Introduction to Volume 2

For this second volume I have chosen seven topics through which to explore the insights and challenges offered by measure theory. Some, like the Radon-Nikodým theorem (Chapter 23) are necessary for any understanding of the structure of the subject; others, like Fourier analysis (Chapter 28) and the discussion of function spaces (Chapter 24) demonstrate the power of measure theory to attack problems in general real and functional analysis. But all have applications outside measure theory, and all have influenced its development. These are the parts of measure theory which any analyst may find himself using.

Every topic is one which ideally one would wish undergraduates to have seen, but the length of this volume makes it plain that no ordinary undergraduate course could include very much of it. It is directed rather at graduate level, where I hope it will be found adequate to support all but the most ambitious courses in measure theory, though it is perhaps a bit too solid to be suitable for direct use as a course text, except with careful selection of the parts to be covered. If you are using it to teach yourself measure theory, I strongly recommend an eclectic approach, looking for particular subjects and theorems that seem startling or useful, and working backwards from them. My other objective, of course, is to provide an account of the central ideas at this level in measure theory, rather fuller than can easily be found in one volume elsewhere. I cannot claim that it is 'definitive', but I do think I cover a good deal of ground in ways that provide a firm foundation for further study. As in Volume 1, I usually do not shrink from giving 'best' results, like Lindeberg's condition for the Central Limit Theorem (§274), or the theory of products of arbitrary measure spaces (§251). If I were teaching this material to students in a PhD programme I would rather accept a limitation in the breadth of the course than leave them unaware of what could be done in the areas discussed.

The topics interact in complex ways – one of the purposes of this book is to exhibit their relationships. There is no canonical linear ordering in which they should be taken. Nor do I think organization charts are very helpful, not least because it may be only two or three paragraphs in a section which are needed for a given chapter later on. I do at least try to lay the material of each section out in an order which makes initial segments useful by themselves. But the order in which to take the chapters is to a considerable extent for you to choose, perhaps after a glance at their individual introductions. I have done my best to pitch the exposition at much the same level throughout the volume, sometimes allowing gradients to steepen in the course of a chapter or a section, but always trying to return to something which anyone who has mastered Volume 1 ought to be able to cope with. (Though perhaps the main theorems of Chapter 26 are harder work than the principal results elsewhere, and §286 is only for the most determined.)

I said there were seven topics, and you will see eight chapters ahead of you. This is because Chapter 21 is rather different from the rest. It is the purest of pure measure theory, and is here only because there are places later in the volume where (in my view) the theorems make sense only in the light of some abstract concepts which are not particularly difficult, but are also not obvious. However it is fair to say that the most important ideas of this volume do not really depend on the work of Chapter 21.

As always, it is a puzzle to know how much prior knowledge to assume in this volume. I do of course call on the results of Volume 1 of this treatise whenever they seem to be relevant. I do not doubt, however, that there will be readers who have learnt the elementary theory from other sources. Provided you can, from first principles, construct Lebesgue measure and prove the basic convergence theorems for integrals on arbitrary measure spaces, you ought to be able to embark on the present volume. Perhaps it would be helpful to have in hand the results-only version of Volume 1, since that includes the most important definitions as well as statements of the theorems.

There is also the question of how much material from outside measure theory is needed. Chapter 21 calls for some non-trivial set theory (given in §2A1), but the more advanced ideas are needed only for the counter-examples in §216, and can be passed over to begin with. The problems become acute in Chapter 24. Here we need a variety of results from functional analysis, some of them depending on non-trivial ideas in general topology. For a full understanding of this material there is no substitute for a course in normed

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spaces up to and including a study of weak compactness. But I do not like to insist on such a preparation, because it is likely to be simultaneously too much and too little. Too much, because I hardly mention linear operators at this stage; too little, because I do ask for some of the theory of non-locally-convex spaces, which is often omitted in first courses on functional analysis. At the risk, therefore, of wasting paper, I have written out condensed accounts of the essential facts (§§2A3-2A5).

Note on second printing, April 2003

For the second printing of this volume, I have made two substantial corrections to inadequate proofs and a large number of minor amendments; I am most grateful to T.D.Austin for his careful reading of the first printing. In addition, I have added a dozen exercises and a handful of straightforward results which turn out to be relevant to the work of later volumes and fit naturally here.

The regular process of revision of this work has led me to make a couple of notational innovations not described explicitly in the early printings of Volume 1. I trust that most readers will find these immediately comprehensible. If, however, you find that there is a puzzling cross-reference which you are unable to match with anything in the version of Volume 1 which you are using, it may be worth while checking the errata pages in `http://www1.essex.ac.uk/maths/people/fremlin/mterr.htm`.

Note on hardback edition, January 2010

For the new ('Lulu') edition of this volume, I have eliminated a number of further errors; no doubt many remain. There are many new exercises, several new theorems and some corresponding rearrangements of material. The new results are mostly additions with little effect on the structure of the work, but there is a short new section (§266) on the Brunn-Minkowski inequality.

Note on second printing of hardback edition, April 2016

There is the usual crop of small mistakes to be corrected, and assorted minor amendments and additions. But my principal reason for issuing a new printed version is a major fault in the proof of Carleson's theorem, where an imprudent move to simplify the argument of Lacey & Thiele 00 was based on an undergraduate error[1]. While the blunder is conspicuous enough, a resolution seems to require an adjustment in a definition, and is not a fair demand on a graduate seminar, the intended readership for this material. Furthermore, the proof was supposed to be a distinguishing feature of not only this volume, but of the treatise as a whole. So, with apologies to any who retired hurt from an encounter with the original version, I present a revision which I hope is essentially sound.

[1] I am most grateful to A.Derighetti for bringing this to my attention.

Version of 17.1.15

*Chapter 21

Taxonomy of measure spaces

I begin this volume with a 'starred chapter'. The point is that I do not really recommend this chapter for beginners. It deals with a variety of technical questions which are of great importance for the later development of the subject, but are likely to be both abstract and obscure for anyone who has not encountered the problems these techniques are designed to solve. On the other hand, if (as is customary) this work is omitted, and the ideas are introduced only when urgently needed, the student is likely to finish with very vague ideas on which theorems can be expected to apply to which types of measure space, and with no vocabulary in which to express those ideas. I therefore take a few pages to introduce the terminology and concepts which can be used to distinguish 'good' measure spaces from others, with a few of the basic relationships. The only paragraphs which are immediately relevant to the theory set out in Volume 1 are those on 'complete', 'σ-finite' and 'semi-finite' measure spaces (211A, 211D, 211F, 211Lc, §212, 213A-213B, 215B), and on Lebesgue measure (211M). For the rest, I think that a newcomer to the subject can very well pass over this chapter for the time being, and return to it for particular items when the text of later chapters refers to it. On the other hand, it can also be used as an introduction to the flavour of the 'purest' kind of measure theory, the study of measure spaces for their own sake, with a systematic discussion of a few of the elementary constructions.

Version of 20.11.03

211 Definitions

I start with a list of definitions, corresponding to the concepts which I have found to be of value in distinguishing different types of measure space. Necessarily, the significance of many of these ideas is likely to be obscure until you have encountered some of the obstacles which arise later on. Nevertheless, you will I hope be able to deal with these definitions on a formal, abstract basis, and to follow the elementary arguments involved in establishing the relationships between them (211L).

In 216C-216E below I will give three substantial examples to demonstrate the rich variety of objects which the definition of 'measure space' encompasses. In the present section, therefore, I content myself with very brief descriptions of sufficient cases to show at least that each of the definitions here discriminates between different spaces (211M-211R).

211A Definition Let (X,Σ,μ) be a measure space. Then μ, or (X,Σ,μ), is (**Carathéodory**) **complete** if whenever $A\subseteq E\in\Sigma$ and $\mu E=0$ then $A\in\Sigma$.

211B Definition Let (X,Σ,μ) be a measure space. Then (X,Σ,μ) is a **probability space** if $\mu X=1$. In this case μ is called a **probability** or **probability measure**.

211C Definition Let (X,Σ,μ) be a measure space. Then μ, or (X,Σ,μ), is **totally finite** if $\mu X<\infty$.

211D Definition Let (X,Σ,μ) be a measure space. Then μ, or (X,Σ,μ), is **σ-finite** if there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure such that $X=\bigcup_{n\in\mathbb{N}}E_n$.

211E Definition Let (X,Σ,μ) be a measure space. Then μ, or (X,Σ,μ), is **strictly localizable** or **decomposable** if there is a partition $\langle X_i\rangle_{i\in I}$ of X into measurable sets of finite measure such that

$$\Sigma=\{E:E\subseteq X,\ E\cap X_i\in\Sigma\ \forall\ i\in I\},$$

$$\mu E=\sum_{i\in I}\mu(E\cap X_i)\text{ for every }E\in\Sigma.$$

I will call such a family $\langle X_i\rangle_{i\in I}$ a **decomposition** of X.

211F Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **semi-finite** if whenever $E \in \Sigma$ and $\mu E = \infty$ there is an $F \subseteq E$ such that $F \in \Sigma$ and $0 < \mu F < \infty$.

211G Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **localizable** if it is semi-finite and, for every $\mathcal{E} \subseteq \Sigma$, there is an $H \in \Sigma$ such that (i) $E \setminus H$ is negligible for every $E \in \mathcal{E}$ (ii) if $G \in \Sigma$ and $E \setminus G$ is negligible for every $E \in \mathcal{E}$, then $H \setminus G$ is negligible. It will be convenient to call such a set H **an essential supremum** of $\mathcal{E}$ in Σ.

211H Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **locally determined** if it is semi-finite and

$$\Sigma = \{E : E \subseteq X,\ E \cap F \in \Sigma \text{ whenever } F \in \Sigma \text{ and } \mu F < \infty\}.$$

211I Definition Let (X, Σ, μ) be a measure space. A set $E \in \Sigma$ is an **atom** for μ if $\mu E > 0$ and whenever $F \in \Sigma$, $F \subseteq E$ one of F, $E \setminus F$ is negligible.

211J Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **atomless** or **diffused** if there are no atoms for μ.

211K Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **purely atomic** if whenever $E \in \Sigma$ and E is not negligible there is an atom for μ included in E.

Remark Every point-supported measure is purely atomic, but not every purely atomic measure is point-supported.

211L Theorem (a) A probability space is totally finite.
(b) A totally finite measure space is σ-finite.
(c) A σ-finite measure space is strictly localizable.
(d) A strictly localizable measure space is localizable and locally determined.
(e) A localizable measure space is semi-finite.
(f) A locally determined measure space is semi-finite.

211M Example: Lebesgue measure Write μ for Lebesgue measure on $\mathbb{R}^r$.

(a) μ is complete.

(b) μ is σ-finite.

(c) μ is strictly localizable, localizable, locally determined and semi-finite.

(d) μ is atomless.

(e) It is now a trivial observation that μ cannot be purely atomic.

211N Counting measure Take X to be any uncountable set, and μ to be counting measure on X.

(a) μ is complete.

(b) μ is not σ-finite. μ is not a probability measure nor totally finite.

(c) μ is strictly localizable. μ is localizable, locally determined and semi-finite.

(d) μ is purely atomic. μ is not atomless.

211O A non-semi-finite space Set $X = \{0\}$, $\Sigma = \{\emptyset, X\}$, $\mu\emptyset = 0$ and $\mu X = \infty$. Then μ is not semi-finite, localizable, locally determined, σ-finite, totally finite nor a probability measure. μ is complete. μ is purely atomic (indeed, it is point-supported).

211P A non-complete space Write $\mathcal{B}$ for the σ-algebra of Borel subsets of $\mathbb{R}$, and μ for the restriction of Lebesgue measure to $\mathcal{B}$. Then $(\mathbb{R}, \mathcal{B}, \mu)$ is atomless, σ-finite and not complete.

211Q Some probability spaces Two constructions of probability spaces are

(a) the subspace measure induced by Lebesgue measure on $[0, 1]$;

(b) the point-supported measure induced on a set X by a function $h : X \to [0, 1]$ such that $\sum_{x \in X} h(x) = 1$, writing $\mu E = \sum_{x \in E} h(x)$ for every $E \subseteq X$; for instance, if X is a singleton $\{x\}$ and $h(x) = 1$, or if $X = \mathbb{N}$ and $h(n) = 2^{-n-1}$.

Of these two, (a) gives an atomless probability measure and (b) gives a purely atomic probability measure.

211R Countable-cocountable measure(a) Let X be any set. Let Σ be the family of those sets $E \subseteq X$ such that either E or $X \setminus E$ is countable. Then Σ is a σ-algebra of subsets of X. Σ is called the **countable-cocountable σ-algebra** of X.

(b) Now consider the function $\mu : \Sigma \to \{0, 1\}$ defined by writing $\mu E = 0$ if E is countable, $\mu E = 1$ if $E \in \Sigma$ and E is not countable. Then μ is a measure. This is the **countable-cocountable measure** on X.

(c) If X is any uncountable set and μ is the countable-cocountable measure on X, then μ is a complete, purely atomic probability measure, but is not point-supported.

Version of 10.9.04

212 Complete spaces

In the next two sections of this chapter I give brief accounts of the theory of measure spaces possessing certain of the properties described in §211. I begin with 'completeness'. I give the elementary facts about complete measure spaces in 212A-212B; then I turn to the notion of 'completion' of a measure (212C) and its relationships with the other concepts of measure theory introduced so far (212D-212G).

212A Proposition Any measure space constructed by Carathéodory's method is complete.

212B Proposition (a) If (X, Σ, μ) is a complete measure space, then any conegligible subset of X is measurable.

(b) Let (X, Σ, μ) be a complete measure space, and f a $[-\infty, \infty]$-valued function defined on a subset of X. If f is virtually measurable, then f is measurable.

(c) Let (X, Σ, μ) be a complete measure space, and f a real-valued function defined on a conegligible subset of X. Then the following are equiveridical, that is, if one is true so are the others:

(i) f is integrable;

(ii) f is measurable and $|f|$ is integrable;

(iii) f is measurable and there is an integrable function g such that $|f| \leq_{\text{a.e.}} g$.

212C The completion of a measure Let (X, Σ, μ) be any measure space.

(a) Let $\hat{\Sigma}$ be the family of those sets $E \subseteq X$ such that there are E', $E'' \in \Sigma$ with $E' \subseteq E \subseteq E''$ and $\mu(E'' \setminus E') = 0$. Then $\hat{\Sigma}$ is a σ-algebra of subsets of X.

(b) For $E \in \hat{\Sigma}$, set

$$\hat{\mu}E = \mu^* E = \min\{\mu F : E \subseteq F \in \Sigma\}.$$

(c) $(X, \hat{\Sigma}, \hat{\mu})$ is a measure space.

(d) $(X, \hat{\Sigma}, \hat{\mu})$ is called the **completion** of the measure space (X, Σ, μ); I will call $\hat{\mu}$ the **completion** of μ, and occasionally I will call $\hat{\Sigma}$ the completion of Σ. Members of $\hat{\Sigma}$ are sometimes called **μ-measurable**.

212D Proposition Let (X,Σ,μ) be any measure space. Then $(X,\hat{\Sigma},\hat{\mu})$, as defined in 212C, is a complete measure space and $\hat{\mu}$ is an extension of μ; and $(X,\hat{\Sigma},\hat{\mu})=(X,\Sigma,\mu)$ iff (X,Σ,μ) is complete.

212E Proposition Let (X,Σ,μ) be a measure space, and $(X,\hat{\Sigma},\hat{\mu})$ its completion.

(a) The outer measures $\hat{\mu}^*$, μ^* defined from $\hat{\mu}$ and μ coincide.

(b) μ, $\hat{\mu}$ give rise to the same negligible and conegligible sets and the same sets of full outer measure.

(c) $\hat{\mu}$ is the only measure with domain $\hat{\Sigma}$ which agrees with μ on Σ.

(d) A subset of X belongs to $\hat{\Sigma}$ iff it is expressible as $F\triangle A$ where $F\in\Sigma$ and A is μ-negligible.

212F Proposition Let (X,Σ,μ) be a measure space and $(X,\hat{\Sigma},\hat{\mu})$ its completion.

(a) A $[-\infty,\infty]$-valued function f defined on a subset of X is $\hat{\Sigma}$-measurable iff it is μ-virtually measurable.

(b) Let f be a $[-\infty,\infty]$-valued function defined on a subset of X. Then $\int f d\mu=\int f d\hat{\mu}$ if either is defined in $[-\infty,\infty]$; in particular, f is μ-integrable iff it is $\hat{\mu}$-integrable.

212G Proposition Let (X,Σ,μ) be a measure space, and $(X,\hat{\Sigma},\hat{\mu})$ its completion.

(a) $(X,\hat{\Sigma},\hat{\mu})$ is a probability space, or totally finite, or σ-finite, or semi-finite, or localizable, iff (X,Σ,μ) is.

(b) $(X,\hat{\Sigma},\hat{\mu})$ is strictly localizable if (X,Σ,μ) is, and any decomposition of X for μ is a decomposition for $\hat{\mu}$.

(c) A set $H\in\hat{\Sigma}$ is an atom for $\hat{\mu}$ iff there is an $E\in\Sigma$ such that E is an atom for μ and $\hat{\mu}(H\triangle E)=0$.

(d) $(X,\hat{\Sigma},\hat{\mu})$ is atomless or purely atomic iff (X,Σ,μ) is.

Version of 13.9.13

213 Semi-finite, locally determined and localizable spaces

In this section I collect a variety of useful facts concerning these types of measure space. I start with the characteristic properties of semi-finite spaces (213A-213B), and continue with complete locally determined spaces (213C) and the concept of 'c.l.d. version' (213D-213H), the most powerful of the universally available methods of modifying a measure space into a better-behaved one. I briefly discuss 'locally determined negligible sets' (213I-213L), and measurable envelopes (213L-213M), and end with results on localizable spaces (213N) and strictly localizable spaces (213O).

213A Lemma Let (X,Σ,μ) be a semi-finite measure space. Then

$$\mu E=\sup\{\mu F : F\in\Sigma,\ F\subseteq E,\ \mu F<\infty\}$$

for every $E\in\Sigma$.

213B Proposition Let (X,Σ,μ) be a semi-finite measure space. Let f be a μ-virtually measurable $[0,\infty]$-valued function defined almost everywhere in X. Then

$$\begin{aligned}\int f&=\sup\{\int g : g \text{ is a simple function, } g\le_{\text{a.e.}} f\}\\&=\sup_{F\in\Sigma,\mu F<\infty}\int_F f\end{aligned}$$

in $[0,\infty]$.

***213C Proposition** Let (X,Σ,μ) be a complete locally determined measure space, and μ^* the outer measure derived from μ. Then the measure defined from μ^* by Carathéodory's method is μ itself.

213D C.l.d. versions: Proposition Let (X, Σ, μ) be a measure space. Write $(X, \hat{\Sigma}, \hat{\mu})$ for its completion and Σ^f for $\{E : E \in \Sigma,\ \mu E < \infty\}$. Set

$$\tilde{\Sigma} = \{H : H \subseteq X,\ H \cap E \in \hat{\Sigma} \text{ for every } E \in \Sigma^f\},$$

and for $H \in \tilde{\Sigma}$ set

$$\tilde{\mu} H = \sup\{\hat{\mu}(H \cap E) : E \in \Sigma^f\}.$$

Then $(X, \tilde{\Sigma}, \tilde{\mu})$ is a complete locally determined measure space.

213E Definition For any measure space (X, Σ, μ), I will call $(X, \tilde{\Sigma}, \tilde{\mu})$, as constructed in 213D, the **c.l.d. version** ('complete locally determined version') of (X, Σ, μ); and $\tilde{\mu}$ will be the **c.l.d. version** of μ.

213F Proposition Let (X, Σ, μ) be any measure space and $(X, \tilde{\Sigma}, \tilde{\mu})$ its c.l.d. version.

(a) $\Sigma \subseteq \tilde{\Sigma}$ and $\tilde{\mu} E = \mu E$ whenever $E \in \Sigma$ and $\mu E < \infty$ – in fact, if $(X, \hat{\Sigma}, \hat{\mu})$ is the completion of (X, Σ, μ), $\hat{\Sigma} \subseteq \tilde{\Sigma}$ and $\tilde{\mu} E = \hat{\mu} E$ whenever $\hat{\mu} E < \infty$.

(b) Writing $\tilde{\mu}^*$ and μ^* for the outer measures defined from $\tilde{\mu}$ and μ respectively, $\tilde{\mu}^* A \le \mu^* A$ for every $A \subseteq X$, with equality if $\mu^* A$ is finite. In particular, μ-negligible sets are $\tilde{\mu}$-negligible; consequently, μ-conegligible sets are $\tilde{\mu}$-conegligible.

(c) If $H \in \tilde{\Sigma}$,

(i) $\tilde{\mu} H = \sup\{\mu F : E \in \Sigma,\ \mu F < \infty,\ F \subseteq H\}$;

(ii) there is an $E \in \Sigma$ such that $E \subseteq H$ and $\mu E = \tilde{\mu} H$, so that if $\tilde{\mu} H < \infty$ then $\tilde{\mu}(H \setminus E) = 0$.

213G Proposition Let (X, Σ, μ) be a measure space, and $(X, \tilde{\Sigma}, \tilde{\mu})$ its c.l.d. version.

(a) If a real-valued function f defined on a subset of X is μ-virtually measurable, it is $\tilde{\Sigma}$-measurable.

(b) If a real-valued function is μ-integrable, it is $\tilde{\mu}$-integrable with the same integral.

(c) If f is a $\tilde{\mu}$-integrable real-valued function, there is a μ-integrable real-valued function which is equal to f $\tilde{\mu}$-almost everywhere.

213H Proposition Let (X, Σ, μ) be a measure space, $(X, \hat{\Sigma}, \hat{\mu})$ its completion and $(X, \tilde{\Sigma}, \tilde{\mu})$ its c.l.d. version.

(a) If (X, Σ, μ) is a probability space, or totally finite, or σ-finite, or strictly localizable, so is $(X, \tilde{\Sigma}, \tilde{\mu})$, and in all these cases $\tilde{\mu} = \hat{\mu}$;

(b) if (X, Σ, μ) is localizable, so is $(X, \tilde{\Sigma}, \tilde{\mu})$, and for every $H \in \tilde{\Sigma}$ there is an $E \in \Sigma$ such that $\tilde{\mu}(E \triangle H) = 0$;

(c) (X, Σ, μ) is semi-finite iff $\tilde{\mu} F = \mu F$ for every $F \in \Sigma$, and in this case $\int f d\tilde{\mu} = \int f d\mu$ whenever the latter is defined in $[-\infty, \infty]$;

(d) a set $H \in \tilde{\Sigma}$ is an atom for $\tilde{\mu}$ iff there is an atom E for μ such that $\mu E < \infty$ and $\tilde{\mu}(H \triangle E) = 0$;

(e) if (X, Σ, μ) is atomless or purely atomic, so is $(X, \tilde{\Sigma}, \tilde{\mu})$;

(f) (X, Σ, μ) is complete and locally determined iff $\tilde{\mu} = \mu$.

213I Locally determined negligible sets: Definition A measure space (X, Σ, μ) has **locally determined negligible sets** if for every non-negligible $A \subseteq X$ there is an $E \in \Sigma$ such that $\mu E < \infty$ and $A \cap E$ is not negligible.

213J Proposition If a measure space (X, Σ, μ) is *either* strictly localizable *or* complete and locally determined, it has locally determined negligible sets.

***213K Lemma** If a measure space (X, Σ, μ) has locally determined negligible sets, and $\mathcal{E} \subseteq \Sigma$ has an essential supremum $H \in \Sigma$, then $H \setminus \bigcup \mathcal{E}$ is negligible.

213L Proposition Let (X, Σ, μ) be a localizable measure space with locally determined negligible sets. Then every subset A of X has a measurable envelope.

213M Corollary (a) If (X,Σ,μ) is σ-finite, then every subset of X has a measurable envelope for μ.

(b) If (X,Σ,μ) is localizable, then every subset of X has a measurable envelope for the c.l.d. version of μ.

213N Theorem Let (X,Σ,μ) be a localizable measure space. Suppose that Φ is a family of measurable real-valued functions, all defined on measurable subsets of X, such that whenever f, $g\in\Phi$ then $f=g$ almost everywhere in $\operatorname{dom} f\cap\operatorname{dom} g$. Then there is a measurable function $h:X\to\mathbb{R}$ such that every $f\in\Phi$ agrees with h almost everywhere in $\operatorname{dom} f$.

213O Proposition Let (X,Σ,μ) be a complete locally determined space.

(a) Suppose that there is a disjoint family $\mathcal{E}\subseteq\Sigma$ such that (α) $\mu E<\infty$ for every $E\in\mathcal{E}$ (β) whenever $F\in\Sigma$ and $\mu F>0$ then there is an $E\in\mathcal{E}$ such that $\mu(E\cap F)>0$. Then (X,Σ,μ) is strictly localizable, $\bigcup\mathcal{E}$ is conegligible, and $\mathcal{E}\cup\{X\setminus\bigcup\mathcal{E}\}$ is a decomposition of X.

(b) Suppose that $\langle X_i\rangle_{i\in I}$ is a partition of X into measurable sets of finite measure such that whenever $E\in\Sigma$ and $\mu E>0$ there is an $i\in I$ such that $\mu(E\cap X_i)>0$. Then (X,Σ,μ) is strictly localizable, and $\langle X_i\rangle_{i\in I}$ is a decomposition of X.

Version of 22.5.09

214 Subspaces

In §131 I described a construction for subspace measures on measurable subsets. It is now time to give the generalization to subspace measures on arbitrary subsets of a measure space. The relationship between this construction and the properties listed in §211 is not quite as straightforward as one might imagine, and in this section I try to give a full account of what can be expected of subspaces in general. I think that for the present volume only (i) general subspaces of σ-finite spaces and (ii) measurable subspaces of general measure spaces will be needed in any essential way, and these do not give any difficulty; but in later volumes we shall need the full theory.

I begin with a general construction for 'subspace measures' (214A-214C), with an account of integration with respect to a subspace measure (214E-214G); these (with 131E-131H) give a solid foundation for the concept of 'integration over a subset' (214D). I present this work in its full natural generality, which will eventually be essential, but even for Lebesgue measure alone it is important to be aware of the ideas here. I continue with answers to some obvious questions concerning subspace measures and the properties of measure spaces so far considered, both for general subspaces (214I) and for measurable subspaces (214K), and I mention a basic construction for assembling measure spaces side-by-side, the 'direct sums' of 214L-214M. At the end of the section I discuss a measure extension problem (214O-214P).

214A Proposition Let (X,Σ,μ) be a measure space, and Y any subset of X. Let μ^* be the outer measure defined from μ, and set $\Sigma_Y=\{E\cap Y:E\in\Sigma\}$; let μ_Y be the restriction of μ^* to Σ_Y. Then (Y,Σ_Y,μ_Y) is a measure space.

214B Definition If (X,Σ,μ) is any measure space and Y is any subset of X, then μ_Y, defined as in 214A, is the **subspace measure** on Y.

214C Lemma Let (X,Σ,μ) be a measure space, Y a subset of X, μ_Y the subspace measure on Y and Σ_Y its domain. Then

(a) for any $F\in\Sigma_Y$, there is an $E\in\Sigma$ such that $F=E\cap Y$ and $\mu E=\mu_Y F$;

(b) for any $A\subseteq Y$, A is μ_Y-negligible iff it is μ-negligible;

(c)(i) if $A\subseteq X$ is μ-conegligible, then $A\cap Y$ is μ_Y-conegligible;

(ii) if $A\subseteq Y$ is μ_Y-conegligible, then $A\cup(X\setminus Y)$ is μ-conegligible;

(d) $(\mu_Y)^*$ agrees with μ^* on $\mathcal{P}Y$;

(e) if $Z\subseteq Y\subseteq X$, then $\Sigma_Z=(\Sigma_Y)_Z$, the subspace σ-algebra of subsets of Z regarded as a subspace of (Y,Σ_Y), and $\mu_Z=(\mu_Y)_Z$ is the subspace measure on Z regarded as a subspace of (Y,μ_Y);

(f) if $Y\in\Sigma$, then μ_Y, as defined here, is exactly the subspace measure on Y defined in 131A-131B.

214D Integration over subsets: Definition Let (X,Σ,μ) be a measure space, Y a subset of X and f a $[-\infty,\infty]$-valued function defined on a subset of X. By $\int_Y f$ I mean $\int (f\restriction Y)d\mu_Y$, if this exists in $[-\infty,\infty]$.

214E Proposition Let (X,Σ,μ) be a measure space, $Y\subseteq X$, and f a $[-\infty,\infty]$-valued function defined on a subset $\operatorname{dom} f$ of X.

(a) If f is μ-integrable then $f\restriction Y$ is μ_Y-integrable, and $\int_Y f\le\int f$ if f is non-negative.

(b) If $\operatorname{dom} f\subseteq Y$ and f is μ_Y-integrable, then there is a μ-integrable function $\tilde f$ on X, extending f, such that $\int_F \tilde f=\int_{F\cap Y} f$ for every $F\in\Sigma$.

214F Proposition Let (X,Σ,μ) be a measure space, Y a subset of X, and f a $[-\infty,\infty]$-valued function such that $\int_X f$ is defined in $[-\infty,\infty]$. If *either* Y has full outer measure in X *or* f is zero almost everywhere in $X\setminus Y$, then $\int_Y f$ is defined and equal to $\int_X f$.

214G Corollary Let (X,Σ,μ) be a measure space, Y a subset of X, and $E\in\Sigma$ a measurable envelope of Y. If f is a $[-\infty,\infty]$-valued function such that $\int_E f$ is defined in $[-\infty,\infty]$, then $\int_Y f$ is defined and equal to $\int_E f$.

214H Subspaces and Carathéodory's method: Lemma Let X be a set, $Y\subseteq X$ a subset, and θ an outer measure on X.

(a) $\theta_Y=\theta\restriction\mathcal{P}Y$ is an outer measure on Y.

(b) Let μ, ν be the measures on X, Y defined by Carathéodory's method from the outer measures θ, θ_Y, and Σ, T their domains; let μ_Y be the subspace measure on Y induced by μ, and Σ_Y its domain. Then

(i) $\Sigma_Y\subseteq\mathrm{T}$ and $\nu F\le\mu_Y F$ for every $F\in\Sigma_Y$;

(ii) if $Y\in\Sigma$ then $\nu=\mu_Y$;

(iii) if $\theta=\mu^*$ then ν extends μ_Y;

(iv) if $\theta=\mu^*$ and $\theta Y<\infty$ then $\nu=\mu_Y$.

214I Theorem Let (X,Σ,μ) be a measure space and Y a subset of X. Let μ_Y be the subspace measure on Y and Σ_Y its domain.

(a) If (X,Σ,μ) is complete, or totally finite, or σ-finite, or strictly localizable, so is (Y,Σ_Y,μ_Y). If $\langle X_i\rangle_{i\in I}$ is a decomposition of X for μ, then $\langle X_i\cap Y\rangle_{i\in I}$ is a decomposition of Y for μ_Y.

(b) Writing $\hat\mu$ for the completion of μ, the subspace measure $\hat\mu_Y=(\hat\mu)_Y$ is the completion of μ_Y.

(c) If (X,Σ,μ) has locally determined negligible sets, then μ_Y is semi-finite.

(d) If (X,Σ,μ) is complete and locally determined, then (Y,Σ_Y,μ_Y) is complete and semi-finite.

(e) If (X,Σ,μ) is complete, locally determined and localizable then so is (Y,Σ_Y,μ_Y).

214J Upper and lower integrals: Proposition Let (X,Σ,μ) be a measure space, A a subset of X and f a real-valued function defined almost everywhere in X. Then

(a) if *either* f is non-negative *or* A has full outer measure in X, $\overline{\int}(f\restriction A)d\mu_A\le\overline{\int}f d\mu$;

(b) if A has full outer measure in X, $\underline{\int}f d\mu\le\underline{\int}(f\restriction A)d\mu_A$.

214K Measurable subspaces: Proposition Let (X,Σ,μ) be a measure space.

(a) Let $E\in\Sigma$ and let μ_E be the subspace measure, with Σ_E its domain. If (X,Σ,μ) is complete, or totally finite, or σ-finite, or strictly localizable, or semi-finite, or localizable, or locally determined, or atomless, or purely atomic, so is (E,Σ_E,μ_E).

(b) Suppose that $\langle X_i\rangle_{i\in I}$ is a partition of X into measurable sets such that

$$\Sigma=\{E:E\subseteq X,\ E\cap X_i\in\Sigma\text{ for every }i\in I\},$$

$$\mu E=\textstyle\sum_{i\in I}\mu(E\cap X_i)\text{ for every }E\in\Sigma.$$

Then (X,Σ,μ) is complete, or strictly localizable, or semi-finite, or localizable, or locally determined, or atomless, or purely atomic, iff $(X_i,\Sigma_{X_i},\mu_{X_i})$ has that property for every $i\in I$.

214L Direct sums Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be any indexed family of measure spaces. Set $X = \bigcup_{i\in I}(X_i \times \{i\})$; for $E \subseteq X$, $i \in I$ set $E_i = \{x : (x, i) \in E\}$. Write

$$\Sigma = \{E : E \subseteq X,\ E_i \in \Sigma_i \text{ for every } i \in I\},$$

$$\mu E = \sum_{i\in I} \mu_i E_i \text{ for every } E \in \Sigma.$$

Then it is easy to check that (X, Σ, μ) is a measure space; I will call it the **direct sum** of the family $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$. Note that if (X, Σ, μ) is any strictly localizable measure space, with decomposition $\langle X_i\rangle_{i\in I}$, then we have a natural isomorphism between (X, Σ, μ) and the direct sum $(X', \Sigma', \mu') = \bigoplus_{i\in I}(X_i, \Sigma_{X_i}, \mu_{X_i})$ of the subspace measures, if we match $(x, i) \in X'$ with $x \in X$ for every $i \in I$ and $x \in X_i$.

214M Proposition Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X, Σ, μ). Let f be a real-valued function defined on a subset of X. For each $i \in I$, set $f_i(x) = f(x, i)$ whenever $(x, i) \in \operatorname{dom} f$.

(a) f is measurable iff f_i is measurable for every $i \in I$.

(b) If f is non-negative, then $\int f d\mu = \sum_{i\in I} \int f_i d\mu_i$ if either is defined in $[0, \infty]$.

214N Corollary Let (X, Σ, μ) be a measure space with a decomposition $\langle X_i\rangle_{i\in I}$. If f is a real-valued function defined on a subset of X, then

(a) f is measurable iff $f\restriction X_i$ is measurable for every $i \in I$,

(b) if $f \geq 0$, then $\int f = \sum_{i\in I} \int_{X_i} f$ if either is defined in $[0, \infty]$.

***214O Lemma** Let (X, Σ, μ) be a measure space, and $\mathcal{I}$ an ideal of subsets of X, that is, a family of subsets of X such that $\emptyset \in \mathcal{I}$, $I \cup J \in \mathcal{I}$ for all I, $J \in \mathcal{I}$, and $I \in \mathcal{I}$ whenever $I \subseteq J \in \mathcal{I}$. Then there is a measure λ on X such that $\Sigma \cup \mathcal{I} \subseteq \operatorname{dom}\lambda$, $\mu E = \lambda E + \sup_{I\in\mathcal{I}} \mu^*(E \cap I)$ for every $E \in \Sigma$, and $\lambda I = 0$ for every $I \in \mathcal{I}$.

***214P Theorem** Let (X, Σ, μ) be a measure space, and $\mathcal{A}$ a family of subsets of X which is well-ordered by the relation $\subseteq$. Then there is an extension of μ to a measure λ on X such that $\lambda(E \cap A)$ is defined and equal to $\mu^*(E \cap A)$ whenever $E \in \Sigma$ and $A \in \mathcal{A}$.

***214Q Proposition** Suppose that (X, Σ, μ) is an atomless measure space and Y a subset of X such that the subspace measure μ_Y is semi-finite. Then μ_Y is atomless.

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215 σ-finite spaces and the principle of exhaustion

I interpolate a short section to deal with some useful facts which might get lost if buried in one of the longer sections of this chapter. The great majority of the applications of measure theory involve σ-finite spaces, to the point that many authors skim over any others. I myself prefer to signal the importance of such concepts by explicitly stating just which theorems apply only to the restricted class of spaces. But undoubtedly some facts about σ-finite spaces need to be grasped early on. In 215B I give a list of properties characterizing σ-finite spaces. Some of these make better sense in the light of the principle of exhaustion (215A). I take the opportunity to include a fundamental fact about atomless measure spaces (215D).

215A The principle of exhaustion: Lemma Let (X, Σ, μ) be any measure space and $\mathcal{E} \subseteq \Sigma$ a non-empty set such that $\sup_{n\in\mathbb{N}} \mu F_n$ is finite for every non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$.

(a) There is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that, for every $E \in \Sigma$, *either* there is an $n \in \mathbb{N}$ such that $E \cup F_n$ is not included in any member of $\mathcal{E}$ *or*, setting $F = \bigcup_{n\in\mathbb{N}} F_n$,

$$\lim_{n\to\infty} \mu(E \setminus F_n) = \mu(E \setminus F) = 0.$$

In particular, if $E \in \mathcal{E}$ and $E \supseteq F$, then $E \setminus F$ is negligible.

(b) If $\mathcal{E}$ is upwards-directed, then there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that, setting $F=\bigcup_{n\in\mathbb{N}}F_n$, $\mu F=\sup_{E\in\mathcal{E}}\mu E$ and $E\setminus F$ is negligible for every $E\in\mathcal{E}$, so that F is an essential supremum of $\mathcal{E}$ in Σ.

(c) If the union of any non-decreasing sequence in $\mathcal{E}$ belongs to $\mathcal{E}$, then there is an $F\in\mathcal{E}$ such that $E\setminus F$ is negligible whenever $E\in\mathcal{E}$ and $F\subseteq E$.

215B Proposition Let (X,Σ,μ) be a semi-finite measure space. Write $\mathcal{N}$ for the family of μ-negligible sets and Σ^f for the family of measurable sets of finite measure. Then the following are equiveridical:

(i) (X,Σ,μ) is σ-finite;

(ii) every disjoint family in $\Sigma^f\setminus\mathcal{N}$ is countable;

(iii) every disjoint family in $\Sigma\setminus\mathcal{N}$ is countable;

(iv) for every $\mathcal{E}\subseteq\Sigma$ there is a countable set $\mathcal{E}_0\subseteq\mathcal{E}$ such that $E\setminus\bigcup\mathcal{E}_0$ is negligible for every $E\in\mathcal{E}$;

(v) for every non-empty upwards-directed $\mathcal{E}\subseteq\Sigma$ there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $E\setminus\bigcup_{n\in\mathbb{N}}F_n$ is negligible for every $E\in\mathcal{E}$;

(vi) for every non-empty $\mathcal{E}\subseteq\Sigma$, there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $E\setminus\bigcup_{n\in\mathbb{N}}F_n$ is negligible whenever $E\in\mathcal{E}$ and $E\supseteq F_n$ for every $n\in\mathbb{N}$;

(vii) *either* $\mu X=0$ *or* there is a probability measure ν on X with the same domain and the same negligible sets as μ;

(viii) there is a measurable integrable function $f:X\to\,]0,1]$;

(ix) either $\mu X=0$ or there is a measurable function $f:X\to\,]0,\infty[$ such that $\int f d\mu=1$.

215C Corollary Let (X,Σ,μ) be a σ-finite measure space, and suppose that $\mathcal{E}\subseteq\Sigma$ is any non-empty set.

(a) There is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that, for every $E\in\Sigma$, *either* there is an $n\in\mathbb{N}$ such that $E\cup F_n$ is not included in any member of $\mathcal{E}$ *or* $E\setminus\bigcup_{n\in\mathbb{N}}F_n$ is negligible.

(b) If $\mathcal{E}$ is upwards-directed, then there is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\bigcup_{n\in\mathbb{N}}F_n$ is an essential supremum of $\mathcal{E}$ in Σ.

(c) If the union of any non-decreasing sequence in $\mathcal{E}$ belongs to $\mathcal{E}$, then there is an $F\in\mathcal{E}$ such that $E\setminus F$ is negligible whenever $E\in\mathcal{E}$ and $F\subseteq E$.

215D Proposition Let (X,Σ,μ) be an atomless measure space. If $E\in\Sigma$ and $0\le\alpha\le\mu E<\infty$, there is an $F\in\Sigma$ such that $F\subseteq E$ and $\mu F=\alpha$.

***215E Proposition** Let (X,Σ,μ) be an atomless measure space and $x\in X$.

(a) If $\mu^*\{x\}$ is finite then $\{x\}$ is negligible.

(b) If μ has locally determined negligible sets then $\{x\}$ is negligible.

(c) If μ is localizable then $\{x\}$ is negligible.

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216 Examples

It is common practice – and, in my view, good practice – in books on pure mathematics, to provide discriminating examples; I mean that whenever we are given a list of new concepts, we expect to be provided with examples to show that we have a fair picture of the relationships between them, and in particular that we are not being kept ignorant of some startling implication. Concerning the concepts listed in 211A-211K, we have ten different properties which some, but not all, measure spaces possess, giving a conceivable total of 2^{10} different types of measure space, classified according to which of these ten properties they have. The list of basic relationships in 211L reduces these 1024 possibilities to 72. Observing that a space can be simultaneously atomless and purely atomic only when the measure of the whole space is 0, we find ourselves with 56 possibilities, being two trivial cases with $\mu X=0$ (because such a measure may or may not be complete) together with $9\times2\times3$ cases, corresponding to the nine classes

probability spaces,
spaces which are totally finite, but not probability spaces,
spaces which are σ-finite, but not totally finite,
spaces which are strictly localizable, but not σ-finite,
spaces which are localizable and locally determined, but not strictly localizable,
spaces which are localizable, but not locally determined,
spaces which are locally determined, but not localizable,
spaces which are semi-finite, but neither locally determined nor localizable,
spaces which are not semi-finite;

the two classes

spaces which are complete,
spaces which are not complete;

and the three classes

spaces which are atomless, not of measure 0,
spaces which are purely atomic, not of measure 0,
spaces which are neither atomless nor purely atomic.

I do not propose to give a complete set of fifty-six examples, particularly as rather fewer than fifty-six different ideas are required. However, I do think that for a proper understanding of abstract measure spaces it is necessary to have seen realizations of some of the critical combinations of properties. I therefore take a few paragraphs to describe three special examples to add to those of 211M-211R.

216A Lebesgue measure(a) Lebesgue measure μ on $\mathbb{R}$ is complete, atomless and σ-finite, therefore strictly localizable, localizable and locally determined.

(b) The subspace measure $\mu_{[0,1]}$ on $[0,1]$ is a complete, atomless probability measure.

(c) The restriction $\mu\restriction\mathcal{B}$ of μ to the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ is atomless, σ-finite and not complete.

***216C A complete, localizable, non-locally-determined space(a)** Let I be any uncountable set, and set $X = \{0,1\} \times I$. For $E \subseteq X$, $y \in \{0,1\}$ set $E[\{y\}] = \{i : (y,i) \in E\} \subseteq I$. Set

$$\Sigma = \{E : E \subseteq X,\ E[\{0\}] \triangle E[\{1\}] \text{ is countable}\}.$$

Then Σ is a σ-algebra of subsets of X.

For $E \in \Sigma$, set $\mu E = \#(E[\{0\}])$ if this is finite, ∞ otherwise; then (X,Σ,μ) is a measure space.

(b) (X,Σ,μ) is complete.

(c) (X,Σ,μ) is semi-finite.

(d) (X,Σ,μ) is localizable.

(e) (X,Σ,μ) is not locally determined.

(f) (X,Σ,μ) is purely atomic.

***216D A complete, locally determined space which is not localizable** We need two sets I, J such that I is uncountable, $I \subseteq J$ and J cannot be expressed as $\bigcup_{i\in I} K_i$ where every K_i is countable.

(a) Let T be the countable-cocountable σ-algebra of J and ν the countable-cocountable measure on J. Set $X = J \times J$ and for $E \subseteq X$ set

$$E[\{\xi\}] = \{\eta : (\xi,\eta) \in E\}, \quad E^{-1}[\{\xi\}] = \{\eta : (\eta,\xi) \in E\}$$

for every $\xi \in J$. Set

$$\Sigma = \{E : E[\{\xi\}] \text{ and } E^{-1}[\{\xi\}] \text{ belong to } \mathrm{T} \text{ for every } \xi \in J\},$$

$$\mu E = \textstyle\sum_{\xi\in J} \nu E[\{\xi\}] + \sum_{\xi\in J} \nu E^{-1}[\{\xi\}]$$

for every $E \in \Sigma$. Σ is a σ-algebra and μ is a measure.

(b) (X,Σ,μ) is complete.

(d) (X, Σ, μ) is semi-finite.

(e) (X, Σ, μ) is locally determined.

(f) (X, Σ, μ) is not localizable.

(g) (X, Σ, μ) is purely atomic.

***216E A complete, locally determined, localizable space which is not strictly localizable(a)** Let C be any set with cardinal greater than $\mathfrak{c}$. Set $I = \mathcal{P}C$ and $X = \{0,1\}^I$. For $\gamma \in C$, define $x_\gamma \in X$ by saying that $x_\gamma(\Gamma) = 1$ if $\gamma \in \Gamma \subseteq C$ and $x_\gamma(\Gamma) = 0$ if $\gamma \notin \Gamma \subseteq C$. Let $\mathcal{K}$ be the family of countable subsets of I, and for $K \in \mathcal{K}$, $\gamma \in C$ set

$$F_{\gamma K} = \{x : x \in X,\, x \restriction K = x_\gamma \restriction K\} \subseteq X.$$

Let

$$\begin{aligned}\Sigma_\gamma = \{E : E \subseteq X, &\text{ either there is a } K \in \mathcal{K} \text{ such that } F_{\gamma K} \subseteq E \\ &\text{or there is a } K \in \mathcal{K} \text{ such that } F_{\gamma K} \subseteq X \setminus E\}.\end{aligned}$$

Then Σ_γ is a σ-algebra of subsets of X.

(b) Set

$$\Sigma = \bigcap_{\gamma \in C} \Sigma_\gamma;$$

then Σ is a σ-algebra of subsets of X. Define $\mu : \Sigma \to [0, \infty]$ by setting

$$\begin{aligned}\mu E &= \#(\{\gamma : x_\gamma \in E\}) \text{ if this is finite,} \\ &= \infty \text{ otherwise;}\end{aligned}$$

then μ is a measure.

(c) It will be convenient later to know something about the sets

$$G_D = \{x : x \in X,\, x(D) = 1\}$$

for $D \subseteq C$. In particular, every G_D belongs to Σ. Also $\{\gamma : x_\gamma \in G_D\} = D$.

(d) (X, Σ, μ) is complete.

(e) (X, Σ, μ) is semi-finite.

(f) (X, Σ, μ) is localizable.

(g) (X, Σ, μ) is not strictly localizable.

(h) (X, Σ, μ) is purely atomic.