_i\cap F_n)=\sum_{i\in I,n\in\mathbb{N}}\nu(E\cap X_i\cap F_n).$$

As E is arbitrary, $\langle X_i\cap F_n\rangle_{i\in I,n\in\mathbb{N}}$ is a decomposition for ν. **Q** So ν is strictly localizable.

(e) If μ is σ-finite, then in (d) we can take I to be countable, so that $I\times\mathbb{N}$ also is countable, and ν will be σ-finite.

(f) If μ is atomless, so is $\hat\mu$ (212Gd). If $E\in\mathrm{T}$ and $\nu E>0$, then $\hat\mu(E\cap G)>0$, so there is an $F\in\hat\Sigma$ such that $F\subseteq E\cap G$ and neither F nor $E\cap G\setminus F$ is $\hat\mu$-negligible. But in this case both $\nu F=\int_F fd\mu$ and $\nu(E\setminus F)=\int_{E\setminus F}fd\mu$ must be greater than 0 (122Rc). As E is arbitrary, ν is atomless.

***234O** For localizable measures, there is a straightforward description of the associated indefinite-integral measures.

Theorem Let (X,Σ,μ) be a localizable measure space. Then a measure ν, with domain $\mathrm{T}\supseteq\Sigma$, is an indefinite-integral measure over μ iff (α) ν is semi-finite and zero on μ-negligible sets (β) ν is the completion of its restriction to Σ (γ) whenever $\nu E>0$ there is an $F\subseteq E$ such that $F\in\Sigma$, $\mu F<\infty$ and $\nu F>0$.

proof (a) If ν is an indefinite-integral measure over ν, then by 234Na, 234Kb and 234Lb it is semi-finite, zero on μ-negligible sets and the completion of its restriction to Σ. Now suppose that $E\in\mathrm{T}$ and $\nu E>0$. Then there is an $E_0\in\Sigma$ such that $E_0\subseteq E$ and $\nu E_0=\nu E$, by 234Lb. If $f:X\to\mathbb{R}$ is a Σ-measurable Radon-Nikodým derivative of ν (234Ka), and $G=\{x:f(x)>0\}$, then $\mu(G\cap E_0)>0$; because μ is semi-finite, there is an $F\in\Sigma$ such that $F\subseteq G\cap E_0$ and $0<\mu F<\infty$, in which case $\nu F>0$.

(b) So now suppose that ν satisfies the conditions.

(i) Set $\mathcal{E}=\{E:E\in\Sigma,\,\nu E<\infty\}$. For each $E\in\mathcal{E}$, consider $\nu_E:\Sigma\to\mathbb{R}$, setting $\nu_E G=\nu(G\cap E)$ for every $G\in\Sigma$. Then ν_E is countably additive and truly continuous with respect to μ. **P** ν_E is countably additive, just as in 231De. Because ν is zero on μ-negligible sets, ν_E must be absolutely continuous with respect to μ, by 232Ba. Since ν_E clearly satisfies condition (γ) of 232Bb, it must be truly continuous. **Q**

By 232E, there is a μ-integrable function f_E such that $\nu_E G=\int_G f_E d\mu$ for every $G\in\Sigma$, and we may suppose that f_E is Σ-measurable (232He). Because ν_E is non-negative, $f_E\ge 0$ μ-almost everywhere.

(ii) If E, $F \in \mathcal{E}$ then $f_E = f_F$ μ-a.e. on $E \cap F$, because

$$\int_G f_E d\mu = \nu G = \int_G f_F d\mu$$

whenever $G \in \Sigma$ and $G \subseteq E \cap F$. Because (X, Σ, μ) is localizable, there is a measurable $f : X \to \mathbb{R}$ such that $f_E = f$ μ-a.e. on E for every $E \in \mathcal{E}$ (213N). Because every f_E is non-negative almost everywhere, we may suppose that f is non-negative, since surely $f_E = f \vee \mathbf{0}$ μ-a.e. on E for every $E \in \mathcal{E}$.

(iii) Let ν' be the indefinite-integral measure defined by f. If $E \in \mathcal{E}$ then

$$\nu E = \int_E f_E d\mu = \int_E f d\mu = \nu' E.$$

For $E \in \Sigma \setminus \mathcal{E}$, we have

$$\nu' E \geq \sup\{\nu' F : F \in \mathcal{E},\, F \subseteq E\} = \sup\{\nu F : F \in \mathcal{E},\, F \subseteq E\} = \nu E = \infty$$

because ν is semi-finite. Thus ν' and ν agree on Σ. But since each is the completion of its restriction to Σ, they must be equal.

234P Ordering measures There are many ways in which one measure can dominate another. Here I will describe one of the simplest.

Definition Let μ, ν be two measures on a set X. I will say that $\mu \leq \nu$ if μE is defined, and $\mu E \leq \nu E$, whenever ν measures E.

234Q Proposition Let X be a set, and write M for the set of all measures on X.

(a) Defining $\leq$ as in 234P, $(\mathrm{M}, \leq)$ is a partially ordered set.

(b) If μ, $\nu \in \mathrm{M}$, then $\mu \leq \nu$ iff there is a $\lambda \in \mathrm{M}$ such that $\mu + \lambda = \nu$.

(c) If $\mu \leq \nu$ in M and f is a $[-\infty, \infty]$-valued function, defined on a subset of X, such that $\int f d\nu$ is defined in $[-\infty, \infty]$, then $\int f d\mu$ is defined; if f is non-negative, $\int f d\mu \leq \int f d\nu$.

proof (a) Of course $\mu \leq \mu$ for every $\mu \in \mathrm{M}$. If $\mu \leq \nu$ and $\nu \leq \lambda$ in M, then $\operatorname{dom}\lambda \subseteq \operatorname{dom}\nu \subseteq \operatorname{dom}\mu$, and $\mu E \leq \nu E \leq \lambda E$ whenever λ measures E. If $\mu \leq \nu$ and $\nu \leq \mu$ then $\operatorname{dom}\mu \subseteq \operatorname{dom}\nu \subseteq \operatorname{dom}\mu$ and $\mu E \leq \nu E \leq \mu E$ for every E in their common domain, so $\mu = \nu$.

(b)(i) If $\mu + \lambda = \nu$, then the definitions in 234G and 234P make it plain that $\mu \leq \nu$.

(ii)($\boldsymbol{\alpha}$) In the reverse direction, if $\mu \leq \nu$, write T for the domain of ν. Define $\lambda : \mathrm{T} \to [0, \infty]$ by setting

$$\lambda G = \sup\{\nu F - \mu F : F \in \mathrm{T},\, F \subseteq G,\, \mu F < \infty\}$$

for $G \in \mathrm{T}$. Then $\lambda \in \mathrm{M}$. **P** Of course $\operatorname{dom}\lambda = \mathrm{T}$ is a σ-algebra, and $\lambda\emptyset = 0$. Suppose that $\langle G_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in T with union G. If $F \in \mathrm{T}$, $F \subseteq G$ and $\mu F < \infty$, then

$$\begin{aligned}\nu F - \mu F &= \sum_{n=0}^{\infty} \nu(F \cap G_n) - \sum_{n=0}^{\infty} \mu(F \cap G_n) \\ &= \sum_{n=0}^{\infty} \nu(F \cap G_n) - \mu(F \cap G_n) \leq \sum_{n=0}^{\infty} \lambda G_n;\end{aligned}$$

as F is arbitrary, $\lambda G \leq \sum_{n=0}^{\infty} \lambda G_n$. If $\gamma < \sum_{n=0}^{\infty} \lambda G_n$, there are an $m \in \mathbb{N}$ such that $\gamma < \sum_{n=0}^{m} \lambda G_n$, and $F_0, \dots, F_m$ such that $F_n \in \mathrm{T}$, $F_n \subseteq G_n$ and $\mu F_n < \infty$ for every $n \leq m$, while $\sum_{n=0}^{m} \nu F_n - \mu F_n \geq \gamma$. Set $F = \bigcup_{n \leq m} F_n$; then $F \in \mathrm{T}$, $F \subseteq G$ and $\mu F < \infty$, so

$$\lambda G \geq \nu F - \mu F = \sum_{n=0}^{m} \nu F_n - \mu F_n \geq \gamma.$$

As γ is arbitrary, $\lambda G \geq \sum_{n=0}^{\infty} \lambda G_n$; as $\langle G_n \rangle_{n \in \mathbb{N}}$ is arbitrary, λ is countably additive. **Q**

($\boldsymbol{\beta}$) Now $\mu + \lambda = \nu$. **P** The domain of $\mu + \lambda$ is $\operatorname{dom}\mu \cap \operatorname{dom}\lambda = \mathrm{T} = \operatorname{dom}\nu$. Take $G \in \mathrm{T}$. If $\mu G = \infty$, then $\nu G = \infty = (\mu + \lambda)G$. Otherwise,

$$(\mu + \lambda)G \geq \mu G + \nu G - \mu G = \nu G.$$

So if $\nu G = \infty$ we shall certainly have $\nu G = (\mu + \lambda)G$. Finally, if $\nu G < \infty$ then

$$\begin{aligned}(\mu+\lambda)G &= \mu G + \sup\{\nu F - \mu F : F \in \mathrm{T},\, F \subseteq G\} \\ &= \sup\{\nu F + \mu(G \setminus F) : F \in \mathrm{T},\, F \subseteq G\} \\ &\le \sup\{\nu F + \nu(G \setminus F) : F \in \mathrm{T},\, F \subseteq G\} = \nu G,\end{aligned}$$

so again we have equality. **Q**

Thus we have an appropriate expression of ν as a sum of measures.

(c)(i) If f is non-negative, put (b) and 234Hc together.

(ii) In general, if $\int f d\nu$ is defined, so are both $\int f^+ d\nu$ and $\int f^- d\nu$, and at most one is infinite; so $\int f^+ d\mu$ and $\int f^- d\mu$ are defined and at most one is infinite.

234X Basic exercises (a) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Let $A \subseteq X$ be a set of full outer measure in X. Show that $\phi[A]$ has full outer measure in Y, and that $\phi \restriction A$ is inverse-measure-preserving for the subspace measures on A and $\phi[A]$.

(b) Let (X, Σ, μ) be a measure space, Y a set and $\phi : X \to Y$ a function. Show that if μ is point-supported, so is the image measure $\mu\phi^{-1}$.

(c) Give an example of a probability space (X, Σ, μ), a set Y, and a function $\phi : X \to Y$ such that the completion of the image measure $\mu\phi^{-1}$ is not the image of the completion of μ. (*Hint*: $\#(X) = 3$.)

(d) Let X, Y be sets, $\phi : X \to Y$ a function and $\langle\mu_i\rangle_{i\in I}$ a family of measures on X with sum μ. Writing $\mu_i\phi^{-1}$, $\mu\phi^{-1}$ for the image measures on Y, show that $\mu\phi^{-1} = \sum_{i\in I} \mu_i\phi^{-1}$.

(e) Let X be a set. (i) Show that if $\langle\mu_i\rangle_{i\in I}$ is a countable family of σ-finite measures on X, and $\mu = \sum_{i\in I} \mu_i$ is semi-finite, then μ is σ-finite. (ii) Show that if $\langle\mu_i\rangle_{i\in I}$ is a family of purely atomic measures on X, and $\mu = \sum_{i\in I} \mu_i$ is semi-finite, then μ is purely atomic. (iii) Show that if $\langle\mu_i\rangle_{i\in I}$ is any family of point-supported measures on X, then $\sum_{i\in I} \mu_i$ is point-supported.

>(f) Let X be a set, and write M for the set of all measures on X. For $\mu \in \mathrm{M}$ and $\alpha \in [0, \infty[$, define $\alpha\mu$ by saying that if $\alpha > 0$ then $(\alpha\mu)(E) = \alpha\mu E$ for $E \in \operatorname{dom}\mu$, while if $\alpha = 0$ then $(\alpha\mu)(E) = 0$ for every $E \subseteq X$. (i) Show that $\alpha\mu \in \mathrm{M}$ for all $\alpha \in [0, \infty[$ and $\mu \in \mathrm{M}$. (ii) Show that $(\alpha + \beta)\mu = \alpha\mu + \beta\mu$, $\alpha(\beta\mu) = (\alpha\beta)\mu$, $\alpha(\mu + \nu) = \alpha\mu + \alpha\nu$ for all α, $\beta \in [0, \infty[$ and μ, $\nu \in \mathrm{M}$.

(g) Let X be a set, and $\langle\mu_i\rangle_{i\in I}$ a family of complete measures on X with sum μ. Show that a $[-\infty, \infty]$-valued function f defined on a subset of X is μ-integrable iff it is μ_i-integrable for every $i \in I$ and $\sum_{i\in I} \int |f| d\mu_i$ is finite.

(h) Let μ be Lebesgue measure on $[0, 1]$, and set $f(x) = \frac{1}{x}$ for $x > 0$. Let ν be the associated indefinite-integral measure. Show that the domain of ν is equal to the domain of μ. Show that for every $\delta \in \left]0, \frac{1}{2}\right]$ there is a measurable set E such that $\mu E = \delta$ but $\nu E = \frac{1}{\delta}$.

(i) Let (X, Σ, μ) be a measure space. (i) Show that if ν_1 and ν_2 are indefinite-integral measures over μ, so is $\nu_1 + \nu_2$. (ii) Show that if $\langle\nu_i\rangle_{i\in I}$ is a countable family of indefinite-integral measures over μ, and $\nu = \sum_{i\in I} \nu_i$ is semi-finite, then ν is an indefinite-integral measure over μ.

(j) Let (X, Σ, μ) be a measure space, and ν an indefinite-integral measure over μ. Show that if μ is purely atomic, so is ν.

(k) Let μ be a point-supported measure. Show that any indefinite-integral measure over μ is point-supported.

(l) Let X be a set, and M the set of measures on X, with the partial ordering defined in 234P. Show that (i) M has greatest and least members (to be described); (ii) if $\langle\mu_i\rangle_{i\in I}$ and $\langle\nu_i\rangle_{i\in I}$ are families in M such that $\mu_i \le \nu_i$ for every i, then $\sum_{i\in I} \mu_i \le \sum_{i\in I} \nu_i$; (iii) if we define scalar multiplication as in 234Xf, then $\alpha\mu \le \mu$ whenever $\mu \in \mathrm{M}$ and $\alpha \in [0, 1]$; (iv) writing $\hat\mu$ for the completion of μ, $\hat\mu \le \mu$ and $\hat\mu \le \hat\nu$ whenever μ, $\nu \in \mathrm{M}$ and $\mu \le \nu$; (v) writing $\tilde\mu$ for the c.l.d. version of μ, $\tilde\mu \le \mu$ for every $\mu \in \mathrm{M}$; (vi) every subset of M has a least upper bound in M (cf. 112Yd); (vii) every subset of M has a greatest lower bound in M.

(m) Write out an elementary direct proof of 234Qc not depending on 234Qb.

(n) Let (X,Σ,μ) and (Y,T,ν) be measure spaces and $\phi:X\to Y$ an inverse-measure-preserving function. Show that if μ is σ-finite and purely atomic then ν is purely atomic.

234Y Further exercises (a) Write ν for Lebesgue measure on $Y=[0,1]$, and T for its domain. Let $A\subseteq[0,1]$ be a set such that $\nu^*A=\nu^*([0,1]\setminus A)=1$, and set $X=[0,1]\cup\{x+1:x\in A\}\cup\{x+2:x\in[0,1]\setminus A\}$. Let μ_{LX} be the subspace measure induced on X by Lebesgue measure, and set $\mu E=\frac{1}{3}\mu_{LX}E$ for $E\in\Sigma=\operatorname{dom}\mu_{LX}$. Define $\phi:X\to Y$ by writing $\phi(x)=x$ if $x\in[0,1]$, $\phi(x)=x-1$ if $x\in X\cap\,]1,2]$ and $\phi(x)=x-2$ if $x\in X\cap\,]2,3]$. Show that ν is the image measure $\mu\phi^{-1}$, but that $\nu^*A>\mu^*\phi^{-1}[A]$.

(b) Look for interesting examples of probability spaces (X,Σ,μ) and (Y,T,ν) for which there are functions $\phi:X\to Y$ such that $\phi[E]\in\mathrm{T}$ and $\nu\phi[E]=\mu E$ for every $E\in\Sigma$. (*Hint*: 254K, 343J.)

(c) Let μ be two-dimensional Lebesgue measure on the unit square $[0,1]^2$, and let $\phi:[0,1]^2\to[0,1]$ be the projection onto the first coordinate, so that $\phi(\xi_1,\xi_2)=\xi_1$ for ξ_1, $\xi_2\in[0,1]$. Show that the image measure $\mu\phi^{-1}$ is Lebesgue one-dimensional measure on $[0,1]$.

(d) In 234F, show that the image measure $\mu\phi^{-1}$ extends ν, and is equal to ν if and only if $F\in\mathrm{T}$ for every $F\subseteq Y\setminus\phi[X]$.

(e) Let (Y,T,ν) be a complete measure space, X a set and $\phi:X\to Y$ a surjection. Set

$$\Sigma=\{E:E\subseteq X,\ \phi[E]\in\mathrm{T},\ \nu(\phi[E]\cap\phi[X\setminus E])=0\},\qquad \mu E=\nu\phi[E]\text{ for }E\in\mathrm{T}.$$

Show that μ is the completion of the measure constructed by the process of 234F.

(f) Let X be a set, and M the set of measures on X. Show that M, with addition as defined for two measures by the formulae of 234G, is a commutative semigroup with identity; describe the identity.

(g) Give an example of a set X, probability measures μ_1, μ_2 on X and a set $A\subseteq X$ such that A is both μ_1-negligible and μ_2-negligible, but is not μ-negligible, where $\mu=\mu_1+\mu_2$.

(h) In 214O, show that if we set $\nu E=\sup_{I\in\mathcal{I}}\mu^*(E\cap I)$ for every $E\in\Sigma$, then ν is a measure, while $\mu=\nu+\lambda$.

(i) Let (X,Σ,μ) be an atomless semi-finite measure space and ν an indefinite-integral measure over μ. Show that the following are equiveridical: (i) for every $\epsilon>0$ there is a $\delta>0$ such that $\nu E\le\epsilon$ whenever $\mu E\le\delta$ (ii) ν has a Radon-Nikodým derivative expressible as the sum of a bounded function and an integrable function.

(j) Let (X,Σ,μ) be a measure space and ν an indefinite-integral measure over μ, with Radon-Nikodým derivative f. Show that the c.l.d. version of ν is the indefinite-integral measure defined by f over the c.l.d. version of μ.

(k) Let (X,Σ,μ) be a semi-finite measure space which is not localizable. Show that there is a measure $\nu:\Sigma\to[0,\infty]$ such that $\nu E\le\mu E$ for every $E\in\Sigma$ but there is no measurable function f such that $\nu E=\int_E f d\mu$ for every $E\in\Sigma$.

(l) Let (X,Σ,μ) be a localizable measure space with locally determined negligible sets. Show that a measure ν, with domain $\mathrm{T}\supseteq\Sigma$, is an indefinite-integral measure over μ iff (α) ν is complete and semi-finite and zero on μ-negligible sets (β) whenever $\nu E>0$ there is an $F\subseteq E$ such that $F\in\Sigma$ and $\mu F<\infty$ and $\nu F>0$.

(m) Give an example of a localizable measure space (X,Σ,μ) and a complete semi-finite measure ν on X, defined on a σ-algebra $\mathrm{T}\supseteq\Sigma$, zero on μ-negligible sets, and such that whenever $\nu E>0$ there is an $F\subseteq E$ such that $F\in\Sigma$ and $\mu F<\infty$ and $\nu F>0$, but ν is not an indefinite-integral measure over μ. (*Hint*: 216Yb.)

(n) Let (X, Σ, μ) be a localizable measure space, and ν a complete localizable measure on X, with domain $\mathrm{T} \supseteq \Sigma$, which is the completion of its restriction to Σ. Show that if we set $\nu_1 F = \sup\{\nu(F \cap E) : E \in \Sigma, \mu E < \infty\}$ for every $F \in \mathrm{T}$, then ν_1 is an indefinite-integral measure over μ, and there is an $H \in \Sigma$ such that $\nu_1 F = \nu(F \cap H)$ for every $F \in \mathrm{T}$.

(o) Let X be a set, and M_{sf} the set of semi-finite measures on X. For μ, $\nu \in M_{\mathrm{sf}}$ say that $\mu \preccurlyeq \nu$ if $\operatorname{dom}\nu \subseteq \operatorname{dom}\mu$, $\mu F \le \nu F$ for every $F \in \operatorname{dom}\nu$, and whenever $E \in \operatorname{dom}\mu$ and $\mu E > 0$ there is an $F \in \operatorname{dom}\nu$ such that $F \subseteq E$ and $0 < \mu F < \infty$. (i) Show that $(M_{\mathrm{sf}}, \preccurlyeq)$ is a partially ordered set. (ii) Show that if $A \subseteq M_{\mathrm{sf}}$ is a non-empty set with an upper bound in M_{sf}, then it has a least upper bound λ defined by saying that $\operatorname{dom}\lambda = \bigcap_{\mu \in A} \operatorname{dom}\mu$ and, for $E \in \operatorname{dom}\lambda$,

$$\begin{aligned}\lambda E &= \sup\{\sum_{i=0}^{n} \mu_i F_i : \mu_0, \dots, \mu_n \in A,\ \langle F_i\rangle_{i\le n} \text{ is a partition of } E,\\ &\qquad\qquad F_i \in \operatorname{dom}\lambda \text{ for every } i \le n\}\\ &= \sup\{\sum_{i=0}^{n} \mu_i F_i : \mu_0, \dots, \mu_n \in A,\ F_0, \dots, F_n \text{ are disjoint},\\ &\qquad\qquad F_i \in \operatorname{dom}\mu_i \text{ and } F_i \subseteq E \text{ for every } i \le n\}.\end{aligned}$$

(iii) Suppose that μ, $\nu \in M_{\mathrm{sf}}$ have completions $\hat\mu$, $\hat\nu$ and c.l.d. versions $\tilde\mu$, $\tilde\nu$. Show that $\tilde\mu \preccurlyeq \hat\mu \preccurlyeq \mu$. Show that if $\mu \preccurlyeq \nu$ then $\hat\mu \preccurlyeq \hat\nu$ and $\tilde\mu \preccurlyeq \tilde\nu$.

234 Notes and comments One of the striking features of measure theory, compared with other comparably abstract branches of pure mathematics, is the relative unimportance of any notion of 'morphism'. The theory of groups, for instance, is dominated by the concept of 'homomorphism', and general topology gives a similar place to 'continuous function'. In my view, the nearest equivalent in measure theory is the idea of 'inverse-measure-preserving function' (234A). I mean in Volumes 3 and 4 to explore this concept more thoroughly. In this volume I will content myself with signalling such functions when they arise, and with the basic facts listed in 234B.

Naturally linked with the idea of inverse-measure-preserving function is the construction of 'image measures' (234C). These appear everywhere in the subject, starting with the not-quite-elementary 234Yc. They are of such importance that it is natural to explore variations, as in 234F and 234Yb, but in my view none are of comparable significance.

Nearly half the section is taken up with 'indefinite-integral measures'. I have taken this part very carefully because the ideas I wish to express here, in so far as they extend the work of §232, rely critically on the details of the formulation in 234I, and it is easy to make a false step once we have left the relatively sheltered context of complete σ-finite measures. I believe that if we take a little trouble at this point we can develop a theory (234K-234N) which will offer a smooth path to later applications; to see what I have in mind, you can refer to the entries under 'indefinite-integral measure' in the index. For the moment I mention only a kind of Radon-Nikodým theorem for localizable measures (234O).

The partial ordering described in 234P-234Q is only one of many which can be considered, and for some purposes it seems unsatisfactory. The most important examples will appear in Chapter 41 of Volume 4, and have a variety of special features for which it might be worth setting out further abstractions. However the version here has the merit of simplicity and supports at least some of the relevant ideas (234Xl). For an alternative notion, see 234Yo.

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235 Measurable transformations

I turn now to a topic which is separate from the Radon-Nikodým theorem, but which seems to fit better here than in either of the next two chapters. I seek to give results which will generalize the basic formula of calculus

$$\int g(y)dy = \int g(\phi(x))\phi'(x)dx$$

in the context of a general transformation ϕ between measure spaces. The principal results are I suppose 235A/235E, which are very similar expressions of the basic idea, and 235J, which gives a general criterion for a stronger result. A formulation from a different direction is in 235R.

235A I start with the basic result, which is already sufficient for a large proportion of the applications I have in mind.

Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $\phi : D_\phi \to Y$, $J : D_J \to [0, \infty[$ functions defined on conegligible subsets D_ϕ, D_J of X such that

$$\int J \times \chi(\phi^{-1}[F])d\mu \text{ exists} = \nu F$$

whenever $F \in \mathrm{T}$ and $\nu F < \infty$. Then

$$\int_{\phi^{-1}[H]} J \times g\phi \, d\mu \text{ exists} = \int_H g \, d\nu$$

for every ν-integrable function g taking values in $[-\infty, \infty]$ and every $H \in \mathrm{T}$, provided that we interpret $(J \times g\phi)(x)$ as 0 when $J(x) = 0$ and $g(\phi(x))$ is undefined. Consequently, interpreting $J \times f\phi$ in the same way,

$$\underline{\int} f d\nu \le \underline{\int} J \times f\phi \, d\mu \le \overline{\int} J \times f\phi \, d\mu \le \overline{\int} f d\nu$$

for every $[-\infty, \infty]$-valued function f defined almost everywhere in Y.

proof (a) If g is a simple function, say $g = \sum_{i=0}^n a_i \chi F_i$ where $\nu F_i < \infty$ for each i, then

$$\int J \times g\phi \, d\mu = \sum_{i=0}^n a_i \int J \times \chi(\phi^{-1}[F_i]) \, d\mu = \sum_{i=0}^n a_i \nu F_i = \int g \, d\nu.$$

(b) If $\nu F = 0$ then $\int J \times \chi(\phi^{-1}[F]) = 0$ so $J = 0$ a.e. on $\phi^{-1}[F]$. So if g is defined ν-a.e., $J = 0$ μ-a.e. on $X \setminus \mathrm{dom}(g\phi) = (X \setminus D_\phi) \cup \phi^{-1}[Y \setminus \mathrm{dom}\, g]$, and, on the convention proposed, $J \times g\phi$ is defined μ-a.e. Moreover, if $\lim_{n\to\infty} g_n = g$ ν-a.e., then $\lim_{n\to\infty} J \times g_n\phi = J \times g\phi$ μ-a.e. So if $\langle g_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions converging almost everywhere to g, $\langle J \times g_n\phi\rangle_{n\in\mathbb{N}}$ will be a non-decreasing sequence of integrable functions converging almost everywhere to $J \times g\phi$; by B.Levi's theorem,

$$\int J \times g\phi \, d\mu \text{ exists} = \lim_{n\to\infty} \int J \times g_n\phi \, d\mu = \lim_{n\to\infty} \int g_n d\nu = \int g \, d\nu.$$

(c) If $g = g^+ - g^-$, where g^+ and g^- are ν-integrable functions, then

$$\int J \times g\phi \, d\mu = \int J \times g^+\phi \, d\mu - \int J \times g^-\phi \, d\mu = \int g^+ d\nu - \int g^- d\nu = \int g \, d\nu.$$

(d) This deals with the case $H = Y$. For the general case, we have

$$\int_H g \, d\nu = \int (g \times \chi H) d\nu$$

(131Fa)

$$= \int J \times (g \times \chi H)\phi \, d\mu = \int J \times g\phi \times \chi(\phi^{-1}[H]) d\mu = \int_{\phi^{-1}[H]} J \times g\phi \, d\mu$$

by 214F.

(e) For the upper and lower integrals, I note first that if F is ν-negligible then $\int J \times \chi(\phi^{-1}[F])d\mu = 0$, so that $J = 0$ μ-a.e. on $\phi^{-1}[F]$. It follows that if f and g are $[-\infty, \infty]$-valued functions on subsets of Y and $f \le_{\text{a.e.}} g$, then $J \times f\phi \le_{\text{a.e.}} J \times g\phi$. Now if $\overline{\int} f d\nu = \infty$, we surely have $\overline{\int} J \times f\phi \, d\mu \le \overline{\int} f d\nu$. Otherwise,

$$\overline{\int} f d\nu = \inf\{\int g \, d\nu : g \text{ is } \nu-\text{integrable and } f \le_{\text{a.e.}} g\}$$

$$= \inf\{\int J \times g\phi \, d\mu : g \text{ is } \nu\text{-integrable and } f \le_{\text{a.e.}} g\}$$

$$\le \inf\{\int h \, d\mu : h \text{ is } \mu\text{-integrable and } J \times f\phi \le_{\text{a.e.}} h\} = \overline{\int} J \times f\phi \, d\mu.$$

Similarly, or applying this argument to $-f$, we have $\underline{\int} J \times f\phi\, d\mu \le \underline{\int} f\, d\nu$.

235B Remarks (a) Note the particular convention

$$0 \times \text{undefined} = 0$$

which I am applying to the interpretation of $J \times g\phi$. This is the first of a number of technical points which will concern us in this section. The point is that if g is defined ν-almost everywhere, then for any extension of g to a function $g_1 : Y \to \mathbb{R}$ we shall, on this convention, have $J \times g\phi = J \times g_1\phi$ except on $\{x : J(x) > 0,\ \phi(x) \in Y \setminus \operatorname{dom} g\}$, which is negligible; so that

$$\int J \times g\phi\, d\mu = \int J \times g_1\phi\, d\mu = \int g_1 d\nu = \int g\, d\nu$$

if g and g_1 are integrable. Thus the convention is appropriate here, and while it adds a phrase to the statements of many of the results of this section, it makes their application smoother. (But I ought to insist that I am using this as a local convention only, and the ordinary rule $0 \times \text{undefined} = \text{undefined}$ will stand elsewhere in this treatise unless explicitly overruled.)

(b) I have had to take care in the formulation of this theorem to distinguish between the hypothesis

$$\int J(x)\chi(\phi^{-1}[F])(x)\mu(dx) \text{ exists} = \nu F \text{ whenever } \nu F < \infty$$

and the perhaps more elegant alternative

$$\int_{\phi^{-1}[F]} J(x)\mu(dx) \text{ exists} = \nu F \text{ whenever } \nu F < \infty,$$

which is not quite adequate for the theorem. (See 235Q below.) Recall that by $\int_A f$ I mean $\int (f\restriction A) d\mu_A$, where μ_A is the subspace measure on A (214D). It is possible for $\int_A (f\restriction A) d\mu_A$ to be defined even when $\int f \times \chi A\, d\mu$ is not; for instance, take μ to be Lebesgue measure on $[0,1]$, A any non-measurable subset of $[0,1]$, and f the constant function with value 1; then $\int_A f = \mu^* A$, but $f \times \chi A = \chi A$ is not μ-integrable. It is however the case that if $\int f \times \chi A\, d\mu$ is defined, then so is $\int_A f$, and the two are equal; this is a consequence of 214F. While 235P shows that in most of the cases relevant to the present volume the distinction can be passed over, it is important to avoid assuming that $\phi^{-1}[F]$ is measurable for every $F \in \mathrm{T}$. A simple example is the following. Set $X = Y = [0,1]$. Let μ be Lebesgue measure on $[0,1]$, and define ν by setting

$$\mathrm{T} = \{F : F \subseteq [0,1],\ F \cap [0, \tfrac{1}{2}] \text{ is Lebesgue measurable}\},$$

$$\nu F = 2\mu(F \cap [0, \tfrac{1}{2}]) \text{ for every } F \in \mathrm{T}.$$

Set $\phi(x) = x$ for every $x \in [0,1]$. Then we have

$$\nu F = \int_F J\, d\mu = \int J \times \chi(\phi^{-1}[F]) d\mu$$

for every $F \in \mathrm{T}$, where $J(x) = 2$ for $x \in [0, \frac{1}{2}]$ and $J(x) = 0$ for $x \in \left]\frac{1}{2}, 1\right]$. But of course there are subsets F of $[\frac{1}{2}, 1]$ which are not Lebesgue measurable (see 134D), and such an F necessarily belongs to T, even though $\phi^{-1}[F]$ does not belong to the domain Σ of μ.

The point here is that if $\nu F_0 = 0$ then we expect to have $J = 0$ on $\phi^{-1}[F_0]$, and it is of no importance whether $\phi^{-1}[F]$ is measurable for $F \subseteq F_0$.

235C Theorem 235A is concerned with integration, and accordingly the hypothesis $\int J \times \chi(\phi^{-1}[F]) d\mu = \nu F$ looks only at sets F of finite measure. If we wish to consider measurability of non-integrable functions, we need a slightly stronger hypothesis. I approach this version more gently, with a couple of lemmas.

Lemma Let Σ, T be σ-algebras of subsets of X and Y respectively. Suppose that $D \subseteq X$ and that $\phi : D \to Y$ is a function such that $\phi^{-1}[F] \in \Sigma_D$, the subspace σ-algebra, for every $F \in \mathrm{T}$. Then $g\phi$ is Σ-measurable for every $[-\infty, \infty]$-valued T-measurable function g defined on a subset of Y.

proof Set $C = \operatorname{dom} g$ and $B = \operatorname{dom} g\phi = \phi^{-1}[C]$. If $a \in \mathbb{R}$, then there is an $F \in \mathrm{T}$ such that $\{y : g(y) \le a\} = F \cap C$. Now there is an $E \in \Sigma$ such that $\phi^{-1}[F] = E \cap D$. So

$$\{x : g\phi(x) \le a\} = B \cap E \in \Sigma_B.$$

As a is arbitrary, $g\phi$ is Σ-measurable.

235D Some of the results below are easier when we can move freely between measure spaces and their completions (212C). The next lemma is what we need.

Lemma Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with completions $(X,\hat{\Sigma},\hat{\mu})$ and $(Y,\hat{\mathrm{T}},\hat{\nu})$. Let $\phi : D_\phi \to Y$, $J : D_J \to [0,\infty[$ be functions defined on conegligible subsets of X.

(a) If $\int J \times \chi(\phi^{-1}[F])d\mu = \nu F$ whenever $F \in \mathrm{T}$ and $\nu F < \infty$, then $\int J \times \chi(\phi^{-1}[F])d\hat{\mu} = \hat{\nu} F$ whenever $F \in \hat{\mathrm{T}}$ and $\hat{\nu} F < \infty$.

(b) If $\int J \times \chi(\phi^{-1}[F])d\mu = \nu F$ whenever $F \in \mathrm{T}$, then $\int J \times \chi(\phi^{-1}[F])d\hat{\mu} = \hat{\nu} F$ whenever $F \in \hat{\mathrm{T}}$.

proof Both rely on the fact that either hypothesis is enough to ensure that $\int J\times\chi(\phi^{-1}[F])d\mu = 0$ whenever $\nu F = 0$. Accordingly, if F is ν-negligible, so that there is an $F' \in \mathrm{T}$ such that $F \subseteq F'$ and $\nu F' = 0$, we shall have

$$\int J \times \chi(\phi^{-1}[F])d\mu = \int J \times \chi(\phi^{-1}[F'])d\mu = 0.$$

But now, given any $F \in \hat{\mathrm{T}}$, there is an $F_0 \in \mathrm{T}$ such that $F_0 \subseteq F$ and $\hat{\nu}(F \setminus F_0) = 0$, so that

$$\begin{aligned}\int J \times \chi(\phi^{-1}[F])d\hat{\mu} &= \int J \times \chi(\phi^{-1}[F])d\mu \\ &= \int J \times \chi(\phi^{-1}[F_0])d\mu + \int J \times \chi(\phi^{-1}[F \setminus F_0])d\mu \\ &= \nu F_0 = \hat{\nu} F,\end{aligned}$$

provided (for part (a)) that $\hat{\nu} F < \infty$.

Remark Thus if we have the hypotheses of any of the principal results of this section valid for a pair of non-complete measure spaces, we can expect to be able to draw some conclusion by applying the result to the completions of the given spaces.

235E Now I come to the alternative version of 235A.

Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $\phi : D_\phi \to Y$, $J : D_J \to [0,\infty[$ two functions defined on conegligible subsets of X such that

$$\int J \times \chi(\phi^{-1}[F])d\mu = \nu F$$

for every $F \in \mathrm{T}$, allowing ∞ as a value of the integral.

(a) $J \times g\phi$ is μ-virtually measurable for every ν-virtually measurable function g defined on a subset of Y.

(b) Let g be a ν-virtually measurable $[-\infty,\infty]$-valued function defined on a conegligible subset of Y. Then $\int J \times g\phi\, d\mu = \int g\, d\nu$ whenever either integral is defined in $[-\infty,\infty]$, if we interpret $(J \times g\phi)(x)$ as 0 when $J(x) = 0$ and $g(\phi(x))$ is undefined.

proof Let $(X,\hat{\Sigma},\hat{\mu})$ and $(Y,\hat{\mathrm{T}},\hat{\nu})$ be the completions of (X,Σ,μ) and (Y,T,ν). By 235D,

$$\int J \times \chi(\phi^{-1}[F])d\hat{\mu} = \hat{\nu} F$$

for every $F \in \hat{\mathrm{T}}$. Recalling that a real-valued function is μ-virtually measurable iff it is $\hat{\Sigma}$-measurable (212Fa), and that $\int f d\mu = \int f d\hat{\mu}$ if either is defined in $[-\infty,\infty]$ (212Fb), the conclusions we are seeking are

(a)′ $J \times g\phi$ is $\hat{\Sigma}$-measurable for every $\hat{\mathrm{T}}$-measurable function g defined on a subset of Y;

(b)′ $\int J \times g\phi\, d\hat{\mu} = \int g\, d\hat{\nu}$ whenever g is a $\hat{\mathrm{T}}$-measurable function defined almost everywhere in Y and either integral is defined in $[-\infty,\infty]$.

(a) When I write

$$\int J \times \chi D_\phi d\mu = \int J \times \chi(\phi^{-1}[Y])d\mu = \nu Y,$$

which is part of the hypothesis of this theorem, I mean to imply that $J \times \chi D_\phi$ is μ-virtually measurable, that is, is $\hat{\Sigma}$-measurable. Because D_ϕ is conegligible, it follows that J is $\hat{\Sigma}$-measurable, and its domain D_J, being conegligible, also belongs to $\hat{\Sigma}$. Set $G = \{x : x \in D_J,\ J(x) > 0\} \in \hat{\Sigma}$. Then for any set $A \subseteq X$, $J\times\chi A$ is $\hat{\Sigma}$-measurable iff $A \cap G \in \hat{\Sigma}$. So the hypothesis is just that $G \cap \phi^{-1}[F] \in \hat{\Sigma}$ for every $F \in \hat{\mathrm{T}}$.

Now let g be a $[-\infty,\infty]$-valued function, defined on a subset C of Y, which is $\hat{\mathrm{T}}$-measurable. Applying 235C to $\phi\restriction G$, we see that $g\phi\restriction G$ is $\hat{\Sigma}$-measurable, so $(J\times g\phi)\restriction G$ is $\hat{\Sigma}$-measurable. On the other hand, $J\times g\phi$ is zero almost everywhere in $X\setminus G$, so (because $G\in\hat{\Sigma}$) $J\times g\phi$ is $\hat{\Sigma}$-measurable, as required.

(b)(i) Suppose first that $g\ge 0$. Then $J\times g\phi\ge 0$, so (a) tells us that $\int J\times g\phi$ is defined in $[0,\infty]$.

($\boldsymbol{\alpha}$) If $\int g\,d\hat{\nu}<\infty$ then $\int J\times g\phi\,d\hat{\mu}=\int g\,d\hat{\nu}$ by 235A.

($\boldsymbol{\beta}$) If there is some $\epsilon>0$ such that $\hat{\nu}H=\infty$, where $H=\{y:g(y)\ge\epsilon\}$, then

$$\int J\times g\phi\,d\hat{\mu}\ge\epsilon\int J\times\chi(\phi^{-1}[H])d\hat{\mu}=\epsilon\hat{\nu}H=\infty,$$

so

$$\int J\times g\phi\,d\hat{\mu}=\infty=\int g\,d\hat{\nu}.$$

($\boldsymbol{\gamma}$) Otherwise,

$$\begin{aligned}\int J\times g\phi\,d\hat{\mu}&\ge\sup\{\int J\times h\phi\,d\hat{\mu}:h\text{ is }\hat{\nu}\text{-integrable, }0\le h\le g\}\\&=\sup\{\int h\,d\hat{\nu}:h\text{ is }\hat{\nu}\text{-integrable, }0\le h\le g\}=\int g\,d\hat{\nu}=\infty,\end{aligned}$$

so once again $\int J\times\phi\,d\hat{\mu}=\int g\,d\hat{\nu}$.

(ii) For general real-valued g, apply (i) to g^+ and g^- where $g^+=\frac{1}{2}(|g|+g)$, $g^-=\frac{1}{2}(|g|-g)$; the point is that $(J\times g\phi)^+=J\times g^+\phi$ and $(J\times g\phi)^-=J\times g^-\phi$, so that

$$\int J\times g\phi=\int J\times g^+\phi-\int J\times g^-\phi=\int g^+-\int g^-=\int g$$

if either side is defined in $[-\infty,\infty]$.

235F Remarks (a) Of course there are two special cases of this theorem which between them carry all its content: the case $J=1$ a.e. and the case in which $X=Y$ and ϕ is the identity function. If $J=\chi X$ we are very close to 235G below, and if ϕ is the identity function we are close to the indefinite-integral measures of §234.

(b) As in 235A, we can strengthen the conclusion of (b) in 235E to

$$\int_{\phi^{-1}[F]}J\times g\phi\,d\mu=\int_F g\,d\nu$$

whenever $F\in\mathrm{T}$ and $\int_F g\,d\nu$ is defined in $[-\infty,\infty]$.

235G Theorem Let (X,Σ,μ) and (Y,T,ν) be measure spaces and $\phi:X\to Y$ an inverse-measure-preserving function. Then

(a) if g is a ν-virtually measurable $[-\infty,\infty]$-valued function defined on a subset of Y, $g\phi$ is μ-virtually measurable;

(b) if g is a ν-virtually measurable $[-\infty,\infty]$-valued function defined on a conegligible subset of Y, $\int g\phi\,d\mu=\int g\,d\nu$ if either integral is defined in $[-\infty,\infty]$;

(c) if g is a ν-virtually measurable $[-\infty,\infty]$-valued function defined on a conegligible subset of Y, and $F\in\mathrm{T}$, then $\int_{\phi^{-1}[F]}g\phi\,d\mu=\int_F g\,d\nu$ if either integral is defined in $[-\infty,\infty]$.

proof (a) This follows immediately from 234Ba and 235C; taking $\hat{\Sigma}$, $\hat{\mathrm{T}}$ to be the domains of the completions of μ, ν respectively, $\phi^{-1}[F]\in\hat{\Sigma}$ for every $F\in\hat{\mathrm{T}}$, so if g is $\hat{\mathrm{T}}$-measurable then $g\phi$ will be $\hat{\Sigma}$-measurable.

(b) Apply 235E with $J=\chi X$; we have

$$\int J\times\chi(\phi^{-1}[F])d\mu=\mu\phi^{-1}[F]=\nu F$$

for every $F\in\mathrm{T}$, so

$$\int g\phi=\int J\times g\phi=\int g$$

if either integral is defined in $[-\infty,\infty]$.

(c) Apply (b) to $g\times\chi F$.

235H The image measure catastrophe Applications of 235A would run much more smoothly if we could say

'$\int g\,d\nu$ exists and is equal to $\int J \times g\phi\,d\mu$ for every $g : Y \to \mathbb{R}$ such that $J \times g\phi$ is μ-integrable'.

Unhappily there is no hope of a universally applicable result in this direction. Suppose, for instance, that ν is Lebesgue measure on $Y = [0,1]$, that $X \subseteq [0,1]$ is a non-Lebesgue-measurable set of outer measure 1 (134D), that μ is the subspace measure ν_X on X, and that $\phi(x) = x$ for $x \in X$. Then

$$\mu\phi^{-1}F = \nu^*(X \cap F) = \nu F$$

for every Lebesgue measurable set $F \subseteq Y$, so we can take $J = \chi X$ and the hypotheses of 235A and 235E will be satisfied. But if we write $g = \chi X : [0,1] \to \{0,1\}$, then $\int g\phi\,d\mu$ is defined even though $\int g\,d\nu$ is not.

The point here is that there is a set $A \subseteq Y$ such that (in the language of 235A/235E) $\phi^{-1}[A] \in \Sigma$ but $A \notin \hat{\mathrm{T}}$. This is the **image measure catastrophe**. The search for contexts in which we can be sure that it does not occur will be one of the motive themes of Volume 4. For the moment, I will offer some general remarks (235I-235J), and describe one of the important cases in which the problem does not arise (235K).

235I Lemma Let Σ, T be σ-algebras of subsets of X, Y respectively, and ϕ a function from a subset D of X to Y. Suppose that $G \subseteq X$ and that

$$\mathrm{T} = \{F : F \subseteq Y,\, G \cap \phi^{-1}[F] \in \Sigma\}.$$

Then a real-valued function g, defined on a member of T, is T-measurable iff $\chi G \times g\phi$ is Σ-measurable.

proof Because surely $Y \in \mathrm{T}$, the hypothesis implies that $G \cap D = G \cap \phi^{-1}[Y]$ belongs to Σ.

Let $g : C \to \mathbb{R}$ be a function, where $C \in \mathrm{T}$. Set $B = \mathrm{dom}(g\phi) = \phi^{-1}[C]$, and for $a \in \mathbb{R}$ set $F_a = \{y : g(y) \ge a\}$,

$$E_a = G \cap \phi^{-1}[F_a] = \{x : x \in G \cap B,\, g\phi(x) \ge a\}.$$

Note that $G \cap B \in \Sigma$ because $C \in \mathrm{T}$.

(i) If g is T-measurable, then $F_a \in \mathrm{T}$ and $E_a \in \Sigma$ for every a. Now

$$G \cap \{x : x \in B,\, g\phi(x) \ge a\} = G \cap \phi^{-1}[F_a] = E_a,$$

so $\{x : x \in B,\, (\chi G \times g\phi)(x) \ge a\}$ is either E_a or $E_a \cup (B \setminus G)$, and in either case is relatively Σ-measurable in B. As a is arbitrary, $\chi G \times g\phi$ is Σ-measurable.

(ii) If $\chi G \times g\phi$ is Σ-measurable, then, for any $a \in \mathbb{R}$,

$$E_a = \{x : x \in G \cap B,\, (\chi G \times g\phi)(x) \ge a\} \in \Sigma$$

because $G \cap B \in \Sigma$ and $\chi G \times g\phi$ is Σ-measurable. So $F_a \in \mathrm{T}$. As a is arbitrary, g is T-measurable.

235J Theorem Let (X, Σ, μ) and (Y, T, ν) be complete measure spaces. Let $\phi : D_\phi \to Y$, $J : D_J \to [0, \infty[$ be functions defined on conegligible subsets of X, and set $G = \{x : x \in D_J,\, J(x) > 0\}$. Suppose that

$$\mathrm{T} = \{F : F \subseteq Y,\, G \cap \phi^{-1}[F] \in \Sigma\},$$

$$\nu F = \int J \times \chi(\phi^{-1}[F]) d\mu \text{ for every } F \in \mathrm{T}.$$

Then, for any real-valued function g defined on a subset of Y, $\int J \times g\phi\,d\mu = \int g\,d\nu$ whenever either integral is defined in $[-\infty, \infty]$, provided that we interpret $(J \times g\phi)(x)$ as 0 when $J(x) = 0$ and $g(\phi(x))$ is undefined.

proof If g is T-measurable and defined almost everywhere, this is a consequence of 235E. So I have to show that if $J \times g\phi$ is measurable and defined almost everywhere, so is g. Set $W = Y \setminus \mathrm{dom}\, g$. Then $J \times g\phi$ is undefined on $G \cap \phi^{-1}[W]$, because $g\phi$ is undefined there and we cannot take advantage of the escape clause available when $J = 0$; so $G \cap \phi^{-1}[W]$ must be negligible, therefore measurable, and $W \in \mathrm{T}$. Next,

$$\nu W = \int J \times \chi(\phi^{-1}[W]) = 0$$

because $J \times \chi(\phi^{-1}[W])$ can be non-zero only on the negligible set $G \cap \phi^{-1}[W]$. So g is defined almost everywhere.

Note that the hypothesis surely implies that $J \times \chi D_\phi = J \times \chi(\phi^{-1}[Y])$ is measurable, so that J is measurable (because D_ϕ is conegligible) and $G \in \Sigma$. Writing $K(x) = 1/J(x)$ for $x \in G$, 0 for $x \in X \setminus G$, the function $K : X \to \mathbb{R}$ is measurable, and

$$\chi G \times g\phi = K \times J \times g\phi$$

is measurable. So 235I tells us that g must be measurable, and we're done.

Remark When $J = \chi X$, the hypothesis of this theorem becomes

$$\mathrm{T} = \{F : F \subseteq Y, \phi^{-1}[F] \in \Sigma\}, \quad \nu F = \mu\phi^{-1}[F] \text{ for every } F \in \mathrm{T};$$

that is, ν is the image measure $\mu\phi^{-1}$ as defined in 234D.

235K Corollary Let (X, Σ, μ) be a complete measure space, and J a non-negative measurable function defined on a conegligible subset of X. Let ν be the associated indefinite-integral measure, and T its domain. Then for any real-valued function g defined on a subset of X, g is T-measurable iff $J \times g$ is Σ-measurable, and $\int g\,d\nu = \int J \times g\,d\mu$ if either integral is defined in $[-\infty, \infty]$, provided that we interpret $(J \times g)(x)$ as 0 when $J(x) = 0$ and $g(x)$ is undefined.

proof Put 235J, taking $Y = X$ and ϕ the identity function, together with 234Ld.

235L Applying the Radon-Nikodým theorem In order to use 235A-235J effectively, we need to be able to find suitable functions J. This can be difficult – some very special examples will take up most of Chapter 26 below. But there are many circumstances in which we can be sure that such J exist, even if we do not know what they are. A minimal requirement is that if $\nu F < \infty$ and $\mu^*\phi^{-1}[F] = 0$ then $\nu F = 0$, because $\int J \times \chi(\phi^{-1}[F])d\mu$ will be zero for any J. A sufficient condition, in the special case of indefinite-integral measures, is in 234O. Another is the following.

235M Theorem Let (X, Σ, μ) be a σ-finite measure space, (Y, T, ν) a semi-finite measure space, and $\phi : D \to Y$ a function such that
(i) D is a conegligible subset of X,
(ii) $\phi^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$;
(iii) $\mu\phi^{-1}[F] > 0$ whenever $F \in \mathrm{T}$ and $\nu F > 0$.
Then there is a Σ-measurable function $J : X \to [0, \infty[$ such that $\int J \times \chi\phi^{-1}[F]\,d\mu = \nu F$ for every $F \in \mathrm{T}$.

proof (a) To begin with (down to the end of (c) below) let us suppose that $D = X$ and that ν is totally finite.

Set $\tilde{\mathrm{T}} = \{\phi^{-1}[F] : F \in \mathrm{T}\} \subseteq \Sigma$. Then $\tilde{\mathrm{T}}$ is a σ-algebra of subsets of X. **P** (i)

$$\emptyset = \phi^{-1}[\emptyset] \in \tilde{\mathrm{T}}.$$

(ii) If $E \in \tilde{\mathrm{T}}$, take $F \in \mathrm{T}$ such that $E = \phi^{-1}[F]$, so that

$$X \setminus E = \phi^{-1}[Y \setminus F] \in \tilde{\mathrm{T}}.$$

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\tilde{\mathrm{T}}$, then for each $n \in \mathbb{N}$ choose $F_n \in \mathrm{T}$ such that $E_n = \phi^{-1}[F_n]$; then

$$\bigcup_{n\in\mathbb{N}} E_n = \phi^{-1}[\bigcup_{n\in\mathbb{N}} F_n] \in \tilde{\mathrm{T}}. \quad \mathbf{Q}$$

Next, we have a totally finite measure $\tilde\nu : \tilde{\mathrm{T}} \to [0, \nu Y]$ given by setting

$$\tilde\nu(\phi^{-1}[F]) = \nu F \text{ for every } F \in \mathrm{T}.$$

P (i) If F, $F' \in \mathrm{T}$ and $\phi^{-1}[F] = \phi^{-1}[F']$, then $\phi^{-1}[F \triangle F'] = \emptyset$, so $\mu(\phi^{-1}[F \triangle F']) = 0$ and $\nu(F \triangle F') = 0$; consequently $\nu F = \nu F'$. This shows that $\tilde\nu$ is well-defined. (ii) Now

$$\tilde\nu\emptyset = \tilde\nu(\phi^{-1}[\emptyset]) = \nu\emptyset = 0.$$

(iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\tilde{\mathrm{T}}$, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in T such that $E_n = \phi^{-1}[F_n]$ for each n; set $F'_n = F_n \setminus \bigcup_{m<n} F_m$ for each n; then $E_n = \phi^{-1}[F'_n]$ for each n, so

$$\tilde\nu(\bigcup_{n\in\mathbb{N}} E_n) = \tilde\nu(\phi^{-1}[\bigcup_{n\in\mathbb{N}} F'_n]) = \nu(\bigcup_{n\in\mathbb{N}} F'_n) = \sum_{n=0}^\infty \nu F'_n = \sum_{n=0}^\infty \tilde\nu E_n. \quad \mathbf{Q}$$

Finally, observe that if $\tilde{\nu}E > 0$ then $\mu E > 0$, because $E = \phi^{-1}[F]$ where $\nu F > 0$.

(b) By 215B(ix) there is a Σ-measurable function $h : X \to \,]0, \infty[$ such that $\int h\,d\mu$ is finite. Define $\tilde{\mu} : \tilde{\mathrm{T}} \to [0, \infty[$ by setting $\tilde{\mu}E = \int_E h\,d\mu$ for every $E \in \tilde{\mathrm{T}}$; then $\tilde{\mu}$ is a totally finite measure. If $E \in \tilde{\mathrm{T}}$ and $\tilde{\mu}E = 0$, then (because h is strictly positive) $\mu E = 0$ and $\tilde{\nu}E = 0$. Accordingly we may apply the Radon-Nikodým theorem to $\tilde{\mu}$ and $\tilde{\nu}$ to see that there is a $\tilde{\mathrm{T}}$-measurable function $g : X \to \mathbb{R}$ such that $\int_E g\,d\tilde{\mu} = \tilde{\nu}E$ for every $E \in \tilde{\mathrm{T}}$. Because $\tilde{\nu}$ is non-negative, we may suppose that $g \ge 0$.

(c) Applying 235A to μ, $\tilde{\mu}$, h and the identity function from X to itself, we see that

$$\int_E g \times h\,d\mu = \int_E g\,d\tilde{\mu} = \tilde{\nu}E$$

for every $E \in \tilde{\mathrm{T}}$, that is, that

$$\int J \times \chi(\phi^{-1}[F])d\mu = \nu F$$

for every $F \in \mathrm{T}$, writing $J = g \times h$.

(d) This completes the proof when ν is totally finite and $D = X$. For the general case, if $Y = \emptyset$ then $\mu X = 0$ and the result is trivial. Otherwise, let $\hat{\phi}$ be any extension of ϕ to a function from X to Y which is constant on $X \setminus D$; then $\hat{\phi}^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$, because $D = \phi^{-1}[Y] \in \Sigma$ and $\hat{\phi}^{-1}[F]$ is always either $\phi^{-1}[F]$ or $(X \setminus D) \cup \phi^{-1}[F]$. Now ν must be σ-finite. **P** Use the criterion of 215B(ii). If $\mathcal{F}$ is a disjoint family in $\{F : F \in \mathrm{T},\, 0 < \nu F < \infty\}$, then $\mathcal{E} = \{\hat{\phi}^{-1}[F] : F \in \mathcal{F}\}$ is a disjoint family in $\{E : \mu E > 0\}$, so $\mathcal{E}$ and $\mathcal{F}$ are countable. **Q**

Let $\langle Y_n \rangle_{n\in\mathbb{N}}$ be a partition of Y into sets of finite ν-measure, and for each $n \in \mathbb{N}$ set $\nu_n F = \nu(F \cap Y_n)$ for every $F \in \mathrm{T}$. Then ν_n is a totally finite measure on Y, and if $\nu_n F > 0$ then $\nu F > 0$ so

$$\mu\hat{\phi}^{-1}[F] = \mu\phi^{-1}[F] > 0.$$

Accordingly μ, $\hat{\phi}$ and ν_n satisfy the assumptions of the theorem together with those of (a) above, and there is a Σ-measurable function $J_n : X \to [0, \infty[$ such that

$$\nu_n F = \int J_n \times \chi(\phi^{-1}[F])d\mu$$

for every $F \in \mathrm{T}$. Now set $J = \sum_{n=0}^{\infty} J_n \times \chi(\phi^{-1}[Y_n])$, so that $J : X \to [0, \infty[$ is Σ-measurable. If $F \in \mathrm{T}$, then

$$\begin{aligned}\int J \times \chi(\phi^{-1}[F])d\mu &= \sum_{n=0}^{\infty} \int J_n \times \chi(\phi^{-1}[Y_n]) \times \chi(\phi^{-1}[F])d\mu \\ &= \sum_{n=0}^{\infty} \int J_n \times \chi(\phi^{-1}[F \cap Y_n])d\mu = \sum_{n=0}^{\infty} \nu(F \cap Y_n) = \nu F,\end{aligned}$$

as required.

235N Remark Theorem 235M can fail if μ is only strictly localizable rather than σ-finite. **P** Let $X = Y$ be an uncountable set, $\Sigma = \mathcal{P}X$, μ counting measure on X (112Bd), T the countable-cocountable σ-algebra of Y, ν the countable-cocountable measure on Y (211R), $\phi : X \to Y$ the identity map. Then $\phi^{-1}[F] \in \Sigma$ and $\mu\phi^{-1}[F] > 0$ whenever $\nu F > 0$. But if J is any μ-integrable function on X, then $F = \{x : J(x) \ne 0\}$ is countable and

$$\nu(Y \setminus F) = 1 \ne 0 = \int_{\phi^{-1}[Y\setminus F]} J\,d\mu. \text{ **Q**}$$

***235O** There are some simplifications in the case of σ-finite spaces; in particular, 235A and 235E become conflated. I will give an adaptation of the hypotheses of 235A which may be used in the σ-finite case. First a lemma.

Lemma Let (X, Σ, μ) be a measure space and f a non-negative integrable function on X. If $A \subseteq X$ is such that $\int_A f + \int_{X\setminus A} f = \int f$, then $f \times \chi A$ is integrable.

proof By 214Eb, there are μ-integrable functions f_1, f_2 such that f_1 extends $f{\restriction}A$, f_2 extends $f{\restriction}X \setminus A$, and

$$\int_E f_1 = \int_{E\cap A} f, \quad \int_E f_2 = \int_{E\setminus A} f$$

for every $E \in \Sigma$. Because f is non-negative, $\int_E f_1$ and $\int_E f_2$ are non-negative for every $E \in \Sigma$, and f_1, f_2 are non-negative a.e. Accordingly we have $f \times \chi A \leq_{\text{a.e.}} f_1$ and $f \times \chi(X \setminus A) \leq_{\text{a.e.}} f_2$, so that $f \leq_{\text{a.e.}} f_1 + f_2$. But also

$$\int f_1 + f_2 = \int_X f_1 + \int_X f_2 = \int_A f + \int_{X\setminus A} f = \int f,$$

so $f =_{\text{a.e.}} f_1 + f_2$. Accordingly

$$f_1 =_{\text{a.e.}} f - f_2 \leq_{\text{a.e.}} f - f \times \chi(X \setminus A) = f \times \chi A \leq_{\text{a.e.}} f_1$$

and $f \times \chi A =_{\text{a.e.}} f_1$ is integrable.

***235P Proposition** Let (X, Σ, μ) be a complete measure space and (Y, T, ν) a complete σ-finite measure space. Suppose that $\phi : D_\phi \to Y$, $J : D_J \to [0, \infty[$ are functions defined on conegligible subsets D_ϕ, D_J of X such that $\int_{\phi^{-1}[F]} J\,d\mu$ exists and is equal to νF whenever $F \in \mathrm{T}$ and $\nu F < \infty$.

(a) $J \times g\phi$ is Σ-measurable for every T-measurable real-valued function g defined on a subset of Y.

(b) If g is a T-measurable real-valued function defined almost everywhere in Y, then $\int J \times g\phi\,d\mu = \int g\,d\nu$ whenever either integral is defined in $[-\infty, \infty]$, interpreting $(J \times g\phi)(x)$ as 0 when $J(x) = 0$, $g(\phi(x))$ is undefined.

proof The point is that the hypotheses of 235E are satisfied. To see this, let us write $\Sigma_C = \{E \cap C : E \in \Sigma\}$ and $\mu_C = \mu^*\restriction\Sigma_C$ for the subspace measure on C, for each $C \subseteq X$. Let $\langle Y_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets with union Y and with $\nu Y_n < \infty$ for every $n \in \mathbb{N}$, starting from $Y_0 = \emptyset$.

(i) Take any $F \in \mathrm{T}$ with $\nu F < \infty$, and set $F_n = F \cup Y_n$ for each $n \in \mathbb{N}$; write $C_n = \phi^{-1}[F_n]$.

Fix n for the moment. Then our hypothesis implies that

$$\int_{C_0} J\,d\mu + \int_{C_n\setminus C_0} J\,d\mu = \nu F + \nu(F_n \setminus F) = \nu F_n = \int_{C_n} J\,d\mu.$$

If we regard the subspace measures on C_0 and $C_n \setminus C_0$ as derived from the measure μ_{C_n} of C_n (214Ce), then 235O tells us that $J \times \chi C_0$ is μ_{C_n}-integrable, and there is a μ-integrable function h_n such that h_n extends $(J \times \chi C_0)\restriction C_n$.

Let E be a μ-conegligible set, included in the domain D_ϕ of ϕ, such that $h_n\restriction E$ is Σ-measurable for every n. Because $\langle C_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with union $\phi^{-1}[\bigcup_{n\in\mathbb{N}} F_n] = D_\phi$,

$$(J \times \chi C_0)(x) = \lim_{n\to\infty} h_n(x)$$

for every $x \in E$, and $(J \times \chi C_0)\restriction E$ is measurable. At the same time, we know that there is a μ-integrable h extending $J\restriction C_0$, and $0 \leq_{\text{a.e.}} J \times \chi C_0 \leq_{\text{a.e.}} |h|$. Accordingly $J \times \chi C_0$ is integrable, and (using 214F)

$$\int J \times \chi\phi^{-1}[F]\,d\mu = \int J \times \chi C_0\,d\mu = \int_{C_0} J\restriction C_0\,d\mu_{C_0} = \nu F.$$

(ii) This deals with F of finite measure. For general $F \in \mathrm{T}$,

$$\int J \times \chi(\phi^{-1}[F])\,d\mu = \lim_{n\to\infty} \int J \times \chi(\phi^{-1}[F \cap Y_n])\,d\mu = \lim_{n\to\infty} \nu(F \cap Y_n) = \nu F.$$

So the hypotheses of 235E are satisfied, and the result follows at once.

***235Q** I remarked in 235Bb that a difficulty can arise in 235A, for general measure spaces, if we speak of $\int_{\phi^{-1}[F]} J\,d\mu$ in the hypothesis, in place of $\int J \times \chi(\phi^{-1}[F])d\mu$. Here is an example.

Example Set $X = Y = [0, 2]$. Write Σ_L for the algebra of Lebesgue measurable subsets of $\mathbb{R}$, and μ_L for Lebesgue measure; write μ_c for counting measure on $\mathbb{R}$. Set

$$\Sigma = \mathrm{T} = \{E : E \subseteq [0, 2],\ E \cap [0, 1[\in \Sigma_L\};$$

of course this is a σ-algebra of subsets of $[0, 2]$. For $E \in \Sigma = \mathrm{T}$, set

$$\mu E = \nu E = \mu_L(E \cap [0, 1[) + \mu_c(E \cap [1, 2]);$$

then μ is a complete measure – in effect, it is the direct sum of Lebesgue measure on $[0, 1[$ and counting measure on $[1, 2]$ (see 214L). It is easy to see that

$$\mu^* B = \mu_L^*(B \cap [0,1[) + \mu_c(B \cap [1,2])$$

for every $B \subseteq [0,2]$. Let $A \subseteq [0,1[$ be a non-Lebesgue-measurable set such that $\mu_L^*(E \setminus A) = \mu_L E$ for every Lebesgue measurable $E \subseteq [0,1[$ (see 134D). Define $\phi : [0,2] \to [0,2]$ by setting $\phi(x) = x + 1$ if $x \in A$, $\phi(x) = x$ if $x \in [0,2] \setminus A$.

If $F \in \Sigma$, then $\mu^*(\phi^{-1}[F]) = \mu F$. **P** **(i)** If $F \cap [1,2]$ is finite, then $\mu F = \mu_L(F \cap [0,1[) + \#(F \cap [1,2])$. Now

$$\phi^{-1}[F] = (F \cap [0,1[\setminus A) \cup (F \cap [1,2]) \cup \{x : x \in A,\, x + 1 \in F\};$$

as the last set is finite, therefore μ-negligible,

$$\mu^*(\phi^{-1}[F]) = \mu_L^*(F \cap [0,1[\setminus A) + \#(F \cap [1,2]) = \mu_L(F \cap [0,1[) + \#(F \cap [1,2]) = \mu F.$$

(ii) If $F \cap [1,2]$ is infinite, so is $\phi^{-1}[F] \cap [1,2]$, so

$$\mu^*(\phi^{-1}[F]) = \infty = \mu F. \text{ } \mathbf{Q}$$

This means that if we set $J(x) = 1$ for every $x \in [0,2]$,

$$\int_{\phi^{-1}[F]} J\,d\mu = \mu_{\phi^{-1}[F]}(\phi^{-1}[F]) = \mu^*(\phi^{-1}[F]) = \mu F$$

for every $F \in \Sigma$, and ϕ, J satisfy the amended hypotheses for 235A. But if we set $g = \chi\,[0,1[$, then g is μ-integrable, with $\int g\,d\mu = 1$, while

$$J(x)g(\phi(x)) = 1 \text{ if } x \in [0,1] \setminus A, \text{ 0 otherwise,}$$

so, because $A \notin \Sigma$, $J \times g\phi$ is not measurable, and therefore (since μ is complete) not μ-integrable.

235R Reversing the burden Throughout the work above, I have been using the formula

$$\int J \times g\phi = \int g,$$

as being the natural extension of the formula

$$\int g = \int g\phi \times \phi'$$

of ordinary advanced calculus. But we can also move the 'derivative' J to the other side of the equation, as follows.

Theorem Let (X, Σ, μ), (Y, T, ν) be measure spaces and $\phi : X \to Y$, $J : Y \to [0, \infty[$ functions such that $\int_F J\,d\nu$ and $\mu\phi^{-1}[F]$ are defined in $[0,\infty]$ and equal for every $F \in \mathrm{T}$. Then $\int g\phi\,d\mu = \int J \times g\,d\nu$ whenever g is ν-virtually measurable and defined ν-almost everywhere and either integral is defined in $[-\infty, \infty]$.

proof Let ν_1 be the indefinite-integral measure over ν defined by J, and $\hat\mu$ the completion of μ. Then ϕ is inverse-measure-preserving for $\hat\mu$ and ν_1. **P** If $F \in \mathrm{T}$, then $\nu_1 F = \int_F J\,d\nu = \mu\phi^{-1}[F]$; that is, ϕ is inverse-measure-preserving for μ and $\nu_1{\restriction}\mathrm{T}$. Since ν_1 is the completion of $\nu_1{\restriction}\mathrm{T}$ (234Lb), ϕ is inverse-measure-preserving for μ and ν_1 (234Ba). **Q**

Of course we can also regard ν_1 as being an indefinite-integral measure over the completion $\hat\nu$ of ν (212Fb). So if g is ν-virtually measurable and defined ν-almost everywhere,

$$\int J \times g\,d\nu = \int J \times g\,d\hat\nu = \int g\,d\nu_1 = \int g\phi\,d\hat\mu = \int g\phi\,d\mu$$

if any of the five integrals is defined in $[-\infty, \infty]$, by 235K, 235Gb and 212Fb again.

235X Basic exercises (a) Explain what 235A tells us when $X = Y$, $\mathrm{T} = \Sigma$, ϕ is the identity function and $\nu E = \alpha\mu E$ for every $E \in \Sigma$.

(b) Let (X, Σ, μ) be a measure space, J an integrable non-negative real-valued function on X, and $\phi : D_\phi \to \mathbb{R}$ a measurable function, where D_ϕ is a conegligible subset of X. Set

$$g(a) = \int_{\{x:\phi(x) \le a\}} J$$

for $a \in \mathbb{R}$, and let μ_g be the Lebesgue-Stieltjes measure associated with g. Show that $\int J \times f\phi\,d\mu = \int f d\mu_g$ for every μ_g-integrable real function f.

(c) Let Σ, T and Λ be σ-algebras of subsets of X, Y and Z respectively. Let us say that a function $\phi : A \to Y$, where $A \subseteq X$, is (Σ, T)-measurable if $\phi^{-1}[F] \in \Sigma_A$, the subspace σ-algebra of A, for every $F \in \mathrm{T}$. Suppose that $A \subseteq X$, $B \subseteq Y$, $\phi : A \to Y$ is (Σ, T)-measurable and $\psi : B \to Z$ is (T, Λ)-measurable. Show that $\psi\phi$ is (Σ, Λ)-measurable. Deduce 235C.

(d) Let (X, Σ, μ) be a measure space and (Y, T, ν) a semi-finite measure space. Let $\phi : D_\phi \to Y$ and $J : D_J \to [0, \infty[$ be functions defined on conegligible subsets D_ϕ, D_J of X such that $\int J \times \chi(\phi^{-1}[F])d\mu$ exists $= \nu F$ whenever $F \in \mathrm{T}$ and $\nu F < \infty$. Let g be a T-measurable real-valued function, defined on a conegligible subset of Y. Show that $J \times g\phi$ is μ-integrable iff g is ν-integrable, and the integrals are then equal, provided we interpret $(J \times g\phi)(x)$ as 0 when $J(x) = 0$ and $g(\phi(x))$ is undefined.

(e) Let (X, Σ, μ) be a measure space and $E \in \Sigma$. Define a measure $\mu \llcorner E$ on X by setting $(\mu \llcorner E)(F) = \mu(E \cap F)$ whenever $F \subseteq X$ is such that $F \cap E \in \Sigma$. Show that, for any function f from a subset of X to $[-\infty, \infty]$, $\int f d(\mu \llcorner E) = \int_E f d\mu$ if either is defined in $[-\infty, \infty]$.

>(f) Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function which is absolutely continuous on every closed bounded interval, and μ_g the associated Lebesgue-Stieltjes measure (114Xa, 225Xd). Write μ for Lebesgue measure on $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that $\int f \times g' \, d\mu = \int f \, d\mu_g$ in the sense that if one of the integrals exists, finite or infinite, so does the other, and they are then equal.

(g) Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function and J a non-negative real-valued μ_g-integrable function, where μ_g is the Lebesgue-Stieltjes measure defined from g. Set $h(x) = \int_{]-\infty, x]} J \, d\mu_g$ for each $x \in \mathbb{R}$, and let μ_h be the Lebesgue-Stieltjes measure associated with h. Show that, for any $f : \mathbb{R} \to \mathbb{R}$, $\int f \times J \, d\mu_g = \int f d\mu_h$, in the sense that if one of the integrals is defined in $[-\infty, \infty]$ so is the other, and they are then equal.

>(h) Let X be a set and λ, μ, ν three measures on X such that μ is an indefinite-integral measure over λ, with Radon-Nikodým derivative f, and ν is an indefinite-integral measure over μ, with Radon-Nikodým derivative g. Show that ν is an indefinite-integral measure over λ, and that $f \times g$ is a Radon-Nikodým derivative of ν with respect to λ, provided we interpret $(f \times g)(x)$ as 0 when $f(x) = 0$ and $g(x)$ is undefined.

(i) In 235M, if ν is not semi-finite, show that we can still find a J such that $\int_{\phi^{-1}[F]} J \, d\mu = \nu F$ for every set F of finite measure. (*Hint*: use the 'semi-finite version' of ν, as described in 213Xc.)

(j) Let (X, Σ, μ) be a σ-finite measure space, and T a σ-subalgebra of Σ. Let $\nu : \mathrm{T} \to \mathbb{R}$ be a countably additive functional such that $\nu F = 0$ whenever $F \in \mathrm{T}$ and $\mu F = 0$. Show that there is a μ-integrable function f such that $\int_F f d\mu = \nu F$ for every $F \in \mathrm{T}$. (*Hint*: use the method of 235M, applied to the positive and negative parts of ν.)

(k) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with completions $(X, \hat{\Sigma}, \hat{\mu})$ and $(Y, \hat{\mathrm{T}}, \hat{\nu})$. Let $\phi : D_\phi \to Y$, $J : D_J \to [0, \infty[$ be functions defined on conegligible subsets of X. Show that if $\int_{\phi^{-1}[F]} J \, d\mu = \nu F$ whenever $F \in \mathrm{T}$ and $\nu F < \infty$, then $\int_{\phi^{-1}[F]} J \, d\mu = \nu F$ whenever $F \in \hat{\mathrm{T}}$ and $\hat{\nu} F < \infty$. Hence, or otherwise, show that 235Pb is valid for non-complete spaces (X, Σ, μ) and (Y, T, ν).

(l) Let (X, Σ, μ) be a complete measure space, Y a set, $\phi : X \to Y$ a function and $\nu = \mu\phi^{-1}$ the corresponding image measure on Y. Let ν_1 be an indefinite-integral measure over ν. Show that there is an indefinite-integral measure μ_1 over μ such that ν_1 is the image measure $\mu_1\phi^{-1}$.

(m) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Let ν_1 be an indefinite-integral measure over ν. Show that there is an indefinite-integral measure μ_1 over μ such that ϕ is inverse-measure-preserving for μ_1 and ν_1.

(n) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Show that $\overline{\int} h\phi \, d\mu \le \overline{\int} h \, d\nu$ for every real-valued function h defined almost everywhere in Y. (Compare 234Bf.)

235Y Further exercises (a) Write T for the algebra of Borel subsets of $Y = [0,1]$, and ν for the restriction of Lebesgue measure to T. Let $A \subseteq [0,1]$ be a set such that both A and $[0,1] \setminus A$ have Lebesgue outer measure 1, and set $X = A \cup [1,2]$. Let Σ be the algebra of relatively Borel subsets of X, and set $\mu E = \mu_A(A \cap E)$ for $E \in \Sigma$, where μ_A is the subspace measure induced on A by Lebesgue measure. Define $\phi : X \to Y$ by setting $\phi(x) = x$ if $x \in A$, $x - 1$ if $x \in X \setminus A$. Show that ν is the image measure $\mu\phi^{-1}$, but that, setting $g = \chi([0,1] \setminus A)$, $g\phi$ is μ-integrable while g is not ν-integrable.

(b) Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ. Let f be a non-negative μ-integrable function with $\int f d\mu = 1$, so that its indefinite-integral measure ν is a probability measure. Let g be a ν-integrable real-valued function and set $h = f \times g$, intepreting $h(x)$ as 0 if $f(x) = 0$ and $g(x)$ is undefined. Let f_1, h_1 be conditional expectations of f, h on T with respect to the measure μ, and set $g_1 = h_1/f_1$, interpreting $g_1(x)$ as 0 if $h_1(x) = 0$ and $f_1(x)$ is either 0 or undefined. Show that g_1 is a conditional expectation of g on T with respect to the measure ν.

235 Notes and comments I see that I have taken up a great deal of space in this section with technicalities; the hypotheses of the theorems vary erratically, with completeness, in particular, being invoked at apparently arbitrary intervals, and ideas repeat themselves in a haphazard pattern. There is nothing deep, and most of the work consists in laboriously verifying details. The trouble with this topic is that it is useful. The results here are abstract expressions of integration-by-substitution; they have applications all over measure theory. I cannot therefore content myself with theorems which will elegantly express the underlying ideas, but must seek formulations which I can quote in later arguments.

I hope that the examples in 235Bb, 235H, 235N, 235Q, 234Ya and 235Ya will go some way to persuade you that there are real traps for the unwary, and that the careful verifications written out at such length are necessary. On the other hand, it is happily the case that in simple contexts, in which the measures μ, ν are σ-finite and the transformations ϕ are Borel isomorphisms, no insuperable difficulties arise, and in particular the image measure catastrophe does not trouble us. But for further work in this direction I refer you to the applications in §263, §265 and §271, and to Volume 4.

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Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

§**234** Section §234 has been rewritten, with a good deal of new material. The former paragraphs 234A-234G, referred to in the 2002 and 2004 editions of Volume 3 and the 2003 and 2006 editions of Volume 4, are now 234I-234O.

§**235** Section §235 has been re-organized, with some material moved to §234. Specifically, 235H, 235I, 235J, 235L, 235M, 235T and 235Xe, referred to in the 2002 and 2004 editions of Volume 3 and the 2003 and 2006 editions of Volume 4, are now dealt with in 234B, 235G, 235H, 235J, 235K, 235R and 234A.

Version of 17.12.12

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