Version of 9.6.16

Chapter 13

Complements

In this chapter I collect a number of results which do not lie in the direct line of the argument from 111A (the definition of 'σ-algebra') to 123C (Lebesgue's Dominated Convergence Theorem), but which nevertheless demand inclusion in this volume, being both relatively elementary, essential for further developments and necessary to a proper comprehension of what has already been done. The longest section is §134, dealing with a few of the elementary special properties of Lebesgue measure; in particular, its translation-invariance, the existence of non-measurable sets and functions, and the Cantor set. The other sections are comparatively lightweight. §131 discusses (measurable) subspaces and the interpretation of the formula $\int_E f$, generalizing the idea of an integral $\int_a^b f$ of a function over an interval. §132 introduces the outer measure associated with a measure, a kind of inverse of Carathéodory's construction of a measure from an outer measure. §§133 and 135 lay out suitable conventions for dealing with 'infinity' and complex numbers (separately! they don't mix well) as values either of integrands or of integrals; at the same time I mention 'upper' and 'lower' integrals. Finally, in §136, I give some theorems on σ-algebras of sets, describing how they can (in some of the most important cases) be generated by relatively restricted operations.

Version of 18.3.05

131 Measurable subspaces

Very commonly we wish to integrate a function over a subset of a measure space; for instance, to form an integral $\int_a^b f(x)dx$, where $a < b$ in $\mathbb{R}$. As often as not, we wish to do this when the function is partly or wholly undefined outside the subset, as in such expressions as $\int_0^1 \ln x\,dx$. The natural framework in which to perform such operations is that of 'subspace measures'. If (X, Σ, μ) is a measure space and $H \in \Sigma$, there is a natural subspace measure μ_H on H, which I describe in this section. I begin with the definition of this subspace measure (131A-131C), with a description of integration with respect to it (131E-131H); this gives a solid foundation for the concept of 'integration over a (measurable) subset' (131D).

131A Proposition Let (X, Σ, μ) be a measure space, and $H \in \Sigma$. Set $\Sigma_H = \{E : E \in \Sigma,\ E \subseteq H\}$ and let μ_H be the restriction of μ to Σ_H. Then (H, Σ_H, μ_H) is a measure space.

proof Of course Σ_H is just $\{E \cap H : E \in \Sigma\}$, and I have noted already (in 121A) that this is a σ-algebra of subsets of H. It is now obvious that μ_H satisfies (iii) of 112A, so that (H, Σ_H, μ_H) is a measure space.

131B Definition If (X, Σ, μ) is any measure space and $H \in \Sigma$, then μ_H, as defined in 131A, is the **subspace measure** on H.

When $X = \mathbb{R}^r$, where $r \geq 1$, and μ is Lebesgue measure on $\mathbb{R}^r$, I will call a subspace measure μ_H **Lebesgue measure on** H.

It is worth noting the following elementary facts.

131C Lemma Let (X, Σ, μ) be a measure space, $H \in \Sigma$, and μ_H the subspace measure on H, with domain Σ_H. Then

(a) for any $A \subseteq H$, A is μ_H-negligible iff it is μ-negligible;

(b) if $G \in \Sigma_H$ then $(\mu_H)_G$, the subspace measure on G when G is regarded as a measurable subset of H, is identical to μ_G, the subspace measure on G when G is regarded as a measurable subset of X.

131D Integration over subsets: Definition Let (X,Σ,μ) be a measure space, $H\in\Sigma$ and f a real-valued function defined on a subset of X. By $\int_H f$ (or $\int_H f(x)\mu(dx)$, etc.) I shall mean $\int(f\restriction H)d\mu_H$, if this exists, following the definitions of 131A-131B and 122M, and taking $\operatorname{dom}(f\restriction H)=H\cap\operatorname{dom}f$, $(f\restriction H)(x)=f(x)$ for $x\in H\cap\operatorname{dom}f$.

131E Proposition Let (X,Σ,μ) be a measure space, $H\in\Sigma$, and f a real-valued function defined on a subset $\operatorname{dom}f$ of H. Set $\tilde f(x)=f(x)$ if $x\in\operatorname{dom}f$, 0 if $x\in X\setminus H$. Then $\int f d\mu_H=\int\tilde f d\mu$ if either is defined in $\mathbb{R}$.

proof (a) If f is μ_H-simple, it is expressible as $\sum_{i=0}^n a_i\chi E_i$, where $E_0,\dots,E_n\in\Sigma_H$, $a_0,\dots,a_n\in\mathbb{R}$ and $\mu_H E_i<\infty$ for each i. Now $\tilde f$ also is equal to $\sum_{i=0}^n a_i\chi E_i$ if this is now interpreted as a function from X to $\mathbb{R}$. So

$$\sum_{i=0}^n a_i\mu_H E_i=\sum_{i=0}^n a_i\mu E_i=\int\tilde f d\mu.$$

(b) If f is a non-negative μ_H-integrable function, there is a non-decreasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of non-negative μ_H-simple functions converging to f μ_H-almost everywhere; now $\langle\tilde f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of μ-simple functions converging to $\tilde f$ μ-a.e. (131Ca), and

$$\sup_{n\in\mathbb{N}}\int\tilde f_n d\mu=\sup_{n\in\mathbb{N}}\int f_n d\mu_H=\int f d\mu_H<\infty,$$

so $\int\tilde f d\mu$ exists and is equal to $\int f d\mu_H$.

(c) If f is μ_H-integrable, it is expressible as f_1-f_2 where f_1 and f_2 are non-negative μ_H-integrable functions, so that $\tilde f=\tilde f_1-\tilde f_2$ and

$$\int\tilde f d\mu=\int\tilde f_1 d\mu-\int\tilde f_2 d\mu=\int f_1 d\mu_H-\int f_2 d\mu_H=\int f d\mu_H.$$

(d) Now suppose that $\tilde f$ is μ-integrable. In this case there is a μ-conegligible $E\in\Sigma$ such that $E\subseteq\operatorname{dom}\tilde f$ and $\tilde f\restriction E$ is Σ-measurable (122P). Of course $\mu(H\setminus E)=0$ so $E\cap H$ is μ_H-conegligible; also, for any $a\in\mathbb{R}$,

$$\{x:x\in E\cap H,\ f(x)\ge a\}=H\cap\{x:x\in E,\ \tilde f(x)\ge a\}\in\Sigma_H,$$

so $f\restriction E\cap H$ is Σ_H-measurable, and f is μ_H-virtually measurable and defined μ_H-a.e. Next, for $\epsilon>0$,

$$\mu_H\{x:x\in E\cap H,\ |f(x)|\ge\epsilon\}=\mu\{x:x\in E,\ |\tilde f(x)|\ge\epsilon\}<\infty,$$

while if g is a μ_H-simple function and $g\le|f|$ μ_H-a.e. then $\tilde g\le|\tilde f|$ μ-a.e. and

$$\int g\,d\mu_H=\int\tilde g\,d\mu\le\int|\tilde f|d\mu<\infty.$$

By the criteria of 122J and 122P, f is μ_H-integrable, so that again we have $\int f d\mu_H=\int\tilde f d\mu$.

131F Corollary Let (X,Σ,μ) be a measure space and f a real-valued function defined on a subset $\operatorname{dom}f$ of X.

(a) If $H\in\Sigma$ and f is defined almost everywhere in X, then $f\restriction H$ is μ_H-integrable iff $f\times\chi H$ is μ-integrable, and in this case $\int_H f=\int f\times\chi H$.

(b) If f is μ-integrable, then $f\ge 0$ a.e. iff $\int_H f\ge 0$ for every $H\in\Sigma$.

(c) If f is μ-integrable, then $f=0$ a.e. iff $\int_H f=0$ for every $H\in\Sigma$.

proof (a) Because $\operatorname{dom}f$ is μ-conegligible, $(f\restriction H)^{\sim}$, as defined in 131E, is equal to $f\times\chi H$ μ-a.e., so that, by 131E,

$$\int_H f d\mu=\int(f\restriction H)^{\sim}d\mu=\int(f\times\chi H)d\mu$$

if any one of the integrals exists.

(b)(i) If $f\ge 0$ μ-a.e., then for any $H\in\Sigma$ we must have $f\restriction H\ge 0$ μ_H-a.e., so $\int_H f=\int(f\restriction H)d\mu_H\ge 0$.

(ii) If $\int_H f\ge 0$ for every $H\in\Sigma$, let $E\in\Sigma$ be a conegligible subset of $\operatorname{dom}f$ such that $f\restriction E$ is measurable. Set $F=\{x:x\in E,\ f(x)<0\}$. Then $\int_F f\ge 0$; by 122Rc, it follows that $f\restriction F=0$ μ_F-a.e., which is possible only if $\mu F=0$, in which case $f\ge 0$ μ-a.e.

(c) Apply (b) to f and to $-f$ to see that $f\le 0\le f$ a.e.

131G Corollary Let (X,Σ,μ) be a measure space and $H\in\Sigma$ a conegligible set. If f is any real-valued function defined on a subset of X, $\int_H f=\int f$ if either is defined.

proof In the language of 131E, $f=(f\restriction H)^{\sim}$ μ-almost everywhere, so that

$$\int f=\int (f\restriction H)^{\sim}=\int_H f$$

if any of the integrals is defined.

131H Corollary Let (X,Σ,μ) be a measure space and f, g two μ-integrable real-valued functions.
(a) If $\int_H f\ge\int_H g$ for every $H\in\Sigma$ then $f\ge g$ a.e.
(b) If $\int_H f=\int_H g$ for every $H\in\Sigma$ then $f=g$ a.e.

proof Apply 131Fb-131Fc to $f-g$.

131X Basic exercises >(a) Let (X,Σ,μ) be a measure space, and f a real-valued function which is integrable over X. For $E\in\Sigma$ set $\nu E=\int_E f$. (i) Show that if E, F are disjoint members of Σ then $\nu(E\cup F)=\nu E+\nu F$. (*Hint*: 131E.) (ii) Show that if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ then $\nu(\bigcup_{n\in\mathbb{N}}E_n)=\sum_{n=0}^{\infty}\nu E_n$. (*Hint*: 123C.) (iii) Show that if f is non-negative then (X,Σ,ν) is a measure space.

>(b) Let μ be Lebesgue measure on $\mathbb{R}$. (i) Show that whenever $a\le b$ in $\mathbb{R}$ and f is a real-valued function with $\operatorname{dom} f\subseteq\mathbb{R}$, then

$$f d\mu=\int_{[a,b[}f d\mu=\int_{]a,b]}f d\mu=\int_{[a,b]}f d\mu$$

if any of these is defined. (*Hint*: apply 131E to four different versions of $\tilde f$.) Write $\int_a^b f d\mu$ for the common value. (ii) Show that if $a\le b\le c$ in $\mathbb{R}$ then, for any real-valued function f, $\int_a^c f d\mu=\int_a^b f d\mu+\int_b^c f d\mu$ if either side is defined. (iii) Show that if f is integrable over $\mathbb{R}$, then $(a,b)\mapsto\int_a^b f d\mu$ is continuous. (*Hint*: *Either* consider simple functions f first *or* consider $\lim_{n\to\infty}\int_{a_n}^b f d\mu$ for monotonic sequences $\langle a_n\rangle_{n\in\mathbb{N}}$.)

(c) Let $g:\mathbb{R}\to\mathbb{R}$ be a non-decreasing function and μ_g the associated Lebesgue-Stieltjes measure (114Xa). (i) Show that if $a\le b\le c$ in $\mathbb{R}$ then, for any real-valued function f, $\int_{[a,c[}f d\mu_g=\int_{[a,b[}f d\mu_g+\int_{[b,c[}f d\mu_g$ if either side is defined. (ii) Show that if f is μ_g-integrable over $\mathbb{R}$, then $(a,b)\mapsto\int_{[a,b[}f d\mu_g$ is continuous on $\{(a,b):a\le b,\ g$ is continuous at both a and $b\}$.

131Y Further exercises (a) Let (X,Σ,μ) be a measure space and $E\in\Sigma$ a measurable set of finite measure. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable real-valued functions, with measurable domains[1], such that $f=\lim_{n\to\infty}f_n$ is defined almost everywhere in E (following the conventions of 121Fa). Show that for every $\epsilon>0$ there is a measurable set $F\subseteq E$ such that $\mu(E\setminus F)\le\epsilon$ and $\langle f_n\rangle_{n\in\mathbb{N}}$ converges uniformly to f on F. (This is **Egorov's theorem**.)

131 Notes and comments If you want a quick definition of $\int_H f$ for measurable H, the simplest seems to be that of 131E, which enables you to avoid the concept of 'subspace measure' entirely. I think however that we really do need to be able to speak of 'Lebesgue measure on $[0,1]$', for instance, meaning the subspace measure $\mu_{[0,1]}$ where μ is Lebesgue measure on $\mathbb{R}$.

This section has a certain amount of detailed technical analysis. The point is that from 131A on we generally have at least two measures in play, and the ordinary language of measure theory – words like 'measurable' and 'integrable' – becomes untrustworthy in such contexts, since it omits the crucial declarations of which σ-algebras or measures are under consideration. Consequently I have to use less elegant and more explicit terminology. I hope however that once you have worked carefully through such results as 131F you will feel that the pattern formed is reasonably coherent. The general rule is that for *measurable* subspaces there are no serious surprises (131Cb, 131Fa).

[1] I am grateful to P.Wallace Thompson for pointing out that this clause, or something with similar effect, is necessary.

I ought to remark that there is also a standard definition of subspace measure on *non-measurable* subsets of a measure space. I have given the definition already in 113Yb; for the theory of integration, extending the results above, I will wait until §214. There are significant extra difficulties and the extra generality is not often needed in elementary applications.

Let me call your attention to 131Fb-131Fc and 131Xa-131Xc; these are first steps to understanding 'indefinite integrals', the functionals $E \mapsto \int_E f : \Sigma \to \mathbb{R}$ where f is an integrable function. I will return to these in Chapters 22 and 23.

Version of 6.4.05

132 Outer measures from measures

The next topic I wish to mention is a simple construction with applications everywhere in measure theory. With any measure there is associated, in a straightforward way, a standard outer measure (132A-132B). If we start with Lebesgue measure we just return to Lebesgue outer measure (132C). I take the opportunity to introduce the idea of 'measurable envelope' (132D-132E).

132A Proposition Let (X, Σ, μ) be a measure space. Define $\mu^* : \mathcal{P}X \to [0, \infty]$ by writing

$$\mu^* A = \inf\{\mu E : E \in \Sigma,\ A \subseteq E\}$$

for every $A \subseteq X$. Then

(a) for every $A \subseteq X$ there is an $E \in \Sigma$ such that $A \subseteq E$ and $\mu^* A = \mu E$;

(b) μ^* is an outer measure on X;

(c) $\mu^* E = \mu E$ for every $E \in \Sigma$;

(d) a set $A \subseteq X$ is μ-negligible iff $\mu^* A = 0$;

(e) $\mu^*(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{n\to\infty} \mu^* A_n$ for every non-decreasing sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of subsets of X;

(f) $\mu^* A = \mu^*(A \cap F) + \mu^*(A \setminus F)$ whenever $A \subseteq X$ and $F \in \Sigma$.

proof (a) For each $n \in \mathbb{N}$ we can choose an $E_n \in \Sigma$ such that $A \subseteq E_n$ and $\mu E_n \le \mu^* A + 2^{-n}$; now $E = \bigcap_{n\in\mathbb{N}} E_n \in \Sigma$, $A \subseteq E$ and

$$\mu^* A \le \mu E \le \inf_{n\in\mathbb{N}} \mu E_n \le \mu^* A.$$

(b)(i) $\mu^*\emptyset = \mu\emptyset = 0$. (ii) If $A \subseteq B \subseteq X$ then $\{E : A \subseteq E \in \Sigma\} \supseteq \{E : B \subseteq E \in \Sigma\}$ so $\mu^* A \le \mu^* B$. (iii) If $\langle A_n \rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{P}X$, then for each $n \in \mathbb{N}$ there is an $E_n \in \Sigma$ such that $A_n \subseteq E_n$ and $\mu E_n = \mu^* A_n$; now $\bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n\in\mathbb{N}} E_n \in \Sigma$ so

$$\mu^*(\bigcup_{n\in\mathbb{N}} A_n) \le \mu(\bigcup_{n\in\mathbb{N}} E_n) \le \sum_{n=0}^{\infty} \mu E_n = \sum_{n=0}^{\infty} \mu^* A_n.$$

(c) This is just because $\mu E \le \mu F$ whenever E, $F \in \Sigma$ and $E \subseteq F$.

(d) By (a), $\mu^* A = 0$ iff there is an $E \in \Sigma$ such that $A \subseteq E$ and $\mu E = 0$; but this is the definition of 'negligible set'.

(e) Of course $\langle \mu^* A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit at most $\mu^* A$, writing $A = \bigcup_{n\in\mathbb{N}} A_n$, just because $\mu^* B \le \mu^* C$ whenever $B \subseteq C \subseteq X$. For each $n \in \mathbb{N}$, let $E_n \in \Sigma$ be such that $A_n \subseteq E_n$ and $\mu E_n = \mu^* A_n$. Set $F_n = \bigcap_{m\ge n} E_m$ for each n; then $\langle F_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ, and $A_n \subseteq F_n \subseteq E_n$, so $\mu^* A_n = \mu F_n$ for each $n \in \mathbb{N}$. Set $F = \bigcup_{n\in\mathbb{N}} F_n$; then $A \subseteq F$ so

$$\mu^* A \le \mu F = \lim_{n\to\infty} \mu F_n = \lim_{n\to\infty} \mu^* A_n.$$

Thus $\mu^* A = \lim_{n\to\infty} \mu^* A_n$, as claimed.

(f) Of course $\mu^* A \le \mu^*(A \cap F) + \mu^*(A \setminus F)$, by (b). On the other hand, there is an $E \in \Sigma$ such that $A \subseteq E$ and $\mu E = \mu^* A$, by (a), and now $A \cap F \subseteq E \cap F \in \Sigma$, $A \setminus F \subseteq E \setminus F \in \Sigma$ so

$$\mu^*(A \cap F) + \mu^*(A \setminus F) \le \mu(E \cap F) + \mu(E \setminus F) = \mu E = \mu^* A.$$

132B Definition If (X, Σ, μ) is a measure space, I will call μ^*, as defined in 132A, **the outer measure defined from** μ.

Remark If we start with an outer measure θ on a set X, construct a measure μ from θ by Carathéodory's method, and then construct the outer measure μ^* from μ, it is not necessarily the case that $\mu^* = \theta$. **P** Take any set X with at least three members, and set $\theta A = 0$ if $A = \emptyset$, 1 if $A = X$, $\frac{1}{2}$ otherwise. Then $\operatorname{dom}\mu = \{\emptyset, X\}$ and $\mu^* A = 1$ for every non-empty $A \subseteq X$. **Q**

However, this problem does not arise with Lebesgue outer measure. I state the next proposition in terms of Lebesgue measure on $\mathbb{R}^r$, but if you skipped §115 I hope that you will still be able to make sense of this, and later results, in terms of Lebesgue measure on $\mathbb{R}$, by setting $r = 1$.

132C Proposition If θ is Lebesgue outer measure on $\mathbb{R}^r$ and μ is Lebesgue measure, then μ^*, as defined in 132A, is equal to θ.

proof Let $A \subseteq \mathbb{R}^r$.

(a) If E is measurable and $A \subseteq E$, then $\theta A \le \theta E = \mu E$; so $\theta A \le \mu^* A$.

(b) If $\epsilon > 0$, there is a sequence $\langle I_n \rangle_{n\in\mathbb{N}}$ of half-open intervals, covering A, with $\sum_{n=0}^\infty \mu I_n \le \theta A + \epsilon$ (using 114G/115G to identify μI_n with the volume λI_n used in the definition of θ), so

$$\mu^* A \le \mu(\bigcup_{n\in\mathbb{N}} I_n) \le \sum_{n=0}^\infty \mu I_n \le \theta A + \epsilon.$$

As ϵ is arbitrary, $\mu^* A \le \theta A$.

Remark Accordingly it will henceforth be unnecessary to distinguish θ from μ^* when speaking of 'Lebesgue outer measure'. (In the language of 132Xa below, Lebesgue outer measure is 'regular'.) In particular (using 132Aa), if $A \subseteq \mathbb{R}^r$ there is a measurable set $E \supseteq A$ such that $\mu E = \theta A$ (compare 134Fc).

132D Measurable envelopes The following is a useful concept in this context. If (X, Σ, μ) is a measure space and $A \subseteq X$, a **measurable envelope** (or **measurable cover**) of A is a set $E \in \Sigma$ such that $A \subseteq E$ and $\mu(F \cap E) = \mu^*(F \cap A)$ for every $F \in \Sigma$. In general, not every set in a measure space has a measurable envelope (I suggest examples in 216Yc in Volume 2). But we do have the following.

132E Lemma Let (X, Σ, μ) be a measure space.

(a) If $A \subseteq E \in \Sigma$, then E is a measurable envelope of A iff $\mu F = 0$ whenever $F \in \Sigma$ and $F \subseteq E \setminus A$.

(b) If $A \subseteq E \in \Sigma$ and $\mu E < \infty$ then E is a measurable envelope of A iff $\mu E = \mu^* A$.

(c) If E is a measurable envelope of A and $H \in \Sigma$, then $E \cap H$ is a measurable envelope of $A \cap H$.

(d) Let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a sequence of subsets of X. Suppose that each A_n has a measurable envelope E_n. Then $\bigcup_{n\in\mathbb{N}} E_n$ is a measurable envelope of $\bigcup_{n\in\mathbb{N}} A_n$.

(e) If $A \subseteq X$ can be covered by a sequence of sets of finite measure, then A has a measurable envelope.

(f) In particular, if μ is Lebesgue measure on $\mathbb{R}^r$, then every subset of $\mathbb{R}^r$ has a measurable envelope for μ.

proof (a) If E is a measurable envelope of A, $F \in \Sigma$ and $F \subseteq E \setminus A$, then

$$\mu F = \mu(F \cap E) = \mu^*(F \cap A) = 0.$$

If E is not a measurable envelope of A, there is an $H \in \Sigma$ such that $\mu^*(A \cap H) < \mu(E \cap H)$. Let $G \in \Sigma$ be such that $A \cap H \subseteq G$ and $\mu G = \mu^*(A \cap H)$, and set $F = E \cap H \setminus G$. Since $\mu G < \mu(E \cap H)$, $\mu F > 0$; but also $F \subseteq E$ and $F \cap A \subseteq H \cap A \setminus G$ is empty.

(b) If E is a measurable envelope of A then we must have

$$\mu^* A = \mu^*(A \cap E) = \mu(E \cap E) = \mu E.$$

If $\mu E = \mu^* A$, and $F \in \Sigma$ is a subset of $E \setminus A$, then $A \subseteq E \setminus F$, so $\mu(E \setminus F) = \mu E$; because μE is finite, it follows that $\mu F = 0$, so the condition of (a) is satisfied and E is a measurable envelope of A.

(c) If $F \in \Sigma$ and $F \subseteq E \cap H \setminus A$, then $F \subseteq E \setminus A$, so $\mu F = 0$, by (a); as F is arbitrary, $E \cap H$ is a measurable envelope of $A \cap H$, by (a) again.

(d) Write A for $\bigcup_{n\in\mathbb{N}} A_n$ and E for $\bigcup_{n\in\mathbb{N}} E_n$. Then $A \subseteq E$. If $F \in \Sigma$ and $F \subseteq E \setminus A$, then, for every $n \in \mathbb{N}$, $F \cap E_n \subseteq E_n \setminus A_n$, so $\mu(F \cap E_n) = 0$, by (a). Consequently $F = \bigcup_{n\in\mathbb{N}} F \cap E_n$ is negligible; as F is arbitrary, E is a measurable envelope of A.

(e) Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of finite measure covering A. For each $n\in\mathbb{N}$, let $E_n\in\Sigma$ be such that $A\cap F_n\subseteq E_n$ and $\mu E_n=\mu^*(A\cap F_n)$ (using 132Aa above); by (b), E_n is a measurable envelope of $A\cap F_n$. By (d), $\bigcup_{n\in\mathbb{N}}E_n$ is a measurable envelope of $\bigcup_{n\in\mathbb{N}}A\cap F_n=A$.

(f) In the case of Lebesgue measure on $\mathbb{R}^r$, of course, the same sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ will work for every A, if we take B_n to be the half-open interval $[-\mathbf{n},\mathbf{n}[$ for each $n\in\mathbb{N}$, writing $\mathbf{n}=(n,\ldots,n)$ as in §115.

132F Full outer measure This is a convenient moment at which to introduce the following term. If (X,Σ,μ) is a measure space, a set $A\subseteq X$ is **of full outer measure** or **thick** if X is a measurable envelope of A; that is, if $\mu^*(F\cap A)=\mu F$ for every $F\in\Sigma$; equivalently, if $\mu F=0$ whenever $F\in\Sigma$ and $F\subseteq X\setminus A$. If $\mu X<\infty$, $A\subseteq X$ has full outer measure iff $\mu^*A=\mu X$.

132X Basic exercises >(a) Let X be a set and θ an outer measure on X; let μ be the measure on X defined by Carathéodory's method from θ, and μ^* the outer measure defined from μ by the construction of 132A. (i) Show that $\mu^*A\geq\theta A$ for every $A\subseteq X$. (ii) θ is said to be a **regular outer measure** if $\theta=\mu^*$. Show that if there is any measure ν on X such that $\theta=\nu^*$ then θ is regular. (iii) Show that if θ is regular and $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X, then $\theta(\bigcup_{n\in\mathbb{N}}A_n)=\lim_{n\to\infty}\theta A_n$.

(b) Let (X,Σ,μ) be a measure space and H any member of Σ. Let μ_H be the subspace measure on H (131B) and μ^*, μ_H^* the outer measures defined from μ, μ_H respectively. Show that $\mu_H^*=\mu^*\restriction\mathcal{P}H$.

(c) Give an example of a measure space (X,Σ,μ) such that the measure $\check{\mu}$ defined by Carathéodory's method from the outer measure μ^* is a proper extension of μ. (*Hint*: take $\mu X=0$.)

>(d) Let (X,Σ,μ) be a measure space and A a subset of X. Suppose that $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ such that $\langle A\cap E_n\rangle_{n\in\mathbb{N}}$ is disjoint. Show that $\mu^*(A\cap\bigcup_{n\in\mathbb{N}}E_n)=\sum_{n=0}^\infty\mu^*(A\cap E_n)$. (*Hint*: replace E_n by $E_n'=E_n\setminus\bigcup_{i<n}E_i$, and use 132Ae-132Af.)

(e) Let (X,Σ,μ) be a measure space and $\langle A_n\rangle_{n\in\mathbb{N}}$ any sequence of subsets of X. Show that the outer measure of $\bigcup_{n\in\mathbb{N}}\bigcap_{i\geq n}A_i$ is at most $\liminf_{n\to\infty}\mu^*A_n$.

(f) Let (X,Σ,μ) be a measure space and suppose that $A\subseteq B\subseteq X$ are such that $\mu^*A=\mu^*B<\infty$. Show that $\mu^*(A\cap E)=\mu^*(B\cap E)$ for every $E\in\Sigma$. (*Hint*: a measurable envelope of B is a measurable envelope of A.)

>(g) Let ν_g be a Lebesgue-Stieltjes measure on $\mathbb{R}$, constructed as in 114Xa from a non-decreasing function $g:\mathbb{R}\to\mathbb{R}$. Show that (i) the outer measure ν_g^* derived from ν_g (132A) coincides with the outer measure θ_g of 114Xa; (ii) if $A\subseteq\mathbb{R}$ is any set, then A has a measurable envelope for the measure ν_g.

>(h) Let $A\subseteq\mathbb{R}^r$ be a set which is not measured by Lebesgue measure μ. Show that there is a bounded measurable set E such that $\mu^*(E\cap A)=\mu^*(E\setminus A)=\mu E>0$. (*Hint*: take $E=E'\cap E''\cap B$, where E' is a measurable envelope for A, E'' is a measurable envelope for $\mathbb{R}^r\setminus A$, and B is a suitable bounded set.)

(i) Let μ be Lebesgue measure on $\mathbb{R}^r$ and Σ its domain, and f a real-valued function, defined on a subset of $\mathbb{R}^r$, which is not Σ-measurable. Show that there are $q<q'$ in $\mathbb{Q}$ and a bounded measurable set E such that

$$\mu^*\{x:x\in E\cap\operatorname{dom}f,\ f(x)\leq q\}=\mu^*\{x:x\in E\cap\operatorname{dom}f,\ f(x)\geq q'\}=\mu E>0.$$

(*Hint*: take E_q, E_q' to be measurable envelopes for $\{x:f(x)\leq q\}$, $\{x:f(x)>q\}$ for each q. Find q such that $\mu(E_q\cap E_q')>0$ and q' such that $\mu(E_q\cap E_{q'}')>0$.)

(j) Check that you can do exercise 113Yc.

(k) Let (X,Σ,μ) be a measure space and μ^* the outer measure defined from μ. Show that $\mu^*(A\cup B)+\mu^*(A\cap B)\leq\mu^*A+\mu^*B$ for all A, $B\subseteq X$.

132Y Further exercises (a) Let (X, Σ, μ) be a measure space and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of real-valued functions defined almost everywhere in X. Suppose that $\langle \epsilon_n \rangle_{n \in \mathbb{N}}$ is a sequence of non-negative real numbers such that

$$\sum_{n=0}^{\infty} \epsilon_n < \infty, \quad \sum_{n=0}^{\infty} \mu^*\{x : |f_{n+1}(x) - f_n(x)| \ge \epsilon_n\} < \infty.$$

Show that $\lim_{n\to\infty} f_n$ is defined (as a real-valued function) almost everywhere.

(b) Let (X, Σ, μ) be a measure space, Y a set and $f : X \to Y$ a function. Let ν be the image measure μf^{-1} (112Xf). Show that $\nu^* f[A] \ge \mu^* A$ for every $A \subseteq X$.

(c) Let (X, Σ, μ) be a measure space with $\mu X < \infty$. Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of subsets of X such that $\bigcup_{n \in \mathbb{N}} A_n$ has full outer measure in X. Show that there is a partition $\langle E_n \rangle_{n \in \mathbb{N}}$ of X into measurable sets such that $\mu E_n = \mu^*(A_n \cap E_n)$ for every $n \in \mathbb{N}$.

(d) Let (X, Σ, μ) be a measure space and $\mathcal{A}$ a family of subsets of X such that $\bigcap_{n \in \mathbb{N}} A_n$ has full outer measure for every sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{A}$. Show that there is a measure ν on X, extending μ, such that every member of $\mathcal{A}$ is ν-conegligible.

(e) Check that you can do exercises 113Yg-113Yh.

(f) Let (X, Σ, μ) be a measure space. Show that $\mu^* : \mathcal{P}X \to [0, \infty]$ is **alternating of all orders**, that is,

$$\sum_{J \subseteq I, \#(J) \text{ is even}} \mu^*(A \cup \bigcup_{i \in J} A_i) \le \sum_{J \subseteq I, \#(J) \text{ is odd}} \mu^*(A \cup \bigcup_{i \in J} A_i)$$

whenever I is a non-empty finite set, $\langle A_i \rangle_{i \in I}$ is a family of subsets of X and A is another subset of X.

(g) Let (X, Σ, μ) be a measure space. Suppose that $A \subseteq B \subseteq C \subseteq X$ and that $\mu^*(B \setminus A) = \mu^* B$. Show that $\mu^*(C \setminus A) = \mu^* C$.

132 Notes and comments Almost the most fundamental fact in measure theory is that in all important measure spaces there are non-measurable sets. (For Lebesgue measure see 134B below.) One can respond to this fact in a variety of ways. An approach which works quite well is just to ignore it. The point is that, for very deep reasons, the sets and functions which arise in ordinary applications nearly always are measurable, or can be made so by elementary manipulations; the only exceptions I know of in applied mathematics appear in generalized control theory. As a pure mathematician I am uncomfortable with such an approach, and as a measure theorist I think it closes the door on some of the most subtle ideas of the theory. In this treatise, therefore, non-measurable sets will always be present, if only subliminally. In this section I have described two of the basic methods of dealing with them: the move from a measure to an outer measure, which at least assigns some sort of size to an arbitrary set, and the idea of 'measurable envelope', which (when defined) describes the region in which the non-measurable set has to be taken into account. In both cases we seek to describe the non-measurable set from the outside, so to speak. There are no real difficulties, and the only points to take note of are that (i) outside the boundary marked by 132Ee measurable envelopes need not exist (ii) Carathéodory's construction of a measure from an outer measure, and the construction here of an outer measure from a measure, are closely related (132C, 132Xg, 113Yc, 132Xa(i)), but are not quite inverses of each other in general (132B, 132Xc).

Version of 29.3.10

133 Wider concepts of integration

There are various contexts in which it is useful to be able to assign a value to the integral of a function which is not quite covered by the basic definition in 122M. In this section I offer suggestions concerning the assignment of the values $\pm\infty$ to integrals of real-valued functions (133A), the integration of complex-valued functions (133C-133H) and upper and lower integrals (133I-133L). In §135 below I will discuss a further elaboration of the ideas of Chapter 12.

133A Infinite integrals It is normal to restrict the phrase 'f is integrable' to functions f to which a finite integral $\int f$ can be assigned (just as a series is called 'summable' only when a finite sum can be assigned to it). But for non-negative functions it is sometimes convenient to write '$\int f=\infty$' if, in some sense, the only way in which f fails to be integrable is that the integral is too large; that is, f is defined almost everywhere, is μ-virtually measurable, and either

$$\{x : x \in \operatorname{dom} f,\ f(x) \ge \epsilon\}$$

includes a set of infinite measure for some $\epsilon > 0$, or

$$\sup\{\int h : h \text{ is simple, } h \le_{\text{a.e.}} f\} = \infty.$$

(Compare 122J.) Under this rule, we shall still have

$$\int f_1 + f_2 = \int f_1 + \int f_2, \quad \int cf = c\int f$$

whenever $c \in [0,\infty[$ and f_1, f_2, f are non-negative functions for which $\int f_1$, $\int f_2$, $\int f$ are defined in $[0,\infty]$.

We can therefore repeat the definition 122M and say that

$$\int f_1 - f_2 = \int f_1 - \int f_2$$

whenever f_1, f_2 are real-valued functions such that $\int f_1$, $\int f_2$ are defined in $[0,\infty]$ and are not both infinite; the last condition being imposed to avoid the possibility of being asked to calculate $\infty-\infty$.

We still have the rules that

$$\int f + g = \int f + \int g, \quad \int (cf) = c\int f, \quad \int |f| \ge |\int f|$$

at least when the right-hand-sides can be interpreted, allowing $0\cdot\infty = 0$, but not allowing any interpretation of $\infty-\infty$; and $\int f \le \int g$ whenever both integrals are defined and $f \le_{\text{a.e.}} g$. (But of course it is now possible to have $f \le g$ and $\int f = \int g = \pm\infty$ without f and g being equal almost everywhere.)

Setting $f^+(x) = \max(f(x),0)$, $f^-(x) = \max(-f(x),0)$ for $x \in \operatorname{dom} f$, then

$$\int f = \infty \iff \int f^+ = \infty \text{ and } f^- \text{ is integrable,}$$

$$\int f = -\infty \iff f^+ \text{ is integrable and } \int f^- = \infty.$$

For further ideas in this direction, see §135 below.

133B Functions with exceptional values It is also convenient to allow as 'integrable' functions f which take occasional values which are not real – typically, where a formula for $f(x)$ allows the value '∞' on some convention. For such a function I will write $\int f = \int \tilde f$ if $\int \tilde f$ is defined, where

$$\operatorname{dom} \tilde f = \{x : x \in \operatorname{dom} f,\ f(x) \in \mathbb{R}\}, \quad \tilde f(x) = f(x) \text{ for } x \in \operatorname{dom} \tilde f.$$

Since in this convention I still require $\tilde f$ to be defined almost everywhere in X, the set $\{x : x \in \operatorname{dom} f,\ f(x) \notin \mathbb{R}\}$ will have to be negligible.

133C Complex-valued functions All the theory of measurable and integrable functions so far developed has been devoted to real-valued functions. There are no substantial new ideas required to deal with complex-valued functions, but perhaps I should spell out some of the details, since there are many applications in which complex-valued functions are the most natural context in which to work.

133D Definitions (a) Let X be a set and Σ a σ-algebra of subsets of X. If $D \subseteq X$ and $f : D \to \mathbb{C}$ is a function, then we say that f is **measurable** if its real and imaginary parts $\mathcal{R}\mathrm{e}\, f$, $\mathcal{I}\mathrm{m}\, f$ are measurable in the sense of 121B-121C.

(b) Let (X,Σ,μ) be a measure space. If f is a complex-valued function defined on a conegligible subset of X, we say that f is **integrable** if its real and imaginary parts are integrable, and then

$$\int f = \int \mathcal{R}\mathrm{e}\, f + i\int \mathcal{I}\mathrm{m}\, f.$$

(c) Let (X,Σ,μ) be a measure space, $H \in \Sigma$ and f a complex-valued function defined on a subset of X. Then $\int_H f$ is $\int (f\restriction H) d\mu_H$ if this is defined in the sense of (b), taking the subspace measure μ_H to be that of 131A-131B.

133E Lemma (a) If X is a set, Σ is a σ-algebra of subsets of X, and f and g are measurable complex-valued functions with domains $\operatorname{dom} f$, $\operatorname{dom} g \subseteq X$, then

(i) $f+g : \operatorname{dom} f \cap \operatorname{dom} g \to \mathbb{C}$ is measurable;

(ii) $cf : \operatorname{dom} f \to \mathbb{C}$ is measurable, for every $c \in \mathbb{C}$;

(iii) $f \times g : \operatorname{dom} f \cap \operatorname{dom} g \to \mathbb{C}$ is measurable;

(iv) $f/g : \{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ g(x) \neq 0\} \to \mathbb{C}$ is measurable;

(v) $|f| : \operatorname{dom} f \to \mathbb{R}$ is measurable.

(b) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of measurable complex-valued functions defined on subsets of X, then $f = \lim_{n\to\infty} f_n$ is measurable, if we take $\operatorname{dom} f$ to be

$$\begin{aligned}\{x : x \in \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \operatorname{dom} f_m,\ \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{C}\}\\ = \operatorname{dom}(\lim_{n\to\infty} \mathcal{R}\mathrm{e}\, f_n) \cap \operatorname{dom}(\lim_{n\to\infty} \mathcal{I}\mathrm{m}\, f_n).\end{aligned}$$

proof (a) All are immediate from 121E, if you write down the formulae for the real and imaginary parts of $f+g, \dots, |f|$ in terms of the real and imaginary parts of f and g.

(b) Use 121Fa.

133F Proposition Let (X, Σ, μ) be a measure space.

(a) If f and g are integrable complex-valued functions defined on conegligible subsets of X, then $f+g$ and cf are integrable, $\int f+g = \int f + \int g$ and $\int cf = c\int f$, for every $c \in \mathbb{C}$.

(b) If f is a complex-valued function defined on a conegligible subset of X, then f is integrable iff $|f|$ is integrable and f is μ-virtually measurable, that is, $\mathcal{R}\mathrm{e}\, f$ and $\mathcal{I}\mathrm{m}\, f$ are μ-virtually measurable.

proof (a) Use 122Oa-122Ob.

(b) The point is that $|\mathcal{R}\mathrm{e}\, f|$, $|\mathcal{I}\mathrm{m}\, f| \leq |f| \leq |\mathcal{R}\mathrm{e}\, f| + |\mathcal{I}\mathrm{m}\, f|$; now we need only apply 122P an adequate number of times.

133G Lebesgue's Dominated Convergence Theorem Let (X, Σ, μ) be a measure space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of integrable complex-valued functions on X such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists in $\mathbb{C}$ for almost every $x \in X$. Suppose moreover that there is a real-valued integrable function g on X such that $|f_n| \leq_{\text{a.e.}} g$ for each n. Then f is integrable and $\lim_{n\to\infty} \int f_n$ exists and is equal to $\int f$.

proof Apply 123C to the sequences $\langle \mathcal{R}\mathrm{e}\, f_n \rangle_{n\in\mathbb{N}}$ and $\langle \mathcal{I}\mathrm{m}\, f_n \rangle_{n\in\mathbb{N}}$.

133H Corollary Let (X, Σ, μ) be a measure space and $]a, b[$ a non-empty open interval in $\mathbb{R}$. Let $f : X \times]a, b[\to \mathbb{C}$ be a function such that

(i) the integral $F(t) = \int f(x,t)dx$ is defined for every $t \in]a, b[$;

(ii) the partial derivative $\frac{\partial f}{\partial t}$ of f with respect to the second variable is defined everywhere in $X \times]a, b[$;

(iii) there is an integrable function $g : X \to [0, \infty[$ such that $|\frac{\partial f}{\partial t}(x,t)| \leq g(x)$ for every $x \in X$, $t \in]a, b[$.

Then the derivative $F'(t)$ and the integral $\int \frac{\partial f}{\partial t}(x,t)dx$ exist for every $t \in]a, b[$, and are equal.

proof Apply 123D to $\mathcal{R}\mathrm{e}\, f$ and $\mathcal{I}\mathrm{m}\, f$.

133I Upper and lower integrals I return now to real-valued functions. Let (X, Σ, μ) be a measure space and f a real-valued function defined almost everywhere in X. Its **upper integral** is

$$\overline{\int} f = \inf\{\int g : \int g \text{ is defined in the sense of 133A and } f \leq_{\text{a.e.}} g\},$$

allowing ∞ for $\inf\{\infty\}$ or $\inf\emptyset$ and $-\infty$ for $\inf\mathbb{R}$. Similarly, the **lower integral** of f is

$$\underline{\int} f = \sup\{\int g : \int g \text{ is defined, } f \geq_{\text{a.e.}} g\},$$

allowing $-\infty$ for $\sup\{-\infty\}$ or $\sup\emptyset$ and ∞ for $\sup\mathbb{R}$.

133J Proposition Let (X, Σ, μ) be a measure space.

(a) Let f be a real-valued function defined almost everywhere in X.

(i) If $\overline{\int} f$ is finite, then there is an integrable g such that $f \leq_{\text{a.e.}} g$ and $\int g = \overline{\int} f$. In this case,

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ g(x) \leq f(x) + g_0(x)\}$$

has full outer measure for every measurable function $g_0 : X \to]0, \infty[$.

(ii) If $\underline{\int} f$ is finite, then there is an integrable h such that $h \leq_{\text{a.e.}} f$ and $\int h = \underline{\int} f$. In this case,

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} h,\ f(x) \leq h(x) + h_0(x)\}$$

has full outer measure for every measurable function $h_0 : X \to]0, \infty[$.

(b) For any real-valued functions f, g defined on conegligible subsets of X and any $c \geq 0$,

(i) $\underline{\int} f \leq \overline{\int} f$,

(ii) $\overline{\int} f + g \leq \overline{\int} f + \overline{\int} g$,

(iii) $\overline{\int} cf = c\overline{\int} f$,

(iv) $\underline{\int}(-f) = -\overline{\int} f$,

(v) $\underline{\int} f + g \geq \underline{\int} f + \underline{\int} g$,

(vi) $\underline{\int} cf = c\underline{\int} f$

whenever the right-hand-sides do not involve adding ∞ to $-\infty$.

(c) If $f \leq_{\text{a.e.}} g$ then $\overline{\int} f \leq \overline{\int} g$ and $\underline{\int} f \leq \underline{\int} g$.

(d) A real-valued function f defined almost everywhere in X is integrable iff

$$\overline{\int} f = \underline{\int} f = a \in \mathbb{R},$$

and in this case $\int f = a$.

(e) $\mu^* A = \overline{\int} \chi A$ for every $A \subseteq X$.

proof (a)(i) For each $n \in \mathbb{N}$, choose a function g_n such that $f \leq_{\text{a.e.}} g_n$ and $\int g_n$ is defined and at most $2^{-n} + \overline{\int} f$; as $\overline{\int} f \leq \int g_n$, $\int g_n$ is finite, so g_n is integrable. Set $h_n = \inf_{i \leq n} g_i$ for each n; then h_n is integrable (because $|h_n - g_0| \leq \sum_{i=0}^{n} |g_i - g_0|$ on $\bigcap_{i \leq n} \operatorname{dom} g_i$), and $f \leq_{\text{a.e.}} h_n$, so

$$\overline{\int} f \leq \int h_n \leq \int g_n \leq 2^{-n} + \overline{\int} f.$$

By B.Levi's theorem (123A), applied to $\langle -h_n \rangle_{n \in \mathbb{N}}$, $g(x) = \inf_{n \in \mathbb{N}} h_n(x) \in \mathbb{R}$ for almost every x, and $\int g = \inf_{n \in \mathbb{N}} \int h_n = \overline{\int} f$; also, of course, $f \leq_{\text{a.e.}} g$.

Now take a measurable function $g_0 : X \to]0, \infty[$, and consider the set

$$A = \{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ g(x) \leq f(x) + g_0(x)\}.$$

? If A does not have full outer measure, there is a non-negligible measurable set $F \subseteq X \setminus A$. Since g_0 is strictly positive, $F = \bigcup_{n \in \mathbb{N}} F_n$ where $F_n = \{x : x \in F,\ g_0(x) \geq 2^{-n}\}$, and there is an $n \in \mathbb{N}$ such that $\mu F_n > 0$. Consider the function $g_1 = g - 2^{-n} \chi F$. Then $f \leq_{\text{a.e.}} g_1$. Also $\int g_1 = \int g - 2^{-n} \mu F_n$ is strictly less than $\int g$, so $\overline{\int} f < \int g$. **X**

(ii) Argue similarly, or use (b-iv).

(b)(i) If either $\underline{\int} f = -\infty$ or $\overline{\int} f = \infty$ this is trivial. Otherwise it follows at once from the fact that if $g \leq_{\text{a.e.}} f \leq_{\text{a.e.}} h$ then $\int g \leq \int h$ if the integrals are defined (in the wide sense).

(ii) If $a > \overline{\int} f + \overline{\int} g$, neither $\overline{\int} f$ nor $\overline{\int} g$ can be ∞, so there must be functions f_1, g_1 such that $f \leq_{\text{a.e.}} f_1$, $g \leq_{\text{a.e.}} g_1$ and $\int f_1 + \int g_1 \leq a$. Now $f + g \leq_{\text{a.e.}} f_1 + g_1$, so

$$\overline{\int} f + g \leq \int f_1 + g_1 \leq a.$$

As a is arbitrary, we have the result.

(iii)(α) If $c = 0$ this is trivial. **(β)** If $c > 0$ and $a > c\overline{\int} f$, there must be an f_1 such that $f \le_{\text{a.e.}} f_1$ and $c\int f_1 \le a$. Now $cf \le_{\text{a.e.}} cf_1$ and $\int cf_1 \le a$, so $\overline{\int} cf \le a$. As a is arbitrary, $\overline{\int} cf \le c\overline{\int} f$. **($\gamma$)** Still supposing that $c > 0$, we also have

$$c\overline{\int} f = c\overline{\int} c^{-1}cf \le cc^{-1}\overline{\int} cf = \overline{\int} cf,$$

so we get equality.

(iv) This is just because $\int(-f_1) = -\int f_1$ for any function f_1 for which either integral is defined.

(v)-(vi) Use (iv) to turn $\underline{\int}$ into $\overline{\int}$, and apply (ii) or (iii).

(c) These are immediate from the definitions, because (for instance) if $g \le_{\text{a.e.}} h$ then $f \le_{\text{a.e.}} h$.

(d) If f is integrable, then

$$\overline{\int} f = \int f = \underline{\int} f$$

by 122Od. If $\overline{\int} f = \underline{\int} f = a \in \mathbb{R}$, then, by (a), there are integrable g, h such that $g \le_{\text{a.e.}} f \le_{\text{a.e.}} h$ and $\int g = \int h = a$, so that $g =_{\text{a.e.}} h$, by 122Rc, $g =_{\text{a.e.}} f =_{\text{a.e.}} h$ and f is integrable, by 122Rb.

(e) If $E \supseteq A$ is measurable, then

$$\mu E = \int \chi E \ge \overline{\int} \chi A;$$

as E is arbitrary, $\mu^* A \ge \overline{\int} \chi A$. If $\int g$ is defined and $\chi A \le_{\text{a.e.}} g$, let $E \subseteq \operatorname{dom} g$ be a conegligible measurable set such that $g\restriction E$ is measurable, and set $F = \{x : x \in E,\ g(x) \ge 1\}$. Then $A \setminus F$ is negligible, so $\mu^* A \le \mu F \le \int g$; as g is arbitrary, $\mu^* A \le \overline{\int} \chi A$.

Remark I hope that the formulae here remind you of lim sup, lim inf.

133K Convergence theorems for upper integrals We have the following versions of B.Levi's theorem and Fatou's Lemma.

Proposition Let (X, Σ, μ) be a measure space, and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions defined almost everywhere in X.

(a) If, for each n, $f_n \le_{\text{a.e.}} f_{n+1}$, and $-\infty < \sup_{n\in\mathbb{N}} \overline{\int} f_n < \infty$, then $f(x) = \sup_{n\in\mathbb{N}} f_n(x)$ is defined in $\mathbb{R}$ for almost every $x \in X$, and $\overline{\int} f = \sup_{n\in\mathbb{N}} \overline{\int} f_n$.

(b) If, for each n, $f_n \ge 0$ a.e., and $\liminf_{n\to\infty} \overline{\int} f_n < \infty$, then $f(x) = \liminf_{n\to\infty} f_n(x)$ is defined in $\mathbb{R}$ for almost every $x \in X$, and $\overline{\int} f \le \liminf_{n\to\infty} \overline{\int} f_n$.

proof (a) Set $c = \sup_{n\in\mathbb{N}} \overline{\int} f_n$. For each n, there is an integrable function g_n such that $f_n \le_{\text{a.e.}} g_n$ and $\int g_n = \overline{\int} f_n$ (133J(a-i)). Set $g'_n = \min(g_n, g_{n+1})$; then g'_n is integrable and $f_n \le_{\text{a.e.}} g'_n \le_{\text{a.e.}} g_n$, so

$$\overline{\int} f_n \le \int g'_n \le \int g_n = \overline{\int} f_n$$

and g'_n must be equal to g_n a.e. Consequently $g_n \le_{\text{a.e.}} g_{n+1}$, for each n, while $\sup_{n\in\mathbb{N}} \int g_n = c < \infty$. By B.Levi's theorem, $g = \sup_{n\in\mathbb{N}} g_n$ is defined, as a real-valued function, almost everywhere in X, and $\int g = c$. Now of course $f(x)$ is defined, and not greater than $g(x)$, for any $x \in \operatorname{dom} g \cap \bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n$ such that $f_n(x) \le g_n(x)$ for every n, that is, for almost every x; so $\overline{\int} f \le \int g = c$. On the other hand, $f_n \le_{\text{a.e.}} f$, so $\overline{\int} f_n \le \overline{\int} f$, for every $n \in \mathbb{N}$; it follows that $\overline{\int} f$ must be at least c, and is therefore equal to c, as required.

(b) The argument follows that of 123B. Set $c = \liminf_{n\to\infty} \overline{\int} f_n$. For each n, set $g_n = \inf_{m\ge n} f_n$; then $\overline{\int} g_n \le \inf_{m\ge n} \overline{\int} f_m \le c$. We have $g_n(x) \le g_{n+1}(x)$ for every $x \in \operatorname{dom} g_n$, that is, almost everywhere, for each n; so, by (a),

$$\overline{\int} g = \sup_{n\in\mathbb{N}} \overline{\int} g_n \le c,$$

where

$$g = \sup_{n\in\mathbb{N}} g_n =_{\text{a.e.}} \liminf_{n\to\infty} f_n,$$

and $\overline{\int} \liminf_{n\to\infty} f_n \le c$, as claimed.

***133L** The following is at a less fundamental level than the results in 133J, but is still important.

Proposition Let (X,Σ,μ) be a measure space and f a real-valued function defined almost everywhere in X. Suppose that h_1, h_2 are non-negative virtually measurable functions defined almost everywhere in X. Then

$$\overline{\int} f \times (h_1 + h_2) = \overline{\int} f \times h_1 + \overline{\int} f \times h_2,$$

where here, for once, we can interpret $\infty + (-\infty)$ or $(-\infty) + \infty$ as ∞ if called for on the right-hand side.

proof (a) If either $\overline{\int} f \times h_1 = \infty$ or $\overline{\int} f \times h_2 = \infty$ then $\overline{\int} f \times (h_1+h_2) = \infty$. **P?** Otherwise, there is a g such that $f \times (h_1+h_2) \le_{\text{a.e.}} g$ and $\int g < \infty$. In this case,

$$f \times h_1 \le_{\text{a.e.}} f^+ \times h_1 \le_{\text{a.e.}} f^+ \times (h_1+h_2) = (f \times (h_1+h_2))^+ \le_{\text{a.e.}} g^+$$

so $\overline{\int} f \times h_1 \le \int g^+ < \infty$. Similarly, $\overline{\int} f \times h_2 < \infty$; contradicting our hypothesis. **XQ** So in this case, under the local rule $\infty + (-\infty) = (-\infty) + \infty = \infty$, we have the result.

(b) Now suppose that the upper integrals $\overline{\int} f \times h_1$ and $\overline{\int} f \times h_2$ are both less than ∞, so that their sum can be interpreted by the usual rules. By 133J(b-ii), $\overline{\int} f \times (h_1+h_2) \le \overline{\int} f \times h_1 + \overline{\int} f \times h_2 < \infty$. In the other direction, suppose that $g \ge_{\text{a.e.}} f \times (h_1+h_2)$ and $\int g < \infty$. For $i = 1$, 2 set

$$\begin{aligned} g_i(x) &= \tfrac{g(x)h_i(x)}{h_1(x)+h_2(x)} \text{ if } x \in \operatorname{dom} g \cap \operatorname{dom} h_1 \cap \operatorname{dom} h_2 \text{ and } h_1(x)+h_2(x) > 0, \\ &= 0 \text{ for other } x \in X. \end{aligned}$$

Then, for both i, g_i is virtually measurable, $g_i^+ \le_{\text{a.e.}} g^+$ and $g_i \ge_{\text{a.e.}} f \times h_i$; while $g \ge_{\text{a.e.}} g_1 + g_2$. **P** The set

$$H = \{x : x \in \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} h_1 \cap \operatorname{dom} h_2,\ g(x) \ge f(x)(h_1(x)+h_2(x))\}$$

is conegligible, and for $x \in H$

$$\begin{aligned} g(x) &= g_1(x) + g_2(x) \text{ if } h_1(x)+h_2(x) > 0, \\ &\ge 0 = g_1(x) + g_2(x) \text{ if } h_1(x)+h_2(x) = 0. \quad \mathbf{Q} \end{aligned}$$

So

$$\overline{\int} f \times h_1 + \overline{\int} f \times h_2 \le \int g_1 + \int g_2 = \int g_1 + g_2 \le \int g$$

(because $\int g_1$ and $\int g_2$ are both at most $\int g^+ < \infty$, so we can add them on the usual rules). As g is arbitrary, $\overline{\int} f \times h_1 + \overline{\int} f \times h_2 \le \overline{\int} f \times (h_1+h_2)$ and we must have equality.

133X Basic exercises >(a) Let (X,Σ,μ) be a measure space, and $f : X \to [0,\infty[$ a measurable function. Show that

$$\begin{aligned} \int f d\mu &= \sup_{n\in\mathbb{N}} 2^{-n} \sum_{k=1}^{4^n} \mu\{x : f(x) \ge 2^{-n}k\} \\ &= \lim_{n\to\infty} 2^{-n} \sum_{k=1}^{4^n} \mu\{x : f(x) \ge 2^{-n}k\} \end{aligned}$$

in $[0,\infty]$.

(b) Let (X,Σ,μ) be a measure space and f a complex-valued function defined on a subset of X. (i) Show that if $E \in \Sigma$, then $f{\restriction}E$ is μ_E-integrable iff $\tilde f$ is μ-integrable, writing μ_E for the subspace measure on E and $\tilde f(x) = f(x)$ if $x \in E \cap \operatorname{dom} f$, 0 if $x \in X \setminus E$; and in this case $\int_E f d\mu_E = \int \tilde f d\mu$. (ii) Show that if $E \in \Sigma$ and f is defined μ-almost everywhere, then $f{\restriction}E$ is μ_E-integrable iff $f \times \chi E$ is μ-integrable, and in this case $\int_E f = \int f \times \chi E$. (iii) Show that if $\int_E f = 0$ for every $E \in \Sigma$, then $f = 0$ a.e.

(c) Suppose that (X,Σ,μ) is a measure space and that G is an open subset of $\mathbb{C}$, that is, a set such that for every $w\in G$ there is a $\delta>0$ such that $\{z:|z-w|<\delta\}\subseteq G$. Let $f:X\times G\to\mathbb{C}$ be a function, and suppose that the derivative $\frac{\partial f}{\partial z}$ of f with respect to the second variable exists for all $x\in X$, $z\in G$. Suppose moreover that (i) $F(z)=\int f(x,z)dx$ exists for every $z\in G$ (ii) there is an integrable function g such that $|\frac{\partial f}{\partial z}(x,z)|\le g(x)$ for every $x\in X$, $z\in G$. Show that the derivative F' of F exists everywhere in G, and $F'(z)=\int\frac{\partial f}{\partial z}(x,z)dx$ for every $z\in G$. (*Hint*: you will need to check that $|f(x,z)-f(x,w)|\le|z-w|g(x)$ whenever $x\in X$, $z\in G$ and w is close to z.)

>(d) Let f be a complex-valued function defined almost everywhere on $[0,\infty[$, endowed as usual with Lebesgue measure. Its **Laplace transform** is the function F defined by writing

$$F(s)=\int_0^\infty e^{-sx}f(x)dx$$

for all those complex numbers s for which the integral is defined in $\mathbb{C}$.

(i) Show that if $s\in\operatorname{dom}F$ and $\mathcal{R}\mathrm{e}\,s'\ge\mathcal{R}\mathrm{e}\,s$ then $s'\in\operatorname{dom}F$ (because $|e^{-s'x}e^{sx}|\le 1$ for all x).

(ii) Show that F is analytic (that is, differentiable as a function of a complex variable) on the interior of its domain. (*Hint*: 133Xc.)

(iii) Show that if F is defined anywhere then $\lim_{\mathcal{R}\mathrm{e}\,s\to\infty}F(s)=0$.

(iv) Show that if f, g have Laplace transforms F, G then the Laplace transform of $f+g$ is $F+G$, at least on $\operatorname{dom}F\cap\operatorname{dom}G$.

>(e) Let f be an integrable complex-valued function defined almost everywhere in $\mathbb{R}$, endowed as usual with Lebesgue measure. Its **Fourier transform** is the function $\hat{f}$ defined by

$$\hat{f}(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-isx}f(x)dx$$

for all real s.

(i) Show that $\hat{f}$ is continuous. (*Hint*: use Lebesgue's Dominated Convergence Theorem on sequences of the form $f_n(x)=e^{-is_nx}f(x)$.)

(ii) Show that if f, g have Fourier transforms $\hat{f}$, $\hat{g}$ then the Fourier transform of $f+g$ is $\hat{f}+\hat{g}$.

(iii) Show that if $\int xf(x)dx$ exists then $\hat{f}$ is differentiable, with $\hat{f}'(s)=-\frac{i}{\sqrt{2\pi}}\int xe^{-isx}f(x)dx$ for every s.

(f) Let (X,Σ,μ) be a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions each defined almost everywhere in X. Suppose that there is an integrable real-valued function g such that $|f_n|\le_{\text{a.e.}}g$ for each n. Show that

$$\overline{\int}\liminf_{n\to\infty}f_n\le\liminf_{n\to\infty}\overline{\int}f_n,\quad\underline{\int}\limsup_{n\to\infty}f_n\ge\limsup_{n\to\infty}\underline{\int}f_n.$$

133Y Further exercises (a) Use the ideas of 133C-133H to develop a theory of measurable and integrable functions taking values in $\mathbb{R}^r$, where $r\ge 2$.

(b) Let X be a set and Σ a σ-algebra of subsets of X. Let Y be a subset of X and $f:Y\to\mathbb{C}$ a Σ_Y-measurable function, where $\Sigma_Y=\{E\cap Y:E\in\Sigma\}$. Show that there is a Σ-measurable function $\tilde{f}:X\to\mathbb{C}$ extending f. (*Hint*: 121I.)

(c) Let f be an integrable complex-valued function defined almost everywhere in $\mathbb{R}^r$, endowed as usual with Lebesgue measure, where $r\ge 1$. Its **Fourier transform** is the function $\hat{f}$ defined by

$$\hat{f}(s)=\frac{1}{(\sqrt{2\pi})^r}\int e^{-is\mathbf{.}x}f(x)dx$$

for all $s\in\mathbb{R}^r$, writing $s\mathbf{.}x$ for $\sigma_1\xi_1+\ldots+\sigma_r\xi_r$ if $s=(\sigma_1,\ldots,\sigma_r)$, $x=(\xi_1,\ldots,\xi_r)\in\mathbb{R}^r$.

(i) Show that $\hat{f}$ is continuous.

(ii) Show that if f, g have Fourier transforms $\hat{f}$, $\hat{g}$ then the Fourier transform of $f+g$ is $\hat{f}+\hat{g}$.

(iii) Show that if $\int \|x\||f(x)|dx$ is finite (taking $\|x\| = \sqrt{\xi_1^2+\ldots+\xi_r^2}$ if $x = (\xi_1,\ldots,\xi_r)$), then $\hat{f}$ is differentiable, with

$$\frac{\partial\hat{f}}{\partial\sigma_k}(s) = -\frac{i}{(\sqrt{2\pi})^r}\int \xi_k e^{-is\,.\,x}f(x)dx$$

for every $s \in \mathbb{R}^r$, $k \le r$.

(d) Recall the definition of 'quasi-simple' function from 122Yd. Show that for any measure space (X,Σ,μ) and any real-valued function f defined almost everywhere in X,

$$\overline{\int} f = \inf\{\int g : g \text{ is quasi-simple, } f \le_{\text{a.e.}} g\},$$

$$\underline{\int} f = \sup\{\int g : g \text{ is quasi-simple, } f \ge_{\text{a.e.}} g\},$$

allowing ∞ for $\inf\emptyset$ and $\sup\mathbb{R}$ and $-\infty$ for $\inf\mathbb{R}$ and $\sup\emptyset$.

(e) State and prove a similar result concerning the 'pseudo-simple' functions of 122Ye.

133 Notes and comments I have spelt this section out in detail, even though there is nothing that can really be called a new idea in it, because it gives us an opportunity to review the previous work, and because the manipulations which are by now, I hope, becoming 'obvious' to you are in fact justifiable only through difficult theorems, and I believe that it is at least some of the time right to look back to the exact points at which justifications were written out.

You may have noticed similarities between results involving 'upper integrals', as described here, and those of §132 concerning 'outer measure' (132Ae and 133Ka, for instance, or 132Xe and 133Kb). These are not a coincidence; an explanation of sorts can be found in 252Ym in Volume 2.

Version of 7.1.04

134 More on Lebesgue measure

The special properties of Lebesgue measure will take up a substantial proportion of this treatise. In this section I present a miscellany of relatively easy basic results. In 134A-134F, r will be a fixed integer greater than or equal to 1, μ will be Lebesgue measure on $\mathbb{R}^r$ and μ^* will be Lebesgue outer measure (see 132C); when I say that a set or a function is 'measurable', then it is to be understood that (unless otherwise stated) this means 'measurable with respect to the σ-algebra of Lebesgue measurable sets', while 'negligible' means 'negligible for Lebesgue measure'. Most of the results will be expressed in terms adapted to the multi-dimensional case; but if you are primarily interested in the real line, you will miss none of the ideas if you read the whole section as if $r = 1$.

134A Proposition Both Lebesgue outer measure and Lebesgue measure are translation-invariant; that is, setting $A + x = \{a + x : a \in A\}$ for $A \subseteq \mathbb{R}^r$, $x \in \mathbb{R}^r$, we have

(a) $\mu^*(A + x) = \mu^*A$ for every $A \subseteq \mathbb{R}^r$, $x \in \mathbb{R}^r$;

(b) whenever $E \subseteq \mathbb{R}^r$ is measurable and $x \in \mathbb{R}^r$, then $E + x$ is measurable, with $\mu(E + x) = \mu E$.

proof The point is that if $I \subseteq \mathbb{R}^r$ is a half-open interval, as defined in 114Aa/115Ab, then so is $I + x$, and $\lambda(I+x) = \lambda I$ for every $x \in \mathbb{R}^r$, where λ is defined as in 114Ab/115Ac; this is immediate from the definition, since $[a, b[+ x = [a + x, b + x[$.

(a) If $A \subseteq \mathbb{R}^r$ and $x \in \mathbb{R}^r$ and $\epsilon > 0$, we can find a sequence $\langle I_j\rangle_{j\in\mathbb{N}}$ of half-open intervals such that $A \subseteq \bigcup_{j\in\mathbb{N}} I_j$ and $\sum_{j=0}^\infty \lambda I_j \le \mu^*A + \epsilon$. Now $A + x \subseteq \bigcup_{j\in\mathbb{N}}(I_j + x)$ so

$$\mu^*(A + x) \le \sum_{j=0}^\infty \lambda(I_j + x) = \sum_{j=0}^\infty \lambda I_j \le \mu^*A + \epsilon.$$

As ϵ is arbitrary, $\mu^*(A + x) \le \mu^*A$. Similarly

$$\mu^*A = \mu^*((A + x) + (-x)) \le \mu^*(A + x),$$

so $\mu^*(A + x) = \mu^*A$, as claimed.

(b) Now suppose that $E \subseteq \mathbb{R}^r$ is measurable and $x \in \mathbb{R}^r$, and that $A \subseteq \mathbb{R}^r$. Then, using (a) repeatedly,

$$\begin{aligned}\mu^*(A \cap (E+x)) + \mu^*(A \setminus (E+x)) &= \mu^*(((A-x)\cap E)+x) + \mu^*(((A-x)\setminus E)+x)\\ &= \mu^*((A-x)\cap E) + \mu^*((A-x)\setminus E)\\ &= \mu^*(A-x) = \mu^*A,\end{aligned}$$

writing $A-x$ for $A+(-x) = \{a-x : a \in A\}$. As A is arbitrary, $E+x$ is measurable. Now

$$\mu(E+x) = \mu^*(E+x) = \mu^*E = \mu E.$$

134B Theorem Not every subset of $\mathbb{R}^r$ is Lebesgue measurable.

proof Set $\mathbf{0} = (0,\dots,0)$, $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^r$. On

$$[\mathbf{0},\mathbf{1}[= \{(\xi_1,\dots,\xi_r) : \xi_i \in [0,1[\text{ for every } i \le r\},$$

consider the relation $\sim$, defined by saying that $x \sim y$ iff $y-x \in \mathbb{Q}^r$. It is easy to see that this is an equivalence relation, so divides $[\mathbf{0},\mathbf{1}[$ into equivalence classes. Choose one point from each of these equivalence classes, and let A be the set of points obtained in this way. Then $\mu^*A \le \mu^*[\mathbf{0},\mathbf{1}[= 1$.

Consider $A+\mathbb{Q}^r = \{a+q : a \in A,\ q \in \mathbb{Q}^r\} = \bigcup_{q\in\mathbb{Q}^r} A+q$. This is equal to $\mathbb{R}^r$. **P** If $x \in \mathbb{R}^r$, there is an $e \in \mathbb{Z}^r$ such that $x-e \in [\mathbf{0},\mathbf{1}[$; there is an $a \in A$ such that $a \sim x-e$, that is, $x-e-a \in \mathbb{Q}^r$; now $x = a+(e+x-e-a) \in A+\mathbb{Q}^r$. **Q** Next, $\mathbb{Q}^r$ is countable (111F(b-iv)), so we have

$$\infty = \mu\mathbb{R}^r \le \textstyle\sum_{q\in\mathbb{Q}^r} \mu^*(A+q),$$

and there must be some $q \in \mathbb{Q}^r$ such that $\mu^*(A+q) > 0$; but as μ^* is translation-invariant (134A), $\mu^*A > 0$.

Take $n \in \mathbb{N}$ such that $n > 2^r/\mu^*A$, and distinct $q_1,\dots,q_n \in [\mathbf{0},\mathbf{1}[\cap \mathbb{Q}^r$. If a, $b \in A$ and $1 \le i < j \le n$, then $a+q_i \ne b+q_j$; for if $a=b$ then $q_i \ne q_j$, while if $a \ne b$ then $a \not\sim b$ so $b-a \ne q_i-q_j$. Thus $A+q_1,\dots,A+q_n$ are disjoint. On the other hand, all are subsets of $[\mathbf{0},\mathbf{2}[$. So we have

$$\textstyle\sum_{i=1}^n \mu^*(A+q_i) = n\mu^*A > 2^r = \mu[\mathbf{0},\mathbf{2}[\ge \mu^*(\bigcup_{1\le i\le n}(A+q_i)).$$

It follows that not all the $A+q_i$ can be measurable; as Lebesgue measure is translation-invariant, we see that A itself is not measurable. In any case we have found a non-measurable set.

***134C Remark** 134B is known as 'Vitali's construction'.

Observe that at the beginning of the proof I asked you to *choose* one member of each of the equivalence classes for $\sim$. This is of course an appeal to the Axiom of Choice. So far I have made rather few appeals to the axiom of choice. One was in (a-iv) of the proof of 114D/115D; an earlier one was in 112Db; yet another in 121A. See also 1A1F. In all of these, only 'countable choice' was involved; that is, I needed to choose simultaneously one member of each of a named sequence of sets. Because there are surely uncountably many equivalence classes for $\sim$, the form of choice needed for the example above is essentially stronger than that needed for the positive results so far. It is in fact the case that very large parts of measure theory can be developed without appealing to the full strength of the axiom of choice.

The significance of this is that it suggests the possibility that there might be a consistent mathematical system in which enough of the axiom of choice is valid to make measure theory possible, without having enough to construct a non-Lebesgue-measurable set. Such a system has indeed been worked out by R.M.Solovay (Solovay 70). (In a formal sense there is room for a residual doubt concerning its consistency. In my view this is of no importance.) In Volume 5 I will return to the question of what Lebesgue measure looks like with a weak axiom of choice, or none at all. For the moment, I have to say that nearly all measure theory continues to proceed in directions at least consistent with the full axiom of choice, so that non-measurable sets are constantly present, at least potentially; and that will be my normal position in this treatise. But I mention the point at this early stage because I believe that it could happen at any time that the focus of interest might switch to systems in which the axiom of choice is false; and in this case measure theory without non-measurable sets might become important to many pure mathematicians, and even to applied mathematicians, who have no reason, other than the convenience of being able to quote results from books like this one, for loyalty to the axiom of choice.

I ought to remark that while we need a fairly strong form of the axiom of choice to construct a non-Lebesgue-measurable set, a non-Borel set can be constructed in much weaker set theories. One possible construction is outlined in §423 in Volume 4.

Of course there is a non-Lebesgue-measurable subset of $\mathbb{R}$ iff there is a non-Lebesgue-measurable function from $\mathbb{R}$ to $\mathbb{R}$; for if every set is measurable, then the definition 121C makes it plain that every real-valued function on any subset of $\mathbb{R}$ is measurable; while if $A \subseteq \mathbb{R}$ is not measurable, then $\chi A : \mathbb{R} \to \mathbb{R}$ is not measurable.

***134D** In fact there are much stronger results than 134B concerning the existence of non-measurable sets (provided, of course, that we allow ourselves to use the axiom of choice). Here I give one which can be reached by a slight refinement of the methods of 134B.

Proposition There is a set $C \subseteq \mathbb{R}^r$ such that $F \cap C$ is not measurable for any measurable set F of non-zero measure; so that both C and its complement have full outer measure in $\mathbb{R}^r$.

proof (a) Start from a set $A \subseteq [\mathbf{0},\mathbf{1}[\subseteq \mathbb{R}^r$ such that $\langle A+q\rangle_{q\in\mathbb{Q}^r}$ is a partition of $\mathbb{R}^r$, as constructed in the proof of 134B. As in 134B, the outer measure μ^*A of A must be greater than 0. The argument there shows in fact that $\mu F = 0$ for every measurable set $F \subseteq A$. **P** For every n we can find distinct $q_1,\dots,q_n \in [\mathbf{0},\mathbf{1}[\cap\mathbb{Q}^r$, and now

$$n\mu F = \mu(\bigcup_{1\le i\le n} F+q_i) \le \mu\,[\mathbf{0},\mathbf{2}[= 2^r,$$

so that $\mu F \le 2^r/n$; as n is arbitrary, $\mu F = 0$. **Q**

(b) Now let $E \subseteq [\mathbf{0},\mathbf{1}[$ be a measurable envelope of A (132Ef). Then $E+q$ is a measurable envelope of $A+q$ for any q. **P** I hope that this will very soon be 'an obvious consequence of the translation-invariance of Lebesgue measure'. In detail: $A+q \subseteq E+q$, $E+q$ is measurable and, for any measurable F,

$$\begin{aligned}\mu(F\cap(E+q)) &= \mu(((F-q)\cap E)+q) = \mu((F-q)\cap E)\\ &= \mu^*((F-q)\cap A) = \mu^*(((F-q)\cap A)+q) = \mu^*(F\cap(A+q)),\end{aligned}$$

using 134A repeatedly. **Q** Also E is a measurable envelope of $A' = E\setminus A$. **P** Of course E is a measurable set including A'. If $F \subseteq E\setminus A'$ is measurable then $F \subseteq A$, so $\mu F = 0$, by (a); now 132Ea tells us that E is a measurable envelope of A'. **Q** It follows that $E+q$ is a measurable envelope of $A'+q$ for every q.

(c) Let $\langle q_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathbb{Q}^r$. Then

$$\bigcup_{n\in\mathbb{N}} E+q_n \supseteq \bigcup_{n\in\mathbb{N}} A+q_n = \mathbb{R}^r.$$

Write E_n for $E+q_n \setminus \bigcup_{i<n} E+q_i$ for $n\in\mathbb{N}$, so that $\langle E_n\rangle_{n\in\mathbb{N}}$ is disjoint and $\bigcup_{n\in\mathbb{N}} E_n = \mathbb{R}^r$.

Now set

$$C = \bigcup_{n\in\mathbb{N}} E_n \cap (A+q_n).$$

This is a set with the required properties.

P (i) Let $F \subseteq \mathbb{R}^r$ be any non-negligible measurable set. Then there must be some $n\in\mathbb{N}$ such that $\mu(F\cap E_n) > 0$. But this means that

$$\mu^*(F\cap E_n\cap C) \ge \mu^*(F\cap E_n\cap(A+q_n)) = \mu(F\cap E_n\cap(E+q_n)) = \mu(F\cap E_n),$$

$$\begin{aligned}\mu^*(F\cap E_n\setminus C) &\ge \mu^*(F\cap E_n\cap((E+q_n)\setminus(A+q_n)))\\ &= \mu(F\cap E_n\cap(E+q_n)) = \mu(F\cap E_n).\end{aligned}$$

Since

$$\mu(F\cap E_n) \le \mu(E+q_n) = \mu E \le 1,$$

$\mu^*(F\cap E_n\cap C) + \mu^*(F\cap E_n\setminus C) > \mu(F\cap E_n)$, and $F\cap C$ cannot be measurable.

(ii) In particular, no measurable subset of $\mathbb{R}^r\setminus C$ can have non-zero measure, and C has full outer measure; similarly, C has no measurable subset of non-zero measure, and $\mathbb{R}^r\setminus C$ has full outer measure. **Q**

Remark In fact it is the case that for any sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^r$ there is a set $C\subseteq\mathbb{R}^r$ such that

$$\mu^*(E\cap D_n\cap C)=\mu^*(E\cap D_n\setminus C)=\mu^*(E\cap D_n)$$

for every measurable set $E\subseteq\mathbb{R}^r$ and every $n\in\mathbb{N}$. But for the proof of this result we must wait for Volume 5.

134E Borel sets and Lebesgue measure on $\mathbb{R}^r$ Recall from 111G that the family $\mathcal{B}$ of Borel sets in $\mathbb{R}^r$ is the σ-algebra generated by the family of open sets. In 114G/115G I showed that every Borel set in $\mathbb{R}^r$ is Lebesgue measurable. It is time we returned to the topic and looked more closely at the very intimate connexion between Borel and measurable sets.

Recall that a set $A\subseteq\mathbb{R}^r$ is **bounded** if there is an M such that $A\subseteq B(\mathbf{0},M)=\{x:\|x\|\le M\}$; equivalently, if $\sup_{x\in A}|\xi_j|<\infty$ for every $j\le r$ (writing $x=(\xi_1,\dots,\xi_r)$, as in §115).

134F Proposition (a) If $A\subseteq\mathbb{R}^r$ is any set, then

$$\mu^*A=\inf\{\mu G:G\text{ is open, }G\supseteq A\}=\min\{\mu H:H\text{ is Borel, }H\supseteq A\}.$$

(b) If $E\subseteq\mathbb{R}^r$ is measurable, then

$$\mu E=\sup\{\mu F:F\text{ is closed and bounded, }F\subseteq E\},$$

and there are Borel sets H_1, H_2 such that $H_1\subseteq E\subseteq H_2$ and

$$\mu(H_2\setminus H_1)=\mu(H_2\setminus E)=\mu(E\setminus H_1)=0.$$

(c) If $A\subseteq\mathbb{R}^r$ is any set, then A has a measurable envelope which is a Borel set.

(d) If f is a Lebesgue measurable real-valued function defined on a subset of $\mathbb{R}^r$, then there is a conegligible Borel set $H\subseteq\mathbb{R}^r$ such that $f\restriction H$ is Borel measurable.

proof (a)(i) First note that if $I\subseteq\mathbb{R}^r$ is a half-open interval, and $\epsilon>0$, then either $I=\emptyset$ is already open, or I is expressible as $[a,b[$ where $a=(\alpha_1,\dots,\alpha_r)$, $b=(\beta_1,\dots,\beta_r)$ and $\alpha_i<\beta_i$ for every i. In the latter case, $G=\,]a-\epsilon(b-a),b[$ is an open set including I, and

$$\mu G=\prod_{i=1}^r(1+\epsilon)(\beta_i-\alpha_i)=(1+\epsilon)^r\mu I,$$

by the formula in 114G/115G.

(ii) Now, given $\epsilon>0$, there is a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of half-open intervals, covering A, such that $\sum_{n=0}^\infty\mu I_n\le\mu^*A+\epsilon$. For each n, let $G_n\supseteq I_n$ be an open set of measure at most $(1+\epsilon)^r\mu I_n$. Then $G=\bigcup_{n\in\mathbb{N}}G_n$ is open (1A2Bd), and $A\subseteq G$; also

$$\mu G\le\sum_{n=0}^\infty\mu G_n\le(1+\epsilon)^r\sum_{n=0}^\infty\mu I_n\le(1+\epsilon)^r(\mu^*A+\epsilon).$$

As ϵ is arbitrary, $\mu^*A\ge\inf\{\mu G:G\text{ is open, }G\supseteq A\}$.

(iii) Next, using (ii), we can choose for each $n\in\mathbb{N}$ an open set $G_n\supseteq A$ such that $\mu G_n\le\mu^*A+2^{-n}$. Set $H_0=\bigcap_{n\in\mathbb{N}}G_n$; then H_0 is a Borel set, $A\subseteq H_0$, and

$$\mu H_0\le\inf_{n\in\mathbb{N}}\mu G_n\le\mu^*A.$$

(iv) On the other hand, we surely have $\mu^*A\le\mu^*H=\mu H$ for every Borel set $H\supseteq A$. So we must have

$$\mu^*A\le\inf\{\mu G:G\text{ is open, }G\supseteq A\},$$

and

$$\mu^*A=\mu H_0=\min\{\mu H:H\text{ is Borel, }H\supseteq A\}.$$

(b)(i) For each $n\in\mathbb{N}$, set $E_n=E\cap B(\mathbf{0},n)$. Let $G_n\supseteq E_n$ be an open set of measure at most μE_n+2^{-n}; then (because $\mu B(\mathbf{0},n)<\infty$) $\mu(G_n\setminus E_n)\le 2^{-n}$. Now, for each n, set $G'_n=\bigcup_{m\ge n}G_m$; then G'_n is open, $E=\bigcup_{m\ge n}E_m\subseteq G'_n$, and

$$\mu(G'_n \setminus E) \le \sum_{m=n}^{\infty} \mu(G_m \setminus E) \le \sum_{m=n}^{\infty} \mu(G_m \setminus E_m) \le \sum_{m=n}^{\infty} 2^{-m} = 2^{-n+1}.$$

Setting $H_2 = \bigcap_{n\in\mathbb{N}} G_n$, we see that H_2 is a Borel set including E and that $\mu(H_2 \setminus E) = 0$.

(ii) Repeating the argument of (i) with $\mathbb{R}^r \setminus E$ in place of E, we obtain a Borel set $\tilde{H}_2 \supseteq \mathbb{R}^r \setminus E$ such that $\mu(\tilde{H}_2 \setminus (\mathbb{R}^r \setminus E)) = 0$; now $H_1 = \mathbb{R}^r \setminus \tilde{H}_2$ is a Borel set included in E and

$$\mu(E \setminus H_1) = \mu(\tilde{H}_2 \setminus (\mathbb{R}^r \setminus E)) = 0.$$

Of course we now also have

$$\mu(H_2 \setminus H_1) = \mu(H_2 \setminus E) + \mu(E \setminus H_1) = 0.$$

(iii) Again using the idea of (i), there is for each $n \in \mathbb{N}$ an open set $\tilde{G}_n \supseteq B(\mathbf{0}, n) \setminus E$ such that

$$\mu(\tilde{G}_n \cap E_n) \le \mu(\tilde{G}_n \setminus (B(\mathbf{0}, n) \setminus E)) \le 2^{-n}.$$

Set

$$F_n = B(\mathbf{0}, n) \setminus \tilde{G}_n = B(\mathbf{0}, n) \cap (\mathbb{R}^r \setminus \tilde{G}_n);$$

then F_n is closed (1A2Fd) and bounded and $F_n \subseteq E_n \subseteq E$. Also

$$\mu E_n = \mu F_n + \mu(E_n \setminus F_n) = \mu F_n + \mu(\tilde{G}_n \cap E_n) \le \mu F_n + 2^{-n}.$$

So

$$\mu E = \lim_{n\to\infty} \mu E_n \le \sup_{n\in\mathbb{N}} \mu F_n \le \sup\{\mu F : F \text{ is closed and bounded, } F \subseteq E\},$$

and

$$\mu E = \sup\{\mu F : F \text{ is closed and bounded, } F \subseteq E\}.$$

(c) Let E be any measurable envelope of A (132Ef), and $H \supseteq E$ a Borel set such that $\mu(H \setminus E) = 0$; then $\mu^*(F \cap A) = \mu(F \cap E) = \mu(F \cap H)$ for every measurable set F, so H is a measurable envelope of A.

(d) Set $D = \operatorname{dom} f$ and write $\mathcal{B}$ for the σ-algebra of Borel sets. For each rational number q, let E_q be a measurable set such that $\{x : f(x) \le q\} = E_q \cap D$. Let H_q, $H'_q \in \mathcal{B}$ be such that $H_q \subseteq E_q \subseteq H'_q$ and $\mu(H'_q \setminus H_q) = 0$. Let H be the conegligible Borel set $\mathbb{R}^r \setminus \bigcup(H'_q \setminus H_q)$. Then

$$\{x : (f\restriction H)(x) \le q\} = H \cap E_q \cap D = H_q \cap D \cap H$$

belongs to the subspace σ-algebra $\mathcal{B}(D)$ for every $q \in \mathbb{Q}$. For irrational $a \in \mathbb{R}$, set $H_a = \bigcap_{q\in\mathbb{Q},q\ge a} H_q$; then $H_a \in \mathcal{B}$, and

$$\{x : (f\restriction H)(x) \le a\} = H_a \cap \operatorname{dom}(f\restriction H).$$

Thus $f\restriction H$ is Borel measurable.

Remark The emphasis on closed *bounded* sets in part (b) of this proposition is on account of their important topological properties, in particular, the fact that they are 'compact'. This is one of the most important facts about Lebesgue measure, as will appear in Volume 4. I will discuss 'compactness' briefly in §2A2 of Volume 2.

134G The Cantor set One of the purposes of the theory of Lebesgue measure and integration is to study rather more irregular sets and functions than can be dealt with by more primitive methods. In the next few paragraphs I discuss *measurable* sets and functions which from the point of view of the present theory are amenable without being trivial. From now on, μ will be Lebesgue measure on $\mathbb{R}$.

(a) The 'Cantor set' $C \subseteq [0, 1]$ is defined as the intersection of a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of sets, constructed as follows. $C_0 = [0, 1]$. Given that C_n consists of 2^n disjoint closed intervals each of length 3^{-n}, take each of these intervals and delete the middle third to produce two closed intervals each of length 3^{-n-1}; take C_{n+1} to be the union of the 2^{n+1} closed intervals so formed, and continue. Observe that $\mu C_n = (\frac{2}{3})^n$ for each n.

Approaching the Cantor set

The **Cantor set** is $C = \bigcap_{n\in\mathbb{N}} C_n$. Its measure is

$$\mu C = \lim_{n\to\infty} \mu C_n = \lim_{n\to\infty} (\frac{2}{3})^n = 0.$$

(b) Each C_n can also be described as the set of real numbers expressible as $\sum_{j=1}^{\infty} 3^{-j}\epsilon_j$ where every ϵ_j is either 0, 1 or 2, and $\epsilon_j \ne 1$ for $j \le n$. Consequently C itself is the set of numbers expressible as $\sum_{j=1}^{\infty} 3^{-j}\epsilon_j$ where every ϵ_j is either 0 or 2; that is, the set of numbers between 0 and 1 expressible in ternary form without 1's. The expression in each case will be unique, so we have a bijection $\phi : \{0,1\}^{\mathbb{N}} \to C$ defined by writing

$$\phi(z) = \frac{2}{3}\sum_{j=0}^{\infty} 3^{-j} z(j)$$

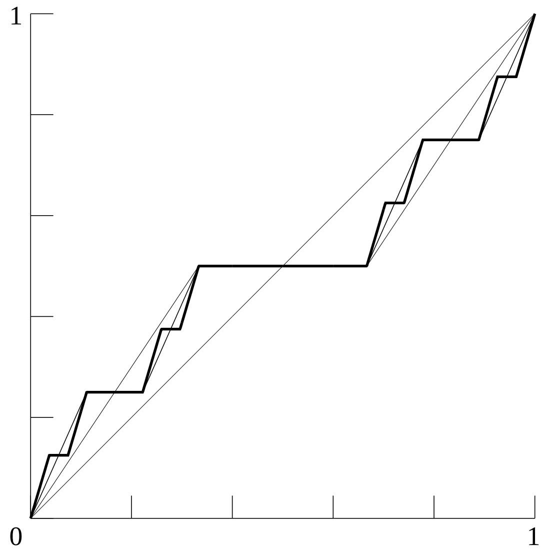
for every $z \in \{0,1\}^{\mathbb{N}}$.

134H The Cantor function Continuing from 134G, we have the following construction.

(a) For each $n \in \mathbb{N}$ we define a function $f_n : [0,1] \to [0,1]$ by setting

$$f_n(x) = (\frac{3}{2})^n \mu(C_n \cap [0,x])$$

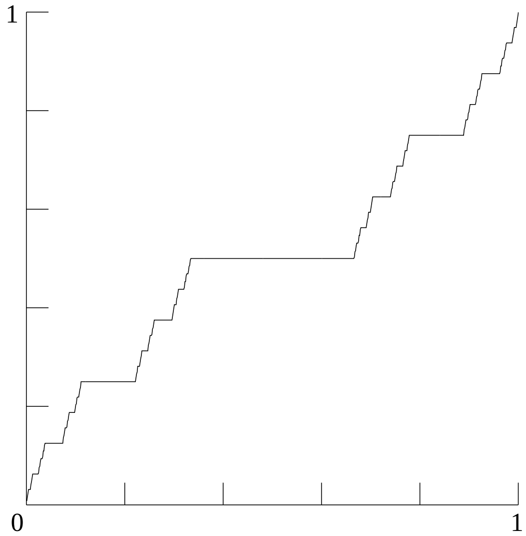
for each $x \in [0,1]$. Because C_n is just a finite union of intervals, f_n is a polygonal function, with $f_n(0) = 0$, $f_n(1) = 1$; f_n is constant on each of the $2^n - 1$ open intervals composing $[0,1] \setminus C_n$, and rises with slope $(\frac{3}{2})^n$ on each of the 2^n closed intervals composing C_n.



Approaching the Cantor function: the functions f_0, f_1, f_2, $\boldsymbol{f_3}$

If the jth interval of C_n, counting from the left, is $[a_{nj}, b_{nj}]$, then $f_n(a_{nj}) = 2^{-n}(j-1)$ and $f_n(b_{nj}) = 2^{-n}j$. Also, $a_{nj} = a_{n+1,2j-1}$ and $b_{nj} = b_{n+1,2j}$; hence, or otherwise, $f_{n+1}(a_{nj}) = f_n(a_{nj})$ and $f_{n+1}(b_{nj}) = f_n(b_{nj})$, and f_{n+1} agrees with f_n on all the endpoints of the intervals of C_n, and therefore on $[0,1] \setminus C_n$.

Within any particular interval $[a_{nj}, b_{nj}]$ of C_n, the greatest difference between $f_n(x)$ and $f_{n+1}(x)$ is at the new endpoints within that interval, viz., $b_{n+1,2j-1}$ and $a_{n+1,2j}$; and the magnitude of the difference is $\frac{1}{6}2^{-n}$ (because, for instance, $f_n(b_{n+1,2j-1}) = \frac{2}{3}f_n(a_{nj}) + \frac{1}{3}f_n(b_{nj})$, while $f_{n+1}(b_{n+1,2j-1}) = \frac{1}{2}f_n(a_{nj}) + \frac{1}{2}f_n(b_{nj})$). Thus we have $|f_{n+1}(x) - f_n(x)| \le \frac{1}{6}2^{-n}$ for every $n \in \mathbb{N}$, $x \in [0,1]$. Because $\sum_{n=0}^{\infty} \frac{1}{6}2^{-n} < \infty$, $\langle f_n \rangle_{n\in\mathbb{N}}$ is uniformly convergent to a function $f : [0,1] \to [0,1]$, and f will be continuous. f is the **Cantor function** or **Devil's Staircase**.



The Cantor function

(b) Because every f_n is non-decreasing, so is f. If $x \in [0,1] \setminus C$, there is an n such that $x \in [0,1] \setminus C_n$; let I be the open interval of $[0,1] \setminus C_n$ containing x; then f_{m+1} agrees on I with f_m for every $m \ge n$, so f agrees on I with f_n, and f is constant on I. Thus, in particular, the derivative $f'(x)$ exists and is 0 for every $x \in [0,1] \setminus C$; so f' is zero almost everywhere in $[0,1]$. Also, of course, $f(0) = 0$ and $f(1) = 1$, because $f_n(0) = 0$, $f_n(1) = 1$ for every n. It follows that $f : [0,1] \to [0,1]$ is surjective (by the Intermediate Value Theorem).

(c) Let $\phi : \{0,1\}^{\mathbb{N}} \to C$ be the function described in 134Gb. Then $f(\phi(z)) = \frac{1}{2}\sum_{j=0}^{\infty} 2^{-j}z(j)$ for every $z \in \{0,1\}^{\mathbb{N}}$. **P** Fix $z = (\zeta_0, \zeta_1, \zeta_2, \dots)$ in $\{0,1\}^{\mathbb{N}}$, and for each n take I_n to be the component interval of C_n containing $\phi(z)$. Then I_{n+1} will be the left-hand third of I_n if $\zeta_n = 0$ and the right-hand third if $\zeta_n = 1$. Taking a_n to be the left-hand endpoint of I_n, we see that

$$a_{n+1} = a_n + \frac{2}{3}3^{-n}\zeta_n, \quad f_{n+1}(a_{n+1}) = f_n(a_n) + \frac{1}{2}2^{-n}\zeta_n$$

for each n. Now

$$\phi(z) = \lim_{n\to\infty} a_n, \quad f(\phi(z)) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f_n(a_n) = \frac{1}{2}\sum_{j=0}^{\infty} 2^{-j}\zeta_j,$$

as claimed. **Q**

In particular, $f[C] = [0,1]$. **P** Any $x \in [0,1]$ is expressible as $\sum_{j=0}^{\infty} 2^{-j-1}z(j) = f(\phi(z))$ for some $z \in \{0,1\}^{\mathbb{N}}$. **Q**

134I The Cantor function modified I continue the argument of 134G-134H.

(a) Consider the formula

$$g(x) = \frac{1}{2}(x + f(x)),$$

where f is the Cantor function, as defined in 134H; this defines a continuous function $g : [0,1] \to [0,1]$ which is strictly increasing (because f is non-decreasing) and has $g(0) = 0$, $g(1) = 1$; consequently, by the Intermediate Value Theorem, g is bijective, and its inverse $g^{-1} : [0,1] \to [0,1]$ is continuous.

Now $g[C]$ is a closed set and $\mu g[C]=\frac{1}{2}$. **P** Because g is a permutation of the points of $[0,1]$, $[0,1]\setminus g[C]=g[\,[0,1]\setminus C]$. For each of the open intervals $I_{nj}=\,]b_{nj},a_{n,j+1}[$ making up $[0,1]\setminus C_n$, we see that $g[I_{nj}]=\,]g(b_{nj}),g(a_{n,j+1})[$ has length just half the length of I_{nj}. Consequently $g[\,[0,1]\setminus C]=\bigcup_{n\ge 1,1\le j<2^n}g[I_{nj}]$ is open, and

$$\mu(g[\,[0,1]\setminus C_n])=\sum_{j=1}^{2^n-1}g(a_{n,j+1})-g(b_{nj})=\frac{1}{2}\sum_{j=1}^{2^n-1}a_{n,j+1}-b_{nj}$$
$$=\frac{1}{2}\mu([0,1]\setminus C_n)=\frac{1}{2}(1-(\frac{2}{3})^n)$$

(134Ga). Because $\langle[0,1]\setminus C_n\rangle_{n\in\mathbb{N}}$ is an increasing sequence of sets with union $[0,1]\setminus C$,

$$\mu g([\,[0,1]\setminus C])=\lim_{n\to\infty}\mu g([\,[0,1]\setminus C_n])=\frac{1}{2}.$$

So $g[C]=[0,1]\setminus g[\,[0,1]\setminus C]$ is closed and $\mu g[C]=\frac{1}{2}$. **Q**

(b) By 134D there is a set $D\subseteq\mathbb{R}$ such that

$$\mu^*(g[C]\cap D)=\mu^*(g[C]\setminus D)=\mu g[C]=\frac{1}{2};$$

set $A=g[C]\cap D$. Of course A cannot be measurable, since $\mu^*A+\mu^*(g[C]\setminus A)>\mu g[C]$. However, $g^{-1}[A]\subseteq C$ must be measurable, because $\mu^*C=0$. This means that if we set $h=\chi(g^{-1}[A]):[0,1]\to\mathbb{R}$, then h is measurable; but $hg^{-1}=\chi A:[0,1]\to\mathbb{R}$ is not.

Thus **the composition of a measurable function with a continuous function need not be measurable**. Contrast this with 121Eg.

134J More examples I think it is worth taking the space to spell out two more of the basic examples of Lebesgue measurable set in detail.

(a) As already observed in 114G, every countable subset of $\mathbb{R}$ is negligible. In particular, $\mathbb{Q}$ is negligible (111Eb). We can say more. Let $\langle q_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathbb{Q}$, and for each $n\in\mathbb{N}$ set

$$I_n=\,]q_n-2^{-n},q_n+2^{-n}[,$$

$$G_n=\bigcup_{k\ge n}I_k.$$

Then G_n is an open set of measure at most $\sum_{k=n}^{\infty}2\cdot 2^{-k}=4\cdot 2^{-n}$, and it contains all but finitely many points of $\mathbb{Q}$, so is dense (that is, meets every non-trivial interval). Set $F_n=\mathbb{R}\setminus G_n$; then F_n is closed, $\mu(\mathbb{R}\setminus F_n)\le 4/2^n$, but F_n does not contain q_k for any $k\ge n$, so F_n cannot include any non-trivial interval. Observe that $\langle G_n\rangle_{n\in\mathbb{N}}$ is non-increasing so $\langle F_n\rangle_{n\in\mathbb{N}}$ is non-decreasing.

(b) We can elaborate the above construction, as follows. There is a measurable set $E\subseteq\mathbb{R}$ such that $\mu(I\cap E)>0$ and $\mu(I\setminus E)>0$ for every non-trivial interval $I\subseteq\mathbb{R}$. **P** First note that if k, $n\in\mathbb{N}$, there is a $j\ge n$ such that $q_j\in I_k$, so that $I_k\cap I_j\ne\emptyset$ and $\mu(I_k\setminus F_n)>0$. Now there must be an $l>n$ such that $\mu G_l<\mu(I_k\setminus F_n)$, so that

$$\mu(I_k\cap F_l\setminus F_n)=\mu((I_k\setminus F_n)\setminus G_l)>0.$$

Choose $n_0<n_1<n_2<\ldots$ as follows. Start with $n_0=0$. Given n_{2k}, where $k\in\mathbb{N}$, choose n_{2k+1}, n_{2k+2} such that

$$\mu(I_k\cap F_{n_{2k+1}}\setminus F_{n_{2k}})>0,\quad \mu(I_k\cap F_{n_{2k+2}}\setminus F_{n_{2k+1}})>0.$$

Continue.

On completing the induction, set

$$E=\bigcup_{k\in\mathbb{N}}F_{n_{2k+1}}\setminus F_{n_{2k}},\quad H=\bigcup_{k\in\mathbb{N}}F_{n_{2k+2}}\setminus F_{n_{2k+1}}.$$

Because $\langle F_k\rangle_{k\in\mathbb{N}}$ is non-decreasing, $E\cap H=\emptyset$. If $k\in\mathbb{N}$, $E\cap I_k$ and $H\cap I_k$ both have positive measure.

Now suppose that $I \subseteq \mathbb{R}$ is an interval with more than one point; suppose that $a, b \in I$ and $a < b$. Then there is an $m \in \mathbb{N}$ such that $4 \cdot 2^{-m} \le b - a$; now there is a $k \ge m$ such that $q_k \in [a + 2^{-m}, b - 2^{-m}]$, so that $I_k \subseteq I$ and

$$\mu(I \cap E) \ge \mu(E \cap I_k) > 0, \quad \mu(I \setminus E) \ge \mu(H \cap I_k) > 0. \ \mathbf{Q}$$

(c) This shows that E and its complement are measurable sets which are not merely both dense (like $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$), but 'essentially' dense in that they meet every non-empty open interval in a set of positive measure, so that (for instance) $E \setminus A$ is dense for every negligible set A.

***134K Riemann integration** I have tried, in writing this book, to assume as little prior knowledge as possible. In particular, it is not necessary to have studied Riemann integration. Nevertheless, if you have worked through the basic theory of the Riemann integral – which is, indeed, not only a splendid training in the techniques of ϵ-δ analysis, but also a continuing source of ideas for the subject – you will, I hope, wish to connect it with the material we are looking at here; both because you will not want to feel that your labour has been wasted, and because you have probably developed a number of intuitions which will continue to be valuable, if suitably adapted to the new context. I therefore give a brief account of the relationship between the Riemann and Lebesgue methods of integration on the real line.

(a) There are many ways of describing the Riemann integral; I choose one of the popular ones. If $[a, b]$ is a non-trivial closed interval in $\mathbb{R}$, then I say that a **dissection** of $[a, b]$ is a finite list $D = (a_0, a_1, \dots, a_n)$, where $n \ge 1$, such that $a = a_0 < a_1 < \dots < a_n = b$. If now f is a real-valued function defined (at least) on $[a, b]$ and bounded on $[a, b]$, the **upper sum** and **lower sum** of f on $[a, b]$ derived from D are

$$S_D(f) = \sum_{i=1}^n (a_i - a_{i-1}) \sup_{x \in]a_{i-1}, a_i[} f(x),$$

$$s_D(f) = \sum_{i=1}^n (a_i - a_{i-1}) \inf_{x \in]a_{i-1}, a_i[} f(x).$$

You have to prove that if D and D' are two dissections of $[a, b]$, then $s_D(f) \le S_{D'}(f)$. Now define the **upper Riemann integral** and **lower Riemann integral** of f to be

$$U_{[a,b]}(f) = \inf\{S_D(f) : D \text{ is a dissection of } [a, b]\},$$

$$L_{[a,b]}(f) = \sup\{s_D(f) : D \text{ is a dissection of } [a, b]\}.$$

Check that $L_{[a,b]}(f)$ is necessarily less than or equal to $U_{[a,b]}(f)$. Finally, declare f to be **Riemann integrable over** $[a, b]$ if $U_{[a,b]}(f) = L_{[a,b]}(f)$, and in this case take the common value to be the **Riemann integral** $\mathrm{R}\!\!\int_a^b f$ of f over $[a, b]$.

(b) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, it is Lebesgue integrable, with the same integral. $\mathbf{P}$ For any dissection $D = (a_0, \dots, a_n)$ of $[a, b]$, define g_D, $h_D : [a, b] \to \mathbb{R}$ by saying

$$g_D(x) = \inf\{f(y) : y \in]a_{i-1}, a_i[\} \text{ if } a_{i-1} < x < a_i, \quad g_D(a_i) = f(a_i) \text{ for each } i,$$

$$h_D(x) = \sup\{f(y) : y \in]a_{i-1}, a_i[\} \text{ if } a_{i-1} < x < a_i, \quad h_D(a_i) = f(a_i) \text{ for each } i.$$

Then g_D and h_D are constant on each interval $]a_{i-1}, a_i[$, so all sets $\{x : g_D(x) \le c\}$, $\{x : h_D(x) \le c\}$ are finite unions of intervals, and g_D and h_D are measurable; moreover,

$$\int g_D d\mu = s_D(f), \quad \int h_D d\mu = S_D(f).$$

Consequently

$$\begin{aligned} \mathrm{R}\!\!\int_a^b f = L_{[a,b]}(f) = \sup_D \int g_D d\mu &\le \underline{\int} f d\mu \\ &\le \overline{\int} f d\mu \le \inf_D \int h_D d\mu = U_{[a,b]}(f) = \mathrm{R}\!\!\int_a^b f, \end{aligned}$$

and $\overline{\int} f d\mu = \underline{\int} f d\nu = \mathrm{R}\!\!\int_a^b f$, so that $\int f d\mu$ exists and is equal to $\mathrm{R}\!\!\int_a^b f$ (133Jd). $\mathbf{Q}$

(c) The discussion above is of the 'proper' Riemann integral, of bounded functions on bounded intervals. For unbounded functions and unbounded intervals, one uses various forms of 'improper' integral; for instance, the improper Riemann integral $\int_0^\infty \frac{\sin x}{x} dx$ is taken to be $\lim_{a\to\infty} \int_0^a \frac{\sin x}{x} dx$, while $\int_0^1 \ln x\,dx$ is taken to be $\lim_{a\downarrow 0} \int_a^1 \ln x\,dx$. Of these, the second exists as a Lebesgue integral, but the first does not, because $\int_0^\infty |\frac{\sin x}{x}| dx = \infty$. The power of the Lebesgue integral to deal directly with 'absolutely integrable' unbounded functions on unbounded domains means that what one might call 'conditionally integrable' functions are pushed into the background of the theory. In Chapter 48 of Volume 4 I will discuss the general theory of such functions, but for the time being I will deal with them individually, on the rare occasions when they arise.

***134L** There is in fact a beautiful characterization of the Riemann integrable functions, as follows.

Proposition If $a < b$ in $\mathbb{R}$, a bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable iff it is continuous almost everywhere in $[a,b]$.

proof (a) Suppose that f is Riemann integrable. For each $x \in [a,b]$, set

$$g(x) = \sup_{\delta>0} \inf_{y\in[a,b],|y-x|\le\delta} f(y),$$

$$h(x) = \inf_{\delta>0} \sup_{y\in[a,b],|y-x|\le\delta} f(y),$$

so that f is continuous at x iff $g(x) = h(x)$. We have $g \le f \le h$, so if D is any dissection of $[a,b]$ then $S_D(g) \le S_D(f) \le S_D(h)$ and $s_D(g) \le s_D(f) \le s_D(h)$. But in fact $S_D(f) = S_D(h)$ and $s_D(g) = s_D(f)$, because on any open interval $]c,d[\subseteq [a,b]$ we must have

$$\inf_{x\in]c,d[} g(x) = \inf_{x\in]c,d[} f(x), \quad \sup_{x\in]c,d[} f(x) = \sup_{x\in]c,d[} h(x).$$

It follows that

$$L_{[a,b]}(f) = L_{[a,b]}(g) \le U_{[a,b]}(g) \le U_{[a,b]}(f),$$

$$L_{[a,b]}(f) \le L_{[a,b]}(h) \le U_{[a,b]}(h) = U_{[a,b]}(f).$$

Because f is Riemann integrable, both g and h must be Riemann integrable, with integrals equal to $\mathrm{R}\!\int_a^b f$. By 134Kb, they are both Lebesgue integrable, with the same integral. But $g \le h$, so $g =_{\text{a.e.}} h$, by 122Rd. Now f is continuous at any point where g and h agree, so f is continuous a.e.

(b) Now suppose that f is continuous a.e. For each $n \in \mathbb{N}$, let D_n be the dissection of $[a,b]$ into 2^n equal portions. Set

$$h_n(x) = \sup_{y\in]c,d[} f(y), \quad g_n(x) = \inf_{y\in]c,d[} f(y)$$

if $]c,d[$ is an open interval of D_n containing x; for definiteness, say $h_n(x) = g_n(x) = f(x)$ if x is one of the points of the list D_n. Then $\langle g_n\rangle_{n\in\mathbb{N}}$, $\langle h_n\rangle_{n\in\mathbb{N}}$ are, respectively, increasing and decreasing sequences of functions, each function constant on each of a finite family of intervals covering $[a,b]$; and $s_{D_n}(f) = \int g_n d\mu$, $S_{D_n}(f) = \int h_n d\mu$. Next,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) = f(x)$$

at any point x at which f is continuous; so $f =_{\text{a.e.}} \lim_{n\to\infty} g_n =_{\text{a.e.}} \lim_{n\to\infty} h_n$. By Lebesgue's Dominated Convergence Theorem (123C),

$$\lim_{n\to\infty} \int g_n d\mu = \int f d\mu = \lim_{n\to\infty} \int h_n d\mu;$$

but this means that

$$L_{[a,b]}(f) \ge \int f d\mu \ge U_{[a,b]}(f),$$

so these are all equal and f is Riemann integrable.

134X Basic exercises >(a) Show that if f is an integrable real-valued function on $\mathbb{R}^r$, then $\int f(x+a)dx$ exists and is equal to $\int f$ for every $a \in \mathbb{R}^r$. (*Hint*: start with simple functions f.)

(b) More generally, show that if $E \subseteq \mathbb{R}^r$ is measurable and f is a real-valued function which is integrable over E in the sense of 131D, then $\int_{E-a} f(x+a)dx$ exists and is equal to $\int_E f$ for every $a \in \mathbb{R}^r$.

(c) Show that if $C \subseteq \mathbb{R}$ is any non-negligible set, it has a non-measurable subset. (*Hint*: use the method of 134B, taking the relation $\sim$ on a suitable bounded subset of C in place of $[\mathbf{0},\mathbf{1}[$.)

>(d) Let ν_g be a Lebesgue-Stieltjes measure on $\mathbb{R}$, constructed as in 114Xa from a non-decreasing function $g : \mathbb{R} \to \mathbb{R}$, and Σ_g its domain. (See also 132Xg.) Show that

(i) if $A \subseteq \mathbb{R}$ is any set, then

$$\begin{aligned}\nu_g^* A &= \inf\{\nu_g G : G \text{ is open, } G \supseteq A\}\\ &= \min\{\nu_g H : H \text{ is Borel, } H \supseteq A\};\end{aligned}$$

(ii) if $E \in \Sigma_g$, then

$$\nu_g E = \sup\{\nu_g F : F \text{ is closed and bounded, } F \subseteq E\},$$

and there are Borel sets H_1, H_2 such that $H_1 \subseteq E \subseteq H_2$ and $\nu_g(H_2 \setminus H_1) = \nu_g(H_2 \setminus E) = \nu_g(E \setminus H_1) = 0$;

(iii) if $A \subseteq \mathbb{R}$ is any set, then A has a measurable envelope which is a Borel set;

(iv) if f is a Σ_g-measurable real-valued function defined on a subset of $\mathbb{R}$, then there is a ν_g-conegligible Borel set $H \subseteq \mathbb{R}$ such that $f{\restriction}H$ is Borel measurable.

(e) Let $E \subseteq \mathbb{R}^r$ be a measurable set, and $\epsilon > 0$. (i) Show that there is an open set $G \supseteq E$ such that $\mu(G \setminus E) \le \epsilon$. (*Hint*: apply 134Fa to each set $E \cap B(\mathbf{0}, n)$.) (ii) Show that there is a closed set $F \subseteq E$ such that $\mu(E \setminus F) \le \epsilon$.

(f) Let $C \subseteq [0,1]$ be the Cantor set. Show that $\{x+y : x, y \in C\} = [0,2]$ and $\{x-y : x, y \in C\} = [-1,1]$.

(g) Let f, g be functions from $\mathbb{R}$ to itself. Show that (i) if f and g are both Borel measurable, so is their composition fg (ii) if f is Borel measurable and g is Lebesgue measurable, then fg is Lebesgue measurable (iii) if f is Lebesgue measurable and g is Borel measurable, then fg need not be Lebesgue measurable.

(h) Show that for any integer $r \ge 1$ there is a measurable set $E \subseteq \mathbb{R}^r$ such that E and $\mathbb{R}^r \setminus E$ both meet every non-empty open interval in a set of strictly positive measure.

(i) Give $[0,1]$ its subspace measure. (i) Show that there is a disjoint sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of subsets of $[0,1]$ all of outer measure 1. (ii) Show that there is a function $f : [0,1] \to]0,1[$ such that $\underline{\int} f = 0$ and $\overline{\int} f = 1$.

(j) Let f be a measurable real function and g a real function such that $\operatorname{dom} g \setminus \operatorname{dom} f$ and $\{x : x \in \operatorname{dom} g \cap \operatorname{dom} f,\ g(x) \ne f(x)\}$ are both negligible. Show that g is measurable.

134Y Further exercises (a) Fix $c > 0$. For $A \subseteq \mathbb{R}^r$ set $cA = \{cx : x \in A\}$. (i) Show that $\mu^*(cA) = c^r \mu^* A$ for every $A \subseteq \mathbb{R}^r$. (ii) Show that cE is measurable for every measurable $E \subseteq \mathbb{R}^r$.

(b) Let $\langle f_{mn} \rangle_{m,n\in\mathbb{N}}$, $\langle f_m \rangle_{m\in\mathbb{N}}$, f be real-valued measurable functions defined almost everywhere in $\mathbb{R}^r$ and such that $f_m =_{\text{a.e.}} \lim_{n\to\infty} f_{mn}$ for each m and $f =_{\text{a.e.}} \lim_{m\to\infty} f_m$. Show that there is a sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ such that $f =_{\text{a.e.}} \lim_{k\to\infty} f_{k,n_k}$. (*Hint*: take n_k such that the measure of $\{x : \|x\| \le k,\ |f_k(x) - f_{k,n_k}(x)| \ge 2^{-k}\}$ is at most 2^{-k} for each k.)

(c) Let f be a measurable real-valued function defined almost everywhere in $\mathbb{R}^r$. Show that there is a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of continuous functions converging to f almost everywhere. (*Hint*: Deal successively with the cases (i) $f = \chi I$ where I is a half-open interval (ii) $f = \chi(\bigcup_{j\le n} I_j)$ where $I_0, \dots, I_n$ are disjoint half-open intervals (iii) $f = \chi E$ where E is a measurable set of finite measure (iv) f is a simple function (v) general f, using 134Yb at steps (iii) and (v).)

(d) Let f be a real-valued function defined on a subset of $\mathbb{R}^r$. Show that the following are equiveridical: (i) f is measurable (ii) whenever $E \subseteq \mathbb{R}^r$ is measurable and $\mu E > 0$, there is a measurable set $F \subseteq E$ such that $\mu F > 0$ and $f{\restriction}F$ is continuous (iii) whenever $E \subseteq \mathbb{R}^r$ is measurable and $\gamma < \mu E$, there is a measurable $F \subseteq E$ such that $\mu F \ge \gamma$ and $f{\restriction}F$ is continuous. (*Hint*: for (i)⇒(iii), use 134Yc and 131Ya; for (ii)⇒(i) use 121D. This is a version of **Lusin's theorem**.)

(e) Let ν be a measure on $\mathbb{R}$ which is translation-invariant in the sense of 134Ab, and such that $\nu[0,1]$ is defined and equal to 1. Show that ν agrees with Lebesgue measure on the Borel sets of $\mathbb{R}$. (*Hint*: Show first that $[a,1]$ belongs to the domain of ν for every $a \in [0,1]$, and hence that every half-open interval of length at most 1 belongs to the domain of ν; show that $\nu\,[a, a+2^{-n}[\, = 2^{-n}$ for every $a \in \mathbb{R}$, $n \in \mathbb{N}$, and hence that $\nu\,[a,b[\, = b-a$ whenever $a < b$.)

(f) Let ν be a measure on $\mathbb{R}^r$ which is translation-invariant in the sense of 134Ab, where $r > 1$, and such that $\nu[\mathbf{0},\mathbf{1}]$ is defined and equal to 1. Show that ν agrees with Lebesgue measure on the Borel sets of $\mathbb{R}^r$.

(g) Show that if f is any real-valued integrable function on $\mathbb{R}$, and $\epsilon > 0$, there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\{x : g(x) \neq 0\}$ is bounded and $\int |f-g| \le \epsilon$. (*Hint*: show that the set Φ of functions f with this property satisfies the conditions of 122Yb.)

(h) Repeat 134Yg for real-valued integrable functions on $\mathbb{R}^r$, where $r > 1$.

(i) Repeat 134Fd, 134Xa, 134Xb, 134Yb, 134Yc, 134Yd, 134Yg and 134Yh for complex-valued functions.

(j) Show that if $G \subseteq \mathbb{R}^r$ is open and not empty, it is expressible as a disjoint union of a sequence of half-open intervals each of the form $\{x : 2^{-m}n_i \le \xi_i < 2^{-m}(n_i+1)$ for every $i \le r\}$ where $m \in \mathbb{N}$, $n_1, \dots, n_r \in \mathbb{Z}$.

(k) Show that a set $E \subseteq \mathbb{R}^r$ is Lebesgue negligible iff there is a sequence $\langle C_n \rangle_{n\in\mathbb{N}}$ of hypercubes in $\mathbb{R}^r$ such that $E \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{k\ge n} C_k$ and $\sum_{k=0}^{\infty} (\operatorname{diam} C_k)^r < \infty$, writing $\operatorname{diam} C_k$ for the diameter of C_k.

(l) Show that there is a continuous function $f : [0,1] \to [0,1]^2$ such that $\mu_1 f^{-1}[E] = \mu_2 E$ for every measurable $E \subseteq [0,1]^2$, writing μ_1, μ_2 for Lebesgue measure on $\mathbb{R}$, $\mathbb{R}^2$ respectively. (*Hint*: for each $n \in \mathbb{N}$, express $[0,1]^2$ as the union of 4^n closed squares of side 2^{-n}; call the set of these squares $\mathcal{D}_n$. Construct continuous $f_n : [0,1] \to [0,1]^2$, families $\langle I_D \rangle_{D\in\mathcal{D}_n}$ inductively in such a way that each I_D is a closed interval of length 4^{-n} and $f_m[I_D] \subseteq D$ whenever $D \in \mathcal{D}_n$ and $m \ge n$. The induction will proceed more smoothly if you suppose that the path f_n enters each square in $\mathcal{D}_n$ at a corner and leaves at an adjacent corner. Take $f = \lim_{n\to\infty} f_n$. This is a special kind of **Peano** or **space-filling** curve.)

(m) Show that if $r \le s$ there is a continuous function $f : [0,1]^r \to [0,1]^s$ such that $\mu_r f^{-1}[E] = \mu_s E$ for every measurable $E \subseteq [0,1]^s$, writing μ_r, μ_s for Lebesgue measure on $\mathbb{R}^r$, $\mathbb{R}^s$ respectively.

(n) Show that there is a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $\mu_1 f^{-1}[E] = \mu_2 E$ for every measurable $E \subseteq \mathbb{R}^2$, writing μ_1, μ_2 for Lebesgue measure on $\mathbb{R}$, $\mathbb{R}^2$ respectively.

(o) Show that the function $f : [0,1] \to [0,1]^2$ of 134Yl may be chosen in such a way that $\mu_2 f[E] = \mu_1 E$ for every Lebesgue measurable set $E \subseteq [0,1]$. (*Hint*: using the construction suggested in 134Yl, and setting $H = f^{-1}[([0,1]\setminus\mathbb{Q})^2]$, $f{\restriction}H$ will be an isomorphism between $(H, \mu_{1,H})$ and $(f[H], \mu_{2,f[H]})$, writing $\mu_{1,H}$ and $\mu_{2,f[H]}$ for the subspace measures.)

(p) Show that $\mathbb{R}$ can be expressed as the union of a disjoint sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of sets of finite measure such that $\mu(I \cap E_n) > 0$ for every non-empty open interval $I \subseteq \mathbb{R}$ and every $n \in \mathbb{N}$.

(q) Show that for any $r \ge 1$, $\mathbb{R}^r$ can be expressed as the union of a disjoint sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of sets of finite measure such that $\mu(G \cap E_n) > 0$ for every non-empty open set $G \subseteq \mathbb{R}^r$ and every $n \in \mathbb{N}$.

(r) Show that there is a disjoint sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}$ such that $\mu^*(A_n \cap E) = \mu E$ for every measurable set E and every $n \in \mathbb{N}$. (*Remark*: in fact there is a disjoint family $\langle A_t \rangle_{t\in\mathbb{R}}$ with this property, but I think a new idea is needed for this extension. See 419I in Volume 4.)

(s) Repeat 134Yr for $\mathbb{R}^r$, where $r > 1$.

(t) Describe a Borel measurable function $f : [0,1] \to [0,1]$ such that $f{\restriction}A$ is discontinuous at every point of A whenever $A \subseteq [0,1]$ is a set of full outer measure.

(u) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence of non-negligible measurable subsets of $\mathbb{R}^r$. Show that there is a measurable set $E\subseteq\mathbb{R}^r$ such that all the sets $E_n\cap E$, $E_n\setminus E$ are non-negligible.

134 Notes and comments Lebesgue measure enjoys an enormous variety of special properties, corresponding to the richness of the real line, with its algebraic and topological and order structures. Here I have only been able to hint at what is possible.

There are many methods of constructing non-measurable sets, all significant; the one I give in 134B is perhaps the most accessible, and shows that translation-invariance is (subject to the axiom of choice) an insuperable barrier to measuring every subset of $\mathbb{R}$.

In 134F I list some of the basic relationships between the measure and the topology of Euclidean space. Others are in 134Yc, 134Yd and 134Yg; see also 134Xd. A systematic analysis of these will take up a large part of Volume 4.

The Cantor set and function (134G-134I) form one of the basic examples in the theory. Here I present them just as an interesting design and as a counter-example to a natural conjecture. But they will reappear in three different chapters of Volume 2 as illustrations of three quite different phenomena.

The relationship between the Lebesgue and Riemann integrals goes a good deal deeper than I wish to explore just at present; the fact that the Lebesgue integral extends the Riemann integral (134Kb) is only a small part of the story, and I should be sorry if you were left with the impression that the Lebesgue integral therefore renders the Riemann integral obsolete. Without going into the details here, I hope that 134F and 134Yg make it plain that the Lebesgue integral is in some sense the canonical extension of the Riemann integral. (This, at least, I shall return to in Chapter 43.) Another way of looking at this is 134Yf; the Lebesgue integral is the basic translation-invariant integral on $\mathbb{R}^r$.

Version of 14.9.04/14.7.07

135 The extended real line

It is often convenient to allow '∞' into our formulae, and in the context of measure theory the appropriate manipulations are sufficiently consistent for it to be possible to develop a theory of the **extended real line**, the set $[-\infty,\infty]=\mathbb{R}\cup\{-\infty,\infty\}$, sometimes written $\overline{\mathbb{R}}$. I give a brief account without full proofs, as I hope that by the time this material becomes necessary to the arguments I use it will all appear thoroughly elementary.

135A The algebraic structure of $[-\infty,\infty]$ **(a)** If we write

$$a+\infty=\infty+a=\infty,\quad a+(-\infty)=(-\infty)+a=-\infty$$

for every $a\in\mathbb{R}$, and

$$\infty+\infty=\infty,\quad (-\infty)+(-\infty)=-\infty,$$

but refuse to define $\infty+(-\infty)$ or $(-\infty)+\infty$, we obtain a partially-defined binary operation on $[-\infty,\infty]$, extending ordinary addition on $\mathbb{R}$. This is *associative* in the sense that

if u, v, $w\in[-\infty,\infty]$ and one of $u+(v+w)$, $(u+v)+w$ is defined, so is the other, and they are then equal,

and *commutative* in the sense that

if u, $v\in[-\infty,\infty]$ and one of $u+v$, $v+u$ is defined, so is the other, and they are then equal.

It has an *identity* 0 such that $u+0=0+u=u$ for every $u\in[-\infty,\infty]$; but ∞ and $-\infty$ lack inverses.

(b) If we define

$$a\cdot\infty=\infty\cdot a=\infty,\quad a\cdot(-\infty)=(-\infty)\cdot a=-\infty$$

for real $a>0$,

$$a\cdot\infty=\infty\cdot a=-\infty,\quad a\cdot(-\infty)=(-\infty)\cdot a=\infty$$

for real $a<0$,

$$\infty\cdot\infty=(-\infty)\cdot(-\infty)=\infty,\quad (-\infty)\cdot\infty=\infty\cdot(-\infty)=-\infty,$$

$$0 \cdot \infty = \infty \cdot 0 = 0 \cdot (-\infty) = (-\infty) \cdot 0 = 0$$

then we obtain a binary operation on $[-\infty, \infty]$ extending ordinary multiplication on $\mathbb{R}$, which is associative and commutative and has an identity 1; 0, ∞ and $-\infty$ lack inverses.

(c) We have a *distributive law*, a little weaker than the associative and commutative laws of addition:

if u, v, $w \in [-\infty, \infty]$ and both $u(v+w)$ and $uv+uw$ are defined, then they are equal.

(But note the problems which arise with such combinations as $\infty(1+(-2))$, $0 \cdot \infty + 0 \cdot (-\infty)$.)

(d) While ∞ and $-\infty$ do not have inverses in the semigroup $([-\infty, \infty], \cdot)$, there seems no harm in writing $a/\infty = a/(-\infty) = 0$ for every $a \in \mathbb{R}$. But of course such an extension of the notion of division must be watched carefully in such formulae as $u \cdot \frac{v}{u}$.

135B The order structure of $[-\infty, \infty]$ **(a)** If we write

$$-\infty \le u \le \infty \text{ for every } u \in [-\infty, \infty],$$

we obtain a relation on $[-\infty, \infty]$, extending the usual ordering of $\mathbb{R}$, which is a *total* ordering, that is,

for any u, v, $w \in [-\infty, \infty]$, if $u \le v$ and $v \le w$ then $u \le w$,

$u \le u$ for every $u \in [-\infty, \infty]$,

for any u, $v \in [-\infty, \infty]$, if $u \le v$ and $v \le u$ then $u = v$,

for any u, $v \in [-\infty, \infty]$, either $u \le v$ or $v \le u$.

Moreover, every subset of $[-\infty, \infty]$ has a supremum and an infimum, if we write $\sup \emptyset = -\infty$, $\inf \emptyset = \infty$.

(b) The ordering is 'translation-invariant' in the weak sense that

if u, v, $w \in [-\infty, \infty]$ and $v \le w$ and $u+v$, $u+w$ are both defined, then $u+v \le u+w$.

It is preserved by non-negative multiplications in the sense that

if u, v, $w \in [-\infty, \infty]$ and $0 \le u$ and $v \le w$, then $uv \le uw$,

while it is reversed by non-positive multiplications in the sense that

if u, v, $w \in [-\infty, \infty]$ and $u \le 0$ and $v \le w$, then $uw \le uv$.

135C The Borel structure of $[-\infty, \infty]$ We say that a set $E \subseteq [-\infty, \infty]$ is a **Borel set** in $[-\infty, \infty]$ if $E \cap \mathbb{R}$ is a Borel subset of $\mathbb{R}$. It is easy to check that the family of such sets is a σ-algebra of subsets of $[-\infty, \infty]$. See also 135Xb below.

135D Convergent sequences in $[-\infty, \infty]$ We can say that a sequence $\langle u_n \rangle_{n \in \mathbb{N}}$ in $[-\infty, \infty]$ **converges** to $u \in [-\infty, \infty]$ if

whenever $v < u$ there is an $n_0 \in \mathbb{N}$ such that $v \le u_n$ for every $n \ge n_0$, and whenever $u < v$ there is an $n_0 \in \mathbb{N}$ such that $u_n \le v$ for every $n \ge n_0$;

alternatively,

either $u \in \mathbb{R}$ and for every $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that $u_n \in [u-\delta, u+\delta]$ for every $n \ge n_0$

or $u = -\infty$ and for every $a \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}$ such that $u_n \le a$ for every $n \ge n_0$

or $u = \infty$ and for every $a \in \mathbb{R}$ there is an $n_0 \in \mathbb{N}$ such that $u_n \ge a$ for every $n \ge n_0$.

(Compare the notion of convergence in 112Ba.)

135E Measurable functions Let X be any set and Σ a σ-algebra of subsets of X.

(a) Let D be a subset of X and Σ_D the subspace σ-algebra (121A). For any function $f : D \to [-\infty, \infty]$, the following are equiveridical:

(i) $\{x : f(x) < u\} \in \Sigma_D$ for every $u \in [-\infty, \infty]$;

(ii) $\{x : f(x) \le u\} \in \Sigma_D$ for every $u \in [-\infty, \infty]$;

(iii) $\{x : f(x) > u\} \in \Sigma_D$ for every $u \in [-\infty, \infty]$;

(iv) $\{x : f(x) \ge u\} \in \Sigma_D$ for every $u \in [-\infty, \infty]$;

(v) $\{x : f(x) \le q\} \in \Sigma_D$ for every $q \in \mathbb{Q}$.

P The proof is almost identical to that of 121B. The only modifications are:

– in (i)⇒(ii), $\{x : f(x) \le \infty\}$ and $\{x : f(x) \le -\infty\}$ are not necessarily equal to $\bigcap_{n\in\mathbb{N}}\{x : f(x) < \infty + 2^{-n}\}$, $\bigcap_{n\in\mathbb{N}}\{x : f(x) < -\infty + 2^{-n}\}$; but the former is D, so surely belongs to Σ_D, and the latter is $\bigcap_{n\in\mathbb{N}}\{x : f(x) < -n\}$, so belongs to Σ_D.

– In (iii)⇒(iv), similarly, we have to use the facts that

$$\{x : f(x) \ge -\infty\} = D \in \Sigma_D, \quad \{x : f(x) \ge \infty\} = \bigcap_{n\in\mathbb{N}}\{x : f(x) > n\} \in \Sigma_D.$$

– Concerning the extra condition (v), of course we have (ii)⇒(v), but also we have (v)⇒(i), because

$$\{x : f(x) < u\} = \bigcup_{q\in\mathbb{Q},q<u}\{x : f(x) \le q\}$$

for every $u \in [-\infty, \infty]$. **Q**

(b) We may therefore say, as in 121C, that a function taking values in $[-\infty, \infty]$ is **measurable** if it satisfies these equivalent conditions.

(c) Note that if $f : D \to [-\infty, \infty]$ is Σ-measurable, then

$$E_\infty(f) = f^{-1}[\{\infty\}] = \{x : f(x) \ge \infty\}, \quad E_{-\infty}(f) = f^{-1}[\{-\infty\}] = \{x : f(x) \le -\infty\}$$

must belong to Σ_D, while $f_{\mathbb{R}} = f\restriction D \setminus (E_\infty(f) \cup E_{-\infty}(f))$, the 'real-valued part of f', is measurable in the sense of 121C.

(d) Conversely, if E_∞ and $E_{-\infty}$ belong to Σ_D, and $f_{\mathbb{R}} : D \setminus (E_\infty \cup E_{-\infty}) \to \mathbb{R}$ is measurable, then $f : D \to [-\infty, \infty]$ will be measurable, where $f(x) = \infty$ if $x \in E_\infty$, $f(x) = -\infty$ if $x \in E_{-\infty}$ and $f(x) = f_{\mathbb{R}}(x)$ for other $x \in D$.

(e) It follows that if f, g are measurable functions from subsets of X to $[-\infty, \infty]$, then $f + g$, $f \times g$ and f/g are measurable. **P** This can be proved either by adapting the arguments of 121Eb, 121Ed and 121Ee, or by applying those results to $f_{\mathbb{R}}$ and $g_{\mathbb{R}}$ and considering separately the sets on which one or both are infinite. **Q**

(f) We can say that a function h from a subset D of $[-\infty, \infty]$ to $[-\infty, \infty]$ is **Borel measurable** if it is measurable (in the sense of (b) above) with respect to the Borel σ-algebra of $[-\infty, \infty]$ (as defined in 135C). Now if X is a set, Σ is a σ-algebra of subsets of X, f is a measurable function from a subset of X to $[-\infty, \infty]$ and h is a Borel measurable function from a subset of $[-\infty, \infty]$ to $[-\infty, \infty]$, then hf is measurable. **P** Apply 121Eg to $h^* f_{\mathbb{R}}$, where $h^* = h\restriction(\mathbb{R} \cap h^{-1}[\mathbb{R}])$, and then look separately at the sets $\{x : f(x) = \pm\infty\}$, $\{x : hf(x) = \pm\infty\}$. **Q**

(g) Let X be a set and Σ a σ-algebra of subsets of X. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of measurable functions from subsets of X to $[-\infty, \infty]$. Then $\lim_{n\to\infty} f_n$, $\sup_{n\in\mathbb{N}} f_n$ and $\inf_{n\in\mathbb{N}} f_n$ are measurable, if, following the principles set out in 121F, we take their domains to be

$$\{x : x \in \bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n} \operatorname{dom} f_m,\ \lim_{n\to\infty} f_n(x) \text{ exists in } [-\infty, \infty]\},$$

$$\bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n.$$

P Follow the method of 121Fa-121Fc. **Q**

135F $[-\infty, \infty]$-valued integrable functions (a) We are surely not going to admit a function as 'integrable' unless it is finite almost everywhere, and for such functions the remarks in 133B are already adequate.

(b) However, it is possible to make a consistent extension of the idea of an infinite integral, elaborating slightly the ideas of 133A. If (X, Σ, μ) is a measure space and f is a function, defined almost everywhere in X, taking values in $[0, \infty]$, and virtually measurable (that is, such that $f\restriction E$ is measurable in the sense of 135E for some conegligible set E), then we can safely write '$\int f = \infty$' whenever f is not integrable. We shall find that for such functions we have $\int f + g = \int f + \int g$ and $\int cf = c\int f$ for every $c \in [0, \infty]$, using the definitions given above for addition and multiplication on $[0,\infty]$. Consequently, as in 122M-122O, we can say that for a general virtually measurable function f, defined almost everywhere in X, taking values in $[-\infty,\infty]$, $\int f=\int f_1-\int f_2$ whenever f is expressible as a difference f_1-f_2 of non-negative functions such that $\int f_1$ and $\int f_2$ are both defined and not both infinite. Now we have, as always, the basic formulae

$$\int f+g=\int f+\int g,\quad \int cf=c\int f,\quad \int|f|\ge|\int f|$$

whenever the right-hand-sides are defined, and $\int f\le\int g$ whenever $f\le_{\text{a.e.}}g$ and both integrals are defined. It is important to note that $\int f$ can be finite, on this definition, only when f is finite almost everywhere.

135G We now have versions of B.Levi's theorem and Fatou's Lemma (compare 133K).

Proposition Let (X,Σ,μ) be a measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of $[-\infty,\infty]$-valued functions defined almost everywhere in X which have integrals defined in $[-\infty,\infty]$.

(a) If $f_n\le_{\text{a.e.}}f_{n+1}$ for every n and $-\infty<\sup_{n\in\mathbb{N}}\int f_n$, then $\int\sup_{n\in\mathbb{N}}f_n=\sup_{n\in\mathbb{N}}\int f_n$.

(b) If, for each n, $f_n\ge 0$ a.e., then $\int\liminf_{n\to\infty}f_n\le\liminf_{n\to\infty}\int f_n$.

proof (a) Note that $f=\sup_{n\in\mathbb{N}}f_n$ is defined everywhere on $\bigcap_{n\in\mathbb{N}}\operatorname{dom}f_n$, which is almost everywhere; and that there is a conegligible set E such that $f_n\restriction E$ is measurable for every n, so that $f\restriction E$ is measurable. Now if $u=\sup_{n\in\mathbb{N}}\int f_n$ is finite, then all but finitely many of the f_n must be finite almost everywhere, and the result is a consequence of B.Levi's theorem for real-valued functions; while if $u=\infty$ then surely $\int\sup_{n\in\mathbb{N}}f_n$ is infinite.

(b) As in 123B or 133Kb, this now follows, applying (a) to $g_n=\inf_{m\ge n}f_m$.

135H Upper and lower integrals again (a) To handle functions taking values in $[-\infty,\infty]$ we need to adapt the definitions in 133I. Let (X,Σ,μ) be a measure space and f a $[-\infty,\infty]$-valued function defined almost everywhere in X. Its **upper integral** is

$$\overline{\int}f=\inf\{\int g:\int g\text{ is defined in the sense of 135F and }f\le_{\text{a.e.}}g\},$$

allowing ∞ for $\inf\{\infty\}$ and $-\infty$ for $\inf\,]-\infty,\infty]$ or $\inf[-\infty,\infty]$. Similarly, the **lower integral** of f is

$$\underline{\int}f=\sup\{\int g:\int g\text{ is defined, }f\ge_{\text{a.e.}}g\}.$$

With this modification, all the results of 133J are valid for functions taking values in $[-\infty,\infty]$ rather than in $\mathbb{R}$.

(b) Corresponding to 133Ka, we have the following. Let (X,Σ,μ) be a measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of $[-\infty,\infty]$-valued functions defined almost everywhere in X.

(i) If $f_n\le_{\text{a.e.}}f_{n+1}$ for every n and $\sup_{n\in\mathbb{N}}\overline{\int}f_n>-\infty$, then $\overline{\int}\sup_{n\in\mathbb{N}}f_n=\sup_{n\in\mathbb{N}}\overline{\int}f_n$.

(ii) If, for each n, $f_n\ge 0$ a.e., then $\overline{\int}\liminf_{n\to\infty}f_n\le\liminf_{n\to\infty}\overline{\int}f_n$.

135I Subspace measures We need to re-examine the ideas of §131 in the new context.

Proposition Let (X,Σ,μ) be a measure space, and $H\in\Sigma$; write Σ_H for the subspace σ-algebra on H and μ_H for the subspace measure. For any $[-\infty,\infty]$-valued function f defined on a subset of H, write $\tilde f$ for the extension of f defined by saying that $\tilde f(x)=f(x)$ if $x\in\operatorname{dom}f$, 0 if $x\in X\setminus H$.

(a) Suppose that f is a $[-\infty,\infty]$-valued function defined on a subset of H.

(i) $\operatorname{dom}f$ is μ_H-conegligible iff $\operatorname{dom}\tilde f$ is μ-conegligible.

(ii) f is μ_H-virtually measurable iff $\tilde f$ is μ-virtually measurable.

(iii) $\int_H f d\mu_H=\int_X\tilde f d\mu$ if either is defined in $[-\infty,\infty]$.

(b) Suppose that h is a $[-\infty,\infty]$-valued function defined almost everywhere in X. Then $\int_H(h\restriction H)d\mu_H=\int h\times\chi H\,d\mu$ if either is defined in $[-\infty,\infty]$.

(c) If h is a $[-\infty,\infty]$-valued function and $\int_X h\,d\mu$ is defined in $[-\infty,\infty]$, then $\int_H(h\restriction H)d\mu_H$ is defined in $[-\infty,\infty]$.

(d) Suppose that h is a $[-\infty,\infty]$-valued function defined almost everywhere in X. Then

$$\overline{\int}_H (h\restriction H) d\mu_H = \overline{\int}_X h \times \chi H d\mu.$$

proof (a)(i) This is immediate from 131Ca, since $H \setminus \operatorname{dom} f = X \setminus \operatorname{dom} \tilde{f}$.

(ii)(α) If f is μ_H-virtually measurable, there is a μ_H-conegligible $E \in \Sigma_H$ such that $f\restriction E$ is Σ_H-measurable. There is an $F \in \Sigma$ such that $E = F \cap H$; now $G = F \cup (X \setminus H)$ belongs to Σ and $E = G \cap H$ and G is μ-conegligible. Also, for $q \in \mathbb{Q}$,

$$\begin{aligned}\{x : x \in G,\, \tilde{f}(x) \le q\} &= \{x : x \in E,\, f(x) \le q\} \in \Sigma_H \subseteq \Sigma \text{ if } q < 0, \\ &= \{x : x \in E,\, f(x) \le q\} \cup (X \setminus H) \in \Sigma \text{ if } q \ge 0,\end{aligned}$$

so $\tilde{f}\restriction G$ is Σ-measurable and $\tilde{f}$ is μ-virtually measurable.

(β) If $\tilde{f}$ is μ-virtually measurable, there is a μ-conegligible $G \in \Sigma$ such that $\tilde{f}\restriction G$ is Σ-measurable. Now $E = G \cap H$ belongs to Σ_H and is μ_H-conegligible, and for $q \in \mathbb{Q}$

$$\{x : x \in E,\, f(x) \le q\} = H \cap \{x : x \in G,\, f(x) \le q\} \in \Sigma_H.$$

So $f\restriction E$ is Σ_H-measurable and f is μ_H-virtually measurable.

(iii) Assume that at least one of the integrals is defined. Then (ii) tells us that there is a μ-conegligible $E \in \Sigma$ such that $\tilde{f}\restriction E$ is Σ-measurable, in which case $f\restriction H \cap E$ is Σ_H-measurable.

(α) Suppose that f is non-negative everywhere on its domain. Then $\int_H f d\mu_H$ and $\int_X \tilde{f} d\mu$ are both defined in $[0, \infty]$. If both are infinite, we can stop. Otherwise,

$$G = \{x : x \in E \cap H,\, f(x) < \infty\} = \{x : x \in E,\, \tilde{f}(x) < \infty\}$$

must be conegligible. Set $g = f\restriction G \cap H$; then $\tilde{g} = \tilde{f}\restriction G$, so $g = f$ μ_H-a.e. and $\tilde{g} = \tilde{f}$ μ-a.e. Accordingly $\int_H f d\mu_H = \int_H g\, d\mu_H$ and $\int_X \tilde{f} d\mu = \int_X \tilde{g}\, d\mu$. Now we are supposing that at least one of these is finite. But in this case we can apply 131E to see that $\int_H g\, d\mu = \int_X \tilde{g}\, d\mu$, so $\int_H f\, d\mu = \int_X \tilde{f}\, d\mu$.

(β) In general, express f as $f^+ - f^-$, where

$$f^+(x) = \max(0, f(x)), \quad f^-(x) = \max(0, -f(x))$$

for $x \in \operatorname{dom} f$. Then $(f^+)^\sim = \tilde{f}^+$ and $(f^-)^\sim = \tilde{f}^-$. So

$$\int_H f d\mu_H = \int_H f^+ d\mu_H - \int_H f^- d\mu_H = \int_X \tilde{f}^+ d\mu - \int_X \tilde{f}^- d\mu = \int_X \tilde{f} d\mu$$

if any of the four expressions is defined in $[-\infty, \infty]$.

(b) Set $f = h\restriction H$; then $(h \times \chi H)(x) = \tilde{f}(x)$ for every $x \in \operatorname{dom} h$, so (a-iii) tells us that

$$\int_X h \times \chi H\, d\mu = \int_X \tilde{f} d\mu = \int_H (h\restriction H) d\mu_H$$

if any of the three is defined in $[-\infty, \infty]$.

(c) Setting $h^+(x) = \max(0, h(x))$ and $h^-(x) = \max(0, -h(x))$ for $x \in \operatorname{dom} h$, both $\int_X h^+ d\mu$ and $\int_X h^- d\mu$ are defined in $[0, \infty]$, and at most one of them is infinite. In particular, both are μ-virtually measurable and defined μ-almost everywhere, so the same is true of $h^+ \times \chi H$ and $h^- \times \chi H$. As $\int_X h^+ \times \chi H d\mu \le \int_X h^+ d\mu$ and $\int_X h^- \times \chi H d\mu \le \int_X h^- d\mu$, at most one of $\int_X h^+ \times \chi H d\mu$, $\int_X h^- \times \chi H d\mu$ is infinite, and

$$\int_X h \times \chi H d\mu = \int_X h^+ \times \chi H d\mu - \int_X h^- \times \chi H d\mu$$

is defined in $[-\infty, \infty]$. By (b) above, $\int_H (h\restriction H) d\mu_H$ is defined in $[-\infty, \infty]$.

(d)(i) Suppose that $\int_X g\, d\mu$ is defined in $[-\infty, \infty]$ and that $h \times \chi H \le g$ μ-a.e. Then

$$\int_H (g\restriction H) d\mu_H = \int_X g \times \chi H d\mu$$

is defined, by (c); and as $g(x) \ge 0$ for μ-almost every $x \in X \setminus H$, $g \times \chi H \le_{\text{a.e.}} g$. So

$$\overline{\int}_H (h\restriction H) d\mu_H \le \int_H (g\restriction H) d\mu_H = \int_X g \times \chi H d\mu \le \int_X g\, d\mu.$$

As g is arbitrary, $\overline{\int}_H (h\restriction H)d\mu_H \le \overline{\int}_X h \times \chi H\, d\mu$.

(ii) Suppose that $\int_H f d\mu_H$ is defined in $[-\infty,\infty]$ and that $h\restriction H \le f$ μ_H-a.e. Then $\int_X \tilde f d\mu$ is defined in $[-\infty,\infty]$ and $h \times \chi H \le \tilde f$ μ-a.e., so

$$\overline{\int}_X h \times \chi H\, d\mu \le \int_X \tilde f d\mu = \int_H f\, d\mu_H.$$

As f is arbitrary, $\overline{\int}_X h \times \chi H\, d\mu \le \overline{\int}_H (h\restriction H)d\mu_H$.

135X Basic exercises (a) We say that a set $G \subseteq [-\infty,\infty]$ is **open** if (i) $G \cap \mathbb{R}$ is open in the usual sense as a subset of $\mathbb{R}$ (ii) if $\infty \in G$, then there is some $a \in \mathbb{R}$ such that $]a,\infty] \subseteq G$ (iii) if $-\infty \in G$ then there is some $a \in \mathbb{R}$ such that $[-\infty, a[\subseteq G$. Show that the family $\mathfrak{T}$ of open subsets of $[-\infty,\infty]$ has the properties corresponding to (a)-(d) of 1A2B.

(b) Show that the Borel sets of $[-\infty,\infty]$ as defined in 135C are precisely the members of the σ-algebra of subsets of $[-\infty,\infty]$ generated by the open sets as defined in 135Xa.

>(c) Define $\phi : [-\infty,\infty] \to [-1,1]$ by setting

$$\phi(-\infty) = -1, \quad \phi(x) = \tanh x = \frac{e^{2x}-1}{e^{2x}+1} \text{ if } -\infty < x < \infty, \quad \phi(\infty) = 1.$$

Show that (i) ϕ is an order-isomorphism between $[-\infty,\infty]$ and $[-1,1]$ (ii) for any sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $[-\infty,\infty]$, $\langle u_n\rangle_{n\in\mathbb{N}} \to u$ iff $\langle \phi(u_n)\rangle_{n\in\mathbb{N}} \to \phi(u)$ (iii) for any set $E \subseteq [-\infty,\infty]$, E is Borel in $[-\infty,\infty]$ iff $\phi[E]$ is a Borel subset of $\mathbb{R}$ (iv) a real-valued function h defined on a subset of $[-\infty,\infty]$ is Borel measurable iff $h\phi^{-1}$ is Borel measurable.

>(d) Let X be a set, Σ a σ-algebra of subsets of X and f a function from a subset of X to $[-\infty,\infty]$. Show that f is measurable iff the composition ϕf is measurable, where ϕ is the function of 135Xc. Use this to reduce 135Ef and 135Eg to the corresponding results in §121.

(e) Let $\phi : [-\infty,\infty] \to [-1,1]$ be the function described in 135Xc. Show that the functions

$$(t,u) \mapsto \phi(\phi^{-1}(t) + \phi^{-1}(u)) : [-1,1]^2 \setminus \{(-1,1),(1,-1)\} \to [-1,1],$$

$$(t,u) \mapsto \phi(\phi^{-1}(t)\phi^{-1}(u)) : [-1,1]^2 \to [-1,1],$$

$$(t,u) \mapsto \phi(\phi^{-1}(t)/\phi^{-1}(u)) : \big([-1,1] \times ([-1,1] \setminus \{0\})\big) \setminus \{(\pm 1, \pm 1)\} \to [-1,1]$$

are Borel measurable. Use this with 121K to prove 135Ee.

(f) Following the conventions of 135Ab and 135Ad, give full descriptions of the cases in which $uu'/vv' = (u/v)(u'/v')$ and in which $uw/vw = u/v$.

(g) Let (X,Σ,μ) be a measure space and suppose that $E \in \Sigma$ has non-zero finite measure. Let f be a virtually measurable $[-\infty,\infty]$-valued function defined on a subset of X and suppose that $f(x)$ is defined and greater than α for almost every $x \in E$. Show that $\int_E f > \alpha\mu E$.

135Y Further exercises (a) Let X be a set and Σ a σ-algebra of subsets of X. Show that if $f : X \to [0,\infty]$ is Σ-measurable, there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ such that $f = \sum_{n=0}^{\infty} \frac{1}{n+1}\chi E_n$.

(b) Let (X,Σ,μ) be a measure space, and f, g two $[-\infty,\infty]$-valued functions, defined on subsets of X, such that $\int f$ and $\int g$ are both defined in $[-\infty,\infty]$. (i) Show that $\int f \vee g$ and $\int f \wedge g$ are defined in $[-\infty,\infty]$, where $(f \vee g)(x) = \max(f(x),g(x))$, $(f \wedge g)(x) = \min(f(x),g(x))$ for $x \in \operatorname{dom} f \cap \operatorname{dom} g$. (ii) Show that $\int f \vee g + \int f \wedge g = \int f + \int g$ in the sense that if one of the sums is defined in $[-\infty,\infty]$ so is the other, and they are then equal.

(c) Let (X,Σ,μ) be a measure space, $f : X \to [-\infty,\infty]$ a function and $g : X \to [0,\infty]$, $h : X \to [0,\infty]$ measurable functions. Show that $\overline{\int} f \times (g+h) = \overline{\int} f \times g + \overline{\int} f \times h$, where here we interpret $\infty + (-\infty)$ as ∞, as in 133L.

135 Notes and comments I have taken this exposition into a separate section partly because of its length, and partly because I wish to emphasize that these techniques are incidental to the principal ideas of this volume. Really all I am trying to do here is give a coherent account of the language commonly used to deal with a variety of peripheral cases. As a general rule, '∞' enters these arguments only as a shorthand for certain types of triviality. When we find ourselves wishing to assign the values $\pm\infty$ to a function, either this happens on a negligible set – in which case it is often right, if slightly less comforting, to think of the function as undefined on that set – or things have got completely out of hand, and the theory has little useful to tell us.

Of course it is not difficult to incorporate the theory of the extended real line directly into the arguments of Chapter 12, so that the results of this section become the basic ones. I have avoided this route partly in an attempt to reduce the number of new ideas needed in the technically very demanding material of Chapter 12 – believing, as I do, that independently of our treatment of $\pm\infty$ it is absolutely necessary to be able to deal with partially-defined functions – and partly because I do not think that the real line should really be regarded as a substructure of the extended real line. I think that they are different structures with different properties, and that the original real line is overwhelmingly more important. But it is fair to say that in terms of the ideas treated in this volume they are so similar that when you are properly familiar with this work you will be able to move freely from one to the other, so freely indeed that you can safely leave the distinction to formal occasions, such as when you are presenting the statement of a theorem.

Version of 22.6.05

*136 The Monotone Class Theorem

For the final section of this volume, I present two theorems on σ-algebras, with some simple corollaries. They are here because I find no natural home for them in Volume 2. While they (especially 136B) are part of the basic technique of measure theory, and have many and widespread applications, they are not central to the particular approach I have chosen, and can if you wish be left on one side until they come to be needed.

136A Lemma Let X be a set, and $\mathcal{A}$ a family of subsets of X. Then the following are equiveridical:

(i) $X \in \mathcal{A}$, $B \setminus A \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$ and $A \subseteq B$, and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$;

(ii) $\emptyset \in \mathcal{A}$, $X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$, and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{A}$.

proof (i)$\Rightarrow$(ii) Suppose that (i) is true. Then of course $\emptyset = X \setminus X$ belongs to $\mathcal{A}$ and $X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$. If $A, B \in \mathcal{A}$ are disjoint, then $A \subseteq X \setminus B \in \mathcal{A}$, so $(X \setminus B) \setminus A$ and its complement $A \cup B$ belong to $\mathcal{A}$. So if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{A}$, $\bigcup_{i\le n} A_i \in \mathcal{A}$ for every n, and $\bigcup_{n\in\mathbb{N}} A_n$ is the union of a non-decreasing sequence in $\mathcal{A}$, so belongs to $\mathcal{A}$. Thus (ii) is true.

(ii)$\Rightarrow$(i) If (ii) is true, then of course $X = X \setminus \emptyset$ belongs to $\mathcal{A}$. If A and B are members of $\mathcal{A}$ such that $A \subseteq B$, then $X \setminus B$ belongs to $\mathcal{A}$ and is disjoint from A, so $A \cup (X \setminus B)$ and its complement $B \setminus A$ belong to $\mathcal{A}$. Thus the second clause of (i) is satisfied. As for the third, if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$, then $A_0, A_1 \setminus A_0, A_2 \setminus A_1, \dots$ is a disjoint sequence in $\mathcal{A}$, so its union $\bigcup_{n\in\mathbb{N}} A_n$ belongs to $\mathcal{A}$.

Definition If $\mathcal{A} \subseteq \mathcal{P}X$ satisfies the conditions of (i) and/or (ii) above, it is called a **Dynkin class** of subsets of X.

136B Monotone Class Theorem Let X be a set and $\mathcal{A}$ a Dynkin class of subsets of X. Suppose that $\mathcal{I} \subseteq \mathcal{A}$ is such that $I \cap J \in \mathcal{I}$ for all $I, J \in \mathcal{I}$. Then $\mathcal{A}$ includes the σ-algebra of subsets of X generated by $\mathcal{I}$.

proof (a) Let $\mathfrak{S}$ be the family of Dynkin classes of subsets of X including $\mathcal{I}$. Then it is easy to check, using either (i) or (ii) of 136A, that the intersection $\Sigma = \bigcap \mathfrak{S}$ also is a Dynkin class (compare 111Ga). Because $\mathcal{A} \in \mathfrak{S}$, $\Sigma \subseteq \mathcal{A}$.

(b) If $H \in \Sigma$, then

$$\Sigma_H = \{E : E \in \Sigma,\, E \cap H \in \Sigma\}$$

is a Dynkin class. **P** (α) $X \cap H = H \in \Sigma$ so $X \in \Sigma_H$. (β) If A, $B \in \Sigma_H$ and $A \subseteq B$ then $A \cap H$, $B \cap H$ belong to Σ and $A \cap H \subseteq B \cap H$; consequently

$$(B \setminus A) \cap H = (B \cap H) \setminus (A \cap H) \in \Sigma$$

and $B \setminus A \in \Sigma_H$. ($\gamma$) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ_H, then $\langle A_n \cap H\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ, so

$$(\bigcup_{n\in\mathbb{N}} A_n) \cap H = \bigcup_{n\in\mathbb{N}}(A_n \cap H) \in \Sigma$$

and $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma_H$. **Q**

It follows that if $I \cap H \in \Sigma$ for every $I \in \mathcal{I}$, so that $\Sigma_H \supseteq \mathcal{I}$, then $\Sigma_H \in \mathfrak{S}$ and must be equal to Σ.

(c) We find next that $G \cap H \in \Sigma$ for all G, $H \in \Sigma$. **P** Take I, $J \in \mathcal{I}$. We know that $I \cap J \in \mathcal{I}$. As I is arbitrary, $\Sigma_J = \Sigma$ and $H \in \Sigma_J$, that is, $H \cap J \in \Sigma$. As J is arbitrary, $\Sigma_H = \Sigma$ and $G \in \Sigma_H$, that is, $G \cap H \in \Sigma$. **Q**

(d) Since Σ is a Dynkin class, $\emptyset = X \setminus X \in \Sigma$. Also

$$G \cup H = X \setminus ((X \setminus G) \cap (X \setminus H)) \in \Sigma$$

for any G, $H \in \Sigma$ (using (c)). So if $\langle G_n\rangle_{n\in\mathbb{N}}$ is any sequence in Σ, $G'_n = \bigcup_{i\le n} G_i \in \Sigma$ for each n (inducing on n). But $\langle G'_n\rangle_{n\in\mathbb{N}}$ is now a non-decreasing sequence in Σ, so

$$\bigcup_{n\in\mathbb{N}} G_n = \bigcup_{n\in\mathbb{N}} G'_n \in \Sigma.$$

This means that Σ satisfies all the conditions of 111A and is a σ-algebra of subsets of X. Since $\mathcal{I} \subseteq \Sigma$, Σ must include the σ-algebra Σ' of subsets of X generated by $\mathcal{I}$. So $\Sigma' \subseteq \Sigma \subseteq \mathcal{A}$, as required.

(Actually, of course, $\Sigma = \Sigma'$, because $\Sigma' \in \mathfrak{S}$.)

Remark I have seen this result called the **Sierpiński Class Theorem** and the **π-λ Theorem**.

136C Corollary Let X be a set, and μ, ν two measures defined on X with domains Σ, T respectively. Suppose that $\mu X = \nu X < \infty$, and that $\mathcal{I} \subseteq \Sigma \cap \mathrm{T}$ is a family of sets such that $\mu I = \nu I$ for every $I \in \mathcal{I}$ and $I \cap J \in \mathcal{I}$ for all I, $J \in \mathcal{I}$. Then $\mu E = \nu E$ for every E in the σ-algebra of subsets of X generated by $\mathcal{I}$.

proof The point is that

$$\mathcal{A} = \{H : H \in \Sigma \cap \mathrm{T},\, \mu H = \nu H\}$$

is a Dynkin class of subsets of X. **P** I work from (ii) of 136A. Of course $\emptyset \in \mathcal{A}$. If $A \in \mathcal{A}$ then

$$\mu(X \setminus A) = \mu X - \mu A = \nu X - \nu A = \nu(X \setminus A)$$

(because $\mu X = \nu X < \infty$, so the subtraction is safe), and $X \setminus A \in \mathcal{A}$. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{A}$, then

$$\mu A = \sum_{n=0}^{\infty} \mu A_n = \sum_{n=0}^{\infty} \nu A_n = \nu A,$$

and $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$. **Q**

Since $\mathcal{I} \subseteq \mathcal{A}$, 136B tells us that the σ-algebra Σ' generated by $\mathcal{I}$ is included in $\mathcal{A}$, that is, μ and ν agree on Σ'.

136D Corollary Let μ, ν be two measures on $\mathbb{R}^r$, where $r \ge 1$, both defined, and agreeing, on all intervals of the form

$$\left]-\infty, a\right] = \{x : x \le a\} = \{(\xi_1, \dots, \xi_r) : \xi_i \le \alpha_i \text{ for every } i \le r\}$$

for $a = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$. Suppose further that $\mu\mathbb{R}^r < \infty$. Then μ and ν agree on all the Borel subsets of $\mathbb{R}^r$.

proof In 136C, take $X = \mathbb{R}^r$ and $\mathcal{I}$ the set of intervals $\left]-\infty, a\right]$. Then $I \cap J \in \mathcal{I}$ for all I, $J \in \mathcal{I}$, since $\left]-\infty, a\right] \cap \left]-\infty, b\right] = \left]-\infty, a \wedge b\right]$, writing $a \wedge b = (\min(\alpha_1, \beta_1), \dots, \min(\alpha_r, \beta_r))$ if $a = (\alpha_1, \dots, \alpha_r)$, $b = (\beta_1, \dots, \beta_r) \in \mathbb{R}^r$. Also, setting $\mathbf{n} = (n, \dots, n)$ for $n \in \mathbb{N}$,

$$\nu\mathbb{R}^r = \lim_{n\to\infty} \nu\,]-\infty, \mathbf{n}] = \lim_{n\to\infty} \mu\,]-\infty, \mathbf{n}] = \mu\mathbb{R}^r.$$

So all the conditions of 136C are satisfied and μ, ν agree on the σ-algebra Σ generated by $\mathcal{I}$. But this is just the σ-algebra of Borel sets, by 121J.

136E Algebras of sets: Definition Let X be a set. A family $\mathcal{E} \subseteq \mathcal{P}X$ is an **algebra** or **field** of subsets of X if

(i) $\emptyset \in \mathcal{E}$;
(ii) for every $E \in \mathcal{E}$, its complement $X \setminus E$ belongs to $\mathcal{E}$;
(iii) for every E, $F \in \mathcal{E}$, $E \cup F \in \mathcal{E}$.

136F Remarks (a) I could very well have introduced this notion in Chapter 11, along with 'σ-algebras'. I omitted it, apart from some exercises, because there seemed to be quite enough new definitions in §111 already, and because I had nothing substantial to say about algebras of sets.

(b) If $\mathcal{E}$ is an algebra of subsets of X, then

$$E \cap F = X \setminus ((X \setminus E) \cup (X \setminus F)), \quad E \setminus F = E \cap (X \setminus F),$$

$$E_0 \cup E_1 \cup \ldots \cup E_n, \quad E_0 \cap E_1 \cap \ldots \cap E_n$$

belong to $\mathcal{E}$ for all E, F, $E_0, \ldots , E_n \in \mathcal{E}$. (Induce on n for the last.)

(c) A σ-algebra of subsets of X is (of course) an algebra of subsets of X.

136G Theorem Let X be a set and $\mathcal{E}$ an algebra of subsets of X. Suppose that $\mathcal{A} \subseteq \mathcal{P}X$ is a family of sets such that

(α) $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-decreasing sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$,
(β) $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-increasing sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$,
(γ) $\mathcal{E} \subseteq \mathcal{A}$.

Then $\mathcal{A}$ includes the σ-algebra of subsets of X generated by $\mathcal{E}$.

proof I use the same ideas as in 136B.

(a) Let $\mathfrak{S}$ be the family of all sets $\mathcal{S} \subseteq \mathcal{P}X$ satisfying (α)-(γ). Then its intersection $\Sigma = \bigcap \mathfrak{S}$ also satisfies the conditions. Because $\mathcal{A} \in \mathfrak{S}$, $\Sigma \subseteq \mathcal{A}$.

(b) If $H \in \Sigma$, then

$$\Sigma_H = \{E : E \in \Sigma,\ E \cap H \in \Sigma\}$$

satisfies conditions (α)-(β). **P** (α) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ_H, then $\langle A_n \cap H\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ, so

$$(\bigcup_{n\in\mathbb{N}} A_n) \cap H = \bigcup_{n\in\mathbb{N}}(A_n \cap H) \in \Sigma$$

and $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma_H$. ($\beta$) Similarly, if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ_H, then $\bigcap_{n\in\mathbb{N}} A_n \cap H \in \Sigma$ so $\bigcap_{n\in\mathbb{N}} A_n \in \Sigma_H$. **Q**

It follows that if $E \cap H \in \Sigma$ for every $E \in \mathcal{E}$, so that Σ_H also satisfies (γ), then $\Sigma_H \in \mathfrak{S}$ and must be equal to Σ.

(c) Consequently $G \cap H \in \Sigma$ for all G, $H \in \Sigma$. **P** Take E, $F \in \mathcal{E}$. We know that $E \cap F \in \mathcal{E}$. As E is arbitrary, $\Sigma_F = \Sigma$ and $H \in \Sigma_F$, that is, $H \cap F \in \Sigma$. As F is arbitrary, $\Sigma_H = \Sigma$ and $G \in \Sigma_H$, that is, $G \cap H \in \Sigma$. **Q**

(d) Next, $\Sigma^* = \{X \setminus H : H \in \Sigma\} \in \mathfrak{S}$. **P** ($\alpha$) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in Σ^*, then $\langle X \setminus A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ, so

$$\bigcup_{n\in\mathbb{N}} A_n = X \setminus \bigcap_{n\in\mathbb{N}}(X \setminus A_n) \in \Sigma^*.$$

(β) Similarly, if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ^*, then

$$\bigcap_{n\in\mathbb{N}} A_n = X \setminus \bigcup_{n\in\mathbb{N}}(X \setminus A_n) \in \Sigma^*.$$

(γ) If $E \in \mathcal{E}$ then $X \setminus E \in \mathcal{E}$ so $X \setminus E \in \Sigma$ and $E \in \Sigma^*$. **Q** It follows that $\Sigma \subseteq \Sigma^*$, that is, that $X \setminus H \in \Sigma$ for every $H \in \Sigma$.

(e) Putting (c) and (d) together with the fact that $X \in \Sigma$ (because $X \in \mathcal{E}$) and the union of a non-decreasing sequence in Σ belongs to Σ (by condition (α)), we see that the same argument as in part (d) of the proof of 136B shows that Σ is a σ-algebra of subsets of X. So, just as in 136B, we conclude that the σ-algebra generated by $\mathcal{E}$ is included in Σ and therefore in $\mathcal{A}$.

***136H Proposition** Let (X, Σ, μ) be a measure space such that $\mu X < \infty$, and $\mathcal{E}$ a subalgebra of Σ; let Σ' be the σ-algebra of subsets of X generated by $\mathcal{E}$. If $F \in \Sigma'$ and $\epsilon > 0$, there is an $E \in \mathcal{E}$ such that $\mu(E \cap F) \le \epsilon$.

proof Let $\mathcal{A}$ be the family of sets $F \in \Sigma$ such that

$$\text{for every } \epsilon > 0 \text{ there is an } E \in \mathcal{E} \text{ such that } \mu(F \triangle E) \le \epsilon.$$

Then $\mathcal{A}$ is a Dynkin class. **P** I check the three conditions of 136A(i). (α) $X \in \mathcal{A}$ because $X \in \mathcal{E}$. (β) If F_1, $F_2 \in \mathcal{A}$ and $\epsilon > 0$, there are E_1, $E_2 \in \mathcal{E}$ such that $\mu(F_i \triangle E_i) \le \frac{1}{2}\epsilon$ for both i; now $E_1 \setminus E_2 \in \mathcal{E}$ and

$$(F_1 \setminus F_2) \triangle (E_1 \setminus E_2) \subseteq (F_1 \triangle E_1) \cup (F_2 \triangle E_2),$$

so

$$\mu((F_1 \setminus F_2) \triangle (E_1 \setminus E_2)) \le \mu(F_1 \triangle E_1) + \mu(F_2 \triangle E_2) \le \epsilon.$$

As ϵ is arbitrary, $F_1 \setminus F_2 \in \mathcal{A}$. ($\gamma$) If $\langle F_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$, with union F, and $\epsilon > 0$, then

$$\lim_{n\to\infty} \mu F_n = \mu F \le \mu X < \infty,$$

so there is an $n \in \mathbb{N}$ such that $\mu(F \setminus F_n) \le \frac{1}{2}\epsilon$. Now there is an $E \in \mathcal{E}$ such that $\mu(F_n \triangle E) \le \frac{1}{2}\epsilon$; as $F \triangle E \subseteq (F \setminus F_n) \cup (F_n \triangle E)$, $\mu(F \triangle E) \le \epsilon$. As ϵ is arbitrary, $F \in \mathcal{A}$. **Q**

Since $\mathcal{E} \subseteq \mathcal{A}$ and $\mathcal{E}$ is closed under $\cap$, $\mathcal{A}$ includes the σ-algebra Σ' generated by $\mathcal{E}$, as claimed.

136X Basic exercises >(a) Let X be a set and $\mathcal{A}$ a family of subsets of X. Show that the following are equiveridical:

(i) $X \in \mathcal{A}$ and $B \setminus A \in \mathcal{A}$ whenever A, $B \in \mathcal{A}$ and $A \subseteq B$;

(ii) $\emptyset \in \mathcal{A}$, $X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$ whenever A, $B \in \mathcal{A}$ are disjoint.

(b) Suppose that X is a set and $\mathcal{A} \subseteq \mathcal{P}X$. Show that $\mathcal{A}$ is a σ-algebra of subsets of X iff it is a Dynkin class and $A \cap B \in \mathcal{A}$ whenever A, $B \in \mathcal{A}$.

(c) Let X be a set, and $\mathcal{I}$ a family of subsets of X such that $I \cap J \in \mathcal{I}$ for all I, $J \in \mathcal{I}$; let Σ be the σ-algebra of subsets of X generated by $\mathcal{I}$. Show that $\mu E = \nu E$ whenever $E \in \Sigma$ is covered by a sequence in $\mathcal{I}$. (*Hint*: For $J \in \mathcal{I}$, set $\mu_J E = \mu(E \cap J)$, $\nu_J E = \nu(E \cap J)$ for $E \in \Sigma$. Use 136C to show that $\mu_J = \nu_J$ for each J.)

>(d) Set $X = \{0, 1, 2, 3\}$, $\mathcal{I} = \{X, \{0, 1\}, \{0, 2\}\}$. Find two distinct measures μ, ν on X, both defined on the σ-algebra $\mathcal{P}X$ and with $\mu I = \nu I < \infty$ for every $I \in \mathcal{I}$.

(e) Let Σ be the family of subsets of $[0, 1[$ expressible as finite unions of half-open intervals $[a, b[$. Show that Σ is an algebra of subsets of $[0, 1[$.

(f) Let X be a set, and $\mathcal{I}$ a family of subsets of X such that $I \cap J \in \mathcal{I}$ whenever I, $J \in \mathcal{I}$. Let Σ be the smallest family of sets such that $X \in \Sigma$, $F \setminus E \in \Sigma$ whenever E, $F \in \Sigma$ and $E \subseteq F$, and $\mathcal{I} \subseteq \Sigma$. Show that Σ is an algebra of subsets of X.

(g) Let X be a set, and $\mathcal{E}$ an algebra of subsets of X. A functional $\nu : \mathcal{E} \to \mathbb{R}$ is called (**finitely**) **additive** if $\nu(E \cup F) = \nu E + \nu F$ whenever E, $F \in \mathcal{E}$ and $E \cap F = \emptyset$. (i) Show that in this case $\nu(E \cup F) + \nu(E \cap F) = \nu E + \nu F$ for all E, $F \in \mathcal{E}$. (ii) Show that if $\nu E \ge 0$ for every $E \in \mathcal{E}$ then $\nu(\bigcup_{i\le n} E_i) \le \sum_{i=0}^{n} \nu E_i$ for all $E_0, \dots, E_n \in \mathcal{E}$.

>(h) Let X be a set, and $\mathcal{A}$ a family of subsets of X such that (α) $\emptyset$, X belong to $\mathcal{A}$ (β) $A \cap B \in \mathcal{A}$ for all A, $B \in \mathcal{A}$ (γ) $A \cup B \in \mathcal{A}$ whenever A, $B \in \mathcal{A}$ and $A \cap B = \emptyset$. Show that $\{A : A \in \mathcal{A}, X \setminus A \in \mathcal{A}\}$ is an algebra of subsets of X.

>(i) Let X be a set, and $\mathcal{A}$ a family of subsets of X such that (α) $\emptyset$, X belong to $\mathcal{A}$ (β) $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ (γ) $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every disjoint sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$. Show that $\{A : A \in \mathcal{A}, X \setminus A \in \mathcal{A}\}$ is a σ-algebra of subsets of X.

>(j) Let $\mathcal{A}$ be a family of subsets of $\mathbb{R}$ such that (i) $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ (ii) $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every disjoint sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ (iii) every open interval $]a,b[$ belongs to $\mathcal{A}$. Show that every Borel subset of $\mathbb{R}$ belongs to $\mathcal{A}$. (*Hint*: show that every half-open interval $[a,b[$, $]a,b]$ belongs to $\mathcal{A}$, and therefore all intervals $]-\infty,a]$, $[a,\infty[$; now use 136Xi.)

>(k) Let X be a set, $\mathcal{E}$ an algebra of subsets of X, and $\mathcal{A}$ a family of subsets of X such that (α) $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-increasing sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ (β) $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every disjoint sequence in $\mathcal{A}$ (γ) $\mathcal{E} \subseteq \mathcal{A}$. Show that the σ-algebra of sets generated by $\mathcal{E}$ is included in $\mathcal{A}$. (*Hint*: use the method of 136B to reduce to the case in which $A \cap B \in \mathcal{A}$ for every A, $B \in \mathcal{A}$; now use 136Xi.)

136Y Further exercises (a) Let X be a set and $\mathcal{E}$ an algebra of subsets of X. Let $\nu : \mathcal{E} \to [0,\infty[$ be a non-negative functional which is additive in the sense of 136Xg. Define $\theta : \mathcal{P}X \to [0,\infty[$ by setting

$$\theta A = \inf\{\textstyle\sum_{n=0}^{\infty} \nu E_n : \langle E_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \mathcal{E} \text{ covering } A\}$$

for every $A \subseteq X$. (i) Show that θ is an outer measure on X and that $\theta E \le \nu E$ for every $E \in \mathcal{E}$. (ii) Let μ be the measure on X defined from θ by Carathéodory's method, and Σ its domain. Show that $\mathcal{E} \subseteq \Sigma$ and that $\mu E \le \nu E$ for every $E \in \mathcal{E}$. (iii) Show that the following are equiveridical: (α) $\mu E = \nu E$ for every $E \in \mathcal{E}$ (β) $\theta X = \nu X$ (γ) whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{E}$ with empty intersection, $\lim_{n\to\infty} \nu E_n = 0$.

(b) Let X be a set, $\mathcal{E}$ an algebra of subsets of X, and ν a non-negative additive functional on $\mathcal{E}$. Let Σ be the σ-algebra of subsets of X generated by $\mathcal{E}$. Show that there is at most one measure μ on X with domain Σ extending ν, and that there is such a measure iff $\lim_{n\to\infty} \nu E_n = 0$ for every non-increasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ with empty intersection.

(c) Let X be a set. Let $\mathcal{G}$ be a family of subsets of X such that (i) $G \cap H \in \mathcal{G}$ for all G, $H \in \mathcal{G}$ (ii) for every $G \in \mathcal{G}$ there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ such that $X \setminus G = \bigcap_{n\in\mathbb{N}} G_n$. Let $\mathcal{A}$ be a family of subsets of X such that (α) $\emptyset$, $X \in \mathcal{A}$ (β) $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-increasing sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$ (γ) $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every disjoint sequence in $\mathcal{A}$ (δ) $\mathcal{G} \subseteq \mathcal{A}$. Show that the σ-algebra of sets generated by $\mathcal{G}$ is included in $\mathcal{A}$.

136 Notes and comments The most useful result here is 136B; it will be needed in Chapter 27, and helpful at various other points in Volume 2, often through its corollaries 136C and 136Xc. Of course 136C, like its corollary 136D and its special case 136Yb, can be used directly only on measures which do not take the value ∞, since we have to know that $\mu(F \setminus E) = \mu F - \mu E$ for measurable sets $E \subseteq F$; that is why it comes into prominence only when we specialize to probability measures (for which the whole space has measure 1). So I include 136Xc to indicate a technique that can take us a step farther. I do not feel that we are really ready for general measures on the Borel sets of $\mathbb{R}^r$, but I mention 136D to show what kind of class $\mathcal{I}$ can appear in 136B.

The two theorems here (136B, 136G) both address the question: given a family of sets $\mathcal{I}$, what operations must we perform in order to build the σ-algebra Σ generated by $\mathcal{I}$? For arbitrary $\mathcal{I}$, of course, we expect to need complements and unions of sequences. The point of the theorems here is that if $\mathcal{I}$ has a certain amount of structure then we can reach Σ with more limited operations; thus if $\mathcal{I}$ is an algebra of sets, then *monotonic* unions and intersections are enough (136G). Of course there are innumerable variations on this theme. I offer 136Xh-136Xj as a typical result which will actually be used in Volume 4, and 136Xk and 136Yc as examples of possible modifications. There is an abstract version of 136B in 313G in Volume 3.

Having once started to consider the extension of an algebra of sets to a σ-algebra, it is natural to ask for conditions under which a functional on an algebra of sets can be extended to a measure. The condition of additivity (136Xg) is obviously necessary, and almost equally obviously not sufficient. I include 136Ya-136Yb as the most important of many necessary and sufficient conditions for an additive functional to be extendable to a measure. We shall have to return to this in Volume 4.

Version of 9.4.05

Concordance for Chapter 13

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

132E Measurable envelopes Parts (d) and (e) of 132E in the 2000 and 2001 editions, referred to in the 2001 edition of Volume 2 and the 2002 edition of Volume 3, are now parts (e) and (f).

132G Pull-back measures Proposition 132G, referred to in the 2006 edition of Volume 4, has been moved to 234F.

Version of 31.5.03

References for Volume 1

In addition to those (very few) works which I have mentioned in the course of this volume, I list some of the books from which I myself learnt measure theory, as a mark of grateful respect, and to give you an opportunity to sample alternative approaches.

Bartle R.G. [66] *The Elements of Integration.* Wiley, 1966.

Cohn D.L. [80] *Measure Theory.* Birkhäuser, 1980.

Facenda J.A. & Freniche F.J. [02] *Integración de funciones de varias variables.* Ediciones Pirámide, 2002.

Halmos P.R. [50] *Measure Theory.* Van Nostrand, 1950.

Lebesgue H. [1901] 'Sur une généralisation de l'intégrale définie', C.R.Acad.Sci. (Paris) 132 (1901) 1025-1027. Reprinted in Lebesgue 72, vol. 1.

Lebesgue H. [1904] *Leçons sur l'intégration et la recherche des fonctions primitives.* Gauthier-Villars, 1904. Reprinted in Lebesgue 72, vol. 2. [Vol. 1 *intro.*]

Lebesgue H. [1918] 'Remarques sur les théories de la mesure et de l'intégration', Ann. Sci. Ecole Norm. Sup. 35 (1918) 191-250. Reprinted in Lebesgue 72, vol. 2. [Vol. 1 *intro.*]

Lebesgue H. [72] *Oeuvres Scientifiques.* L'Enseignement Mathématique, Institut de Mathématiques, Univ. de Genève, 1972.

Munroe M.E. [53] *Introduction to Measure and Integration.* Addison-Wesley, 1953.

Royden H.L. [63] *Real Analysis.* Macmillan, 1963.

Solovay R.M. [70] 'A model of set theory in which every set of reals is Lebesgue measurable', Annals of Math. 92 (1970) 1-56. [134C.]

Tao T. [11] *An introduction to measure theory.* Amer. Math. Soc., 2011; `terrytao.files.wordpress.com/2011/01/me` `book1.pdf`.

Widom H. [69] *Lectures on Measure and Integration.* Van Nostrand Reinhold, 1969.

Williamson J.H. [62] *Lebesgue Integration.* Holt, Rinehart & Winston, 1962.