

MEASURE THEORY

D.H.Fremlin

University of Essex, Colchester, England

Introduction In this treatise I aim to give a comprehensive description of modern abstract measure theory, with some indication of its principal applications. The first two volumes are set at an introductory level; they are intended for students with a solid grounding in the concepts of real analysis, but possibly with rather limited detailed knowledge. As the book proceeds, the level of sophistication and expertise demanded will increase; thus for the volume on topological measure spaces, familiarity with general topology will be assumed. The emphasis throughout is on the mathematical ideas involved, which in this subject are mostly to be found in the details of the proofs.

My intention is that the book should be usable both as a first introduction to the subject and as a reference work. For the sake of the first aim, I try to limit the ideas of the early volumes to those which are really essential to the development of the basic theorems. For the sake of the second aim, I try to express these ideas in their full natural generality, and in particular I take care to avoid suggesting any unnecessary restrictions in their applicability. Of course these principles are to some extent contradictory. Nevertheless, I find that most of the time they are very nearly reconcilable, *provided* that I indulge in a certain degree of repetition. For instance, right at the beginning, the puzzle arises: should one develop Lebesgue measure first on the real line, and then in spaces of higher dimension, or should one go straight to the multidimensional case? I believe that there is no single correct answer to this question. Most students will find the one-dimensional case easier, and it therefore seems more appropriate for a first introduction, since even in that case the technical problems can be daunting. But certainly every student of measure theory must at a fairly early stage come to terms with Lebesgue area and volume as well as length; and with the correct formulations, the multidimensional case differs from the one-dimensional case only in a definition and a (substantial) lemma. So what I have done is to write them both out (in §§114-115), so that you can pass over the higher dimensions at first reading (by omitting §115) and at the same time have a complete and uncluttered argument for them (if you omit §114). In the same spirit, I have been uninhibited, when setting out exercises, by the fact that many of the results I invite students to look for will appear in later chapters; I believe that throughout mathematics one has a better chance of understanding a theorem if one has previously attempted something similar alone.

The original plan of the work was as follows:

Volume 1: The Irreducible Minimum

(first edition May 2000, reprinted September 2001 and February 2004, hardback
(‘Lulu’) edition January 2011)

Volume 2: Broad Foundations

(first edition May 2001, reprinted April 2003, hardback edition January 2010,
reprinted April 2016)

Volume 3: Measure Algebras

(first edition May 2002, reprinted May 2004, hardback edition December 2012)

Volume 4: Topological Measure Spaces

(first edition November 2003, reprinted February 2006, hardback edition October 2013)

Volume 5: Set-theoretic Measure Theory

Extract from MEASURE THEORY, results-only version, by D.H.FREMLIN, University of Essex, Colchester. This material is copyright. It is issued under the terms of the Design Science License as published in <http://dsl.org/copyleft/dsl.txt>. This is a development version and the source files are not permanently archived, but current versions are normally accessible through <https://www1.essex.ac.uk/math/people/fremlin/mt.htm>. For further information contact david@fremlin.org.

© 2016 D. H. Fremlin

(first edition December 2008, reprinted January 2015).

Volume 1 is intended for those with no prior knowledge of measure theory, but competent in the elementary techniques of real analysis. I hope that it will be found useful by undergraduates meeting Lebesgue measure for the first time. Volume 2 aims to lay out some of the fundamental results of pure measure theory (the Radon-Nikodým theorem, Fubini's theorem), but also gives short introductions to some of the most important applications of measure theory (probability theory, Fourier analysis). While I should like to believe that most of it is written at a level accessible to anyone who has mastered the contents of Volume 1, I should not myself have the courage to try to cover it in an undergraduate course, though I would certainly attempt to include some parts of it. Volumes 3 and 4 are set at a rather higher level, suitable to postgraduate courses; while Volume 5 assumes a wide-ranging competence over large parts of real analysis and set theory.

In 2011 I embarked on a project to develop a version of stochastic calculus on the same principles as the main text sketched above. This has evolved into a supplementary Volume 6, with contents as listed below. It is not in fact ready for publication. You will find a great many faults, some obvious, some not, in the presentation as it stands. But I am now eighty years old and it is unlikely that I shall ever achieve the standards of coherence and accuracy which I set myself for the first five volumes. My intention here is just to make my current formulation accessible to any who have the hardihood to follow me along an idiosyncratic path into some fascinating mathematics.

There is a disclaimer which I ought to make in a place where you might see it in time to avoid paying for this book. I make no real attempt to describe the history of the subject. This is not because I think the history uninteresting or unimportant; rather, it is because I have no confidence of saying anything which would not be seriously misleading. Indeed I have very little confidence in anything I have ever read concerning the history of ideas. So while I am happy to honour the names of Lebesgue and Kolmogorov and Maharam in more or less appropriate places, and I try to include in the bibliographies the works which I have myself consulted, I leave any consideration of the details to those bolder and better qualified than myself.

I do not wish to admit that the length of this treatise is excessive, but it has certainly taken a very long time to write. Moreover, I continue to make regular corrections and additions. I am therefore presenting a version on the Internet; for details see <https://www1.essex.ac.uk/math/people/fremlin/mt.htm>. Each chapter is available separately, and with an elementary knowledge of the \TeX language you will be able to extract individual sections for printing. In addition, I am offering the material in two forms. Apart from the 'full' version, there is a 'results-only' version, omitting proofs, exercises and notes. I hope that this will be found useful for reference and revision, while saving printing costs and easing handling and storage.

For the time being, at least, printing will be in short runs. I hope that readers will be energetic in commenting on errors and omissions, since it should be possible to correct these relatively promptly. An inevitable consequence of this is that paragraph references may go out of date rather quickly. I shall be most flattered if anyone chooses to rely on this book as a source for basic material; and I am willing to attempt to maintain a concordance to such references, indicating where migratory results have come to rest for the moment, if authors will supply me with copies of papers which use them. On the web page given above you will find a link to 'errata'. Under this heading I offer postscript and pdf files listing not only corrections to published volumes, but also changes which I have made from previous printings.

I mention some minor points concerning the layout of the material. Most sections conclude with lists of 'basic exercises' and 'further exercises', which I hope will be generally instructive and occasionally entertaining. How many of these you should attempt must be for you and your teacher, if any, to decide, as no two students will have quite the same needs. I mark with a $>$ those which seem to me to be particularly important. But while you may not need to write out solutions to all the 'basic exercises', if you are in any doubt as to your capacity to do so you should take this as a warning to slow down a bit. The 'further exercises' are unbounded in difficulty, and are unified only by a presumption that each has at least one solution based on ideas already introduced. Occasionally I add a final 'problem', a question to which I do not know the answer and which seems to arise naturally in the course of the work.

The impulse to write this book is in large part a desire to present a unified account of the subject. Cross-references are correspondingly abundant and wide-ranging. (I apologise for the way in which the piecemeal process of writing and revising renders some of them inaccurate.) In order to be able to refer freely across the whole text, I have chosen a reference system which gives the same code name to a paragraph wherever it

is being called from. Thus 244Pc is the third subparagraph of the sixteenth paragraph in the fourth section of the fourth chapter of Volume 2, and is referred to by that name throughout. Let me emphasize that cross-references are supposed to help the reader, not distract her. Do not take the interpolation '(324D)' as an instruction, or even a recommendation, to lift Volume 3 off the shelf and hunt for §324. If you are happy with an argument as it stands, independently of the reference, then carry on. If, however, I seem to have made rather a large jump, or my language has suddenly become opaque, local cross-references may help you to fill in the gaps. If a cross-reference between different volumes is particularly obscure, it may be worth checking the errata files mentioned above, in case you have run into a significant change between editions.

Each volume has an appendix of 'useful facts', in which I set out material which is called on somewhere in that volume, and which I do not feel I can take for granted. Typically the arrangement of material in these appendices is directed very narrowly at the particular applications I have in mind, and is unlikely to be a satisfactory substitute for conventional treatments of the topics touched on. Moreover, the ideas may well be needed only on rare and isolated occasions. So as a rule I advise you to ignore the appendices until you have some direct reason to suppose that a fragment may be useful to you.

During the extended gestation of this project I have been helped by many people, and I hope that my friends and colleagues will be pleased when they recognise their ideas scattered through the pages below. But I am especially grateful to those who have taken the trouble to read through earlier drafts and comment on obscurities and errors.

There is a particular debt which may not be obvious from the text, and which I ought to acknowledge. From 1984 to 2006 the biennial CARTEMI conferences, organized by the Department of Mathematics of the University Federico II of Naples, were the principal meeting place of European measure theorists, and a clearing house for ideas from all over the world. I had the good fortune to attend nearly all the meetings from 1988 onwards. I do not think it is a coincidence that I should have started work on this book in 1992; and from then on every meeting contributed something to its content. It would have been very different, probably shorter, but certainly duller, without this regular stimulation. Now the CARTEMI conferences, while of course dependent on the energies and talents of many people, were essentially the creation of one man, whose vision and determination maintained a consistent level of quality and variety. So while the dedication on the title page must remain to my wife, without whose support and forbearance the project would have been simply impossible, I should like to offer a second dedication here, to my friend Paulo de Lucia.

Contents

I list the material which is at present available in some form.

Introduction

Volume 1: The Irreducible Minimum

Introduction to Volume 1

Chapter 11: Measure Spaces

111 σ -algebras

112 Measure spaces

113 Outer measures and Carathéodory's construction

114 Lebesgue measure on \mathbb{R}

115 Lebesgue measure on \mathbb{R}^r

Chapter 12: Integration

121 Measurable functions

122 Definition of the integral

123 The convergence theorems

Chapter 13: Complements

131 Measurable subspaces

132 Outer measures from measures

- 133 Wider concepts of integration
- 134 More on Lebesgue measure
- 135 The extended real line
- *136 The Monotone Class Theorem

Appendix to Volume 1

- 1A1 Set theory
- 1A2 Open and closed sets in \mathbb{R}^r
- 1A3 Lim sups and lim infs

Concordance

References for Volume 1

Index to Volume 1

- Principal topics and results
- General index

Volume 2: Broad Foundations

Introduction to Volume 2

*Chapter 21: Taxonomy of measure spaces

- 211 Definitions
- 212 Complete spaces
- 213 Semi-finite, locally determined and localizable spaces
- 214 Subspaces
- 215 σ -finite spaces and the principle of exhaustion
- *216 Examples

Chapter 22: The Fundamental Theorem of Calculus

- 221 Vitali's theorem in \mathbb{R}
- 222 Differentiating an indefinite integral
- 223 Lebesgue's density theorems
- 224 Functions of bounded variation
- 225 Absolutely continuous functions
- *226 The Lebesgue decomposition of a function of bounded variation

Chapter 23: The Radon-Nikodým theorem

- 231 Countably additive functionals
- 232 The Radon-Nikodým theorem
- 233 Conditional expectations
- 234 Operations on measures
- 235 Measurable transformations

Chapter 24: Function spaces

- 241 \mathcal{L}^0 and L^0
- 242 L^1
- 243 L^∞
- 244 L^p
- 245 Convergence in measure
- 246 Uniform integrability
- 247 Weak compactness in L^1

Chapter 25: Product measures

- 251 Finite products
- 252 Fubini's theorem
- 253 Tensor products
- 254 Infinite products
- 255 Convolutions of functions
- 256 Radon measures on \mathbb{R}^r
- 257 Convolutions of measures

Chapter 26: Change of variable in the integral

- 261 Vitali's theorem in \mathbb{R}^r
- 262 Lipschitz and differentiable functions
- 263 Differentiable transformations in \mathbb{R}^r
- 264 Hausdorff measures
- 265 Surface measures
- *266 The Brunn-Minkowski inequality

Chapter 27: Probability theory

- 271 Distributions
- 272 Independence
- 273 The strong law of large numbers
- 274 The Central Limit Theorem
- 275 Martingales
- 276 Martingale difference sequences

Chapter 28: Fourier analysis

- 281 The Stone-Weierstrass theorem
- 282 Fourier series
- 283 Fourier transforms I
- 284 Fourier transforms II
- 285 Characteristic functions
- *286 Carleson's theorem

Appendix to Volume 2

- 2A1 Set theory
- 2A2 The topology of Euclidean space
- 2A3 General topology
- 2A4 Normed spaces
- 2A5 Linear topological spaces
- 2A6 Factorization of matrices

Concordance

References for Volume 2

Index to Volumes 1 and 2

- Principal topics and results
- General index

Volume 3: Measure Algebras

Part I

Introduction to Volume 3

Chapter 31: Boolean algebras

- 311 Boolean algebras
- 312 Homomorphisms
- 313 Order-continuity
- 314 Order-completeness
- 315 Products and free products
- 316 Further topics

Chapter 32: Measure algebras

- 321 Measure algebras
- 322 Taxonomy of measure algebras
- 323 The topology of a measure algebra
- 324 Homomorphisms
- 325 Free products and product measures
- 326 Additive functionals on Boolean algebras
- 327 Additive functionals on measure algebras
- *328 Reduced products and other constructions

Chapter 33: Maharam's theorem

- 331 Maharam types and homogeneous measure algebras
- 332 Classification of localizable measure algebras
- 333 Closed subalgebras
- 334 Products

Chapter 34: Liftings

- 341 The lifting theorem
- 342 Compact measure spaces
- 343 Realization of homomorphisms
- 344 Realization of automorphisms
- 345 Translation-invariant liftings
- 346 Consistent liftings

Concordance to part I

Part II

Chapter 35: Riesz spaces

- 351 Partially ordered linear spaces
- 352 Riesz spaces
- 353 Archimedean and Dedekind complete Riesz spaces
- 354 Banach lattices
- 355 Spaces of linear operators
- 356 Dual spaces

Chapter 36: Function spaces

- 361 S
- 362 S^\sim
- 363 L^∞
- 364 L^0
- 365 L^1
- 366 L^p
- 367 Convergence in measure
- 368 Embedding Riesz spaces in L^0
- 369 Banach function spaces

Chapter 37: Linear operators between function spaces

- 371 The Chacon-Krengel theorem
- 372 The ergodic theorem
- 373 Decreasing rearrangements
- 374 Rearrangement-invariant spaces
- 375 Kwapien's theorem
- 376 Integral operators
- 377 Function spaces of reduced products

Chapter 38: Automorphisms

- 381 Automorphisms of Boolean algebras
- 382 Factorization of automorphisms
- 383 Automorphism groups of measure algebras
- 384 Outer automorphisms
- 385 Entropy
- 386 More about entropy
- 387 Ornstein's theorem
- 388 Dye's theorem

Chapter 39: Measurable algebras

- 391 Kelley's theorem
- 392 Submeasures
- 393 Maharam algebras
- 394 Talagrand's example
- 395 Kawada's theorem
- 396 The Hajian-Ito theorem

Appendix to Volume 3

- 3A1 Set theory
- 3A2 Rings
- 3A3 General topology
- 3A4 Uniformities
- 3A5 Normed spaces
- 3A6 Groups

Concordance to part II

References for Volume 3

Index to Volumes 1-3

- Principal topics and results
- General index

Volume 4: Topological Measure Spaces

Part I

Introduction to Volume 4

Chapter 41: Topologies and measures I

- 411 Definitions
- 412 Inner regularity
- 413 Inner measure constructions
- 414 τ -additivity
- 415 Quasi-Radon measure spaces
- 416 Radon measure spaces
- 417 τ -additive product measures
- 418 Measurable functions and almost continuous functions
- 419 Examples

- Chapter 42: Descriptive set theory
 421 Souslin's operation
 422 K-analytic spaces
 423 Analytic spaces
 424 Standard Borel spaces
 *425 Realization of automorphisms
- Chapter 43: Topologies and measures II
 431 Souslin's operation
 432 K-analytic spaces
 433 Analytic spaces
 434 Borel measures
 435 Baire measures
 436 Representation of linear functionals
 437 Spaces of measures
 438 Measure-free cardinals
 439 Examples
- Chapter 44: Topological groups
 441 Invariant measures on locally compact spaces
 442 Uniqueness of Haar measure
 443 Further properties of Haar measure
 444 Convolutions
 445 The duality theorem
 446 The structure of locally compact groups
 447 Translation-invariant liftings
 448 Polish group actions
 449 Amenable groups
- Chapter 45: Perfect measures, disintegrations and processes
 451 Perfect, compact and countably compact measures
 452 Integration and disintegration of measures
 453 Strong liftings
 454 Measures on product spaces
 455 Markov and Lévy processes
 456 Gaussian distributions
 457 Simultaneous extension of measures
 458 Relative independence and relative products
 459 Symmetric measures and exchangeable random variables

Concordance to Part I

Part II

- Chapter 46: Pointwise compact sets of measurable functions
 461 Barycenters and Choquet's theorem
 462 Pointwise compact sets of continuous functions
 463 \mathfrak{T}_p and \mathfrak{T}_m
 464 Talagrand's measure
 465 Stable sets
 466 Measures on linear topological spaces
 *467 Locally uniformly rotund norms

Chapter 47: Geometric measure theory	
471 Hausdorff measures	
472 Besicovitch's Density Theorem	
473 Poincaré's inequality	
474 The distributional perimeter	
475 The essential boundary	
476 Concentration of measure	
477 Brownian motion	
478 Harmonic functions	
479 Newtonian capacity	
Chapter 48: Gauge integrals	
481 Tagged partitions	
482 General theory	
483 The Henstock integral	
484 The Pfeffer integral	
Chapter 49: Further topics	
491 Equidistributed sequences	
492 Combinatorial concentration of measure	
493 Extremely amenable groups	
494 Groups of measure-preserving automorphisms	
495 Poisson point processes	
496 Maharam submeasures	
497 Tao's proof of Szemerédi's theorem	
498 Cubes in product spaces	
Appendix to Volume 4	
4A1 Set theory	
4A2 General topology	
4A3 Topological σ -algebras	
4A4 Locally convex spaces	
4A5 Topological groups	
4A6 Banach algebras	
Concordance to Part II	
References for Volume 4	
Index to Volumes 1-4	
Principal topics and results	
General index	

Volume 5: Set-Theoretic Measure Theory

Part I

Introduction to Volume 5	
Chapter 51: Cardinal functions	
511 Definitions	
512 Galois-Tukey connections	
513 Partially ordered sets	
514 Boolean algebras	
515 The Balcar-Franěk theorem	
516 Precalibers	
517 Martin numbers	
518 Freese-Nation numbers	

- Chapter 52: Cardinal functions of measure theory
- 521 Basic theory
 - 522 Cichoń's diagram
 - 523 The measure of $\{0, 1\}^I$
 - 524 Radon measures
 - 525 Precalibers of measure algebras
 - 526 Asymptotic density zero
 - 527 Skew products of ideals
 - 528 Amoeba algebras
 - 529 Further partially ordered sets of analysis
- Chapter 53: Topologies and measures III
- 531 Maharam types of Radon measures
 - 532 Completion regular measures on $\{0, 1\}^I$
 - 533 Special topics
 - 534 Hausdorff measures, strong measure zero and Rothberger's property
 - 535 Liftings
 - 536 Alexandra Bellow's problem
 - 537 Sierpiński sets, shrinking numbers and strong Fubini theorems
 - 538 Filters and limits
 - 539 Maharam submeasures

Concordance to Part I

Part II

- Chapter 54: Real-valued-measurable cardinals
- 541 Saturated ideals
 - 542 Quasi-measurable cardinals
 - 543 The Gitik-Shelah theorem
 - 544 Measure theory with an atomlessly-measurable cardinal
 - 545 PMEA and NMA
 - 546 Power set σ -quotient algebras
 - 547 Cohen algebras and σ -measurable algebras
 - 548 Selectors and disjoint refinements
- Chapter 55: Possible worlds
- 551 Forcing with quotient algebras
 - 552 Random reals I
 - 553 Random reals II
 - 554 Cohen reals
 - 555 Solovay's construction of real-valued-measurable cardinals
 - 556 Forcing with Boolean subalgebras
- Chapter 56: Choice and Determinacy
- 561 Analysis without choice
 - 562 Borel codes
 - 563 Borel measures without choice
 - 564 Integration without choice
 - 565 Lebesgue measure without choice
 - 566 Countable choice
 - 567 Determinacy

Appendix to Volume 5

- 5A1 Set theory
- 5A2 Pcf theory
- 5A3 Forcing
- 5A4 General topology
- 5A5 Real analysis
- 5A6 Special axioms

References for Volume 5

Index to Volumes 1-5

- Principal topics and results
- General index

Volume 6: Stochastic Calculus

Part I

Introduction to Volume 6

Chapter 61: The Riemann-sum integral

- 611 Stopping times
- 612 Fully adapted processes
- 613 Definition of the integral
- 614 Simple and order-processes and bounded variation
- 615 Moderately oscillatory processes
- 616 Integrating interval functions
- 617 Integral identities and quadratic variations
- 618 Jump-free processes
- 619 Itô's formula

Chapter 62: Martingales

- 621 Finite martingales
- 622 Fully adapted martingales
- 623 Virtually local martingales
- 624 Quadratic variation
- 625 Changing the measure
- 626 Submartingales and previsible variations
- 627 Integrators and semimartingales
- *628 Refining a martingale inequality

Part II

Chapter 63: Structural alterations

- 631 Near-simple processes
- 632 Right-continuous filtrations
- 633 Separating sublattices
- 634 Changing the algebra
- 635 Changing the filtration

Chapter 64: The S-integral

- 641 Previsible versions
- 642 Previsible processes
- 643 The fundamental theorem of martingales
- 644 Pointwise convergence
- 645 Construction of the S-integral
- 646 Basic properties of the S-integral
- 647 Changing the filtration II
- 648 Changing the algebra II
- 649 Pathwise integration

Chapter 65: Applications

- 651 Exponential processes
- 652 Lévy processes
- 653 Brownian processes
- 654 Picard's theorem
- 655 The Black-Scholes model

Appendix to Volume 6

- 6A1 Real analysis

References for Volume 6

Index to Volume 6

Errata

Volume 1, 2000 edition
Volume 1, 2001 printing
Volume 1, 2004 printing
Volume 1, 2011 edition
Volume 2, 2001 edition
Volume 2, 2003 printing
Volume 2, 2010 edition
Volume 2, 2016 printing
Volume 3, 2002 edition
Volume 3, 2004 printing
Volume 3, 2012 edition
Volume 4, 2003 edition
Volume 4, 2006 printing
Volume 4, 2013 edition
Volume 5, 2008 edition
Volume 5, 2015 printing