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MEASURE THEORY

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Introduction In this treatise I aim to give a comprehensive description of modern abstract measure theory, with some indication of its principal applications. The first two volumes are set at an introductory level; they are intended for students with a solid grounding in the concepts of real analysis, but possibly with rather limited detailed knowledge. As the book proceeds, the level of sophistication and expertise demanded will increase; thus for the volume on topological measure spaces, familiarity with general topology will be assumed. The emphasis throughout is on the mathematical ideas involved, which in this subject are mostly to be found in the details of the proofs.

My intention is that the book should be usable both as a first introduction to the subject and as a reference work. For the sake of the first aim, I try to limit the ideas of the early volumes to those which are really essential to the development of the basic theorems. For the sake of the second aim, I try to express these ideas in their full natural generality, and in particular I take care to avoid suggesting any unnecessary restrictions in their applicability. Of course these principles are to some extent contradictory. Nevertheless, I find that most of the time they are very nearly reconcilable, *provided* that I indulge in a certain degree of repetition. For instance, right at the beginning, the puzzle arises: should one develop Lebesgue measure first on the real line, and then in spaces of higher dimension, or should one go straight to the multidimensional case? I believe that there is no single correct answer to this question. Most students will find the one-dimensional case easier, and it therefore seems more appropriate for a first introduction, since even in that case the technical problems can be daunting. But certainly every student of measure theory must at a fairly early stage come to terms with Lebesgue area and volume as well as length; and with the correct formulations, the multidimensional case differs from the one-dimensional case only in a definition and a (substantial) lemma. So what I have done is to write them both out (in §§114-115), so that you can pass over the higher dimensions at first reading (by omitting §115) and at the same time have a complete and uncluttered argument for them (if you omit §114). In the same spirit, I have been uninhibited, when setting out exercises, by the fact that many of the results I invite students to look for will appear in later chapters; I believe that throughout mathematics one has a better chance of understanding a theorem if one has previously attempted something similar alone.

The original plan of the work was as follows:

Volume 1: The Irreducible Minimum
  (first edition May 2000, reprinted September 2001 and February 2004, hardback ('Lulu') edition January 2011)

Volume 2: Broad Foundations
  (first edition May 2001, reprinted April 2003, hardback edition January 2010, reprinted April 2016)

Volume 3: Measure Algebras
  (first edition May 2002, reprinted May 2004, hardback edition December 2012)

Volume 4: Topological Measure Spaces
  (first edition November 2003, reprinted February 2006, hardback edition October 2013)

Volume 5: Set-theoretic Measure Theory

(first edition December 2008, reprinted January 2015).

Volume 1 is intended for those with no prior knowledge of measure theory, but competent in the elementary techniques of real analysis. I hope that it will be found useful by undergraduates meeting Lebesgue measure for the first time. Volume 2 aims to lay out some of the fundamental results of pure measure theory (the Radon-Nikodým theorem, Fubini's theorem), but also gives short introductions to some of the most important applications of measure theory (probability theory, Fourier analysis). While I should like to believe that most of it is written at a level accessible to anyone who has mastered the contents of Volume 1, I should not myself have the courage to try to cover it in an undergraduate course, though I would certainly attempt to include some parts of it. Volumes 3 and 4 are set at a rather higher level, suitable to postgraduate courses; while Volume 5 assumes a wide-ranging competence over large parts of real analysis and set theory.

In 2011 I embarked on a project to develop a version of stochastic calculus on the same principles as the main text sketched above. This has evolved into a supplementary Volume 6, with contents as listed below. It is not in fact ready for publication. You will find a great many faults, some obvious, some not, in the presentation as it stands. But I am now eighty years old and it is unlikely that I shall ever achieve the standards of coherence and accuracy which I set myself for the first five volumes. My intention here is just to make my current formulation accessible to any who have the hardihood to follow me along an idiosyncratic path into some fascinating mathematics.

There is a disclaimer which I ought to make in a place where you might see it in time to avoid paying for this book. I make no real attempt to describe the history of the subject. This is not because I think the history uninteresting or unimportant; rather, it is because I have no confidence of saying anything which would not be seriously misleading. Indeed I have very little confidence in anything I have ever read concerning the history of ideas. So while I am happy to honour the names of Lebesgue and Kolmogorov and Maharam in more or less appropriate places, and I try to include in the bibliographies the works which I have myself consulted, I leave any consideration of the details to those bolder and better qualified than myself.

I do not wish to admit that the length of this treatise is excessive, but it has certainly taken a very long time to write. Moreover, I continue to make regular corrections and additions. I am therefore presenting a version on the Internet; for details see `https://www1.essex.ac.uk/maths/people/fremlin/mt.htm`. Each chapter is available separately, and with an elementary knowledge of the TEX language you will be able to extract individual sections for printing. In addition, I am offering the material in two forms. Apart from the 'full' version, there is a 'results-only' version, omitting proofs, exercises and notes. I hope that this will be found useful for reference and revision, while saving printing costs and easing handling and storage.

For the time being, at least, printing will be in short runs. I hope that readers will be energetic in commenting on errors and omissions, since it should be possible to correct these relatively promptly. An inevitable consequence of this is that paragraph references may go out of date rather quickly. I shall be most flattered if anyone chooses to rely on this book as a source for basic material; and I am willing to attempt to maintain a concordance to such references, indicating where migratory results have come to rest for the moment, if authors will supply me with copies of papers which use them. On the web page given above you will find a link to 'errata'. Under this heading I offer postscript and pdf files listing not only corrections to published volumes, but also changes which I have made from previous printings.

I mention some minor points concerning the layout of the material. Most sections conclude with lists of 'basic exercises' and 'further exercises', which I hope will be generally instructive and occasionally entertaining. How many of these you should attempt must be for you and your teacher, if any, to decide, as no two students will have quite the same needs. I mark with a **>** those which seem to me to be particularly important. But while you may not need to write out solutions to all the 'basic exercises', if you are in any doubt as to your capacity to do so you should take this as a warning to slow down a bit. The 'further exercises' are unbounded in difficulty, and are unified only by a presumption that each has at least one solution based on ideas already introduced. Occasionally I add a final 'problem', a question to which I do not know the answer and which seems to arise naturally in the course of the work.

The impulse to write this book is in large part a desire to present a unified account of the subject. Cross-references are correspondingly abundant and wide-ranging. (I apologise for the way in which the piecemeal process of writing and revising renders some of them inaccurate.) In order to be able to refer freely across the whole text, I have chosen a reference system which gives the same code name to a paragraph wherever it is being called from. Thus 244Pc is the third subparagraph of the sixteenth paragraph in the fourth section of the fourth chapter of Volume 2, and is referred to by that name throughout. Let me emphasize that cross-references are supposed to help the reader, not distract her. Do not take the interpolation '(324D)' as an instruction, or even a recommendation, to lift Volume 3 off the shelf and hunt for §324. If you are happy with an argument as it stands, independently of the reference, then carry on. If, however, I seem to have made rather a large jump, or my language has suddenly become opaque, local cross-references may help you to fill in the gaps. If a cross-reference between different volumes is particularly obscure, it may be worth checking the errata files mentioned above, in case you have run into a significant change between editions.

Each volume has an appendix of 'useful facts', in which I set out material which is called on somewhere in that volume, and which I do not feel I can take for granted. Typically the arrangement of material in these appendices is directed very narrowly at the particular applications I have in mind, and is unlikely to be a satisfactory substitute for conventional treatments of the topics touched on. Moreover, the ideas may well be needed only on rare and isolated occasions. So as a rule I advise you to ignore the appendices until you have some direct reason to suppose that a fragment may be useful to you.

During the extended gestation of this project I have been helped by many people, and I hope that my friends and colleagues will be pleased when they recognise their ideas scattered through the pages below. But I am especially grateful to those who have taken the trouble to read through earlier drafts and comment on obscurities and errors.

There is a particular debt which may not be obvious from the text, and which I ought to acknowledge. From 1984 to 2006 the biennial CARTEMI conferences, organized by the Department of Mathematics of the University Federico II of Naples, were the principal meeting place of European measure theorists, and a clearing house for ideas from all over the world. I had the good fortune to attend nearly all the meetings from 1988 onwards. I do not think it is a coincidence that I should have started work on this book in 1992; and from then on every meeting contributed something to its content. It would have been very different, probably shorter, but certainly duller, without this regular stimulation. Now the CARTEMI conferences, while of course dependent on the energies and talents of many people, were essentially the creation of one man, whose vision and determination maintained a consistent level of quality and variety. So while the dedication on the title page must remain to my wife, without whose support and forbearance the project would have been simply impossible, I should like to offer a second dedication here, to my friend Paulo de Lucia.

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I list the material which is at present available in some form, with dates of the most recent full revisions.

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369 Banach function spaces (23.11.16)

Riesz spaces separated by their order-continuous duals; representing $U^\times$ when $U\subseteq L^0$; Kakutani's representation of L-spaces as L^1 spaces; extended Fatou norms; associate norms; $L^{\tau'}\cong(L^\tau)^\times$; Fatou norms and convergence in measure; $M^{\infty,1}$ and $M^{1,\infty}$, $\| \|_{\infty,1}$ and $\| \|_{1,\infty}$; $L^{\tau_1}+L^{\tau_2}$.

Chapter 37: Linear operators between function spaces

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371 The Chacon-Krengel theorem (13.12.06)

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372 The ergodic theorem (7.12.08)

The Maximal Ergodic Theorem and the Ergodic Theorem for operators in $\mathcal{T}^{(0)}_{\bar\mu,\bar\mu}$; for inverse-measure-preserving functions $\phi:X\to X$; limit operators as conditional expectations; applications to continued fractions; mixing and ergodic transformations.

373 Decreasing rearrangements (25.5.16)

The classes $\mathcal{T}$, $\mathcal{T}^\times$; the space $M^{0,\infty}$; decreasing rearrangements u^*; $\|u^*\|_p=\|u\|_p$; $\int|Tu\times v|\le\int u^*\times v^*$ if $T\in\mathcal{T}$; the very weak operator topology and compactness of $\mathcal{T}$; v is expressible as Tu, where $T\in\mathcal{T}$, iff $\int_0^t v^*\le\int_0^t u^*$ for every t; finding T such that $\int Tu\times v=\int u^*\times v^*$; the adjoint operator from $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$ to $\mathcal{T}^{(0)}_{\bar\nu,\bar\mu}$.

374 Rearrangement-invariant spaces (15.6.09)

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375 Kwapien's theorem (30.1.10)

Linear operators on L^0 spaces; if $\mathfrak{B}$ is measurable, a positive linear operator from $L^0(\mathfrak{A})$ to $L^0(\mathfrak{B})$ can be assembled from Riesz homomorphisms.

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377 Function spaces of reduced products (30.12.09)

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381 Automorphisms of Boolean algebras (19.7.06)

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382 Factorization of automorphisms (15.8.06)

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383 Automorphism groups of measure algebras (9.11.14)

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385 Entropy (21.10.03)

Entropy of a partition of unity in a probability algebra; conditional entropy; entropy of a measure-preserving homomorphism; calculation of entropy (Kolmogorov-Sinaǐ theorem); Bernoulli shifts; isomorphic homomorphisms and conjugacy classes in $\operatorname{Aut}_{\bar\mu}\mathfrak{A}$; almost isomorphic inverse-measure-preserving functions.

Errata

Volume 1, 2000 edition (16.2.17)
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