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Chapter 12

Integration

If you look along the appropriate shelf of your college's library, you will see that the words 'measure' and 'integration' go together like Siamese twins. The linkage is both more complex and more intimate than any simple explanation can describe. But if we say that one of the concepts on which integration is based is that of 'area under a curve', then it is clear that any method of determining 'areas' ought to correspond to a method of integrating functions; and this has from the beginning been an essential part of the Lebesgue theory. For a literal description of the integral of a non-negative function in terms of the area of its ordinate set, I think it best to wait until Chapter 25 in Volume 2. In the present chapter I seek to give a concise description of the standard integral of a real-valued function on a general measure space, with the half-dozen most important theorems concerning this integral.

The construction bristles with technical difficulties at every step, and you will find it easy to understand why it was not done before 1901. What may be less clear is why it was ever done at all. So perhaps you should immediately read the statements of 123A-123D below. It is the case (some of the details will appear, rather late, in §436 in Volume 4) that any theory of integration powerful enough to have theorems of this kind must essentially encompass all the ideas of this chapter, and nearly all the ideas of the last.

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121 Measurable functions

In this section, I take a step back to develop ideas relating to σ-algebras of sets, following §111; there will be no mention of 'measures' here, except in the exercises. The aim is to establish the concept of 'measurable function' (121C) and a variety of associated techniques. The best single example of a σ-algebra to bear in mind when reading this chapter is probably the σ-algebra of Borel subsets of $\mathbb{R}$; the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ is a good second.

Throughout the exposition here (starting with 121A) I seek to deal with functions which are not defined on the whole of the space X under consideration. I believe that there are compelling reasons for facing up to such functions at an early stage; but undeniably they add to the technical difficulties, and it would be fair to read through the chapter once with the mental reservation that all functions are taken to be defined everywhere, before returning to deal with the general case.

121A Lemma Let X be a set and Σ a σ-algebra of subsets of X. Let D be any subset of X and write

$$\Sigma_D = \{E \cap D : E \in \Sigma\}.$$

Then Σ_D is a σ-algebra of subsets of D.

Notation I will call Σ_D the **subspace σ-algebra** of subsets of D, and I will say that its members are **relatively measurable** in D. Σ_D is also sometimes called the **trace** of Σ on D.

121B Proposition Let X be a set, Σ a σ-algebra of subsets of X, and D a subset of X. Write Σ_D for the subspace σ-algebra of subsets of D. Then for any function $f : D \to \mathbb{R}$ the following assertions are equiveridical:

(i) $\{x : f(x) < a\} \in \Sigma_D$ for every $a \in \mathbb{R}$;
(ii) $\{x : f(x) \le a\} \in \Sigma_D$ for every $a \in \mathbb{R}$;
(iii) $\{x : f(x) > a\} \in \Sigma_D$ for every $a \in \mathbb{R}$;
(iv) $\{x : f(x) \ge a\} \in \Sigma_D$ for every $a \in \mathbb{R}$.

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121C Definition Let X be a set, Σ a σ-algebra of subsets of X, and D a subset of X. A function $f : D \to \mathbb{R}$ is called **measurable** (or **Σ-measurable**) if it satisfies any, or equivalently all, of the conditions (i)-(iv) of 121B.

If X is $\mathbb{R}$ or $\mathbb{R}^r$, and Σ is its Borel σ-algebra, a Σ-measurable function is called **Borel measurable**. If X is $\mathbb{R}$ or $\mathbb{R}^r$, and Σ is the σ-algebra of Lebesgue measurable sets, a Σ-measurable function is called **Lebesgue measurable**.

121D Proposition Let X be $\mathbb{R}^r$ for some $r \geq 1$, D a subset of X, and $g : D \to \mathbb{R}$ a function.

(a) If g is Borel measurable it is Lebesgue measurable.

(b) If g is continuous it is Borel measurable.

(c) If $r = 1$ and g is monotonic it is Borel measurable.

121E Theorem Let X be a set and Σ a σ-algebra of subsets of X. Let f and g be real-valued functions defined on domains $\operatorname{dom} f$, $\operatorname{dom} g \subseteq X$.

(a) If f is constant it is measurable.

(b) If f and g are measurable, so is $f + g$, where $(f + g)(x) = f(x) + g(x)$ for $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

(c) If f is measurable and $c \in \mathbb{R}$, then cf is measurable, where $(cf)(x) = c \cdot f(x)$ for $x \in \operatorname{dom} f$.

(d) If f and g are measurable, so is $f \times g$, where $(f \times g)(x) = f(x) \times g(x)$ for $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

(e) If f and g are measurable, so is f/g, where $(f/g)(x) = f(x)/g(x)$ when $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and $g(x) \neq 0$.

(f) If f is measurable and $E \subseteq \mathbb{R}$ is a Borel set, then there is an $F \in \Sigma$ such that $f^{-1}[E] = \{x : f(x) \in E\}$ is equal to $F \cap \operatorname{dom} f$.

(g) If f is measurable and h is a Borel measurable function from a subset $\operatorname{dom} h$ of $\mathbb{R}$ to $\mathbb{R}$, then hf is measurable, where $(hf)(x) = h(f(x))$ for $x \in \operatorname{dom}(hf) = \{y : y \in \operatorname{dom} f,\ f(y) \in \operatorname{dom} h\}$.

(h) If f is measurable and A is any set, then $f \restriction A$ is measurable, where $\operatorname{dom}(f \restriction A) = A \cap \operatorname{dom} f$ and $(f \restriction A)(x) = f(x)$ for $x \in A \cap \operatorname{dom} f$.

121F Theorem Let X be a set and Σ a σ-algebra of subsets of X. Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a sequence of Σ-measurable real-valued functions with domains included in X.

(a) Define a function $\lim_{n \to \infty} f_n$ by writing

$$(\lim_{n \to \infty} f_n)(x) = \lim_{n \to \infty} f_n(x)$$

for all those $x \in \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \operatorname{dom} f_m$ for which the limit exists in $\mathbb{R}$. Then $\lim_{n \to \infty} f_n$ is Σ-measurable.

(b) Define a function $\sup_{n \in \mathbb{N}} f_n$ by writing

$$(\sup_{n \in \mathbb{N}} f_n)(x) = \sup_{n \in \mathbb{N}} f_n(x)$$

for all those $x \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} f_n$ for which the supremum exists in $\mathbb{R}$. Then $\sup_{n \in \mathbb{N}} f_n$ is Σ-measurable.

(c) Define a function $\inf_{n \in \mathbb{N}} f_n$ by writing

$$(\inf_{n \in \mathbb{N}} f_n)(x) = \inf_{n \in \mathbb{N}} f_n(x)$$

for all those $x \in \bigcap_{n \in \mathbb{N}} \operatorname{dom} f_n$ for which the infimum exists in $\mathbb{R}$. Then $\inf_{n \in \mathbb{N}} f_n$ is Σ-measurable.

(d) Define a function $\limsup_{n \to \infty} f_n$ by writing

$$(\limsup_{n \to \infty} f_n)(x) = \limsup_{n \to \infty} f_n(x)$$

for all those $x \in \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \operatorname{dom} f_m$ for which the $\limsup$ exists in $\mathbb{R}$. Then $\limsup_{n \to \infty} f_n$ is Σ-measurable.

(e) Define a function $\liminf_{n \to \infty} f_n$ by writing

$$(\liminf_{n \to \infty} f_n)(x) = \liminf_{n \to \infty} f_n(x)$$

for all those $x \in \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \operatorname{dom} f_m$ for which the $\liminf$ exists in $\mathbb{R}$. Then $\liminf_{n \in \mathbb{N}} f_n$ is Σ-measurable.

121H Proposition Let X be a set and Σ a σ-algebra of subsets of X; let f, g and f_n, for $n \in \mathbb{N}$, be Σ-measurable real-valued functions whose domains belong to Σ. Then all the functions

$$f + g, \quad f \times g, \quad f/g,$$

$$\sup_{n\in\mathbb{N}} f_n, \quad \inf_{n\in\mathbb{N}} f_n, \quad \lim_{n\to\infty} f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n$$

have domains belonging to Σ. Moreover, if h is a Borel measurable real-valued function defined on a Borel subset of $\mathbb{R}$, then $\operatorname{dom} hf \in \Sigma$.

***121I Proposition** Let X be a set and Σ a σ-algebra of subsets of X. Let D be a subset of X and $f : D \to \mathbb{R}$ a function. Then f is measurable iff there is a measurable function $h : X \to \mathbb{R}$ extending f.

***121J Lemma** Let $r \ge 1$ be an integer, and write $\mathcal{J}$ for the family of subsets of $\mathbb{R}^r$ of the form $\{x : \xi_i \le \alpha\}$ where $i \le r$, $\alpha \in \mathbb{R}$, writing $x = (\xi_1, \dots, \xi_r)$, as in §115. Then the σ-algebra of subsets of $\mathbb{R}^r$ generated by $\mathcal{J}$ is precisely the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}^r$.

***121K Proposition** Let X be a set and Σ a σ-algebra of subsets of X. Let $r \ge 1$ be an integer, and $f_1, \dots, f_r$ measurable functions defined on subsets of X. Set $D = \bigcap_{i\le r} \operatorname{dom} f_i$ and for $x \in D$ set $f(x) = (f_1(x), \dots, f_r(x)) \in \mathbb{R}^r$. Then

(a) for any Borel set $E \subseteq \mathbb{R}^r$, $f^{-1}[E]$ belongs to the subspace σ-algebra Σ_D;

(b) if h is a Borel measurable function from a subset $\operatorname{dom} h$ of $\mathbb{R}^r$ to $\mathbb{R}$, then the composition hf is measurable.

Version of 4.1.04

122 Definition of the integral

I set out the definition of ordinary integration for real-valued functions defined on an arbitrary measure space, with its most basic properties.

122A Definitions Let (X, Σ, μ) be a measure space.

(a) For any set $A \subseteq X$, I write χA for the **indicator function** or **characteristic function** of A, the function from X to $\{0, 1\}$ given by setting $\chi A(x) = 1$ if $x \in A$, 0 if $x \in X \setminus A$. χA is Σ-measurable iff $A \in \Sigma$.

(b) Now a **simple** function on X is a function of the form $\sum_{i=0}^{n} a_i \chi E_i$, where $E_0, \dots, E_n$ are measurable sets of finite measure and $a_0, \dots, a_n$ belong to $\mathbb{R}$.

122B Lemma Let (X, Σ, μ) be a measure space.

(a) Every simple function on X is measurable.

(b) If $f, g : X \to \mathbb{R}$ are simple functions, so is $f + g$.

(c) If $f : X \to \mathbb{R}$ is a simple function and $c \in \mathbb{R}$, then $cf : X \to \mathbb{R}$ is a simple function.

(d) The constant zero function is simple.

122C Lemma Let (X, Σ, μ) be a measure space.

(a) If $E_0, \dots, E_n$ are measurable sets of finite measure, there are disjoint measurable sets $G_0, \dots, G_m$ of finite measure such that each E_i is expressible as a union of some of the G_j.

(b) If $f : X \to \mathbb{R}$ is a simple function, it is expressible in the form $\sum_{j=0}^{m} b_j \chi G_j$ where $G_0, \dots, G_m$ are disjoint measurable sets of finite measure.

(c) If $E_0, \dots, E_n$ are measurable sets of finite measure, and $a_0, \dots, a_n \in \mathbb{R}$ are such that $\sum_{i=0}^{n} a_i \chi E_i(x) \ge 0$ for every $x \in X$, then $\sum_{i=0}^{n} a_i \mu E_i \ge 0$.

122D Corollary Let (X, Σ, μ) be a measure space. If

$$\sum_{i=0}^{m} a_i \chi E_i = \sum_{j=0}^{n} b_j \chi F_j,$$

where all the E_i and F_j are measurable sets of finite measure and the a_i, b_j are real numbers, then

$$\sum_{i=0}^{m} a_i \mu E_i = \sum_{j=0}^{n} b_j \mu F_j.$$

122E Definition Let (X, Σ, μ) be a measure space. Then we may define the **integral** $\int f$ of f, for simple functions $f : X \to \mathbb{R}$, by saying that $\int f = \sum_{i=0}^{m} a_i \mu E_i$ whenever $f = \sum_{i=0}^{m} a_i \chi E_i$ and every E_i is a measurable set of finite measure.

122F Proposition Let (X, Σ, μ) be a measure space.
(a) If f, $g : X \to \mathbb{R}$ are simple functions, then $f + g$ is a simple function and $\int f + g = \int f + \int g$.
(b) If f is a simple function and $c \in \mathbb{R}$, then cf is a simple function and $\int cf = c \int f$.
(c) If f, g are simple functions and $f(x) \leq g(x)$ for every $x \in X$, then $\int f \leq \int g$.

122G Lemma Let (X, Σ, μ) be a measure space. If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of simple functions which is non-decreasing (in the sense that $f_n(x) \leq f_{n+1}(x)$ for every $n \in \mathbb{N}$, $x \in X$) and f is a simple function such that $f(x) \leq \sup_{n\in\mathbb{N}} f_n(x)$ for almost every $x \in X$ (allowing $\sup_{n\in\mathbb{N}} f_n(x) = \infty$ in this formula), then $\int f \leq \sup_{n\in\mathbb{N}} \int f_n$.

122H Definition Let (X, Σ, μ) be a measure space. For the rest of this section, I will write U for the set of functions f such that

(i) the domain of f is a conegligible subset of X and $f(x) \in [0, \infty[$ for each $x \in \operatorname{dom} f$,

(ii) there is a non-decreasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative simple functions such that $\sup_{n\in\mathbb{N}} \int f_n < \infty$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$.

122I Lemma If f and $\langle f_n \rangle_{n\in\mathbb{N}}$ are as in 122H, then

$$\sup_{n\in\mathbb{N}} \int f_n = \sup\{\int g : g \text{ is a simple function and } g \leq_{\text{a.e.}} f\}.$$

122J Lemma Let (X, Σ, μ) be a measure space.
(a) If f is a function defined on a conegligible subset of X and taking values in $[0, \infty[$, then $f \in U$ iff there is a conegligible measurable set $E \subseteq \operatorname{dom} f$ such that

(α) $f \restriction E$ is measurable,

(β) for every $\epsilon > 0$, $\mu\{x : x \in E,\ f(x) \geq \epsilon\} < \infty$,

(γ) $\sup\{\int g : g \text{ is a simple function, } g \leq_{\text{a.e.}} f\} < \infty$.

(b) Suppose that $f \in U$ and that h is a function defined on a conegligible subset of X and taking values in $[0, \infty[$. Suppose that $h \leq_{\text{a.e.}} f$ and there is a conegligible $F \subseteq X$ such that $h \restriction F$ is measurable. Then $h \in U$.

122K Definition Let (X, Σ, μ) be a measure space. For $f \in U$, set

$$\int f = \sup\{\int g : g \text{ is a simple function and } g \leq_{\text{a.e.}} f\}.$$

122L Lemma Let (X, Σ, μ) be a measure space.
(a) If f, $g \in U$ then $f + g \in U$ and $\int f + g = \int f + \int g$.
(b) If $f \in U$ and $c \geq 0$ then $cf \in U$ and $\int cf = c \int f$.
(c) If f, $g \in U$ and $f \leq_{\text{a.e.}} g$ then $\int f \leq \int g$.
(d) If $f \in U$ and g is a function with domain a conegligible subset of X, taking values in $[0, \infty[$, and equal to f almost everywhere, then $g \in U$ and $\int g = \int f$.
(e) If f_1, g_1, f_2, $g_2 \in U$ and $f_1 - f_2 = g_1 - g_2$, then $\int f_1 - \int f_2 = \int g_1 - \int g_2$.

122M Definition Let (X, Σ, μ) be a measure space. A real-valued function f is **integrable**, or **integrable over** X, or μ**-integrable over** X, if it is expressible as $f_1 - f_2$ with f_1, $f_2 \in U$, and in this case its **integral** is

$$\int f = \int f_1 - \int f_2.$$

122O Theorem Let (X, Σ, μ) be a measure space.

(a) If f and g are integrable over X then $f+g$ is integrable and $\int f+g = \int f + \int g$.

(b) If f is integrable over X and $c \in \mathbb{R}$ then cf is integrable and $\int cf = c\int f$.

(c) If f is integrable over X and $f \geq 0$ a.e. then $\int f \geq 0$.

(d) If f and g are integrable over X and $f \leq_{\text{a.e.}} g$ then $\int f \leq \int g$.

122P Theorem Let (X, Σ, μ) be a measure space and f a real-valued function defined on a conegligible subset of X. Then the following are equiveridical:

(i) f is integrable;

(ii) $|f| \in U$ and there is a conegligible set $E \subseteq X$ such that $f\restriction E$ is measurable;

(iii) there are a $g \in U$ and a conegligible set $E \subseteq X$ such that $|f| \leq_{\text{a.e.}} g$ and $f\restriction E$ is measurable.

122Q Remark The condition 'there is a conegligible set E such that $f\restriction E$ is measurable' recurs so often that I think it worth having a phrase for it; I will call such functions **virtually measurable**, or **μ-virtually measurable** if it seems necessary to specify the measure.

122R Corollary Let (X, Σ, μ) be a measure space.

(a) A non-negative real-valued function, defined on a subset of X, is integrable iff it belongs to U.

(b) If f is integrable over X and h is a real-valued function, defined on a conegligible subset of X and equal to f almost everywhere, then h is integrable, with $\int h = \int f$.

(c) If f is integrable over X, $f \geq 0$ a.e. and $\int f \leq 0$, then $f = 0$ a.e.

(d) If f and g are integrable over X, $f \leq_{\text{a.e.}} g$ and $\int g \leq \int f$, then $f =_{\text{a.e.}} g$.

(e) If f is integrable over X, so is $|f|$, and $|\int f| \leq \int |f|$.

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123 The convergence theorems

The great labour we have gone through so far has not yet been justified by any theorems powerful enough to make it worth while. We come now to the heart of the modern theory of integration, the 'convergence theorems', describing conditions under which we can integrate the limit of a sequence of integrable functions.

123A B.Levi's theorem Let (X, Σ, μ) be a measure space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions, all integrable over X, such that (i) $f_n \leq_{\text{a.e.}} f_{n+1}$ for every $n \in \mathbb{N}$ (ii) $\sup_{n\in\mathbb{N}} \int f_n < \infty$. Then $f = \lim_{n\to\infty} f_n$ is integrable, and $\int f = \lim_{n\to\infty} \int f_n$.

Remarks The statement 'f is integrable' includes the assertion 'f is defined, as a real number, almost everywhere'.

123B Fatou's Lemma Let (X, Σ, μ) be a measure space, and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions, all integrable over X. Suppose that every f_n is non-negative a.e., and that $\liminf_{n\to\infty} \int f_n < \infty$. Then $\liminf_{n\to\infty} f_n$ is integrable, and $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$.

123C Lebesgue's Dominated Convergence Theorem Let (X, Σ, μ) be a measure space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions, all integrable over X, such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}$ for almost every $x \in X$. Suppose moreover that there is an integrable function g such that $|f_n| \leq_{\text{a.e.}} g$ for every n. Then f is integrable, and $\lim_{n\to\infty} \int f_n$ exists and is equal to $\int f$.

123D Corollary Let (X, Σ, μ) be a measure space and $]a, b[$ a non-empty open interval in $\mathbb{R}$. Let $f : X \times]a, b[\to \mathbb{R}$ be a function such that

(i) the integral $F(t) = \int f(x, t)dx$ is defined for every $t \in]a, b[$;

(ii) the partial derivative $\frac{\partial f}{\partial t}$ of f with respect to the second variable is defined everywhere in $X \times]a, b[$;

(iii) there is an integrable function $g : X \to [0, \infty[$ such that $|\frac{\partial f}{\partial t}(x, t)| \leq g(x)$ for every $x \in X$ and $t \in]a, b[$.

Then the derivative $F'(t)$ and the integral $\int \frac{\partial f}{\partial t}(x, t)dx$ exist for every $t \in]a, b[$, and are equal.

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Concordance for Chapter 12

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

121Yb (Σ, T)**-measurable functions** Exercise 121Yb in the 2000 and 2001 editions, referred to in the 2001 and 2003 editions of Volume 2, has been moved to 121Yc.

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