Fremlin 85

Séminaire Initiation à l'Analyse
G. CHOQUET, M. ROGALSKI, J. SAINT RAYMOND
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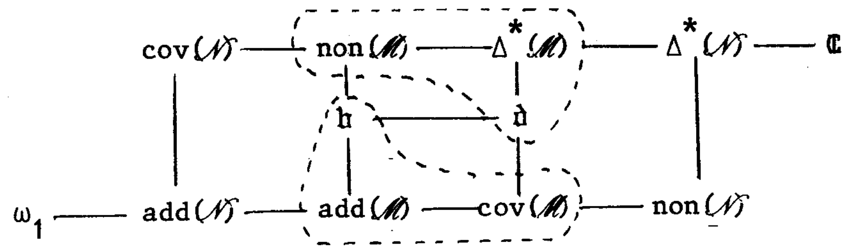
CICHOŃ'S DIAGRAM

par D.H. FREMLIN

In this note I discuss the relationships between ten cardinal numbers lying between ω_1 and $\mathbb{C}$, related to category and measure. I learnt this material from J. Cichoń (University of Wrocław), who was closely involved with its evolution into its present form.

The principal results are encapsulated in the diagram of § 1 ; this is interpreted in §§ 2-3 and proved in §§ 4-17.

1. THE DIAGRAM.



2. DEFINITIONS. (a) $\mathcal{N}$ is the ideal of Lebesgue negligible subsets of $\mathbb{R}$: $\mathcal{M}$ is the ideal of meagre subsets of $\mathbb{R}$.

(b) For $\mathcal{I} = \mathcal{M}$ or $\mathcal{I} = \mathcal{N}$,

$\mathrm{add}(\mathcal{I}) = \min\{\#(\mathcal{E}) : \mathcal{E} \subseteq \mathcal{I}, \bigcup\mathcal{E} \notin \mathcal{I}\}$

(the additivity of the ideal $\mathcal{I}$) ;

$\mathrm{cov}(\mathcal{I}) = \min\{\#(\mathcal{E}) : \mathcal{E} \subseteq \mathcal{I}, \bigcup\mathcal{E} = \mathbb{R}\}$;

$\mathrm{non}(\mathcal{I}) = \min\{\#(A) : A \subseteq R , A \notin \mathcal{I}\}$.

(c) For any partially ordered set P ,

$\Delta^*(P) = \min\{\#(Q) : Q \subseteq P$ is cofinal with $P\}$,

$\Delta_*(P) = \min\{\#(Q) : Q \subseteq P$ is coinitial with $P\}$.

Thus if $\mathcal{I} = \mathcal{M}$ of $\mathcal{I} = \mathcal{N}$,

$\Delta^*(\mathcal{I}) = \min\{\#(\mathcal{E}) : \mathcal{I} = \bigcup_{E\in\mathcal{E}} \mathcal{P}E\}$.

(d) For $f , g \in \mathbb{N}^{\mathbb{N}}$ say that $f \leq^* g$ (" g eventually dominates f ") if $\{n : g(n) < f(n)\}$ is finite. Now

$\mathfrak{b} = \min\{\#(B) : B \subseteq \mathbb{N}^{\mathbb{N}} , \nexists\, g \in \mathbb{N}^{\mathbb{N}} , f \leq^* g \quad \forall f \in B\}$,

$\mathfrak{d} = \min\{\#(D) : D \subseteq \mathbb{N}^{\mathbb{N}} , \forall f \in \mathbb{N}^{\mathbb{N}} \ \exists g \in D , f \leq^* g\}$

$= \Delta^*(\mathbb{N}^{\mathbb{N}})$.

3. INTERPRETATION OF THE DIAGRAM. The cardinals increase (not necessarily strictly) from south-west to north-east ; thus

$$\omega_1 \leq \mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{M}) \leq \mathfrak{b} \leq \mathfrak{d} \leq \Delta^*(\mathcal{M}) \leq \dots ,$$

but nothing is said about the relative sizes of $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$. The two closed curves, indicated by lines of dashes, represent the further known relations

$$\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) , \ \Delta^*(\mathcal{M}) = \max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) .$$

4. *PROPOSITION*. *(a)* $\omega_1 \leq \mathrm{add}(\mathcal{N})$. *(b)* $\Delta^*(\mathcal{N}) \leq \mathfrak{c}$.

Proof (a) This says just that Lebesgue measure is countably additive.

(b) This is because $\mathcal{B} \cap \mathcal{N}$ is cofinal with $\mathcal{N}$, where $\mathcal{B}$ is the algebra of Borel subsets of $\mathbb{R}$, and $\#(\mathcal{B}) = \mathfrak{c}$.

5. *PROPOSITION*. *For* $\mathcal{I} = \mathcal{M}$ *or* $\mathcal{I} = \mathcal{N}$,

(a) $\mathrm{add}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{I})$, *(b)* $\mathrm{non}(\mathcal{I}) \leq \Delta^*(\mathcal{I})$.

Proof (a) This is because $\mathbb{R} \notin \mathcal{I}$. (b) Let $\mathcal{E} \subseteq \mathcal{I}$ be a cofinal set of cardinal $\Delta^*(\mathcal{I})$; for each $E \in \mathcal{E}$ choose $\alpha_E \in R \setminus E$; then $\{\alpha_E : E \in \mathcal{E}\} \notin \mathcal{I}$.

6. LOCALIZATION. I write

$$\mathcal{S}_0 = \{S : S \subseteq \mathbb{N} \times \mathbb{N} , \#\{j : (i,j) \in S\} \leq i \quad \forall\, i \in \mathbb{N}\} .$$

For $f \in \mathbb{N}^{\mathbb{N}}$, $S \subseteq \mathbb{N} \times \mathbb{N}$ I write $f \subseteq^* S$ if

$$\{i : i \in \mathbb{N} , (i,f(i)) \notin S\}$$

is finite.

7. *LEMMA. There are functions* $f \mapsto V_f : \mathbb{N}^{\mathbb{N}} \to \mathcal{N}$ *and* $E \to R_E : \mathcal{N} \to \mathcal{S}_0$ *such that* $f \subseteq^* R_E$ *whenever* $V_f \subseteq E$.

Proof. Write

$$\mathcal{S}_1 = \{S : S \subseteq \mathbb{N} \times \mathbb{N} , \#\{j : (i,j) \in S\} \leq (i+1)^2 \ \forall\, i \in \mathbb{N}\} .$$

Let μ_L be Lebesgue measure. Take any μ_L-independent double sequence $\langle G_{ij} \rangle_{i,j \in \mathbb{N}}$ of open subsets of $]0,1[$ such that $\mu_L G_{ij} = 1/(i+1)^2$ for all $i,j \in \mathbb{N}$. For each $f \in \mathbb{N}^{\mathbb{N}}$ set

$$V'_f = \bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} G_{i,f(i)} .$$

Because $\Sigma_{i \in \mathbb{N}} \mu_L G_{i,f(i)} < \infty$, $V'_f \in \mathcal{N}$. For each $E \in \mathcal{N}$ choose a compact non-empty set $K_E \subseteq \,]0,1[\setminus E$ which is "supporting", i.e. such that $\mu_L(K_E \cap U) > 0$ whenever U is an open set meeting K_E . Enumerate as $\langle U_n^E \rangle_{n \in \mathbb{N}}$ a base for the relative topology of K_E which does not contain $\emptyset$. Set

$$A(E,n,i) = \{j : U_n^E \cap G_{ij} = \emptyset\} \ \forall\, i,n \in \mathbb{N} .$$

Then

$$0 < \mu_L U_n^E \leq \Pi_{i \in \mathbb{N}} \Pi_{j \in A(E,n,i)} \mu_L(]0,1[\setminus G_{ij})$$

because the G_{ij} are independent subsets of the probability space $]0,1[$; that is,

$$0 < \Pi_{i \in \mathbb{N}} \left(1 - \frac{1}{(i+1)^2}\right)^{\#(A(E,n,i))} ,$$

and $\Sigma_{i \in \mathbb{N}} \#(A(E,n,i))/(i+1)^2 < \infty$. Let $k(E,n) \in \mathbb{N}$ be such that $\#(A(E,n,i)) \leq (i+1)^2/2^{n+1}$ for $i \geq k(E,n)$. Set

$$R'_E = \bigcup_{n \in \mathbb{N}} \{(i,j) : i \geq k(E,n) , j \in A(E,n,i)\} .$$

It is easy to see that $R'_E \in \mathcal{S}_1$.

Now suppose that $f \in \mathbb{N}^{\mathbb{N}}$ and $E \in \mathcal{N}$ and $V'_f \subseteq E$. Then

$$K_E \cap \bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} G_{i,f(i)} = \emptyset ;$$

by Baire's theorem, there are m , $n \in \mathbb{N}$ such that

$$U_n^E \cap \bigcup_{i\geq m} G_{i,f(i)} = \emptyset .$$

Thus $f(i) \in A(E,n,i)$ for every $i \geq m$, and $(i,f(i)) \in R'_E$ for every $i \geq \max(m,k(E,n))$; so that $f \subseteq^* R'_E$.

This, in effect, proves the lemma with $\mathcal{S}_1$ in place of $\mathcal{S}_o$. To convert to $\mathcal{S}_o$, set

$$L(n) = \{i : i \in \mathbb{N} , (n+1)^2 \leq i < (n+2)^2\}$$

and let $\theta_n : \mathbb{N}^{L(n)} \to \mathbb{N}$ be in injection, for each $n \in \mathbb{N}$. For $E \in \mathcal{N}$, set

$$R_E = \bigcup_{n\in\mathbb{N}}\{(i,j) : i \in L(n) , \exists\, h \in \mathbb{N}^{L(n)} , \; h(i) = j \text{ and } (n,\theta_n(h)) \in R'_E\} ;$$

then $R_E \in \mathcal{S}_o$ because $R'_E \in \mathcal{S}_1$. For $f \in \mathbb{N}^{\mathbb{N}}$ set $V_f = V'_g$ where $g(n) = \theta_n(f \restriction L(n))$ for each $n \in \mathbb{N}$. If $V_f \subseteq E$ then $g \subseteq^* R'_E$, i.e. there is an $m \in \mathbb{N}$ such that $(n,\theta_n(f \restriction L(n))) \in R'_E$ for every $n \geq m$; now $(i,f(i)) \in R_E$ for every $i \geq (m+1)^2$, so $f \subseteq^* R_E$.

8. ___LEMMA___. *Let $U \subseteq \mathbb{R}$ be a non-empty open set, and $n \in \mathbb{N}$. Then there is a countable family $\mathcal{V}$ of open subsets of U such that (i) every dense open subset of $\mathbb{R}$ includes some member of $\mathcal{V}$ (ii) $\bigcap_{i\leq n} V_i \neq \emptyset$ whenever $V_o,\ldots,V_n \in \mathcal{V}$.*

__Proof__. Let $\langle U_n \rangle_{n\in\mathbb{N}}$ enumerate a countable base for the relative topology of U , not containing $\emptyset$, and closed under finite unions. For $k \in \mathbb{N}$, set

$$A_k = \{n : n > k , U_n \cap \bigcap_{i\in I} U_i \neq \emptyset \text{ whenever } I \subseteq k+1 \text{ and } \bigcap_{i\in I} U_i \neq \emptyset\} .$$

Set

$$\mathcal{V} = \{\bigcup_{i\leq n} U_{m_i} : m_o \in \mathbb{N} , m_{i+1} \in A_{m_i} \; \forall\, i < n\} .$$

(i) If G is a dense open subset of $\underset{\sim}{R}$, then (because $\{U_n : n \in \underset{\sim}{N}\}$ is closed under finite unions) A_k meets $\{n : U_n \subseteq G\}$ for every $k \in \underset{\sim}{N}$; so we can choose $< m_i >_{i \leq n}$ inductively such that

$$U_{m_i} \subseteq G \quad \forall\, i \leq n \ , \ m_{i+1} \in A_{m_i} \quad \forall\, i < n \ .$$

Now $G \supseteq \bigcup_{i \leq n} U_{m_i} \in \mathscr{V}$.

(ii) If $V_o, \ldots, V_n \in \mathscr{V}$, express each V_j as $\bigcup_{i \leq n} U_{m(j,i)}$ where $m(j,i+1) \in A_{m(j,i)}$ for $i < n$; re-order the V_j if necessary so that $m(i,i) \leq m(j,i)$ if $i \leq j \leq n$. In this case

$$m(i+1,i+1) \in A_{m(i+1,i)} \subseteq A_{m(i,i)} \quad \forall\, i < n$$

and (inducing on k)

$$\bigcap_{i \leq k} U_{m(i,i)} \neq \emptyset \quad \forall\, k \leq n \ .$$

So

$$\bigcap_{j \leq n} V_j \supseteq \bigcap_{j \leq n} U_{m(j,j)} \neq \emptyset \ ,$$

as required.

9. <u>LEMMA</u>. *There are functions* $F \mapsto g_F : \mathscr{M} \to \underset{\sim}{N}^{\underset{\sim}{N}}$ *and* $S \mapsto W_S : \mathscr{S}_o \to \mathscr{M}$ *such that* $F \subseteq W_S$ *whenever* $g_F \subseteq^* S$.

<u>Proof</u>. Let $< U_n >_{n \in N}$ enumerate a countable base for the topology of $\underset{\sim}{R}$, not containing $\emptyset$. For each $n \in \underset{\sim}{N}$, construct $\mathscr{V}_n$ from U_n and n as in Lemma 8. Let $< V(n,m) >_{m \in \underset{\sim}{N}}$ enumerate $\mathscr{V}_n$.

For $F \in \mathscr{M}$ express F as $\bigcup_{n \in N} H_n^F$ where $< H_n^F >_{n \in \underset{\sim}{N}}$ is an increasing sequence of nowhere dense sets, and choose $g_F : \underset{\sim}{N} \to \underset{\sim}{N}$ such that $H_n^F \cap V(n, g_F(n)) = \emptyset$ for each $n \in \underset{\sim}{N}$ (using (i) of Lemma 8). For $S \in \mathscr{S}_o$ set

$$W_S = \underset{\sim}{R} \setminus \bigcap_{n \in \underset{\sim}{N}} \bigcup_{m \geq n} \bigcap_{(m,i) \in S} V(m,i) \ .$$

Because

$$\emptyset \neq \bigcap_{(m,i) \in S} V(m,i) \subseteq U_m$$

for every $m \in \underset{\sim}{N}$, $\bigcup_{m \geq n} \bigcap_{(m,i) \in S} V(m,i)$ is dense for each $n \in \underset{\sim}{N}$, and $W_S \in \mathscr{M}$.

If $g_F \subseteq^* S$ there in an n_o such that $(m, g_F(m)) \in S$ for every $m \geq n_o$; now

$$\bigcup_{m \geq n} \bigcap_{(m,i) \in S} V(m,i) \subseteq \bigcup_{m \geq n} V(m, g_F(m))$$

does not meet H_n^F for any $n \geq n_o$, so $W_S \supseteq F$.

10. <u>*COROLLARY*</u>. *There are functions* $F \mapsto V_F^* : \mathscr{M} \to \mathscr{N}$ *and* $E \to W_E^* : \mathscr{N} \to \mathscr{M}$ *such that* $F \subseteq W_E^*$ *whenever* $V_F^* \subseteq E$.

<u>Proof</u>. Compose the functions of Lemma 7 and 9.

11. <u>*THEOREM*</u>. *(a)* $\mathrm{add}(\mathscr{N}) \leq \mathrm{add}(\mathscr{M})$. *(b)* $\Delta^*(\mathscr{M}) \leq \Delta^*(\mathscr{N})$.

<u>Proof</u>. Take V_F^* , W_E^* from Corollary 10.

(a) If $\mathscr{F} \subseteq \mathscr{M}$ and $\#(\mathscr{F}) < \mathrm{add}(\mathscr{N})$, then

$$E = \bigcup_{F \in \mathscr{F}} V_F^* \in \mathscr{N},$$

so $\bigcup \mathscr{F} \subseteq W_E^* \in \mathscr{M}$.

(b) Let $\mathscr{E} \subseteq \mathscr{N}$ be a cofinal subset with $\#(\mathscr{E}) = \Delta^*(\mathscr{N})$; then $\{W_E^* : E \in \mathscr{E}\}$ is cofinal with $\mathscr{M}$.

12. <u>*LEMMA*</u>. *There are functions* $f \mapsto C_f : \underset{\sim}{N}^{\underset{\sim}{N}} \to \mathscr{M}$ *and* $F \mapsto h_F : \mathscr{M} \to \underset{\sim}{N}^{\underset{\sim}{N}}$ *such that* $f \leq^* h_F$ *whenever* $C_f \subseteq F$.

<u>Proof</u>. (a) Give $\mathscr{P}\underset{\sim}{N}$ the compact metrizable topology obtained by identifying it with $\{0,1\}^{\underset{\sim}{N}}$. Then the set $[\underset{\sim}{N}]^\omega$ of infinite subsets of $\underset{\sim}{N}$ is a dense G_δ subset of $\mathscr{P}N$ with dense complement, so is homeomorphic to $\underset{\sim}{R} \setminus \underset{\sim}{Q}$ ([6], § 36.II, Theorem 3). Let $\varphi : \underset{\sim}{R} \setminus \underset{\sim}{Q} \to [\underset{\sim}{N}]^\omega$ be a homeomorphism.

(b) If $H \subseteq [\underset{\sim}{N}]^\omega$ is nowhere dense and $n \in \underset{\sim}{N}$, there are an $m > n$ and a set $I \subseteq m \setminus n$ such that

$$H \cap \{a : a \in [N]^\omega , a \cap m \setminus n = I\} = \emptyset .$$

To see this, observe that $a \to a \Delta J : [\underset{\sim}{N}]^\omega \to [\underset{\sim}{N}]^\omega$ is a self-inverse homeormorphism for every finite $J \subseteq \underset{\sim}{N}$, so that

$$\tilde{H} = \{a \Delta J : a \in H , J \subseteq n\}$$

is a finite union of nowhere dense sets and is nowhere dense. Consequently there are an $m > n$ and an $I_o \subseteq m$ such that

$$\tilde{H} \cap \{a : a \cap m = I_o\} = \emptyset .$$

Take $I_o = I \setminus n$ and see that if $a \cap m \setminus n = I$ there is a $J \subseteq n$ such that $(a \Delta J) \cap m = I_o$, so that $a \notin H$.

(c) For $f \in \mathbb{N}^{\mathbb{N}}$ define $\tilde{f}(n) = n + \max_{i \leq 2n} f(i)$ for each $n \in \mathbb{N}$, so that $\tilde{f}$ is strictly increasing. Set

$$C_f = \{\alpha : \alpha \in \mathbb{R} \setminus \mathbb{Q} , \varphi(\alpha) \supseteq \tilde{f}[N]\} ,$$

so that C_f is the inverse image of a compact nowhere dense subset of $[\mathbb{N}]^{\omega}$, and is nowhere dense in $\mathbb{R} \setminus \mathbb{Q}$, therefore nowhere dense in $\mathbb{R}$.

(d) For $F \in \mathcal{M}$ let $\langle H_n^F \rangle_{n \in N}$ be an increasing sequence of nowhere dense subsets of $[\mathbb{N}]^{\omega}$ such that $\bigcup_{n \in \mathbb{N}} H_n^F = \varphi[F \setminus \mathbb{Q}]$. Using (b) , choose inductively $\langle h_F(n) \rangle_{n \in \mathbb{N}}$ and $\langle I_n^F \rangle_{n \in \mathbb{N}}$ such that

$$h_F(0) = 0 ,$$

$$h_F(n+1) > h_F(n) , \quad I_n^F \subseteq h_F(n+1) \setminus h_F(n) ,$$

$$H_n^F \cap \{a : a \cap h_F(n+1) \setminus h_F(n) = I_n^F\} = \emptyset$$

for every $n \in \mathbb{N}$.

(e) If now $f \in \mathbb{N}^{\mathbb{N}}$, $F \in \mathcal{M}$ and $C_f \subseteq F$, set

$$a = \tilde{f}[\mathbb{N}] \cup \bigcup_{n \in \mathbb{N}} I_n^F \in [N]^{\omega} .$$

As $a \supseteq \tilde{f}[\mathbb{N}]$, $\varphi^{-1}(a) \in C_f \subseteq F$, and there is an r such that $a \in H_r^F$; now $a \cap h_F(n+1) \setminus h_F(n) \neq I_n^F$ for every $n \geq r$, so

$$\tilde{f}[\mathbb{N}] \cap h_F(n+1) \setminus h_F(n) \neq \emptyset \quad \forall\, n \geq r .$$

Consequently $\tilde{f}(k) < h_F(k+r+1)$ for every $k \in \mathbb{N}$, and $f \leq^* h_F$.

13. *PROPOSITION. (a) $\mathrm{add}(\mathcal{M}) \leq \mathfrak{b}$. (b) $\mathfrak{d} \leq \Delta^*(\mathcal{M})$.*

Proof. (a) If $B \subseteq \mathbb{N}^{\mathbb{N}}$ and $\#(B) < \mathrm{add}(\mathcal{M})$, set $F = \bigcup_{f\in B} C_f$ in Lemma 12, and see that $f \le^* h_F$ for every $f \in B$.

(b) Let $\mathcal{F} \subseteq \mathcal{M}$ be a cofinal set with $\#(\mathcal{F}) = \Delta^*(\mathcal{M})$; set $D = \{h_F : F \in \mathcal{F}\}$; then for every $f \in \mathbb{N}^{\mathbb{N}}$ there is an $h \in D$ with $f \le^* h$.

14. *PROPOSITION.* *(a)* $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) \le \mathrm{add}(\mathcal{M})$. *(b)* $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) \ge \Delta^*(\mathcal{M})$.

Proof. Enumerate $\mathbb{Q}$ as $\langle q_n \rangle_{n\in\mathbb{N}}$. For $\alpha \in \mathbb{R}$, $f \in \mathbb{N}^{\mathbb{N}}$ set

$$W_{\alpha f} = \mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n}]\alpha+q_n-2^{-f(n)}, \alpha+q_n+2^{-f(n)}[\in \mathcal{M}.$$

If H is an F_σ set and $\alpha \in \mathbb{R}\setminus(H+\mathbb{Q})$, there is a $g \in \mathbb{N}^{\mathbb{N}}$ such that $H \subseteq W_{\alpha f}$ whenever $g \le^* f$. To see this, express H as $\bigcup_{n\in\mathbb{N}} H_n$ where $\langle H_n \rangle_{n\in\mathbb{N}}$ is an increasing sequence of closed sets, and choose g such that

$$H_n \cap]\alpha+q_n-2^{-g(n)}, \alpha+q_n+2^{-g(n)}[= \emptyset \ \forall\, n \in \mathbb{N}.$$

(a) If $\mathcal{F} \subseteq \mathcal{M}$ and $\#(\mathcal{F}) < \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$, then choose for each $F \in \mathcal{F}$ a meagre F_σ set $H_F \supseteq F$. As $\#(\mathcal{F}) < \mathrm{cov}(\mathcal{M})$, there is an $\alpha \in \mathbb{R}\setminus \bigcup_{F\in\mathcal{F}, q\in\mathbb{Q}} (H_F+q)$. Next, for each $F \in \mathcal{F}$, there is a $g_F \in \mathbb{N}^{\mathbb{N}}$ such that $H_F \subseteq W_{\alpha f}$ whenever $g_F \le^* f$. But as $\#(\mathcal{F}) < \mathfrak{b}$, there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that $g_F \le^* f$ for every $F \in \mathcal{F}$, and now $\bigcup\mathcal{F} \subseteq W_{\alpha f} \in \mathcal{M}$.

(b) Let $A \subseteq \mathbb{R}$, $D \subseteq \mathbb{N}^{\mathbb{N}}$ be such that

$$\#(A) = \mathrm{non}(\mathcal{M}),\ A \notin \mathcal{M},$$

$$\#(D) = \mathfrak{d},\ \forall\, g \in \mathbb{N}^{\mathbb{N}}\ \exists\, f \in D,\ g \le^* f.$$

Set $\mathcal{F} = \{W_{\alpha f} : \alpha \in A, f \in D\}$; then every meagre F_σ set is included in some member of $\mathcal{F}$, so $\mathcal{F}$ is cofinal with $\mathcal{M}$, and

$$\Delta^*(\mathcal{M}) \le \#(\mathcal{F}) \le \max(\mathrm{non}(\mathcal{M}), \mathfrak{d}).$$

15. *PROPOSITION.* *(a)* $\mathrm{cov}(\mathcal{M}) \le \mathrm{non}(\mathcal{N})$. *(b)* $\mathrm{cov}(\mathcal{N}) \le \mathrm{non}(\mathcal{M})$.

Proof. There is a comeagre $E \in \mathcal{N}$ (take $E = R\setminus W_{0f}$ in Proposition 14, where $f(n) = n$ for every $n \in \mathbb{N}$).

(a) Take $A \subseteq \mathbb{R}$ such that $\#(A) = \mathrm{non}(\mathcal{M})$ and $A \notin \mathcal{M}$. Then $(A+\beta) \cap E \neq \emptyset$ for every $\beta \in \mathbb{R}$, so $E - A = \mathbb{R}$ i.e. $\{E - \alpha : \alpha \in A\}$ is a cover of $\mathbb{R}$ by negligible sets, and $\mathrm{cov}(\mathcal{N}) \leq \#(A) = \mathrm{non}(\mathcal{M})$.

(b) Similarly, using $\mathbb{R} \setminus E \in \mathcal{M}$ in place of $E \in \mathcal{N}$.

16. PROPOSITION. (a) $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$. *(b)* $\mathfrak{d} \geq \mathrm{cov}(\mathcal{M})$.

Proof. Let $\varphi : \mathbb{N}^{\mathbb{N}} \to \mathbb{R} \setminus \mathbb{Q}$ be a homeomorphism ([6], § 36.II, Theorem 3).

(a) Let $A \subseteq \mathbb{R}$ be a non-meagre set of cardinal $\mathrm{non}(\mathcal{M})$. Set $B = \varphi^{-1}[A] \subseteq \mathbb{N}^{\mathbb{N}}$. As $A \setminus Q \notin \mathcal{M}$, there is no K_σ subset of $\mathbb{R} \setminus \mathbb{Q}$ including $A \setminus \mathbb{Q}$, and there is no K_σ subset of $\mathbb{N}^{\mathbb{N}}$ including B. But

$$M_g = \{f : f \leq^* g\}$$

is K_σ in $\mathbb{N}^{\mathbb{N}}$ for every $g \in \mathbb{N}^{\mathbb{N}}$. So there is no g with $f \leq^* g$ for every $f \in B$, and $\mathfrak{b} \leq \#(B) = \mathrm{non}(\mathcal{M})$.

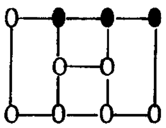
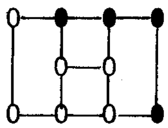
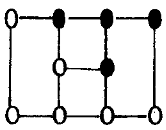
(b) Let $D \subseteq \mathbb{N}^{\mathbb{N}}$ be a cofinal set of cardinal $\mathfrak{d}$. Then $\mathcal{F} = \{\varphi[M_g] : g \in D\} \cup \{Q\} \subseteq \mathcal{M}$ is a cover of $\mathbb{R}$, so $\mathrm{cov}(\mathcal{M}) \leq \#(\mathcal{F}) = \mathfrak{d}$.

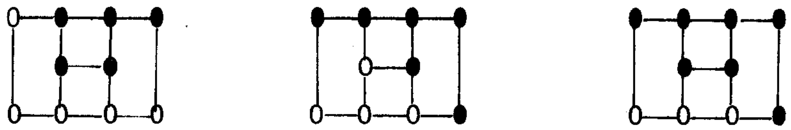
17. PROPOSITION. $\mathfrak{b} \leq \mathfrak{d}$.

Proof. Evident.

18. REMARK. I have deliberately organized this proof in such a way as to emphasiez the symmetry of the diagram about its centre. I should note like to suggest, however, that there is anything definitive about this apparent symmetry.

19. ON THE POSSIBLE VALUES OF THESE CARDINALS. (a) If we suppose that $\mathfrak{c} = \omega_2$ then the diagram allows 23 assignments of the values ω_1 , ω_2 to the ten cardinals. I understand from J. Cichoń and A.W. Miller that models have been found for all but the following :



(In the patterns above, 0 stands for ω_1 and ● stands for ω_2) .

(b) Apart from the restrictions on the relative magnitudes of the ten cardinals encoded in the diagram, there are restrictions on their cofinalities, as follows.

(i) If $\mathscr{I}$ is either $\mathscr{M}$ or $\mathscr{N}$, then $\mathrm{add}(\mathscr{I})$ must be regular. Moreover, both $\mathrm{cf}(\mathrm{non}(\mathscr{I}))$ and $\mathrm{cf}(\Delta^*(\mathscr{I}))$ must be at least $\mathrm{add}(\mathscr{I})$. (For the latter, suppose that $\mathrm{cf}(\Delta^*(\mathscr{I})) = \lambda$. Then there is a cofinal $\mathscr{E} \subseteq \mathscr{I}$ expressible as $\bigcup_{\xi<\lambda} \mathscr{E}_\xi$ where $\#(\mathscr{E}_\xi) < \Delta^*(\mathscr{I})$ for each $\xi < \lambda$. For each $\xi < \lambda$ let E_ξ be a member of $\mathscr{I}$ not included in any member of $\mathscr{E}_\xi$. Then $\bigcup_{\xi<\lambda} E_\xi$ does not belong to $\mathscr{I}$. A similar technique shows that $\mathrm{cf}(\mathrm{non}(\mathscr{I})) \geq \mathrm{add}(\mathscr{I})$.) The same ideas can be used to show that $\mathfrak{h}$ is regular and that $\mathrm{cf}(\mathfrak{d}) > \mathfrak{h}$ (see also 20e below).

(ii) Another restriction appears as follows : if $\mathscr{I}$ is either $\mathscr{M}$ or $\mathscr{N}$, and if $\mathrm{cov}(\mathscr{I}) = \Delta^*(\mathscr{I}) = \kappa$, then $\mathrm{cf}(\kappa) \geq \mathrm{non}(\mathscr{I})$. To see this, observe that under these conditions each E_ξ in the argument of the last paragraph can be taken to be a singleton.

(iii) It follows that every cardinal in the diagram has uncountable cofinality except perhaps $\mathrm{cov}(\mathscr{M})$ and $\mathrm{cov}(\mathscr{N})$. (Of course $\mathrm{cf}(\mathfrak{c}) \geq \omega_1$ by König's theorem). A.W. Miller has shown ([8]) that $\mathrm{cf}(\mathrm{cov}(\mathscr{M})) \geq \omega_1$.

I see no reason to believe that this list of restrictions on cofinalities is complete.

20. ALTERNATIVE DEFINITIONS OF THE CARDINALS. (a) The cardinal $\mathrm{add}(\mathscr{N})$ can be characterized as

$$\min \{\#(B) : B \subseteq \mathbb{N}^{\mathbb{N}} , \forall S \in \mathscr{S}_0 \ \exists f \in B , f \not\subseteq^* S\} .$$

This result is essentially due to [1]; the arguments are given in [3] .

(b) The cardinal $\mathrm{cov}(\mathscr{M})$ can be characterized as the largest cardinal κ with the property that whenever P is a non-empty countable partially ordered set and $\mathcal{Q}$ is a family of cofinal subsets of P with $\#(\mathcal{Q}) < \kappa$, there is an upwards-directed subset of P which meets every member of $\mathcal{Q}$.

See [8] or [4].

(c) The cardinal $\Delta^*(\mathcal{N})$ can be characterized in any of the following ways.

(i) It is

$\min\{\#(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{S}_0 , \forall f \in \mathbb{N}^{\mathbb{N}} \ \exists S \in \mathcal{S} , f \subseteq^* S\}$.

(ii) It is $\Delta_*(\Sigma_L \setminus \mathcal{N})$, where Σ_L is the algebra of Lebesgue measurable sets.

(iii) It is $\Delta_*(\mathfrak{A}_L \setminus \{0\})$, where $\mathfrak{A}_L = \Sigma_L / \mathcal{N}$ is the Lebesgue measure algebra.

See [2].

(d) $\Delta^*(\mathcal{M}) = \Delta_*(\hat{\mathcal{B}} \setminus \mathcal{M})$ where $\hat{\mathcal{B}}$ is the algebra of subsets of $\mathbb{R}$ with the Baire property. (Note however that $\Delta_*((\hat{\mathcal{B}}/\mathcal{M}) \setminus \{0\}) = \omega$) .

(e) Let $\mathcal{K}$ be the σ-ideal of subsets of $\mathbb{N}^{\mathbb{N}}$ generated by the compact sets ; then $\mathfrak{b} = \mathrm{add}(\mathcal{K}) = \mathrm{non}(\mathcal{K})$ and $\mathfrak{d} = \mathrm{cov}(\mathcal{K}) = \Delta^*(\mathcal{K})$. Other characterizations of $\mathfrak{d}$ are given in [5].

21. <u>SOURCES</u>. The oldest results here are those of F. Rothberger ; $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{N})$ and $\mathrm{cov}(\mathcal{N}) \leq \mathrm{non}(\mathcal{M})$ are given in [12], and $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$ is given in [13]. For the next forty years it was widely supposed that the relationship between $\mathcal{M}$ and $\mathcal{N}$ was essentially symmetrical, though I understand that K. Kunen conjectured in the 1970s that $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{M})$. The fact that $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) \leq \mathrm{add}(\mathcal{M})$ is implicit in [14]; A.W. Miller ([7]) made it explicit, and observed that they are in fact equal. Miller also found that $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$, and began work on the opposite corner of the diagram with $\mathfrak{d} \leq \Delta^*(\mathcal{N})$ ([9]) ; but the results which made the diagram planar, $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{M})$ and $\Delta^*(\mathcal{M}) \leq \Delta^*(\mathcal{N})$, were due independently to T. Bartoszyński ([1]) and J. Raisonnier & J. Stern ([11]) . Both relied on the essential idea of Lemma 7. The arguments of Lemma 8-9 were given to me by J. Pawlikowski ([10]) . Lemma 12 is based on ideas in [7].

22. PROBLEMS. (a) Can $\mathrm{cov}(\mathcal{N})$ have countable cofinality ? (See [8]).

(b) Are any of the configurations of 19a impossible ?

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D.H. FREMLIN

D. Lect.

Department of Mathematics

UNIVERSITY OF ESSEX

COLCHESTER Essex

England U.K.

University of Essex

Department of Mathematics
Wivenhoe Park
Colchester CO4 3SQ

from D.H.Fremlin

Tel: Colchester 44144 (STD Code 020 6)
Telegraphic address: University Colchester
Telex: 98440 (UNILIB COLCHSTR)

Postscript (7.9.90) to "Cichoń's Diagram"

If I were re-writing this now I should use cf($\mathcal{M}$), cf($\mathcal{N}$) in place of $\Delta^*(\mathcal{M})$, $\Delta^*(\mathcal{N})$.

References to this paper should note that the "Seminaire Choquet-Rogalski-St Raymond" was held at the Université Pierre et Marie Curie, Paris.

The full references for [3], [11] are

D.H.Fremlin, "On the additivity and cofinality of Radon measures", Mathematatika 31 (1984) 323-335.

J.Raisonnier & J.Stern, "The strength of measurability hypotheses", Israel J. Math. 50 (1985) 337-349.

I understand from J.Ihoda (letter of 20.10.87) that he and S.Shelah have shown the consistency of all the six diagrams of pp. 5-10 and 5-11.

T.Bartoszyński, "On covering of real line by null sets", Pacific J.Math. 131 (1988) 1-12, has shown that if cof($\mathcal{N}$) $\leq \aleph$ then cf(cov($\mathcal{N}$)) > ω (see 22a).

I develop the idea of Lemma 9 in my paper "The partially ordered sets of measure theory and Tukey's ordering", submitted to Dissertationes Math., 1989.

Further work along the lines of this note and of [3] may be found in my article "Measure Algebras" in the Handbook of Boolean Algebra, (ed. J.Monk), North-Holland, 1989.

The full reference for [2] is

J.Cichoń, T.Kamburelis & J.Pawlikowski, "On dense subsets of the measure algebra", Proc. Amer. Math. Soc. 94 (1985) 142-146.

The Ihoda-Shelah work has been written up as

T.Bartoszyński, H.Judah & S.Shelah, "The Cichoń diagram", preprint, 1989.