Version of 24.2.16

Introduction to Volume 2

For this second volume I have chosen seven topics through which to explore the insights and challenges offered by measure theory. Some, like the Radon-Nikodým theorem (Chapter 23) are necessary for any understanding of the structure of the subject; others, like Fourier analysis (Chapter 28) and the discussion of function spaces (Chapter 24) demonstrate the power of measure theory to attack problems in general real and functional analysis. But all have applications outside measure theory, and all have influenced its development. These are the parts of measure theory which any analyst may find himself using.

Every topic is one which ideally one would wish undergraduates to have seen, but the length of this volume makes it plain that no ordinary undergraduate course could include very much of it. It is directed rather at graduate level, where I hope it will be found adequate to support all but the most ambitious courses in measure theory, though it is perhaps a bit too solid to be suitable for direct use as a course text, except with careful selection of the parts to be covered. If you are using it to teach yourself measure theory, I strongly recommend an eclectic approach, looking for particular subjects and theorems that seem startling or useful, and working backwards from them. My other objective, of course, is to provide an account of the central ideas at this level in measure theory, rather fuller than can easily be found in one volume elsewhere. I cannot claim that it is 'definitive', but I do think I cover a good deal of ground in ways that provide a firm foundation for further study. As in Volume 1, I usually do not shrink from giving 'best' results, like Lindeberg's condition for the Central Limit Theorem (§274), or the theory of products of arbitrary measure spaces (§251). If I were teaching this material to students in a PhD programme I would rather accept a limitation in the breadth of the course than leave them unaware of what could be done in the areas discussed.

The topics interact in complex ways – one of the purposes of this book is to exhibit their relationships. There is no canonical linear ordering in which they should be taken. Nor do I think organization charts are very helpful, not least because it may be only two or three paragraphs in a section which are needed for a given chapter later on. I do at least try to lay the material of each section out in an order which makes initial segments useful by themselves. But the order in which to take the chapters is to a considerable extent for you to choose, perhaps after a glance at their individual introductions. I have done my best to pitch the exposition at much the same level throughout the volume, sometimes allowing gradients to steepen in the course of a chapter or a section, but always trying to return to something which anyone who has mastered Volume 1 ought to be able to cope with. (Though perhaps the main theorems of Chapter 26 are harder work than the principal results elsewhere, and §286 is only for the most determined.)

I said there were seven topics, and you will see eight chapters ahead of you. This is because Chapter 21 is rather different from the rest. It is the purest of pure measure theory, and is here only because there are places later in the volume where (in my view) the theorems make sense only in the light of some abstract concepts which are not particularly difficult, but are also not obvious. However it is fair to say that the most important ideas of this volume do not really depend on the work of Chapter 21.

As always, it is a puzzle to know how much prior knowledge to assume in this volume. I do of course call on the results of Volume 1 of this treatise whenever they seem to be relevant. I do not doubt, however, that there will be readers who have learnt the elementary theory from other sources. Provided you can, from first principles, construct Lebesgue measure and prove the basic convergence theorems for integrals on arbitrary measure spaces, you ought to be able to embark on the present volume. Perhaps it would be helpful to have in hand the results-only version of Volume 1, since that includes the most important definitions as well as statements of the theorems.

There is also the question of how much material from outside measure theory is needed. Chapter 21 calls for some non-trivial set theory (given in §2A1), but the more advanced ideas are needed only for the counter-examples in §216, and can be passed over to begin with. The problems become acute in Chapter 24. Here we need a variety of results from functional analysis, some of them depending on non-trivial ideas in general topology. For a full understanding of this material there is no substitute for a course in normed spaces up to and including a study of weak compactness. But I do not like to insist on such a preparation, because it is likely to be simultaneously too much and too little. Too much, because I hardly mention linear operators at this stage; too little, because I do ask for some of the theory of non-locally-convex spaces, which is often omitted in first courses on functional analysis. At the risk, therefore, of wasting paper, I have written out condensed accounts of the essential facts (§§2A3-2A5).

Note on second printing, April 2003

For the second printing of this volume, I have made two substantial corrections to inadequate proofs and a large number of minor amendments; I am most grateful to T.D.Austin for his careful reading of the first printing. In addition, I have added a dozen exercises and a handful of straightforward results which turn out to be relevant to the work of later volumes and fit naturally here.

The regular process of revision of this work has led me to make a couple of notational innovations not described explicitly in the early printings of Volume 1. I trust that most readers will find these immediately comprehensible. If, however, you find that there is a puzzling cross-reference which you are unable to match with anything in the version of Volume 1 which you are using, it may be worth while checking the errata pages in `http://www1.essex.ac.uk/maths/people/fremlin/mterr.htm`.

Note on hardback edition, January 2010

For the new ('Lulu') edition of this volume, I have eliminated a number of further errors; no doubt many remain. There are many new exercises, several new theorems and some corresponding rearrangements of material. The new results are mostly additions with little effect on the structure of the work, but there is a short new section (§266) on the Brunn-Minkowski inequality.

Note on second printing of hardback edition, April 2016

There is the usual crop of small mistakes to be corrected, and assorted minor amendments and additions. But my principal reason for issuing a new printed version is a major fault in the proof of Carleson's theorem, where an imprudent move to simplify the argument of Lacey & Thiele 00 was based on an undergraduate error[1]. While the blunder is conspicuous enough, a resolution seems to require an adjustment in a definition, and is not a fair demand on a graduate seminar, the intended readership for this material. Furthermore, the proof was supposed to be a distinguishing feature of not only this volume, but of the treatise as a whole. So, with apologies to any who retired hurt from an encounter with the original version, I present a revision which I hope is essentially sound.

[1] I am most grateful to A.Derighetti for bringing this to my attention.

Version of 17.1.15

*Chapter 21

Taxonomy of measure spaces

I begin this volume with a 'starred chapter'. The point is that I do not really recommend this chapter for beginners. It deals with a variety of technical questions which are of great importance for the later development of the subject, but are likely to be both abstract and obscure for anyone who has not encountered the problems these techniques are designed to solve. On the other hand, if (as is customary) this work is omitted, and the ideas are introduced only when urgently needed, the student is likely to finish with very vague ideas on which theorems can be expected to apply to which types of measure space, and with no vocabulary in which to express those ideas. I therefore take a few pages to introduce the terminology and concepts which can be used to distinguish 'good' measure spaces from others, with a few of the basic relationships. The only paragraphs which are immediately relevant to the theory set out in Volume 1 are those on 'complete', 'σ-finite' and 'semi-finite' measure spaces (211A, 211D, 211F, 211Lc, §212, 213A-213B, 215B), and on Lebesgue measure (211M). For the rest, I think that a newcomer to the subject can very well pass over this chapter for the time being, and return to it for particular items when the text of later chapters refers to it. On the other hand, it can also be used as an introduction to the flavour of the 'purest' kind of measure theory, the study of measure spaces for their own sake, with a systematic discussion of a few of the elementary constructions.

Version of 20.11.03

211 Definitions

I start with a list of definitions, corresponding to the concepts which I have found to be of value in distinguishing different types of measure space. Necessarily, the significance of many of these ideas is likely to be obscure until you have encountered some of the obstacles which arise later on. Nevertheless, you will I hope be able to deal with these definitions on a formal, abstract basis, and to follow the elementary arguments involved in establishing the relationships between them (211L).

In 216C-216E below I will give three substantial examples to demonstrate the rich variety of objects which the definition of 'measure space' encompasses. In the present section, therefore, I content myself with very brief descriptions of sufficient cases to show at least that each of the definitions here discriminates between different spaces (211M-211R).

211A Definition Let (X,Σ,μ) be a measure space. Then μ, or (X,Σ,μ), is (**Carathéodory**) **complete** if whenever $A\subseteq E\in\Sigma$ and $\mu E=0$ then $A\in\Sigma$; that is, if every negligible subset of X is measurable.

211B Definition Let (X,Σ,μ) be a measure space. Then (X,Σ,μ) is a **probability space** if $\mu X=1$. In this case μ is called a **probability** or **probability measure**.

211C Definition Let (X,Σ,μ) be a measure space. Then μ, or (X,Σ,μ), is **totally finite** if $\mu X<\infty$.

211D Definition Let (X,Σ,μ) be a measure space. Then μ, or (X,Σ,μ), is **σ-finite** if there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure such that $X=\bigcup_{n\in\mathbb{N}}E_n$.

Remark Note that in this case we can set

$$F_n=E_n\setminus\bigcup_{i<n}E_i,\quad G_n=\bigcup_{i\le n}E_i$$

for each n, to obtain a partition $\langle F_n\rangle_{n\in\mathbb{N}}$ of X (that is, a disjoint cover of X) into measurable sets of finite measure, and a non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of sets of finite measure covering X.

211E Definition Let (X,Σ,μ) be a measure space. Then μ, or (X,Σ,μ), is **strictly localizable** or **decomposable** if there is a partition $\langle X_i\rangle_{i\in I}$ of X into measurable sets of finite measure such that

$$\Sigma = \{E : E \subseteq X,\ E \cap X_i \in \Sigma\ \forall\ i \in I\},$$

$$\mu E = \sum_{i \in I} \mu(E \cap X_i) \text{ for every } E \in \Sigma.$$

I will call such a family $\langle X_i \rangle_{i \in I}$ a **decomposition** of X.

Remark In this context, we can interpret the sum $\sum_{i \in I} \mu(E \cap X_i)$ simply as

$$\sup\{\sum_{i \in J} \mu(E \cap X_i) : J \text{ is a finite subset of } I\},$$

taking $\sum_{i \in \emptyset} \mu(E \cap X_i) = 0$, because we are concerned only with sums of non-negative terms (cf. 112Bd).

211F Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **semi-finite** if whenever $E \in \Sigma$ and $\mu E = \infty$ there is an $F \subseteq E$ such that $F \in \Sigma$ and $0 < \mu F < \infty$.

211G Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **localizable** if it is semi-finite and, for every $\mathcal{E} \subseteq \Sigma$, there is an $H \in \Sigma$ such that (i) $E \setminus H$ is negligible for every $E \in \mathcal{E}$ (ii) if $G \in \Sigma$ and $E \setminus G$ is negligible for every $E \in \mathcal{E}$, then $H \setminus G$ is negligible. It will be convenient to call such a set H **an essential supremum** of $\mathcal{E}$ in Σ.

Remark The definition here is clumsy, because really the concept applies to measure *algebras* rather than to measure *spaces* (see 211Ya-211Yb). However, the present definition can be made to work (see 213N, 241G, 243G below) and enables us to proceed without a formal introduction to the concept of 'measure algebra' before the time comes to do the job properly in Volume 3.

211H Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **locally determined** if it is semi-finite and

$$\Sigma = \{E : E \subseteq X,\ E \cap F \in \Sigma \text{ whenever } F \in \Sigma \text{ and } \mu F < \infty\};$$

that is to say, for any $E \in \mathcal{P}X \setminus \Sigma$ there is an $F \in \Sigma$ such that $\mu F < \infty$ and $E \cap F \notin \Sigma$.

211I Definition Let (X, Σ, μ) be a measure space. A set $E \in \Sigma$ is an **atom** for μ if $\mu E > 0$ and whenever $F \in \Sigma$, $F \subseteq E$ one of F, $E \setminus F$ is negligible.

211J Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **atomless** or **diffused** if there are no atoms for μ. (Note that this is *not* the same thing as saying that all finite sets are negligible; see 211R below. Some authors use the word **continuous** in this context.)

211K Definition Let (X, Σ, μ) be a measure space. Then μ, or (X, Σ, μ), is **purely atomic** if whenever $E \in \Sigma$ and E is not negligible there is an atom for μ included in E.

Remark Recall that a measure μ on a set X is **point-supported** if μ measures every subset of X and $\mu E = \sum_{x \in E} \mu\{x\}$ for every $E \subseteq X$ (112Bd). Every point-supported measure is purely atomic, because $\{x\}$ must be an atom whenever $\mu\{x\} > 0$, but not every purely atomic measure is point-supported (211R).

211L The relationships between the concepts above are in a sense very straightforward; all the direct implications in which one property implies another are given in the next theorem.

Theorem (a) A probability space is totally finite.

(b) A totally finite measure space is σ-finite.

(c) A σ-finite measure space is strictly localizable.

(d) A strictly localizable measure space is localizable and locally determined.

(e) A localizable measure space is semi-finite.

(f) A locally determined measure space is semi-finite.

proof (a), (b), (e) and (f) are trivial.

(c) Let (X, Σ, μ) be a σ-finite measure space; let $\langle F_n \rangle_{n\in\mathbb{N}}$ be a disjoint sequence of measurable sets of finite measure covering X (see the remark in 211D). If $E \in \Sigma$, then of course $E \cap F_n \in \Sigma$ for every $n \in \mathbb{N}$, and

$$\mu E = \sum_{n=0}^{\infty} \mu(E \cap F_n) = \sum_{n\in\mathbb{N}} \mu(E \cap F_n).$$

If $E \subseteq X$ and $E \cap F_n \in \Sigma$ for every $n \in \mathbb{N}$, then

$$E = \bigcup_{n\in\mathbb{N}} E \cap F_n \in \Sigma.$$

So $\langle F_n \rangle_{n\in\mathbb{N}}$ is a decomposition of X and (X, Σ, μ) is strictly localizable.

(d) Let (X, Σ, μ) be a strictly localizable measure space; let $\langle X_i \rangle_{i\in I}$ be a decomposition of X.

(i) Let $\mathcal{E}$ be a family of measurable subsets of X. Let $\mathcal{F}$ be the family of measurable sets $F \subseteq X$ such that $\mu(F \cap E) = 0$ for every $E \in \mathcal{E}$. Note that $\emptyset \in \mathcal{F}$ and, if $\langle F_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{F}$, then $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{F}$. For each $i \in I$, set $\gamma_i = \sup\{\mu(F \cap X_i) : F \in \mathcal{F}\}$ and choose a sequence $\langle F_{in} \rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\lim_{n\to\infty} \mu(F_{in} \cap X_i) = \gamma_i$; set

$$F_i = \bigcup_{n\in\mathbb{N}} F_{in} \in \mathcal{F}.$$

Set

$$F = \bigcup_{i\in I} F_i \cap X_i \subseteq X$$

and $H = X \setminus F$.

We see that $F \cap X_i = F_i \cap X_i$ for each $i \in I$ (because $\langle X_i \rangle_{i\in I}$ is disjoint), so $F \in \Sigma$ and $H \in \Sigma$. For any $E \in \mathcal{E}$,

$$\mu(E \setminus H) = \mu(E \cap F) = \sum_{i\in I} \mu(E \cap F \cap X_i) = \sum_{i\in I} \mu(E \cap F_i \cap X_i) = 0$$

because every F_i belongs to $\mathcal{F}$. Thus $F \in \mathcal{F}$. If $G \in \Sigma$ and $\mu(E \setminus G) = 0$ for every $E \in \mathcal{E}$, then $X \setminus G$ and $F' = F \cup (X \setminus G)$ belong to $\mathcal{F}$. So $\mu(F' \cap X_i) \le \gamma_i$ for each $i \in I$. But also $\mu(F \cap X_i) \ge \sup_{n\in\mathbb{N}} \mu(F_{in} \cap X_i) = \gamma_i$, so $\mu(F \cap X_i) = \mu(F' \cap X_i)$ for each i. *Because μX_i is finite,* it follows that $\mu((F' \setminus F) \cap X_i) = 0$, for each i. Summing over i, $\mu(F' \setminus F) = 0$, that is, $\mu(H \setminus G) = 0$.

Thus H is an essential supremum for $\mathcal{E}$ in Σ. As $\mathcal{E}$ is arbitrary, (X, Σ, μ) is localizable.

(ii) If $E \in \Sigma$ and $\mu E = \infty$, then there is some $i \in I$ such that

$$0 < \mu(E \cap X_i) \le \mu X_i < \infty;$$

so (X, Σ, μ) is semi-finite. If $E \subseteq X$ and $E \cap F \in \Sigma$ whenever $\mu F < \infty$, then, in particular, $E \cap X_i \in \Sigma$ for every $i \in I$, so $E \in \Sigma$; thus (X, Σ, μ) is locally determined.

211M Example: Lebesgue measure Let us consider Lebesgue measure in the light of the concepts above. Write μ for Lebesgue measure on $\mathbb{R}^r$ and Σ for its domain.

(a) μ is complete, because it is constructed by Carathéodory's method; if $A \subseteq E$ and $\mu E = 0$, then $\mu^* A = \mu^* E = 0$ (writing μ^* for Lebesgue outer measure), so, for any $B \subseteq \mathbb{R}$,

$$\mu^*(B \cap A) + \mu^*(B \setminus A) \le 0 + \mu^* B = \mu^* B,$$

and A must be measurable.

(b) μ is σ-finite, because $\mathbb{R}^r = \bigcup_{n\in\mathbb{N}} [-\mathbf{n}, \mathbf{n}]$, writing $\mathbf{n}$ for the vector $(n, \dots, n)$, and $\mu[-\mathbf{n}, \mathbf{n}] = (2n)^r < \infty$ for every n. Of course μ is neither totally finite nor a probability measure.

(c) Because μ is σ-finite, it is strictly localizable (211Lc), localizable (211Ld), locally determined (211Ld) and semi-finite (211Le or 211Lf).

(d) μ is atomless. $\mathbf{P}$ Suppose that $E \in \Sigma$. Consider the function

$$a \mapsto f(a) = \mu(E \cap [-\mathbf{a}, \mathbf{a}]) : [0, \infty[\to \mathbb{R}$$

We have

$$f(a) \le f(b) \le f(a) + \mu[-\mathbf{b}, \mathbf{b}] - \mu[-\mathbf{a}, \mathbf{a}] = f(a) + (2b)^r - (2a)^r$$

whenever $a \le b$ in $[0, \infty[$, so f is continuous. Now $f(0) = 0$ and $\lim_{n\to\infty} f(n) = \mu E > 0$. By the Intermediate Value Theorem there is an $a \in [0, \infty[$ such that $0 < f(a) < \mu E$. So we have

$$0 < \mu(E \cap [-\mathbf{a}, \mathbf{a}]) < \mu E.$$

As E is arbitrary, μ is atomless. **Q**

(e) It is now a trivial observation that μ cannot be purely atomic, because $\mathbb{R}^r$ itself is a set of positive measure not including any atom.

211N Counting measure Take X to be any uncountable set (e.g., $X = \mathbb{R}$), and μ to be counting measure on X (112Bd).

(a) μ is complete, because if $A \subseteq E$ and $\mu E = 0$ then

$$A = E = \emptyset \in \Sigma.$$

(b) μ is not σ-finite, because if $\langle E_n \rangle_{n\in\mathbb{N}}$ is any sequence of sets of finite measure then every E_n is finite, therefore countable, and $\bigcup_{n\in\mathbb{N}} E_n$ is countable (1A1F), so cannot be X. *A fortiori,* μ is not a probability measure nor totally finite.

(c) μ is strictly localizable. **P** Set $X_x = \{x\}$ for every $x \in X$. Then $\langle X_x \rangle_{x\in X}$ is a partition of X, and for any $E \subseteq X$

$$\mu(E \cap X_x) = 1 \text{ if } x \in E, \quad 0 \text{ otherwise.}$$

By the definition of μ,

$$\mu E = \textstyle\sum_{x\in X} \mu(E \cap X_x)$$

for every $E \subseteq X$, and $\langle X_x \rangle_{x\in X}$ is a decomposition of X. **Q**

Consequently μ is localizable, locally determined and semi-finite.

(d) μ is purely atomic. **P** $\{x\}$ is an atom for every $x \in X$, and if $\mu E > 0$ then surely E includes $\{x\}$ for some x. **Q** Obviously, μ is not atomless.

211O A non-semi-finite space Set $X = \{0\}$, $\Sigma = \{\emptyset, X\}$, $\mu\emptyset = 0$ and $\mu X = \infty$. Then μ is not semi-finite, as $\mu X = \infty$ but X has no subset of non-zero finite measure. It follows that μ cannot be localizable, locally determined, σ-finite, totally finite nor a probability measure. Because $\Sigma = \mathcal{P}X$, μ is complete. X is an atom for μ, so μ is purely atomic (indeed, it is point-supported).

211P A non-complete space Write $\mathcal{B}$ for the σ-algebra of Borel subsets of $\mathbb{R}$ (111G), and μ for the restriction of Lebesgue measure to $\mathcal{B}$ (recall that by 114G every Borel subset of $\mathbb{R}$ is Lebesgue measurable). Then $(\mathbb{R}, \mathcal{B}, \mu)$ is atomless, σ-finite and not complete.

proof (a) To see that μ is not complete, recall that there is a continuous, strictly increasing permutation $g : [0, 1] \to [0, 1]$ such that $\mu g[C] > 0$, where C is the Cantor set, so that there is a set $A \subseteq g[C]$ which is not Lebesgue measurable (134Ib). Now $g^{-1}[A] \subseteq C$ cannot be a Borel set, since $\chi A = \chi(g^{-1}[A]) \circ g^{-1}$ is not Lebesgue measurable, therefore not Borel measurable, and the composition of two Borel measurable functions is Borel measurable (121Eg); so $g^{-1}[A]$ is a non-measurable subset of the negligible set C.

(b) The rest of the arguments of 211M apply to μ just as well as to true Lebesgue measure, so μ is σ-finite and atomless.

***Remark** The argument offered in (a) could give rise to a seriously false impression. The set A referred to there can be constructed only with the use of a strong form of the axiom of choice. No such device is necessary for the result here. There are many methods of constructing non-Borel subsets of the Cantor set, all illuminating in different ways. In the absence of any form of the axiom of choice, there are difficulties with the concept of 'Borel set', and others with the concept of 'Lebesgue measure', which I will come to in Chapter 56; but countable choice is quite sufficient for the existence of a non-Borel subset of $\mathbb{R}$. For details of a possible approach see 423M in Volume 4.

211Q Some probability spaces Two obvious constructions of probability spaces, restricting myself to the methods described in Volume 1, are

(a) the subspace measure induced by Lebesgue measure on $[0,1]$ (131B);

(b) the point-supported measure induced on a set X by a function $h : X \to [0,1]$ such that $\sum_{x\in X} h(x) = 1$, writing $\mu E = \sum_{x\in E} h(x)$ for every $E \subseteq X$; for instance, if X is a singleton $\{x\}$ and $h(x) = 1$, or if $X = \mathbb{N}$ and $h(n) = 2^{-n-1}$.

Of these two, (a) gives an atomless probability measure and (b) gives a purely atomic probability measure.

211R Countable-cocountable measure The following is one of the basic constructions to keep in mind when considering abstract measure spaces.

(a) Let X be any set. Let Σ be the family of those sets $E \subseteq X$ such that either E or $X \setminus E$ is countable. Then Σ is a σ-algebra of subsets of X. **P** (i) $\emptyset$ is countable, so belongs to Σ. (ii) The condition for E to belong to Σ is symmetric between E and $X \setminus E$, so $X \setminus E \in \Sigma$ for every $E \in \Sigma$. (iii) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be any sequence in Σ, and set $E = \bigcup_{n\in\mathbb{N}} E_n$. If every E_n is countable, then E is countable, so belongs to Σ. Otherwise, there is some n such that $X \setminus E_n$ is countable, in which case $X \setminus E \subseteq X \setminus E_n$ is countable, so again $E \in \Sigma$. **Q** Σ is called the **countable-cocountable σ-algebra** of X.

(b) Now consider the function $\mu : \Sigma \to \{0,1\}$ defined by writing $\mu E = 0$ if E is countable, $\mu E = 1$ if $E \in \Sigma$ and E is not countable. Then μ is a measure. **P** (i) $\emptyset$ is countable so $\mu\emptyset = 0$. (ii) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in Σ, and E its union. (α) If every E_m is countable, then so is E, so

$$\mu E = 0 = \sum_{n=0}^{\infty} \mu E_n.$$

(β) If some E_m is uncountable, then $E \supseteq E_m$ also is uncountable, and $\mu E = \mu E_m = 1$. But in this case, because $E_m \in \Sigma$, $X \setminus E_m$ is countable, so E_n, being a subset of $X \setminus E_m$, is countable for every $n \neq m$; thus $\mu E_n = 0$ for every $n \neq m$, and

$$\mu E = 1 = \sum_{n=0}^{\infty} \mu E_n.$$

As $\langle E_n\rangle_{n\in\mathbb{N}}$ is arbitrary, μ is a measure. **Q** This is the **countable-cocountable measure** on X.

(c) If X is any uncountable set and μ is the countable-cocountable measure on X, then μ is a complete, purely atomic probability measure, but is not point-supported. **P** (i) If $A \subseteq E$ and $\mu E = 0$, then E is countable, so A also is countable and belongs to Σ. Thus μ is complete. (ii) Because X is uncountable, $\mu X = 1$ and μ is a probability measure. (iii) If $\mu E > 0$, then $\mu F = \mu E = 1$ whenever F is a non-negligible measurable subset of E, so E is itself an atom; thus μ is purely atomic. (iv) $\mu X = 1 > 0 = \sum_{x\in X} \mu\{x\}$, so μ is not point-supported. **Q**

211X Basic exercises >(a) Let (X,Σ,μ) be a semi-finite measure space. Show that μ is σ-finite iff there is a totally finite measure ν on X with the same measurable sets and the same negligible sets as μ.

>(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function and μ_g the associated Lebesgue-Stieltjes measure (114Xa). Show that μ_g is complete and σ-finite. Show that

(i) μ_g is totally finite iff g is bounded;

(ii) μ_g is a probability measure iff $\lim_{x\to\infty} g(x) - \lim_{x\to-\infty} g(x) = 1$;

(iii) μ_g is atomless iff g is continuous;

(iv) if E is any atom for μ_g, there is a point $x \in E$ such that $\mu_g E = \mu_g\{x\}$;

(v) μ_g is purely atomic iff it is point-supported.

>(c) Let μ be counting measure on a set X. Show that μ is always complete, strictly localizable and purely atomic, and that it is σ-finite iff X is countable, totally finite iff X is finite, a probability measure iff X is a singleton, and atomless iff X is empty.

(d) Show that a point-supported measure is always complete, and is strictly localizable iff it is semi-finite.

(e) Let X be a set. Show that for any σ-ideal $\mathcal{I}$ of subsets of X (definition: 112Db), the set

$$\Sigma = \mathcal{I} \cup \{X \setminus A : A \in \mathcal{I}\}$$

is a σ-algebra of subsets of X, and that there is a measure $\mu : \Sigma \to \{0, 1\}$ given by setting

$$\mu E = 0 \text{ if } E \in \mathcal{I}, \quad \mu E = 1 \text{ if } E \in \Sigma \setminus \mathcal{I}.$$

Show that $\mathcal{I}$ is precisely the null ideal of μ, that μ is complete, totally finite and purely atomic, and is a probability measure iff $X \notin \mathcal{I}$.

(g) Let (X, Σ, μ) be a measure space such that $\mu X > 0$. Show that the set of conegligible subsets of X is a filter on X.

211Y Further exercises (a) Let (X, Σ, μ) be a measure space, and for E, $F \in \Sigma$ write $E \sim F$ if $\mu(E \triangle F) = 0$. Show that $\sim$ is an equivalence relation on Σ. Let $\mathfrak{A}$ be the set of equivalence classes in Σ for $\sim$; for $E \in \Sigma$, write $E^\bullet \in \mathfrak{A}$ for its equivalence class. Show that there is a partial ordering $\subseteq$ on $\mathfrak{A}$ defined by saying that, for E, $F \in \Sigma$,

$$E^\bullet \subseteq F^\bullet \iff \mu(E \setminus F) = 0.$$

Show that μ is localizable iff for every $A \subseteq \mathfrak{A}$ there is an $h \in \mathfrak{A}$ such that (i) $a \subseteq h$ for every $a \in A$ (ii) whenever $g \in \mathfrak{A}$ is such that $a \subseteq g$ for every $a \in \mathfrak{A}$, then $h \subseteq g$.

(b) Let (X, Σ, μ) be a measure space, and construct $\mathfrak{A}$ as in 211Ya. Show that there are operations $\cup$, $\cap$, $\setminus$ on $\mathfrak{A}$ defined by saying that

$$E^\bullet \cap F^\bullet = (E \cap F)^\bullet, \quad E^\bullet \cup F^\bullet = (E \cup F)^\bullet, \quad E^\bullet \setminus F^\bullet = (E \setminus F)^\bullet$$

for all E, $F \in \Sigma$. Show that if $A \subseteq \mathfrak{A}$ is any *countable* set, then there is an $h \in \mathfrak{A}$ such that (i) $a \subseteq h$ for every $a \in A$ (ii) whenever $g \in \mathfrak{A}$ is such that $a \subseteq g$ for every $a \in \mathfrak{A}$, then $h \subseteq g$. Show that there is a functional $\bar{\mu} : \mathfrak{A} \to [0, \infty]$ defined by saying that $\bar{\mu}(E^\bullet) = \mu E$ for every $E \in \Sigma$. ($(\mathfrak{A}, \bar{\mu})$ is called the **measure algebra** of (X, Σ, μ).)

(c) Let (X, Σ, μ) be a semi-finite measure space. Show that it is atomless iff whenever $\epsilon > 0$, $E \in \Sigma$ and $\mu E < \infty$, then there is a finite partition of E into measurable sets of measure at most ϵ.

(d) Let (X, Σ, μ) be a strictly localizable measure space. Show that the following are equiveridical: (i) (X, Σ, μ) is atomless; (ii) for every $\epsilon > 0$ there is a decomposition of X consisting of sets of measure at most ϵ; (iii) there is a measurable function $f : X \to \mathbb{R}$ such that $\mu f^{-1}[\{t\}] = 0$ for every $t \in \mathbb{R}$.

(e) Let Σ be the countable-cocountable σ-algebra of $\mathbb{R}$. Show that $[0, \infty[\notin \Sigma$. Let μ be the restriction of counting measure to Σ. Show that $(\mathbb{R}, \Sigma, \mu)$ is complete, semi-finite and purely atomic, but not localizable nor locally determined.

211 Notes and comments The list of definitions in 211A-211K probably strikes you as quite long enough, even though I have omitted many occasionally useful ideas. The concepts here vary widely in importance, and the importance of each varies widely with context. My own view is that it is absolutely necessary, when studying any measure space, to know its classification under the eleven discriminating features listed here, and to be able to describe any atoms which are present. Fortunately, for most 'ordinary' measure spaces, the classification is fairly quick, because if (for instance) the space is σ-finite, and you know the measure of the whole space, the only remaining questions concern completeness and atoms. The distinctions between spaces which are, or are not, strictly localizable, semi-finite, localizable and locally determined are relevant only for spaces which are not σ-finite, and do not arise in elementary applications.

I think it is also fair to say that the notions of 'complete' and 'locally determined' measure space are technical; I mean, that they do not correspond to significant features of the essential structure of a space, though there are some interesting problems concerning incomplete measures. One manifestation of this is the existence of canonical constructions for rendering spaces complete or complete and locally determined (212C, 213D-213E). In addition, measure spaces which are not semi-finite do not really belong to measure theory, but rather to the more general study of σ-algebras and σ-ideals. The most important classifications, in terms of the behaviour of a measure space, seem to me to be 'σ-finite', 'localizable' and 'strictly localizable'; these are the critical features which cannot be forced by elementary constructions.

If you know anything about Borel subsets of the real line, the argument of part (a) of the proof of 211P must look very clumsy. But 'better' proofs rely on ideas which we shall not need until Volume 4, and the proof here is based on a construction which we have to understand for other reasons.

Version of 10.9.04

212 Complete spaces

In the next two sections of this chapter I give brief accounts of the theory of measure spaces possessing certain of the properties described in §211. I begin with 'completeness'. I give the elementary facts about complete measure spaces in 212A-212B; then I turn to the notion of 'completion' of a measure (212C) and its relationships with the other concepts of measure theory introduced so far (212D-212G).

212A Proposition Any measure space constructed by Carathéodory's method is complete.

proof Recall that 'Carathéodory's method' starts from an arbitrary outer measure $\theta : \mathcal{P}X \to [0,\infty]$ and sets

$$\Sigma = \{E : E \subseteq X,\, \theta A = \theta(A \cap E) + \theta(A \setminus E) \text{ for every } A \subseteq X\}, \quad \mu = \theta\restriction\Sigma$$

(113C). In this case, if $B \subseteq E \in \Sigma$ and $\mu E = 0$, then $\theta B = \theta E = 0$ (113A(ii)), so for any $A \subseteq X$ we have

$$\theta(A \cap B) + \theta(A \setminus B) = \theta(A \setminus B) \le \theta A \le \theta(A \cap B) + \theta(A \setminus B),$$

and $B \in \Sigma$.

212B Proposition (a) If (X,Σ,μ) is a complete measure space, then any conegligible subset of X is measurable.

(b) Let (X,Σ,μ) be a complete measure space, and f a $[-\infty,\infty]$-valued function defined on a subset of X. If f is virtually measurable (that is, there is a conegligible set $E \subseteq X$ such that $f\restriction E$ is measurable), then f is measurable.

(c) Let (X,Σ,μ) be a complete measure space, and f a real-valued function defined on a conegligible subset of X. Then the following are equiveridical, that is, if one is true so are the others:

(i) f is integrable;

(ii) f is measurable and $|f|$ is integrable;

(iii) f is measurable and there is an integrable function g such that $|f| \le_{\text{a.e.}} g$.

proof (a) If E is conegligible, then $X \setminus E$ is negligible, therefore measurable, and E is measurable.

(b) Let $a \in \mathbb{R}$. Then there is an $H \in \Sigma$ such that

$$\{x : (f\restriction E)(x) \le a\} = H \cap \operatorname{dom}(f\restriction E) = H \cap E \cap \operatorname{dom} f.$$

Now $F = \{x : x \in \operatorname{dom} f \setminus E,\, f(x) \le a\}$ is a subset of the negligible set $X \setminus E$, so is measurable, and

$$\{x : f(x) \le a\} = (F \cup H) \cap \operatorname{dom} f \in \Sigma_{\operatorname{dom} f},$$

writing $\Sigma_D = \{D \cap E : E \in \Sigma\}$, as in 121A. As a is arbitrary, f is measurable (135E).

(c)(i)$\Rightarrow$(ii) If f is integrable, then by 122P f is virtually measurable and by 122Re $|f|$ is integrable. By (b) here, f is measurable, so (ii) is true.

(ii)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(i) If f is measurable and g is integrable and $|f| \le_{\text{a.e.}} g$, then f is virtually measurable, $|g|$ is integrable and $|f| \le_{\text{a.e.}} |g|$, so 122P tells us that f is integrable.

212C The completion of a measure Let (X,Σ,μ) be any measure space.

(a) Let $\hat{\Sigma}$ be the family of those sets $E \subseteq X$ such that there are E', $E'' \in \Sigma$ with $E' \subseteq E \subseteq E''$ and $\mu(E'' \setminus E') = 0$. Then $\hat{\Sigma}$ is a σ-algebra of subsets of X. **P** (i) Of course $\emptyset$ belongs to $\hat{\Sigma}$, because we can take $E' = E'' = \emptyset$. (ii) If $E \in \hat{\Sigma}$, take E', $E'' \in \Sigma$ such that $E' \subseteq E \subseteq E''$ and $\mu(E'' \setminus E') = 0$. Then

$$X \setminus E'' \subseteq X \setminus E \subseteq X \setminus E', \quad \mu((X \setminus E') \setminus (X \setminus E'')) = \mu(E'' \setminus E') = 0,$$

so $X \setminus E \in \hat{\Sigma}$. (iii) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\hat{\Sigma}$, then for each n choose E'_n, $E''_n \in \Sigma$ such that $E'_n \subseteq E_n \subseteq E''_n$ and $\mu(E''_n \setminus E'_n) = 0$. Set

$$E = \bigcup_{n\in\mathbb{N}} E_n, \quad E' = \bigcup_{n\in\mathbb{N}} E'_n, \quad E'' = \bigcup_{n\in\mathbb{N}} E''_n;$$

then $E' \subseteq E \subseteq E''$ and $E'' \setminus E' \subseteq \bigcup_{n\in\mathbb{N}}(E''_n \setminus E'_n)$ is negligible, so $E \in \hat{\Sigma}$. **Q**

(b) For $E \in \hat{\Sigma}$, set

$$\hat{\mu}E = \mu^*E = \min\{\mu F : E \subseteq F \in \Sigma\}$$

(132A). It is worth remarking at once that if $E \in \hat{\Sigma}$, E', $E'' \in \Sigma$, $E' \subseteq E \subseteq E''$ and $\mu(E'' \setminus E') = 0$, then $\mu E' = \hat{\mu}E = \mu E''$; this is because

$$\mu E' = \mu^* E' \le \mu^* E \le \mu^* E'' = \mu E'' = \mu E' + \mu(E'' \setminus E) = \mu E'$$

(recalling from 132A, or noting now, that $\mu^* A \le \mu^* B$ whenever $A \subseteq B \subseteq X$, and that μ^* agrees with μ on Σ).

(c) We now find that $(X, \hat{\Sigma}, \hat{\mu})$ is a measure space. **P** (i) Of course $\hat{\mu}$, like μ, takes values in $[0, \infty]$. (ii) $\hat{\mu}\emptyset = \mu\emptyset = 0$. (iii) Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\hat{\Sigma}$, with union E. For each $n \in \mathbb{N}$ choose E'_n, $E''_n \in \Sigma$ such that $E'_n \subseteq E_n \subseteq E''_n$ and $\mu(E''_n \setminus E'_n) = 0$. Set $E' = \bigcup_{n\in\mathbb{N}} E'_n$, $E'' = \bigcup_{n\in\mathbb{N}} E''_n$. Then (as in (a-iii) above) $E' \subseteq E \subseteq E''$ and $\mu(E'' \setminus E') = 0$, so

$$\hat{\mu}E = \mu E' = \sum_{n=0}^{\infty} \mu E'_n = \sum_{n=0}^{\infty} \hat{\mu} E_n$$

because $\langle E'_n \rangle_{n\in\mathbb{N}}$, like $\langle E_n \rangle_{n\in\mathbb{N}}$, is disjoint. **Q**

(d) The measure space $(X, \hat{\Sigma}, \hat{\mu})$ is called the **completion** of the measure space (X, Σ, μ); equally, I will call $\hat{\mu}$ the **completion** of μ, and occasionally (if it seems plain which null ideal is under consideration) I will call $\hat{\Sigma}$ the completion of Σ. Members of $\hat{\Sigma}$ are sometimes called **μ-measurable**.

212D There is something I had better check at once.

Proposition Let (X, Σ, μ) be any measure space. Then $(X, \hat{\Sigma}, \hat{\mu})$, as defined in 212C, is a complete measure space and $\hat{\mu}$ is an extension of μ; and $(X, \hat{\Sigma}, \hat{\mu}) = (X, \Sigma, \mu)$ iff (X, Σ, μ) is complete.

proof (a) Suppose that $A \subseteq E \in \hat{\Sigma}$ and $\hat{\mu}E = 0$. Then (by 212Cb) there is an $E'' \in \Sigma$ such that $E \subseteq E''$ and $\mu E'' = 0$. Accordingly we have

$$\emptyset \subseteq A \subseteq E'', \quad \mu(E'' \setminus \emptyset) = 0,$$

so $A \in \hat{\Sigma}$. As A is arbitrary, $\hat{\mu}$ is complete.

(b) If $E \in \Sigma$, then of course $E \in \hat{\Sigma}$, because $E \subseteq E \subseteq E$ and $\mu(E \setminus E) = 0$; and $\hat{\mu}E = \mu^*E = \mu E$. Thus $\Sigma \subseteq \hat{\Sigma}$ and $\hat{\mu}$ extends μ.

(c) If $\mu = \hat{\mu}$ then of course μ must be complete. If μ is complete, and $E \in \hat{\Sigma}$, then there are E', $E'' \in \Sigma$ such that $E' \subseteq E \subseteq E''$ and $\mu(E'' \setminus E') = 0$. But now $E \setminus E' \subseteq E'' \setminus E'$, so (because (X, Σ, μ) is complete) $E \setminus E' \in \Sigma$ and $E = E' \cup (E \setminus E') \in \Sigma$. As E is arbitrary, $\hat{\Sigma} \subseteq \Sigma$ and $\hat{\Sigma} = \Sigma$ and $\mu = \hat{\mu}$.

212E The importance of this construction is such that it is worth spelling out some further elementary properties.

Proposition Let (X, Σ, μ) be a measure space, and $(X, \hat{\Sigma}, \hat{\mu})$ its completion.

(a) The outer measures $\hat{\mu}^*$, μ^* defined from $\hat{\mu}$ and μ coincide.

(b) μ, $\hat{\mu}$ give rise to the same negligible and conegligible sets and the same sets of full outer measure.

(c) $\hat{\mu}$ is the only measure with domain $\hat{\Sigma}$ which agrees with μ on Σ.

(d) A subset of X belongs to $\hat{\Sigma}$ iff it is expressible as $F\triangle A$ where $F\in\Sigma$ and A is μ-negligible.

proof (a) Take any $A\subseteq X$. (i) If $A\subseteq F\in\Sigma$, then $F\in\hat{\Sigma}$ and $\mu F=\hat{\mu}F$, so

$$\hat{\mu}^*A\le\hat{\mu}F=\mu F;$$

as F is arbitrary, $\hat{\mu}^*A\le\mu^*A$. (ii) If $A\subseteq E\in\hat{\Sigma}$, there is an $E''\in\Sigma$ such that $E\subseteq E''$ and $\mu E''=\hat{\mu}E$, so

$$\mu^*A\le\mu E''=\hat{\mu}E;$$

as E is arbitrary, $\mu^*A\le\hat{\mu}^*A$.

(b) Now, for $A\subseteq X$,

$$A\text{ is }\mu\text{-negligible}\iff\mu^*A=0\iff\hat{\mu}^*A=0\iff A\text{ is }\hat{\mu}\text{-negligible},$$

$$\begin{aligned}A\text{ is }\mu\text{-conegligible}&\iff\mu^*(X\setminus A)=0\\&\iff\hat{\mu}^*(X\setminus A)=0\iff A\text{ is }\hat{\mu}\text{-conegligible.}\end{aligned}$$

If A has full outer measure for μ, $F\in\hat{\Sigma}$ and $F\cap A=\emptyset$, then there is an $F'\in\Sigma$ such that $F'\subseteq F$ and $\mu F'=\hat{\mu}F$; as $F'\cap A=\emptyset$, F' is μ-negligible and F is $\hat{\mu}$-negligible; as F is arbitrary, A has full outer measure for $\hat{\mu}$. In the other direction, of course, if A has full outer measure for $\hat{\mu}$ then

$$\mu^*(F\cap A)=\hat{\mu}^*(F\cap A)=\hat{\mu}F=\mu F$$

for every $F\in\Sigma$, so A has full outer measure for μ.

(c) If $\tilde{\mu}$ is any measure with domain $\hat{\Sigma}$ extending μ, we must have

$$\mu E'\le\tilde{\mu}E\le\mu E'',\quad \mu E'=\hat{\mu}E=\mu E'',$$

so $\tilde{\mu}E=\hat{\mu}E$, whenever E', $E''\in\Sigma$, $E'\subseteq E\subseteq E''$ and $\mu(E''\setminus E')=0$.

(d)(i) If $E\in\hat{\Sigma}$, take E', $E''\in\Sigma$ such that $E'\subseteq E\subseteq E''$ and $\mu(E''\setminus E')=0$. Then $E\setminus E'\subseteq E''\setminus E'$, so $E\setminus E'$ is μ-negligible, and $E=E'\triangle(E\setminus E')$ is the symmetric difference of a member of Σ and a negligible set.

(ii) If $E=F\triangle A$, where $F\in\Sigma$ and A is μ-negligible, take $G\in\Sigma$ such that $\mu G=0$ and $A\subseteq G$; then $F\setminus G\subseteq E\subseteq F\cup G$ and $\mu((F\cup G)\setminus(F\setminus G))=\mu G=0$, so $E\in\hat{\Sigma}$.

212F Now let us consider integration with respect to the completion of a measure.

Proposition Let (X,Σ,μ) be a measure space and $(X,\hat{\Sigma},\hat{\mu})$ its completion.

(a) A $[-\infty,\infty]$-valued function f defined on a subset of X is $\hat{\Sigma}$-measurable iff it is μ-virtually measurable.

(b) Let f be a $[-\infty,\infty]$-valued function defined on a subset of X. Then $\int fd\mu=\int fd\hat{\mu}$ if either is defined in $[-\infty,\infty]$; in particular, f is μ-integrable iff it is $\hat{\mu}$-integrable.

proof (a)(i) Suppose that f is a $[-\infty,\infty]$-valued $\hat{\Sigma}$-measurable function. For $q\in\mathbb{Q}$ let $E_q\in\hat{\Sigma}$ be such that $\{x:f(x)\le q\}=\operatorname{dom}f\cap E_q$, and choose E'_q, $E''_q\in\Sigma$ such that $E'_q\subseteq E_q\subseteq E''_q$ and $\mu(E''_q\setminus E'_q)=0$. Set $H=X\setminus\bigcup_{q\in\mathbb{Q}}(E''_q\setminus E'_q)$; then H is μ-conegligible. For $a\in\mathbb{R}$ set

$$G_a=\bigcup_{q\in\mathbb{Q},q<a}E'_q\in\Sigma;$$

then

$$\{x:x\in\operatorname{dom}(f{\restriction}H),\,(f{\restriction}H)(x)<a\}=G_a\cap\operatorname{dom}(f{\restriction}H).$$

This shows that $f{\restriction}H$ is Σ-measurable, so that f is μ-virtually measurable.

(ii) If f is μ-virtually measurable, then there is a μ-conegligible set $H\subseteq X$ such that $f{\restriction}H$ is Σ-measurable. Since $\Sigma\subseteq\hat{\Sigma}$, $f{\restriction}H$ is also $\hat{\Sigma}$-measurable. And H is $\hat{\mu}$-conegligible, by 212Eb. But this means that f is $\hat{\mu}$-virtually measurable, therefore $\hat{\Sigma}$-measurable, by 212Bb.

(b)(i) Let $f : D \to [-\infty, \infty]$ be a function, where $D \subseteq X$. If either of $\int f d\mu$, $\int f d\hat{\mu}$ is defined in $[-\infty, \infty]$, then f is virtually measurable, and defined almost everywhere, for one of the appropriate measures, and therefore for both (putting (a) above together with 212Bb).

(ii) Now suppose that f is non-negative and integrable either with respect to μ or with respect to $\hat{\mu}$. Let $E \in \Sigma$ be a conegligible set included in $\operatorname{dom} f$ such that $f \restriction E$ is Σ-measurable. For $n \in \mathbb{N}$, $k \geq 1$ set

$$E_{nk} = \{x : x \in E,\ f(x) \geq 2^{-n}k\};$$

then each E_{nk} belongs to Σ and is of finite measure for both μ and $\hat{\mu}$. (If f is μ-integrable,

$$\hat{\mu}E_{nk} = \mu E_{nk} \leq 2^n \int f d\mu;$$

if f is $\hat{\mu}$-integrable,

$$\mu E_{nk} = \hat{\mu}E_{nk} \leq 2^n \int f d\hat{\mu}.)$$

So

$$f_n = \sum_{k=1}^{4^n} 2^{-n}\chi E_{nk}$$

is both μ-simple and $\hat{\mu}$-simple, and $\int f_n d\mu = \int f_n d\hat{\mu}$. Observe that, for $x \in E$,

$$\begin{aligned} f_n(x) &= 2^{-n}k \text{ if } k < 4^n \text{ and } 2^{-n}k \leq f(x) < 2^{-n}(k+1), \\ &= 2^n \text{ if } f(x) \geq 2^n. \end{aligned}$$

Thus $\langle f_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of functions converging to f at every point of E, that is, both μ-almost everywhere and $\hat{\mu}$-almost everywhere. So we have, for any $c \in \mathbb{R}$,

$$\begin{aligned} \int f d\mu = c &\iff \lim_{n\to\infty} \int f_n d\mu = c \\ &\iff \lim_{n\to\infty} \int f_n d\hat{\mu} = c \iff \int f d\hat{\mu} = c. \end{aligned}$$

(iii) As for infinite integrals, recall that for a non-negative function I write '$\int f = \infty$' just when f is defined almost everywhere, is virtually measurable, and is not integrable. So (i) and (ii) together show that $\int f d\mu = \int f d\hat{\mu}$ whenever f is non-negative and either integral is defined in $[0, \infty]$.

(iv) Since both μ, $\hat{\mu}$ agree that $\int f$ is to be interpreted as $\int f^+ - \int f^-$ just when this can be defined in $[-\infty, \infty]$, writing $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$ for $x \in \operatorname{dom} f$, the result for general real-valued f follows at once.

212G I turn now to the question of the effect of the construction on the properties listed in 211B-211K.

Proposition Let (X, Σ, μ) be a measure space, and $(X, \hat{\Sigma}, \hat{\mu})$ its completion.

(a) $(X, \hat{\Sigma}, \hat{\mu})$ is a probability space, or totally finite, or σ-finite, or semi-finite, or localizable, iff (X, Σ, μ) is.

(b) $(X, \hat{\Sigma}, \hat{\mu})$ is strictly localizable if (X, Σ, μ) is, and any decomposition of X for μ is a decomposition for $\hat{\mu}$.

(c) A set $H \in \hat{\Sigma}$ is an atom for $\hat{\mu}$ iff there is an $E \in \Sigma$ such that E is an atom for μ and $\hat{\mu}(H \triangle E) = 0$.

(d) $(X, \hat{\Sigma}, \hat{\mu})$ is atomless or purely atomic iff (X, Σ, μ) is.

proof (a)(i) Because $\hat{\mu}X = \mu X$, $(X, \hat{\Sigma}, \hat{\mu})$ is a probability space, or totally finite, iff (X, Σ, μ) is.

(ii)(α) If (X, Σ, μ) is σ-finite, there is a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$, covering X, with $\mu E_n < \infty$ for each n. Now $\hat{\mu}E_n < \infty$ for each n, so $(X, \hat{\Sigma}, \hat{\mu})$ is σ-finite.

(β) If $(X, \hat{\Sigma}, \hat{\mu})$ is σ-finite, there is a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$, covering X, with $\hat{\mu}E_n < \infty$ for each n. Now we can find, for each n, an $E_n'' \in \Sigma$ such that $\mu E_n'' < \infty$ and $E_n \subseteq E_n''$; so that $\langle E_n'' \rangle_{n\in\mathbb{N}}$ witnesses that (X, Σ, μ) is σ-finite.

(iii)(α) If (X,Σ,μ) is semi-finite and $\hat\mu E=\infty$, then there is an $E'\in\Sigma$ such that $E'\subseteq E$ and $\mu E'=\infty$. Next, there is an $F\in\Sigma$ such that $F\subseteq E'$ and $0<\mu F<\infty$. Of course we now have $F\in\hat\Sigma$, $F\subseteq E$ and $0<\hat\mu F<\infty$. As E is arbitrary, $(X,\hat\Sigma,\mu)$ is semi-finite.

(β) If $(X,\hat\Sigma,\hat\mu)$ is semi-finite and $\mu E=\infty$, then $\hat\mu E=\infty$, so there is an $F\subseteq E$ such that $0<\hat\mu F<\infty$. Next, there is an $F'\in\Sigma$ such that $F'\subseteq F$ and $\mu F'=\hat\mu F$. Of course we now have $F'\subseteq E$ and $0<\mu F'<\infty$. As E is arbitrary, (X,Σ,μ) is semi-finite.

(iv)(α) If (X,Σ,μ) is localizable and $\mathcal{E}\subseteq\hat\Sigma$, then set

$$\mathcal{F}=\{F:F\in\Sigma,\ \exists\ E\in\mathcal{E},\ F\subseteq E\}.$$

Let H be an essential supremum of $\mathcal{F}$ in Σ, as in 211G.

If $E\in\mathcal{E}$, there is an $E'\in\Sigma$ such that $E'\subseteq E$ and $E\setminus E'$ is negligible; now $E'\in\mathcal{F}$, so

$$\hat\mu(E\setminus H)\le\hat\mu(E\setminus E')+\mu(E'\setminus H)=0.$$

If $G\in\hat\Sigma$ and $\hat\mu(E\setminus G)=0$ for every $E\in\mathcal{E}$, let $G''\in\Sigma$ be such that $G\subseteq G''$ and $\hat\mu(G''\setminus G)=0$; then, for any $F\in\mathcal{F}$, there is an $E\in\mathcal{E}$ including F, so that

$$\mu(F\setminus G'')\le\hat\mu(E\setminus G)=0.$$

As F is arbitrary, $\mu(H\setminus G'')=0$ and $\hat\mu(H\setminus G)=0$. This shows that H is an essential supremum of $\mathcal{E}$ in $\hat\Sigma$. As $\mathcal{E}$ is arbitrary, $(X,\hat\Sigma,\hat\mu)$ is localizable.

(β) Suppose that $(X,\hat\Sigma,\hat\mu)$ is localizable and that $\mathcal{E}\subseteq\Sigma$. Working in $(X,\hat\Sigma,\hat\mu)$, let H be an essential supremum for $\mathcal{E}$ in $\hat\Sigma$. Let $H'\in\Sigma$ be such that $H'\subseteq H$ and $\hat\mu(H\setminus H')=0$. Then

$$\mu(E\setminus H')\le\hat\mu(E\setminus H)+\hat\mu(H\setminus H')=0$$

for every $E\in\mathcal{E}$; while if $G\in\Sigma$ and $\mu(E\setminus G)=0$ for every $E\in\mathcal{E}$, we must have

$$\mu(H'\setminus G)\le\hat\mu(H\setminus G)=0.$$

Thus H' is an essential supremum of $\mathcal{E}$ in Σ. As $\mathcal{E}$ is arbitrary, (X,Σ,μ) is localizable.

(b) Let $\langle X_i\rangle_{i\in I}$ be a decomposition of X for μ, as in 211E. Of course it is a partition of X into sets of finite $\hat\mu$-measure. If $H\subseteq X$ and $H\cap X_i\in\hat\Sigma$ for every i, choose for each $i\in I$ sets E_i', $E_i''\in\Sigma$ such that

$$E_i'\subseteq H\cap X_i\subseteq E_i'',\quad \mu(E_i''\setminus E_i')=0.$$

Set $E'=\bigcup_{i\in I}E_i'$, $E''=\bigcup_{i\in I}(E_i''\cap X_i)$. Then $E'\cap X_i=E_i'$, $E''\cap X_i=E_i''\cap X_i$ for each i, so E' and E'' belong to Σ. Also

$$\mu(E''\setminus E')=\sum_{i\in I}\mu(E_i''\cap X_i\setminus E_i')=0.$$

As $E'\subseteq H\subseteq E''$, $H\in\hat\Sigma$ and

$$\hat\mu H=\mu E'=\sum_{i\in I}\mu E_i'=\sum_{i\in I}\hat\mu(H\cap X_i).$$

As H is arbitrary, $\langle X_i\rangle_{i\in I}$ is a decomposition of X for $\hat\mu$.

Accordingly, $(X,\hat\Sigma,\hat\mu)$ is strictly localizable if such a decomposition exists, which is so if (X,Σ,μ) is strictly localizable.

(c)-(d)(i) Suppose that $E\in\hat\Sigma$ is an atom for $\hat\mu$. Let $E'\in\Sigma$ be such that $E'\subseteq E$ and $\hat\mu(E\setminus E')=0$. Then $\mu E'=\hat\mu E>0$. If $F\in\Sigma$ and $F\subseteq E'$, then $F\subseteq E$, so either $\mu F=\hat\mu F=0$ or $\mu(E'\setminus F)=\hat\mu(E\setminus F)=0$. As F is arbitrary, E' is an atom for μ, and $\hat\mu(E\triangle E')=\hat\mu(E\setminus E')=0$.

(ii) Suppose that $E\in\Sigma$ is an atom for μ, and that $H\in\hat\Sigma$ is such that $\hat\mu(H\triangle E)=0$. Then $\hat\mu H=\mu E>0$. If $F\in\hat\Sigma$ and $F\subseteq H$, let $F'\subseteq F$ be such that $F'\in\Sigma$ and $\hat\mu(F\setminus F')=0$. Then $E\cap F'\subseteq E$ and $\hat\mu(F\triangle(E\cap F'))=0$, so either $\hat\mu F=\mu(E\cap F')=0$ or $\hat\mu(H\setminus F)=\mu(E\setminus F')=0$. As F is arbitrary, H is an atom for $\hat\mu$.

(iii) It follows at once that $(X,\hat\Sigma,\hat\mu)$ is atomless iff (X,Σ,μ) is.

(iv)(α) On the other hand, if (X,Σ,μ) is purely atomic and $\hat\mu H>0$, there is an $E\in\Sigma$ such that $E\subseteq H$ and $\mu E>0$, and an atom F for μ such that $F\subseteq E$; but F is also an atom for $\hat\mu$. As H is arbitrary, $(X,\hat\Sigma,\hat\mu)$ is purely atomic.

(β) And if $(X,\hat\Sigma,\hat\mu)$ is purely atomic and $\mu E>0$, then there is an $H\subseteq E$ which is an atom for $\hat\mu$; now let $F\in\Sigma$ be such that $F\subseteq H$ and $\hat\mu(H\setminus F)=0$, so that F is an atom for μ and $F\subseteq E$. As E is arbitary, (X,Σ,μ) is purely atomic.

212X Basic exercises >(a) Let (X,Σ,μ) be a complete measure space. Suppose that $A\subseteq E\in\Sigma$ and that $\mu^*A+\mu^*(E\setminus A)=\mu E<\infty$. Show that $A\in\Sigma$.

>(b) Let μ and ν be two measures on a set X, with completions $\hat\mu$ and $\hat\nu$. Show that the following are equiveridical: (i) the outer measures μ^*, ν^* defined from μ and ν coincide; (ii) $\hat\mu E=\hat\nu E$ whenever either is defined and finite; (iii) $\int f d\mu=\int f d\nu$ whenever f is a real-valued function such that either integral is defined and finite. (*Hint*: for (i)$\Rightarrow$(ii), if $\hat\mu E<\infty$, take a measurable envelope F of E for ν and calculate $\nu^*E+\nu^*(F\setminus E)$.)

(c) Let μ be the restriction of Lebesgue measure to the Borel σ-algebra of $\mathbb{R}$, as in 211P. Show that its completion is Lebesgue measure itself. (*Hint*: 134F.)

(d) Repeat 212Xc for (i) Lebesgue measure on $\mathbb{R}^r$ (ii) Lebesgue-Stieltjes measures on $\mathbb{R}$ (114Xa).

(e) Let X be a set and Σ a σ-algebra of subsets of X. Let $\mathcal{I}$ be a σ-ideal of subsets of X (112Db). (i) Show that $\Sigma_1=\{E\triangle A: E\in\Sigma,\ A\in\mathcal{I}\}$ is a σ-algebra of subsets of X. (ii) Let Σ_2 be the family of sets $E\subseteq X$ such that there are E', $E''\in\Sigma$ with $E'\subseteq E\subseteq E''$ and $E''\setminus E'\in\mathcal{I}$. Show that Σ_2 is a σ-algebra of subsets of X and that $\Sigma_2\subseteq\Sigma_1$. (iii) Show that $\Sigma_2=\Sigma_1$ iff every member of $\mathcal{I}$ is included in a member of $\Sigma\cap\mathcal{I}$.

(f) Let (X,Σ,μ) be a measure space, Y any set and $\phi:X\to Y$ a function. Set $\theta B=\mu^*\phi^{-1}[B]$ for every $B\subseteq Y$. (i) Show that θ is an outer measure on Y. (ii) Let ν be the measure defined from θ by Carathéodory's method, and T its domain. Show that if $C\subseteq Y$ and $\phi^{-1}[C]\in\Sigma$ then $C\in\mathrm{T}$. (iii) Suppose that (X,Σ,μ) is complete and totally finite. Show that ν is the image measure $\mu\phi^{-1}$.

(g) Let g, h be two non-decreasing functions from $\mathbb{R}$ to itself, and μ_g, μ_h the associated Lebesgue-Stieltjes measures. Show that a real-valued function f defined on a subset of $\mathbb{R}$ is μ_{g+h}-integrable iff it is both μ_g-integrable and μ_h-integrable, and that then $\int f d\mu_{g+h}=\int f d\mu_g+\int f d\mu_h$. (*Hint*: 114Yb).

(h) Let (X,Σ,μ) be a measure space, and $\mathcal{I}$ a σ-ideal of subsets of X; set $\Sigma_1=\{E\triangle A:E\in\Sigma,\ A\in\mathcal{I}\}$, as in 212Xe. Show that if every member of $\Sigma\cap\mathcal{I}$ is μ-negligible, then there is a unique extension of μ to a measure μ_1 with domain Σ_1 such that $\mu_1 A=0$ for every $A\in\mathcal{I}$.

(i) Let (X,Σ,μ) be a complete measure space such that $\mu X>0$, Y a set, $f:X\to Y$ a function and μf^{-1} the image measure on Y. Show that if $\mathcal{F}$ is the filter of μ-conegligible subsets of X, then the image filter $f[[\mathcal{F}]]$ (2A1Ib) is the filter of μf^{-1}-conegligible subsets of Y.

(j) Let (X,Σ,μ) be a complete measure space and $f:X\to\mathbb{R}$ a function such that $\overline{\int} f d\mu<\infty$. Show that there is a measurable function $g:X\to\mathbb{R}$ such that $f(x)\le g(x)$ for every $x\in X$ and $\int g\,d\mu=\overline{\int} f d\mu$.

212Y Further exercises (a) Let X be a set and ϕ an inner measure on X, that is, a functional from $\mathcal{P}X$ to $[0,\infty]$ such that

$\phi\emptyset=0$,

$\phi(A\cup B)\ge\phi A+\phi B$ if $A\cap B=\emptyset$,

$\phi(\bigcap_{n\in\mathbb{N}}A_n)=\lim_{n\to\infty}\phi A_n$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X and $\phi A_0<\infty$,

if $\phi A=\infty$, $a\in\mathbb{R}$ there is a $B\subseteq A$ such that $a\le\phi B<\infty$.

Let μ be the measure defined from ϕ, that is, $\mu=\phi\restriction\Sigma$, where

$$\Sigma=\{E:\phi(A)=\phi(A\cap E)+\phi(A\setminus E)\ \forall\ A\subseteq X\}$$

(113Yg). Show that μ must be complete.

(b) Let (X,Σ,μ) be a strictly localizable measure space. Suppose that for every $n\in\mathbb{N}$ there is a disjoint family $\langle D_i\rangle_{i<n}$ of subsets of full outer measure. Show that there is a disjoint sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of sets of full outer measure.

212 Notes and comments The process of completion is so natural, and so universally applicable, and so convenient, that over large parts of measure theory it is reasonable to use only complete measure spaces. Indeed many authors so phrase their definitions that, explicitly or implicitly, only complete measure spaces are considered. In this treatise I avoid taking quite such a large step, even though it would simplify the statements of many of the theorems in this volume (for instance). I did take the trouble, in Volume 1, to give a definition of 'integrable function' which, in effect, looks at integrability with respect to the completion of a measure (212Fb). There are non-complete measure spaces which are worthy of study (for example, the restriction of Lebesgue measure to the Borel σ-algebra of $\mathbb{R}$ – see 211P), and some interesting questions to be dealt with in Volumes 3 and 5 apply to them. At the cost of rather a lot of verbal manoeuvres, therefore, I prefer to write theorems out in a form in which they can be applied to arbitrary measure spaces, without assuming completeness. But it would be reasonable, and indeed would sharpen your technique, if you regularly sought the alternative formulations which become natural if you are interested only in complete spaces.

Version of 13.9.13

213 Semi-finite, locally determined and localizable spaces

In this section I collect a variety of useful facts concerning these types of measure space. I start with the characteristic properties of semi-finite spaces (213A-213B), and continue with complete locally determined spaces (213C) and the concept of 'c.l.d. version' (213D-213H), the most powerful of the universally available methods of modifying a measure space into a better-behaved one. I briefly discuss 'locally determined negligible sets' (213I-213L), and measurable envelopes (213L-213M), and end with results on localizable spaces (213N) and strictly localizable spaces (213O).

213A Lemma Let (X,Σ,μ) be a semi-finite measure space. Then

$$\mu E=\sup\{\mu F: F\in\Sigma,\ F\subseteq E,\ \mu F<\infty\}$$

for every $E\in\Sigma$.

proof Set $c=\sup\{\mu F: F\in\Sigma,\ F\subseteq E,\ \mu F<\infty\}$. Then surely $c\le\mu E$, so if $c=\infty$ we can stop. If $c<\infty$, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable subsets of E, of finite measure, such that $\lim_{n\to\infty}\mu F_n=c$; set $F=\bigcup_{n\in\mathbb{N}}F_n$. For each $n\in\mathbb{N}$, $\bigcup_{k\le n}F_k$ is a measurable set of finite measure included in E, so $\mu(\bigcup_{k\le n}F_k)\le c$, and

$$\mu F=\lim_{n\to\infty}\mu(\bigcup_{k\le n}F_k)\le c.$$

Also

$$\mu F\ge\sup_{n\in\mathbb{N}}\mu F_n\ge c,$$

so $\mu F=c$.

If F' is a measurable subset of $E\setminus F$ and $\mu F'<\infty$, then $F\cup F'$ has finite measure and is included in E, so has measure at most $c=\mu F$; it follows that $\mu F'=0$. But this means that $\mu(E\setminus F)$ cannot be infinite, since then, because (X,Σ,μ) is semi-finite, it would have to include a measurable set of non-zero finite measure. So $E\setminus F$ has finite measure, and is therefore in fact negligible; and $\mu E=c$, as claimed.

213B Proposition Let (X,Σ,μ) be a semi-finite measure space. Let f be a μ-virtually measurable $[0,\infty]$-valued function defined almost everywhere in X. Then

$$\begin{aligned}\int f &= \sup\{\int g : g \text{ is a simple function, } g \le_{\text{a.e.}} f\}\\ &= \sup_{F\in\Sigma,\mu F<\infty} \int_F f\end{aligned}$$

in $[0,\infty]$.

proof (a) For any measure space (X,Σ,μ), a $[0,\infty]$-valued function defined on a subset of X is integrable iff there is a conegligible set E such that

(α) $E \subseteq \operatorname{dom} f$ and $f\restriction E$ is measurable,

(β) $\sup\{\int g : g$ is a simple function, $g \le_{\text{a.e.}} f\}$ is finite,

(γ) for every $\epsilon > 0$, $\{x : x \in E,\ f(x) \ge \epsilon\}$ has finite measure,

(δ) f is finite almost everywhere

(see 122Ja, 133B). But if μ is semi-finite, (γ) and (δ) are consequences of the rest. **P** Let $\epsilon > 0$. Set

$$E_\epsilon = \{x : x \in E,\ f(x) \ge \epsilon\},$$

$$c = \sup\{\textstyle\int g : g \text{ is a simple function, } g \le_{\text{a.e.}} f\};$$

we are supposing that c is finite. If $F \subseteq E_\epsilon$ is measurable and $\mu F < \infty$, then $\epsilon\chi F$ is a simple function and $\epsilon\chi F \le_{\text{a.e.}} f$, so

$$\epsilon\mu F = \textstyle\int \epsilon\chi F \le c, \quad \mu F \le \frac{c}{\epsilon}.$$

As F is arbitrary, 213A tells us that $\mu E_\epsilon \le \frac{c}{\epsilon}$ is finite. As ϵ is arbitrary, (γ) is satisfied.

As for (δ), if $F = \{x : x \in E,\ f(x) = \infty\}$ then μF is finite (by (γ)) and $n\chi F \le_{\text{a.e.}} f$, so $n\mu F \le c$, for every $n \in \mathbb{N}$, so $\mu F = 0$. **Q**

(b) Now suppose that $f : D \to [0,\infty]$ is a μ-virtually measurable function, where $D \subseteq X$ is conegligible, so that $\int f$ is defined in $[0,\infty]$ (135F). Then (a) tells us that

$$\int f = \sup_{\substack{g \text{ is simple}\\ g\le f \text{ a.e.}}} \int g$$

(if either is finite, and therefore also if either is infinite)

$$= \sup_{\substack{g \text{ is simple}\\ g\le f \text{ a.e.}\\ \mu F<\infty}} \int_F g \le \sup_{\mu F<\infty} \int_F f \le \int f,$$

so we have the equalities we seek.

***213C Proposition** Let (X,Σ,μ) be a complete locally determined measure space, and μ^* the outer measure derived from μ (132A-132B). Then the measure defined from μ^* by Carathéodory's method is μ itself.

proof Write $\check\mu$ for the measure defined by Carathéodory's method from μ^*, and $\check\Sigma$ for its domain.

(a) If $E \in \Sigma$ and $A \subseteq X$ then $\mu^*(A \cap E) + \mu^*(A \setminus E) = \mu^*A$ (132Af), so $E \in \check\Sigma$. Now $\check\mu E = \mu^*E = \mu E$ (132Ac). Thus $\Sigma \subseteq \check\Sigma$ and $\mu = \check\mu\restriction\Sigma$.

(b) Now suppose that $H \in \check\Sigma$. Let $E \in \Sigma$ be such that $\mu E < \infty$. Then $H \cap E \in \Sigma$. **P** Let E_1, $E_2 \in \Sigma$ be measurable envelopes of $E \cap H$, $E \setminus H$ respectively, both included in E (132Ee). Because $H \in \check\Sigma$,

$$\mu E_1 + \mu E_2 = \mu^*(E \cap H) + \mu^*(E \setminus H) = \mu^*E = \mu E.$$

As $E_1 \cup E_2 = E$,

$$\mu(E_1 \cap E_2) = \mu E_1 + \mu E_2 - \mu E = 0.$$

Now $E_1 \setminus (E \cap H) \subseteq E_1 \cap E_2$; because μ is complete, $E_1 \setminus (E \cap H)$ and $E \cap H$ belong to Σ. **Q**

As E is arbitrary, and μ is locally determined, $H \in \Sigma$. As H is arbitrary, $\check{\Sigma} = \Sigma$ and $\check{\mu} = \mu$.

213D C.l.d. versions: Proposition Let (X, Σ, μ) be a measure space. Write $(X, \hat{\Sigma}, \hat{\mu})$ for its completion (212C) and Σ^f for $\{E : E \in \Sigma,\ \mu E < \infty\}$. Set

$$\tilde{\Sigma} = \{H : H \subseteq X,\ H \cap E \in \hat{\Sigma} \text{ for every } E \in \Sigma^f\},$$

and for $H \in \tilde{\Sigma}$ set

$$\tilde{\mu} H = \sup\{\hat{\mu}(H \cap E) : E \in \Sigma^f\}.$$

Then $(X, \tilde{\Sigma}, \tilde{\mu})$ is a complete locally determined measure space.

proof (a) I check first that $\tilde{\Sigma}$ is a σ-algebra. **P** (i) $\emptyset \cap E = \emptyset \in \hat{\Sigma}$ for every $E \in \Sigma^f$, so $\emptyset \in \tilde{\Sigma}$. (ii) If $H \in \tilde{\Sigma}$ then

$$(X \setminus H) \cap E = E \setminus (E \cap H) \in \hat{\Sigma}$$

for every $E \in \Sigma^f$, so $X \setminus H \in \tilde{\Sigma}$. (iii) If $\langle H_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\tilde{\Sigma}$ with union H, then

$$H \cap E = \bigcup_{n \in \mathbb{N}} H_n \cap E \in \hat{\Sigma}$$

for every $E \in \Sigma^f$, so $H \in \tilde{\Sigma}$. **Q**

(b) It is obvious that $\tilde{\mu}\emptyset = 0$. If $\langle H_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\tilde{\Sigma}$ with union H, then

$$\begin{aligned}\tilde{\mu} H &= \sup\{\hat{\mu}(H \cap E) : E \in \Sigma^f\} \\ &= \sup\{\sum_{n=0}^{\infty} \hat{\mu}(H_n \cap E) : E \in \Sigma^f\} \le \sum_{n=0}^{\infty} \tilde{\mu} H_n.\end{aligned}$$

On the other hand, given $a < \sum_{n=0}^{\infty} \tilde{\mu} H_n$, there is an $m \in \mathbb{N}$ such that $a < \sum_{n=0}^{m} \tilde{\mu} H_n$; now we can find $E_0, \dots, E_m \in \Sigma^f$ such that $a \le \sum_{n=0}^{m} \hat{\mu}(H_n \cap E_n)$. Set $E = \bigcup_{n \le m} E_n \in \Sigma^f$; then

$$\tilde{\mu} H \ge \hat{\mu}(H \cap E) = \sum_{n=0}^{\infty} \hat{\mu}(H_n \cap E) \ge \sum_{n=0}^{m} \hat{\mu}(H_n \cap E_n) \ge a.$$

As a is arbitrary, $\tilde{\mu} H \ge \sum_{n=0}^{\infty} \tilde{\mu} H_n$ and $\tilde{\mu} H = \sum_{n=0}^{\infty} \tilde{\mu} H_n$.

(c) Thus $(X, \tilde{\Sigma}, \tilde{\mu})$ is a measure space. To see that it is semi-finite, note first that $\hat{\Sigma} \subseteq \tilde{\Sigma}$ (because if $H \in \hat{\Sigma}$ then surely $H \cap E \in \hat{\Sigma}$ for every $E \in \Sigma^f$), and that $\tilde{\mu} H = \hat{\mu} H$ whenever $\hat{\mu} H < \infty$ (because then, by the definition in 212Ca, there is an $E \in \Sigma^f$ such that $H \subseteq E$, so that $\tilde{\mu} H = \hat{\mu}(H \cap E) = \hat{\mu} H$). Now suppose that $H \in \tilde{\Sigma}$ and that $\tilde{\mu} H = \infty$. There is surely an $E \in \Sigma^f$ such that $\hat{\mu}(H \cap E) > 0$; but now $0 < \hat{\mu}(H \cap E) < \infty$, so $0 < \tilde{\mu}(H \cap E) < \infty$.

(d) Thus $(X, \tilde{\Sigma}, \tilde{\mu})$ is a semi-finite measure space. To see that it is locally determined, let $H \subseteq X$ be such that $H \cap G \in \tilde{\Sigma}$ whenever $G \in \tilde{\Sigma}$ and $\tilde{\mu} G < \infty$. Then, in particular, we must have $H \cap E \in \tilde{\Sigma}$ for every $E \in \Sigma^f$. But this means in fact that $H \cap E \in \hat{\Sigma}$ for every $E \in \Sigma^f$, so that $H \in \tilde{\Sigma}$. As H is arbitrary, (X, Σ, μ) is locally determined.

(e) To see that $(X, \tilde{\Sigma}, \tilde{\mu})$ is complete, suppose that $A \subseteq H \in \tilde{\Sigma}$ and that $\tilde{\mu} H = 0$. Then for every $E \in \Sigma^f$ we must have $\hat{\mu}(H \cap E) = 0$. Because $(X, \hat{\Sigma}, \hat{\mu})$ is complete, and $A \cap E \subseteq H \cap E$, $A \cap E \in \hat{\Sigma}$. As E is arbitrary, $A \in \tilde{\Sigma}$.

213E Definition For any measure space (X, Σ, μ), I will call $(X, \tilde{\Sigma}, \tilde{\mu})$, as constructed in 213D, the **c.l.d. version** ('complete locally determined version') of (X, Σ, μ); and $\tilde{\mu}$ will be the **c.l.d. version** of μ.

213F Following the same pattern as in 212E-212G, I start with some elementary remarks to facilitate manipulation of this construction.

Proposition Let (X,Σ,μ) be any measure space and $(X,\tilde{\Sigma},\tilde{\mu})$ its c.l.d. version.

(a) $\Sigma\subseteq\tilde{\Sigma}$ and $\tilde{\mu}E=\mu E$ whenever $E\in\Sigma$ and $\mu E<\infty$ – in fact, if $(X,\hat{\Sigma},\hat{\mu})$ is the completion of (X,Σ,μ), $\hat{\Sigma}\subseteq\tilde{\Sigma}$ and $\tilde{\mu}E=\hat{\mu}E$ whenever $\hat{\mu}E<\infty$.

(b) Writing $\tilde{\mu}^*$ and μ^* for the outer measures defined from $\tilde{\mu}$ and μ respectively, $\tilde{\mu}^*A\le\mu^*A$ for every $A\subseteq X$, with equality if μ^*A is finite. In particular, μ-negligible sets are $\tilde{\mu}$-negligible; consequently, μ-conegligible sets are $\tilde{\mu}$-conegligible.

(c) If $H\in\tilde{\Sigma}$,

(i) $\tilde{\mu}H=\sup\{\mu F:E\in\Sigma,\ \mu F<\infty,\ F\subseteq H\}$;

(ii) there is an $E\in\Sigma$ such that $E\subseteq H$ and $\mu E=\tilde{\mu}H$, so that if $\tilde{\mu}H<\infty$ then $\tilde{\mu}(H\setminus E)=0$.

proof (a) This is already covered by remarks in the proof of 213D.

(b) If $\mu^*A=\infty$ then surely $\tilde{\mu}^*A\le\mu^*A$. If $\mu^*A<\infty$, take $E\in\Sigma$ such that $A\subseteq E$ and $\mu E=\mu^*A$ (132Aa). Then

$$\tilde{\mu}^*A\le\tilde{\mu}E=\mu E=\mu^*A.$$

On the other hand, if $A\subseteq H\in\tilde{\Sigma}$, then

$$\tilde{\mu}H\ge\hat{\mu}(H\cap E)\ge\hat{\mu}^*A=\mu^*A,$$

using 212Ea. So $\mu^*A\le\tilde{\mu}^*A$ and $\mu^*A=\tilde{\mu}^*A$.

(c) Write Σ^f for $\{E:E\in\Sigma,\ \mu E<\infty\}$; then, by the definition in 213D, $\tilde{\mu}H=\sup\{\hat{\mu}(H\cap E):E\in\Sigma^f\}$. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ^f such that $\tilde{\mu}H=\sup_{n\in\mathbb{N}}\hat{\mu}(H\cap E_n)$. For each $n\in\mathbb{N}$ there is an $F_n\in\Sigma$ such that $F_n\subseteq H\cap E_n$ and $\mu F_n=\hat{\mu}(H\cap E_n)$ (212C). Set $E=\bigcup_{n\in\mathbb{N}}F_n$. Then $E\in\Sigma$, $E\subseteq H$ and

$$\tilde{\mu}H=\sup_{n\in\mathbb{N}}\mu F_n\le\lim_{n\to\infty}\mu(\bigcup_{i\le n}F_i)=\mu E=\lim_{n\to\infty}\tilde{\mu}(\bigcup_{i\le n}F_i)\le\tilde{\mu}H,$$

so $\mu E=\tilde{\mu}H$, and if $\tilde{\mu}H<\infty$ then $\tilde{\mu}(H\setminus E)=0$. At the same time,

$$\tilde{\mu}H=\sup_{n\in\mathbb{N}}\mu F_n\le\sup_{F\in\Sigma^f,F\subseteq H}\mu F=\sup_{F\in\Sigma^f,F\subseteq H}\tilde{\mu}F$$

(by (a) again)

$$\le\tilde{\mu}H,$$

so we have equality here too.

213G The next step is to look at functions which are measurable or integrable with respect to $\tilde{\mu}$.

Proposition Let (X,Σ,μ) be a measure space, and $(X,\tilde{\Sigma},\tilde{\mu})$ its c.l.d. version.

(a) If a real-valued function f defined on a subset of X is μ-virtually measurable, it is $\tilde{\Sigma}$-measurable.

(b) If a real-valued function is μ-integrable, it is $\tilde{\mu}$-integrable with the same integral.

(c) If f is a $\tilde{\mu}$-integrable real-valued function, there is a μ-integrable real-valued function which is equal to f $\tilde{\mu}$-almost everywhere.

proof Write Σ^f for $\{E:E\in\Sigma,\ \mu E<\infty\}$. By 213Fa, $\tilde{\mu}$ and μ agree on Σ^f.

(a) By 212Fa, f is $\hat{\Sigma}$-measurable, where $\hat{\Sigma}$ is the domain of the completion of μ; but since $\hat{\Sigma}\subseteq\tilde{\Sigma}$, f is $\tilde{\Sigma}$-measurable.

(b)(i) If f is a μ-simple function it is $\tilde{\mu}$-simple, and $\int f d\mu=\int f d\tilde{\mu}$, because $\tilde{\mu}E=\mu E$ for every $E\in\Sigma^f$.

(ii) If f is a non-negative μ-integrable function, there is a non-decreasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of μ-simple functions converging to f μ-almost everywhere; now (by 213Fb) μ-negligible sets are $\tilde{\mu}$-negligible, so $\langle f_n\rangle_{n\in\mathbb{N}}$ converges to f $\tilde{\mu}$-a.e. and (by B.Levi's theorem, 123A) f is $\tilde{\mu}$-integrable, with

$$\int f d\tilde{\mu}=\lim_{n\to\infty}\int f_n d\tilde{\mu}=\lim_{n\to\infty}\int f_n d\mu=\int f d\mu.$$

(iii) In general, if $\int f d\mu$ is defined in $\mathbb{R}$, we have

$$\int f d\tilde{\mu} = \int f^+ d\tilde{\mu} - \int f^- d\tilde{\mu} = \int f^+ d\mu - \int f^- d\mu = \int f d\mu,$$

writing f^+ for $f \vee 0$ and f^- for $(-f) \vee 0$.

(c)(i) Let f be a $\tilde{\mu}$-simple function. Express it as $\sum_{i=0}^n a_i \chi H_i$ where $\tilde{\mu} H_i < \infty$ for each i. Choose $E_0, \dots, E_n \in \Sigma$ such that $E_i \subseteq H_i$ and $\tilde{\mu}(H_i \setminus E_i) = 0$ for each i (using 213Fc above). Then $g = \sum_{i=0}^n a_i \chi E_i$ is μ-simple, $g = f$ $\tilde{\mu}$-a.e., and $\int g\, d\mu = \int f d\tilde{\mu}$.

(ii) Let f be a non-negative $\tilde{\mu}$-integrable function. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of $\tilde{\mu}$-simple functions converging $\tilde{\mu}$-almost everywhere to f. For each n, choose a μ-simple function g_n equal $\tilde{\mu}$-almost everywhere to f_n. Then $\{x : g_{n+1}(x) < g_n(x)\}$ belongs to Σ^f and is $\tilde{\mu}$-negligible, therefore μ-negligible. So $\langle g_n \rangle_{n\in\mathbb{N}}$ is non-decreasing μ-almost everywhere. Because

$$\lim_{n\to\infty} \int g_n d\mu = \lim_{n\to\infty} \int f_n d\tilde{\mu} = \int f d\tilde{\mu},$$

B.Levi's theorem tells us that $\langle g_n \rangle_{n\in\mathbb{N}}$ converges μ-almost everywhere to a μ-integrable function g; because μ-negligible sets are $\tilde{\mu}$-negligible,

$$\begin{aligned}(X \setminus \operatorname{dom} f) &\cup (X \setminus \operatorname{dom} g)\\ &\cup \bigcup_{n\in\mathbb{N}} \{x : f_n(x) \neq g_n(x)\}\\ &\cup \{x : x \in \operatorname{dom} f,\ f(x) \neq \sup_{n\in\mathbb{N}} f_n(x)\}\\ &\cup \{x : x \in \operatorname{dom} g,\ g(x) \neq \sup_{n\in\mathbb{N}} g_n(x)\}\end{aligned}$$

is $\tilde{\mu}$-negligible, and $f = g$ $\tilde{\mu}$-a.e.

(iii) If f is $\tilde{\mu}$-integrable, express it as $f_1 - f_2$ where f_1 and f_2 are $\tilde{\mu}$-integrable and non-negative; then there are μ-integrable functions g_1, g_2 such that $f_1 = g_1$, $f_2 = g_2$ $\tilde{\mu}$-a.e., so that $g = g_1 - g_2$ is μ-integrable and equal to f $\tilde{\mu}$-a.e.

213H Thirdly, I turn to the effect of the construction here on the other properties being considered in this chapter.

Proposition Let (X, Σ, μ) be a measure space, $(X, \hat{\Sigma}, \hat{\mu})$ its completion and $(X, \tilde{\Sigma}, \tilde{\mu})$ its c.l.d. version.

(a) If (X, Σ, μ) is a probability space, or totally finite, or σ-finite, or strictly localizable, so is $(X, \tilde{\Sigma}, \tilde{\mu})$, and in all these cases $\tilde{\mu} = \hat{\mu}$;

(b) if (X, Σ, μ) is localizable, so is $(X, \tilde{\Sigma}, \tilde{\mu})$, and for every $H \in \tilde{\Sigma}$ there is an $E \in \Sigma$ such that $\tilde{\mu}(E \triangle H) = 0$;

(c) (X, Σ, μ) is semi-finite iff $\tilde{\mu} F = \mu F$ for every $F \in \Sigma$, and in this case $\int f d\tilde{\mu} = \int f d\mu$ whenever the latter is defined in $[-\infty, \infty]$;

(d) a set $H \in \tilde{\Sigma}$ is an atom for $\tilde{\mu}$ iff there is an atom E for μ such that $\mu E < \infty$ and $\tilde{\mu}(H \triangle E) = 0$;

(e) if (X, Σ, μ) is atomless or purely atomic, so is $(X, \tilde{\Sigma}, \tilde{\mu})$;

(f) (X, Σ, μ) is complete and locally determined iff $\tilde{\mu} = \mu$.

proof (a)(i) I start by showing that if (X, Σ, μ) is strictly localizable, then $\tilde{\mu} = \hat{\mu}$. **P** Let $\langle X_i \rangle_{i\in I}$ be a decomposition of X for μ; then it is also a decomposition for $\hat{\mu}$ (212Gb). If $H \in \tilde{\Sigma}$, we shall have $H \cap X_i \in \hat{\Sigma}$ for every i, and therefore $H \in \hat{\Sigma}$; moreover,

$$\begin{aligned}\hat{\mu} H &= \sum_{i\in I} \hat{\mu}(H \cap X_i) = \sup\{\sum_{i\in J} \hat{\mu}(H \cap X_i) : J \subseteq I \text{ is finite}\}\\ &\le \sup\{\hat{\mu}(H \cap E) : E \in \Sigma,\ \mu E < \infty\} = \tilde{\mu} H \le \hat{\mu} H.\end{aligned}$$

So $\hat{\mu} H = \tilde{\mu} H$ for every $H \in \tilde{\Sigma}$ and $\hat{\mu} = \tilde{\mu}$. **Q**

(ii) Consequently, if (X, Σ, μ) is a probability space, or totally finite, or σ-finite, or strictly localizable, so is $(X, \tilde{\Sigma}, \tilde{\mu})$, using 212Ga-212Gb to see that $(X, \hat{\Sigma}, \hat{\mu})$ has the property involved.

(b) If (X, Σ, μ) is localizable, let $\mathcal{H}$ be any subset of $\tilde{\Sigma}$. Set

$$\mathcal{E} = \{E : E \in \Sigma^f,\ \exists\ H \in \mathcal{H},\ E \subseteq H\}$$

where $\Sigma^f = \{E : \mu E < \infty\}$ as usual. Working in (X, Σ, μ), let $F \in \Sigma$ be an essential supremum for $\mathcal{E}$.

(i) ? Suppose, if possible, that there is an $H \in \mathcal{H}$ such that $\tilde{\mu}(H \setminus F) > 0$. By 213F(c-i), there is an $E \in \Sigma^f$ such that $E \subseteq H \setminus F$ and $\mu E > 0$. This E belongs to $\mathcal{E}$ and $\mu(E \setminus F) = \mu E > 0$; which is impossible if F is an essential supremum of $\mathcal{E}$. **X**

(ii) Thus $\tilde{\mu}(H \setminus F) = 0$ for every $H \in \mathcal{H}$. Now take any $G \in \tilde{\Sigma}$ such that $\tilde{\mu}(H \setminus G) = 0$ for every $H \in \mathcal{H}$. **?** If $\tilde{\mu}(F \setminus G) > 0$, there is an $E_0 \in \Sigma^f$ such that $E_0 \subseteq F \setminus G$ and $\mu E_0 > 0$. If $E \in \mathcal{E}$, there is an $H \in \mathcal{H}$ such that $E \subseteq H$, so that $E \setminus (F \setminus E_0) \subseteq H \setminus (F \cap G)$, while $\mu(E \setminus (F \setminus E_0)) < \infty$; so

$$\mu(E \setminus (F \setminus E_0)) \le \tilde{\mu}(H \setminus (F \cap G)) \le \tilde{\mu}(H \setminus F) + \tilde{\mu}(H \setminus G) = 0.$$

Because F is an essential supremum for $\mathcal{E}$ in Σ,

$$0 = \mu(F \setminus (F \setminus E_0)) = \mu E_0. \textbf{ X}$$

This shows that F is an essential supremum for $\mathcal{H}$ in $\tilde{\Sigma}$. As $\mathcal{H}$ is arbitrary, $(X, \tilde{\Sigma}, \tilde{\mu})$ is localizable.

(iii) The argument of (i)-(ii) shows in fact that if $\mathcal{H} \subseteq \tilde{\Sigma}$ then $\mathcal{H}$ has an essential supremum F in $\tilde{\Sigma}$ such that F actually belongs to Σ. Taking $\mathcal{H} = \{H\}$, we see that if $H \in \tilde{\Sigma}$ there is an $F \in \Sigma$ such that $\mu(H \triangle F) = 0$.

(c) We already know that $\tilde{\mu}E \le \mu E$ for every $E \in \Sigma$, with equality if $\mu E < \infty$, by 213Fa.

(i) If (X, Σ, μ) is semi-finite, then 213A and 213F(c-i) tell us that for any $F \in \Sigma$ we have

$$\mu F = \sup\{\mu E : E \in \Sigma,\ E \subseteq F,\ \mu E < \infty\} = \tilde{\mu} F.$$

(ii) Suppose that $\tilde{\mu}F = \mu F$ for every $F \in \Sigma$. If $\mu F = \infty$, then $\tilde{\mu}F = \infty$ so (by 213F(c-i) again) there must be an $E \in \Sigma^f$ such that $E \subseteq F$ and $\mu E > 0$; as F is arbitrary, (X, Σ, μ) is semi-finite.

(iii) If f is non-negative and $\int f d\mu = \infty$, then f is μ-virtually measurable, therefore $\tilde{\Sigma}$-measurable (213Ga), and defined μ-almost everywhere, therefore $\tilde{\mu}$-almost everywhere. Now

$$\begin{aligned}\int f d\tilde{\mu} &= \sup\{\int g\, d\tilde{\mu} : g \text{ is } \tilde{\mu}\text{-simple},\ 0 \le g \le f\ \tilde{\mu}\text{-a.e.}\}\\ &\ge \sup\{\int g\, d\mu : g \text{ is } \mu\text{-simple},\ 0 \le g \le f\ \mu\text{-a.e.}\} = \infty\end{aligned}$$

by 213B. With 213Gb, this shows that $\int f d\tilde{\mu} = \int f d\mu$ whenever f is non-negative and $\int f d\mu$ is defined in $[0, \infty]$. Applying this to the positive and negative parts of f, we see that $\int f d\tilde{\mu} = \int f d\mu$ whenever the latter is defined in $[-\infty, \infty]$.

(d)(i) If $H \in \tilde{\Sigma}$ is an atom for $\tilde{\mu}$, then there is an $E \in \Sigma^f$ such that $E \subseteq H$ and $0 < \mu E < \infty$. In this case, $\tilde{\mu}E > 0$ so $\tilde{\mu}(H \setminus E)$ must be zero. If $F \in \Sigma$ and $F \subseteq E$, then either $\mu F = \tilde{\mu} F = 0$ or $\mu(E \setminus F) = \tilde{\mu}(H \setminus F) = 0$. Thus $E \in \Sigma$ is an atom for μ with $\tilde{\mu}(H \triangle E) = 0$ and $\mu E < \infty$.

(ii) If $H \in \tilde{\Sigma}$ and there is an atom E for μ such that $\mu E < \infty$ and $\tilde{\mu}(H \triangle E) = 0$, let $G \in \tilde{\Sigma}$ be a subset of H with $\tilde{\mu}G > 0$. We have $\tilde{\mu}(E \cap G) = \tilde{\mu}(H \cap G) > 0$, so there is an $F \in \Sigma$ such that $F \subseteq E \cap G$ and $\mu F > 0$. Now $\mu(E \setminus F)$ must be zero, so

$$\tilde{\mu}(H \setminus G) \le \tilde{\mu}(H \setminus F) = \tilde{\mu}(E \setminus F) = \mu(E \setminus F) = 0.$$

As G is arbitrary, H is an atom for $\tilde{\mu}$.

(e) If (X, Σ, μ) is atomless, then $(X, \tilde{\Sigma}, \tilde{\mu})$ must be atomless, by (d).

If (X, Σ, μ) is purely atomic, $H \in \tilde{\Sigma}$ and $\tilde{\mu}H > 0$, then there is an $E \in \Sigma^f$ such that $E \subseteq H$ and $\mu E > 0$. There is an atom F for μ such that $F \subseteq E$; now $\mu F < \infty$ so F is an atom for $\tilde{\mu}$, by (d). Also $F \subseteq H$. As H is arbitrary, $(X, \tilde{\Sigma}, \tilde{\mu})$ is purely atomic.

(f) If $\mu = \tilde{\mu}$, then of course (X, Σ, μ) must be complete and locally determined, because $(X, \tilde{\Sigma}, \tilde{\mu})$ is. If (X, Σ, μ) is complete and locally determined, then $\hat{\mu} = \mu$ so (using the definition in 213D) $\tilde{\Sigma} \subseteq \Sigma$ and $\tilde{\mu} = \mu$, by (c) above.

213I Locally determined negligible sets The following simple idea is occasionally useful.

Definition A measure space (X, Σ, μ) has **locally determined negligible sets** if for every non-negligible $A \subseteq X$ there is an $E \in \Sigma$ such that $\mu E < \infty$ and $A \cap E$ is not negligible.

213J Proposition If a measure space (X, Σ, μ) is *either* strictly localizable *or* complete and locally determined, it has locally determined negligible sets.

proof Let $A \subseteq X$ be a set such that $A \cap E$ is negligible whenever $\mu E < \infty$; I need to show that A is negligible.

(i) If μ is strictly localizable, let $\langle X_i \rangle_{i \in I}$ be a decomposition of X. For each $i \in I$, $A \cap X_i$ is negligible, so there we can choose a negligible $E_i \in \Sigma$ such that $A \cap X_i \subseteq E_i$. Set $E = \bigcup_{i \in I} E_i \cap X_i$. Then $\mu E = \sum_{i \in I} \mu(E_i \cap X_i) = 0$ and $A \subseteq E$, so A is negligible.

(ii) If μ is complete and locally determined, take any measurable set E of finite measure. Then $A \cap E$ is negligible, therefore measurable; as E is arbitrary, A is measurable; as μ is semi-finite, A is negligible.

***213K Lemma** If a measure space (X, Σ, μ) has locally determined negligible sets, and $\mathcal{E} \subseteq \Sigma$ has an essential supremum $H \in \Sigma$ in the sense of 211G, then $H \setminus \bigcup \mathcal{E}$ is negligible.

proof Set $A = H \setminus \bigcup \mathcal{E}$. Take any $F \in \Sigma$ such that $\mu F < \infty$. Then $F \cap A$ has a measurable envelope V say (132Ee again). If $E \in \mathcal{E}$, then

$$\mu(E \setminus (X \setminus V)) = \mu(E \cap V) = \mu^*(E \cap F \cap A) = 0,$$

so $H \cap V = H \setminus (X \setminus V)$ is negligible and $F \cap A$ is negligible. As F is arbitrary and μ has locally determined negligible sets, A is negligible, as claimed.

213L Proposition Let (X, Σ, μ) be a localizable measure space with locally determined negligible sets. Then every subset A of X has a measurable envelope.

proof Set

$$\mathcal{E} = \{E : E \in \Sigma,\ \mu^*(A \cap E) = \mu E < \infty\}.$$

Let G be an essential supremum for $\mathcal{E}$ in Σ.

(i) $A \setminus G$ is negligible. **P** Let F be any set of finite measure for μ. Let E be a measurable envelope of $A \cap F$. Then $E \in \mathcal{E}$ so $E \setminus G$ is negligible. But $F \cap A \setminus G \subseteq E \setminus G$, so $F \cap A \setminus G$ is negligible. Because μ has locally determined negligible sets, this is enough to show that $A \setminus G$ is negligible. **Q**

(ii) Let E_0 be a negligible measurable set including $A \setminus G$, and set $\tilde{G} = E_0 \cup G$, so that $\tilde{G} \in \Sigma$, $A \subseteq \tilde{G}$ and $\mu(\tilde{G} \setminus G) = 0$. **?** Suppose, if possible, that there is an $F \in \Sigma$ such that $\mu^*(A \cap F) < \mu(\tilde{G} \cap F)$. Let $F_1 \subseteq F$ be a measurable envelope of $A \cap F$. Set $H = X \setminus (F \setminus F_1)$; then $A \subseteq H$. If $E \in \mathcal{E}$ then

$$\mu E = \mu^*(A \cap E) \le \mu(H \cap E),$$

so $E \setminus H$ is negligible; as E is arbitrary, $G \setminus H$ is negligible and $\tilde{G} \setminus H$ is negligible. But $\tilde{G} \cap F \setminus F_1 \subseteq \tilde{G} \setminus H$ and

$$\mu(\tilde{G} \cap F \setminus F_1) = \mu(\tilde{G} \cap F) - \mu^*(A \cap F) > 0. \text{ **X**}$$

This shows that $\tilde{G}$ is a measurable envelope of A, as required.

213M Corollary (a) If (X, Σ, μ) is σ-finite, then every subset of X has a measurable envelope for μ.

(b) If (X, Σ, μ) is localizable, then every subset of X has a measurable envelope for the c.l.d. version of μ.

proof (a) Use 132Ee, or 213L, 211Lc and 213J.

(b) Use 213L and the fact that the c.l.d. version of μ is localizable as well as being complete and locally determined (213Hb).

213N When we come to use the concept of 'localizability', it will frequently be through the following property, which in fact characterizes localizable spaces (213Xm).

Theorem Let (X,Σ,μ) be a localizable measure space. Suppose that Φ is a family of measurable real-valued functions, all defined on measurable subsets of X, such that whenever f, $g\in\Phi$ then $f=g$ almost everywhere in $\operatorname{dom}f\cap\operatorname{dom}g$. Then there is a measurable function $h:X\to\mathbb{R}$ such that every $f\in\Phi$ agrees with h almost everywhere in $\operatorname{dom}f$.

proof For $q\in\mathbb{Q}$, $f\in\Phi$ set

$$E_{fq}=\{x:x\in\operatorname{dom}f,\ f(x)\ge q\}\in\Sigma.$$

For each $q\in\mathbb{Q}$, let E_q be an essential supremum of $\{E_{fq}:f\in\Phi\}$ in Σ. Set

$$h^*(x)=\sup\{q:q\in\mathbb{Q},\ x\in E_q\}\in[-\infty,\infty]$$

for $x\in X$, taking $\sup\emptyset=-\infty$ if necessary.

If f, $g\in\Phi$ and $q\in\mathbb{Q}$, then

$$\begin{aligned}E_{fq}\setminus(X\setminus(\operatorname{dom}g\setminus E_{gq}))&=E_{fq}\cap\operatorname{dom}g\setminus E_{gq}\\&\subseteq\{x:x\in\operatorname{dom}f\cap\operatorname{dom}g,\ f(x)\ne g(x)\}\end{aligned}$$

is negligible; as f is arbitrary,

$$E_q\cap\operatorname{dom}g\setminus E_{gq}=E_q\setminus(X\setminus(\operatorname{dom}g\setminus E_{gq}))$$

is negligible. Also $E_{gq}\setminus E_q$ is negligible, so $E_{gq}\triangle(E_q\cap\operatorname{dom}g)$ is negligible. Set $H_g=\bigcup_{q\in\mathbb{Q}}E_{gq}\triangle(E_q\cap\operatorname{dom}g)$; then H_g is negligible. But if $x\in\operatorname{dom}g\setminus H_g$, then, for every $q\in\mathbb{Q}$, $x\in E_q\iff x\in E_{gq}$; it follows that for such x, $h^*(x)=g(x)$. Thus $h^*=g$ almost everywhere in $\operatorname{dom}g$; and this is true for every $g\in\Phi$.

The function h^* is not necessarily real-valued. But it is measurable, because

$$\{x:h^*(x)>a\}=\bigcup\{E_q:q\in\mathbb{Q},\ q>a\}\in\Sigma$$

for every real a. So if we modify it by setting

$$\begin{aligned}h(x)&=h^*(x)\text{ if }h(x)\in\mathbb{R},\\&=0\text{ if }h^*(x)\in\{-\infty,\infty\},\end{aligned}$$

we shall get a measurable real-valued function $h:X\to\mathbb{R}$; and for any $g\in\Phi$, $h(x)$ will be equal to $g(x)$ at least whenever $h^*(x)=g(x)$, which is true for almost every $x\in\operatorname{dom}g$. Thus h is a suitable function.

213O There is an interesting and useful criterion for a space to be strictly localizable which I introduce at this point, though it will be used rarely in this volume.

Proposition Let (X,Σ,μ) be a complete locally determined space.

(a) Suppose that there is a disjoint family $\mathcal{E}\subseteq\Sigma$ such that (α) $\mu E<\infty$ for every $E\in\mathcal{E}$ (β) whenever $F\in\Sigma$ and $\mu F>0$ then there is an $E\in\mathcal{E}$ such that $\mu(E\cap F)>0$. Then (X,Σ,μ) is strictly localizable, $\bigcup\mathcal{E}$ is conegligible, and $\mathcal{E}\cup\{X\setminus\bigcup\mathcal{E}\}$ is a decomposition of X.

(b) Suppose that $\langle X_i\rangle_{i\in I}$ is a partition of X into measurable sets of finite measure such that whenever $E\in\Sigma$ and $\mu E>0$ there is an $i\in I$ such that $\mu(E\cap X_i)>0$. Then (X,Σ,μ) is strictly localizable, and $\langle X_i\rangle_{i\in I}$ is a decomposition of X.

proof (a)(i) The first thing to note is that if $F\in\Sigma$ and $\mu F<\infty$, there is a countable $\mathcal{E}'\subseteq\mathcal{E}$ such that $\mu(F\setminus\bigcup\mathcal{E}')=0$. **P** Set

$$\mathcal{E}'_n=\{E:E\in\mathcal{E},\ \mu(F\cap E)\ge 2^{-n}\}\text{ for each }n\in\mathbb{N},$$

$$\mathcal{E}'=\bigcup_{n\in\mathbb{N}}\mathcal{E}'_n=\{E:E\in\mathcal{E},\ \mu(F\cap E)>0\}.$$

Because $\mathcal{E}$ is disjoint, we must have

$$\#(\mathcal{E}'_n)\le 2^n\mu F$$

for every $n \in \mathbb{N}$, so that every $\mathcal{E}'_n$ is finite and $\mathcal{E}'$, being the union of a sequence of countable sets, is countable. Set $E' = \bigcup \mathcal{E}'$ and $F' = F \setminus E'$, so that both E' and F' belong to Σ. If $E \in \mathcal{E}'$, then $E \subseteq E'$ so $\mu(E \cap F') = \mu\emptyset = 0$; if $E \in \mathcal{E} \setminus \mathcal{E}'$, then $\mu(E \cap F') = \mu(E \cap F) = 0$. Thus $\mu(E \cap F') = 0$ for every $E \in \mathcal{E}$. By the hypothesis (β) on $\mathcal{E}$, $\mu F' = 0$, so $\mu(F \setminus \bigcup \mathcal{E}') = 0$, as required. **Q**

(ii) Now suppose that $H \subseteq X$ is such that $H \cap E \in \Sigma$ for every $E \in \mathcal{E}$. In this case $H \in \Sigma$. **P** Let $F \in \Sigma$ be such that $\mu F < \infty$. Let $\mathcal{E}' \subseteq \mathcal{E}$ be a countable set such that $\mu(F \setminus E') = 0$, where $E' = \bigcup \mathcal{E}'$. Then $H \cap (F \setminus E') \in \Sigma$ because (X, Σ, μ) is complete. But also $H \cap E' = \bigcup_{E \in \mathcal{E}'} H \cap E \in \Sigma$. So

$$H \cap F = (H \cap (F \setminus E')) \cup (F \cap (H \cap E')) \in \Sigma.$$

As F is arbitrary and (X, Σ, μ) is locally determined, $H \in \Sigma$. **Q**

(iii) We find also that $\mu H = \sum_{E \in \mathcal{E}} \mu(H \cap E)$ for every $H \in \Sigma$. **P** (α) Because $\mathcal{E}$ is disjoint, we must have $\sum_{E \in \mathcal{E}'} \mu(H \cap E) \le \mu H$ for every finite $\mathcal{E}' \subseteq \mathcal{E}$, so

$$\textstyle\sum_{E \in \mathcal{E}} \mu(H \cap E) = \sup\{\sum_{E \in \mathcal{E}'} \mu(H \cap E) : \mathcal{E}' \subseteq \mathcal{E} \text{ is finite}\} \le \mu H.$$

(β) For the reverse inequality, consider first the case $\mu H < \infty$. By (i), there is a countable $\mathcal{E}' \subseteq \mathcal{E}$ such that $\mu(H \setminus \bigcup \mathcal{E}') = 0$, so that

$$\textstyle\mu H = \mu(H \cap \bigcup \mathcal{E}') = \sum_{E \in \mathcal{E}'} \mu(H \cap E) \le \sum_{E \in \mathcal{E}} \mu(H \cap E).$$

(γ) In general, because (X, Σ, μ) is semi-finite,

$$\begin{aligned} \mu H &= \sup\{\mu F : F \subseteq H,\, \mu F < \infty\} \\ &\le \sup\{\sum_{E \in \mathcal{E}} \mu(F \cap E) : F \subseteq H,\, \mu F < \infty\} \le \sum_{E \in \mathcal{E}} \mu(H \cap E). \end{aligned}$$

So in all cases we have $\mu H \le \sum_{E \in \mathcal{E}} \mu(H \cap E)$, and the two are equal. **Q**

(iv) In particular, setting $E_0 = X \setminus \bigcup \mathcal{E}$, $E_0 \in \Sigma$ and $\mu E_0 = 0$; that is, $\bigcup \mathcal{E}$ is conegligible. Consider $\mathcal{E}^* = \mathcal{E} \cup \{E_0\}$. This is a partition of X into sets of finite measure (now using the hypothesis (α) on $\mathcal{E}$). If $H \subseteq X$ is such that $H \cap E \in \Sigma$ for every $E \in \mathcal{E}^*$, then $H \in \Sigma$ and

$$\textstyle\mu H = \sum_{E \in \mathcal{E}} \mu(H \cap E) = \sum_{E \in \mathcal{E}^*} \mu(H \cap E).$$

Thus $\mathcal{E}^*$ (or, if you prefer, the indexed family $\langle E \rangle_{E \in \mathcal{E}^*}$) is a decomposition witnessing that (X, Σ, μ) is strictly localizable.

(b) Apply (a) with $\mathcal{E} = \{X_i : i \in I\}$, noting that E_0 in (iv) is empty, so can be dropped.

213X Basic exercises (a) Let (X, Σ, μ) be any measure space, μ^* the outer measure defined from μ, and $\check{\mu}$ the measure defined by Carathéodory's method from μ^*; write $\check{\Sigma}$ for the domain of $\check{\mu}$. Show that (i) $\check{\mu}$ extends the completion $\hat{\mu}$ of μ; (ii) if $H \subseteq X$ is such that $H \cap F \in \check{\Sigma}$ whenever $F \in \Sigma$ and $\mu F < \infty$, then $H \in \check{\Sigma}$; (iii) $(\check{\mu})^* = \mu^*$, so that the integrable functions for $\check{\mu}$ and μ are the same (212Xb); (iv) if μ is strictly localizable then $\check{\mu} = \hat{\mu}$.

>(b) Let μ be counting measure restricted to the countable-cocountable σ-algebra of a set X (211R, 211Ye). (i) Show that the c.l.d. version $\tilde{\mu}$ of μ is just counting measure on X. (ii) Show that $\check{\mu}$, as defined in 213Xa, is equal to $\tilde{\mu}$, and in particular strictly extends the completion of μ if X is uncountable.

(c) Let (X, Σ, μ) be any measure space. For $E \in \Sigma$ set

$$\mu_{\mathrm{sf}} E = \sup\{\mu(E \cap F) : F \in \Sigma,\, \mu F < \infty\}.$$

(i) Show that $(X, \Sigma, \mu_{\mathrm{sf}})$ is a semi-finite measure space, and is equal to (X, Σ, μ) iff (X, Σ, μ) is semi-finite.

(ii) Show that a μ-integrable real-valued function f is μ_{sf}-integrable, with the same integral.

(iii) Show that if $E \in \Sigma$ then E can be expressed as $E_1 \cup E_2$ where E_1, $E_2 \in \Sigma$, $\mu E_1 = \mu_{\mathrm{sf}} E_1$ and $\mu_{\mathrm{sf}} E_2 = 0$.

(iv) Show that if f is a μ_{sf}-integrable real-valued function on X, it is equal μ_{sf}-almost everywhere to a μ-integrable function.

(v) Show that if $(X,\Sigma,\mu_{\text{sf}})$ is complete, so is (X,Σ,μ).
(vi) Show that μ and μ_{sf} have identical c.l.d. versions.

(d) Let (X,Σ,μ) be any measure space. Define $\check\mu$ as in 213Xa. Show that $(\check\mu)_{\text{sf}}$, as constructed in 213Xc, is precisely the c.l.d. version $\tilde\mu$ of μ, so that $\check\mu=\tilde\mu$ iff $\check\mu$ is semi-finite.

(e) Let (X,Σ,μ) be a measure space. For $A\subseteq X$ set $\mu_*A=\sup\{\mu E: E\in\Sigma,\ \mu E<\infty,\ E\subseteq A\}$, as in 113Yh. (i) Show that the measure constructed from μ_* by the method of 113Yg/212Ya is just the c.l.d. version $\tilde\mu$ of μ. (ii) Show that $\tilde\mu_*=\mu_*$. (iii) Show that if ν is another measure on X, with domain T, then $\tilde\mu=\tilde\nu$ iff $\mu_*=\nu_*$.

(f) Let X be a set and θ an outer measure on X. Show that θ_{sf}, defined by writing

$$\theta_{\text{sf}}A=\sup\{\theta B: B\subseteq A,\ \theta B<\infty\}$$

is also an outer measure on X. Show that the measures defined by Carathéodory's method from θ, θ_{sf} have the same domains.

(g) Let (X,Σ,μ) be any measure space. Set

$$\mu^*_{\text{sf}}A=\sup\{\mu^*(A\cap E): E\in\Sigma,\ \mu E<\infty\}$$

for every $A\subseteq X$.

(i) Show that

$$\mu^*_{\text{sf}}A=\sup\{\mu^*B: B\subseteq A,\ \mu^*B<\infty\}$$

for every A.

(ii) Show that μ^*_{sf} is an outer measure.

(iii) Show that if $A\subseteq X$ and $\mu^*_{\text{sf}}A<\infty$, there is an $E\in\Sigma$ such that $\mu^*_{\text{sf}}A=\mu^*(A\cap E)=\mu E$, $\mu^*_{\text{sf}}(A\setminus E)=0$. (*Hint*: take a non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure such that $\mu^*_{\text{sf}}A=\lim_{n\to\infty}\mu^*(A\cap E_n)$, and let $E\subseteq\bigcup_{n\in\mathbb{N}}E_n$ be a measurable envelope of $A\cap\bigcup_{n\in\mathbb{N}}E_n$.)

(iv) Show that the measure defined from μ^*_{sf} by Carathéodory's method is precisely the c.l.d. version $\tilde\mu$ of μ.

(v) Show that $\mu^*_{\text{sf}}=\tilde\mu^*$, so that if μ is complete and locally determined then $\mu^*_{\text{sf}}=\mu^*$.

>(h) Let (X,Σ,μ) be a measure space with locally determined measurable sets. Show that it is semi-finite.

>(i) Let (X,Σ,μ) be a measure space, $\hat\mu$ the completion of μ, $\tilde\mu$ the c.l.d. version of μ and $\check\mu$ the measure defined by Carathéodory's method from μ^*. Show that the following are equiveridical: (i) μ has locally determined negligible sets; (ii) μ and $\tilde\mu$ have the same negligible sets; (iii) $\check\mu=\tilde\mu$; (iv) $\hat\mu$ and $\tilde\mu$ have the same sets of finite measure; (v) μ and $\tilde\mu$ have the same integrable functions; (vi) $\tilde\mu^*=\mu^*$; (vii) the outer measure μ^*_{sf} of 213Xg is equal to μ^*.

>(j) Let (X,Σ,μ) be a strictly localizable measure space with a decomposition $\langle X_i\rangle_{i\in I}$. Show that $\mu^*A=\sum_{i\in I}\mu^*(A\cap X_i)$ for every $A\subseteq X$.

>(k) Let (X,Σ,μ) be a complete locally determined measure space, and let $A\subseteq X$ be such that $\min(\mu^*(E\cap A),\mu^*(E\setminus A))<\mu E$ whenever $E\in\Sigma$ and $0<\mu E<\infty$. Show that $A\in\Sigma$. (*Hint*: given $\mu F<\infty$, consider the intersection E of measurable envelopes of $F\cap A$, $F\setminus A$ to see that $\mu^*(F\cap A)+\mu^*(F\setminus A)=\mu F$.)

(l) Let us say that a measure space (X,Σ,μ) has the **measurable envelope property** if every subset of X has a measurable envelope. (i) Show that a semi-finite space with the measurable envelope property has locally determined negligible sets. (ii) Show that a complete semi-finite space with the measurable envelope property is locally determined.

(m) Let (X,Σ,μ) be a semi-finite measure space, and suppose that it satisfies the conclusion of Theorem 213N. Show that it is localizable. (*Hint*: given $\mathcal{E}\subseteq\Sigma$, set $\mathcal{F}=\{F: F\in\Sigma,\ E\cap F$ is negligible for every $E\in\mathcal{E}\}$. Let Φ be the set of functions f from subsets of X to $\{0,1\}$ such that $f^{-1}[\{1\}]\in\mathcal{E}$ and $f^{-1}[\{0\}]\in\mathcal{F}$.)

(n) Let (X,Σ,μ) be a measure space. Show that its c.l.d. version is strictly localizable iff there is a disjoint family $\mathcal{E}\subseteq\Sigma$ such that $\mu E<\infty$ for every $E\in\mathcal{E}$ and whenever $F\in\Sigma$ and $0<\mu F<\infty$ there is an $E\in\mathcal{E}$ such that $\mu(E\cap F)>0$.

(o) Show that the c.l.d. version of any point-supported measure is point-supported.

213Y Further exercises (a) Let (X,Σ,μ) be a measure space. Show that μ is semi-finite iff there is a family $\mathcal{E}\subseteq\Sigma$ such that $\mu E<\infty$ for every $E\in\mathcal{E}$ and $\mu F=\sum_{E\in\mathcal{E}}\mu(F\cap E)$ for every $F\in\Sigma$. (*Hint*: take $\mathcal{E}$ maximal subject to the intersection of any two elements being negligible.)

(b) Set $X=\mathbb{N}$, and for $A\subseteq X$ set

$$\theta A=\sqrt{\#(A)}\text{ if }A\text{ is finite, }\infty\text{ if }A\text{ is infinite.}$$

Show that θ is an outer measure on X, that $\theta A=\sup\{\theta B:B\subseteq A,\,\theta B<\infty\}$ for every $A\subseteq X$, but that the measure μ defined from θ by Carathéodory's method is not semi-finite. Show that if $\check\mu$ is the measure defined by Carathéodory's method from μ^* (213Xa), then $\check\mu\neq\mu$.

(c) Set $X=[0,1]\times\{0,1\}$, and let Σ be the family of those subsets E of X such that

$$\{x:x\in[0,1],\,E[\{x\}]\neq\emptyset,\,E[\{x\}]\neq\{0,1\}\}$$

is countable, writing $E[\{x\}]=\{y:(x,y)\in E\}$ for each $x\in[0,1]$. Show that Σ is a σ-algebra of subsets of X. For $E\in\Sigma$, set $\mu E=\#(\{x:(x,1)\in E\})$ if this is finite, ∞ otherwise. Show that μ is a complete semi-finite measure. Show that the measure $\check\mu$ defined from μ^* by Carathéodory's method (213Xa) is not semi-finite. Show that the domain of the c.l.d. version of μ is the whole of $\mathcal{P}X$.

(d) Set $X=\mathbb{N}$, and for $A\subseteq X$ set

$$\phi A=\#(A)^2\text{ if }A\text{ is finite, }\infty\text{ if }A\text{ is infinite.}$$

Show that ϕ satisfies the conditions of 212Ya, but that the measure defined from ϕ by the method there is not semi-finite.

(e) Let (X,Σ,μ) be a complete locally determined measure space. Suppose that $D\subseteq X$ and that $f:D\to\mathbb{R}$ is a function. Show that the following are equiveridical: (i) f is measurable; (ii)

$$\mu^*\{x:x\in D\cap E,\,f(x)\le a\}+\mu^*\{x:x\in D\cap E,\,f(x)\ge b\}\le\mu E$$

whenever $a<b$ in $\mathbb{R}$, $E\in\Sigma$ and $\mu E<\infty$ (iii)

$$\min(\mu^*\{x:x\in D\cap E,\,f(x)\le a\},\mu^*\{x:x\in D\cap E,\,f(x)\ge b\})<\mu E$$

whenever $a<b$ in $\mathbb{R}$ and $0<\mu E<\infty$. (*Hint*: for (iii)$\Rightarrow$(i), show that if $E\subseteq X$ then

$$\mu^*\{x:x\in D\cap E,\,f(x)>a\}=\sup_{b>a}\mu^*\{x:x\in D\cap E,\,f(x)\ge b\},$$

and use 213Xk above.)

(f) Let (X,Σ,μ) be a complete locally determined measure space and suppose that $\mathcal{E}\subseteq\Sigma$ is such that $\mu E<\infty$ for every $E\in\mathcal{E}$ and whenever $F\in\Sigma$ and $\mu F<\infty$ there is a countable $\mathcal{E}_0\subseteq\mathcal{E}$ such that $F\setminus\bigcup\mathcal{E}_0$, $F\cap\bigcup(\mathcal{E}\setminus\mathcal{E}_0)$ are negligible. Show that (X,Σ,μ) is strictly localizable.

213 Notes and comments I think it is fair to say that if the definition of 'measure space' were re-written to exclude all spaces which are not semi-finite, nothing significant would be lost from the theory. There are solid reasons for not taking such a drastic step, starting with the fact that it would confuse everyone (if you say to an unprepared audience 'let (X,Σ,μ) be a measure space', there is a danger that some will imagine that you mean 'σ-finite measure space', but very few will suppose that you mean 'semi-finite measure space'). But the whole point of measure theory is that we distinguish between sets by their measures, and if every subset of E is either non-measurable, or negligible, or of infinite measure, the classification is too crude to support most of the usual ideas, starting, of course, with ordinary integration.

Let us say that a measurable set E is **purely infinite** if E itself and all its non-negligible measurable subsets have infinite measure. On the definition of the integral which I chose in Volume 1, every simple function, and therefore every integrable function, must be zero almost everywhere in E. This means that the whole theory of integration will ignore E entirely. Looking at the definition of 'c.l.d. version' (213D-213E), you will see that the c.l.d. version of the measure will render E negligible, as does the 'semi-finite version' described in 213Xc. These amendments do not, however, affect sets of finite measure, and consequently leave integrable functions integrable, with the same integrals.

The strongest reason we have yet seen for admitting non-semi-finite spaces into consideration is that Carathéodory's method does not always produce semi-finite spaces. (I give examples in 213Yb-213Yc; more important ones are the Hausdorff measures of §§264-265 below.) In practice the right thing to do is often to take the c.l.d. version of the measure produced by Carathéodory's construction.

It is a reasonable general philosophy, in measure theory, to say that we wish to measure as many sets, and integrate as many functions, as we can manage in a canonical way – I mean, without making blatantly arbitrary choices about the values we assign to our measure or integral. The revision of a measure μ to its c.l.d. version $\tilde{\mu}$ is about as far as we can go with an arbitrary measure space in which we have no other structure to guide our choices.

You will observe that $\tilde{\mu}$ is not as close to μ as the completion $\hat{\mu}$ of μ is; naturally so, because if $E \in \Sigma$ is purely infinite for μ then we have to choose between setting $\tilde{\mu}E = 0 \neq \mu E$ and finding some way of fitting many sets of finite measure into E; which if E is a singleton will be actually impossible, and in any case would be an arbitrary process. However the integrable functions for $\tilde{\mu}$, while not always the same as those for μ (since $\tilde{\mu}$ turns purely infinite sets into negligible ones, so that their indicator functions become integrable), are 'nearly' the same, in the sense that any $\tilde{\mu}$-integrable function can be changed into a μ-integrable function by adjusting it on a $\tilde{\mu}$-negligible set. This corresponds, of course, to the fact that any set of finite measure for $\tilde{\mu}$ is the symmetric difference of a set of finite measure for μ and a $\tilde{\mu}$-negligible set. For sets of infinite measure this can fail, unless μ is localizable (213Hb, 213Xb).

If (X, Σ, μ) is semi-finite, or localizable, or strictly localizable, then of course it is correspondingly closer to $(X, \tilde{\Sigma}, \tilde{\mu})$, as detailed in 213Ha-c.

It is worth noting that while the measure $\check{\mu}$ obtained by Carathéodory's method directly from the outer measure μ^* defined from μ may fail to be semi-finite, even when μ is (213Yc), a simple modification of μ^* (213Xg) yields the c.l.d. version $\tilde{\mu}$ of μ, which can also be obtained from an appropriate inner measure (213Xe). The measure $\check{\mu}$ is of course related in other ways to $\tilde{\mu}$; see 213Xd.

Version of 22.5.09

214 Subspaces

In §131 I described a construction for subspace measures on measurable subsets. It is now time to give the generalization to subspace measures on arbitrary subsets of a measure space. The relationship between this construction and the properties listed in §211 is not quite as straightforward as one might imagine, and in this section I try to give a full account of what can be expected of subspaces in general. I think that for the present volume only (i) general subspaces of σ-finite spaces and (ii) measurable subspaces of general measure spaces will be needed in any essential way, and these do not give any difficulty; but in later volumes we shall need the full theory.

I begin with a general construction for 'subspace measures' (214A-214C), with an account of integration with respect to a subspace measure (214E-214G); these (with 131E-131H) give a solid foundation for the concept of 'integration over a subset' (214D). I present this work in its full natural generality, which will eventually be essential, but even for Lebesgue measure alone it is important to be aware of the ideas here. I continue with answers to some obvious questions concerning subspace measures and the properties of measure spaces so far considered, both for general subspaces (214I) and for measurable subspaces (214K), and I mention a basic construction for assembling measure spaces side-by-side, the 'direct sums' of 214L-214M. At the end of the section I discuss a measure extension problem (214O-214P).

214A Proposition Let (X, Σ, μ) be a measure space, and Y any subset of X. Let μ^* be the outer measure defined from μ (132A-132B), and set $\Sigma_Y = \{E \cap Y : E \in \Sigma\}$; let μ_Y be the restriction of μ^* to Σ_Y. Then (Y, Σ_Y, μ_Y) is a measure space.

proof (a) I have noted in 121A that Σ_Y is a σ-algebra of subsets of Y.

(b) Of course $\mu_Y F \in [0, \infty]$ for every $F \in \Sigma_Y$.

(c) $\mu_Y \emptyset = \mu^* \emptyset = 0$.

(d) If $\langle F_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in Σ_Y with union F, then choose E_n, E'_n, $E \in \Sigma$ such that $F_n = Y \cap E_n$, $F_n \subseteq E'_n$ and $\mu_Y F_n = \mu E'_n$ for each n, $F \subseteq E$ and $\mu_Y F = \mu E$ (using 132Aa repeatedly). Set $G_n = E_n \cap E'_n \cap E \setminus \bigcup_{m<n} E_m$ for each $n \in \mathbb{N}$; then $\langle G_n \rangle_{n \in \mathbb{N}}$ is disjoint and $F_n \subseteq G_n \subseteq E'_n$ for each n, so $\mu_Y F_n = \mu G_n$. Also $F \subseteq \bigcup_{n \in \mathbb{N}} G_n \subseteq E$, so

$$\mu_Y F = \mu(\bigcup_{n \in \mathbb{N}} G_n) = \sum_{n=0}^{\infty} \mu G_n = \sum_{n=0}^{\infty} \mu_Y F_n.$$

As $\langle F_n \rangle_{n \in \mathbb{N}}$ is arbitrary, μ_Y is a measure.

214B Definition If (X, Σ, μ) is any measure space and Y is any subset of X, then μ_Y, defined as in 214A, is the **subspace measure** on Y.

It is worth noting the following.

214C Lemma Let (X, Σ, μ) be a measure space, Y a subset of X, μ_Y the subspace measure on Y and Σ_Y its domain. Then

(a) for any $F \in \Sigma_Y$, there is an $E \in \Sigma$ such that $F = E \cap Y$ and $\mu E = \mu_Y F$;

(b) for any $A \subseteq Y$, A is μ_Y-negligible iff it is μ-negligible;

(c)(i) if $A \subseteq X$ is μ-conegligible, then $A \cap Y$ is μ_Y-conegligible;

(ii) if $A \subseteq Y$ is μ_Y-conegligible, then $A \cup (X \setminus Y)$ is μ-conegligible;

(d) $(\mu_Y)^*$, the outer measure on Y defined from μ_Y, agrees with μ^* on $\mathcal{P}Y$;

(e) if $Z \subseteq Y \subseteq X$, then $\Sigma_Z = (\Sigma_Y)_Z$, the subspace σ-algebra of subsets of Z regarded as a subspace of (Y, Σ_Y), and $\mu_Z = (\mu_Y)_Z$ is the subspace measure on Z regarded as a subspace of (Y, μ_Y);

(f) if $Y \in \Sigma$, then μ_Y, as defined here, is exactly the subspace measure on Y defined in 131A-131B; that is, $\Sigma_Y = \Sigma \cap \mathcal{P}Y$ and $\mu_Y = \mu \restriction \Sigma_Y$.

proof (a) By the definition of Σ_Y, there is an $E_0 \in \Sigma$ such that $F = E_0 \cap Y$. By 132Aa, there is an $E_1 \in \Sigma$ such that $F \subseteq E_1$ and $\mu^* F = \mu E_1$. Set $E = E_0 \cap E_1$; this serves.

(b)(i) If A is μ_Y-negligible, there is a set $F \in \Sigma_Y$ such that $A \subseteq F$ and $\mu_Y F = 0$; now $\mu^* A \le \mu^* F = 0$ so A is μ-negligible, by 132Ad. (ii) If A is μ-negligible, there is an $E \in \Sigma$ such that $A \subseteq E$ and $\mu E = 0$; now $A \subseteq E \cap Y \in \Sigma_Y$ and $\mu_Y(E \cap Y) = 0$, so A is μ_Y-negligible.

(c) If $A \subseteq X$ is μ-conegligible, then $A \cap Y$ is μ_Y-conegligible, because $Y \setminus A = Y \cap (X \setminus A)$ is μ-negligible, therefore μ_Y-negligible. If $A \subseteq Y$ is μ_Y-conegligible, then $A \cup (X \setminus Y)$ is μ-conegligible because $X \setminus (A \cup (X \setminus Y)) = Y \setminus A$ is μ_Y-negligible, therefore μ-negligible.

(d) Let $A \subseteq Y$. (i) If $A \subseteq E \in \Sigma$, then $A \subseteq E \cap Y \in \Sigma_Y$, so $\mu_Y^* A \le \mu_Y(E \cap Y) \le \mu E$; as E is arbitrary, $\mu_Y^* A \le \mu^* A$. (ii) If $A \subseteq F \in \Sigma_Y$, there is an $E \in \Sigma$ such that $F \subseteq E$ and $\mu_Y F = \mu^* F = \mu E$; now $A \subseteq E$ so $\mu^* A \le \mu E = \mu_Y F$. As F is arbitrary, $\mu^* A \le \mu_Y^* A$.

(e) That $\Sigma_Z = (\Sigma_Y)_Z$ is immediate from the definition of Σ_Y, etc.; now

$$(\mu_Y)_Z = \mu_Y^* \restriction \Sigma_Z = \mu^* \restriction \Sigma_Z = \mu_Z$$

by (d).

(f) This is elementary, because $E \cap Y \in \Sigma$ and $\mu^*(E \cap Y) = \mu(E \cap Y)$ for every $E \in \Sigma$.

214D Integration over subsets: Definition Let (X, Σ, μ) be a measure space, Y a subset of X and f a $[-\infty, \infty]$-valued function defined on a subset of X. By $\int_Y f$ (or $\int_Y f(x)\mu(dx)$, etc.) I mean $\int (f \restriction Y) d\mu_Y$, if this exists in $[-\infty, \infty]$, following the definitions of 214A-214B, 133A and 135F, and taking $\operatorname{dom}(f \restriction Y) = Y \cap \operatorname{dom} f$, $(f \restriction Y)(x) = f(x)$ for $x \in Y \cap \operatorname{dom} f$. (Compare 131D.)

214E Proposition Let (X, Σ, μ) be a measure space, $Y \subseteq X$, and f a $[-\infty, \infty]$-valued function defined on a subset $\operatorname{dom} f$ of X.

(a) If f is μ-integrable then $f \restriction Y$ is μ_Y-integrable, and $\int_Y f \le \int f$ if f is non-negative.

(b) If $\operatorname{dom} f \subseteq Y$ and f is μ_Y-integrable, then there is a μ-integrable function $\tilde{f}$ on X, extending f, such that $\int_F \tilde{f} = \int_{F \cap Y} f$ for every $F \in \Sigma$.

proof (a)(i) If f is μ-simple, it is expressible as $\sum_{i=0}^n a_i \chi E_i$, where $E_0, \dots, E_n \in \Sigma$, $a_0, \dots, a_n \in \mathbb{R}$ and $\mu E_i < \infty$ for each i. Now $f \restriction Y = \sum_{i=0}^n a_i \chi_Y(E_i \cap Y)$, where $\chi_Y(E_i \cap Y) = (\chi E_i) \restriction Y$ is the indicator function of $E_i \cap Y$ regarded as a subset of Y; and each $E_i \cap Y$ belongs to Σ_Y, with $\mu_Y(E_i \cap Y) \le \mu E_i < \infty$, so $f \restriction Y : Y \to \mathbb{R}$ is μ_Y-simple.

If $f : X \to \mathbb{R}$ is a non-negative simple function, it is expressible as $\sum_{i=0}^n a_i \chi E_i$ where $E_0, \dots, E_n$ are disjoint sets of finite measure (122Cb). Now $f \restriction Y = \sum_{i=0}^n a_i \chi_Y(E_i \cap Y)$ and

$$\int (f \restriction Y) d\mu_Y = \sum_{i=0}^n a_i \mu_Y(E_i \cap Y) \le \sum_{i=0}^n a_i \mu E_i = \int f d\mu$$

because $a_i \ge 0$ whenever $E_i \ne \emptyset$, so that $a_i \mu_Y(E_i \cap Y) \le a_i \mu E_i$ for every i.

(ii) If f is a non-negative μ-integrable function, there is a non-decreasing sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of non-negative μ-simple functions converging to f μ-almost everywhere; now $\langle f_n \restriction Y \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of μ_Y-simple functions increasing to $f \restriction Y$ μ_Y-a.e. (by 214Cb), and

$$\sup_{n \in \mathbb{N}} \int (f_n \restriction Y) d\mu_Y \le \sup_{n \in \mathbb{N}} \int f_n d\mu = \int f d\mu < \infty,$$

so $\int (f \restriction Y) d\mu_Y$ exists and is at most $\int f d\mu$.

(iii) Finally, if f is any μ-integrable real-valued function, it is expressible as $f_1 - f_2$ where f_1 and f_2 are non-negative μ-integrable functions, so that $f \restriction Y = (f_1 \restriction Y) - (f_2 \restriction Y)$ is μ_Y-integrable.

(b) Let us say that if f is a μ_Y-integrable function, then an 'enveloping extension' of f is a μ-integrable function $\tilde{f}$, extending f, real-valued on $X \setminus Y$, such that $\int_F \tilde{f} = \int_{F \cap Y} f$ for every $F \in \Sigma$.

(i) If f is of the form χH, where $H \in \Sigma_Y$ and $\mu_Y H < \infty$, let $E_0 \in \Sigma$ be such that $H = Y \cap E_0$ and $E_1 \in \Sigma$ a measurable envelope for H (132Ee); then $E = E_0 \cap E_1$ is a measurable envelope for H and $H = E \cap Y$. Set $\tilde{f} = \chi E$, regarded as a function from X to $\{0, 1\}$. Then $\tilde{f} \restriction Y = f$, and for any $F \in \Sigma$ we have

$$\int_F \tilde{f} = \mu_F(E \cap F) = \mu(E \cap F) = \mu^*(H \cap F) = \mu_{Y \cap F}(H \cap F) = \int_{Y \cap F} f.$$

So $\tilde{f}$ is an enveloping extension of f.

(ii) If f, g are μ_Y-integrable functions with enveloping extensions $\tilde{f}$, $\tilde{g}$, and a, $b \in \mathbb{R}$, then $a\tilde{f} + b\tilde{g}$ extends $af + bg$ and

$$\begin{aligned} \int_F a\tilde{f} + b\tilde{g} &= a \int_F \tilde{f} + b \int_F \tilde{g} \\ &= a \int_{F \cap Y} f + b \int_{F \cap Y} g = \int_{F \cap Y} af + bg \end{aligned}$$

for every $F \in \Sigma$, so $a\tilde{f} + b\tilde{g}$ is an enveloping extension of $af + bg$.

(iii) Putting (i) and (ii) together, we see that every μ_Y-simple function f has an enveloping extension.

(iv) Now suppose that $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of non-negative μ_Y-simple functions converging μ_Y-almost everywhere to a μ_Y-integrable function f. For each $n \in \mathbb{N}$ let $\tilde{f}_n$ be an enveloping extension of f_n. Then $\tilde{f}_n \le_{\text{a.e.}} \tilde{f}_{n+1}$. **P** If $F \in \Sigma$ then

$$\int_F \tilde{f}_n = \int_{F \cap Y} f_n \le \int_{F \cap Y} f_{n+1} = \int_F \tilde{f}_{n+1}.$$

So $\tilde{f}_n \le_{\text{a.e.}} \tilde{f}_{n+1}$, by 131Ha. **Q** Also

$$\lim_{n \to \infty} \int_F \tilde{f}_n = \lim_{n \to \infty} \int_{F \cap Y} f_n = \int_{F \cap Y} f$$

for every $F \in \Sigma$. Taking $F = X$ to begin with, B.Levi's theorem tells us that $h = \lim_{n\to\infty} \tilde{f}_n$ is defined (as a real-valued function) μ-almost everywhere; now letting F vary, we have $\int_F h = \int_{F\cap Y} f$ for every $F \in \Sigma$, because $h\restriction F = \lim_{n\to\infty} \tilde{f}_n\restriction F$ μ_F-a.e. (I seem to be using 214Cb here.) Now $h\restriction Y = f$ μ_Y-a.e., by 214Cb again. If we define $\tilde{f}$ by setting

$$\tilde{f}(x) = f(x) \text{ for } x \in \operatorname{dom} f,\ h(x) \text{ for } x \in \operatorname{dom} h \setminus \operatorname{dom} f,\ 0 \text{ for other } x \in X,$$

then $\tilde{f}$ is defined everywhere in X and is equal to h μ-almost everywhere; so that if $F \in \Sigma$, $\tilde{f}\restriction F$ will be equal to $h\restriction F$ μ_F-almost everywhere, and

$$\int_F \tilde{f} = \int_F h = \int_{F\cap Y} f.$$

As F is arbitrary, $\tilde{f}$ is an enveloping extension of f.

(v) Thus every non-negative μ_Y-integrable function has an enveloping extension. Using (ii) again, every μ_Y-integrable function has an enveloping extension, as claimed.

214F Proposition Let (X, Σ, μ) be a measure space, Y a subset of X, and f a $[-\infty, \infty]$-valued function such that $\int_X f$ is defined in $[-\infty, \infty]$. If *either* Y has full outer measure in X *or* f is zero almost everywhere in $X \setminus Y$, then $\int_Y f$ is defined and equal to $\int_X f$.

proof (a) Suppose first that f is non-negative, Σ-measurable and defined everywhere in X. In this case $f\restriction Y$ is Σ_Y-measurable. Set $F_{nk} = \{x : x \in X,\ f(x) \ge 2^{-n}k\}$ for $k, n \in \mathbb{N}$, $f_n = \sum_{k=1}^{4^n} 2^{-n}\chi F_{nk}$ for $n \in \mathbb{N}$, so that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of real-valued measurable functions converging everywhere to f, and $\int_X f = \lim_{n\to\infty} \int_X f_n$. For each $n \in \mathbb{N}$ and $k \ge 1$,

$$\mu_Y(F_{nk} \cap Y) = \mu^*(F_{nk} \cap Y) = \mu F_{nk}$$

either because $F_{nk} \setminus Y$ is negligible or because X is a measurable envelope of Y. So

$$\begin{aligned}\int_Y f &= \lim_{n\to\infty} \int_Y f_n = \lim_{n\to\infty} \sum_{k=1}^{4^n} 2^{-n}\mu_Y(F_{nk} \cap Y)\\ &= \lim_{n\to\infty} \sum_{k=1}^{4^n} 2^{-n}\mu F_{nk} = \lim_{n\to\infty} \int_X f_n = \int_X f.\end{aligned}$$

(b) Now suppose that f is non-negative, defined almost everywhere in X and μ-virtually measurable. In this case there is a conegligible measurable set $E \subseteq \operatorname{dom} f$ such that $f\restriction E$ is measurable. Set $\tilde{f}(x) = f(x)$ for $x \in E$, 0 for $x \in X \setminus E$; then $\tilde{f}$ satisfies the conditions of (a) and $f = \tilde{f}$ μ-a.e. Accordingly $f\restriction Y = \tilde{f}\restriction Y$ μ_Y-a.e. (214Cc), and

$$\int_Y f = \int_Y \tilde{f} = \int_X \tilde{f} = \int_X f.$$

(c) Finally, for the general case, we can apply (b) to the positive and negative parts f^+, f^- of f to get

$$\int_Y f = \int_Y f^+ - \int_Y f^- = \int_X f^+ - \int_X f^- = \int_X f.$$

214G Corollary Let (X, Σ, μ) be a measure space, Y a subset of X, and $E \in \Sigma$ a measurable envelope of Y. If f is a $[-\infty, \infty]$-valued function such that $\int_E f$ is defined in $[-\infty, \infty]$, then $\int_Y f$ is defined and equal to $\int_E f$.

proof By 214Ce, we can identify the subspace measure μ_Y with the subspace measure $(\mu_E)_Y$ induced by the subspace measure on E. Now, regarded as a subspace of E, Y has full outer measure, so 214F gives the result.

214H Subspaces and Carathéodory's method The following easy technical results will occasionally be useful.

Lemma Let X be a set, $Y \subseteq X$ a subset, and θ an outer measure on X.

(a) $\theta_Y = \theta\restriction\mathcal{P}Y$ is an outer measure on Y.

(b) Let μ, ν be the measures on X, Y defined by Carathéodory's method from the outer measures θ, θ_Y, and Σ, T their domains; let μ_Y be the subspace measure on Y induced by μ, and Σ_Y its domain. Then

(i) $\Sigma_Y \subseteq \mathrm{T}$ and $\nu F \le \mu_Y F$ for every $F \in \Sigma_Y$;

(ii) if $Y \in \Sigma$ then $\nu = \mu_Y$;

(iii) if $\theta = \mu^*$ (that is, θ is 'regular') then ν extends μ_Y;

(iv) if $\theta = \mu^*$ and $\theta Y < \infty$ then $\nu = \mu_Y$.

proof (a) You have only to read the definition of 'outer measure' (113A).

(b)(i) Suppose that $F \in \Sigma_Y$. Then it is of the form $E \cap Y$ where $E \in \Sigma$. If $A \subseteq Y$, then

$$\theta_Y(A \cap F) + \theta_Y(A \setminus F) = \theta(A \cap F) + \theta(A \setminus F) = \theta(A \cap E) + \theta(A \setminus E) = \theta A = \theta_Y A,$$

so $F \in \mathrm{T}$. Now

$$\nu F = \theta_Y F = \theta F \le \mu^* F = \mu_Y F.$$

(ii) Suppose that $F \in \mathrm{T}$. If $A \subseteq X$, then

$$\begin{aligned}\theta A &= \theta(A \cap Y) + \theta(A \setminus Y) = \theta_Y(A \cap Y) + \theta(A \setminus Y)\\ &= \theta_Y(A \cap Y \cap F) + \theta_Y(A \cap Y \setminus F) + \theta(A \setminus Y)\\ &= \theta(A \cap F) + \theta(A \cap Y \setminus F) + \theta(A \setminus Y)\\ &= \theta(A \cap F) + \theta((A \setminus F) \cap Y) + \theta((A \setminus F) \setminus Y) = \theta(A \cap F) + \theta(A \setminus F);\end{aligned}$$

as A is arbitrary, $F \in \Sigma$ and therefore $F \in \Sigma_Y$. Also

$$\mu_Y F = \mu F = \theta F = \theta_Y F = \nu F.$$

Putting this together with (i), we see that μ_Y and ν are identical.

(iii) Let $F \in \Sigma_Y$. Then $F \in \mathrm{T}$, by (i). Now $\nu F = \theta F = \mu^* F = \mu_Y F$. As F is arbitrary, ν extends μ_Y.

(iv) Now suppose that $F \in \mathrm{T}$. Because $\mu^* Y = \theta Y < \infty$, we have measurable envelopes E_1, E_2 of F and $Y \setminus F$ for μ (132Ee). Then

$$\begin{aligned}\theta Y &= \theta_Y Y = \theta_Y F + \theta_Y(Y \setminus F) = \theta F + \theta(Y \setminus F)\\ &= \mu^* F + \mu^*(Y \setminus F) = \mu E_1 + \mu E_2 \ge \mu(E_1 \cup E_2) = \theta(E_1 \cup E_2) \ge \theta Y,\end{aligned}$$

so $\mu E_1 + \mu E_2 = \mu(E_1 \cup E_2)$ and

$$\mu(E_1 \cap E_2) = \mu E_1 + \mu E_2 - \mu(E_1 \cup E_2) = 0.$$

As μ is complete (212A) and $E_1 \cap Y \setminus F \subseteq E_1 \cap E_2$ is μ-negligible, therefore belongs to Σ, $F = Y \cap (E_1 \setminus (E_1 \cap Y \setminus F))$ belongs to Σ_Y. Thus $\mathrm{T} \subseteq \Sigma_Y$; putting this together with (iii), we see that $\nu = \mu_Y$.

214I I now turn to the relationships between subspace measures and the classification of measure spaces developed in this chapter.

Theorem Let (X, Σ, μ) be a measure space and Y a subset of X. Let μ_Y be the subspace measure on Y and Σ_Y its domain.

(a) If (X, Σ, μ) is complete, or totally finite, or σ-finite, or strictly localizable, so is (Y, Σ_Y, μ_Y). If $\langle X_i \rangle_{i \in I}$ is a decomposition of X for μ, then $\langle X_i \cap Y \rangle_{i \in I}$ is a decomposition of Y for μ_Y.

(b) Writing $\hat\mu$ for the completion of μ, the subspace measure $\hat\mu_Y = (\hat\mu)_Y$ is the completion of μ_Y.

(c) If (X, Σ, μ) has locally determined negligible sets, then μ_Y is semi-finite.

(d) If (X, Σ, μ) is complete and locally determined, then (Y, Σ_Y, μ_Y) is complete and semi-finite.

(e) If (X, Σ, μ) is complete, locally determined and localizable then so is (Y, Σ_Y, μ_Y).

proof (a)(i) Suppose that (X, Σ, μ) is complete. If $A \subseteq U \in \Sigma_Y$ and $\mu_Y U = 0$, there is an $E \in \Sigma$ such that $U = E \cap Y$ and $\mu E = \mu_Y U = 0$; now $A \subseteq E$ so $A \in \Sigma$ and $A = A \cap Y \in \Sigma_Y$.

(ii) $\mu_Y Y = \mu^* Y \le \mu X$, so μ_Y is totally finite if μ is.

(iii) If $\langle X_n\rangle_{n\in\mathbb{N}}$ is a sequence of sets of finite measure for μ which covers X, then $\langle X_n \cap Y\rangle_{n\in\mathbb{N}}$ is a sequence of sets of finite measure for μ_Y which covers Y. So (Y, Σ_Y, μ_Y) is σ-finite if (X, Σ, μ) is.

(iv) Suppose that $\langle X_i\rangle_{i\in I}$ is a decomposition of X for μ. Then $\langle X_i \cap Y\rangle_{i\in I}$ is a decomposition of Y for μ_Y. **P** Because $\mu_Y(X_i \cap Y) \le \mu X_i < \infty$ for each i, $\langle X_i \cap Y\rangle_{i\in I}$ is a partition of Y into sets of finite measure. Suppose that $U \subseteq Y$ is such that $U_i = U \cap X_i \cap Y \in \Sigma_Y$ for every i. For each $i \in I$, choose $E_i \in \Sigma$ such that $U_i = E_i \cap Y$ and $\mu E_i = \mu_Y U_i$; we may of course suppose that $E_i \subseteq X_i$. Set $E = \bigcup_{i\in I} E_i$. Then $E \cap X_i = E_i \in \Sigma$ for every i, so $E \in \Sigma$ and $\mu E = \sum_{i\in I} \mu E_i$. Now $U = E \cap Y$ so $U \in \Sigma_Y$ and

$$\mu_Y U \le \mu E = \textstyle\sum_{i\in I} \mu E_i = \sum_{i\in I} \mu_Y U_i.$$

On the other hand, $\mu_Y U$ is surely greater than or equal to $\sum_{i\in I} \mu_Y U_i = \sup_{J\subseteq I \text{ is finite}} \sum_{i\in J} \mu_Y U_i$, so they are equal. As U is arbitrary, $\langle X_i \cap Y\rangle_{i\in I}$ is a decomposition of Y for μ_Y. **Q**

Consequently (Y, Σ_Y, μ_Y) is strictly localizable if (X, Σ, μ) is.

(b) The domain of the completion $(\mu_Y)\hat{}$ is

$$\begin{aligned}\hat\Sigma_Y &= \{F \triangle A : F \in \Sigma_Y,\ A \subseteq Y \text{ is } \mu_Y\text{-negligible}\}\\ &= \{(E \cap Y)\triangle(A \cap Y) : E \in \Sigma,\ A \subseteq X \text{ is } \mu\text{-negligible}\}\\ &= \{(E \triangle A) \cap Y : E \in \Sigma,\ A \text{ is } \mu\text{-negligible}\} = \operatorname{dom} \hat\mu_Y. \qquad \text{(214Cb)}\end{aligned}$$

If $H \in \hat\Sigma_Y$ then

$$(\mu_Y)\hat{}(H) = \mu_Y^* H = \mu^* H = (\hat\mu)^* H = \hat\mu_Y H,$$

using 214Cd for the second step, and 212Ea for the third.

(c) Take $U \in \Sigma_Y$ such that $\mu_Y U > 0$. Then there is an $E \in \Sigma$ such that $\mu E < \infty$ and $\mu^*(E \cap U) > 0$. **P?** Otherwise, $E \cap U$ is μ-negligible whenever $\mu E < \infty$; because μ has locally determined negligible sets, U is μ-negligible and $\mu_Y U = \mu^* U = 0$. **XQ** Now $E \cap U \in \Sigma_Y$ and

$$0 < \mu^*(E \cap U) = \mu_Y(E \cap U) \le \mu E < \infty.$$

(d) By (a), μ_Y is complete; by 213J and (c) here, it is semi-finite.

(e) By (d), μ_Y is complete and semi-finite. To see that it is locally determined, take any $U \subseteq Y$ such that $U \cap V \in \Sigma_Y$ whenever $V \in \Sigma_Y$ and $\mu_Y V < \infty$. By 213J and 213L, there is a measurable envelope E of U for μ; of course $E \cap Y \in \Sigma_Y$.

I claim that $\mu(E \cap Y \setminus U) = 0$. **P** Take any $F \in \Sigma$ with $\mu F < \infty$. Then $F \cap U \in \Sigma_Y$, so

$$\mu_Y(F \cap E \cap Y) \le \mu(F \cap E) = \mu^*(F \cap U) = \mu_Y(F \cap U) \le \mu_Y(F \cap E \cap Y);$$

thus $\mu_Y(F \cap E \cap Y) = \mu_Y(F \cap U)$ and

$$\mu^*(F \cap E \cap Y \setminus U) = \mu_Y(F \cap E \cap Y \setminus U) = 0.$$

Because μ is complete, $\mu(F \cap E \cap Y \setminus U) = 0$; because μ is locally determined and F is arbitrary, $\mu(E \cap Y \setminus U) = 0$. **Q** But this means that $E \cap Y \setminus U \in \Sigma_Y$ and $U \in \Sigma_Y$. As U is arbitrary, μ_Y is locally determined.

To see that μ_Y is localizable, let $\mathcal{U}$ be any family in Σ_Y. Set

$$\mathcal{E} = \{E : E \in \Sigma,\ \mu E < \infty,\ \mu E = \mu^*(E \cap U) \text{ for some } U \in \mathcal{U}\},$$

and let $G \in \Sigma$ be an essential supremum for $\mathcal{E}$ in Σ. I claim that $G \cap Y$ is an essential supremum for $\mathcal{U}$ in Σ_Y. **P** (i) **?** If $U \in \mathcal{U}$ and $U \setminus (G \cap Y)$ is not negligible, then (because μ_Y is semi-finite) there is a $V \in \Sigma_Y$ such that $V \subseteq U \setminus G$ and $0 < \mu_Y V < \infty$. Now there is an $E \in \Sigma$ such that $V \subseteq E$ and $\mu E = \mu^* V$. We have $\mu^*(E \cap U) \ge \mu^* V = \mu E$, so $E \in \mathcal{E}$ and $E \setminus G$ must be negligible; but $V \subseteq E \setminus G$ is not negligible. **X** Thus $U \setminus (G \cap Y)$ is negligible for every $U \in \mathcal{U}$. (ii) If $W \in \Sigma_Y$ is such that $U \setminus W$ is negligible for every $U \in \mathcal{U}$, express W as $H \cap Y$ where $H \in \Sigma$. If $E \in \mathcal{E}$, there is a $U \in \mathcal{U}$ such that $\mu E = \mu^*(E \cap U)$; now $\mu^*(E \cap U \setminus W) = 0$, so $\mu E = \mu^*(E \cap U \cap W) \le \mu(E \cap H)$ and $E \setminus H$ is negligible. As E is arbitrary, H is an essential upper bound for $\mathcal{E}$ and $G \setminus H$ is negligible; but this means that $G \cap Y \setminus W$ is negligible. As W is arbitrary, $G \cap Y$ is an essential supremum for $\mathcal{U}$. **Q**

As $\mathcal{U}$ is arbitrary, μ_Y is localizable.

214J Upper and lower integrals The following elementary facts are sometimes useful.

Proposition Let (X, Σ, μ) be a measure space, A a subset of X and f a real-valued function defined almost everywhere in X. Then

(a) if *either* f is non-negative *or* A has full outer measure in X, $\overline{\int}(f \restriction A) d\mu_A \le \overline{\int} f d\mu$;

(b) if A has full outer measure in X, $\underline{\int} f d\mu \le \underline{\int}(f \restriction A) d\mu_A$.

proof (a)(i) Suppose that f is non-negative. If $\overline{\int} f d\mu = \infty$, the result is trivial. Otherwise, there is a μ-integrable function g such that $f \le g$ μ-a.e. and $\overline{\int} f d\mu = \int g\,d\mu$, by 133J(a-i). Now $f \restriction A \le g \restriction A$ μ_A-a.e., by 214Cb, and $\int (g \restriction A)\,d\mu_A$ is defined and less than or equal to $\int g\,d\mu$, by 214Ea; so

$$\overline{\int}(f \restriction A) d\mu_A \le \int (g \restriction A) d\mu_A \le \int g\,d\mu = \overline{\int} f d\mu.$$

(ii) Now suppose that A has full outer measure in X. If g is such that $f \le g$ μ-a.e. and $\int g\,d\mu$ is defined in $[-\infty, \infty]$, then $f \restriction A \le g \restriction A$ μ_A-a.e. and $\int (g \restriction A) d\mu_A = \int g\,d\mu$, by 214F. So $\overline{\int}(f \restriction A) d\mu_A \le \int g\,d\mu$. As g is arbitrary, $\overline{\int}(f \restriction A) d\mu_A \le \overline{\int} f\,d\mu$.

(b) Apply (a) to $-f$, and use 133J(b-iv).

214K Measurable subspaces: Proposition Let (X, Σ, μ) be a measure space.

(a) Let $E \in \Sigma$ and let μ_E be the subspace measure, with Σ_E its domain. If (X, Σ, μ) is complete, or totally finite, or σ-finite, or strictly localizable, or semi-finite, or localizable, or locally determined, or atomless, or purely atomic, so is (E, Σ_E, μ_E).

(b) Suppose that $\langle X_i \rangle_{i \in I}$ is a partition of X into measurable sets (not necessarily of finite measure) such that

$$\Sigma = \{E : E \subseteq X,\ E \cap X_i \in \Sigma \text{ for every } i \in I\},$$

$$\mu E = \textstyle\sum_{i \in I} \mu(E \cap X_i) \text{ for every } E \in \Sigma.$$

Then (X, Σ, μ) is complete, or strictly localizable, or semi-finite, or localizable, or locally determined, or atomless, or purely atomic, iff $(X_i, \Sigma_{X_i}, \mu_{X_i})$ has that property for every $i \in I$.

proof I really think that if you have read attentively up to this point, you ought to find this easy. If you are in any doubt, this makes a very suitable set of sixteen exercises to do.

214L Direct sums Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be any indexed family of measure spaces. Set $X = \bigcup_{i \in I}(X_i \times \{i\})$; for $E \subseteq X$, $i \in I$ set $E_i = \{x : (x, i) \in E\}$. Write

$$\Sigma = \{E : E \subseteq X,\ E_i \in \Sigma_i \text{ for every } i \in I\},$$

$$\mu E = \textstyle\sum_{i \in I} \mu_i E_i \text{ for every } E \in \Sigma.$$

Then it is easy to check that (X, Σ, μ) is a measure space; I will call it the **direct sum** of the family $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$. Note that if (X, Σ, μ) is any strictly localizable measure space, with decomposition $\langle X_i \rangle_{i \in I}$, then we have a natural isomorphism between (X, Σ, μ) and the direct sum $(X', \Sigma', \mu') = \bigoplus_{i \in I}(X_i, \Sigma_{X_i}, \mu_{X_i})$ of the subspace measures, if we match $(x, i) \in X'$ with $x \in X$ for every $i \in I$ and $x \in X_i$.

For some of the elementary properties (to put it plainly, I know of no properties which are not elementary) of direct sums, see 214M and 214Xh-214Xk.

214M Proposition Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of measure spaces, with direct sum (X, Σ, μ). Let f be a real-valued function defined on a subset of X. For each $i \in I$, set $f_i(x) = f(x, i)$ whenever $(x, i) \in \operatorname{dom} f$.

(a) f is measurable iff f_i is measurable for every $i \in I$.

(b) If f is non-negative, then $\int f d\mu = \sum_{i \in I} \int f_i d\mu_i$ if either is defined in $[0, \infty]$.

proof (a) For $a \in \mathbb{R}$, set $F_a = \{(x,i) : (x,i) \in \operatorname{dom} f,\ f(x,i) \ge a\}$. (i) If f is measurable, $i \in I$ and $a \in \mathbb{R}$, then there is an $E \in \Sigma$ such that $F_a = E \cap \operatorname{dom} f$; now

$$\{x : f_i(x) \ge a\} = \operatorname{dom} f_i \cap \{x : (x,i) \in E\}$$

belongs to the subspace σ-algebra on $\operatorname{dom} f_i$ induced by Σ_i. As a is arbitrary, f_i is measurable. (ii) If every f_i is measurable and $a \in \mathbb{R}$, then for each $i \in I$ there is an $E_i \in \Sigma_i$ such that $\{x : (x,i) \in F_a\} = E_i \cap \operatorname{dom} f$; setting $E = \{(x,i) : i \in I,\ x \in E_i\}$, $F_a = \operatorname{dom} f \cap E$ belongs to the subspace σ-algebra on $\operatorname{dom} f$. As a is arbitrary, f is measurable.

(b)(i) Suppose first that f is measurable and defined everywhere. Set $F_{nk} = \{(x,i) : (x,i) \in X,\ f(x,i) \ge 2^{-n}k\}$ for k, $n \in \mathbb{N}$, $g_n = \sum_{k=1}^{4^n} 2^{-n}\chi F_{nk}$ for $n \in \mathbb{N}$, $F_{nki} = \{x : (x,i) \in F_{nk}\}$ for k, $n \in \mathbb{N}$ and $i \in I$, $g_{ni}(x) = g_n(x,i)$ for $i \in I$, $x \in X_i$. Then

$$\begin{aligned}\int f d\mu &= \lim_{n\to\infty} \int g_n d\mu = \sup_{n\in\mathbb{N}} \sum_{k=1}^{4^n} 2^{-n}\mu F_{nk} \\ &= \sup_{n\in\mathbb{N}} \sum_{k=1}^{4^n} \sum_{i\in I} 2^{-n}\mu F_{nki} = \sum_{i\in I} \sup_{n\in\mathbb{N}} \sum_{k=1}^{4^n} 2^{-n}\mu F_{nki} \\ &= \sum_{i\in I} \sup_{n\in\mathbb{N}} \int g_{ni} d\mu_i = \sum_{i\in I} \int f_i d\mu_i.\end{aligned}$$

(ii) Generally, if $\int f d\mu$ is defined, there are a measurable $g : X \to [0,\infty[$ and a conegligible measurable set $E \subseteq \operatorname{dom} f$ such that $g = f$ on E. Now $E_i = \{x : (x,i) \in X_i\}$ belongs to Σ_i for each i, and $\sum_{i\in I} \mu_i(X_i \setminus E_i) = \mu(X \setminus E) = 0$, so E_i is μ_i-conegligible for every i. Setting $g_i(x) = g(x,i)$ for $x \in X_i$, (i) tells us that

$$\textstyle\sum_{i\in I} \int f_i d\mu_i = \sum_{i\in I} \int g_i d\mu_i = \int g\, d\mu = \int f d\mu.$$

(iii) On the other hand, if $\int f_i d\mu_i$ is defined for every $i \in I$, then for each $i \in I$ we can find a measurable function $g_i : X_i \to [0,\infty[$ and a μ_i-conegligible measurable set $E_i \subseteq \operatorname{dom} f_i$ such that $g_i = f_i$ on E_i. Setting $g(x,i) = g_i(x)$ for $i \in I$, $x \in X_i$, (a) tells us that g is measurable, while $g = f$ on $\{(x,i) : i \in I,\ x \in E_i\}$, which is conegligible (by the calculation in (ii) just above); so

$$\textstyle\int f d\mu = \int g\, d\mu = \sum_{i\in I} \int g_i d\mu_i = \sum_{i\in I} \int f_i d\mu_i,$$

again using (i) for the middle step.

214N Corollary Let (X,Σ,μ) be a measure space with a decomposition $\langle X_i\rangle_{i\in I}$. If f is a real-valued function defined on a subset of X, then

(a) f is measurable iff $f{\restriction}X_i$ is measurable for every $i \in I$,

(b) if $f \ge 0$, then $\int f = \sum_{i\in I} \int_{X_i} f$ if either is defined in $[0,\infty]$.

proof Apply 214M to the direct sum of $\langle(X_i, \Sigma_{X_i}, \mu_{X_i})\rangle_{i\in I}$, identified with (X,Σ,μ) as in 214L.

***214O** I make space here for a general theorem which puts rather heavy demands on the reader. So I ought to say that I advise skipping it on first reading. It will not be quoted in this volume, in the full form here I do not expect to use it anywhere in this treatise, only the special case of 214Xm is at all often applied, and the proof depends on a concept ('ideal of sets') and a technique ('transfinite induction', part (d) of the proof of 214P) which are used nowhere else in this volume. However, 'extension of measures' is one of the central themes of Volume 4, and this result may help to make sense of some of the patterns which will appear there.

Lemma Let (X,Σ,μ) be a measure space, and $\mathcal{I}$ an ideal of subsets of X, that is, a family of subsets of X such that $\emptyset \in \mathcal{I}$, $I \cup J \in \mathcal{I}$ for all I, $J \in \mathcal{I}$, and $I \in \mathcal{I}$ whenever $I \subseteq J \in \mathcal{I}$. Then there is a measure λ on X such that $\Sigma \cup \mathcal{I} \subseteq \operatorname{dom}\lambda$, $\mu E = \lambda E + \sup_{I\in\mathcal{I}} \mu^*(E \cap I)$ for every $E \in \Sigma$, and $\lambda I = 0$ for every $I \in \mathcal{I}$.

proof (a) Let Λ be the set of those $F \subseteq X$ such that there are $E \in \Sigma$ and a countable $\mathcal{J} \subseteq \mathcal{I}$ such that $E \triangle F \subseteq \bigcup \mathcal{J}$. Then Λ is a σ-algebra of subsets of X including $\Sigma \cup \mathcal{I}$. **P** $\Sigma \subseteq \Lambda$ because $E \triangle E \subseteq \bigcup \emptyset$ for every $E \in \Sigma$. $\mathcal{I} \subseteq \Lambda$ because $\emptyset \triangle I \subseteq \bigcup \{I\}$ for every $I \in \mathcal{I}$. In particular, $\emptyset \in \Lambda$. If $F \in \Lambda$, let $E \in \Sigma$ and $\mathcal{J} \subseteq \mathcal{I}$ be such that $\mathcal{J}$ is countable and $F \triangle E \subseteq \bigcup \mathcal{J}$; then $(X \setminus F) \triangle (X \setminus E) \subseteq \bigcup \mathcal{J}$ so $X \setminus F \in \Lambda$. If $\langle F_n \rangle_{n \in \mathbb{N}}$ is a sequence in Λ with union F, then for each $n \in \mathbb{N}$ choose $E_n \in \Sigma$, $\mathcal{J}_n \subseteq \mathcal{I}$ such that $\mathcal{J}_n$ is countable and $E_n \triangle F_n \subseteq \bigcup \mathcal{J}_n$; then $E = \bigcup_{n \in \mathbb{N}} E_n$ belongs to Σ, $\mathcal{J} = \bigcup_{n \in \mathbb{N}} \mathcal{J}_n$ is a countable subset of $\mathcal{I}$ and $E \triangle F \subseteq \bigcup \mathcal{J}$, so $F \in \Lambda$. Thus Λ is a σ-algebra. **Q**

(b) For $F \in \Lambda$ set

$$\lambda F = \sup\{\mu E : E \in \Sigma,\, E \subseteq F,\, \mu^*(E \cap I) = 0 \text{ for every } I \in \mathcal{I}\}.$$

Then λ is a measure. **P** The only subset of $\emptyset$ is $\emptyset$, so $\lambda\emptyset = 0$. Let $\langle F_n \rangle_{n \in \mathbb{N}}$ be a disjoint sequence in Λ with union F and set $u = \sum_{n=0}^{\infty} \lambda F_n$. (i) For each $n \in \mathbb{N}$ take $E_n \in \Sigma$ and a countable $\mathcal{J}_n \subseteq \mathcal{I}$ such that $E_n \triangle F_n \subseteq \bigcup \mathcal{J}_n$. Set $\mathcal{J} = \bigcup_{n \in \mathbb{N}} \mathcal{J}_n$. If $E \in \Sigma$, $E \subseteq F$ and $\mu^*(E \cap I) = 0$ for every $I \in \mathcal{I}$, then $\mu^*(E \cap \bigcup \mathcal{J}) = 0$; let $H \in \Sigma$ be a μ-negligible set including $E \cap \bigcup \mathcal{J}$. If $n \in \mathbb{N}$, then $(E \cap E_n) \triangle (E \cap F_n) \subseteq H$ so $E \cap F_n \setminus H = E \cap E_n \setminus H$ belongs to Σ, while $\mu^*((E \cap F_n \setminus H) \cap I) = 0$ for every $I \in \mathcal{I}$. Now

$$\mu E = \mu(E \cap F \setminus H) = \textstyle\sum_{n=0}^{\infty} \mu(E \cap F_n \setminus H) \le \sum_{n=0}^{\infty} \lambda F_n = u.$$

As E is arbitrary, $\lambda F \le u$. (ii) Take any $\gamma < u$. For $n \in \mathbb{N}$, set $\gamma_n = \lambda F_n - 2^{-n-1} \min(1, u - \gamma)$ if λF_n is finite, γ otherwise. For each n, we can find an $E_n \in \Sigma$ such that $E_n \subseteq F_n$, $\mu^*(E_n \cap I) = 0$ for every $I \in \mathcal{I}$, and $\mu E_n \ge \gamma_n$. Set $E = \bigcup_{n \in \mathbb{N}} E_n$; then $E \subseteq F$ and $E \cap I = \bigcup_{n \in \mathbb{N}} E_n \cap I$ is μ-negligible for every $I \in \mathcal{I}$, so $\lambda F \ge \mu E = \sum_{n=0}^{\infty} \mu E_n \ge \gamma$. As γ is arbitrary, $\lambda F \ge u$. (iii) As $\langle F_n \rangle_{n \in \mathbb{N}}$ is arbitrary, λ is a measure. **Q**

(c) Now take any $E \in \Sigma$ and set $u = \sup_{I \in \mathcal{I}} \mu^*(E \cap I)$. If $u = \infty$ then we certainly have $\mu E = \infty = \lambda E + u$. Otherwise, let $\langle I_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{I}$ such that $\lim_{n \to \infty} \mu^*(E \cap I_n) = u$; replacing I_n by $\bigcup_{m \le n} I_m$ for each n if necessary, we may suppose that $\langle I_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. Set $A = E \cap \bigcup_{n \in \mathbb{N}} I_n$; because $E \cap I_n$ has finite outer measure for each n, A can be covered by a sequence of sets of finite measure, and has a measurable envelope H for μ included in E (132Ee). Observe that

$$\mu H = \mu^* A = \sup_{n \in \mathbb{N}} \mu^*(E \cap I_n) = u$$

by 132Ae.

Set $G = E \setminus H$. Then $\mu^*(G \cap I) = 0$ for every $I \in \mathcal{I}$. **P** For any $n \in \mathbb{N}$ there is an $F \in \Sigma$ such that $F \supseteq E \cap (I_n \cup I)$ and $\mu F \le u$; in which case

$$\mu^*(G \cap I) + \mu^*(E \cap I_n) \le \mu(F \setminus H) + \mu(F \cap H) \le u.$$

As n is arbitrary, $\mu^*(G \cap I) = 0$. **Q** Accordingly

$$u + \lambda E \ge \mu H + \mu G = \mu E.$$

On the other hand, if $F \in \Sigma$ is such that $F \subseteq E$ and $\mu^*(F \cap I) = 0$ for every $I \in \mathcal{I}$, then

$$\mu^*(E \cap I_n) \le \mu(E \setminus F) + \mu^*(F \cap I_n) = \mu(E \setminus F)$$

for every n, so

$$u + \mu F \le \mu(E \setminus F) + \mu F = \mu E;$$

as F is arbitrary, $u + \lambda E \le \mu E$.

(d) If $J \in \mathcal{I}$, $F \in \Sigma$, $F \subseteq J$ and $\mu^*(F \cap I) = 0$ for every $I \in \mathcal{I}$, then $F \cap J = F$ is μ-negligible; as F is arbitrary, $\lambda J = 0$. Thus λ has all the required properties.

***214P Theorem** Let (X, Σ, μ) be a measure space, and $\mathcal{A}$ a family of subsets of X which is well-ordered by the relation $\subseteq$. Then there is an extension of μ to a measure λ on X such that $\lambda(E \cap A)$ is defined and equal to $\mu^*(E \cap A)$ whenever $E \in \Sigma$ and $A \in \mathcal{A}$.

proof (a) Adding $\emptyset$ and X to $\mathcal{A}$ if necessary, we may suppose that $\mathcal{A}$ has $\emptyset$ as its least member and X as its greatest member. By 2A1Dg, $\mathcal{A}$ is isomorphic, as ordered set, to some ordinal; since $\mathcal{A}$ has a greatest member, this ordinal is a successor, expressible as $\zeta + 1$; let $\xi \mapsto A_\xi : \zeta + 1 \to \mathcal{A}$ be the order-isomorphism, so that $\langle A_\xi \rangle_{\xi \le \zeta}$ is a non-decreasing family of subsets of X, $A_0 = \emptyset$ and $A_\zeta = X$.

(b) For each ordinal $\xi \le \zeta$, write μ_ξ for the subspace measure on A_ξ, Σ_ξ for its domain and $\mathcal{I}_\xi$ for $\bigcup_{\eta<\xi} \mathcal{P}A_\eta$. Because $A_\eta \cup A_{\eta'} = A_{\max(\eta,\eta')}$ for η, $\eta' < \xi$, $\mathcal{I}_\xi$ is an ideal of subsets of A_ξ. By 214O, we have a measure λ_ξ on A_ξ, with domain $\Lambda_\xi \supseteq \Sigma_\xi \cup \mathcal{I}_\xi$, such that $\mu_\xi E = \lambda_\xi E + \sup_{I\in\mathcal{I}_\xi} \mu^*_\xi(E\cap I)$ for every $E \in \Sigma_\xi$ and $\lambda_\xi I = 0$ for every $I \in \mathcal{I}_\xi$. Because every member of $\mathcal{I}_\xi$ is included in A_η for some $\eta < \xi$, we have

$$\mu^*(E\cap A_\xi) = \lambda_\xi(E\cap A_\xi) + \sup_{\eta<\xi} \mu^*_\xi(E\cap A_\eta) = \lambda_\xi(E\cap A_\xi) + \sup_{\eta<\xi} \mu^*(E\cap A_\eta)$$

(214Cd) for every $E \in \Sigma$. Also, of course, $\lambda_\xi A_\eta = 0$ for every $\eta < \xi$.

(c) Now set

$$\Lambda = \{F : F \subseteq X,\ F\cap A_\xi \in \Lambda_\xi \text{ for every } \xi \le \zeta\},$$

$$\lambda F = \sum_{\xi\le\zeta} \lambda_\xi(F\cap A_\xi)$$

for every $F \in \Lambda$. Because Λ_ξ is a σ-algebra of subsets of A_ξ for each ξ, Λ is a σ-algebra of subsets of X; because every λ_ξ is a measure, so is λ. If $E \in \Sigma$, then

$$E\cap A_\xi \in \Sigma_\xi \subseteq \Lambda_\xi$$

for each ξ, so $E \in \Lambda$. If $\eta \le \zeta$, then for each $\xi \le \zeta$ either $\eta < \xi$ and

$$A_\eta \cap A_\xi = A_\eta \in \mathcal{I}_\xi \subseteq \Lambda_\xi$$

or $\eta \ge \xi$ and $A_\eta \cap A_\xi = A_\xi$ belongs to Λ_ξ. So $A_\eta \in \Lambda$ for every $\eta \le \zeta$.

(d) Finally, $\lambda(E\cap A_\xi) = \mu^*(E\cap A_\xi)$ whenever $E \in \Sigma$ and $\xi \le \zeta$. **P?** Otherwise, because the ordinal $\zeta + 1$ is well-ordered, there is a least ξ such that $\lambda(E\cap A_\xi) \ne \mu^*(E\cap A_\xi)$. As $A_0 = \emptyset$ we surely have $\lambda(E\cap A_0) = \mu^*(E\cap A_0)$ and $\xi > 0$. Note that if $\eta > \xi$, then $\lambda_\eta(E\cap A_\xi) = 0$; so

$$\lambda(E\cap A_\xi) = \sum_{\eta\le\xi} \lambda_\eta(E\cap A_\xi\cap A_\eta) = \sum_{\eta\le\xi} \lambda_\eta(E\cap A_\eta).$$

Now

$$\mu^*(E\cap A_\xi) = \lambda_\xi(E\cap A_\xi) + \sup_{\xi'<\xi} \mu^*(E\cap A_{\xi'})$$

((b) above)

$$= \lambda_\xi(E\cap A_\xi) + \sup_{\xi'<\xi} \sum_{\eta\le\xi'} \lambda_\eta(E\cap A_\eta)$$

(because ξ was the first problematic ordinal)

$$= \lambda_\xi(E\cap A_\xi) + \sup_{\xi'<\xi}\ \sup_{K\subseteq\xi'+1 \text{ is finite}} \sum_{\eta\in K} \lambda_\eta(E\cap A_\eta)$$

(see the definition of 'sum' in 112Bd, or 226A below)

$$= \lambda_\xi(E\cap A_\xi) + \sup_{K\subseteq\xi \text{ is finite}} \sum_{\eta\in K} \lambda_\eta(E\cap A_\eta)$$

$$= \sup_{K\subseteq\xi+1 \text{ is finite}} \sum_{\eta\in K} \lambda_\eta(E\cap A_\eta) = \sum_{\eta\le\xi} \lambda_\eta(E\cap A_\eta) \ne \mu^*(E\cap A_\xi)$$

by the choice of ξ; but this is absurd. **XQ**

In particular,

$$\lambda E = \lambda(E\cap A_\zeta) = \mu^*(E\cap A_\zeta) = \mu E$$

for every $E \in \Sigma$. This completes the proof of the theorem.

***214Q Proposition** Suppose that (X, Σ, μ) is an atomless measure space and Y a subset of X such that the subspace measure μ_Y is semi-finite. Then μ_Y is atomless.

proof Let $F \subseteq Y$ be such that $\mu_Y F$ is defined and not 0. Because μ_Y is semi-finite, there is an $F' \subseteq F$ such that $\mu_Y F'$ is defined, finite and not zero. In this case, $\mu^* F' = \mu_Y F'$ is finite, so F' has a measurable envelope E say with respect to μ. Because μ is atomless, there is an $E_1 \in \Sigma$ such that $E_1 \subseteq E$ and neither E_1 nor $E \setminus E_1$ is μ-negligible. Now $F \cap E_1$ is measured by μ_Y and

$$\mu_Y(F \cap E_1) \geq \mu^*(F' \cap E_1) = \mu(E \cap E_1) > 0,$$

$$\mu_Y(F \setminus E_1) \geq \mu^*(F' \setminus E_1) = \mu(E \setminus E_1) > 0.$$

As F is arbitrary, μ_Y is atomless.

214X Basic exercises (a) Let (X, Σ, μ) be a localizable measure space. Show that there is an $E \in \Sigma$ such that the subspace measure μ_E is purely atomic and $\mu_{X \setminus E}$ is atomless.

(b) Let X be a set, θ a regular outer measure on X, and Y a subset of X. Let μ be the measure on X defined by Carathéodory's method from θ, μ_Y the subspace measure on Y, and ν the measure on Y defined by Carathéodory's method from $\theta \restriction \mathcal{P}Y$. Show that if μ_Y is locally determined (in particular, if μ is locally determined and localizable) then $\nu = \mu_Y$.

(c) Let (X, Σ, μ) be a localizable measure space, and Y a subset of X such that the subspace measure μ_Y is semi-finite. Show that μ_Y is localizable.

>(d) Let (X, Σ, μ) be a measure space, and Y a subset of X such that the subspace measure μ_Y is semi-finite. (i) Show that a set $F \subseteq Y$ is an atom for μ_Y iff it is of the form $E \cap Y$ where E an atom for μ. (ii) Show that if μ is purely atomic, so is μ_Y.

(e) Let (X, Σ, μ) be a localizable measure space, and Y any subset of X. Show that the c.l.d. version of the subspace measure on Y is localizable.

(f) Let (X, Σ, μ) be a measure space with locally determined negligible sets, and Y a subset of X, with its subspace measure μ_Y. Show that μ_Y has locally determined negligible sets.

>(g) Let (X, Σ, μ) be a measure space. Show that (X, Σ, μ) has locally determined negligible sets iff the subspace measure μ_Y is semi-finite for every $Y \subseteq X$.

>(h) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ be a family of measure spaces, with direct sum (X, Σ, μ) (214L). Set $X_i' = X_i \times \{i\} \subseteq X$ for each $i \in I$. Show that X_i', with the subspace measure, is isomorphic to (X_i, Σ_i, μ_i). Under what circumstances is $\langle X_i' \rangle_{i \in I}$ a decomposition of X? Show that μ is complete, or strictly localizable, or localizable, or locally determined, or semi-finite, or atomless, or purely atomic iff every μ_i is. Show that a measure space is strictly localizable iff it is isomorphic to a direct sum of totally finite spaces.

>(i) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ be a family of measure spaces, and (X, Σ, μ) their direct sum. Show that the completion of (X, Σ, μ) can be identified with the direct sum of the completions of the (X_i, Σ_i, μ_i), and that the c.l.d. version of (X, Σ, μ) can be identified with the direct sum of the c.l.d. versions of the (X_i, Σ_i, μ_i).

(j) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ be a family of measure spaces. Show that their direct sum has locally determined negligible sets iff every μ_i has.

(k) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ be a family of measure spaces, and (X, Σ, μ) their direct sum. Show that (X, Σ, μ) has the measurable envelope property (213Xl) iff every (X_i, Σ_i, μ_i) has.

(l) Let (X, Σ, μ) be a measure space, Y a subset of X, and $f : X \to [0, \infty]$ a function such that $\int_Y f$ is defined in $[0, \infty]$. Show that $\int_Y f = \overline{\int} f \times \chi Y \, d\mu$.

>(m) Write out a direct proof of 214P in the special case in which $\mathcal{A} = \{A\}$. (*Hint*: for E, $F \in \Sigma$,

$$\lambda((E \cap A) \cup (F \setminus A)) = \mu^*(E \cap A) + \sup\{\mu G : G \in \Sigma, \, G \subseteq F \setminus A\}.)$$

>(n) Let (X, Σ, μ) be a measure space and $\mathcal{A}$ a finite family of subsets of X. Show that there is a measure on X, extending μ, which measures every member of $\mathcal{A}$.

>(o) Let (X,Σ,μ) be a measure space and $\langle X_n\rangle_{n\in\mathbb{N}}$ a sequence of subsets of X such that $\bigcup_{n\in\mathbb{N}} X_n$ has full outer measure on X. Suppose that for each $n\in\mathbb{N}$ we have a set $A_n\subseteq X_n$ of full outer measure for the subspace measure on X_n, Show that $\bigcup_{n\in\mathbb{N}} A_n$ has full outer measure in X.

214Y Further exercises (a) Let (X,Σ,μ) be a measure space and A a subset of X such that the subspace measure on A is semi-finite. Set $\alpha=\sup\{\mu E: E\in\Sigma,\ E\subseteq A\}$. Show that if $\alpha\le\gamma\le\mu^*A$ then there is a measure λ on X, extending μ, such that $\lambda A=\gamma$.

(b) Let (X,Σ,μ) be a measure space and $\langle A_n\rangle_{n\in\mathbb{Z}}$ a double-ended sequence of subsets of X such that $A_m\subseteq A_n$ whenever $m\le n$ in $\mathbb{Z}$. Show that there is a measure on X, extending μ, which measures every A_n. (*Hint*: use 214P twice.)

(c) Let X be a set and $\mathcal{A}$ a family of subsets of X. Show that the following are equiveridical: (i) for every measure μ on X there is a measure on X extending μ and measuring every member of $\mathcal{A}$; (ii) for every totally finite measure μ on X there is a measure on X extending μ and measuring every member of $\mathcal{A}$. (*Hint*: 213Xa.)

(d) For this exercise only, I will say that a measure μ on a set X is **nowhere all-measuring** if whenever $A\subseteq X$ is not μ-negligible there is a subset of A which is not measured by the subspace measure on A. Show that if X is a set and $\mu_0,\dots,\mu_n$ are nowhere all-measuring complete totally finite measures on X, then there are disjoint $A_0,\dots,A_n\subseteq X$ such that $\mu_i^*A_i=\mu_iX$ for every $i\le n$. (*Hint*: start with the case $n=1$, $\mu_0=\mu_1$.)

(e) Let (X,Σ,μ) be a measure space and $\mathcal{A}$ a disjoint family of subsets of X. Show that there is a measure on X, extending μ, which measures every member of $\mathcal{A}$.

214 Notes and comments I take the first part of the section, down to 214H, slowly and carefully, because while none of the arguments are deep (214Eb is the longest) the patterns formed by the results are not always easy to predict. There is a counter-example to a tempting extension of 214H/214Xb in 216Xb.

The message of the second part of the section (214I-214L, 214Q) is that subspaces inherit many, but not all, of the properties of a measure space; and in particular there can be a difficulty with semi-finiteness, unless we have locally determined negligible sets (214Xg). (I give an example in 216Xa.) Of course 213Hb shows that if we start with a localizable space, we can convert it into a complete locally determined localizable space without doing great violence to the structure of the space, so the difficulty is ordinarily superable.

By far the most important case of 214P is when $\mathcal{A}=\{A\}$ is a singleton, so that the argument simplifies dramatically (214Xm). In §439 of Volume 4 I will return to the problem of extending a measure to a given larger σ-algebra in the absence of any helpful auxiliary structure. That section will mostly offer counter-examples, in particular showing that there is no general theorem extending 214Xn from finite families to countable families, and that the special conditions in 214P and 214Yb are there for good reasons. But in Chapter 54 and §552 of Volume 5 I will discuss mathematical systems in which much stronger extension theorems are true, at least if we start from Lebesgue measure.

Version of 13.11.13

215 σ-finite spaces and the principle of exhaustion

I interpolate a short section to deal with some useful facts which might get lost if buried in one of the longer sections of this chapter. The great majority of the applications of measure theory involve σ-finite spaces, to the point that many authors skim over any others. I myself prefer to signal the importance of such concepts by explicitly stating just which theorems apply only to the restricted class of spaces. But undoubtedly some facts about σ-finite spaces need to be grasped early on. In 215B I give a list of properties characterizing σ-finite spaces. Some of these make better sense in the light of the principle of exhaustion (215A). I take the opportunity to include a fundamental fact about atomless measure spaces (215D).

215A The principle of exhaustion The following is an example of the use of one of the most important methods in measure theory.

Lemma Let (X, Σ, μ) be any measure space and $\mathcal{E} \subseteq \Sigma$ a non-empty set such that $\sup_{n \in \mathbb{N}} \mu F_n$ is finite for every non-decreasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$.

(a) There is a non-decreasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ such that, for every $E \in \Sigma$, *either* there is an $n \in \mathbb{N}$ such that $E \cup F_n$ is not included in any member of $\mathcal{E}$ *or*, setting $F = \bigcup_{n \in \mathbb{N}} F_n$,

$$\lim_{n \to \infty} \mu(E \setminus F_n) = \mu(E \setminus F) = 0.$$

In particular, if $E \in \mathcal{E}$ and $E \supseteq F$, then $E \setminus F$ is negligible.

(b) If $\mathcal{E}$ is upwards-directed, then there is a non-decreasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ such that, setting $F = \bigcup_{n \in \mathbb{N}} F_n$, $\mu F = \sup_{E \in \mathcal{E}} \mu E$ and $E \setminus F$ is negligible for every $E \in \mathcal{E}$, so that F is an essential supremum of $\mathcal{E}$ in Σ in the sense of 211G.

(c) If the union of any non-decreasing sequence in $\mathcal{E}$ belongs to $\mathcal{E}$, then there is an $F \in \mathcal{E}$ such that $E \setminus F$ is negligible whenever $E \in \mathcal{E}$ and $F \subseteq E$.

proof (a) Choose $\langle F_n \rangle_{n \in \mathbb{N}}$, $\langle \mathcal{E}_n \rangle_{n \in \mathbb{N}}$ and $\langle u_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. Take F_0 to be any member of $\mathcal{E}$. Given $F_n \in \mathcal{E}$, set $\mathcal{E}_n = \{E : F_n \subseteq E \in \mathcal{E}\}$ and $u_n = \sup\{\mu E : E \in \mathcal{E}_n\}$ in $[0, \infty]$, and choose $F_{n+1} \in \mathcal{E}_n$ such that $\mu F_{n+1} \geq \min(n, u_n - 2^{-n})$; continue.

Observe that this construction yields a non-decreasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$. Since $\mathcal{E}_{n+1} \subseteq \mathcal{E}_n$ for every n, $\langle u_n \rangle_{n \in \mathbb{N}}$ is non-increasing, and has a limit u in $[0, \infty]$. Since $\min(n, u - 2^{-n}) \leq \mu F_{n+1} \leq u_n$ for every n, $\lim_{n \to \infty} \mu F_n = u$. Our hypothesis on $\mathcal{E}$ now tells us that u is finite.

If $E \in \Sigma$ is such that for every $n \in \mathbb{N}$ there is an $E_n \in \mathcal{E}$ such that $E \cup F_n \subseteq E_n$, then $E_n \in \mathcal{E}_n$, so

$$\mu F_n \leq \mu(E \cup F_n) \leq \mu E_n \leq u_n$$

for every n, and $\lim_{n \to \infty} \mu(E \cup F_n) = u$. But this means that

$$\mu(E \setminus F) \leq \lim_{n \to \infty} \mu(E \setminus F_n) = \lim_{n \to \infty} \mu(E \cup F_n) - \mu F_n = 0,$$

as stated. In particular, this is so if $E \in \mathcal{E}$ and $E \supseteq F$.

(b) Take $\langle F_n \rangle_{n \in \mathbb{N}}$ from (a). If $E \in \mathcal{E}$, then (because $\mathcal{E}$ is upwards-directed) $E \cup F_n$ is included in some member of $\mathcal{E}$ for every $n \in \mathbb{N}$; so we must have the second alternative of (a), and $E \setminus F$ is negligible. It follows that

$$\sup_{E \in \mathcal{E}} \mu E \leq \mu F = \lim_{n \to \infty} \mu F_n \leq \sup_{E \in \mathcal{E}} \mu E,$$

so $\mu F = \sup_{E \in \mathcal{E}} \mu E$.

If G is any measurable set such that $E \setminus G$ is negligible for every $E \in \mathcal{E}$, then $F_n \setminus G$ is negligible for every n, so that $F \setminus G$ is negligible; thus F is an essential supremum for $\mathcal{E}$.

(c) Again take $\langle F_n \rangle_{n \in \mathbb{N}}$ from (a), and set $F = \bigcup_{n \in \mathbb{N}} E_n$. Our hypothesis now is that $F \in \mathcal{E}$, so has both the properties declared.

215B σ-finite spaces are so important that I think it is worth spelling out the following facts.

Proposition Let (X, Σ, μ) be a semi-finite measure space. Write $\mathcal{N}$ for the family of μ-negligible sets and Σ^f for the family of measurable sets of finite measure. Then the following are equiveridical:

(i) (X, Σ, μ) is σ-finite;

(ii) every disjoint family in $\Sigma^f \setminus \mathcal{N}$ is countable;

(iii) every disjoint family in $\Sigma \setminus \mathcal{N}$ is countable;

(iv) for every $\mathcal{E} \subseteq \Sigma$ there is a countable set $\mathcal{E}_0 \subseteq \mathcal{E}$ such that $E \setminus \bigcup \mathcal{E}_0$ is negligible for every $E \in \mathcal{E}$;

(v) for every non-empty upwards-directed $\mathcal{E} \subseteq \Sigma$ there is a non-decreasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ such that $E \setminus \bigcup_{n \in \mathbb{N}} F_n$ is negligible for every $E \in \mathcal{E}$;

(vi) for every non-empty $\mathcal{E} \subseteq \Sigma$, there is a non-decreasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ such that $E \setminus \bigcup_{n \in \mathbb{N}} F_n$ is negligible whenever $E \in \mathcal{E}$ and $E \supseteq F_n$ for every $n \in \mathbb{N}$;

(vii) *either* $\mu X = 0$ *or* there is a probability measure ν on X with the same domain and the same negligible sets as μ;

(viii) there is a measurable integrable function $f : X \to \,]0, 1]$;

(ix) either $\mu X = 0$ or there is a measurable function $f : X \to]0,\infty[$ such that $\int f d\mu = 1$.

proof (i)⇒(vii) and (viii) If $\mu X = 0$, (vii) is trivial and we can take $f = \chi X$ in (viii). Otherwise, let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in Σ^f covering X. Then it is easy to see that there is a sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ of strictly positive real numbers such that $\sum_{n=0}^\infty \alpha_n\mu E_n = 1$. Set $\nu E = \sum_{n=0}^\infty \alpha_n\mu(E\cap E_n)$ for $E\in\Sigma$; then ν is a probability measure with domain Σ and the same negligible sets as μ. Also $f = \sum_{n=0}^\infty \min(1,\alpha_n)\chi E_n$ is a strictly positive measurable integrable function.

(vii)⇒(vi) and (v) Assume (vii), and let $\mathcal{E}$ be a non-empty family of measurable sets. If $\mu X = 0$ then (vi) and (v) are certainly true. Otherwise, let ν be a probability measure with domain Σ and the same negligible sets as μ. Since $\sup_{E\in\mathcal{E}}\nu E \le 1$ is finite, we can apply 215Aa to find a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $E\setminus\bigcup_{n\in\mathbb{N}}F_n$ is negligible whenever $E\in\mathcal{E}$ includes $\bigcup_{n\in\mathbb{N}}F_n$; and if $\mathcal{E}$ is upwards-directed, $E\setminus\bigcup_{n\in\mathbb{N}}F_n$ will be negligible for every $E\in\mathcal{E}$, as in 215Ab.

(vi)⇒(iv) Assume (vi), and let $\mathcal{E}$ be any subset of Σ. Set

$$\mathcal{H} = \{\bigcup\mathcal{E}_0 : \mathcal{E}_0\subseteq\mathcal{E} \text{ is countable}\}.$$

By (vi), there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that $H\setminus\bigcup_{n\in\mathbb{N}}H_n$ is negligible whenever $H\in\mathcal{H}$ and $H\supseteq H_n$ for every $n\in\mathbb{N}$. Now we can express each H_n as $\bigcup\mathcal{E}_n'$, where $\mathcal{E}_n'\subseteq\mathcal{E}$ is countable; setting $\mathcal{E}_0=\bigcup_{n\in\mathbb{N}}\mathcal{E}_n'$, $\mathcal{E}_0$ is countable. If $E\in\mathcal{E}$, then $E\cup\bigcup_{n\in\mathbb{N}}H_n = \bigcup(\{E\}\cup\mathcal{E}_0)$ belongs to $\mathcal{H}$ and includes every H_n, so that $E\setminus\bigcup\mathcal{E}_0 = E\setminus\bigcup_{n\in\mathbb{N}}H_n$ is negligible. So $\mathcal{E}_0$ has the property we need, and (iv) is true.

(iv)⇒(iii) Assume (iv). If $\mathcal{E}$ is a disjoint family in $\Sigma\setminus\mathcal{N}$, take a countable $\mathcal{E}_0\subseteq\mathcal{E}$ such that $E\setminus\bigcup\mathcal{E}_0$ is negligible for every $E\in\mathcal{E}$. Then $E = E\setminus\bigcup\mathcal{E}_0$ is negligible for every $E\in\mathcal{E}\setminus\mathcal{E}_0$; but this just means that $\mathcal{E}\setminus\mathcal{E}_0$ is empty, so that $\mathcal{E}=\mathcal{E}_0$ is countable.

(iii)⇒(ii) is trivial.

(ii)⇒(i) Assume (ii). Let $\mathfrak{P}$ be the set of all disjoint subsets of $\Sigma^f\setminus\mathcal{N}$, ordered by $\subseteq$. Then $\mathfrak{P}$ is a partially ordered set, not empty (as $\emptyset\in\mathfrak{P}$), and if $\mathfrak{Q}\subseteq\mathfrak{P}$ is non-empty and totally ordered then it has an upper bound in $\mathfrak{P}$. **P** Set $\mathcal{E}=\bigcup\mathfrak{Q}$, the union of all the disjoint families belonging to $\mathfrak{Q}$. If $E\in\mathcal{E}$ then $E\in\mathcal{C}$ for some $\mathcal{C}\in\mathfrak{Q}$, so $E\in\Sigma^f\setminus\mathcal{N}$. If E, $F\in\mathcal{E}$ and $E\ne F$, then there are $\mathcal{C}$, $\mathcal{D}\in\mathfrak{Q}$ such that $E\in\mathcal{C}$, $F\in\mathcal{D}$; now $\mathfrak{Q}$ is totally ordered, so one of $\mathcal{C}$, $\mathcal{D}$ is larger than the other, and in either case $\mathcal{C}\cup\mathcal{D}$ is a member of $\mathfrak{Q}$ containing both E and F. But since any member of $\mathfrak{Q}$ is a disjoint collection of sets, $E\cap F=\emptyset$. As E and F are arbitrary, $\mathcal{E}$ is a disjoint family of sets and belongs to $\mathfrak{P}$. And of course $\mathcal{C}\subseteq\mathcal{E}$ for every $\mathcal{C}\in\mathfrak{Q}$, so $\mathcal{E}$ is an upper bound for $\mathfrak{Q}$ in $\mathfrak{P}$. **Q**

By Zorn's Lemma (2A1M), $\mathfrak{P}$ has a maximal element $\mathcal{E}$ say. By (ii), $\mathcal{E}$ must be countable, so $\bigcup\mathcal{E}\in\Sigma$. Now $H = X\setminus\bigcup\mathcal{E}$ is negligible. **P?** Suppose, if possible, otherwise. Because (X,Σ,μ) is semi-finite, there is a set G of finite measure such that $G\subseteq H$ and $\mu G>0$, that is, $G\in\Sigma^f\setminus\mathcal{N}$ and $G\cap E=\emptyset$ for every $E\in\mathcal{E}$. But this means that $\{G\}\cup\mathcal{E}$ is a member of $\mathfrak{P}$ strictly larger than $\mathcal{E}$, which is supposed to be impossible. **XQ**

Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{E}\cup\{H\}$. Then $\langle X_n\rangle_{n\in\mathbb{N}}$ is a cover of X by a sequence of measurable sets of finite measure, so (X,Σ,μ) is σ-finite.

(v)⇒(i) If (v) is true, then we have a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ^f such that $E\setminus\bigcup_{n\in\mathbb{N}}E_n$ is negligible for every $E\in\Sigma^f$. Because μ is semi-finite, $X\setminus\bigcup_{n\in\mathbb{N}}E_n$ must be negligible, so X is covered by a countable family of sets of finite measure and μ is σ-finite.

(viii)⇒(ix) If $\mu X=0$ this is trivial. Otherwise, if f is a strictly positive measurable integrable function, then $c=\int f>0$ (122Rc), so $\frac{1}{c}f$ is a strictly positive measurable function with integral 1.

(ix)⇒(i) If $f : X\to]0,\infty[$ is measurable and integrable, $\langle\{x : f(x)\ge 2^{-n}\}\rangle_{n\in\mathbb{N}}$ is a sequence of sets of finite measure covering X.

215C Corollary Let (X,Σ,μ) be a σ-finite measure space, and suppose that $\mathcal{E}\subseteq\Sigma$ is any non-empty set.

(a) There is a non-decreasing sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that, for every $E\in\Sigma$, *either* there is an $n\in\mathbb{N}$ such that $E\cup F_n$ is not included in any member of $\mathcal{E}$ *or* $E\setminus\bigcup_{n\in\mathbb{N}}F_n$ is negligible.

(b) If $\mathcal{E}$ is upwards-directed, then there is a non-decreasing sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ such that $\bigcup_{n\in\mathbb{N}} F_n$ is an essential supremum of $\mathcal{E}$ in Σ.

(c) If the union of any non-decreasing sequence in $\mathcal{E}$ belongs to $\mathcal{E}$, then there is an $F \in \mathcal{E}$ such that $E \setminus F$ is negligible whenever $E \in \mathcal{E}$ and $F \subseteq E$.

proof By 215B, there is a totally finite measure ν on X with the same measurable sets and the same negligible sets as μ. Since $\sup_{E\in\mathcal{E}} \nu E$ is finite, we can apply 215A to ν to obtain the results.

215D As a further example of the use of the principle of exhaustion, I give a fundamental fact about atomless measure spaces.

Proposition Let (X, Σ, μ) be an atomless measure space. If $E \in \Sigma$ and $0 \le \alpha \le \mu E < \infty$, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\mu F = \alpha$.

proof (a) We need to know that if $G \in \Sigma$ is non-negligible and $n \in \mathbb{N}$, then there is an $H \subseteq G$ such that $0 < \mu H \le 2^{-n}\mu G$. **P** Induce on n. For $n = 0$ this is trivial. For the inductive step to $n+1$, use the inductive hypothesis to find $H \subseteq G$ such that $0 < \mu H \le 2^{-n}\mu G$. Because μ is atomless, there is an $H' \subseteq H$ such that $\mu H'$, $\mu(H \setminus H')$ are both defined and non-zero. Now at least one of them has measure less than or equal to $\frac{1}{2}\mu H$, so gives us a subset of G of non-zero measure less than or equal to $2^{-n-1}\mu G$. **Q**

It follows that if $G \in \Sigma$ has non-zero finite measure and $\epsilon > 0$, there is a measurable set $H \subseteq G$ such that $0 < \mu H \le \epsilon$.

(b) Let $\mathcal{H}$ be the family of all those $H \in \Sigma$ such that $H \subseteq E$ and $\mu H \le \alpha$. If $\langle H_n \rangle_{n\in\mathbb{N}}$ is any non-decreasing sequence in $\mathcal{H}$, then $\mu(\bigcup_{n\in\mathbb{N}} H_n) = \lim_{n\to\infty} \mu H_n \le \alpha$, so $\bigcup_{n\in\mathbb{N}} H_n \in \mathcal{H}$. So 215Ac tells us that there is an $F \in \mathcal{H}$ such that $H \setminus F$ is negligible whenever $H \in \mathcal{H}$ and $F \subseteq H$. **?** Suppose, if possible, that $\mu F < \alpha$. By (a), there is an $H \subseteq E \setminus F$ such that $0 < \mu H \le \alpha - \mu F$. But in this case $H \cup F \in \mathcal{H}$ and $\mu((H \cup F) \setminus F) > 0$, which is impossible. **X**

So we have found an appropriate set F.

***215E** One of the basic properties of Lebesgue measure is that singleton subsets (and therefore countable subsets) are negligible. This is of course associated with the fact that Lebesgue measure is atomless (211Md). It is easy to construct measures for which singleton sets are negligible but there are atoms (e.g., the countable-cocountable measures of 211R). It takes a little more ingenuity to construct atomless measures in which not all singleton sets are negligible (216Ye). The following result gives conditions under which this can't happen.

Proposition Let (X, Σ, μ) be an atomless measure space and $x \in X$.

(a) If $\mu^*\{x\}$ is finite then $\{x\}$ is negligible.

(b) If μ has locally determined negligible sets then $\{x\}$ is negligible.

(c) If μ is localizable then $\{x\}$ is negligible.

proof (a) ? Otherwise, $0 < \mu^*\{x\} < \infty$. Let E be a set containing x such that $\mu E < 2\mu^*\{x\}$. By 215D, there is an $F \subseteq E$ such that

$$\mu F = \tfrac{1}{2}\mu E = \mu(E \setminus F) < \mu^*\{x\}.$$

So neither F nor $E \setminus F$ can contain x; but $x \in E$. **X**

(b) For any set E of finite measure, $E \cap \{x\}$ is negligible; for if it is not empty then $x \in E$ and we can apply (a). As μ has locally determined negligible sets, $\{x\}$ is negligible.

(c) Let $\mathcal{E} \subseteq \Sigma^f$ be a maximal family such that $\mu E < \infty$ for every $E \in \mathcal{E}$ and $\mu(E \cap F) = \emptyset$ whenever E, $F \in \mathcal{E}$ are distinct. For each $E \in \mathcal{E}$ and $n \in \mathbb{N}$, use 215D n times to find a partition $E_{n0}, E_{n1}, \dots, E_{nn}$ of E such that $\mu E_{ni} = \frac{1}{n+1}\mu E$ for every $i \le n$. Next, for $i \le n \in \mathbb{N}$, let H_{ni} be an essential supremum of $\{E_{ni} : E \in \mathcal{E}\}$. For each $n \in \mathbb{N}$ and $E \in \mathcal{E}$.

$$\mu(E \setminus \bigcup_{i\le n} H_{ni}) \le \sum_{i=0}^{n} \mu(E_{ni} \setminus H_{ni}) = 0.$$

So if $F \in \Sigma$, $\mu F < \infty$ and $F \cap \bigcup_{i\le n} H_{ni} = \emptyset$, we shall have $\mu(E \cap F) = 0$ for every $E \in \mathcal{E}$; as $\mathcal{E}$ was maximal, F must be negligible. We are supposing that μ is semi-finite, so $X \setminus \bigcup_{i\le n} H_{ni}$ is negligible.

It follows that if there is any n such that $x \notin \bigcup_{i \le n} H_{ni}$ then $\{x\}$ is negligible and we can stop. Consider $H = \bigcap\{H_{ni} : i \le n \in \mathbb{N},\ x \in H_{ni}\}$. **?** If $\mu H > 0$, there is an $F \subseteq H$ such that $0 < \mu F < \infty$. By the maximality of $\mathcal{E}$, there is an $E^* \in \mathcal{E}$ such that $\mu(F \cap E^*) > 0$. Let n be such that $\mu(F \cap E^*) > \frac{1}{n+1}\mu E^*$. There is an $i \le n$ such that $x \in H_{ni}$, so that $F \subseteq H_{ni}$, while $\mu(F \cap E^*_{ni}) \le \frac{1}{n+1}\mu E^*$, so $F' = F \cap E^* \setminus E^*_{ni}$ has non-zero measure. Consider $G = H_{ni} \setminus F'$. If $E \in \mathcal{E}$, then either $E = E^*$ and

$$E_{ni} \setminus G = E^*_{ni} \setminus G = E^*_{ni} \setminus H_{ni}$$

is negligible, or $E \ne E^*$ and

$$E_{ni} \setminus G \subseteq (E_{ni} \setminus H_{ni}) \cup (E \cap E^*)$$

is negligible. Because H_{ni} is an essential supremum of $\{E_{ni} : E \in \mathcal{E}\}$, $H_{ni} \setminus G$ must be negligible and F' is negligible; which is impossible. **Q**

Thus H is a negligible set containing x and $\{x\}$ is negligible in this case also.

215X Basic exercises (a) Let (X, Σ, μ) be a measure space and Φ a non-empty set of μ-integrable real-valued functions from X to $\mathbb{R}$. Suppose that $\sup_{n\in\mathbb{N}} \int f_n$ is finite for every sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in Φ such that $f_n \le_{\text{a.e.}} f_{n+1}$ for every n. Show that there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in Φ such that $f_n \le_{\text{a.e.}} f_{n+1}$ for every n and, for every integrable real-valued function f on X, *either* $f \le_{\text{a.e.}} \sup_{n\in\mathbb{N}} f_n$ *or* there is an $n \in \mathbb{N}$ such that no member of Φ is greater than or equal to $\max(f, f_n)$ almost everywhere.

>(b) Let (X, Σ, μ) be a measure space. (i) Suppose that $\mathcal{E}$ is a non-empty upwards-directed subset of Σ such that $c = \sup_{E\in\mathcal{E}} \mu E$ is finite. Show that $E \setminus \bigcup_{n\in\mathbb{N}} F_n$ is negligible whenever $E \in \mathcal{E}$ and $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{E}$ such that $\lim_{n\to\infty} \mu F_n = c$. (ii) Let Φ be a non-empty set of integrable functions on X which is upwards-directed in the sense that for all f, $g \in \Phi$ there is an $h \in \Phi$ such that $\max(f, g) \le_{\text{a.e.}} h$, and suppose that $c = \sup_{f\in\Phi} \int f$ is finite. Show that $f \le_{\text{a.e.}} \sup_{n\in\mathbb{N}} f_n$ whenever $f \in \Phi$ and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in Φ such that $\lim_{n\to\infty} \int f_n = c$.

(c) Use 215A to shorten the proof of 211Ld.

(d) Give an example of a (non-semi-finite) measure space (X, Σ, μ) satisfying conditions (ii)-(iv) of 215B, but not (i).

>(e) Let (X, Σ, μ) be an atomless σ-finite measure space. Show that for any $\epsilon > 0$ there is a disjoint sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of measurable sets with measure at most ϵ such that $X = \bigcup_{n\in\mathbb{N}} E_n$.

(f) Let (X, Σ, μ) be an atomless strictly localizable measure space. Show that for any $\epsilon > 0$ there is a decomposition $\langle X_i\rangle_{i\in I}$ of X such that $\mu X_i \le \epsilon$ for every $i \in I$.

215Y Further exercises (a) Let (X, Σ, μ) be a σ-finite measure space and $\langle f_{mn}\rangle_{m,n\in\mathbb{N}}$, $\langle f_m\rangle_{m\in\mathbb{N}}$, f measurable real-valued functions defined almost everywhere in X and such that $\langle f_{mn}\rangle_{n\in\mathbb{N}} \to f_m$ a.e. for each m and $\langle f_m\rangle_{m\in\mathbb{N}} \to f$ a.e. Show that there is a strictly increasing sequence $\langle n_m\rangle_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $\langle f_{m,n_m}\rangle_{m\in\mathbb{N}} \to f$ a.e. (Compare 134Yb.)

(b) Let (X, Σ, μ) be a σ-finite measure space. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable real-valued functions such that $f = \lim_{n\to\infty} f_n$ is defined almost everywhere in X. Show that there is a non-decreasing sequence $\langle X_k\rangle_{k\in\mathbb{N}}$ of measurable subsets of X such that $\bigcup_{k\in\mathbb{N}} X_k$ is conegligible in X and $\langle f_n\rangle_{n\in\mathbb{N}} \to f$ uniformly on every X_k, in the sense that for any $\epsilon > 0$ there is an $m \in \mathbb{N}$ such that $|f_j(x) - f(x)|$ is defined and less than or equal to ϵ whenever $j \ge m$ and $x \in X_k$.

(This is a version of Egorov's theorem.)

(c) Let (X, Σ, μ) be a totally finite measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$, f measurable real-valued functions defined almost everywhere in X. Show that $\langle f_n\rangle_{n\in\mathbb{N}} \to f$ a.e. iff there is a sequence $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ of strictly positive real numbers, converging to 0, such that

$$\lim_{n\to\infty} \mu^*(\bigcup_{k\ge n}\{x : x \in \operatorname{dom} f_k \cap \operatorname{dom} f,\ |f_k(x) - f(x)| \ge \epsilon_n\}) = 0.$$

(d) Find a direct proof of (v)⇒(vi) in 215B. (*Hint*: given $\mathcal{E} \subseteq \Sigma$, use Zorn's Lemma to find a maximal totally ordered $\mathcal{E}' \subseteq \mathcal{E}$ such that $E \triangle F \notin \mathcal{N}$ for any distinct E, $F \in \mathcal{E}'$, and apply (v) to $\mathcal{E}'$.)

215 Notes and comments The common ground of 215A, 215B(vi), 215C and 215Xa is actually one of the most fundamental ideas in measure theory. It appears in such various forms that it is often easier to prove an application from first principles than to explain how it can be reduced to the versions here. But I will try henceforth to signal such applications as they arise, calling the method (the proof of 215Aa or 215Xa) the 'principle of exhaustion'. One point which is perhaps worth noting here is the inductive construction of the sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ in the proof of 215Aa. Each F_{n+1} is chosen *after* the preceding one. It is this which makes it possible, in the proof of 215B(vii)⇒(vi), to extract a suitable sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ directly. In many applications (starting with what is surely the most important one in the elementary theory, the Radon-Nikodým theorem of §232, or with part (i) of the proof of 211Ld), this refinement is not needed; we are dealing with an upwards-directed set, as in 215B(v), and can choose the whole sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ at once, no term interacting with any other, as in 215Xb. The axiom of 'dependent choice', which asserts that we can construct sequences term-by-term, is known to be stronger than the axiom of 'countable choice', which asserts only that we can choose countably many objects simultaneously.

In 215B I try to indicate the most characteristic properties of σ-finiteness; in particular, the properties which distinguish σ-finite measures from other strictly localizable measures. This result is in a way more abstract than the manipulations in the rest of the section. Note that it makes an essential use of the axiom of choice in the form of Zorn's Lemma. I spent a paragraph in 134C commenting on the distinction between 'countable choice', which is needed for anything which looks like the standard theory of Lebesgue measure, and the full axiom of choice, which is relatively little used in the elementary theory. The implication (ii)⇒(i) of 215B is one of the points where we do need something beyond countable choice. (I should perhaps remark that the whole theory of non-σ-finite measure spaces looks very odd without the general axiom of choice.) Note also that in 215B the proofs of (i)⇒(vii) and (vii)⇒(vi) are the only points where anything so vulgar as a number appears. The conditions (iii), (iv), (v) and (vi) are linked in ways that have nothing to do with measure theory, and involve only with the structure $(X, \Sigma, \mathcal{N})$. (See 215Yd here, and 316D-316E in Volume 3.) There are similar conditions relating to measurable functions rather than measurable sets; for a fairly abstract example, see 241Ye.

In 215Ya-215Yc are three more standard theorems on almost-everywhere-convergent sequences which depend on σ- or total finiteness.

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216 Examples

It is common practice – and, in my view, good practice – in books on pure mathematics, to provide discriminating examples; I mean that whenever we are given a list of new concepts, we expect to be provided with examples to show that we have a fair picture of the relationships between them, and in particular that we are not being kept ignorant of some startling implication. Concerning the concepts listed in 211A-211K, we have ten different properties which some, but not all, measure spaces possess, giving a conceivable total of 2^{10} different types of measure space, classified according to which of these ten properties they have. The list of basic relationships in 211L reduces these 1024 possibilities to 72. Observing that a space can be simultaneously atomless and purely atomic only when the measure of the whole space is 0, we find ourselves with 56 possibilities, being two trivial cases with $\mu X = 0$ (because such a measure may or may not be complete) together with $9 \times 2 \times 3$ cases, corresponding to the nine classes

probability spaces,
spaces which are totally finite, but not probability spaces,
spaces which are σ-finite, but not totally finite,
spaces which are strictly localizable, but not σ-finite,
spaces which are localizable and locally determined, but not strictly localizable,
spaces which are localizable, but not locally determined,
spaces which are locally determined, but not localizable,

spaces which are semi-finite, but neither locally determined nor localizable,
spaces which are not semi-finite;
the two classes
spaces which are complete,
spaces which are not complete;
and the three classes
spaces which are atomless, not of measure 0,
spaces which are purely atomic, not of measure 0,
spaces which are neither atomless nor purely atomic.

I do not propose to give a complete set of fifty-six examples, particularly as rather fewer than fifty-six different ideas are required. However, I do think that for a proper understanding of abstract measure spaces it is necessary to have seen realizations of some of the critical combinations of properties. I therefore take a few paragraphs to describe three special examples to add to those of 211M-211R.

216A Lebesgue measure Before turning to the new ideas, let me mention Lebesgue measure again. As remarked in 211M, 211P and 211Qa,

(a) Lebesgue measure μ on $\mathbb{R}$ is complete, atomless and σ-finite, therefore strictly localizable, localizable and locally determined.

(b) The subspace measure $\mu_{[0,1]}$ on $[0,1]$ is a complete, atomless probability measure.

(c) The restriction $\mu\restriction\mathcal{B}$ of μ to the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ is atomless, σ-finite and not complete.

216B I now embark on the description of three 'counter-examples'; meaning spaces built specifically for the purpose of showing that there are no unexpected implications among the ten properties under consideration here. Even by the standards of this chapter these must be regarded as dispensable by the student who wants to get on with the real business of understanding the big theorems of the subject. Neither the existence of these examples, nor the techniques needed in constructing them, are vital for anything else we shall look at before Volume 5. But if you are going to take abstract measure theory seriously at all, sooner or later you will need to form some kind of mental picture of the nature of the spaces possessing the different properties here, and a minimal requirement of such a picture is that it should include the discriminations witnessed by these examples.

***216C A complete, localizable, non-locally-determined space** The first example hardly needs an idea beyond what we already have, but it does call for more manipulations than it seems fair to set as an exercise, and may therefore be useful as a demonstration of technique.

(a) Let I be any uncountable set, and set $X=\{0,1\}\times I$. For $E\subseteq X$, $y\in\{0,1\}$ set $E[\{y\}]=\{i:(y,i)\in E\}\subseteq I$. Set

$$\Sigma=\{E:E\subseteq X,\ E[\{0\}]\triangle E[\{1\}]\text{ is countable}\}.$$

Then Σ is a σ-algebra of subsets of X. **P** (i) $\emptyset[\{0\}]\triangle\emptyset[\{1\}]=\emptyset$ is countable, so $\emptyset\in\Sigma$. (ii) If $E\in\Sigma$ then

$$(X\setminus E)[\{0\}]\triangle(X\setminus E)[\{1\}]=E[\{0\}]\triangle E[\{1\}]$$

is countable. (iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ and $E=\bigcup_{n\in\mathbb{N}}E_n$, then

$$E[\{0\}]\triangle E[\{1\}]\subseteq\bigcup_{n\in\mathbb{N}}E_n[\{0\}]\triangle E_n[\{1\}]$$

is countable. **Q**

For $E\in\Sigma$, set $\mu E=\#(E[\{0\}])$ if this is finite, ∞ otherwise; then (X,Σ,μ) is a measure space.

(b) (X,Σ,μ) is complete. **P** If $A\subseteq E\in\Sigma$ and $\mu E=0$, then $(0,i)\notin E$ for every i. So

$$A[\{0\}]\triangle A[\{1\}]=A[\{1\}]\subseteq E[\{1\}]=E[\{1\}]\triangle E[\{0\}]$$

must be countable, and $A\in\Sigma$. **Q**

(c) (X,Σ,μ) is semi-finite. **P** If $E\in\Sigma$ and $\mu E>0$, there is an $i\in I$ such that $(0,i)\in E$; now $F=\{(0,i)\}\subseteq E$ and $\mu F=1$. **Q**

(d) (X,Σ,μ) is localizable. **P** Let $\mathcal{E}$ be any subset of Σ. Set

$$J = \bigcup_{E\in\mathcal{E}} E[\{0\}], \quad G = \{0,1\} \times J.$$

Then $G \in \Sigma$. If $H \in \Sigma$, then

$$\begin{aligned}\mu(E \setminus H) = 0 \text{ for every } E \in \mathcal{E}&\\ &\iff E[\{0\}] \subseteq H[\{0\}] \text{ for every } E \in \mathcal{E}\\ &\iff (0,i) \in H \text{ for every } i \in J\\ &\iff \mu(G \setminus H) = 0.\end{aligned}$$

Thus G is an essential supremum for $\mathcal{E}$ in Σ; as $\mathcal{E}$ is arbitrary, μ is localizable. **Q**

(e) (X,Σ,μ) is not locally determined. **P** Consider $H = \{0\} \times I$. Then $H \notin \Sigma$ because $H[\{0\}]\triangle H[\{1\}] = I$ is uncountable. But let $E \in \Sigma$ be any set such that $\mu E < \infty$. Then

$$(E \cap H)[\{0\}]\triangle(E \cap H)[\{1\}] = (E \cap H)[\{0\}] \subseteq E[\{0\}]$$

is finite, so $E \cap H \in \Sigma$. As E is arbitrary, H witnesses that μ is not locally determined. **Q**

(f) (X,Σ,μ) is purely atomic. **P** Let $E \in \Sigma$ be any set of non-zero measure. Let $i \in I$ be such that $(0,i) \in E$. Then $(0,i) \in E$ and $F = \{(0,i)\}$ is a set of measure 1, included in E; because F is a singleton set, it must be an atom for μ; as E is arbitrary, μ is purely atomic. **Q**

(g) Thus the construction here yields a complete, localizable, purely atomic, non-locally-determined space.

***216D A complete, locally determined space which is not localizable** The next construction requires a little set theory. We need two sets I, J such that I is uncountable (more strictly, I cannot be expressed as the union of countably many countable sets), $I \subseteq J$ and J cannot be expressed as $\bigcup_{i\in I} K_i$ where every K_i is countable. The most natural way of doing this, subject to the axiom of choice, is to take $I = \omega_1$, the first uncountable ordinal, and J to be ω_2, the first ordinal from which there is no injection into ω_1 (see 2A1Fc); but in case you prefer other formulations (e.g., $I = \{\{x\} : x \in \mathbb{R}\}$ and $J = \mathcal{P}\mathbb{R}$), I will write the following argument in terms of I and J, and you can pick your own pair.

(a) Let T be the countable-cocountable σ-algebra of J and ν the countable-cocountable measure on J (211R). Set $X = J \times J$ and for $E \subseteq X$ set

$$E[\{\xi\}] = \{\eta : (\xi,\eta) \in E\}, \quad E^{-1}[\{\xi\}] = \{\eta : (\eta,\xi) \in E\}$$

for every $\xi \in J$. Set

$$\Sigma = \{E : E[\{\xi\}] \text{ and } E^{-1}[\{\xi\}] \text{ belong to } \mathrm{T} \text{ for every } \xi \in J\},$$

$$\mu E = \sum_{\xi\in J} \nu E[\{\xi\}] + \sum_{\xi\in J} \nu E^{-1}[\{\xi\}]$$

for every $E \in \Sigma$. It is easy to check that Σ is a σ-algebra and that μ is a measure.

(b) (X,Σ,μ) is complete. **P** If $A \subseteq E \in \Sigma$ and $\mu E = 0$, then all the sets $E[\{\xi\}]$ and $E^{-1}[\{\xi\}]$ are countable, so the same is true of all the sets $A[\{\xi\}]$ and $A^{-1}[\{\xi\}]$, and $A \in \Sigma$. **Q**

(d) (X,Σ,μ) is semi-finite. **P** For each $\zeta \in J$, set

$$G_\zeta = \{\zeta\} \times J, \quad \tilde{G}_\zeta = J \times \{\zeta\}.$$

Then all the sections $G_\zeta[\{\xi\}]$, $G_\zeta^{-1}[\{\xi\}]$, $\tilde{G}_\zeta[\{\xi\}]$ and $\tilde{G}_\zeta^{-1}[\{\xi\}]$ are either J or $\emptyset$ or $\{\zeta\}$, so belong to T, and all the G_ζ, $\tilde{G}_\zeta$ belong to Σ, with μ-measure 1.

Suppose that $E \in \Sigma$ is a set of strictly positive measure. Then there must be some $\xi \in J$ such that

$$0 < \nu E[\{\xi\}] + \nu E^{-1}[\{\xi\}] = \mu(E \cap G_\xi) + \mu(E \cap \tilde{G}_\xi) < \infty,$$

and one of the sets $E \cap G_\xi$, $E \cap \tilde{G}_\xi$ is a set of non-zero finite measure included in E. **Q**

(e) (X,Σ,μ) is locally determined. **P** Suppose that $H \subseteq X$ is such that $H \cap E \in \Sigma$ whenever $E \in \Sigma$ and $\mu E < \infty$. Then, in particular, $H \cap G_\zeta$ and $H \cap \tilde{G}_\zeta$ belong to Σ, so

$$H[\{\zeta\}] = (H \cap \tilde{G}_\zeta)[\{\zeta\}] \in \mathrm{T},$$

$$H^{-1}[\{\zeta\}] = (H \cap G_\zeta)^{-1}[\{\zeta\}] \in \mathrm{T},$$

for every $\zeta \in J$. This shows that $H \in \Sigma$. As H is arbitrary, μ is locally determined. **Q**

(f) (X, Σ, μ) is not localizable. **P** Set $\mathcal{E} = \{G_\zeta : \zeta \in J\}$. **?** Suppose, if possible, that $G \in \Sigma$ is an essential supremum for $\mathcal{E}$. Then

$$\nu(J \setminus G[\{\xi\}]) = \mu(G_\xi \setminus G) = 0$$

and $J \setminus G[\{\xi\}]$ is countable, for every $\xi \in J$. Consequently $J \neq \bigcup_{\xi \in I}(J \setminus G[\{\xi\}])$, and there is an η belonging to $J \setminus \bigcup_{\xi \in I}(J \setminus G[\{\xi\}]) = \bigcap_{\xi \in I} G[\{\xi\}]$. This means just that $(\xi, \eta) \in G$ for every $\xi \in I$, that is, that $I \subseteq G^{-1}[\{\eta\}]$. Accordingly $G^{-1}[\{\eta\}]$ is uncountable, so that $\nu G^{-1}[\{\eta\}] = \mu(G \cap \tilde{G}_\eta) = 1$. But observe that $\mu(G_\xi \cap \tilde{G}_\eta) = \mu\{(\xi, \eta)\} = 0$ for every $\xi \in J$. This means that, setting $H = X \setminus \tilde{G}_\eta$, $E \setminus H$ is negligible, for every $E \in \mathcal{E}$; so that we must have $0 = \mu(G \setminus H) = \mu(G \cap \tilde{G}_\eta) = 1$, which is absurd. **X**

Thus $\mathcal{E}$ has no essential supremum in Σ, and μ cannot be localizable. **Q**

(g) (X, Σ, μ) is purely atomic. **P** If $E \in \Sigma$ has non-zero measure, there must be some $\xi \in J$ such that one of $E[\{\xi\}]$, $E^{-1}[\{\xi\}]$ is not countable; that is, such that one of $E \cap G_\xi$, $E \cap \tilde{G}_\xi$ is not negligible. But if now $H \in \Sigma$ and $H \subseteq E \cap G_\xi$, either $H[\{\xi\}]$ is countable, and $\mu H = 0$, or $J \setminus H[\{\xi\}]$ is countable, and $\mu(G_\xi \setminus H) = 0$; similarly, if $H \subseteq E \cap \tilde{G}_\xi$, one of μH, $\mu(\tilde{G}_\xi \setminus H)$ must be 0, according to whether $H^{-1}[\{\xi\}]$ is countable or not. Thus $E \cap G_\xi$ and $E \cap \tilde{G}_\xi$, if not negligible, must be atoms, and E must include an atom. As E is arbitrary, μ is purely atomic. **Q**

(h) Thus (X, Σ, μ) is complete, locally determined and purely atomic, but is not localizable.

***216E A complete, locally determined, localizable space which is not strictly localizable** For the last, and most interesting, construction, we need a non-trivial result in infinitary combinatorics, which I have written out in 2A1P: if I is any set, and $\langle f_\alpha \rangle_{\alpha \in A}$ is a family in $\{0,1\}^I$, the set of functions from I to $\{0,1\}$, with $\#(A)$ strictly greater than $\mathfrak{c}$, the cardinal of the continuum, and if $\langle K_\alpha \rangle_{\alpha \in A}$ is any family of countable subsets of I, then there must be distinct α, $\beta \in A$ such that f_α and f_β agree on $K_\alpha \cap K_\beta$.

Armed with this fact, I proceed as follows.

(a) Let C be any set with cardinal greater than $\mathfrak{c}$. Set $I = \mathcal{P}C$ and $X = \{0,1\}^I$. For $\gamma \in C$, define $x_\gamma \in X$ by saying that $x_\gamma(\Gamma) = 1$ if $\gamma \in \Gamma \subseteq C$ and $x_\gamma(\Gamma) = 0$ if $\gamma \notin \Gamma \subseteq C$. Let $\mathcal{K}$ be the family of countable subsets of I, and for $K \in \mathcal{K}$, $\gamma \in C$ set

$$F_{\gamma K} = \{x : x \in X,\ x{\restriction}K = x_\gamma{\restriction}K\} \subseteq X.$$

Let

$$\begin{aligned}\Sigma_\gamma = \{E : E \subseteq X, &\text{ either there is a } K \in \mathcal{K} \text{ such that } F_{\gamma K} \subseteq E \\ &\text{ or there is a } K \in \mathcal{K} \text{ such that } F_{\gamma K} \subseteq X \setminus E\}.\end{aligned}$$

Then Σ_γ is a σ-algebra of subsets of X. **P** (i) $F_{\gamma\emptyset} \subseteq X \setminus \emptyset$ so $\emptyset \in \Sigma_\gamma$. (ii) The definition of Σ_γ is symmetric between E and $X \setminus E$, so $X \setminus E \in \Sigma_\gamma$ whenever $E \in \Sigma_\gamma$. (iii) Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence in Σ_γ, with union E. (α) If there are $n \in \mathbb{N}$, $K \in \mathcal{K}$ such that $F_{\gamma K} \subseteq E_n$, then $F_{\gamma K} \subseteq E$, so $E \in \Sigma_\gamma$. (β) Otherwise, there is for each $n \in \mathbb{N}$ a $K_n \in \mathcal{K}$ such that $F_{\gamma, K_n} \subseteq X \setminus E_n$. Set $K = \bigcup_{n \in \mathbb{N}} K_n \in \mathcal{K}$. Then

$$\begin{aligned}F_{\gamma K} &= \{x : x{\restriction}K = x_\gamma{\restriction}K\} = \{x : x{\restriction}K_n = x_\gamma{\restriction}K_n \text{ for every } n \in \mathbb{N}\} \\ &= \bigcap_{n \in \mathbb{N}} F_{\gamma, K_n} \subseteq \bigcap_{n \in \mathbb{N}} X \setminus E_n = X \setminus E,\end{aligned}$$

so again $E \in \Sigma_\gamma$. As $\langle E_n \rangle_{n \in \mathbb{N}}$ is arbitrary, Σ_γ is a σ-algebra. **Q**

(b) Set

$$\Sigma = \bigcap_{\gamma \in C} \Sigma_\gamma;$$

then Σ, being an intersection of σ-algebras, is a σ-algebra of subsets of X (see 111Ga). Define $\mu : \Sigma \to [0,\infty]$ by setting

$$\begin{aligned}\mu E &= \#(\{\gamma : x_\gamma \in E\}) \text{ if this is finite,}\\ &= \infty \text{ otherwise;}\end{aligned}$$

then μ is a measure.

(c) It will be convenient later to know something about the sets

$$G_D = \{x : x \in X,\, x(D) = 1\}$$

for $D \subseteq C$. In particular, every G_D belongs to Σ. **P** If $\gamma \in D$, then $x_\gamma(D) = 1$ so $G_D = F_{\gamma,\{D\}} \in \Sigma_\gamma$. If $\gamma \in C \setminus D$, then $x_\gamma(D) = 0$ so $G_D = X \setminus F_{\gamma,\{D\}} \in \Sigma_\gamma$. **Q** Also, of course, $\{\gamma : x_\gamma \in G_D\} = D$.

(d) (X,Σ,μ) is complete. **P** Suppose that $A \subseteq E \subseteq \Sigma$ and that $\mu E = 0$. For every $\gamma \in C$, $E \in \Sigma_\gamma$ and $x_\gamma \notin E$, so $F_{\gamma K} \not\subseteq E$ for any $K \in \mathcal{K}$ and there is a $K \in \mathcal{K}$ such that

$$F_{\gamma K} \subseteq X \setminus E \subseteq X \setminus A.$$

Thus $A \in \Sigma_\gamma$; as γ is arbitrary, $A \in \Sigma$. As A is arbitrary, μ is complete. **Q**

(e) (X,Σ,μ) is semi-finite. **P** Let $E \in \Sigma$ be a set of positive measure. Then there must be some $\gamma \in C$ such that $x_\gamma \in E$. Consider $E' = E \cap G_{\{\gamma\}}$. As $x_\gamma \in E'$, $\mu E' \ge 1 > 0$. On the other hand, $\mu G_{\{\gamma\}} = \#(\{\delta : \delta \in \{\gamma\}\}) = 1$, so $\mu E' = 1$. As E is arbitrary, μ is semi-finite. **Q**

(f) (X,Σ,μ) is localizable. **P** Let $\mathcal{E}$ be any subset of Σ. Set $D = \{\delta : \delta \in C,\, x_\delta \in \bigcup\mathcal{E}\}$. Consider G_D. For $H \in \Sigma$,

$$\begin{aligned}\mu(E \setminus H) = 0 \text{ for every } E \in \mathcal{E}\\ &\iff x_\gamma \notin E \setminus H \text{ for every } E \in \mathcal{E}, \gamma \in C\\ &\iff x_\gamma \in H \text{ for every } \gamma \in D\\ &\iff x_\gamma \notin G_D \setminus H \text{ for every } \gamma \in C\\ &\iff \mu(G_D \setminus H) = 0.\end{aligned}$$

Thus G_D is an essential supremum for $\mathcal{E}$ in Σ. As $\mathcal{E}$ is arbitrary, μ is localizable. **Q**

(g) (X,Σ,μ) is not strictly localizable. **P?** Suppose, if possible, that $\langle X_j\rangle_{j\in J}$ is a decomposition of (X,Σ,μ). Set $J' = \{j : j \in J,\, \mu X_j > 0\}$. For each $j \in J'$, the set $C_j = \{\gamma : x_\gamma \in X_j\}$ must be finite and non-empty. Moreover, for each $\gamma \in C$, there must be some $j \in J$ such that $\mu(G_{\{\gamma\}} \cap X_j) > 0$, and in this case $j \in J'$ and $\gamma \in C_j$. Thus $C = \bigcup_{j\in J'} C_j$. Because $\#(C) > \mathfrak{c}$, $\#(J') > \mathfrak{c}$ (2A1Ld).

For each $j \in J'$, choose $\gamma_j \in C_j$. Then

$$x_{\gamma_j} \in X_j \in \Sigma \subseteq \Sigma_{\gamma_j},$$

so there must be a $K_j \in \mathcal{K}$ such that $F_{\gamma_j,K_j} \subseteq X_j$.

At this point I finally turn to the result cited at the start of this example. Because $\#(J') > \mathfrak{c}$, there must be distinct j, $k \in J'$ such that x_{γ_j} and x_{γ_k} agree on $K_j \cap K_k$. We may therefore define $x \in X$ by saying that

$$\begin{aligned}x(\delta) &= x_{\gamma_j}(\delta) \text{ if } \delta \in K_j,\\ &= x_{\gamma_k}(\delta) \text{ if } \delta \in K_k,\\ &= 0 \text{ if } \delta \in C \setminus (K_j \cup K_j).\end{aligned}$$

Now

$$x \in F_{\gamma_j,K_j} \cap F_{\gamma_k,K_k} \subseteq X_j \cap X_k,$$

and $X_j \cap X_k \ne \emptyset$; contradicting the assumption that the X_j formed a decomposition of X. **XQ**

(h) (X,Σ,μ) is purely atomic. **P** If $E \in \Sigma$ and $\mu E > 0$, then (as remarked in (e) above) there is a $\gamma \in C$ such that $\mu(E \cap G_{\{\gamma\}}) = 1$; now $E \cap G_{\{\gamma\}}$ must be an atom. **Q**

(i) Accordingly (X,Σ,μ) is a complete, locally determined, localizable, purely atomic measure space which is not strictly localizable.

216X Basic exercises (a) In the construction of 216C, show that the subspace measure on $\{1\}\times I$ is not semi-finite.

(b) Suppose, in 216D, that $I=\omega_1$. (i) Show that the set $\{(\xi,\eta):\xi\le\eta<\omega_1\}$ is measured by the measure constructed by Carathéodory's method from $\mu^*{\restriction}\mathcal{P}(I\times I)$, but not by the subspace measure on $I\times I$. (ii) Hence, or otherwise, show that the subspace measure on $I\times I$ is not locally determined.

(c) In 216Ya, 252Yq and 252Ys below, I indicate how to construct atomless versions of 216C, 216D and 216E, that is, atomless complete measure spaces of which the first is localizable but not locally determined, the second is locally determined spaces but not localizable, and the third is locally determined and localizable but not strictly localizable. Show how direct sums of these, together with counting measure and the examples described in this chapter, can be assembled to provide all 56 examples called for by the discussion in the introduction to this section.

216Y Further exercises (a) Let λ be Lebesgue measure on $[0,1]$, and Λ its domain. Set $Y=[0,1]\times\{0,1\}$ and write

$$\mathrm{T}=\{F:F\subseteq Y,\ F^{-1}[\{0\}]\in\Lambda\},$$

$$\nu F=\lambda F^{-1}[\{0\}]\text{ for every }F\in\mathrm{T}.$$

Set

$$\mathrm{T}_0=\{F:F\in\mathrm{T},\ F^{-1}[\{0\}]\triangle F^{-1}[\{1\}]\text{ is }\lambda\text{-negligible}\}.$$

Let I be an uncountable set. Set $X=Y\times I$,

$$\Sigma=\{E:E\subseteq X,\ E^{-1}[\{i\}]\in\mathrm{T}\text{ for every }i\in I,\ \{i:E^{-1}[\{i\}]\notin\mathrm{T}_0\}\text{ is countable}\},$$

$$\mu E=\textstyle\sum_{i\in I}\nu E^{-1}[\{i\}]\text{ for }E\in\Sigma.$$

(i) Show that (Y,T,ν) and $(Y,\mathrm{T}_0,\nu{\restriction}\mathrm{T}_0)$ are complete probability spaces, and that for every $F\in\mathrm{T}$ there is an $F'\in\mathrm{T}_0$ such that $\nu(F\triangle F')=0$. (ii) Show that (X,Σ,μ) is an atomless complete localizable measure space which is not locally determined.

(b) Define a measure μ on $X=\omega_2\times\omega_2$ as follows. Take Σ to be the σ-algebra of subsets of X generated by

$$\{A\times\omega_2:A\subseteq\omega_2\}\cup\{\omega_2\times\alpha:\alpha<\omega_2\}.$$

For $E\in\Sigma$ set

$$W(E)=\{\xi:\xi<\omega_2,\ \sup E[\{\xi\}]=\omega_2\},$$

and set $\mu E=\#(W(E))$ if this is finite, ∞ otherwise. Show that μ is a measure on X, is localizable and locally determined, but does not have locally determined negligible sets. Find a subspace Y of X such that the subspace measure on Y is not semi-finite.

(c) Show that in the space described in 216E every set has a measurable envelope, but that this is not true in the spaces of 216C and 216D.

(d) Set $X=\omega_1\times\omega_2$. For $E\subseteq X$ set

$$A(E)=\{\zeta:\text{for some }\xi,\text{ just one of }(\xi,\zeta),\ (\xi,\zeta+1)\text{ belongs to }E\},$$

$$B(E)=\{\zeta:\text{there are }\xi,\ \zeta'\text{ such that }\zeta<\zeta'<\omega_2\text{ and just one of }(\xi,\zeta),\ (\xi,\zeta')\text{ belongs to }E\},$$

$$W(E)=\{\xi:\#(E[\{\xi\}])=\omega_2\}.$$

Let Σ be the set of subsets E of X such that $A(E)$ is countable and $\#(B(E)) \le \omega_1$. For $E \in \Sigma$, set $\mu E = \#(W(E))$ if this is finite, ∞ otherwise. (i) Show that (X, Σ, μ) is a measure space. (ii) Show that if $\hat{\mu}$ is the completion of μ, then its domain is the set of subsets E of X such that $A(E)$ is countable, and $\hat{\mu}$ is strictly localizable. (iii) Show that μ is not strictly localizable.

(e) Show that there is a complete atomless semi-finite measure space with a singleton subset which is not negligible. (*Hint*: set $X = (\omega_1 \times [0,1]) \cup \{\omega_1\}$ and let Σ be the σ-algebra of subsets of X generated by $\{\{\xi\} \times E : \xi < \omega_1,\ E \subseteq [0,1]$ is Lebesgue measurable$\}$).

216 Notes and comments The examples 216C-216E are designed to form, with Lebesgue measure, a basis for constructing a complete set of examples for the concepts listed in 211A-211K. One does not really expect to encounter these phenomena in applications, but a clear understanding of the possibilities demonstrated by these examples is part of a proper appreciation of their rarity. Of course, if we add further properties to our list – for instance, the property of having locally determined negligible sets (213I), or the property that every subset should have a measurable envelope (213Xl) – then there are further positive results to complement 211L, and more examples to hunt for, like 216Yb. But it is time, perhaps past time, that we returned to the classical theorems which apply to the measure spaces at the centre of the subject.

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Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

211Ya Countable-cocountable algebra of $\mathbb{R}$ This exercise, referred to in the 2002 edition of Volume 3, has been moved to 211Ye.

213Y Inner measures Exercise 213Yc, referred to in the 2003 and 2006 editions of Volume 4, is now 213Yd.

214J Subspace measures on measurable subspaces, direct sums 214J-214M, referred to in the 2002 and 2004 editions of Volume 3, the 2003 and 2006 editions of Volume 4, and the 2008 edition of Volume 5, have been moved to 214K-214N.

214N Upper and lower integrals This result, referred to in the 2008 edition of Volume 5, has been moved to 214J.

215Yc Measurable envelopes This exercise, referred to in the 2000 edition of Volume 1, has been moved to 216Yc.

Version of 17.12.12

References for Volume 2

Alexits G. [78] (ed.) *Fourier Analysis and Approximation Theory.* North-Holland, 1978 (Colloq. Math. Soc. Janos Bolyai 19).

Antonov N.Yu. [96] 'Convergence of Fourier series', East J. Approx. 7 (1996) 187-196. [§286 *notes*.]

Arias de Reyna J. [02] *Pointwise Convergence of Fourier Series.* Springer, 2002 (Lecture Notes in Mathematics 1785). [§286 *notes*.]

Baker R. [04] '"Lebesgue measure" on $\mathbb{R}^\infty$, II', Proc. Amer. Math. Soc. 132 (2004) 2577-2591. [254Yb.]

Bergelson V., March P. & Rosenblatt J. [96] (eds.) *Convergence in Ergodic Theory and Probability.* de Gruyter, 1996.

Bogachev V.I. [07] *Measure theory.* Springer, 2007.

du Bois-Reymond P. [1876] 'Untersuchungen über die Convergenz und Divergenz der Fouriersche Darstellungformeln', Abh. Akad. München 12 (1876) 1-103. [§282 *notes*.]

Bourbaki N. [66] *General Topology.* Hermann/Addison-Wesley, 1968. [2A5F.]

Bourbaki N. [87] *Topological Vector Spaces.* Springer, 1987. [2A5E.]

Carleson L. [66] 'On convergence and growth of partial sums of Fourier series', Acta Math. 116 (1966) 135-157. [§282 *notes*, §286 *intro.*, §286 *notes*.]

Clarkson J.A. [1936] 'Uniformly convex spaces', Trans. Amer. Math. Soc. 40 (1936) 396-414. [244O.]

Defant A. & Floret K. [93] *Tensor Norms and Operator Ideals*, North-Holland, 1993. [§253 *notes*.]

Doob J.L. [53] *Stochastic Processes.* Wiley, 1953.

Dudley R.M. [89] *Real Analysis and Probability.* Wadsworth & Brooks/Cole, 1989. [§282 *notes*.]

Dunford N. & Schwartz J.T. [57] *Linear Operators I.* Wiley, 1957 (reprinted 1988). [§244 *notes*, 2A5J.]

Enderton H.B. [77] *Elements of Set Theory.* Academic, 1977. [§2A1.]

Engelking R. [89] *General Topology.* Heldermann, 1989 (Sigma Series in Pure Mathematics 6). [2A5F.]

Etemadi N. [96] 'On convergence of partial sums of independent random variables', pp. 137-144 in Bergelson March & Rosenblatt 96. [272V.]

Evans L.C. & Gariepy R.F. [92] *Measure Theory and Fine Properties of Functions.* CRC Press, 1992. [263Ec, §265 *notes*.]

Federer H. [69] *Geometric Measure Theory.* Springer, 1969 (reprinted 1996). [262C, 263Ec, §264 *notes*, §265 *notes*, §266 *notes*.]

Feller W. [66] *An Introduction to Probability Theory and its Applications,* vol. II. Wiley, 1966. [Chap. 27 *intro.*, 274H, 275Xc, 285N.]

Fremlin D.H. [74] *Topological Riesz Spaces and Measure Theory.* Cambridge U.P., 1974. [§232 *notes*, 241F, §244 *notes*, §245 *notes*, §247 *notes*.]

Fremlin D.H. [93] 'Real-valued-measurable cardinals', pp. 151-304 in Judah 93. [232H.]

Haimo D.T. [67] (ed.) *Orthogonal Expansions and their Continuous Analogues.* Southern Illinois University Press, 1967.

Hall P. [82] *Rates of Convergence in the Central Limit Theorem.* Pitman, 1982. [274H.]

Halmos P.R. [50] *Measure Theory.* Van Nostrand, 1950. [§251 *notes*, §252 *notes*, 255Yn.]

Halmos P.R. [60] *Naive Set Theory.* Van Nostrand, 1960. [§2A1.]

Hanner O. [56] 'On the uniform convexity of L^p and l^p', Arkiv för Matematik 3 (1956) 239-244. [244O.]

Henle J.M. [86] *An Outline of Set Theory.* Springer, 1986. [§2A1.]

Hoeffding W. [63] 'Probability inequalities for sums of bounded random variables', J. Amer. Statistical Association 58 (1963) 13-30. [272W.]

Hunt R.A. [67] 'On the convergence of Fourier series', pp. 235-255 in Haimo 67. [§286 *notes*.]

Jorsbøe O.G. & Mejlbro L. [82] *The Carleson-Hunt Theorem on Fourier Series.* Springer, 1982 (Lecture Notes in Mathematics 911). [§286 *notes*.]

Judah H. [93] (ed.) Proceedings of the Bar-Ilan Conference on Set Theory and the Reals, 1991. Amer. Math. Soc. (Israel Mathematical Conference Proceedings 6), 1993.

Kelley J.L. [55] *General Topology.* Van Nostrand, 1955. [2A5F.]

Kelley J.L. & Namioka I. [76] *Linear Topological Spaces.* Springer, 1976. [2A5C.]

Kirszbraun M.D. [1934] 'Über die zusammenziehenden und Lipschitzian Transformationen', Fund. Math. 22 (1934) 77-108. [262C.]

Kolmogorov A.N. [1926] 'Une série de Fourier-Lebesgue divergente partout', C. R. Acad. Sci. Paris 183 (1926) 1327-1328. [§282 *notes.*]

Komlós J. [67] 'A generalization of a problem of Steinhaus', Acta Math. Acad. Sci. Hung. 18 (1967) 217-229. [276H.]

Körner T.W. [88] *Fourier Analysis.* Cambridge U.P., 1988. [§282 *notes.*]

Köthe G. [69] *Topological Vector Spaces I.* Springer, 1969. [2A5C, 2A5E, 2A5J.]

Krivine J.-L. [71] *Introduction to Axiomatic Set Theory.* D. Reidel, 1971. [§2A1.]

Lacey M.T. [05] *Carleson's Theorem: Proof, Complements, Variations.* `http://arxiv.org/pdf/math/0307008v4.pdf`. [§286 *notes.*]

Lacey M.T. & Thiele C.M. [00] 'A proof of boundedness of the Carleson operator', Math. Research Letters 7 (2000) 1-10. [§286 *intro.*, 286H.]

Lebesgue H. [72] *Oeuvres Scientifiques.* L'Enseignement Mathématique, Institut de Mathématiques, Univ. de Genève, 1972. [Chap. 27 *intro.*]

Liapounoff A. [1901] 'Nouvelle forme du théorème sur la limite de probabilité', Mém. Acad. Imp. Sci. St-Pétersbourg 12(5) (1901) 1-24. [274Xh.]

Lighthill M.J. [59] *Introduction to Fourier Analysis and Generalised Functions.* Cambridge U.P., 1959. [§284 *notes.*]

Lindeberg J.W. [1922] 'Eine neue Herleitung des Exponentialgesetzes in der Wahrscheinlichkeitsrechnung', Math. Zeitschrift 15 (1922) 211-225. [274H, §274 *notes.*]

Lipschutz S. [64] *Set Theory and Related Topics.* McGraw-Hill, 1964 (Schaum's Outline Series). [§2A1.]

Loève M. [77] *Probability Theory I.* Springer, 1977. [Chap. 27 *intro.*, 274H.]

Luxemburg W.A.J. & Zaanen A.C. [71] *Riesz Spaces I.* North-Holland, 1971. [241F.]

Mozzochi C.J. [71] *On the Pointwise Convergence of Fourier Series.* Springer, 1971 (Lecture Notes in Mathematics 199). [§286 *notes.*]

Naor A. [04] 'Proof of the uniform convexity lemma', `http://www.cims.nyu.edu/~naor/homepage files/inequality.pdf`, 26.2.04. [244O.]

Rényi A. [70] *Probability Theory.* North-Holland, 1970. [274H.]

Roitman J. [90] *An Introduction to Set Theory.* Wiley, 1990. [§2A1.]

Roselli P. & Willem M. [02] 'A convexity inequality', Amer. Math. Monthly 109 (2002) 64-70. [244Ym.]

Saks S. [1924] 'Sur les nombres dérivés des fonctions', Fund

Schipp F. [78] 'On Carleson's method', pp. 679-695 in ALEXITS 78. [§286 *notes.*]

Semadeni Z. [71] *Banach spaces of continuous functions I.* Polish Scientific Publishers, 1971. [§253 *notes.*]

Shiryayev A.N. [84] *Probability.* Springer, 1984. [285N.]

Steele J.M. [86] 'An Efron-Stein inequality of nonsymmetric statistics', Annals of Statistics 14 (1986) 753-758. [274Ya.]

Zaanen A.C. [83] *Riesz Spaces II.* North-Holland, 1983. [241F.]

Zygmund A. [59] *Trigonometric Series.* Cambridge U.P, 1959. [§244 *notes*, §282 *notes*, 284Xk.]