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Fremlin 81

MEASURABLE FUNCTIONS AND ALMOST CONTINUOUS FUNCTIONS

D.H.Fremlin

I show that if (X,μ) is a Radon measure space and Y is a metric space, then a function from X to Y is μ-measurable iff it is almost continuous (= Lusin measurable). I discuss other cases in which measurable functions are almost continuous.

Introduction. A *topological measure space* is a quadruple $(X,\mathfrak{T},\Sigma,\mu)$ where (X,Σ,μ) is a measure space and $\mathfrak{T}$ is a topology on X such that $\mathfrak{T} \subseteq \Sigma$ (i.e. every open set is measurable). If $(Y,\mathfrak{S})$ is another topological space, I shall say that a function $f : X \to Y$ is *measurable* (corresponding to "Borel μ-measurable" in [14]) if $f^{-1}[G] \in \Sigma$ for every $G \in \mathfrak{S}$, and *almost continuous* (corresponding to "Lusin μ-measurable" in [14], or "μ-measurable" in [1b]) if whenever $E \in \Sigma$ and $\alpha < \mu E$ there is an $F \in \Sigma$ such that $F \subseteq E$, $\mu F \geq \alpha$ and $f|_F$ is continuous.

The known cases in which these concepts are related to each other seem to be the following.

(a) If (X,Σ,μ) is *complete* (i.e. $E \in \Sigma$, $\mu E = 0$ and $F \subseteq E$ imply that $F \in \Sigma$) and *locally determined* (i.e. if $A \subseteq X$ is such that $A \cap E \in \Sigma$ whenever $E \in \Sigma$ and $\mu E < \infty$, then $A \in \Sigma$ and $\mu A = \sup\{ \mu E : E \in \Sigma , E \subseteq A , \mu E < \infty \}$), then, for any topological space Y, every almost continuous function from X to Y will be measurable; this follows directly from the definitions. Observe (i) that this is a condition on X alone (ii) that it can always be satisfied by suitably adjusting μ (see [4], 64Ja-b) (iii) that, on the definitions I use, the condition is almost necessary as well as

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sufficient. I will use the abbreviation *c.l.d.* for "complete and locally determined".

(b) In the other direction, we have the following fundamental result: if μ is inner regular for the closed sets of finite measure and $\mathfrak{S}$ has a countable base, then every measurable function from X to Y is almost continuous. (I say that a measure μ on a set X is *inner regular* for a class $\mathcal{K}$ of subsets of X if $\mu E = \sup\{ \mu F : F \in \Sigma \cap \mathcal{K}, F \subseteq E \}$ for every $E \in \Sigma$, where Σ is the domain of μ.) Proofs of this result may be found in [14], p. 26, Theorem 5 and [1a], §5, no. 5, Théorème 4; the hypotheses there are more restrictive but it is easy to see that the same arguments apply.

(c) One possible generalization of this is given in [14], p. 129, Theorem 14: if μ is inner regular for the closed sets of finite measure, and Y is Souslin (i.e. a Hausdorff continuous image of a Polish space), then every measurable function from X to Y is almost continuous. The condition on Y can be relaxed, without changing the proof, to: Y is Hausdorff and Radon (i.e. every finite Borel measure on Y is inner regular for the compact sets) and the compact sets of Y are metrizable.

(d) I shall follow [4] in saying that a *Radon measure space* is a c.l.d. topological measure space $(X,\mathfrak{T},\Sigma,\mu)$ such that (i) μ is inner regular for the compact sets (ii) $\mathfrak{T}$ is Hausdorff (iii) every point of X has a neighbourhood of finite measure. (From the viewpoint of [14] or [1b], Σ should be interpreted as the algebra of μ-measurable sets.) Now (a) and (b) together show that if $(X,\mathfrak{T},\Sigma,\mu)$ is a Radon measure space and $(Y,\mathfrak{S})$ is a separable metrizable space, then a function $f : X \to Y$ is measurable iff it is almost continuous. The main new result of this paper (Theorem 2B) is that this remains true for non-separable Y.

(e) Subject to special axioms, rather stronger results are already known. (i) If there are no measurable cardinals, then all metric spaces are Radon, so we can apply (c), thus greatly weakening the hypotheses required for X. (ii) If there are no 2-valued measurable cardinals, then the arguments can be shortened and generalized to topological measure spaces $(X,\mathfrak{T},\Sigma,\mu)$ such that μ is inner regular for the closed sets of finite measure and (X,Σ,μ) is "perfect"; see 2E below.

(f) If we consider the question of further generalizations, there are two obvious directions in which to move. (i) We can ask, given some class $\mathfrak{X}$ of topological measure spaces, which topological spaces Y will have the property that if $X \in \mathfrak{X}$ and $f : X \to Y$ is measurable, then f is almost continuous. (ii) Assuming the existence of measurable cardinals, we can ask which topological measure spaces X will have the property that every measurable function from X to a metric space will be almost continuous. I give examples and partial results related to these questions in §3.

1. Preliminary results

I begin with a description of some of the more or less well-known facts about topological measure spaces that I wish to call upon, with an easy, but useful, new result on Radon probability spaces (1D).

1A Hyperstonean spaces. Among the technical devices I need the following is particularly important. Let (X,Σ,μ) be any measure space of finite magnitude (i.e. $\mu X < \infty$). Then its measure algebra $\mathfrak{A} = \Sigma/\{ E : \mu E = 0 \}$ is a Dedekind complete Boolean algebra, so can be identified with the algebra of clopen sets of an extremally disconnected compact Hausdorff space Z. For $E \in \Sigma$ let $\hat{E}$ be the clopen set in Z corresponding with the image $E^\bullet$ of E in $\mathfrak{A}$. Then there is a unique Radon measure $\hat{\mu}$ on Z such that $\hat{\mu}\hat{E} = \mu E$ for every $E \in \Sigma$. (See [9], p. 120, Theorem 3.) I shall call $(Z,\hat{\mu})$ the *hyperstonean space* of (X,μ).

1B Measure algebra embeddings, inverse-measure-preserving functions. Let (X,Σ,μ) and (Y,T,ν) be measure spaces with measure algebras $\mathfrak{A}$, $\mathfrak{B}$ respectively; I shall again write μ, ν for the induced functionals on $\mathfrak{A}$, $\mathfrak{B}$. A function $f : X \to Y$ is *inverse-measure-preserving* if $f^{-1}[F] \in \Sigma$ and $\mu f^{-1}[F] = \nu F$ for every $F \in T$. In this case f gives rise to a map $\phi : \mathfrak{B} \to \mathfrak{A}$ given by $\phi(F^\bullet) = (f^{-1}[F])^\bullet$ for every $F \in T$, and ϕ can be regarded as embedding $\mathfrak{B}$ in $\mathfrak{A}$. Conversely, we have the following important result.

1C THEOREM. Let (X,Σ,μ) be a complete measure space and $(Y,\mathfrak{S},T,\nu)$ a Radon measure space. Write $\mathfrak{A}$, $\mathfrak{B}$ for their measure algebras; suppose that $\mu X = \nu Y < \infty$, and let $\phi : \mathfrak{B} \to \mathfrak{A}$ be a measure-preserving embedding. Then ϕ is induced by some inverse-measure-preserving $f : X \to Y$.

proof. Apply [6], Theorem 1 to the map $F \mapsto \phi(F^{\bullet}) : T \to \mathfrak{A}$. In [6], it is shown only that $f^{-1}[F] \in \Sigma$ for Borel sets $F \subseteq Y$; but since every member of T is sandwiched between two Borel sets of the same measure, and (X,Σ,μ) is complete, it follows at once that f is inverse-measure-preserving in the sense I use here.

1D If we put this together with Maharam's representation theorem for measure algebras we obtain the following corollary.

PROPOSITION. Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon probability space. Then there is a cardinal κ and an inverse-measure-preserving function $f : \{0,1\}^{\kappa} \to X$, where $\{0,1\}^{\kappa}$ is given its usual Radon measure (i.e. the product measure for which each coordinate takes the values 0, 1 with equal probability $\frac{1}{2}$).

proof. By Theorem 1C, it is enough to find an embedding of the measure algebra $\mathfrak{A}$ of X into the measure algebra $\mathfrak{B}_{\kappa}$ of $\{0,1\}^{\kappa}$ for some κ. But this is easy to construct from Maharam's decomposition of $\mathfrak{A}$ into a countable number of pieces each isomorphic, up to a scalar multiple of the measure, to some $\mathfrak{B}_{\kappa}$; taking each piece of $\mathfrak{A}$ to be the measure algebra of $\{u_i\}\times\{0,1\}^{\kappa(i)}$, where the $\{u_i\}$ are atoms with measures summing to 1, we can embed $\mathfrak{A}$ into the measure algebra of $\{0,1\}^{\mathbb{N}}\times\{0,1\}^{\kappa}$ where $\kappa = \sup_i \kappa(i)$. (See [9], §14 for a precise statement, with proof, of Maharam's theorem.)

REMARK. An alternative proof can be got from the main theorem of [10].

1E Finally, I spell out a well-known result on measure spaces which are *diffuse* i.e. such that if $\mu E > 0$ there is a measurable set $F \subseteq E$ such that $\mu F > 0$, $\mu(E\setminus F) > 0$.

PROPOSITION. Let (X,Σ,μ) be a diffuse measure space of finite magnitude. Then there is an inverse-measure-preserving function $f : X \to [0,\mu X]$ where the interval $[0,\mu X]$ is given Lebesgue measure.

2. The main results

The key idea of this section is Lemma 2A; the rest is made up of easy corollaries.

2A LEMMA. Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space of finite magnitude and $\langle X_\iota\rangle_{\iota\in I}$ a partition of X which is *completely measurable* i.e. such that $\bigcup_{\iota\in A} X_\iota \in \Sigma$ for every $A \subseteq I$. Then $\mu X = \Sigma_{\iota\in I}\mu X_\iota$.

proof. I suppose rather that there is an example in which

(i) $\mu X > \Sigma_{\iota\in I}\mu X_\iota$

and seek to derive a contradiction. The first half of the argument proceeds through a series of reductions which will be familiar to anyone who has studied the famous measurable cardinal problem.

(a) We may of course suppose that I is of the least possible cardinal for which such an $(X,\mathfrak{T},\Sigma,\mu)$ and $\langle X_\iota\rangle_{\iota\in I}$ can be found, and that I is itself a cardinal (= initial ordinal) κ. In this case we find that

(ii) If $\mathcal{A}$ is a disjoint family of subsets of κ and $\#(\mathcal{A}) < \kappa$ and $\mu(\bigcup_{\xi\in A} X_\xi) = 0$ $\forall$ $A \in \mathcal{A}$, then $\mu(\bigcup_{\xi\in\bigcup\mathcal{A}} X_\xi) = 0$.

P Let Y be $\bigcup_{\xi\in\bigcup\mathcal{A}} X_\xi$ and, for each $A \in \mathcal{A}$, set $Y_A = \bigcup_{\xi\in A} X_\xi$; then $\langle Y_A\rangle_{A\in\mathcal{A}}$ is a completely measurable partition of Y, so (because $\#(I)$ is minimal) $\mu Y = \Sigma_{A\in\mathcal{A}}\mu Y_A = 0$. (The point is that Y, with the induced topology and measure, is a Radon measure space.) **Q**

(b) Next, we can suppose that

(iii) $\forall$ $A \subseteq \kappa$, $\mu(\bigcup_{\xi\in A} X_\xi)$ is either 0 or μX.

P Consider the measure ν on κ given by

$\nu A = \mu(\bigcup_{\xi\in A} X_\xi)$ $\forall$ $A \subseteq \kappa$.

? Suppose, if possible, that ν is diffuse. Then by 1E there is an inverse-measure-preserving function $g : \kappa \to [0,\nu\kappa] = [0,\mu X]$. Define $f : X \to [0,\mu X]$ by saying that $f(t) = g(\xi)$ if $t \in X_\xi$. Then f is also inverse-measure-preserving; in particular, it is measurable. Because $[0,\mu X]$ is a separable metric space, f is almost continuous. It follows that the measure μf^{-1}, defined on $T = \{ E : f^{-1}[E] \in \Sigma \}$ by $(\mu f^{-1})(E) = \mu(f^{-1}[E])$ for every $E \in T$, is a Radon measure ([14], p. 32, Theorem 8); as it agrees with Lebesgue measure on the compact sets, it must actually be Lebesgue measure; but μf^{-1} is defined on every subset of $[0,\mu X]$, while Lebesgue measure is not. **X**

Consequently there must be an atom for ν, i.e. a set $A \subseteq \kappa$ such that $\nu A > 0$ and $\nu B = 0$ or νA for every $B \subseteq A$. Repeating the argument above for measurable subsets of X, we see that κ must be a countable union of atoms. As $\nu\kappa = \mu X > \Sigma_{\xi\in\kappa}\mu X_\xi = \Sigma_{\xi\in\kappa}\nu\{\xi\}$, at least one of the atoms A for ν must be such that $\nu A > \Sigma_{\xi\in A}\nu\{\xi\}$. Set $X' = \bigcup_{\xi\in A}X_\xi$. Then $\langle X_\xi\rangle_{\xi\in A}$ is a completely measurable partition of X'; $\mu X' > \Sigma_{\xi\in A}\mu X_\xi$; and if $B \subseteq A$, then $\mu(\bigcup_{\xi\in B}X_\xi) = \nu B$ is either $\nu A = \mu X'$ or 0. Of course $\#(A) \leqslant \kappa$ so $\#(A)$ must be actually equal to κ and, replacing X by X' and re-enumerating $\langle X_\xi\rangle_{\xi\in A}$, we obtain an example in which (i)-(iii) all hold. **Q**

(c) Thirdly, we may suppose that, for some cardinal λ,

(iv) $(X,\mathfrak{T},\Sigma,\mu)$ is $\{0,1\}^\lambda$ with its usual measure.

P Normalizing the measure, we may obviously take μX to be 1. By Proposition 1D, there is a λ and an inverse-measure-preserving $f : \{0,1\}^\lambda \to X$; now $\langle f^{-1}[X_\xi]\rangle_{\xi\in\kappa}$ is a completely measurable partition of $\{0,1\}^\lambda$ still satisfying (i)-(iii). **Q**

(d) We are now in a position to move towards the kill, using a standard measurable-cardinal argument. Assume that we have (i)-(iv). Let $\mathcal{F}$ be

$$\{ A : A \subseteq \kappa ,\ \mu(\bigcup_{\xi\in A}X_\xi) = 1 \} .$$

Then $\mathcal{F}$ is a filter on κ which is an ultrafilter by (iii) and is closed under intersections of fewer than κ members, by (ii). Of course $\kappa > \aleph_0$, so that κ is 2-valued-measurable; by Ulam's theorem ([3], chap. 6, Theorem 1.2) $\kappa > \mathfrak{c}$.

For each $\xi < \kappa$, $\mu(\bigcup_{\eta\geqslant\xi}X_\eta) = 1$. Now the measure of $\{0,1\}^\lambda$ is *completion regular* i.e. inner regular for the closed G_δ sets (see e.g. [2], Theorem 3). So there is a closed G_δ set $C_\xi \subseteq \bigcup_{\eta\geqslant\xi}X_\eta$

with $\mu C_\xi > 0$; now C_ξ can be regarded as $C_\xi'\times\{0,1\}^{\lambda\setminus I(\xi)}$ for some countable $I(\xi) \subseteq \lambda$ and non-empty $C_\xi' \subseteq \{0,1\}^{I(\xi)}$; choose $t_\xi \in C_\xi'$.

Consider

$$J = \lim_{\xi\to\mathcal{F}}I(\xi) = \{ \iota : \iota\in\lambda ,\ \{ \xi : \iota\in I(\xi) \} \in \mathcal{F} \} .$$

Note first that J is countable. **P?** If not, there is a set $K \subseteq J$ with $\#(K) = \aleph_1$; now

$$\emptyset = \{ \xi : K \subseteq I(\xi) \} = \bigcap_{\iota\in K}\{ \xi : \iota\in I(\xi) \} \in \mathcal{F} ,$$

because $\#(K) \leqslant \mathfrak{c} < \kappa$; which is impossible. **XQ** So

$$A = \{ \xi : J \subseteq I(\xi) \} = \bigcap_{\iota\in J}\{ \xi : \iota\in I(\xi) \} \in \mathcal{F} .$$

For $\xi\in A$, let $s_\xi = t_\xi|_J \in \{0,1\}^J$. As $\#(\{0,1\}^J) \leqslant \mathfrak{c} < \kappa$, there is an $s\in\{0,1\}^J$ such that $B = \{ \xi : \xi\in A ,\ s_\xi = s \} \in \mathcal{F}$ (for otherwise $\emptyset = \bigcap_s\{ \xi : s_\xi \neq s \}$ would have to belong to $\mathcal{F}$).

Now choose $\langle\zeta(\xi)\rangle_{\xi<\kappa}$ inductively, as follows. Given $\langle\zeta(\eta)\rangle_{\eta<\xi}$, set $J(\xi) = \bigcup_{\eta<\xi}I(\zeta(\eta))\setminus J$. Then $\#(J(\xi)) < \kappa$, so

$$B(\xi) = \{ \eta : I(\eta)\cap J(\xi) = \emptyset \}$$
$$= \bigcap_{\iota\in J(\xi)}\{ \eta : \iota\notin I(\eta) \} \in \mathcal{F} ,$$

and there is a $\zeta(\xi) \in B\cap B(\xi)\cap[\xi,\kappa[$. Thus we shall have

$$I(\zeta(\xi))\cap I(\zeta(\eta)) = J ,\quad t_{\zeta(\xi)}|_J = s$$

whenever $\xi\neq\eta$ in κ. Accordingly there is a $t\in X$ such that

$$t|_{I(\zeta(\xi))} = t_{\zeta(\xi)} \quad \forall\ \xi<\kappa$$

and $t\in C_{\zeta(\xi)} \subseteq \bigcup_{\eta\geqslant\zeta(\xi)}X_\eta \subseteq \bigcup_{\eta\geqslant\xi}X_\eta$ for every $\xi<\kappa$; which is impossible, as $\langle X_\xi\rangle_{\xi<\kappa}$ is supposed to be a partition of X.

Here at last is the required contradiction.

2B <u>THEOREM</u>. <u>If X is a Radon measure space and Y is a metric space, then a function $f : X\to Y$ is measurable iff it is almost continuous.</u>

<u>proof.</u> (a) Of course (by the definition I am using of "Radon" measure space) X is c.l.d.; so if f is almost continuous it is measurable.

(b) It will be enough to consider the case in which $\mu X < \infty$; for if the restriction of f to every set of finite measure is almost continuous, f will be almost continuous.

(c) Now take X to be of finite magnitude, and f measurable.

In this case, for any $\varepsilon > 0$, there is a locally finite cover of Y by open sets of diameter $\leqslant \varepsilon$, by A.H.Stone's theorem; let $\langle G_\xi \rangle_{\xi<\kappa}$ enumerate such a cover. Writing $E_\xi = G_\xi \setminus \bigcup_{\eta<\xi} G_\eta$, we see that $\langle E_\xi \rangle_{\xi<\kappa}$ is a partition of Y such that the union of any subfamily is G_δ. Consequently $\langle f^{-1}[E_\xi] \rangle_{\xi<\kappa}$ is a completely measurable partition of X. By Lemma 2A, $\mu X = \Sigma_{\xi<\kappa} \mu f^{-1}[E_\xi]$, and there is a countable $A \subseteq \kappa$ such that $\mu X = \mu(\bigcup_{\xi \in A} f^{-1}[E_\xi])$.

(d) Accordingly we can find, for each $n \in \mathbb{N}$, a Borel set $B_n \subseteq Y$ such that $\mu X = \mu f^{-1}[B_n]$ and B_n can be covered by countably many sets of radius $\leqslant 2^{-n}$. Setting $X_o = \bigcap_{n \in \mathbb{N}} f^{-1}[B_n]$, $\mu X_o = \mu X$ and $f[X_o]$ is separable. Consequently the restriction of f to X_o is almost continuous and f is almost continuous.

2C COROLLARY. *Let X be a Radon measure space.*

(a) If Y is a metric space and $f : X \to Y$ is measurable and if $Y = \bigcup \{ G : G \subseteq Y$ open, $\mu f^{-1}[G] < \infty \}$, then there is a Radon measure on Y for which f is inverse-measure-preserving.

(b) If $\langle Y_n \rangle_{n \in \mathbb{N}}$ is a sequence of metric spaces with product Y, and if $f_n : X \to Y_n$ is measurable for each $n \in \mathbb{N}$, then $t \mapsto \langle f_n(t) \rangle_{n \in \mathbb{N}} : X \to Y$ is measurable.

(c) If Y is a metrizable linear topological space, then the set of measurable functions from X to Y is a linear subspace of Y^X.

proof. All of these are true for almost continuous functions without the metrizability hypothesis on Y ([14], p. 31; [1b], §2, no. 3, Proposition 4; [1a], §5, no. 3, Théorème 1 & Corollaire 3).

2D COROLLARY. *If X is a Radon measure space in which every subset is measurable, then the measure on X consists entirely of singletons.*

proof. Apply Lemma 2A to the partitions of compact subsets of X into singletons.

2E REMARKS. (a) Let us say that a measure space (X,Σ,μ) has the *completely measurable partition property*, or *c.m.p.p.*, if, for any completely measurable partition $\langle X_\iota \rangle_{\iota \in I}$ of X,

$$\mu E = \Sigma_{\iota \in I} \mu(E \cap X_\iota) \quad \forall \ E \in \Sigma .$$

Thus Lemma 2A becomes: every Radon measure space of finite magnitude (and hence every Radon measure space) has the c.m.p.p. The content of Theorem 2B is in effect that if a semi-finite topological measure space has the c.m.p.p., and every measurable function to a separable metric space is almost continuous, then every measurable function to any metric space is almost continuous. (A measure space (X,μ) is *semi-finite* if μ is inner regular for the sets of finite measure.) Moreover, 2C clearly applies to any c.l.d. measure space with the c.m.p.p.; we can avoid using the auxiliary notion of almost continuity by observing that 2C applies to arbitrary measure spaces of finite magnitude if all the Y_n, Y are separable metric spaces. (Strictly speaking, we need μ complete in 2C(a).)

(b) There are great simplifications to be made if we assume that there are no measurable cardinals, i.e. that Corollary 2D applies to all measure spaces of finite magnitude. In this case, every semi-finite measure space has the c.m.p.p.; in Lemma 2A, we have only to consider the measure ν on I given by $\nu A = \mu(\bigcup_{\iota \in I} X_\iota)$ for every $A \subseteq I$. Consequently 2B applies to any topological measure space in which the measure is inner regular for the closed sets, and 2C applies to any c.l.d. measure space. (See [15].)

(c) If we assume only that there are no 2-valued measurable cardinals, then we can still dispense with parts (c) and (d) of the proof of Lemma 2A, and see that a c.l.d. measure space (X,Σ,μ) will have the c.m.p.p. if it is *perfect* (i.e. whenever $f : X \to \mathbb{R}$ is measurable, the measure μf^{-1} with domain $\{ E : f^{-1}[E] \in \Sigma \}$ is inner regular for the compact sets), for this is the essential property of Radon measure spaces required in part (b) of the proof. This is done in [8], Theorem 2.5.

(d) Taking up the idea of (c), it is plain that even if there are 2-valued measurable cardinals, then the conditions on X suggested in (c) will suffice if we require also that $\#(X)$ is smaller than the least 2-valued measurable cardinal; as the latter must be unthinkably large, this is likely to cover any X arising in ordinary applications.

(e) Collecting these together, we see that a c.l.d. topological measure space $(X,\mathfrak{T},\Sigma,\mu)$ will have the property that every measurable function from X to a metric space Y is almost continuous if

<u>either</u> $(X,\mathfrak{T},\Sigma,\mu)$ is a Radon measure space;

<u>or</u> (X,Σ,μ) is perfect, μ is inner regular for the closed sets of finite measure, and $\#(X)$ is less than any 2-valued measurable cardinal;

<u>or</u> μ is inner regular for the closed sets of finite measure, there is no real-valued measurable cardinal, and $\#(X)$ is less than any 2-valued measurable cardinal.

In 3B I shall give a further case, based on an entirely different argument.

3. <u>Further elaborations</u>

Following the program suggested in (f) of the Introduction, I seek further special cases and examples.

3A <u>Quasi-Radon measure spaces</u>. It will be helpful to use the following idea from [4]. A *quasi-Radon measure space* is a c.l.d. topological measure space $(X,\mathfrak{T},\Sigma,\mu)$ such that (i) μ is inner regular for the closed sets (ii) if $\mathcal{G}\subseteq\mathfrak{T}$ is upwards-directed, then $\mu(\bigcup\mathcal{G}) = \sup_{G\in\mathcal{G}}\mu G$ (iii) if $\mu E > 0$ then there is an open set G such that $\mu G < \infty$ and $\mu(E\cap G) > 0$. Some of the basic properties of these spaces are set out in [4]; those of principal interest to us here are (a) every Radon measure space is quasi-Radon ([4], 73B) (b) if $(X,\mathfrak{T},\Sigma,\mu)$ is a quasi-Radon measure space and $Y\subseteq X$ is any set, then $(Y,\mathfrak{T}_Y,\Sigma_Y,\mu_Y)$ is a quasi-Radon measure space, where $\mathfrak{T}_Y$ is the induced topology on Y , $\Sigma_Y = \{ E\cap Y : E\in\Sigma \}$, and $\mu_Y F = \inf\{ \mu E : E\in\Sigma , E\supseteq F \}$ for every $F\in\Sigma_Y$. (Crucial for (b) are the facts that quasi-Radon measure spaces are decomposable, and that if in a quasi-Radon measure space we have an upwards-directed family $\mathcal{G}$ of open sets, then $\mu(E\cap\bigcup\mathcal{G}) = \sup_{G\in\mathcal{G}}\mu(E\cap G)$ for every measurable set E ; see [4], 72B and 72Gd. For general facts concerning subspaces, see [5], §9.) Thus quasi-Radon measure spaces arise fairly naturally as non-measurable subspaces of Radon measure spaces, just as Radon measure spaces arise as measurable subspaces of compact Hausdorff spaces.

In my first result, however, I shall be dealing with second-countable quasi-Radon measure spaces, where the topological structure is merely a matter of notational convenience, since it can be described in terms of a sequence of measurable sets. The argument here is lifted directly from [12]; I spell it out as I am generalizing it in a new direction.

3B <u>PROPOSITION</u>. <u>Let</u> $(X,\mathfrak{T},\Sigma,\mu)$ <u>be a second-countable quasi-Radon measure space</u>. <u>Then</u> X <u>has the c.m.p.p.</u>

<u>proof</u> (a) As X is locally determined, and every subspace of X is also a second-countable quasi-Radon measure space, it will be enough to consider the case in which $\mu X < \infty$. In this case μ will be outer regular for the open sets (i.e. $\mu E = \inf\{ \mu G : G \text{ open}, G\supseteq E \}$ for every $E\in\Sigma$).

(b) The essential idea is this: if $(Y,\mathcal{P}Y,\nu)$ is any measure space of finite magnitude in which every set is measurable, and $A\subseteq X\times Y$ is such that $A_u = \{ t : (t,u)\in A \}\in\Sigma$ for every $u\in Y$, then A is measurable for the ordinary completed product measure $\mu\times\nu$ on $X\times Y$. **P** Enumerate as $\langle G_n\rangle_{n\in\mathbb{N}}$ a base for $\mathfrak{T}$. Let ρ be the usual product outer measure on $X\times Y$. Let $\varepsilon > 0$. For each $u\in Y$, there is an open set $H_u\supseteq A_u$ such that $\mu H_u \leqslant \mu A_u + \varepsilon$; set $I(u) = \{ n : G_n\subseteq H_u \}$. Set $D_n = \{ u : n\in I(u) \}$ and $H = \{ (t,u) : t\in H_u \} = \bigcup_{n\in\mathbb{N}} G_n\times D_n$. Then H is $\mu\times\nu$-measurable because every D_n is ν-measurable. So

$$\rho A \leqslant (\mu\times\nu)(H) = \int\mu H_u\,\nu(du) \leqslant \int\mu A_u\,\nu(du) + \varepsilon\nu Y ;$$

the last integral exists because ν is defined on every subset of Y . As ε is arbitrary, $\rho A \leqslant \int\mu A_u\,\nu(du)$. Similarly, $\rho((X\times Y)\setminus A) \leqslant \int\mu(X\setminus A_u)\nu(du)$. So $\rho A + \rho((X\times Y)\setminus A) \leqslant \mu X.\nu Y$ and A must be measurable for the measure derived from ρ viz. $\mu\times\nu$. **Q**

(c) Consequently, if $\langle X_\xi\rangle_{\xi<\kappa}$ is a completely measurable partition of X , where κ is a cardinal, and $\mu X > 0$, we can define ν on κ by writing $\nu A = \mu(\bigcup_{\xi\in A}X_\xi)$ for $A\subseteq\kappa$, and consider

$$\{ (t,\xi) : t\in\bigcup_{\eta\leqslant\xi}X_\eta \} \subseteq X\times\kappa .$$

By (b), this is a measurable set; applying Fubini's theorem to it in each direction we obtain

$$\int\mu(\bigcup_{\eta\leqslant\xi}X_\eta)\nu(d\xi) = \int\nu([h(t),\kappa[)\mu(dt)$$

where $h(t) = \xi$ for $t \in X_\xi$. From this it is plain that $\mu(\bigcup_{\eta\leq\xi} X_\eta) > 0$ for some $\xi < \kappa$.

(d) Now the arguments used at the beginning of the proof of 2A show that X has the c.m.p.p.

3C Thus the results of §2 apply to some quasi-Radon measure spaces as well as to all Radon measure spaces. But they do not apply (in the presence of measurable cardinals) to all quasi-Radon measure spaces, or even to all subspaces of Radon probability spaces, as the following example shows.

Example. Assume that there is a set I and a measure ν defined on every subset of I, such that $\nu I = 1$ and $\nu\{\xi\} = 0$ for every $\xi \in I$. Let $(Z, \mathfrak{V})$ be the hyperstonean space of $(I, \mathcal{P}I, \nu)$. By 1D above, the canonical isomorphism between the measure algebras of Z and of I is generated by an inverse-measure-preserving function $f : I \to Z$, so that $\nu(J \triangle f^{-1}[\hat{J}]) = 0$ for every $J \subseteq I$.

Let $\mathcal{N}$ be $\{ J : J \subseteq I, \nu J = 0 \}$, and set $W = [0,1]^{\mathcal{N}} \times Z$ with the product measure μ (each copy of $[0,1]$ being given Lebesgue measure). Then μ is completion regular ([2], Theorem 3).

For $\xi \in I$, let $X_\xi \subseteq W$ be

$$\{ (t, f(\xi)) : \forall \ A \in \mathcal{N}, t(A) = 1 \Leftrightarrow \xi \in A \}.$$

Set $X = \bigcup_{\xi\in I} X_\xi$. I claim that X, with the subspace measure described in 3A above, has $\langle X_\xi \rangle_{\xi\in I}$ as a completely measurable partition, and that $\mu_X X = 1 > 0 = \Sigma_{\xi\in I}\mu_X X_\xi$.

I have to check the following points.

(i) $\langle X_\xi \rangle_{\xi\in I}$ is a partition of X because, if ξ and η are distinct members of I, then $t(\{\xi\}) = 1$ whenever $(t,u) \in X_\xi$, while $t(\{\xi\}) < 1$ whenever $(t,u) \in X_\eta$; so $X_\xi \cap X_\eta = \emptyset$.

(ii) $\mu_X X = 1$ because X meets every set in W of positive measure. **P** Take any $E \subseteq W$ with $\mu E > 0$; because μ is completion regular, there is a closed G_δ set $F \subseteq E$ with $\mu F > 0$. Let $\mathcal{A} \subseteq \mathcal{N}$ be a countable set such that F is expressible as $F' \times [0,1]^{\mathcal{N}\setminus\mathcal{A}}$, where $F' \subseteq [0,1]^{\mathcal{A}} \times Z$, and set $B = \bigcup\mathcal{A} \in \mathcal{N}$. Now $\nu(I \setminus B) = 1$, so $I \setminus B$ must meet $f^{-1}[H]$ whenever $\mathfrak{V}H > 0$, and $f[I\setminus B]$ must have outer measure 1 in Z; consequently $[0,1[^{\mathcal{N}} \times f[I \setminus B]$ has outer measure 1 in W, and meets F in $(s, f(\xi))$ say, where $s(A) < 1$ for every $A \in \mathcal{N}$ and $\xi \in I \setminus B$. Take $t \in [0,1]^{\mathcal{N}}$ such that

$$t(A) = s(A) \quad \forall \ A \in \mathcal{A},$$
$$t(A) = 1 \Leftrightarrow \xi \in A, \quad \forall \ A \in \mathcal{N}.$$

Then $(t|_{\mathcal{A}}, f(\xi)) = (s|_{\mathcal{A}}, f(\xi))$ so $(t, f(\xi)) \in F$; also $(t, f(\xi)) \in X_\xi$, so X meets F and X meets E. **Q**

(iii) To see that $\langle X_\xi \rangle_{\xi\in I}$ is completely measurable in X, let $J \subseteq I$ be any set. Then $A = J \triangle f^{-1}[\hat{J}] \in \mathcal{N}$. Set $F_J = [0,1]^{\mathcal{N}} \times \hat{J}$. Now if $\xi \in J$ and $(t,u) \in X_\xi \setminus F_J$, we have $u = f(\xi) \notin \hat{J}$, so $\xi \in A$ and $t(A) = 1$. Thus $\bigcup_{\xi\in J} X_\xi \setminus F_J \subseteq \{ (t,u) : t(A) = 1 \}$, which has measure 0 in W. Similarly

$$(X \cap F_J) \setminus \bigcup_{\xi\in J} X_\xi = \bigcup_{\xi\in I\setminus J} X_\xi \cap F_J = \bigcup_{\xi\in I\setminus J} X_\xi \setminus F_{I\setminus J}$$

has zero measure. So $\mu(X \cap (F_J \triangle \bigcup_{\xi\in J} X_\xi)) = 0$ and $\bigcup_{\xi\in J} X_\xi$ is equivalent in X to the measurable set $X \cap F_J$.

(iv) Finally, $\mu_X X_\xi \leq \mu\{ (t,u) : t(\{\xi\}) = 1 \} = 0$, for each $\xi \in I$.

This completes the proof that X does not have the c.m.p.p. To see that it does not have the property of Theorem 2B, take $g : X \to I$ given by $g(t,u) = \xi$ whenever $(t,u) \in X_\xi$. If I is given the discrete topology, g cannot be continuous on any set $F \subseteq X$ of positive measure, because if it were then $\{ F \cap g^{-1}[J] : J \subseteq I \text{ finite} \}$ would be an upwards-directed family of relatively open sets of zero measure covering F; but the induced measure on F has to be quasi-Radon, so this is impossible.

3D Remarks. I should like now to turn to the other part of the program in (f) of the introduction. If X is any topological measure space of finite magnitude, we can write $\mathcal{Y}(X)$ for the class of topological spaces Y such that every measurable function from X to Y is almost continuous. It is clear that $\mathcal{Y}$ is always closed under countable products and subspaces; that $\mathcal{Y}$ contains $\{0,1\}$ iff it contains all separable metric spaces iff it contains all Souslin spaces iff it is closed under countable discrete unions; and that $\mathcal{Y}$ is closed under arbitrary discrete unions whenever X has the c.m.p.p., sufficient conditions for which are set out in 2E(e). Judicious use of these facts will produce many "new" pairs X , Y such that every measurable $f : X \to Y$ is almost continuous; but they seem mostly to be of little importance. The barriers to finding new examples among familiar spaces are indicated by the following example. Take X to be $[0,1]$ with the usual topology and Lebesgue measure, take Y to be $[0,1]$ with the half-open-interval topology generated by $\{ Y \cap [\alpha,\beta[: \alpha < \beta \}$, and f the identity map. Then f is measurable but is not continuous on any uncountable set. As Y is completely regular, it can be embedded in compact Hausdorff spaces, e.g. the split interval $I^{\|}$ (i.e. $[0,1]\times\{-,+\}$ with the compact Hausdorff order topology derived from its lexicographic ordering). Now $I^{\|}$ is a Radon hereditarily Lindelöf hereditarily separable compact Hausdorff space. Thus most of the obvious approximations to metrizability are insufficient to ensure membership of $\mathcal{Y}([0,1])$. However, subject to special axioms, I have found a few cases; I describe one which depends on Martin's Axiom and another which depends on the Generalized Continuum Hypothesis.

3E <u>PROPOSITION</u>. <u>Assume that Martin's Axiom is true. Then for</u> $\#(I) < \mathfrak{c}$, <u>a function</u> $f : [0,1] \to [0,1]^I$ <u>is measurable iff it is almost continuous iff all the coordinate functionals</u> $f_\xi : [0,1] \to [0,1]$ <u>are measurable, where</u> $\xi \in I$.

<u>proof</u>. Clearly the only thing we need to check is that if every f_ξ is measurable then f is almost continuous. I use the technique of [11], §4, Theorem 1. Let $\varepsilon > 0$ and let $\mathcal{G}$ be the set of open subsets of $[0,1]$ of measure $< \varepsilon$, ordered by $\subseteq$. Then if $\mathcal{H} \subseteq \mathcal{G}$ is uncountable, there are distinct $G , H \in \mathcal{H}$ such that $G \cup H \in \mathcal{G}$, because $L^1([0,1])$ is separable (see [11], p. 168). For each $\xi \in I$ let $\mathcal{G}_\xi = \{ G : G \in \mathcal{G} , f_\xi|[0,1]\setminus G$ is continuous $\}$; then $\mathcal{G}_\xi$ is upwards-cofinal in $\mathcal{G}$. By Martin's axiom, there is an upwards-directed $\mathcal{H} \subseteq \mathcal{G}$ which meets every $\mathcal{G}_\xi$. Set $H = \bigcup\mathcal{H}$; then $\mu H \leq \varepsilon$ and $f_\xi|[0,1]\setminus H$ is continuous for every $\xi \in I$, i.e. $f|[0,1]\setminus H$ is continuous. As ε is arbitrary, f is almost continuous.

<u>Remarks</u>. What I have really shown here is that if X is a quasi-Radon measure space with a separable measure algebra, and if Martin's axiom is true, then $\mathcal{Y}(X)$ is closed under products of fewer than $\mathfrak{c}$

members. Note that $[0,1]^{\mathfrak{c}}$ never belongs to $\mathcal{Y}([0,1])$ as $I^{\|}$ embeds into $[0,1]^{\mathfrak{c}}$. Moreover, if X has non-separable measure algebra, then $\mathcal{Y}(X)$ will not normally be closed under products of $\aleph_1$ factors, as the next example shows.

3F <u>Example</u>. Set $X = [0,1]^I$ with its usual Radon measure (the product of Lebesgue measure on each factor), where I is any uncountable set. Then there is a measurable function $f : X \to X$ which is not almost continuous. **P** Set $f(t)(\xi) = t(\xi)$ if $0 < t(\xi) < 1$, $1 - t(\xi)$ otherwise, for $t = \langle t(\xi)\rangle_{\xi\in I} \in X$. Then $\mu f^{-1}[E]$ exists $= \mu E$ for every measurable cylinder set in X , therefore for every Baire set $E \subseteq X$. Because μ is completion regular, f is inverse-measure-preserving, therefore measurable. But f is not continuous on any non-empty Baire set of X , so cannot be almost continuous. **Q**

I conclude with another theorem on general Radon measure spaces.

3G <u>THEOREM</u>. <u>Assume the Generalized Continuum Hypothesis. If</u> $(X,\mathfrak{T},\Sigma,\mu)$ <u>is a Radon measure space and</u> ζ <u>is an ordinal, every measurable function</u> $f : X \to \zeta$ <u>is almost continuous</u>.

<u>proof</u>. (<u>a</u>) Observe that

(i) as usual, it will be enough to consider the case $\mu X < \infty$;

(ii) it will be enough to show that if $\mu X > 0$, there is a $\xi < \zeta$ such that $\mu f^{-1}[\{\xi\}] > 0$; for then, if $\mu X < \infty$, an exhaustion argument will give a countable $A \subseteq \zeta$ such that $\mu f^{-1}[A] = \mu X$, and now the restriction of f to $f^{-1}[A]$ is almost continuous, so that f is almost continuous.

(<u>b</u>) So I take $0 < \mu X < \infty$ and seek to show, by induction on ζ , that if $f : X \to \zeta$ is measurable there is a $\xi < \zeta$ such that $\mu f^{-1}[\{\xi\}] > 0$. "Most" cases are easy. (i) For countable ζ the result is trivial. (ii) If $\zeta = \eta + 1$ then either $\mu f^{-1}[\{\eta\}] > 0$ or $\mu f^{-1}[\eta] > 0$ and (using the inductive hypothesis on the restriction of f to $f^{-1}[\eta]$) there is a $\xi < \eta$ such that $\mu f^{-1}[\{\xi\}] > 0$. (iii) If ζ is a limit ordinal for which $\kappa = cf(\zeta) < \zeta$, let $\langle\zeta_\eta\rangle_{\eta<\kappa}$ be a strictly increasing family in ζ with supremum ζ . Define $g : \zeta \to \kappa$ by $g(\xi) = \min\{ \eta : \xi \leqslant \zeta_\eta \}$; then g is continuous so gf is measurable. By the inductive hypothesis, there is an $\eta < \kappa$ such that $0 < \mu(gf)^{-1}[\{\eta\}] \leqslant \mu f^{-1}[\zeta_\eta + 1]$. Now apply the inductive hypothesis again to see that there is a $\xi \leqslant \zeta_\eta$ such that $\mu f^{-1}[\{\xi\}] > 0$.

(c) We are thus left with the case in which ζ is an uncountable regular cardinal; as we are assuming GCH, we have $\zeta \geqslant \mathfrak{c}$ and $2^\zeta = \zeta^+$. Observe that we may take $\mu X = 1$ and also, using Proposition 1D again, we may suppose that $X = \{0,1\}^I$ for some set I . Let $\mathcal{F}$ be the filter of closed unbounded sets in ζ ; enumerate $\mathcal{F}$ as $\langle V_\xi \rangle_{\xi<\lambda}$, where $\lambda = 2^\zeta = \zeta^+$. For $U , V \in \mathcal{F}$ say $U \subseteq_{ess} V$ if $\#(U \setminus V) < \zeta$. If $\mathcal{A} \subseteq \mathcal{F}$ and $\#(\mathcal{A}) \leqslant \zeta$, there is a $U \in \mathcal{F}$ such that $U \subseteq_{ess} V$ for every $V \in \mathcal{A}$; consequently we can choose $\langle U_\xi \rangle_{\xi<\lambda}$ inductively in $\mathcal{F}$ such that $U_\xi \subseteq_{ess} U_\eta$ whenever $\eta < \xi$ and $U_\xi \subseteq_{ess} V_\xi$ for every $\xi < \lambda$.

(d) From this point to the end of the proof let us suppose, if possible, that we have $\mu f^{-1}[\{\xi\}] = 0$ for every $\xi < \zeta$; by the inductive hypothesis, we have $\mu f^{-1}[\xi] = 0$ for every $\xi < \zeta$. It follows that $\mu f^{-1}[V] = 1$ for every $V \in \mathcal{F}$. **P** If $V \in \mathcal{F}$, enumerate V in ascending order as $\langle \theta_\xi \rangle_{\xi<\zeta}$. Write $A_\xi =]\theta_\xi, \theta_{\xi+1}[$ for each $\xi < \zeta$. Then

$$\mathcal{A} = \{\theta_0\} \cup \{ A_\xi : \xi < \zeta \} \cup \{V\}$$

is a partition of ζ such that $\bigcup \mathcal{B}$ is Borel for every $\mathcal{B} \subseteq \mathcal{A}$. So $\{ f^{-1}[A] : A \in \mathcal{A} \}$ is a completely measurable partition of X , and by Lemma 2A

$$1 = \mu f^{-1}[\theta_0] + \Sigma_{\xi<\zeta} \mu f^{-1}[A_\xi] + \mu f^{-1}[V] = \mu f^{-1}[V] \text{ . } \mathbf{Q}$$

(e) Write $\mathcal{E}$ for the set of conegligible Baire sets in X ; as μ is completion regular and $\mu f^{-1}[\zeta \setminus \xi] = 1$ for each $\xi < \zeta$, we can choose $H_\xi \in \mathcal{E}$ such that $H_\xi \subseteq f^{-1}[\zeta \setminus \xi]$. Note that each H_ξ factorizes through $\{0,1\}^J$ for some countable $J \subseteq I$. Now there is a $K \subseteq I$ with $\#(K) \leqslant \zeta$ such that every H_ξ factorizes through $\{0,1\}^K$ and whenever $V \in \mathcal{F}$, $J \subseteq I \setminus K$ and $\#(J) \leqslant \zeta$ there is an $E \in \mathcal{E}$ factorizing through $\{0,1\}^{I \setminus J}$ with $E \subseteq f^{-1}[V]$. **P ?** Otherwise, choose $W_\eta \in \mathcal{F}$, $K(\eta) \subseteq I$, $J(\eta) \subseteq I \setminus K(\eta)$ inductively, for $\eta < \omega_1$, as follows. $K(0)$ is to be such that every H_ξ factors through $\{0,1\}^{K(0)}$ and $\#(K(0)) \leqslant \zeta$. Given

$K(\eta)$, choose $W_\eta \in \mathcal{F}$ and $J(\eta) \subseteq I \setminus K(\eta)$ such that $\#(J(\eta)) \leqslant \zeta$ and there is no $E \in \mathcal{E}$ factorizing through $\{0,1\}^{I \setminus J(\eta)}$ with $E \subseteq f^{-1}[W_\eta]$. For $\eta > 0$, set $K(\eta) = K(0) \cup \bigcup_{\xi<\eta} J(\xi)$; then we shall always have $\#(K(\eta)) \leqslant \zeta$, so that the induction will proceed.

Now there is a $W \in \mathcal{F}$ such that $W \subseteq_{ess} W_\eta$ for every $\eta < \omega_1$, and $\mu f^{-1}[W] = 1$, so there is an $E \in \mathcal{E}$ such that $E \subseteq f^{-1}[W]$. There is a countable $J \subseteq I$ such that E factors through $\{0,1\}^J$; now the $J(\eta)$ are disjoint, so there is an $\eta < \omega_1$ such that $J \cap J(\eta) = \emptyset$. We have $W \setminus W_\eta \subseteq \xi$ for some $\xi < \zeta$, so that $f^{-1}[W_\eta] \supseteq f^{-1}[W] \cap H_\xi \supseteq E \cap H_\xi$. But $E \cap H_\xi \in \mathcal{E}$ and factors through $\{0,1\}^{I \setminus J(\eta)}$. **X Q**

(f) Take a suitable $K \subseteq I$ as in (e) above. For $\xi < \lambda$, take U_ξ from (c). Choose $E_\xi \in \mathcal{E}$, countable $J(\xi) \subseteq I$ inductively for $\xi < \lambda$ such that

$$E_\xi \subseteq f^{-1}[U_\xi] \text{ , } E_\xi \text{ factors through } \{0,1\}^{J(\xi)} \text{ ,}$$
$$J(\xi) \cap (\bigcup_{\eta<\xi} J(\eta) \setminus K) = \emptyset$$

for each $\xi < \lambda$.

We can regard $\{0,1\}^I$ as $\{0,1\}^K \times \{0,1\}^{I \setminus K}$; let μ_K , $\mu_{I \setminus K}$ be the corresponding factor measures. For each $\xi < \lambda$ there is a Baire set $B_\xi \subseteq \{0,1\}^K$ such that $\mu_K(B_\xi) = 1$ and

$$\mu_{I \setminus K}\{ u : (t,u) \in E_\xi \} = 1 \quad \forall \ t \in B_\xi \text{ .}$$

Since $\zeta \geqslant \mathfrak{c}$, the collection of Baire sets of $\{0,1\}^K$ has cardinal $\leqslant \zeta$; so there must be a $B \subseteq \{0,1\}^K$ such that

$$A = \{ \xi : \xi < \lambda , B_\xi = B \}$$

has cardinal λ .

(g) Consider $Y = \bigcap_{\xi \in A} E_\xi$. Then $\mu^* Y = 1$. **P** For $t \in B$,

$$\{ u : (t,u) \in Y \} = \bigcap_{\xi \in A} \{ u : (t,u) \in E_\xi \} \text{ .}$$

Since $\{ u : (t,u) \in E_\xi \}$ factors through $\{0,1\}^{J(\xi) \setminus K}$ and the $J(\xi) \setminus K$ are disjoint,

$$\mu^*_{I \setminus K}(\bigcap_{\xi \in A} \{ u : (t,u) \in E_\xi \}) = \prod_{\xi \in A} \mu_{I \setminus K}\{ u : (t,u) \in E_\xi \} = 1 \text{ .}$$

Of course $\mu_K B = 1$, so

$$\mu^* Y \geqslant \int \mu^*_{I \setminus K}\{ u : (t,u) \in Y \} \mu_K(dt) = 1 \text{ . } \mathbf{Q}$$

(h) Consequently $Y \not\subseteq f^{-1}[\xi]$ for any $\xi < \zeta$, and $\#(f[Y]) = \zeta$. There is therefore a $V \in \mathcal{F}$ such that $\#(f[Y] \setminus V) = \zeta$. Let $\eta < \lambda$ be such that $V = V_\eta$; let $\xi \in A \cap [\eta, \lambda[$, so that

$U_\xi \subsetneq_{ess} V$; then we have $f[Y] \not\subseteq U_\xi$. But $Y \subseteq E_\xi \subseteq f^{-1}[U_\xi]$.

This contradiction shows that $\mu f^{-1}[\{\xi\}] > 0$ for some $\xi < \zeta$, so that the induction continues.

3H Problems. The following questions are left open by the work above.

(a) Let X be a subspace of a Radon measure space such that every subset of X is measurable for the induced quasi-Radon measure. Does the measure on X have to consist of point masses? (This problem arises only if we assume the existence of a real-valued measurable cardinal; 2-valued measurable cardinals are irrelevant because an atom in a T_0 quasi-Radon measure space must be a point mass.)

NO. (DHF 1988)

(b) Does Theorem 3G really require GCH? For all I know, it is true independently of special axioms. If $X = [0,1]$, $\zeta = \omega_1$ then of course much weaker axioms are sufficient (e.g. Martin's Axiom, or the Definable Forcing Axiom of van Douwen and Fleissner); but even in this case I have no proof in ZFC alone.

NO (DHF 1984)

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Mathematics Department
University of Essex
Colchester co4 3sq
England

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