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# Chapter 22

## The Fundamental Theorem of Calculus

In this chapter I address one of the most important properties of the Lebesgue integral. Given an integrable function $f : [a, b] \to \mathbb{R}$, we can form its indefinite integral $F(x) = \int_a^x f(t)dt$ for $x \in [a, b]$. Two questions immediately present themselves. (i) Can we expect to have the derivative $F'$ of $F$ equal to $f$? (ii) Can we identify which functions $F$ will appear as indefinite integrals? Reasonably satisfactory answers may be found for both of these questions: $F' = f$ almost everywhere (222E) and indefinite integrals are the absolutely continuous functions (225E). In the course of dealing with them, we need to develop a variety of techniques which lead to many striking results both in the theory of Lebesgue measure and in other, apparently unrelated, topics in real analysis.

The first step is 'Vitali's theorem' (§221), a remarkable argument – it is more a method than a theorem – which uses the geometric nature of the real line to extract disjoint subfamilies from collections of intervals. It is the foundation stone not only of the results in §222 but of all geometric measure theory, that is, measure theory on spaces with a geometric structure. I use it here to show that monotonic functions are differentiable almost everywhere (222A). Following this, Fatou's Lemma and Lebesgue's Dominated Convergence Theorem are enough to show that the derivative of an indefinite integral is almost everywhere equal to the integrand. We find that some innocent-looking manipulations of this fact take us surprisingly far; I present these in §223.

I begin the second half of the chapter with a discussion of functions 'of bounded variation', that is, expressible as the difference of bounded monotonic functions (§224). This is one of the least measure-theoretic sections in the volume; only in 224I and 224J are measure and integration even mentioned. But this material is needed for Chapter 28 as well as for the next section, and is also one of the basic topics of twentieth-century real analysis. §225 deals with the characterization of indefinite integrals as the 'absolutely continuous' functions. In fact this is now quite easy; it helps to call on Vitali's theorem again, but everything else is a straightforward application of methods previously used. The second half of the section introduces some new ideas in an attempt to give a deeper intuition into the essential nature of absolutely continuous functions. §226 returns to functions of bounded variation and their decomposition into 'saltus' and 'absolutely continuous' and 'singular' parts, the first two being relatively manageable and the last looking something like the Cantor function.

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## 221 Vitali's theorem in $\mathbb{R}$

I give the first theorem of this chapter a section to itself. It occupies a position between measure theory and geometry (it is, indeed, one of the fundamental results of 'geometric measure theory'), and its proof involves both the measure and the geometry of the real line.

**221A Vitali's theorem** Let $A$ be a bounded subset of $\mathbb{R}$ and $\mathcal{I}$ a family of non-singleton closed intervals in $\mathbb{R}$ such that every point of $A$ belongs to arbitrarily short members of $\mathcal{I}$. Then there is a countable set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that (i) $\mathcal{I}_0$ is disjoint, that is, $I \cap I' = \emptyset$ for all distinct $I$, $I' \in \mathcal{I}_0$ (ii) $\mu(A \setminus \bigcup \mathcal{I}_0) = 0$, where $\mu$ is Lebesgue measure on $\mathbb{R}$.

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## 222 Differentiating an indefinite integral

I come now to the first of the two questions mentioned in the introduction to this chapter: if $f$ is an integrable function on $[a,b]$, what is $\frac{d}{dx}\int_a^x f$? It turns out that this derivative exists and is equal to $f$ almost everywhere (222E). The argument is based on a striking property of monotonic functions: they are differentiable almost everywhere (222A), and we can bound the integrals of their derivatives (222C).

**222A Theorem** Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a monotonic function. Then $f$ is differentiable almost everywhere in $I$.

**222B Remarks** If $(X, \Sigma, \mu)$ is a measure space, $K$ is a countable set, and $\langle E_k\rangle_{k\in K}$ is a family in $\Sigma$,

$$\mu(\bigcup_{k\in K} E_k) \le \sum_{k\in K} \mu E_k,$$

with equality if $\langle E_k\rangle_{k\in K}$ is disjoint.

**222C Lemma** Suppose that $a \le b$ in $\mathbb{R}$, and that $F : [a,b] \to \mathbb{R}$ is a non-decreasing function. Then $\int_a^b F'$ exists and is at most $F(b) - F(a)$.

**Remark** I write $\int_a^x f$ to mean $\int_{[a,x[} f$.

**222D Lemma** Suppose that $a < b$ in $\mathbb{R}$, and that $f$, $g$ are real-valued functions, both integrable over $[a,b]$, such that $\int_a^x f = \int_a^x g$ for every $x \in [a,b]$. Then $f = g$ almost everywhere in $[a,b]$.

**222E Theorem** Suppose that $a \le b$ in $\mathbb{R}$ and that $f$ is a real-valued function which is integrable over $[a,b]$. Then $F(x) = \int_a^x f$ exists in $\mathbb{R}$ for every $x \in [a,b]$, and the derivative $F'(x)$ exists and is equal to $f(x)$ for almost every $x \in [a,b]$.

**222F Corollary** Suppose that $f$ is any real-valued function which is integrable over $\mathbb{R}$, and set $F(x) = \int_{-\infty}^x f$ for every $x \in \mathbb{R}$. Then $F'(x)$ exists and is equal to $f(x)$ for almost every $x \in \mathbb{R}$.

**222G Corollary** Suppose that $E \subseteq \mathbb{R}$ is a measurable set and that $f$ is a real-valued function which is integrable over $E$. Set $F(x) = \int_{E\cap]-\infty,x[} f$ for $x \in \mathbb{R}$. Then $F'(x) = f(x)$ for almost every $x \in E$, and $F'(x) = 0$ for almost every $x \in \mathbb{R} \setminus E$.

**222H Proposition** Suppose that $a \le b$ in $\mathbb{R}$ and that $f$ is a real-valued function which is integrable over $[a,b]$. Set $F(x) = \int_a^x f$ for $x \in [a,b]$. Then $F'(x)$ exists and is equal to $f(x)$ at any point $x \in \operatorname{dom}(f) \cap ]a,b[$ at which $f$ is continuous.

**222I Complex-valued functions (a)** If $a \le b$ in $\mathbb{R}$ and $f$ is a complex-valued function which is integrable over $[a,b]$, then $F(x) = \int_a^x f$ is defined in $\mathbb{C}$ for every $x \in [a,b]$, and its derivative $F'(x)$ exists and is equal to $f(x)$ for almost every $x \in [a,b]$; moreover, $F'(x) = f(x)$ whenever $x \in \operatorname{dom}(f) \cap ]a,b[$ and $f$ is continuous at $x$.

**(b)** If $f$ is a complex-valued function which is integrable over $\mathbb{R}$, and $F(x) = \int_{-\infty}^x f$ for each $x \in \mathbb{R}$, then $F'$ exists and is equal to $f$ almost everywhere in $\mathbb{R}$.

**(c)** If $E \subseteq \mathbb{R}$ is a measurable set and $f$ is a complex-valued function which is integrable over $E$, and $F(x) = \int_{E\cap]-\infty,x[} f$ for each $x \in \mathbb{R}$, then $F'(x) = f(x)$ for almost every $x \in E$ and $F'(x) = 0$ for almost every $x \in \mathbb{R} \setminus E$.

***222J The Denjoy-Young-Saks theorem: Definition** Let $f$ be any real function, and $A \subseteq \mathbb{R}$ its domain. Write

$$\tilde{A}^+ = \{x : x \in A,\ ]x, x+\delta] \cap A \neq \emptyset \text{ for every } \delta > 0\},$$

$$\tilde{A}^- = \{x : x \in A,\ [x-\delta, x[ \cap A \neq \emptyset \text{ for every } \delta > 0\}.$$

Set

$$(\overline{D}^+ f)(x) = \limsup_{y \in A, y \downarrow x} \frac{f(y)-f(x)}{y-x} = \inf_{\delta>0} \sup_{y \in A, x<y\le x+\delta} \frac{f(y)-f(x)}{y-x},$$

$$(\underline{D}^+ f)(x) = \liminf_{y \in A, y \downarrow x} \frac{f(y)-f(x)}{y-x} = \sup_{\delta>0} \inf_{y \in A, x<y\le x+\delta} \frac{f(y)-f(x)}{y-x}$$

for $x \in \tilde{A}^+$, and

$$(\overline{D}^- f)(x) = \limsup_{y \in A, y \uparrow x} \frac{f(y)-f(x)}{y-x} = \inf_{\delta>0} \sup_{y \in A, x-\delta\le y<x} \frac{f(y)-f(x)}{y-x},$$

$$(\underline{D}^- f)(x) = \liminf_{y \in A, y \uparrow x} \frac{f(y)-f(x)}{y-x} = \sup_{\delta>0} \inf_{y \in A, x-\delta\le y<x} \frac{f(y)-f(x)}{y-x}$$

for $x \in \tilde{A}^-$, all defined in $[-\infty, \infty]$. (These are the four **Dini derivates** of $f$.)

Note that we surely have $(\underline{D}^+ f)(x) \le (\overline{D}^+ f)(x)$ for every $x \in \tilde{A}^+$, while $(\underline{D}^- f)(x) \le (\overline{D}^- f)(x)$ for every $x \in \tilde{A}^-$. The ordinary derivative $f'(x)$ is defined and equal to $c \in \mathbb{R}$ iff $(\alpha)$ $x$ belongs to some open interval included in $A$ $(\beta)$ $(\overline{D}^+ f)(x) = (\underline{D}^+ f)(x) = (\overline{D}^- f)(x) = (\underline{D}^- f)(x) = c$.

***222K Lemma** For $A \subseteq \mathbb{R}$, define $\tilde{A}^+$ and $\tilde{A}^-$ as in 222J. Then $A \setminus \tilde{A}^+$ and $A \setminus \tilde{A}^-$ are countable, therefore negligible.

***222L Theorem** Let $f$ be any real function, and $A$ its domain. Then for almost every $x \in A$

*either* all four Dini derivates of $f$ at $x$ are defined, finite and equal
*or* $(\overline{D}^+ f)(x) = (\underline{D}^- f)(x)$ is finite, $(\underline{D}^+ f)(x) = -\infty$ and $(\overline{D}^- f)(x) = \infty$
*or* $(\underline{D}^+ f)(x) = (\overline{D}^- f)(x)$ is finite, $(\overline{D}^+ f)(x) = \infty$ and $(\underline{D}^- f)(x) = -\infty$
*or* $(\overline{D}^+ f)(x) = (\overline{D}^- f)(x) = \infty$ and $(\underline{D}^+ f)(x) = (\underline{D}^- f)(x) = -\infty$.

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## 223 Lebesgue's density theorems

I now turn to a group of results which may be thought of as corollaries of Theorem 222E, but which also have a vigorous life of their own, including the possibility of significant generalizations which will be treated in Chapter 26. The idea is that any measurable function $f$ on $\mathbb{R}$ is almost everywhere 'continuous' in a variety of very weak senses; for almost every $x$, the value $f(x)$ is determined by the behaviour of $f$ near $x$, in the sense that $f(y) \simeq f(x)$ for 'most' $y$ near $x$. I should perhaps say that while I recommend this work as a preparation for Chapter 26, and I also rely on it in Chapter 28, I shall not refer to it again in the present chapter, so that readers in a hurry to characterize indefinite integrals may proceed directly to §224.

**223A Lebesgue's Density Theorem: integral form** Let $I$ be an interval in $\mathbb{R}$, and let $f$ be a real-valued function which is integrable over $I$. Then

$$f(x) = \lim_{h\downarrow 0} \frac{1}{h} \int_x^{x+h} f = \lim_{h\downarrow 0} \frac{1}{h} \int_{x-h}^{x} f = \lim_{h\downarrow 0} \frac{1}{2h} \int_{x-h}^{x+h} f$$

for almost every $x \in I$.

**223B Corollary** Let $E \subseteq \mathbb{R}$ be a measurable set. Then

$$\lim_{h\downarrow 0} \frac{1}{2h}\mu(E \cap [x-h, x+h]) = 1 \text{ for almost every } x \in E,$$

$$\lim_{h\downarrow 0} \frac{1}{2h}\mu(E \cap [x-h, x+h]) = 0 \text{ for almost every } x \in \mathbb{R} \setminus E.$$

**223C Corollary** Let $f$ be a measurable real-valued function defined almost everywhere in $\mathbb{R}$. Then for almost every $x \in \mathbb{R}$,

$$\lim_{h\downarrow 0} \frac{1}{2h}\mu\{y : y \in \operatorname{dom} f,\ |y-x| \le h,\ |f(y)-f(x)| \le \epsilon\} = 1,$$

$$\lim_{h\downarrow 0} \frac{1}{2h}\mu\{y : y \in \operatorname{dom} f,\ |y-x| \le h,\ |f(y)-f(x)| \ge \epsilon\} = 0$$

for every $\epsilon > 0$.

**223D Theorem** Let $I$ be an interval in $\mathbb{R}$, and let $f$ be a real-valued function which is integrable over $I$. Then

$$\lim_{h\downarrow 0} \frac{1}{2h}\int_{x-h}^{x+h} |f(y)-f(x)|dy = 0$$

for almost every $x \in I$.

**Remark** The set

$$\{x : x \in \operatorname{dom} f,\ \lim_{h\downarrow 0} \frac{1}{2h}\int_{x-h}^{x+h} |f(y)-f(x)|dy = 0\}$$

is sometimes called the **Lebesgue set** of $f$.

**223E Complex-valued functions**

**(a)** Let $I$ be an interval in $\mathbb{R}$, and let $f$ be a complex-valued function which is integrable over $I$. Then

$$f(x) = \lim_{h\downarrow 0} \frac{1}{h}\int_{x}^{x+h} f = \lim_{h\downarrow 0} \frac{1}{h}\int_{x-h}^{x} f = \lim_{h\downarrow 0} \frac{1}{2h}\int_{x-h}^{x+h} f$$

for almost every $x \in I$.

**(b)** Let $f$ be a measurable complex-valued function defined almost everywhere in $\mathbb{R}$. Then for almost every $x \in \mathbb{R}$,

$$\lim_{h\downarrow 0} \frac{1}{2h}\mu\{y : y \in \operatorname{dom} f,\ |y-x| \le h,\ |f(y)-f(x)| \ge \epsilon\} = 0$$

for every $\epsilon > 0$.

**(c)** Let $I$ be an interval in $\mathbb{R}$, and let $f$ be a complex-valued function which is integrable over $I$. Then

$$\lim_{h\downarrow 0} \frac{1}{2h}\int_{x-h}^{x+h} |f(y)-f(x)|dy = 0$$

for almost every $x \in I$.

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## 224 Functions of bounded variation

I turn now to the second of the two problems to which this chapter is devoted: the identification of those real functions which are indefinite integrals. I take the opportunity to offer a brief introduction to the theory of functions of bounded variation, which are interesting in themselves and will be important in Chapter 28. I give the basic characterization of these functions as differences of monotonic functions (224D), with a representative sample of their elementary properties.

**224A Definition** Let $f$ be a real-valued function and $D$ a subset of $\mathbb{R}$. I define $\mathrm{Var}_D(f)$, the **(total) variation of** $f$ **on** $D$, as follows. If $D \cap \operatorname{dom} f = \emptyset$, $\mathrm{Var}_D(f) = 0$. Otherwise, $\mathrm{Var}_D(f)$ is

$$\sup\{\sum_{i=1}^{n} |f(a_i) - f(a_{i-1})| : a_0, a_1, \dots, a_n \in D \cap \operatorname{dom} f,\ a_0 \le a_1 \le \dots \le a_n\},$$

allowing $\mathrm{Var}_D(f) = \infty$. If $\mathrm{Var}_D(f)$ is finite, we say that $f$ is **of bounded variation** on $D$. I may write $\mathrm{Var}\, f$ for $\mathrm{Var}_{\operatorname{dom} f}(f)$, and say that $f$ is simply '**of bounded variation**' if this is finite.

**224B Remarks**

$$\mathrm{Var}_D(f) = \mathrm{Var}_{D \cap \operatorname{dom} f}(f) = \mathrm{Var}(f{\restriction}D)$$

for all $D$, $f$.

**224C Proposition** (a) If $f$, $g$ are two real-valued functions and $D \subseteq \mathbb{R}$, then

$$\mathrm{Var}_D(f+g) \le \mathrm{Var}_D(f) + \mathrm{Var}_D(g).$$

(b) If $f$ is a real-valued function, $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ then $\mathrm{Var}_D(cf) = |c| \mathrm{Var}_D(f)$.

(c) If $f$ is a real-valued function, $D \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ then

$$\mathrm{Var}_D(f) \ge \mathrm{Var}_{D \cap ]-\infty, x]}(f) + \mathrm{Var}_{D \cap [x, \infty[}(f),$$

with equality if $x \in D \cap \operatorname{dom} f$.

(d) If $f$ is a real-valued function and $D \subseteq D' \subseteq \mathbb{R}$ then $\mathrm{Var}_D(f) \le \mathrm{Var}_{D'}(f)$.

(e) If $f$ is a real-valued function and $D \subseteq \mathbb{R}$, then $|f(x) - f(y)| \le \mathrm{Var}_D(f)$ for all $x$, $y \in D \cap \operatorname{dom} f$; so if $f$ is of bounded variation on $D$ then $f$ is bounded on $D \cap \operatorname{dom} f$ and (if $D \cap \operatorname{dom} f \ne \emptyset$)

$$\sup_{y \in D \cap \operatorname{dom} f} |f(y)| \le |f(x)| + \mathrm{Var}_D(f)$$

for every $x \in D \cap \operatorname{dom} f$.

(f) If $f$ is a monotonic real-valued function and $D \subseteq \mathbb{R}$ meets $\operatorname{dom} f$, then

$$\mathrm{Var}_D(f) = \sup_{x \in D \cap \operatorname{dom} f} f(x) - \inf_{x \in D \cap \operatorname{dom} f} f(x).$$

**224D Theorem** For any real-valued function $f$ and any set $D \subseteq \mathbb{R}$, the following are equiveridical:

(i) there are two bounded non-decreasing functions $f_1$, $f_2 : \mathbb{R} \to \mathbb{R}$ such that $f = f_1 - f_2$ on $D \cap \operatorname{dom} f$;

(ii) $f$ is of bounded variation on $D$;

(iii) there are bounded non-decreasing functions $f_1$, $f_2 : \mathbb{R} \to \mathbb{R}$ such that $f = f_1 - f_2$ on $D \cap \operatorname{dom} f$ and $\mathrm{Var}_D(f) = \mathrm{Var}\, f_1 + \mathrm{Var}\, f_2$.

**224E Corollary** Let $f$ be a real-valued function and $D$ any subset of $\mathbb{R}$. If $f$ is of bounded variation on $D$, then

$$\lim_{x \downarrow a} \mathrm{Var}_{D \cap ]a, x]}(f) = \lim_{x \uparrow a} \mathrm{Var}_{D \cap [x, a[}(f) = 0$$

for every $a \in \mathbb{R}$, and

$$\lim_{a \to -\infty} \mathrm{Var}_{D \cap ]-\infty, a]}(f) = \lim_{a \to \infty} \mathrm{Var}_{D \cap [a, \infty[}(f) = 0.$$

**224F Corollary** Let $f$ be a real-valued function of bounded variation on $[a, b]$, where $a < b$. If $\operatorname{dom} f$ meets every interval $]a, a+\delta]$ with $\delta > 0$, then

$$\lim_{t \in \operatorname{dom} f, t \downarrow a} f(t)$$

is defined in $\mathbb{R}$. If $\operatorname{dom} f$ meets $[b-\delta, b[$ for every $\delta > 0$, then

$$\lim_{t\in\operatorname{dom} f, t\uparrow b} f(t)$$

is defined in $\mathbb{R}$.

**224G Corollary** Let $f$, $g$ be real functions and $D$ a subset of $\mathbb{R}$. If $f$ and $g$ are of bounded variation on $D$, so is $f \times g$.

**224H Proposition** Let $f : D \to \mathbb{R}$ be a function of bounded variation, where $D \subseteq \mathbb{R}$. Then $f$ is continuous at all except countably many points of $D$.

**224I Theorem** Let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ a function of bounded variation. Then $f$ is differentiable almost everywhere in $I$, and $f'$ is integrable over $I$, with

$$\int_I |f'| \le \operatorname{Var}_I(f).$$

**224J Proposition** Let $f$, $g$ be real-valued functions defined on subsets of $\mathbb{R}$, and suppose that $g$ is integrable over an interval $[a, b]$, where $a < b$, and $f$ is of bounded variation on $]a, b[$ and defined almost everywhere in $]a, b[$. Then $f \times g$ is integrable over $[a, b]$, and

$$\Bigl|\int_a^b f \times g\Bigr| \le \bigl(\lim_{x\in\operatorname{dom} f, x\uparrow b} |f(x)| + \operatorname{Var}_{]a,b[}(f)\bigr) \sup_{c\in[a,b]} \Bigl|\int_a^c g\Bigr|.$$

**224K Complex-valued functions**

**(a)** Let $D$ be a subset of $\mathbb{R}$ and $f$ a complex-valued function. The **variation** of $f$ on $D$, $\operatorname{Var}_D(f)$, is zero if $D \cap \operatorname{dom} f = \emptyset$, and otherwise is

$$\sup\{\textstyle\sum_{j=1}^n |f(a_j) - f(a_{j-1})| : a_0 \le a_1 \le \ldots \le a_n \text{ in } D \cap \operatorname{dom} f\},$$

allowing $\infty$. If $\operatorname{Var}_D(f)$ is finite, we say that $f$ is **of bounded variation** on $D$.

**(b)** A complex-valued function of bounded variation must be bounded, and

$$\operatorname{Var}_D(f + g) \le \operatorname{Var}_D(f) + \operatorname{Var}_D(g),$$

$$\operatorname{Var}_D(cf) = |c| \operatorname{Var}_D(f),$$

$$\operatorname{Var}_D(f) \ge \operatorname{Var}_{D\cap]-\infty,x]}(f) + \operatorname{Var}_{D\cap[x,\infty[}(f)$$

for every $x \in \mathbb{R}$, with equality if $x \in D \cap \operatorname{dom} f$,

$$\operatorname{Var}_D(f) \le \operatorname{Var}_{D'}(f) \text{ whenever } D \subseteq D'.$$

**(c)** A complex-valued function is of bounded variation iff its real and imaginary parts are both of bounded variation. So a complex-valued function $f$ is of bounded variation on $D$ iff there are bounded non-decreasing functions $f_1, \ldots , f_4 : \mathbb{R} \to \mathbb{R}$ such that $f = f_1 - f_2 + if_3 - if_4$ on $D$.

**(d)** Let $f$ be a complex-valued function and $D$ any subset of $\mathbb{R}$. If $f$ is of bounded variation on $D$, then

$$\lim_{x\downarrow a} \operatorname{Var}_{D\cap]a,x]}(f) = \lim_{x\uparrow a} \operatorname{Var}_{D\cap[x,a[}(f) = 0$$

for every $a \in \mathbb{R}$, and

$$\lim_{a\to-\infty} \operatorname{Var}_{D\cap]-\infty,a]}(f) = \lim_{a\to\infty} \operatorname{Var}_{D\cap[a,\infty[}(f) = 0.$$

**(e)** Let $f$ be a complex-valued function of bounded variation on $[a, b]$, where $a < b$. If $\operatorname{dom} f$ meets every interval $]a, a + \delta]$ with $\delta > 0$, then $\lim_{t\in\operatorname{dom} f,t\downarrow a} f(t)$ is defined in $\mathbb{C}$. If $\operatorname{dom} f$ meets $[b - \delta, b[$ for every $\delta > 0$, then $\lim_{t\in\operatorname{dom} f,t\uparrow b} f(t)$ is defined in $\mathbb{C}$.

**(f)** Let $f$, $g$ be complex functions and $D$ a subset of $\mathbb{R}$. If $f$ and $g$ are of bounded variation on $D$, so is $f \times g$.

**(g)** Let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{C}$ a function of bounded variation. Then $f$ is differentiable almost everywhere in $I$, and $\int_I |f'| \le \mathrm{Var}_I(f)$.

**(h)** Let $f$ and $g$ be complex-valued functions defined on subsets of $\mathbb{R}$, and suppose that $g$ is integrable over an interval $[a, b]$, where $a < b$, and $f$ is of bounded variation on $]a, b[$ and defined almost everywhere in $]a, b[$. Then $f \times g$ is integrable over $[a, b]$, and

$$|\int_a^b f \times g| \le \big(\lim_{x \in \mathrm{dom}\, f, x \uparrow b} |f(x)| + \mathrm{Var}_{]a,b[}(f)\big) \sup_{c \in [a,b]} |\int_a^c g|.$$

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## 225 Absolutely continuous functions

We are now ready for a full characterization of the functions that can appear as indefinite integrals (225E, 225Xf). The essential idea is that of 'absolute continuity' (225B). In the second half of the section (225G-225N) I describe some of the relationships between this concept and those we have already seen.

**225A Absolute continuity of the indefinite integral: Theorem** Let $(X, \Sigma, \mu)$ be any measure space and $f$ an integrable real-valued function defined on a conegligible subset of $X$. Then for any $\epsilon > 0$ there are a measurable set $E$ of finite measure and a real number $\delta > 0$ such that $\int_F |f| \le \epsilon$ whenever $F \in \Sigma$ and $\mu(F \cap E) \le \delta$.

**225B Absolutely continuous functions on $\mathbb{R}$: Definition** If $[a, b]$ is a non-empty closed interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is a function, we say that $f$ is **absolutely continuous** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\sum_{i=1}^n |f(b_i) - f(a_i)| \le \epsilon$ whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^n b_i - a_i \le \delta$.

**225C Proposition** Let $[a, b]$ be a non-empty closed interval in $\mathbb{R}$.

(a) If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, it is uniformly continuous.

(b) If $f : [a, b] \to \mathbb{R}$ is absolutely continuous it is of bounded variation on $[a, b]$, so is differentiable almost everywhere in $[a, b]$, and its derivative is integrable over $[a, b]$.

(c) If $f$, $g : [a, b] \to \mathbb{R}$ are absolutely continuous, so are $f + g$ and $cf$, for every $c \in \mathbb{R}$.

(d) If $f$, $g : [a, b] \to \mathbb{R}$ are absolutely continuous so is $f \times g$.

(e) If $g : [a, b] \to [c, d]$ and $f : [c, d] \to \mathbb{R}$ are absolutely continuous, and $g$ is non-decreasing, then the composition $fg : [a, b] \to \mathbb{R}$ is absolutely continuous.

**225D Lemma** Let $[a, b]$ be a non-empty closed interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ an absolutely continuous function which has zero derivative almost everywhere in $[a, b]$. Then $f$ is constant on $[a, b]$.

**225E Theorem** Let $[a, b]$ be a non-empty closed interval in $\mathbb{R}$ and $F : [a, b] \to \mathbb{R}$ a function. Then the following are equiveridical:

(i) there is an integrable real-valued function $f$ such that $F(x) = F(a) + \int_a^x f$ for every $x \in [a, b]$;

(ii) $\int_a^x F'$ exists and is equal to $F(x) - F(a)$ for every $x \in [a, b]$;

(iii) $F$ is absolutely continuous.

**225F Integration by parts: Theorem** Let $f$ be a real-valued function which is integrable over an interval $[a, b] \subseteq \mathbb{R}$, and $g : [a, b] \to \mathbb{R}$ an absolutely continuous function. Suppose that $F$ is an indefinite integral of $f$, so that $F(x) - F(a) = \int_a^x f$ for $x \in [a, b]$. Then

$$\int_a^b f \times g = F(b)g(b) - F(a)g(a) - \int_a^b F \times g'.$$

**225G Proposition** Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ an absolutely continuous function.

(a) $f[A]$ is negligible for every negligible set $A\subseteq\mathbb{R}$.

(b) $f[E]$ is measurable for every measurable set $E\subseteq\mathbb{R}$.

**225H Semi-continuous functions** If $D\subseteq\mathbb{R}^r$, a function $g:D\to[-\infty,\infty]$ is **lower semi-continuous** if $\{x:g(x)>u\}$ is an open subset of $D$ for every $u\in[-\infty,\infty]$. Any lower semi-continuous function is Borel measurable, therefore Lebesgue measurable.

**225I Proposition** Suppose that $r\ge 1$ and that $f$ is a real-valued function, defined on a subset $D$ of $\mathbb{R}^r$, which is integrable over $D$. Then for any $\epsilon>0$ there is a lower semi-continuous function $g:\mathbb{R}^r\to[-\infty,\infty]$ such that $g(x)\ge f(x)$ for every $x\in D$ and $\int_D g$ is defined and not greater than $\epsilon+\int_D f$.

**225J Theorem** Let $D$ be a subset of $\mathbb{R}$ and $f:D\to\mathbb{R}$ any function. Then

$$E=\{x:x\in D,\ f\text{ is continuous at }x\}$$

is relatively Borel measurable in $D$, and

$$F=\{x:x\in D,\ f\text{ is differentiable at }x\}$$

is Borel measurable in $\mathbb{R}$; moreover, $f':F\to\mathbb{R}$ is Borel measurable.

**225K Proposition** Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$, and $f:[a,b]\to\mathbb{R}$ a function. Set $F=\{x:x\in\,]a,b[\,,\ f'(x)\text{ is defined}\}$. Then $f$ is absolutely continuous iff (i) $f$ is continuous (ii) $f'$ is integrable over $F$ (iii) $f[\,[a,b]\setminus F]$ is negligible.

**225L Corollary** Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$. Let $f:[a,b]\to\mathbb{R}$ be a continuous function which is differentiable on the open interval $]a,b[$. If its derivative $f'$ is integrable over $[a,b]$, then $f$ is absolutely continuous, and $f(b)-f(a)=\int_a^b f'$.

**225M Corollary** Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$, and $f:[a,b]\to\mathbb{R}$ a continuous function. Then $f$ is absolutely continuous iff it is continuous and of bounded variation and $f[A]$ is negligible for every negligible $A\subseteq[a,b]$.

**225N The Cantor function** Let $C\subseteq[0,1]$ be the Cantor set. Recall that the 'Cantor function' is a non-decreasing continuous function $f:[0,1]\to[0,1]$ such that $f'(x)$ is defined and equal to zero for every $x\in[0,1]\setminus C$, but $f(0)=0<1=f(1)$. $f$ is of bounded variation and not absolutely continuous. $C$ is negligible and $f[C]=[0,1]$ is not. If $x\in C$, then for every $n\in\mathbb{N}$ there is an interval of length $3^{-n}$, containing $x$, on which $f$ increases by $2^{-n}$; so $f$ cannot be differentiable at $x$, and the set $F=\operatorname{dom}f'$ of 225K is precisely $[0,1]\setminus C$, so that $f[\,[0,1]\setminus F]=[0,1]$.

**225O Complex-valued functions (a)** Let $(X,\Sigma,\mu)$ be any measure space and $f$ an integrable complex-valued function defined on a conegligible subset of $X$. Then for any $\epsilon>0$ there are a measurable set $E$ of finite measure and a real number $\delta>0$ such that $\int_F|f|\le\epsilon$ whenever $F\in\Sigma$ and $\mu(F\cap E)\le\delta$.

**(b)** If $[a,b]$ is a non-empty closed interval in $\mathbb{R}$ and $f:[a,b]\to\mathbb{C}$ is a function, we say that $f$ is **absolutely continuous** if for every $\epsilon>0$ there is a $\delta>0$ such that $\sum_{i=1}^n|f(b_i)-f(a_i)|\le\epsilon$ whenever $a\le a_1\le b_1\le a_2\le b_2\le\ldots\le a_n\le b_n\le b$ and $\sum_{i=1}^n b_i-a_i\le\delta$. Observe that $f$ is absolutely continuous iff its real and imaginary parts are both absolutely continuous.

**(c)** Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$.

(i) If $f:[a,b]\to\mathbb{C}$ is absolutely continuous it is of bounded variation on $[a,b]$, so is differentiable almost everywhere in $[a,b]$, and its derivative is integrable over $[a,b]$.

(ii) If $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous, so are $f+g$ and $\zeta f$, for any $\zeta\in\mathbb{C}$, and $f\times g$.

(iii) If $g:[a,b]\to[c,d]$ is monotonic and absolutely continuous, and $f:[c,d]\to\mathbb{C}$ is absolutely continuous, then $fg:[a,b]\to\mathbb{C}$ is absolutely continuous.

**(d)** Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$ and $F:[a,b]\to\mathbb{C}$ a function. Then the following are equiveridical:

(i) there is an integrable complex-valued function $f$ such that $F(x)=F(a)+\int_a^x f$ for every $x\in[a,b]$;

(ii) $\int_a^x F'$ exists and is equal to $F(x)-F(a)$ for every $x\in[a,b]$;

(iii) $F$ is absolutely continuous.

**(e)** Let $f$ be an integrable complex-valued function on an interval $[a,b]\subseteq\mathbb{R}$, and $g:[a,b]\to\mathbb{C}$ an absolutely continuous function. Set $F(x)=\int_a^x f$ for $x\in[a,b]$. Then

$$\int_a^b f\times g=F(b)g(b)-F(a)g(a)-\int_a^b F\times g'.$$

**(f)** Let $f$ be a continuous complex-valued function on a closed interval $[a,b]\subseteq\mathbb{R}$, and suppose that $f$ is differentiable at every point of the open interval $]a,b[$, with $f'$ integrable over $[a,b]$. Then $f$ is absolutely continuous.

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## 226 The Lebesgue decomposition of a function of bounded variation

I end this chapter with some notes on a method of analysing a general function of bounded variation which may help to give a picture of what such functions can be, though (apart from 226A) it is hardly needed in this volume.

**226A Sums over arbitrary index sets (a)** If $I$ is any set and $\langle a_i\rangle_{i\in I}$ any family in $[0,\infty]$, we set

$$\sum_{i\in I}a_i=\sup\{\sum_{i\in K}a_i:K\text{ is a finite subset of }I\},$$

with the convention that $\sum_{i\in\emptyset}a_i=0$. For general $a_i\in[-\infty,\infty]$, we can set

$$\sum_{i\in I}a_i=\sum_{i\in I}a_i^+-\sum_{i\in I}a_i^-$$

if this is defined in $[-\infty,\infty]$, where $a^+=\max(a,0)$ and $a^-=\max(-a,0)$ for each $a$. If $\sum_{i\in I}a_i$ is defined and finite, we say that $\langle a_i\rangle_{i\in I}$ is **summable**.

**(b)** For any set $I$, we have the corresponding 'counting measure' $\mu$ on $I$. Every family $\langle a_i\rangle_{i\in I}$ of real numbers is a measurable real-valued function on $I$. A real-valued function $f$ on $I$ is 'simple' if $K=\{i:f(i)\ne0\}$ is finite. Now a general function $f:I\to\mathbb{R}$ is integrable iff $\sum_{i\in I}|f(i)|<\infty$, and in this case

$$\int f d\mu=\sum_{i\in I}f(i),$$

Thus we have

$$\sum_{i\in I}a_i=\int_I a_i\mu(di),$$

and the standard rules under which we allow $\infty$ as the value of an integral match the interpretations in (a) above.

**(c)** I observe here that this notion of summability is 'absolute'; a family $\langle a_i\rangle_{i\in I}$ is summable iff it is absolutely summable.

**(d)** If $\langle a_i\rangle_{i\in I}$ is an (absolutely) summable family of real numbers, then for every $\epsilon>0$ there is a finite $K\subseteq I$ such that $\sum_{i\in I\setminus K}|a_i|\le\epsilon$. Consequently, for any family $\langle a_i\rangle_{i\in I}$ of real numbers and any $s\in\mathbb{R}$, the following are equiveridical:

(i) $\sum_{i\in I}a_i=s$;

(ii) for every $\epsilon>0$ there is a finite $K\subseteq I$ such that $|s-\sum_{i\in J}a_i|\le\epsilon$ whenever $J$ is finite and $K\subseteq J\subseteq I$.

**(e)** If $\sum_{i\in I}|a_i|<\infty$, then

$$J=\{i:a_i\ne 0\}=\bigcup_{n\in\mathbb{N}}\{i:|a_i|\ge 2^{-n}\}$$

is countable. If $J$ is finite, then $\sum_{i\in I}a_i=\sum_{i\in J}a_i$ reduces to a finite sum. Otherwise, we can enumerate $J$ as $\langle j_n\rangle_{n\in\mathbb{N}}$, and we shall have

$$\sum_{i\in I}a_i=\sum_{i\in J}a_i=\lim_{n\to\infty}\sum_{k=0}^{n}a_{j_k}=\sum_{n=0}^{\infty}a_{j_n}\ .$$

Conversely, if $\langle a_i\rangle_{i\in I}$ is such that there is a countably infinite $J\subseteq\{i:a_i\ne 0\}$ enumerated as $\langle j_n\rangle_{n\in\mathbb{N}}$, and if $\sum_{n=0}^{\infty}|a_{j_n}|<\infty$, then $\sum_{i\in I}a_i$ will be $\sum_{n=0}^{\infty}a_{j_n}$.

**(f)** Let $I$ and $J$ be sets and $\langle a_{ij}\rangle_{i\in I,j\in J}$ a family in $[0,\infty]$. Then

$$\sum_{(i,j)\in I\times J}a_{ij}=\sum_{i\in I}(\sum_{j\in J}a_{ij})=\sum_{j\in J}(\sum_{i\in I}a_{ij}).$$

**226B Saltus functions** Suppose that $a<b$ in $\mathbb{R}$.

**(a)** A (real) **saltus function** on $[a,b]$ is a function $F:[a,b]\to\mathbb{R}$ expressible in the form

$$F(x)=\sum_{t\in[a,x[}u_t+\sum_{t\in[a,x]}v_t$$

for $x\in[a,b]$, where $\langle u_t\rangle_{t\in[a,b[}$, $\langle v_t\rangle_{t\in[a,b]}$ are real-valued families such that $\sum_{t\in[a,b[}|u_t|$ and $\sum_{t\in[a,b]}|v_t|$ are finite.

**(b)** For any function $F:[a,b]\to\mathbb{R}$ we can write

$$F(x^+)=\lim_{y\downarrow x}F(y) \text{ if } x\in[a,b[ \text{ and the limit exists,}$$

$$F(x^-)=\lim_{y\uparrow x}F(y) \text{ if } x\in]a,b] \text{ and the limit exists.}$$

Observe that if $F$ is a saltus function, as defined in (b), with associated families $\langle u_t\rangle_{t\in[a,b[}$ and $\langle v_t\rangle_{t\in[a,b]}$, then $v_a=F(a)$, $v_x=F(x)-F(x^-)$ for $x\in]a,b]$ and $u_x=F(x^+)-F(x)$ for $x\in[a,b[$.

$F$ is continuous at $x\in]a,b[$ iff $u_x=v_x=0$, while $F$ is continuous at $a$ iff $u_a=0$ and $F$ is continuous at $b$ iff $v_b=0$. In particular, $\{x:x\in[a,b],\ F$ is not continuous at $x\}$ is countable.

**(c)** If $F$ is a saltus function defined on $[a,b]$, with associated families $\langle u_t\rangle_{t\in[a,b[}$ and $\langle v_t\rangle_{t\in[a,b]}$, then $F$ is of bounded variation on $[a,b]$, and

$$\mathrm{Var}_{[a,b]}(F)\le\sum_{t\in[a,b[}|u_t|+\sum_{t\in]a,b]}|v_t|.$$

**(d)** The inequality in (c) is actually an equality.

**(e)** Because a saltus function is of bounded variation, it is differentiable almost everywhere. In fact its derivative is zero almost everywhere.

**226C The Lebesgue decomposition of a function of bounded variation** Take $a$, $b\in\mathbb{R}$ with $a<b$.

**(a)** If $F:[a,b]\to\mathbb{R}$ is non-decreasing, set $v_a=0$, $v_t=F(t)-F(t^-)$ for $t\in]a,b]$, $u_t=F(t^+)-F(t)$ for $t\in[a,b[$. Then all the $v_t$, $u_t$ are non-negative, and $\sum_{t\in[a,b[}u_t$ and $\sum_{t\in[a,b]}v_t$ are both finite. Let $F_p$ be the corresponding saltus function. $F_p$ and $F_c=F-F_p$ are non-decreasing. $F_c$ is continuous.

Clearly this expression of $F=F_p+F_c$ as the sum of a saltus function and a continuous function is unique, except that we can freely add a constant to one if we subtract it from the other.

**(b)** Set $F_{ac}(x)=F(a)+\int_a^x F'$ for each $x\in[a,b]$. $F_{cs}=F_c-F_{ac}$ is still non-decreasing; $F_{cs}$ is continuous; $F'_{cs}=0$ a.e.

Again, the expression of $F_c=F_{ac}+F_{cs}$ as the sum of an absolutely continuous function and a function with zero derivative almost everywhere is unique, except for the possibility of moving a constant from one to the other.

**(c)** Putting these together: if $F : [a,b] \to \mathbb{R}$ is any non-decreasing function, it is expressible as $F_p + F_{ac} + F_{cs}$, where $F_p$ is a saltus function, $F_{ac}$ is absolutely continuous, and $F_{cs}$ is continuous and differentiable, with zero derivative, almost everywhere; all three components are non-decreasing; and the expression is unique if we say that $F_{ac}(a) = F(a)$ and $F_p(a) = F_{cs}(a) = 0$.

The Cantor function $f : [0,1] \to [0,1]$ is continuous and $f' = 0$ a.e., so $f_p = f_{ac} = 0$ and $f = f_{cs}$. Setting $g(x) = \frac{1}{2}(x + f(x))$ for $x \in [0,1]$, we get $g_p(x) = 0$, $g_{ac}(x) = \frac{x}{2}$ and $g_{cs}(x) = \frac{1}{2}f(x)$.

**(d)** Now suppose that $F : [a,b] \to \mathbb{R}$ is of bounded variation. Then it is expressible as a difference $G - H$ of non-decreasing functions. So writing $F_p = G_p - H_p$, etc., we can express $F$ as a sum $F_p + F_{cs} + F_{ac}$, where $F_p$ is a saltus function, $F_{ac}$ is absolutely continuous, $F_{cs}$ is continuous, $F'_{cs} = 0$ a.e., $F_{ac}(a) = F(a)$ and $F_{cs}(a) = F_p(a) = 0$. Under these conditions the expression is unique.

This is a **Lebesgue decomposition** of the function $F$. I will call $F_p$ the **saltus part** of $F$.

**226D Complex functions (a)** If $I$ is any set and $\langle a_j \rangle_{j \in I}$ is a family of complex numbers, then the following are equiveridical:

(i) $\sum_{j \in I} |a_j| < \infty$;

(ii) there is an $s \in \mathbb{C}$ such that for every $\epsilon > 0$ there is a finite $K \subseteq I$ such that $|s - \sum_{j \in J} a_j| \le \epsilon$ whenever $J$ is finite and $K \subseteq J \subseteq I$.

In this case

$$s = \sum_{j \in I} \mathcal{R}\mathrm{e}(a_j) + i \sum_{j \in I} \mathcal{I}\mathrm{m}(a_j) = \int_I a_j \mu(dj),$$

where $\mu$ is counting measure on $I$, and we write $s = \sum_{j \in I} a_j$.

**(b)** If $a < b$ in $\mathbb{R}$, a complex **saltus function** on $[a,b]$ is a function $F : [a,b] \to \mathbb{C}$ expressible in the form

$$F(x) = \sum_{t \in [a,x[} u_t + \sum_{t \in [a,x]} v_t$$

for $x \in [a,b]$, where $\langle u_t \rangle_{t \in [a,b[}$, $\langle v_t \rangle_{t \in [a,b]}$ are complex-valued families such that $\sum_{t \in [a,b[} |u_t|$ and $\sum_{t \in [a,b]} |v_t|$ are finite. In this case $F$ is continuous except at countably many points and differentiable, with zero derivative, almost everywhere in $[a,b]$, and

$$u_x = \lim_{t \downarrow x} F(t) - F(x) \text{ for every } x \in [a,b[,$$

$$v_x = \lim_{t \uparrow x} F(x) - F(t) \text{ for every } x \in ]a,b].$$

$F$ is of bounded variation, and its variation is

$$\mathrm{Var}_{[a,b]}(F) = \sum_{t \in [a,b[} |u_t| + \sum_{t \in ]a,b]} |v_t|.$$

**(c)** If $F : [a,b] \to \mathbb{C}$ is a function of bounded variation, where $a < b$ in $\mathbb{R}$, it is uniquely expressible as $F = F_p + F_{cs} + F_{ac}$, where $F_p$ is a saltus function, $F_{ac}$ is absolutely continuous, $F_{cs}$ is continuous and has zero derivative almost everywhere, and $F_{ac}(a) = F(a)$, $F_p(a) = F_{cs}(a) = 0$.

**226E Proposition** Let $(X, \Sigma, \mu)$ be a measure space, $I$ a *countable* set, and $\langle f_i \rangle_{i \in I}$ a family of $\mu$-integrable real- or complex-valued functions such that $\sum_{i \in I} \int |f_i| d\mu$ is finite. Then $f(x) = \sum_{i \in I} f_i(x)$ is defined almost everywhere and $\int f d\mu = \sum_{i \in I} \int f_i d\mu$.