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Chapter 12

Integration

If you look along the appropriate shelf of your college's library, you will see that the words 'measure' and 'integration' go together like Siamese twins. The linkage is both more complex and more intimate than any simple explanation can describe. But if we say that one of the concepts on which integration is based is that of 'area under a curve', then it is clear that any method of determining 'areas' ought to correspond to a method of integrating functions; and this has from the beginning been an essential part of the Lebesgue theory. For a literal description of the integral of a non-negative function in terms of the area of its ordinate set, I think it best to wait until Chapter 25 in Volume 2. In the present chapter I seek to give a concise description of the standard integral of a real-valued function on a general measure space, with the half-dozen most important theorems concerning this integral.

The construction bristles with technical difficulties at every step, and you will find it easy to understand why it was not done before 1901. What may be less clear is why it was ever done at all. So perhaps you should immediately read the statements of 123A-123D below. It is the case (some of the details will appear, rather late, in §436 in Volume 4) that any theory of integration powerful enough to have theorems of this kind must essentially encompass all the ideas of this chapter, and nearly all the ideas of the last.

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121 Measurable functions

In this section, I take a step back to develop ideas relating to σ-algebras of sets, following §111; there will be no mention of 'measures' here, except in the exercises. The aim is to establish the concept of 'measurable function' (121C) and a variety of associated techniques. The best single example of a σ-algebra to bear in mind when reading this chapter is probably the σ-algebra of Borel subsets of $\mathbb{R}$ (111G); the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ (114E) is a good second.

Throughout the exposition here (starting with 121A) I seek to deal with functions which are not defined on the whole of the space X under consideration. I believe that there are compelling reasons for facing up to such functions at an early stage (see 121G); but undeniably they add to the technical difficulties, and it would be fair to read through the chapter once with the mental reservation that all functions are taken to be defined everywhere, before returning to deal with the general case.

121A Lemma Let X be a set and Σ a σ-algebra of subsets of X. Let D be any subset of X and write

$$\Sigma_D = \{E \cap D : E \in \Sigma\}.$$

Then Σ_D is a σ-algebra of subsets of D.

proof (i) $\emptyset = \emptyset \cap D \in \Sigma_D$ because $\emptyset \in \Sigma$.

(ii) If $F \in \Sigma_D$, there is an $E \in \Sigma$ such that $F = E \cap D$; now $D \setminus F = (X \setminus E) \cap D \in \Sigma_D$ because $X \setminus E \in \Sigma$.

(iii) If $\langle F_n \rangle_{n\in\mathbb{N}}$ is any sequence in Σ_D, then for each $n \in \mathbb{N}$ we may choose an $E_n \in \Sigma$ such that $F_n = E_n \cap D$; now $\bigcup_{n\in\mathbb{N}} F_n = (\bigcup_{n\in\mathbb{N}} E_n) \cap D \in \Sigma_D$ because $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$.

Notation I will call Σ_D the **subspace σ-algebra** of subsets of D, and I will say that its members are **relatively measurable** in D. Σ_D is also sometimes called the **trace** of Σ on D.

121B Proposition Let X be a set, Σ a σ-algebra of subsets of X, and D a subset of X. Write Σ_D for the subspace σ-algebra of subsets of D. Then for any function $f : D \to \mathbb{R}$ the following assertions are equiveridical, that is, if one of them is true so are all the others:

(i) $\{x : f(x) < a\} \in \Sigma_D$ for every $a \in \mathbb{R}$;
(ii) $\{x : f(x) \le a\} \in \Sigma_D$ for every $a \in \mathbb{R}$;
(iii) $\{x : f(x) > a\} \in \Sigma_D$ for every $a \in \mathbb{R}$;
(iv) $\{x : f(x) \ge a\} \in \Sigma_D$ for every $a \in \mathbb{R}$.

proof (i)⇒(ii) Assume (i), and let $a \in \mathbb{R}$. Then

$$\{x : f(x) \le a\} = \bigcap_{n\in\mathbb{N}} \{x : f(x) < a + 2^{-n}\} \in \Sigma_D$$

because $\{x : f(x) < a + 2^{-n}\} \in \Sigma_D$ for every n and Σ_D is closed under countable intersections (111Dd). Because a is arbitrary, (ii) is true.

(ii)⇒(iii) Assume (ii), and let $a \in \mathbb{R}$. Then

$$\{x : f(x) > a\} = D \setminus \{x : f(x) \le a\} \in \Sigma_D$$

because $\{x : f(x) \le a\} \in \Sigma_D$ and Σ_D is closed under complementation. Because a is arbitrary, (iii) is true.

(iii)⇒(iv) Assume (iii), and let $a \in \mathbb{R}$. Then

$$\{x : f(x) \ge a\} = \bigcap_{n\in\mathbb{N}} \{x : f(x) > a - 2^{-n}\} \in \Sigma_D$$

because $\{x : f(x) > a - 2^{-n}\} \in \Sigma_D$ for every n and Σ_D is closed under countable intersections. Because a is arbitrary, (iv) is true.

(iv)⇒(i) Assume (iv), and let $a \in \mathbb{R}$. Then

$$\{x : f(x) < a\} = D \setminus \{x : f(x) \ge a\} \in \Sigma_D$$

because $\{x : f(x) \ge a\} \in \Sigma_D$ and Σ_D is closed under complementation. Because a is arbitrary, (i) is true.

121C Definition Let X be a set, Σ a σ-algebra of subsets of X, and D a subset of X. A function $f : D \to \mathbb{R}$ is called **measurable** (or **Σ-measurable**) if it satisfies any, or equivalently all, of the conditions (i)-(iv) of 121B.

If X is $\mathbb{R}$ or $\mathbb{R}^r$, and Σ is its Borel σ-algebra (111G), a Σ-measurable function is called **Borel measurable**. If X is $\mathbb{R}$ or $\mathbb{R}^r$, and Σ is the σ-algebra of Lebesgue measurable sets (114E, 115E), a Σ-measurable function is called **Lebesgue measurable**.

Remark Naturally the principal case here is when $D = X$. However, partially-defined functions are so common, and so important, in analysis (consider, for instance, the real function $\ln \sin$) that it seems worth while, from the beginning, to establish techniques for handling them efficiently.

Many authors develop a theory of 'extended real numbers' at this point, working with $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$, and defining measurability for functions taking values in this set. I outline such a theory in §135 below.

121D Proposition Let X be $\mathbb{R}^r$ for some $r \ge 1$, D a subset of X, and $g : D \to \mathbb{R}$ a function.

(a) If g is Borel measurable it is Lebesgue measurable.
(b) If g is continuous it is Borel measurable.
(c) If $r = 1$ and g is monotonic it is Borel measurable.

proof (a) This is immediate from the definitions in 121C, if we recall that the Borel σ-algebra is included in the Lebesgue σ-algebra (114G, 115G).

(b) Take $a \in \mathbb{R}$. Set

$$\mathcal{G} = \{G : G \subseteq \mathbb{R}^r \text{ is open}, \ g(x) < a \ \forall\, x \in G \cap D\},$$

$$G_0 = \bigcup \mathcal{G} = \{x : \exists\ G \in \mathcal{G}, \, x \in G\}.$$

Then G_0 is a union of open sets, therefore open (1A2Bd). Next,

$$\{x : g(x) < a\} = G_0 \cap D.$$

P (i) If $g(x) < a$, then (because g is continuous) there is a $\delta > 0$ such that $|g(y) - g(x)| < a - g(x)$ whenever $y \in D$ and $\|y - x\| < \delta$. But $\{y : \|y - x\| < \delta\}$ is open (1A2D), so belongs to $\mathcal{G}$ and is included in G_0, and $x \in G_0 \cap D$. **(ii)** If $x \in G_0 \cap D$, then there is a $G \in \mathcal{G}$ such that $x \in G$; now $g(y) < a$ for every $y \in G \cap D$, so, in particular, $g(x) < a$. **Q**

Finally, G_0, being open, is a Borel set. As a is arbitrary, g is Borel measurable.

(c) Suppose first that g is non-decreasing. Let $a \in \mathbb{R}$ and write $E = \{x : g(x) < a\}$. If $E = D$ or $E = \emptyset$ then of course it is the intersection of D with a Borel set. Otherwise, E is non-empty and bounded above in $\mathbb{R}$, so has a supremum $c \in \mathbb{R}$. Now E must be either $D \cap \left]-\infty, c\right[$ or $D \cap \left]-\infty, c\right]$, according to whether $c \in E$ or not, and in either case is the intersection of D with a Borel set (see 114G).

Similarly, if g is non-increasing, $\{x : g(x) > a\}$ will again be the intersection of D with either $\emptyset$ or $\mathbb{R}$ or $\left]-\infty, c\right]$ or $\left]-\infty, c\right[$ for some c. So in this case 121B(iii) will be satisfied.

Remark I see that in part (b) of the above proof I use some basic facts about open sets in $\mathbb{R}^r$. These are covered in detail in §1A2. If they are new to you it would probably be sensible to rehearse the arguments with $r = 1$, so that $D \subseteq \mathbb{R}$, before embracing the general case.

121E Theorem Let X be a set and Σ a σ-algebra of subsets of X. Let f and g be real-valued functions defined on domains $\operatorname{dom} f$, $\operatorname{dom} g \subseteq X$.

(a) If f is constant it is measurable.

(b) If f and g are measurable, so is $f + g$, where $(f + g)(x) = f(x) + g(x)$ for $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

(c) If f is measurable and $c \in \mathbb{R}$, then cf is measurable, where $(cf)(x) = c \cdot f(x)$ for $x \in \operatorname{dom} f$.

(d) If f and g are measurable, so is $f \times g$, where $(f \times g)(x) = f(x) \times g(x)$ for $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

(e) If f and g are measurable, so is f/g, where $(f/g)(x) = f(x)/g(x)$ when $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and $g(x) \neq 0$.

(f) If f is measurable and $E \subseteq \mathbb{R}$ is a Borel set, then there is an $F \in \Sigma$ such that $f^{-1}[E] = \{x : f(x) \in E\}$ is equal to $F \cap \operatorname{dom} f$.

(g) If f is measurable and h is a Borel measurable function from a subset $\operatorname{dom} h$ of $\mathbb{R}$ to $\mathbb{R}$, then hf is measurable, where $(hf)(x) = h(f(x))$ for $x \in \operatorname{dom}(hf) = \{y : y \in \operatorname{dom} f,\ f(y) \in \operatorname{dom} h\}$.

(h) If f is measurable and A is any set, then $f\restriction A$ is measurable, where $\operatorname{dom}(f\restriction A) = A \cap \operatorname{dom} f$ and $(f\restriction A)(x) = f(x)$ for $x \in A \cap \operatorname{dom} f$.

proof For any $D \subseteq X$ write Σ_D for the subspace σ-algebra of subsets of D.

(a) If $f(x) = c$ for every $x \in \operatorname{dom} f$, then $\{x : f(x) < a\} = \operatorname{dom} f$ if $c < a$, $\emptyset$ otherwise, and in either case belongs to $\Sigma_{\operatorname{dom} f}$.

(b) Write $D = \operatorname{dom}(f + g) = \operatorname{dom} f \cap \operatorname{dom} g$. If $a \in \mathbb{R}$ then set $K = \{(q, q') : q,\ q' \in \mathbb{Q},\ q + q' \leq a\}$. Then K is a subset of $\mathbb{Q} \times \mathbb{Q}$, so is countable (111Fb, 1A1E). For $q \in \mathbb{Q}$ choose sets F_q, $G_q \in \Sigma$ such that

$$\{x : f(x) < q\} = F_q \cap \operatorname{dom} f, \quad \{x : g(x) < q\} = G_q \cap \operatorname{dom} g.$$

For each $(q, q') \in K$, the set

$$E_{qq'} = \{x : f(x) < q,\ g(x) < q'\} = F_q \cap G_{q'} \cap D$$

belongs to Σ_D. Finally, if $(f + g)(x) < a$, then we can find $q \in \left]f(x), a - g(x)\right[$, $q' \in \left]g(x), a - q\right]$, so that $(q, q') \in K$ and $x \in E_{qq'}$; while if $(q, q') \in K$ and $x \in E_{qq'}$, then $(f + g)(x) < q + q' \leq a$. Thus

$$\{x : (f + g)(x) < a\} = \bigcup_{(q,q') \in K} E_{qq'} \in \Sigma_D$$

by 111Fa. As a is arbitrary, $f + g$ is measurable.

(c) Write $D = \operatorname{dom} f$. Let $a \in \mathbb{R}$. If $c > 0$, then

$$\{x : cf(x) < a\} = \{x : f(x) < \frac{a}{c}\} \in \Sigma_D.$$

If $c < 0$, then

$$\{x : cf(x) < a\} = \{x : f(x) > \frac{a}{c}\} \in \Sigma_D.$$

While if $c = 0$, then $\{x : cf(x) < a\}$ is either D or $\emptyset$, as in (a) above, so belongs to Σ_D. As a is arbitrary, cf is measurable.

(d) Write $D = \operatorname{dom}(f \times g) = \operatorname{dom} f \cap \operatorname{dom} g$. Let $a \in \mathbb{R}$. Let K be

$$\{(q_1, q_2, q_3, q_4) : q_1, \dots , q_4 \in \mathbb{Q},\ uv < a \text{ whenever } u \in]q_1, q_2[\,,\ v \in]q_3, q_4[\}.$$

Then K is countable. For $q \in \mathbb{Q}$ choose sets F_q, F'_q, G_q, $G'_q \in \Sigma$ such that

$$\{x : f(x) < q\} = F_q \cap \operatorname{dom} f, \quad \{x : f(x) > q\} = F'_q \cap \operatorname{dom} f,$$

$$\{x : g(x) < q\} = G_q \cap \operatorname{dom} g, \quad \{x : g(x) > q\} = G'_q \cap \operatorname{dom} g.$$

For $(q_1, q_2, q_3, q_4) \in K$ set

$$\begin{aligned} E_{q_1q_2q_3q_4} &= \{x : f(x) \in]q_1, q_2[\,,\ g(x) \in]q_3, q_4[\} \\ &= D \cap F'_{q_1} \cap F_{q_2} \cap G'_{q_3} \cap G_{q_4} \in \Sigma_D; \end{aligned}$$

then $E = \bigcup_{(q_1,q_2,q_3,q_4)\in K} E_{q_1q_2q_3q_4} \in \Sigma_D$.

Now $E = \{x : (f \times g)(x) < a\}$. **P (i)** If $(f \times g)(x) < a$, set $u = f(x)$, $v = g(x)$. Set

$$\eta = \min(1, \tfrac{a-uv}{1+|u|+|v|}) > 0.$$

Take $q_1, \dots , q_4 \in \mathbb{Q}$ such that

$$u - \eta \le q_1 < u < q_2 \le u + \eta, \quad v - \eta \le q_3 < v < q_4 \le v + \eta.$$

If $u' \in]q_1, q_2[$, $v' \in]q_3, q_4[$, then $|u' - u| < \eta$ and $|v' - v| < \eta$, so

$$\begin{aligned} u'v' - uv &= (u' - u)(v' - v) + (u' - u)v + u(v' - v) \\ &< \eta^2 + \eta|v| + |u|\eta \le \eta(1 + |u| + |v|) \le a - uv, \end{aligned}$$

and $u'v' < a$. Accordingly $(q_1, q_2, q_3, q_4) \in K$. Also $x \in E_{q_1q_2q_3q_4}$, so $x \in E$. Thus $\{x : (f \times g)(x) < a\} \subseteq E$. **(ii)** On the other hand, if $x \in E$, there are $q_1, \dots , q_4$ such that $(q_1, q_2, q_3, q_4) \in K$ and $x \in E_{q_1q_2q_3q_4}$, so that $f(x) \in]q_1, q_2[$ and $g(x) \in]q_3, q_4[$ and $f(x)g(x) < a$. So $E \subseteq \{x : (f \times g)(x) < a\}$. **Q**

Thus $\{x : (f \times g)(x) < a\} \in \Sigma_D$. As a is arbitrary, $f \times g$ is measurable.

(e) In view of (d), it will be enough to show that $1/g$ is measurable. Now if $a > 0$, $\{x : 1/g(x) < a\} = \{x : g(x) > 1/a\} \cup \{x : g(x) < 0\}$; if $a < 0$, then $\{x : 1/g(x) < a\} = \{x : 1/a < g(x) < 0\}$; and if $a = 0$, then $\{x : 1/g(x) < a\} = \{x : g(x) < 0\}$. And all of these belong to $\Sigma_{\operatorname{dom} 1/g}$.

(f) Write $D = \operatorname{dom} f$ and consider the set

$$\mathrm{T} = \{E : E \subseteq \mathbb{R},\ f^{-1}[E] \in \Sigma_D\}.$$

Then T is a σ-algebra of subsets of $\mathbb{R}$. **P** (i) $f^{-1}[\emptyset] = \emptyset \in \Sigma_D$, so $\emptyset \in \mathrm{T}$. (ii) If $E \in \mathrm{T}$, then $f^{-1}[\mathbb{R} \setminus E] = D \setminus f^{-1}[E] \in \Sigma_D$ so $\mathbb{R} \setminus E \in \mathrm{T}$. (iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in T, then $f^{-1}[\bigcup_{n\in\mathbb{N}} E_n] = \bigcup_{n\in\mathbb{N}} f^{-1}[E_n] \in \Sigma_D$ because Σ_D is a σ-algebra, so $\bigcup_{n\in\mathbb{N}} E_n \in \mathrm{T}$. **Q**

Next, T contains all sets of the form $H_a =]-\infty, a[$ for $a \in \mathbb{R}$, by the definition of measurability of f. The result follows by arguments already used in 114G above. First, all open subsets of $\mathbb{R}$ belong to T. **P** Let $G \subseteq \mathbb{R}$ be open. Let $K \subseteq \mathbb{Q} \times \mathbb{Q}$ be the set of pairs (q, q') of rational numbers such that $[q, q'[\subseteq G$. K is countable. Also, every $[q, q'[$ belongs to T, being $H_{q'} \setminus H_q$. So $G' = \bigcup_{(q,q')\in K} [q, q'[\in \mathrm{T}$.

By the definition of K, $G' \subseteq G$. On the other hand, if $x \in G$, there is a $\delta > 0$ such that $]x - \delta, x + \delta[\subseteq G$. Now there are rational numbers $q \in]x - \delta, x]$ and $q' \in]x, x + \delta]$, so that $(q, q') \in K$ and $x \in [q, q'[\subseteq G'$. As x is arbitrary, $G = G'$ and $G \in \mathrm{T}$. **Q**

Finally, T is a σ-algebra of subsets of $\mathbb{R}$ including the family of open sets, so must contain every Borel set, by the definition of Borel set (111G).

(g) If $a \in \mathbb{R}$, then $\{y : h(y) < a\}$ is of the form $E \cap \operatorname{dom} h$, where E is a Borel subset of $\mathbb{R}$. Next, $f^{-1}[E]$ is of the form $F \cap \operatorname{dom} f$, where $F \in \Sigma$, by (f) above. So

$$\{x : (hf)(x) < a\} = F \cap \operatorname{dom} hf \in \Sigma_{\operatorname{dom} hf}.$$

As a is arbitrary, hf is measurable.

(h) The point is that $\Sigma_{A\cap\mathrm{dom}\, f} = \{E\cap A : E \in \Sigma_{\mathrm{dom}\, f}\}$. So if $a\in\mathbb{R}$,

$$\{x : (f\restriction A)(x) < a\} = A\cap\{x : f(x) < a\} \in \Sigma_{\mathrm{dom}(f\restriction A)}.$$

Remarks Of course part (c) of this theorem is just a matter of putting (a) and (d) together, while (e) is a consequence of (d), (g) and the fact that continuous functions are Borel measurable (121Db).

I hope you will recognise the technique in the proof of part (d) as a version of arguments which may be used to prove that the limit of a product is the product of the limits, or that the product of continuous functions is continuous. In fact (b) and (d) here, as well as the theorems on sums and products of limits, are consequences of the fact that addition and multiplication are continuous functions. In 121K I give a general result which may be used to exploit such facts.

Really, part (f) here is the essence of the concept of 'measurable' real-valued function. The point of the definition in 121B-121C is that the Borel σ-algebra of $\mathbb{R}$ can be generated by any of the families $\{]-\infty,a[: a\in\mathbb{R}\}$, $\{]-\infty,a] : a\in\mathbb{R}\},\dots.$ (See 121Yc(ii).) There are many routes covering this territory in rather fewer words than I have used, at the cost of slightly greater abstraction.

121F Theorem Let X be a set and Σ a σ-algebra of subsets of X. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of Σ-measurable real-valued functions with domains included in X.

(a) Define a function $\lim_{n\to\infty} f_n$ by writing

$$(\lim_{n\to\infty} f_n)(x) = \lim_{n\to\infty} f_n(x)$$

for all those $x\in\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\mathrm{dom}\, f_m$ for which the limit exists in $\mathbb{R}$. Then $\lim_{n\to\infty} f_n$ is Σ-measurable.

(b) Define a function $\sup_{n\in\mathbb{N}} f_n$ by writing

$$(\sup_{n\in\mathbb{N}} f_n)(x) = \sup_{n\in\mathbb{N}} f_n(x)$$

for all those $x\in\bigcap_{n\in\mathbb{N}}\mathrm{dom}\, f_n$ for which the supremum exists in $\mathbb{R}$. Then $\sup_{n\in\mathbb{N}} f_n$ is Σ-measurable.

(c) Define a function $\inf_{n\in\mathbb{N}} f_n$ by writing

$$(\inf_{n\in\mathbb{N}} f_n)(x) = \inf_{n\in\mathbb{N}} f_n(x)$$

for all those $x\in\bigcap_{n\in\mathbb{N}}\mathrm{dom}\, f_n$ for which the infimum exists in $\mathbb{R}$. Then $\inf_{n\in\mathbb{N}} f_n$ is Σ-measurable.

(d) Define a function $\limsup_{n\to\infty} f_n$ by writing

$$(\limsup_{n\to\infty} f_n)(x) = \limsup_{n\to\infty} f_n(x)$$

for all those $x\in\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\mathrm{dom}\, f_m$ for which the $\limsup$ exists in $\mathbb{R}$. Then $\limsup_{n\to\infty} f_n$ is Σ-measurable.

(e) Define a function $\liminf_{n\to\infty} f_n$ by writing

$$(\liminf_{n\to\infty} f_n)(x) = \liminf_{n\to\infty} f_n(x)$$

for all those $x\in\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\mathrm{dom}\, f_m$ for which the $\liminf$ exists in $\mathbb{R}$. Then $\liminf_{n\in\mathbb{N}} f_n$ is Σ-measurable.

proof For $n\in\mathbb{N}$, $a\in\mathbb{R}$ choose $H_n(a)\in\Sigma$ such that $\{x : f_n(x)\le a\} = H_n(a)\cap\mathrm{dom}\, f_n$. The proofs are now a matter of observing the following facts:

(a) $\{x : (\lim_{n\to\infty} f_n)(x)\le a\} = \mathrm{dom}(\lim_{n\to\infty} f_n)\cap\bigcap_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n} H_m(a+2^{-k})$;

(b) $\{x : (\sup_{n\in\mathbb{N}} f_n)(x)\le a\} = \mathrm{dom}(\sup_{n\in\mathbb{N}} f_n)\cap\bigcap_{n\in\mathbb{N}} H_n(a)$;

(c) $\inf_{n\in\mathbb{N}} f_n = -\sup_{n\in\mathbb{N}}(-f_n)$;

(d) $\limsup_{n\to\infty} f_n = \lim_{n\to\infty}\sup_{m\in\mathbb{N}} f_{m+n}$;

(e) $\liminf_{n\to\infty} f_n = -\limsup_{n\to\infty}(-f_n)$.

121G Remarks It is at this point that we first encounter clearly the problem of functions which are not defined everywhere. (The quotient f/g of 121Ee also need not be defined everywhere on the common domain of f and g, but it is less important and more easily dealt with.) The whole point of the theory of measure and integration, since Lebesgue, is that we can deal with limits of sequences of functions; and the set on which $\lim_{n\to\infty} f_n(x)$ exists can be decidedly irregular, even for apparently well-behaved functions f_n. (If you have encountered the theory of Fourier series, then an appropriate example to think of is the sequence of partial sums $f_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$ of a Fourier series in which $\sum_{k=1}^{\infty} |a_k| + |b_k| = \infty$, so that the series is not uniformly absolutely summable, but may be conditionally summable at certain points.)

I have tried to make it clear what domains I mean to attach to the functions $\sup_{n\in\mathbb{N}} f_n$, $\lim_{n\to\infty} f_n$, etc. The guiding principle is that they should be the set of all $x \in X$ for which the defining formulae $\sup_{n\in\mathbb{N}} f_n(x)$, $\lim_{n\to\infty} f_n(x)$ can be interpreted as real numbers. (As I noted in 121C, I am for the time being avoiding '∞' as a value of a function, though it gives little difficulty, and some formulae are more naturally interpreted by allowing it.) But in the case of lim, lim sup, lim inf it should be noted that I am using the restrictive definition, that $\lim_{n\to\infty} a_n$ can be regarded as existing only when there is some $n \in \mathbb{N}$ such that a_m is defined for every $m \geq n$. There are occasions when it would be more natural to admit the limit when we know only that a_m is defined for infinitely many m; but such a convention could make 121Fa false, unless care was taken.

As in 111E-111F, we can use the ideas of parts (b), (c) here to discuss functions of the form $\sup_{k\in K} f_k$, $\inf_{k\in K} f_k$ for any family $\langle f_k\rangle_{k\in K}$ of measurable functions indexed by a non-empty countable set K.

In this theorem and the last, the functions f, g, f_n have been permitted to have arbitrary domains, and consequently there is nothing that can be said about the domains of the constructed functions. However, it is of course the case that if the original functions have measurable domains, so do the functions constructed from them by the rules I propose. I spell out the details in the next proposition.

121H Proposition Let X be a set and Σ a σ-algebra of subsets of X; let f, g and f_n, for $n \in \mathbb{N}$, be Σ-measurable real-valued functions whose domains belong to Σ. Then all the functions

$$f + g, \quad f \times g, \quad f/g,$$

$$\sup_{n\in\mathbb{N}} f_n, \quad \inf_{n\in\mathbb{N}} f_n, \quad \lim_{n\to\infty} f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n$$

have domains belonging to Σ. Moreover, if h is a Borel measurable real-valued function defined on a Borel subset of $\mathbb{R}$, then $\operatorname{dom} hf \in \Sigma$.

proof For the first two, we have $\operatorname{dom}(f + g) = \operatorname{dom}(f \times g) = \operatorname{dom} f \cap \operatorname{dom} g$. Next, if E is a Borel subset of $\mathbb{R}$, there is an $H \in \Sigma$ such that $f^{-1}[E] = H \cap \operatorname{dom} f$; so $f^{-1}[E] \in \Sigma$. Thus

$$\operatorname{dom} hf = f^{-1}[\operatorname{dom} h] \in \Sigma.$$

Setting $h(a) = 1/a$ for $a \in \mathbb{R} \setminus \{0\}$, we see that $\operatorname{dom}(1/f) \in \Sigma$. ($\operatorname{dom} h = \mathbb{R} \setminus \{0\}$ is a Borel set because it is open.) Similarly, $\operatorname{dom}(1/g)$ and $\operatorname{dom}(f/g) = \operatorname{dom} f \cap \operatorname{dom}(1/g)$ belong to Σ.

Now for the infinite combinations. Set $H_n(a) = \{x : x \in \operatorname{dom} f_n,\ f_n(x) < a\}$ for $n \in \mathbb{N}$, $a \in \mathbb{R}$; as just explained, every $H_n(a)$ belongs to Σ. Now

$$\operatorname{dom}(\sup_{n\in\mathbb{N}} f_n) = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} H_n(m) \in \Sigma.$$

Next, $|f_m - f_n|$ is measurable, with domain in Σ, for all m, $n \in \mathbb{N}$ (applying the results above to $-f_n = -1 \cdot f_n$, $f_m - f_n = f_m + (-f_n)$ and $|f_m - f_n| = |\ | \circ (f_m - f_n)$), so

$$G_{mnk} = \{x : x \in \operatorname{dom} f_m \cap \operatorname{dom} f_n,\ |f_m(x) - f_n(x)| \leq 2^{-k}\} \in \Sigma$$

for all m, n, $k \in \mathbb{N}$. Accordingly

$$\operatorname{dom}(\lim_{n\to\infty} f_n) = \{x : \exists\ n,\ \langle f_m(x)\rangle_{m\geq n} \text{ is Cauchy}\} = \bigcap_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} G_{mnk} \in \Sigma.$$

Manipulating the above results as in (c), (d) and (e) of the proof of 121F, we easily complete the proof.

Remark Note the use of the General Principle of Convergence in the proof above. I am not sure whether this will strike you as 'natural', and there are alternative methods; but the formula

$$\{x : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\} = \{x : \langle f_n(x)\rangle_{n\in\mathbb{N}} \text{ is Cauchy}\}$$

is one worth storing in your long-term memory.

***121I** I end this section with two results which can be safely passed by on first reading, but which you will need at some point to master if you wish to go farther into measure theory than the present chapter, as both are essential parts of the concept of 'measurable function'.

Proposition Let X be a set and Σ a σ-algebra of subsets of X. Let D be a subset of X and $f : D \to \mathbb{R}$ a function. Then f is measurable iff there is a measurable function $h : X \to \mathbb{R}$ extending f.

proof (a) If $h : X \to \mathbb{R}$ is measurable and $f = h\restriction D$, then f is measurable by 121Eh.

(b) Now suppose that f is measurable.

(i) For each $q \in \mathbb{Q}$, the set $D_q = \{x : x \in D,\ f(x) \le q\}$ belongs to the subspace σ-algebra Σ_D, that is, there is an $E_q \in \Sigma$ such that $D_q = E_q \cap D$. Set

$$F = X \setminus \bigcup_{q\in\mathbb{Q}} E_q,$$

$$G = \bigcap_{n\in\mathbb{N}} \bigcup_{q\in\mathbb{Q},q\le -n} E_q;$$

then both F and G belong to Σ, and are disjoint from D. **P** If $x \in D$, there is a $q \in \mathbb{Q}$ such that $f(x) \le q$, so that $x \in E_q$ and $x \notin F$. Also there is an $n \in \mathbb{N}$ such that $f(x) > -n$, so that $x \notin E_{q'}$ for $q' \le -n$ and $x \notin G$. **Q**

Set $H = X \setminus (F \cup G) \in \Sigma$. For $x \in H$,

$$\{q : q \in \mathbb{Q},\ x \in E_q\}$$

is non-empty and bounded below, so we may set

$$h(x) = \inf\{q : x \in E_q\};$$

for $x \in F \cup G$, set $h(x) = 0$. This defines $h : X \to \mathbb{R}$.

(ii) $h(x) = f(x)$ for $x \in D$. **P** As remarked above, $x \in H$. If $q \in \mathbb{Q}$ and $x \in E_q$, then $f(x) \le q$; consequently $h(x) \ge f(x)$. On the other hand, given $\epsilon > 0$, there is a $q \in \mathbb{Q} \cap [f(x), f(x) + \epsilon]$, and now $x \in E_q$, so $h(x) \le q \le f(x) + \epsilon$; as ϵ is arbitrary, $h(x) \le f(x)$. **Q**

(iii) h is measurable. **P** If $a > 0$ then

$$\{x : h(x) < a\} = (H \cap \bigcup_{q<a} E_q) \cup (F \cup G) \in \Sigma,$$

while if $a \le 0$

$$\{x : h(x) < a\} = H \cap \bigcup_{q<a} E_q \in \Sigma. \ \mathbf{Q}$$

This completes the proof.

***121J** The next proposition may illuminate 121E, as well as being indispensable for the work of Volume 2. I start with a useful description of the Borel sets of $\mathbb{R}^r$.

Lemma Let $r \ge 1$ be an integer, and write $\mathcal{J}$ for the family of subsets of $\mathbb{R}^r$ of the form $\{x : \xi_i \le \alpha\}$ where $i \le r$, $\alpha \in \mathbb{R}$, writing $x = (\xi_1, \dots, \xi_r)$, as in §115. Then the σ-algebra of subsets of $\mathbb{R}^r$ generated by $\mathcal{J}$ is precisely the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}^r$.

proof (a) All the sets in $\mathcal{J}$ are closed, so must belong to $\mathcal{B}$; writing Σ for the σ-algebra generated by $\mathcal{J}$, we must have $\Sigma \subseteq \mathcal{B}$.

(b) The next step is to observe that all half-open intervals of the form

$$]a, b] = \{x : \alpha_i < \xi_i \le \beta_i \ \forall\, i \le r\}$$

belong to Σ; this is because

$$]a, b] = \bigcap_{i\le r}(\{x : \xi_i \le \beta_i\} \setminus \{x : \xi_i \le \alpha_i\}).$$

It follows that all open sets belong to Σ. **P** (Compare the proof of 121Ef.) Let $G \subseteq \mathbb{R}^r$ be an open set. Let $\mathcal{I}$ be the set of all intervals of the form $]q, q']$ which are included in G, where q, $q' \in \mathbb{Q}^r$. Then $\mathcal{I}$ is a countable subset of Σ, so (because Σ is a σ-algebra) $\bigcup \mathcal{I} \in \Sigma$. By the definition of $\mathcal{I}$, $\bigcup \mathcal{I} \subseteq G$. But also, if $x \in G$, there is a $\delta > 0$ such that the open ball $U(x, \delta)$ with centre x and radius δ is included in G (1A2A). Now, for each $i \le r$, we can find rational numbers α_i, β_i such that

$$\xi_i - \frac{\delta}{r} \le \alpha_i < \xi_i \le \beta_i < \xi_i + \frac{\delta}{r},$$

so that

$$x \in]a,b] \subseteq U(x,\delta) \subseteq G$$

and $x \in]a,b] \in \mathcal{I}$. Thus $x \in \bigcup \mathcal{I}$. As x is arbitrary, $G \subseteq \bigcup \mathcal{I}$ and $G = \bigcup \mathcal{I} \in \Sigma$. **Q**

(c) Thus Σ is a σ-algebra of sets containing every open set, and must include $\mathcal{B}$, the smallest such σ-algebra.

Remark Compare the proof of 115G.

***121K Proposition** Let X be a set and Σ a σ-algebra of subsets of X. Let $r \ge 1$ be an integer, and $f_1, \dots, f_r$ measurable functions defined on subsets of X. Set $D = \bigcap_{i \le r} \operatorname{dom} f_i$ and for $x \in D$ set $f(x) = (f_1(x), \dots, f_r(x)) \in \mathbb{R}^r$. Then

(a) for any Borel set $E \subseteq \mathbb{R}^r$, $f^{-1}[E]$ belongs to the subspace σ-algebra Σ_D;

(b) if h is a Borel measurable function from a subset $\operatorname{dom} h$ of $\mathbb{R}^r$ to $\mathbb{R}$, then the composition hf is measurable.

proof (a)(i) Consider the set

$$\mathrm{T} = \{E : E \subseteq \mathbb{R}^r,\ f^{-1}[E] \in \Sigma_D\}.$$

Then T is a σ-algebra of subsets of $\mathbb{R}^r$. **P** (Compare 121Ef.) **(α)** $f^{-1}[\emptyset] = \emptyset \in \Sigma_D$, so $\emptyset \in \mathrm{T}$. **(β)** If $E \in \mathrm{T}$, then $f^{-1}[\mathbb{R}^r \setminus E] = D \setminus f^{-1}[E] \in \Sigma_D$ so $\mathbb{R}^r \setminus E \in \mathrm{T}$. **($\gamma$)** If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence in T, then $f^{-1}[\bigcup_{n\in\mathbb{N}} E_n] = \bigcup_{n\in\mathbb{N}} f^{-1}[E_n] \in \Sigma_D$ because Σ_D is a σ-algebra, so $\bigcup_{n\in\mathbb{N}} E_n \in \mathrm{T}$. **Q**

(ii) Next, for any $i \le r$ and $\alpha \in \mathbb{R}$, $J = \{x : \xi_i \le \alpha\}$ belongs to T, because

$$f^{-1}[J] = \{x : x \in D,\ f_i(x) \le \alpha\} \in \Sigma_D.$$

So T includes the family $\mathcal{J}$ of 121J and therefore includes the σ-algebra $\mathcal{B}$ generated by $\mathcal{J}$, that is, contains every Borel subset of $\mathbb{R}^r$.

(b) Now the rest follows by the argument of 121Eg. If $a \in \mathbb{R}$, then $\{y : y \in \operatorname{dom} h,\ h(y) < a\}$ is of the form $E \cap \operatorname{dom} h$, where E is a Borel subset of $\mathbb{R}^r$, so $\{x : x \in \operatorname{dom}(hf),\ (hf)(x) < a\} = f^{-1}[E] \cap \operatorname{dom}(hf)$ belongs to $\Sigma_{\operatorname{dom} hf}$.

121X Basic exercises >(a) Let X be a set, Σ a σ-algebra of subsets of X, and $D \subseteq X$. Let $\langle D_n \rangle_{n\in\mathbb{N}}$ be a partition of D into relatively measurable sets and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of measurable real-valued functions such that $D_n \subseteq \operatorname{dom} f_n$ for each n. Define $f : D \to \mathbb{R}$ by setting $f(x) = f_n(x)$ whenever $n \in \mathbb{N}$, $x \in D_n$. Show that f is measurable.

(b) Let X be a set and Σ a σ-algebra of subsets of X. If f and g are measurable real-valued functions defined on subsets of X, show that f^+, f^-, $f \wedge g$ and $f \vee g$ are measurable, where

$$f^+(x) = \max(f(x), 0) \text{ for } x \in \operatorname{dom} f,$$

$$f^-(x) = \max(-f(x), 0) \text{ for } x \in \operatorname{dom} f,$$

$$(f \vee g)(x) = \max(f(x), g(x)) \text{ for } x \in \operatorname{dom} f \cap \operatorname{dom} g,$$

$$(f \wedge g)(x) = \min(f(x), g(x)) \text{ for } x \in \operatorname{dom} f \cap \operatorname{dom} g.$$

>(c) Let (X, Σ, μ) be a measure space. Write $\mathcal{L}^0$ for the set of real-valued functions f such that (α) $\operatorname{dom} f$ is a conegligible subset of X (β) there is a conegligible set $E \subseteq X$ such that $f \restriction E$ is measurable. (i) Show that the set E of clause (β) in the last sentence may be taken to belong to Σ and be included in $\operatorname{dom} f$. (ii) Show that if f, $g \in \mathcal{L}^0$ and $c \in \mathbb{R}$, then $f + g$, cf, $f \times g$, $|f|$, f^+, f^-, $f \wedge g$, $f \vee g$ all belong to $\mathcal{L}^0$. (iii) Show that if f, $g \in \mathcal{L}^0$ and $g \ne 0$ a.e. then $f/g \in \mathcal{L}^0$. (iv) Show that if $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^0$ then the functions

$$\lim_{n\to\infty} f_n, \quad \sup_{n\in\mathbb{N}} f_n, \quad \inf_{n\in\mathbb{N}} f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n$$

belong to $\mathcal{L}^0$ whenever they are defined almost everywhere as real-valued functions. (v) Show that if $f \in \mathcal{L}^0$ and $h : \mathbb{R} \to \mathbb{R}$ is Borel measurable then $hf \in \mathcal{L}^0$.

>(d) Consider the following four families of subsets of $\mathbb{R}$:

$$\mathcal{A}_1 = \{]-\infty, a[: a \in \mathbb{R}\}, \quad \mathcal{A}_2 = \{]-\infty, a] : a \in \mathbb{R}\},$$

$$\mathcal{A}_3 = \{]a, \infty[: a \in \mathbb{R}\}, \quad \mathcal{A}_4 = \{[a, \infty[: a \in \mathbb{R}\}.$$

Show that for each j the σ-algebra of subsets of $\mathbb{R}$ generated by $\mathcal{A}_j$ is the σ-algebra of Borel sets.

(e) Let D be any subset of $\mathbb{R}^r$, where $r \geq 1$. Write $\mathfrak{T}_D$ for the set $\{G \cap D : G \subseteq \mathbb{R}^r \text{ is open}\}$. (i) Show that $\mathfrak{T}_D$ satisfies the properties of open sets listed in 1A2B. (ii) Let $\mathcal{B}$ be the σ-algebra of Borel sets in $\mathbb{R}^r$, and $\mathcal{B}(D)$ the subspace σ-algebra on D. Show that $\mathcal{B}(D)$ is just the σ-algebra of subsets of D generated by $\mathfrak{T}_D$. (*Hint*: (α) observe that $\mathfrak{T}_D \subseteq \mathcal{B}(D)$ (β) consider $\{E : E \subseteq \mathbb{R}^r, E \cap D$ belongs to the σ-algebra generated by $\mathfrak{T}_D\}$.)

(f) Let (X, Σ, μ) be a measure space and define $\mathcal{L}^0$ as in 121Xc. Show that if $f_1, \dots, f_r$ belong to $\mathcal{L}^0$ and $h : \mathbb{R}^r \to \mathbb{R}$ is Borel measurable then $h(f_1, \dots, f_r)$ belongs to $\mathcal{L}^0$.

121Y Further exercises (a) Let X and Y be sets, Σ a σ-algebra of subsets of X, $\phi : X \to Y$ a function and g a real-valued function defined on a subset of Y. Set $\mathrm{T} = \{F : F \subseteq Y, \phi^{-1}[F] \in \Sigma\}$; then T is a σ-algebra of subsets of Y (see 111Xc). (i) Show that if g is T-measurable then $g\phi$ is Σ-measurable. (ii) Give an example in which $g\phi$ is Σ-measurable but g is not T-measurable. (iii) Show that if $g\phi$ is Σ-measurable and *either* ϕ is injective *or* $\mathrm{dom}(g\phi) \in \Sigma$ *or* $\phi[X] \subseteq \mathrm{dom}\, g$, then g is T-measurable.[1]

(b) Let X and Y be sets, T a σ-algebra of subsets of Y and $\phi : X \to Y$ a function. Set $\Sigma = \{\phi^{-1}[F] : F \in \mathrm{T}\}$, as in 111Xd. Show that a function $f : X \to \mathbb{R}$ is Σ-measurable iff there is a T-measurable function $g : Y \to \mathbb{R}$ such that $f = g\phi$.

(c) Let X and Y be sets and Σ, T σ-algebras of subsets of X, Y respectively. I say that a function $\phi : X \to Y$ is (Σ, T)**-measurable** if $\phi^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$. (i) Show that if Σ, T, Υ are σ-algebras of subsets of X, Y, Z respectively, and $\phi : X \to Y$ is (Σ, T)-measurable, $\psi : Y \to Z$ is (T, Υ)-measurable, then $\psi\phi : X \to Z$ is (Σ, Υ)-measurable. (ii) Suppose that $\mathcal{A} \subseteq \mathrm{T}$ is such that T is the σ-algebra of subsets of Y generated by $\mathcal{A}$ (111Gb). Show that $\phi : X \to Y$ is (Σ, T)-measurable iff $\phi^{-1}[A] \in \Sigma$ for every $A \in \mathcal{A}$. (iii) For $r \geq 1$, write $\mathcal{B}_r$ for the σ-algebra of Borel subsets of $\mathbb{R}^r$. Show that if X is any set and Σ is a σ-algebra of subsets of X, then a function $f : X \to \mathbb{R}^r$ is $(\Sigma, \mathcal{B}_r)$-measurable iff $\pi_i f : X \to \mathbb{R}$ is $(\Sigma, \mathcal{B}_1)$-measurable for every $i \leq r$, writing $\pi_i(x) = \xi_i$ for $i \leq r$, $x = (\xi_1, \dots, \xi_r) \in \mathbb{R}^r$. (iv) Rewrite these ideas for partially-defined functions.

(d) Let X be a set and Σ a σ-algebra of subsets of X. For $r \geq 1$, $D \subseteq X$ say that a function $\phi : D \to \mathbb{R}^r$ is **measurable** if $\phi^{-1}[G]$ is relatively measurable in D for every open set $G \subseteq \mathbb{R}^r$. If $X = \mathbb{R}^s$ and Σ is the σ-algebra $\mathcal{B}_s$ of Borel subsets of $\mathbb{R}^s$, say that ϕ is **Borel measurable**. (i) Show that ϕ is measurable in this sense iff all its coordinate functions $\phi_i : D \to \mathbb{R}$ are measurable in the sense of 121C, taking $\phi(x) = (\phi_i(x), \dots, \phi_r(x))$ for $x \in D$. (In particular, this definition agrees with 121C when $r = 1$.) (ii) Show that $\phi : D \to \mathbb{R}^r$ is measurable iff it is $(\Sigma, \mathcal{B}_r)$-measurable in the sense of 121Yc. (iii) Show that if $\phi : D \to \mathbb{R}^r$ is measurable and $\psi : E \to \mathbb{R}^s$ is Borel measurable, where $E \subseteq \mathbb{R}^r$, then $\psi\phi : \phi^{-1}[E] \to \mathbb{R}^s$ is measurable. (iv) Show that any continuous function from a subset of $\mathbb{R}^s$ to $\mathbb{R}^r$ is Borel measurable.

(e) Let X be a set and θ an outer measure on X; let μ be the measure defined from θ by Carathéodory's method, and Σ its domain. Suppose that $f : X \to \mathbb{R}$ is a function such that

$$\theta\{x : x \in A,\, f(x) \leq a\} + \theta\{x : x \in A,\, f(x) \geq b\} \leq \theta A$$

whenever $A \subseteq X$ and $a < b$ in $\mathbb{R}$. Show that f is Σ-measurable. (*Hint*: suppose that $a \in \mathbb{R}$ and $\theta A < \infty$. Set

$$B_k = \{x : x \in A,\, a + \tfrac{1}{2k+2} \leq f(x) \leq a + \tfrac{1}{2k+1}\},$$

[1] I am grateful to P.Wallace Thompson for pointing out the error in the original version of this exercise.

$$B'_k = \{x : x \in A,\ a + \tfrac{1}{2k+3} \le f(x) \le a + \tfrac{1}{2k+2}\}$$

for $k \in \mathbb{N}$. Show that $\sum_{k=0}^{\infty} \theta B_k \le \theta A$, and check a similar result for B'_k. Hence show that

$$\theta\{x : x \in A,\ f(x) > a\} = \lim_{k\to\infty} \theta\{x : x \in A,\ f(x) \ge a + \tfrac{1}{k}\}.)$$

121 Notes and comments I find myself offering no fewer than three definitions of 'measurable function', in 121C, 121Yc and 121Yd. It is in fact the last which is probably the most important and the best guide to further ideas. Nevertheless, the overwhelming majority of applications refer to real-valued functions, and the four equivalent conditions of 121B are the most natural and most convenient to use. The fact that they all coincide with the condition of 121Yd corresponds to the fact that they are all of the form

$$f^{-1}[E] \in \Sigma_D \text{ for every } E \in \mathcal{A}$$

where $\mathcal{A}$ is a family of subsets of $\mathbb{R}$ generating the Borel σ-algebra (121Xd, 121Yc(ii)).

The class of measurable functions may well be the widest you have yet seen, not counting the family of all real-valued functions; all easily describable functions between subsets of $\mathbb{R}$ are measurable. Not only is the space of measurable functions closed under addition and multiplication and composition with continuous functions (121E), but any natural operation acting on a sequence of measurable functions will produce a measurable function (121F, 121Xb, 121Xa). It is *not* however the case that the composition of two Lebesgue measurable functions from $\mathbb{R}$ to itself is always Lebesgue measurable; I offer a counter-example in 134Ib. The point here is that a function is called 'measurable' if it is $(\Sigma, \mathcal{B})$-measurable, in the language of 121Yc, where $\mathcal{B}$ is the σ-algebra of Borel sets. Such a function can well fail to be (Σ, Σ)-measurable, if Σ properly includes $\mathcal{B}$, so the natural argument for compositions (121Yc(i)) fails. Nevertheless, for reasons which I will hint at in §134, non-Lebesgue-measurable functions are hard to come by, and only in the most rarefied kinds of real analysis do they appear in any natural way. You may therefore approach the question of whether a particular function is Lebesgue measurable with reasonable confidence that it is, and that the proof is merely a challenge to your technique.

You will observe that the results of 121E are mostly covered by 121I-121K, which also include large parts of 114G and 115G; and that 121Kb is covered by 121Yd(iii). You can count yourself as having mastered this part of the subject when you find my exposition tediously repetitive. Of course, in order to deduce 121Ed from 121K, for instance, you have to know that multiplication, regarded as a function from $\mathbb{R}^2$ to $\mathbb{R}$, is continuous, therefore Borel measurable; the proof of this is embedded in the proof I give of 121Ed (look at the formula for η half way through).

Version of 4.1.04

122 Definition of the integral

I set out the definition of ordinary integration for real-valued functions defined on an arbitrary measure space, with its most basic properties.

122A Definitions Let (X, Σ, μ) be a measure space.

(a) For any set $A \subseteq X$, I write χA for the **indicator function** or **characteristic function** of A, the function from X to $\{0, 1\}$ given by setting $\chi A(x) = 1$ if $x \in A$, 0 if $x \in X \setminus A$. (Of course this notation depends on it being understood which is the 'universal' set X under consideration; perhaps I should call it the 'indicator function of A as a subset of X'.) Observe that χA is Σ-measurable, in the sense of 121C above, iff $A \in \Sigma$ (because $A = \{x : \chi A(x) > 0\}$).

(b) Now a **simple** function on X is a function of the form $\sum_{i=0}^{n} a_i \chi E_i$, where $E_0, \dots, E_n$ are measurable sets of finite measure and $a_0, \dots, a_n$ belong to $\mathbb{R}$. **Warning!** Some authors allow arbitrary sets E_i, so that a simple function on X is any function taking only finitely many values.

122B Lemma Let (X, Σ, μ) be a measure space.

(a) Every simple function on X is measurable.

(b) If f, $g : X \to \mathbb{R}$ are simple functions, so is $f + g$.

(c) If $f : X \to \mathbb{R}$ is a simple function and $c \in \mathbb{R}$, then $cf : X \to \mathbb{R}$ is a simple function.

(d) The constant zero function is simple.

proof (a) comes from the facts that χE is measurable for measurable E, and that sums and scalar multiples of measurable functions are measurable (121Eb-121Ec). (b)-(d) are trivial.

122C Lemma Let (X, Σ, μ) be a measure space.

(a) If $E_0, \dots , E_n$ are measurable sets of finite measure, there are disjoint measurable sets $G_0, \dots , G_m$ of finite measure such that each E_i is expressible as a union of some of the G_j.

(b) If $f : X \to \mathbb{R}$ is a simple function, it is expressible in the form $\sum_{j=0}^m b_j \chi G_j$ where $G_0, \dots , G_m$ are disjoint measurable sets of finite measure.

(c) If $E_0, \dots , E_n$ are measurable sets of finite measure, and $a_0, \dots , a_n \in \mathbb{R}$ are such that $\sum_{i=0}^n a_i \chi E_i(x) \ge 0$ for every $x \in X$, then $\sum_{i=0}^n a_i \mu E_i \ge 0$.

proof (a) Set $m = 2^{n+1} - 2$, and enumerate the non-empty subsets of $\{0, \dots , n\}$ as $I_0, \dots , I_m$. For each $j \le m$, set

$$G_j = \bigcap_{i \in I_j} E_i \setminus \bigcup_{i \le n, i \notin I_j} E_i.$$

Then every G_j is a measurable set, being obtained from finitely many measurable sets by the operations $\cup$, $\cap$ and $\setminus$, and has finite measure, because $I_j \ne \emptyset$ and $G_j \subseteq E_i$ if $i \in I_j$. Moreover, the G_j are disjoint, for if $i \in I_j \setminus I_k$ then $G_j \subseteq E_i$ and $G_k \cap E_i = \emptyset$. Finally, if $k \le n$ and $x \in E_k$, there is a $j \le m$ such that $I_j = \{i : i \le n, x \in E_i\}$, and in this case $x \in G_j \subseteq E_k$; thus E_k is the union of those G_j which it includes.

(b) Express f as $\sum_{i=0}^n a_i \chi E_i$ where $E_0, \dots , E_n$ are measurable sets of finite measure and $a_0, \dots , a_n$ are real numbers. Let $G_0, \dots , G_m$ be disjoint measurable sets of finite measure such that every E_i is expressible as a union of appropriate G_j. Set $c_{ij} = 1$ if $G_j \subseteq E_i$, 0 otherwise, so that, because the G_j are disjoint, $\chi E_i = \sum_{j=0}^m c_{ij} \chi G_j$ for each i. Then

$$f = \sum_{i=0}^n a_i \chi E_i = \sum_{i=0}^n \sum_{j=0}^m a_i c_{ij} \chi G_j = \sum_{j=0}^m b_j \chi G_j,$$

setting $b_j = \sum_{i=0}^n a_i c_{ij}$ for each $j \le m$.

(c) Set $f = \sum_{i=0}^n a_i \chi E_i$, and take G_j, c_{ij}, b_j as in (b). Then $b_j \mu G_j \ge 0$ for every j. **P** If $G_j = \emptyset$, this is trivial. Otherwise, let $x \in G_j$; then

$$0 \le f(x) = \sum_{i=0}^n b_i \chi G_i(x) = b_j \chi G_j(x) = b_j,$$

so again $b_j \mu G_j \ge 0$. **Q** Next, because the G_j are disjoint,

$$\mu E_i = \sum_{j=0}^m c_{ij} \mu G_j$$

for each i, so

$$\sum_{i=0}^n a_i \mu E_i = \sum_{i=0}^n \sum_{j=0}^m a_i c_{ij} \mu G_j = \sum_{j=0}^m b_j \mu G_j \ge 0,$$

as required.

122D Corollary Let (X, Σ, μ) be a measure space. If

$$\sum_{i=0}^m a_i \chi E_i = \sum_{j=0}^n b_j \chi F_j,$$

where all the E_i and F_j are measurable sets of finite measure and the a_i, b_j are real numbers, then

$$\sum_{i=0}^m a_i \mu E_i = \sum_{j=0}^n b_j \mu F_j.$$

proof Apply 122Cc to $\sum_{i=0}^m a_i \chi E_i + \sum_{j=0}^n (-b_j) \chi F_j$ to see that $\sum_{i=0}^m a_i \mu E_i - \sum_{j=0}^n b_j \mu F_j \ge 0$; now reverse the roles of the two sums to get the opposite inequality.

122E Definition Let (X, Σ, μ) be a measure space. Then we may define the **integral** $\int f$ of f, for simple functions $f : X \to \mathbb{R}$, by saying that $\int f = \sum_{i=0}^m a_i \mu E_i$ whenever $f = \sum_{i=0}^m a_i \chi E_i$ and every E_i is a measurable set of finite measure; 122D promises us that it won't matter which representation of f we pick on for the calculation.

122F Proposition Let (X, Σ, μ) be a measure space.
(a) If f, $g : X \to \mathbb{R}$ are simple functions, then $f + g$ is a simple function and $\int f + g = \int f + \int g$.
(b) If f is a simple function and $c \in \mathbb{R}$, then cf is a simple function and $\int cf = c \int f$.
(c) If f, g are simple functions and $f(x) \le g(x)$ for every $x \in X$, then $\int f \le \int g$.

proof (a) and (b) are immediate from the formula given for $\int f$ in 122E. As for (c), observe that $g - f$ is a non-negative simple function, so that $\int g - f \ge 0$, by 122Cc; but this means that $\int g - \int f \ge 0$.

122G Lemma Let (X, Σ, μ) be a measure space. If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a sequence of simple functions which is non-decreasing (in the sense that $f_n(x) \le f_{n+1}(x)$ for every $n \in \mathbb{N}$, $x \in X$) and f is a simple function such that $f(x) \le \sup_{n\in\mathbb{N}} f_n(x)$ for almost every $x \in X$ (allowing $\sup_{n\in\mathbb{N}} f_n(x) = \infty$ in this formula), then $\int f \le \sup_{n\in\mathbb{N}} \int f_n$.

proof Note that $f - f_0$ is a simple function, so $H = \{x : (f - f_0)(x) \ne 0\}$ is a finite union of sets of finite measure, and $\mu H < \infty$; also $f - f_0$ is bounded, so there is an $M \ge 0$ such that $(f - f_0)(x) \le M$ for every $x \in X$.

Let $\epsilon > 0$. For each $n \in \mathbb{N}$, set $H_n = \{x : (f - f_n)(x) \ge \epsilon\}$. Then each H_n is measurable (by 121E), and $\langle H_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets with intersection

$$\bigcap_{n\in\mathbb{N}} H_n = \{x : f(x) \ge \epsilon + \sup_{n\in\mathbb{N}} f_n(x)\} \subseteq \{x : f(x) > \sup_{n\in\mathbb{N}} f_n(x)\}.$$

Because $f(x) \le \sup_{n\in\mathbb{N}} f_n(x)$ for almost every x, $\{x : f(x) > \sup_{n\in\mathbb{N}} f_n(x)\}$ and $\bigcap_{n\in\mathbb{N}} H_n$ are negligible. Also $\mu H_0 < \infty$, because $H_0 \subseteq H$. Consequently

$$\lim_{n\to\infty} \mu H_n = \mu(\bigcap_{n\in\mathbb{N}} H_n) = 0$$

(112Cf). Let n be so large that $\mu H_n \le \epsilon$.

Consider the simple function $g = f_n + \epsilon\chi H + M\chi H_n$. Then $f \le g$, so

$$\int f \le \int g = \int f_n + \epsilon\mu H + M\mu H_n \le \int f_n + \epsilon(M + \mu H).$$

As ϵ is arbitrary, $\int f \le \sup_{n\in\mathbb{N}} \int f_n$.

122H Definition Let (X, Σ, μ) be a measure space. For the rest of this section, I will write U for the set of functions f such that

(i) the domain of f is a conegligible subset of X and $f(x) \in [0, \infty[$ for each $x \in \operatorname{dom} f$,

(ii) there is a non-decreasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative simple functions such that $\sup_{n\in\mathbb{N}} \int f_n < \infty$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$.

122I Lemma If f and $\langle f_n \rangle_{n\in\mathbb{N}}$ are as in 122H, then

$$\sup_{n\in\mathbb{N}} \int f_n = \sup\{\int g : g \text{ is a simple function and } g \le_{\text{a.e.}} f\}.$$

proof Of course

$$\sup_{n\in\mathbb{N}} \int f_n \le \sup\{\int g : g \text{ is a simple function and } g \le_{\text{a.e.}} f\}$$

because $f_n \le_{\text{a.e.}} f$ for each n. On the other hand, if g is a simple function and $g \le_{\text{a.e.}} f$, then $g(x) \le \sup_{n\in\mathbb{N}} f_n(x)$ for almost every x, so $\int g \le \sup_{n\in\mathbb{N}} \int f_n$ by 122G. Thus

$$\sup_{n\in\mathbb{N}} \int f_n \ge \sup\{\int g : g \text{ is a simple function and } g \le_{\text{a.e.}} f\},$$

as required.

122J Lemma Let (X, Σ, μ) be a measure space, and define U as in 122H.

(a) If f is a function defined on a conegligible subset of X and taking values in $[0, \infty[$, then $f \in U$ iff there is a conegligible measurable set $E \subseteq \operatorname{dom} f$ such that

(α) $f \restriction E$ is measurable,

(β) for every $\epsilon > 0$, $\mu\{x : x \in E,\ f(x) \ge \epsilon\} < \infty$,

(γ) $\sup\{\int g : g \text{ is a simple function, } g \le_{\text{a.e.}} f\} < \infty$.

(b) Suppose that $f \in U$ and that h is a function defined on a conegligible subset of X and taking values in $[0, \infty[$. Suppose that $h \leq_{\text{a.e.}} f$ and there is a conegligible $F \subseteq X$ such that $h \restriction F$ is measurable. Then $h \in U$.

proof (a)(i) Suppose that $f \in U$. Then there is a non-decreasing sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of non-negative simple functions such that $f =_{\text{a.e.}} \lim_{n\to\infty} f_n$ and $\sup_{n \in \mathbb{N}} \int f_n = c < \infty$. The set $\{x : f(x) = \lim_{n\to\infty} f_n(x)\}$ is conegligible, so includes a measurable conegligible set E say. Now $f \restriction E = (\lim_{n\to\infty} f_n) \restriction E$ is measurable, by 121Fa and 121Eh; thus (α) is satisfied. Next, given $\epsilon > 0$, set $H_n = \{x : x \in E,\ f_n(x) \geq \frac{1}{2}\epsilon\}$; then $f_n \geq \frac{1}{2}\epsilon\chi H_n$, so

$$\frac{1}{2}\epsilon\mu H_n = \int \tfrac{1}{2}\epsilon\chi H_n \leq \int f_n \leq c,$$

for each n. Now $\langle H_n \rangle_{n \in \mathbb{N}}$ is non-decreasing, so

$$\mu(\textstyle\bigcup_{n \in \mathbb{N}} H_n) = \sup_{n \in \mathbb{N}} \mu H_n \leq 2c/\epsilon,$$

by 112Ce. Accordingly

$$\mu\{x : x \in E,\ f(x) \geq \epsilon\} \leq \mu(\textstyle\bigcup_{n \in \mathbb{N}} H_n) \leq 2c/\epsilon < \infty.$$

As ϵ is arbitrary, (β) is satisfied. Finally, (γ) is satisfied by 122I.

(ii) Now suppose that the conditions (α)-(γ) are satisfied. Take an appropriate conegligible $E \in \Sigma$, and for each $n \in \mathbb{N}$ define $f_n : X \to \mathbb{R}$ by setting

$$\begin{aligned} f_n(x) &= 2^{-n}k \text{ if } x \in E,\ 0 \leq k < 4^n,\ 2^{-n}k \leq f(x) < 2^{-n}(k+1), \\ &= 0 \text{ if } x \in X \setminus E, \\ &= 2^n \text{ if } x \in E \text{ and } f(x) \geq 2^n. \end{aligned}$$

Then f_n is a non-negative simple function, being expressible as

$$f_n = \textstyle\sum_{k=1}^{4^n} 2^{-n}\chi\{x : x \in E,\ f(x) \geq 2^{-n}k\};$$

all the sets $\{x : x \in E,\ f(x) \geq 2^{-n}k\}$ being measurable (because $f \restriction E$ is measurable) and of finite measure, by (β). Also it is easy to see that $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence which converges to f at every point of E, so that $f =_{\text{a.e.}} \lim_{n\to\infty} f_n$. Finally,

$$\lim_{n\to\infty} \int f_n = \sup_{n \in \mathbb{N}} \int f_n \leq \sup\{\int g : g \leq f \text{ is simple}\} < \infty,$$

by (γ). So $f \in U$.

(b) Let E be a set as in (a). The sets E, F and $\{x : h(x) \leq f(x)\}$ are all conegligible, so there is a conegligible measurable set E' included in their intersection. Now $E' \subseteq \operatorname{dom} h$, $h \restriction E'$ is measurable,

$$\mu\{x : x \in E',\ h(x) \geq \epsilon\} \leq \mu\{x : x \in E,\ f(x) \geq \epsilon\} < \infty$$

for every $\epsilon > 0$, and

$$\sup\{\int g : g \text{ is simple},\ g \leq_{\text{a.e.}} h\} \leq \sup\{\int g : g \text{ is simple},\ g \leq_{\text{a.e.}} f\} < \infty.$$

So $h \in U$.

122K Definition Let (X, Σ, μ) be a measure space, and define U as in 122H. For $f \in U$, set

$$\int f = \sup\{\int g : g \text{ is a simple function and } g \leq_{\text{a.e.}} f\}.$$

By 122I, we see that $\int f = \lim_{n\to\infty} \int f_n$ whenever $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of simple functions converging to f almost everywhere; in particular, if $f \in U$ is itself a simple function, then $\int f$, as defined here, agrees with the original definition of $\int f$ in 122E, since we may take $f_n = f$ for every n.

122L Lemma Let (X, Σ, μ) be a measure space.
(a) If $f, g \in U$ then $f + g \in U$ and $\int f + g = \int f + \int g$.
(b) If $f \in U$ and $c \geq 0$ then $cf \in U$ and $\int cf = c\int f$.

(c) If f, $g \in U$ and $f \leq_{\text{a.e.}} g$ then $\int f \leq \int g$.

(d) If $f \in U$ and g is a function with domain a conegligible subset of X, taking values in $[0, \infty[$, and equal to f almost everywhere, then $g \in U$ and $\int g = \int f$.

(e) If f_1, g_1, f_2, $g_2 \in U$ and $f_1 - f_2 = g_1 - g_2$, then $\int f_1 - \int f_2 = \int g_1 - \int g_2$.

proof (a) We know that there are non-decreasing sequences $\langle f_n \rangle_{n\in\mathbb{N}}$, $\langle g_n \rangle_{n\in\mathbb{N}}$ of non-negative simple functions such that $f =_{\text{a.e.}} \lim_{n\to\infty} f_n$, $g =_{\text{a.e.}} \lim_{n\to\infty} g_n$, $\sup_{n\in\mathbb{N}} \int f_n < \infty$ and $\sup_{n\in\mathbb{N}} \int g_n < \infty$. Now $\langle f_n + g_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions converging to $f + g$ a.e., and

$$\sup_{n\in\mathbb{N}} \int f_n + g_n = \lim_{n\to\infty} \int f_n + g_n = \lim_{n\to\infty} \int f_n + \lim_{n\to\infty} \int g_n = \int f + \int g.$$

Accordingly $f + g \in U$, and also, as remarked in 122K,

$$\int f + g = \lim_{n\to\infty} \int f_n + g_n = \int f + \int g.$$

(b) We know that there is a non-decreasing sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of non-negative simple functions such that $f =_{\text{a.e.}} \lim_{n\to\infty} f_n$ and $\sup_{n\in\mathbb{N}} \int f_n < \infty$. Now $\langle cf_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions converging to cf a.e., and

$$\sup_{n\in\mathbb{N}} \int cf_n = \lim_{n\to\infty} \int cf_n = c \lim_{n\to\infty} \int f_n = c\int f.$$

Accordingly $cf \in U$, and also, as remarked in 122K,

$$\int cf = \lim_{n\to\infty} \int cf_n = c\int f.$$

(c) This is obvious from 122K.

(d) If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions converging to f a.e., then it also converges to g a.e.; so $g \in U$ and

$$\int g = \lim_{n\to\infty} \int f_n = \int f.$$

(e) By (a), $f_1 + g_2$ and $f_2 + g_1$ both belong to U. Also, they are equal at any point at which all four functions are defined, which is almost everywhere. So

$$\int f_1 + \int g_2 = \int f_1 + g_2 = \int f_2 + g_1 = \int f_2 + \int g_1,$$

using (a) and (d). Shifting $\int g_2$ and $\int f_2$ across the equation, we have the result.

122M Definition Let (X, Σ, μ) be a measure space. Define U as in 122H. A real-valued function f is **integrable**, or **integrable over** X, or μ**-integrable over** X, if it is expressible as $f_1 - f_2$ with f_1, $f_2 \in U$, and in this case its **integral** is

$$\int f = \int f_1 - \int f_2.$$

122N Remarks (a) We see from 122Le that the integral $\int f$ is uniquely defined by the formula in 122M. Secondly, if $f \in U$, then $f = f - \mathbf{0}$ is integrable, and the integral here agrees with that defined in 122K. Finally, if f is a simple function, then it can be expressed as $f_1 - f_2$ where f_1, f_2 are non-negative simple functions (if $f = \sum_{i=0}^{n} a_i \chi E_i$, where each E_i is measurable and of finite measure, set

$$f_1 = \sum_{i=0}^{n} a_i^+ \chi E_i, \quad f_2 = \sum_{i=0}^{n} a_i^- \chi E_i,$$

writing $a_i^+ = \max(a_i, 0)$, $a_i^- = \max(-a_i, 0)$); so that

$$\int f = \int f_1 - \int f_2 = \sum_{i=0}^{n} a_i \mu E_i,$$

and the definition of 122M is consistent with the definition of 122E.

(b) Alternative notations which I will use for $\int f$ are $\int_X f$, $\int f d\mu$, $\int f(x)\mu(dx)$, $\int f(x)dx$, $\int_X f(x)\mu(dx)$, etc., according to which aspects of the context seem due for emphasis.

When μ is Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^r$ we say that $\int f$ is the **Lebesgue integral** of f, and that f is **Lebesgue integrable** if this is defined.

(c) Note that when I say, in 122M, that 'f can be expressed as $f_1 - f_2$', I mean to interpret $f_1 - f_2$ according to the rules set out in 121E, so that $\operatorname{dom} f$ must be $\operatorname{dom}(f_1 - f_2) = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, and is surely conegligible.

122O Theorem Let (X, Σ, μ) be a measure space.

(a) If f and g are integrable over X then $f + g$ is integrable and $\int f + g = \int f + \int g$.

(b) If f is integrable over X and $c \in \mathbb{R}$ then cf is integrable and $\int cf = c \int f$.

(c) If f is integrable over X and $f \geq 0$ a.e. then $\int f \geq 0$.

(d) If f and g are integrable over X and $f \leq_{\text{a.e.}} g$ then $\int f \leq \int g$.

proof (a) Express f as $f_1 - f_2$ and g as $g_1 - g_2$ where f_1, f_2, g_1 and g_2 belong to U, as defined in 122H. Then $f + g = (f_1 + g_1) - (f_2 + g_2)$ is integrable because U is closed under addition (122La), and

$$\int f + g = \int f_1 + g_1 - \int f_2 + g_2 = \int f_1 + \int g_1 - \int f_2 - \int g_2 = \int f + \int g.$$

(b) Express f as $f_1 - f_2$ where f_1, f_2 belong to U. If $c \geq 0$ then $cf = cf_1 - cf_2$ is integrable because U is closed under multiplication by non-negative scalars (122Lb), and

$$\int cf = \int cf_1 - \int cf_2 = c\int f_1 - c\int f_2 = c\int f.$$

If $c = -1$ then $-f = f_2 - f_1$ is integrable and

$$\int (-f) = \int f_2 - \int f_1 = -\int f.$$

Putting these together we get the result for $c < 0$.

(c) Express f as $f_1 - f_2$ where f_1, $f_2 \in U$. Then $f_2 \leq_{\text{a.e.}} f_1$, so $\int f_2 \leq \int f_1$ (122Lc), and $\int f \geq 0$.

(d) Apply (c) to $g - f$.

122P Theorem Let (X, Σ, μ) be a measure space and f a real-valued function defined on a conegligible subset of X. Then the following are equiveridical:

(i) f is integrable;

(ii) $|f| \in U$, as defined in 122H, and there is a conegligible set $E \subseteq X$ such that $f\restriction E$ is measurable;

(iii) there are a $g \in U$ and a conegligible set $E \subseteq X$ such that $|f| \leq_{\text{a.e.}} g$ and $f\restriction E$ is measurable.

proof (i)$\Rightarrow$(iii) Suppose that f is integrable. Let f_1, $f_2 \in U$ be such that $f = f_1 - f_2$. Then there are conegligible sets E_1, E_2 such that $f_1\restriction E_1$ and $f_2\restriction E_2$ are measurable; set $E = E_1 \cap E_2$, so that E also is a conegligible set. Now $f\restriction E = f_1\restriction E - f_2\restriction E$ is measurable. Next, $f_1 + f_2 \in U$ (122La) and $|f|(x) \leq f_1(x) + f_2(x)$ for every $x \in \operatorname{dom} f$, so we may take $g = f_1 + f_2$.

(iii)$\Rightarrow$(ii) If $f\restriction E$ is measurable, so is $|f|\restriction E = |f\restriction E|$ (121Eg); so if $g \in U$ and $|f| \leq_{\text{a.e.}} g$, then $|f| \in U$ by 122Jb.

(ii)$\Rightarrow$(i) Suppose that f satisfies the conditions of (ii). Set $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$. Of course $|f|\restriction E$, $f^+\restriction E$ and $f^-\restriction E$ are all measurable. Also $0 \leq f^+(x) \leq |f|(x)$ and $0 \leq f^-(x) \leq |f|(x)$ for every $x \in \operatorname{dom} f$, while $|f| \in U$ by hypothesis, so f^+ and f^- belong to U by 122Jb. Finally, $f = f^+ - f^-$, so f is integrable.

122Q Remark The condition 'there is a conegligible set E such that $f\restriction E$ is measurable' recurs so often that I think it worth having a phrase for it; I will call such functions **virtually measurable**, or **μ-virtually measurable** if it seems necessary to specify the measure.

122R Corollary Let (X, Σ, μ) be a measure space.

(a) A non-negative real-valued function, defined on a subset of X, is integrable iff it belongs to U, as defined in 122H.

(b) If f is integrable over X and h is a real-valued function, defined on a conegligible subset of X and equal to f almost everywhere, then h is integrable, with $\int h = \int f$.

(c) If f is integrable over X, $f \geq 0$ a.e. and $\int f \leq 0$, then $f = 0$ a.e.

(d) If f and g are integrable over X, $f \leq_{\text{a.e.}} g$ and $\int g \leq \int f$, then $f =_{\text{a.e.}} g$.
(e) If f is integrable over X, so is $|f|$, and $|\int f| \leq \int |f|$.

proof (a) If f is integrable then $f = |f| \in U$, by 122P(ii). If $f \in U$ then $f = f - \mathbf{0}$ is integrable, by 122M.

(b) Let E, F be conegligible sets such that $f\restriction E$ is measurable and $h\restriction F = f\restriction F$; then $E \cap F$ is conegligible and $h\restriction(E \cap F) = (f\restriction E)\restriction F$ is measurable. Next, there is a $g \in U$ such that $|f| \leq_{\text{a.e.}} g$, and of course $|h| \leq_{\text{a.e.}} g$. So h is integrable by 122P(iii). By 122Od, applied to f and h and then to h and f, $\int h = \int f$.

(c) ? Suppose, if possible, otherwise. Let $E \subseteq X$ be a conegligible set such that $f\restriction E$ is measurable (122P(ii)), and $E' \subseteq E \cap \operatorname{dom} f$ a conegligible measurable set. Then $F = \{x : x \in E', f(x) > 0\}$ must be non-negligible. Set $F_k = \{x : x \in E', f(x) \geq 2^{-k}\}$ for each $k \in \mathbb{N}$, so that $F = \bigcup_{k\in\mathbb{N}} F_k$ and there is a k such that $\mu F_k > 0$. But consider $g = 2^{-k}\chi F_k$. Because $f \geq 0$ a.e. and $f \geq 2^{-k}$ on F_k, $f \geq_{\text{a.e.}} g$, so that

$$0 < 2^{-k}\mu F_k = \int g \leq \int f,$$

by 122Od; which is impossible. **X**

(d) Apply (c) to $g - f$.

(e) By (i)⇒(ii) of 122P, $|f|$ is integrable. Now $f^+ = \frac{1}{2}(|f| + f)$ and $f^- = \frac{1}{2}(|f| - f)$ are both integrable and non-negative, so have non-negative integrals, and

$$|\int f| = |\int f^+ - \int f^-| \leq \int f^+ + \int f^- = \int |f|.$$

122X Basic exercises (a) Let (X, Σ, μ) be a measure space. (i) Show that if $f : X \to \mathbb{R}$ is simple so is $|f|$, setting $|f|(x) = |f(x)|$ for $x \in \operatorname{dom} f = X$. (ii) Show that if f, $g : X \to \mathbb{R}$ are simple functions so are $f \vee g$ and $f \wedge g$, as defined in 121Xb.

>(b) Let (X, Σ, μ) be a measure space and f a real-valued function which is integrable over X. Show that for every $\epsilon > 0$ there is a simple function $g : X \to \mathbb{R}$ such that $\int |f - g| \leq \epsilon$. (*Hint*: consider non-negative f first.)

(c) Let (X, Σ, μ) be a measure space, and write $\mathcal{L}^1$ for the set of all real-valued functions which are integrable over X. Let $\Phi \subseteq \mathcal{L}^1$ be such that

(i) $\chi E \in \Phi$ whenever $E \in \Sigma$ and $\mu E < \infty$;
(ii) $f + g \in \Phi$ for all f, $g \in \Phi$;
(iii) $cf \in \Phi$ whenever $c \in \mathbb{R}$, $f \in \Phi$;
(iv) $f \in \Phi$ whenever $f \in \mathcal{L}^1$ is such that there is a non-decreasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in Φ for which $\lim_{n\to\infty} f_n = f$ almost everywhere.

Show that $\Phi = \mathcal{L}^1$.

>(d) Let μ be counting measure on $\mathbb{N}$ (112Bd). Show that a function $f : \mathbb{N} \to \mathbb{R}$ (that is, a sequence $\langle f(n)\rangle_{n\in\mathbb{N}}$) is μ-integrable iff it is absolutely summable, and in this case

$$\int f d\mu = \int_{\mathbb{N}} f(n)\mu(dn) = \sum_{n=0}^{\infty} f(n).$$

>(e) Let (X, Σ, μ) be a measure space and f, g two virtually measurable real-valued functions defined on subsets of X. (i) Show that $f + g$, $f \times g$ and f/g, defined as in 121E, are all virtually measurable. (ii) Show that if h is a Borel measurable real-valued function defined on any subset of $\mathbb{R}$, then the composition hf is virtually measurable.

>(f) Let (X, Σ, μ) be a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of virtually measurable real-valued functions defined on subsets of X. Show that $\lim_{n\to\infty} f_n$, $\sup_{n\in\mathbb{N}} f_n$, $\inf_{n\in\mathbb{N}} f_n$, $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$, defined as in 121F, are virtually measurable.

>(g) Let (X, Σ, μ) be a measure space and f, g real-valued functions which are integrable over X. Show that $f \wedge g$ and $f \vee g$, as defined in 121Xb, are integrable.

>(h) Let (X,Σ,μ) be a measure space, f a real-valued function which is integrable over X, and g a bounded real-valued virtually measurable function defined on a conegligible subset of X. Show that $f\times g$, defined as in 121Ed, is integrable.

(i) Let X be a set, Σ a σ-algebra of subsets of X, and μ_1, μ_2 two measures with domain Σ. Set $\mu E=\mu_1E+\mu_2E$ for $E\in\Sigma$. Show that for any real-valued function f defined on a subset of X, $\int f d\mu=\int f d\mu_1+\int f d\mu_2$ in the sense that if one side is defined as a real number so is the other, and they are then equal. (*Hint*: (α) Check that a subset of X is μ-conegligible iff it is μ_i-conegligible for both i. (β) Check the result for simple functions f. (γ) Now consider general non-negative f.)

122Y Further exercises (a) Let (X,Σ,μ) be a 'complete' measure space, that is, one in which all negligible sets are measurable (see, for instance, 113Xa). Show that if f is a virtually measurable real-valued function defined on a subset of X, then f is measurable. Use this fact to find appropriate simplifications of 122J and 122P for such spaces (X,Σ,μ).

(b) Write $\mathcal{L}^1$ for the set of all Lebesgue integrable real-valued functions on $\mathbb{R}$. Let $\Phi\subseteq\mathcal{L}^1$ be such that

(i) $\chi I\in\Phi$ whenever I is a bounded half-open interval in $\mathbb{R}$;

(ii) $f+g\in\Phi$ for all $f,\,g\in\Phi$;

(iii) $cf\in\Phi$ whenever $c\in\mathbb{R}$, $f\in\Phi$;

(iv) $f\in\Phi$ whenever $f\in\mathcal{L}^1$ is such that there is a non-decreasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in Φ for which $\lim_{n\to\infty}f_n=f$ almost everywhere.

Show that $\Phi=\mathcal{L}^1$. (*Hint*: show that (α) $\chi E\in\Phi$ whenever E is expressible as the union of finitely many half-open intervals (β) $\chi E\in\Phi$ whenever E has finite measure and is expressible as the union of a sequence of half-open intervals (γ) $\chi E\in\Phi$ whenever E is measurable and has finite measure.)

(c) Let X be any set, and let μ be counting measure on X. Let $f:X\to\mathbb{R}$ be a function; set $f^+(x)=\max(0,f(x))$, $f^-(x)=\max(0,-f(x))$ for $x\in X$. Show that the following are equiveridical: (i) $\int f d\mu$ exists in $\mathbb{R}$, and is equal to s; (ii) for every $\epsilon>0$ there is a finite $K\subseteq X$ such that $|s-\sum_{i\in I}f(i)|\le\epsilon$ whenever $I\subseteq X$ is a finite set including K (iii) $\sum_{x\in X}f^+(x)$ and $\sum_{x\in X}f^-(x)$, defined as in 112Bd, are finite, and $s=\sum_{x\in X}f^+(x)-\sum_{x\in X}f^-(x)$.

(d) Let (X,Σ,μ) be a measure space. Let us say that a function $g:X\to\mathbb{R}$ is **quasi-simple** if it is expressible as $\sum_{i=0}^\infty a_i\chi G_i$, where $\langle G_i\rangle_{i\in\mathbb{N}}$ is a partition of X into measurable sets, $\langle a_i\rangle_{i\in\mathbb{N}}$ is a sequence in $\mathbb{R}$, and $\sum_{i=0}^\infty|a_i|\mu G_i<\infty$, counting $0\cdot\infty$ as 0, so that there can be G_i of infinite measure provided that the corresponding a_i are zero.

(i) Show that if g and h are quasi-simple functions so are $g+h$, $|g|$ and cg, for any $c\in\mathbb{R}$. (*Hint*: for $g+h$ you will need 111F(b-ii) or its equivalent.)

(ii) Show from first principles (I mean, without using anything later than 122F in this chapter) that if $g=\sum_{i=0}^\infty a_i\chi G_i$ and $h=\sum_{i=0}^\infty b_i\chi H_i$ are quasi-simple functions, and $g\le_{\text{a.e.}}h$, then $\sum_{i=0}^\infty a_i\mu G_i\le\sum_{i=0}^\infty b_i\mu H_i$.

(iii) Hence show that we have a functional I defined by saying that $I(g)=\sum_{i=0}^\infty a_i\mu G_i$ whenever g is a quasi-simple function represented as $\sum_{i=0}^\infty a_i\chi G_i$.

(iv) Show that if g and h are quasi-simple functions and $c\in\mathbb{R}$, then $I(g+h)=I(g)+I(h)$ and $I(cg)=cI(g)$, and that $I(g)\le I(h)$ if $g\le_{\text{a.e.}}h$.

(v) Show that if g is a quasi-simple function then g is integrable and $\int g=I(g)$. (I do now allow you to use 122G-122R.)

(vi) Show that a real-valued function f, defined on a conegligible subset of X, is integrable iff for every $\epsilon>0$ there are quasi-simple functions g, h such that $g\le_{\text{a.e.}}f\le_{\text{a.e.}}h$ and $I(h)-I(g)\le\epsilon$.

(e) Let μ be Lebesgue measure on $\mathbb{R}$. Let us say (for this exercise only) that a real-valued function g with $\operatorname{dom}g\subseteq\mathbb{R}$ is 'pseudo-simple' if it is expressible as $\sum_{i=0}^\infty a_i\chi J_i$, where $\langle J_i\rangle_{i\in\mathbb{N}}$ is a sequence of bounded half-open intervals (*not* necessarily disjoint) and $\sum_{i=0}^\infty|a_i|\mu J_i<\infty$. (Interpret the infinite sum $\sum_{i=0}^\infty a_i\chi J_i$ as in 121F, so that

$$\operatorname{dom}(\textstyle\sum_{i=0}^\infty a_i\chi J_i)=\{x:\lim_{n\to\infty}\sum_{i=0}^n a_i(\chi J_i)(x)\text{ exists in }\mathbb{R}\}.)$$

(i) Show that if g, h are pseudo-simple functions so are $g + h$ and cg, for any $c \in \mathbb{R}$.

(ii) Show that if g is a pseudo-simple function then g is integrable.

(iii) Show that a real-valued function f, defined on a conegligible subset of $\mathbb{R}$, is integrable iff for every $\epsilon > 0$ there are pseudo-simple functions g, h such that $g \leq_{\text{a.e.}} f \leq_{\text{a.e.}} h$ and $\int h - g\,d\mu \leq \epsilon$. (*Hint*: Take Φ to be the set of integrable functions with this property, and show that it satisfies the conditions of 122Yb.)

(f) Repeat 122Yb and 122Ye for Lebesgue measure on $\mathbb{R}^r$, where $r > 1$.

(g) Let (X, Σ, μ) be a measure space, and assume that there is at least one partition of X into infinitely many non-empty measurable sets. Let $f : X \to \mathbb{R}$ be a function, and $a \in \mathbb{R}$. Show that the following are equiveridical:

(i) f is integrable, with $\int f = a$;

(ii) for every $\epsilon > 0$ there is a partition $\langle E_n \rangle_{n\in\mathbb{N}}$ of X into non-empty measurable sets such that

$$\sum_{n=0}^{\infty} |f(t_n)|\mu E_n < \infty, \quad |a - \sum_{n=0}^{\infty} f(t_n)\mu E_n| \leq \epsilon$$

whenever $\langle t_n \rangle_{n\in\mathbb{N}}$ is a sequence such that $t_n \in E_n \cap \operatorname{dom} f$ for every n. (As usual, take $0 \cdot \infty = 0$ in these formulae.) (*Hint*: use 122Yd.)

(h) Find a re-formulation of (g) which covers the case of measure spaces which can *not* be partitioned into sequences of non-empty measurable sets.

(i) Let X be a set, Σ a σ-algebra of subsets of X, and $\langle \mu_n \rangle_{n\in\mathbb{N}}$ a sequence of measures with domain Σ. Set $\mu E = \sum_{n=0}^{\infty} \mu_n E$ for $E \in \Sigma$. (i) Show that μ is a measure. (ii) Show that for any real-valued function f defined on a subset of X, f is μ-integrable iff it is μ_n-integrable for every n and $\sum_{n=0}^{\infty} \int |f| d\mu_n$ is finite, and that then $\int f d\mu = \sum_{n=0}^{\infty} \int f d\mu_n$.

(j) Let X be a set, Σ a σ-algebra of subsets of X, and $\langle \mu_i \rangle_{i\in I}$ a family of measures with domain Σ. Set $\mu E = \sum_{i\in I} \mu_i E$ for $E \in \Sigma$. (i) Show that μ is a measure. (ii) Show that for any Σ-measurable function $f : X \to \mathbb{R}$, f is μ-integrable iff it is μ_i-integrable for every i and $\sum_{i\in I} \int |f| d\mu_i$ is finite.

122 Notes and comments Just as in §121, some extra technical problems are caused by my insistence on trying to integrate (i) functions which are not defined on the whole of the measure space under consideration (ii) functions which are not, strictly speaking, measurable, but are only measurable on some conegligible set. There is nothing in the present section to justify either of these elaborations. In the next section, however, we shall be looking at the limits of sequences of functions, and these limits need not be defined at every point; and the examples in which the limits are not everywhere defined are not in any sense pathological, but are central to the most important applications of the theory.

The question of integrating not-quite-measurable functions is more disputable. I will discuss this point further after formally introducing 'complete' measure spaces in Chapter 21. For the moment, I will say only that I think it is worth taking the trouble to have a formalisation which integrates as many functions as is reasonably possible; the original point of the Lebesgue integral being, in part, that it enables us to integrate more functions than its predecessors.

The definition of 'integration' here proceeds in three distinguishable stages: (i) integration of simple functions (122A-122G); (ii) integration of non-negative functions (122H-122L); (iii) integration of general real-valued functions (122M-122R). I have taken each stage slowly, passing to non-negative integrable functions only when I have a full set of the requisite lemmas on simple functions, for instance. There are, of course, innumerable alternative routes; see, for instance, 122Yd, which offers a definition using two steps rather than three. I prefer the longer, gentler climb partly because (to my eye) it gives a clearer view of the ideas and partly because it corresponds to an almost canonical method of proving properties of integrable functions: we prove them first for simple functions, then for non-negative integrable functions, then for general integrable functions. (The hint I give for 122Yb conforms to this philosophy. See also 122Xc; but I do not give this as a formally expressed theorem, because the exact order of proof varies from case to case, and I think it is best remembered as a method of attack rather than as a specific result to apply.)

You have a right to feel that this section has been singularly abstract, and gives very little idea of which of your favourite functions are likely to be integrable, let alone what the integrals are. I hope that Chapter

13 will provide some help in this direction, though I have to say that for really useful methods for calculating integrals we must wait for Chapters 22, 25 and 26 in the next volume. If you want to know the true centre of the arguments of this section, I would myself locate it in 122G, 122H and 122K. The point is that the ideas of 122A-122F apply to a much wider class of structures (X, Σ, μ), because they involve only operations on finitely many members of Σ at a time; there is no mention of sequences of sets. The key that makes all the rest possible is 122G, which is founded on 112Cf. And after 122G-122K, the rest of the section, although by no means elementary, really is no more than a careful series of checks to ensure that the functional defined in 122K behaves as we expect it to.

Many of the results of this section (including the key one, 122G) will be superseded by stronger results in the following section. But I should remark on Lemma 122Ja, which will periodically recur as a most useful criterion for integrability of non-negative functions (see 122Ra).

There is another point about the standard integral as defined here. It is an 'absolute' integral, meaning that if f is integrable so is $|f|$ (122P). This means that although the Lebesgue integral extends the 'proper' Riemann integral (see 134K below), there are functions with finite 'improper' Riemann integrals which are not Lebesgue integrable; a typical example is $f(x) = \frac{\sin x}{x}$, where $\lim_{a\to\infty} \int_0^a f$ exists in $\mathbb{R}$, while $\lim_{a\to\infty} \int_0^a |f| = \infty$, so that f is not integrable, in the sense defined here, over the whole interval $]0, \infty[$. (For full proofs of these assertions, see 283D and 282Xm in Volume 2.) If you have encountered the theory of 'absolutely' and 'conditionally' summable series, you will be aware that the latter can exhibit very confusing behaviour, and will appreciate that restricting the notion of 'integrable' to mean 'absolutely integrable' is a great convenience.

Indeed, it is more than just a convenience; it is necessary to make the definition work at the level of abstraction used in this chapter. Consider the example of counting measure μ on $\mathbb{N}$ (112Bd, 122Xd). The structure $(\mathbb{N}, \mathcal{P}\mathbb{N}, \mu)$ is invariant under permutations; that is, $\mu(\pi[A]) = \mu A$ for every $A \subseteq \mathbb{N}$ and every permutation $\pi : \mathbb{N} \to \mathbb{N}$. Consequently, *any* definition of integration which depends only on the structure $(\mathbb{N}, \mathcal{P}\mathbb{N}, \mu)$ must also be invariant under permutations, that is,

$$\int f(\pi(n))\mu(dn) = \int f(n)\mu(dn)$$

for any integrable function f and any permutation π. But of course (as I hope you have been told) a series $\langle f(n)\rangle_{n\in\mathbb{N}}$ such that $\sum_{n=0}^{\infty} f(\pi(n)) = \sum_{n=0}^{\infty} f(n) \in \mathbb{R}$ for any permutation π must be absolutely summable. Thus if we are to define an integral on an abstract measure space (X, Σ, μ) in terms depending only on Σ and μ, we are nearly inevitably forced to define an absolute integral.

Naturally there are important contexts in which this restriction is an embarrassment, and in which some kind of 'improper' integral seems appropriate. A typical one is the theory of Fourier transforms, in which we find ourselves looking at $\lim_{a\to\infty} \int_{-a}^{a} f$ in place of $\int_{-\infty}^{\infty} f$ (see §283). A vast number of more or less abstract forms of improper integral have been proposed; many are interesting and some are important; but none rivals the 'standard' integral as described in this chapter. (For an attempt at a systematic examination of a particular class of such improper integrals, see Chapter 48 in Volume 4.)

Much less work has been done on the integration of non-measurable functions – to speak more exactly, of functions which are not equal almost everywhere to a measurable integrable function. I am sure that this is simply because there are too few important problems to show us which way to turn. In 134C below I mention the question of whether there is *any* non-measurable real-valued function on $\mathbb{R}$. The standard answer is 'yes', but no such function can possibly arise as a result of any ordinary construction. Consequently the majority of questions concerning non-measurable functions appear in very special contexts, and so far I have seen none which gives any useful hint of what a generally appropriate extension of the notion of 'integrability' might be.

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123 The convergence theorems

The great labour we have gone through so far has not yet been justified by any theorems powerful enough to make it worth while. We come now to the heart of the modern theory of integration, the 'convergence theorems', describing conditions under which we can integrate the limit of a sequence of integrable functions.

123A B.Levi's theorem Let (X, Σ, μ) be a measure space and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions, all integrable over X, such that (i) $f_n \le_{\text{a.e.}} f_{n+1}$ for every $n \in \mathbb{N}$ (ii) $\sup_{n\in\mathbb{N}} \int f_n < \infty$. Then $f = \lim_{n\to\infty} f_n$ is integrable, and $\int f = \lim_{n\to\infty} \int f_n$.

Remarks I ought to repeat at once the conventions I am following here. Each of the functions f_n is taken to be defined on a conegligible set $\operatorname{dom} f_n \subseteq X$, as in 122Nc, and the limit function f is taken to have domain

$$\{x : x \in \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} \operatorname{dom} f_m,\ \lim_{n\to\infty} f_n(x) \text{ is defined in } \mathbb{R}\},$$

as in 121Fa. You would miss no important idea if you supposed that every f_n was defined everywhere on X; but the statement 'f is integrable' includes the assertion 'f is defined, as a real number, almost everywhere', and this is an essential part of the theorem.

proof (a) Let us first deal with the case in which $f_0 = 0$ a.e. Write $c = \sup_{n\in\mathbb{N}} \int f_n = \lim_{n\to\infty} \int f_n$ (noting that, by 122Od, $\langle \int f_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence).

(i) All the sets $\operatorname{dom} f_n$, $\{x : f_0(x) = 0\}$, $\{x : f_n(x) \le f_{n+1}(x)\}$ are conegligible, so their intersection F also is. For each $n \in \mathbb{N}$ there is a conegligible set E_n such that $f_n \restriction E_n$ is measurable (122P); let E^* be a measurable conegligible set included in the conegligible set $F \cap \bigcap_{n\in\mathbb{N}} E_n$.

(ii) For $a > 0$ and $n \in \mathbb{N}$ set $H_n(a) = \{x : x \in E^*,\ f_n(x) \ge a\}$; then $H_n(a)$ is measurable because $f_n \restriction E_n$ is measurable and E^* is a measurable subset of E_n. Also $a\chi H_n(a) \le f_n$ everywhere on E^*, so

$$a\mu H_n(a) = \int a\chi H_n(a) \le \int f_n \le c.$$

Because $f_n(x) \le f_{n+1}(x)$ for every $x \in E^*$, $H_n(a) \subseteq H_{n+1}(a)$ for every $n \in \mathbb{N}$, and writing $H(a) = \bigcup_{n\in\mathbb{N}} H_n(a)$, we have

$$\mu H(a) = \lim_{n\to\infty} \mu H_n(a) \le \frac{c}{a}$$

(112Ce). In particular, $\mu H(a) < \infty$ for every a. Furthermore,

$$\mu(\bigcap_{k\ge 1} H(k)) \le \inf_{k\ge 1} \mu H(k) \le \inf_{k\ge 1} \frac{c}{k} = 0.$$

Set $E = E^* \setminus \bigcap_{k\ge 1} H(k)$; then E is conegligible.

(iii) If $x \in E$, there is some k such that $x \notin H(k)$, that is, $x \notin \bigcup_{n\in\mathbb{N}} H_n(k)$, that is, $f_n(x) < k$ for every n; moreover, $\langle f_n(x) \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence, so $f(x) = \lim_{n\to\infty} f_n(x) = \sup_{n\in\mathbb{N}} f_n(x)$ is defined in $\mathbb{R}$. Thus the limit function f is defined almost everywhere. Because every $f_n \restriction E$ is measurable (121Eh), so is $f \restriction E = \lim_{n\to\infty} f_n \restriction E$ (121Fa). If $\epsilon > 0$ then $\{x : x \in E,\ f(x) \ge \epsilon\}$ is included in $H(\frac{1}{2}\epsilon)$, so has finite measure.

(iv) Now suppose that g is a simple function and that $g \le_{\text{a.e.}} f$. As in the proof of 122G, $H = \{x : g(x) \ne 0\}$ has finite measure, and g is bounded above by M say.

Let $\epsilon > 0$. For each $n \in \mathbb{N}$ set $G_n = \{x : x \in E,\ (g - f_n)(x) \ge \epsilon\}$. Then each G_n is measurable, and $\langle G_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with intersection

$$\{x : x \in E,\ g(x) \ge \epsilon + \sup_{n\in\mathbb{N}} f_n(x)\} \subseteq \{x : g(x) > f(x)\},$$

which is negligible. Also $\mu G_0 < \infty$ because $G_0 \subseteq H$. Consequently $\lim_{n\to\infty} \mu G_n = 0$ (112Cf). Let n be such that $\mu G_n \le \epsilon$. Then, for any $x \in E$,

$$g(x) \le f_n(x) + \epsilon\chi H(x) + M\chi G_n(x),$$

so

$$g \le_{\text{a.e.}} f_n + M\chi G_n + \epsilon\chi H$$

and

$$\int g \le \int f_n + M\mu G_n + \epsilon\mu H \le c + \epsilon(M + \mu H).^2$$

As ϵ is arbitrary, $\int g \le c$.

[2] I am grateful to P.Wallace Thompson for noticing a fault at this stage in previous editions.

(v) Accordingly, $f\restriction E$ (which is non-negative) satisfies the conditions of Lemma 122Ja, and is integrable. Moreover, its integral is at most c, by Definition 122K. Because $f =_{\text{a.e.}} f\restriction E$, f also is integrable, with the same integral (122Rb). On the other hand, $f \ge_{\text{a.e.}} f_n$ for each n, so $\int f \ge \sup_{n\in\mathbb{N}} \int f_n = c$, by 122Od.

This completes the proof when $f_0 = 0$ a.e.

(b) For the general case, consider the sequence $\langle f'_n\rangle_{n\in\mathbb{N}} = \langle f_n - f_0\rangle_{n\in\mathbb{N}}$. By (a), $f' = \lim_{n\to\infty} f'_n$ is integrable, and $\int f' = \lim_{n\to\infty} \int f'_n$; now $\lim_{n\to\infty} f_n =_{\text{a.e.}} f' + f_0$, so is integrable, with integral $\int f' + \int f_0 = \lim_{n\to\infty} \int f_n$.

Remark You may have observed, without surprise, that the argument of (a-iv) in the proof here repeats that used for the special case 122G.

123B Fatou's Lemma Let (X, Σ, μ) be a measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions, all integrable over X. Suppose that every f_n is non-negative a.e., and that $\liminf_{n\to\infty} \int f_n < \infty$. Then $\liminf_{n\to\infty} f_n$ is integrable, and $\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n$.

Remark Once again, this theorem includes the assertion that $\liminf_{n\to\infty} f_n(x)$ is defined in $\mathbb{R}$ for almost every $x \in X$.

proof Set $c = \liminf_{n\to\infty} \int f_n$ and $f = \liminf_{n\to\infty} f_n$. For each $n \in \mathbb{N}$, let E_n be a conegligible set such that $f'_n = f_n\restriction E_n$ is measurable and non-negative. Set $g_n = \inf_{m\ge n} f'_m$; then each g_n is measurable (121Fc), non-negative and defined on the conegligible set $\bigcap_{m\ge n} E_m$, and $g_n \le_{\text{a.e.}} f_n$; by 122Re and 122Ra, $|f_n|$ belongs to U, as defined in 122H, while $|g_n| \le_{\text{a.e.}} |f_n|$, so g_n is integrable (122P) with $\int g_n \le \inf_{m\ge n} \int f_m \le c$. Next, $g_n(x) \le g_{n+1}(x)$ for every $x \in \operatorname{dom} g_n$, so $\langle g_n\rangle_{n\in\mathbb{N}}$ satisfies the conditions of B.Levi's theorem (123A), and $g = \lim_{n\to\infty} g_n$ is integrable, with $\int g = \lim_{n\to\infty} \int g_n \le c$. Finally, because every f'_n is equal to f_n almost everywhere, $g = \liminf_{n\to\infty} f'_n =_{\text{a.e.}} f$, and $\int f$ exists, equal to $\int g \le c$.

123C Lebesgue's Dominated Convergence Theorem Let (X, Σ, μ) be a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions, all integrable over X, such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}$ for almost every $x \in X$. Suppose moreover that there is an integrable function g such that $|f_n| \le_{\text{a.e.}} g$ for every n. Then f is integrable, and $\lim_{n\to\infty} \int f_n$ exists and is equal to $\int f$.

proof Consider $\tilde f_n = f_n + g$ for each $n \in \mathbb{N}$. Then $0 \le \tilde f_n \le 2g$ a.e. for each n, so $\tilde c = \liminf_{n\to\infty} \int \tilde f_n$ exists in $\mathbb{R}$, and $\tilde f = \liminf_{n\to\infty} \tilde f_n$ is integrable, with $\int \tilde f \le \tilde c$, by Fatou's Lemma (123B). But observe that $f =_{\text{a.e.}} \tilde f - g$, since $f(x) = \tilde f(x) - g(x)$ at least whenever $f(x)$ and $g(x)$ are both defined, so f is integrable, with

$$\int f = \int \tilde f - \int g \le \liminf_{n\to\infty} \int \tilde f_n - \int g = \liminf_{n\to\infty} \int f_n.$$

Similarly, considering $\langle -f_n\rangle_{n\in\mathbb{N}}$,

$$\int(-f) \le \liminf_{n\to\infty} \int(-f_n),$$

that is,

$$\int f \ge \limsup_{n\to\infty} \int f_n.$$

So $\lim_{n\to\infty} \int f_n$ exists and is equal to $\int f$.

Remark We have at last reached the point where the technical problems associated with partially-defined functions are reducing, or rather, are being covered efficiently by the conventions I am using concerning the interpretation of such formulae as '$\limsup$'.

123D To try to show the power of these theorems, I give a result here which is one of the standard applications of the theory.

Corollary Let (X, Σ, μ) be a measure space and $]a, b[$ a non-empty open interval in $\mathbb{R}$. Let $f : X \times]a, b[\to \mathbb{R}$ be a function such that

(i) the integral $F(t) = \int f(x, t)dx$ is defined for every $t \in]a, b[$;

(ii) the partial derivative $\frac{\partial f}{\partial t}$ of f with respect to the second variable is defined everywhere in $X \times]a,b[$;

(iii) there is an integrable function $g : X \to [0,\infty[$ such that $|\frac{\partial f}{\partial t}(x,t)| \le g(x)$ for every $x \in X$ and $t \in]a,b[$.

Then the derivative $F'(t)$ and the integral $\int \frac{\partial f}{\partial t}(x,t)dx$ exist for every $t \in]a,b[$, and are equal.

proof (a) Let t be any point of $]a,b[$, and $\langle t_n\rangle_{n\in\mathbb{N}}$ any sequence in $]a,b[\setminus \{t\}$ converging to t. Consider

$$\frac{F(t_n)-F(t)}{t_n-t} = \int \frac{f(x,t_n)-f(x,t)}{t_n-t}\mu(dx)$$

for each n. (This step uses 122O.) If we set

$$f_n(x) = \frac{f(x,t_n)-f(x,t)}{t_n-t},$$

for $x \in X$, then we see from the Mean Value Theorem that there is a τ (depending, of course, on both n and x), lying between t_n and t, such that $f_n(x) = \frac{\partial f}{\partial t}(x,\tau)$, so that $|f_n(x)| \le g(x)$. At the same time, $\lim_{n\to\infty} f_n(x) = \frac{\partial f}{\partial t}(x,t)$ for every x. So Lebesgue's Dominated Convergence Theorem tells us that $\int \frac{\partial f}{\partial t}(x,t)dx$ exists and is equal to

$$\lim_{n\to\infty} \int f_n(x)dx = \lim_{n\to\infty} \frac{F(t_n)-F(t)}{t_n-t}.$$

(b) Because $\langle t_n\rangle_{n\in\mathbb{N}}$ is arbitrary,

$$\lim_{s\to t} \frac{F(s)-F(t)}{s-t} = \int \frac{\partial f}{\partial t}(x,t)dx,$$

as claimed.

Remark In the next volume I offer a variation on this theorem, with both hypotheses and conclusion weakened (252Ye).

123X Basic exercises >(a) Let (X,Σ,μ) be a measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions, all integrable over X, such that $\sum_{n=0}^\infty \int |f_n|$ is finite. Show that $f(x) = \sum_{n=0}^\infty f_n(x)$ is defined in $\mathbb{R}$ for almost every $x \in X$, and that $\int f = \sum_{n=0}^\infty \int f_n$. (*Hint*: consider first the case in which every f_n is non-negative.)

(b) Let (X,Σ,μ) be a measure space. Suppose that T is any subset of $\mathbb{R}$, and $\langle f_t\rangle_{t\in T}$ a family of functions, all integrable over X, such that, for any $t \in T$,

$$f_t(x) = \lim_{s\in T, s\to t} f_s(x)$$

for almost every $x \in X$. Suppose moreover that there is an integrable function g such that $|f_t| \le_{\text{a.e.}} g$ for every $t \in T$. Show that $t \mapsto \int f_t : T \to \mathbb{R}$ is continuous.

>(c) Let f be a real-valued function defined everywhere on $[0,\infty[$, endowed with Lebesgue measure. Its (real) **Laplace transform** is the function F defined by

$$F(s) = \int_0^\infty e^{-sx} f(x)dx$$

for all those real numbers s for which the integral is defined.

(i) Show that if $s \in \operatorname{dom} F$ and $s' \ge s$ then $s' \in \operatorname{dom} F$ (because $e^{-s'x}e^{sx} \le 1$ for all x). (How do you know that $x \mapsto e^{-s'x}e^{sx}$ is measurable?)

(ii) Show that F is differentiable on the interior of its domain. (*Hint*: note that if $a_0 \in \operatorname{dom} F$ and $a_0 < a < b$ then there is some M such that $xe^{-sx}|f(x)| \le Me^{-a_0x}|f(x)|$ whenever $x \in [0,\infty[$, $s \in [a,b]$.)

(iii) Show that if F is defined anywhere then $\lim_{s\to\infty} F(s) = 0$. (*Hint*: use Lebesgue's Dominated Convergence Theorem to show that $\lim_{n\to\infty} F(s_n) = 0$ whenever $\lim_{n\to\infty} s_n = \infty$.)

(iv) Show that if f, g have Laplace transforms F, G then the Laplace transform of $f+g$ is $F+G$, at least on $\operatorname{dom} F \cap \operatorname{dom} G$.

(d) Let (X,Σ,μ) be a measure space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of real-valued functions, all integrable over X, such that there is an integrable function g such that $|f_n|\le_{\text{a.e.}} g$ for every n. Show that $\limsup_{n\to\infty} f_n$ is integrable and that $\int \limsup_{n\to\infty} f_n \ge \limsup_{n\to\infty}\int f_n$.

123Y Further exercises (a) Let (X,Σ,μ) be a measure space, Y any set and $\phi: X\to Y$ any function; let $\mu\phi^{-1}$ be the image measure on Y (112Xf). Show that if $h: Y\to\mathbb{R}$ is $\mu\phi^{-1}$-integrable then $h\phi$ is μ-integrable, and the integrals are then equal.

(b) Explain how to adapt 123Xc to the case in which f is undefined on a negligible subset of $\mathbb{R}$.

(c) Let (X,Σ,μ) be a measure space and $a<b$ in $\mathbb{R}$. Let $f: X\times\,]a,b[\,\to[0,\infty[$ be a function such that $\int f(x,t)dx$ is defined for every $t\in\,]a,b[$ and $t\mapsto f(x,t)$ is continuous for every $x\in X$. Suppose that $c\in\,]a,b[$ is such that $\liminf_{t\to c}\int f(x,t)dx<\infty$. Show that $\int\liminf_{t\to c} f(x,t)dx$ is defined and less than or equal to $\liminf_{t\to c}\int f(x,t)dx$.

(d) Show that there is a function $f:\mathbb{R}^2\to\{0,1\}$ such that (i) the Lebesgue integral $\int f(x,t)dx$ is defined and equal to 1 for every $t\ne 0$ (ii) the function $x\mapsto\liminf_{t\to 0} f(x,t)$ is not Lebesgue measurable. (*Remark*: you will of course have to start your construction from a non-measurable subset of $\mathbb{R}$; see 134B for such a set.)

(e) Let (Y,T,ν) be a measure space. Let X be a set, Σ a σ-algebra of subsets of X, and $\langle\mu_y\rangle_{y\in Y}$ a family of measures on X such that $\mu_y X$ is finite for every y and $\mu E=\int\mu_y E\,\nu(dy)$ is defined for every $E\in\Sigma$. (i) Show that $\mu:\Sigma\to[0,\infty[$ is a measure. (ii) Show that if $f: X\to[0,\infty[$ is a Σ-measurable function, then f is μ-integrable iff it is μ_y-integrable for almost every $y\in Y$ and $\int\bigl(\int f d\mu_y\bigr)\nu(dy)$ is defined, and that this is then $\int f d\mu$.

(f) Let (X,Σ,μ) be a measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of virtually measurable real-valued functions all defined almost everywhere in X. Suppose that $\sum_{n=0}^\infty\int|f_n(x)-1|\mu(dx)<\infty$. Show that $\prod_{n=0}^\infty f_n(x)$ is defined in $\mathbb{R}$ for almost every $x\in X$.

123 Notes and comments I hope that 123D and its special case 123Xc will help you to believe that the theory here has useful applications.

All the theorems of this section can be thought of as 'exchange of limit' theorems, setting out conditions under which

$$\lim_{n\to\infty}\int f_n=\int\lim_{n\to\infty} f_n,$$

or

$$\frac{\partial}{\partial t}\int f\,dx=\int\frac{\partial f}{\partial t}dx.$$

Even for functions which are accessible to much more primitive methods of integration (e.g., the Riemann integral), theorems of this type can involve laborious validation of inequalities. The power of Lebesgue's integral is that it gives general theorems which cover a reasonable proportion of the important cases which arise in practice. (I have to admit, however, that nothing is more typical of applied analysis than its need for special results which are related to, but not derivable from, the standard general theorems.) For instance, in 123Xc, the fact that the range of integration is the unbounded interval $[0,\infty[$ adds no difficulty. Of course this is connected with the fact that we consider only integrals of functions with integrable absolute values.

The limits used in 123A-123C are all limits of sequences; it is of course part of the essence of measure theory that we expect to be able to handle countable families of sets or functions, but that anything larger is alarming. Nevertheless, there are many contexts in which we can take other types of limit. I describe some in 123D, 123Xb and 123Xc(iii). The point is that in such limits as $\lim_{t\to u}\phi(t)$, where $u\in[-\infty,\infty]$, we shall have $\lim_{t\to u}\phi(t)=a$ iff $\lim_{n\to\infty}\phi(t_n)=a$ whenever $\langle t_n\rangle_{n\in\mathbb{N}}$ converges to u; so that when seeking a limit $\lim_{t\to u}\int f_t$, for some family $\langle f_t\rangle_{t\in T}$ of functions, it will be sufficient if we can find $\lim_{n\to\infty}\int f_{t_n}$ for enough sequences $\langle t_n\rangle_{n\in\mathbb{N}}$. This type of argument will be effective for any of the standard limits $\lim_{t\uparrow a}$, $\lim_{t\downarrow a}$, $\lim_{t\to a}$, $\lim_{t\to\infty}$, $\lim_{t\to-\infty}$ of basic calculus, and can be used in conjunction either with B.Levi's theorem or with Lebesgue's theorem. I should perhaps remark that a difficulty arises with a similar extension of Fatou's lemma (123Yc-123Yd).

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Concordance for Chapter 12

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this chapter, and which have since been changed.

121Yb (Σ, T)**-measurable functions** Exercise 121Yb in the 2000 and 2001 editions, referred to in the 2001 and 2003 editions of Volume 2, has been moved to 121Yc.

Version of 31.5.03

References for Volume 1

In addition to those (very few) works which I have mentioned in the course of this volume, I list some of the books from which I myself learnt measure theory, as a mark of grateful respect, and to give you an opportunity to sample alternative approaches.

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