

# MEASURE THEORY

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**Introduction** In this treatise I aim to give a comprehensive description of modern abstract measure theory, with some indication of its principal applications. The first two volumes are set at an introductory level; they are intended for students with a solid grounding in the concepts of real analysis, but possibly with rather limited detailed knowledge. As the book proceeds, the level of sophistication and expertise demanded will increase; thus for the volume on topological measure spaces, familiarity with general topology will be assumed. The emphasis throughout is on the mathematical ideas involved, which in this subject are mostly to be found in the details of the proofs.

My intention is that the book should be usable both as a first introduction to the subject and as a reference work. For the sake of the first aim, I try to limit the ideas of the early volumes to those which are really essential to the development of the basic theorems. For the sake of the second aim, I try to express these ideas in their full natural generality, and in particular I take care to avoid suggesting any unnecessary restrictions in their applicability. Of course these principles are to some extent contradictory. Nevertheless, I find that most of the time they are very nearly reconcilable, *provided* that I indulge in a certain degree of repetition. For instance, right at the beginning, the puzzle arises: should one develop Lebesgue measure first on the real line, and then in spaces of higher dimension, or should one go straight to the multidimensional case? I believe that there is no single correct answer to this question. Most students will find the one-dimensional case easier, and it therefore seems more appropriate for a first introduction, since even in that case the technical problems can be daunting. But certainly every student of measure theory must at a fairly early stage come to terms with Lebesgue area and volume as well as length; and with the correct formulations, the multidimensional case differs from the one-dimensional case only in a definition and a (substantial) lemma. So what I have done is to write them both out (in §§114-115), so that you can pass over the higher dimensions at first reading (by omitting §115) and at the same time have a complete and uncluttered argument for them (if you omit §114). In the same spirit, I have been uninhibited, when setting out exercises, by the fact that many of the results I invite students to look for will appear in later chapters; I believe that throughout mathematics one has a better chance of understanding a theorem if one has previously attempted something similar alone.

The plan of the work is as follows:

Volume 1: The Irreducible Minimum

(first edition May 2000, reprinted September 2001 and February 2004, hardback ('Lulu') edition January 2011)

Volume 2: Broad Foundations

(first edition May 2001, reprinted April 2003, hardback edition January 2010)

Volume 3: Measure Algebras

(first edition May 2002, reprinted May 2004, hardback edition December 2012)

Volume 4: Topological Measure Spaces

(first edition November 2003, reprinted February 2006, hardback edition October 2013)

Volume 5: Set-theoretic Measure Theory

(first edition December 2008, reprinted January 2015).

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Volume 1 is intended for those with no prior knowledge of measure theory, but competent in the elementary techniques of real analysis. I hope that it will be found useful by undergraduates meeting Lebesgue measure for the first time. Volume 2 aims to lay out some of the fundamental results of pure measure theory (the Radon-Nikodým theorem, Fubini's theorem), but also gives short introductions to some of the most important applications of measure theory (probability theory, Fourier analysis). While I should like to believe that most of it is written at a level accessible to anyone who has mastered the contents of Volume 1, I should not myself have the courage to try to cover it in an undergraduate course, though I would certainly attempt to include some parts of it. Volumes 3 and 4 are set at a rather higher level, suitable to postgraduate courses; while Volume 5 assumes a wide-ranging competence over large parts of real analysis and set theory.

There is a disclaimer which I ought to make in a place where you might see it in time to avoid paying for this book. I make no real attempt to describe the history of the subject. This is not because I think the history uninteresting or unimportant; rather, it is because I have no confidence of saying anything which would not be seriously misleading. Indeed I have very little confidence in anything I have ever read concerning the history of ideas. So while I am happy to honour the names of Lebesgue and Kolmogorov and Maharam in more or less appropriate places, and I try to include in the bibliographies the works which I have myself consulted, I leave any consideration of the details to those bolder and better qualified than myself.

I do not wish to admit that the length of this treatise is excessive, but it has certainly taken a very long time to write. Moreover, I continue to make regular corrections and additions. I am therefore presenting revised versions in draft form on the Internet; for details see <http://www.essex.ac.uk/math/people/fremlin/mt.htm>. Each chapter is available separately, and with an elementary knowledge of the T<sub>E</sub>X language you will be able to extract individual sections for printing. In addition, I am offering the material in two forms. Apart from the 'full' version, there is a 'results-only' version, omitting proofs, exercises and notes. I hope that this will be found useful for reference and revision, while saving printing costs and easing handling and storage.

For the time being, at least, printing will be in short runs. I hope that readers will be energetic in commenting on errors and omissions, since it should be possible to correct these relatively promptly. An inevitable consequence of this is that paragraph references may go out of date rather quickly. I shall be most flattered if anyone chooses to rely on this book as a source for basic material; and I am willing to attempt to maintain a concordance to such references, indicating where migratory results have come to rest for the moment, if authors will supply me with copies of papers which use them. On the web page given above you will find a link to 'errata'. Under this heading I have prepared Postscript files listing not only corrections to published volumes, but also changes which I have made from previous editions.

I mention some minor points concerning the layout of the material. Most sections conclude with lists of 'basic exercises' and 'further exercises', which I hope will be generally instructive and occasionally entertaining. How many of these you should attempt must be for you and your teacher, if any, to decide, as no two students will have quite the same needs. I mark with a > those which seem to me to be particularly important. But while you may not need to write out solutions to all the 'basic exercises', if you are in any doubt as to your capacity to do so you should take this as a warning to slow down a bit. The 'further exercises' are unbounded in difficulty, and are unified only by a presumption that each has at least one solution based on ideas already introduced. Occasionally I add a final 'problem', a question to which I do not know the answer and which seems to arise naturally in the course of the work.

The impulse to write this book is in large part a desire to present a unified account of the subject. Cross-references are correspondingly abundant and wide-ranging. (I apologise for the way in which the piecemeal process of writing and revising renders some of them inaccurate.) In order to be able to refer freely across the whole text, I have chosen a reference system which gives the same code name to a paragraph wherever it is being called from. Thus 244Pc is the third subparagraph of the sixteenth paragraph in the fourth section of the fourth chapter of Volume 2, and is referred to by that name throughout. Let me emphasize that cross-references are supposed to help the reader, not distract her. Do not take the interpolation '(324D)' as an instruction, or even a recommendation, to lift Volume 3 off the shelf and hunt for §324. If you are happy with an argument as it stands, independently of the reference, then carry on. If, however, I seem to have made rather a large jump, or my language has suddenly become opaque, local cross-references may help you to fill in the gaps. If a cross-reference between different volumes is particularly obscure, it may be worth checking the errata files mentioned above, in case you have run into a significant change between editions.

Each volume will have an appendix of ‘useful facts’, in which I set out material which is called on somewhere in that volume, and which I do not feel I can take for granted. Typically the arrangement of material in these appendices is directed very narrowly at the particular applications I have in mind, and is unlikely to be a satisfactory substitute for conventional treatments of the topics touched on. Moreover, the ideas may well be needed only on rare and isolated occasions. So as a rule I advise you to ignore the appendices until you have some direct reason to suppose that a fragment may be useful to you.

During the extended gestation of this project I have been helped by many people, and I hope that my friends and colleagues will be pleased when they recognise their ideas scattered through the pages below. But I am especially grateful to those who have taken the trouble to read through earlier drafts and comment on obscurities and errors.

There is a particular debt which may not be obvious from the text, and which I ought to acknowledge. From 1984 to 2006 the biennial CARTEMI conferences, organized by the Department of Mathematics of the University Federico II of Naples, were the principal meeting place of European measure theorists, and a clearing house for ideas from all over the world. I had the good fortune to attend nearly all the meetings from 1988 onwards. I do not think it is a coincidence that I should have started work on this book in 1992; and from then on every meeting has contributed something to its content. It would have been very different, probably shorter, but certainly duller, without this regular stimulation. Now the CARTEMI conferences, while of course dependent on the energies and talents of many people, were essentially the creation of one man, whose vision and determination maintained a consistent level of quality and variety. So while the dedication on the title page must remain to my wife, without whose support and forbearance the project would have been simply impossible, I should like to offer a second dedication here, to my friend Paulo de Lucia.

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246 Uniform integrability (17.11.06)

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253 Tensor products (18.4.08)

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256 Radon measures on  $\mathbb{R}^r$  (6.8.15)

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257 Convolutions of measures (14.8.13)

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262 Lipschitz and differentiable functions (11.8.15)

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263 Differentiable transformations in  $\mathbb{R}^r$  (4.4.13)

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264 Hausdorff measures (12.5.03)

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265 Surface measures (3.9.13)

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\*266 The Brunn-Minkowski inequality (28.1.09)

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271 Distributions (11.12.08)

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272 Independence (3.4.09)

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273 The strong law of large numbers (2.12.09)

$\frac{1}{n+1} \sum_{i=0}^n X_i \rightarrow 0$  a.e. if the  $X_n$  are independent with zero expectation and either (i)  $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \text{Var}(X_n) < \infty$  or (ii)  $\sum_{n=0}^{\infty} \mathbb{E}(|X_n|^{1+\delta}) < \infty$  for some  $\delta > 0$  or (iii) the  $X_n$  are identically distributed.

## 274 The Central Limit Theorem (13.4.10)

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## 275 Martingales (3.12.12)

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## 276 Martingale difference sequences (16.4.13)

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## 281 The Stone-Weierstrass theorem (4.12.12)

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## 282 Fourier series (24.9.09)

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## 283 Fourier transforms I (31.3.13)

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## 284 Fourier transforms II (30.8.13)

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## 285 Characteristic functions (18.9.14)

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## \*286 Carleson's theorem (23.4.10)

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## 2A2 The topology of Euclidean space (30.11.09)

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## 2A3 General topology (25.7.07)

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## 2A4 Normed spaces (4.3.14)

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## 2A5 Linear topological spaces (13.11.07)

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## 2A6 Factorization of matrices (10.11.14)

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312 Homomorphisms (29.5.07)

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313 Order-continuity (8.6.11)

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314 Order-completeness (26.7.07)

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315 Products and free products (13.11.12)

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316 Further topics (26.1.09)

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321 Measure algebras (3.1.11)

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322 Taxonomy of measure algebras (24.4.06)

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323 The topology of a measure algebra (20.7.06)

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324 Homomorphisms (6.2.02)

Homomorphisms induced by measurable functions; order-continuous and continuous homomorphisms; the topology of a semi-finite measure algebra is determined by the algebraic structure; measure-preserving homomorphisms.

325 Free products and product measures (30.8.06)

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326 Additive functionals on Boolean algebras (21.5.11)

Additive, countably additive and completely additive functionals; Jordan decomposition; Hahn decomposition; Liapounoff's convexity theorem; the region  $\llbracket \mu > \nu \rrbracket$ .



## 327 Additive functionals on measure algebras (13.7.11)

Absolutely continuous and continuous additive functionals; Radon-Nikodým theorem; the standard extension of a continuous additive functional on a closed subalgebra.

## \*328 Reduced products and other constructions (2.6.09)

Reduced products of probability algebras; inductive and projective limits; converting homomorphisms into automorphisms.

## Chapter 33: Maharam's theorem

## Introduction (17.11.10)

## 331 Maharam types and homogeneous measure algebras (1.2.05)

Relatively atomless algebras; one-step extension of measure-preserving homomorphisms; Maharam type of a measure algebra; Maharam-type-homogeneous probability algebras of the same Maharam type are isomorphic; the measure algebra of  $\{0, 1\}^\kappa$  is homogeneous.

## 332 Classification of localizable measure algebras (19.3.05)

Any localizable measure algebra is isomorphic to a simple product of homogeneous totally finite algebras; complete description of isomorphism types; closed subalgebras.

## 333 Closed subalgebras (27.6.08)

Relative Maharam types; extension of measure-preserving Boolean homomorphisms; complete classification of closed subalgebras of probability algebras as triples  $(\mathfrak{A}, \bar{\mu}, \mathfrak{C})$ ; fixed-point subalgebras.

## 334 Products (26.9.08)

Maharam types of product measures; infinite powers of probability spaces are Maharam-type-homogeneous.

## Chapter 34: Liftings

## Introduction (28.5.07)

## 341 The lifting theorem (9.4.10)

Liftings and lower densities; strictly localizable spaces have lower densities; construction of a lifting from a density; complete strictly localizable spaces have liftings; liftings and Stone spaces.

## 342 Compact measure spaces (9.7.10)

Inner regular measures; compact classes; compact and locally compact measures; perfect measures.

## 343 Realization of homomorphisms (17.11.10)

Representing homomorphisms between measure algebras by functions; possible when target measure space is locally compact; countably separated measures and uniqueness of representing functions; the split interval; perfect measures.

## 344 Realization of automorphisms (22.3.06)

Simultaneously representing groups of automorphisms of measure algebras by functions – Stone spaces, countably separated measure spaces, measures on  $\{0, 1\}^I$ ; characterization of Lebesgue measure as a measure space; strong homogeneity of usual measure on  $\{0, 1\}^I$ .

## 345 Translation-invariant liftings (27.6.06)

Translation-invariant liftings on  $\mathbb{R}^r$  and  $\{0, 1\}^I$ ; there is no t.-i. Borel lifting on  $\mathbb{R}$ .

## 346 Consistent liftings (17.12.10)

Liftings of product measures which respect the product structure; translation-invariant liftings on  $\{0, 1\}^I$ ; products of Maharam-type-homogeneous probability spaces; lower densities respecting product structures; consistent liftings; the Stone space of Lebesgue measure.

## Concordance to part I

## Part II

## Chapter 35: Riesz spaces

## Introduction (4.9.09)

## 351 Partially ordered linear spaces (16.10.07)

Partially ordered linear spaces; positive cones; suprema and infima; positive linear operators; order-continuous linear operators; Riesz homomorphisms; quotient spaces; reduced powers; representation of p.o.l.ss as subspaces of reduced powers of  $\mathbb{R}$ ; Archimedean spaces.

## 352 Riesz spaces (22.9.03)

Riesz spaces; identities; general distributive laws; Riesz homomorphisms; Riesz subspaces; order-dense subspaces and order-continuous operators; bands; the algebra of complemented bands; the algebra of projection bands; principal bands;  $f$ -algebras.

## 353 Archimedean and Dedekind complete Riesz spaces (26.1.02)

Order-dense subspaces; bands; Dedekind ( $\sigma$ )-complete spaces; order-closed Riesz subspaces; order units;  $f$ -algebras.

## 354 Banach lattices (18.8.08)

Riesz norms; Fatou norms; the Levi property; order-continuous norms; order-unit norms;  $M$ -spaces; are isomorphic to  $C(X)$  for compact Hausdorff  $X$ ;  $L$ -spaces; uniform integrability in  $L$ -spaces.

## 355 Spaces of linear operators (1.12.07)

Order-bounded linear operators; the space  $L^\sim(U; V)$ ; order-continuous operators; extension of order-continuous operators; the space  $L^\times(U; V)$ ; order-continuous norms.

## 356 Dual spaces (26.1.02)

The spaces  $U^\sim, U^\times, U^*$ ; biduals, embeddings  $U \rightarrow V^\times$  where  $V \subseteq U^\sim$ ; perfect Riesz spaces;  $L$ - and  $M$ -spaces; uniformly integrable sets in the dual of an  $M$ -space; relative weak compactness in  $L$ -spaces.

## Chapter 36: Function spaces

## Introduction (2.2.02)

361  $S$  (6.2.08)

Additive functions on Boolean rings; the space  $S(\mathfrak{A})$ ; universal mapping theorems for linear operators on  $S$ ; the map  $T_\pi : S(\mathfrak{A}) \rightarrow S(\mathfrak{B})$  induced by a ring homomorphism  $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ ; projection bands in  $S(\mathfrak{A})$ ; identifying  $S(\mathfrak{A})$  when  $\mathfrak{A}$  is a quotient of an algebra of sets.

362  $S^\sim$  (31.12.10)

Bounded additive functionals on  $\mathfrak{A}$  identified with order-bounded linear functionals on  $S(\mathfrak{A})$ ; the  $L$ -space  $S^\sim$  and its bands; countably additive, completely additive, absolutely continuous and continuous functionals; uniform integrability in  $S^\sim$ .

363  $L^\infty$  (4.3.08)

The space  $L^\infty(\mathfrak{A})$ , as an  $M$ -space and  $f$ -algebra; universal mapping theorems for linear operators on  $L^\infty$ ;  $T_\pi : L^\infty(\mathfrak{A}) \rightarrow L^\infty(\mathfrak{B})$ ; representing  $L^\infty$  when  $\mathfrak{A}$  is a quotient of an algebra of sets; integrals with respect to finitely additive functionals; projection bands in  $L^\infty$ ;  $(L^\infty)^\sim$  and its bands; Dedekind completeness of  $\mathfrak{A}$  and  $L^\infty$ ; representing  $\sigma$ -complete  $M$ -spaces; the generalized Hahn-Banach theorem; the Banach-Ulam problem.

364  $L^0$  (16.7.11)

The space  $L^0(\mathfrak{A})$ ; representing  $L^0$  when  $\mathfrak{A}$  is a quotient of a  $\sigma$ -algebra of sets; algebraic operations on  $L^0$ ; action of Borel measurable functions on  $L^0$ ; identifying  $L^0(\mathfrak{A})$  with  $L^0(\mu)$  when  $\mathfrak{A}$  is a measure algebra; embedding  $S$  and  $L^\infty$  in  $L^0$ ; suprema and infima in  $L^0$ ; Dedekind completeness in  $\mathfrak{A}$  and  $L^0$ ; multiplication in  $L^0$ ; projection bands;  $T_\pi : L^0(\mathfrak{A}) \rightarrow L^0(\mathfrak{B})$ ; when  $\pi$  represented by a  $(T, \Sigma)$ -measurable function; simple products; \*regular open algebras; \*the space  $C^\infty(X)$ .

365  $L^1$  (20.1.15)

The space  $L^1(\mathfrak{A}, \bar{\mu})$ ; identification with  $L^1(\mu)$ ;  $\int_a u$ ; the Radon-Nikodým theorem again;  $\int w \times T_\pi u d\bar{\nu} = \int u d\bar{\mu}$ ; additive functions on  $\mathfrak{A}$  and linear operators on  $L^1$ ; the duality between  $L^1$  and  $L^\infty$ ;  $T_\pi : L^1(\mathfrak{A}, \bar{\mu}) \rightarrow L^1(\mathfrak{B}, \bar{\nu})$  and  $P_\pi : L^1(\mathfrak{B}, \bar{\nu}) \rightarrow L^1(\mathfrak{A}, \bar{\mu})$ ; conditional expectations; bands in  $L^1$ ; varying  $\bar{\mu}$ .

366  $L^p$  (10.11.08)

The spaces  $L^p(\mathfrak{A}, \bar{\mu})$ ; identification with  $L^p(\mu)$ ;  $L^q$  as the dual of  $L^p$ ; the spaces  $M^0$  and  $M^{1,0}$ ;  $T_\pi : M^0(\mathfrak{A}, \bar{\mu}) \rightarrow M^0(\mathfrak{B}, \bar{\nu})$  and  $P_\pi : M^{1,0}(\mathfrak{B}, \bar{\nu}) \rightarrow M^{1,0}(\mathfrak{A}, \bar{\mu})$ ; conditional expectations; the case  $p = 2$ ; spaces  $L^p_{\mathbb{C}}(\mathfrak{A}, \bar{\mu})$ .

## 367 Convergence in measure (18.9.03)

Order\*-convergence of sequences in lattices; in Riesz spaces; in Banach lattices; in quotients of spaces of measurable functions; in  $C(X)$ ; Lebesgue's Dominated Convergence Theorem and Doob's Martingale Theorem; convergence in measure in  $L^0(\mathfrak{A})$ ; and pointwise convergence; defined by the Riesz space structure; positive linear operators on  $L^0$ ; convergence in measure and the canonical projection  $(L^1)^{**} \rightarrow L^1$ ; the set of independent families of random variables.

368 Embedding Riesz spaces in  $L^0$  (16.9.09)

Extension of order-continuous Riesz homomorphisms into  $L^0$ ; representation of Archimedean Riesz spaces as subspaces of  $L^0$ ; Dedekind completion of Riesz spaces; characterizing  $L^0$  spaces as Riesz spaces; weakly  $(\sigma, \infty)$ -distributive Riesz spaces.

## 369 Banach function spaces (5.11.03)

Riesz spaces separated by their order-continuous duals; representing  $U^\times$  when  $U \subseteq L^0$ ; Kakutani's representation of  $L$ -spaces as  $L^1$  spaces; extended Fatou norms; associate norms;  $L^{\tau'} \cong (L^\tau)^\times$ ; Fatou norms and convergence in measure;  $M^{\infty,1}$  and  $M^{1,\infty}$ ,  $\|\cdot\|_{\infty,1}$  and  $\|\cdot\|_{1,\infty}$ ;  $L^{\tau_1} + L^{\tau_2}$ .

## Chapter 37: Linear operators between function spaces

Introduction (6.11.03)

371 The Chacon-Krengel theorem (13.12.06)

$L^\sim(U;V) = L^\times(U;V) = B(U;V)$  for  $L$ -spaces  $U$  and  $V$ ; the class  $\mathcal{T}_{\bar{\mu},\bar{\nu}}^{(0)}$  of  $\|\cdot\|_1$ -decreasing,  $\|\cdot\|_\infty$ -decreasing linear operators from  $M^{1,0}(\mathfrak{A},\bar{\mu})$  to  $M^{1,0}(\mathfrak{B},\bar{\nu})$ .

372 The ergodic theorem (7.12.08)

The Maximal Ergodic Theorem and the Ergodic Theorem for operators in  $\mathcal{T}_{\bar{\mu},\bar{\mu}}^{(0)}$ ; for inverse-measure-preserving functions  $\phi : X \rightarrow X$ ; limit operators as conditional expectations; applications to continued fractions; mixing and ergodic transformations.

373 Decreasing rearrangements (10.2.02)

The classes  $\mathcal{T}$ ,  $\mathcal{T}^\times$ ; the space  $M^{0,\infty}$ ; decreasing rearrangements  $u^*$ ;  $\|u^*\|_p = \|u\|_p$ ;  $\int |Tu \times v| \leq \int u^* \times v^*$  if  $T \in \mathcal{T}$ ; the very weak operator topology and compactness of  $\mathcal{T}$ ;  $v$  is expressible as  $Tu$ , where  $T \in \mathcal{T}$ , iff  $\int_0^t v^* \leq \int_0^t u^*$  for every  $t$ ; finding  $T$  such that  $\int Tu \times v = \int u^* \times v^*$ ; the adjoint operator from  $\mathcal{T}_{\bar{\mu},\bar{\nu}}^{(0)}$  to  $\mathcal{T}_{\bar{\nu},\bar{\mu}}^{(0)}$ .

374 Rearrangement-invariant spaces (15.6.09)

$\mathcal{T}$ -invariant subspaces of  $M^{1,\infty}$ , and  $\mathcal{T}$ -invariant extended Fatou norms; relating  $\mathcal{T}$ -invariant norms on different spaces; rearrangement-invariant sets and norms; when rearrangement-invariance implies  $\mathcal{T}$ -invariance.

375 Kwapien's theorem (30.1.10)

Linear operators on  $L^0$  spaces; if  $\mathfrak{B}$  is measurable, a positive linear operator from  $L^0(\mathfrak{A})$  to  $L^0(\mathfrak{B})$  can be assembled from Riesz homomorphisms.

376 Integral operators (8.4.10)

Kernel operators; free products of measure algebras and tensor products of  $L^0$  spaces; tensor products of  $L^1$  spaces; abstract integral operators (i) as a band in  $L^\times(U;V)$  (ii) represented by kernels belonging to  $L^0(\mathfrak{A} \otimes \mathfrak{B})$  (iii) as operators converting weakly convergent sequences into order\*-convergent sequences; operators into  $M$ -spaces or out of  $L$ -spaces.

377 Function spaces of reduced products (30.12.09)

Measure-bounded Boolean homomorphisms on products of probability algebras; associated maps on subspaces of  $\prod_{i \in I} L^0(\mathfrak{A}_i)$  and  $\prod_{i \in I} L^p(\mathfrak{A}_i)$ ; reduced powers; universal mapping theorems for function spaces on projective and inductive limits of probability algebras.

## Chapter 38: Automorphisms

Introduction (15.8.08)

381 Automorphisms of Boolean algebras (19.7.06)

Assembling an automorphism; elements supporting an automorphism; periodic and aperiodic parts; full and countably full subgroups; recurrence; induced automorphisms of principal ideals; Stone spaces; cyclic automorphisms.

382 Factorization of automorphisms (15.8.06)

Separators and transversals; Frolík's theorem; exchanging involutions; expressing an automorphism as the product of three involutions; subgroups of  $\text{Aut } \mathfrak{A}$  with many involutions; normal subgroups of full groups with many involutions; simple automorphism groups.

383 Automorphism groups of measure algebras (9.11.14)

Measure-preserving automorphisms as products of involutions; normal subgroups of  $\text{Aut } \mathfrak{A}$  and  $\text{Aut}_{\bar{\mu}} \mathfrak{A}$ ; conjugacy in  $\text{Aut } \mathfrak{A}$  and  $\text{Aut}_{\bar{\mu}} \mathfrak{A}$ .

384 Outer automorphisms (5.11.14)

If  $G \leq \text{Aut } \mathfrak{A}$ ,  $H \leq \text{Aut } \mathfrak{B}$  have many involutions, any isomorphism between  $G$  and  $H$  arises from an isomorphism between  $\mathfrak{A}$  and  $\mathfrak{B}$ ; if  $\mathfrak{A}$  is nowhere rigid,  $\text{Aut } \mathfrak{A}$  has no outer automorphisms; applications to localizable measure algebras.

385 Entropy (21.10.03)

Entropy of a partition of unity in a probability algebra; conditional entropy; entropy of a measure-preserving homomorphism; calculation of entropy (Kolmogorov-Sinai theorem); Bernoulli shifts; isomorphic homomorphisms and conjugacy classes in  $\text{Aut}_{\bar{\mu}} \mathfrak{A}$ ; almost isomorphic inverse-measure-preserving functions.

386 More about entropy (20.8.15)

The Halmos-Rokhlin-Kakutani lemma; the Shannon-McMillan-Breiman theorem; the Csiszár-Kullback inequality; various lemmas.

387 Ornstein's theorem (9.3.16)

Bernoulli partitions; finding Bernoulli partitions with elements of given measure (Sinai's theorem); adjusting Bernoulli partitions; Ornstein's theorem (Bernoulli shifts of the same finite entropy are isomorphic); Ornstein's and Sinai's theorems in the case of infinite entropy.

## 388 Dye's theorem (14.1.04)

Orbits of inverse-measure-preserving functions; von Neumann transformations; von Neumann transformations generating a given full subgroup; classification of full subgroups generated by a single automorphism.

## Chapter 39: Measurable algebras

## Introduction (17.11.10)

## 391 Kelley's theorem (5.9.07)

Measurable algebras; strictly positive additive functionals and weak  $(\sigma, \infty)$ -distributivity; additive functionals subordinate to or dominating a given functional; intersection numbers; existence of strictly positive additive functionals.

## 392 Submeasures (11.2.08)

Submeasures; exhaustive, uniformly exhaustive and Maharam submeasures; the Kalton-Roberts theorem (a strictly positive uniformly exhaustive submeasure provides a strictly positive additive functional); strictly positive submeasures, associated metrics and metric completions of algebras; products of submeasures.

## 393 Maharam algebras (11.5.08)

Maharam submeasures; Maharam algebras; topologies on Boolean algebras; order-sequential topologies; characterizations of Maharam algebras.

## 394 Talagrand's example (13.6.11)

An exhaustive submeasure which is not uniformly exhaustive; a non-measurable Maharam algebra; control measures.

## 395 Kawada's theorem (15.6.08)

Full local semigroups;  $\tau$ -equidecomposability; fully non-paradoxical subgroups of  $\text{Aut } \mathfrak{A}$ ;  $\lfloor b : a \rfloor$  and  $\lceil b : a \rceil$ ; invariant additive functions from  $\mathfrak{A}$  to  $L^\infty(\mathfrak{C})$ , where  $\mathfrak{C}$  is the fixed-point subalgebra of a group; invariant additive functionals and measures; ergodic fully non-paradoxical groups.

## 396 The Hajian-Ito theorem (15.8.08)

Invariant measures on measurable algebras; weakly wandering elements.

## Appendix to Volume 3

## Introduction (13.3.08)

## 3A1 Set theory (31.10.07)

Calculation of cardinalities; cofinal sets, cofinalities; notes on the use of Zorn's Lemma; the natural numbers as finite ordinals; lattice homomorphisms; the Marriage Lemma.

## 3A2 Rings (22.11.07)

Rings; subrings, ideals, homomorphisms, quotient rings, the First Isomorphism Theorem; products.

## 3A3 General topology (14.12.07)

Hausdorff, regular, completely regular, zero-dimensional, extremally disconnected, compact and locally compact spaces; continuous functions; dense subsets; meager sets; Baire's theorem for locally compact spaces; products; Tychonoff's theorem; the usual topologies on  $\{0, 1\}^I$ ,  $\mathbb{R}^I$ ; cluster points of filters; topology bases; uniform convergence of sequences of functions; one-point compactifications; topologies defined from sequential convergences.

## 3A4 Uniformities (30.1.08)

Uniform spaces; and pseudometrics; uniform continuity; subspaces; product uniformities; Cauchy filters and completeness; extending uniformly continuous functions; completions.

## 3A5 Normed spaces (22.5.11)

The Hahn-Banach theorem in analytic and geometric forms; cones and convex sets; weak and weak\* topologies; reflexive spaces; Uniform Boundedness Theorem; strong operator topologies; completions; normed algebras; compact linear operators; Hilbert spaces; bounded sets in linear topological spaces.

## 3A6 Groups (6.8.08)

Involutions; inner and outer automorphisms; normal subgroups.

## Concordance to part II

## References for Volume 3 (9.4.05)

## Index to Volumes 1-3

Principal topics and results

General index

## Volume 4: Topological Measure Spaces

## Part I

## Introduction to Volume 4 (14.12.06)

## Chapter 41: Topologies and measures I

## Introduction (17.4.10)

## 411 Definitions (31.12.08)

Topological, inner regular,  $\tau$ -additive, outer regular, locally finite, effectively locally finite, quasi-Radon, Radon, completion regular, Baire, Borel and strictly positive measures; measurable and almost continuous functions; self-supporting sets and supports of measures; Stone spaces; Dieudonné's measure.

## 412 Inner regularity (31.1.05)

Exhaustion; Baire measures; Borel measures on metrizable spaces; completions and c.l.d. versions; complete locally determined spaces; inverse-measure-preserving functions; subspaces; indefinite-integral measures; products; outer regularity.

## 413 Inner measure constructions (25.2.05)

Inner measures; constructing a measure from an inner measure; the inner measure defined by a measure; complete locally determined spaces; extension of functionals to measures; countably compact classes; constructing measures dominating given functionals.

414  $\tau$ -additivity (26.1.10)

Semi-continuous functions; supports; strict localizability; subspace measures; regular topologies; density topologies; lifting topologies.

## 415 Quasi-Radon measure spaces (7.4.05)

Strict localizability; subspaces; regular topologies; hereditarily Lindelöf spaces; products of separable metrizable spaces; comparison and specification of quasi-Radon measures; construction of quasi-Radon measures extending given functionals; indefinite-integral measures;  $L^p$  spaces; Stone spaces.

## 416 Radon measure spaces (24.6.05)

Radon and quasi-Radon measures; specification of Radon measures; c.l.d. versions of Borel measures; locally compact topologies; constructions of Radon measures extending or dominating given functionals; additive functionals on Boolean algebras and Radon measures on Stone spaces; subspaces; products; Stone spaces of measure algebras; compact and perfect measures; representation of homomorphisms of measure algebras.

417  $\tau$ -additive product measures (9.3.10)

The product of two effectively locally finite  $\tau$ -additive measures; the product of many  $\tau$ -additive probability measures; Fubini's theorem; generalized associative law; measures on subproducts as image measures; products of strictly positive measures; quasi-Radon and Radon product measures; when 'ordinary' product measures are  $\tau$ -additive; continuous functions and Baire  $\sigma$ -algebras in product spaces.

## 418 Measurable functions and almost continuous functions (19.8.05)

Measurable functions; into (separable) metrizable spaces; and image measures; almost continuous functions; continuity, measurability, image measures; expressing Radon measures as images of Radon measures; Prokhorov's theorem on projective limits of Radon measures; representing measurable functions into  $L^0$  spaces.

## 419 Examples (2.12.05)

A nearly quasi-Radon measure; a Radon measure space in which the Borel sets are inadequate; a nearly Radon measure; the Stone space of the Lebesgue measure algebra; measures with domain  $\mathcal{P}\omega_1$ ; notes on Lebesgue measure; the split interval.

## Chapter 42: Descriptive set theory

## Introduction (30.9.08)

## 421 Souslin's operation (14.12.07)

Souslin's operation; is idempotent; as a projection operator; Souslin-F sets; \*constituents.

## 422 K-analytic spaces (4.12.04)

Usco-compact relations; K-analytic sets; and Souslin-F sets; \*First Separation Theorem.

## 423 Analytic spaces (13.8.05)

Analytic spaces; are K-analytic spaces with countable networks; Souslin-F sets; Borel measurable functions; injective images of Polish spaces; non-Borel analytic sets; a von Neumann-Jankow selection theorem; \*constituents of coanalytic sets.

## 424 Standard Borel spaces (21.3.08)

Basic properties; isomorphism types; subspaces; Borel measurable actions of Polish groups.

## \*425 Realization of automorphisms (9.8.13)

Extending group actions; Törnquist's theorem.

## Chapter 43: Topologies and measures II

## Introduction (21.8.15)

## 431 Souslin's operation (4.8.15)

The domain of a complete locally determined measure is closed under Souslin's operation; the kernel of a Souslin scheme is approximable from within; Baire-property algebras;  $\omega_1$ -saturated ideals.

## 432 K-analytic spaces (2.10.13)

Topological measures on K-analytic spaces; extensions to Radon measures; expressing Radon measures as images of Radon measures; Choquet capacities.

## 433 Analytic spaces (27.6.10)

Measures on spaces with countable networks; inner regularity of Borel measures; expressing Radon measures as images of Radon measures; measurable and almost continuous functions; the von Neumann-Jankow selection theorem; products; extension of measures on  $\sigma$ -subalgebras; standard Borel spaces.

## 434 Borel measures (18.1.14)

Classification of Borel measures; Radon spaces; universally measurable sets and functions; Borel-measure-compact, Borel-measure-complete and pre-Radon spaces; countable compactness and countable tightness; quasi-dyadic spaces and completion regular measures; first-countable spaces and Borel product measures.

## 435 Baire measures (16.8.08)

Classification of Baire measures; extension of Baire measures to Borel measures (Mařík's theorem); measure-compact spaces.

## 436 Representation of linear functionals (9.5.11)

Smooth and sequentially smooth linear functionals; measures and sequentially smooth functionals; Baire measures; sequential spaces and products of Baire measures; quasi-Radon measures and smooth functionals; locally compact spaces and Radon measures.

## 437 Spaces of measures (5.11.12)

Smooth and sequentially smooth duals; signed measures; embedding spaces of measurable functions in the bidual of  $C_b(X)$ ; vague and narrow topologies; product measures; extreme points; uniform tightness; total variation metric, Kantorovich-Rubinshtein metric; invariant probability measures; Prokhorov spaces.

## 438 Measure-free cardinals (13.12.06)

Measure-free cardinals; point-finite families of sets with measurable unions; measurable functions into metrizable spaces; Radon and measure-compact metric spaces; metacompact spaces; hereditarily weakly  $\theta$ -refinable spaces; when  $c$  is measure-free.

## 439 Examples (7.7.14)

Measures on  $[0, 1]$  not extending to Borel measures; universally negligible sets; Hausdorff measures are rarely semi-finite; a smooth linear functional not expressible as an integral; a first-countable non-Radon space; Baire measures not extending to Borel measures;  $\mathbb{N}^c$  is not Borel-measure-compact; the Sorgenfrey line;  $\mathbb{Q}$  is not a Prokhorov space.

## Chapter 44: Topological groups

## Introduction (18.4.08)

## 441 Invariant measures on locally compact spaces (3.1.06)

Measures invariant under homeomorphisms; Haar measures; measures invariant under isometries.

## 442 Uniqueness of Haar measure (21.3.07)

Two (left) Haar measures are multiples of each other; left and right Haar measures; Haar measurable and Haar negligible sets; the modular function of a group; formulae for  $\int f(x^{-1})dx$ ,  $\int f(xy)dx$ .

## 443 Further properties of Haar measure (14.1.13)

The Haar measure algebra of a group carrying Haar measures; actions of the group on the Haar measure algebra; locally compact groups; actions of the group on  $L^0$  and  $L^p$ ; the bilateral uniformity; Borel sets are adequate; completing the group; expressing an arbitrary Haar measure in terms of a Haar measure on a locally compact group; completion regularity of Haar measure; invariant measures on the set of left cosets of a closed subgroup of a locally compact group; modular functions of subgroups and quotient groups; transitive actions of compact groups on compact spaces.

## 444 Convolutions (23.7.07)

Convolutions of quasi-Radon measures; the Banach algebra of signed  $\tau$ -additive measures; continuous actions and corresponding actions on  $L^0(\nu)$  for an arbitrary quasi-Radon measure  $\nu$ ; convolutions of measures and functions; indefinite-integral measures over a Haar measure  $\mu$ ; convolutions of functions;  $L^p(\mu)$ ; approximate identities; convolution in  $L^2(\mu)$ .

## 445 The duality theorem (20.3.08)

Dual groups; Fourier-Stieltjes transforms; Fourier transforms; identifying the dual group with the maximal ideal space of  $L^1$ ; the topology of the dual group; positive definite functions; Bochner's theorem; the Inversion Theorem; the Plancherel Theorem; the Duality Theorem.

## 446 The structure of locally compact groups (8.10.13)

Finite-dimensional representations separate the points of a compact group; groups with no small subgroups have  $B$ -sequences; chains of subgroups.

## 447 Translation-invariant liftings (7.1.10)

Translation-invariant liftings and lower densities; Vitali's theorem and a density theorem for groups with  $B$ -sequences; Haar measures have translation-invariant liftings.

## 448 Polish group actions (12.4.13)

Countably full local semigroups of  $\text{Aut } \mathfrak{A}$ ;  $\sigma$ -equidecomposability; countably non-paradoxical groups;  $G$ -invariant additive functions from  $\mathfrak{A}$  to  $L^\infty(\mathfrak{C})$ ; measures invariant under Polish group actions (the Nadkarni-Becker-Kechris theorem); measurable liftings of  $L^0$ ; the Borel structure of  $L^0$ ; representing a Borel measurable action on a measure algebra by a Borel measurable action on a Polish space (Mackey's theorem).

## 449 Amenable groups (13.6.13)

Amenable groups; permanence properties; the greatest ambit of a topological group; locally compact amenable groups; Tarski's theorem; discrete amenable groups; isometry-invariant extensions of Lebesgue measure.

## Chapter 45: Perfect measures, disintegrations and processes

## Introduction (5.6.09)

## 451 Perfect, compact and countably compact measures (8.11.07)

Basic properties of the three classes; subspaces, completions, c.l.d. versions, products; measurable functions from compact measure spaces to metrizable spaces; \*weakly  $\alpha$ -favourable spaces.

## 452 Integration and disintegration of measures (6.11.08)

Integrating families of measures;  $\tau$ -additive and Radon measures; disintegrations and regular conditional probabilities; disintegrating countably compact measures; disintegrating Radon measures; \*images of countably compact measures.

## 453 Strong liftings (22.3.10)

Strong and almost strong liftings; existence; on product spaces; disintegrations of Radon measures over spaces with almost strong liftings; Stone spaces; Losert's example.

## 454 Measures on product spaces (6.12.05)

Perfect, compact and countably compact measures on product spaces; extension of finitely additive functions with perfect countably additive marginals; Kolmogorov's extension theorem; measures defined from conditional distributions; distributions of random processes; measures on  $C(T)$  for Polish  $T$ ; completion regular product measures.

## 455 Markov and Lévy processes (18.1.09)

Realization of a Markov process with given conditional distributions; the Markov property for stopping times taking countably many values – disintegrations and conditional expectations; Radon conditional distributions; narrowly continuous and uniformly time-continuous systems of conditional distributions; càdlàg and càllàl functions; extending the distribution of a process to a Radon measure; when the subspace measure on the càdlàg functions is quasi-Radon; general stopping times, hitting times; the strong Markov property; independent increments, Lévy processes; expressing the strong Markov property with an inverse-measure-preserving function.

## 456 Gaussian distributions (19.5.10)

Gaussian distributions and processes; covariance matrices, correlation and independence; supports; universal Gaussian distributions; cluster sets of  $n$ -dimensional processes;  $\tau$ -additivity.

## 457 Simultaneous extension of measures (18.1.13)

Extending families of finitely additive functionals; Strassen's theorem; extending families of measures; examples; the Wasserstein metric.

## 458 Relative independence and relative products (16.1.07)

Relatively independent algebras of measurable sets; relative distributions and relatively independent random variables; relatively independent subalgebras of a probability algebra; relative free products of probability algebras; relative products of probability spaces; existence of relative products.

## 459 Symmetric measures and exchangeable random variables (7.12.10)

Exchangeable families of inverse-measure-preserving functions; De Finetti's theorem; countably compact symmetric measures on product spaces disintegrate into product measures; symmetric quasi-Radon measures; other actions of symmetric groups.

## Concordance to Part I

## Part II

## Chapter 46: Pointwise compact sets of measurable functions

Introduction (26.8.13)

461 Barycenters and Choquet's theorem (9.7.08)

Barycenters; elementary properties; sufficient conditions for existence; closed convex hulls of compact sets; Krein's theorem; existence and uniqueness of measures on sets of extreme points; ergodic functions and extreme measures.

462 Pointwise compact sets of continuous functions (30.6.07)

Angelic spaces; the topology of pointwise convergence on  $C(X)$ ; weak convergence and weakly compact sets in  $C_0(X)$ ; Radon measures on  $C(X)$ ; separately continuous functions; convex hulls.463  $\mathfrak{T}_p$  and  $\mathfrak{T}_m$  (1.2.13)

Pointwise convergence and convergence in measure on spaces of measurable functions; compact and sequentially compact sets; perfect measures and Fremlin's Alternative; separately continuous functions.

464 Talagrand's measure (25.5.13)

The usual measure on  $\mathcal{P}I$ ; the intersection of a sequence of non-measurable filters; Talagrand's measure; the  $L$ -space of additive functionals on  $\mathcal{P}I$ ; measurable and purely non-measurable functionals.

465 Stable sets (28.1.06)

Stable sets of functions; elementary properties; pointwise compactness; pointwise convergence and convergence in measure; a law of large numbers; stable sets and uniform convergence in the strong law of large numbers; convex hulls; stable sets in  $L^0$  and  $L^1$ ; \*R-stable sets.

466 Measures on linear topological spaces (2.8.13)

Quasi-Radon measures for weak and strong topologies; Kadec norms; constructing weak-Borel measures; characteristic functions of measures on locally convex spaces; universally measurable linear operators; Gaussian measures on linear topological spaces.

\*467 Locally uniformly rotund norms (13.1.10)

Locally uniformly rotund norms; separable normed spaces; long sequences of projections; K-countably determined spaces; weakly compactly generated spaces; Banach lattices with order-continuous norms; Eberlein compacta and Schachermeyer's theorem.

## Chapter 47: Geometric measure theory

Introduction (8.4.13)

471 Hausdorff measures (10.2.16)

Metric outer measures; Increasing Sets Lemma; analytic spaces; inner regularity; Vitali's theorem and a density theorem; Howroyd's theorem.

472 Besicovitch's Density Theorem (22.3.11)

Besicovitch's Covering Lemma; Besicovitch's Density Theorem; \*a maximal theorem.

473 Poincaré's inequality (25.7.11)

Differentiable and Lipschitz functions; smoothing by convolution; the Gagliardo-Nirenberg-Sobolev inequality; Poincaré's inequality for balls.

474 The distributional perimeter (17.11.12)

The divergence of a vector field; sets with locally finite perimeter, perimeter measures and outward-normal functions; the reduced boundary; invariance under isometries; isoperimetric inequalities; Federer exterior normals; the Compactness Theorem.

475 The essential boundary (24.1.13)

Essential interior, closure and boundary; the reduced boundary, the essential boundary and perimeter measures; characterizing sets with locally finite perimeter; the Divergence Theorem; calculating perimeters from cross-sectional counts, and an integral-geometric formula; Cauchy's Perimeter Theorem; the Isoperimetric Theorem for convex sets.

476 Concentration of measure (10.11.07)

Vietoris and Fell topologies; concentration by partial reflection; concentration of measure in  $\mathbb{R}^r$ ; the Isoperimetric Theorem; concentration of measure on spheres.

477 Brownian motion (2.1.10)

Brownian motion as a stochastic process; Wiener measure on  $C([0, \infty)_0)$ ; \*as a limit of random walks; Brownian motion in  $\mathbb{R}^r$ ; invariant transformations of Wiener measure on  $C([0, \infty[; \mathbb{R}^r)_0$ ; Wiener measure is strictly positive; the strong Markov property; hitting times; almost every Brownian path is nowhere differentiable; almost every Brownian path has zero two-dimensional Hausdorff measure.

478 Harmonic functions (4.6.09)

Harmonic and superharmonic functions; a maximal principle;  $f$  is superharmonic iff  $\nabla^2 f \leq 0$ ; the Poisson kernel and harmonic functions with given values on a sphere; smoothing by convolution; Brownian motion and Dynkin's formula; Brownian motion and superharmonic functions; recurrence and divergence of Brownian motion; harmonic measures and Dirichlet's problem; disintegrating harmonic measures over intermediate boundaries; hitting probabilities.



## 479 Newtonian capacity (15.2.10)

Defining Newtonian capacity from Brownian hitting probabilities, and equilibrium measures from harmonic measures; submodularity and sequential order-continuity; extending Newtonian capacity to Choquet-Newton capacity; Newtonian potential and energy of a Radon measure; Riesz kernels and their Fourier transforms; energy and  $(r-1)$ -potentials; alternative definitions of capacity and equilibrium measures; analytic sets of finite capacity; polar sets; general sets of finite capacity; Brownian hitting probabilities and equilibrium potentials; Hausdorff measure; self-intersecting Brownian paths; a discontinuous equilibrium potential; yet another definition of Newtonian capacity; capacity and volume; a measure on the set of closed subsets of  $\mathbb{R}^r$ .

## Chapter 48: Gauge integrals

## Introduction (9.5.11)

## 481 Tagged partitions (4.9.09)

Tagged partitions and Riemann sums; gauge integrals; gauges; residual sets; subdivisions; examples (the Riemann integral, the Henstock integral, the symmetric Riemann-complete integral, the McShane integral, box products, the approximately continuous Henstock integral).

## 482 General theory (11.5.10)

Saks-Henstock lemma; when gauge-integrable functions are measurable; when integrable functions are gauge-integrable;  $L_\nu(f \times \chi_H)$ ; improper integrals; integrating derivatives; B.Levi's theorem; Fubini's theorem.

## 483 The Henstock integral (6.9.10)

The Henstock and Lebesgue integrals; indefinite Henstock integrals; Saks-Henstock lemma; Fundamental Theorem of Calculus; the Perron integral;  $\|f\|_H$  and  $HL^1$ ;  $AC_*$  and  $ACG_*$  functions.

## 484 The Pfeffer integral (21.1.10)

The Tamanini-Giacomelli theorem; a family of tagged-partition structures; the Pfeffer integral; the Saks-Henstock indefinite integral of a Pfeffer integrable function; Pfeffer's Divergence Theorem; differentiating the indefinite integral; invariance under lipeomorphisms.

## Chapter 49: Further topics

## Introduction (31.8.09)

## 491 Equidistributed sequences (14.8.08)

The asymptotic density ideal  $\mathcal{Z}$ ; equidistributed sequences; when equidistributed sequences exist;  $\mathfrak{I} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ ; effectively regular measures; equidistributed sequences and induced embeddings of measure algebras in  $\mathfrak{I}$ .

## 492 Combinatorial concentration of measure (30.12.06)

The Hamming metric; concentration of measure in product spaces; concentration of measure in permutation groups.

## 493 Extremely amenable groups (4.1.13)

Extremely amenable groups; concentrating additive functionals; measure algebras under  $\Delta$ ;  $L^0$ ; isometry groups of spheres in inner product spaces; locally compact groups.

## 494 Groups of measure-preserving automorphisms (17.5.13)

Weak and uniform topologies on  $\text{Aut}_{\bar{\mu}}\mathfrak{A}$ ; a weakly mixing automorphism which is not mixing; full subgroups and fixed-point subalgebras; extreme amenability; automatic continuity; algebraic cofinality.

## 495 Poisson point processes (20.12.08)

Poisson distributions; Poisson point processes; disintegrations; transforming disjointness into stochastic independence; representing Poisson point processes by Radon measures; exponential distributions and Poisson point processes on  $[0, \infty[$ .

## 496 Maharam submeasures (27.5.09)

Submeasures; totally finite Radon submeasures; Souslin's operation; (K-)analytic spaces; product submeasures.

## 497 Tao's proof of Szemerédi's theorem (7.12.10)

T-removable intersections; and relative independence; permutation-invariant measures on  $\mathcal{P}([I]^{<\omega})$ ; and T-removable intersections; the Hypergraph Removal Lemma; Szemerédi's theorem; a multiple recurrence theorem.

## 498 Cubes in product spaces (4.9.08)

Subsets of measure algebras with non-zero infima; product sets included in given sets of positive measure.

## Appendix to Volume 4

## Introduction (3.9.13)

## 4A1 Set theory (27.1.13)

Cardinals; closed cofinal sets and stationary sets;  $\Delta$ -system lemma; free sets; Ramsey's theorem; the Marriage Lemma again; filters; normal ultrafilters; Ostaszewski's  $\clubsuit$ ; the size of  $\sigma$ -algebras.

## 4A2 General topology (21.4.13)

Glossary; general constructions;  $F_\sigma$ ,  $G_\delta$ , zero and cozero sets; weight; countable chain condition; separation axioms; compact and locally compact spaces; Lindelöf spaces; Stone-Čech compactifications; uniform spaces; first-countable, sequential, countably tight, metrizable spaces; countable networks; second-countable spaces; separable metrizable spaces; Polish spaces; order topologies; Vietoris and Fell topologies.

4A3 Topological  $\sigma$ -algebras (11.10.07)

Borel  $\sigma$ -algebras; measurable functions; hereditarily Lindelöf spaces; second-countable spaces; Polish spaces;  $\omega_1$ ; Baire  $\sigma$ -algebras; product spaces; compact spaces; Baire-property algebras; cylindrical  $\sigma$ -algebras; spaces of càdlàg functions.

## 4A4 Locally convex spaces (19.6.13)

Linear topological spaces; locally convex spaces; Hahn-Banach theorem; normed spaces; inner product spaces; max-flow min-cut theorem.

## 4A5 Topological groups (4.8.13)

Group actions; topological groups; uniformities; quotient groups; metrizable groups.

## 4A6 Banach algebras (8.12.10)

Stone-Weierstrass theorem (fourth form); multiplicative linear functionals; spectral radius; invertible elements; exponentiation; Arens multiplication.

## 4A7 'Later editions only' (5.10.13)

Items recently interpolated into other volumes.

## Concordance to Part II

## References for Volume 4 (9.2.13)

## Index to Volumes 1-4

Principal topics and results

General index

## Volume 5: Set-Theoretic Measure Theory

## Part I

## Introduction to Volume 5 (9.1.15)

## Chapter 51: Cardinal functions

## Introduction (3.1.15)

## 511 Definitions (10.10.13)

Cardinal functions of partially ordered sets, topological spaces, Boolean algebras and measures; precalibers; ideals of sets.

## 512 Galois-Tukey connections (27.11.13)

Supported relations; Galois-Tukey connections; covering numbers, additivity, saturation, linking numbers; simple products; sequential composition of supported relations.

## 513 Partially ordered sets (23.2.14)

Saturation and the Erdős-Tarski theorem; cofinalities of cardinal functions; Tukey functions; Tukey equivalence of directed sets;  $\sigma$ -additivities; \*metrizable compactly based directed sets; \*measurable Tukey functions.

## 514 Boolean algebras (16.5.14)

Stone spaces; cardinal functions of Boolean algebras; order-preserving functions of Boolean algebras; regular open algebras; regular open algebras of partially ordered sets; finite-support products.

## 515 The Balcar-Franěk theorem (29.8.14)

Boolean-independent sets; free subalgebras; refining systems; the 0Balcar-Franěk theorem; the Pierce-Koppelberg theorem; regular open algebras of powers of  $\{0, 1\}$ .

## 516 Precalibers (9.10.14)

Precalibers of supported relations; and Galois-Tukey connections; partially ordered sets, topological spaces and Boolean algebras; saturation and linking numbers; saturation of product spaces.

## 517 Martin numbers (14.11.14)

Characterizations of  $\mathfrak{m}(P)$ ; regular open algebras, Stone spaces and Novák numbers; precalibers, saturation and weak distributivity;  $\mathfrak{m}$ ,  $\mathfrak{m}_{\text{countable}}$ ,  $\mathfrak{p}$  and  $\mathfrak{m}_{\aleph}$ .

## 518 Freese-Nation numbers (24.12.14)

Freese-Nation numbers of partially ordered sets; Boolean algebras; upper and lower bounds for  $\text{FN}(\mathfrak{A})$  under special axioms; tight filtrations and Geschke systems; large algebras are not tightly filtered.

## Chapter 52: Cardinal functions of measure theory

## Introduction (7.10.13)

## 521 Basic theory (3.3.14)

$\text{add } \mu$  and  $\text{add } \mathcal{N}(\mu)$ ; measure algebras and function spaces; the topological density of a measure algebra; shrinking numbers;  $\pi(\mu)$ ; subspace measures, direct sums, image measures, products; perfect measures, compact measures; complete locally determined measure spaces and strict localizability; magnitudes; bounds on the Maharam type of a measure; countably separated spaces; measurable additive functionals on  $\mathcal{P}I$ .

## 522 Cichoń's diagram (31.3.10)

The cardinals  $\mathfrak{b}$  and  $\mathfrak{d}$ ; inequalities linking them with the additivity, cofinality, uniformity and covering numbers of measure and category in the real line; the localization relation;  $\mathfrak{m}_{\text{countable}}$  and other Martin numbers;  $\text{FN}(\mathcal{P}\mathbb{N})$ ; cofinalities of the cardinals.

523 The measure of  $\{0, 1\}^I$  (23.6.10)

The additivity, covering number, uniformity, shrinking number and cofinality of the usual measure on  $\{0, 1\}^I$ ; Kraszewski's theorems; what happens with GCH.

## 524 Radon measures (29.9.10)

The additivity, covering number, uniformity and cofinality of a Radon measure;  $\ell^1(\kappa)$  and localization; cardinal functions of measurable algebras; countably compact and quasi-Radon measures.

## 525 Precalibers of measure algebras (11.9.13)

Precalibers of measurable algebras; measure-precalsibers of probability algebras; (quasi-)Radon measure spaces; under GCH; precaliber triples  $(\kappa, \kappa, k)$ .

## 526 Asymptotic density zero (24.1.14)

$\mathcal{Z}$  is metrizable compactly based;  $\mathbb{N}^{\mathbb{N}} \preceq_{\text{T}} \mathcal{Z} \preceq_{\text{T}} \ell^1 \preceq_{\text{GT}} \mathbb{N}^{\mathbb{N}} \times \mathcal{Z}$ ; cardinal functions of  $\mathcal{Z}$ ; meager sets and nowhere dense sets; sets with negligible closures;  $\mathcal{N}\text{wd} \not\preceq_{\text{T}} \mathcal{Z}$  and  $\mathcal{Z} \not\preceq_{\text{T}} \mathcal{N}\text{wd}$ .

## 527 Skew products of ideals (10.1.10)

$\mathcal{N} \times_{\mathcal{B}} \mathcal{N}$  and Fubini's theorem;  $\mathcal{M} \times_{\mathcal{B}} \mathcal{M}$  and the Kuratowski-Ulam theorem;  $\mathcal{M} \times_{\mathcal{B}} \mathcal{N}$ ;  $\mathcal{N} \times_{\mathcal{B}} \mathcal{M}$ ; harmless Boolean algebras.

## 528 Amoeba algebras (10.2.11)

Amoeba algebras; variable-measure amoeba algebras; isomorphic amoeba algebras; regular embeddings of amoeba algebras; localization posets; Martin numbers and other cardinal functions; algebras with countable Maharam type.

## 529 Further partially ordered sets of analysis (26.5.11)

$L^p$  and  $L^0$ ;  $L$ -spaces; the localization poset and the regular open algebra of  $\{0, 1\}^c$ ; the Novák numbers  $n(\{0, 1\}^I)$ ; the reaping numbers  $\mathfrak{r}(\omega_1, \lambda)$ .

## Chapter 53: Topologies and measures III

## Introduction (30.8.14)

## 531 Maharam types of Radon measures (19.2.11)

Topological and measure-theoretic cardinal functions; the set  $\text{Mah}_{\mathbb{R}}(X)$  of Maharam types of homogeneous Radon measures on  $X$ ;  $\text{Mah}_{\mathbb{R}}(X)$ , precalibers and continuous surjections onto  $[0, 1]^{\kappa}$ ;  $\text{Mah}_{\mathbb{R}}(X)$  and  $\chi(X)$ ; a perfectly normal hereditarily separable space under CH; when  $\mathfrak{m}_{\aleph} > \omega_1$ .

532 Completion regular measures on  $\{0, 1\}^I$  (1.6.13)

The set  $\text{Mah}_{\text{cfr}}(X)$  of Maharam types of homogeneous completion regular Radon measures on  $X$ ; products of quasi-dyadic spaces; convexity of the relation ' $\lambda \in \text{Mah}_{\text{cfr}}(\{0, 1\}^{\kappa})$ '; the measure algebra of  $\{0, 1\}^{\lambda}$ ;  $\mathfrak{d}$ ,  $\text{cov } \mathcal{N}$ ,  $\text{FN}(\mathcal{P}\mathbb{N})$ ,  $\text{add } \mathcal{N}$  and the case  $\lambda = \omega$ ;  $\square$ , Chang's conjecture and the case  $\text{cf } \lambda = \omega$ .

## 533 Special topics (4.1.14)

$\text{add } \mathcal{N}$  and (quasi-)Radon measures of countable Maharam type; uniformly regular measures; when  $\mathbb{R}^{\kappa}$  is measure-compact.

## 534 Hausdorff measures and strong measure zero (1.8.10)

Cardinal functions of Hausdorff measures; strong measure zero in uniform spaces; Rothberger's property;  $\sigma$ -compact groups; non  $\text{Smz}$ ,  $\text{add Smz}$ ;  $\text{Smz}$ -equivalence; uncountable sets with strong measure zero.

## 535 Liftings (1.6.11)

Liftings of non-complete measure spaces; Baire liftings for usual measures on  $\{0, 1\}^{\kappa}$ ; tightly  $\omega_1$ -filtered measure algebras; Mokobodzki's theorems; strong Borel liftings; Borel liftings for Radon measures on metrizable spaces; linear liftings; problems.

## 536 Alexandra Bellow's problem (20.2.12)

The problem; consequences of a negative solution.

## 537 Sierpiński sets, shrinking numbers and strong Fubini theorems (12.8.13)

Sierpiński and strongly Sierpiński sets; entangled totally ordered sets; non-ccc products; scalarly measurable functions; repeated integrals of separately measurable functions; changing the order of integration in multiply repeated integrals;  $\text{shr}^+$ ,  $\text{cov}$  and repeated upper and lower integrals.

## 538 Filters and limits (18.2.14)

Filters on  $\mathbb{N}$ ; the Rudin-Keisler ordering; products and iterated products; Ramsey ultrafilters; measure-centering ultrafilters; extending perfect measures with measure-centering ultrafilters; Benedikt's theorem; measure-converging filters; the Fatou property; medial functionals and limits.

## 539 Maharam submeasures (24.5.14)

Maharam algebras; Maharam-algebra topology, pre-ordered set of partitions of unity, weak distributivity,  $\pi$ -weight, centering number, precalibers; null ideals of Maharam submeasures; splitting reals; Quickert's ideal; Todorčević's  $p$ -ideal dichotomy; a consistent characterization of Maharam algebras; Souslin algebras; reflection principles; exhaustivity rank, Maharam submeasure rank.

## Concordance to Part I

## Part II

## Chapter 54: Real-valued-measurable cardinals

## Introduction (23.10.14)

## 541 Saturated ideals (10.12.12)

$\kappa$ -saturated  $\kappa^+$ -additive ideals;  $\kappa$ -saturated  $\kappa$ -additive ideals;  $\text{Tr}_T(X; Y)$ ; normal ideals;  $\kappa$ -saturated normal ideals; two-valued-measurable and weakly compact cardinals; the Tarski-Solovay dichotomy;  $\text{cov}_{\text{Sh}}(2^\gamma, \kappa, \delta^+, \delta)$ .

## 542 Quasi-measurable cardinals (8.7.13)

Definition and basic properties;  $\omega_1$ -saturated  $\sigma$ -ideals; and pcf theory; and cardinal arithmetic; cardinals of quotient algebras; cofinality of  $[\kappa]^{<\theta}$ ; cofinality of product partial orders.

## 543 The Gitik-Shelah theorem (11.11.13)

Real-valued-measurable and atomlessly-measurable cardinals; Ulam's dichotomy; a Fubini inequality; Maharam types of witnessing probabilities; compact measures, inverse-measure-preserving functions and extensions of measures.

## 544 Measure theory with an atomlessly-measurable cardinal (31.12.13)

Covering numbers of null ideals; repeated integrals; measure-precalibers; functions from  $[\kappa]^{<\omega}$  to null ideals; Sierpiński sets; uniformities of null ideals; weakly  $\Pi_1^1$ -indescribable cardinals; Cichoń's diagram.

## 545 PMEA and NMA (10.2.14)

The product measure extension axiom; the normal measure axiom; Boolean algebras with many measurable subalgebras.

546 Power set  $\sigma$ -quotient algebras (10.5.14)

Power set  $\sigma$ -quotient algebras; harmless algebras and skew products of ideals; the Gitik-Shelah theorem for category algebras; the category algebra  $\mathfrak{G}_\omega$  of  $\{0, 1\}^\omega$ ; completed free products of probability algebras with  $\mathfrak{G}_\omega$ .

## 547 Disjoint refinements of sequences of sets (26.8.14)

Refining a sequence of sets to a disjoint sequence without changing outer measures; other results on simultaneous partitions.

## Chapter 55: Possible worlds

## Introduction (7.1.15)

## 551 Forcing with quotient algebras (2.12.13)

Measurable spaces with negligibles; associated forcing notions; representing names for members of  $\{0, 1\}^I$ ; representing names for Baire sets in  $\{0, 1\}^I$ ; the usual measure on  $\{0, 1\}^I$ ; re-interpreting Baire sets in the forcing model; representing Baire measurable functions; representing measure algebras; iterated forcing; extending filters.

## 552 Random reals I (29.1.14)

Random real forcing notions; calculating  $2^\kappa$ ;  $\mathfrak{b}$  and  $\mathfrak{d}$ ; preservation of outer measure; Sierpiński sets; cardinal functions of the usual measure on  $\{0, 1\}^\lambda$ ; Carlson's theorem on extending measures; iterated random real forcing.

## 553 Random reals II (3.5.14)

Rothberger's property; non-scattered compact sets; Haydon's property; rapid  $p$ -point ultrafilters; products of ccc partially ordered sets; Aronszajn and Souslin trees; medial limits; universally measurable sets.

## 554 Cohen reals (2.9.14)

Calculating  $2^{\aleph_1}$ ; Lusin sets; precaliber pairs of measure algebras; Freese-Nation numbers; Borel liftings for Lebesgue measure.

## 555 Solovay's construction of real-valued-measurable cardinals (12.4.08)

Measurable cardinals are quasi-measurable after ccc forcing, real-valued-measurable after random real forcing; Maharam-type-homogeneity; covering number of product measure; power set  $\sigma$ -quotient algebras can have countable centering number or Maharam type; supercompact cardinals and the normal measure axiom.

## 556 Forcing with Boolean subalgebras (3.1.15)

Forcing names over a Boolean subalgebra; Boolean operations, ring homomorphisms; when the subalgebra is regularly embedded; upper bounds, suprema, saturation, Maharam type; quotient forcing; Dedekind completeness;  $L^0$ ; probability algebras; relatively independent subalgebras; strong law of large numbers; Dye's theorem; Kawada's theorem; the Dedekind completion of the asymptotic density algebra.

## Chapter 56: Choice and Determinacy

## Introduction (3.1.15)

## 561 Analysis without choice (8.9.13)

Elementary facts; Tychonoff's theorem; Baire's theorem; Stone's theorem; Haar measure; Kakutani's representation of  $L$ -spaces; Hilbert space.

## 562 Borel codes (20.10.13)

Coding sets with trees; codable Borel sets; in a Polish space, a set is analytic and coanalytic iff it is a codable Borel set; resolvable sets are self-coding; codable families of codable sets; codable Borel functions, codable Borel equivalence; real-valued functions; codable families of codable functions; codable Baire sets and functions for general topological spaces.

## 563 Borel measures without choice (3.12.13)

Borel-coded measures on second-countable spaces; construction of measures; inner and outer regularity; analytic sets are universally measurable; Baire-coded measures on general topological spaces; measure algebras.

## 564 Integration without choice (9.2.14)

Integration with respect to Baire-coded measures; convergence theorems for codable sequences of functions; Riesz representation theorem; when  $L^1$  is a Banach space; Radon-Nikodým theorem; conditional expectations; products of measures on second-countable spaces.

## 565 Lebesgue measure without choice (25.4.14)

Construction of Lebesgue measure as a Borel-coded measure; Vitali's theorem; Fundamental Theorem of Calculus; Hausdorff measures as Borel-coded measures.

## 566 Countable choice (22.8.14)

Basic measure theory survives; exhaustion;  $\sigma$ -finite spaces and algebras; atomless countably additive functionals; Vitali's theorem; bounded additive functionals; infinite products without DC; topological product measures; the Loomis-Sikorski theorem; the usual measure on  $\{0,1\}^{\mathbb{N}}$  and its measure algebra; weak compactness; automorphisms of measurable algebras; Baire  $\sigma$ -algebras; dependent choice.

## 567 Determinacy (31.10.14)

Infinite games; closed games are determined; the axiom of determinacy;  $AC(\mathbb{R};\omega)$ ; universal measurability and the Baire property; automatic continuity of group homomorphisms and linear operators; countable additivity of functionals; reflexivity of  $L$ -spaces;  $\omega_1$  is two-valued-measurable; surjections from  $\mathcal{P}\mathbb{N}$  onto ordinals; two-valued-measurable cardinals and determinacy in ZFC; measurability of PCA sets.

## Appendix to Volume 5

## Introduction (21.11.13)

## 5A1 Set theory (23.2.11)

Ordinal and cardinal arithmetic; trees; cofinalities;  $\Delta$ -systems and free sets; partition calculus; transversals.

## 5A2 Pcf theory (6.6.11)

Reduced products of partially ordered sets; cofinalities of reduced products;  $\text{cov}_{\text{Sh}}(\alpha, \beta, \gamma, \delta)$ ;  $\Theta(\alpha, \gamma)$ .

## 5A3 Forcing (18.4.08)

Forcing notions; forcing languages; the forcing relation; the forcing theorem; Boolean truth values; names for functions; regular open algebras; discriminating names;  $L^0$  and names for real numbers; forcing with Boolean algebras; ordinals and cardinals; iterated forcing; Martin's axiom; countably closed forcings.

## 5A4 General topology (15.8.13)

Cardinal functions; compactness; Vietoris topologies; category and the Baire property; normal and paracompact spaces.

## 5A5 Real analysis (3.10.13)

Real-entire functions.

## 5A6 Special axioms (4.8.10)

GCH,  $V=L$ ,  $0^\sharp$  and the Covering Lemma, squares, Chang's transfer principle, Todorćević's  $p$ -ideal dichotomy, the filter dichotomy.

## References for Volume 5 (6.9.13)

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## Index (31.12.10)

## Errata

Volume 1, 2000 edition (27.2.10)

Volume 1, 2001 printing (14.1.10)

Volume 1, 2004 printing (27.11.13)

Volume 2, 2001 edition (30.12.09)

Volume 2, 2003 printing (14.6.11)

Volume 2, 2010 edition (29.11.10)

Volume 3, 2002 edition (7.8.15)

Volume 3, 2004 printing (14.8.10)

Volume 4, 2003 edition (21.5.11)

Volume 4, 2006 printing (20.5.10)

Volume 5, 2008 edition (21.2.11)