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Chapter 22

The Fundamental Theorem of Calculus

In this chapter I address one of the most important properties of the Lebesgue integral. Given an integrable function $f : [a,b] \to \mathbb{R}$, we can form its indefinite integral $F(x) = \int_a^x f(t)dt$ for $x \in [a,b]$. Two questions immediately present themselves. (i) Can we expect to have the derivative F' of F equal to f? (ii) Can we identify which functions F will appear as indefinite integrals? Reasonably satisfactory answers may be found for both of these questions: $F' = f$ almost everywhere (222E) and indefinite integrals are the absolutely continuous functions (225E). In the course of dealing with them, we need to develop a variety of techniques which lead to many striking results both in the theory of Lebesgue measure and in other, apparently unrelated, topics in real analysis.

The first step is 'Vitali's theorem' (§221), a remarkable argument – it is more a method than a theorem – which uses the geometric nature of the real line to extract disjoint subfamilies from collections of intervals. It is the foundation stone not only of the results in §222 but of all geometric measure theory, that is, measure theory on spaces with a geometric structure. I use it here to show that monotonic functions are differentiable almost everywhere (222A). Following this, Fatou's Lemma and Lebesgue's Dominated Convergence Theorem are enough to show that the derivative of an indefinite integral is almost everywhere equal to the integrand. We find that some innocent-looking manipulations of this fact take us surprisingly far; I present these in §223.

I begin the second half of the chapter with a discussion of functions 'of bounded variation', that is, expressible as the difference of bounded monotonic functions (§224). This is one of the least measure-theoretic sections in the volume; only in 224I and 224J are measure and integration even mentioned. But this material is needed for Chapter 28 as well as for the next section, and is also one of the basic topics of twentieth-century real analysis. §225 deals with the characterization of indefinite integrals as the 'absolutely continuous' functions. In fact this is now quite easy; it helps to call on Vitali's theorem again, but everything else is a straightforward application of methods previously used. The second half of the section introduces some new ideas in an attempt to give a deeper intuition into the essential nature of absolutely continuous functions. §226 returns to functions of bounded variation and their decomposition into 'saltus' and 'absolutely continuous' and 'singular' parts, the first two being relatively manageable and the last looking something like the Cantor function.

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221 Vitali's theorem in $\mathbb{R}$

I give the first theorem of this chapter a section to itself. It occupies a position between measure theory and geometry (it is, indeed, one of the fundamental results of 'geometric measure theory'), and its proof involves both the measure and the geometry of the real line.

221A Vitali's theorem Let A be a bounded subset of $\mathbb{R}$ and $\mathcal{I}$ a family of non-singleton closed intervals in $\mathbb{R}$ such that every point of A belongs to arbitrarily short members of $\mathcal{I}$. Then there is a countable set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that (i) $\mathcal{I}_0$ is disjoint, that is, $I \cap I' = \emptyset$ for all distinct I, $I' \in \mathcal{I}_0$ (ii) $\mu(A \setminus \bigcup \mathcal{I}_0) = 0$, where μ is Lebesgue measure on $\mathbb{R}$.

proof (a) If there is a finite disjoint set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $A \subseteq \bigcup \mathcal{I}_0$ (including the possibility that $A = \mathcal{I}_0 = \emptyset$), we can stop. So let us suppose henceforth that there is no such $\mathcal{I}_0$.

Let μ^* be Lebesgue outer measure on $\mathbb{R}$. Suppose that $|x| < M$ for every $x \in A$, and set

$$\mathcal{I}' = \{I : I \in \mathcal{I},\, I \subseteq [-M, M]\}.$$

(b) In this case, if $\mathcal{I}_0$ is any finite disjoint subset of $\mathcal{I}'$, there is a $J \in \mathcal{I}'$ which is disjoint from any member of $\mathcal{I}_0$. **P** Take $x \in A \setminus \bigcup \mathcal{I}_0$. Now there is a $\delta > 0$ such that $[x-\delta, x+\delta]$ does not meet any member of $\mathcal{I}_0$, and as $|x| < M$ we can suppose that $[x-\delta, x+\delta] \subseteq [-M, M]$. Let J be a member of $\mathcal{I}$, containing x, and of length at most δ; then $J \in \mathcal{I}'$ and $J \cap \bigcup \mathcal{I}_0 = \emptyset$. **Q**

(c) We can now choose a sequence $\langle \gamma_n \rangle_{n\in\mathbb{N}}$ of real numbers and a disjoint sequence $\langle I_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{I}'$ inductively, as follows. Given $\langle I_j \rangle_{j<n}$ (if $n = 0$, this is the empty sequence, with no members), with $I_j \in \mathcal{I}'$ for each $j < n$, and $I_j \cap I_k = \emptyset$ for $j < k < n$, set

$$\mathcal{J}_n = \{I : I \in \mathcal{I}', \, I \cap I_j = \emptyset \text{ for every } j < n\}.$$

We know from (b) that $\mathcal{J}_n \neq \emptyset$. Set

$$\gamma_n = \sup\{\mu I : I \in \mathcal{J}_n\};$$

then $0 < \gamma_n \leq 2M$. We may therefore choose a set $I_n \in \mathcal{J}_n$ such that $\mu I_n \geq \frac{1}{2}\gamma_n$, and this continues the induction.

(e) Because the I_n are disjoint Lebesgue measurable subsets of $[-M, M]$, we have

$$\sum_{n=0}^{\infty} \gamma_n \leq 2\sum_{n=0}^{\infty} \mu I_n \leq 4M < \infty,$$

and $\lim_{n\to\infty} \gamma_n = 0$. Now define I'_n to be the closed interval with the same midpoint as I_n but five times the length, so that it projects past each end of I_n by at least γ_n. I claim that, for any n,

$$A \subseteq \bigcup_{j<n} I_j \cup \bigcup_{j\geq n} I'_j.$$

P? Suppose, if possible, otherwise. Take any x belonging to $A \setminus (\bigcup_{j<n} I_j \cup \bigcup_{j\geq n} I'_j)$. Let $\delta > 0$ be such that

$$[x-\delta, x+\delta] \subseteq [-M, M] \setminus \bigcup_{j<n} I_j,$$

and let $J \in \mathcal{I}$ be such that

$$x \in J \subseteq [x-\delta, x+\delta].$$

Then

$$\mu J > 0 = \lim_{m\to\infty} \gamma_m;$$

let m be the least integer greater than or equal to n such that $\gamma_m < \mu J$. In this case J cannot belong to $\mathcal{J}_m$, so there must be some $k < m$ such that $J \cap I_k \neq \emptyset$, because certainly $J \in \mathcal{I}'$. By the choice of δ, k cannot be less than n, so $n \leq k < m$, and $\gamma_k \geq \mu J$. In this case, the distance from x to the nearest endpoint of I_k is at most $\mu J \leq \gamma_k$. But the ends of I'_k project beyond the ends of I_k by at least γ_k, so $x \in I'_k$; which contradicts the choice of x. **XQ**

(f) It follows that

$$\mu^*(A \setminus \bigcup_{j<n} I_j) \leq \mu(\bigcup_{j\geq n} I'_j) \leq \sum_{j=n}^{\infty} \mu I'_j \leq 5\sum_{j=n}^{\infty} \mu I_j.$$

As

$$\sum_{j=0}^{\infty} \mu I_j \leq 2M < \infty,$$

we must have

$$\lim_{n\to\infty} \mu^*(A \setminus \bigcup_{j<n} I_j) = 0,$$

and

$$\mu(A \setminus \bigcup_{j\in\mathbb{N}} I_j) = \mu^*(A \setminus \bigcup_{j\in\mathbb{N}} I_j) \leq \inf_{n\in\mathbb{N}} \mu^*(A \setminus \bigcup_{j<n} I_j) = 0.$$

Thus in this case we may set $\mathcal{I}_0 = \{I_n : n \in \mathbb{N}\}$ to obtain a countable disjoint family in $\mathcal{I}$ with $\mu(A \setminus \bigcup \mathcal{I}_0) = 0$.

221B Remarks (a) I have expressed this theorem in the form 'there is a countable set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that ...' in an attempt to find a concise way of expressing the three possibilities

(i) $A = \mathcal{I} = \emptyset$, so that we must take $\mathcal{I}_0 = \emptyset$;

(ii) there are disjoint $I_0, \ldots, I_n \in \mathcal{I}$ such that $A \subseteq I_0 \cup \ldots \cup I_n$, so that we can take $\mathcal{I}_0 = \{I_0, \ldots, I_n\}$;

(iii) there is a disjoint sequence $\langle I_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $\mu(A \setminus \bigcup_{n\in\mathbb{N}} I_n) = 0$, so that we can take $\mathcal{I}_0 = \{I_n : n \in \mathbb{N}\}$.

Of course many applications, like the proof of 221A itself, will use forms of these three alternatives.

(b) The actual theorem here, as stated, will be used in the next section. But quite as important as the statement of the theorem is the principle of its proof. The I_n are chosen 'greedily', that is, when we come to choose I_n we look at the family $\mathcal{J}_n$ of possible intervals, given the choices $I_0, \ldots, I_{n-1}$ already made, and choose an $I_n \in \mathcal{J}_n$ which is 'about' as big as it could be. The supremum of the possibilities for μI_n is γ_n; but since we do not know that there is any $I \in \mathcal{J}_n$ such that $\mu I = \gamma_n$, we must settle for a little less. I follow the standard formula in taking $\mu I_n \geq \frac{1}{2}\gamma_n$, but of course I could have taken $\mu I_n \geq \frac{99}{100}\gamma_n$, or $\mu I_n \geq (1 - 2^{-n})\gamma_n$, if that had helped later on. The remarkable thing is that this works; we can choose the I_n without foresight and without considering their interrelationships (for that matter, without examining the set A) beyond the minimal requirement that $I_n \cap I_j = \emptyset$ for $j < n$, and even this arbitrary and casual procedure yields a suitable sequence.

(c) I have stated the theorem in terms of bounded sets A and closed intervals, which is adequate for our needs, but very small changes in the proof suffice to deal with arbitrary (non-singleton) intervals, and another refinement handles unbounded sets A. (See 221Ya.)

221X Basic exercises (a) Let $\alpha \in \,]0, 1[$. Suppose, in part (c) of the proof of 221A, we take $\mu I_n \geq \alpha\gamma_n$ for each $n \in \mathbb{N}$, rather than $\mu I_n \geq \frac{1}{2}\gamma_n$. What will be the appropriate constant to take in place of 5 in defining the sets I'_j of part (e)?

221Y Further exercises (a) Let A be a subset of $\mathbb{R}$ and $\mathcal{I}$ a family of non-singleton intervals in $\mathbb{R}$ such that every point of A belongs to arbitrarily short members of $\mathcal{I}$. Show that there is a countable disjoint set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $A \setminus \bigcup \mathcal{I}_0$ is Lebesgue negligible. (*Hint*: apply 221A to the sets $A \cap \,]n, n+1[$, $\{\overline{I} : I \in \mathcal{I}, \overline{I} \subseteq \,]n, n+1[\}$, writing $\overline{I}$ for the closed interval with the same endpoints as I.)

(b) Let $\mathcal{J}$ be any family of non-singleton intervals in $\mathbb{R}$. Show that $\bigcup \mathcal{J}$ is Lebesgue measurable. (*Hint*: apply (a) to $A = \bigcup \mathcal{J}$ and the family $\mathcal{I}$ of non-singleton subintervals of members of $\mathcal{J}$.)

(c) Let (X, ρ) be a metric space, A a subset of X, and $\mathcal{I}$ a family of closed balls of non-zero radius in X such that every point of A belongs to arbitrarily small members of $\mathcal{I}$. (I say here that a set is a 'closed ball of non-zero radius' if it is expressible in the form $B(x, \delta) = \{y : \rho(y, x) \leq \delta\}$ where $x \in X$ and $\delta > 0$. Of course it is possible for such a ball to be a singleton $\{x\}$.) Show that either A can be covered by a finite disjoint family in $\mathcal{I}$ or there is a disjoint sequence $\langle B(x_n, \delta_n)\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that

$$A \subseteq \bigcup_{m\leq n} B(x_m, \delta_m) \cup \bigcup_{m>n} B(x_m, 5\delta_m) \text{ for every } n \in \mathbb{N}$$

or there is a disjoint sequence $\langle B(x_n, \delta_n)\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}$ such that $\inf_{n\in\mathbb{N}} \delta_n > 0$.

(d) Give an example of a family $\mathcal{I}$ of open intervals such that every point of $\mathbb{R}$ belongs to arbitrarily small members of $\mathcal{I}$, but if $\langle I_n\rangle_{n\in\mathbb{N}}$ is any disjoint sequence in $\mathcal{I}$, and for each $n \in \mathbb{N}$ we write I'_n for the closed interval with the same centre as I_n and ten times the length, then there is an n such that $]0, 1[\not\subseteq \bigcup_{m<n} I_m \cup \bigcup_{m\geq n} I'_m$.

(e)(i) Show that if $\mathcal{I}$ is a *finite* family of intervals in $\mathbb{R}$ there are $\mathcal{I}_0$, $\mathcal{I}_1 \subseteq \mathcal{I}$ such that $\bigcup(\mathcal{I}_0 \cup \mathcal{I}_1) = \bigcup \mathcal{I}$ and both $\mathcal{I}_0$ and $\mathcal{I}_1$ are disjoint families. (*Hint*: induce on $\#(\mathcal{I})$.) (ii) Suppose that $\mathcal{I}$ is a family of non-singleton intervals, of length at most 1, covering a bounded set $A \subseteq \mathbb{R}$, and that $\epsilon > 0$. Show that there is a disjoint subfamily $\mathcal{I}_0$ of $\mathcal{I}$ such that $\mu^*(A \setminus \bigcup \mathcal{I}_0) \leq \frac{1}{2}\mu^* A + \epsilon$. (*Hint*: replacing each member of $\mathcal{I}$ by a slightly longer one with rational endpoints, reduce to the case in which $\mathcal{I}$ is countable and thence to the case in which $\mathcal{I}$ is finite; now use (i).) (iii) Use (ii) to prove Vitali's theorem. (I learnt this argument from J.Aldaz.)

221 Notes and comments I have headed this section 'Vitali's theorem in $\mathbb{R}$' because there is an r-dimensional version, which will appear in Chapter 26 below. There is an anomaly in the position of this theorem. It is an indispensable element of the proofs of some of the most important theorems in measure theory; on the other hand, the ideas involved in its own proof are not used elsewhere in the elementary theory. I have therefore myself sometimes omitted the proof when teaching this material, and would not reproach any student who left it to one side for the moment. At some stage, of course, any measure theorist must master the method, not just for the sake of completeness, but in order to gain an intuition for possible variations. I must emphasize that it is the *principle* of the proof, rather than its details, which is important, because there are innumerable forms of 'Vitali's theorem'. (I offer some variations in the exercises here and in §261 below, and there are many others which are important in more advanced work; one will appear in §472 in Volume 4.) This principle is, I suppose, that

(i) we choose the I_n greedily, according to some more or less natural criterion applicable to each I_n as we come to choose it, without attempting to look ahead;

(ii) we prove that their sizes tend to zero, even though we seemed to do nothing to ensure that they would (but note the shift from $\mathcal{I}$ to $\mathcal{I}'$ in part (a) of the proof of 221A, which is exactly what is needed to make this step work);

(iii) we check that for a suitable definition of I'_n, enlarging I_n, we shall have $A \subseteq \bigcup_{m<n} I_m \cup \bigcup_{m\ge n} I'_m$ for every n, while $\sum_{n=0}^{\infty} \mu I'_n < \infty$.

In a way, we have to count ourselves lucky every time this works. The reason for studying as many variations as possible of a technique of this kind is to learn to guess when we might be lucky.

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222 Differentiating an indefinite integral

I come now to the first of the two questions mentioned in the introduction to this chapter: if f is an integrable function on $[a,b]$, what is $\frac{d}{dx}\int_a^x f$? It turns out that this derivative exists and is equal to f almost everywhere (222E). The argument is based on a striking property of monotonic functions: they are differentiable almost everywhere (222A), and we can bound the integrals of their derivatives (222C).

222A Theorem Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a monotonic function. Then f is differentiable almost everywhere in I.

Remark If I seem to be speaking of a measure on $\mathbb{R}$ without naming it, as here, I mean Lebesgue measure.

proof As usual, write μ^* for Lebesgue outer measure on $\mathbb{R}$, μ for Lebesgue measure.

(a) To begin with (down to the end of (c) below), let us suppose that f is non-decreasing and I is a bounded open interval on which f is bounded; say $|f(x)| \le M$ for $x \in I$. For any closed subinterval $J = [a,b]$ of I, write $f^*(J)$ for the open interval $]f(a), f(b)[$. For $x \in I$, write

$$\overline{D}f(x) = \limsup_{h\to 0} \tfrac{1}{h}(f(x+h) - f(x)), \quad \underline{D}f(x) = \liminf_{h\to 0} \tfrac{1}{h}(f(x+h) - f(x)),$$

allowing the value ∞ in both cases. Then f is differentiable at x iff $\overline{D}f(x) = \underline{D}f(x) \in \mathbb{R}$. Because surely $\overline{D}f(x) \ge \underline{D}f(x) \ge 0$, f will be differentiable at x iff $\overline{D}f(x)$ is finite and $\overline{D}f(x) \le \underline{D}f(x)$.

I therefore have to show that the sets

$$\{x : x \in I,\, \overline{D}f(x) = \infty\}, \quad \{x : x \in I,\, \overline{D}f(x) > \underline{D}f(x)\}$$

are negligible.

(b) Let us take $A = \{x : x \in I,\, \overline{D}f(x) = \infty\}$ first. Fix an integer $m \ge 1$ for the moment, and set

$$A_m = \{x : x \in I,\, \overline{D}f(x) > m\} \supseteq A.$$

Let $\mathcal{I}$ be the family of non-trivial closed intervals $[a,b] \subseteq I$ such that $f(b) - f(a) \ge m(b-a)$; then $\mu f^*(J) \ge m\mu J$ for every $J \in \mathcal{I}$. If $x \in A_m$, then for any $\delta > 0$ we have an h with $0 < |h| \le \delta$ and $\frac{1}{h}(f(x+h) - f(x)) > m$, so that

$$[x, x+h] \in \mathcal{I} \text{ if } h > 0, \quad [x+h, x] \in \mathcal{I} \text{ if } h < 0;$$

thus every member of A_m belongs to arbitrarily small intervals in $\mathcal{I}$. By Vitali's theorem (221A), there is a countable disjoint set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\mu(A \setminus \bigcup \mathcal{I}_0) = 0$. Now, because f is non-decreasing, $\langle f^*(J) \rangle_{J \in \mathcal{I}_0}$ is disjoint, and all the $f^*(J)$ are included in $[-M, M]$, so $\sum_{J \in \mathcal{I}_0} \mu f^*(J) \le 2M$ and $\sum_{J \in \mathcal{I}_0} \mu J \le 2M/m$. Because $A_m \setminus \bigcup \mathcal{I}_0$ is negligible,

$$\mu^* A \le \mu^* A_m \le \frac{2M}{m}.$$

As m is arbitrary, $\mu^* A = 0$ and A is negligible.

(c) Now consider $B = \{x : x \in I, \overline{D}f(x) > \underline{D}f(x)\}$. For q, $q' \in \mathbb{Q}$ with $0 \le q < q'$, set

$$B_{qq'} = \{x : x \in I, \underline{D}f(x) < q, \overline{D}f(x) > q'\}.$$

Fix such q, q' for the moment, and write $\gamma = \mu^* B_{qq'}$. Take any $\epsilon > 0$, and let G be an open set including $B_{qq'}$ such that $\mu G \le \gamma + \epsilon$ (134Fa). Let $\mathcal{J}$ be the set of non-trivial closed intervals $[a, b] \subseteq I \cap G$ such that $f(b) - f(a) \le q(b - a)$; this time $\mu f^*(J) \le q \mu J$ for $J \in \mathcal{J}$. Then every member of $B_{qq'}$ is included in arbitrarily small members of $\mathcal{J}$, so there is a countable disjoint $\mathcal{J}_0 \subseteq \mathcal{J}$ such that $B_{qq'} \setminus \bigcup \mathcal{J}_0$ is negligible. Let L be the set of endpoints of members of $\mathcal{J}_0$; then L is a countable union of doubleton sets, so is countable, therefore negligible. Set

$$C = B_{qq'} \cap \bigcup \mathcal{J}_0 \setminus L;$$

then $\mu^* C = \gamma$. Let $\mathcal{I}$ be the set of non-trivial closed intervals $J = [a, b]$ such that (i) J is included in one of the members of $\mathcal{J}_0$ (ii) $f(b) - f(a) \ge q'(b - a)$; now $\mu f^*(J) \ge q' \mu J$ for every $J \in \mathcal{I}$. Once again, because every member of C is an interior point of some member of $\mathcal{J}_0$, every point of C belongs to arbitrarily small members of $\mathcal{I}$; so there is a countable disjoint $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\mu(C \setminus \bigcup \mathcal{I}_0) = 0$.

As in (b) above,

$$\gamma q' \le q' \mu(\bigcup \mathcal{I}_0) = \sum_{I \in \mathcal{I}_0} q' \mu I \le \sum_{I \in \mathcal{I}_0} \mu f^*(I) = \mu(\bigcup_{I \in \mathcal{I}_0} f^*(I)).$$

On the other hand,

$$\mu(\bigcup_{J \in \mathcal{J}_0} f^*(J)) = \sum_{J \in \mathcal{J}_0} \mu f^*(J) \le q \sum_{J \in \mathcal{J}_0} \mu J = q\mu(\bigcup \mathcal{J}_0)$$
$$\le q\mu(\bigcup \mathcal{J}) \le q \mu G \le q(\gamma + \epsilon).$$

But $\bigcup_{I \in \mathcal{I}_0} f^*(I) \subseteq \bigcup_{J \in \mathcal{J}_0} f^*(J)$, because every member of $\mathcal{I}_0$ is included in a member of $\mathcal{J}_0$, so $\gamma q' \le q(\gamma + \epsilon)$ and $\gamma \le \epsilon q/(q' - q)$. As ϵ is arbitrary, $\gamma = 0$.

Thus every $B_{qq'}$ is negligible. Consequently $B = \bigcup_{q, q' \in \mathbb{Q}, 0 \le q < q'} B_{qq'}$ is negligible.

(d) This deals with the case of a bounded open interval on which f is bounded and non-decreasing. Still for non-decreasing f, but for an arbitrary interval I, observe that $K = \{(q, q') : q, q' \in I \cap \mathbb{Q}, q < q'\}$ is countable and that $I \setminus \bigcup_{(q,q') \in K}]q, q'[$ has at most two points (the endpoints of I, if any), so is negligible. If we write S for the set of points of I at which f is not differentiable, then from (a)-(c) we see that $S \cap]q, q'[$ is negligible for every $(q, q') \in K$, so that $S \cap \bigcup_{(q,q') \in K}]q, q'[$ is negligible and S is negligible.

(e) Thus we are done if f is non-decreasing. For non-increasing f, apply the above to $-f$, which is differentiable at exactly the same points as f.

222B Remarks (a) I note that in the above argument I am using such formulae as $\sum_{J \in \mathcal{I}_0} \mu f^*(J)$. This is because Vitali's theorem leaves it open whether the families $\mathcal{I}_0$ will be finite or infinite. The sum must be interpreted along the lines laid down in 112Bd in Volume 1; generally, $\sum_{k \in K} a_k$, where K is an arbitrary set and every $a_k \ge 0$, is to be $\sup_{L \subseteq K \text{ is finite}} \sum_{k \in L} a_k$, with the convention that $\sum_{k \in \emptyset} a_k = 0$. Now, in this context, if (X, Σ, μ) is a measure space, K is a countable set, and $\langle E_k \rangle_{k \in K}$ is a family in Σ,

$$\mu(\bigcup_{k \in K} E_k) \le \sum_{k \in K} \mu E_k,$$

with equality if $\langle E_k \rangle_{k \in K}$ is disjoint. **P** If $K = \emptyset$, this is trivial. Otherwise, let $n \mapsto k_n : \mathbb{N} \to K$ be a surjection, and set

$$K_n = \{k_i : i \le n\}, \quad G_n = \bigcup_{i \le n} E_{k_i} = \bigcup_{k \in K_n} E_k$$

for each $n \in \mathbb{N}$. Then $\langle G_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with union $E = \bigcup_{k \in K} E_k$, so

$$\mu E = \lim_{n \to \infty} \mu G_n = \sup_{n \in \mathbb{N}} \mu G_n;$$

and

$$\mu G_n \le \sum_{k \in K_n} \mu E_k \le \sum_{k \in K} \mu E_k$$

for every n, so $\mu E \le \sum_{k \in K} \mu E_k$. If the E_k are disjoint, then μG_n is precisely $\sum_{k \in K_n} \mu E_k$ for each n; but as $\langle K_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of sets with union K, every finite subset of K is included in some K_n, and

$$\sum_{k \in K} \mu E_k = \sup_{n \in \mathbb{N}} \sum_{k \in K_n} \mu E_k = \sup_{n \in \mathbb{N}} \mu G_n = \mu E,$$

as required. **Q**

(b) Some readers will prefer to re-index sets regularly, so that all the sums they need to look at will be of the form $\sum_{i=0}^{n}$ or $\sum_{i=0}^{\infty}$. In effect, that is what I did in Volume 1, in the proof of 114Da/115Da, when showing that Lebesgue outer measure is indeed an outer measure. The disadvantage of this procedure in the context of 222A is that we must continually check that it doesn't matter whether we have a finite or infinite sum at any particular moment. I believe that it is worth taking the trouble to learn the technique sketched here, because it very frequently happens that we wish to consider unions of sets indexed by sets other than $\mathbb{N}$ and $\{0, \dots, n\}$.

(c) Of course the argument above can be shortened if you know a tiny bit more about countable sets than I have explicitly stated so far. But note that the value assigned to $\sum_{k \in K} a_k$ must not depend on which enumeration $\langle k_n \rangle_{n \in \mathbb{N}}$ we pick on.

222C Lemma Suppose that $a \le b$ in $\mathbb{R}$, and that $F : [a, b] \to \mathbb{R}$ is a non-decreasing function. Then $\int_a^b F'$ exists and is at most $F(b) - F(a)$.

Remark I discussed integration over subsets at length in §131 and §214. For measurable subsets, which are sufficient for our needs in this chapter, we have a simple description: if (X, Σ, μ) is a measure space, $E \in \Sigma$ and f is a real-valued function, then $\int_E f = \int \tilde{f}$ if the latter integral exists, where $\operatorname{dom} \tilde{f} = (E \cap \operatorname{dom} f) \cup (X \setminus E)$ and $\tilde{f}(x) = f(x)$ if $x \in E \cap \operatorname{dom} f$, 0 if $x \in X \setminus E$ (apply 131Fa to $\tilde{f}$). It follows at once that if now $F \in \Sigma$ and $F \subseteq E$, $\int_F f = \int_E f \times \chi F$.

I write $\int_a^x f$ to mean $\int_{[a,x[} f$, which (because $[a, x[$ is measurable) can be dealt with as described above. Note that, as long as we are dealing with Lebesgue measure, so that $[a, x] \setminus]a, x[= \{a, x\}$ is negligible, there is no need to distinguish between $\int_{[a,x]}$, $\int_{]a,x[}$, $\int_{[a,x[}$, $\int_{]a,x]}$; for other measures on $\mathbb{R}$ we may need to take more care. I use half-open intervals to make it obvious that $\int_a^x f + \int_x^y f = \int_a^y f$ if $a \le x \le y$, because

$$f \times \chi [a, y[= f \times \chi [a, x[+ f \times \chi [x, y[.$$

proof (a) The result is trivial if $a = b$; let us suppose that $a < b$. By 222A, F' is defined almost everywhere in $[a, b]$.

(b) For each $n \in \mathbb{N}$, define a simple function $g_n : [a, b[\to \mathbb{R}$ as follows. For $0 \le k < 2^n$, set $a_{nk} = a + 2^{-n} k(b - a)$, $b_{nk} = a + 2^{-n}(k + 1)(b - a)$, $I_{nk} = [a_{nk}, b_{nk}[$. For each $x \in [a, b[$, take that $k < 2^n$ such that $x \in I_{nk}$, and set

$$g_n(x) = \frac{2^n}{b-a}(F(b_{nk}) - F(a_{nk}))$$

for $x \in I_{nk}$, so that g_n gives the slope of the chord of the graph of F defined by the endpoints of I_{nk}. Then

$$\int_a^b g_n = \sum_{k=0}^{2^n - 1} F(b_{nk}) - F(a_{nk}) = F(b) - F(a).$$

(c) On the other hand, if we set

$$C = \{x : x \in]a, b[\,, F'(x) \text{ exists}\},$$

then $[a,b]\setminus C$ is negligible, by 222A, and $F'(x)=\lim_{n\to\infty}g_n(x)$ for every $x\in C$. **P** Let $\epsilon>0$. Then there is a $\delta>0$ such that $x+h\in[a,b]$ and $|F(x+h)-F(x))-hF'(x)|\le\epsilon|h|$ whenever $|h|\le\delta$. Let $n\in\mathbb{N}$ be such that $2^{-n}(b-a)\le\delta$. Let $k<2^n$ be such that $x\in I_{nk}$. Then

$$x-\delta\le a_{nk}\le x<b_{nk}\le x+\delta,\quad g_n(x)=\frac{2^n}{b-a}(F(b_{nk})-F(a_{nk})).$$

Now we have

$$\begin{aligned}|g_n(x)-F'(x)|&=|\frac{2^n}{b-a}(F(b_{nk})-F(a_{nk}))-F'(x)|\\&=\frac{2^n}{b-a}|F(b_{nk})-F(a_{nk})-(b_{nk}-a_{nk})F'(x)|\\&\le\frac{2^n}{b-a}\bigl(|F(b_{nk})-F(x)-(b_{nk}-x)F'(x)|\\&\qquad\qquad+|F(x)-F(a_{nk})-(x-a_{nk})F'(x)|\bigr)\\&\le\frac{2^n}{b-a}(\epsilon|b_{nk}-x|+\epsilon|x-a_{nk}|)=\epsilon.\end{aligned}$$

And this is true whenever $2^{-n}\le\delta$, that is, for all n large enough. As ϵ is arbitrary, $F'(x)=\lim_{n\to\infty}g_n(x)$. **Q**

(d) Thus $g_n\to F'$ almost everywhere in $[a,b]$. By Fatou's Lemma,

$$\int_a^b F'=\int_a^b\liminf_{n\to\infty}g_n\le\liminf_{n\to\infty}\int_a^b g_n=\lim_{n\to\infty}\int_a^b g_n=F(b)-F(a),$$

as required.

Remark There is a generalization of this result in 224I.

222D Lemma Suppose that $a<b$ in $\mathbb{R}$, and that f, g are real-valued functions, both integrable over $[a,b]$, such that $\int_a^x f=\int_a^x g$ for every $x\in[a,b]$. Then $f=g$ almost everywhere in $[a,b]$.

proof The point is that

$$\int_E f=\int_a^b f\times\chi E=\int_a^b g\times\chi E=\int_E g$$

for any measurable set $E\subseteq[a,b[$.

P (i) If $E=[c,d[$ where $a\le c\le d\le b$, then

$$\int_E f=\int_a^d f-\int_a^c f=\int_a^d g-\int_a^c g=\int_E g.$$

(ii) If $E=[a,b[\cap G$ for some open set $G\subseteq\mathbb{R}$, then for each $n\in\mathbb{N}$ set

$$K_n=\{k:k\in\mathbb{Z},\,|k|\le 4^n,\,[2^{-n}k,2^{-n}(k+1)[\subseteq G\},$$

$$H_n=\bigcup_{k\in K_n}[2^{-n}k,2^{-n}(k+1)[\cap[a,b[;$$

then $\langle H_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of measurable sets with union E, so $f\times\chi E=\lim_{n\to\infty}f\times\chi H_n$, and (by Lebesgue's Dominated Convergence Theorem, because $|f\times\chi H_n|\le|f|$ almost everywhere for every n, and $|f|$ is integrable)

$$\int_E f=\lim_{n\to\infty}\int_{H_n}f.$$

At the same time, each H_n is a finite disjoint union of half-open intervals in $[a,b[$, so

$$\int_{H_n}f=\sum_{k\in K_n}\int_{[2^{-n}k,2^{-n}(k+1)[\cap[a,b[}f=\sum_{k\in K_n}\int_{[2^{-n}k,2^{-n}(k+1)[\cap[a,b[}g=\int_{H_n}g,$$

and

$$\int_E g=\lim_{n\to\infty}\int_{H_n}g=\lim_{n\to\infty}\int_{H_n}f=\int_E f.$$

(iii) For general measurable $E\subseteq[a,b[$, we can choose for each $n\in\mathbb{N}$ an open set $G_n\supseteq E$ such that $\mu G_n\le\mu E+2^{-n}$ (134Fa). Set $G_n'=\bigcap_{m\le n}G_m$, $E_n=[a,b[\cap G_n'$ for each n,

$$F = [a,b[\,\cap \bigcap_{n\in\mathbb{N}} G_n = \bigcap_{n\in\mathbb{N}} [a,b[\,\cap G'_n = \bigcap_{n\in\mathbb{N}} E_n.$$

Then $E \subseteq F$ and

$$\mu F \le \inf_{n\in\mathbb{N}} \mu G_n = \mu E,$$

so $F \setminus E$ is negligible and $f \times \chi(F \setminus E)$ is zero almost everywhere; consequently $\int_{F\setminus E} f = 0$ and $\int_F f = \int_E f$. On the other hand,

$$f \times \chi F = \lim_{n\to\infty} f \times \chi E_n,$$

so by Lebesgue's Dominated Convergence Theorem again

$$\int_E f = \int_F f = \lim_{n\to\infty} \int_{E_n} f.$$

Similarly

$$\int_E g = \lim_{n\to\infty} \int_{E_n} g.$$

But by part (ii) we have $\int_{E_n} g = \int_{E_n} f$ for every n, so $\int_E g = \int_E f$, as required. **Q**

By 131Hb, $f = g$ almost everywhere in $[a,b[$, and therefore almost everywhere in $[a,b]$.

222E Theorem Suppose that $a \le b$ in $\mathbb{R}$ and that f is a real-valued function which is integrable over $[a,b]$. Then $F(x) = \int_a^x f$ exists in $\mathbb{R}$ for every $x \in [a,b]$, and the derivative $F'(x)$ exists and is equal to $f(x)$ for almost every $x \in [a,b]$.

proof (a) For most of this proof (down to the end of (c) below) I suppose that f is non-negative. In this case,

$$F(y) = F(x) + \int_x^y f \ge F(x)$$

whenever $a \le x \le y \le b$; thus F is non-decreasing and therefore differentiable almost everywhere in $[a,b]$, by 222A.

By 222C we know also that $\int_a^x F'$ exists and is less than or equal to $F(x) - F(a) = F(x)$ for every $x \in [a,b]$.

(b) Now suppose, in addition, that f is bounded; say $0 \le f(t) \le M$ for every $t \in \operatorname{dom} f$. Then $M - f$ is integrable over $[a,b]$; let G be its indefinite integral, so that $G(x) = M(x-a) - F(x)$ for every $x \in [a,b]$. Applying (a) to $M - f$ and G, we have $\int_a^x G' \le G(x)$ for every $x \in [a,b]$; but of course $G' = M - F'$, so $M(x-a) - \int_a^x F' \le M(x-a) - F(x)$, that is, $\int_a^x F' \ge F(x)$ for every $x \in [a,b]$. Thus $\int_a^x F' = \int_a^x f$ for every $x \in [a,b]$. Now 222D tells us that $F' = f$ almost everywhere in $[a,b]$.

(c) Thus for bounded, non-negative f we are done. For unbounded f, let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of non-negative simple functions converging to f almost everywhere in $[a,b]$, and let $\langle F_n\rangle_{n\in\mathbb{N}}$ be the corresponding indefinite integrals. Then for any n and any x, y with $a \le x \le y \le b$, we have

$$F(y) - F(x) = \int_x^y f \ge \int_x^y f_n = F_n(y) - F_n(x),$$

so that $F'(x) \ge F'_n(x)$ for any $x \in\,]a,b[$ where both are defined, and $F'(x) \ge f_n(x)$ for almost every $x \in [a,b]$. This is true for every n, so $F' \ge f$ almost everywhere, and $F' - f \ge 0$ almost everywhere. On the other hand, as noted in (a),

$$\int_a^b F' \le F(b) - F(a) = \int_a^b f,$$

so $\int_a^b F' - f \le 0$. It follows that $F' =_{\text{a.e.}} f$ (that is, that $F' = f$ almost everywhere in $[a,b]$)(122Rd).

(d) This completes the proof for non-negative f. For general f, we can express f as $f_1 - f_2$ where f_1, f_2 are non-negative integrable functions; now $F = F_1 - F_2$ where F_1, F_2 are the corresponding indefinite integrals, so $F' =_{\text{a.e.}} F'_1 - F'_2 =_{\text{a.e.}} f_1 - f_2$, and $F' =_{\text{a.e.}} f$.

222F Corollary Suppose that f is any real-valued function which is integrable over $\mathbb{R}$, and set $F(x) = \int_{-\infty}^x f$ for every $x \in \mathbb{R}$. Then $F'(x)$ exists and is equal to $f(x)$ for almost every $x \in \mathbb{R}$.

proof For each $n \in \mathbb{N}$, set

$$F_n(x) = \int_{-n}^{x} f$$

for $x \in [-n, n]$. Then $F_n'(x) = f(x)$ for almost every $x \in [-n, n]$. But $F(x) = F(-n) + F_n(x)$ for every $x \in [-n, n]$, so $F'(x)$ exists and is equal to $F_n'(x)$ for every $x \in \,]-n, n[$ for which $F_n'(x)$ is defined; and $F'(x) = f(x)$ for almost every $x \in [-n, n]$. As n is arbitrary, $F' =_{\text{a.e.}} f$.

222G Corollary Suppose that $E \subseteq \mathbb{R}$ is a measurable set and that f is a real-valued function which is integrable over E. Set $F(x) = \int_{E\cap]-\infty,x[} f$ for $x \in \mathbb{R}$. Then $F'(x) = f(x)$ for almost every $x \in E$, and $F'(x) = 0$ for almost every $x \in \mathbb{R} \setminus E$.

proof Apply 222F to $\tilde{f}$, where $\tilde{f}(x) = f(x)$ for $x \in E \cap \operatorname{dom} f$ and $\tilde{f}(x) = 0$ for $x \in \mathbb{R} \setminus E$, so that $F(x) = \int_{-\infty}^{x} \tilde{f}$ for every $x \in \mathbb{R}$.

222H The result that $\frac{d}{dx}\int_a^x f = f(x)$ for almost every x is satisfying, but is no substitute for the more elementary result that this equality is valid at any point at which f is continuous.

Proposition Suppose that $a \le b$ in $\mathbb{R}$ and that f is a real-valued function which is integrable over $[a, b]$. Set $F(x) = \int_a^x f$ for $x \in [a, b]$. Then $F'(x)$ exists and is equal to $f(x)$ at any point $x \in \operatorname{dom}(f) \cap \,]a, b[$ at which f is continuous.

proof Set $c = f(x)$. Let $\epsilon > 0$. Let $\delta > 0$ be such that $\delta \le \min(b - x, x - a)$ and $|f(t) - c| \le \epsilon$ whenever $t \in \operatorname{dom} f$ and $|t - x| \le \delta$. If $x < y \le x + \delta$, then

$$\Bigl|\frac{F(y)-F(x)}{y-x} - c\Bigr| = \frac{1}{y-x}\Bigl|\int_x^y f - c\Bigr| \le \frac{1}{y-x}\int_x^y |f - c| \le \epsilon.$$

Similarly, if $x - \delta \le y < x$,

$$\Bigl|\frac{F(y)-F(x)}{y-x} - f(x)\Bigr| = \frac{1}{x-y}\Bigl|\int_y^x f - c\Bigr| \le \frac{1}{x-y}\int_y^x |f - c| \le \epsilon.$$

As ϵ is arbitrary,

$$F'(x) = \lim_{y\to x} \frac{F(y)-F(x)}{y-x} = c,$$

as required.

222I Complex-valued functions So far in this section, I have taken every f to be real-valued. The extension to complex-valued f is just a matter of applying the above results to the real and imaginary parts of f. Specifically, we have the following.

(a) If $a \le b$ in $\mathbb{R}$ and f is a complex-valued function which is integrable over $[a, b]$, then $F(x) = \int_a^x f$ is defined in $\mathbb{C}$ for every $x \in [a, b]$, and its derivative $F'(x)$ exists and is equal to $f(x)$ for almost every $x \in [a, b]$; moreover, $F'(x) = f(x)$ whenever $x \in \operatorname{dom}(f) \cap \,]a, b[$ and f is continuous at x.

(b) If f is a complex-valued function which is integrable over $\mathbb{R}$, and $F(x) = \int_{-\infty}^{x} f$ for each $x \in \mathbb{R}$, then F' exists and is equal to f almost everywhere in $\mathbb{R}$.

(c) If $E \subseteq \mathbb{R}$ is a measurable set and f is a complex-valued function which is integrable over E, and $F(x) = \int_{E\cap]-\infty,x[} f$ for each $x \in \mathbb{R}$, then $F'(x) = f(x)$ for almost every $x \in E$ and $F'(x) = 0$ for almost every $x \in \mathbb{R} \setminus E$.

***222J The Denjoy-Young-Saks theorem** The next result will not be used, on present plans, anywhere in this treatise. It is however a classical part of the real analysis for which this volume is supposed to be a foundation, and while the argument requires a certain sophistication it is not really a large step from Lebesgue's theorem 222A. I must begin with some notation.

Definition Let f be any real function, and $A \subseteq \mathbb{R}$ its domain. Write

$$\tilde{A}^+ = \{x : x \in A,\ ]x, x + \delta] \cap A \ne \emptyset \text{ for every } \delta > 0\},$$

$$\tilde{A}^- = \{x : x \in A,\ [x-\delta, x[\, \cap A \neq \emptyset \text{ for every } \delta > 0\}.$$

Set

$$(\overline{D}^+ f)(x) = \limsup_{y \in A, y \downarrow x} \frac{f(y)-f(x)}{y-x} = \inf_{\delta>0} \sup_{y\in A, x<y\le x+\delta} \frac{f(y)-f(x)}{y-x},$$

$$(\underline{D}^+ f)(x) = \liminf_{y \in A, y \downarrow x} \frac{f(y)-f(x)}{y-x} = \sup_{\delta>0} \inf_{y\in A, x<y\le x+\delta} \frac{f(y)-f(x)}{y-x}$$

for $x \in \tilde{A}^+$, and

$$(\overline{D}^- f)(x) = \limsup_{y \in A, y \uparrow x} \frac{f(y)-f(x)}{y-x} = \inf_{\delta>0} \sup_{y\in A, x-\delta\le y<x} \frac{f(y)-f(x)}{y-x},$$

$$(\underline{D}^- f)(x) = \liminf_{y \in A, y \uparrow x} \frac{f(y)-f(x)}{y-x} = \sup_{\delta>0} \inf_{y\in A, x-\delta\le y<x} \frac{f(y)-f(x)}{y-x}$$

for $x \in \tilde{A}^-$, all defined in $[-\infty, \infty]$. (These are the four **Dini derivates** of f. You will also see D^+, d^+, D^-, d^- used in place of my $\overline{D}^+$, $\underline{D}^+$, $\overline{D}^-$ and $\underline{D}^-$.)

Note that we surely have $(\underline{D}^+ f)(x) \le (\overline{D}^+ f)(x)$ for every $x \in \tilde{A}^+$, while $(\underline{D}^- f)(x) \le (\overline{D}^- f)(x)$ for every $x \in \tilde{A}^-$. The ordinary derivative $f'(x)$ is defined and equal to $c \in \mathbb{R}$ iff (α) x belongs to some open interval included in A (β) $(\overline{D}^+ f)(x) = (\underline{D}^+ f)(x) = (\overline{D}^- f)(x) = (\underline{D}^- f)(x) = c$.

***222K Lemma** For $A \subseteq \mathbb{R}$, define $\tilde{A}^+$ and $\tilde{A}^-$ as in 222J. Then $A \setminus \tilde{A}^+$ and $A \setminus \tilde{A}^-$ are countable, therefore negligible.

proof We have

$$A \setminus \tilde{A}^+ = \bigcup_{q \in \mathbb{Q}} \{x : x \in A,\ x < q,\ A \cap\,]x, q] = \emptyset\}.$$

But for any $q \in \mathbb{Q}$ there can be at most one $x \in A$ such that $x < q$ and $]x, q]$ does not meet A, so $A \setminus \tilde{A}^+$ is a countable union of finite sets and is countable. Similarly,

$$A \setminus \tilde{A}^- = \bigcup_{q \in \mathbb{Q}} \{x : x \in A,\ q < x,\ A \cap [q, x[\, = \emptyset\}$$

is countable.

***222L Theorem** Let f be any real function, and A its domain. Then for almost every $x \in A$

either all four Dini derivates of f at x are defined, finite and equal
or $(\overline{D}^+ f)(x) = (\underline{D}^- f)(x)$ is finite, $(\underline{D}^+ f)(x) = -\infty$ and $(\overline{D}^- f)(x) = \infty$
or $(\underline{D}^+ f)(x) = (\overline{D}^- f)(x)$ is finite, $(\overline{D}^+ f)(x) = \infty$ and $(\underline{D}^- f)(x) = -\infty$
or $(\overline{D}^+ f)(x) = (\overline{D}^- f)(x) = \infty$ and $(\underline{D}^+ f)(x) = (\underline{D}^- f)(x) = -\infty$.

proof[1] **(a)(i)** Suppose that $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ are such that

$$E_{qn} = \{x : x \in A,\ x < q,\ f(y) \ge f(x) - n(y-x)\} \text{ for every } y \in A \cap [x, q]\}$$

is not empty. Set $\beta_{qn} = \sup E_{qn} \in\,]-\infty, q]$, $\alpha_{qn} = \inf E_{qn} \in [-\infty, \beta_{qn}]$ and $I_{qn} =]\alpha_{qn}, \beta_{qn}[$; now for $x \in I_{qn}$ set $f_{qn}(x) = \inf\{f(y) + ny : y \in A \cap [x, q]\}$. Note that if $x \in E_{qn} \setminus \{\alpha_{qn}, \beta_{qn}\}$ then $f_{qn}(x) = f(x) + nx$ is finite; also f_{qn} is non-decreasing, therefore finite everywhere in I_{qn}, and of course $f_{qn}(x) \le f(x) + nx$ for every $x \in A \cap I_{qn}$.

Set $F_{qn} = E_{qn} \cap \operatorname{dom} f'_{qn} \subseteq I_{qn}$, and $g_{qn}(x) = f_{qn}(x) - nx$ for $x \in I_{qn}$; then g_{qn} is differentiable at every point of F_{qn}, while $g_{qn}(x) \le f(x)$ for $x \in A \cap I_{qn}$ and $g_{qn}(x) = f(x)$ for $x \in E_{qn} \cap I_{qn}$.

(ii) Take $x \in \tilde{F}^+_{qn} \cap \tilde{F}^-_{qn}$. Then $x \in I_{qn} \cap \tilde{A}^+ \cap \tilde{A}^-$ so the Dini derivates $(\underline{D}^+ f)(x)$ and $(\overline{D}^- f)(x)$ are defined in $[-\infty, \infty]$, while $g_{qn}(x) = f(x)$.

($\boldsymbol{\alpha}$) $(\underline{D}^+ f)(x) = g'_{qn}(x)$. **P**

[1] I am indebted to P.Klinger Monteiro for pointing out a blunder in the original version of this proof.

$$(\underline{D}^+f)(x) = \sup_{\delta>0} \inf_{y\in A\cap]x,x+\delta]} \tfrac{f(y)-f(x)}{y-x} \le \sup_{\delta>0} \inf_{y\in E_{qn}\cap]x,x+\delta]} \tfrac{f(y)-f(x)}{y-x}$$
$$= \sup_{\delta>0} \inf_{y\in E_{qn}\cap]x,x+\delta]} \tfrac{g_{qn}(y)-g_{qn}(x)}{y-x} = g'_{qn}(x)$$

(because $x \in \tilde{E}^+_{qn}$)

$$= \sup_{0<\delta<\beta_{qn}-x} \inf_{y\in]x,x+\delta]} \tfrac{g_{qn}(y)-g_{qn}(x)}{y-x}$$
$$\le \sup_{0<\delta<\beta_{qn}-x} \inf_{y\in A\cap]x,x+\delta]} \tfrac{g_{qn}(y)-f(x)}{y-x}$$
$$\le \sup_{0<\delta<\beta_{qn}-x} \inf_{y\in A\cap]x,x+\delta]} \tfrac{f(y)-f(x)}{y-x}$$

(because $g_{qn}(y) \le f(y)$ for $y \in A \cap I_{qn}$)

$$\le \sup_{\delta>0} \inf_{y\in A\cap]x,x+\delta]} \tfrac{f(y)-f(x)}{y-x} = (\underline{D}^+f)(x). \textbf{ Q}$$

(β) $(\overline{D}^-f)(x) = g'_{qn}(x)$. **P**

$$(\overline{D}^-f)(x) = \inf_{\delta>0} \sup_{y\in A\cap[x-\delta,x[} \tfrac{f(y)-f(x)}{y-x} = \inf_{\delta>0} \sup_{y\in A\cap[x-\delta,x[} \tfrac{f(x)-f(y)}{x-y}$$
$$\le \inf_{0<\delta<x-\alpha_{qn}} \sup_{y\in A\cap[x-\delta,x[} \tfrac{f(x)-f(y)}{x-y}$$
$$\le \inf_{0<\delta<x-\alpha_{qn}} \sup_{y\in A\cap[x-\delta,x[} \tfrac{g_{qn}(x)-g_{qn}(y)}{x-y}$$

(because $g_{qn}(y) \le f(y)$ for $y \in A \cap I_{qn}$)

$$\le \inf_{0<\delta<x-\alpha_{qn}} \sup_{y\in[x-\delta,x[} \tfrac{g_{qn}(y)-g_{qn}(x)}{y-x} = g'_{qn}(x)$$
$$= \inf_{\delta>0} \sup_{y\in E_{qn}\cap[x-\delta,x[} \tfrac{g_{qn}(y)-g_{qn}(x)}{y-x}$$

(because $x \in \tilde{E}^-_{qn}$)

$$= \inf_{\delta>0} \sup_{y\in E_{qn}\cap[x-\delta,x[} \tfrac{f(y)-f(x)}{y-x}$$
$$\le \inf_{\delta>0} \sup_{y\in A\cap[x-\delta,x[} \tfrac{f(y)-f(x)}{y-x} = (\overline{D}^-f)(x). \textbf{ Q}$$

(γ) Putting these together, we see that if $x \in \tilde{F}^+_{qn} \cap \tilde{F}^-_{qn}$ then $(\underline{D}^+f)(x) = (\overline{D}^-f)(x) = g'_{qn}(x)$ is finite.

(iii) Conventionally setting $F_{qn} = \emptyset$ if E_{qn} is empty, the last sentence is true for all $q \in \mathbb{Q}$ and $n \in \mathbb{N}$. Now we know that $A\setminus\tilde{A}^+$ is negligible (222K), as are $F_{qn}\setminus\tilde{F}^+_{qn}$, $F_{qn}\setminus\tilde{F}^-_{qn}$ and $I_{qn}\setminus\operatorname{dom} f'_{qn}$ (222A), whenever $q \in \mathbb{Q}$ and $n \in \mathbb{N}$. So $H = (A \setminus \tilde{A}^+) \cup \bigcup_{q\in\mathbb{Q},n\in\mathbb{N}}((E_{qn} \setminus F_{qn}) \cup (F_{qn} \setminus (\tilde{F}^+_{qn} \cap \tilde{F}^-_{qn})))$ is negligible. And if $x \in A \setminus H$ and $(\overline{D}^+f(x)) > -\infty$, then $(\underline{D}^+f)(x) = (\overline{D}^-f)(x) \in \mathbb{R}$. **P** As $A \setminus \tilde{A}^+ \subseteq H$, $x \in \tilde{A}^+$. Let $n \in \mathbb{N}$ be such that $(\overline{D}^+f)(x) > -n$. Then there is a $\delta > 0$ such that $\frac{f(y)-f(x)}{y-x} > -n$ whenever $y \in A$ and $x < y \le x+\delta$. Let $q \in \mathbb{Q}$ be such that $x < q \le x+\delta$; then $f(y) \ge f(x) - n(y-x)$ whenever $y \in A \cap [x, q]$, and $x \in E_{qn} \setminus H$. So $x \in F_{qn} \setminus H \subseteq \tilde{F}^+_{qn} \cap \tilde{F}^-_{qn}$. So (ii) above tells us that $(\underline{D}^+f)(x) = (\overline{D}^-f)(x) = g'_{qn}(x)$ is finite. **Q**

(b) Thus for almost every $x \in A$,

$$\text{either } (\overline{D}^+ f)(x) = -\infty \text{ or } (\overline{D}^+ f)(x) = (\underline{D}^- f)(x) \in \mathbb{R}.$$

Applying (a) to $x \mapsto f(-x) : -A \to \mathbb{R}$, $x \mapsto -f(x) : A \to \mathbb{R}$ and $x \mapsto -f(-x) : -A \to \mathbb{R}$, we see that, for almost every $x \in A$,

$$\text{either } (\overline{D}^- f)(x) = \infty \text{ or } (\overline{D}^- f)(x) = (\underline{D}^+ f)(x) \in \mathbb{R},$$

$$\text{either } (\overline{D}^+ f)(x) = \infty \text{ or } (\overline{D}^+ f)(x) = (\underline{D}^- f)(x) \in \mathbb{R},$$

$$\text{either } (\underline{D}^- f)(x) = -\infty \text{ or } (\underline{D}^- f)(x) = (\overline{D}^+ f)(x) \in \mathbb{R}.$$

For almost every $x \in A$, therefore,

$$(\underline{D}^+ f)(x) > -\infty \Longrightarrow (\overline{D}^- f)(x) < \infty \Longrightarrow (\underline{D}^+ f)(x) < \infty$$

and in this case $(\underline{D}^+ f)(x) = (\overline{D}^- f)(x)$ is finite; similarly,

$$(\underline{D}^- f)(x) > -\infty \iff (\overline{D}^+ f)(x) < \infty \iff (\underline{D}^- f)(x) = (\overline{D}^+ f)(x) \in \mathbb{R}.$$

Thus we have

$$\text{either } (\underline{D}^+ f)(x) = (\overline{D}^- f)(x) \text{ is finite or } (\underline{D}^+ f)(x) = -\infty \text{ and } (\overline{D}^- f)(x) = \infty,$$

$$\text{either } (\underline{D}^- f)(x) = (\overline{D}^+ f)(x) \text{ is finite or } (\underline{D}^- f)(x) = -\infty \text{ and } (\overline{D}^+ f)(x) = \infty.$$

These two dichotomies lead to four possibilities; and since

$$(\underline{D}^+ f)(x) = (\overline{D}^- f)(x) \text{ is finite}, \quad (\underline{D}^- f)(x) = (\overline{D}^+ f)(x) \text{ is finite}$$

together imply that

$$(\underline{D}^- f)(x) \le (\overline{D}^- f)(x) = (\underline{D}^+ f)(x) \le (\overline{D}^+ f)(x) = (\underline{D}^- f)(x),$$

this can happen only when all four derivates are equal and finite, so we have the four cases listed in the statement of the theorem.

222X Basic exercises >(a) Let $F : [0,1] \to [0,1]$ be the Cantor function (134H). Show that $\int_0^1 F' = 0 < F(1) - F(0)$.

>(b) Suppose that $a < b$ in $\mathbb{R}$ and that h is a real-valued function such that $\int_x^y h$ exists and is non-negative whenever $a \le x \le y \le b$. Show that $h \ge 0$ almost everywhere in $[a,b]$.

>(c) Suppose that $a < b$ in $\mathbb{R}$ and that f, g are integrable complex-valued functions on $[a,b]$ such that $\int_a^x f = \int_a^x g$ for every $x \in [a,b]$. Show that $f = g$ almost everywhere in $[a,b]$.

>(d) Suppose that $a < b$ in $\mathbb{R}$ and that f is a real-valued function which is integrable over $[a,b]$. Show that the indefinite integral $x \mapsto \int_a^x f$ is continuous.

222Y Further exercises (a) Let $\langle F_n \rangle_{n \in \mathbb{N}}$ be a sequence of non-negative, non-decreasing functions on $[0,1]$ such that $F(x) = \sum_{n=0}^\infty F_n(x)$ is finite for every $x \in [0,1]$. Show that $\sum_{n=0}^\infty F_n'(x) = F'(x)$ for almost every $x \in [0,1]$. (*Hint*: take $\langle n_k \rangle_{k \in \mathbb{N}}$ such that $\sum_{k=0}^\infty F(1) - G_k(1) < \infty$, where $G_k = \sum_{j=0}^{n_k} F_j$, and set $H(x) = \sum_{k=0}^\infty F(x) - G_k(x)$. Observe that $\sum_{k=0}^\infty F'(x) - G_k'(x) \le H'(x)$ whenever all the derivatives are defined, so that $F' = \lim_{k \to \infty} G_k'$ almost everywhere.)

(b) Let $F : [0,1] \to \mathbb{R}$ be a continuous non-decreasing function. (i) Show that if $c \in \mathbb{R}$ then $C = \{(x,y) : x, y \in [0,1], F(y) - F(x) = c\}$ is connected. (*Hint*: A set $A \subseteq \mathbb{R}^r$ is **connected** if there is no continuous surjection $h : A \to \{0,1\}$. Show that if $h : C \to \{0,1\}$ is continuous then it is of the form $(x,y) \mapsto h_1(x)$ for some continuous function h_1.) (ii) Now suppose that $F(0) = 0$, $F(1) = 1$ and that $G : [0,1] \to [0,1]$ is a second continuous non-decreasing function with $G(0) = 0$, $G(1) = 1$. Show that for any $n \ge 1$ there are x, $y \in [0,1]$ such that $F(y) - F(x) = G(y) - G(x) = \frac{1}{n}$.

(c) Let f, g be non-negative integrable functions on $\mathbb{R}$, and $n \ge 1$. Show that there are $u < v$ in $[-\infty, \infty]$ such that $\int_u^v f = \frac{1}{n}\int f$ and $\int_u^v g = \frac{1}{n}\int g$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Show that $H = \operatorname{dom} f'$ is a measurable set and that f' is a measurable function.

(e) Suppose that $A \subseteq B \subseteq \mathbb{R}$, $f : B \to \mathbb{R}$ is a function and $f_A = f\restriction A$. Show that (i) if $x \in \tilde{A}^+$, as defined in 222J, then $(\underline{D}^+ f)(x) \le (\underline{D}^+ f_A)(x) \le (\overline{D}^+ f_A)(x) \le (\overline{D}^+ f)(x)$ (ii) if $x \in \tilde{A}^-$, then $(\underline{D}^- f)(x) \le (\underline{D}^- f_A)(x) \le (\overline{D}^- f_A)(x) \le (\overline{D}^- f)(x)$.

(f) Construct a Borel measurable function $f : [0,1] \to \{-1, 0, 1\}$ such that each of the four possibilities described in Theorem 222L occurs on a set of measure $\frac{1}{4}$.

222 Notes and comments I have relegated to an exercise (222Xd) the fundamental fact that an indefinite integral $x \mapsto \int_a^x f$ is always continuous; this is not strictly speaking needed in this section, and a much stronger result is given in 225E. There is also much more to be said about monotonic functions, to which I will return in §224. What we need here is the fact that they are differentiable almost everywhere (222A), which I prove by applying Vitali's theorem three times, once in part (b) of the proof and twice in part (c). Following this, the arguments of 222C-222E form a fine series of exercises in the central ideas of Volume 1, using the concept of integration over a (measurable) subset, Fatou's Lemma (part (d) of the proof of 222C), Lebesgue's Dominated Convergence Theorem (parts (ii) and (iii) of the proof of 222D) and the approximation of Lebesgue measurable sets by open sets (part (iii) of the proof of 222D). Of course knowing that $\frac{d}{dx}\int_a^x f = f(x)$ almost everywhere is not at all the same thing as knowing that this holds for any particular x, and when we come to differentiate any particular indefinite integral we generally turn to 222H first; the point of 222E is that it applies to wildly discontinuous functions, for which more primitive methods give no information at all.

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223 Lebesgue's density theorems

I now turn to a group of results which may be thought of as corollaries of Theorem 222E, but which also have a vigorous life of their own, including the possibility of significant generalizations which will be treated in Chapter 26. The idea is that any measurable function f on $\mathbb{R}$ is almost everywhere 'continuous' in a variety of very weak senses; for almost every x, the value $f(x)$ is determined by the behaviour of f near x, in the sense that $f(y) \simeq f(x)$ for 'most' y near x. I should perhaps say that while I recommend this work as a preparation for Chapter 26, and I also rely on it in Chapter 28, I shall not refer to it again in the present chapter, so that readers in a hurry to characterize indefinite integrals may proceed directly to §224.

223A Lebesgue's Density Theorem: integral form Let I be an interval in $\mathbb{R}$, and let f be a real-valued function which is integrable over I. Then

$$f(x) = \lim_{h\downarrow 0}\frac{1}{h}\int_x^{x+h} f = \lim_{h\downarrow 0}\frac{1}{h}\int_{x-h}^{x} f = \lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h} f$$

for almost every $x \in I$.

proof Setting $F(x) = \int_{I\cap]-\infty,x[} f$, we know from 222G that

$$\begin{aligned}f(x) = F'(x) &= \lim_{h\downarrow 0}\frac{1}{h}(F(x+h) - F(x)) = \lim_{h\downarrow 0}\frac{1}{h}\int_x^{x+h} f\\ &= \lim_{h\downarrow 0}\frac{1}{h}(F(x) - F(x-h)) = \lim_{h\downarrow 0}\frac{1}{h}\int_{x-h}^{x} f\\ &= \lim_{h\downarrow 0}\frac{1}{2h}(F(x+h) - F(x-h)) = \lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h} f\end{aligned}$$

for almost every $x \in I$.

223B Corollary Let $E \subseteq \mathbb{R}$ be a measurable set. Then

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu(E\cap[x-h,x+h]) = 1 \text{ for almost every } x \in E,$$

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu(E\cap[x-h,x+h]) = 0 \text{ for almost every } x \in \mathbb{R}\setminus E.$$

proof Take $n \in \mathbb{N}$. Applying 223A to $f = \chi(E\cap[-n,n])$, we see that

$$\lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h} f = \lim_{h\downarrow 0}\frac{1}{2h}\mu(E\cap[x-h,x+h])$$

whenever $x \in \left]-n,n\right[$ and either limit exists, so that

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu(E\cap[x-h,x+h]) = 1 \text{ for almost every } x \in E\cap[-n,n],$$

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu(E\cap[x-h,x+h]) = 0 \text{ for almost every } x \in [-n,n]\setminus E.$$

As n is arbitrary, we have the result.

Remark For a measurable set $E \subseteq \mathbb{R}$, a point x such that $\lim_{h\downarrow 0}\frac{1}{2h}\mu(E\cap[x-h,x+h]) = 1$ is sometimes called a **density point** of E.

223C Corollary Let f be a measurable real-valued function defined almost everywhere in $\mathbb{R}$. Then for almost every $x \in \mathbb{R}$,

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu\{y : y \in \operatorname{dom} f,\ |y-x| \le h,\ |f(y)-f(x)| \le \epsilon\} = 1,$$

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu\{y : y \in \operatorname{dom} f,\ |y-x| \le h,\ |f(y)-f(x)| \ge \epsilon\} = 0$$

for every $\epsilon > 0$.

proof For q, $q' \in \mathbb{Q}$, set

$$D_{qq'} = \{x : x \in \operatorname{dom} f,\ q \le f(x) < q'\},$$

so that $D_{qq'}$ is measurable,

$$C_{qq'} = \{x : x \in D_{qq'},\ \lim_{h\downarrow 0}\frac{1}{2h}\mu(D_{qq'}\cap[x-h,x+h]) = 1\},$$

so that $D_{qq'}\setminus C_{qq'}$ is negligible, by 223B; now set

$$C = \operatorname{dom} f \setminus \bigcup_{q,q'\in\mathbb{Q}}(D_{qq'}\setminus C_{qq'}),$$

so that C is conegligible. If $x \in C$ and $\epsilon > 0$, then there are q, $q' \in \mathbb{Q}$ such that $f(x)-\epsilon \le q \le f(x) < q' \le f(x)+\epsilon$, so that x belongs to $D_{qq'}$ and therefore to $C_{qq'}$, and now

$$\liminf_{h\downarrow 0}\frac{1}{2h}\mu\{y : y \in \operatorname{dom} f\cap[x-h,x+h],\ |f(y)-f(x)| \le \epsilon\}$$

$$\ge \liminf_{h\downarrow 0}\frac{1}{2h}\mu(D_{qq'}\cap[x-h,x+h]) = 1,$$

so

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu\{y : y \in \operatorname{dom} f\cap[x-h,x+h],\ |f(y)-f(x)| \le \epsilon\} = 1.$$

It follows at once that

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu\{y : y \in \operatorname{dom} f\cap[x-h,x+h],\ |f(y)-f(x)| > \epsilon\} = 0$$

for almost every x; but since ϵ is arbitrary, this is also true of $\frac{1}{2}\epsilon$, so in fact

$$\lim_{h\downarrow 0}\frac{1}{2h}\mu\{y : y \in \operatorname{dom} f \cap [x-h,x+h],\ |f(y)-f(x)| \ge \epsilon\} = 0$$

for almost every x.

223D Theorem Let I be an interval in $\mathbb{R}$, and let f be a real-valued function which is integrable over I. Then

$$\lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h}|f(y)-f(x)|dy = 0$$

for almost every $x \in I$.

proof (a) Suppose first that I is a bounded open interval $]a,b[$. For each $q \in \mathbb{Q}$, set $g_q(x) = |f(x)-q|$ for $x \in I \cap \operatorname{dom} f$; then g is integrable over I, and

$$\lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h} g_q = g_q(x)$$

for almost every $x \in I$, by 223A. Setting

$$E_q = \{x : x \in I \cap \operatorname{dom} f,\ \lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h} g_q = g_q(x)\},$$

we have $I \setminus E_q$ negligible, so $I \setminus E$ is negligible, where $E = \bigcap_{q\in\mathbb{Q}} E_q$. Now

$$\lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h}|f(y)-f(x)|dy = 0$$

for every $x \in E$. **P** Take $x \in E$ and $\epsilon > 0$. Then there is a $q \in \mathbb{Q}$ such that $|f(x)-q| \le \epsilon$, so that

$$|f(y)-f(x)| \le |f(y)-q| + \epsilon = g_q(y) + \epsilon$$

for every $y \in I \cap \operatorname{dom} f$, and

$$\begin{aligned}\limsup_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h}|f(y)-f(x)|dy &\le \limsup_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h} g_q(y)+\epsilon\,dy \\ &= \epsilon + g_q(x) \le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary,

$$\lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h}|f(y)-f(x)|dy = 0,$$

as required. **Q**

(b) If I is an unbounded open interval, apply (a) to the intervals $I_n = I \cap \,]-n,n[$ to see that the limit is zero almost everywhere in every I_n, and therefore on I. If I is an arbitrary interval, note that it differs by at most two points from an open interval, and that since we are looking only for something to happen almost everywhere we can ignore these points.

Remark The set

$$\{x : x \in \operatorname{dom} f,\ \lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h}|f(y)-f(x)|dy = 0\}$$

is sometimes called the **Lebesgue set** of f.

223E Complex-valued functions I have expressed the results above in terms of real-valued functions, this being the most natural vehicle for the ideas. However there are applications of great importance in which the functions involved are complex-valued, so I spell out the relevant statements here. In all cases the proof is elementary, being nothing more than applying the corresponding result (223A, 223C or 223D) to the real and imaginary parts of the function f.

(a) Let I be an interval in $\mathbb{R}$, and let f be a complex-valued function which is integrable over I. Then

$$f(x) = \lim_{h\downarrow 0} \frac{1}{h} \int_x^{x+h} f = \lim_{h\downarrow 0} \frac{1}{h} \int_{x-h}^{x} f = \lim_{h\downarrow 0} \frac{1}{2h} \int_{x-h}^{x+h} f$$

for almost every $x \in I$.

(b) Let f be a measurable complex-valued function defined almost everywhere in $\mathbb{R}$. Then for almost every $x \in \mathbb{R}$,

$$\lim_{h\downarrow 0} \frac{1}{2h}\mu\{y : y \in \operatorname{dom} f,\ |y-x| \le h,\ |f(y)-f(x)| \ge \epsilon\} = 0$$

for every $\epsilon > 0$.

(c) Let I be an interval in $\mathbb{R}$, and let f be a complex-valued function which is integrable over I. Then

$$\lim_{h\downarrow 0} \frac{1}{2h}\int_{x-h}^{x+h} |f(y)-f(x)|dy = 0$$

for almost every $x \in I$.

223X Basic exercises >(a) Let $E \subseteq [0,1]$ be a measurable set for which there is an $\alpha > 0$ such that $\mu(E \cap [a,b]) \ge \alpha(b-a)$ whenever $0 \le a \le b \le 1$. Show that $\mu E = 1$.

(b) Let $A \subseteq \mathbb{R}$ be any set. Show that $\lim_{h\downarrow 0} \frac{1}{2h}\mu^*(A \cap [x-h, x+h]) = 1$ for almost every $x \in A$. (*Hint*: apply 223B to a measurable envelope E of A.)

(c) Let E, $F \subseteq \mathbb{R}$ be measurable sets, and $x \in \mathbb{R}$ a point which is a density point of both. Show that x is a density point of $E \cap F$.

(d) Let $E \subseteq \mathbb{R}$ be a non-negligible measurable set. Show that for any $n \in \mathbb{N}$ there is a $\delta > 0$ such that $\bigcap_{i\le n} E + x_i$ is non-empty whenever $x_0, \dots, x_n \in \mathbb{R}$ are such that $|x_i - x_j| \le \delta$ for all i, $j \le n$. (*Hint*: find a non-trivial interval I such that $\mu(E \cap I) > \frac{n}{n+1}\mu I$.)

(e) Let f be any real-valued function defined almost everywhere in $\mathbb{R}$. Show that $\lim_{h\downarrow 0} \frac{1}{2h}\mu^*\{y : y \in \operatorname{dom} f,\ |y-x| \le h,\ |f(y)-f(x)| \le \epsilon\} = 1$ for almost every $x \in \mathbb{R}$. (*Hint*: use the argument of 223C, but with 223Xb in place of 223B.)

>(f) Let I be an interval in $\mathbb{R}$, and let f be a real-valued function which is integrable over I. Show that $\lim_{h\downarrow 0} \frac{1}{h}\int_x^{x+h} |f(y)-f(x)|dy = 0$ for almost every $x \in I$.

(g) Let E, $F \subseteq \mathbb{R}$ be measurable sets, and suppose that F is bounded and of non-zero measure. Let $x \in \mathbb{R}$ be such that $\lim_{h\downarrow 0} \frac{1}{2h}\mu(E \cap [x-h, x+h]) = 1$. Show that $\lim_{h\downarrow 0} \frac{\mu(E\cap(x+hF))}{h\,\mu F} = 1$. (*Hint:* it helps to know that $\mu(hF) = h\mu F$ (134Ya, 263A). Show that if $F \subseteq [-M, M]$, then

$$\frac{1}{2hM}\mu(E \cap [x-hM, x+hM]) \le 1 - \frac{\mu F}{2M}\bigl(1 - \frac{\mu(E\cap(x+hF))}{h\,\mu F}\bigr).)$$

(Compare 223Ya.)

(h) Let f be a real-valued function which is integrable over $\mathbb{R}$, and E be the Lebesgue set of f. Show that $\lim_{h\downarrow 0} \frac{1}{2h}\int_{x-h}^{x+h} |f(t)-c|dt = |f(x)-c|$ for every $x \in E$ and $c \in \mathbb{R}$.

(i) Let f be an integrable real-valued function defined almost everywhere in $\mathbb{R}$. Let $x \in \operatorname{dom} f$ be such that $\lim_{n\to\infty} \frac{n}{2}\int_{x-1/n}^{x+1/n} |f(y)-f(x)| = 0$. Show that x belongs to the Lebesgue set of f.

(j) Let f be an integrable real-valued function defined almost everywhere in $\mathbb{R}$, and x any point of the Lebesgue set of f. Show that for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever I is a non-trivial interval and $x \in I \subseteq [x-\delta, x+\delta]$, then $|f(x) - \frac{1}{\mu I}\int_I f| \le \epsilon$.

223Y Further exercises (a) Let E, $F \subseteq \mathbb{R}$ be measurable sets, and suppose that $0 < \mu F < \infty$. Let $x \in \mathbb{R}$ be such that $\lim_{h\downarrow 0} \frac{1}{2h}\mu(E \cap [x-h, x+h]) = 1$. Show that

$$\lim_{h\downarrow 0} \frac{\mu(E\cap(x+hF))}{h\,\mu F} = 1.$$

(*Hint*: apply 223Xg to sets of the form $F \cap [-M, M]$.)

(b) Let $\mathfrak{T}$ be the family of measurable sets $G \subseteq \mathbb{R}$ such that every point of G is a density point of G. (i) Show that $\mathfrak{T}$ is a topology on $\mathbb{R}$. (*Hint*: take $\mathcal{G} \subseteq \mathfrak{T}$. By 215B(iv) there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\mu(G \setminus \bigcup \mathcal{G}_0) = 0$ for every $G \in \mathcal{G}$. Show that

$$\bigcup \mathcal{G} \subseteq \{x : \limsup_{h\downarrow 0} \frac{1}{2h}\mu(\bigcup \mathcal{G}_0 \cap [x-h, x+h]) > 0\},$$

so that $\mu(\bigcup \mathcal{G} \setminus \bigcup \mathcal{G}_0) = 0$.) (ii) Show that a function $f : \mathbb{R} \to \mathbb{R}$ is measurable iff it is $\mathfrak{T}$-continuous at almost every $x \in \mathbb{R}$. ($\mathfrak{T}$ is the **density topology** on $\mathbb{R}$. See 414P in Volume 4.)

(c) Show that if $f : [a, b] \to \mathbb{R}$ is bounded and continuous for the density topology on $\mathbb{R}$, then $f(x) = \frac{d}{dx}\int_a^x f$ for every $x \in]a, b[$.

(d) Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous for the density topology at $x \in \mathbb{R}$ iff $\lim_{h\downarrow 0} \frac{1}{2h}\mu^*\{y : y \in [x-h, x+h], |f(y) - f(x)| \ge \epsilon\} = 0$ for every $\epsilon > 0$.

(e) A set $A \subseteq \mathbb{R}$ is **porous** at a point $x \in \mathbb{R}$ if $\limsup_{y\to x} \frac{\rho(y,A)}{|y-x|} > 0$, where $\rho(y, A) = \inf_{a\in A} |y - a|$. (Take $\rho(y, \emptyset) = \infty$.) Show that if A is porous at every $x \in A$ then A is negligible.

(f) For a measurable set $E \subseteq \mathbb{R}$ write int*E for the set of its density points. Show that if E, $F \subseteq \mathbb{R}$ are measurable then (i) $\text{int}^*(E \cap F) = \text{int}^*E \cap \text{int}^*F$ (ii) $\text{int}^*E \subseteq \text{int}^*F$ iff $\mu(E \setminus F) = 0$ (iii) $\mu(E \triangle \text{int}^*E) = 0$ (iv) $\text{int}^*(\text{int}^*E) = \text{int}^*E$ (v) for every compact set $K \subseteq \text{int}^*E$ there is a compact set $L \subseteq K \cup E$ such that $K \subseteq \text{int}^*L$.

(g) Let f be an integrable real-valued function defined almost everywhere in $\mathbb{R}$, and x any point of the Lebesgue set of f. Show that for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x)\int g - \int f \times g| \le \epsilon \int g$ whenever $g : \mathbb{R} \to [0, \infty[$ is such that g is non-decreasing on $]-\infty, x]$, non-increasing on $[x, \infty[$ and zero outside $[x - \delta, x + \delta]$. (*Hint*: express g as a limit almost everywhere of functions of the form $\frac{g(x)}{n+1}\sum_{i=0}^n \chi\,]a_i, b_i[$, where $x - \delta \le a_0 \le \ldots \le a_n \le x \le b_n \le \ldots \le b_0 \le x + \delta$.)

(h) For each integrable real-valued function f defined almost everywhere in $\mathbb{R}$, let E_f be the Lebesgue set of f. Show that $E_f \cap E_g \subseteq E_{f+g}$, $E_f \subseteq E_{|f|}$ for all integrable f, g.

(i) Let $E \subseteq \mathbb{R}$ be a non-negligible measurable set. Show that 0 belongs to the interior of $E - E = \{x - y : x, y \in E\}$.

223 Notes and comments The results of this section can be thought of as saying that a measurable function is in some sense 'almost continuous'; indeed, 223Yb is an attempt to make this notion precise. For an integrable function we have stronger results, of which the furthest-reaching seems to be 223D/223Ec.

There are r-dimensional versions of all these theorems, using balls centered on x in place of intervals $[x - h, x + h]$; I give these in 261C-261E. A new idea is needed for the r-dimensional version of Lebesgue's density theorem (261C), but the rest of the generalization is straightforward. A less natural, and less important, extension, also in §261, involves functions defined on non-measurable sets (compare 223Xb-223Xe).

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224 Functions of bounded variation

I turn now to the second of the two problems to which this chapter is devoted: the identification of those real functions which are indefinite integrals. I take the opportunity to offer a brief introduction to the theory of functions of bounded variation, which are interesting in themselves and will be important in Chapter 28. I give the basic characterization of these functions as differences of monotonic functions (224D), with a representative sample of their elementary properties.

224A Definition Let f be a real-valued function and D a subset of $\mathbb{R}$. I define $\operatorname{Var}_D(f)$, the **(total) variation of** f **on** D, as follows. If $D \cap \operatorname{dom} f = \emptyset$, $\operatorname{Var}_D(f) = 0$. Otherwise, $\operatorname{Var}_D(f)$ is

$$\sup\{\sum_{i=1}^{n} |f(a_i) - f(a_{i-1})| : a_0, a_1, \ldots, a_n \in D \cap \operatorname{dom} f,\ a_0 \le a_1 \le \ldots \le a_n\},$$

allowing $\operatorname{Var}_D(f) = \infty$. If $\operatorname{Var}_D(f)$ is finite, we say that f is **of bounded variation** on D. If the context seems clear, I may write $\operatorname{Var} f$ for $\operatorname{Var}_{\operatorname{dom} f}(f)$, and say that f is simply '**of bounded variation**' if this is finite.

224B Remarks (a) In the present chapter, we shall virtually exclusively be concerned with the case in which D is a bounded closed interval included in $\operatorname{dom} f$. The general formulation will be useful for some technical questions arising in Chapter 28; but if it makes you more comfortable, you will lose nothing by supposing for the moment that D is an interval.

(b) Clearly

$$\operatorname{Var}_D(f) = \operatorname{Var}_{D \cap \operatorname{dom} f}(f) = \operatorname{Var}(f \restriction D)$$

for all D, f.

224C Proposition (a) If f, g are two real-valued functions and $D \subseteq \mathbb{R}$, then

$$\operatorname{Var}_D(f + g) \le \operatorname{Var}_D(f) + \operatorname{Var}_D(g).$$

(b) If f is a real-valued function, $D \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ then $\operatorname{Var}_D(cf) = |c| \operatorname{Var}_D(f)$.

(c) If f is a real-valued function, $D \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ then

$$\operatorname{Var}_D(f) \ge \operatorname{Var}_{D \cap]-\infty, x]}(f) + \operatorname{Var}_{D \cap [x, \infty[}(f),$$

with equality if $x \in D \cap \operatorname{dom} f$.

(d) If f is a real-valued function and $D \subseteq D' \subseteq \mathbb{R}$ then $\operatorname{Var}_D(f) \le \operatorname{Var}_{D'}(f)$.

(e) If f is a real-valued function and $D \subseteq \mathbb{R}$, then $|f(x) - f(y)| \le \operatorname{Var}_D(f)$ for all x, $y \in D \cap \operatorname{dom} f$; so if f is of bounded variation on D then f is bounded on $D \cap \operatorname{dom} f$ and (if $D \cap \operatorname{dom} f \ne \emptyset$)

$$\sup_{y \in D \cap \operatorname{dom} f} |f(y)| \le |f(x)| + \operatorname{Var}_D(f)$$

for every $x \in D \cap \operatorname{dom} f$.

(f) If f is a monotonic real-valued function and $D \subseteq \mathbb{R}$ meets $\operatorname{dom} f$, then

$$\operatorname{Var}_D(f) = \sup_{x \in D \cap \operatorname{dom} f} f(x) - \inf_{x \in D \cap \operatorname{dom} f} f(x).$$

proof (a) If $D \cap \operatorname{dom}(f + g) = \emptyset$ this is trivial, because $\operatorname{Var}_D(f)$ and $\operatorname{Var}_D(g)$ are surely non-negative. Otherwise, if $a_0 \le \ldots \le a_n$ in $D \cap \operatorname{dom}(f + g)$, then

$$\begin{aligned}\sum_{i=1}^{n} |(f+g)(a_i) - (f+g)(a_{i-1})| &\le \sum_{i=1}^{n} |f(a_i) - f(a_{i-1})| + \sum_{i=1}^{n} |g(a_i) - g(a_{i-1})| \\ &\le \operatorname{Var}_D(f) + \operatorname{Var}_D(g);\end{aligned}$$

as $a_0, \ldots, a_n$ are arbitrary, $\operatorname{Var}_D(f + g) \le \operatorname{Var}_D(f) + \operatorname{Var}_D(g)$.

(b)

$$\sum_{i=1}^{n}|(cf)(a_i)-(cf)(a_{i-1})|=|c|\sum_{i=1}^{n}|f(a_i)-f(a_{i-1})|$$

whenever $a_0\le\ldots\le a_n$ in $D\cap\operatorname{dom}f$.

(c)(i) If either $D\cap\,]-\infty,x]\cap\operatorname{dom}f$ or $D\cap[x,\infty[\,\cap\operatorname{dom}f$ is empty, this is trivial. If $a_0\le\ldots\le a_m$ in $D\cap\,]-\infty,x]\cap\operatorname{dom}f$, $b_0\le\ldots\le b_n$ in $D\cap[x,\infty[\,\cap\operatorname{dom}f$, then

$$\begin{aligned}\sum_{i=1}^{m}|f(a_i)-f(a_{i-1})|+\sum_{j=1}^{n}|f(b_i)-f(b_{i-1})|&\le\sum_{i=1}^{m+n+1}|f(a_i)-f(a_{i-1})|\\&\le\operatorname{Var}_D(f),\end{aligned}$$

if we write $a_i=b_{i-m-1}$ for $m+1\le i\le m+n+1$. So

$$\operatorname{Var}_{D\cap]-\infty,x]}(f)+\operatorname{Var}_{D\cap[x,\infty[}(f)\le\operatorname{Var}_D(f).$$

(ii) Now suppose that $x\in D\cap\operatorname{dom}f$. If $a_0\le\ldots\le a_n$ in $D\cap\operatorname{dom}f$, and $a_0\le x\le a_n$, let k be such that $x\in[a_{k-1},a_k]$; then

$$\begin{aligned}\sum_{i=1}^{n}|f(a_i)-f(a_{i-1})|\le\sum_{i=1}^{k-1}&|f(a_i)-f(a_{i-1})|+|f(x)-f(a_{k-1})|\\&+|f(a_k)-f(x)|+\sum_{i=k+1}^{n}|f(a_i)-f(a_{i-1})|\\\le\operatorname{Var}&_{D\cap]-\infty,x]}(f)+\operatorname{Var}_{D\cap[x,\infty[}(f)\end{aligned}$$

(counting empty sums $\sum_{i=1}^{0}$, $\sum_{i=n+1}^{n}$ as 0). If $x\le a_0$ then $\sum_{i=1}^{n}|f(a_i)-f(a_{i-1})|\le\operatorname{Var}_{D\cap[x,\infty[}(f)$; if $x\ge a_n$ then $\sum_{i=1}^{n}|f(a_i)-f(a_{i-1})|\le\operatorname{Var}_{D\cap]-\infty,x]}(f)$. Thus

$$\sum_{i=1}^{n}|f(a_i)-f(a_{i-1})|\le\operatorname{Var}_{D\cap]-\infty,x]}(f)+\operatorname{Var}_{D\cap[x,\infty[}(f)$$

in all cases; as $a_0,\ldots,a_n$ are arbitrary,

$$\operatorname{Var}_D(f)\le\operatorname{Var}_{D\cap]-\infty,x]}(f)+\operatorname{Var}_{D\cap[x,\infty[}(f).$$

So the two sides are equal.

(d) is trivial.

(e) If x, $y\in D\cap\operatorname{dom}f$ and $x\le y$ then

$$|f(x)-f(y)|=|f(y)-f(x)|\le\operatorname{Var}_D(f)$$

by the definition of Var_D; and the same is true if $y\le x$. So of course $|f(y)|\le|f(x)|+\operatorname{Var}_D(f)$.

(f) If f is non-decreasing, then

$$\begin{aligned}\operatorname{Var}_D(f)&=\sup\{\sum_{i=1}^{n}|f(a_i)-f(a_{i-1})|:a_0,a_1,\ldots,a_n\in D\cap\operatorname{dom}f,\ a_0\le a_1\le\ldots\le a_n\}\\&=\sup\{\sum_{i=1}^{n}f(a_i)-f(a_{i-1}):a_0,a_1,\ldots,a_n\in D\cap\operatorname{dom}f,\ a_0\le a_1\le\ldots\le a_n\}\\&=\sup\{f(b)-f(a):a,\ b\in D\cap\operatorname{dom}f,\ a\le b\}\\&=\sup_{b\in D\cap\operatorname{dom}f}f(b)-\inf_{a\in D\cap\operatorname{dom}f}f(a).\end{aligned}$$

If f is non-increasing then

$$\begin{aligned}\operatorname{Var}_D(f) &= \sup\{\sum_{i=1}^{n}|f(a_i)-f(a_{i-1})| : a_0,a_1,\dots,a_n\in D\cap\operatorname{dom}f,\ a_0\le a_1\le\dots\le a_n\}\\ &= \sup\{\sum_{i=1}^{n}f(a_{i-1})-f(a_i) : a_0,a_1,\dots,a_n\in D\cap\operatorname{dom}f,\ a_0\le a_1\le\dots\le a_n\}\\ &= \sup\{f(a)-f(b) : a,\,b\in D\cap\operatorname{dom}f,\ a\le b\}\\ &= \sup_{a\in D\cap\operatorname{dom}f}f(a)-\inf_{b\in D\cap\operatorname{dom}f}f(b).\end{aligned}$$

224D Theorem For any real-valued function f and any set $D\subseteq\mathbb{R}$, the following are equiveridical:

(i) there are two bounded non-decreasing functions $f_1,\,f_2:\mathbb{R}\to\mathbb{R}$ such that $f=f_1-f_2$ on $D\cap\operatorname{dom}f$;

(ii) f is of bounded variation on D;

(iii) there are bounded non-decreasing functions $f_1,\,f_2:\mathbb{R}\to\mathbb{R}$ such that $f=f_1-f_2$ on $D\cap\operatorname{dom}f$ and $\operatorname{Var}_D(f)=\operatorname{Var}f_1+\operatorname{Var}f_2$.

proof (i)⇒(ii) If $f:\mathbb{R}\to\mathbb{R}$ is bounded and non-decreasing, then $\operatorname{Var}f=\sup_{x\in\mathbb{R}}f(x)-\inf_{x\in\mathbb{R}}f(x)$ is finite. So if f agrees on $D\cap\operatorname{dom}f$ with f_1-f_2 where f_1 and f_2 are bounded and non-decreasing, then

$$\begin{aligned}\operatorname{Var}_D(f)=\operatorname{Var}_{D\cap\operatorname{dom}f}(f)&\le\operatorname{Var}_{D\cap\operatorname{dom}f}(f_1)+\operatorname{Var}_{D\cap\operatorname{dom}f}(f_2)\\ &\le\operatorname{Var}f_1+\operatorname{Var}f_2<\infty,\end{aligned}$$

using (a), (b) and (d) of 224C.

(ii)⇒(iii) Suppose that f is of bounded variation on D. Set $D'=D\cap\operatorname{dom}f$. If $D'=\emptyset$ we can take both f_j to be the zero function, so henceforth suppose that $D'\ne\emptyset$. Write

$$g(x)=\operatorname{Var}_{D\cap]-\infty,x]}(f)$$

for $x\in D'$. Then $g_1=g+f$ and $g_2=g-f$ are both non-decreasing. **P** If $a,\,b\in D'$ and $a\le b$, then

$$g(b)=g(a)+\operatorname{Var}_{D\cap[a,b]}(f)\ge g(a)+|f(b)-f(a)|.$$

So

$$g_1(b)-g_1(a)=g(b)-g(a)+f(b)-f(a),\quad g_2(b)-g_2(a)=g(b)-g(a)-f(b)+f(a)$$

are both non-negative. **Q**

Now there are non-decreasing functions $h_1,\,h_2:\mathbb{R}\to\mathbb{R}$, extending $g_1,\,g_2$ respectively, such that $\operatorname{Var}h_j=\operatorname{Var}g_j$ for both j. **P** f is bounded on D, by 224Ce, and g is bounded just because $\operatorname{Var}_D(f)<\infty$, so that g_j is bounded. Set $c_j=\inf_{x\in D'}g_j(x)$ and

$$h_j(x)=\sup(\{c_j\}\cup\{g_j(y):y\in D',\,y\le x\})$$

for every $x\in\mathbb{R}$; this works. **Q** Observe that for $x\in D'$,

$$h_1(x)+h_2(x)=g_1(x)+g_2(x)=g(x)+f(x)+g(x)-f(x)=2g(x),$$

$$h_1(x)-h_2(x)=2f(x).$$

Now, because g_1 and g_2 are non-decreasing,

$$\sup_{x\in D'}g_1(x)+\sup_{x\in D'}g_2(x)=\sup_{x\in D'}g_1(x)+g_2(x)=2\sup_{x\in D'}g(x),$$

$$\inf_{x\in D'}g_1(x)+\inf_{x\in D'}g_2(x)=\inf_{x\in D'}g_1(x)+g_2(x)=2\inf_{x\in D'}g(x)\ge 0.$$

But this means that

$$\operatorname{Var}h_1+\operatorname{Var}h_2=\operatorname{Var}g_1+\operatorname{Var}g_2=2\operatorname{Var}g\le 2\operatorname{Var}_D(f),$$

using 224Cf three times. So if we set $f_j(x)=\frac{1}{2}h_j(x)$ for $j\in\{1,2\}$ and $x\in\mathbb{R}$, we shall have non-decreasing functions such that

$$f_1(x) - f_2(x) = f(x) \text{ for } x \in D', \quad \operatorname{Var} f_1 + \operatorname{Var} f_2 = \frac{1}{2}\operatorname{Var} h_1 + \frac{1}{2}\operatorname{Var} h_2 \le \operatorname{Var}_D(f).$$

Since we surely also have

$$\operatorname{Var}_D(f) \le \operatorname{Var}_D(f_1) + \operatorname{Var}_D(f_2) \le \operatorname{Var} f_1 + \operatorname{Var} f_2,$$

we see that $\operatorname{Var}_D(f) = \operatorname{Var} f_1 + \operatorname{Var} f_2$, and (iii) is true.

(iii)⇒(i) is trivial.

224E Corollary Let f be a real-valued function and D any subset of $\mathbb{R}$. If f is of bounded variation on D, then

$$\lim_{x\downarrow a} \operatorname{Var}_{D\cap]a,x]}(f) = \lim_{x\uparrow a} \operatorname{Var}_{D\cap[x,a[}(f) = 0$$

for every $a \in \mathbb{R}$, and

$$\lim_{a\to-\infty} \operatorname{Var}_{D\cap]-\infty,a]}(f) = \lim_{a\to\infty} \operatorname{Var}_{D\cap[a,\infty[}(f) = 0.$$

proof (a) Consider first the case in which $D = \operatorname{dom} f = \mathbb{R}$ and f is a bounded non-decreasing function. Then

$$\operatorname{Var}_{D\cap]a,x]}(f) = \sup_{y\in]a,x]} f(x) - f(y) = f(x) - \inf_{y>a} f(y) = f(x) - \lim_{y\downarrow a} f(y),$$

so of course

$$\lim_{x\downarrow a} \operatorname{Var}_{D\cap]a,x]}(f) = \lim_{x\downarrow a} f(x) - \lim_{y\downarrow a} f(y) = 0.$$

In the same way

$$\lim_{x\uparrow a} \operatorname{Var}_{D\cap[x,a[}(f) = \lim_{y\uparrow a} f(y) - \lim_{x\uparrow a} f(x) = 0,$$

$$\lim_{a\to-\infty} \operatorname{Var}_{D\cap]-\infty,a]}(f) = \lim_{a\to-\infty} f(a) - \lim_{y\to-\infty} f(y) = 0,$$

$$\lim_{a\to\infty} \operatorname{Var}_{D\cap[a,\infty[}(f) = \lim_{y\to\infty} f(y) - \lim_{a\to\infty} f(a) = 0.$$

(b) For the general case, define f_1, f_2 from f and D as in 224D. Then for every interval I we have

$$\operatorname{Var}_{D\cap I}(f) \le \operatorname{Var}_I(f_1) + \operatorname{Var}_I(f_2),$$

so the results for f follow from those for f_1 and f_2 as established in part (a) of the proof.

224F Corollary Let f be a real-valued function of bounded variation on $[a,b]$, where $a < b$. If $\operatorname{dom} f$ meets every interval $]a, a+\delta]$ with $\delta > 0$, then

$$\lim_{t\in\operatorname{dom} f, t\downarrow a} f(t)$$

is defined in $\mathbb{R}$. If $\operatorname{dom} f$ meets $[b-\delta, b[$ for every $\delta > 0$, then

$$\lim_{t\in\operatorname{dom} f, t\uparrow b} f(t)$$

is defined in $\mathbb{R}$.

proof Let f_1, $f_2 : \mathbb{R} \to \mathbb{R}$ be non-decreasing functions such that $f = f_1 - f_2$ on $[a,b] \cap \operatorname{dom} f$. Then

$$\lim_{t\in\operatorname{dom} f, t\downarrow a} f(t) = \lim_{t\downarrow a} f_1(t) - \lim_{t\downarrow a} f_2(t) = \inf_{t>a} f_1(t) - \inf_{t>a} f_2(t),$$

$$\lim_{t\in\operatorname{dom} f, t\uparrow b} f(t) = \lim_{t\uparrow b} f_1(t) - \lim_{t\uparrow b} f_2(t) = \sup_{t<b} f_1(t) - \sup_{t<b} f_2(t).$$

224G Corollary Let f, g be real functions and D a subset of $\mathbb{R}$. If f and g are of bounded variation on D, so is $f \times g$.

proof (a) The point is that there are *non-negative* bounded non-decreasing functions f_1, $f_2 : \mathbb{R} \to \mathbb{R}$ such that $f = f_1 - f_2$ on $D \cap \operatorname{dom} f$. **P** We know that there are bounded non-decreasing h_1, h_2 such that $f = h_1 - h_2$ on $D \cap \operatorname{dom} f$. Set $\gamma_i = \inf_{x\in\mathbb{R}} h_i(x)$ for $i = 1, 2$,

$$\beta_1 = \max(\gamma_1 - \gamma_2, 0), \quad \beta_2 = \max(\gamma_2 - \gamma_1, 0),$$

$$f_1 = h_1 - \gamma_1 + \beta_1, \quad f_2 = h_1 - \gamma_2 + \beta_2;$$

this works. **Q**

(b) Now taking similar functions g_1, g_2 such that $g = g_1 - g_2$ on $D \cap \operatorname{dom} g$, we have

$$f \times g = f_1 \times g_1 - f_2 \times g_1 - f_1 \times g_2 + f_2 \times g_2$$

everywhere in $D \cap \operatorname{dom}(f \times g) = D \cap \operatorname{dom} f \cap \operatorname{dom} g$; but all the $f_i \times g_j$ are bounded non-decreasing functions, so of bounded variation, and $f \times g$ must be of bounded variation on D.

224H Proposition Let $f : D \to \mathbb{R}$ be a function of bounded variation, where $D \subseteq \mathbb{R}$. Then f is continuous at all except countably many points of D.

proof For $n \geq 1$ set

$$A_n = \{x : x \in D, \text{ for every } \delta > 0 \text{ there is a } y \in D \cap [x - \delta, x + \delta]$$
$$\text{such that } |f(y) - f(x)| \geq \frac{1}{n}\}.$$

Then $\#(A_n) \leq n \operatorname{Var} f$. **P?** Otherwise, we can find distinct $x_0, \dots, x_k \in A_n$ with $k + 1 > n \operatorname{Var} f$. Order these so that $x_0 < x_1 < \dots < x_k$. Set $\delta = \frac{1}{2} \min_{1 \leq i \leq k} x_i - x_{i-1} > 0$. For each i, there is a $y_i \in D \cap [x_i - \delta, x_i + \delta]$ such that $|f(y_i) - f(x_i)| \geq \frac{1}{n}$. Take x'_i, y'_i to be x_i, y_i in order, so that $x'_i < y'_i$. Now

$$x'_0 \leq y'_0 \leq x'_1 \leq y'_1 \leq \dots \leq x'_k \leq y'_k,$$

and

$$\operatorname{Var} f \geq \sum_{i=0}^{k} |f(y'_i) - f(x'_i)| = \sum_{i=0}^{k} |f(y_i) - f(x_i)| \geq \frac{1}{n}(k+1) > \operatorname{Var} f,$$

which is impossible. **XQ**

It follows that $A = \bigcup_{n \in \mathbb{N}} A_n$ is countable, being a countable union of finite sets. But A is exactly the set of points of D at which f is not continuous.

224I Theorem Let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$ a function of bounded variation. Then f is differentiable almost everywhere in I, and f' is integrable over I, with

$$\int_I |f'| \leq \operatorname{Var}_I(f).$$

proof (a) Let f_1 and f_2 be non-decreasing functions such that $f = f_1 - f_2$ everywhere in I (224D). Then f_1 and f_2 are differentiable almost everywhere (222A). At any point of I except possibly its endpoints, if any, f will be differentiable if f_1 and f_2 are, so $f'(x)$ is defined for almost every $x \in I$.

(b) Set $F(x) = \operatorname{Var}_{I \cap]-\infty, x]} f$ for $x \in \mathbb{R}$. If x, $y \in I$ and $x \leq y$, then

$$F(y) - F(x) = \operatorname{Var}_{[x,y]} f \geq |f(y) - f(x)|,$$

by 224Cc; so $F'(x) \geq |f'(x)|$ whenever x is an interior point of I and both derivatives exist, which is almost everywhere. So $\int_I |f'| \leq \int_I F'$. But if a, $b \in I$ and $a \leq b$,

$$\int_a^b F' \leq F(b) - F(a) \leq F(b) \leq \operatorname{Var} f.$$

Now I is expressible as $\bigcup_{n \in \mathbb{N}} [a_n, b_n]$ where $a_{n+1} \leq a_n \leq b_n \leq b_{n+1}$ for every n. So

$$\begin{aligned} \int_I |f'| \leq \int_I F' &= \int F' \times \chi I \\ &= \int \sup_{n \in \mathbb{N}} F' \times \chi[a_n, b_n] = \sup_{n \in \mathbb{N}} \int F' \times \chi[a_n, b_n] \end{aligned}$$

(by B.Levi's theorem)

$$= \sup_{n\in\mathbb{N}} \int_{a_n}^{b_n} F' \le \mathrm{Var}_I(f).$$

224J The next result is not needed in this chapter, but is one of the most useful properties of functions of bounded variation, and will be used repeatedly in Chapter 28.

Proposition Let f, g be real-valued functions defined on subsets of $\mathbb{R}$, and suppose that g is integrable over an interval $[a,b]$, where $a<b$, and f is of bounded variation on $]a,b[$ and defined almost everywhere in $]a,b[$. Then $f\times g$ is integrable over $[a,b]$, and

$$|\int_a^b f\times g| \le \big(\lim_{x\in\mathrm{dom}\, f, x\uparrow b} |f(x)| + \mathrm{Var}_{]a,b[}(f)\big) \sup_{c\in[a,b]} |\int_a^c g|.$$

proof (a) By 224F, $l=\lim_{x\in\mathrm{dom}\, f,x\uparrow b} f(x)$ is defined. Write $M=|l|+\mathrm{Var}_{]a,b[}(f)$. Note that if y is any point of $\mathrm{dom}\, f\cap\,]a,b[$,

$$|f(y)| \le |f(x)|+|f(x)-f(y)| \le |f(x)|+\mathrm{Var}_{]a,b[}(f) \to M$$

as $x\uparrow b$ in $\mathrm{dom}\, f$, so $|f(y)|\le M$. Moreover, f is measurable on $]a,b[$, because there are bounded monotonic functions f_1, $f_2:\mathbb{R}\to\mathbb{R}$ such that $f=f_1-f_2$ everywhere in $]a,b[\,\cap\,\mathrm{dom}\, f$. So $f\times g$ is measurable and dominated by $M|g|$, and is integrable over $]a,b[$ or $[a,b]$.

(b) For $n\in\mathbb{N}$, $k\le 2^n$ set $a_{nk}=a+2^{-n}k(b-a)$, and for $1\le k\le 2^n$ choose $x_{nk}\in\mathrm{dom}\, f\cap\,]a_{n,k-1},a_{nk}]$. Define $f_n:\,]a,b]\to\mathbb{R}$ by setting $f_n(x)=f(x_{nk})$ if $1\le k\le 2^n$ and $x\in\,]a_{n,k-1},a_{nk}]$. Then $f(x)=\lim_{n\to\infty} f_n(x)$ whenever $x\in\,]a,b[\,\cap\,\mathrm{dom}\, f$ and f is continuous at x, which must be almost everywhere (224H). Note next that all the f_n are measurable, and that they are uniformly bounded, in modulus, by M. So $\{f_n\times g:n\in\mathbb{N}\}$ is dominated by the integrable function $M|g|$, and Lebesgue's Dominated Convergence Theorem tells us that

$$\int_a^b f\times g = \lim_{n\to\infty}\int_a^b f_n\times g.$$

(c) Fix $n\in\mathbb{N}$ for the moment. Set $K=\sup_{c\in[a,b]}|\int_a^c g|$. (Note that K is finite because $c\mapsto\int_a^c g$ is continuous.) Then

$$\begin{aligned}
|\int_a^b f_n\times g| &= |\sum_{k=1}^{2^n}\int_{a_{n,k-1}}^{a_{nk}} f_n\times g| \\
&= |\sum_{k=1}^{2^n} f(x_{nk})(\int_a^{a_{nk}} g-\int_a^{a_{n,k-1}} g)| \\
&= |\sum_{k=1}^{2^n-1}(f(x_{nk})-f(x_{n,k+1}))\int_a^{a_{nk}} g + f(x_{n,2^n})\int_a^b g| \\
&\le |f(x_{n,2^n})||\int_a^b g| + \sum_{k=1}^{2^n-1}|f(x_{n,k+1})-f(x_{nk})||\int_a^{a_{nk}} g| \\
&\le (|f(x_{n,2^n})|+\mathrm{Var}_{]a,b[}(f))K \to MK
\end{aligned}$$

as $n\to\infty$.

(d) Now

$$|\int_a^b f\times g| = \lim_{n\to\infty}|\int_a^b f_n\times g| \le MK,$$

as required.

224K Complex-valued functions So far I have taken all functions to be real-valued. This is adequate for the needs of the present chapter, but in Chapter 28 we shall need to look at complex-valued functions of bounded variation, and I should perhaps spell out the (elementary) adaptations involved in the extension to the complex case.

(a) Let D be a subset of $\mathbb{R}$ and f a complex-valued function. The **variation** of f on D, $\mathrm{Var}_D(f)$, is zero if $D \cap \mathrm{dom}\, f = \emptyset$, and otherwise is

$$\sup\{\sum_{j=1}^{n} |f(a_j) - f(a_{j-1})| : a_0 \le a_1 \le \ldots \le a_n \text{ in } D \cap \mathrm{dom}\, f\},$$

allowing ∞. If $\mathrm{Var}_D(f)$ is finite, we say that f is **of bounded variation** on D.

(b) Just as in the real case, a complex-valued function of bounded variation must be bounded, and

$$\mathrm{Var}_D(f+g) \le \mathrm{Var}_D(f) + \mathrm{Var}_D(g),$$

$$\mathrm{Var}_D(cf) = |c| \mathrm{Var}_D(f),$$

$$\mathrm{Var}_D(f) \ge \mathrm{Var}_{D\cap]-\infty,x]}(f) + \mathrm{Var}_{D\cap[x,\infty[}(f)$$

for every $x \in \mathbb{R}$, with equality if $x \in D \cap \mathrm{dom}\, f$,

$$\mathrm{Var}_D(f) \le \mathrm{Var}_{D'}(f) \text{ whenever } D \subseteq D';$$

the arguments of 224C go through unchanged.

(c) A complex-valued function is of bounded variation iff its real and imaginary parts are both of bounded variation (because

$$\max(\mathrm{Var}_D(\mathcal{R}\mathrm{e}\, f), \mathrm{Var}_D(\mathcal{I}\mathrm{m}\, f)) \le \mathrm{Var}_D(f) \le \mathrm{Var}_D(\mathcal{R}\mathrm{e}\, f) + \mathrm{Var}_D(\mathcal{I}\mathrm{m}\, f).)$$

So a complex-valued function f is of bounded variation on D iff there are bounded non-decreasing functions $f_1, \ldots, f_4 : \mathbb{R} \to \mathbb{R}$ such that $f = f_1 - f_2 + if_3 - if_4$ on D (224D).

(d) Let f be a complex-valued function and D any subset of $\mathbb{R}$. If f is of bounded variation on D, then

$$\lim_{x\downarrow a} \mathrm{Var}_{D\cap]a,x]}(f) = \lim_{x\uparrow a} \mathrm{Var}_{D\cap[x,a[}(f) = 0$$

for every $a \in \mathbb{R}$, and

$$\lim_{a\to-\infty} \mathrm{Var}_{D\cap]-\infty,a]}(f) = \lim_{a\to\infty} \mathrm{Var}_{D\cap[a,\infty[}(f) = 0.$$

(Apply 224E to the real and imaginary parts of f.)

(e) Let f be a complex-valued function of bounded variation on $[a,b]$, where $a < b$. If $\mathrm{dom}\, f$ meets every interval $]a, a+\delta]$ with $\delta > 0$, then $\lim_{t\in\mathrm{dom}\, f, t\downarrow a} f(t)$ is defined in $\mathbb{C}$. If $\mathrm{dom}\, f$ meets $[b-\delta, b[$ for every $\delta > 0$, then $\lim_{t\in\mathrm{dom}\, f, t\uparrow b} f(t)$ is defined in $\mathbb{C}$. (Apply 224F to the real and imaginary parts of f.)

(f) Let f, g be complex functions and D a subset of $\mathbb{R}$. If f and g are of bounded variation on D, so is $f \times g$. (For $f \times g$ is expressible as a linear combination of the four products $\mathcal{R}\mathrm{e}\, f \times \mathcal{R}\mathrm{e}\, g, \ldots, \mathcal{I}\mathrm{m}\, f \times \mathcal{I}\mathrm{m}\, g$, to each of which we can apply 224G.)

(g) Let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{C}$ a function of bounded variation. Then f is differentiable almost everywhere in I, and $\int_I |f'| \le \mathrm{Var}_I(f)$. (As 224I.)

(h) Let f and g be complex-valued functions defined on subsets of $\mathbb{R}$, and suppose that g is integrable over an interval $[a,b]$, where $a < b$, and f is of bounded variation on $]a,b[$ and defined almost everywhere in $]a,b[$. Then $f \times g$ is integrable over $[a,b]$, and

$$\Bigl|\int_a^b f \times g\Bigr| \le \bigl(\lim_{x\in\mathrm{dom}\, f, x\uparrow b} |f(x)| + \mathrm{Var}_{]a,b[}(f)\bigr) \sup_{c\in[a,b]} \Bigl|\int_a^c g\Bigr|.$$

(The argument of 224J applies virtually unchanged.)

224X Basic exercises >(a) Set $f(x) = x^2 \sin \frac{1}{x^2}$ for $x \neq 0$, $f(0) = 0$. Show that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and uniformly continuous, but is not of bounded variation on any non-trivial interval containing 0.

(b) Give an example of a non-negative function $g : [0,1] \to [0,1]$, of bounded variation, such that $\sqrt{g}$ is not of bounded variation.

(c) Show that if f is any real-valued function defined on a subset of $\mathbb{R}$, there is a function $\tilde{f} : \mathbb{R} \to \mathbb{R}$, extending f, such that $\operatorname{Var} \tilde{f} = \operatorname{Var} f$. Under what circumstances is $\tilde{f}$ unique?

(d) Let $f : D \to \mathbb{R}$ be a function of bounded variation, where $D \subseteq \mathbb{R}$ is a non-empty set. Show that if $\inf_{x \in D} |f(x)| > 0$ then $1/f$ is of bounded variation.

(e) Let $f : [a,b] \to \mathbb{R}$ be a continuous function, where $a \le b$ in $\mathbb{R}$. Show that if $c < \operatorname{Var} f$ then there is a $\delta > 0$ such that $\sum_{i=1}^n |f(a_i) - f(a_{i-1})| \ge c$ whenever $a = a_0 \le a_1 \le \ldots \le a_n = b$ and $\max_{1 \le i \le n} a_i - a_{i-1} \le \delta$.

(f) Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence of real functions, and set $f(x) = \lim_{n \to \infty} f_n(x)$ whenever the limit is defined. Show that $\operatorname{Var} f \le \liminf_{n \to \infty} \operatorname{Var} f_n$.

(g) Let f be a real-valued function which is integrable over an interval $[a,b] \subseteq \mathbb{R}$. Set $F(x) = \int_a^x f$ for $x \in [a,b]$. Show that $\operatorname{Var} F = \int_a^b |f|$. (*Hint*: start by checking that $\operatorname{Var} F \le \int |f|$; for the reverse inequality, consider the case $f \ge 0$ first.)

(h) Show that if f is a real-valued function defined on a non-empty set $D \subseteq \mathbb{R}$, then

$$\operatorname{Var} f = \sup\{|\sum_{i=1}^n (-1)^i (f(a_i) - f(a_{i-1}))| : a_0 \le a_1 \le \ldots \le a_n \text{ in } D\}.$$

(i) Let f be a real-valued function which is integrable over a bounded interval $[a,b] \subseteq \mathbb{R}$. Show that

$$\int_a^b |f| = \sup\{|\sum_{i=1}^n (-1)^i \int_{a_{i-1}}^{a_i} f| : a = a_0 \le a_1 \le a_2 \le \ldots \le a_n = b\}.$$

(*Hint*: put 224Xg and 224Xh together.)

(j) Let f and g be real-valued functions defined on subsets of $\mathbb{R}$, and suppose that g is integrable over an interval $[a,b]$, where $a < b$, and f is of bounded variation on $]a,b[$ and defined almost everywhere in $]a,b[$. Show that

$$|\int_a^b f \times g| \le (\lim_{x \in \operatorname{dom} f, x \downarrow a} |f(x)| + \operatorname{Var}_{]a,b[}(f)) \sup_{c \in [a,b]} |\int_c^b g|.$$

(k) Suppose that $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function. Show that f is expressible as a difference of non-decreasing functions iff $\operatorname{Var}_{D \cap [a,b]}(f)$ is finite whenever $a \le b$ in D.

(l) Suppose that $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is a continuous function of bounded variation. Show that f is expressible as the difference of two continuous non-decreasing functions.

(m) Suppose that $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is a function of bounded variation which is continuous on the right, that is, whenever $x \in D$ and $\epsilon > 0$ there is a $\delta > 0$ such that $|f(t) - f(x)| \le \epsilon$ for every $t \in D \cap [x, x + \delta]$. Show that f is expressible as the difference of two non-decreasing functions which are continuous on the right.

224Y Further exercises (a) Show that if f is any complex-valued function defined on a subset of $\mathbb{R}$, there is a function $\tilde{f} : \mathbb{R} \to \mathbb{C}$, extending f, such that $\operatorname{Var} \tilde{f} = \operatorname{Var} f$. Under what circumstances is $\tilde{f}$ unique?

(b) Let D be any non-empty subset of $\mathbb{R}$, and let $\mathcal{V}$ be the space of functions $f : D \to \mathbb{R}$ of bounded variation. For $f \in \mathcal{V}$ set

$$\|f\| = \sup\{|f(t_0)| + \sum_{i=1}^n |f(t_i) - f(t_{i-1})| : t_0 \le \ldots \le t_n \in D\}.$$

Show that (i) $\| \|$ is a norm on $\mathcal{V}$ (ii) $\mathcal{V}$ is complete under $\| \|$ (iii) $\|f \times g\| \le \|f\| \|g\|$ for all f, $g \in \mathcal{V}$, so that $\mathcal{V}$ is a Banach algebra.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Show that there is a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of differentiable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$, $\lim_{n\to\infty} \int |f_n - f| = 0$, and $\operatorname{Var}(f_n) \le \operatorname{Var}(f)$ for every $n \in \mathbb{N}$. (*Hint*: start with non-decreasing f.)

(d) For any partially ordered set X and any function $f : X \to \mathbb{R}$, say that $\operatorname{Var}_X(f) = 0$ if $X = \emptyset$ and otherwise

$$\operatorname{Var}_X(f) = \sup\{\sum_{i=1}^n |f(a_i) - f(a_{i-1})| : a_0, a_1, \dots, a_n \in X,\ a_0 \le a_1 \le \dots \le a_n\}.$$

State and prove results in this framework generalizing 224D and 224Yb. (*Hints*: f will be 'non-decreasing' if $f(x) \le f(y)$ whenever $x \le y$; interpret $]-\infty, x]$ as $\{y : y \le x\}$.)

(e) Let (X, ρ) be a metric space and $f : [a, b] \to X$ a function, where $a \le b$ in $\mathbb{R}$. Set $\operatorname{Var}_{[a,b]}(f) = \sup\{\sum_{i=1}^n \rho(f(a_i), f(a_{i-1})) : a \le a_0 \le \dots \le a_n \le b\}$. (i) Show that $\operatorname{Var}_{[a,b]}(f) = \operatorname{Var}_{[a,c]}(f) + \operatorname{Var}_{[c,b]}(f)$ for every $c \in [a, b]$. (ii) Show that if $\operatorname{Var}_{[a,b]}(f)$ is finite then f is continuous at all but countably many points of $[a, b]$. (iii) Show that if X is complete and $\operatorname{Var}_{[a,b]}(f) < \infty$ then $\lim_{t\uparrow x} f(t)$ is defined for every $x \in]a, b]$. (iv) Show that if X is complete then $\operatorname{Var}_{[a,b]}(f)$ is finite iff f is expressible as a composition gh, where $h : [a, b] \to \mathbb{R}$ is non-decreasing and $g : \mathbb{R} \to X$ is 1-Lipschitz, that is, $\rho(g(c), g(d)) \le |c - d|$ for all c, $d \in \mathbb{R}$.

(f) Let U be a normed space and $a \le b$ in $\mathbb{R}$. For functions $f : [a, b] \to U$ define $\operatorname{Var}_{[a,b]}(f)$ as in 224Ye, using the standard metric $\rho(x, y) = \|x - y\|$ for x, $y \in U$. (i) Show that $\operatorname{Var}_{[a,b]}(f + g) \le \operatorname{Var}_{[a,b]}(f) + \operatorname{Var}_{[a,b]}(g)$, $\operatorname{Var}_{[a,b]}(cf) = |c| \operatorname{Var}_{[a,b]}(f)$ for all f, $g : [a, b] \to U$ and all $c \in \mathbb{R}$. (ii) Show that if V is another normed space and $T : U \to V$ is a bounded linear operator then $\operatorname{Var}_{[a,b]}(Tf) \le \|T\| \operatorname{Var}_{[a,b]}(f)$ for every $f : [a, b] \to U$.

(g) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. For $y \in \mathbb{R}$ set $h(y) = \#(f^{-1}[\{y\}])$ if this is finite, ∞ otherwise. Show that (if we allow ∞ as a value of the integral) $\operatorname{Var}_{[0,1]}(f) = \int h$. (*Hint*: for $n \in \mathbb{N}$, $i < 2^n$ set $c_{ni} = \sup\{f(x) - f(y) : x, y \in [2^{-n}i, 2^{-n}(i + 1)]\}$, $h_{ni}(y) = 1$ if $y \in f[\,[2^{-n}i, 2^{-n}(i + 1)[\,]$, 0 otherwise. Show that $c_{ni} = \int h_{ni}$, $\lim_{n\to\infty} \sum_{i=0}^{2^n-1} c_{ni} = \operatorname{Var} f$, $\lim_{n\to\infty} \sum_{i=0}^{2^n-1} h_{ni} = h$.) (See also 226Yc.)

(h) Let ν be any Lebesgue-Stieltjes measure on $\mathbb{R}$, $I \subseteq \mathbb{R}$ an interval (which may be either open or closed, bounded or unbounded), and $D \subseteq I$ a non-empty set. Let $\mathcal{V}$ be the space of functions of bounded variation from D to $\mathbb{R}$, and $\| \ \|$ the norm of 224Yb on $\mathcal{V}$. Let $g : D \to \mathbb{R}$ be a function such that $\int_{[a,b]\cap D} g\,d\nu$ exists whenever $a \le b$ in I, and $K = \sup_{a,b\in I, a\le b} |\int_{[a,b]\cap D} g\,d\nu|$. Show that $|\int_D f \times g\,d\nu| \le K\|f\|$ for every $f \in \mathcal{V}$.

(i) Explain how to apply 224Yh with $D = \mathbb{N}$ to obtain Abel's theorem that the product of a monotonic sequence converging to 0 with a series which has bounded partial sums is summable.

(j) Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of sets covering I. Let $f : I \to \mathbb{R}$ be continuous. Show that $\operatorname{Var} f \le \sum_{n=0}^\infty \operatorname{Var}_{A_n} f$. (*Hint*: reduce to the case of closed sets A_n; use Baire's theorem (4A2Ma).)

(k) Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}$. Show that the following are equiveridical: (α) $\lim_{n\to\infty} f(t_n)$ is defined for every montonic sequence $\langle t_n \rangle_{n\in\mathbb{N}}$ in D; (β) for every $\epsilon > 0$ there is a function $g : D \to \mathbb{R}$ of bounded variation such that $|f(t) - g(t)| \le \epsilon$ for every $t \in D$.

224 Notes and comments I have taken the ideas above rather farther than we need immediately; for the present chapter, it is enough to consider the case in which $D = \operatorname{dom} f = [a, b]$ for some interval $[a, b] \subseteq \mathbb{R}$. The extension to functions with irregular domains will be useful in Chapter 28, and the extension to irregular sets D, while not important to us here, is of some interest – for instance, taking $D = \mathbb{N}$, we obtain the notion of 'sequence of bounded variation', which is surely relevant to problems of convergence and summability.

The central result of the section is of course the fact that a function of bounded variation can be expressed as the difference of monotonic functions (224D); indeed, one of the objects of the concept is to characterize the linear span of the monotonic functions. Nearly everything else here can be derived as easy consequences of this, as in 224E-224G. In 224I and 224Xg we go a little deeper, and indeed some measure theory appears;

this is where the ideas here begin to connect with the real business of this chapter, to be continued in the next section. Another result which is easy enough in itself, but contains the germs of important ideas, is 224Yg.

In 224Yb I mention a natural development in functional analysis, and in 224Yd-224Yf I suggest further wide-ranging generalizations.

Version of 16.8.15

225 Absolutely continuous functions

We are now ready for a full characterization of the functions that can appear as indefinite integrals (225E, 225Xf). The essential idea is that of 'absolute continuity' (225B). In the second half of the section (225G-225N) I describe some of the relationships between this concept and those we have already seen.

225A Absolute continuity of the indefinite integral I begin with an easy fundamental result from general measure theory.

Theorem Let (X,Σ,μ) be any measure space and f an integrable real-valued function defined on a conegligible subset of X. Then for any $\epsilon>0$ there are a measurable set E of finite measure and a real number $\delta>0$ such that $\int_F|f|\le\epsilon$ whenever $F\in\Sigma$ and $\mu(F\cap E)\le\delta$.

proof There is a non-decreasing sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ of non-negative simple functions such that $|f| =_{\text{a.e.}} \lim_{n\to\infty}g_n$ and $\int|f|=\lim_{n\to\infty}\int g_n$. Take $n\in\mathbb{N}$ such that $\int g_n\ge\int|f|-\frac{1}{2}\epsilon$. Let $M>0$, $E\in\Sigma$ be such that $\mu E<\infty$ and $g_n\le M\chi E$; set $\delta=\epsilon/2M$. If $F\in\Sigma$ and $\mu(F\cap E)\le\delta$, then

$$\int_F g_n=\int g_n\times\chi F\le M\mu(F\cap E)\le\frac{1}{2}\epsilon;$$

consequently

$$\int_F|f|=\int_F g_n+\int_F|f|-g_n\le\frac{1}{2}\epsilon+\int|f|-g_n\le\epsilon.$$

225B Absolutely continuous functions on $\mathbb{R}$: Definition If $[a,b]$ is a non-empty closed interval in $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ is a function, we say that f is **absolutely continuous** if for every $\epsilon>0$ there is a $\delta>0$ such that $\sum_{i=1}^n|f(b_i)-f(a_i)|\le\epsilon$ whenever $a\le a_1\le b_1\le a_2\le b_2\le\ldots\le a_n\le b_n\le b$ and $\sum_{i=1}^n b_i-a_i\le\delta$.

Remark The phrase 'absolutely continuous' is used in various senses in measure theory, closely related (if you look at them in the right way) but not identical; you will need to keep the context of each definition in clear focus.

225C Proposition Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$.

(a) If $f:[a,b]\to\mathbb{R}$ is absolutely continuous, it is uniformly continuous.

(b) If $f:[a,b]\to\mathbb{R}$ is absolutely continuous it is of bounded variation on $[a,b]$, so is differentiable almost everywhere in $[a,b]$, and its derivative is integrable over $[a,b]$.

(c) If f, $g:[a,b]\to\mathbb{R}$ are absolutely continuous, so are $f+g$ and cf, for every $c\in\mathbb{R}$.

(d) If f, $g:[a,b]\to\mathbb{R}$ are absolutely continuous so is $f\times g$.

(e) If $g:[a,b]\to[c,d]$ and $f:[c,d]\to\mathbb{R}$ are absolutely continuous, and g is non-decreasing, then the composition $fg:[a,b]\to\mathbb{R}$ is absolutely continuous.

proof (a) Let $\epsilon>0$. Then there is a $\delta>0$ such that $\sum_{i=1}^n|f(b_i)-f(a_i)|\le\epsilon$ whenever $a\le a_1\le b_1\le a_2\le b_2\le\ldots\le a_n\le b_n\le b$ and $\sum_{i=1}^n b_i-a_i\le\delta$; but of course now $|f(y)-f(x)|\le\epsilon$ whenever x, $y\in[a,b]$ and $|x-y|\le\delta$. As ϵ is arbitrary, f is uniformly continuous.

(b) Let $\delta>0$ be such that $\sum_{i=1}^n|f(b_i)-f(a_i)|\le 1$ whenever $a\le a_1\le b_1\le a_2\le b_2\le\ldots\le a_n\le b_n\le b$ and $\sum_{i=1}^n b_i-a_i\le\delta$. If $a\le c=c_0\le c_1\le\ldots\le c_n\le d\le\min(b,c+\delta)$, then $\sum_{i=1}^n|f(c_i)-f(c_{i-1})|\le 1$, so

$\mathrm{Var}_{[c,d]}(f) \le 1$; accordingly (inducing on k, using 224Cc for the inductive step) $\mathrm{Var}_{[a,\min(a+k\delta,b)]}(f) \le k$ for every k, and

$$\mathrm{Var}_{[a,b]}(f) \le \lceil (b-a)/\delta \rceil < \infty.$$

It follows that f' is integrable, by 224I.

(c)(i) Let $\epsilon > 0$. Then there are δ_1, $\delta_2 > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \le \frac{1}{2}\epsilon$$

whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta_1$,

$$\sum_{i=1}^{n} |g(b_i) - g(a_i)| \le \frac{1}{2}\epsilon$$

whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta_2$. Set $\delta = \min(\delta_1, \delta_2) > 0$, and suppose that $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta$. Then

$$\sum_{i=1}^{n} |(f+g)(b_i) - (f+g)(a_i)| \le \sum_{i=1}^{n} |f(b_i) - f(a_i)| + \sum_{i=1}^{n} |g(b_i) - g(a_i)| \le \epsilon.$$

As ϵ is arbitrary, $f + g$ is absolutely continuous.

(ii) Let $\epsilon > 0$. Then there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \le \frac{\epsilon}{1+|c|}$$

whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta$. Now

$$\sum_{i=1}^{n} |(cf)(b_i) - (cf)(a_i)| \le \epsilon$$

whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta$. As ϵ is arbitrary, cf is absolutely continuous.

(d) By either (a) or (b), f and g are bounded; set $M = \sup_{x\in[a,b]} |f(x)|$, $M' = \sup_{x\in[a,b]} |g(x)|$. Let $\epsilon > 0$. Then there are δ_1, $\delta_2 > 0$ such that

$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \le \epsilon$ whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta_1$,

$\sum_{i=1}^{n} |g(b_i) - g(a_i)| \le \epsilon$ whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta_2$.

Set $\delta = \min(\delta_1, \delta_2) > 0$ and suppose that $a \le a_1 \le b_1 \le \ldots \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta$. Then

$$\begin{aligned}
\sum_{i=1}^{n} |f(b_i)g(b_i) - f(a_i)g(a_i)| &= \sum_{i=1}^{n} |(f(b_i) - f(a_i))g(b_i) + f(a_i)(g(b_i) - g(a_i))| \\
&\le \sum_{i=1}^{n} |f(b_i) - f(a_i)||g(b_i)| + |f(a_i)||g(b_i) - g(a_i)| \\
&\le \sum_{i=1}^{n} |f(b_i) - f(a_i)|M' + M|g(b_i) - g(a_i)| \\
&\le \epsilon M' + M\epsilon = \epsilon(M + M').
\end{aligned}$$

As ϵ is arbitrary, $f \times g$ is absolutely continuous.

(e) Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\sum_{i=1}^{n} |f(d_i) - f(c_i)| \le \epsilon$ whenever $c \le c_1 \le d_1 \le \ldots \le c_n \le d_n \le d$ and $\sum_{i=1}^{n} d_i - c_i \le \delta$; and there is an $\eta > 0$ such that $\sum_{i=1}^{n} |g(b_i) - g(a_i)| \le \delta$ whenever $a \le a_1 \le b_1 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \eta$. Now suppose that $a \le a_1 \le b_1 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \eta$. Because g is non-decreasing, we have $c \le g(a_1) \le \ldots \le g(b_n) \le d$ and $\sum_{i=1}^{n} g(b_i) - g(a_i) \le \delta$, so $\sum_{i=1}^{n} |f(g(b_i)) - f(g(a_i))| \le \epsilon$; as ϵ is arbitrary, fg is absolutely continuous.

225D Lemma Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ an absolutely continuous function which has zero derivative almost everywhere in $[a,b]$. Then f is constant on $[a,b]$.

proof Let $x \in [a,b]$, $\epsilon > 0$. Let $\delta > 0$ be such that $\sum_{i=1}^{n} |f(b_i) - f(a_i)| \le \epsilon$ whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta$. Set $A = \{t : a < t < x,\ f'(t) \text{ exists } = 0\}$; then $\mu A = x - a$, writing μ for Lebesgue measure. Let $\mathcal{I}$ be the set of non-empty non-singleton closed intervals $[c,d] \subseteq [a,x]$ such that $|f(d) - f(c)| \le \epsilon(d-c)$; then every member of A belongs to arbitrarily short members of $\mathcal{I}$. By Vitali's theorem (221A), there is a countable disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\mu(A \setminus \bigcup \mathcal{I}_0) = 0$, that is,

$$x - a = \mu(\bigcup \mathcal{I}_0) = \sum_{I \in \mathcal{I}_0} \mu I.$$

Now there is a finite $\mathcal{I}_1 \subseteq \mathcal{I}_0$ such that

$$\mu(\bigcup \mathcal{I}_1) = \sum_{I \in \mathcal{I}_1} \mu I \ge x - a - \delta.$$

If $\mathcal{I}_1 = \emptyset$, then $x \le a + \delta$ and $|f(x) - f(a)| \le \epsilon$. Otherwise, express $\mathcal{I}_1$ as $\{[c_0, d_0], \ldots, [c_n, d_n]\}$, where $a \le c_0 < d_0 < c_1 < d_1 < \ldots < c_n < d_n \le x$. Then

$$(c_0 - a) + \sum_{i=1}^{n} (c_i - d_{i-1}) + (x - d_n) = \mu([a,x] \setminus \bigcup \mathcal{I}_1) \le \delta,$$

so

$$|f(c_0) - f(a)| + \sum_{i=1}^{n} |f(c_i) - f(d_{i-1})| + |f(x) - f(d_n)| \le \epsilon.$$

On the other hand, $|f(d_i) - f(c_i)| \le \epsilon(d_i - c_i)$ for each i, so

$$\sum_{i=0}^{n} |f(d_i) - f(c_i)| \le \epsilon \sum_{i=0}^{n} d_i - c_i \le \epsilon(x - a).$$

Putting these together,

$$\begin{aligned}
|f(x) - f(a)| &\le |f(c_0) - f(a)| + |f(d_0) - f(c_0)| + |f(c_1) - f(d_0)| + \ldots \\
&\qquad\qquad + |f(d_n) - f(c_n)| + |f(x) - f(d_n)| \\
&= |f(c_0) - f(a)| + \sum_{i=1}^{n} |f(c_i) - f(d_{i-1})| \\
&\qquad\qquad + |f(x) - f(d_n)| + \sum_{i=0}^{n} |f(d_i) - f(c_i)| \\
&\le \epsilon + \epsilon(x - a) = \epsilon(1 + x - a).
\end{aligned}$$

As ϵ is arbitrary, $f(x) = f(a)$. As x is arbitrary, f is constant.

225E Theorem Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$ and $F : [a,b] \to \mathbb{R}$ a function. Then the following are equiveridical:

(i) there is an integrable real-valued function f such that $F(x) = F(a) + \int_a^x f$ for every $x \in [a,b]$;

(ii) $\int_a^x F'$ exists and is equal to $F(x) - F(a)$ for every $x \in [a,b]$;

(iii) F is absolutely continuous.

Remark Here, and for the rest of the section (except in 225Oa), integrals will be taken with respect to Lebesgue measure on $\mathbb{R}$.

proof (i)$\Rightarrow$(iii) Assume (i). Let $\epsilon > 0$. By 225A, there is a $\delta > 0$ such that $\int_H |f| \le \epsilon$ whenever $H \subseteq [a,b]$ and $\mu H \le \delta$, writing μ for Lebesgue measure as usual. Now suppose that $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^{n} b_i - a_i \le \delta$. Consider $H = \bigcup_{1 \le i \le n} [a_i, b_i[$. Then $\mu H \le \delta$ and

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| = \sum_{i=1}^{n} |\int_{[a_i,b_i[} f| \le \sum_{i=1}^{n} \int_{[a_i,b_i[} |f| = \int_F |f| \le \epsilon.$$

As ϵ is arbitrary, F is absolutely continuous.

(iii)$\Rightarrow$(ii) If F is absolutely continuous, then it is of bounded variation (225Cb), so $\int_a^b F'$ exists (224I). Set $G(x) = \int_a^x F'$ for $x \in [a,b]$; then $G' =_{\text{a.e.}} F'$ (222E) and G is absolutely continuous (by (i)$\Rightarrow$(iii) just proved). Accordingly $G - F$ is absolutely continuous (225Cc) and is differentiable, with zero derivative, almost everywhere. It follows that $G - F$ must be constant (225D). But as $G(a) = 0$, $G = F - F(a)$; just as required by (ii).

(ii)$\Rightarrow$(i) is trivial.

225F Integration by parts As an application of this result, I give a justification of a familiar formula.

Theorem Let f be a real-valued function which is integrable over an interval $[a,b] \subseteq \mathbb{R}$, and $g : [a,b] \to \mathbb{R}$ an absolutely continuous function. Suppose that F is an indefinite integral of f, so that $F(x) - F(a) = \int_a^x f$ for $x \in [a,b]$. Then

$$\int_a^b f \times g = F(b)g(b) - F(a)g(a) - \int_a^b F \times g'.$$

proof Set $h = F \times g$. Because F is absolutely continuous (225E), so is h (225Cd). Consequently $h(b) - h(a) = \int_a^b h'$, by (iii)$\Rightarrow$(ii) of 225E. But $h' = F' \times g + F \times g'$ wherever F' and g' are defined, which is almost everywhere, and $F' =_{\text{a.e.}} f$, by 222E again; so $h' =_{\text{a.e.}} f \times g + F \times g'$. Finally, g and F are continuous, therefore measurable, and bounded, while f and g' are integrable (using 225E yet again), so $f \times g$ and $F \times g'$ are integrable, and

$$F(b)g(b) - F(a)g(a) = h(b) - h(a) = \int_a^b h' = \int_a^b f \times g + \int_a^b F \times g',$$

as required.

225G I come now to a group of results at a rather deeper level than most of the work of this chapter, being closer to the ideas of Chapter 26.

Proposition Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ an absolutely continuous function.

(a) $f[A]$ is negligible for every negligible set $A \subseteq \mathbb{R}$.

(b) $f[E]$ is measurable for every measurable set $E \subseteq \mathbb{R}$.

proof (a) Let $\epsilon > 0$. Then there is a $\delta > 0$ such that $\sum_{i=1}^n |f(b_i) - f(a_i)| \le \epsilon$ whenever $a \le a_1 \le b_1 \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^n b_i - a_i \le \delta$. Now there is a sequence $\langle I_k \rangle_{k\in\mathbb{N}}$ of closed intervals, covering A, with $\sum_{k=0}^\infty \mu I_k \le \delta$. For each $m \in \mathbb{N}$, let F_m be $[a,b] \cap \bigcup_{k\le m} I_k$. Then $\mu f[F_m] \le \epsilon$. **P** F_m must be expressible as $\bigcup_{i\le n}[c_i, d_i]$ where $n \le m$ and $a \le c_0 \le d_0 \le \ldots \le c_n \le d_n \le b$. For each $i \le n$ choose x_i, y_i such that $c_i \le x_i$, $y_i \le d_i$ and

$$f(x_i) = \min_{x\in[c_i,d_i]} f(x), \quad f(y_i) = \max_{x\in[c_i,d_i]} f(x);$$

such exist because f is continuous (225Ca), so is bounded and attains its bounds on $[c_i, d_i]$. Set $a_i = \min(x_i, y_i)$, $b_i = \max(x_i, y_i)$, so that $c_i \le a_i \le b_i \le d_i$. Then

$$\sum_{i=0}^n b_i - a_i \le \sum_{i=0}^n d_i - c_i = \mu F_m \le \mu(\bigcup_{k\in\mathbb{N}} I_k) \le \delta,$$

so

$$\begin{aligned}\mu f[F_m] = \mu(\bigcup_{i\le m} f[\,[c_i,d_i]\,]) &\le \sum_{i=0}^n \mu(f[\,[c_i,d_i]\,]) \\ = \sum_{i=0}^n \mu[f(x_i), f(y_i)] &= \sum_{i=0}^n |f(b_i) - f(a_i)| \le \epsilon. \quad \mathbf{Q}\end{aligned}$$

But $\langle f[F_m]\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence covering $f[A]$, so

$$\mu^* f[A] \le \mu(\bigcup_{m\in\mathbb{N}} f[F_m]) = \sup_{m\in\mathbb{N}} \mu f[F_m] \le \epsilon.$$

As ϵ is arbitrary, $f[A]$ is negligible, as claimed.

(b) By 134Fb, there is a sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ of closed subsets of $E \cap [a,b]$ such that $\lim_{n\to\infty} \mu F_n = \mu(E \cap [a,b])$. For each n, F_n is closed and bounded, therefore compact (2A2F); as f is continuous, $f[F_n]$ is compact (2A2Eb), therefore closed (2A2F, in the other direction) and measurable (114G). Next, setting $A = E \cap [a,b] \setminus \bigcup_{n\in\mathbb{N}} F_n$, A is negligible, so $f[A]$ is negligible, by (a) here, therefore measurable. Consequently

$$f[E] = f[E \cap [a,b]] = f[\bigcup_{n\in\mathbb{N}} F_n \cup A] = \bigcup_{n\in\mathbb{N}} f[F_n] \cup f[A]$$

is measurable, as claimed.

225H Semi-continuous functions In preparation for the last main result of this section, I give a general result concerning measurable real-valued functions on subsets of $\mathbb{R}$. It will be convenient here, for once, to consider functions taking values in $[-\infty, \infty]$. If $D \subseteq \mathbb{R}^r$, a function $g : D \to [-\infty, \infty]$ is **lower semi-continuous** if $\{x : g(x) > u\}$ is an open subset of D (for the subspace topology, see 2A3C) for every $u \in [-\infty, \infty]$. Any lower semi-continuous function is Borel measurable, therefore Lebesgue measurable (121B-121D). Now we have the following result.

225I Proposition Suppose that $r \geq 1$ and that f is a real-valued function, defined on a subset D of $\mathbb{R}^r$, which is integrable over D. Then for any $\epsilon > 0$ there is a lower semi-continuous function $g : \mathbb{R}^r \to [-\infty, \infty]$ such that $g(x) \geq f(x)$ for every $x \in D$ and $\int_D g$ is defined and not greater than $\epsilon + \int_D f$.

Remarks This is a result of great general importance, so I give it in a fairly general form; but for the present chapter all we need is the case $r = 1$, $D = [a, b]$ where $a \leq b$.

proof (a) We can enumerate $\mathbb{Q}$ as $\langle q_n \rangle_{n \in \mathbb{N}}$. By 225A, there is a $\delta > 0$ such that $\int_F |f| \leq \frac{1}{2}\epsilon$ whenever $\mu_D F \leq \delta$, where μ_D is the subspace measure on D, so that $\mu_D F = \mu^* F$, the outer Lebesgue measure of F, for every $F \in \Sigma_D$, the domain of μ_D (214A-214B). For each $n \in \mathbb{N}$, set

$$\delta_n = 2^{-n-1} \min(\tfrac{\epsilon}{1+2|q_n|}, \delta),$$

so that $\sum_{n=0}^{\infty} \delta_n |q_n| \leq \frac{1}{2}\epsilon$ and $\sum_{n=0}^{\infty} \delta_n \leq \delta$. For each $n \in \mathbb{N}$, let $E_n \subseteq \mathbb{R}^r$ be a Lebesgue measurable set such that $\{x : f(x) \geq q_n\} = D \cap E_n$, and choose an open set $G_n \supseteq E_n \cap B(\mathbf{0}, n)$ such that $\mu G_n \leq \mu(E_n \cap B(\mathbf{0}, n)) + \delta_n$ (134Fa), writing $B(\mathbf{0}, n)$ for the ball $\{x : \|x\| \leq n\}$. For $x \in \mathbb{R}^r$, set

$$g(x) = \sup\{q_n : x \in G_n\},$$

allowing $-\infty$ as $\sup \emptyset$ and ∞ as the supremum of a set with no upper bound in $\mathbb{R}$.

(b) Now check the properties of g.

(i) g is lower semi-continuous. **P** If $u \in [-\infty, \infty]$, then

$$\{x : g(x) > u\} = \bigcup\{G_n : q_n > u\}$$

is a union of open sets, therefore open. **Q**

(ii) $g(x) \geq f(x)$ for every $x \in D$. **P** If $x \in D$ and $\eta > 0$, there is an $n \in \mathbb{N}$ such that $\|x\| \leq n$ and $f(x) - \eta \leq q_n \leq f(x)$; now $x \in E_n \subseteq G_n$ so $g(x) \geq q_n \geq f(x) - \eta$. As η is arbitrary, $g(x) \geq f(x)$. **Q**

(iii) Consider the functions h_1, $h_2 : D \to \,]-\infty, \infty]$ defined by setting

$$\begin{aligned} h_1(x) &= |f(x)| \text{ if } x \in D \cap \bigcup_{n \in \mathbb{N}} (G_n \setminus E_n), \\ &= 0 \text{ for other } x \in D, \\ h_2(x) &= \sum_{n=0}^{\infty} |q_n| \chi(G_n \setminus E_n)(x) \text{ for every } x \in D. \end{aligned}$$

Setting $F = \bigcup_{n \in \mathbb{N}} G_n \setminus E_n$,

$$\mu F \leq \textstyle\sum_{n=0}^{\infty} \mu(G_n \setminus E_n) \leq \delta,$$

so

$$\int_D h_1 = \int_{D \cap F} |f| \leq \frac{1}{2}\epsilon$$

by the choice of δ. As for h_2, we have (by B.Levi's theorem)

$$\int_D h_2 = \textstyle\sum_{n=0}^{\infty} |q_n| \mu_D(D \cap G_n \setminus F_n) \leq \sum_{n=0}^{\infty} |q_n| \mu(G_n \setminus F_n) \leq \displaystyle\frac{1}{2}\epsilon$$

– because this is finite, $h_2(x) < \infty$ for almost every $x \in D$. Thus $\int_D h_1 + h_2 \leq \epsilon$.

(iv) The point is that $g \le f + h_1 + h_2$ everywhere in D. **P** Take any $x \in D$. If $n \in \mathbb{N}$ and $x \in G_n$, then either $x \in E_n$, in which case

$$f(x) + h_1(x) + h_2(x) \ge f(x) \ge q_n,$$

or $x \in G_n \setminus E_n$, in which case

$$f(x) + h_1(x) + h_2(x) \ge f(x) + |f(x)| + |q_n| \ge q_n.$$

Thus

$$f(x) + h_1(x) + h_2(x) \ge \sup\{q_n : x \in G_n\} \ge g(x). \quad \mathbf{Q}$$

So $g \le f + h_1 + h_2$ everywhere in D.

(v) Putting (iii) and (iv) together,

$$\int_D g \le \int_D f + h_1 + h_2 \le \epsilon + \int_D f,$$

as required.

225J We need some results on Borel measurable sets and functions which are of independent interest.

Theorem Let D be a subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ any function. Then

$$E = \{x : x \in D,\ f \text{ is continuous at } x\}$$

is relatively Borel measurable in D, and

$$F = \{x : x \in D,\ f \text{ is differentiable at } x\}$$

is Borel measurable in $\mathbb{R}$; moreover, $f' : F \to \mathbb{R}$ is Borel measurable.

proof (a) For $k \in \mathbb{N}$ set

$$\mathcal{G}_k = \{]a, b[: a,\ b \in \mathbb{R},\ |f(x) - f(y)| \le 2^{-k} \text{ for all } x,\ y \in D \cap]a, b[\}.$$

Then $G_k = \bigcup \mathcal{G}_k$ is an open set, so $E_0 = \bigcap_{k\in\mathbb{N}} G_k$ is a Borel set. But $E = D \cap E_0$, so E is a relatively Borel subset of D.

(b)(i) I should perhaps say at once that when interpreting the formula $f'(x) = \lim_{h\to 0}(f(x+h) - f(x))/h$, I insist on the restrictive definition

$$a = \lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h}$$

if

for every $\epsilon > 0$ there is a $\delta > 0$ such that $\frac{f(x+h)-f(x)}{h}$ is defined and

$$|\tfrac{f(x+h)-f(x)}{h} - a| \le \epsilon \text{ whenever } 0 < |h| \le \delta.$$

So $f'(x)$ can be defined only if there is some $\delta > 0$ such that the whole interval $[x - \delta, x + \delta]$ lies within the domain D of f.

(ii) For $p,\ q,\ q' \in \mathbb{Q}$ and $k \in \mathbb{N}$ set

$$\begin{aligned} H(k, p, q, q') &= \{x : x \in E \cap]q, q'[\,,\ |f(y) - f(x) - p(y - x)| \le 2^{-k}|y - x| \text{ for every } y \in]q, q'[\} \\ &\qquad\qquad\qquad\qquad \text{if }]q, q'[\subseteq D \\ &= \emptyset \text{ otherwise.} \end{aligned}$$

Then $H(k, p, q, q') = E \cap]q, q'[\cap \overline{H(k, p, q, q')}$. **P** If $x \in E \cap]q, q'[\cap \overline{H(k, p, q, q')}$ there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $H(k, p, q, q')$ converging to x. Because f is continuous at x,

$$\begin{aligned} |f(y) - f(x) - p(y - x)| &= \lim_{n\to\infty} |f(y) - f(x_n) - p(y - x_n)| \\ &\le 2^{-k} \lim_{n\to\infty} 2^{-k}|y - x_n| = 2^{-k}|y - x| \end{aligned}$$

for every $y \in \,]q,q'[$, so that $x \in H(k,p,q,q')$. **Q** Consequently $H(k,p,q,q')$ is a Borel set. **P** There is a Borel set E_0 such that $E = E_0 \cap D$, by (a), so that if $]q,q'[\, \subseteq D$ then

$$H(k,p,q,q') = E \cap \,]q,q'[\, \cap \overline{H(k,p,q,q')} = E_0 \cap \,]q,q'[\, \cap \overline{H(k,p,q,q')}$$

is Borel. Otherwise, of course, $H(k,p,q,q')$ is Borel because it is empty. **Q**

(iii) Now

$$F = \bigcap_{k\in\mathbb{N}} \bigcup_{p,q,q'\in\mathbb{Q}} H(k,p,q,q').$$

P (α) Suppose $x \in F$, that is, $f'(x)$ is defined; say $f'(x) = a$. Take any $k \in \mathbb{N}$. Then there are $p \in \mathbb{Q}$, $\delta > 0$ such that $|p - a| \le 2^{-k-1}$ and $[x-\delta, x+\delta] \subseteq D$ and $|\frac{f(x+h)-f(x)}{h} - a| \le 2^{-k-1}$ whenever $0 < |h| \le \delta$; now take $q \in \mathbb{Q} \cap [x-\delta, x[$, $q' \in \mathbb{Q} \cap \,]x, x+\delta]$ and see that $x \in H(k,p,q,q')$. As x is arbitrary, $F \subseteq \bigcap_{k\in\mathbb{N}} \bigcup_{p,q,q'\in\mathbb{Q}} H(k,p,q,q')$. ($\beta$) If $x \in \bigcap_{k\in\mathbb{N}} \bigcup_{p,q,q'\in\mathbb{Q}} H(k,p,q,q')$, then for each $k \in \mathbb{N}$ choose p_k, q_k, $q'_k \in \mathbb{Q}$ such that $x \in H(k,p_k,q_k,q'_k)$. If $h \ne 0$, $x+h \in \,]q_k, q'_k[$ then $|\frac{f(x+h)-f(x)}{h} - p_k| \le 2^{-k}$. But this means, first, that $|p_k - p_l| \le 2^{-k} + 2^{-l}$ for every k, l (since surely there is some $h \ne 0$ such that $x+h \in \,]q_k, q'_k[\cap]q_l, q'_l[$), so that $\langle p_k\rangle_{k\in\mathbb{N}}$ is a Cauchy sequence, with limit a say; and, second, that $|\frac{f(x+h)-f(x)}{h} - a| \le 2^{-k} + |a - p_k|$ whenever $h \ne 0$ and $x + h \in \,]q_k, q'_k[$, so that $f'(x)$ is defined and equal to a. **Q**

(iv) Because $\mathbb{Q}$ is countable, all the unions $\bigcup_{p,q,q'\in\mathbb{Q}} H(k,p,q,q')$ are Borel sets, so F also is.

(v) Now enumerate $\mathbb{Q}^3$ as $\langle(p_i, q_i, q'_i)\rangle_{i\in\mathbb{N}}$, and set $H'_{ki} = H(k,p_i,q_i,q'_i) \setminus \bigcup_{j<i} H(k,p_j,q_j,q'_j)$ for each k, $i \in \mathbb{N}$. Every H'_{ki} is Borel measurable, $\langle H'_{ki}\rangle_{i\in\mathbb{N}}$ is disjoint, and

$$\bigcup_{i\in\mathbb{N}} H'_{ki} = \bigcup_{i\in\mathbb{N}} H(k,p_i,q_i,q'_i) \supseteq F$$

for each k. Note that $|f'(x) - p| \le 2^{-k}$ whenever $x \in F \cap H(k,p,q,q')$, so if we set $f_k(x) = p_i$ for every $x \in H'_{ki}$ we shall have a Borel measurable function f_k such that $|f(x) - f_k(x)| \le 2^{-k}$ for every $x \in F$. Accordingly $f' = \lim_{k\to\infty} f_k\restriction F$ is Borel measurable.

225K Proposition Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$, and $f : [a,b] \to \mathbb{R}$ a function. Set $F = \{x : x \in \,]a,b[, f'(x) \text{ is defined}\}$. Then f is absolutely continuous iff (i) f is continuous (ii) f' is integrable over F (iii) $f[\,[a,b] \setminus F]$ is negligible.

proof (a) Suppose first that f is absolutely continuous. Then f is surely continuous (225Ca) and f' is integrable over $[a,b]$, therefore over F (225E); also $[a,b] \setminus F$ is negligible, so $f[\,[a,b] \setminus F]$ is negligible, by 225G.

(b) So now suppose that f satisfies the conditions. Set $f^*(x) = |f'(x)|$ for $x \in F$, 0 for $x \in [a,b] \setminus F$. Then $f(b) \le f(a) + \int_a^b f^*$.

P (i) Because F is a Borel set and f' is a Borel measurable function (225J), f^* is measurable. Let $\epsilon > 0$. Let G be an open subset of $\mathbb{R}$ such that $f[\,[a,b] \setminus F] \subseteq G$ and $\mu G \le \epsilon$ (134Fa again). Let $g : \mathbb{R} \to [0,\infty]$ be a lower semi-continuous function such that $f^*(x) \le g(x)$ for every $x \in [a,b]$ and $\int_a^b g \le \int_a^b f^* + \epsilon$ (225I). Consider

$$A = \{x : a \le x \le b,\ \mu([f(a), f(x)] \setminus G) \le 2\epsilon(x-a) + \int_a^x g\},$$

interpreting $[f(a), f(x)]$ as $\emptyset$ if $f(x) < f(a)$. Then $a \in A \subseteq [a,b]$, so $c = \sup A$ is defined and belongs to $[a,b]$.

Because f is continuous, the function $x \mapsto \mu([f(a), f(x)] \setminus G)$ is continuous; also $x \mapsto 2\epsilon(x-a) + \int_a^x g$ is certainly continuous, so $c \in A$.

(ii) ? If $c \in F$, so that $f^*(c) = |f'(c)|$, then there is a $\delta > 0$ such that

$$a \le c - \delta \le c + \delta \le b,$$

$$g(x) \ge g(c) - \epsilon \ge |f'(c)| - \epsilon \text{ whenever } |x - c| \le \delta,$$

$$|\frac{f(x)-f(c)}{x-c} - f'(c)| \le \epsilon \text{ whenever } |x - c| \le \delta.$$

Consider $x = c + \delta$. Then $c < x \le b$ and

$$\begin{aligned}\mu([f(a), f(x)] \setminus G) &\le \mu([f(a), f(c)] \setminus G) + |f(x) - f(c)| \\ &\le 2\epsilon(c-a) + \int_a^c g + \epsilon(x-c) + |f'(c)|(x-c) \\ &\le 2\epsilon(c-a) + \int_a^c g + \epsilon(x-c) + \int_c^x (g+\epsilon)\end{aligned}$$

(because $g(t) \ge |f'(c)| - \epsilon$ whenever $c \le t \le x$)

$$= 2\epsilon(x-a) + \int_a^x g,$$

so $x \in A$; but c is supposed to be an upper bound of A. **X**

Thus $c \in [a,b] \setminus F$.

(iii) ? Now suppose, if possible, that $c < b$. We know that $f(c) \in G$, so there is an $\eta > 0$ such that $[f(c) - \eta, f(c) + \eta] \subseteq G$; now there is a $\delta > 0$ such that $|f(x) - f(c)| \le \eta$ whenever $x \in [a,b]$ and $|x - c| \le \delta$. Set $x = \min(c + \delta, b)$; then $c < x \le b$ and $[f(c), f(x)] \subseteq G$, so

$$\mu([f(a), f(x)] \setminus G) = \mu([f(a), f(c)] \setminus G) \le 2\epsilon(c-a) + \int_a^c g \le 2\epsilon(x-a) + \int_a^x g$$

and once again $x \in A$, even though $x > \sup A$. **X**

(iv) We conclude that $c = b$, so that $b \in A$. But this means that

$$\begin{aligned}f(b) - f(a) &\le \mu([f(a), f(b)]) \le \mu([f(a), f(b)] \setminus G) + \mu G \\ &\le 2\epsilon(b-a) + \int_a^b g + \epsilon \le 2\epsilon(b-a) + \int_a^b f^* + \epsilon + \epsilon \\ &= 2\epsilon(1+b-a) + \int_a^b f^*.\end{aligned}$$

As ϵ is arbitrary, $f(b) - f(a) \le \int_a^b f^*$, as claimed. **Q**

(c) Similarly, or applying (b) to $-f$, $f(a) - f(b) \le \int_a^b f^*$, so that $|f(b) - f(a)| \le \int_a^b f^*$.

Of course the argument applies equally to any subinterval of $[a,b]$, so $|f(d) - f(c)| \le \int_c^d f^*$ whenever $a \le c \le d \le b$. Now let $\epsilon > 0$. By 225A once more, there is a $\delta > 0$ such that $\int_E f^* \le \epsilon$ whenever $E \subseteq [a,b]$ and $\mu E \le \delta$. Suppose that $a \le a_1 \le b_1 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^n b_i - a_i \le \delta$. Then

$$\sum_{i=1}^n |f(b_i) - f(a_i)| \le \sum_{i=1}^n \int_{a_i}^{b_i} f^* = \int_{\bigcup_{i \le n}[a_i, b_i]} f^* \le \epsilon.$$

So f is absolutely continuous, as claimed.

225L Corollary Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$. Let $f : [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on the open interval $]a,b[$. If its derivative f' is integrable over $[a,b]$, then f is absolutely continuous, and $f(b) - f(a) = \int_a^b f'$.

proof $f[[a,b] \setminus F] = \{f(a), f(b)\}$ is surely negligible, so f is absolutely continuous, by 225K; consequently $f(b) - f(a) = \int_a^b f'$, by 225E.

225M Corollary Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$, and $f : [a,b] \to \mathbb{R}$ a continuous function. Then f is absolutely continuous iff it is continuous and of bounded variation and $f[A]$ is negligible for every negligible $A \subseteq [a,b]$.

proof (a) Suppose that f is absolutely continuous. By 225C(a-b) it is continuous and of bounded variation, and by 225G we have $f[A]$ negligible for every negligible $A \subseteq [a,b]$.

(b) So now suppose that f satisfies the conditions. Set $F = \{x : x \in]a,b[\,,\ f'(x) \text{ is defined}\}$. By 224I once more, $[a,b] \setminus F$ is negligible, so $f[[a,b] \setminus F]$ is negligible. Moreover, also by 224I, f' is integrable over $[a,b]$ or F. So the conditions of 225K are satisfied and f is absolutely continuous.

225N The Cantor function I should mention the standard example of a continuous function of bounded variation which is not absolutely continuous. Let $C \subseteq [0,1]$ be the Cantor set (134G). Recall that the 'Cantor function' is a non-decreasing continuous function $f : [0,1] \to [0,1]$ such that $f'(x)$ is defined and equal to zero for every $x \in [0,1] \setminus C$, but $f(0) = 0 < 1 = f(1)$ (134H). Of course f is of bounded variation and not absolutely continuous. C is negligible and $f[C] = [0,1]$ is not. If $x \in C$, then for every $n \in \mathbb{N}$ there is an interval of length 3^{-n}, containing x, on which f increases by 2^{-n}; so f cannot be differentiable at x, and the set $F = \operatorname{dom} f'$ of 225K is precisely $[0,1] \setminus C$, so that $f[\,[0,1] \setminus F] = [0,1]$.

225O Complex-valued functions As usual, I spell out the results above in the forms applicable to complex-valued functions.

(a) Let (X, Σ, μ) be any measure space and f an integrable complex-valued function defined on a conegligible subset of X. Then for any $\epsilon > 0$ there are a measurable set E of finite measure and a real number $\delta > 0$ such that $\int_F |f| \le \epsilon$ whenever $F \in \Sigma$ and $\mu(F \cap E) \le \delta$. (Apply 225A to $|f|$.)

(b) If $[a,b]$ is a non-empty closed interval in $\mathbb{R}$ and $f : [a,b] \to \mathbb{C}$ is a function, we say that f is **absolutely continuous** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\sum_{i=1}^n |f(b_i) - f(a_i)| \le \epsilon$ whenever $a \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n \le b$ and $\sum_{i=1}^n b_i - a_i \le \delta$. Observe that f is absolutely continuous iff its real and imaginary parts are both absolutely continuous.

(c) Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$.

(i) If $f : [a,b] \to \mathbb{C}$ is absolutely continuous it is of bounded variation on $[a,b]$, so is differentiable almost everywhere in $[a,b]$, and its derivative is integrable over $[a,b]$.

(ii) If f, $g : [a,b] \to \mathbb{C}$ are absolutely continuous, so are $f + g$ and ζf, for any $\zeta \in \mathbb{C}$, and $f \times g$.

(iii) If $g : [a,b] \to [c,d]$ is monotonic and absolutely continuous, and $f : [c,d] \to \mathbb{C}$ is absolutely continuous, then $fg : [a,b] \to \mathbb{C}$ is absolutely continuous.

(d) Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$ and $F : [a,b] \to \mathbb{C}$ a function. Then the following are equiveridical:

(i) there is an integrable complex-valued function f such that $F(x) = F(a) + \int_a^x f$ for every $x \in [a,b]$;

(ii) $\int_a^x F'$ exists and is equal to $F(x) - F(a)$ for every $x \in [a,b]$;

(iii) F is absolutely continuous.

(Apply 225E to the real and imaginary parts of F.)

(e) Let f be an integrable complex-valued function on an interval $[a,b] \subseteq \mathbb{R}$, and $g : [a,b] \to \mathbb{C}$ an absolutely continuous function. Set $F(x) = \int_a^x f$ for $x \in [a,b]$. Then

$$\int_a^b f \times g = F(b)g(b) - F(a)g(a) - \int_a^b F \times g'.$$

(Apply 225F to the real and imaginary parts of f and g.)

(f) Let f be a continuous complex-valued function on a closed interval $[a,b] \subseteq \mathbb{R}$, and suppose that f is differentiable at every point of the open interval $]a,b[$, with f' integrable over $[a,b]$. Then f is absolutely continuous. (Apply 225L to the real and imaginary parts of f.)

(g) For a result corresponding to 225M, see 264Yp.

225X Basic exercises (a) Show directly from the definition in 225B (without appealing to 225E) that any absolutely continuous real-valued function on a closed interval $[a,b]$ is expressible as the difference of non-decreasing absolutely continuous functions.

(b) Show directly from the definition in 225B and the Mean Value Theorem (without appealing to 225K) that if a function f is continuous on a closed interval $[a,b]$, differentiable on the open interval $]a,b[$, and has bounded derivative in $]a,b[$, then f is absolutely continuous, so that $f(x) = f(a) + \int_a^x f'$ for every $x \in [a,b]$.

(c) Show that if $f : [a,b] \to \mathbb{R}$ is absolutely continuous, then $\operatorname{Var} f = \int_a^b |f'|$. (*Hint*: put 224I and 225E together.)

(d) Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function which is absolutely continuous on every bounded interval; let μ_g be the associated Lebesgue-Stieltjes measure (114Xa), and Σ_g its domain. Show that $\int_E g' = \mu_g E$ for any $E \in \Sigma_g$, if we allow ∞ as a value of the integral. (*Hint*: start with intervals E.)

(e) Let $g : [a,b] \to \mathbb{R}$ be a non-decreasing absolutely continuous function, and $f : [g(a), g(b)] \to \mathbb{R}$ a continuous function. Show that $\int_{g(a)}^{g(b)} f(t)dt = \int_a^b f(g(t))g'(t)dt$. (*Hint*: set $F(x) = \int_{g(a)}^x f$, $G = Fg$ and consider $\int_a^b G'(t)dt$. See also 263J.)

(f) Suppose that $I \subseteq \mathbb{R}$ is any non-trivial interval (bounded or unbounded, open, closed or half-open, but not empty or a singleton), and $f : I \to \mathbb{R}$ a function. Show that f is absolutely continuous on every closed bounded subinterval of I iff there is a function g such that $\int_a^b g = f(b) - f(a)$ whenever $a \le b$ in I, and in this case g is integrable iff f is of bounded variation on I.

(g) Show that $\int_0^1 \frac{\ln x}{x-1} dx = \sum_{n=1}^\infty \frac{1}{n^2}$. (*Hint*: use 225F to find $\int_0^1 x^n \ln x\, dx$, and recall that $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$ for $0 \le x < 1$.)

(h)(i) Show that $\int_0^1 t^a dt$ is finite for every $a > -1$. (ii) Show that $\int_1^\infty t^a e^{-t} dt$ is finite for every $a \in \mathbb{R}$. (*Hint*: show that there is an M such that $t^a \le Me^{t/2}$ for $t \ge 1$.) (iii) Show that $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$ is defined for every $a > 0$. (iv) Show that $\Gamma(a+1) = a\Gamma(a)$ for every $a > 0$. (v) Show that $\Gamma(n+1) = n!$ for every $n \in \mathbb{N}$.

(Γ is of course the **gamma function**.)

(i) Show that if $b > 0$ then $\int_0^\infty u^{b-1} e^{-u^2/2} du = 2^{(b-2)/2}\Gamma(\frac{b}{2})$. (*Hint*: consider $f(t) = t^{(b-2)/2} e^{-t}$, $g(u) = u^2/2$ in 225Xe.)

(j) Suppose that f, g are lower semi-continuous functions, defined on subsets of $\mathbb{R}^r$, and taking values in $]-\infty, \infty]$. (i) Show that $f+g$, $f \wedge g$ and $f \vee g$ are lower semi-continuous, and that αf is lower semi-continuous for every $\alpha \ge 0$. (ii) Show that if f and g are non-negative, then $f \times g$ is lower semi-continuous. (iii) Show that if f is non-negative and g is continuous, then $f \times g$ is lower semi-continuous. (iv) Show that if f is non-decreasing then the composition fg is lower semi-continuous.

(k) Let A be a non-empty family of lower semi-continuous functions defined on subsets of $\mathbb{R}^r$ and taking values in $[-\infty, \infty]$. Set $g(x) = \sup\{f(x) : f \in A, x \in \operatorname{dom} f\}$ for $x \in D = \bigcup_{f \in A} \operatorname{dom} f$. Show that g is lower semi-continuous.

(l) Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function, where $a \le b$. (i) Show that $|f| : [a,b] \to \mathbb{R}$ is absolutely continuous. (ii) Show that gf is absolutely continuous whenever $g : f[\,[a,b]\,] \to \mathbb{R}$ is absolutely continuous and g' is bounded. (iii) Show that if $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $\inf_{x \in [a,b]} |g(x)| > 0$ then f/g is absolutely continuous.

(m) Suppose that $f : [a,b] \to \mathbb{R}$ is continuous, and differentiable at all but countably many points of $[a,b]$. Show that f is absolutely continuous iff it is of bounded variation.

(n) Let $f : [0, \infty[\to \mathbb{C}$ be a function which is absolutely continuous on $[0,a]$ for every $a \in [0, \infty[$ and has Laplace transform $F(s) = \int_0^\infty e^{-sx} f(x) dx$ defined on $\{s : \mathcal{R}\mathrm{e}\, s > S\}$. Suppose also that $\lim_{x \to \infty} e^{-Sx} f(x) = 0$. Show that f' has Laplace transform $sF(s) - f(0)$ defined whenever $\mathcal{R}\mathrm{e}\, s > S$. (*Hint*: show that

$$f(x)e^{-sx} - f(0) = \int_0^x \frac{d}{dt}(f(t)e^{-st}) dt$$

for every $x \ge 0$.)

225Y Further exercises (a) Show that the composition of two absolutely continuous functions need not be absolutely continuous. (*Hint*: 224Xb.)

(b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, where $a < b$. Set $G = \{x : x \in \,]a, b[\,, \exists\, y \in \,]x, b]$ such that $f(x) < f(y)\}$. Show that G is open and is expressible as a disjoint union of intervals $]c, d[$ where $f(c) \le f(d)$. Use this to prove 225D without calling on Vitali's theorem.

(c) Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation and $\gamma > 0$. Show that there is an absolutely continuous function $g : [a, b] \to \mathbb{R}$ such that $|g'(x)| \le \gamma$ wherever the derivative is defined and $\{x : x \in [a, b],\ f(x) \ne g(x)\}$ has measure at most $\frac{1}{\gamma}\operatorname{Var} f$. (*Hint*: reduce to the case of non-decreasing f. Apply 225Yb to the function $x \mapsto f(x) - \gamma x$ and show that $\gamma\mu G \le \operatorname{Var} f$. Set $g(x) = f(x)$ for $x \in \,]a, b[\, \setminus G$.)

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is absolutely continuous on every bounded interval. Show that $\operatorname{Var} f \le \frac{1}{2}\operatorname{Var} f' + \int |f|$.

(e) Let f be a non-negative measurable real-valued function defined on a subset D of $\mathbb{R}^r$, where $r \ge 1$. Show that for any $\epsilon > 0$ there is a lower semi-continuous function $g : \mathbb{R}^r \to [-\infty, \infty]$ such that $g(x) \ge f(x)$ for every $x \in D$ and $\int_D g - f \le \epsilon$.

(f) Let f be a measurable real-valued function defined on a subset D of $\mathbb{R}^r$, where $r \ge 1$. Show that for any $\epsilon > 0$ there is a lower semi-continuous function $g : \mathbb{R}^r \to [-\infty, \infty]$ such that $g(x) \ge f(x)$ for every $x \in D$ and $\mu^*\{x : x \in D,\ g(x) > f(x)\} \le \epsilon$. (*Hint*: 134Yd, 134Fb.)

(g)(i) Show that if f is a Lebesgue measurable real function then all its Dini derivates are Lebesgue measurable. (ii) Show that if f is a Borel measurable real function then all its Dini derivates are Borel measurable.

225 Notes and comments There is a good deal more to say about absolutely continuous functions; I will return to the topic in the next section and in Chapter 26. I shall rarely make direct use of the results from 225H on in their full strengths, but it seems to me that this kind of investigation is necessary for any clear picture of the relationships between such concepts as absolute continuity and bounded variation. Of course, in order to apply these results, we do need a store of simple kinds of absolutely continuous function, differentiable functions with bounded derivative forming the most important class (225Xb). A larger family of the same kind is the class of 'Lipschitz' functions (262Bc).

The definition of 'absolutely continuous function' is ordinarily set out for closed bounded intervals, as in 225B. The point is that for other intervals the simplest generalizations of this formulation do not seem quite appropriate. In 225Xf I try to suggest the kind of demands one might make on functions defined on other types of interval.

I should remark that the real prize is still not quite within our grasp. I have been able to give a reasonably satisfactory formulation of simple integration by parts (225F), at least for bounded intervals – a further limiting process is necessary to deal with unbounded intervals. But a companion method from advanced calculus, integration by substitution, remains elusive. The best I think we can do at this point is 225Xe, which insists on a continuous integrand f. It is the case that the result is valid for general integrable f, but there are some further subtleties to be mastered on the way; the necessary ideas are given in the much more general results 235A and 263D below, and applied to the one-dimensional case in 263J.

On the way to the characterization of absolutely continuous functions in 225K, I find myself calling on one of the fundamental relationships between Lebesgue measure and the topology of $\mathbb{R}^r$ (225I). The technique here can be adapted to give many variations of the result; see 225Ye-225Yf. If you have not seen semi-continuous functions before, 225Xj-225Xk give a partial idea of their properties. In 225J I give a fragment of 'descriptive set theory', the study of the kinds of set which can arise from the formulae of analysis. These ideas too will re-surface elsewhere (compare 225Yg and also the proof of 262M below) and will be of great importance in Volumes 4 and 5.

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226 The Lebesgue decomposition of a function of bounded variation

I end this chapter with some notes on a method of analysing a general function of bounded variation which may help to give a picture of what such functions can be, though (apart from 226A) it is hardly needed in this volume.

226A Sums over arbitrary index sets To get a full picture of this fragment of real analysis, a bit of preparation will be helpful. This concerns the notion of a sum over an arbitrary index set, which I have rather been skirting around so far.

(a) If I is any set and $\langle a_i\rangle_{i\in I}$ any family in $[0,\infty]$, we set

$$\sum_{i\in I} a_i = \sup\{\sum_{i\in K} a_i : K \text{ is a finite subset of } I\},$$

with the convention that $\sum_{i\in\emptyset} a_i = 0$. (See 112Bd, 222Ba.) For general $a_i \in [-\infty,\infty]$, we can set

$$\sum_{i\in I} a_i = \sum_{i\in I} a_i^+ - \sum_{i\in I} a_i^-$$

if this is defined in $[-\infty,\infty]$, that is, at least one of $\sum_{i\in I} a_i^+$, $\sum_{i\in I} a_i^-$ is finite, where $a^+ = \max(a,0)$ and $a^- = \max(-a,0)$ for each a. If $\sum_{i\in I} a_i$ is defined and finite, we say that $\langle a_i\rangle_{i\in I}$ is **summable**.

(b) Since this is a book on measure theory, I will immediately describe the relationship between this kind of summability and an appropriate notion of integration. For any set I, we have the corresponding 'counting measure' μ on I (112Bd). Every subset of I is measurable, so every family $\langle a_i\rangle_{i\in I}$ of real numbers is a measurable real-valued function on I. A subset of I has finite measure iff it is finite; so a real-valued function f on I is 'simple' if $K = \{i : f(i) \ne 0\}$ is finite. In this case,

$$\int f d\mu = \sum_{i\in K} f(i) = \sum_{i\in I} f(i)$$

as defined in part (a). The measure μ is semi-finite (211Nc) so a non-negative function f is integrable iff $\int f = \sup_{\mu K<\infty} \int_K f$ is finite (213B); but of course this supremum is precisely

$$\sup\{\sum_{i\in K} f(i) : K \subseteq I \text{ is finite}\} = \sum_{i\in I} f(i).$$

Now a general function $f : I \to \mathbb{R}$ is integrable iff it is measurable and $\int |f| d\mu < \infty$, that is, iff $\sum_{i\in I} |f(i)| < \infty$, and in this case

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu = \sum_{i\in I} f(i)^+ - \sum_{i\in I} f(i)^- = \sum_{i\in I} f(i),$$

writing $f^\pm(i) = f(i)^\pm$ for each i. Thus we have

$$\sum_{i\in I} a_i = \int_I a_i \mu(di),$$

and the standard rules under which we allow ∞ as the value of an integral (133A, 135F) match well with the interpretations in (a) above.

(c) Accordingly, and unsurprisingly, the operation of summation is a linear operation on the linear space of summable families of real numbers.

I observe here that this notion of summability is 'absolute'; a family $\langle a_i\rangle_{i\in I}$ is summable iff it is absolutely summable. This is necessary because it must also be 'unconditional'; we have no structure on an arbitrary set I to guide us to take the sum in any particular order. See 226Xa. In particular, I distinguish between '$\sum_{n\in\mathbb{N}} a_n$', which in this book will always be interpreted as in 226A above, and '$\sum_{n=0}^{\infty} a_n$' which (if it makes a difference) should be interpreted as $\lim_{m\to\infty} \sum_{n=0}^{m} a_n$. So, for instance, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2$, while $\sum_{n\in\mathbb{N}} \frac{(-1)^n}{n+1}$ is undefined. Of course $\sum_{n=0}^{\infty} a_n = \sum_{n\in\mathbb{N}} a_n$ whenever the latter is defined in $[-\infty,\infty]$.

(d) There is another, and very important, approach to the sum described here. If $\langle a_i\rangle_{i\in I}$ is an (absolutely) summable family of real numbers, then for every $\epsilon > 0$ there is a finite $K \subseteq I$ such that $\sum_{i\in I\setminus K} |a_i| \le \epsilon$. **P** This is nothing but a special case of 225A; there is a set K with $\mu K < \infty$ such that $\int_{I\setminus K} |a_i|\mu(di) \le \epsilon$, but

$$\int_{I\setminus K}|a_i|\mu(di)=\sum_{i\in I\setminus K}|a_i|.\ \mathbf{Q}$$

(Of course there are 'direct' proofs of this result from the definition in (a), not mentioning measures or integrals. But I think you will see that they rely on the same idea as that in the proof of 225A.) Consequently, for any family $\langle a_i\rangle_{i\in I}$ of real numbers and any $s\in\mathbb{R}$, the following are equiveridical:

(i) $\sum_{i\in I}a_i=s$;

(ii) for every $\epsilon>0$ there is a finite $K\subseteq I$ such that $|s-\sum_{i\in J}a_i|\le\epsilon$ whenever J is finite and $K\subseteq J\subseteq I$.

P (i)⇒(ii) Take K such that $\sum_{i\in I\setminus K}|a_i|\le\epsilon$. If $K\subseteq J\subseteq I$, then

$$|s-\sum_{i\in J}a_i|=|\sum_{i\in I\setminus J}a_i|\le\sum_{i\in I\setminus K}|a_i|\le\epsilon.$$

(ii)⇒(i) Let $\epsilon>0$, and let $K\subseteq I$ be as described in (ii). If $J\subseteq I\setminus K$ is any finite set, then set $J_1=\{i:i\in J,\,a_i\ge 0\}$, $J_2=J\setminus J_1$. We have

$$\begin{aligned}\sum_{i\in J}|a_i|&=|\sum_{i\in J_1\cup K}a_i-\sum_{i\in J_2\cup K}a_i|\\&\le|s-\sum_{i\in J_1\cup K}a_i|+|s-\sum_{i\in J_2\cup K}a_i|\le 2\epsilon.\end{aligned}$$

As J is arbitrary, $\sum_{i\in I\setminus K}|a_i|\le 2\epsilon$ and

$$\sum_{i\in I}|a_i|\le\sum_{i\in K}|a_i|+2\epsilon<\infty.$$

Accordingly $\sum_{i\in I}a_i$ is well-defined in $\mathbb{R}$. Also

$$|s-\sum_{i\in I}a_i|\le|s-\sum_{i\in K}a_i|+|\sum_{i\in I\setminus K}a_i|\le\epsilon+\sum_{i\in I\setminus K}|a_i|\le 3\epsilon.$$

As ϵ is arbitrary, $\sum_{i\in I}a_i=s$, as required. **Q**

In this way, we express $\sum_{i\in I}a_i$ directly as a limit; we could write it as

$$\sum_{i\in I}a_i=\lim_{K\uparrow I}\sum_{i\in K}a_i,$$

on the understanding that we look at finite sets K in the right-hand formula.

(e) Yet another approach is through the following fact. If $\sum_{i\in I}|a_i|<\infty$, then for any $\delta>0$ the set $\{i:|a_i|\ge\delta\}$ is finite, indeed can have at most $\frac{1}{\delta}\sum_{i\in I}|a_i|$ members; consequently

$$J=\{i:a_i\ne 0\}=\bigcup_{n\in\mathbb{N}}\{i:|a_i|\ge 2^{-n}\}$$

is countable (1A1F). If J is finite, then of course $\sum_{i\in I}a_i=\sum_{i\in J}a_i$ reduces to a finite sum. Otherwise, we can enumerate J as $\langle j_n\rangle_{n\in\mathbb{N}}$, and we shall have

$$\sum_{i\in I}a_i=\sum_{i\in J}a_i=\lim_{n\to\infty}\sum_{k=0}^n a_{j_k}=\sum_{n=0}^\infty a_{j_n}$$

(using (d) to reduce the sum $\sum_{i\in J}a_i$ to a limit of finite sums). Conversely, if $\langle a_i\rangle_{i\in I}$ is such that there is a countably infinite $J\subseteq\{i:a_i\ne 0\}$ enumerated as $\langle j_n\rangle_{n\in\mathbb{N}}$, and if $\sum_{n=0}^\infty|a_{j_n}|<\infty$, then $\sum_{i\in I}a_i$ will be $\sum_{n=0}^\infty a_{j_n}$.

(f) It will be useful later to have a fragment of general theory. Let I and J be sets and $\langle a_{ij}\rangle_{i\in I,j\in J}$ a family in $[0,\infty]$. Then

$$\sum_{(i,j)\in I\times J}a_{ij}=\sum_{i\in I}(\sum_{j\in J}a_{ij})=\sum_{j\in J}(\sum_{i\in I}a_{ij}).$$

P (i) If $\sum_{(i,j)\in I\times J}a_{ij}>u$, then there is a finite set $M\subseteq I\times J$ such that $\sum_{(i,j)\in M}a_{ij}>u$. Now $K=\{i:(i,j)\in M\}$ and $L=\{j:(i,j)\in M\}$ are finite, so

$$\sum_{i\in I}\sum_{j\in J}a_{ij}\ge\sum_{i\in K}\sum_{j\in J}a_{ij}\ge\sum_{i\in K}\sum_{j\in L}a_{ij}$$

(because $\sum_{j\in J}a_{ij}\ge\sum_{j\in L}a_{ij}$ for every i)

$$= \sum_{(i,j)\in K\times L} a_{ij} \ge \sum_{(i,j)\in M} a_{ij} > u.$$

As u is arbitrary, $\sum_{i\in I}\sum_{j\in J} a_{ij} \ge \sum_{(i,j)\in I\times J} a_{ij}$. (ii) If $\sum_{i\in I}\sum_{j\in J} a_{ij} > u \ge 0$, there is a non-empty finite set $K \subseteq I$ such that $\sum_{i\in K}\sum_{j\in J} a_{ij} > u$. Let $\epsilon \in\,]0,1[$ be such that $\sum_{i\in K}\sum_{j\in J} a_{ij} > u+\epsilon$, and set $\delta = \frac{\epsilon}{\#(K)}$. For each $i \in K$ set $\gamma_i = \min(u+1, \sum_{j\in J} a_{ij}) - \delta$; then

$$\epsilon + \sum_{i\in K}\gamma_i = \sum_{i\in K}\min(u+1,\sum_{j\in J} a_{ij}) \ge \min(u+1, \sum_{i\in K}\sum_{j\in J} a_{ij}) > u+\epsilon,$$

so $\sum_{i\in K}\gamma_i > u$. For each $i \in K$, $\gamma_i < \sum_{j\in J} a_{ij}$, so there is a finite $L_i \subseteq J$ such that $\sum_{j\in L_i} a_{ij} \ge \gamma_i$. Set $M = \{(i,j) : i \in K,\, j \in L_i\}$, so that M is a finite subset of $I \times J$; then

$$\sum_{(i,j)\in I\times J} a_{ij} \ge \sum_{(i,j)\in M} a_{ij} = \sum_{i\in K}\sum_{j\in L_i} a_{ij} \ge \sum_{i\in K}\gamma_i > u.$$

As u is arbitrary, $\sum_{(i,j)\in I\times J} a_{ij} \ge \sum_{i\in I}\sum_{j\in J} a_{ij}$ and these two sums are equal. (iii) Similarly, $\sum_{(i,j)\in I\times J} a_{ij} = \sum_{j\in J}\sum_{i\in I} a_{ij}$. **Q**

226B Saltus functions Now we are ready for a special type of function of bounded variation on $\mathbb{R}$. Suppose that $a < b$ in $\mathbb{R}$.

(a) A (real) **saltus function** on $[a,b]$ is a function $F : [a,b] \to \mathbb{R}$ expressible in the form

$$F(x) = \sum_{t\in[a,x[} u_t + \sum_{t\in[a,x]} v_t$$

for $x \in [a,b]$, where $\langle u_t\rangle_{t\in[a,b[}$, $\langle v_t\rangle_{t\in[a,b]}$ are real-valued families such that $\sum_{t\in[a,b[} |u_t|$ and $\sum_{t\in[a,b]} |v_t|$ are finite.

(b) For any function $F : [a,b] \to \mathbb{R}$ we can write

$$F(x^+) = \lim_{y\downarrow x} F(y) \text{ if } x \in [a,b[\text{ and the limit exists,}$$

$$F(x^-) = \lim_{y\uparrow x} F(y) \text{ if } x \in\,]a,b] \text{ and the limit exists.}$$

(I hope that this will not lead to confusion with the alternative interpretation of x^+ as $\max(x,0)$.) Observe that if F is a saltus function, as defined in (b), with associated families $\langle u_t\rangle_{t\in[a,b[}$ and $\langle v_t\rangle_{t\in[a,b]}$, then $v_a = F(a)$, $v_x = F(x) - F(x^-)$ for $x \in\,]a,b]$ and $u_x = F(x^+) - F(x)$ for $x \in [a,b[$. **P** Let $\epsilon > 0$. As remarked in 226Ad, there is a finite $K \subseteq [a,b]$ such that

$$\sum_{t\in[a,b[\setminus K} |u_t| + \sum_{t\in[a,b]\setminus K} |v_t| \le \epsilon.$$

Given $x \in [a,b]$, let $\delta > 0$ be such that $[x-\delta, x+\delta]$ contains no point of K except perhaps x. In this case, if $\max(a, x-\delta) \le y < x$, we must have

$$\begin{aligned}|F(y) - (F(x) - v_x)| &= |\sum_{t\in[y,x[} u_t + \sum_{t\in]y,x[} v_t| \\ &\le \sum_{t\in[a,b[\setminus K} |u_t| + \sum_{t\in[a,b]\setminus K} |v_t| \le \epsilon,\end{aligned}$$

while if $x < y \le \min(b, x+\delta)$ we shall have

$$\begin{aligned}|F(y) - (F(x) + u_x)| &= |\sum_{t\in]x,y[} u_t + \sum_{t\in]x,y]} v_t| \\ &\le \sum_{t\in[a,b[\setminus K} |u_t| + \sum_{t\in[a,b]\setminus K} |v_t| \le \epsilon.\end{aligned}$$

As ϵ is arbitrary, we get $F(x^-) = F(x) - v_x$ (if $x > a$) and $F(x^+) = F(x) + u_x$ (if $x < b$). **Q**

It follows that F is continuous at $x \in\,]a,b[$ iff $u_x = v_x = 0$, while F is continuous at a iff $u_a = 0$ and F is continuous at b iff $v_b = 0$. In particular, $\{x : x \in [a,b],\ F \text{ is not continuous at } x\}$ is countable (see 226Ae).

(c) If F is a saltus function defined on $[a,b]$, with associated families $\langle u_t\rangle_{t\in[a,b[}$ and $\langle v_t\rangle_{t\in[a,b]}$, then F is of bounded variation on $[a,b]$, and

$$\mathrm{Var}_{[a,b]}(F)\le\sum_{t\in[a,b[}|u_t|+\sum_{t\in]a,b]}|v_t|.$$

P If $a\le x<y\le b$, then

$$F(y)-F(x)=u_x+\sum_{t\in]x,y[}(u_t+v_t)+v_y,$$

so

$$|F(y)-F(x)|\le\sum_{t\in[x,y[}|u_t|+\sum_{t\in]x,y]}|v_t|.$$

If $a\le a_0\le a_1\le\ldots\le a_n\le b$, then

$$\begin{aligned}\sum_{i=1}^n|F(a_i)-F(a_{i-1})|&\le\sum_{i=1}^n\Bigl(\sum_{t\in[a_{i-1},a_i[}|u_t|+\sum_{t\in]a_{i-1},a_i]}|v_t|\Bigr)\\&\le\sum_{t\in[a,b[}|u_t|+\sum_{t\in]a,b]}|v_t|.\end{aligned}$$

Consequently

$$\mathrm{Var}_{[a,b]}(F)\le\sum_{t\in[a,b[}|u_t|+\sum_{t\in]a,b]}|v_t|<\infty.\ \mathbf{Q}$$

(d) The inequality in (c) is actually an equality. To see this, note first that if $a\le x<y\le b$, then $\mathrm{Var}_{[x,y]}(F)\ge|u_x|+|v_y|$. **P** I noted in (b) that $u_x=\lim_{t\downarrow x}F(t)-F(x)$ and $v_y=F(y)-\lim_{t\uparrow y}F(t)$. So, given $\epsilon>0$, we can find t_1, t_2 such that $x<t_1\le t_2<y$ and

$$|F(t_1)-F(x)|\ge|u_x|-\epsilon,\quad |F(y)-F(t_2)|\ge|v_y|-\epsilon.$$

Now

$$\mathrm{Var}_{[x,y]}(F)\ge|F(t_1)-F(x)|+|F(t_2)-F(t_1)|+|F(y)-F(t_2)|\ge|u_x|+|v_y|-2\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

Now, given $a\le t_0<t_1<\ldots<t_n\le b$, we must have

$$\mathrm{Var}_{[a,b]}(F)\ge\sum_{i=1}^n\mathrm{Var}_{[t_{i-1},t_i]}(F)$$

(using 224Cc)

$$\ge\sum_{i=1}^n|u_{t_{i-1}}|+|v_{t_i}|.$$

As $t_0,\ldots,t_n$ are arbitrary,

$$\mathrm{Var}_{[a,b]}(F)\ge\sum_{t\in[a,b[}|u_t|+\sum_{t\in]a,b]}|v_t|,$$

as required.

(e) Because a saltus function is of bounded variation ((c) above), it is differentiable almost everywhere (224I). In fact its derivative is zero almost everywhere. **P** Let $F:[a,b]\to\mathbb{R}$ be a saltus function, with associated families $\langle u_t\rangle_{t\in[a,b[}$, $\langle v_t\rangle_{t\in[a,b]}$. Let $\epsilon>0$. Let $K\subseteq[a,b]$ be a finite set such that

$$\sum_{t\in[a,b[\setminus K}|u_t|+\sum_{t\in[a,b]\setminus K}|v_t|\le\epsilon.$$

Set

$$\begin{aligned} u'_t &= u_t \text{ if } t \in [a,b[\,\cap K, \\ &= 0 \text{ if } t \in [a,b[\,\setminus K, \\ v'_t &= v_t \text{ if } t \in K, \\ &= 0 \text{ if } t \in [a,b] \setminus K, \\ u''_t &= u_t - u'_t \text{ for } t \in [a,b[\,, \\ v''_t &= v_t - v'_t \text{ for } t \in [a,b]. \end{aligned}$$

Let G, H be the saltus functions corresponding to $\langle u'_t\rangle_{t\in[a,b[}$, $\langle v'_t\rangle_{t\in[a,b]}$ and $\langle u''_t\rangle_{t\in[a,b[}$ $\langle v''_t\rangle_{t\in[a,b]}$, so that $F = G + H$. Then $G'(t) = 0$ for every $t \in \,]a,b[\,\setminus K$, since $]a,b[\,\setminus K$ comprises a finite number of open intervals on each of which G is constant. So $G' = 0$ a.e. and $F' =_{\text{a.e.}} H'$. On the other hand,

$$\int_a^b |H'| \le \mathrm{Var}_{[a,b]}(H) = \sum_{t\in[a,b[\setminus K} |u_t| + \sum_{t\in]a,b]\setminus K} |v_t| \le \epsilon,$$

using 224I and (d) above. So

$$\int_a^b |F'| = \int_a^b |H'| \le \epsilon.$$

As ϵ is arbitrary, $\int_a^b |F'| = 0$ and $F' = 0$ a.e., as claimed. **Q**

226C The Lebesgue decomposition of a function of bounded variation Take $a, b \in \mathbb{R}$ with $a < b$.

(a) If $F : [a,b] \to \mathbb{R}$ is non-decreasing, set $v_a = 0$, $v_t = F(t) - F(t^-)$ for $t \in \,]a,b]$, $u_t = F(t^+) - F(t)$ for $t \in [a,b[$, defining $F(t^+)$, $F(t^-)$ as in 226Bb. Then all the v_t, u_t are non-negative, and if $a < t_0 < t_1 < \ldots < t_n < b$, then

$$\sum_{i=0}^n (u_{t_i} + v_{t_i}) = \sum_{i=0}^n (F(t_i^+) - F(t_i^-)) \le F(b) - F(a).$$

Accordingly $\sum_{t\in[a,b[} u_t$ and $\sum_{t\in[a,b]} v_t$ are both finite. Let F_p be the corresponding saltus function, as defined in 226Ba, so that

$$F_p(x) = F(a^+) - F(a) + \sum_{t\in]a,x[}(F(t^+) - F(t^-)) + F(x) - F(x^-)$$

if $a < x \le b$. If $a \le x < y \le b$ then

$$\begin{aligned} F_p(y) - F_p(x) &= F(x^+) - F(x) + \sum_{t\in]x,y[} (F(t^+) - F(t^-)) + F(y) - F(y^-) \\ &\le F(y) - F(x) \end{aligned}$$

because if $x = t_0 < t_1 < \ldots < t_n < t_{n+1} = y$ then

$$\begin{aligned} F(x^+) - F(x) &+ \sum_{i=1}^n (F(t_i^+) - F(t_i^-)) + F(y) - F(y^-) \\ &= F(y) - F(x) - \sum_{i=1}^{n+1} (F(t_i^-) - F(t_{i-1}^+)) \le F(y) - F(x). \end{aligned}$$

Accordingly both F_p and $F_c = F - F_p$ are non-decreasing. Also, because

$$F_p(a) = 0 = v_a,$$

$$F_p(t) - F_p(t^-) = v_t = F(t) - F(t^-) \text{ for } t \in \,]a,b],$$

$$F_p(t^+) - F_p(t) = u_t = F(t^+) - F(t) \text{ for } t \in [a,b[,$$

we shall have

$$F_c(a) = F(a),$$

$$F_c(t) = F_c(t^-) \text{ for } t \in \,]a,b],$$

$$F_c(t) = F_c(t^+) \text{ for } t \in [a,b[,$$

and F_c is continuous.

Clearly this expression of $F = F_p + F_c$ as the sum of a saltus function and a continuous function is unique, except that we can freely add a constant to one if we subtract it from the other.

(b) Still taking $F : [a,b] \to \mathbb{R}$ to be non-decreasing, we know that F' is integrable (222C); moreover, $F' =_{\text{a.e.}} F'_c$, by 226Be. Set $F_{ac}(x) = F(a) + \int_a^x F'$ for each $x \in [a,b]$. We have

$$F_{ac}(y) - F_{ac}(x) = \int_x^y F'_c \le F_c(y) - F_c(x)$$

for $a \le x \le y \le b$ (222C again), so $F_{cs} = F_c - F_{ac}$ is still non-decreasing; F_{ac} is continuous (225A), so F_{cs} is continuous; $F'_{ac} =_{\text{a.e.}} F' =_{\text{a.e.}} F'_c$ (222E), so $F'_{cs} = 0$ a.e.

Again, the expression of $F_c = F_{ac} + F_{cs}$ as the sum of an absolutely continuous function and a function with zero derivative almost everywhere is unique, except for the possibility of moving a constant from one to the other, because two absolutely continuous functions whose derivatives are equal almost everywhere must differ by a constant (225D).

(c) Putting all these together: if $F : [a,b] \to \mathbb{R}$ is any non-decreasing function, it is expressible as $F_p + F_{ac} + F_{cs}$, where F_p is a saltus function, F_{ac} is absolutely continuous, and F_{cs} is continuous and differentiable, with zero derivative, almost everywhere; all three components are non-decreasing; and the expression is unique if we say that $F_{ac}(a) = F(a)$ and $F_p(a) = F_{cs}(a) = 0$.

The Cantor function $f : [0,1] \to [0,1]$ (134H) is continuous and $f' = 0$ a.e. (134Hb), so $f_p = f_{ac} = 0$ and $f = f_{cs}$. Setting $g(x) = \frac{1}{2}(x + f(x))$ for $x \in [0,1]$, as in 134I, we get $g_p(x) = 0$, $g_{ac}(x) = \frac{x}{2}$ and $g_{cs}(x) = \frac{1}{2}f(x)$.

(d) Now suppose that $F : [a,b] \to \mathbb{R}$ is of bounded variation. Then it is expressible as a difference $G - H$ of non-decreasing functions (224D). So writing $F_p = G_p - H_p$, etc., we can express F as a sum $F_p + F_{cs} + F_{ac}$, where F_p is a saltus function, F_{ac} is absolutely continuous, F_{cs} is continuous, $F'_{cs} = 0$ a.e., $F_{ac}(a) = F(a)$ and $F_{cs}(a) = F_p(a) = 0$. Under these conditions the expression is unique, because (for instance) $F_p(t^+) - F_p(t) = F(t^+) - F(t)$ for $t \in [a,b[$, while $F'_{ac} =_{\text{a.e.}} (F - F_p)' =_{\text{a.e.}} F'$.

This is a **Lebesgue decomposition** of the function F. (I have to say 'a' Lebesgue decomposition because of course the assignments $F_{ac}(a) = F(a)$, $F_p(a) = F_{cs}(a) = 0$ are arbitrary.) I will call F_p the **saltus part** of F.

226D Complex functions The modifications needed to deal with complex functions are elementary.

(a) If I is any set and $\langle a_j \rangle_{j \in I}$ is a family of complex numbers, then the following are equiveridical:

(i) $\sum_{j \in I} |a_j| < \infty$;

(ii) there is an $s \in \mathbb{C}$ such that for every $\epsilon > 0$ there is a finite $K \subseteq I$ such that $|s - \sum_{j \in J} a_j| \le \epsilon$ whenever J is finite and $K \subseteq J \subseteq I$.

In this case

$$s = \sum_{j \in I} \mathcal{R}\mathrm{e}(a_j) + i \sum_{j \in I} \mathcal{I}\mathrm{m}(a_j) = \int_I a_j \mu(dj),$$

where μ is counting measure on I, and we write $s = \sum_{j \in I} a_j$.

(b) If $a < b$ in $\mathbb{R}$, a complex **saltus function** on $[a,b]$ is a function $F : [a,b] \to \mathbb{C}$ expressible in the form

$$F(x) = \sum_{t \in [a,x[} u_t + \sum_{t \in [a,x]} v_t$$

for $x \in [a,b]$, where $\langle u_t \rangle_{t \in [a,b[}$, $\langle v_t \rangle_{t \in [a,b]}$ are complex-valued families such that $\sum_{t \in [a,b[} |u_t|$ and $\sum_{t \in [a,b]} |v_t|$ are finite; that is, if the real and imaginary parts of F are saltus functions. In this case F is continuous except at countably many points and differentiable, with zero derivative, almost everywhere in $[a,b]$, and

$$u_x = \lim_{t \downarrow x} F(t) - F(x) \text{ for every } x \in [a,b[,$$

$$v_x = \lim_{t \uparrow x} F(x) - F(t) \text{ for every } x \in]a,b]$$

(apply the results of 226B to the real and imaginary parts of F). F is of bounded variation, and its variation is

$$\mathrm{Var}_{[a,b]}(F) = \sum_{t\in[a,b[} |u_t| + \sum_{t\in]a,b]} |v_t|$$

(repeat the arguments of 226Bc-d).

(c) If $F : [a, b] \to \mathbb{C}$ is a function of bounded variation, where $a < b$ in $\mathbb{R}$, it is uniquely expressible as $F = F_p + F_{cs} + F_{ac}$, where F_p is a saltus function, F_{ac} is absolutely continuous, F_{cs} is continuous and has zero derivative almost everywhere, and $F_{ac}(a) = F(a)$, $F_p(a) = F_{cs}(a) = 0$. (Apply 226C to the real and imaginary parts of F.)

226E As an elementary exercise in the language of 226A, I interpolate a version of a theorem of B.Levi which is sometimes useful.

Proposition Let (X, Σ, μ) be a measure space, I a *countable* set, and $\langle f_i\rangle_{i\in I}$ a family of μ-integrable real- or complex-valued functions such that $\sum_{i\in I} \int |f_i| d\mu$ is finite. Then $f(x) = \sum_{i\in I} f_i(x)$ is defined almost everywhere and $\int f d\mu = \sum_{i\in I} \int f_i d\mu$.

proof If I is finite this is elementary. Otherwise, since there must be a bijection between I and $\mathbb{N}$, we may take it that $I = \mathbb{N}$. Setting $g_n = \sum_{i=0}^{n} |f_i|$ for each n, we have a non-decreasing sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ of integrable functions such that $\int g_n \le \sum_{i\in\mathbb{N}} \int |f_i|$ for every n, so that $g = \sup_{n\in\mathbb{N}} g_n$ is integrable, by B.Levi's theorem as stated in 123A. In particular, g is finite almost everywhere. Now if $x \in X$ is such that $g(x)$ is defined and finite, $\sum_{i\in J} |f_i(x)| \le g(x)$ for every finite $J \subseteq \mathbb{N}$, so $\sum_{i\in\mathbb{N}} |f_i(x)|$ and $\sum_{i\in\mathbb{N}} f_i(x)$ are defined. In this case, of course, $\sum_{i\in\mathbb{N}} f_i(x) = \lim_{n\to\infty} \sum_{i=0}^{n} f_i(x)$. But $|\sum_{i=0}^{n} f_i| \le_{\text{a.e.}} g$ for each n, so Lebesgue's Dominated Convergence Theorem tells us that

$$\int \sum_{i\in\mathbb{N}} f_i = \lim_{n\to\infty} \int \sum_{i=0}^{n} f_i = \lim_{n\to\infty} \sum_{i=0}^{n} \int f_i = \sum_{i\in\mathbb{N}} \int f_i.$$

226X Basic exercises >(a) Suppose that I and J are sets and that $\langle a_i\rangle_{i\in I}$ is a summable family of real numbers. (i) Show that if $f : J \to I$ is injective then $\langle a_{f(j)}\rangle_{j\in J}$ is summable. (ii) Show that if $g : I \to J$ is any function, then $\sum_{j\in J} \sum_{i\in g^{-1}[\{j\}]} a_i$ is defined and equal to $\sum_{i\in I} a_i$.

>(b) A **step-function** on an interval $[a, b]$ is a function F such that, for suitable $t_0, \dots, t_n$ with $a = t_0 \le \dots \le t_n = b$, F is constant on each interval $]t_{i-1}, t_i[$. Show that $F : [a, b] \to \mathbb{R}$ is a saltus function iff for every $\epsilon > 0$ there is a step-function $G : [a, b] \to \mathbb{R}$ such that $\mathrm{Var}_{[a,b]}(F - G) \le \epsilon$.

(c) Let F, G be real-valued functions of bounded variation defined on an interval $[a, b] \subseteq \mathbb{R}$. Show that, in the language of 226C,

$$(F + G)_p = F_p + G_p, \quad (F + G)_c = F_c + G_c,$$

$$(F + G)_{cs} = F_{cs} + G_{cs}, \quad (F + G)_{ac} = F_{ac} + G_{ac}.$$

>(d) Let F be a real-valued function of bounded variation on an interval $[a, b] \subseteq \mathbb{R}$. Show that, in the language of 226C,

$$\mathrm{Var}_{[a,b]}(F) = \mathrm{Var}_{[a,b]}(F_p) + \mathrm{Var}_{[a,b]}(F_c) = \mathrm{Var}_{[a,b]}(F_p) + \mathrm{Var}_{[a,b]}(F_{cs}) + \mathrm{Var}_{[a,b]}(F_{ac}).$$

(e) Let F be a real-valued function of bounded variation on an interval $[a, b] \subseteq \mathbb{R}$. Show that F is absolutely continuous iff $\mathrm{Var}_{[a,b]}(F) = \int_a^b |F'|$.

(f) Consider the function g of 134I/226Cc. Show that $g^{-1} : [0, 1] \to [0, 1]$ is differentiable almost everywhere in $[0, 1]$, and find $\mu\{x : (g^{-1})'(x) \le a\}$ for each $a \in \mathbb{R}$.

>(g)(i) Show that a continuous bijection $f : [0, 1] \to [0, 1]$ is either strictly increasing or strictly decreasing, and that its inverse is continuous. (ii) Show that if $f : [0, 1] \to [0, 1]$ is a continuous bijection, then $f' = 0$ a.e. in $[0, 1]$ iff there is a conegligible set $E \subseteq [0, 1]$ such that $f[E]$ is negligible, and that in this case f^{-1} has the same property. (iii) Construct a function satisfying the conditions of (ii). (*Hint*: try $f = \sum_{n=0}^{\infty} 2^{-n-1} f_n$ where each f_n is a variation on the Cantor function.) (iv) Repeat (iii) with $f = f^{-1}$.

226Y Further exercises (a) Show that a set I is countable iff there is a summable family $\langle a_i\rangle_{i\in I}$ of non-zero real numbers.

(b) Explain modifications which might be appropriate in the description of the Lebesgue decomposition of a function of bounded variation if we wish to consider functions on open or half-open intervals, including unbounded intervals.

(c) Suppose that $F:[a,b]\to\mathbb{R}$ is a function of bounded variation, and set $h(y)=\#(F^{-1}[\{y\}])$ for $y\in\mathbb{R}$. Show that $\int h=\mathrm{Var}_{[a,b]}(F_c)$, where F_c is the 'continuous part' of F as defined in 226Ca/226Cd.

(d) Suppose that $a<b$ in $\mathbb{R}$, and that $F:[a,b]\to\mathbb{R}$ is a function of bounded variation; let F_p be its saltus part. Show that $|F(b)-F(a)|\le\mu F[\,[a,b]\,]+\mathrm{Var}_{[a,b]}F_p$, where μ is Lebesgue measure on $\mathbb{R}$.

226 Notes and comments In 232I and 232Yh below I will revisit these ideas, linking them to a decomposition of the Lebesgue-Stieltjes measure corresponding to a non-decreasing real function, and thence to more general measures. All this work is peripheral to the main concerns of this volume, but I think it is illuminating, and certainly it is part of the basic knowledge assumed of anyone working in real analysis.

Version of 17.8.15

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

225Xi This exercise, referred to in the 2002, 2004 and 2012 printings of Volume 3, is now 225Xg.

225Xj This exercise, referred to in the 2003, 2006 and 2013 printings of Volume 4, is now 225Xh.

Version of 17.12.12

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