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Appendix to Volume 1

Useful Facts

Each volume of this treatise will have an appendix, containing very brief accounts of material which many readers will have met before but some may not, and which is relevant to some topic dealt with in the volume. For this first volume the appendix is short, partly because the volume itself is short, but mostly because the required basic knowledge of analysis is so fundamental that it must be done properly from a regular textbook or in a regular course. However I do set out a few details that might be omitted from some first courses in analysis, describing some not-quite-standard notation and the elementary theory of countable sets (§1A1), open and closed sets in Euclidean space (§1A2) and upper and lower limits of sequences and functions (§1A3).

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1A1 Set theory

In 111E-111F I briefly discussed 'countable' sets. The approach there was along what seemed to be the shortest path to the facts immediately needed, and it is perhaps right that I should here indicate a more conventional route. I take the opportunity to list some notation which I find convenient but is not universally employed.

1A1A Square bracket notations(a) For a, $b \in \mathbb{R}$, I write

$$[a,b] = \{x : a \le x \le b\}, \quad]a,b[= \{x : a < x < b\},$$

$$[a,b[= \{x : a \le x < b\}, \quad]a,b] = \{x : a < x \le b\}.$$

It is natural, when these formulae appear, to jump to the conclusion that $a < b$; but just occasionally it is useful to interpret them when $b \le a$, in which case

$$[a,a] = \{a\}, \quad]a,a[= [a,a[=]a,a] = \emptyset,$$

$$[a,b] =]a,b[= [a,b[=]a,b] = \emptyset \text{ if } b < a.$$

(b)

$$]-\infty,b[= \{x : x < b\}, \quad]a,\infty[= \{x : a < x\}, \quad]-\infty,\infty[= \mathbb{R},$$

$$[a,\infty[= \{x : x \ge a\}, \quad]-\infty,b] = \{x : x \le b\},$$

$$[0,\infty] = \{x : x \in \mathbb{R},\, x \ge 0\} \cup \{\infty\}, \quad [-\infty,\infty] = \mathbb{R} \cup \{-\infty,\infty\}.$$

1A1B Direct and inverse images(a) If f is a function and A is a set, I write

$$f[A] = \{f(x) : x \in A \cap \operatorname{dom} f\}$$

for the **direct image** of A under f.

(b) If f is a function and B is a set, I write

$$f^{-1}[B] = \{x : x \in \operatorname{dom} f,\, f(x) \in B\}$$

for the **inverse image** of B under f.

(c) Now suppose that R is a relation, that is, a set of ordered pairs, and A, B are sets. Then

$$R[A] = \{y : \exists\, x \in A \text{ such that } (x, y) \in R\},$$

$$R^{-1}[B] = \{x : \exists\, y \in B \text{ such that } (x, y) \in R\}.$$

1A1D Proposition Let K be a set. Then the following are equiveridical:
(i) either K is empty or there is a surjection from $\mathbb{N}$ onto K;
(ii) either K is finite or there is a bijection between $\mathbb{N}$ and K;
(iii) there is an injection from K to $\mathbb{N}$.

1A1E Properties of countable sets(a) If K is countable and $\phi : K \to L$ is a surjection, then L is countable.

(b) If K is countable and $\phi : L \to K$ is an injection, then L is countable.

(c) In particular, any subset of a countable set is countable.

(d) The Cartesian product of finitely many countable sets is countable.

(e) $\mathbb{Z}$ is countable.

(f) $\mathbb{Q}$ is countable.

1A1F Theorem If $\mathcal{K}$ is a countable collection of countable sets, then

$$\bigcup \mathcal{K} = \{x : \exists\, K \in \mathcal{K},\ x \in K\}$$

is countable.

1A1H Some uncountable sets(a) There is no surjection from $\mathbb{N}$ onto $\mathbb{R}$.
Thus $\mathbb{R}$ is uncountable.

(b) There is no surjection from $\mathbb{N}$ onto its power set $\mathcal{P}\mathbb{N}$.
Thus $\mathcal{P}\mathbb{N}$ is uncountable.

1A1J Notation I will say that a family $\mathcal{A}$ of sets is a **partition** of a set X whenever $\mathcal{A}$ is a disjoint cover of X, that is, $X = \bigcup \mathcal{A}$ and $A \cap A' = \emptyset$ for all distinct A, $A' \in \mathcal{A}$. Similarly, an indexed family $\langle A_i \rangle_{i \in I}$ is a **partition** partition of X if $\bigcup_{i \in I} A_i = X$ and $A_i \cap A_j = \emptyset$ for all distinct i, $j \in I$.

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1A2 Open and closed sets in $\mathbb{R}^r$

In 111G I gave the definition of an open set in $\mathbb{R}$ or $\mathbb{R}^r$, and in 121D I used, in passing, some of the basic properties of these sets; perhaps it will be helpful if I set out a tiny part of the elementary theory.

1A2A Open sets Recall that a set $G \subseteq \mathbb{R}$ is **open** if for every $x \in G$ there is a $\delta > 0$ such that $]x - \delta, x + \delta[\subseteq G$; similarly, a set $G \subseteq \mathbb{R}^r$ is **open** if for every $x \in G$ there is a $\delta > 0$ such that $U(x, \delta) \subseteq G$, where $U(x, \delta) = \{y : \|y - x\| < \delta\}$, writing $\|z\|$ for $\sqrt{\zeta_1^2 + \ldots + \zeta_r^2}$ if $z = (\zeta_1, \ldots, \zeta_r)$.

1A2B The family of all open sets Let $\mathfrak{T}$ be the family of open sets of $\mathbb{R}^r$.
(a) $\emptyset \in \mathfrak{T}$.
(b) $\mathbb{R}^r \in \mathfrak{T}$.
(c) If G, $H \in \mathfrak{T}$ then $G \cap H \in \mathfrak{T}$.
(d) If $\mathcal{G} \subseteq \mathfrak{T}$, then

$$\bigcup \mathcal{G} = \{x : \exists\, G \in \mathcal{G},\ x \in G\} \in \mathfrak{T}.$$

1A2C Cauchy's inequality: Proposition For all x, $y \in \mathbb{R}^r$, $\|x+y\| \le \|x\| + \|y\|$.

1A2D Corollary $U(x,\delta)$ is open, for every $x \in \mathbb{R}^r$ and $\delta > 0$.

1A2E Closed sets: Definition A set $F \subseteq \mathbb{R}^r$ is **closed** if $\mathbb{R}^r \setminus F$ is open.

1A2F Proposition Let $\mathcal{F}$ be the family of closed subsets of $\mathbb{R}^r$.
(a) $\emptyset \in \mathcal{F}$.
(b) $\mathbb{R}^r \in \mathcal{F}$.
(c) If E, $F \in \mathcal{F}$ then $E \cup F \in \mathcal{F}$.
(d) If $\mathcal{E} \subseteq \mathcal{F}$ is a *non-empty* family of closed sets, then

$$\bigcap \mathcal{E} = \{x : x \in F \ \forall \ F \in \mathcal{E}\} \in \mathcal{F}.$$

1A2G Lemma If $x \in \mathbb{R}^r$ and $\delta \ge 0$ then $B(x,\delta) = \{y : \|y-x\| \le \delta\}$ is closed.

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1A3 Lim sups and lim infs

It occurs to me that not every foundation course in real analysis has time to deal with the concepts lim sup and lim inf.

1A3A Definition (a) For a real sequence $\langle a_n \rangle_{n\in\mathbb{N}}$, write

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\ge n} a_m = \inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m,$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\ge n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\ge n} a_m;$$

if we allow the values $\pm\infty$, both for suprema and infima and for limits, these will always be defined.

(c) For $u \in [-\infty,\infty]$, we can say that

$\limsup_{n\to\infty} a_n = u$ iff (i) for every $v > u$ (if any) there is an $n_0 \in \mathbb{N}$ such that $a_n \le v$ for every $n \ge n_0$ (ii) for every $v < u$, $n_0 \in \mathbb{N}$ there is an $n \ge n_0$ such that $a_n \ge v$,

$\liminf_{n\to\infty} a_n = u$ iff (i) for every $v < u$ there is an $n_0 \in \mathbb{N}$ such that $a_n \ge v$ for every $n \ge n_0$ (ii) for every $v > u$, $n_0 \in \mathbb{N}$ there is an $n \ge n_0$ such that $a_n \le v$.

1A3B Proposition For any sequences $\langle a_n \rangle_{n\in\mathbb{N}}$, $\langle b_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$,
(a) $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$,
(b) $\lim_{n\to\infty} a_n = u \in [-\infty,\infty]$ iff $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = u$,
(c) $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$,
(d) $\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$,
(e) $\liminf_{n\to\infty}(a_n + b_n) \ge \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$,
(f) $\limsup_{n\to\infty} ca_n = c \limsup_{n\to\infty} a_n$ if $c \ge 0$,
(g) $\liminf_{n\to\infty} ca_n = c \liminf_{n\to\infty} a_n$ if $c \ge 0$,
with the proviso in (d) and (e) that we must be able to interpret the right-hand-side of the inequality according to the rules in 135A, while in (f) and (g) we take $0 \cdot \infty = 0 \cdot (-\infty) = 0$.

***1A3D Other expressions of the same idea** The concepts of lim sup and lim inf may be applied in any context in which we can consider the limit of a real-valued function. For instance, if f is a real-valued function defined (at least) on a punctured interval of the form $\{x : 0 < |c - x| \le \epsilon\}$ where $c \in \mathbb{R}$ and $\epsilon > 0$, then

$$\limsup_{t\to c} f(t) = \lim_{\delta\downarrow 0} \sup_{0<|t-c|\le\delta} f(t) = \inf_{0<\delta\le\epsilon} \sup_{0<|t-c|\le\delta} f(t),$$

$$\liminf_{t\to c} f(t) = \lim_{\delta\downarrow 0} \inf_{0<|t-c|\le\delta} f(t) = \sup_{0<\delta\le\epsilon} \inf_{0<|t-c|\le\delta} f(t),$$

allowing ∞ and $-\infty$ whenever they seem called for. Or if f is defined on the half-open interval $]c, c+\epsilon]$, we can say

$$\limsup_{t\downarrow c} f(t) = \lim_{\delta\downarrow 0} \sup_{c<t\le c+\delta} f(t) = \inf_{0<\delta\le\epsilon} \sup_{c<t\le c+\delta} f(t),$$

$$\liminf_{t\downarrow c} f(t) = \lim_{\delta\downarrow 0} \inf_{c<t\le c+\delta} f(t) = \sup_{0<\delta\le\epsilon} \inf_{c<t\le c+\delta} f(t).$$

Similarly, if f is defined on $[M, \infty[$ for some $M \in \mathbb{R}$, we have

$$\limsup_{t\to\infty} f(t) = \lim_{a\to\infty} \sup_{t\ge a} f(t) = \inf_{a\ge M} \sup_{t\ge a} f(t),$$

$$\liminf_{t\to\infty} f(t) = \lim_{a\to\infty} \inf_{t\ge a} f(t) = \sup_{a\ge M} \inf_{t\ge a} f(t).$$