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SATURATING ULTRAFILTERS ON N

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Abstract. We discuss saturating ultrafilters on N, relating them to other types of nonprincipal ultrafilter. (a) There is an $(\omega, \mathfrak{c})$-saturating ultrafilter on N iff $2^\lambda \le \mathfrak{c}$ for every $\lambda < \mathfrak{c}$ and there is no cover of **R** by fewer than $\mathfrak{c}$ nowhere dense sets. (b) Assume Martin's axiom. Then, for any cardinal κ, a nonprincipal ultrafilter on N is (ω, κ)-saturating iff it is almost κ-good. In particular, (i) $p(\kappa)$-point ultrafilters are (ω, κ)-saturating, and (ii) the set of (ω, κ)-saturating ultrafilters is invariant under homeomorphisms of $\beta N \setminus N$. (c) It is relatively consistent with ZFC to suppose that there is a Ramsey $p(\mathfrak{c})$-point ultrafilter which is not $(\omega, \mathfrak{c})$-saturating.

1. Introduction. We must begin by recalling the definitions used in the abstract. Let $\mathscr{F}$ be a nonprincipal ultrafilter on N, and κ a cardinal. Write $A \subseteq^* B$ to mean that $A \setminus B$ is finite.

(i) We say that $\mathscr{F}$ is *(ω, κ)-saturating* if for every family $\mathscr{B}$ of subsets of $N \times N$ such that $\#(\mathscr{B}) < \kappa$ and $\pi_1[\bigcap \mathscr{B}']$ (the projection of $\bigcap \mathscr{B}'$ onto the first coordinate) belongs to $\mathscr{F}$ for every nonempty finite $\mathscr{B}' \subseteq \mathscr{B}$, there is a function $f: N \to N$ such that $\pi_1[f \cap B] = \{i: (i, f(i)) \in B\}$ belongs to $\mathscr{F}$ for every $B \in \mathscr{B}$. (The name "(ω, κ)-saturating" is chosen because of certain properties of ultrapowers defined from such ultrafilters; see [4, §6.1], and [10, A3D]. But in this paper it will simplify matters to work exclusively from the definition just given.)

(ii) $\mathscr{F}$ is *almost κ-good* if whenever $\lambda < \kappa$ and $I \mapsto A_I: [\lambda]^{<\omega} \to \mathscr{F}$ is a function from the set of finite subsets of λ to $\mathscr{F}$, there is a family $\langle F_\xi \rangle_{\xi < \lambda}$ in $\mathscr{F}$ such that $\bigcap_{\xi \in I} F_\xi \subseteq^* A_I$ for every nonempty finite $I \subseteq \lambda$. ($\mathscr{F}$ would be "κ-good" in the sense of [4, p. 307] if we could conclude "$\bigcap_{\xi \in I} F_\xi \subseteq A_I$ for every nonempty finite $I \subseteq \lambda$".)

(iii) $\mathscr{F}$ is *$<\kappa$-OK* if whenever $\lambda < \kappa$ and $\langle A_n \rangle_{n \in N}$ is a sequence in $\mathscr{F}$, there is a family $\langle F_\xi \rangle_{\xi < \lambda}$ in $\mathscr{F}$ such that $\bigcap_{\xi \in I} F_\xi \subseteq^* A_n$ whenever $n \ge 1$ and $I \in [\lambda]^n$. (See [14] or [17, §4.1].)

(iv) $\mathscr{F}$ is a *$p(\kappa)$-point ultrafilter* if whenever $\mathscr{A} \subseteq \mathscr{F}$ and $\#(\mathscr{A}) < \kappa$ there is an $F \in \mathscr{F}$ such that $F \subseteq^* A$ for every $A \in \mathscr{A}$. $\mathscr{F}$ is a *p-point ultrafilter* if it is a $p(\omega_1)$-point ultrafilter. $\mathscr{F}$ is a *weak p-point ultrafilter* if it is not a cluster point, for the topology of βN, of any sequence of distinct nonprincipal ultrafilters.

(v) $\mathscr{F}$ is *Ramsey* (or "selective") if whenever $r \in N$ and $\mathscr{S}$ is a finite cover of $[N]^r$ there are an $F \in \mathscr{F}$ and an $S \in \mathscr{S}$ such that $[F]^r \subseteq S$ (see [5, §9]).

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It has been known almost from the beginning that Martin's axiom implies the existence of many $p(\mathfrak{c})$-point ultrafilters [3], many Ramsey ultrafilters [3], [2], and many $(\omega, \mathfrak{c})$-saturating ultrafilters [9]. In [10] the first author unified these arguments (using ideas from [12]) to show that if Martin's axiom is true there are $2^{\mathfrak{c}}$ $(\omega, \mathfrak{c})$-saturating Ramsey $p(\mathfrak{c})$-point ultrafilters. It seems, however, that no attempt has been made to integrate classes of saturating ultrafilters into the elaborate hierarchy of types of ultrafilter that has been explored in the last fifteen or twenty years (see [17]). In this paper we show that there is a variety of nontrivial relationships, at least in the presence of special axioms.

The first problem is the question of determining precisely when (ω, κ)-saturating ultrafilters exist. It is not hard to check that every nonprincipal ultrafilter is (ω, ω_1)-saturating (see [4, Theorem 6.1.1]) and that no ultrafilter on N can be $(\omega, \mathfrak{c}^+)$-saturating (see Proposition 3(a) below), so we are concerned only with the case $\omega_1 < \kappa \leq \mathfrak{c}$. (In particular, we have nothing interesting to say if the continuum hypothesis is true.) In Theorem 6 we deal with the case $\kappa = \mathfrak{c}$; the case $\omega_1 < \kappa < \mathfrak{c}$ remains problematic.

Now for relationships between the types of ultrafilter described above. It is easy to see that a $p(\kappa)$-point ultrafilter is almost κ-good, that an almost κ-good ultrafilter is $<\kappa$-OK, that a p-point ultrafilter is a weak p-point ultrafilter, and that an ω_1-OK (="$<\omega_2$-OK") ultrafilter is a weak p-point ultrafilter. (See [6] or [17, 4.3.1(a)].) We find that every (ω, κ)-saturating ultrafilter is $<\kappa$-OK (Proposition 3(b)) and that it is consistent to suppose that an ultrafilter is (ω, κ)-saturating iff it is almost κ-good (Theorem 9); in this case every $p(\kappa)$-point ultrafilter will be (ω, κ)-saturating. (We ought to say that throughout these remarks we are considering only nonprincipal ultrafilters on N.) On the other hand, if there is any $(\omega, \mathfrak{c})$-saturating ultrafilter, there is one which is not a p-point (Theorem 6); and it is also consistent to suppose that there is a $p(\mathfrak{c})$-point ultrafilter which is not an $(\omega, \mathfrak{c})$-saturating ultrafilter (Proposition 12).

It will be convenient to use the following terminology from [10]. We write $\mathfrak{m}$ for the least cardinal such that MA($\mathfrak{m}$) is false, $\mathfrak{p}$ for the greatest cardinal such that $P(\mathfrak{p})$ is true (see [8]), and $\mathfrak{m}_{\text{countable}}$ for the least cardinal of any cover of the real line $\mathbf{R}$ by nowhere dense sets. (We choose this notation because it is also the least cardinal for which MA(countable partially ordered sets, $\mathfrak{m}_{\text{countable}}$) is false; see [19].) Thus $\omega_1 \leq \mathfrak{m} \leq \mathfrak{p} \leq \mathfrak{m}_{\text{countable}} \leq \mathfrak{c}$; "$\mathfrak{m} = \mathfrak{c}$" is Martin's axiom; "$\mathfrak{p} = \mathfrak{c}$" is "$P$" or "$P(\mathfrak{c})$" or "Booth's lemma"; and "$\mathfrak{m}_{\text{countable}} = \mathfrak{c}$" is "$B(\mathfrak{c})$" of [18] and [1], or "MAC" of [21].

We write $2^{<\kappa} = \sup\{2^\lambda : \lambda < \kappa\}$. We shall systematically identify functions with their graphs (cf. "$\pi_1[f \cap B]$" in (i) above).

2. We set out explicitly an elementary lemma that will simplify some of our arguments.

LEMMA. *Suppose that κ is a cardinal and that $\mathscr{F}$ is an (ω, κ)-saturating ultrafilter on* N. *Let $\langle K_n \rangle_{n \in \mathrm{N}}$ be a sequence of nonempty countable sets and write $Z = \bigcup_{n \in \mathrm{N}}(\{n\} \times K_n)$. Suppose that $\mathscr{B} \subseteq \mathscr{P}Z$ is such that $\pi_1[\bigcap \mathscr{B}'] \in \mathscr{F}$ for every nonempty finite $\mathscr{B}' \subseteq \mathscr{B}$. Then there is a function $f \in \prod_{n \in \mathrm{N}} K_n$ such that $\pi_1[f \cap B] \in \mathscr{F}$ for every $B \in \mathscr{B}$.*

PROOF. This is just a matter of recoding each K_n as a subset of N, so that Z becomes a subset of N × N, and we can apply the definition of 1(i).

3. PROPOSITION. *Suppose that κ is a cardinal and that $\mathscr{F}$ is an (ω,κ)-saturating nonprincipal ultrafilter on* N. *Then*

(a) $\kappa \leq \mathfrak{m}_{\text{countable}}$;

(b) $\mathscr{F}$ *is* $<\kappa$-*OK*;

(c) $2^{<\kappa} \leq \mathfrak{c}$;

(d) *if $\mathscr{F}$ is a $p(\kappa)$-point ultrafilter then $\kappa \leq \mathfrak{p}$; and*

(e) *the totally ordered set* $\mathbf{N}^\mathbf{N}/\mathscr{F}$ *has cofinality at least* κ.

PROOF. (a) We need the following characterization $\mathfrak{m}_{\text{countable}}$, given as Corollary 1.8 of [1]: $\mathfrak{m}_{\text{countable}}$ is the least cardinal of any set $G \subseteq \mathbf{N}^\mathbf{N}$ such that for every $h \in \mathbf{N}^\mathbf{N}$ there is a $g \in G$ such that $h \cap g$ is finite.

Given this, take a $G \subseteq \mathbf{N}^\mathbf{N}$ such that $\#(G) = \mathfrak{m}_{\text{countable}}$ and for every $h \in \mathbf{N}^\mathbf{N}$ there is a $g \in G$ such that $h \cap g$ is finite. Let $\langle L(n)\rangle_{n\in\mathbf{N}}$ be a disjoint sequence of subsets of N such that $\#(L(n)) = n$ for every n. For each $n \in \mathbf{N}$ let K_n be the countable set $[\mathbf{N}^{L(n)}]^{\leq n}$. Set $Z = \bigcup_{n\in\mathbf{N}}(\{n\} \times K_n)$. For $g \in G$ set

$$B_g = \{(n, I): n \in \mathbf{N},\ g \restriction L(n) \in I \in K_n\} \subseteq Z,$$

and consider $\mathscr{B} = \{B_g: g \in G\}$.

If $\mathscr{B}' \subseteq \mathscr{B}$ is a nonempty finite set, it is of the form $\{B_g: g \in J\}$ for some finite $J \subseteq G$. Now for any $n \geq \#(J)$,

$$(n, \{g \restriction L(n): g \in J\}) \in \bigcap_{g\in J} B_g,$$

so $\pi_1[\bigcap\mathscr{B}'] \supseteq \{n: n \geq \#(J)\} \in \mathscr{F}$.

Suppose, if possible, that there were a function $f \in \prod_{n\in\mathbf{N}} K_n$ such that $\pi_1[f \cap B] \in \mathscr{F}$ for every $B \in \mathscr{B}$. Choose $h: \mathbf{N} \to \mathbf{N}$ such that whenever $n \in \mathbf{N}$ and $h' \in f(n)$ there is an $i \in L(n)$ such that $h(i) = h'(i)$; this is possible because $\#(f(n)) \leq n = \#(L(n))$. If $g \in G$ then

$$\begin{aligned}\{n: \exists i \in L(n), g(i) = h(i)\}\\ \supseteq \{n: g \restriction L(n) \in f(n)\} = \pi_1[f \cap B_g] \in \mathscr{F},\end{aligned}$$

so $g \cap h$ must be infinite. But this contradicts the choice of G.

So there is no such f, and $\kappa \leq \#(\mathscr{B}) \leq \#(G) = \mathfrak{m}_{\text{countable}}$ by Lemma 2.

(b) Let $\langle A_n\rangle_{n\in\mathbf{N}}$ be any sequence in $\mathscr{F}$ and λ a cardinal less than κ. There is a family $\langle h_\xi\rangle_{\xi<\lambda}$ in $\mathbf{N}^\mathbf{N}$ such that $h_\xi \cap h_\eta$ is finite for all $\xi \neq \eta$ (because $\lambda \leq \mathfrak{c}$). For each $r \in \mathbf{N}$ set $m(r) = \min(\{r\} \cup \{n: r \notin A_{n+1}\})$ and $K_r = [\mathbf{N}]^{\leq m(r)}$. Set $Z = \bigcup_{r\in\mathbf{N}}(\{r\} \times K_r)$, and for $\xi < \lambda$ set $B_\xi = \{(n, I): n \in \mathbf{N}, h_\xi(n) \in I \in K_n\}$. Write $\mathscr{B} = \{B_\xi: \xi < \lambda\}$. As in (a) we see that if $\mathscr{B}' \in [\mathscr{B}]^k$, where $k \geq 1$, then

$$\pi_1[\bigcap\mathscr{B}'] \supseteq \{r: r \in \mathbf{N}, m(r) \geq k\} = \bigcap_{1\leq n\leq k} A_n\backslash k \in \mathscr{F}.$$

So there is a function $f \in \prod_{r\in\mathbf{N}} K_r$ such that $F_\xi = \pi_1[f \cap B_\xi] \in \mathscr{F}$ for every $\xi < \lambda$. If $k \geq 1$ and $J \in [\lambda]^k$ then there is an $n \geq k$ such that $h_\xi(i) \neq h_\eta(i)$ whenever $i \geq n$ and ξ, η are distinct members of J. If $r \in \bigcap_{\xi\in J} F_\xi\backslash n$ then $(r, f(r)) \in \bigcap_{\xi\in J} B_\xi$, i.e. $h_\xi(r) \in f(r) \in K_r$ for every $\xi \in J$; so $m(r) \geq \#(f(r)) \geq k$ and $r \in A_k$. Thus $\bigcap_{\xi\in J} F_\xi\backslash A_k \subseteq n$ is finite. As λ and $\langle A_n\rangle_{n\in\mathbf{N}}$ are arbitrary, $\mathscr{F}$ is $<\kappa$-OK.

(c) We need to know that there is a family $\langle A_\xi\rangle_{\xi<\mathfrak{c}}$ of subsets of N such that $\bigcap_{\xi\in I} A_\xi \setminus \bigcup_{\eta\in J} A_\eta$ is infinite for all disjoint finite subsets I, J of $\mathfrak{c}$ [17, 3.1.2(b)]. Now suppose that $\lambda < \kappa$. For each set $M \subseteq \lambda$ let $\mathscr{B}_M$ be the set

$$\{\mathrm{N} \times A_\xi : \xi \in M\} \cup \{\mathrm{N} \times (\mathrm{N}\setminus A_\eta) : \eta \in \lambda\setminus M\}.$$

Then we see that $\pi_1[\bigcap\mathscr{B}'] = \mathrm{N} \in \mathscr{F}$ for every nonempty finite $\mathscr{B}' \subseteq \mathscr{B}_M$. So there is an $f_M \in \mathrm{N}^{\mathrm{N}}$ such that $\pi_1[f_M \cap B] \in \mathscr{F}$ for every $B \in \mathscr{B}_M$, i.e. $f_M^{-1}[A_\xi] \in \mathscr{F}$ and $f_M^{-1}[\mathrm{N}\setminus A_\eta] \in \mathscr{F}$ for all $\xi \in M$ and $\eta \in \lambda\setminus M$. But this means that

$$M = \{\xi : \xi < \lambda, f_M^{-1}[A_\xi] \in \mathscr{F}\},$$

so that $M \mapsto f_M : \mathscr{P}\lambda \to \mathrm{N}^{\mathrm{N}}$ is injective, and $2^\lambda \le \mathfrak{c}$.

(d) Suppose that $\mathscr{F}$ is a $p(\kappa)$-point ultrafilter. Let $\mathscr{A}$ be a nonempty family of subsets of N such that $\#(\mathscr{A}) < \kappa$ and $\bigcap\mathscr{A}'$ is infinite for every finite $\mathscr{A}' \subseteq \mathscr{A}$. We seek an infinite $C \subseteq \mathrm{N}$ such that $C \subseteq^* A$ for every $A \in \mathscr{A}$; we may of course suppose that $\mathscr{A}$ is infinite, that $\kappa \ge \omega_1$ and that $\mathrm{N}\setminus n \in \mathscr{A}$ for every $n \in \mathrm{N}$. Set $\mathscr{B} = \{\mathrm{N} \times A : A \in \mathscr{A}\}$, so that $\pi_1[\bigcap\mathscr{B}'] = \mathrm{N} \in \mathscr{F}$ for every nonempty finite $\mathscr{B}' \subseteq \mathscr{B}$. Let $f \in \mathrm{N}^{\mathrm{N}}$ be such that $\pi_1[f \cap B] \in \mathscr{F}$ for every $B \in \mathscr{B}$, i.e. $f^{-1}[A] \in \mathscr{F}$ for every $A \in \mathscr{A}$. Because $\mathscr{F}$ is a $p(\kappa)$-point ultrafilter there is an $F \in \mathscr{F}$ such that $F \subseteq^* f^{-1}[A]$ for every $A \in \mathscr{A}$. Set $C = f[F]$; then $C \subseteq^* A$ for every $A \in \mathscr{A}$. But also F is infinite and $F \subseteq^* f^{-1}[\mathrm{N}\setminus n]$ for every $n \in \mathrm{N}$; so C is infinite. Thus we have a suitable C. As $\mathscr{A}$ is arbitrary, $\mathfrak{p} \ge \kappa$.

(e) Finally, suppose that $D \subseteq \mathrm{N}^{\mathrm{N}}/\mathscr{F}$ and $\#(D) < \kappa$. For each $d \in D$ let $f_d \in \mathrm{N}^{\mathrm{N}}$ be such that $d = f_d^{\cdot}$, the equivalence class of f_d in $\mathrm{N}^{\mathrm{N}}/\mathscr{F}$. Consider $\mathscr{B} = \{B_d : d \in D\}$, where

$$B_d = \{(n, i) : i > f_d(n)\} \subseteq \mathrm{N} \times \mathrm{N}.$$

Then $\pi_1[\bigcap\mathscr{B}'] = \mathrm{N} \in \mathscr{F}$ for every nonempty finite $\mathscr{B}' \subseteq \mathscr{B}$, so there is an $f : \mathrm{N} \to \mathrm{N}$ such that

$$\{n : f(n) > f_d(n)\} = \pi_1[f \cap B_d] \in \mathscr{F}$$

for every $d \in D$. In this case $f^{\cdot} > f_d^{\cdot}$ for every $d \in D$, so D is not cofinal with $\mathrm{N}^{\mathrm{N}}/\mathscr{F}$.

4. Remarks. Of course (e) is just a simple special case of the fact that the model $\mathrm{N}^{\mathrm{N}}/\mathscr{F}$ of the first-order theory of totally ordered sets is κ-saturated if $\mathscr{F}$ is (ω, κ)-saturating; (c) also has a model-theoretic formulation in terms of incompatible types.

It is a consequence of (e) that if there is an (ω, κ)-saturating ultrafilter on N then $\mathrm{cf}(\mathrm{N}^{\mathrm{N}})$, which is called $\mathfrak{d}$ in [8] and [11], is greater than or equal to κ; this was observed also by B. Balcar. But as $\mathfrak{m}_{\text{countable}} \le \mathfrak{d}$ (this follows immediately from Bartoszyński's characterization of $\mathfrak{m}_{\text{countable}}$ mentioned in the proof of 3(a) above, or by direct methods, as in [11, 16(b)]), with strict inequality possible (add ω_1 random reals to a model of $\mathfrak{m} = \mathfrak{c} = \omega_2$; see [15]), our 3(a) is a sharper result.

5. Corollary. (a) *An (ω, ω_2)-saturating ultrafilter on* N *is a weak p-point ultrafilter.*

(b) *Not every nonprincipal ultrafilter on* N *is (ω, ω_2)-saturating.*

Proof. (a) By 3(b), it is ω_1-OK, therefore a weak p-point ultrafilter, as remarked in §1. (b) Immediate from (a).

6. THEOREM. *If one of the following is true, so are the others:*

(i) *There is an* $(\omega, \mathfrak{c})$*-saturating ultrafilter on* $\mathbf{N}$.

(ii) $\mathfrak{m}_{\text{countable}} = 2^{<\mathfrak{c}} = \mathfrak{c}$.

(iii) *There is an* $(\omega, \mathfrak{c})$*-saturating ultrafilter on* $\mathbf{N}$ *which is not a p-point ultrafilter.*

PROOF. (a) (iii) $\Rightarrow$ (i) is trivial, and (i) $\Rightarrow$ (ii) is covered by 3(a) and 3(c) above. So we assume (ii) and seek to prove (iii).

(b) Our aim is to generate a filter $\mathscr{F}$ from an increasing family $\langle \mathscr{F}_\xi \rangle_{\xi<\mathfrak{c}}$ of filter bases. Each step from $\mathscr{F}_\xi$ to $\mathscr{F}_{\xi+1}$ will seek to deal with some $\mathscr{B} \subseteq \mathscr{P}(\mathbf{N} \times \mathbf{N})$. In order to ensure that $\mathscr{F}$ is not a p-point ultrafilter we shall require $\mathscr{F}_\xi \cap \mathscr{I} = \emptyset$ for every $\xi < \mathfrak{c}$, where $\mathscr{I}$ is an ideal of sets to be defined shortly; and in order to retain control of the induction we require $\#(\mathscr{F}_\xi) < \mathfrak{c}$ for every $\xi < \mathfrak{c}$.

(c) Before starting we must set up the following. Let $\langle A_n \rangle_{n\in\mathbf{N}}$ be a fixed partition of $\mathbf{N}$ into infinite sets, and write

$$\mathscr{I} = \{A : A \subseteq \mathbf{N}, \{n : A \cap A_n \text{ is infinite}\} \text{ is finite}\},$$

so that $\mathscr{I} \lhd \mathscr{P}\mathbf{N}$. Let $\langle \mathscr{B}_\xi \rangle_{\xi<\mathfrak{c}}$ enumerate $[\mathscr{P}(\mathbf{N} \times \mathbf{N})]^{<\mathfrak{c}}$; such an enumeration exists because $2^{<\mathfrak{c}} = \mathfrak{c}$.

(d) To start the induction, set

$$\mathscr{F}_0 = \left\{\mathbf{N} \setminus \bigcup_{i<n} A_i : n \in \mathbf{N}\right\},$$

so that $\mathscr{F}_0$ is a filter base, $\mathscr{F}_0 \cap \mathscr{I} = \emptyset$, and $\#(\mathscr{F}_0) = \omega < \mathfrak{c}$.

(e) To construct $\mathscr{F}_{\xi+1}$ from $\mathscr{F}_\xi$, consider two cases.

(i) If there are a finite $\mathscr{B}' \subseteq \mathscr{B}_\xi$ and an $F \in \mathscr{F}_\xi$ such that $F \cap \pi_1[\bigcap \mathscr{B}'] \in \mathscr{I}$, set $\mathscr{F}_{\xi+1} = \mathscr{F}_\xi$, and proceed.

(ii) Otherwise, let W_ξ be the set of quadruples $(m, n, \mathscr{B}', F)$ such that $m, n \in \mathbf{N}$, $\mathscr{B}' \in [\mathscr{B}_\xi]^{<\omega}$, $F \in \mathscr{F}_\xi$ and $A_n \cap \pi_1[\bigcap \mathscr{B}'] \cap F$ is infinite. For $w = (m, n, \mathscr{B}', F) \in W_\xi$ set

$$\begin{aligned} G_w &= \{h : h \in \mathbf{N}^{\mathbf{N}}, \pi_1[h \cap \textstyle\bigcap \mathscr{B}'] \cap A_n \cap F \not\subseteq m\} \\ &= \{h : \exists i \in A_n \cap F \setminus m \text{ such that } (i, h(i)) \in \textstyle\bigcap \mathscr{B}'\}. \end{aligned}$$

Then each G_w is a dense open set in the Polish space $\mathbf{N}^{\mathbf{N}}$. Because $\mathbf{N}^{\mathbf{N}}$ is homeomorphic to the set $\mathbf{R} \setminus \mathbf{Q}$, which is comeagre in $\mathbf{R}$, it cannot be covered by fewer than $\mathfrak{m}_{\text{countable}}$ nowhere dense sets; since $\#(W_\xi) \leq \max(\omega, \#(\mathscr{B}_\xi), \#(\mathscr{F}_\xi)) < \mathfrak{c} = \mathfrak{m}_{\text{countable}}$, there is an $h_\xi \in \bigcap \{G_w : w \in W_\xi\}$.

For each $\mathscr{B}' \in [\mathscr{B}_\xi]^{<\omega}$, set $E(\xi, \mathscr{B}') = \pi_1[h_\xi \cap \bigcap \mathscr{B}']$. We claim that $F \cap E(\xi, \mathscr{B}') \notin \mathscr{I}$ for every $F \in \mathscr{F}_\xi$ and $\mathscr{B}' \in [\mathscr{B}_\xi]^{<\omega}$. For given F and $\mathscr{B}'$ we know that $F \cap \pi_1[\bigcap \mathscr{B}'] \notin \mathscr{I}$, so that the set

$$J = \{n : n \in \mathbf{N}, F \cap \pi_1[\textstyle\bigcap \mathscr{B}'] \cap A_n \text{ is infinite}\}$$

is infinite, and $(m, n, \mathscr{B}', F) \in W$ for every $m \in \mathbf{N}$ and $n \in J$. So $\pi_1[h_\xi \cap \bigcap \mathscr{B}'] \cap A_n \cap F \not\subseteq m$ for $m \in \mathbf{N}$ and $n \in J$, and $E(\xi, \mathscr{B}') \cap F \cap A_n$ is infinite for every $n \in J$, so that $E(\xi, \mathscr{B}') \cap F \notin \mathscr{I}$.

It follows that we may set

$$\mathscr{F}_{\xi+1} = \{F \cap E(\xi, \mathscr{B}') : \mathscr{B}' \in [\mathscr{B}_\xi]^{<\omega}, F \in \mathscr{F}_\xi\}$$

and obtain a new filter base, including $\mathscr{F}_\xi$, not meeting $\mathscr{I}$, and of cardinal less than $\mathfrak{c}$.

(f) For nonzero limit ordinals $\xi < \mathfrak{c}$ set $\mathscr{F}_\xi = \bigcup_{\eta<\xi} \mathscr{F}_\eta$; $\mathscr{F}_\xi$ is a filter base because $\langle \mathscr{F}_\eta \rangle_{\eta<\xi}$ is an increasing family of filter bases, and has cardinal less than $\mathfrak{c}$ because $\mathfrak{c}$ is regular, by König's theorem [13, I.10.41] and the assumption that $2^{<\mathfrak{c}} = \mathfrak{c}$.

(g) Now consider $\mathscr{F} = \bigcup_{\xi<\mathfrak{c}} \mathscr{F}_\xi$. This is a filter base. In fact it is an ultrafilter. For if $A \subseteq \mathrm{N}$ there is a $\xi < \mathfrak{c}$ such that $\mathscr{B}_\xi = \{A \times \mathrm{N}\}$. Now the construction of (e) will ensure that either $A \cap F \in \mathscr{I}$ for some $\mathscr{F} \in \mathscr{F}_\xi$, or $A = E(\xi, B_\xi) \in \mathscr{F}_{\xi+1}$; so that either $A \in \mathscr{F}$ or $A \cap F \in \mathscr{I}$ for some $F \in \mathscr{F}$. The same is true for $\mathrm{N} \backslash A$, so in fact either $A \in \mathscr{F}$ or $\mathrm{N}\backslash A \in \mathscr{F}$, as required.

Of course $\mathscr{F} \cap \mathscr{I} = \varnothing$, while $\mathrm{N}\backslash A_n \in \mathscr{F}$ for every $n \in \mathrm{N}$. So $\mathscr{F}$ is not a p-point ultrafilter.

Finally, $\mathscr{F}$ is $(\omega, \mathfrak{c})$-saturating. For suppose that $\mathscr{B} \in [\mathscr{P}(\mathrm{N} \times \mathrm{N})]^{<\omega}$ and that $\pi_1[\bigcap \mathscr{B}'] \in \mathscr{F}$ for every nonempty finite $\mathscr{B}' \subseteq \mathscr{B}$. Then $\mathscr{B} = \mathscr{B}_\xi$ for some $\xi < \mathfrak{c}$. As $F \cap \pi_1[\bigcap \mathscr{B}'] \in \mathscr{F} \subseteq \mathscr{P}\mathrm{N}\backslash\mathscr{I}$ for every $\mathscr{B}' \in [\mathscr{B}_\xi]^{<\omega}$ and $F \in \mathscr{F}_\xi$, we must have used the construction of (e–ii) at this ξ. Accordingly

$$\pi_1[h_\xi \cap B] = E(\xi, \{B\}) \in \mathscr{F}_{\xi+1} \subseteq \mathscr{F}$$

for every $B \in \mathscr{B}_\xi = \mathscr{B}$. As $\mathscr{B}$ is arbitrary, $\mathscr{F}$ is $(\omega, \mathfrak{c})$-saturating.

7. REMARK. K. Kunen has pointed out that (ii) above is also sufficient to prove that there is an $(\omega, \mathfrak{c})$-saturating ultrafilter which is a p-point ultrafilter.

8. We turn now to the characterization of (ω, κ)-saturating ultrafilters for $\kappa \leq \mathfrak{m}$. Our first lemma may be of independent interest.

LEMMA. *Suppose that $\lambda < \mathfrak{m}$ and that $I \mapsto A_I: [\lambda]^{<\omega} \to \mathscr{P}\mathrm{N}$ is a decreasing function (i.e. that $A_I \subseteq A_J$ whenever $J \subseteq I$). Then there is a family $\langle B_\xi \rangle_{\xi<\lambda}$ of subsets of $\mathrm{N} \times \mathrm{N}$ such that $A_I \Delta \pi_1[\bigcap_{\xi \in I} B_\xi]$ is finite for every nonempty finite $I \subseteq \lambda$.*

PROOF. (a) Let P be the set of pairs (m, f), where $m \in \mathrm{N}$ and f is a function from a finite subset of λ to $[m \times \mathrm{N}]^{<\omega}$. Say that $(m, f) \leq (n, g)$ if $m \leq n$, $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$, $f(\xi) = g(\xi) \cap (m \times \mathrm{N})$ for every $\xi \in \mathrm{dom}(f)$, and

$$A_J \cap n\backslash m = \pi_1\left[\bigcap_{\xi \in J} g(\xi)\right] \cap n\backslash m$$

for every nonempty subset J of $\mathrm{dom}(f)$. It is easy to check that $\leq$ is a partial order on P.

(b) Now P is upwards-ccc. For suppose that $\langle (m_\xi, f_\xi) \rangle_{\xi<\omega_1}$ is any family in P. There is an $m \in \mathrm{N}$ such that $A = \{\xi : \xi < \omega_1, m_\xi = m\}$ is uncountable, there is an uncountable $B \subseteq A$ such that $\langle \mathrm{dom}(f_\xi) \rangle_{\xi \in B}$ is a Δ-system with root I (say), and there is a function $h: I \to [m \times \mathrm{N}]^{<\omega}$ such that $C = \{\xi : \xi \in B, f_\xi \restriction I = h\}$ is uncountable. Now $(m, f_\xi \cup f_\eta)$ is a common upper bound in P of (m_ξ, f_ξ) and (m_η, f_η) for any $\xi, \eta \in C$.

(c) If $\xi < \lambda$ then $Q_\xi = \{(m, f) : (m, f) \in P, \xi \in \mathrm{dom}(f)\}$ is cofinal with P because $(m, f) \leq (m, f \cup \{(\xi, \varnothing)\})$ if $\xi \notin \mathrm{dom}(f)$. Moreover, if $n \in \mathrm{N}$, $Q'_n = \{(m, f) : (m, f) \in P, n \leq m\}$ is cofinal with P. To see this, take any $(m, f) \in P \backslash Q'_n$. Set $I = \mathrm{dom}(f)$. Let $J \mapsto r_J : \mathscr{P}_I \to \mathrm{N}$ be injective. Define g by saying that $\mathrm{dom}(g) = \mathrm{dom}(f)$ and

$$g(\xi) = f(\xi) \cup \{(i, r_J) : \xi \in J \subseteq I, i \in A_J \cap n\backslash m\}$$

for $\xi \in \mathrm{dom}(f)$. Then $(n, g) \in Q'_n$. If J is a nonempty subset of I then

$$\begin{aligned}\pi_1\left[\bigcap_{\xi\in J} g(\xi)\right] \cap n\backslash m \\ &= \{i : \exists K \subseteq I, i \in A_K \cap n\backslash m, \xi \in K\ \forall \xi \in J\} \\ &= A_J \cap n\backslash m\end{aligned}$$

because $A_J \supseteq A_K$ if $J \subseteq K$. This is what we need to know to see that $(m, f) \leq (n, g)$.

(d) Because $\lambda < \mathfrak{m}$ there is an upwards-directed $R \subseteq P$ meeting every Q'_n and every Q_ξ. Set $B_\xi = \bigcup\{f(\xi) : (m, f) \in R, \xi \in \mathrm{dom}(f)\}$. Note that if $(n, g) \in R$ and $\xi \in \mathrm{dom}(g)$ then $g(\xi) = B_\xi \cap (n \times \mathrm{N})$; this is because $g(\xi) = f(\xi) \cap (n \times \mathrm{N})$ whenever $(n, g) \leq (m, f)$ in P.

If $I \subseteq \lambda$ is a nonempty finite set, then because R is upwards-directed and meets every Q_ξ there is an $(m, f) \in R$ such that $I \subseteq \mathrm{dom}(f)$. Now suppose that $i \geq m$. There is an $(n, g) \in R \cap Q'_{i+1}$ such that $(m, f) \leq (n, g)$. In this case $B_\xi \cap (n \times \mathrm{N}) = g(\xi)$ for every $\xi \in I$, so that

$$i \in \pi_1\left[\bigcap_{\xi\in I} B_\xi\right] \Leftrightarrow i \in \pi_1\left[\bigcap_{\xi\in I} g(\xi)\right] \Leftrightarrow i \in A_I$$

by the definition of the order of P. This shows that $A_I \triangle \pi_1[\bigcap_{\xi\in I} B_\xi] \subseteq m$ is finite, as required.

9. Theorem. *If $\kappa \leq \mathfrak{m}$ then a nonprincipal ultrafilter $\mathscr{F}$ on* N *is (ω, κ)-saturating iff it is almost κ-good.*

Proof. (a) Suppose that $\mathscr{F}$ is (ω, κ)-saturating and that $I \mapsto A_I : [\lambda]^{<\omega} \to \mathscr{F}$ is a function, where $\lambda < \kappa$. Set $A'_I = \bigcap_{J\subseteq I} A_J$ for each $I \in [\lambda]^{<\omega}$, so that $I \mapsto A'_I : [\lambda]^{<\omega} \to \mathscr{F}$ is a decreasing function. By Lemma 8 there is a family $\langle B_\xi\rangle_{\xi<\lambda}$ in $\mathscr{P}(\mathrm{N} \times \mathrm{N})$ such that $A'_I \triangle \pi_1[\bigcap_{\xi\in I} B_\xi]$ is finite for every nonempty finite $I \subseteq \lambda$. Because $\mathscr{F}$ is (ω, κ)-saturating, there is an $f : \mathrm{N} \to \mathrm{N}$ such that $F_\xi = \pi_1[f \cap B_\xi] \in \mathscr{F}$ for every $\xi < \kappa$. But now

$$\bigcap_{\xi\in I} F_\xi \backslash A_I \subseteq \pi_1\left[\bigcap_{\xi\in I} B_\xi\right]\backslash A'_I$$

is finite for every nonempty finite $I \subseteq \lambda$. As $I \mapsto A_I$ is arbitrary, $\mathscr{F}$ is almost κ-good.

(b) Suppose that $\mathscr{F}$ is almost κ-good.

(i) Let $\langle B_\xi\rangle_{\xi<\lambda}$ be a family of subsets of $\mathrm{N} \times \mathrm{N}$, where $\lambda < \kappa$, such that $A_I = \pi_1[\bigcap_{\xi\in I} B_\xi] \in \mathscr{F}$ for every nonempty finite $I \subseteq \lambda$. Because $\mathscr{F}$ is almost κ-good, there is a family $\langle F_\xi\rangle_{\xi<\kappa}$ in $\mathscr{F}$ such that $\bigcap_{\xi\in I} F_\xi \subseteq^* A_I$ for every nonempty finite $I \subseteq \lambda$.

(ii) Let P be the set of triples (m, D, I) such that $m \in \mathrm{N}$, $D \subseteq m \times \mathrm{N}$, each vertical section of D has at most two members, I is a finite subset of λ, and $\bigcap_{\xi\in K} F_\xi \backslash A_K \subseteq m$ for every nonempty subset K of I. Say that $(m, D, I) \leq (n, E, J)$ if $m \leq n$, $D \subseteq E$, $I \subseteq J$ and

$$\pi_1\left[E \cap \bigcap_{\xi\in K} B_\xi\right] \supseteq \bigcap_{\xi\in K} F_\xi \cap n\backslash m$$

for all nonempty $K \subseteq I$. It is easy to check that $\leq$ is a partial order on P.

(iii) P is upwards-ccc. For suppose that $S \subseteq P$ is uncountable. Then there must be distinct members of S with the same first two coordinates; suppose that (m, D, I) and (m, D, J) both belong to S. Take $n \geq m$ such that $\bigcap_{\xi \in K} F_\xi \setminus A_k \subseteq n$ for every nonempty finite $K \subseteq I \cup J$. For $i \in n \setminus m$ set

$$K_i = \{\xi : \xi \in I, i \in F_\xi\}, \qquad K'_i = \{\xi : \xi \in J, i \in F_\xi\}.$$

Choose $r_i \in \mathbb{N}$ such that $(i, r_i) \in \bigcap \{B_\xi : \xi \in K_i\}$; this is possible because if $K_i \neq \varnothing$ then $\bigcap_{\xi \in K_i} F_\xi \setminus \pi_1[\bigcap_{\xi \in K_i} B_\xi] \subseteq m$, so $i \in \pi_1[\bigcap_{\xi \in K_i} B_\xi]$. Similarly there is an r'_i such that $(i, r'_i) \in \bigcap \{B_\xi : \xi \in K'_i\}$. Now set

$$E = D \cup \{(i, r_i) : m \leq i < n\} \cup \{(i, r'_i) : m \leq i < n\}.$$

It is easy to see that $(n, E, I \cup J)$ is a common upper bound for (m, D, I) and (m, D, J) in P. As S is arbitrary, P is upwards-ccc.

(iv) For each $\xi < \lambda$ and $n \in \mathbb{N}$ the sets

$$Q_\xi = \{(m, D, I) : \xi \in I\}, \qquad Q'_n = \{(m, D, I) : m > n\}$$

are cofinal with P; the argument follows that of (iii) just above. As $\lambda < \mathfrak{m}$ there is an upwards-directed $R \subseteq P$ meeting every Q'_n and every Q_ξ. Set $H = \bigcup \{D : (m, D, I) \in R\}$. Then vertical sections of H all have at most two members, so there are two functions $h_0, h_1 \in \mathbb{N}^\mathbb{N}$ such that $H \subseteq h_0 \cup h_1$.

(v) Suppose, if possible, that there are $\xi, \eta < \lambda$ such that $\pi_1[h_0 \cap B_\xi] \notin \mathscr{F}$ and $\pi_1[h_1 \cap B_\eta] \notin \mathscr{F}$. Then $\pi_1[H \cap B_\xi \cap B_\eta] \notin \mathscr{F}$. Because R is upwards-directed and meets both Q_ξ and Q_η, there is an $(m, D, I) \in R$ such that ξ and η both belong to I. Now $F_\xi \cap F_\eta \in \mathscr{F}$, so there is an $i \geq m$ such that

$$i \in F_\xi \cap F_\eta \setminus \pi_1[H \cap B_\xi \cap B_\eta].$$

Because R also meets Q'_i, there is an $(n, E, J) \in R$ such that $i < n$ and $(m, D, I) \leq (n, E, J)$. But in this case

$$i \in F_\xi \cap F_\eta \cap n \setminus m \subseteq \pi_1[E \cap B_\xi \cap B_\eta] \subseteq \pi_1[H \cap B_\xi \cap B_\eta]$$

by the definition of the order $\leq$ on P. So there can be no such ξ and η.

(vi) Accordingly, either $\pi_1[h_0 \cap B_\xi] \in \mathscr{F}\ \forall \xi < \lambda$, or $\pi_1[h_1 \cap B_\xi] \in \mathscr{F}\ \forall \xi < \lambda$. Since $\langle B_\xi \rangle_{\xi < \lambda}$ is arbitrary, this shows that $\mathscr{F}$ is (ω, κ)-saturating.

10. REMARKS. Theorem 6.1.8 of [4] shows that a nonprincipal κ-good ultrafilter is (ω, κ)-saturating, and indeed much more. But there are no ω_2-good nonprincipal ultrafilters on $\mathbb{N}$ [4, Exercise 6.1.3].

Note that the partial orders of Lemma 8 and Theorem 9 above are both in well-recognised special classes of ccc partial order. That of Lemma 8 satisfies Knaster's condition [10, 11A], while that of Theorem 9 is σ-linked [10, B1D]. Consequently Theorem 9 remains true if we replace $\mathfrak{m}$ by the cardinal $\mathfrak{m}_K$ of [10, 11D].

11. COROLLARY. *Suppose that $\kappa \leq \mathfrak{m}$.*

(a) *Every $p(\kappa)$-point ultrafilter on $\mathbb{N}$ is (ω, κ)-saturating.*

(b) *If $\mathscr{F}$ is an (ω, κ)-saturating ultrafilter on $\mathbb{N}$ and φ: $\beta\mathbb{N} \setminus \mathbb{N} \to \beta\mathbb{N} \setminus \mathbb{N}$ is a homeomorphism, then $\varphi(\mathscr{F})$ is (ω, κ)-saturating.*

PROOF. (a) We need only remember that $p(\kappa)$-point ultrafilters are almost κ-good, as remarked in §1.

(b) The point is that the property of being "almost κ-good" is a topological one. We could say that a point x of a topological space X is "κ-good" if for every $\lambda < \kappa$ and for every function $I \mapsto U_I: [\lambda]^{<\omega} \to \mathfrak{N}(x)$, where $\mathfrak{N}(x)$ is the filter of neighbourhoods of x, there is a family $\langle V_\xi \rangle_{\xi < \lambda}$ in $\mathfrak{N}(x)$ such that $\bigcap_{\xi \in I} V_\xi \subseteq U_I$ for every nonempty finite $I \subseteq \lambda$. Now a nonprincipal ultrafilter on N is almost κ-good in the sense of 1(ii) iff it is κ-good when regarded as a point in the topological space $\beta\mathrm{N}\setminus\mathrm{N}$. So our result follows from Theorem 9 at once.

12. PROPOSITION. *It is relatively consistent with ZFC to suppose that there is a Ramsey $p(\mathfrak{c})$-point ultrafilter on* N *which is not $(\omega, \mathfrak{c})$-saturating.*

PROOF. (a) We use the following construction due to P. Dordal [7]. Let M be a countable transitive model of ZFC + GCH. In M, let P be a ccc partially ordered set forcing $\mathfrak{m} = \mathfrak{c} = \omega_2$ [13, VIII.6.3], and let Q be $\mathrm{Fn}(\omega_3, 2, \omega_1)$, the set of functions $f \subseteq \omega_3 \times \{0,1\}$ with countable domains, ordered in reverse. Take $G \subseteq P \times Q$ to be a $(P \times Q)$-generic filter over M; then $G = G_1 \times G_2$, where G_1 is a P-generic filter over M and G_2 is a Q-generic filter over M [13, VIII.1.3].

The facts we need (mostly given in [7], and readily proved by techniques in [13]) are as follows:

(α) M, $M[G_1]$ and $M[G]$ have the same cardinals.

(β) $M[G_1] \models$ "$\mathfrak{m} = \mathfrak{c} = \omega_2$".

(γ) $\mathscr{P}\mathrm{N} \cap M[G] = \mathscr{P}\mathrm{N} \cap M[G_1]$.

(δ) $M[G] \models$ "$2^{\omega_1} = \omega_3$".

(b) Now we argue as follows. Start in $M[G_1]$. Because $\mathfrak{m} = \mathfrak{c} = \omega_2$ there are a Ramsey $p(\omega_2)$-point ultrafilter $\mathscr{F}$ on N and a family $\langle F_\xi \rangle_{\xi < \omega_2}$ in $\mathscr{F}$ such that $\{F_\xi: \xi < \omega_2\}$ is a base for $\mathscr{F}$ and $F_\xi \setminus F_\eta$ is finite whenever $\eta \leq \xi < \omega_2$.

Move to $M[G] \supseteq M[G_1]$. Because $\mathscr{P}\mathrm{N} \cap M[G] \subseteq M[G_1]$, $\mathscr{F}$ is still a Ramsey ultrafilter on N when examined in $M[G]$. What is more, it is still a $p(\omega_2)$-point. For suppose that $\langle A_\xi \rangle_{\xi < \omega_1} \in M[G]$ is a family in $\mathscr{F}$. (Recall that "ω_1" and "ω_2" have the same interpretations in $M[G]$ and $M[G_1]$, by fact (α) of (a) above.) Then for each $\xi < \omega_1$ there is an $\alpha(\xi) < \omega_2$ such that $A_\xi \supseteq F_{\alpha(\xi)}$. Take $\gamma = \sup_{\xi < \omega_1} \alpha(\xi) < \omega_2$; then $F_\gamma \in \mathscr{F}$ and $F_\gamma \setminus A_\xi$ is finite for every $\xi < \omega_1$.

On the other hand, in $M[G]$, $\mathfrak{c} = \omega_2$ (because $\mathscr{P}\mathrm{N} \cap M[G] = \mathscr{P}\mathrm{N} \cap M[G_1]$) and $2^{\omega_1} = \omega_3 > \mathfrak{c}$. So $\mathscr{F}$ cannot be $(\omega, \mathfrak{c})$-saturating, by 3(c).

Thus $M[G] \models$ "there is a Ramsey $p(\mathfrak{c})$-point ultrafilter which is not $(\omega, \mathfrak{c})$-saturating".

13. CONCLUDING REMARKS. (a) In [10, 26Ma], the question was raised: is every Ramsey $p(\mathfrak{c})$-point ultrafilter $(\omega, \mathfrak{c})$-saturating? This question is shown to be undecidable in ZFC by 11(a) and 12 above. But the point of the question was that the preceding Theorem 26E of [10] had referred, under the assumption that $\mathfrak{p} = \mathfrak{c}$, to $(\omega, \mathfrak{c})$-saturating Ramsey $p(\mathfrak{c})$-point ultrafilters; and if $\mathfrak{p} = \mathfrak{c}$ implies that Ramsey $p(\mathfrak{c})$-point ultrafilters are $(\omega, \mathfrak{c})$-saturating, there is a misleading redundancy. If we assume that $\mathfrak{p} = \mathfrak{c} > \mathfrak{m}$, which is the relevant context, the question remains open. In fact, as remarked in §10, we are interested in the case $\mathfrak{p} = \mathfrak{c} > \mathfrak{m}_{\sigma\text{-linked}}$. Another way of putting the remaining question is: can the conclusion of 11(a) be reached using a

σ-centered partially ordered set in place of the σ-linked partially ordered set of Theorem 9? because by Bell's theorem $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centered}}$ [10, 14C].

(b) Some further questions arise naturally from the results above. For instance, is the set of (ω, κ)-saturating ultrafilters always topologically invariant in $\beta N \backslash N$? It is surely topologically invariant in βN (because autohomeomorphisms of βN always arise from bijections from N to N); but that is not the same thing. There are models of set theory in which all autohomeomorphisms of $\beta N \backslash N$ are derived from functions from N to itself (see [20, Chapter IV]), and in these the set of (ω, κ)-saturating ultrafilters is always topologically invariant, as it is if $\kappa \leq \mathfrak{m}$ (Corollary 11(b) above). But J. Steprāns reports that in any model obtained by adding ω_2 Cohen reals to a model of ZFC + GCH the set of $(\omega, \mathfrak{c})$-saturating ultrafilters is not topologically invariant.

(c) Can Theorem 6 be refined to give a general description of those cardinals κ for which there is some (ω, κ)-saturating ultrafilter on N? We surely need $\kappa \leq \mathfrak{m}_{\text{countable}}$ and $2^{<\kappa} \leq \mathfrak{c}$, by Proposition 3; but there seems no reason to believe that these will be enough in general. We are inclined to suppose, for instance, that there need not be an $(\omega, \mathfrak{m})$-saturating ultrafilter, if $\omega_1 < \mathfrak{m} < \mathfrak{c}$.

14. Acknowledgements. The partially ordered set P of Theorem 9 is based on a method due to H. Woodin; P. Komjáth independently used a similar device. (The point is that if we try to work with finite functions instead of the sets D, we find ourselves with a partially ordered set which is not ccc.) We should also like to thank B. Balcar, P. Dordal, K. Kunen and J. Steprāns for helpful remarks.

REFERENCES

[1] T. Bartoszyński, *Combinatorial aspects of measure and category*, ***Fundamenta Mathematicae***, vol. 127 (1987), pp. 225–239.

[2] A. Blass, *The Rudin-Keisler ordering of P-points*, ***Transactions of the American Mathematical Society***, vol. 179 (1973), pp. 145–166.

[3] D. Booth, *Ultrafilters on a countable set*, ***Annals of Mathematical Logic***, vol. 2 (1970/71), pp. 1–24.

[4] C. C. Chang and H. J. Keisler, ***Model theory***, North-Holland, Amsterdam, 1973.

[5] W. W. Comfort and S. Negrepontis, ***The theory of ultrafilters***, Springer-Verlag, Berlin, 1974.

[6] Á. Császár (editor), ***Topology (proceedings of the fourth colloquium, Budapest, 1978)***. Vols. I, II, Colloquia Mathematica Societatis János Bolyai, vol. 23, North-Holland, Amsterdam, 1980.

[7] P. Dordal, *Towers in* $[\omega_1]^\omega$ *and* ${}^\omega\omega$, preprint, 1986.

[8] E. K. van Douwen, *The integers and topology*, in [16], pp. 111–168.

[9] E. Ellentuck and R. V. B. Rucker, *Martin's axiom and saturated models*, ***Proceedings of the American Mathematical Society***, vol. 34 (1972), pp. 243–249.

[10] D. H. Fremlin, ***Consequences of Martin's axiom***, Cambridge University Press, Cambridge, 1984.

[11] ———, *Cichoń's diagram*, ***Séminaire d'initiation à l'analyse (G. Choquet–M. Rogalski–J. Saint-Raymond), 23ème annee: 1983/1984***, Université Pierre et Marie Curie (Paris-VI), Paris, 1984, Exposé 5.

[12] A. Kucia and A. Szymański, *Absolute points in* $\beta N \backslash N$, ***Czechoslovak Mathematical Journal***, vol. 26 (1976), pp. 381–387.

[13] K. Kunen, ***Set theory***, North-Holland, Amsterdam, 1980.

[14] ———, *Weak p-points in* N*, in [6], pp. 741–749.

[15] ———, *Random and Cohen reals*, in [16], pp. 887–911.

[16] K. Kunen and J. E. Vaughan (editors), ***Handbook of set-theoretic topology***, North-Holland, Amsterdam, 1984.

[17] J. van Mill, *An introduction to* $\beta\omega$, in [16], pp. 503–568.

[18] A. W. MILLER, *Some properties of measure and category*, ***Transactions of the American Mathematical Society***, vol. 266 (1981), pp. 93–114.

[19] ———, *A characterization of the least cardinal for which the Baire category theorem fails*, ***Proceedings of the American Mathematical Society***, vol. 86 (1982), pp. 498–502.

[20] S. SHELAH, ***Proper forcing***, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin, 1982.

[21] W. WEISS, *Versions of Martin's axiom*, in [16], pp. 827–886.

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