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Appendix to Volume 1

Useful Facts

Each volume of this treatise will have an appendix, containing very brief accounts of material which many readers will have met before but some may not, and which is relevant to some topic dealt with in the volume. For this first volume the appendix is short, partly because the volume itself is short, but mostly because the required basic knowledge of analysis is so fundamental that it must be done properly from a regular textbook or in a regular course. However I do set out a few details that might be omitted from some first courses in analysis, describing some not-quite-standard notation and the elementary theory of countable sets (§1A1), open and closed sets in Euclidean space (§1A2) and upper and lower limits of sequences and functions (§1A3).

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1A1 Set theory

In 111E-111F I briefly discussed 'countable' sets. The approach there was along what seemed to be the shortest path to the facts immediately needed, and it is perhaps right that I should here indicate a more conventional route. I take the opportunity to list some notation which I find convenient but is not universally employed.

1A1A Square bracket notations I use square brackets [and] in a variety of ways; the context will I hope always make it clear what interpretation is expected.

(a) For a, $b \in \mathbb{R}$, I write

$$[a,b] = \{x : a \le x \le b\}, \quad]a,b[= \{x : a < x < b\},$$

$$[a,b[= \{x : a \le x < b\}, \quad]a,b] = \{x : a < x \le b\}.$$

It is natural, when these formulae appear, to jump to the conclusion that $a < b$; but just occasionally it is useful to interpret them when $b \le a$, in which case I follow the formulae above literally, so that

$$[a,a] = \{a\}, \quad]a,a[= [a,a[=]a,a] = \emptyset,$$

$$[a,b] =]a,b[= [a,b[=]a,b] = \emptyset \text{ if } b < a.$$

(b) We can interpret the formulae with infinite a or b; for example,

$$]-\infty,b[= \{x : x < b\}, \quad]a,\infty[= \{x : a < x\}, \quad]-\infty,\infty[= \mathbb{R},$$

$$[a,\infty[= \{x : x \ge a\}, \quad]-\infty,b] = \{x : x \le b\},$$

and even

$$[0,\infty] = \{x : x \in \mathbb{R},\, x \ge 0\} \cup \{\infty\}, \quad [-\infty,\infty] = \mathbb{R} \cup \{-\infty,\infty\}.$$

(c) With some circumspection – since further choices have to be made, which it is safer to set out explicitly when the occasion arises – we can use similar formulae for 'intervals' in multidimensional space $\mathbb{R}^r$; see, for instance, 115A or 136D; and even in general partially ordered sets, though these will not be important to us before Volume 3.

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(d) Perhaps I owe you an explanation for my choice of $]a,b[$, $[a,b[$ in favour of (a,b), $[a,b)$, which are both commoner and more pleasing to the eye. In the first instance it is simply because the formula

$$(1,2) \in \,]0,2[\times]1,3[$$

makes better sense than its translation. Generally, it leads to a slightly better balance in the number of appearances of (and [, even allowing for the further uses of $[\ldots]$ which I am about to specify.

1A1B Direct and inverse images I now describe an entirely different use of square brackets, belonging to abstract set theory rather than to the theory of the real number system.

(a) If f is a function and A is a set, I write

$$f[A] = \{f(x) : x \in A \cap \operatorname{dom} f\}$$

for the **direct image** of A under f. Note that while A will often be a subset of the domain of f, this is not assumed.

(b) If f is a function and B is a set, I write

$$f^{-1}[B] = \{x : x \in \operatorname{dom} f,\ f(x) \in B\}$$

for the **inverse image** of B under f. This time, it is important to note that there is no presumption that f is injective, or that f^{-1} is a function; the formula $f^{-1}[\,]$ is being given a meaning independent of any meaning of the expression f^{-1}. But it is easy to see that when f is injective, so that we have a true inverse function f^{-1} (defined on the set of values of f, $f[\operatorname{dom} f]$), then $f^{-1}[B]$, as defined here, agrees with its interpretation under (a).

(c) Now suppose that R is a relation, that is, a set of ordered pairs, and A, B are sets. Then I write

$$R[A] = \{y : \exists\, x \in A \text{ such that } (x,y) \in R\},$$

$$R^{-1}[B] = \{x : \exists\, y \in B \text{ such that } (x,y) \in R\}.$$

If we write

$$R^{-1} = \{(y,x) : (x,y) \in R\},$$

then we have an alternative interpretation of $R^{-1}[B]$ which agrees with the one just given. Moreover, if R is the graph of a function f, that is, if for every x there is at most one y such that $(x,y) \in R$, then the formulae here agree with those of (a)-(b) above.

(d) (The following is addressed exclusively to readers who have been taught to distinguish between the words 'set' and 'class'.) I have used the word 'set' more than once above. But that was purely for euphony. The same formulae can be used with arbitrary classes, though in some set theories the expressions involved may not be recognised as 'terms' in the technical sense.

1A1C Countable sets In 111Fa I defined 'countable set' as follows: a set K is countable if either it is empty or there is a surjective function from $\mathbb{N}$ to K. A commoner formulation is to say that a set K is countable iff either it is finite or there is a bijection between $\mathbb{N}$ and K. So I should check at once that these two formulations agree.

1A1D Proposition Let K be a set. Then the following are equiveridical:
(i) either K is empty or there is a surjection from $\mathbb{N}$ onto K;
(ii) either K is finite or there is a bijection between $\mathbb{N}$ and K;
(iii) there is an injection from K to $\mathbb{N}$.

proof (a)(i)⇒(iii) Assume (i). If K is empty, then the empty function is an injection from K to $\mathbb{N}$. Otherwise, there is a surjection $\phi : \mathbb{N} \to K$. Now, for each $k \in K$, set

$$\psi(k) = \min\{n : n \in \mathbb{N},\ \phi(n) = k\};$$

this is always well-defined because ϕ is surjective, so that $\{n : \phi(n) = k\}$ is never empty, and must have a least member. Because $\phi\psi(k) = k$ for every k, ψ must be injective, so is the required injection from K to $\mathbb{N}$.

(b)(iii)⇒(ii) Assume (iii); let $\psi : K \to \mathbb{N}$ be an injection, and set $A = \psi[K] \subseteq \mathbb{N}$. Then ψ is a bijection between K and A. If K is finite, then of course (ii) is satisfied. Otherwise, A must also be infinite. Define $\phi : A \to \mathbb{N}$ by setting

$$\phi(m) = \#(\{i : i \in A,\, i < m\}),$$

the number of elements of A less than m, for each $m \in A$; thus $\phi(m)$ is the position of m if the elements of A are listed from the bottom, starting at 0 for the least element of A. Then $\phi : A \to \mathbb{N}$ is a bijection, because A is infinite, and $\phi\psi : K \to \mathbb{N}$ is a bijection.

(c)(ii)⇒(i) If K is empty, surely it satisfies (i). If K is finite and not empty, list its members as $k_0, \dots , k_n$; now set $\phi(i) = k_i$ for $i \le n$, k_0 for $i > n$ to get a surjection $\phi : \mathbb{N} \to K$. If K is infinite, there is a bijection from $\mathbb{N}$ to K, which is of course also a surjection from $\mathbb{N}$ to K. So (i) is true in all cases.

Remark I referred to the 'empty function' in the proof above. This is the function with domain $\emptyset$; having said this, any, or no, rule for calculating the function will have the same effect, since it will never be applied. By examining your feelings about this construction you can learn something about your basic attitude to mathematics. You may feel that it is an artificial irrelevance, or you may feel that it is as necessary as the number 0. Both are entirely legitimate feelings, and the fully rounded mathematician alternates between them; but I have to say that I myself tend to the latter more often than the former, and that when I say 'function' in this treatise the empty function will generally be in the back of my mind as a possibility.

1A1E Properties of countable sets Let me recapitulate the basic properties of countable sets:

(a) If K is countable and $\phi : K \to L$ is a surjection, then L is countable. **P** If K is empty then so is L. Otherwise there is a surjection $\psi : \mathbb{N} \to K$, so $\phi\psi$ is a surjection from $\mathbb{N}$ onto L, and L is countable. **Q**

(b) If K is countable and $\phi : L \to K$ is an injection, then L is countable. **P** By 1A1D(iii), there is an injection $\psi : K \to \mathbb{N}$; now $\psi\phi : L \to \mathbb{N}$ is injective, so L is countable. **Q**

(c) In particular, any subset of a countable set is countable (as in 111F(b-i)).

(d) The Cartesian product of finitely many countable sets is countable (111Fb(iii)-(iv)).

(e) $\mathbb{Z}$ is countable. **P** The map $(m, n) \mapsto m - n : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ is surjective. **Q**

(f) $\mathbb{Q}$ is countable. **P** The map $(m, n) \mapsto \frac{m}{n+1} : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ is surjective. **Q**

1A1F Another fundamental property is worth distinguishing from these, as it relies on a slightly deeper argument.

Theorem If $\mathcal{K}$ is a countable collection of countable sets, then

$$\bigcup \mathcal{K} = \{x : \exists\, K \in \mathcal{K},\, x \in K\}$$

is countable.

proof Set

$$\mathcal{K}' = \mathcal{K} \setminus \{\emptyset\} = \{K : K \in \mathcal{K},\, K \ne \emptyset\};$$

then $\mathcal{K}' \subseteq \mathcal{K}$, so is countable, and $\bigcup \mathcal{K}' = \bigcup \mathcal{K}$. If $\mathcal{K}' = \emptyset$, then

$$\bigcup \mathcal{K} = \bigcup \mathcal{K}' = \emptyset$$

is surely countable. Otherwise, let $m \mapsto K_m : \mathbb{N} \to \mathcal{K}'$ be a surjection. For each $m \in \mathbb{N}$, K_m is a non-empty countable set, so there is a surjection $n \mapsto k_{mn} : \mathbb{N} \to K_m$. Now $(m, n) \mapsto k_{mn} : \mathbb{N} \times \mathbb{N} \to \bigcup \mathcal{K}$ is a surjection (if $k \in \bigcup \mathcal{K}$, there is a $K \in \mathcal{K}'$ such that $k \in K$; there is an $m \in \mathbb{N}$ such that $K = K_m$; there is an $n \in \mathbb{N}$ such that $k = k_{mn}$). So $\bigcup \mathcal{K}$ is countable, as required.

***1A1G Remark** I divide this result from the 'elementary' facts in 1A1E partly because it uses a different principle of argument from any necessary for the earlier work. In the middle of the proof I wrote 'so there is a surjection $n \mapsto k_{mn} : \mathbb{N} \to K_m$'. That there is a surjection from $\mathbb{N}$ onto K_m does indeed follow from the immediately preceding statement 'K_m is a non-empty countable set'. The sleight of hand lies in immediately naming such a surjection as '$n \mapsto k_{mn}$'. There may of course be many surjections from $\mathbb{N}$ to K_m – as a rule, indeed, there will be uncountably many – and what I am in effect doing here is picking arbitrarily on one of them. The choice has to be arbitrary, because I am working in a totally abstract context, and while in any particular application of this theorem there may be some natural surjection to use, I have no way of forecasting what approach, if any, might offer a criterion for distinguishing a particular function here. Now it has been a basic method of mathematical argument, from Euclid's time at least, that we are willing to give a name to an object, a 'general point' or an 'arbitrary number', without specifying exactly which object we are naming. But here I am picking out simultaneously infinitely many objects, all arbitrary members of certain sets. This is a use of the **Axiom of Choice**.

I do not recall ever having had a student criticise an argument in the form of that in 1A1F on the grounds that it uses a new, and possibly illegitimate, principle; I am sure that it never occurred to me that anything exceptionable was being done in these cases, until someone pointed it out. If you find that discussions of this kind are irrelevant to your own mathematical interests, you can certainly pass them by, at least until you reach Volume 5. Mathematical systems have been studied in which the axiom of choice is false; they are of great interest but so far remain peripheral to the subject. Systems in which the axiom of choice is so false that the union of countably many countable sets is sometimes uncountable have a character all of their own, and in particular the theory of Lebesgue measure is transformed; I will come to this possibility in Chapter 56 of Volume 5.

For a brief comment on other ways of using the axiom of choice, see 134C.

1A1H Some uncountable sets Of course not all sets are countable. In 114G/115G I remark that all countable subsets of Euclidean space are negligible for Lebesgue measure; consequently, any set which is not negligible – for instance, any non-trivial interval – must be uncountable. But perhaps it will be helpful if I offer here elementary arguments to show that $\mathbb{R}$ and $\mathcal{P}\mathbb{N}$ are not countable.

(a) There is no surjection from $\mathbb{N}$ onto $\mathbb{R}$. **P** Let $n \mapsto a_n : \mathbb{N} \to \mathbb{R}$ be any function. For each $n \in \mathbb{N}$, express a_n in decimal form as

$$a_n = k_n + 0 \cdot \epsilon_{n1}\epsilon_{n2} \ldots = k_n + \sum_{i=1}^{\infty} 10^{-i}\epsilon_{ni},$$

where $k_n \in \mathbb{Z}$ is the greatest integer not greater than a_n, and each ϵ_{ni} is an integer between 0 and 9; for definiteness, if a_n happens to be an exact decimal, use the terminating expansion, so that the ϵ_{ni} are eventually 0 rather than eventually 9.

Now define ϵ_i, for $i \ge 1$, by saying that

$$\begin{aligned} \epsilon_i &= 6 \text{ if } \epsilon_{ii} < 6, \\ &= 5 \text{ if } \epsilon_{ii} \ge 6. \end{aligned}$$

Consider $a = k_0 + 1 + \sum_{i=1}^{\infty} 10^{-i}\epsilon_i$, so that $a = k_0 + 1 + 0 \cdot \epsilon_1\epsilon_2 \ldots$ in decimal form. I claim that $a \ne a_n$ for every n. Of course $a \ne a_0$ because $a_0 < k_0 + 1 \le a$. If $n \ge 1$, then $\epsilon_n \ne \epsilon_{nn}$; because no ϵ_i is either 0 or 9, there is no alternative decimal expansion of a, so the expansion $a_n = k_n + 0 \cdot \epsilon_{n1}\epsilon_{n2} \ldots$ cannot represent a, and $a \ne a_n$.

Thus I have constructed a real number which is not in the list $a_0, a_1, \ldots$. As $\langle a_n \rangle_{n\in\mathbb{N}}$ is arbitrary, there is no surjection from $\mathbb{N}$ onto $\mathbb{R}$. **Q**

Thus $\mathbb{R}$ is uncountable.

(b) There is no surjection from $\mathbb{N}$ onto its power set $\mathcal{P}\mathbb{N}$. **P** Let $n \mapsto A_n : \mathbb{N} \to \mathcal{P}\mathbb{N}$ be any function. Set

$$A = \{n : n \in \mathbb{N},\, n \notin A_n\}.$$

If $n \in \mathbb{N}$, then

either $n \in A_n$, in which case $n \notin A$,

or $n \notin A_n$, in which case $n \in A$.

Thus in both cases we have $n \in A\triangle A_n$, so that $A \ne A_n$. As n is arbitrary, $A \notin \{A_n : n \in \mathbb{N}\}$ and $n \mapsto A_n$ is not a surjection. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, there is no surjection from $\mathbb{N}$ onto $\mathcal{P}\mathbb{N}$. **Q**

Thus $\mathcal{P}\mathbb{N}$ is also uncountable.

1A1I Remark In fact it is the case that there is a bijection between $\mathbb{R}$ and $\mathcal{P}\mathbb{N}$ (2A1Ha); so that the uncountability of both can be established by just one of the arguments above.

1A1J Notation For definiteness, I remark here that I will say that a family $\mathcal{A}$ of sets is a **partition** of a set X whenever $\mathcal{A}$ is a disjoint cover of X, that is, $X = \bigcup\mathcal{A}$ and $A \cap A' = \emptyset$ for all distinct A, $A' \in \mathcal{A}$; in particular, the empty set may or may not belong to $\mathcal{A}$. Similarly, an indexed family $\langle A_i\rangle_{i\in I}$ is a **partition** partition of X if $\bigcup_{i\in I} A_i = X$ and $A_i \cap A_j = \emptyset$ for all distinct i, $j \in I$; again, one or more of the A_i may be empty.

1A1 Notes and comments The ideas of 1A1C-1A1I are essentially due to G.F.Cantor. These concepts are fundamental to modern set theory, and indeed to very large parts of modern pure mathematics. The notes above hardly begin to suggest the extraordinary fertility of these ideas, which need a book of their own for their proper expression; my only aim here has been to try to make sense of those tiny parts of the subject which are needed in the present volume. In later volumes I will present results which call on substantially more advanced ideas, which I will discuss in appendices to those volumes.

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1A2 Open and closed sets in $\mathbb{R}^r$

In 111G I gave the definition of an open set in $\mathbb{R}$ or $\mathbb{R}^r$, and in 121D I used, in passing, some of the basic properties of these sets; perhaps it will be helpful if I set out a tiny part of the elementary theory.

1A2A Open sets Recall that a set $G \subseteq \mathbb{R}$ is **open** if for every $x \in G$ there is a $\delta > 0$ such that $]x-\delta, x+\delta[\subseteq G$; similarly, a set $G \subseteq \mathbb{R}^r$ is **open** if for every $x \in G$ there is a $\delta > 0$ such that $U(x,\delta) \subseteq G$, where $U(x,\delta) = \{y : \|y-x\| < \delta\}$, writing $\|z\|$ for $\sqrt{\zeta_1^2+\ldots+\zeta_r^2}$ if $z = (\zeta_1,\ldots,\zeta_r)$. Henceforth I give the arguments for general r; if you are at present interested only in the one-dimensional case, you should have no difficulty in reading them as if $r = 1$ throughout.

1A2B The family of all open sets Let $\mathfrak{T}$ be the family of open sets of $\mathbb{R}^r$. Then $\mathfrak{T}$ has the following properties.

(a) $\emptyset \in \mathfrak{T}$, that is, the empty set is open. **P** Because the definition of '$\emptyset$ is open' begins with 'for every $x \in \emptyset$, ...', it must be vacuously satisfied by the empty set. **Q**

(b) $\mathbb{R}^r \in \mathfrak{T}$, that is, the whole space under consideration is an open set. **P** $U(x,1) \subseteq \mathbb{R}^r$ for every $x \in \mathbb{R}^r$. **Q**

(c) If G, $H \in \mathfrak{T}$ then $G \cap H \in \mathfrak{T}$; that is, the intersection of two open sets is always an open set. **P** Let $x \in G \cap H$. Then there are δ_1, $\delta_2 > 0$ such that $U(x,\delta_1) \subseteq G$ and $U(x,\delta_2) \subseteq H$. Set $\delta = \min(\delta_1,\delta_2) > 0$; then

$$U(x,\delta) = \{y : \|y-x\| < \min(\delta_1,\delta_2)\} = U(x,\delta_1) \cap U(x,\delta_2) \subseteq G \cap H.$$

As x is arbitrary, $G \cap H$ is open. **Q**

(d) If $\mathcal{G} \subseteq \mathfrak{T}$, then

$$\bigcup\mathcal{G} = \{x : \exists\, G \in \mathcal{G},\ x \in G\} \in \mathfrak{T};$$

that is, the union of any family of open sets is open. **P** Let $x \in \bigcup\mathcal{G}$. Then there is a $G \in \mathcal{G}$ such that $x \in G$. Because $G \in \mathfrak{T}$, there is a $\delta > 0$ such that

$$U(x,\delta) \subseteq G \subseteq \bigcup\mathcal{G}.$$

As x is arbitrary, $\bigcup\mathcal{G} \in \mathfrak{T}$. **Q**

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1A2C Cauchy's inequality: Proposition For all x, $y \in \mathbb{R}^r$, $\|x+y\| \le \|x\| + \|y\|$.

proof Express x as $(\xi_1, \dots, \xi_r)$, y as $(\eta_1, \dots, \eta_r)$; set $\alpha = \|x\|$, $\beta = \|y\|$. Then both α and β are non-negative. If $\alpha = 0$ then $\sum_{j=1}^{r} \xi_j^2 = 0$ so every $\xi_j = 0$ and $x = \mathbf{0}$, so $\|x+y\| = \|y\| = \|x\| + \|y\|$; if $\beta = 0$, then $y = \mathbf{0}$ and $\|x+y\| = \|x\| = \|x\| + \|y\|$. Otherwise, consider

$$\begin{aligned}
\alpha\beta\|x+y\|^2 &\le \alpha\beta\|x+y\|^2 + \|\alpha y - \beta x\|^2 \\
&= \alpha\beta\sum_{j=1}^{r}(\xi_j + \eta_j)^2 + \sum_{j=1}^{r}(\alpha\eta_j - \beta\xi_j)^2 \\
&= \sum_{j=1}^{r}\alpha\beta\xi_j^2 + \alpha\beta\eta_j^2 + \alpha^2\eta_j^2 + \beta^2\xi_j^2 \\
&= \alpha^3\beta + \alpha\beta^3 + \alpha^2\beta^2 + \beta^2\alpha^2 \\
&= \alpha\beta(\alpha+\beta)^2 = \alpha\beta(\|x\| + \|y\|)^2.
\end{aligned}$$

Dividing both sides by $\alpha\beta$ and taking square roots we have the result.

1A2D Corollary $U(x,\delta)$, as defined in 1A2A, is open, for every $x \in \mathbb{R}^r$ and $\delta > 0$.

proof If $y \in U(x,\delta)$, then $\eta = \delta - \|y - x\| > 0$. Now if $z \in U(y,\eta)$,

$$\|z - x\| = \|(z-y) + (y-x)\| \le \|z-y\| + \|y-x\| < \eta + \|y-x\| = \delta,$$

and $z \in U(x,\delta)$; thus $U(y,\eta) \subseteq U(x,\delta)$. As y is arbitrary, $U(x,\delta)$ is open.

1A2E Closed sets: Definition A set $F \subseteq \mathbb{R}^r$ is **closed** if $\mathbb{R}^r \setminus F$ is open. (*Warning!* 'Most' subsets of $\mathbb{R}^r$ are neither open nor closed; two subsets of $\mathbb{R}^r$, viz., $\emptyset$ and $\mathbb{R}^r$, are both open and closed.) Corresponding to the list in 1A2B, we have the following properties of the family $\mathcal{F}$ of closed subsets of $\mathbb{R}^r$.

1A2F Proposition Let $\mathcal{F}$ be the family of closed subsets of $\mathbb{R}^r$.
(a) $\emptyset \in \mathcal{F}$ (because $\mathbb{R}^r \in \mathfrak{T}$).
(b) $\mathbb{R}^r \in \mathcal{F}$ (because $\emptyset \in \mathfrak{T}$).
(c) If E, $F \in \mathcal{F}$ then $E \cup F \in \mathcal{F}$, because

$$\mathbb{R}^r \setminus (E \cup F) = (\mathbb{R}^r \setminus E) \cap (\mathbb{R}^r \setminus F) \in \mathfrak{T}.$$

(d) If $\mathcal{E} \subseteq \mathcal{F}$ is a *non-empty* family of closed sets, then

$$\bigcap\mathcal{E} = \{x : x \in F \ \forall\ F \in \mathcal{E}\} = \mathbb{R}^r \setminus \bigcup_{F\in\mathcal{E}}(\mathbb{R}^r \setminus F) \in \mathcal{F}.$$

Remark In (d), we need to assume that $\mathcal{E} \ne \emptyset$ to ensure that $\bigcap\mathcal{E} \subseteq \mathbb{R}^r$.

1A2G Corresponding to 1A2D, we have the following fact:

Lemma If $x \in \mathbb{R}^r$ and $\delta \ge 0$ then $B(x,\delta) = \{y : \|y - x\| \le \delta\}$ is closed.

proof Set $G = \mathbb{R}^r \setminus B(x,\delta)$. If $y \in G$, then $\eta = \|y - x\| - \delta > 0$; if $z \in U(y,\eta)$, then

$$\delta + \eta = \|y - x\| \le \|y - z\| + \|z - x\| < \eta + \|z - x\|,$$

so $\|z - x\| > \delta$ and $z \in G$. So $U(y,\eta) \subseteq G$. As y is arbitrary, G is open and $B(x,\delta)$ is closed.

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1A3 Lim sups and lim infs

It occurs to me that not every foundation course in real analysis has time to deal with the concepts lim sup and lim inf.

1A3A Definition (a) For a real sequence $\langle a_n\rangle_{n\in\mathbb{N}}$, write

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\ge n} a_m = \inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m,$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{m\ge n} a_m = \sup_{n\in\mathbb{N}} \inf_{m\ge n} a_m;$$

if we allow the values $\pm\infty$, both for suprema and infima and for limits (see 112Ba), these will always be defined, because the sequences

$$\langle \sup_{m\ge n} a_m\rangle_{n\in\mathbb{N}}, \quad \langle \inf_{m\ge n} a_m\rangle_{n\in\mathbb{N}}$$

are monotonic.

(b) Explicitly:

$$\limsup_{n\to\infty} a_n = \infty \iff \{a_n : n\in\mathbb{N}\} \text{ is unbounded above,}$$

$$\limsup_{n\to\infty} a_n = -\infty \iff \lim_{n\to\infty} a_n = -\infty,$$

that is, if and only if for every $a\in\mathbb{R}$ there is an $n_0\in\mathbb{N}$ such that $a_n \le a$ for every $n\ge n_0$;

$$\liminf_{n\to\infty} a_n = -\infty \iff \{a_n : n\in\mathbb{N}\} \text{ is unbounded below,}$$

$$\liminf_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} a_n = \infty,$$

that is, if and only if for every $a\in\mathbb{R}$ there is an $n_0\in\mathbb{N}$ such that $a_n \ge a$ for every $n\ge n_0$.

(c) For finite $a\in\mathbb{R}$, we have

$\limsup_{n\to\infty} a_n = a$ iff (i) for every $\epsilon>0$ there is an $n_0\in\mathbb{N}$ such that $a_n\le a+\epsilon$ for every $n\ge n_0$ (ii) for every $\epsilon>0$, $n_0\in\mathbb{N}$ there is an $n\ge n_0$ such that $a_n\ge a-\epsilon$,

while

$\liminf_{n\to\infty} a_n = a$ iff (i) for every $\epsilon>0$ there is an $n_0\in\mathbb{N}$ such that $a_n\ge a-\epsilon$ for every $n\ge n_0$ (ii) for every $\epsilon>0$, $n_0\in\mathbb{N}$ there is an $n\ge n_0$ such that $a_n\le a+\epsilon$.

Generally, for $u\in[-\infty,\infty]$, we can say that

$\limsup_{n\to\infty} a_n = u$ iff (i) for every $v>u$ (if any) there is an $n_0\in\mathbb{N}$ such that $a_n\le v$ for every $n\ge n_0$ (ii) for every $v<u$, $n_0\in\mathbb{N}$ there is an $n\ge n_0$ such that $a_n\ge v$,

$\liminf_{n\to\infty} a_n = u$ iff (i) for every $v<u$ there is an $n_0\in\mathbb{N}$ such that $a_n\ge v$ for every $n\ge n_0$ (ii) for every $v>u$, $n_0\in\mathbb{N}$ there is an $n\ge n_0$ such that $a_n\le v$.

1A3B We have the following basic results.

Proposition For any sequences $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$,

(a) $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$,

(b) $\lim_{n\to\infty} a_n = u\in[-\infty,\infty]$ iff $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = u$,

(c) $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$,

(d) $\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$,

(e) $\liminf_{n\to\infty}(a_n+b_n) \ge \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$,

(f) $\limsup_{n\to\infty} ca_n = c\limsup_{n\to\infty} a_n$ if $c\ge 0$,

(g) $\liminf_{n\to\infty} ca_n = c\liminf_{n\to\infty} a_n$ if $c\ge 0$,

with the proviso in (d) and (e) that we must be able to interpret the right-hand-side of the inequality according to the rules in 135A, while in (f) and (g) we take $0\cdot\infty = 0\cdot(-\infty) = 0$.

proof (a) $\sup_{m\ge n} a_m \ge \inf_{m\ge n} a_m$ for every n, so

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{m\ge n} a_m \ge \lim_{n\to\infty} \inf_{m\ge n} a_m = \liminf_{n\to\infty} a_n.$$

(b) Using the last description of $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ in 1A3Ac, and a corresponding description of $\lim_{n\to\infty}$, we have

$$\begin{aligned}
&\lim_{n\to\infty} a_n = u\\
&\qquad\iff \text{ for every } v > u \text{ there is an } n_1 \in \mathbb{N} \text{ such that } a_n \le v \text{ for every } n \ge n_1\\
&\qquad\quad \text{and for every } v < u \text{ there is an } n_2 \in \mathbb{N} \text{ such that } a_n \ge v \text{ for every } n \ge n_2\\
&\qquad\iff \text{ for every } v > u \text{ there is an } n_1 \in \mathbb{N} \text{ such that } a_n \le v \text{ for every } n \ge n_1\\
&\qquad\quad \text{and for every } v < u,\ n_0 \in \mathbb{N} \text{ there is an } n \ge n_0 \text{ such that } a_n \ge v\\
&\qquad\quad \text{and for every } v < u \text{ there is an } n_2 \in \mathbb{N} \text{ such that } a_n \ge v \text{ for every } n \ge n_2\\
&\qquad\quad \text{and for every } v > u,\ n_0 \in \mathbb{N} \text{ there is an } n \ge n_0 \text{ such that } a_n \le v\\
&\qquad\iff \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = u.
\end{aligned}$$

(c) This is just a matter of turning the formulae upside down:

$$\begin{aligned}
\liminf_{n\to\infty} a_n &= \sup_{n\in\mathbb{N}} \inf_{m\ge n} a_m = \sup_{n\in\mathbb{N}}(-\sup_{m\ge n}(-a_m))\\
&= -\inf_{n\in\mathbb{N}} \sup_{m\ge n}(-a_m) = -\limsup_{n\to\infty}(-a_n).
\end{aligned}$$

(d) If $v > \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$, there are v_1, v_2 such that $v_1 > \limsup_{n\to\infty} a_n$, $v_2 > \limsup_{n\to\infty} b_n$ and $v_1 + v_2 = v$. Now there are n_1, $n_2 \in \mathbb{N}$ such that $\sup_{m\ge n_1} a_n \le v_1$ and $\sup_{m\ge n_2} b_n \le v_2$; so that

$$\begin{aligned}
\sup_{m\ge\max(n_1,n_2)} a_m + b_m &\le \sup_{m\ge\max(n_1,n_2)} a_m + \sup_{m\ge\max(n_1,n_2)} b_m\\
&\le \sup_{m\ge n_1} a_m + \sup_{m\ge n_2} b_m \le v_1 + v_2 = v.
\end{aligned}$$

As v is arbitrary,

$$\limsup\nolimits_{n\to\infty} a_n + b_n = \inf\nolimits_{n\in\mathbb{N}} \sup\nolimits_{m\ge n} a_m + b_m \le \limsup\nolimits_{n\to\infty} a_n + \limsup\nolimits_{n\to\infty} b_n.$$

(e) Putting (c) and (d) together,

$$\begin{aligned}
\liminf_{n\to\infty} a_n + b_n &= -\limsup_{n\to\infty}(-a_n) + (-b_n)\\
&\ge -\limsup_{n\to\infty}(-a_n) - \limsup_{n\to\infty}(-b_n) = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.
\end{aligned}$$

(f) Because $c \ge 0$,

$$\begin{aligned}
\limsup_{n\to\infty} ca_n &= \inf_{n\in\mathbb{N}} \sup_{m\ge n} ca_m = \inf_{n\in\mathbb{N}} c \sup_{m\ge n} a_m\\
&= c \inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m = c \limsup_{n\to\infty} a_n.
\end{aligned}$$

(g) Finally,

$$\liminf\nolimits_{n\to\infty} ca_n = -\limsup\nolimits_{n\to\infty} c(-a_n) = -c\limsup\nolimits_{n\to\infty}(-a_n) = c\liminf\nolimits_{n\to\infty} a_n.$$

1A3C Remark Of course the familiar results that $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$, $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$ are immediate corollaries of 1A3B.

***1A3D Other expressions of the same idea** The concepts of lim sup and lim inf may be applied in any context in which we can consider the limit of a real-valued function. For instance, if f is a real-valued function defined (at least) on a punctured interval of the form $\{x : 0 < |c - x| \le \epsilon\}$ where $c \in \mathbb{R}$ and $\epsilon > 0$, then

$$\limsup_{t\to c} f(t) = \lim_{\delta\downarrow 0} \sup_{0<|t-c|\le\delta} f(t) = \inf_{0<\delta\le\epsilon} \sup_{0<|t-c|\le\delta} f(t),$$

$$\liminf_{t\to c} f(t) = \lim_{\delta\downarrow 0} \inf_{0<|t-c|\le\delta} f(t) = \sup_{0<\delta\le\epsilon} \inf_{0<|t-c|\le\delta} f(t),$$

allowing ∞ and $-\infty$ whenever they seem called for. Or if f is defined on the half-open interval $]c, c+\epsilon]$, we can say

$$\limsup_{t\downarrow c} f(t) = \lim_{\delta\downarrow 0} \sup_{c<t\le c+\delta} f(t) = \inf_{0<\delta\le\epsilon} \sup_{c<t\le c+\delta} f(t),$$

$$\liminf_{t\downarrow c} f(t) = \lim_{\delta\downarrow 0} \inf_{c<t\le c+\delta} f(t) = \sup_{0<\delta\le\epsilon} \inf_{c<t\le c+\delta} f(t).$$

Similarly, if f is defined on $[M, \infty[$ for some $M \in \mathbb{R}$, we have

$$\limsup_{t\to\infty} f(t) = \lim_{a\to\infty} \sup_{t\ge a} f(t) = \inf_{a\ge M} \sup_{t\ge a} f(t),$$

$$\liminf_{t\to\infty} f(t) = \lim_{a\to\infty} \inf_{t\ge a} f(t) = \sup_{a\ge M} \inf_{t\ge a} f(t).$$

A further extension of the idea is examined briefly in 2A3S in Volume 2.

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References for Volume 1

In addition to those (very few) works which I have mentioned in the course of this volume, I list some of the books from which I myself learnt measure theory, as a mark of grateful respect, and to give you an opportunity to sample alternative approaches.

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